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UNIFORMITIES UNIQUELY DETERMINED BY THEIR UNIFORMLY CONTINUOUS SELF-MAPS

E. MAKAI, JR.

Abstract

We prove for a class of uniform (proximity, resp. topological) spaces that any space of this class is uniquely determined (among all uniform, proximity, resp. topological spaces) by its uniformly (proximally) continuous (resp. continuous) self-maps. This class contains e.g. all Peano continua and the long line ([2], exercises 6J, 15R, 16H) (with the unique uniformity), resp. all countable precompact uniform spaces with discrete topology and zero-dimensional metric completion. Our results largely parallel analogous results for topological spaces (cf. [17]). In fact most of our auxiliary results treat the topological case — usually considering instead of $C(X, X)$ the more general case $C(X, Y)$. As applications we determine the coarsest concrete functors between some subcategories of uniform spaces.

§ 0

Uniform and proximity spaces are not assumed to be separated. The category of uniform, resp. proximity spaces is denoted by Unif resp. Prox.

Definition. A uniform (or proximity) space X is called special if for any uniform (or proximity) space Y on the same underlying set $\hom(Y, Y)=\hom(X, X)\Rightarrow Y=X$.

Remark. [17] defines similarly special topologies. Restricting our attention to non-empty spaces by [11], p. 197, this property is equivalent to the following: the existence of a semigroup-isomorphism i: $\hom(X, X)\to\hom(Y, Y)$, for any space Y, implies the existence of an isomorphism j: $X\to Y$, with $i(f)(y)=j[f(j^{-1}(y))]$.

First we recall some concepts. A topological space is Fréchet—Urysohn if $x\in X\supset A$, $x\in\bar{A}\Rightarrow\exists x_n\in A$, $(n\in N)$, $x_n\to x$, and it is sequential if sequentially closed sets are closed. A topological space is S_1 (S_2, $S_{3\frac{1}{2}}$) if its T_0-reflection is T_1 (T_2, $T_{3\frac{1}{2}}$). A Peano continuum is a connected, locally connected compact metric space. Equivalently (if it is not empty), it is a T_2 continuous image of $[0, 1]$ ([10], § 45, II. 2, p. 185, [1], § 2.10, Prop. 17). Peano continua are arcwise connected ([10], § 45, II. 1, p. 184 and I. 2, p. 182). Pseudocompact is meant to imply $S_{3\frac{1}{2}}$. Pseudocompact spaces with the fine uniformity are just the precompact fine spaces ([7], p. 135). For a uniform space X the generated topology is denoted by τX and the precompact reflection by pX. Proximity spaces will be identified with precompact uniform spaces. The covering character (cov char) of a uniform space X is $\min\{\alpha \mid X$ has a discrete subspace of cardinality $\beta\Rightarrow\beta<\alpha\}+\aleph_0$ ($=\min\{\alpha \mid X$ has a basis of coverings of cardi-

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nalities $<\alpha\}+\aleph_0$) ([7], p. 134). A topology is saturated if arbitrary intersections of open sets are open. γ denotes completion. A concrete functor is a functor between concrete categories commuting with the respective underlying set functors. A map betwen uniform (etc.) spaces is always meant as a morphism of the category in question.

§ 1

The following proposition follows the lines of [17].

PROPOSITION 1. *Let X, Y be topological spaces. Let Y have a family of subspaces $\{Y_\alpha\}$, and for each α a family of filters $\{\mathcal{F}_{\alpha\beta}\}$ on Y_α such that $B\subset Y$ is closed iff* [$\forall\alpha,\beta$, $B\cap Y_\alpha\in\mathcal{F}_{\alpha\beta}\to y\in Y_\alpha$ *implies* $B\cap Y_\alpha\ni y$]. *Let further* $\forall x\in X\ \forall\alpha\ \forall Z\subset Y_\alpha$ *with* ($\exists\beta, y$, $Z\in\mathcal{F}_{\alpha\beta}\to y\in Y_\alpha$, $Z\not\ni y$) $\exists y^*\in Y_\alpha\cap(\bar{Z}\setminus Z)$, $\exists n\in N$, $\exists X_1,\dots,X_n$, $X\setminus\{x\}=\bigcup_1^n X_i$, $\forall i\,(1\leq i\leq n)$ $\exists f\in C(X,Y)$, $f(X_i)\subset Z$, $f(x)=y^*$. *Let X', Y' be other topologies on the underlying sets of X resp. Y, with $C(X,Y)\subset C(X',Y')$. Then either X' is a discrete space or Y' is coarser than Y.*

PROOF. Let $B\subset Y$ be closed in Y'. We show it is closed in Y, too, i.e. $\forall\alpha,\beta$ $B\cap Y_\alpha\in\mathcal{F}_{\alpha\beta}\to y\in Y_\alpha$ implies $B\cap Y_\alpha\ni y$. Denote $Z=B\cap Y_\alpha$, and suppose this implication is false for Z. Suppose X' is not discrete. Then $\exists x\in X$, $\overline{X\setminus\{x\}}^{X'}\ni x$. Thus for one of the X_i-s $(1\leq i\leq n)$, assured by hypothesis, we have $\bar{X}_i^{X'}\ni x$. Since $\exists f\in C(X,Y)\subset C(X',Y')$, $f(X_i)\subset Z$, $f(x)=y^*$, therefore $\bar{B}^{Y'}\supset\bar{Z}^{Y'}\supset\overline{f(X_i)}^{Y'}\ni f(x)=y^*$. Thus $y^*\in\bar{B}^{Y'}\setminus B$, contradicting our assumption $\bar{B}^{Y'}=B$.

COROLLARY 1. *Let X be a $T_{3\frac{1}{2}}$ space in which every point is a G_δ-set. Let Y be a space with unique limits of such sequences which are contained in Peano continua $\subset Y$, and let Y have the weak topology w. r. t. its subspaces which are Peano continua (e.g. Y is T_2, first countable and locally arcwise connected). Then the hypothesis and statement of Proposition 1 hold. The same is valid if X is a zero-dimensional space in which every point is a G_δ-set, and Y is sequential.*

PROOF. In the first case let $\{Y_\alpha\}=\{$subspaces of Y homeomorphic to N^*(= =one point compactification of a countable discrete space)$\}$, $\forall\alpha\ \{\mathcal{F}_{\alpha\beta}\}=\{$cofinite filter on $Y_\alpha\}$, which evidently satisfy the property in Proposition 1. Choose for $x\in X$ an $h\in C(X,[0,1])$ with $h^{-1}(0)=\{x\}$. Let $n=2$,

$$X_1=h^{-1}\Big(\bigcup_{j=1}^{\infty}(2^{-2j+1},2^{-2j+2}]\Big),\quad X_2=h^{-1}\Big(\bigcup_{j=1}^{\infty}(2^{-2j},2^{-2j+1}]\Big).$$

Let $Z\subset Y_\alpha$ satisfy the condition on Z from Proposition 1. Then $Z=\{y_k\}$, in Y_α $\exists\lim y_k=y^*$, $\bar{Z}^{Y_\alpha}=\{y_k\}\cup\{y^*\}$. Since $\{y_k\}$ is not closed in Y, for some Peano continuum $g([0,1])\subset Y$ (where g: $[0,1]\to Y$) $Z=\{y_k\}\cap g([0,1])$ is not closed in $g([0,1])$. Denote this infinite subsequence once more by $\{y_k\}$. Choose $u_k\in g^{-1}(y_k)$. We may suppose $\exists\lim u_k=u$ (otherwise choose a subsequence). Thus by condition $g(u)=y^*$. Let ψ: $[0,1]\to[0,1]$ map $(2^{-2k+1},2^{-2k+2}]$ to u_k; thus $\psi(0)=u$. So $g\psi\big(\bigcup_{j=1}^{\infty}(2^{-2j+1},2^{-2j+2}]\big)\subset\{y_k\}$, $g\psi(0)=y^*$. Then $f=g\psi h$: $X\to Y$ satisfies $f(X_1)\subset Z$, $f(x)=y^*$. In case of X_2 we proceed analogously.

In the second case let $\{Y_\alpha\}=$ {convergent sequences with any of their limits}, $\{\mathcal{F}_{\alpha\beta}\}=$ {filter generated by the tails of the respective convergent sequence}. Choose for $x\in X$ an $h\in C(X, N^*)$ with $h^{-1}(\infty)=\{x\}$. Let $n=1$, $X_1=X\setminus\{x\}$. If $Y_\alpha=$ $=\{y_k\}\cup\{y'\}$, where $y_k\to y$, and Z satisfies the hypothesis of Proposition 1, then Z contains an infinite subsequence $\{y_{i(k)}\}$ of $\{y_k\}$. Let now $\varphi\colon N^*\to Y$, $\varphi(k)=y_{i(k)}$, $\varphi(\infty)=y$. Then $f=\varphi h$ satisfies $f(X_1)\subset Z$, $f(x)=y$.

The remark in the brackets follows from considering an $y\in\bar{A}\setminus A$ $(A\subset Y)$, then choosing $y_n\in A, y_n\to y$, and choosing arcs joining y to y_n, in elements of a neighbourhood base of y.

Proposition 2. *Let X, Y be uniform spaces. Let Y have a family of subspaces $\{Y_\alpha\}$ such that $Z_1\delta Z_2\Leftrightarrow\exists\alpha\ (Z_1\cap Y_\alpha)\delta(Z_2\cap Y_\alpha)$. Let $\mathcal{U}=\{\{A_\beta, B_\beta\}\}$ be a subbasis of uniform coverings of the discrete proximity on X $(A_\beta, B_\beta\neq X)$. Let $\forall\{A_\beta, B_\beta\}\in\mathcal{U}$ $\forall\alpha\ \forall Z_1, Z_2\subset Y_\alpha$ with $Z_1\delta Z_2\ \exists n, m\in N$, $\exists A_1, \dots, A_n$, $B_1, \dots, B_m$, $X\setminus A_\beta=\bigcup_1^n A_i$, $X\setminus B_\beta=\bigcup_1^m B_j$, $\forall i\ (1\leqq i\leqq n)$, $\forall j\ (1\leqq j\leqq m)\ \exists f\in U(X, Y)\ f(A_i)\subset Z_1$, $f(B_j)\subset Z_2$. Let X', Y' be other uniformities on the underlying sets of X resp. Y, with $U(X, Y)\subset$ $\subset U(X', Y')$. Then either pX' is a discrete proximity or pY' is coarser than pY.*

Proof. We have to show pY' is coarser than pY, i.e. $Z_1\delta_Y Z_2\Rightarrow Z_1\delta_{Y'} Z_2$. By the hypothesis on $\{Y_\alpha\}$ $\exists\alpha$ $(Z_1\cap Y_\alpha)\delta_Y(Z_2\cap Y_\alpha)$. Hence for proving the above implication Z_1, Z_2 can be replaced by $Z_1'=Z_1\cap Y_\alpha$, $Z_2'=Z_2\cap Y_\alpha$. Suppose pX' is not a discrete proximity. Since $\mathcal{U}=\{\{A_\beta, B_\beta\}\}$ is a subbasis of the discrete proximity, some $\{A_\beta, B_\beta\}$ is not a uniform cover of pX', i.e. $(X\setminus A_\beta)\delta_{X'}(X\setminus B_\beta)$. For this A_β, B_β we have by hypothesis $X\setminus A_\beta=\bigcup_1^n A_i$, $X\setminus B_\beta=\bigcup_1^m B_j$. Therefore $\exists i, j$ with $A_i\delta_{X'}B_j$. By hypothesis $\exists f\in U(X, Y)\subset U(X', Y')$, $f(A_i)\subset Z_1'$, $f(B_j)\subset Z_2'$. Thus by $A_i\delta_{X'}B_j$ we have $Z_1'\delta_{Y'}Z_2'$, which was to be shown.

Corollary 2. *Let X, Y be uniform spaces. Let $\forall A\subset X$, $\emptyset\neq A\neq X$ $\exists g\colon X\to$ $\to\{1/k\}$ $(\subset[0, 1]$ — where $k\in N)$, $g(A)\cap g(X\setminus A)=\emptyset$. (This class of spaces contains each countable X with τX discrete and is closed under taking subspaces, finer uniformities and sums.) Let for the completion γpY of pY hold: $Z_1, Z_2\subset Y$, $Z_1\delta Z_2\Rightarrow\exists y\in\gamma pY$, $\exists y_{1,l}\in Z_1$, $\exists y_{2,l}\in Z_2$ $(l\in N)$, $y_{1,l}\to y$, $y_{2,l}\to y$. Then the hypothesis and statement of Proposition 2 hold.*

Proof. Let $Z_1, Z_2\subset Y$, $Z_1\delta Z_2$. Then $\exists y_{1,l}\in Z_1$, $y_{2,l}\in Z_2$, $y_{1,l}\to y$, $y_{2,l}\to y$. Thus choose $\{Y_\alpha\}=\{\{y_n\}\subset Y\mid\exists y\in\gamma pY,\ y_n\to y\}$, and also $\mathcal{U}=$ {two-element partitions of X}. Let $A\subset X$ $(\emptyset\neq A\neq X)$. Thus $\{A, X\setminus A\}=\{A_\beta, B_\beta\}\in\mathcal{U}$. By hypothesis $\exists g\colon X\to\{1/k\}$ with the desired properties. Let $m=n=1$. We can suppose $g(X\setminus A_\beta)\subset$ $\subset\{1/(2l-1)\}$, $g(X\setminus B_\beta)\subset\{1/2l\}$. Let $h(1/(2l-1))=y_{1,l}$, $h(1/2l)=y_{2,l}$. Then $f=hg$ satisfies $f(X\setminus A_\beta)\subset Z_1$, $f(X\setminus B_\beta)\subset Z_2$.

The remark in the brackets follows from the fact that, for τX discrete, X is finer than the proximity on X corresponding to the one-point compactification.

Remark 1. The condition of Corollary 2 implies τX is discrete. It would be interesting to determine the class of spaces X satisfying the above condition. A proposition similar to Proposition 2 can be formulated, for uniform spaces, using non-vanishing (=near) systems ([7], p. 86, [6]), or micromerous collections ([15], [9]).

Before the next proposition which goes to the other direction we prove six lemmas, on the lines of [17] (e.g. proof of Theorem 4.1) and [14] (Lemma 2).

LEMMA 1. *Let X, X', resp. Y, Y' be uniformities on the same respective underlying sets, $U(X, Y)\subset U(X', Y')$ (or only $U(X, Y)\subset C(\tau X', \tau Y')$). If τX is discrete or τY is indiscrete, then $\tau X'$ is discrete or $\tau Y'$ is indiscrete. The same holds for p instead of τ (supposing $U(X, Y)\subset U(pX', pY')$).*

PROOF. Supposing $|X|, |Y|>1$, take any $x\in X$, $y_1\neq y_2\in Y$. Then $\exists f\in U(X, Y)$, $f(x)=y_1$, $f(X\setminus\{x\})=\{y_2\}$. Since $f\in C(\tau X', \tau Y')$, either y_1, y_2 are not separated by $\tau Y'$ or x is isolated. For p take, instead of $x\in X$, $\emptyset\neq A\neq X$.

REMARK 2. For the uniform case one can show similarly $U(X, Y)=Y^X\Leftrightarrow$ [$\exists$ infinite cardinal α, each cover of X of cardinality $<\alpha$ is uniform and cov char Y $\leqq\alpha$]$\vee$[Y is indiscrete]. (Use functions $f\colon X\to Y$, with $f(X)\subset Y$ discrete.) In particular $U(X, X)=X^X\Leftrightarrow$[$\exists$ infinite cardinal α, X has for base all partitions of cardinality $<\alpha$]$\vee$[X is indiscrete].

LEMMA 2. *Let X, X', resp. Y, Y' be uniformities on the same respective underlying sets, $U(X, Y)\subset U(X', Y')$ (or only $U(X, Y)\subset C(\tau X', \tau Y')$). Then each pair of points separated by Y' is separated by Y, too, unless $\tau X'$ is a discrete topology. If, moreover $U(X, Y)=U(X', Y')$ (or only $U(X', Y')\subset C(\tau X, \tau Y)$) then the only exceptional case is $\tau X'$ is a discrete topology and either τX is a discrete topology or τY is indiscrete. The same holds for p instead of τ (supposing $U(X, Y)\subset U(pX', pY')$ etc.).*

PROOF. Let Y not separate $y_1\neq y_2\in Y$, separated by Y'. Supposing $|X|>1$, take any $x\in X$. Then $\exists f\in U(X, Y), f(x)=y_1, f(X\setminus\{x\})=\{y_2\}$. Since $f\in C(\tau X', \tau Y')$, $\tau X'$ is a discrete topology. For the remainder use Lemma 1.

REMARK 3. Statements corresponding to Lemmas 1 and 2 hold also for topological spaces (for Lemma 1 comp. [17], quoted just above: for topologies X, X' resp. Y, Y' on the same respective underlying sets $C(X, Y)\subset C(X', Y')$, X is discrete or Y is indiscrete $\Rightarrow X'$ is discrete or Y' is indiscrete; in Lemma 2 separation meaning T_0-separation).

A sharpening of Lemma 2 for the topological case is

LEMMA 3. *Let X, X', resp. Y, Y' be topologies on the same respective underlying sets, $C(X, Y)\subset C(X', Y')$. Then $y_1, y_2\in Y$, $\overline{\{y_1\}}^{Y'}\not\ni y_2\Rightarrow\overline{\{y_1\}}^{Y}\not\ni y_2$, unless (1) X' is finer than the topology with open base {closed sets of X}. If $C(X, Y)=C(X', Y')$, the only exceptional cases are (1.a) X' is discrete and either X is discrete or Y indiscrete, (1.b) the T_0-reflection of X is discrete and X' is the topological sum of a system of its connected subspaces, these connected subspaces of X' being the minimal non-empty open subspaces of X, and (1.c) X is saturated and {open sets of X}={closed sets of X'}.*

PROOF. Let $\overline{\{y_1\}}^{Y'}\not\ni y_2$, $\overline{\{y_1\}}^{Y}\ni y_2$. Let $\emptyset\neq G\subsetneqq X$ be open in X. Then $\exists f\in$ $\in C(X, Y), f(G)=\{y_1\}, f(X\setminus G)=\{y_2\}$. Since $f\in C(X', Y')$, G is closed in X'. Suppose now $C(X, Y)=C(X', Y')$.Denote $Y_0\subset Y$ ($Y'_0\subset Y'$) the subspace $\{y_1, y_2\}$. If $\overline{\{y_2\}}^Y\ni y_1$, then $C(X, Y)\supset Y_0^X$, hence $C(X', Y')\supset Y_0'^{X'}$ and then (1.a) holds. If $\overline{\{y_2\}}^Y\not\ni y_1$, $\overline{\{y_2\}}^{Y'}\not\ni y_1$, then {open sets of X}= {clopen sets of X'}, so in X {open sets}= {closed sets}. Hence $\forall x\in X$ among the open sets of X containing x there is a minimal one, and these form for $\forall x\in X$ an open partition. Thus the T_0-reflection of X is discrete, and (1.b) holds. If $\overline{\{y_2\}}^Y\not\ni y_1$, $\overline{\{y_2\}}^{Y'}\ni y_1$, then (1.c) holds.

For separation in sense of T_1 one proves similarly

Lemma 4. *Let X, X', resp. Y, Y' be topologies on the same respective underlying sets, $C(X, Y)\subset C(X', Y')$. Then each two-point discrete subspace of Y' is a two-point discrete subspace of Y, too, unless* (1) *X' is finer than the topology with open subbase {open sets of X}$\cup$ {closed sets of X}. If $C(X, Y)=C(X', Y')$, the only exceptional cases are* (1.a) *and* (1.b) *of Lemma 3.*

Lemma 5. *Let X, X', resp. Y, Y' be topologies on the same respective underlying sets, $C(X, Y)\subset C(X', Y')$. Let $\{\{A_\lambda, B_\lambda\}|\lambda\in\Lambda\}$ be a set of pairs of subsets of X. Suppose for any choice of $a_\lambda\in A_\lambda$, $b_\lambda\in B_\lambda$, $a_\lambda\neq b_\lambda$ the transitive hull of the relation $\{(f(a_\lambda), f(b_\lambda))|f\in C(X, Y),\ \lambda\in\Lambda\}$ (resp. for each λ of the relation $\{(f(a_\lambda), f(b_\lambda))|f\in C(X, Y)\}$) is Y^2. Then either $\exists\lambda\in\Lambda$ (resp. $\forall\lambda\in\Lambda$) $\forall a\in A_\lambda$ $\forall b\in B_\lambda$ $a\neq b\Rightarrow \overline{\{a\}}^{X'}\not\ni b$, or Y' is indiscrete.*

Proof. Suppose $\forall\lambda\in\Lambda$ $\exists a_\lambda\in A_\lambda$, $\exists b_\lambda\in B_\lambda$, $a_\lambda\neq b_\lambda$, $\overline{\{a_\lambda\}}^{X'}\ni b_\lambda$. Then $\forall f\in C(X', Y')$ $\overline{\{f(a_\lambda)\}}^{Y'}\ni f(b_\lambda)$. Since the transitive hull of the relation $\{(f(a_\lambda), f(b_\lambda))|f\in C(X', Y'),\ \lambda\in\Lambda\}\supset\{(f(a_\lambda), f(b_\lambda))|f\in C(X, Y), \lambda\in\Lambda\}$ is Y^2, Y' is indiscrete. The other case is similar.

Lemma 6. *Let X, X', resp. Y, Y' be uniformities on the same respective underlying sets, $U(X, Y)\subset C(\tau X', \tau Y')$. Let $\{\{A_\lambda, B_\lambda\}|\lambda\in\Lambda\}$ be a set of pairs of subsets of X. Suppose for any choice of $a_\lambda\in A_\lambda$, $b_\lambda\in B_\lambda$, $a_\lambda\neq b_\lambda$ the minimal equivalence relation containing $\{(f(a_\lambda), f(b_\lambda))|\ f\in U(X, Y),\ \lambda\in\Lambda\}$ (resp. for each λ the minimal equivalence relation containing $\{(f(a_\lambda), f(b_\lambda))|f\in U(X, Y)\}$) is the indiscrete one. Then either $\exists\lambda\in\Lambda$ (resp. $\forall\lambda\in\Lambda$) $[a\in A_\lambda, b\in B_\lambda, a\neq b\Rightarrow \overline{\{a\}}^{\tau X}\not\ni b]$, or Y' is indiscrete.*

Proof. Suppose $\forall\lambda\in\Lambda$, $\exists a_\lambda\in A_\lambda$, $\exists b_\lambda\in B_\lambda$, $a_\lambda\neq b_\lambda$, $\overline{\{a_\lambda\}}^{\tau X'}\ni b_\lambda$. Then $\forall f\in C(\tau X', \tau Y')$ $\overline{\{f(a_\lambda)\}}^{\tau Y'}\ni f(b_\lambda)$. Since the minimal equivalence relation containing $\{(f(a_\lambda), f(b_\lambda))|f\in C(\tau X', \tau Y'),\ \lambda\in\Lambda\}\supset\{(f(a_\lambda), f(b_\lambda))|f\in U(X, Y), \lambda\in\Lambda\}$ is Y^2, Y' is indiscrete. The other case is similar.

Remark 4. In Lemmas 1—4 (and Remark 3) we only used functions assuming two values. The conditions could have been weakened accordingly (cf. [13]). Higher (resp. any) separation properties of Y (resp. X) are not reflected similarly by $C(X, Y)$, cf. [4], where for each T_1-space Y a T_3-space X is constructed such that $C(X, Y)=$ = {constant functions: $X\to Y$} (resp. for any connected X choose Y=two-point discrete space).

Proposition 3. *Let X, Y be uniform spaces. Let $A\subset Y\times Y$ consist of pairs separated by Y. Let $\{(f^{-1}(y_1), f^{-1}(y_2))|(y_1, y_2)\in A, f\in U(X, Y)\}$ be a proximity subbase for far pairs of sets in pX. Let X', Y' be other uniformities on the underlying sets of X and Y with $U(X, Y)\subset U(pX', pY')$. Let Y' separate all pairs of points in A*

or $U(pX, pY)\supset U(X', Y')$ *(or only* (1) $C(\tau X, \tau Y)\supset U(X', Y'))$ *or* (2) $X'=Y'$ *and* $\forall (y_1, y_2)\in A$ *the minimal equivalence relation containing* $\{(f(y_1), f(y_2))|f\in U(X, Y)\}$ *is the indiscrete one. Then either* pX *is a discrete proximity and* Y' *is indiscrete (or in case* (1) *only* τX *is discrete and either* Y' *is indiscrete or* $\tau X'$ *is discrete or in case* (2) *only* Y' *is indiscrete) or* X' *is finer than* pX.

PROOF. $U(pX, pY)\supset U(X', Y')$ and $U(X, Y)\subset U(pX', pY')$ imply by Lemma 2 that each pair $(y_1, y_2)\in A$ is separated by Y', too, unless pX is a discrete proximity and either pX' is a discrete proximity (in which case X' is finer than pX) or Y' is indiscrete. If only (1) $C(\tau X, \tau Y)\supset U(X', Y')$ and $U(X, Y)\subset U(pX', pY')$, the same holds unless τX is discrete and either $\tau X'$ is discrete or Y' is indiscrete. In case (2) by Lemma 6 Y' is indiscrete or each pair $(y_1, y_2)\in A$ is separated by $X'=Y'$. Let B_1, B_2 be non-empty far sets in X from the mentioned proximity subbase for X. Then $\exists f\in U(X, Y)\subset U(pX', pY')$ with $f(B_1)=\{y_1\}$, $f(B_2)=\{y_2\}$, $(y_1, y_2)\in A$. Hence B_1, B_2 are far in X', too. Thus X' is finer than pX.

COROLLARY 3. *Let* X, Y *be uniform spaces,* $Y\supset[0, 1]$. *Let* X', Y' *be other uniformities on the underlying sets of* X *and* Y, $U(X, Y)\subset U(pX', pY')$, Y' *separating some pair of points in* $[0, 1]$ *or* $U(pX, pY)\supset U(X', Y')$ *(or only* (1) $C(\tau X, \tau Y)\supset$ $\supset U(X', Y'))$. *Then the hypothesis and statement of Proposition 3 hold. If the* T_0-*reflection of* τY *is arcwise connected,* $U(X, Y)\subset U(pX', pY')$, *then either* Y' *is indiscrete or* X' *is finer than* pX. *The same holds for* $\delta dX=0$ *and* $U(X, Y)\subset U(pX', pY')$.

PROOF. In the last two cases if Y is indiscrete, use Lemma 1. If Y is not indiscrete, supposing Y' also not indiscrete, $\exists y_1\neq y_2\in Y$ separated both by Y and Y'. (In fact, suppose each y_1, y_2 separated by Y is not separated by Y'. Then for each y_1, y_2 not separated by Y we have a y_3, separated by Y from y_1, y_2. Thus by hypothesis, in Y' y_1, y_3, resp. y_2, y_3 are not separated, hence y_1, y_2 are not separated either.) Let now $A=\{(y_1, y_2)\}$. If the T_0-reflection of τY is arcwise connected, there is an arc in τY joining y_1 and y_2 and thus the subbase hypothesis is evidently satisfied. If $\delta dX=0$, any two far sets can be separated by a uniformly continuous function to $\{y_1, y_2\}$, thus the subbase hypothesis is satisfied once more. Now we can finish the proof like at Proposition 3 (taking B_1, B_2, etc.).

The following proposition is related to [17], Theorem 1.1, [11], Ch. 1, Theorem 2.3.

PROPOSITION 4. *Let* X, Y *be topological spaces. Let* $A\subset B\subset Y$, *and* $a\in A\Rightarrow$ $\Rightarrow \overline{\{a\}}^Y\cap B=\{a\}$. *Let* $\{f^{-1}(y)|y\in A, f\in C(X, Y), f(X)\subset B\}$ *be a closed subbase of* X. *Let* X', Y' *be other topologies on the underlying sets of* X *and* Y, $C(X, Y)\subset C(X', Y')$. *Let* $\forall a\in A$ $\overline{\{a\}}^{Y'}\cap B=\{a\}$, *or* $C(X, Y)=C(X', Y')$, or (1) $X'=Y'$, $\forall a\in A$, $\forall b\in B$, $b\neq a$ *the transitive hull of the relation* $\{(f(a), f(b))|f\in C(X, Y)\}$ *be* Y^2. *Then* X *is discrete and* Y' *indiscrete or* X *is saturated,* $\{$*open sets of* $X\}=\{$*closed sets of* $X'\}$, *or* X' *is finer than* X *(or in case* (1) Y' *is indiscrete or* X' *is finer than* X).

PROOF. Analogously to Proposition 3, we apply Lemmas 3 and 5. (Note that X' saturated, {open sets of X'}={closed sets of X} imply X is saturated, too, and {open sets of X}={closed sets of X'}.) We obtain that — apart from the cases listed in the statement of the proposition (except the last ones, i.e. X' finer than X) and apart from the case (2): X is the topological sum of its non-empty connected subspaces

X_α, X' being the topological sum of X'_α-s, where X'_α is X_α with indiscrete topology — we have $\forall a \in A\ \overline{\{a\}}^Y \cap B = \{a\}$. By hypothesis then in fact X' is finer than X.

Now we show in case (2) X' is finer than X. We have $C(X, Y) = C(X', Y')$. This implies $\forall a\ C(X_\alpha, Y) = C(X'_\alpha, Y')$. If $|B| = 1$, by hypothesis X is indiscrete, hence X' is finer than X. If $|B| > 1$, then $\{f_\alpha^{-1}(y) | y \in A,\ f_\alpha \in C(X_\alpha, Y),\ f_\alpha(X_\alpha) \subset B\}$ is a closed subbase of X_α (since for $f \in C(X, Y)$, $f(X) \subset B$ we have $f^{-1}(y) \cap X_\alpha = f_1^{-1}(y) = (f|_{X_\alpha})^{-1}(y)$, where $f_1|_{X_\alpha} = f|_{X_\alpha}$, $f_1(X \setminus X_\alpha) = \{y_1\} \neq \{y\}$, $y_1 \in B$).

Denote $\{Y'_\beta\}$ the maximal subsets of Y', not (T_0-) separated by Y', and Y_β the corresponding subspaces of Y. Then $C(X'_\alpha, Y') = \bigcup_\beta Y_\beta'^{X'_\alpha}$, so $C(X_\alpha, Y) = \bigcup_\beta Y_\beta^{X_\alpha}$. This implies $C(X_\alpha, Y_\beta) = Y_\beta^{X_\alpha}$, hence by the topological version of Lemma 1 (cf. Remark 3) $\forall \beta \forall \alpha\ Y_\beta$ is indiscrete or X_α is discrete (and non-empty connected), i.e. $|X_\alpha| = 1$. If $\forall \alpha\ |X_\alpha| = 1$, then $X = X' =$ discrete space, thus X' is finer than X. Otherwise

(*) $\forall \beta\ Y_\beta$ is indiscrete.

If $\forall \alpha\ X_\alpha$ is indiscrete, $X = X'$, thus we are done. Suppose $\exists \alpha$, X_α is not indiscrete. Since $\{f_\alpha^{-1}(y) | y \in A,\ f_\alpha \in C(X_\alpha, Y),\ f_\alpha(X_\alpha) \subset B\}$ is a subbase of X_α, $\exists f_\alpha \in C(X_\alpha, Y)$, $B \supset f_\alpha(X_\alpha) \supsetneqq \{y\}$, for some $y \in A$, $\overline{\{y\}}^Y \cap B = \{y\}$. But $f_\alpha \in C(X'_\alpha, Y')$, hence $f_\alpha(X'_\alpha) \subset Y'$ is indiscrete. By (*) $f_\alpha(X_\alpha) \subset Y$ is indiscrete, too, which contradicts $f_\alpha(X_\alpha) \supsetneqq \{y\}$, $\overline{\{y\}}^Y \cap f_\alpha(X_\alpha) = \{y\}$.

Corollary 4. *Let X, Y be topological spaces, $X\ S_{3\frac{1}{2}}$, $Y \supset [0, 1]$. Let X', Y' be other topologies on the underlying sets of X and Y, $C(X, Y) \subset C(X', Y')$. Let some $y \in [0, 1]$ be closed in the subspace of Y' corresponding to $[0, 1]$, or $C(X, Y) = C(X', Y')$, or (1) $X' = Y'$, $X\ T_{3\frac{1}{2}}$, $\forall y_1, y_2 \in Y$ being contained in some image of $[0, 1]$ in Y. Then the hypothesis and statement of Proposition 4 hold. If X is zero-dimensional, $C(X, Y) \subset C(X', Y')$, then Y' is indiscrete or X' is finer than X.*

Proof. In case (1) $\forall x_1 \neq x_2 \in X\ \ \forall c_1, c_2 \in [0, 1]\ \ \exists g \colon X \to [0, 1]$, $g(x_1) = c_1$, $g(x_2) = c_2$. Since $\forall y_1, y_2 \in Y$ we have $y_1, y_2 \in \varphi[0, 1]$, $\varphi \colon [0, 1] \to Y$, thus for some $c_1, c_2 \in [0, 1]$ $y_i = \varphi(c_i) = \varphi g(x_i)$, $\varphi g \colon X \to Y$. Thus $\{(f(x_1), f(x_2)) | f \in C(X, Y)\} = Y^2$.

For zero-dimensional X, supposing Y' not indiscrete, $\exists y_1 \neq y_2$ separated by Y'. Let $B = \{y_1, y_2\}$, and suppose e.g. $\overline{\{y_1\}}^{Y'} \not\ni y_2$.

If $\overline{\{y_1\}}^Y \not\ni y_2$, apply Proposition 4 with $A = \{y_1\}$. If $\overline{\{y_1\}}^Y \ni y_2$, by Lemma 3 X' is finer than the topology with open base {closed sets of X}. However, X is zero-dimensional, hence the clopen sets constitute a base for X. Thus X' is finer than the topology with open base {clopen sets of X}, i.e. X.

§ 2

Theorem 1. *Let X be a uniform space (resp. α) precompact fine uniform space), and let X' be another uniformity on the same underlying set. Let τX satisfy the hypotheses of Proposition 1 (τX playing the role of both X and Y). Let $C(\tau X, \tau X) \subset C(\tau X', \tau X')$ or let X be fine. Let X, X' satisfy with some $A \subset X \times X$ the hypotheses of Proposition 3 (with $Y = X$, $Y' = X'$). Then either pX is a discrete proximity and X'*

is indiscrete (or in case (1) *only τX is discrete and $\tau X'$ is either indiscrete or discrete or in case* (2) *only X' is indiscrete) or $\tau X'$ is discrete or $\tau X=\tau X'$ and X' is finer than pX (resp. in case* α) *even $X'=X$). If $C(\tau X, \tau X)\supset U(X', X')$, the case $\tau X'$ is discrete can be replaced by pX' is a discrete proximity and X is indiscrete.*

PROOF. X' is finer than pX by Proposition 3, unless pX is a discrete proximity and X' is indiscrete (or correspondingly in cases (1) and (2)). If X is fine, $C(\tau X, \tau X)=$ $=U(X, X)\subset U(pX', pX')\subset C(\tau X', \tau X')$. Hence anyway by Proposition 1 $\tau X'$ is coarser than τX, or else $\tau X'$ is discrete. If $C(\tau X, \tau X)\supset U(X', X')$ and $\tau X'$ is discrete, then by Lemma 1 τX is discrete (thus $\tau X'$ is coarser than τX) or indiscrete; if here τX is indiscrete, then $X^X=U(pX', pX')$ and by Lemma 1 pX' is indiscrete or is a discrete proximity. Lastly, if X' is finer than pX and $\tau X'$ is coarser than τX, then $\tau X=\tau X'$, and if X is precompact fine, then $X=X'$.

Taking into account that non-degenerate Peano continua contain arcs, [$\delta dX=$ $=0\Rightarrow\tau X$ is zero-dimensional], and [X is $S_{3\frac{1}{2}}$, $\forall x\in X$ $\{x\}$ is $G_\delta\Rightarrow X$ is T_0, hence $T_{3\frac{1}{2}}$], we have

COROLLARY 5. *Let X be a pseudocompact space with fine uniformity, which is not a finite discrete space or an indiscrete space. Let each point of X be a G_δ-set, and* A) *let τX have the weak topology w.r.t. its subspaces which are Peano continua (e.g. be T_2, first countable and locally arcwise connected, cf. Corollary 1) or* B) *let $\delta dX=0$ and τX be sequential. Let X' be another uniformity on the underlying set of X, $U(X', X')\subset$ $\subset U(X, X)\subset U(pX', pX')$. Then $X'=X$, thus X is a special uniform space. In case* A) *if X is also arcwise connected and in case* B) *even $U(X, X)\subset U(pX', pX')$, $\tau X'$ non-discrete, non-indiscrete imply $X'=X$.*

Analogously one has

THEOREM 2. *Let X be a uniform space, and let X' be another uniformity on the same underlying set (resp.* α) *let X be precompact and either X have the finest uniformity compatible with its proximity — i.e. by* [8] *X has no subspace X_1 which is a countable discrete proximity space, and also is a retract of a proximal neighbourhood of itself — or X' be a precompact uniform space, too). Let X satisfy the hypotheses of Proposition 2 (X playing the role of both X and Y), and X, X' satisfy with some $A\subset X\times X$ the hypotheses of Proposition 3 (with $X=Y$, $X'=Y'$, the case* (1) *in Proposition 3 omitted). Then either pX is a discrete proximity and X' is indiscrete (or in case* (2) *only X' is indiscrete) or pX' is a discrete proximity or $pX'=pX$ (resp. in case* α) *$X'=X$). If $U(pX, pX)\supset U(X', X')$ holds, the case pX' is a discrete proximity can be replaced by pX' is a discrete proximity and X is indiscrete.*

PROOF. pX' is finer than pX by Proposition 3, unless pX is a discrete proximity and X' is indiscrete (and correspondingly in case (2)). By Proposition 2 (applied for X playing the role of X, Y, and pX' playing the role of X', Y') either pX' is a discrete proximity or pX' is coarser than pX. If $U(pX, pX)\supset U(X', X')$ and pX' is a discrete proximity, then by Lemma 1 pX is a discrete or indiscrete proximity. The case α) is evident.

Take into account the characterization in [8] mentioned in Theorem 2 and the fact that X precompact, $\tau\gamma X$ sequential and $X\supset X_1$, X_1 countable discrete proximity lead to the contradiction $\exists\{x_i\}\subset X_1$, $x_i\to x\in\gamma X$. Considering also Remark 1, we have

COROLLARY 6. *Let X be a precompact uniform space, which is not a discrete proximity space. Let X satisfy the hypotheses for both X and Y in Corollary 2, $\delta dX=0$. Let X' be another uniformity on the underlying set of X. Let either $\tau\gamma X$ be sequential, or X have the finest uniformity compatible with its proximity, or X' be precompact. Let $U(X, X)\subset U(pX', pX')$, pX' non-discrete, non-indiscrete proximity. Then $X'=X$, thus X is a special uniform space.*

REMARK 5. We do not know if in the last sentence of Corollary 5, resp. in Corollary 6 $\tau X'$ discrete topology, resp. pX' discrete proximity implies $\exists\alpha$, X' has for base all partitions of cardinality $<\alpha$.

THEOREM 3. *Let X, Y be uniform spaces, and let X', Y' be other uniformities on the underlying sets of X and Y. Let $\tau X, \tau Y$ satisfy the conditions of Proposition 1 (playing the roles of X and Y) and let Y be compact. Let either X be a fine uniformity and $U(X, Y)\subset C(\tau X', \tau Y')$ or pX be a fine proximity and $U(pX, pY)\subset C(\tau X', \tau Y')$ or $C(\tau X, \tau Y)\subset C(\tau X', \tau Y')$. Let either Y' separate all pairs of points separated by Y or let $U(pX, pY)\supset U(X', Y')$ (or only α) $C(\tau X, \tau Y)\supset U(X', Y')$). Then either pX is a discrete proximity (in case α) only τX is discrete) and Y' is indiscrete or $\tau X'$ is discrete or $Y'=Y$.*

PROOF. The conditions imply $C(\tau X, \tau Y)\subset C(\tau X', \tau Y')$. Thus by Proposition 1 $\tau Y'$ is coarser than τY, or else $\tau X'$ is discrete. $U(pX, pY)\supset U(X', Y')$ (resp. $C(\tau X, \tau Y)\supset U(X', Y')$) implies by Lemma 2 that each pair of points separated by Y is separated by Y', too, unless pX is a discrete proximity (resp. τX is discrete). If pX is a discrete proximity or only τX is discrete, by the topological version of Lemma 1 (cf. Remark 3) $C(\tau X, \tau Y)\subset C(\tau X', \tau Y')\Rightarrow \tau X'$ is discrete or Y' is indiscrete. If each pair of points separated by Y is separated by Y', too, by the compactness of Y $Y'=Y$.

Turning to the topological case one has

THEOREM 4. *Let X be a topological space. Let X satisfy the hypotheses of Proposition 1 (X playing the role of both X and Y). Let X' be another topology on the same underlying set, X, X' satisfying with some $A\subset B\subset X$ the hypotheses of Proposition 4 (with $X=Y$, $X'=Y'$). Then either X' is discrete or X is discrete with X' indiscrete or X is saturated with $\{$open sets of $X'\}=\{$closed sets of $X\}$ or $X=X'$ (or in case* (1) *X' is discrete or indiscrete or $X'=X$). If $C(X, X)=C(X', X')$ holds, the case X' is discrete can be replaced by X' is discrete and X is indiscrete.*

PROOF. Apply Propositions 1 and 4, and for the last sentence the topological analogue of Lemma 1.

From here, using Corollaries 1 and 4 we have

COROLLARY 7. *Let X be a non-discrete, non-indiscrete topological space. Let each point of X be a G_δ-set, and* A) *let X be $T_{3\frac{1}{2}}$, and have the weak topology w.r.t. its subspaces which are Peano continua or* B) *let X be zero-dimensional and sequential. Then X is special. In case* A) *if also X is arcwise connected and in case* B) *even $C(X, X)\subset C(X', X')$ (X' another non-discrete, non-indiscrete topology on the underlying set of X) implies $X=X'$.*

REMARK 6. Theorem 4 is closely related to [17], Theorem 4.5. In Corollary 7 A) is related to [17], Theorems 4.6, 3.3 and 4.7, while B) is a generalization of [17], Theorem 4.9.

REMARK 7. For separated uniform (proximity) spaces, resp. T_1 topological spaces the assumptions of our Propositions 3, 4, Theorems and Corollaries simplify, instead of two-sided inclusions we only need to suppose the inclusions of the type $\hom(FX, FY) \subset \hom(F'X', F'Y')$. Opposite inclusions do not suffice in general, e.g. $C(X, X)$ (or $U(X, X)) \supset C(X', X')$ for any rigid topology X' (i.e. $C(X', X') = \{\text{identity}\} \cup \{\text{constant maps}\}$, cf. [5]).

§ 3

Using the above results we can determine the coarsest concrete functors between any two of the categories {uniform spaces}, {proximity spaces}, $\{S_{3\frac{1}{2}}$ spaces} (separated or not). We consider $S_{3\frac{1}{2}}$ spaces as embedded in Unif by fine proximities. A subcategory $\mathscr{C}$ is meant to be a full one.

PROPOSITION 5. *Let $\mathscr{C} \subset$ {separated uniform spaces}, $\exists Y \in \mathrm{Ob}\,\mathscr{C}$, $[0, 1] \subset Y$. Then for every concrete functor F: $\mathscr{C} \to$ {separated uniform spaces} we have: $\forall X \in \mathrm{Ob}\,\mathscr{C}$ FX is finer than pX.*

PROOF. Using Corollary 3 we see FY is finer than pY (it cannot be indiscrete). For each $X \in \mathrm{Ob}\,\mathscr{C}$ pX is generated by $U(X, [0, 1]) \subset U(X, Y)$. We have $U(X, Y) \subset U(FX, FY)$. Denote Z the subspace of FY corresponding to $[0, 1]$; thus Z is finer than $p[0, 1] = [0, 1]$. We have $U(X, [0, 1]) \subset U(FX, Z) \subset U(FX, [0, 1])$. Thus the statement follows.

REMARK 8. Proposition 5 implies a similar "extremal property" of the precompact reflection — among concrete reflections in {separated uniform spaces} — which is given in [16], Theorem 3.5.

PROPOSITION 6. *Let $\mathscr{C} \subset$ Unif, $\exists Y \in \mathrm{Ob}\,\mathscr{C}$, Y is not indiscrete. Then for every concrete functor F: $\mathscr{C} \to$ Unif we have: $\forall X \in \mathrm{Ob}\,\mathscr{C}$ FX is indiscrete, or $\forall X \in \mathrm{Ob}\,\mathscr{C}$ FX is finer than the reflection $RX =$ [uniformity on X with base the finite uniform partitions on X] of X to $\{X' | X'$ has a basis consisting of finite partitions}.*

PROOF. Choose a non-indiscrete $Y \in \mathrm{Ob}\,\mathscr{C}$ with non-indiscrete FY and $y_1 \neq y_2 \in Y$ separated both by Y and FY, like in the proof of Proposition 3. This choice is possible since otherwise for each non-indiscrete $Y \in \mathrm{Ob}\,\mathscr{C}$ — or for at least one space Y with $pFY \neq$ discrete proximity — and each indiscrete $X \in \mathrm{Ob}\,\mathscr{C}$ $U(Y, X) \subset U(FY, FX)$, which implies by Lemma 1 FX is indiscrete. Now for any $X \in \mathrm{Ob}\,\mathscr{C}$ RX is generated by $U(X, \{y_1, y_2\}) \subset U(X, Y)$. However, RFX is generated by $U(FX, \{y_1, y_2\})$, or by the hypothesis on y_1, y_2 by $U(FX, Z) \subset U(FX, FY)$ where Z is the subspace of FY consisting of y_1, y_2. By $U(X, Y) \subset U(FX, FY)$ we have $U(X, \{y_1, y_2\}) \subset U(FX, Z)$, too. Here $U(X, \{y_1, y_2\})$ generates RX, $U(FX, Z)$ generates RFX, thus RX is coarser than RFX, which is coarser than FX.

PROPOSITION 7. *Let $\mathscr{C} \subset$ Unif, $\exists Y \in \mathrm{Ob}\,\mathscr{C}$, Y is not separated. Then for every concrete functor F: $\mathscr{C} \to$ {separated uniform spaces} we have: $\forall X \in \mathrm{Ob}\,\mathscr{C}$ FX is finer than the discrete proximity.*

PROOF. Let $y_1 \neq y_2 \in Y$ be not separated by Y. Let $X \in \mathrm{Ob}\,\mathscr{C}$. For any $\emptyset \neq A \subsetneq X$ define $f \in U(X, Y) \subset U(FX, FY)$, $f(A) = \{y_1\}$, $f(X \setminus A) = \{y_2\}$. Since FY is separated, this means $\{A, X \setminus A\}$ is a uniform cover of FX, which implies the statement.

REMARK 9. An (in fact most general) example is FX=[coarsest common refinement of GX and the discrete proximity on X], where G is any concrete functor to Unif (comp. [7], p. 79) (note that FX is separated). Also in Propositions 5, 6, 7 if the codomain of the functor is (separated) proximity resp. $S_{3\frac{1}{2}}$ spaces, rather than (separated) uniform spaces, the coarsest non-indiscrete concrete functor is the same, resp. $\tau\circ$[the same].

PROPOSITION 8. *Let $\mathscr{C} \subset$ {indiscrete spaces} (resp.* Set*), and $F: \mathscr{C} \to$* Unif *be a concrete functor. Then $\forall X \in \mathrm{Ob}\,\mathscr{C}$ FX=indiscrete space over X, or $\forall X \in \mathrm{Ob}\,\mathscr{C}$ FX=discrete space over X, or $\forall X \in \mathrm{Ob}\,\mathscr{C}$ FX has for base all partitions of X of cardinality $<\alpha$, α an infinite cardinal depending only on F.*

PROOF. Suppose some FY is not indiscrete, say, separates $y_1 \neq y_2$. Then $\forall X \in \in \mathrm{Ob}\,\mathscr{C}$ and $\forall \emptyset \neq A \subsetneq X$ $f \in U(X, Y) \subset U(FX, FY)$, where $f(A) = \{y_1\}$, $f(X \setminus A) = = \{y_2\}$, thus FX is separated. $U(X, X) \subset U(FX, FX)$ implies thus by Remark 2 that $FX = X_\alpha$=uniformity on X with base all covers of cardinality $<\alpha$, for some infinite cardinal $\alpha = \alpha(X)$. If for each X $\alpha(X) > |X|$, X_α is discrete for each X.

Now for any X, Y we have $Y^X = U(X, Y) \subset U(FX, FY) = U(X_{\alpha(X)}, Y_{\alpha(Y)})$. Note that the inverse images of partitions of Y of cardinalities $<\alpha(Y)$ with any functions $f \in Y^X$ are just the partitions of X of cardinalities $<\min(\alpha(Y), |X|^+)$. Thus we have $\min(\alpha(Y), |X|^+) \leqq \alpha(X)$, and also conversely $\min(\alpha(X), |Y|^+) \leqq \leqq \alpha(Y)$. This means one of the following possibilities hold: $[|X|^+ \leqq \alpha(X)$ and $|Y|^+ \leqq \alpha(Y)]$, $|X|^+ \leqq \alpha(X) \leqq \alpha(Y)$, $|Y|^+ \leqq \alpha(Y) \leqq \alpha(X)$ or $\alpha(X) = \alpha(Y)$. We may suppose Y satisfies $\alpha(Y) \leqq |Y|$, while X is arbitrary. This leaves the only possibilities $\alpha(X) = \alpha(Y)$ and $|X|^+ \leqq \alpha(X) \leqq \alpha(Y) \leqq |Y|$. In the first case we are done. In the second case $X_{\alpha(X)}$ is discrete, and equals $X_{\alpha(Y)}$, thus we showed the assertion.

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ON FINITE FIXED CENSORING

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1. Introduction

In the usual censoring problem $X_1, ..., X_n, ...$ are i.i.d. random variables with unknown distribution function F on the probability space $(\Omega, \mathscr{A}, \mathsf{P})$. There is given an other so called censoring sequence $L_1, ..., L_n, ...$ which is either a sequence of numbers or a sequence of random variables. If the censoring sequence is random than it is assumed to be independent of $\{X_i\}$. Set $Z_i = \min\{X_i, L_i\}$, $\delta_i = [X_i \leqq L_i]$ for $i = 1, ..., n$ where $[A]$ denotes the indicator function of the set A. One way to estimate F from the sample $\{(Z_i, \delta_i)\}_{i=1}^n$ is by means of the F_n^* product limit (PL) estimator (Kaplan—Meier [4]). It is known that the PL estimator is the maximum likelihood estimator of the distribution F. In this paper it will be considered such a case when the censoring is fixed and having finite values on the interval $(-\infty; T]$. A paper of P. Meier [5] deals with the fixed censorship model. He pointed out the fact that this model is more applicable than the random censorship one. At the same time if a theorem is valid for the fixed censorship case, it can be proved for the random censorship case and the assumption of the independency of $\{Y_i\}$ is not necessary.

In the first part of the paper a Glivenko—Cantelli type theorem is given for the finite valued fixed censoring case, and then as a consequence of it, a similar theorem is given for the pairwise independent not identically distributed random censoring case. To estimate F on the interval $(-\infty; T]$ it is necessary that $\sum_{i=1}^n [L_i > t] \to \infty$ for all $t \in (-\infty; T]$ but further condition on the order of the above sum is superfluous for a Glivenko—Cantelli type theorem, as it is shown by Theorem 1.1. Corollary 1.1 states the same result for random pairwise independent censoring sequence. In the second part an exponential bound is given for the probability $\mathsf{P}(\sup |F_n^* - F| > \varepsilon)$, and a convergence rate of $\sup |F_n^* - F|$ is given under the condition that $\dfrac{\log n}{\sum_{i=1}^n [L_i > t]} \to 0$. Similar statements hold for the independent random censoring case, as it is shown in Corollaries 2.1 and 2.2. The condition $\dfrac{\log n}{\sum_{i=1}^n [Y_i > T]} \to 0$ is supposed in the random censoring case by Földes [1] dealing with the rate of convergence of

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$\sup|F_n^*-F|$. The really interesting results in this fixed censoring model are that this condition on the order of $\sum_{i=1}^{n}[L_i>T]$ is not necessary, and — as a consequence — the independency of the censoring random variables can be dropped in random censoring models.

1. Glivenko—Cantelli type theorem

Let $X_1, X_2, \dots, X_n, \dots$ independent i.i.d. r.v.-s with right continuous distribution function $F(x)=\mathrm{P}(X_k\leqq x)$, and survival function $\bar{F}(x)=1-F(x)$, $L_1, L_2, \dots, L_n$ is the censoring sequence. $Z_i=\min(X_i, L_i)$, $\delta_i=[X_i\leqq L_i]$, where $[A]$ denotes the indicator function of the set A.

For the definition of the Kaplan—Meier estimator let $Z_{(1)}\leqq Z_{(2)}\leqq\dots\leqq Z_{(n)}$ be the ordered sample where the ordering means that if for $i<j$ $Z_{(i)}=Z_{(j)}$ then $\delta_{(i)}\geqq\delta_{(j)}$, $\delta_{(l)}$ denotes the δ belonging to $Z_{(l)}$.

Definition ([4]). The *product limit* (PL) *estimate* of F, from the sample $\{(Z_i, \delta_i)\}_{i=1}^{n}$ is the following

$$\bar{F}_n^*(t)=1-F_n^*(t)=\begin{cases}\prod\limits_{j:Z_{(j)}\leqq t}\left(\dfrac{n-j}{n-j+1}\right)^{\delta_{(j)}} & \text{if } t\leqq Z_{(n)},\\ 0 & \text{if } t>Z_{(n)}.\end{cases}$$

To state the theorem we need

$$L(t)=L(t,n)=\sum_{i=1}^{n}[L_i\geqq t],\qquad L^+(t)=L^+(t,n)=\sum_{i=1}^{n}[L_i>t]$$

(1.1)

$$N(t)=N(t,n)=\sum_{i=1}^{n}[Z_i\geqq t],\qquad N^+(t)=N^+(t,n)=\sum_{i=1}^{n}[Z_i>t].$$

Theorem 1.1. *Suppose that on the interval* $(-\infty; T]$ *the following conditions hold:*
(i) $1-F(T^-)>0$;
(ii) *the censoring sequence* $\{L_i\}_{i=1}^{\infty}$ *has* $0\leqq K<+\infty$ *different values on* $(-\infty; T]$;
(iii) $\lim\limits_{n\to\infty} L(T,n)=+\infty$.

Then

$$\sup_{-\infty<x\leqq T}|F_n^*(x)-F(x)|\xrightarrow{a.s.}0.$$

In the *random censoring model* we suppose that $\{L_i\}_{i=1}^{\infty}$ is a realization of a sequence $\{Y_i\}_{i=1}^{\infty}$ of random variables. It is also assumed that $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ are independent sequences. In this case the statistician can observe the sequence of pairs

$$(Z_i=\min(X_i; Y_i);\quad \delta_i=[X_i\leqq Y_i])\quad i=1,\dots,n,$$

and F_n^* is the same as above.

It is not necessary to suppose that $\{Y_i\}_{i=1}^{\infty}$ is an independent sequence, we need only condition (iii) of Theorem 1.1, with probability 1. We shall show that the Erdős—Rényi form of Borel—Cantelli theorem reduces (iii) to a more plausible form, i.e. condition (i) and (iv) of Corollary 1.1. The condition (iii) of Theorem 1.1 holds in many other cases, i.e. if we have a law of large numbers for the random variables $\{[Y_i \geqq T]\}_{i=1}^{\infty}$.

COROLLARY 1.1. *Suppose that in the random censoring model the following conditions hold:*

(i) *the r.v.-s* $\{Y_i\}_{i=1}^{\infty}$ *are pairwise independent and the random pairs* $\{(Z_i;\ \delta_i)\}_{i=1}^{\infty}$ *are independent;*

(ii) *at the point* $T > -\infty$, $1 - F(T^-) > 0$;

(iii) *the r.v.-s* $\{Y_i\}_{i=1}^{\infty}$ *have* $0 \leqq K < +\infty$ *different values on the interval* $(-\infty;\ T]$;

(iv) $\sum_{i=1}^{\infty} \mathsf{P}(Y_i \geqq T) = \sum_{i=1}^{\infty} \bar{G}_i(T^-) = +\infty$.

Then with probability 1

$$\sup_{-\infty < x \leqq T} |F_n^*(x) - F(x)| \to 0.$$

PROOF of the Corollary 1.1. Consider the sample $\{(Z_i;\ \delta_i)\}_{i=1}^{n}$ and the PL estimator $F_n^*(x)$ under the condition that given the sequence of censoring variables. If

$$\{Y_i(\omega)\}_{i=1}^{\infty} = \{L_i\}_{i=1}^{\infty}, \quad \sum_{i=1}^{n} [L_i \geqq T] = \sum_{i=1}^{n} [Y_i(\omega) \geqq T] \to \infty$$

then using Theorem 1.1 we have

$$\mathsf{P}\big(\lim_{n\to\infty} \sup_{-\infty < x \leqq T} |F_n^*(x) - F(x)| = 0 \big| \{Y_i(\omega)\}_{i=1}^{\infty}\big) = 1.$$

The Erdős—Rényi form of the Borel—Cantelli theorem (see, e.g. Rényi [6]) can be applied for the events $[Y_i \geqq T]$. It follows by Conditions (i) and (iv) that with probability 1 $\sum_{i=1}^{\infty} [Y_i(\omega) \geqq T] = +\infty$. Hence Condition (iii) of Theorem 1.1 fulfils with probability 1, for the sequences $\{Y_i(\omega)\}_{i=1}^{\infty}$. Thus

$$\mathsf{P}\big(\lim_{n\to\infty} \sup_{-\infty < x \leqq T} |F_n^*(x) - F(x)| = 0\big) =$$
$$= \int_{\Omega} \mathsf{P}\big(\lim_{n\to\infty} \sup_{-\infty < x \leqq T} |F_n^*(x) - F(x)| = 0 \big| \{Y_i(\omega)\}_{i=1}^{\infty}\big)\, d\mathsf{P}(\omega) = 1. \quad \square$$

The proving method of Theorem 1.1 is similar to that of the papers [2], [3]. We shall use similar notations for the proof.

Let $x \in (-\infty;\ T]$ arbitrary. A *partition belonging to* x means a partition

$$\xi_0 = -\infty < \xi_1 < \ldots < \xi_J = x$$

of the interval $(-\infty;\ x]$, which contains all of the different values in $(-\infty;\ x]$ of the censoring sequence as a point of the partition.

Set

$$p_j = \mathsf{P}(X \geqq \xi_j | X > \xi_{j-1}), \quad \tilde{p}_j = \mathsf{P}(X > \xi_j | X \geqq \xi_j) \quad (j = 1, \ldots, J)$$

$$1 - F(x) = \mathsf{P}(X > x) = \prod_{j=1}^{J} p_j \tilde{p}_j, \tag{1.2}$$

$$1 - F(x^-) = \mathsf{P}(X \geqq x) = p_J \prod_{j=1}^{J-1} p_j \tilde{p}_j.$$

Using the notation

$$(1.3)\qquad \hat{p}_j = \prod_{\substack{l:1\le l\le n\\ \xi_{j-1}<Z_{(l)}<\xi_j}} \left(\frac{n-l}{n-l+1}\right)^{\delta_{(l)}}, \quad \tilde{p}_j = \prod_{\substack{l:1\le l\le n\\ Z_{(l)}=\xi_j}} \left(\frac{n-l}{n-l+1}\right)^{\delta_{(l)}}$$

$$(j = 1, \dots, J),$$

it is easy to see, from the definition of the PL estimator that

$$(1.4)\qquad \bar{F}_n^*(x) = \prod_{j=1}^{J} \hat{p}_j\tilde{p}_j, \quad \bar{F}_n^*(x^-) = \hat{p}_J \prod_{j=1}^{J-1} \hat{p}_j\tilde{p}_j.$$

In the above notations, and further on 0/0 is interpreted as 1. Finally, set

$$(1.5)\qquad \begin{aligned} N_j &= N^+(\xi_{j-1}) = \sum_{l=1}^{n}[Z_l > \xi_{j-1}], \quad \tilde{N}_j = N(\xi_j) = \sum_{l=1}^{n}[Z_l \geqq \xi_j], \\ D_j &= \sum_{l=1}^{n}[\xi_{j-1} < Z_l < \xi_j, \delta_l = 1], \quad \tilde{D}_j = \sum_{l=1}^{n}[Z_l = \xi_j, \delta_l = 1] \quad (j = 1, \dots, J). \end{aligned}$$

For the proof we need two lemmas. The first one is really simple but essential.

Lemma 1.1. *For fixed t and n, the random variables $N(t)$ and $N^+(t)$ are binomially distributed with parameters $(L(t), \bar{F}(t^-))$ and $(L^+(t), \bar{F}(t))$.*

Proof. The statements follow from the equalities

$$N(t) = \sum_{i=1}^{n}[Z_i \geqq t] = \sum_{i:L_i\geqq t}[X_i \geqq t],$$

$$N^+(t) = \sum_{i=1}^{n}[Z_i > t] = \sum_{i:L_i>t}[X_i > t]. \quad \square$$

Lemma 1.2. *Let us consider a partition belonging to $x\in(-\infty; T]$. Then*

(i) $\tilde{p}_j = \dfrac{\tilde{N}_j-\tilde{D}_j}{\tilde{N}_j}$, $\quad \hat{p}_j = \dfrac{N_j-D_j}{N_j}$

(ii) $\tilde{p}_j \to \dfrac{\bar{F}(\xi_j)}{\bar{F}(\xi_j^-)}$ *and* $\hat{p}_j \to \dfrac{\bar{F}(\xi_j^-)}{\bar{F}(\xi_{j-1})}$ *with probability* 1,

if $L(T)\to\infty$, where $1\leqq j\leqq J$ arbitrary.

Proof. (i) If there is no uncensored sample element at the point ξ_j, then using (1.3) $\tilde{p}_j=1$. In this case $\tilde{D}_j=0$, thus $\tilde{p}_j=\dfrac{\tilde{N}_j}{\tilde{N}_j}=1$. If we have uncensored sample element at ξ_j, then the statement follows easily using the ordering of the sample elements, i.e. that the censored sample elements are followed by the uncensored ones. Again, it is easy to see the second equality if $D_j=0$. Suppose that $D_j>0$, and the sample elements $Z_{(k+1)}, \dots, Z_{(k+D_j)}$ are in the interval (ξ_{j-1}, ξ_j). In this case $N_j = n-k$. In the considered partition these sample elements are uncensored. Hence if $Z_{(l)} \in (\xi_{j-1}, \xi_j)$ then $\delta_{(l)} = 1$. Thus, from (1.3) we get

$$\hat{p}_j = \frac{n-k-1}{n-k} \frac{n-k-2}{n-k-1} \cdots \frac{n-k-D_j}{n-k-D_j+1} = \frac{n-k-D_j}{n-k} = \frac{N_j - D_j}{N_j}.$$

(ii) By Lemma 1.1 $\tilde{N}_j$ is binomially distributed with parameters $(L(\xi_j), \bar{F}(\xi_j^-))$. Furthermore

$$\tilde{N}_j - \tilde{D}_j = \sum_{i: L_i \geqq \xi_j} [X_i > \xi_j],$$

thus $\tilde{N}_j - \tilde{D}_j$ is binomially distributed with parameters $(L(\xi_j), \bar{F}(\xi_j))$, and the first statement follows from the strong law of large numbers. To prove the second one, for the fixed n and the partition one can see that

$$N_j - D_j = \tilde{N}_j \quad \text{and} \quad L(\xi_j) = L^+(\xi_{j-1}).$$

Thus using Lemma 1.1 and the strong law of large numbers we get the statement. □

Proof of Theorem 1.1. We prove that for an arbitrary $0 < \varepsilon < 1$, $\sup_{-\infty < x \leqq T} |F_n^* - F| < 2\varepsilon$ holds with probability 1. Consider a fixed ε, without loss of generality we can suppose that $\varepsilon \leqq 1/K$, where K is the number of different elements of the sequence $\{L_i\}_{i=1}^{\infty}$ on the interval $(-\infty; T]$. Furthermore, there exists an $\tilde{\Omega} \subset \Omega$, $\mathsf{P}(\tilde{\Omega}) = 1$ and if $\omega \in \tilde{\Omega}$ then there exists $n_1(\omega)$ such that

$$T \leqq \max\{Z_1(\omega), \dots, Z_n(\omega)\}$$

for $n \geqq n_1(\omega)$. This is a consequence of the fact that $0 < \bar{F}(T^-)$. Consider a partition belonging to T. Let us choose the $\xi_0 = -\infty < \xi_1 < \dots < \xi_{J(\varepsilon)} = T$ points satisfying the following conditions:

(a) all of the different elements of the censoring sequence $\{L_i\}_{i=1}^{\infty}$ are among the ξ_j-s;

(b) $F(\xi_i^-) - F(\xi_{i-1}) \leqq \varepsilon/2$;

(c) $J(\varepsilon) \leqq 4/\varepsilon$.

Using Lemma 1.2 it can be supposed that

$$|\hat{p}_j - \tilde{p}_j| < \frac{\varepsilon}{4J} \quad \text{and} \quad |\tilde{p}_j - p_j| < \frac{\varepsilon}{4J} \quad (j = 1, \dots, J(\varepsilon)), \tag{1.6}$$

i.e. let us consider the set $\Omega_0 \subseteqq \tilde{\Omega}$, $\mathsf{P}(\Omega_0) = 1$, where for all $\omega \in \Omega_0$ there exists $n_2(\omega)$, such that if $n \geqq n_2(\omega)$ then (1.6) holds for all $1 \leqq j \leqq J(\varepsilon)$. Thus, consider a sample $\{(Z_i(\omega); \delta_i(\omega))\}_{i=1}^n$, $\omega \in \Omega_0$ and $n > n_2(\omega)$. Then the sample element $Z_{(n)} \geqq T$. Hence

$$\sup_{-\infty < x \leqq T} |F_n^*(x) - F(x)| = \max_{1 \leqq j \leqq J} \sup_{\xi_{j-1} \leqq x \leqq \xi_j} |\bar{F}_n^*(x) - \bar{F}(x)|. \tag{1.7}$$

Using condition (b) about the partition, for arbitrary $x \in [\xi_{i-1}, \xi_i)$ we have

$$\bar{F}_n^*(\xi_i^-) - \bar{F}(\xi_i^-) - \frac{\varepsilon}{2} \leqq \bar{F}_n^*(x) - \bar{F}(x) \leqq F_n^*(\xi_{i-1}) - F(\xi_{i-1}) + \frac{\varepsilon}{2}. \tag{1.8}$$

Thus we have to examine the differences $|F_n^*(\xi_i) - F(\xi_i)|$ and $|F_n^*(\xi_i^-) - F(\xi_i^-)|$

($i=1, \dots, J$). For this we need the following inequality: If $|u_k|\leqq 1$ and $|v_k|\leqq 1$ $k=1, \dots, K$ then

$$(1.9)\qquad \left|\prod_{i=1}^{K} u_i - \prod_{i=1}^{K} v_i\right| \leqq \sum_{i=1}^{K} |u_i - v_i|.$$

In view of (1.2), (1.4), (1.9) and (1.6) we have that

$$(1.10)\qquad \begin{aligned} |\bar F_n^*(\xi_i)-\bar F(\xi_i)| &\leqq \sum_{j=1}^{i} |\hat p_j - p_j| + \sum_{j=1}^{i} |\hat{\tilde p}_j - \tilde p_j| \leqq \frac{\varepsilon}{2} \\ |\bar F_n^*(\xi_i^-)-\bar F(\xi_i^-)| &\leqq \sum_{j=1}^{i} |\hat p_j - p_j| + \sum_{j=1}^{i-1} |\hat{\tilde p}_j - \tilde p_j| \leqq \frac{\varepsilon}{2}. \end{aligned}$$

By (1.8) $|\bar F_n^*(x)-\bar F(x)|\leqq\varepsilon$ for arbitrary $x\in[\xi_{i-1}, \xi_i]$. Hence the statement follows using (1.7). □

2. The rate of convergence

In this part, for the sake of completeness we give two theorems, the first states an exponential bound, the second gives the convergence rate. The proof of Theorems 2.1, 2.2 are similar to that of Lemma 2 and Theorem 1 of papeı [2].

THEOREM 2.1. *Suppose that conditions* (i) *and* (ii) *of Theorem 1.1 are fulfilled, furthermore* $L(T, n)\geqq 1$. *Then for arbitrary* $0\leqq\varepsilon\leqq 1$

$$\mathsf{P}\Big(\sup_{-\infty<x\leqq T} |F_n^*(x)-F(x)| > \varepsilon\Big) \leqq \frac{32(K+1)}{\varepsilon} \exp\left\{-\frac{3}{128}\,\frac{L(T)\bar F(T^-)\varepsilon^2}{(K+1)^2}\right\}.$$

THEOREM 2.2. *Suppose that conditions* (i) *and* (ii) *of Theorem 1.1 are fulfilled, furthermore*

$$\frac{\log n}{L(T, n)} \to 0.$$

Then

$$\sup_{-\infty<x\leqq T} |F_n^*(x)-F(x)| \leqq c\,\frac{\log n}{L(T, n)}$$

with probability 1, *where* $c=1+\dfrac{256(K+1)^2}{3F(T^-)}$.

We sketch the proof only stating the basic lemmas without proof. The proof of Lemma 2.1 goes on the same way as the proof of Lemma 3.3 of paper [3], while the proof of Lemma 2.2 is similar to that of Lemma 1 of paper [2].

LEMMA 2.1. *Suppose that* $L(T)\geqq 1$, $\bar F(T^-)>0$. *Then for arbitrary partition belonging to an arbitrary* $x\in(-\infty; T]$ *the following hold:*

$$\left.\begin{aligned} &\mathsf{P}(|\hat p_j - p_j| > t) \\ &\mathsf{P}(|\hat{\tilde p}_j - \tilde p_j| > t) \end{aligned}\right\} \leqq 2\exp\left\{-\frac{3}{8}\,t^2 L(t)\bar F(T^-)\right\}$$

for all $0\leqq t\leqq 1$ *and* $1\leqq j\leqq J(x)$.

LEMMA 2.2. *Suppose that conditions* (i) *and* (ii) *of Theorem 1.1 are fulfilled, furthermore* $L(T,n)\geqq 1$. *Then for arbitrary* $x\in(-\infty; T]$ *and* $0<\varepsilon<2$ *the following hold:*

$$\left.\begin{aligned}&\mathsf{P}(|F_n^*(x)-F(x)|>\varepsilon)\\&\mathsf{P}(|F_n^*(x^-)-F(x^-)|>\varepsilon)\end{aligned}\right\}\leqq 4(K+1)\exp\left\{-\frac{3}{32}L(T)\bar{F}(T^-)\frac{\varepsilon^2}{(K+1)^2}\right\}.$$

Now the proof of Theorem 2.1 follows from Lemmas 1.1, 1.2, 2.1, 2.2 using inequality 1.9. The proof of Theorem 2.2 follows from Theorem 2.1 via the Borel—Cantelli lemma.

Further on we state two corollaries, for independent stochastic censoring, where the censoring sequence is not necessarily identically distributed. The second one is a special case of Theorem 1 of [1], if we suppose the continuity of F. For these corollaries the continuity of F is not necessary.

COROLLARY 2.1. *Suppose that in the random censoring model the following conditions hold:*

(i) $0<\bar{F}(T^-)$;

(ii) *the r.v.-s* $\{Y_i\}_{i=1}^{\infty}$ *have* $0\leqq K<+\infty$ *different values on the interval* $(-\infty; T]$;

(iii) $\sum_{i=1}^{\infty}\mathsf{P}(Y_i\geqq T)=\sum_{i=1}^{\infty}\bar{G}_i(T^-)=+\infty.$

Then

$$\mathsf{P}\Big(\sup_{-\infty<X\leqq T}|F_n^*(x)-F(x)|>\varepsilon\Big)\leqq$$

$$\leqq\frac{32(K+1)}{\varepsilon}\exp\left\{-\frac{3\varepsilon^2\bar{F}(T^-)}{356(K+1)^2}\sum_{i=1}^{n}\bar{G}_i(T^-)\right\}+\exp\left\{-\sum_{i=1}^{n}\bar{G}_i(T^-)\right\}.$$

COROLLARY 2.2. *Suppose that in the random censoring model the conditions* (i)—(iii) *of Corollary 2.2 are fulfilled. Then with probability* 1

$$\sup_{-\infty<X\leqq T}|F_n^*(x)-F(x)|=O\left(\sqrt{\frac{\log n}{\sum_{i=1}^{n}\bar{G}_i(T^-)}}\right).$$

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CUBE-LATTICES WITH GOOD DISTRIBUTION BEHAVIOUR

JÓZSEF BECK

Abstract

In this note we prove the following theorem: For arbitrary natural numbers r, n and real $\varepsilon > 0$ there exists a threshold $k_0(r, n, \varepsilon)$ such that given any n measurable subsets of the r-dimensional unit cube, one can find an aligned r-dimensional cube-lattice of size $k \times \ldots \times k$ with $k < k_0(r, n, \varepsilon)$ so that its "discrepancy" is less than ε relative to each of the given sets. In spite of appearance, this result is far from being a triviality. The proof needs some "advanced" ideas, namely a version of the "second-moment method".

1. Introduction

Let U^r denote the r-dimensional unit cube $0 \leqq x_1 \leqq 1, \ldots, 0 \leqq x_r \leqq 1$. We say that Q is an r-dimensional *cube-lattice* of *order* k if it can be written in the form

$$\{(t_1 + j_1 b, t_2 + j_2 b, \ldots, t_r + j_r b) \colon j_i = 0, 1, \ldots, k-1; \ 1 \leqq i \leqq r\},$$

where $\mathbf{t} = (t_1, \ldots, t_r)$ is an arbitrary r-dimensional vector and b is a real number. Denote by λ_r the r-dimensional normed Lebesgue measure, i.e. $\lambda_r(U^r) = 1$. $|X|$ denotes the cardinality of the set X.

THEOREM 1.1. *There is a universal threshold function $k_0(r, n, \varepsilon)$ such that, given any n measurable subsets $A_1, \ldots, A_n$ of U^r, one can find an r-dimensional cube-lattice $Q \subset U^r$ of order $k < k_0(r, n, \varepsilon)$ with the property*

$$\left| \frac{|Q \cap A_i|}{|Q|} - \lambda_r(A_i) \right| < \varepsilon \quad \text{for all } i, \quad 1 \leqq i \leqq n.$$

Let $[0, N]^r$ denote the set of the integer coordinate points $(a_1, a_2, \ldots, a_r)$ where $a_i = 0, 1, \ldots, N$ and $1 \leqq i \leqq r$. Using Lebesgue's measure theory one can easily deduce Theorem 1.1 from the following purely combinatorial result.

THEOREM 1.2. *There is a threshold $k_1(r, n, \varepsilon)$ such that, given any natural number N and any n subsets $B_1, \ldots, B_n$ of $[0, N]^r$, one can find an r-dimensional cube-lattice $Q \subset [0, N]^r$ of order $k < k_1(r, n, \varepsilon)$ with the property*

$$\left| \frac{|Q \cap B_i|}{|Q|} - \frac{|B_i|}{(N+1)^r} \right| < \varepsilon \quad \text{for all } i, \ 1 \leqq i \leqq n.$$

For the sake of completeness, here we give a deduction of Theorem 1.1 from Theorem 1.2.

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If all A_i'-s are the union of finitely many r-dimensional balls, then we are ready as follows. Choosing any sufficiently dense aligned cube-lattice R we can guarantee that each A_i has *discrepancy* less than $\varepsilon/2$ relative to R, i.e.,

$$\left|\frac{|A_i \cap R|}{(N+1)^r} - \lambda_r(A_i)\right| < \frac{\varepsilon}{2}$$

where $N+1$ denotes the order of the lattice R (of course, N can be arbitrarily large). Then applying Theorem 1.2 to R, we obtain the existence of a cube-lattice $Q \subset R$ such that the order of Q is less than $k_1(r, n, \varepsilon/2)$ and for each i, $1 \leqq i \leqq n$,

$$\left|\frac{|Q \cap B_i|}{|Q|} - \frac{|B_i|}{(N+1)^r}\right| < \frac{\varepsilon}{2} \quad \text{where} \quad B_i = A_i \cap R.$$

Now in this particular case the proof of Theorem 1.1 is complete, since

$$\left|\frac{|Q \cap A_i|}{|Q|} - \lambda_r(A_i)\right| \leqq \left|\frac{|Q \cap A_i|}{|Q|} - \frac{|R \cap A_i|}{(N+1)^r}\right| + \left|\frac{|R \cap A_i|}{(N+1)^r} - \lambda_r(A_i)\right| <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In the general case there are some minor technical difficulties. Let G_i $(1 \leqq i \leqq n)$ be a union of finitely many r-dimensional balls such that $\lambda_r\left(\bigcup_{i=1}^{n} (G_i \triangle A_i)\right) < \frac{\varepsilon}{5}$ where $\triangle$ denotes the symmetric difference.

Assume that one can find an aligned cube-lattice R such that each G_i has discrepancy less than $\varepsilon/5$ relative to R and $|R \cap A_0|/|R| < \varepsilon/5$ where $A_0 = \bigcup_{i=1}^{n} (G_i \triangle A_i)$. Then we are ready by the argument above. Indeed, applying Theorem 1.2 to this lattice R and to the sets $B_0 = R \cap A_0$, $B_i = R \cap G_i$, $1 \leqq i \leqq n$, we obtain the existence of a sublattice $Q \subset R$ such that the order of Q is less than $k_1(r, n+1, \varepsilon/5)$ and for each i, $1 \leqq i \leqq n$,

$$\left|\frac{|Q \cap B_i|}{Q} - \frac{|B_i|}{|R|}\right| < \frac{\varepsilon}{5} \quad \text{and} \quad \frac{|Q \cap B_0|}{|Q|} < \frac{|R \cap A_0|}{|R|} + \frac{\varepsilon}{5} < \frac{2}{5}\varepsilon.$$

Now Theorem 1.1 follows, since

$$\left|\frac{|Q \cap A_i|}{|Q|} - \lambda_r(A_i)\right| \leqq \left|\frac{|Q \cap A_i|}{|Q|} - \frac{|Q \cap G_i|}{|Q|}\right| + \left|\frac{|Q \cap G_i|}{|Q|} - \frac{|R \cap G_i|}{|R|}\right| +$$

$$+ \left|\frac{|R \cap G_i|}{|R|} - \lambda_r(G_i)\right| + |\lambda_r(G_i) - \lambda_r(A_i)| <$$

$$< \frac{|Q \cap (G_i \triangle A_i)|}{|Q|} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \lambda_r(G_i \triangle A_i) \leqq$$

$$\leqq \frac{|Q \cap B_0|}{|Q|} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \lambda_r(A_0) < 2\frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.$$

Therefore, it suffices to find the desired cube-lattice T. The first requirement of R (i.e. each G_i has discrepancy $<\varepsilon/5$ relative to R) is automatically satisfied if R is sufficiently dense (we recall that G_i is a finite union of balls). Consequently, it is enough to find an aligned cube-lattice R such that the order of R is greater than a threshold $N_0=N_0(G_1, G_2, \dots, G_n)$ and $|R\cap A_0|/|R|<\varepsilon/5$ where $A_0=\bigcup_{i=1}^{n}(G_i\triangle A_i)$. Let $N>N_0$ and let R_N denote the set of points

$$\left(\frac{a_1}{N}, \frac{a_2}{N}, \dots, \frac{a_r}{N}\right) \quad \text{where} \quad a_i = 0, 1, 2, \dots, N-1; \ 1 \leqq i \leqq r.$$

Denote by $R_N+\mathbf{v}$ the translated image of R_N by the vector $\mathbf{v}$, and for any real number μ let $\mu\cdot U^r$ denote the set of vectors $\mu\mathbf{v}$, $\mathbf{v}\in U^r$. Observe that

$$\int_{(1/N)U^r} |(R_N+\mathbf{v})\cap A_0|\, d\mathbf{v} = \lambda_r(A_0) < \frac{\varepsilon}{5}.$$

Since $\lambda_r\left(\frac{1}{N}U^r\right)=\frac{1}{N^r}$, there must exist a $\mathbf{v}_0\in\frac{1}{N}U^r$ such that

$$|(R_N+\mathbf{v}_0)\cap A_0| < \frac{\varepsilon}{5}N^r,$$

i.e.,

$$\frac{|(R_N+\mathbf{v}_0)\cap A_0|}{|R_N+\mathbf{v}_0|} < \frac{\varepsilon}{5}.$$

This completes the deduction of Theorem 1.1 from Theorem 1.2.

The proof of Theorem 1.2 will be based on a multidimensional "large sieve" type estimate (see inequality (9)).

2. Proof of Theorem 1.2

Given an arbitrary *prime number* p and an integer coordinate vector $\mathbf{a}=(a_1, \dots, a_r)$ with $0\leqq a_i\leqq p-1$, $1\leqq i\leqq r$, let $Q_{p,\mathbf{a}}$ denote the cube-lattice

$$\{(a_1+j_1p, a_2+j_2p, \dots, a_r+j_rp)\colon 0 \leqq a_i+j_ip \leqq N, \ 1 \leqq i \leqq r\}.$$

Let $f_{p,\mathbf{a}}$ denote the characteristic function of the cube-lattice $Q_{p,\mathbf{a}}$, i.e., $f_{p,\mathbf{a}}(\mathbf{b})=1$ if $\mathbf{b}\in Q_{p,\mathbf{a}}$ and 0 if $\mathbf{b}\in[0, N]^r\setminus Q_{p,\mathbf{a}}$. Finally, introduce the function $\Phi_{p,\mathbf{a}}$ defined on $[0, N]^r$ as follows:

$$\Phi_{p,\mathbf{a}} = \frac{(N+1)^r}{|Q_{p,\mathbf{a}}|} f_{p,\mathbf{a}}-1.$$

The following lemma express the quasi-orthogonality property of $\Phi_{p,\mathbf{a}}$'s in quantitative form.

LEMMA 2.1. *If p and q are distinct primes such that $p \cdot q < N$, then*

(1) $$\Big|\sum_{\mathbf{b}\in[0,N]^r} \Phi_{p,\mathbf{a}_1}(\mathbf{b})\Phi_{q,\mathbf{a}_2}(\mathbf{b})\Big| = O\left(\frac{pq}{N}\right)(N+1)^r.$$

If $\mathbf{a}_1 \neq \mathbf{a}_2$, *then*

(2) $$\sum_{\mathbf{b}\in[0,N]^r} \Phi_{p,\mathbf{a}_1}(\mathbf{b})\Phi_{p,\mathbf{a}_2}(\mathbf{b}) = -(N+1)^r.$$

Finally,

(3) $$\sum_{\mathbf{b}\in[0,N]^r} \Phi^2_{p,\mathbf{a}}(\mathbf{b}) = O(p^r(N+1)^r).$$

Here and in what follows the implicit constants depend only on the dimension r.

PROOF. We start with the verification of (1). We have

$$\sum_{\mathbf{b}} \Phi_{p,\mathbf{a}_1}(\mathbf{b})\Phi_{q,\mathbf{a}_2}(\mathbf{b}) =$$

(4) $$= \sum_{\mathbf{b}}\left(\frac{(N+1)^r}{|Q_{p,\mathbf{a}_1}|}f_{p,\mathbf{a}_1}(\mathbf{b})-1\right)\left(\frac{(N+1)^r}{|Q_{q,\mathbf{a}_2}|}f_{q,\mathbf{a}_2}(\mathbf{b})-1\right) =$$

$$= \sum_{\mathbf{b}}\left\{\frac{(N+1)^{2r}}{|Q_{p,\mathbf{a}_1}||Q_{q,\mathbf{a}_2}|}f_{p,\mathbf{a}_1}(\mathbf{b})f_{q,\mathbf{a}_2}(\mathbf{b})+1-\frac{(N+1)^r}{|Q_{p,\mathbf{a}_1}|}f_{p,\mathbf{a}_1}(\mathbf{b})-\frac{(N+1)^r}{|Q_{q,\mathbf{a}_2}|}f_{q,\mathbf{a}_2}(\mathbf{b})\right\} =$$

$$= \frac{(N+1)^{2r}}{|Q_{p,\mathbf{a}_1}||Q_{q,\mathbf{a}_2}|}|Q_{p,\mathbf{a}_1}\cap Q_{q,\mathbf{a}_2}|-(N+1)^r.$$

Observe that

$$|Q_{p,\mathbf{a}_1}| = \left(\frac{(N+1)}{p}\right)^r\left(1+O\left(\frac{p}{N}\right)\right), \quad |Q_{q,\mathbf{a}_2}| = \left(\frac{(N+1)}{q}\right)^r\left(1+O\left(\frac{q}{N}\right)\right),$$

and using the fact that p, q are prime to each other,

$$|Q_{p,\mathbf{a}_1}\cap Q_{q,\mathbf{a}_2}| = \left(\frac{N+1}{pq}\right)^r\left(1+O\left(\frac{pq}{N}\right)\right).$$

Hence

$$\frac{(N+1)^{2r}}{|Q_{p,\mathbf{a}_1}||Q_{q,\mathbf{a}_2}|}|Q_{p,\mathbf{a}_1}\cap Q_{q,\mathbf{a}_1}| =$$

$$= \frac{(N+1)^{3r}(1+O(pq/N))}{(N+1)^{2r}(1+O(p/N))(1+O(q/N))} = (N+1)^r(1+O(pq/N)).$$

Returning to (4) we obtain the validity of (1).

In order to check (2) observe that $Q_{p,\mathbf{a}_1}\cap Q_{p,\mathbf{a}_2}=\emptyset$ if $\mathbf{a}_1\neq\mathbf{a}_2$. Therefore, by (4) we are done.

Finally, to verify (3) observe that $|Q_{p,\mathbf{a}}|=O\left(\left(\frac{N+1}{p}\right)^r\right)$, thus by (4) we obtain

$$\sum_{\mathbf{b}\in[0,N]^r} \Phi^2_{p,\mathbf{a}}(\mathbf{b}) = \frac{(N+1)^{2r}}{|Q_{p,\mathbf{a}}|} - (N+1)^r = O\big(p^r(N+1)^r\big).$$

Lemma 2.1 is complete. □

We will require an appropriately modified form of the classical Bessel inequality applicable to quasi-orthogonal systems.

LEMMA 2.2 (A. Selberg). *Let $\xi, \varphi_1, \ldots, \varphi_h$ be elements of an inner product space over the real numbers. Then*

$$\sum_{i=1}^{h} (\xi, \varphi_i)^2 \left(\sum_{j=1}^{h} |(\varphi_i, \varphi_j)|\right)^{-1} \leqq \|\xi\|^2.$$

For completeness we include here the simple and elegant *proof* (cf. [1] p. 8). We have $\|\xi - \sum_i c_i\varphi_i\|^2 \geqq 0$ for real numbers c_i, that is to say

$$\|\xi\|^2 - 2\sum_i c_i(\xi, \varphi_i) + \sum_{i,j} c_ic_j(\varphi_i, \varphi_j) \geqq 0.$$

Using

$$|c_ic_j| \leqq \frac{1}{2}(|c_i|^2 + |c_j|^2),$$

we obtain

(5)
$$\Big|\sum_{i,j} c_ic_j(\varphi_i, \varphi_j)\Big| \leqq \sum_i |c_i|^2 \sum_{j=1}^{h} |(\varphi_i, \varphi_j)|.$$

From (5) we have

$$2\sum_i c_i(\xi, \varphi_i) \leqq \|\xi\|^2 + \sum_i |c_i|^2 \sum_{j=1}^{h} |(\varphi_i, \varphi_j)|.$$

If we now take $c_i = (\xi, \varphi_i)\big(\sum_{j=1}^{h} |(\varphi_i, \varphi_j)|\big)^{-1}$ the result follows. □

We remark that A. Rényi was the first to realize that inequality like Selberg's lemma above could be used in proving the "large sieve" in number theory, see [1].

In our application, we will be concerned with inner products of type

$$(\varphi, \psi) = \sum_{\mathbf{b}\in[0,N]^r} \varphi(\mathbf{b})\psi(\mathbf{b})$$

where φ, ψ are real functions over $[0, N]^r$, and of course $\|\varphi\|^2 = (\varphi, \varphi)$.

Let g_i denote the characteristic function of the subset $B_i \subset [0, N]^r$.

Clearly,

(6)
$$(g_i, \Phi_{p,\mathbf{a}}) = \frac{(N+1)^r}{|Q_{p,\mathbf{a}}|}|Q_{p,\mathbf{a}} \cap B_i| - |B_i| =$$
$$= (N+1)^r\left\{\frac{|Q_{p,\mathbf{a}} \cap B_i|}{|Q_{p,\mathbf{a}}|} - \frac{|B_i|}{(N+1)^r}\right\}.$$

By Lemma 2.2

$$\sum_{M\le p\le 2M}\ \sum_{\substack{\mathbf{a}=(a_1,\dots,a_r)\\ 0\le a_i\le p-1}} |(g_i, \Phi_{p,\mathbf{a}})|^2 \Big(\sum_{M\le q\le 2M}\ \sum_{\substack{\mathbf{b}=(b_1,\dots,b_r)\\ 0\le b_i\le q-1}} |(\Phi_{p,\mathbf{a}}, \Phi_{q,\mathbf{b}})|\Big)^{-1} \le \tag{7}$$

$$\le \|g_i\|^2 = |B_i| \le (N+1)^r,$$

where p, q are prime numbers belonging to the interval $[M, 2M]$ and M will be specified later. By Lemma 2.1, for each $\Phi_{p,\mathbf{a}}$

$$\sum_{M\le q\le 2M}\ \sum_{\substack{\mathbf{b}=(b_1,\dots,b_r)\\ 0\le b_i\le q-1}} |(\Phi_{p,\mathbf{a}}, \Phi_{q,\mathbf{b}})| \le$$

$$\le \|\Phi_{p,\mathbf{a}}\|^2 + \sum_{\substack{\mathbf{b}:\mathbf{b}\ne\mathbf{a}\\ 0\le b_i\le p-1}} |(\Phi_{p,\mathbf{a}}, \Phi_{p,\mathbf{b}})| + \sum_{\substack{M\le q\le 2M\\ q\ne p}}\ \sum_{\substack{\mathbf{b}=(b_1,\dots,b_r)\\ 0\le b_i\le q-1}} |(\Phi_{p,\mathbf{a}}, \Phi_{q,\mathbf{b}})| = \tag{8}$$

$$= O\big(p^r(N+1)^r\big) + p^r O\big((N+1)^r\big) + MM^r O\left(\frac{M^2}{N}\right)(N+1)^r =$$

$$= O\big((N+1)^r\big)\left(M^r + \frac{M^{r+3}}{N}\right).$$

By (6), (7) and (8)

$$\sum_{M\le p\le 2M}\ \sum_{\substack{\mathbf{a}=(a_1,\dots,a_r)\\ 0\le a_i\le p-1}} (N+1)^{2r}\left\{\frac{|Q_{p,\mathbf{a}}\cap B_i|}{|Q_{p,\mathbf{a}}|} - \frac{|B_i|}{(N+1)^r}\right\}^2 =$$

$$= O\big((N+1)^r\big)\left(M^r + \frac{M^{r+3}}{N}\right)(N+1)^r.$$

Divided both sides by $(N+1)^{2r}$ we have

$$\sum_{M\le p\le 2M}\ \sum_{\substack{\mathbf{a}=(a_1,\dots,a_r)\\ 0\le a_i\le p-1}} \left\{\frac{|Q_{p,\mathbf{a}}\cap B_i|}{|Q_{p,\mathbf{a}}|} - \frac{|B_i|}{(N+1)^r}\right\}^2 = O\left(M^r + \frac{M^{r+3}}{N}\right).$$

Choosing $M = N^{1/3}$ and summing by i, $1\le i\le n$ we obtain

$$\sum_{i=1}^{n}\ \sum_{N^{1/3}\le p\le 2N^{1/3}}\ \sum_{\substack{\mathbf{a}=(a_1,\dots,a_r)\\ 0\le a_i\le p-1}} \left\{\frac{|Q_{p,\mathbf{a}}\cap B_i|}{|Q_{p,\mathbf{a}}|} - \frac{|B_i|}{(N+1)^r}\right\}^2 = O(nN^{r/3}). \tag{9}$$

Since the number of primes in the interval $[M, 2M]$ is greater than $c_0 M/\log M$ with some constant $c_0 > 0$, (9) immediately yields the existence of a cube-lattice $Q_{p_0,\mathbf{a}_0}$ with $N^{1/3}\le p_0 < 2N^{1/3}$, such that

$$\sum_{i=1}^{n} \left\{\frac{|Q_{p_0,\mathbf{a}_0}\cap B_i|}{|Q_{p_0,\mathbf{a}_0}|} - \frac{|B_i|}{(N+1)^r}\right\}^2 = O\left(\frac{n\log N}{N^{1/3}}\right). \tag{10}$$

From (10) it follows that

$$\left|\frac{|Q_{p_0,a_0}\cap B_i|}{|Q_{p_0,a_0}|}-\frac{|B_i|}{(N+1)^r}\right|< c_1(r,n)\frac{(\log N)^{1/2}}{N^{1/6}} \quad \text{for all } i,\ 1\leqq i\leqq n.$$

Thus we have proved the following

LEMMA 2.3. *Given any natural number N and any n subsets $B_1, \ldots, B_n$ of the cube-lattice* $[0, N]^r$, *there exists an r-dimensional cube-lattice* $Q_1\subset[0,N]^r$ *of order* $\leqq N^{2/3}$ *with the property*

$$\left|\frac{|Q_1\cap B_i|}{|Q_1|}-\frac{|B_i|}{(N+1)^r}\right|< c_2(r,n)N^{-1/7} \quad \textit{for all } i,\quad 1\leqq i\leqq n. \quad \square$$

Let N_1 denote the order of Q_1. Applying Lemma 2.3 to Q_1 we conclude that there exists a cube-lattice $Q_2\subset Q_1$ of order $N_2\leqq N_1^{2/3}$, with the property

$$\left|\frac{|Q_2\cap B_i|}{|Q_2|}-\frac{|Q_1\cap B_i|}{Q_1}\right|< c_2(r,n)N_1^{-1/7} \quad \text{for all } i,\ 1\leqq i\leqq n.$$

By repeated application of Lemma 2.3 we obtain the existence of a sequence $Q_0=$ $=[0,N]^r\supset Q_1\supset Q_2\supset\ldots\supset Q_j\supset\ldots$ of cube-lattices with the properties $N_j\leqq N_{j-1}^{2/3}$ where N_j denotes the order of Q_j, and

$$\left|\frac{|Q_j\cap B_i|}{|Q_j|}-\frac{|B_i|}{(N+1)^r}\right|< c_2(r,n)\sum_{l=0}^{j-1} N_l^{-1/7} \quad \text{for all } i,\ 1\leqq i\leqq n.$$

Elementary calculation shows that $\sum_{l=0}^{j-1} N_l^{-1/7}<\delta$ if only $N_j>c_3(\delta)$ where $c_3(\delta)$ is a sufficiently large constant depending only on $\delta>0$. This completes the proof of Theorem 1.2. □

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ON THE ORDER OF CONVERGENCE OF A FINITE ELEMENT METHOD FOR MIXED BOUNDARY VALUE PROBLEMS

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Weisel obtained in [1] error bounds for a finite element approximation of the mixed boundary value problem for second order elliptic equations in the case when the boundary is a polygon. In the present paper we shall generalize Weisel's results to regions with curved boundaries.

1. Let R be a bounded open plane region whose boundary C consists of a finite number of piecewise analytic simple closed curves. For the sake of simplicity we shall assume that the boundary C consists of two analytic arcs C^1 and C^2 which meet at the corners A_1 and A_2 and form the interior angles $\pi\alpha_1$ and $\pi\alpha_2$ $(0<\alpha_i<2)$ there, respectively (see Fig. 1). The general case can be treated in the same way.

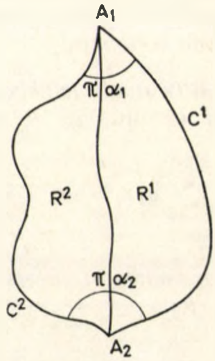

Fig. 1

We consider the mixed boundary value problem

$$Lu \equiv \nabla(p\nabla u) - qu = f \quad \text{in} \quad R,$$

$$u = 0 \quad \text{on} \ C^2,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \ C^1.$$

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Let the coefficients $p=p(x, y)$, $q=q(x, y)$ and the right-hand side $f=f(x, y)$ be infinitely differentiable in R. We assume that $p(x, y)\geqq c_1>0$ (c_1 is a constant), $q(x, y)\geqq 0$ for all $(x, y)\in R$. It is well-known that under the above assumptions the mixed boundary value problem (1) has a unique solution $u(x, y)$.

Let Ω be a bounded open region in the plane of R. We denote by $W_2^{(m)}(\Omega)$ the Hilbert space of all functions which, together with their generalized partial derivatives up to the mth order, belong to $L_2(\Omega)$. The norm is given by

(1)
$$\|v\|^2_{m,\Omega} = \sum_{j=0}^{m} |v|_{j,\Omega},$$

where

$$|v|^2_{j,\Omega} = \sum_{|i|=j} \|D^i v\|^2_{L_2(\Omega)}.$$

Here $i=(i_1, i_2)$, $|i|=i_1+i_2$, $D^i v=\dfrac{\partial^{|i|} v}{\partial x^{i_1}\partial y^{i_2}}$.

It is well-known that the solution $u(x, y)$ of the boundary value problem (1) minimizes the functional

(2)
$$F(v) = \iint_R \left\{p\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + qv^2+2fv\right\} dx\, dy$$

over the subspace of $W_2^{(1)}(R)$ formed by the functions $v(x, y)$ such that $v|_{C^2}=0$.

In the sequel we shall use the following

LEMMA. *Let D_{A_i} be a sufficiently small neighbourhood of the corner A_i. If $2\alpha_i$ is not an odd integer, then for all $(x, y)\in D_{A_i}\cap R$ we have*

(3)
$$u(x, y) = \sum_{\substack{k<2\alpha_i\\ k\,\text{odd}}} a_k r_i^{\frac{k}{2\alpha_i}} \sin^{\frac{k}{2\alpha_i}} \vartheta_i + w(x, y),$$

where r_i and ϑ_i are the polar coordinates of the point (x, y), the coefficients a_k are constant and $w(x, y)\in W_2^{(2)}(D_{A_i}\cap R)$.

For a proof, see [1], p. 36.

2. The line A_1A_2 subdivides the region R into two disjoint subregions R^1 and R^2 (see Fig. 1). Let h be a sufficiently small positive real number. We approximate the region R^2 by the Oganesjan polygon R_h^2 (see [2], p. 1042). We triangulate R_h^2, i.e. we subdivide R_h^2 into a finite number of triangles such that any two triangles are either disjoint or have a common vertex or a common side. Denote by M_h^2 the set of all triangles of the triangulation of R_h^2. Similarly, we cover R^1 by a finite number of arbitrary triangles such that any two triangles are either disjoint or have a common vertex or a common side. We retain only those triangles T for which

$$\iint_{T\cap R^1} dx\, dy > 0,$$

i.e. for which T and R^1 have some common area. Denote by M_k^1 the set of triangles covering R^1 and by R_h^1 the union of all triangles $T \in M_h^1$. In the sequel we assume that

$$h < c_2 \bar{h}, \quad h^* \leqq h, \quad \vartheta \geqq \vartheta_0 > 0, \tag{4}$$

where h^* is the largest side, $\bar{h}$ is the smallest side and ϑ is the smallest angle of all triangles $T \in M_h^1 \cup M_h^2$; c_2 and ϑ_0 do not depend on h. Moreover, we assume that if $T_1 \in M_h^1$ and $T_2 \in M_h^2$ then T_1 and T_2 are either disjoint or have a common vertex or a common side on the line $A_1 A_2$ (see Fig.2).

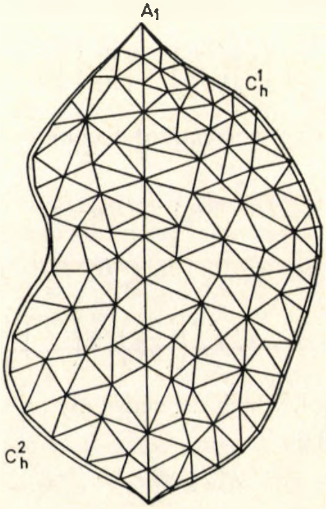

Fig. 2

Denote by C_h^1 and C_h^2 the boundary of the polygon R_h^1 and R_h^2, respectively, excluding the interval $\overline{A_1 A_2}$. Let $H(\bar{R}_h)$ be the set of all functions which are continuous on the closed region $\bar{R}_h = R \cup C \cup R_h^1 \cup C_h^1$ and linear over each triangle $T \in M_h^1 \cup M_h^2$. Denote by $\mathring{H}_{C_h^2}(\bar{R}_h)$ the set of functions from $H(\bar{R}_h)$ which vanish on C_h^2 and in $R^2 - R_h^2$. The solution $u(x, y)$ of the problem (1) is approximated by the function $u_h(x, y)$ which minimizes the functional (2) over the space $\mathring{H}_{C_h^2}(\bar{R}_h)$.

3. Theorem. *Let $u(x, y)$ be the solution of the boundary value problem* (1) *and let $u_h(x, y)$ be the function which minimizes the functional* (2) *over the space $\mathring{H}_{C_h^2}(\bar{R}_h)$. Assume that $2\alpha_i$ $(i = 1, 2)$ is not an odd integer. Then for sufficiently small h we have*

$$\|u - u_h\|_{1,R} < c_3 h^\beta \tag{5}$$

and

$$\max_{(x,y) \in R} |u(x, y) - u_h(x, y)| < c_4 h^\beta |\log h|^{1/2} \tag{6}$$

where

$$\beta = \begin{cases} \dfrac{1}{2\max(\alpha_1, \alpha_2)} & \textit{if} \quad \max(\alpha_1, \alpha_2) > \dfrac{1}{2}, \\ 1 & \textit{if} \quad \max(\alpha_1, \alpha_2) < \dfrac{1}{2}, \end{cases}$$

c_3 and c_4 are positive constants which depend only on the coefficients $p(x, y)$, $q(x, y)$, the right-hand side $f(x, y)$ and the region R.

Proof. Let the functional $D(v)$ be defined by

$$D(v) = \iint_R \left\{ p\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + qv^2 \right\} dx\,dy$$

for all $v \in W_2^{(1)}(R)$. Then, using a standard technique (see, for example, [3], p. 6), we can easily prove that

$$D(u-u_h) \leqq D(u-z) \tag{7}$$

for all $z \in \mathring{H}_{C_h^2}(\bar{R}_h)$. Mihlin proved that if $v \in W_2^{(1)}(R)$ and $v|_{C^2}$ 0, then

$$\|v\|^2_{L_2(R)} \leqq c_5 \iint_R \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dx\,dy$$

where c_5 is a positive constant, which depends only on the region R (see [4], p. 144). Thus from (7) it follows that

$$\|u-u_h\|_{1,R} \leqq c_6 \|u-z\|_{1,R} \tag{8}$$

for all $z \in \mathring{H}_{C_h^2}(\bar{R}_h)$; here c_6 is a positive constant which depends only on the region R and the coefficients of the operator L.

By (3) we have for all $(x, y) \in D_{A_i} \cap R$

$$u(x, y) = \sum_{\substack{k < 2\alpha_i \\ k \text{ odd}}} a_k r_i^{\frac{k}{2\alpha_i}} \sin^{\frac{k}{2\alpha_i}} \vartheta_i + w(x, y),$$

where $w(x, y) \in W_2^{(2)}(D_{A_i} \cap R)$. From the Calderón extension theorem (see, for example, [5], p. 171) it follows that there exists a function $w_{\text{ext}}(x, y) \in W_2^{(2)}(D_{A_i} \cap R_h)$ such that $w_{\text{ext}}(x, y) = w(x, y)$ for all $(x, y) \in D_{A_i} \cap R$ and

$$\|w_{\text{ext}}\|_{2, D_{A_i} \cap R_h} \leqq c_7 \|w\|_{2, D_{A_i} \cap R}$$

where c_7 is a positive constant which depends only on the region R. Let the function $u_{\text{ext}}(x, y)$ be defined by

$$u_{\text{ext}}(x, y) = \sum_{\substack{k < 2\alpha_i \\ k \text{ odd}}} a_k r_i^{\frac{k}{2\alpha_i}} \sin^{\frac{k}{2\alpha_i}} \vartheta_i + w_{\text{ext}}(x, y)$$

for all $(x, y) \in D_{A_i} \cap \bar{R}_h$. It is well-known that under our assumptions $u(x, y) \in W_2^{(2)}(R - (D_{A_1} \cup D_{A_2}))$. Let $u_{\text{ext}}(x, y)$ be for all $(x, y) \in \bar{R}_h - (D_{A_1} \cup D_{A_2})$ the Calderón extension of $u(x, y)$ onto the region $\overline{R}_h-(D_{A_1}\cup D_{A_2})$. Thus we have defined $u_{ext}(x, y)$ on the closed region $\overline{R}_h$. It is easy to show that $u_{ext}(x, y)$ is continuous in $\overline{R}_h$. Let $\varrho_h u_{ext}(x, y)$ be the function from $\mathring{H}_{C_h^2}(\overline{R}_h)$ which assumes the same values as $u_{ext}(x, y)$ at the vertices of the triangles $T\in M_h^1\cup M_h^2$ excluding the vertices on C_h^2. Evidently,

$$\|u-\varrho_h u_{ext}\|_{1,R} \leqq \|u-\varrho_h u_{ext}\|_{1,R^2}+\|u_{ext}-\varrho_h u_{ext}\|_{1,R_h^1} \tag{9}$$

The first term on the right of (9) can be estimated, using a theorem of Wigley (see [8], p. 551) in the same way as in the case of the Dirichlet problem (see [6]). Thus we obtain that

$$\|u-\varrho_h u_{ext}\|_{1,R^2} = O(h^\beta). \tag{10}$$

The second term on the right of (9) can be estimated in the same way as in [1] (see [1], p. 62). Thus we obtain that

$$\|u_{ext}-\varrho_h u_{ext}\|_{1,R^1} = O(h^\beta). \tag{11}$$

Substituting (10) and (11) into (9) and then into (8) we get the inequality (5). The inequality (6) immediately follows from (5) and a theorem of V.P. Il'in (see [7], p. 101). This completes the proof of our Theorem.

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ON SUMS OF INTEGERS HAVING SMALL PRIME FACTORS, I

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1. Throughout this paper, we use the following notation: $c_1, c_2, \dots, N_0, N_1, \dots$ denote positive absolute constants. We write $e^x = \exp(x)$ and $e^{2\pi i\alpha} = e(\alpha)$. The distance from α to the nearest integer is denoted by $\|\alpha\|$ so that $\|\alpha\| = \min(\alpha - [\alpha], [\alpha] + 1 - \alpha)$. We put

$$\min\left(A, \frac{1}{0}\right) = A.$$

We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$. $\nu(n)$ denotes the number of all the prime factors of n, while $\tau(n)$ denotes the divisor function:

$$\nu(n) = \sum_{p^\alpha | n,\, p^{\alpha+1} \nmid n} \alpha, \qquad \tau(n) = \sum_{d | n} 1.$$

2. In this series, we study the representations of a positive integer N in the form

$$n_1 + n_2 + \dots + n_k = N$$

where $P(n_1 n_2 \dots n_k)$ is possibly small in terms of N; this problem has been raised by P. Erdős. In particular, here we study the special case $k=3$. In fact, this paper is devoted to the proof of the following

THEOREM. *If $N > N_0$ then N can be written in the form*

$$n_1 + n_2 + n_3 = N$$

where

$$P(n_1 n_2 n_3) \leqq \exp\{3(\log N \log\log N)^{1/2}\}.$$

(Note that a recent result of A. Fujii yields this assertion with the much weaker estimate $P(n_1 n_2 n_3) < N^\varepsilon$ in place of the last inequality; see [1].)

In Part II the analogous binary problem will be studied.

3. In order to prove our theorem, we use the Hardy—Littlewood method; also, we adapt some ideas from [4].

Let y denote any real number satisfying $\exp\{3(\log N \log\log N)^{1/2}\} \leqq y < N^{2/3}$,

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and put

$$z = \frac{1}{2} y^{1/2},$$

$$Q = \frac{N}{z} = 2\frac{N}{y^{1/2}}$$

and

$$U = \left[2\frac{N}{y}\right]+1.$$

Let $\mathcal{A}$ denote the set of the integers k such that $\frac{3}{5}\frac{N}{y}<k\leqq\frac{N}{y}$ and $z<p(k)$, $P(k)\leqq y$. We write

$$A = \sum_{k\in\mathcal{A}} 1,$$

$$d_n = \sum_{\substack{mk=n\\ m\leqq y\\ k\in\mathcal{A}}} 1 \quad (\text{for} \quad 1\leqq n\leqq N),$$

$$D = \sum_{n=1}^{N} d_n^2,$$

$$S_x(\alpha) = \sum_{n\leqq x} d_n e(n\alpha) \quad (\text{for} \quad 0\leqq x\leqq N),$$

$$S(\alpha) = S_N(\alpha) = \sum_{n=1}^{N} d_n e(n\alpha),$$

$$S = S(0) = \sum_{n=1}^{N} d_n,$$

$$U(\alpha) = \sum_{n=0}^{U-1} e(n\alpha)$$

and

$$S(\alpha)U(\alpha) = \sum_{n=1}^{N+U-1} g_n e(n\alpha)$$

so that

$$g_n = \sum_{n-U<j\leqq n} d_j.$$

We start out from the integral

$$J = \int_0^1 (S(\alpha))^3 e(-N\alpha)\,d\alpha =$$

$$= \int_0^1 \Big(\sum_{1\leqq n_1,n_2,n_3\leqq N} d_{n_1}d_{n_2}d_{n_3}e((n_1+n_2+n_3-N)\alpha)\Big)\,d\alpha =$$

$$= \sum_{n_1+n_2+n_3=N} d_{n_1}d_{n_2}d_{n_3}.$$

Obviously, $d_n > 0$ implies that

$$P(n) \leqq y.$$

Thus it is sufficient to show that

$$J = \sum_{n_1+n_2+n_3=N} d_{n_1} d_{n_2} d_{n_3} > 0, \tag{1}$$

since choosing $y = \exp\{3(\log N \log\log N)^{1/2}\}$, this implies the existence of integers n_1, n_2, n_3 of the desired properties. In order to prove (1), we need some lemmas.

(We note that one of the most important ideas in the proof is the use of the weights d_n which help to keep under control both the "major arcs" with $q \neq 1$ and the "minor arcs". Also, the estimate of the integral on the interval $-1/Q < \alpha < +1/Q$ is different from the usual one; in fact, the estimate of this integral is based on some ideas from [4].)

4. In this section, we assert some preliminary lemmas.

Lemma 1. *If M is a positive integer, α a real number then we have*

$$\left|\sum_{n=0}^{M-1} e(n\alpha) - M\right| < 4M^2|\alpha|.$$

Proof. With respect to the well-known inequality

$$|1 - e(\beta)| \leqq 2\pi|\beta| \tag{2}$$

we have

$$\left|\sum_{n=0}^{M-1} e(n\alpha) - M\right| \leqq \sum_{n=0}^{M-1} |e(n\alpha) - 1| \leqq \sum_{n=0}^{M-1} 2\pi n|\alpha| =$$

$$= \pi(M-1)M|\alpha| \leqq 4M^2|\alpha|.$$

Lemma 2. *For arbitrary real numbers α, x we have*

$$\left|\sum_{1 \leqq m \leqq x} e(m\alpha)\right| \leqq \min\left(x, \frac{1}{2\|\alpha\|}\right).$$

See e.g. [2], p. 9.

Lemma 3. *If α, V are real numbers and a, q, f are integers such that $q > 0$, $(a, q) = 1$ and $\left|\alpha - \frac{a}{q}\right| \leqq \frac{1}{q^2}$ then we have*

$$\sum_{x=f+1}^{f+q} \min\left(V, \frac{1}{2\|\alpha x\|}\right) \leqq 6V + q\log q.$$

See e.g. [2], p. 23.

LEMMA 4. *If* α, M, V *are real numbers and* a, q *are integers such that* $M \geqq 1$, $q > 0$, $(a, q) = 1$ *and* $\left|\alpha - \frac{a}{q}\right| \leqq \frac{1}{q^2}$ *then we have*

$$\sum_{x \leqq M} \min\left(V, \frac{1}{2\|\alpha x\|}\right) \leqq \left(\frac{M}{q}+1\right)(6V + q\log q).$$

PROOF. With respect to Lemma 3, we have

$$\sum_{x \leqq M} \min\left(V, \frac{1}{2\|\alpha x\|}\right) \leqq \sum_{k=1}^{[M/q]+1} \sum_{x=(k-1)q+1}^{kq} \min\left(V, \frac{1}{2\|\alpha x\|}\right) \leqq$$

$$\leqq \sum_{k=1}^{[M/q]+1} (6V + q\log q) = \left(\left[\frac{M}{q}\right]+1\right)(6V+q\log q) <$$

$$< \left(\frac{M}{q}+1\right)(6V+q\log q).$$

LEMMA 5. *For* $x \geqq 2$ *we have*

$$\sum_{n \leqq x} (\tau(n))^2 < c_1 x (\log x)^3.$$

See e.g. [3], p. 26.

5. In this section, we estimate D, S, $S(\alpha)$, g_n and A.

LEMMA 6. *We have*

$$S \leqq N.$$

PROOF.

$$S = \sum_{n=1}^{N} d_n = \sum_{n=1}^{N} \sum_{\substack{mk=n \\ m \leqq y \\ k \in \mathcal{A}}} 1 = \sum_{m \leqq y} \sum_{\substack{k \leqq N/m \\ k \in \mathcal{A}}} 1 \leqq$$

$$\leqq \sum_{m \leqq y} \sum_{k \in \mathcal{A}} 1 \leqq \sum_{m \leqq y} \sum_{k \leqq N/y} 1 \leqq \sum_{m \leqq y} \frac{N}{y} \leqq y \frac{N}{y} = N.$$

LEMMA 7. *For* $N \geqq 2$ *we have*

$$D < c_1 N (\log N)^3.$$

PROOF. With respect to Lemma 5, we have

$$D = \sum_{n=1}^{N} d_n^2 = \sum_{n=1}^{N} \Big(\sum_{\substack{mk=n \\ m \leqq y \\ k \in \mathcal{A}}} 1\Big)^2 \leqq \sum_{n=1}^{N} (\tau(n))^2 < c_1 N (\log N)^3.$$

LEMMA 8. *If* $1 \leqq u \leqq N$ *and* a, q *are integers such that* $2 \leqq q \leqq z$ *and* $(a, q) = 1$ *then we have*

$$|S_u(a/q)| \leqq \frac{Nq}{y}.$$

PROOF. We have

$$S_u(a/q) = \sum_{n \leqq u} d_n e(na/q) = \sum_{b=1}^{q} \Big(\sum_{\substack{n \leqq u \\ nx \equiv b (\mathrm{mod}\ q)}} d_n \Big) e(b/q). \tag{3}$$

Here the inner sum can be rewritten in the following form:

$$\begin{aligned} \sum_{\substack{n \leqq u \\ na \equiv b (\mathrm{mod}\ q)}} d_n &= \sum_{\substack{n \leqq u \\ na \equiv b (\mathrm{mod}\ q)}} \sum_{\substack{mk = n \\ m \leqq y \\ k \in \mathscr{A}}} 1 = \\ &= \sum_{\substack{mk \leqq u \\ mka \equiv b (\mathrm{mod}\ q) \\ m \leqq y \\ k \in \mathscr{A}}} 1 = \sum_{\substack{k \leqq u \\ k \in \mathscr{A}}} \sum_{\substack{m \leqq u/k \\ mka \equiv b (\mathrm{mod}\ q) \\ m \leqq y}} 1 = \\ &= \sum_{\substack{k \leqq u/y \\ k \in \mathscr{A}}} \sum_{\substack{m \leqq y \\ mka \equiv b (\mathrm{mod}\ q)}} 1 + \sum_{\substack{u/y < k \leqq u \\ k \in \mathscr{A}}} \sum_{\substack{m \leqq u/k \\ mka \equiv b (\mathrm{mod}\ q)}} 1. \end{aligned} \tag{4}$$

$p(k) > z \geqq q$ and $(a, q) = 1$ imply that $(ka, q) = 1$ hence

$$\Big| \sum_{\substack{m \leqq y \\ mka \equiv b (\mathrm{mod}\ q)}} 1 - \frac{y}{q} \Big| \leqq 1$$

and

$$\Big| \sum_{\substack{m \leqq u/k \\ mka \equiv b (\mathrm{mod}\ q)}} 1 - \frac{u}{kq} \Big| \leqq 1$$

so that we obtain from (4) that

$$\begin{aligned} &\Big| \sum_{\substack{n \leqq u \\ na \equiv b (\mathrm{mod}\ q)}} d_n - \Big(\sum_{\substack{k \leqq u/y \\ k \in \mathscr{A}}} \frac{y}{q} + \sum_{\substack{u/y < k \leqq u \\ k \in \mathscr{A}}} \frac{u}{kq} \Big) \Big| = \\ &= \Big| \sum_{\substack{k \leqq u/y \\ k \in \mathscr{A}}} \Big(\sum_{\substack{m \leqq y \\ mka \equiv b (\mathrm{mod}\ q)}} 1 - \frac{y}{q} \Big) + \sum_{\substack{u/y < k \leqq u \\ k \in \mathscr{A}}} \Big(\sum_{\substack{m \leqq u/k \\ mka \equiv b\ (\mathrm{mod}\ q)}} 1 - \frac{u}{kq} \Big) \Big| \leqq \\ &\leqq \sum_{\substack{k \leqq u/y \\ k \in \mathscr{A}}} 1 + \sum_{\substack{u/y < k \leqq u \\ k \in \mathscr{A}}} 1 = \sum_{\substack{k \leqq u \\ k \in \mathscr{A}}} 1 \leqq \sum_{k \in \mathscr{A}} 1 \leqq \sum_{k \leqq N/y} 1 = \frac{N}{y}. \end{aligned} \tag{5}$$

By $q \geqq 2$,

$$\sum_{b=1}^{q} e(b/q) = 0.$$

Thus (3) and (5) yield that

$$|S_u(a/q)| = \Bigg|\sum_{b=1}^{q}\Bigg\{\sum_{\substack{n\le u\\ na\equiv b(\mathrm{mod}\, q)}} d_n - \Bigg(\sum_{\substack{k\le u/q\\ k\in\mathcal{A}}}\frac{y}{q} + \sum_{\substack{u/y<k\le u\\ k\in\mathcal{A}}}\frac{u}{kq}\Bigg)\Bigg\} e(b/q) +$$

$$+\Bigg(\sum_{\substack{k\le u/q\\ k\in\mathcal{A}}}\frac{y}{q} + \sum_{\substack{u/y<k\le u\\ k\in\mathcal{A}}}\frac{u}{kq}\Bigg)\sum_{b=1}^{q} e(b/q)\Bigg| \le$$

$$\le \sum_{b=1}^{q}\Bigg|\sum_{\substack{n\le u\\ na\equiv b(\mathrm{mod}\, q)}} d_n - \Bigg(\sum_{\substack{k\le u/q\\ k\in\mathcal{A}}}\frac{y}{q} + \sum_{\substack{u/y<k\le u\\ k\in\mathcal{A}}}\frac{u}{kq}\Bigg)\Bigg| \le \sum_{b=1}^{q}\frac{N}{y} = \frac{Nq}{y}.$$

LEMMA 9. *If* α *is a real number and* a, q *are integers such that* $2\le q\le z$, $(a, q)=1$ *and* $\left|\alpha-\frac{a}{q}\right|<\frac{1}{qQ}$ *then we have*

$$|S(\alpha)| < \frac{4N}{y^{1/2}}.$$

PROOF. We write $\beta=\alpha-a/q$ so that

$$|\beta| = \left|\alpha-\frac{a}{q}\right| < \frac{1}{qQ}.$$

Then by using Lemma 8 and (2), we obtain by partial summation that

$$|S(\alpha)| = \left|\sum_{n=1}^{N}\big(S_n(a/q)-S_{n-1}(a/q)\big)e(n\beta)\right| =$$

$$= \left|\sum_{n=1}^{N} S_n(a/q)\big(e(n\beta)-e((n+1)\beta)\big)+S_N(a/q)e((N+1)\beta)\right| \le$$

$$\le \sum_{n=1}^{N}|S_n(a/q)|\,|1-e(\beta)|+|S_N(a/q)| \le$$

$$\le \sum_{n=1}^{N}\frac{Nq}{y}2\pi|\beta|+\frac{Nq}{y} = \frac{Nq}{y}(1+2\pi N|\beta|) < \frac{Nq}{y}\left(1+7N\frac{1}{qQ}\right) =$$

$$= \frac{Nq}{y}\left(\frac{N}{zQ}+\frac{7N}{qQ}\right) \le \frac{Nq}{y}\,\frac{8N}{qQ} = \frac{8N^2}{yQ} = \frac{4N}{y^{1/2}}.$$

LEMMA 10. *If* α *is a real number and* a, q *are integers such that* $z<q\le Q$, $(a, q)=1$ *and* $\left|\alpha-\frac{a}{q}\right|\le\frac{1}{q^2}$ *then for* $N>N_1$ *we have*

$$|S(\alpha)| < 5\frac{N}{y^{1/2}}\log N.$$

PROOF. For $k \in \mathscr{A}$ we have

$$\frac{N}{k} \geqq \frac{N}{N/y} = y.$$

Thus by using Lemmas 2 and 4, we obtain for large N that

$$|S(\alpha)| = \Big|\sum_{n=1}^{N} d_n e(n\alpha)\Big| = \Big|\sum_{\substack{mk \leqq N \\ m \leqq y \\ k \in \mathscr{A}}} e(mk\alpha)\Big| =$$

$$= \Big|\sum_{k \in \mathscr{A}} \Big(\sum_{\substack{m \leqq N/k \\ m \leqq y}} e(mk\alpha)\Big)\Big| = \Big|\sum_{k \in \mathscr{A}} \Big(\sum_{m \leqq y} e(mk\alpha)\Big)\Big| \leqq$$

$$\leqq \sum_{k \in \mathscr{A}} \Big|\sum_{m \leqq y} e(mk\alpha)\Big| \leqq \sum_{k \in \mathscr{A}} \min\left(y, \frac{1}{2\|k\alpha\|}\right) \leqq \sum_{k \leqq N/y} \min\left(y, \frac{1}{2\|k\alpha\|}\right) \leqq$$

$$\leqq \left(\frac{N}{qy}+1\right)(6y+q\log q) = \frac{6N}{q}+6y+\frac{N}{y}\log q+q\log q <$$

$$< \frac{6N}{z}+6\frac{y^{3/2}}{y^{1/2}}+\frac{N}{y}\log N+Q\log N <$$

$$< \frac{6N}{z}+6\frac{N}{y^{1/2}}+\frac{N}{z}\log N+\frac{N}{z}\log N = (2+o(1))\frac{N}{z}\log N+O\left(\frac{N}{y^{1/2}}\right) =$$

$$= (4+o(1))\frac{N}{y^{1/2}}\log N < 5\frac{N}{y^{1/2}}\log N.$$

LEMMA 11. *If*

$$\frac{1}{Q} < \alpha < 1-\frac{1}{Q} \tag{6}$$

then for $N > N_2$ *we have*

$$|S(\alpha)| < 5\frac{N}{y^{1/2}}\log N. \tag{7}$$

PROOF. By Dirichlet's theorem, there exist integers a, q such that $1 \leqq q \leqq Q$, $(a, q)=1$ and

$$\left|\alpha-\frac{a}{q}\right| < \frac{1}{qQ}\left(\leqq \frac{1}{q^2}\right).$$

(6) implies that $q>1$. If $2 \leqq q \leqq z$ then (7) is a consequence of Lemma 9 while if $z<q \leqq Q$ then (7) holds by Lemma 10.

LEMMA 12. *If n is a positive integer satisfying $U \leqq n \leqq 3N/5$ then we have*

$$g_n \geqq A.$$

PROOF. For $U \leqq n \leqq 3N/5$ we have

$$g_n = \sum_{j=n-U+1}^{n} d_j = \sum_{j=n-U+1}^{n} \sum_{\substack{mk=j \\ m \leqq y \\ k \in \mathcal{A}}} 1 =$$

$$= \sum_{\substack{n-U<mk\leqq n \\ m\leqq y \\ k\in\mathcal{A}}} 1 = \sum_{k\in\mathcal{A}} \sum_{\substack{\frac{n-U}{k}<m\leqq\frac{n}{k} \\ m\leqq y}} 1 = \sum_{k\in\mathcal{A}} \sum_{\frac{n-U}{k}<m\leqq\frac{n}{k}} 1 \geqq$$

$$\geqq \sum_{k\in\mathcal{A}} \left(\frac{U}{k}-1\right) \geqq \sum_{k\in\mathcal{A}} \left(\frac{U}{k}-\frac{U}{2N/y}\right) \geqq \sum_{k\in\mathcal{A}} \left(\frac{U}{k}-\frac{U}{2k}\right) =$$

$$= \frac{U}{2} \sum_{k\in\mathcal{A}} \frac{1}{k} \geqq \frac{U}{2} \sum_{k\in\mathcal{A}} \frac{1}{N/y} = \frac{Uy}{2N} \sum_{k\in\mathcal{A}} 1 = \frac{Uy}{2N} A > A$$

since for $k\in\mathcal{A}$ and $n\leqq 3N/5$,

$$\frac{n}{k} < \frac{3N/5}{3N/5y} = y.$$

LEMMA 13. *For* $t>0$ *and* $j=1, 2, \ldots,$ *let*

$$A_j(t) = \sum_{\substack{3t/5<k\leqq t \\ z<p(k)\leqq P(k)\leqq y \\ \nu(k)\leqq j}} 1.$$

If $N>N_3$ *and* $2\leqq j$ *then for*

(8) $$2z < t \leqq \frac{y^j}{2^{j-2}}$$

we have

(9) $$A_j(t) > \frac{t}{j!(7\log y)^j}.$$

PROOF. We prove the assertion by induction (on j). Assume first that $j=2$ so that

$$2z < t \leqq y^2.$$

If

$$2z < t \leqq y$$

then for large N (then also y is large) we have

(10)
$$A_2(t) = \sum_{\substack{3t/5<k\leqq t \\ z<p(k)\leqq P(k)\leqq y \\ \nu(k)\leqq 2}} 1 \geqq \sum_{\substack{3t/5<p\leqq t \\ z<p\leqq y}} 1 =$$

$$= \sum_{3t/5<p\leqq t} 1 > \frac{1}{2}\frac{t}{\log t} \geqq \frac{1}{2}\frac{t}{\log y} > \frac{t}{2!(7\log y)^2}$$

while for

$$y < t \leqq y^2$$

and large N we have

$$A_2(t) = \sum_{\substack{3t/5<k\leq t\\ z<p(k)\leq P(k)\leq y\\ \nu(k)\leq 2}} 1 \geq \sum_{\substack{3t/5<pq\leq t\\ z<p\leq q\leq y}} 1 \geq$$

(11)
$$\geq \frac{1}{2}\Bigg(\sum_{\sqrt{3t/5}<p\leq\sqrt{t}} 1\Bigg)^2 \geq \frac{1}{2}\left(\frac{1}{5}\frac{\sqrt{t}}{\log\sqrt{t}}\right)^2 > \frac{1}{20}\frac{t}{\log^2 t} \geq$$

$$\geq \frac{1}{80}\frac{t}{\log^2 y} > \frac{t}{2!(7\log y)^2}$$

(since $\sqrt{3t/5}>\sqrt{3y/5}=\sqrt{3(2z)^2/5}>z$ and $\sqrt{t}\leq y$).

(10) and (11) yield (9) (with $j=2$) in both cases.

Assume that (9) holds for all t satisfying (8). We have to show that

$$2z < t \leq \frac{y^{j+1}}{2^{j-1}}$$

implies

(12)
$$A_{j+1}(t) > \frac{t}{(j+1)!(7\log y)^{j+1}}.$$

If

$$2z < t \leq \frac{y^j}{2^{j-2}}$$

then this is a consequence of (9) and the trivial inequality $A_j(t)\leq A_{j+1}(t)$. (Note that the right-hand side of (9) is decreasing function of j.) Thus it is sufficient to study the case

(13)
$$\frac{y^j}{2^{j-2}} < t \leq \frac{y^{j+1}}{2^{j-1}}.$$

Then we have

$$A_{j+1}(t) = \sum_{\substack{3t/5<k\leq t\\ z<p(k)\leq P(k)\leq y\\ \nu(k)\leq j+1}} 1 \geq$$

(14)

$$\geq \frac{1}{j+1}\sum_{y/2<p\leq y}\ \sum_{\substack{3t/5<pl\leq t\\ z<p(l)\leq P(l)\leq y\\ \nu(l)\leq j}} 1 = \frac{1}{j+1}\sum_{y/2<p\leq y} A_j(t/p).$$

If t satisfies (13) and $y/2<p\leq y$ then

$$\frac{t}{p} \geq \frac{y^j/2^{j-2}}{y} \geq \frac{y^2/2^{2-2}}{y} = y > 2z$$

and

$$\frac{t}{p} < \frac{y^{j+1}/2^{j-1}}{y/2} = \frac{y^j}{2^{j-2}}$$

so that (8) holds and thus (9) can be used in order to estimate $A_j(t/p)$. We obtain from (14) that for large N,

$$A_{j+1}(t) \geqq \frac{1}{j+1} \sum_{y/2<p\leqq y} A_j(t/p) >$$

$$> \frac{1}{j+1} \sum_{y/2<p\leqq y} \frac{t/p}{j!(7\log y)^j} = \frac{t}{(j+1)!(7\log y)^j} \sum_{y/2<p\leqq y} \frac{1}{p} \geqq$$

$$\geqq \frac{t}{(j+1)!(7\log y)^j} \frac{1}{y} \sum_{y/2<p\leqq y} 1 >$$

$$> \frac{t}{(j+1)!(7\log y)^j} \frac{1}{y} \frac{1}{3} \frac{y}{\log y} > \frac{t}{(j+1)!(7\log y)^{j+1}}$$

which proves (12) and this completes the proof of Lemma 13.

LEMMA 14. *For* $N>N_3$ *we have*

$$A > \frac{N}{y} \exp\left\{-\frac{6}{5} \frac{\log N}{\log y} \log\log N\right\}.$$

PROOF. Define the positive integer j by

$$\frac{y^{j-1}}{2^{j-3}} < \frac{N}{y} \leqq \frac{y^j}{2^{j-2}}$$

so that

$$\left(\frac{y}{2}\right)^j < N/8 < N,$$

$$j < \frac{\log N}{\log y/2}.$$

Then for large N, Lemma 13 yields that

$$A = \sum_{\substack{3N/5y<k\leqq N/y \\ z<p(k)\leqq P(k)\leqq y}} 1 \geqq \sum_{\substack{3N/5y<k\leqq N/y \\ z<p(k)\leqq P(k)\leqq y \\ \nu(k)\leqq j}} 1 = A_j(N/y) >$$

$$> \frac{N/y}{j!(7\log y)^j} > \frac{N}{y} \frac{1}{(7y\log y)^j} = \frac{N}{y} \exp\{-j\log(7j\log y)\} >$$

$$> \frac{N}{y} \exp\left\{-\frac{\log N}{\log y/2} \log\left(7\frac{\log N}{\log y/2} \log y\right)\right\} =$$

$$= \frac{N}{y} \exp\left\{-(1+o(1))\frac{\log N}{\log y} \log\log N\right\} >$$

$$> \frac{N}{y} \exp\left(-\frac{6}{5} \frac{\log N}{\log y} \log\log N\right).$$

6. In this section, we complete the proof of Theorem 1.

For $|a|\leqq 1$ we have

$$|1-a^3| = |1-a|\,|1+a+a^2| \leqq 3\,|1-a|.$$

Thus by using Lemmas 1, 6, 7, 11 and Parseval's formula, we obtain that

$$\left|J-\frac{1}{U^3}\int_0^1 (S(\alpha)U(\alpha))^3 e(-N\alpha)\,d\alpha\right| =$$

$$= \left|\int_{-1/Q}^{1-1/Q} (S(\alpha))^3 e(-N\alpha)\,d\alpha - \frac{1}{U^3}\int_{-1/Q}^{1-1/Q} (S(\alpha)U(\alpha))^3 e(-N\alpha)\,d\alpha\right| =$$

$$= \left|\int_{-1/Q}^{1/Q} (S(\alpha))^3\left(1-\left(\frac{U(\alpha)}{U}\right)^3\right)e(-N\alpha)\,d\alpha + \int_{1/Q}^{1-1/Q} (S(\alpha))^3\left(1-\left(\frac{U(\alpha)}{U}\right)^3\right)e(-N\alpha)\,d\alpha\right| \leqq$$

$$\leqq \int_{-1/Q}^{1/Q} |S(\alpha)^3|\,3\left|\frac{U-U(\alpha)}{U}\right| d\alpha + \int_{1/Q}^{1-1/Q} |S(\alpha)|^3\left(1+\left|\frac{U(\alpha)}{U}\right|^3\right) d\alpha \leqq$$

$$\leqq \int_{-1/Q}^{+1/Q} S\,|S(\alpha)|^2\,3\,\frac{4U^2|\alpha|}{U}\,d\alpha + \int_{-1/Q}^{1-1/Q} 2\,|S(\alpha)|^3\,d\alpha \leqq \tag{15}$$

$$\leqq 12SU\int_{-1/Q}^{1/Q} |S(\alpha)|^2\frac{1}{Q}\,d\alpha + 2\left(\max_{1/Q<\alpha<1-1/Q}|S(\alpha)|\right)\int_{-1/Q}^{1-1/Q} |S(\alpha)|^2\,d\alpha \leqq$$

$$\leqq 12\,\frac{SU}{Q}\int_0^1 |S(\alpha)|^2\,d\alpha + 2\cdot 5\,\frac{N}{y^{1/2}}\log N\int_0^1 |S(\alpha)|^2\,d\alpha =$$

$$= \left(12\,\frac{SU}{Q}+10\,\frac{N}{y^{1/2}}\log N\right)D <$$

$$< 12\left(\frac{N\cdot 3N/y}{2N/y^{1/2}}+\frac{N}{y^{1/2}}\log N\right)c_1N(\log N)^3 < c_2\,\frac{N^2(\log N)^4}{y^{1/2}}.$$

Furthermore, by Lemma 12 and since $g_n\geqq 0$ for all n, for large N we have

$$\int_0^1 (S(\alpha)U(\alpha))^3 e(-N\alpha)\,d\alpha =$$

$$= \int_0^1\left(\sum_{n=1}^{N+U-1} g_n e(n\alpha)\right)^3 e(-N\alpha)\,d\alpha =$$

$$= \sum_{\substack{n_1+n_2+n_3=N\\ 1\leqq n_1,n_2,n_3\leqq N+U-1}} g_{n_1}g_{n_2}g_{n_3} = \sum_{\substack{n_1+n_2+n_3=N\\ 1\leqq n_1,n_2,n_3\leqq N}} g_{n_1}g_{n_2}g_{n_3} \geqq \tag{16}$$

$$\geqq \sum_{\substack{n_1+n_2+n_3=N\\ N/5<n_1,n_2,n_3\leqq N}} g_{n_1}g_{n_2}g_{n_3} \geqq \sum_{\substack{n_1+n_2+n_3=N\\ N/5<n_1,n_2,n_3\leqq N}} \left(\min_{N/5<n\leqq 3N/5} g_n\right)^3 \geqq$$

$$\geqq A^3 \sum_{\substack{n_1+n_2+n_3=N\\ N/5<n_1,n_2,n_3\leqq N}} 1.$$

In order to estimate the last sum, put

$$m_i = n_i-\left[\frac{N}{5}\right] \quad (\text{for } i = 1, 2, 3).$$

Then for large N we have

$$\sum_{\substack{n_1+n_2+n_3=N \\ N/5<n_1,n_2,n_3\le N}} 1 = \sum_{\substack{m_1+m_2+m_3=N-3[N/5] \\ 0<m_1,m_2,m_3}} 1 = \tag{17}$$

$$= \binom{N-3[N/5]}{2} > \frac{1}{2}\left(N-4\frac{N}{5}\right)^2 = \frac{1}{50}N^2.$$

We obtain from (16) and (17) that

$$\int_0^1 (S(\alpha)U(\alpha))^3 e(-N\alpha)\,d\alpha > \frac{1}{50}N^2A^3. \tag{18}$$

By Lemma 14, (15) and (18) yield for large N that

$$|J| \geqq \left|\frac{1}{U^3}\int_0^1 (S(\alpha)U(\alpha))^3 e(-N\alpha)\,d\alpha\right| -$$

$$-\left|J-\frac{1}{U^3}\int_0^1 (S(\alpha)U(\alpha))^3 e(-N\alpha)\,d\alpha\right| \geqq$$

$$\geqq \frac{1}{U^3}\frac{1}{50}N^2A^3 - c_2\frac{N^2(\log N)^4}{y^{1/2}} >$$

$$> \frac{1}{\left(3\frac{N}{y}\right)^3}\frac{1}{50}N^2A^3 - c_2\frac{N^2(\log N)^4}{y^{1/2}} >$$

$$> \frac{1}{2000}N^2\left(\left(\frac{A}{N/y}\right)^3 - c_3\frac{(\log N)^4}{y^{1/2}}\right) >$$

$$> \frac{1}{2000}N\left\{\exp\left(-3\frac{6}{5}\frac{\log N}{\log y}\log\log N\right) - c_3\exp\left(4\log\log N - \frac{1}{2}\log y\right)\right\} \geqq$$

$$\geqq \frac{1}{2000}N^2\left\{\exp\left(-\frac{18}{5}\frac{\log N}{3(\log N\log\log N)^{1/2}}\log\log N\right) -\right.$$

$$\left. -c_3\exp\left(4\log\log N - \frac{1}{2}3(\log N\log\log N)^{1/2}\right)\right\} =$$

$$= \frac{1}{2000}N^2\left\{\exp\left(-\frac{6}{5}(\log N\log\log N)^{1/2}\right) -\right.$$

$$\left. -c_3\exp\left(-\left(\frac{3}{2}+o(1)\right)(\log N\log\log N)^{1/2}\right)\right\} =$$

$$= \frac{1}{2000}(1+o(1))N^2\exp\left(-\frac{6}{5}(\log N\log\log N)^{1/2}\right) > 0$$

which proves (1) and this completes the proof of our theorem.

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PACKING AND COVERING WITH INCONGRUENT CIRCLES

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We shall denote a domain and its area with the same symbol and the power mean $\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\varepsilon}\right)^{1/\varepsilon}$ of exponent ε of the quantities $x_1, \ldots, x_n$ with $M_\varepsilon(x_1, \ldots, x_n)$. We shall prove the following theorems.

THEOREM 1. *Let $\alpha_0=0.91249\ldots$ be the positive root of the equation*

$$\left(\frac{1}{3}\cot\frac{\pi}{3}\right)^{\frac{x}{1-x}} - 4\left(\frac{1}{6}\cot\frac{\pi}{6}\right)^{\frac{x}{1-x}} + 3\left(\frac{1}{7}\cot\frac{\pi}{7}\right)^{\frac{x}{1-x}} = 0. \tag{1}$$

If the circles $C_1, \ldots, C_n$ are packed into a convex polygon P with at most six sides then for any $\alpha\leqq\alpha_0$ the density $d=(C_1+\ldots+C_n)/P$ of the packing satisfies the inequality

$$d \leqq \frac{\pi}{\sqrt{12}}\frac{M_1(C_1, \ldots, C_n)}{M_\alpha(C_1, \ldots, C_n)}. \tag{2}$$

THEOREM 2. *Let $\beta_0=1.22540\ldots$ be the positive root of the equation*

$$\left(\frac{1}{3}\sec\frac{2\pi}{3}\right)^{\frac{x}{1-x}} - 4\left(\frac{1}{6}\sec\frac{2\pi}{6}\right)^{\frac{x}{1-x}} + 3\left(\frac{1}{7}\sec\frac{2\pi}{7}\right)^{\frac{x}{1-x}} = 0. \tag{3}$$

If the circles $C_1, \ldots, C_n$ cover a convex polygon P with at most six sides then for any $\beta\geqq\beta_0$ the density $D=(C_1+\ldots+C_n)/P$ of the covering satisfies the inequality

$$D \geqq \frac{2\pi}{\sqrt{27}}\frac{M_1(C_1, \ldots, C_n)}{M_\beta(C_1, \ldots, C_n)}. \tag{4}$$

It is easy to check that the equations (1) and (3) have only one positive root. The inequalities (2) and (4) have been proved earlier for $\alpha\leqq 0.77\ldots$ and $\beta\geqq 2.11\ldots$ [3]. The packing consisting of the face-incircles of the Archimedean tiling (3, 12, 12) and the covering consisting of the face-circumcircles of (4, 8, 8) show that for $\alpha>0.9487\ldots$ and $\beta<1.1049\ldots$ (2) and (4) do not hold any more. Recently, L. Fejes Tóth [5] proved that the densities d and D occurring in Theorems 1 and 2 satisfy

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the inequalities

(5) $$d \leqq \left\{1+\frac{\sqrt{12}-\pi}{\pi}\frac{M_{1/3}(C_1, \dots, C_n)}{M_1(C_1, \dots, C_n)}\right\}^{-1}$$

and

(6) $$D \geqq \left\{1-\frac{1}{4}\left(1-\frac{\sqrt{27}}{4\pi}\right)\frac{M_{1/3}(C_1, \dots, C_n)}{M_1(C_1, \dots, C_n)}\right\}^{-1}.$$

We note that in some cases the inequalities (2) and (4), in other cases the inequalities (5) and (6) are stronger. As an example we consider a set of circles of areas 1 and 10 with ten times as many big circles as small ones. Now (2) and (4) imply that $d \leqq 0.91214\dots$ and $D \geqq 1.19426\dots$, while the inequalities (5) and (6) yield only $d \leqq 0.91220\dots$ and $D \geqq 1.15938\dots$. On the other hand, if the ratio of the number of big circles to the number of small circles is 1:10, (5) and (6) are stronger than (2) and (4).

Now we turn to the proof of Theorems 1 and 2. Since $M_\varepsilon(x_1, \dots, x_n)$ is an increasing function of ε, it suffices to prove (2) for $\alpha=\alpha_0$ and (4) for $\beta=\beta_0$. Let D_i be the Dirichlet cell of C_i with respect to P, i.e. the set of those points of P whose power with respect to C_i is less than their power with respect to any other circle C_j, $j=1, \dots, n$, $j\neq i$. It is known that the sets D_i are convex polygons which tile P [4]. Denoting with p_i the number of vertices of D_i we have, as a simple consequence of Euler's formula,

(7) $$p_1+\dots+p_n \leqq 6n.$$

Furthermore, we have $C_i \subset D_i$ if the circles are packed into P and $C_i \supset D_i$ if the circles cover P. We write

$$\varphi(p) = \frac{\pi}{p}\cot\frac{\pi}{p} \quad \text{and} \quad \psi(p) = \frac{2\pi}{p}\sec\frac{2\pi}{p}.$$

We note that $\varphi(p)$ is equal to the density of a circle with respect to the regular p-gon circumscribed about it. Analogously, $\psi(p)$ is the density of a circle with respect to the regular n-gon inscribed into it. Thus we have $C_i \leqq D_i\varphi(p_i)$, $i=1, \dots, n$, in the case of the packing and $C_i \geqq D_i\psi(p_i)$, $i=1, \dots, n$, in the case of the covering. Combining these relations with Hölder's inequality and with the relation $\sum_{i=1}^{n} D_i = P$, we obtain

(8) $$\sum_{i=1}^{n} C_i^{\alpha_0} \leqq \sum_{i=1}^{n} D_i^{\alpha_0}[\varphi(p_i)]^{\alpha_0} \leqq P^{\alpha_0}\left\{\sum_{i=1}^{n}[\varphi(p_i)]^{\frac{\alpha_0}{1-\alpha_0}}\right\}^{1-\alpha_0}$$

and

(9) $$\sum_{i=1}^{n} C_i^{\beta_0} \geqq \sum_{i=1}^{n} D_i^{\beta_0}[\psi(p_i)]^{\beta_0} \geqq P^{\beta_0}\left\{\sum_{i=1}^{n}(p_i)^{\frac{\beta_0}{1-\beta_0}}\right\}^{1-\beta_0},$$

respectively. Here we used Hölder's inequality in the form as stated in Theorem 13 on p. 28 of [6].

We are going to give upper bounds for the sums

$$\sum_{i=1}^{n}[\varphi(p_i)]^a, \quad a=\frac{\alpha_0}{1-\alpha_0} \quad \text{and} \quad \sum_{i=1}^{n}[\psi(p_i)]^b, \quad b=\frac{\beta_0}{1-\beta_0}.$$

We consider the functions

$$f(p)=[\varphi(6)]^a+(p-6)\{[\varphi(7)]^a-[\varphi(6)]^a\}$$

and

$$g(p)=[\psi(6)]^b+(p-6)\{[\psi(7)]^b-[\psi(6)]^b\}.$$

The relations (1) and (3), the definitions of $f(p)$ and $g(p)$ and some numerical computations show that

$$\begin{aligned}
f(3)-[\varphi(3)]^a &= g(3)-[\psi(3)]^b = f(6)-[\varphi(6)]^a = g(6)-[\psi(6)]^b = \\
&= f(7)-[\varphi(7)]^a = g(7)-[\psi(7)]^b = 0,\\
f(4)-[\varphi(4)]^a &= 0.04\ldots>0, \quad g(4)-[\psi(4)]^b = 0.03\ldots>0,\\
f(5)-[\varphi(5)]^a &= 0.02\ldots>0, \quad g(5)-[\psi(5)]^b = 0.02\ldots>0,\\
f(8)-[\varphi(8)]^a &= 0.02\ldots>0, \quad g(8)-[\psi(8)]^b = 0.02\ldots>0,\\
f(9)-[\varphi(9)]^a &= 0.06\ldots>0, \quad g(9)-[\psi(9)]^b = 0.06\ldots>0,\\
f(10)-[\varphi(10)]^a &= 0.13\ldots>0, \quad g(10)-[\psi(10)]^b = 0.12\ldots>0,\\
f(11)-[\varphi(11)]^a &= 0.20\ldots>0, \quad g(11)-[\psi(11)]^b = 0.19\ldots>0,\\
f(12) &= 1.07\ldots>1, \qquad g(12) = 1.05\ldots>1.
\end{aligned}$$

Since $[\varphi(p)]^a<1$ and $[\psi(p)]^b<1$ for any $p\geqq 3$ and since the functions $f(p)$ and $g(p)$ are increasing, the relations above imply that $[\varphi(p)]^a\leqq f(p)$ and $[\psi(p)]^b\leqq g(p)$ for any $p\geqq 3$. Using inequality (7) and the fact that the functions $f(p)$ and $g(p)$ are linear and increasing we conclude that

$$\sum_{i=1}^{n}[\varphi(p_i)]^a \leqq \sum_{i=1}^{n} f(p_i) \leqq nf(6) = n[\varphi(6)]^a = n\left[\frac{\pi}{\sqrt{12}}\right]^a$$

and

$$\sum_{i=1}^{n}[\psi(p_i)]^b \leqq \sum_{i=1}^{n} g(p_i) \leqq ng(6) = n[\psi(6)]^b = n\left[\frac{2\pi}{\sqrt{27}}\right]^b.$$

Combining these inequalities with (8) and (9) we obtain

$$\sum_{i=1}^{n} C_i^{\alpha_0} \leqq n^{1+\alpha_0}\left[\frac{\pi P}{\sqrt{12}}\right]^{\alpha_0}$$

and

$$\sum_{i=1}^{n} C_i^{\beta_0} \geqq n^{1-\beta_0}\left[\frac{2\pi P}{\sqrt{27}}\right]^{\beta_0},$$

which can easily be seen to be equivalent with (2) and (4).

Theorem 1 and Theorem 2 can be generalized as follows:

THEOREM 1*. *Let* $\varphi_C(p)$ *be the density of a convex domain* C *with respect to the* p*-gon of minimal area circumscribed about* C. *Suppose that* α_0 *is a positive number and* $f(p)$ *is a linear function such that* $[\varphi_C(p)]^{\alpha_0/(1-\alpha_0)} \leqq f(p)$ *for* $p=3, 4, \ldots$ *and* $[\varphi_C(6)]^{\alpha_0/(1-\alpha_0)} = f(6)$. *If* $C_1, \ldots, C_n$ *are affine images of* C *forming a packing into a convex polygon* P *with at most six sides then for any* $\alpha \leqq \alpha_0$ *the density*

$$d = (C_1 + \ldots + C_n)/P$$

satisfies the inequality

$$d \leqq \varphi_C(6) \frac{M_1(C_1, \ldots, C_n)}{M_\alpha(C_1, \ldots, C_n)}. \tag{10}$$

THEOREM 2*. *Let* $\psi_C(p)$ *be the density of a convex domain* C *with respect to the* p*-gon of maximal area inscribed into* C. *Suppose that* β_0 *is a positive number and* $g(p)$ *is a linear function such that* $[\psi_C(p)]^{\beta_0/(1-\beta_0)} \leqq g(p)$ *for* $p=3, 4, \ldots$ *and* $[\psi_C(6)]^{\beta_0/(1-\beta_0)} = g(6)$. *If* $C_1, \ldots, C_n$ *are affine images of* C *covering a convex polygon* P *with at most six sides without crossing each other, then for any* $\beta \geqq \beta_0$ *the density* $D = (C_1 + \ldots + C_n)/P$ *satisfies the inequality*

$$D \geqq \psi_C(6) \frac{M_1(C_1, \ldots, C_n)}{M_\beta(C_1, \ldots, C_n)}. \tag{11}$$

The sets A and B *cross* each other if neither $A-B$ nor $B-A$ is connected. It is conjectured that Theorem 2* remains true without the condition that the sets do not cross.

The proof of these theorems is analogous to the proof of Theorems 1 and 2. The part of the polygons D_i is played by certain polygons defined by known constructions [1, 2, 3].

It may be conjectured that there are absolute constants $0 < \bar{\alpha} < 1$ and $1 < \bar{\beta} < \infty$ such that (10) and (11) hold for any convex set C and any $\alpha \leqq \bar{\alpha}$ and $\beta \geqq \bar{\beta}$, respectively. In particular, it is likely that (10) holds for any convex set and any $\alpha \leqq 1/2$.

Let C be a centro-symmetric convex domain. We consider a packing consisting of similar replicas of C with a given number-density. How should these copies of C be chosen and arranged so as to maximize the perimeter-density of them? The last conjecture would imply that we have to choose congruent copies of C and arrange them so as to form a densest lattice packing.

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LOWER ESTIMATES OF THE BÔCHER'S PAIRS WITH RESPECT TO EQUATION $y''+p(x)y=0$

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In the works [1—2] lower estimates were given for the zeros of the solution y of the differential equation

$$y''+p(x)y=0, \quad p>0, \quad \sigma=\sqrt{p}, \quad p\in C^1(I), \quad I=(0,\infty) \tag{1}$$

and for the zeros of the derivative y' by means of the functional $\int\limits_{x_0}^{x}\sqrt{p(z)}\,dz$. It was the function

$$\alpha(x)=\operatorname{arc\,tg}\frac{\sigma(x)y(x)}{y'(x)} \tag{2}$$

by the study of which this investigation started and was carried out [1].

In this paper we look for a natural generalization of these studies and results. Instead of y and y' and of $\alpha(x)$ in (2) let us consider the Bôcher's pair

$$\Phi=\varphi_1 y-\varphi_2\frac{y'}{\sigma}, \quad \Psi=\psi_1 y-\psi_2\frac{y'}{\sigma} \tag{3}$$

$$\varphi_i,\psi_i\in C_1(I), \quad D=\varphi_1\psi_2-\varphi_2\psi_1\neq 0, \quad (i=1,2)$$

$$\alpha(x)=\operatorname{arc\,tg}\frac{\Phi(x)}{\Psi(x)}, \quad \alpha(x)\in C_1(I) \tag{4}$$

which in a particular case[1] reduce to $\left(y,\frac{y'}{\sigma}\right)$ and to the above $\alpha(x)$. As is well-known (see [4]), the functions Φ and Ψ have no common zeros and no double zero and their zeros — if any — separate each other provided

$$\{\varphi_1,\varphi_2\}\neq 0, \quad \{\psi_1,\psi_2\}\neq 0 \quad \text{for} \quad x\in I, \tag{5}$$

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[1] $\varphi_1=-\psi_2=1,\ \varphi_2=\psi_1=0$

where

$$\{\varphi_1, \varphi_2\} = \varphi_1'\varphi_2 - \varphi_2'\varphi_1 + \frac{\sigma'}{\sigma}\varphi_1\varphi_2 + \sigma(\varphi_1^2+\varphi_2^2) \tag{6}$$

and $\{\psi_1, \psi_2\}$ is formed in a similar way.

Let us denote the zeros of Φ and Ψ by x_i $(i=0, 1, 2, \dots)$ and x_i' $(i=1, 2, 3, \dots)$, respectively, where

$$0 \leqq x_0 < x_1' < x_1 < x_2' < x_2 \dots, \tag{7}$$

and choose that branch of (4) where $\alpha(x_0)=0$. — In order to get an expression for $\alpha'(x) = \dfrac{\Phi'\Psi - \Psi'\Phi}{\Phi^2+\Psi^2}$ we start from the identity

$$\Phi\left(\psi_1 y - \psi_2\frac{y'}{\sigma}\right) - \Psi\left(\varphi_1 y - \varphi_2\frac{y'}{\sigma}\right) = 0$$

or

$$(\Phi\psi_1 - \Psi\varphi_1)y - (\Phi\psi_2 - \Psi\varphi_2)\frac{y'}{\sigma} = 0. \tag{8}$$

After differentiation of (8) and by (1) we have

$$Ay - B\frac{y'}{\sigma} = 0 \tag{9}$$

where A and B involve expressions of $\varphi_i, \psi_i, \varphi_i', \psi_i', \Phi, \Psi, \Phi', \Psi', \sigma, \sigma'$, but do not contain y and y'. Since $y^2+y'^2>0$, (8) and (9) imply

$$\begin{vmatrix} \Phi\psi_1 - \Psi\varphi_1 & \Phi\psi_2 - \Psi\varphi_2 \\ A & B \end{vmatrix} = 0,$$

whence — as the result of a lengthy but elementary computation — we get for $\Delta = \Phi'\Psi - \Psi'\Phi$

$$D\Delta = -a\Phi^2 - b\Psi^2 + c\Phi\Psi \tag{10}$$

where a and b are the above functions $\{\psi_1, \psi_2\}$ and $\{\varphi_1, \varphi_2\}$, respectively, and

(11)

$$c = [\varphi, \psi] = \varphi_1'\psi_2 - \psi_2'\varphi_1 + \psi_1'\varphi_2 - \varphi_2'\psi_1 + \frac{\sigma'}{\sigma}(\varphi_1\psi_2 + \varphi_2\psi_1) + 2\sigma(\varphi_1\psi_1 + \varphi_2\psi_2).$$

Taking into account that

$$\operatorname{tg}\alpha = \frac{\Phi}{\Psi},$$

relation (10) and the above formula of $\alpha'(x)$ imply

$$D\alpha' = -a\sin^2\alpha - b\cos^2\alpha + \frac{c}{2}\sin 2\alpha. \tag{12}$$

Let the following conditions be assumed:

$$(C_1) \qquad D < 0, \quad a > 0, \quad b > 0, \quad c < 0, \qquad (x \in I)$$

$$(C_2) \qquad \mathscr{D} = c^2 - 4ab < 0,$$

the study and analysis of which — to conserve the easy survey — we postpone to later time. — By these assumptions $\alpha' > 0$, α is increasing $(x \in I)$ and by the choice $\alpha(x_0) = 0$ we have

$$(13) \qquad \begin{aligned} \alpha(x_i) &= i\pi, \qquad i = 0, 1, 2, \ldots \\ \alpha(x_i') &= \left(i - \frac{1}{2}\right)\pi, \qquad i = 1, 2, \ldots. \end{aligned}$$

Notice that without (C_2) $\alpha(x)$ is not necessarily increasing, however, by (12), it takes on all the values $i\pi$, $\left(i - \frac{1}{2}\right)\pi$ increasingly, thus only once and

$$(14) \qquad \begin{matrix} x_i < x < x_{i+1} & & i\pi < \alpha(x) < (i+1)\pi, & i = 0, 1, 2 \ldots \\ x_i' < x < x_{i+1}' & \text{involves} & \left(i - \frac{1}{2}\right)\pi < \alpha(x) < \left(i + \frac{1}{2}\right)\pi, & i = 1, 2, \ldots. \end{matrix}$$

By integration of (12) we have

$$(15) \qquad \alpha(x) = J(x) + F(x),$$

where

$$(16) \qquad J(x) = -\int_0^x \frac{1}{D(z)} [a(z) \sin^2 \alpha(z) + b(z) \cos^2 \alpha(z)]\, dz,$$

$$(17) \qquad F(x) = \frac{1}{2} \int_0^x \frac{1}{D(z)} c(z) \sin 2\alpha(z)\, dz.$$

As is easy to see, the function $F(x)$ assumes its local maxima at x_i'. It will be shown that these maxima decrease (do not increase), i.e. the first of them — $F(x_1')$ — is the largest one, involving

$$(18) \qquad F(x) \leqq F(x_1') \qquad x \geqq 0.$$

To prove this assertion it is sufficient to show that $\Delta_i \leqq 0$ $(i = 1, 2, \ldots)$ where

$$(19) \qquad \Delta_i = F(x_{i+1}') - F(x_i') = \frac{1}{2} \int_{x_i'}^{x_{i+1}'} \frac{c(z)}{D(z)} \sin 2\alpha(z)\, dz.$$

This value can be written in the form

$$(20) \qquad \Delta_i = \int_{x_i'}^{x_{i+1}'} \frac{\sin 2\alpha(x) \alpha'(x)}{f(x) + \sin 2\alpha(x)} dx$$

where

$$f(x) = -\frac{2}{c}(a \sin^2 \alpha + b \cos^2 \alpha). \tag{21}$$

Since $u = \alpha(x)$ is increasing, its inverse $x = v(u)$ exists and is also increasing and the substitution $\alpha(x) = u$ in (20) can be carried out, giving

$$\Delta_i = \int_{\left(i-\frac{1}{2}\right)\pi}^{\left(i+\frac{1}{2}\right)\pi} \frac{\sin 2u}{g(u)+\sin 2u}\, du, \qquad g(u) = f(v(u)). \tag{22}$$

By decomposing Δ_i in two parts we have

$$\Delta_i = \int_{\left(i-\frac{1}{2}\right)\pi}^{i\pi} + \int_{i\pi}^{\left(i+\frac{1}{2}\right)\pi} .$$

In the first integral put $u = z - \pi/2$ and thereafter $z = u$ again, obtaining

$$\Delta_i = \int_{i\pi}^{\left(i+\frac{1}{2}\right)\pi} \sin 2u \, \frac{2 \sin 2u + g(u) - g\left(u - \frac{\pi}{2}\right)}{[\sin 2u + g(u)]\left[\sin 2u - g\left(u - \frac{\pi}{2}\right)\right]}\, du. \tag{23}$$

Here $\sin 2u \geqq 0$, $\sin 2u + g(u) > 0$ and $\sin 2u - g\left(u - \frac{\pi}{2}\right) < 0$ $\left(\text{namely } \sin 2u - g\left(u - \frac{\pi}{2}\right) = -\frac{D(x)}{c(x)}\alpha'(x)\right)$, therefore $\Delta_i \leqq 0$ provided the condition

(C_3) $\qquad f(x)$ is increasing $\quad (x > 0)$

(which involves

$$g(u) - g\left(u - \frac{\pi}{2}\right) > 0 \quad (u > 0))$$

is satisfied, but not only in this case, as we shall see later. Continuing our reasoning we have

$$F(x) \leqq F(x_1') \leqq F(x_1') + J(x_1') = \alpha(x_1') = \frac{\pi}{2},$$

involving

$$J(x) = \alpha(x) - F(x) > \alpha(x) - \frac{\pi}{2}, \tag{24}$$

whence

$$J(x_i) > \alpha(x_i) - \frac{\pi}{2} = i\pi - \frac{\pi}{2}, \quad i = 0, 1, 2, \ldots$$

$$J(x_i') > \alpha(x_i') - \frac{\pi}{2} = \left(i - \frac{1}{2}\right)\pi - \frac{\pi}{2}, \quad i = 1, 2, \ldots \tag{25}$$

which is the lower estimate we looked for.

EXAMPLE 1. If $\varphi_1 = -\psi_2 = 1$, $\varphi_2 = \psi_1 = 0$, then

$$(26) \qquad a = b = \sigma, \quad c = -\frac{\sigma'}{\sigma}, \quad f(x) = \frac{2\sigma^2}{\sigma'}, \quad D = -1, \quad \mathscr{D} = \left(\frac{\sigma'}{\sigma}\right)^2 - 4\sigma^2,$$

furthermore if

$$(27) \qquad \frac{\sigma'}{\sigma} \text{ decreases and } 0 < \frac{\sigma'}{\sigma^2} < 2,$$

then $f(x)$ is increasing and $\mathscr{D}<0$. On the other hand, if $\sigma \in C^2(I)$ then it can be proved that instead of (27) the unique condition

$$(28) \qquad 2\sigma'^2 - \sigma\sigma'' > 0$$

is sufficient (see Á. Elbert [1]). Summarizing: the function

$$(29) \qquad J(x) = \int_0^x \sigma(z)\, dz$$

satisfies (25) under the conditions (27) or (28).

REMARK. Since $\alpha' = \sigma + \frac{\sigma'}{2\sigma} \sin 2\alpha$, we have

$$\sigma - \frac{\sigma'}{2\sigma} < \alpha' < \sigma + \frac{\sigma'}{2\sigma}$$

provided $\sigma' > 0$, involving

$$\int_{z_i}^{z_{i+1}} \sigma\, dx - \frac{1}{2} \log \frac{\sigma(z_{i+1})}{\sigma(z_i)} < \pi < \int_{z_i}^{z_{i+1}} \sigma\, dx + \frac{1}{2} \log \frac{\sigma(z_{i+1})}{\sigma(z_i)} \quad \begin{pmatrix} z_k = x_k & \text{or} & x_k' \\ k = i, i+1 & & \end{pmatrix}.$$

EXAMPLE 2. This example will show that a similar result can be obtained also in the case $c > 0$ provided the rest of conditions (C_1) and (C_2) remains valid. — Applying the well-known relation (see [5] p. 44)

$$(x^\nu Z_\nu(x))' = x^\nu Z_{\nu-1}(x)$$

to the Bessel function of the first or second kind $Z_\nu(x)$ and $\nu = n + 1/2$ ($n \in \mathbf{Z}$), we have

$$(x^{n+(1/2)} Z_{n+(1/2)}(x))' = (x^n x^{1/2} Z_{n+(1/2)}(x))' = x^{n+(1/2)} Z_{n-(1/2)}(x),$$

whence with the notation $y = x^{1/2} Z_{n+(1/2)}(x)$,

$$nx^{n-1} y + x^n y' = x^{n+(1/2)} Z_{n-(1/2)}(x)$$

or

$$nx^{-1} y + y' = x^{1/2} Z_{n-(1/2)}(x).$$

The function y satisfies (1) with

$$p = 1 + \left[\frac{1}{4} - \left(n + \frac{1}{2}\right)\right]^2 x^{-2} = 1 - n(n+1)x^{-2} = \sigma^2.$$

Consider the Bôcher's pair

$$\Phi = y = x^{1/2} Z_{n+(1/2)}(x),$$

$$\Psi = nx^{-1}y + y' = x^{1/2} Z_{n-(1/2)}(x).$$

Here $\varphi_1 = 1$, $\varphi_2 = 0$, $\psi_1 = nx^{-1}$, $\psi_2 = -\sigma$, involving in turn $D = -\sigma$,

$$a = b = \sigma, \quad c = 2n\sigma x^{-1}, \quad \mathscr{D} = 4\sigma^2(n^2x^{-2} - 1) < 0 \quad (x > n)$$

$$\alpha' = 1 - nx^{-1} \sin 2\alpha.$$

Obviously, $\alpha(x)$ increases for $x > n$. If $x_0 > n$, then

$$\alpha(x) = J(x) + F(x),$$

where

$$J(x) = x - x_0, \quad F(x) = -n \int_{x_0}^{x} z^{-1} \sin 2\alpha(z)\, dz.$$

Here x_0 means the first zero of y greater than n. The function $F(x)$ assumes its maxima at x_i $(i = 0, 1, 2, \ldots)$, and we shall see that the first of them is the greatest one, i.e. $F(x) \leqq F(x_0) = 0$, $x \geqq x_0$. To this end it is sufficient to show that $\Delta_i = F(x_{i+1}) - F(x_i) \leqq$ $\leqq 0$, $i = 0, 1, 2, \ldots$. Viz.

$$\Delta_i = -n \int_{x_i}^{x_{i+1}} x^{-1} \sin 2\alpha(x)\, dx = \int_{x_i}^{x_{i+1}} \frac{\sin 2\alpha(x)\alpha'(x)\, dx}{-\frac{1}{n} x\alpha'(x)} =$$

$$= \int_{x_i}^{x_{i+1}} \frac{\sin 2\alpha(x)\alpha'(x)\, dx}{\sin 2\alpha(x) - \frac{x}{n}}.$$

By the substitution $u = \alpha(x)$, $x = g(u)$ and the notation $h(u) = (1/n)g(u)$,

$$\Delta_i = \int_{i\pi}^{(i+1)\pi} \frac{\sin 2u\, du}{\sin 2u - h(u)}.$$

Here $\sin 2u - h(u) = -\frac{x\alpha'(x)}{n} < 0$. Decompose Δ_i as follows

$$\Delta_i = \int_{i\pi}^{\left(i+\frac{1}{2}\right)\pi} + \int_{\left(i+\frac{1}{2}\right)\pi}^{(i+1)\pi} = I_1 + I_2.$$

Putting in I_1 $u = z - \pi/2$ and again $z = u$ we have

$$I_1 = \int_{\left(i+\frac{1}{2}\right)\pi}^{(i+1)\pi} \frac{-\sin 2u\, du}{-\sin 2u - h\left(u - \frac{\pi}{2}\right)} = \int_{\left(i+\frac{1}{2}\right)\pi}^{(i+1)\pi} \frac{\sin 2u\, du}{\sin 2u + h\left(u - \frac{\pi}{2}\right)}$$

and

$$\Delta_i = \int_{\left(i+\frac{1}{2}\right)\pi}^{(i+1)\pi} \frac{\sin 2u\left[2\sin 2u + h\left(u-\frac{\pi}{2}\right) - h(u)\right]}{\left[\sin 2u + h\left(u-\frac{\pi}{2}\right)\right][\sin 2u - h(u)]}\,du,$$

which shows that $\Delta_i \leqq 0$, since $\sin 2u < 0$ in $\left(\left(i+\frac{1}{2}\right)\pi,\ (i+1)\pi\right)$ and the factors of the denominator are in turn positive and negative, respectively, and $h(u)$ is increasing. Thus

$$J(x) = \alpha(x) - F(x) \geqq \alpha(x) - F(x_0) \geqq \alpha(x) - [F(x_0) + J(x_0)] = \alpha(x) - \alpha(x_0) = \alpha(x).$$

Putting here $x = x_n$ and $x = x_n'$ we get

$$x_n - x_0 \geqq n\pi, \qquad n = 1, 2, \ldots$$

$$x_n' - x_0 \geqq \left(n - \frac{1}{2}\right)\pi, \qquad n = 1, 2, 3, \ldots .$$

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ZEROS OF THE BÔCHER-FUNCTION AND ITS DERIVATIVE WITH RESPECT TO DIFFERENTIAL EQUATION $y''+p(x)y=0$, II

I. BIHARI

In part I [1] lower estimates were given for the zeros in question. In this second part upper and lower bounds will be obtained for them.

Let us consider the differential equation

$$y''+p(x)y = 0, \quad \left(' = \frac{d}{dx}\right) \tag{1}$$

$$x\in I = [x_0, \infty), \quad x_0\in \mathbf{R}, \quad p > 0, \quad p\in C_3(I)$$

and the "Bôcher-function"

$$\Phi = \varphi_1 y-\varphi_2 y', \quad \varphi_i\in C_4(I), \quad (i = 1, 2) \tag{2}$$

and its pair

$$\Phi'(x) = \psi_1 y-\psi_2 y', \quad \psi_1 = \varphi_1'+\varphi_2 p, \quad \psi_2 = \varphi_2'-\varphi_1 \tag{3}$$

where y is a non-trivial solution of (1), and suppose

$$\{\varphi_1, \varphi_2\} \neq 0 \quad x\in I. \tag{4}$$

Here the symbol $\{u, v\}$ is defined by

$$\{u, v\} = u'v-v'u+u^2+v^2p. \tag{4'}$$

Then

$$-\begin{vmatrix} \varphi_1 & \varphi_2 \\ \psi_1 & \psi_2 \end{vmatrix} = \{\varphi_1, \varphi_2\} \neq 0, \quad x\in I. \tag{5}$$

Then we have the following results 1°–4° stated in a more general situation in an earlier work [2].

1° Φ and Φ' do not vanish simultaneously, i.e., Φ has no multiple zeros and its zeros do not accumulate at a finite point.

2° In addition we assume $\{\psi_1, \psi_2\}\neq 0$. Then the zeros of Φ and Φ' separate each other (they are interlacing). Of course, between two consecutive zeros of Φ, Φ'

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has a zero, but — just be the separation — no more than one. The same property excludes that Φ' is oscillatory and Φ is not.

3° Φ (and Φ') are not oscillatory provided

$$\{\varphi_1, \varphi_2\}\{\psi_1, \psi_2\} < 0, \quad x \in I; \tag{6}$$

then only one of Φ and Φ' can vanish, moreover, at most once.

4° y and Φ are oscillatory or non-oscillatory at the same time provided

$$\mathscr{D} = (\psi_1 - \varphi_2)^2 + 4\varphi_1\psi_2 < 0, \quad \varphi_1 \neq 0 \quad \text{and} \quad \frac{\psi_1}{\varphi_1} \quad \text{is bounded as} \quad x \to \infty.$$

Remark 1. It can happen that y oscillates and Φ does not. Namely, if y_1, y_2 are linearly independent solutions of (1), then

$$y_1' y_2 - y_2' y_1 = c = \text{const.} \neq 0, \quad x \in I.$$

Let us choose

$$\varphi_1 = y_1', \quad \varphi_2 = y_1 \quad \text{and} \quad y = y_2,$$

then

$$\Phi = \varphi_1 y - \varphi_2 y' = c,$$

which does not oscillate even if y is oscillatory. N. B. now $\{\varphi_1, \varphi_2\} = 0$! Conversely, if y is not oscillatory, then e.g.

$$\Phi = \varphi y - y' = \sin x$$

is oscillatory provided

$$\varphi = \frac{y' + \sin x}{y}, \quad y \neq 0, \quad x \in I.$$

Remark 2. The meaning of (4) can be expressed by saying that φ_1/φ_2 does not satisfy in any point the Riccati equation

$$u' + u^2 + p = 0 \tag{7}$$

belonging to (1). Namely, (4) can be written as

$$\varphi_2^2 \left[\left(\frac{\varphi_1}{\varphi_2}\right)' + \left(\frac{\varphi_1}{\varphi_2}\right)^2 + p\right] \neq 0, \quad \varphi_2 \neq 0. \tag{8}$$

In the same way, if $\varphi_1 \neq 0$ then φ_2/φ_1 cannot satisfy the equation

$$-u' + 1 + u^2 p = 0, \tag{7'}$$

which is satisfied by $y/y' (y' \neq 0)$.

Now let us form Φ''

$$\Phi'' = (\psi_1' + \psi_2 p) y - (\psi_2' - \psi_1) y' \tag{9}$$

and eliminate y and y' from (2), (3), (9). Then we obtain

$$\begin{vmatrix} \varphi_1 & \varphi_2 & \Phi \\ \psi_1 & \psi_2 & \Phi' \\ \psi_1' + \psi_2 p & \psi_2' - \psi_1 & \Phi'' \end{vmatrix} = 0, \tag{10}$$

in expanded form

$$\{\varphi_1, \varphi_2\}\Phi'' - [\varphi, \psi]\Phi' + \{\psi_1, \psi_2\}\Phi = 0, \tag{11}$$

where

$$[\varphi, \psi] = (\psi_1' + \psi_2 p)\varphi_2 - (\psi_2' - \psi_1)\varphi_1 + \varphi_1''\varphi_2 - \varphi_2''\varphi_1 + 2(\varphi_1\varphi_1' + \varphi_2\varphi_2' p) + \varphi_2' p'.$$

The equation (11) is a non-singular second order homogeneous linear differential equation for Φ. Thus from an alternative point of view we see again that Φ and Φ' cannot vanish simultaneously provided Φ is a non-trivial solution of (11). The function Φ can be trivial only when $y \equiv 0$. Indeed, in this case either $\varphi_1/\varphi_2 = y'/y\,(\varphi_2 \neq 0)$ or $\varphi_2 = 0$. The first case is excluded by the fact that y'/y is a solution of (7) while φ_1/φ_2 is not. In the second case $\varphi_1 \neq 0$ and $\Phi = \varphi_1 y = 0$ involves $y \equiv 0$.

Let y_1, y_2 be two linearly independent solutions of (1), i.e.,

$$\Delta = y_1' y_2 - y_2' y_1 = c = \text{const.} \neq 0, \quad x \in I,$$

then $\Phi_i = \varphi_1 y_i - \varphi_2 y_i'$ $(i = 1, 2)$ are linearly independent solutions of (11), since

$$\Phi_1' \Phi_2 - \Phi_2' \Phi_1 = \{\varphi_1, \varphi_2\}\Delta \neq 0, \quad x \in I. \tag{11a}$$

Therefore the zeros of Φ_1 and Φ_2 separate each other (see [2]).

Conversely, if Φ_1, Φ_2 are linearly independent solutions of (11), then from the system of linear equations

$$\begin{aligned} \Phi_1 &= \varphi_1 y_1 - \varphi_2 y_1' \\ \Phi_1' &= \psi_1 y_1 - \psi_2 y_1' \\ \Phi_2 &= \varphi_1 y_2 - \varphi_2 y_2' \\ \Phi_2' &= \psi_1 y_2 - \psi_2 y_2' \end{aligned}$$

the four unknown y_i, y_i' $(i = 1, 2)$ can be determined uniquely, since the determinant

$$\begin{vmatrix} \varphi_1 & \varphi_2 & 0 & 0 \\ \psi_1 & \psi_2 & 0 & 0 \\ 0 & 0 & \varphi_1 & \varphi_2 \\ 0 & 0 & \psi_1 & \psi_2 \end{vmatrix} = a^2 \neq 0, \qquad a = \{\varphi_1, \varphi_2\},$$

namely

$$y_1 = -\frac{1}{a}(\psi_2 \Phi_1 - \varphi_2 \Phi_1'), \qquad y_2 = -\frac{1}{a}(\psi_2 \Phi_2 - \varphi_2 \Phi_2'),$$

$$y_1' = -\frac{1}{a}(\psi_1 \Phi_1 - \varphi_1 \Phi_1'), \qquad y_2' = -\frac{1}{a}(\psi_1 \Phi_2 - \varphi_1 \Phi_2').$$

Now (11a) and $\Phi_1' \Phi_2 - \Phi_2' \Phi_1 \neq 0$ involve $\Delta \neq 0$, i.e., if Φ_1, Φ_2 are linearly independent solutions of (11), then they can always be derived — according to the given formulae — from two linearly independent solutions of (1).

Using the notations

$$a = \{\varphi_1, \varphi_2\}, \quad b = -[\varphi, \psi], \quad c = \{\psi_1, \psi_2\},$$

the equation (11) is of the form

(12) $$aY''+bY'+cY=0 \quad \text{or} \quad Y''+\frac{b}{a}Y'+\frac{c}{a}Y=0,$$

which, multiplied by $\exp\left(2\int\limits_{x_0}^{x}\frac{b}{a}\right)$, gives

$$e^{\int\frac{b}{a}}\left(e^{\int\frac{b}{a}}Y'\right)'+\frac{c}{a}e^{2\int\frac{b}{a}}Y=0.$$

Substituting here

(13) $$\xi=\int\limits_{x_0} e^{-\int\limits_{x_0}\frac{b}{a}}dx, \qquad Y(x)=\bar{Y}(\xi),$$

we have

(14) $$\frac{d^2\bar{Y}}{d\xi^2}+\bar{\varrho}(\xi)\bar{Y}=0, \qquad \bar{\varrho}(\xi)=\varrho(x)=\frac{c}{a}e^{2\int\limits_{x_0}^{x}\frac{b}{a}}.$$

Now the following theorem of E. Makai ([4]) can be applied to (14).

THEOREM of E. Makai. *Let us assume that in* $J=[\xi_0,\infty)$

(i) $\bar{\varrho}(\xi)>0, \quad \bar{\varrho}\in C_2(J)$,

(ii) $r(\bar{\varrho})=5\dot{\bar{\varrho}}^2-4\bar{\varrho}\ddot{\bar{\varrho}}\geqq 0, \quad \left(\cdot=\frac{d}{d\xi}\right)$

(iii) $\xi_1<\xi_2$ *are consecutive zeros of* $\bar{Y}(\xi)$ *(which is a non-trivial solution of* (14)*) and at least at one point of* (ξ_1,ξ_2) *the sign* $>$ *holds in* (ii), *then*

(15) $$\int\limits_{\xi_1}^{\xi_2}\sqrt{\bar{\varrho}(\xi)}\,d\xi<\pi.$$

If in (ii) $\geqq$ *is replaced by* $\leqq$, *then the sign* $<$ *in* (15) *turns into* $>$.

As an application we have

THEOREM 1. *In the present case*

(16) $$\int\limits_{\xi_1}^{\xi_2}\sqrt{\frac{c}{a}e^{\int\frac{b}{a}}}\,d\xi=\int\limits_{x_1}^{x_2}\sqrt{\frac{c}{a}}\,dx=\int\limits_{x_1}^{x_2}\sqrt{\frac{\{\psi_1,\psi_2\}}{\{\varphi_1,\varphi_2\}}}\,dx<\pi \quad (>\pi)$$

provided Conditions (i)—(iii) *are satisfied. Here* $x_1<x_2$ *are consecutive zeros of* $Y(x)=\Phi(x)$ *corresponding to* ξ_1 *and* ξ_2. — *Condition*[2] (ii) *is satisfied with* $\geqq$ *if, e.g.* $\ddot{\bar{\varrho}}(\xi)<0$,

[2] Condition (i) is fulfilled when $\{\varphi_1,\varphi_2\}$ and $\{\psi_1,\psi_2\}$ have equal signs.

i.e., ϱ — as a function of ξ — is concave. Expressed by x, condition (ii) *takes the form*

$$\text{(ii)}' \quad r(\varrho) = 5\varrho'^2 - 4\varrho\left(\varrho'' + \frac{b}{a}\varrho'\right) \geqq 0 \qquad (\text{N. B. } p \in C_3(I),\ \varphi_i \in C_4(I)),$$

to the analysis of which we shall return.

The equation (14) is of the form

$$\ddot{\bar{Y}} + \bar{\varrho}(\xi)\bar{Y} = 0, \qquad \bar{Y} = \bar{Y}(\xi),$$

whence by differentiation (omitting the bars)

$$\dddot{Y} + \dot{\varrho}Y + \varrho\dot{Y} = \dddot{Y} - \frac{\dot{\varrho}}{\varrho}\ddot{Y} + \varrho\dot{Y} = 0,$$

or by the notation $Z = \dot{Y}$

$$\text{(17)} \qquad \ddot{Z} - \frac{\dot{\varrho}}{\varrho}\dot{Z} + \varrho Z = 0,$$

which, multiplied by ϱ^{-2}, gives

$$\varrho^{-1}(\varrho^{-1}\dot{Z})^{\cdot} + \varrho^{-1}Z = 0.$$

Putting

$$\text{(18)} \qquad \eta = \int_{\xi_0}^{\xi} \varrho(\xi)\,d\xi, \qquad \tilde{Z}(\eta) = Z(\xi),$$

the equation (17) turns into

$$\text{(19)} \qquad \frac{d^2\tilde{Z}}{d\eta^2} + (\tilde{\varrho})^{-1}\tilde{Z} = 0, \qquad \tilde{\varrho}(\eta) = \varrho(\xi).$$

Let $s(\eta) = \big(\tilde{\varrho}(\eta)\big)^{-1}$ satisfy, as a function of η, the conditions (i)—(ii). Then — if $\eta_1 < \eta_2$ are consecutive zeros of $\tilde{Z}(\eta)$ and at least in one point of (η_1, η_2) condition (ii) holds with sign $>$ — we have

$$\text{(20)} \qquad \int_{\eta_1}^{\eta_2} (\tilde{\varrho}(\eta))^{-1}\,d\eta = \int_{\xi_1'}^{\xi_2'} \sqrt{\varrho^{-1}(\xi)}\,\varrho(\xi)\,d\xi = \int_{\xi_1'}^{\xi_2'} \sqrt{\varrho(\xi)}\,d\xi < \pi \quad (> \pi)$$

where $\xi_1' < \xi_2'$ are adjacent zeros of $\dot{Y}(\xi)$. Finally, we have as above

$$\text{(21)} \qquad \int_{x_1'}^{x_2'} \sqrt{\frac{\{\psi_1, \psi_2\}}{\{\varphi_1, \varphi_2\}}}\,dx < \pi \quad (> \pi)$$

where $x_1' < x_2'$ are adjacent zeros of $\Phi'(x)$. — The conditions corresponding to (i)—(ii) are now

$$\text{(i)}_\eta \quad \tilde{\varrho}(\eta) > 0,$$

$$\text{(ii)}_\eta \quad 5\left(\frac{ds}{d\eta}\right)^2 - 4s\frac{d^2s}{d\eta^2} \geqq 0 \quad (\leqq 0).$$

Condition (i)$_\eta$ requires nothing new, but (ii)$_\eta$ states

$$\text{(ii)}'' \quad R(\eta) = -7\dot{\varrho}^2 + 4\varrho\ddot{\varrho} \geqq 0, \quad (\leqq 0), \quad \left(\cdot = \frac{d}{d\xi}\right)$$

which is in some sense the counterpart of (ii), because $\ddot{\varrho}<0$ involves $r(e)\geqq 0$, $R(\varrho)\leqq 0$ which, in turn, imply

$$\text{(22)} \qquad \int_{x_1}^{x_2} \sqrt{\frac{\{\psi_1, \psi_2\}}{\{\varphi_1, \varphi_2\}}}\, dx < \pi, \qquad \int_{x_1'}^{x_2'} \sqrt{\frac{\{\psi_1, \psi_2\}}{\{\varphi_1, \varphi_2\}}}\, dx > \pi.$$

(The statement concerning $x_1' < x_2'$ goes back to J. H. E. Cohn [8].) Now several problems arise:

1) Can it happen that we have $r(\varrho)\leqq 0$, $R(\varrho)\geqq 0$ for some $\ddot{\varrho}\geqq 0$? In this case the inequalities (22) will be reversed.

2) Can $r(\varrho)$ and $R(\varrho)$ have the same sign? In this case the sign $<$ (or $>$) holds in both inequalities of (22).

These problems can be answered by analysing the conditions (ii) and (ii)″. We postpone answering Problems 1) and 2) to a later time.

First the case $\ddot{\varrho}\leqq 0$, i.e., $\varrho'' + (b/a)\varrho' \leqq 0$, will be discussed. This can be written as

$$\left(\varrho' \exp\left(\int_{x_0}^{x} \frac{b}{a}\right)\right)' \leqq 0.$$

By integration we obtain

$$\text{(23)} \qquad \varrho - \varrho_0 \leqq \varrho_0' \int_{x_0}^{x} \exp\left(-\int_{x_0}^{\tau} \frac{b(z)}{a(z)}\, dz\right) d\tau, \qquad \varrho_0 = \varrho(x_0), \quad \varrho_0' = \varrho'(x_0).$$

Here $\varrho = \dfrac{c}{a} e^{2\int_{x_0}^{x} \frac{a}{b}}$, therefore $\ddot{\varrho}\leqq 0$ implies

$$\text{(24)} \qquad \frac{c}{a} e^{2\int_{x_0}^{x} \frac{b}{a}} - \frac{c(x_0)}{a(x_0)} \leqq \left[\left(\frac{c}{a}\right)' + \frac{2bc}{a^2}\right]_{x=x_0} \int_{x_0}^{x} \exp\left[-\int_{x_0}^{\tau} \frac{b(z)}{a(z)}\, dz\right] d\tau.$$

With regard to the involved form of a, b, c a general analysis of (24) seems hopeless. We have to find suitable examples satisfying (ii) and (ii)′. Otherwise, (ii) (or (ii)′) can be integrated in the special case $\dot{\varrho}>0$ (or $\dot{\varrho}<0$). Namely, in this case (ii) can be written as

$$5\frac{\dot{\varrho}}{\varrho} - 4\frac{\ddot{\varrho}}{\dot{\varrho}} \geqq 0,$$

whence

$$5 \log \varrho - 4 \log \dot{\varrho} = \log \frac{\varrho^5}{\dot{\varrho}^4} \geqq \log k, \qquad k = \frac{\varrho_0^5}{\dot{\varrho}_0^4} = \text{const} > 0,$$

$$\dot{\varrho}\varrho^{-5/4} \leqq K = k^{1/4},$$

$$\text{(25)} \qquad \varrho \leqq (A + B\xi)^{-4} \qquad (A, B \text{ are constant}).$$

E.g., if $\varrho=\xi^{-4}$ then $\dot{\varrho}<0$, $r(\varrho)\equiv 0$ and in (16) the sign $=$ holds and $Y(\xi)=\xi\sin 1/\xi$ is a solution of (14) which is oscillatory in the neighbourhood of the origin and, in fact, we have for consecutive zeros $\xi_1<\xi_2$ of $Y(\xi)$

$$\int_{\xi_1}^{\xi_2}\sqrt{\varrho(\xi)}\,d\xi=\frac{1}{\xi_1}-\frac{1}{\xi_2}=\pi.$$

In the same way we get from (ii)″ $\varrho\geqq(A+B\xi)^{-4/3}$ provided $\dot{\varrho}>0$. But

(26) $\qquad (A_1+B_1\xi)^{-4/3}\leqq\varrho\leqq(A_2+B_2\xi)^{-4}, \qquad \xi\geqq\xi_0 \quad (A_i, B_i$ are constant)

cannot hold, i.e., both (ii) and (ii)′ cannot be valid at the same time with the sign $\geqq 0$. — In the opposite case — when $\dot{\varrho}<0$ — the answer is positive, because the converse of inequality (26) is possible.

Now it remains to find convincing examples.

EXAMPLE 1. The function

(27) $$u_\nu(x)=x^{1/2}Z_\nu(x) \qquad (\nu>0)$$

— where $Z_\nu(x)$ is a Bessel function of the first or second kind — satisfies (1) with $p=1+(1/4-\nu^2)x^{-2}$. Choose as $\Phi(x)$ the function

(28) $$\Phi(x)=\left(\nu+\frac{1}{2}\right)x^{-1/2}u_\nu(x)-x^{-1/2}u_\nu'(x).$$

It is well-known that

(29) $$\Phi(x)=x^{1/2}u_{\nu+1}(x),$$

hence the zeros x_i of $\Phi(x)$ are those of $u_{\nu+1}(x)$ and $Z_{\nu+1}(x)$. In the present case

$$\varphi_1=\left(\nu+\frac{1}{2}\right)x^{-1/2}, \quad \varphi_2=x^{1/2},$$

involving in turn

$$\psi_1=x^{1/2}-\left(\nu+\frac{1}{2}\right)\nu x^{-3/2}, \qquad \psi_2=-\nu x^{-1/2},$$

$$a=\{\varphi_1,\varphi_2\}=x, \qquad -b=[\varphi,\psi]=1, \qquad c=\{\psi_1,\psi_2\}=x-\nu(\nu+2)x^{-1},$$

$$\frac{c}{a}=1-\nu(\nu+2)x^{-2}, \qquad -\int\frac{b}{a}=\log x, \qquad \varrho=\frac{c}{a}e^{-2\log x}=x^{-2}-\nu(\nu+2)x^{-4},$$

(30) $$r(\varrho)=-12x^{-6}+48\nu(\nu+2)x^{-8}-16\nu^2(\nu+2)^2x^{-10}<0$$

(31) $$R(\varrho)=4x^{-6}-72\nu(\nu+2)x^{-8}-116\nu^2(\nu+2)^2x^{-10}>0$$

for x large enough.

(Here we have a positive answer to the above Problem 1).) It is an easy exercise to determine the X_0, X_0' beyond which the last inequalities are satisfied. We obtain

$$X_0=\sqrt{\left(2+\frac{2}{3}\sqrt{6}\right)\nu(\nu+2)}, \qquad X_0'=\sqrt{(9+\sqrt{110})\nu(\nu+2)}.$$

Using the inequalities

$$1-u<\sqrt{1-u}<1-\frac{u}{2}\qquad(0<u<1)$$

we have

$$I_1=\int_A^B\left(1-\nu(\nu+2)x^{-2}\right)dx<\int_A^B\sqrt{\frac{c}{a}}\,dx=\int_A^B\sqrt{1-\nu(\nu+2)x^{-2}}\,dx<$$

$$<\int_A^B\left[1-\frac{1}{2}\nu(\nu+2)x^{-2}\right]dx=I_2,$$

but by (30)—(31)

$$\int_{x_i}^{x_{i+1}}\sqrt{\frac{c}{a}}\,dx>\pi,\qquad\int_{x_i'}^{x_{i+1}'}\sqrt{\frac{c}{a}}\,dx<\pi,$$

consequently

(32) $$I_2=x_{i+1}-x_i-\frac{1}{2}\nu(\nu+2)\left(\frac{1}{x_i}-\frac{1}{x_{i+1}}\right)>\pi\qquad(A=x_i,\ B=x_{i+1})$$

and

(33) $$I_1=x_{i+1}'-x_i'-\nu(\nu+2)\left(\frac{1}{x_i'}-\frac{1}{x_{i+1}'}\right)<\pi\qquad(A=x_i',\ B=x_{i+1}').$$

We have to interprete the result (32)—(33). Both of $\Delta_i=x_{i+1}-x_i$ and $\Delta_i'=x_{i+1}'-x_i'$ tend to π as $i\to\infty$, where x_i is the common zero of $\Phi(x)$ and $u_{\nu+1}(x)$ and x_i' is the zero of $\Phi'(x)$. Furthermore,

$$\Delta_i\gtrless\pi\begin{cases}\nu>\dfrac{1}{2}\\[2mm]\nu<\dfrac{1}{2}\end{cases}$$

in consequence of another theorem of E. Makai [6] saying that

$$\Delta_i\quad\text{is}\begin{Bmatrix}\text{decreasing}\\\text{increasing}\end{Bmatrix}\text{if}\ \ p\begin{Bmatrix}\text{increases}\\\text{decreases}\end{Bmatrix}\begin{matrix}\left(\nu>\dfrac{1}{2}\right)\\[2mm]\left(\nu<\dfrac{1}{2}\right)\end{matrix}\text{with increasing } i.$$

In our case $\nu>0$, $\nu+1>1/2$, thus $\Delta_i>\pi$ and (32) states that subtracting the positive quantity $\dfrac{\nu(\nu+2)}{2}\left(\dfrac{1}{x_i}-\dfrac{1}{x_{i+1}}\right)$ from Δ_i the result still remains greater than π, while the first mentioned theorem of Makai gives

(34) $$\Delta_i-\left[\nu(\nu+2)+\frac{3}{4}\right]\left(\frac{1}{x_i}-\frac{1}{x_{i+1}}\right)<\pi$$

since

$$r(p)=24\left[(\nu+1)^2-\frac{1}{4}\right]x^{-4}-k^2x^{-6}>0\qquad(k>0).$$

This shows that both estimates are accurate enough. The inequalities (32) and (34) together yield

$$\frac{1}{2}\nu(\nu+2)x_{i+1}^{-2} < \frac{\Delta_i-\pi}{\Delta_i} < \left[\nu(\nu+2)+\frac{3}{4}\right]x_i^{-2}.$$

As to Δ_i', the situation is the following. If y is a non-trivial solution of (1), then $z=y'$ satisfies

$$z''-\frac{p'}{p}z'+pz=0$$

and substituting $\xi=\int_{x_0}^{x} p\,dx$ this becomes

$$\ddot{v}+p^{-1}(\xi)v=0, \quad v(\xi)=z(x), \quad p(\xi)=p(x), \quad \left(\cdot=\frac{d}{d\xi}\right).$$

In our case $p^{-1}(\xi)$ is decreasing, involving the increase of $\Delta_i^{(\xi)}=\xi_{i+1}'-\xi_i'$ where ξ_i' is the zero of $v(\xi)$. We have now

$$\Delta_i^{(\xi)}=\int_{x_i'}^{x_{i+1}'} p(x)\,dx > \Delta_i'-\left[\nu(\nu+2)+\frac{3}{4}\right]\left(\frac{1}{x_i'}-\frac{1}{x_{i+1}'}\right)=\Delta_i'A,$$

$$A=1-\left[\nu(\nu+2)+\frac{3}{4}\right](x_i'x_{i+1}')^{-1}.$$

Since $x_i'\to\infty$ as $i\to\infty$, A increases with i and it cannot be asserted that Δ_i' is increasing, too, and — in this way — the accuracy of (33) cannot be decided. If $\Delta_i'<\pi$ then (33) gives nothing new.

EXAMPLE 2. Consider the case $p=x^\lambda$ in (1) with $\lambda>0$, then the solution of (1) is

$$u_\nu(x)=\sqrt{x}\,Z_\nu(2\nu x^{1/2\nu}), \qquad \nu=\frac{1}{\lambda+2} \quad \left(<\frac{1}{2}\right)$$

where $Z_\nu(x)$ has the above meaning. Concerning $Z_\nu(x)$ the formulae

$$\nu Z_\nu(x)-xZ_\nu'(x)=xZ_{\nu+1}(x),$$

$$\nu Z_\nu(x)+xZ_\nu'(x)=xZ_{\nu-1}(x)$$

are valid (see [5], p. 45). The first of them reads, transcribed in terms of $u_\nu(x)$, as

$$\Phi(x)=u_\nu(x)-xu_\nu'(x)=\left(\frac{\nu+1}{\nu}\right)^{\nu+1}u_{\nu+1}\left(\left(\frac{\nu}{\nu+1}\right)^{2(\nu+1)}x^{(\nu+1)/\nu}\right).$$

Namely, if $y=2\nu x^{1/2\nu}$, $x=\left(\frac{y}{2\nu}\right)^{2\nu}$, then

$$Z_\nu(y)=x^{-1/2}u_\nu(x)=\left(\frac{y}{2\nu}\right)^{-\nu}u_\nu\left(\left(\frac{y}{2\nu}\right)^{2\nu}\right)$$

and the left-hand side of the formula $\nu Z_\nu(y)-yZ'_\nu(y)=yZ^{(y)}_{\nu+1}$ is of the form

$$\nu Z_\nu(y)-yZ'_\nu(y)=\nu x^{-1/2}u_\nu(x)-2\nu x^{1/2\nu}\times \tag{36}$$

$$\times\left[-\frac{1}{2}x^{-3/2}u_\nu(x)+x^{-1/2}u'_\nu(x)\right]x^{1-(1/2\nu)}=2\nu x^{-1/2}u_\nu(x)-2\nu x^{1/2}u'_\nu(x)$$

while the right member of the same formula reads as

$$yZ_{\nu+1}(y)=y\left(\frac{y}{2(\nu+1)}\right)^{-(\nu+1)}u_{\nu+1}\left(\left(\frac{y}{2(\nu+1)}\right)^{2(\nu+1)}\right).$$

By putting here $y=2\nu x^{1/2\nu}$ we obtain

$$yZ_{\nu+1}(y)=2\nu\left(\frac{\nu+1}{\nu}\right)^{\nu+1}x^{-1/2}u_{\nu+1}\left(\left(\frac{\nu}{\nu+1}\right)^{2(\nu+1)}x^{(\nu+1)/\nu}\right). \tag{37}$$

Finally (36)—(37) involve (35).

Here $\varphi_1=1$, $\varphi_2=x$ and in turn $\psi_1=x^{\lambda+1}$, $\psi_2=0$,

$$a=x^{\lambda+2},\quad c=x^{2(\lambda+1)},\quad -b=(\lambda+2)x^{\lambda+1},$$

$$\varrho=x^{-\lambda-4},\quad r(\varrho)=-(\lambda+4)(3\lambda+8)x^{-2\lambda-10}<0.$$

If the zeros of $\Phi(x)$ are $x_{\nu,i}$ and those of $Z_\nu(x)$ are $j_{\nu,i}$, respectively, then they are related to each other by

$$j_{\nu+1,i}=2\nu x^{1/2\nu}_{\nu,i} \tag{38}$$

(viz. if the zeros of $u_\nu(x)$ are $y_{\nu,i}$, then we have

$$\left(\frac{\nu}{\nu+1}\right)^{2(\nu+1)}x^{(\nu+1)/\nu}_{\nu i}=y_{\nu+1,i}\quad\text{and}\quad 2\nu y^{1/2\nu}_{\nu i}=j_{\nu,i}$$

or

$$2(\nu+1)y^{1/(2(\nu+1))}_{\nu+1,i}=j_{\nu+1,i},$$

whence (38) follows by eliminating $y_{\nu+1,i}$) and by our result

$$\int_{x_{\nu,i}}^{x_{\nu,i+1}}x^{\lambda/2}\,dx=\left[\frac{x^{\lambda/2+1}}{\lambda/2+1}\right]_{x_{\nu,i}}^{x_{\nu,i+1}}=2\nu\,[x^{1/2\nu}]_{x_{\nu,i}}^{x_{\nu,i+1}}>\pi,$$

i.e.

$$j_{\nu+1,i+1}-j_{\nu+1,i}>\pi.$$

EXAMPLE 3. Choose $p=\lambda^{-1}x^{-2}$ ($\lambda>0$, $x>0$) and assume the solution in the form $y=x^\alpha$. Then we have

$$\alpha(\alpha-1)+\lambda^{-1}=0,\quad \alpha=\frac{1}{2}\pm\sqrt{\frac{\lambda-4}{4\lambda}},$$

which gives one/two power functions provided $\lambda=4/\lambda>4$ and two oscillatory solutions provided $0<\lambda<4$, namely

$$y_1=x^{1/2}\sin u,\quad y_2=x^{1/2}\cos u,\quad u=\omega\log x,\quad \omega=\sqrt{\frac{4-\lambda}{4\lambda}}$$

Take $y=y_1$, then

$$y'=x^{-1/2}\left(\frac{1}{2}\sin u+\omega\cos u\right)=x^{-1/2}A\sin(u+\delta),\qquad A^2=\frac{1}{4}+\omega^2,\qquad \operatorname{tg}\delta=2\omega.$$

If the zeros of y and y' are x_i and x_i', respectively, then

$$\omega\log\frac{x_{i+1}}{x_i}=\pi,\qquad \omega\log\frac{x'_{i+1}}{x'_i}=\pi,$$

$$\frac{x_{i+1}}{x_i}=e^{\pi/\omega},\qquad \frac{x'_{i+1}}{x'_i}=e^{\pi/\omega}.$$

Since $p=\lambda^{-1}x^{-2}$, we have

$$r(p)=5p'^2-4pp''=-4\lambda^{-2}x^{-6}<0,$$
$$R(p)=-7p'^2+4pp''=-4\lambda^{-2}x^{-6}<0.$$

Consequently, by Makai's theorem,

$$\int_{z_i}^{z_{i+1}}\sqrt{p}=\lambda^{-1/2}\log\frac{z_{i+1}}{z_i}>\pi\qquad \begin{pmatrix} z_i=x_i & \text{or} & x_i' \\ z_{i+1}=x_{i+1} & \text{or} & x'_{i+1}, \end{pmatrix},$$

which is true since in fact $\frac{1}{\omega}>\lambda^{-1/2}$.

Now let us see how our method works in the following case.

$$\Phi(x)=y+ky'=x^{-1/2}\left[\left(\frac{k}{2}+x\right)\sin u+k\omega\cos u\right],\quad u=\omega\log x,\quad k=\text{const}>0$$

and its zeros x_i, then

$$\operatorname{ctg}u_i=-\frac{\frac{k}{2}+x_i}{k\omega},\qquad u_i=\omega\log x_i$$

and the figure shows clearly that $u_{i+1}-u_i=\omega\log\frac{x_{i+1}}{x_i}>\pi$,

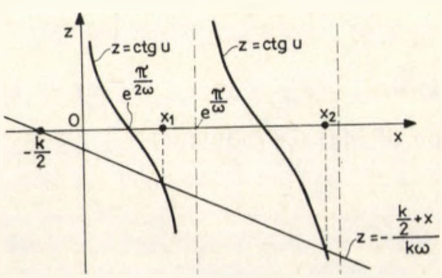

Fig. 1

Now in turn

$$\varphi_1 = 1, \quad \varphi_2 = -k, \quad \psi_1 = -k\lambda^{-1}x^{-2}, \quad \psi_2 = -1,$$

$$a = 1+k^2\lambda^{-1}x^{-2}, \quad c = \lambda^{-1}x^{-2}-2k\lambda^{-1}x^{-3}+k^2\lambda^{-2}x^{-4}, \quad b = 2k^2\lambda^{-1}x^{-3},$$

$$\frac{c}{a} = c[1-k^2\lambda^{-1}x^{-2}+O(x^{-4})] = \lambda^{-1}x^{-2}[1-2kx^{-1}+O(x^{-3})],$$

$$\sqrt{\frac{c}{a}} = \lambda^{-1/2}x^{-1}[1-kx^{-1}+O(x^{-3})] = \lambda^{-1/2}x^{-1}-k\lambda^{-1/2}x^{-2}+O(x^{-4}),$$

$$I = \int\sqrt{\frac{c}{a}} = \lambda^{-1/2}\log x+\frac{k}{2}\lambda^{-1/2}x^{-1}+O(x^{-3}),$$

$$\frac{b}{a} = 2k^2\lambda^{-1}x^{-3}[1-k^2\lambda^{-1}x^{-2}+O(x^{-4})] = 2k^2\lambda^{-1}x^{-3}-2k^4\lambda^{-2}x^{-5}+O(x^{-7}),$$

$$\int\frac{b}{a} = -k^2\lambda^{-1}x^{-2}+\frac{1}{2}k^4\lambda^{-2}x^{-4}+O(x^{-6}),$$

$$\varrho = \frac{c}{a}e^{2\int\frac{b}{a}} = \lambda^{-1}x^{-2}[1-2kx^{-1}+O(x^{-3})]\exp[-2k^2\lambda^{-1}x^{-2}+O(x^{-4})] =$$

$$= \lambda^{-1}x^{-2}[1-2kx^{-1}+O(x^{-2})] = \lambda^{-1}x^{-2}-2k\lambda^{-1}x^{-3}+O(x^{-4}),$$

$$\varrho' = -2\lambda^{-1}x^{-3}+6k\lambda^{-1}x^{-4}+O(x^{-5}),$$

$$\varrho'' = 6\lambda^{-1}x^{-4}-24k\lambda^{-1}x^{-5}+O(x^{-6}),$$

which gives

$$r(\varrho) = 5\varrho'^2-4\varrho\left(\varrho''+\frac{b}{a}\varrho'\right) = -4\lambda^{-2}x^{-6}+O(x^{-7}) < 0,$$

$$R(\varrho) = -7\varrho'^2+4\varrho\left(\varrho''+\frac{b}{a}\varrho'\right) = -4\lambda^{-2}x^{-6}+O(x^{-7}) < 0.$$

(At the same time this is a positive answer to question 2) above.)

All these involves

$$\int_{z_i}^{z_{i+1}}\sqrt{\frac{c}{a}} \approx \lambda^{-1/2}\log\frac{z_{i+1}}{z_i}-\frac{k\lambda^{-1}}{2}\left(\frac{1}{z_i}-\frac{1}{z_{i+1}}\right) > \pi,$$

where $z_j = x_j$ or x_j' $(j = i, i+1)$,

which is a generalization of Makai's estimate [4].

Appendix. Generalization of some theorems of Wintner, Liapunov, Makai, etc.[3]

1. Wintner's theorem applied to (14) reads as follows: Equation (14) (and (11)) is oscillatory provided $\int^{\infty} \varrho(\xi)\,d\xi = +\infty$, i.e.,

$$\int^{\infty} \frac{c}{a} e^{2\int_{\xi_0}^{\xi} \frac{b}{a}} d\xi = \infty$$

or

$$\int^{\infty} \frac{\{\psi_1, \psi_2\}}{\{\varphi_1, \varphi_2\}} \exp\left[-\int_{x_0}^{x} \frac{[\varphi, \psi]}{\{\varphi_1, \varphi_2\}}\right] dx = +\infty.$$

2. Liapunov's theorem states: If some solution of (14) has at least two zeros in $[A, B]$, then

$$\int_{A}^{B} \varrho^{+}(\xi)\,d\xi > \frac{4}{B-A}, \qquad \varrho^{+} = \max(\varrho, 0).$$

If in the variable x, the interval $[\alpha, \beta]$ corresponds to $[A, B]$, then $B-A = \int_{\alpha}^{\beta} e^{-\int \frac{a}{b}} dx$. Therefore, if some solution of (11) has at least two zeros in $[\alpha, \beta]$ then

$$\int_{\alpha}^{\beta} \left(\frac{c}{a}\right)^{+} e^{\int_{x_0}^{x} \frac{b}{a}} dx \int_{\alpha}^{\beta} e^{-\int_{x_0}^{x} \frac{b}{a}} dx > 4,$$

i.e.,

$$\int_{\alpha}^{\beta} \left(\frac{\{\psi_1, \psi_2\}}{\{\varphi_1, \varphi_2\}}\right)^{+} \exp\left(-\int_{x_0}^{x} \frac{[\varphi, \psi]}{\{\varphi_1, \varphi_2\}}\right) dx \int_{\alpha}^{\beta} \exp\left(\int_{x_0}^{x} \frac{[\varphi, \psi]}{\{\varphi_1, \varphi_2\}}\right) dx > 4.$$

3. If $\varrho' = e^{\int \frac{b}{a}} \dot{\varrho} > 0$, then $\Phi(x) = Y(x) = \overline{Y}(\xi)$ is oscillatory and by the theorem of Sonin and Polya the amplitudes of $\Phi(x)$ are decreasing. At the same time — since $\frac{d\varrho^{-1}}{d\eta} = -\frac{\varrho'}{\varrho^3} e^{\int \frac{b}{a}} < 0$ — the amplitudes of $\dot{\Phi} = \Phi' e^{\int \frac{b}{a}}$ are increasing. Furthermore by certain theorems of Makai [6] and Bihari [7], the integrals

$$\int_{\xi_i}^{\xi_{i+1}} |\Phi(\xi)|\,d\xi = \int_{x_i}^{x_{i+1}} |\Phi(x)| e^{-\int_{x_0}^{x}} dx$$

are decreasing with i and

$$\int_{\xi_i}^{\xi_i'} |\Phi(\xi)|\,d\xi > \int_{\xi_i'}^{\xi_{i+1}} |\Phi(\xi)|\,d\xi > \int_{\xi_{i+1}}^{\xi_{i+1}'} |\Phi(\xi)|\,d\xi$$

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[3] Some theorems concerning oscillation, zeros and monotonicity of the solutions can be extended to the equation (11).

or

$$\int_{x_i}^{x_i'} \Psi(x)\,dx > \int_{x_i'}^{x_{i+1}} \Psi(x)\,dx > \int_{x_{i+1}}^{x_{i+1}'} \Psi(x)\,dx, \qquad \Psi(x) = |\Phi(x)| \exp\left(-\int_{x_0}^{x} \frac{b}{a}\right).$$

Here $\xi_i'(x_i')$ means the zero of $\dot{\bar{\Phi}}(\xi)\big(\Phi'(x)\big)$ consecutive to the zero $\xi_i(x_i)$ of $\bar{\Phi}(\xi)\big(\Phi(x)\big)$. By putting here $|\bar{\Phi}(\xi)|=|\Phi(x)|=1$ these formulae remain valid, since they reduce to the lengths of the intervals

$$[\xi_i, \xi_{i+1}], \quad [\xi_i, \xi_i'], \quad [\xi_i', \xi_{i+1}], \quad [\xi_{i+1}, \xi_{i+1}']$$

situated on the line $-\infty<\xi<\infty$.

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A MEAN ERGODIC THEOREM WITH A LOOK AT MARTINGALES

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Since von Neumann proved his statistical ergodic theorem a vast collection of ergodic theorems has appeared. The theorems presented here are general ergodic theorems for operators on Banach spaces. The special feature of our main result (Theorem 1) is that it draws near martingale convergence and ergodic theorems to each other. The similarity of martingale and ergodic theorems is striking and has been known for a long time. Theorem 1 is something like a unified martingale and ergodic theorem, at least on the level of mean convergence, the pointwise convergence is not touched here, however, it does not cover the most general martingales. The whole paper is motivated by a theorem of N. Dunford ([2]) and by the idea of F. Riesz: to divide the space into two subspaces and to prove the convergence on each of them.

Let X be a Banach space, $\{P_n\}\subset\mathcal{B}(X)$ a sequence of bounded operators and $\Phi\subset X^*$ a subspace. We shall assume the following conditions:

(i) for $x\in X$ $\|x\|=\sup\{|\varphi(x)|:\ \varphi\in\Phi$ and $\|\varphi\|\leq 1\}$,
(ii) $\sup\|P_n\|=K<+\infty$,
(iii) the family $\{P_n\}$ is equicontinuous with respect to the $\sigma(X,\Phi)$ topology,
(iv) for every $m\in\mathbf{N}$ and $x\in X$ we have $P_n(I-P_m)x\to 0$ and $(I-P_m)P_nx\to 0$ as $n\to\infty$,
(v) $\{P_nx\}$ is relatively sequentially compact with respect to $\sigma(X,\Phi)$ for every $x\in X$.

Let $X_1=\bigcap\limits_n \operatorname{Ker}(I-P_n)$, $X(2)=\bigcup\limits_n \operatorname{Rng}(I-P_n)$ and $X_2=\overline{X(2)}^{\Phi}$. Now we are in a position to state the following

THEOREM 1. *With the notations above and under the conditions* (i)—(v) *there is a $\sigma(X,\Phi)$-continuous bounded projection E of X onto X_1 such that* $\operatorname{Ker} E=X_2$, *and for every $x\in X$ we have $P_nx\xrightarrow{\Phi}Ex$. Moreover, there is a $\sigma(X,\Phi)$-dense subspace $X_0\subset X$ such that $P_nx\to Ex$ in norm for $x\in X_0$.*

PROOF. First we check that $X_1\cap X_2=\{0\}$. If $x\in X(2)$ then $P_nx\to 0$ by condition (iv). Since $\{P_n\}$ is equicontinuous $P_nx\xrightarrow{\Phi}0$ is valid for $x\in X_2$, as well. On the other hand, $P_nx=x$ for $x\in X_1$ and therefore $X_1\cap X_2=\{0\}$.

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Define Ex as a $\sigma(X, \Phi)$-cluster point of the sequence $\{P_n x\}$. Assume that $P_{n_k}x \xrightarrow{\Phi} y_1$ and $P_{m_k} \xrightarrow{\Phi} y_2$ as $k\to\infty$. Then $P_{n_k}-P_{m_k}x=(P_{n_k}x-x)-(P_{m_k}x-x)\in X_2$ and $y_1-y_2\in X_2$. Condition (iv) gives that $P_m y_1=y_1$ and $P_m y_2=y_2$ for every $m\in\mathbf{N}$. So $y_1-y_2\in X_1$ and Ex is well defined. We also have obtained that $\operatorname{Rng} E=X_1$, $E|X_1=$identity and $X_2\subset\operatorname{Ker} E$. According to (i) and (ii) we have $\|E\|<K$ and the $\sigma(X, \Phi)$-continuity of E.

Suppose that $Ex=0$ and $x\notin X_2$. Then there is a $\varphi\in\Phi$ such that $\varphi|X_2\equiv 0$ and $\varphi(x)\neq 0$. Since $Ex=\Phi-\lim P_n x$ one obtains $\varphi(Ex)=\lim\varphi(P_n x)$. However, $\varphi(P_n x-x)=0$ and $\varphi(Ex)=\varphi(x)$ is a contradiction. Hence $\operatorname{Ker} E=X_2$ is proved.

Since Ex is the only cluster point of the sequence $\{P_n x\}$ we have $P_n x \xrightarrow{\Phi} Ex$. $X_0=X_1\oplus\overline{X(2)}$ is $\sigma(X, \Phi)$-dense in X which is a direct sum $X_1\oplus X_2$. For $x_1\in X_1$ $P_n x_1\to x_1$ is evident and for $x_2\in X(2)$ $P_n x_2\to 0$ by condition (iv). Since $\|P_n\|<K$ we have $P_n x\to Ex$ for every $x_0\in X_0$.

If $\Phi=X^*$ then condition (i) is fulfilled and (ii) implies (iii). In this case $X_0=X$ and $P_n x\to Ex$ in norm for every $x\in X$. We note that if $\Phi=X^*$ and X reflexive then condition (ii) implies conditions (iii) and (v).

Now we are going to use Theorem 1 to deduce ergodic and martingale mean convergence theorems. The following result is a slight generalization of Lorch's theorem ([5]).

THEOREM 2. *Let X be a Banach space and $T\in\mathcal{B}(X)$ a power bounded operator (i.e. there is $K>0$ such that $\|T^n\|<K$ for every $n\in\mathbf{N}$). Set $P_n=n^{-1}\sum_{i=0}^{n-1} T^i$ and assume that $\{T^i x\colon i\in\mathbf{N}\}$ is relatively weakly sequentially compact for every $x\in X$ — i.e., any subsequence of $\{T^i x\}$ has a subsequence which converges weakly to an element of X. Then there is a projection E onto the subspace $\{x\in X\colon Tx=x\}$ such that $P_n\to E$ pointwise.*

PROOF. We apply Theorem 1 taking $\Phi=X^*$. Since the convex hull of a relatively weakly sequentially compact set is itself of this kind, condition (v) is satisfied (see [3] V. 6. 1 and V. 6. 4). In order to show condition (iv) we need a lemma.

LEMMA 1. $\|P_n(I-P_m)\|\leqq 2K\,m/n$ *if $n>m$ and K is a bound for* $\{\|T^n\|\colon n\in\mathbf{N}\}$.

This follows from the identity

$$nP_n(I-P_m)=\sum_{i=0}^{m-2} T^i-\sum_{i=0}^{m-2}\left(\frac{i+1}{m}T^i+\frac{m-i-1}{m}T^{n+i}\right)$$

that can be checked easily.

By the notice after Theorem 1 the proof is complete.

The following assertion is a continuous version of Theorem 2. For the sake of simpler formulation we state it for a reflexive space.

THEOREM 3. *Let X be a reflexive Banach space and $(T_t)_{t\in\mathbf{R}^+}$ a strongly continuous one-parameter semigroup of operators. Assume that $\|T_t\|\leqq K$ for every $t\in\mathbf{R}^+$ and set $A(\alpha)=\frac{1}{\alpha}\int_0^\alpha T_t\,dt$ for $\alpha>0$. Then $A(\alpha)$ converges pointwise to a projection onto the subspace $\{x\in X\colon T_t x=x$ for every $t\in\mathbf{R}^+\}$ as $\alpha\to+\infty$.*

PROOF. It is sufficient to prove that $A(\alpha_n)$ converges pointwise for every sequence $\alpha_n \to +\infty$. So the proof is completely similar to that of Theorem 2 but instead of Lemma 1 we need its continuous form.

LEMMA 2. *If* $\beta > \alpha > 0$ *then*

$$\|A(\beta)(I - A(\alpha))\| \leqq 2K^2 \alpha \beta^{-1}.$$

PROOF. First assume that $\alpha = n\delta$ and $\beta = k\delta$ with some integers n and k. Then

$$A(\beta)(x - A(\alpha)x) = P_k^\delta x_\delta - P_K^\delta P_n^\delta x_\delta + P_K^\delta A(\alpha)x - \frac{1}{\beta}\int_0^\beta T_t A(\alpha)x\,dt$$

where $x_\delta = \delta^{-1}\int_0^\delta T_t x dt$ and $P_m^\delta = m^{-1}\sum_{i=0}^{m-1} T_{i\delta}$. On the one hand

$$\|P_k^\delta x_\delta - P_0^\delta P_n^\delta x_\delta\| \leqq \frac{2nK\|x_\delta\|}{k} \leqq \frac{2\alpha K^2\|x\|}{\beta}$$

by Lemma 1, on the other hand

$$\left\|P_k^\delta A(\alpha)x - \frac{1}{\beta}\int_0^\beta T_t A(\alpha)x\,dt\right\| \to 0$$

as $\delta \to 0$. So we have obtained the estimation for rational α/β. If it is irrational one can use an approximation argument.

Concerning several one-parameter semigroups we have the following Dunford—Schwartz—Zygmund type theorem (cf. [2] VIII. 7. 10).

THEOREM 4. *Let X be a reflexive Banach space and $(T_t^i)_{t\in\mathbf{R}^+}$ a strongly continuous one-parameter semigroup of operators $(i \leqq k)$. Assume that $\|T_t^i\| \leqq K$ for $i \leqq k$ and $t \in \mathbf{R}^+$. Let $A_i(\alpha) = \frac{1}{\alpha}\int_0^\alpha T_t^i dt$ $(\alpha > 0, i \leqq k)$. Then*

$$A_k(\alpha_k)\dots A_1(\alpha_1)x \to E_k \dots E_1 x \quad (x \in X)$$

as $\alpha_1 \to +\infty, \dots, \alpha_k \to +\infty$ independently where E_i is a projection onto $\{x \in X\colon T_t^i x = x, t \in \mathbf{R}^+\}$.

PROOF. We assume that $k=2$. The general case can be settled similarly. Theorem 3 gives that

$$x = (x - A_1(s)x) + E_1 x + x(s)$$

$$E_1 x = (E_1 x - A_2(s)E_1 x) + \bar{x}(s)$$

where $x(s) \to 0$ and $\bar{x}(s) \to 0$ as $s \to +\infty$. Hence

$$A_2(\alpha_2)A_1(\alpha_1)x - E_2 E_1 x = A_2(\alpha_2)A_1(\alpha_1)(x - A_1(s)x) + A_2(\alpha_2)(E_1 x - A_2(s)E_1 x) +$$

$$+ A_2(\alpha_2)A_1(\alpha_1)x(s) + A_2(\alpha_2)\bar{x}(s).$$

If D_j denotes the j-th term on the right-hand side then

$$\|D_1\| \leqq K^3 \frac{2s}{\alpha_1} \|x\|, \quad \|D_2\| \leqq K^3 \frac{2s}{\alpha_2} \|x\|$$

$$\|D_3\| \leqq K^2 \|x(s)\|, \quad \|D_4\| \leqq K \|\bar{x}(s)\|.$$

Consequently, $\sum_{j=1}^{4} \|D_j\|$ is small if s, α_1, α_2 are big enough.

We note that $E_k \dots E_1$ is not a projection in general but it will be a projection if the semigroups are commuting. Now we turn to martingales.

Let X be a Banach space and $(E_n) \subset \mathscr{B}(X)$ a sequence of projections with the properties $\sup \|E_n\| < +\infty$ and $E_n E_m = E_m E_n = E_{m \wedge n}$. (Here $m \wedge n$ stands for $\min(m, n)$.) We say that (X, E_n, f_n) is an abstract martingale if $(f_n) \subset X$ and $E_n f_m = f_{n \wedge m}$.

THEOREM 4. *Let (X, E_n, f_n) be an abstract martingale and assume that X is reflexive and $\underline{\lim} \|f_n\| < +\infty$. Then there is an $f \in X$ such that $f_n \to f$ and $E_n f = f_n$.*

PROOF. We intend to apply Theorem 1 with $\Phi = X^*$ and $P_n = I - E_n$. We have $P_n(I - P_m) = E_m - E_{n \wedge m} = 0$ if $n > m$. There is a convergent subsequence of (f_n), say $f_{k_n} \xrightarrow{w} f$. So $E_m f_{k_n} \xrightarrow{w} E_m f$ and $f_m = E_m f$. But $E_m f = f - P_m f \to f - Ef$ for some projection E given by Theorem 1. Hence $Ef = 0$ and $P_m f = f_m \to f$.

This assertion covers the vector valued martingale mean convergence in $L_p(\mu, Y)$ whenever $1 < p < \infty$ and Y is a reflexive Banach space. In this case $L_p(\mu, Y) = X$ is reflexive, too, and E_m's are conditional expectations. (See [2] p. 126 or [1].)

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ON SUMS OF INTEGERS HAVING SMALL PRIME FACTORS, II

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1. Throughout this paper, we use the following notations: $c_1, c_2, \ldots, N_0, N_1, \ldots$ denote positive absolute constants. We write $e^x = \exp(x)$ and $e^{2\pi i\alpha} = e(\alpha)$. The distance from α to the nearest integer is denoted by $\|\alpha\|$ so that $\|\alpha\| = \min(\alpha - [\alpha], [\alpha] + 1 - \alpha)$. We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$.

2. In Part I (see [1]), we proved the following theorem:

THEOREM 1. *If* $N > N_0$ *then* N *can be written in the form* $n_1 + n_2 + n_3 = N$ *where*

$$P(n_1 n_2 n_3) \leqq \exp\{3(\log N \log\log N)^{1/2}\}.$$

In this paper, we study the analogous *binary* problem. In fact, we prove the following theorem:

THEOREM 2. *There exist absolute constants* M_0, c_1 (>0) *such that if* $M > M_0$ *and*

$$\exp\{5(\log M \log\log M)^{1/2}\} \leqq y < M^{1/3} \tag{1}$$

then

$$n_1 + n_2 = n, \quad P(n_1 n_2) \leqq y \tag{2}$$

can be solved for all but $c_1 \dfrac{M}{y} \exp\left(10 \dfrac{\log M}{\log y} \log\log M\right) \left(< \dfrac{M}{y^{1/2}}\right)$ *integers* $n \leqq M$.

We conjecture that if $\varepsilon > 0$, $N > N_1(\varepsilon)$ *then*

$$n_1 + n_2 = N, \quad P(n_1 n_2) \leqq N^{\varepsilon}$$

can be solved; in fact, this is, perhaps, true also with $\exp\left(c(\log N \log\log N)^{1/2}\right)$ in place of N^{ε}. Unfortunately, we can prove only the following much weaker theorem:

THEOREM 3. *If* $N > N_2$ *then* N *can be written in the form*

$$n_1 + n_2 = N$$

where

$$P(n_1 n_2) \leqq 2N^{2/5}.$$

The proof of Theorem 3 will be based on studying the distribution of bm^2 in short intervals. Using deep results on trigonometrical sums the exponent 2/5 in Theorem 3 can be slightly improved.

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3. In order to prove Theorem 2, we use the Hardy—Littlewood method. The proof is based on the estimates given in Part I, and we use also an idea suggested by [6].

Let y be any real number satisfying (1), and let $\mathcal{R}$ denote the set of the integers n such that (2) cannot be solved.

Let N be a positive integer such that

$$M^{2/3} \leqq N \leqq M. \tag{3}$$

Then (1) implies that

$$\exp\{5(\log N \log\log N)^{1/2}\} \leqq y < N^{1/2} \tag{4}$$

(so that y and N satisfy the condition given in Part I and thus the lemmas proved there can be used also here). Put

$$z = \frac{1}{2} y^{1/2},$$

$$Q = \frac{N}{z} = 2\frac{N}{y^{1/2}}$$

and

$$U = \left[2\frac{N}{y}\right] + 1.$$

Let $\mathcal{A}$ denote the set of the integers k such that $\frac{3}{5}\frac{N}{y} < k \leqq \frac{N}{y}$ and $z < p(k)$, $P(k) \leqq y$. We write

$$A = \sum_{k \in \mathcal{A}} 1,$$

$$d_n = \sum_{\substack{mk=n \\ m \leqq y \\ k \in \mathcal{A}}} 1 \quad (\text{for } 1 \leqq n \leqq N),$$

$$D = \sum_{n=1}^{N} d_n^2,$$

$$S(\alpha) = \sum_{n=1}^{N} d_n e(n\alpha),$$

$$S = S(0) = \sum_{n=1}^{N} d_n,$$

$$U(\alpha) = \sum_{n=1}^{U-1} e(n\alpha),$$

$$S(\alpha)U(\alpha) = \sum_{n=1}^{N+U-1} g_n e(n\alpha),$$

$$S^2(\alpha) = \sum_{n=1}^{2N} r_n e(n\alpha),$$

$$\left(\frac{1}{U} U(\alpha)S(\alpha)\right)^2 = \sum_{n=1}^{2(N+U-1)} t_n e(n\alpha)$$

so that

$$g_n = \sum_{n-U<j\leq n} d_j,$$

$$r_n = \sum_{n_1+n_2=n} d_{n_1} d_{n_2}$$

and

$$t_n = \frac{1}{U^2} \sum_{n_1+n_2=n} g_{n_1} g_{n_2}.$$

Obviously, if $d_n>0$ then we have $P(n)\leqq y$ so that

(5) $$n \in \mathcal{R} \text{ implies that } r_n = \sum_{n_1+n_2=n} d_{n_1} d_{n_2} = 0.$$

We start out from the integral

$$J = \int_0^1 \left| (S(\alpha))^2 - \left(\frac{U(\alpha)S(\alpha)}{U} \right)^2 \right|^2 d\alpha.$$

4. In order to estimate the integral J, we need some lemmas from Part I of this paper.

LEMMA 1. *If T is a positive integer, α a real number then we have*

$$\left| \sum_{n=0}^{T-1} e(n\alpha) - T \right| < 4T^2 |\alpha|.$$

This is identical with Lemma 1 in [1].

LEMMA 2. *We have*

$$S \leqq N.$$

This is Lemma 6 in [1].

LEMMA 3. *For $N\geqq 2$ we have*

$$D < c_2 N (\log N)^3.$$

This is Lemma 7 in [1].

LEMMA 4. *If*

$$\frac{1}{Q} < \alpha < 1 - \frac{1}{Q}$$

then for $N>N_3$ we have

$$|S(\alpha)| < 5 \frac{N}{y^{1/2}} \log N.$$

This is Lemma 11 in [1].

LEMMA 5. *If n is a positive integer satisfying $U\leqq n\leqq 3N/5$ then we have*

$$g_n \geqq A.$$

This is Lemma 12 in [1].

LEMMA 6. *For* $N>N_4$ *we have*

$$A > \frac{N}{y}\exp\left(-\frac{6}{5}\frac{\log N}{\log y}\log\log N\right).$$

This is Lemma 14 in [1].

5. In this section, we derive a lower estimate for the integral J.

By Parseval's identity and (5) (and writing $r_{2N+1}=r_{2N+2}=\ldots=0$), we have

$$J=\int_0^1\left|(S(\alpha))^2-\left(\frac{U(\alpha)S(\alpha)}{U}\right)^2\right|^2 d\alpha=$$

$$=\int_0^1\Big|\sum_{n=1}^{2N} r_n e(n\alpha)-\sum_{n=1}^{2(N+U-1)} t_n e(n\alpha)\Big|^2 d\alpha=$$

(6)

$$=\int_0^1\Big|\sum_{n=1}^{2(N+U-1)}(r_n-t_n)e(n\alpha)\Big|^2 d\alpha=\sum_{n=1}^{2(N+U-1)}(r_n-t_n)^2\geqq$$

$$\geqq\sum_{\substack{\frac{4}{5}N<n\leqq N\\ r_n=0}}(r_n-t_n)^2=\sum_{\substack{\frac{4}{5}N<n\leqq N\\ r_n=0}} t_n^2\geqq\sum_{\substack{\frac{4}{5}N<n\leqq N\\ n\in\mathcal{R}}} t_n^2.$$

With respect to Lemma 5 (and since $g_n\geqq 0$ for all n), for $4N/5<n\leqq N$ we have

$$t_n=\frac{1}{U^2}\sum_{n_1+n_2=n} g_{n_1}g_{n_2}\geqq$$

$$\geqq\frac{1}{U^2}\sum_{\substack{n_1+n_2=n\\ n_1\leqq 3N/5\\ n_2\leqq 3N/5}} g_{n_1}g_{n_2}\geqq\frac{1}{U^2}\sum_{\substack{n_1+n_2=n\\ N/5<n_1\leqq 3N/5}} g_{n_1}g_{n_2}\geqq$$

$$\geqq\frac{1}{U^2}\sum_{\substack{n_1+n_2=n\\ N/5<n_1\leqq 3N/5}} A^2=\frac{A^2}{U^2}\sum_{N/5<n_1\leqq 3N/5} 1=\frac{A^2}{U^2}\left(\left[\frac{3N}{5}\right]-\left[\frac{N}{5}\right]\right)>$$

$$>\frac{A^2}{(3N/y)^2}\frac{N}{5}=\frac{A^2y^2}{45N}.$$

Thus we obtain from (6) that

$$J\geqq\sum_{\substack{\frac{4}{5}N<n\leqq N\\ n\in\mathcal{R}}} t_n^2>\sum_{\substack{\frac{4}{5}N<n\leqq N\\ n\in\mathcal{R}}}\left(\frac{A^2y^2}{45N}\right)^2>$$

(7)

$$>\frac{1}{2500}\frac{A^4y^4}{N^2}\sum_{\substack{\frac{4}{5}N<n\leqq N\\ n\in\mathcal{R}}} 1.$$

6. In this section, we give an upper estimate for J.
By using Lemmas 1, 2, 3 and 4, we obtain that

$$J=\int_0^1 |S(\alpha)|^4\left|1-\frac{(U(\alpha))^2}{U^2}\right|^2 d\alpha =$$

$$=\int_{-1/Q}^{+1/Q} |S(\alpha)|^4\left|\frac{U-U(\alpha)}{U}\right|^2\left|1+\frac{U(\alpha)}{U}\right|^2 d\alpha+$$

$$+\int_{+1/Q}^{1-1/Q} |S(\alpha)|^4\left|1-\frac{(U(\alpha))^2}{U^2}\right|^2 d\alpha \leqq$$

$$\leqq \int_{-1/Q}^{+1/Q} S^2|S(\alpha)|^2(4U|\alpha|)^2\left(1+\frac{|U(\alpha)|}{U}\right)^2 d\alpha+$$

$$+\int_{+1/Q}^{1-1/Q} \Big(\max_{+1/Q\leqq\alpha\leqq 1-1/Q}|S(\alpha)|\Big)^2|S(\alpha)|^2\left(1+\frac{|U(\alpha)|^2}{U^2}\right)^2 d\alpha \leqq$$

$$\leqq \int_{-1/Q}^{+1/Q} S^2|S(\alpha)|^2 16U^2\frac{1}{Q^2}(1+1)^2\,d\alpha+ \tag{8}$$

$$+\int_{+1/Q}^{1-1/Q}\left(5\frac{N}{y^{1/2}}\log N\right)^2|S(\alpha)|^2(1+1)^2\,d\alpha \leqq$$

$$\leqq 64\frac{S^2U^2}{Q^2}\int_0^1 |S(\alpha)|^2 d\alpha+100\,\frac{N^2(\log N)^2}{y}\int_0^1 |S(\alpha)|^2 d\alpha \leqq$$

$$\leqq 100\left(\frac{N^2(3N/y)^2}{(2N/y^{1/2})^2}+\frac{N^2(\log N)^2}{y}\right)\int_0^1 |S(\alpha)|^2 d\alpha =$$

$$=100\left(\frac{9}{4}\frac{N^2}{y}+\frac{N^2(\log N)^2}{y}\right)D<c_3\frac{N^2}{y}(\log N)^2N(\log N)^3=$$

$$=c_3\frac{N^3(\log N)^5}{y}.$$

7. In this section, we complete the proof of Theorem 2. (7) and (8) yield that

$$\frac{1}{2500}\frac{A^4y^4}{N^2}\sum_{\substack{\frac{4}{5}N<n\leqq N\\ n\in\mathcal{R}}}1<J<c_3\frac{N^3(\log N)^5}{y}.$$

Thus with respect to Lemma 6,

$$\sum_{\substack{\frac{4}{5}N<n\leq N\\ n\in\mathcal{R}}} 1 < c_4\frac{1}{A^4}\frac{N^5(\log N)^5}{y^5} <$$

$$< c_4\frac{y^4}{N^4}\exp\left(\frac{24}{5}\frac{\log N}{\log y}\log\log N\right)\frac{N^5(\log N)^5}{y^5} =$$

(9)

$$= c_4\frac{N}{y}\exp\left(\left(\frac{24}{5}\frac{\log N}{\log y}+5\right)\log\log N\right) < c_4\frac{N}{y}\exp\left(10\frac{\log N}{\log y}\log\log N\right) \leq$$

$$\leq c_4\frac{N}{y}\exp\left(10\frac{\log M}{\log y}\log\log M\right)$$

and this holds for all N satisfying (3).

Define the positive integer k by

$$\left(\frac{5}{6}\right)^k M < M^{2/3} \leq \left(\frac{5}{6}\right)^{k-1} M,$$

i.e.,

$$k = \left[\frac{\log M}{3\log 6/5}\right]+1$$

so that

$$k < 2\log M.$$

Then by (1) and (9) we have

$$\sum_{\substack{1\leq n\leq M\\ n\in\mathcal{R}}} 1 = \sum_{\substack{1\leq n\leq M^{2/3}\\ n\in\mathcal{R}}} 1 + \sum_{\substack{M^{2/3}<n\leq M\\ n\in\mathcal{R}}} 1 \leq$$

$$\leq \sum_{1\leq n\leq M^{2/3}} 1 + \sum_{j=1}^{k}\Big(\sum_{\substack{\left(\frac{5}{6}\right)^j M<n\leq\left(\frac{5}{6}\right)^{j-1}M\\ n\in\mathcal{R}}} 1\Big) \leq$$

$$\leq M^{2/3} + \sum_{j=1}^{k}\Big(\sum_{\substack{\frac{4}{5}\left[\left(\frac{5}{6}\right)^{j-1}M\right]<n\leq\left[\left(\frac{5}{6}\right)^{j-1}M\right]\\ n\in\mathcal{R}}} 1\Big) <$$

$$< M^{2/3} + \sum_{j=1}^{k} c_4\frac{\left(\frac{5}{6}\right)^{j-1}M}{y}\exp\left(10\frac{\log M}{\log y}\log\log M\right) <$$

$$< M^{2/3}+c_5\frac{M}{y}\exp\left(10\frac{\log M}{\log y}\log\log M\right) <$$

$$< c_6\frac{M}{y}\exp\left(10\frac{\log M}{\log y}\log\log M\right)$$

which completes the proof of Theorem 2.

8. In this section, we prove Theorem 3 (by using an idea from [2]). In order to prove this theorem, it is sufficient to show that there exist integers a, b, c, d such that $N=a+bcd$ and $0 \leqq a, b, c, d \leqq 2N^{2/5}$, i.e., there exist integers b, c, d such that

$$N-2N^{2/5} \leqq bcd < N \quad \text{and} \quad 0 \leqq b,\ c,\ d \leqq 2N^{2/5}. \tag{10}$$

Put $b=\left[\frac{1}{2}N^{1/5}\right]$, $m=[(N/b)^{1/2}+1]$ and let k be the smallest integer satisfying $b(m^2-k^2)<N$. Obviously, $bm^2>N$, hence $k \geqq 1$. By the definition of k we have $N \leqq b(m^2-(k-1)^2)$, so that

$$(k-1)^2 \leqq m^2-\frac{N}{b} \leqq ((N/b)^{1/2}+1)^2-\frac{N}{b} = 2(N/b)^{1/2}+1 = \big(1+o(1)\big)2^{3/2}N^{2/5}. \tag{11}$$

Thus for large N,

$$\begin{aligned} &(N>)\, b(m^2-k^2)=b\big(m^2-(k-1)^2\big)-2bk+b \geqq N-2b(k-1)-b \geqq \\ &\geqq N-b\big(2(2(N/b)^{1/2}+1)^{1/2}+1\big) = N-\big(1+o(1)\big)2^{3/4}N^{2/5} > N-2N^{2/5}. \end{aligned} \tag{12}$$

This inequality shows that $b\left(=\left(\frac{1}{2}+o(1)\right) N^{1/5}\right)$, $c=m-k$ $\left(=(\sqrt{2}+o(1))N^{2/5}\right)$ and $d=m+k$ $\left(=(\sqrt{2}+o(1))N^{2/5}\right)$ satisfy (10) which completes the proof of Theorem 3.

9. The method used in Section 8 can be generalized in the following way: Let X, Y be real numbers for which $100X \leqq Y \leqq N$, and

$$N < bm^2 \leqq N+Y, \quad X < b \leqq 2X \tag{13}$$

can be solved (in integers b, m). Let k be the smallest integer with $b(m^2-k^2)<N$. Similarly to (11) and (12),

$$(k-1)^2 \leqq m^2-\frac{N}{b} \leqq \frac{Y}{b} \tag{14}$$

and

$$(N>)\, b(m^2-k^2) \geqq N-2b(k-1)-b \geqq N-3\sqrt{XY}, \tag{15}$$

and we get from (13) and (14) that

$$m-k < m+k \leqq 3\sqrt{\frac{N}{X}}. \tag{16}$$

Thus (15) implies Theorem 3 with

$$3\max\left(\sqrt{XY},\ \sqrt{\frac{N}{X}}\right) \tag{17}$$

in place of $2N^{2/5}$.

Taking as usually

$$\Psi(x) = x-[x]-\frac{1}{2},$$

the number of solutions of (13) can be written in the form

$$S = \sum_{X<b\leq 2X} \left(\left[\sqrt{\frac{N+Y}{b}}\right] - \left[\sqrt{\frac{N}{b}}\right]\right) = \tag{18}$$
$$= \frac{Y}{\sqrt{N+Y}+\sqrt{N}} \sum_{X<b\leq 2X} \frac{1}{\sqrt{b}} - \sum_{X<b\leq 2X} \Psi\left(\sqrt{\frac{N+Y}{b}}\right) + \sum_{X<b\leq 2X} \Psi\left(\sqrt{\frac{N}{b}}\right).$$

Thus in order to improve Theorem 3, we have to show the solvability of (13) for X, Y such that the maximum in (17) is possibly small; and the study of the solvability of (13) leads to the estimate of S, i.e., of the last two sums in S. These sums can be estimated by using the method of exponent pairs (see [4], [5]), and the best bounds for these sums can be obtained by using the recent result of G. Kolesnik [3] concerning exponential sums of two variables. But this rather complicated method leads to very slight improvement only, namely we can replace $N^{2/5}$ in Theorem 3 by $N^{0.392\dots}$; thus we do not work out the details.

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ON PRIMES IN SHORT INTERVALS II

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1. Refining a method developed by M. Jutila and H. Iwaniec [4], D. R. Heath-Brown and Iwaniec [3] proved in 1979 the inequality

$$\pi(x)-\pi(x-y)\gg\frac{y}{\log x},\quad y=x^{\theta}\tag{1.1}$$

for $\theta>11/20=0.55$. This method was the combination of the linear sieve and weighted density theorems for the zeta-zeros. In part I [8] we have shown that their main lemma could be improved slightly and this will enable us to prove the following

THEOREM. *If* $\theta\geqq 17/31-c_1$ $(=0.548387...-c_1)$ *then*

$$\pi(x)-\pi(x-y)\geqq c_2\frac{y}{\log x}\tag{1.2}$$

where c_1 and c_2 are explicitly calculable positive absolute constants. In particular, for $x>c_3$ we have

$$\pi(x)-\pi(x-x^{17/31})>\frac{x^{17/31}}{3000\log x}.\tag{1.3}$$

COROLLARY.

$$p_{n+1}-p_n\ll p_n^{17/31-c_1}.$$

We note here that Heath-Brown developed a method which makes possible a slight improvement of $\theta=11/20$, too; further, Iwaniec [6] proved independently essentially the same result, namely for $\theta>17/31$, $x>x(\theta)$

$$\pi(x)-\pi(x-y)>\frac{y}{45\log x}\tag{1.4}$$

and consequently

$$p_{n+1}-p_n\ll p_n^{17/31+\varepsilon}\tag{1.5}$$

and he mentioned that the constant 17/31 could be reduced slightly by elaborating his method. His method depends also finally on the Deshouillers—Iwaniec theorem for power-mean values of the zeta-function [1] (cf. (2.1) in [8]).

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Now following the arguments of Heath-Brown and Iwaniec [3] we shall sketch the needed modifications compared to [3] which will lead to the proof of the Theorem. The only change compared to [3] is that we improve slightly their Lemma 6. (This improvement was not needed in [3], since there the limit of the method, $\theta > 11/20$ was required for the main Lemma.) However, to make the paper understandable in itself, we shall sketch the arguments of Heath-Brown and Iwaniec, but for the details the reader is referred to [3]. We remark further that improving the present analytic arguments (using Vaughan's method and the method described in [8]) a better value, e.g. $\theta = 1/2 + 1/21 = 23/42 = 0{,}547619\ldots$ can be obtained, too (if we modify slightly the main Lemma in part I [8] additionally). This will be proved in a forthcoming joint paper with H. Iwaniec.

2. Let

$$P(z) = \prod_{p<z} p, \quad V(z) = \prod_{p<z}\left(1-\frac{1}{p}\right) \tag{2.1}$$

where p runs over primes. Let

$$\begin{gathered}\mathscr{A} = \{n,\ x-y < n \leqq x\}, \quad \mathscr{A}_d = \{n \in \mathscr{A},\ d \mid n\},\\ S(\mathscr{A}, z) = \{n \in \mathscr{A},\ (n, P(z)) = 1\},\\ W^-(\mathscr{A}, z, D) = S(\mathscr{A}, z) - \sum_{(D/p)^{1/3} \leqq q < p < z} S(\mathscr{A}_{qp};\ q)\end{gathered} \tag{2.2}$$

where q and p run over primes and $\sqrt{z} \leqq D \leqq z^4$. Using the Buchstab identity $(z_1 \leqq z_2)$

$$S(\mathscr{A}, z_2) = S(\mathscr{A}, z_1) - \sum_{z_1 \leqq p < z_2} S(\mathscr{A}_p, p) \tag{2.3}$$

we can write

$$\begin{aligned}\pi(x) - \pi(x-y) &= S(\mathscr{A}, \sqrt{x}) =\\ &= S(\mathscr{A}, z) - \sum_{z \leqq p < \sqrt{D_\theta}} S(\mathscr{A}_p, p) - \sum_{\sqrt{D_\theta} \leqq p < \sqrt{x}} S(\mathscr{A}_p, p) =\\ &= W^-(\mathscr{A}, z, D) + \sum_{(D/p)^{1/3} \leqq q < p < z} S(\mathscr{A}_{pq}, q) - \sum_{z \leqq p < \sqrt{D_\theta}} S(\mathscr{A}_p, p) -\\ &\quad - \sum_{\sqrt{D_\theta} \leqq p < \sqrt{x}} S(\mathscr{A}_p, (D/p)^{1/3}) + \sum_{\substack{\sqrt{D_\theta} \leqq p < \sqrt{x}\\ (D/p)^{1/3} \leqq q < p}} S(\mathscr{A}_{qp}, q) =:\\ &=: \textstyle\sum_1 + \sum_2 - \sum_3 - \sum_4 + \sum_5.\end{aligned} \tag{2.4}$$

As to our parameters we shall require (in accordance with [3] and [8])

$$\begin{gathered}0 < \eta < 10^{-3}, \quad \theta = 0.55 - \delta, \quad y = x^{\theta + 5\eta}, \quad x > x(\theta, \eta),\\ D = x^{0.92 + B - 2\eta} > x^{0.9}, \quad z = T^{16/5} x^{-1}, \quad D_\theta = x^2 T^{-12/5}\end{gathered} \tag{2.5}$$

where

$$T = x^{1+3\eta} y^{-1} = x^{1-\theta-2\eta} = x^{0.45+\delta-2\eta} \tag{2.6}$$

and η will be chosen sufficiently small ($\eta \to 0$ as $x \to \infty$); c will denote a positive absolute constant that need not be the same at each appearance.

The starting point is the following sieve result of Iwaniec [5].

LEMMA 1. *With the standard functions $F(s)$ and $f(s)$ (cf., e.g. Halberstam and Richert* [2]*)*

$$S(\mathscr{A}, z) \leqq yV(z)\{F(s)+c\eta\}+R^{+} \tag{2.7}$$

$$W^{-}(\mathscr{A}, z, D) \geqq yV(z)\{f(s)-c\eta\}-R^{-} \tag{2.8}$$

where $s=\log D/\log z$ and $R^{\pm}$ have the special form given by Iwaniec [5] (cf. also [3] or [8]). The main result of part I [8] was that under given circumstances the remainder terms $R^{\pm}$ are negligible compared to the main terms. This we formulate as:

LEMMA 2. *If we have additionally*

$$0 \leqq \delta \leqq \frac{7}{2980}, \quad |B| \leqq 0.016 \tag{2.9}$$

$$\frac{5}{2}B+\frac{24}{5}\delta \leqq 0.04 \tag{2.10}$$

$$B+9\delta \leqq 0.03 \tag{2.11}$$

$$12B+263\delta \leqq 0.61 \tag{2.12}$$

then

$$R^{\pm} \ll y\exp(-\log^{1/5} x). \tag{2.13}$$

According to this we shall choose B as

$$B=\frac{0.4-48\delta}{25}=\frac{48\theta-26}{25} \tag{2.14}$$

and then easy calculation shows that all the inequalities (2.9)–(2.12) hold if

$$0 \leqq \delta \leqq \frac{1}{620}+c_1, \quad \text{i.e.} \quad \frac{11}{20} \geqq \theta \geqq \frac{17}{31}-c_1. \tag{2.15}$$

3. Similarly to Heath-Brown and Iwaniec [3] we get from Lemmas 1—2 with $s_1=\log D/\log z=(48\theta-3)/(55-80\theta)+O(\eta)$

$$\sum\nolimits_1 \geqq \left(\frac{2\log(s_1-1)-c\eta}{\log D}\right)y, \tag{3.1}$$

i.e.

$$\sum\nolimits_1 \geqq (c_1(\theta)-c\eta)\frac{y}{\log x} \tag{3.2}$$

where

$$c_1(\theta)=\frac{50}{48\theta-3}\log\frac{128\theta-58}{55-80\theta}. \tag{3.3}$$

$\sum_4$ can be treated analogously by Vaughan's method as in Heath-Brown and Iwaniec [3]. The only change is that here the main Lemma, our Lemma 2, is not so general as in [3]. But the proof of Lemma 2 (cf. [8]) gives that for the critical quantity S_4 of $\sum_4$ in § 4 of [3] the inequality, analogous to (2.13),

$$S_4 \ll y \exp(-\log^{1/5} x) \tag{3.4}$$

is again valid, because S_4 can be written in a multilinear form having the same essential properties as $R^{\pm}$, which were used in Part I [8] to prove (2.13). So we get in our case

$$\begin{aligned}\sum_4 &< (2+c\eta)y \sum_{\sqrt{D_0}\leqq n\leqq 2\sqrt{x}} \frac{\Lambda(n)}{n\log n\log(D/n)} < \\ &< (c_4(\theta)+c\eta)\frac{y}{\log x}\end{aligned} \tag{3.5}$$

with

$$c_4(\theta) = \frac{50}{48\theta-3}\log\frac{90\theta+10}{(96\theta-31)(6\theta-1)}. \tag{3.6}$$

The term $\sum_3$ being the same as in [3] we obtain from [3], § 5, that

$$\sum_3 = (c_3(\theta)+O(\eta))\frac{y}{\log x} \tag{3.7}$$

where

$$c_3(\theta) = \log\frac{(6\theta-1)(8\theta-3)}{3(1-\theta)(11-16\theta)}. \tag{3.8}$$

4. Now we shall consider weighted density estimates for the evaluation of some parts of $\sum_2$ and $\sum_5$. Let, as in [3],

$$M(s) = \sum_{M<m\leqq 2M} a_m m^{-s}, \qquad N(s) = \sum_{N<n\leqq 2N} b_n n^{-s}, \tag{4.1}$$

where $M, N \geqq 2$ and $|a_m|, |b_n| \leqq 1$. The sums below run over zeros $\varrho=\beta+i\gamma$ of $\zeta(s)$ and τ lies in the range $X^{4/9}\leqq\tau\leqq X^{1/2}$.

Lemma 3. *Let* $\tau^{18/5}X^{-1}\leqq MN\leqq X\tau^{-3/5}$ *and* $\tau^{13/5}X^{-1}\leqq M/N\leqq X\tau^{-18/5}$. *Then*

$$\sum_{\beta\geqq\sigma,\,|\gamma|\leqq\tau} |M(\varrho)N(\varrho)| \ll_\eta X^{(1-\sigma)(1+\eta)} \tag{4.2}$$

uniformly for $0\leqq\sigma\leqq 1$.

Lemma 4. *Let* $\tau^{86/23}X^{-20/23}\leqq MN\leqq X$, $MN^{-23/20}\leqq X\tau^{-43/20}$ *and* $MN^{-23/20}\leqq$ $\leqq X\tau^{-43/20}$. *Then* (4.2) *holds uniformly in* σ.

Lemma 5. *Let* $\tau^{52/15}X^{-1}\leqq MN\leqq X\tau^{-4/5}$ *and* $\tau^{14/5}X^{-1}\leqq M/N\leqq X\tau^{-22/15}$. *Then* (4.2) *holds uniformly in* σ.

Lemmas 3 and 5 are the same as Lemmas 5 and 7 in Heath-Brown and Iwaniec [3]. Lemma 4 slightly improves Lemma 6 of [3] which is needed here to reach the exponent 17/31 in the main problem. To prove Lemma 4 we shall use the density theorem of Jutila [7], according to which for $\sigma \geqq 33/43$,

$$N(\sigma, T) = \sum_{\beta \geqq \sigma, |\gamma| \leqq \tau} 1 \ll_\eta \tau^{\frac{43}{20}(1-\sigma)\left(1+\frac{\eta}{2}\right)} \tag{4.3}$$

(This follows from (1.8) of Jutila, choosing $k=5$ and $\alpha \geqq 33/43$.)

Analogously to Heath-Brown and Iwaniec [3] we obtain from the mean-value theorem of Dirichlet polynomials for $\sigma \leqq 33/43$

$$\begin{aligned} H := \big(X^{(\sigma-1)(1+\eta)} & \sum_{\beta \geqq \sigma, |\gamma| \leqq \tau} |M(\varrho)N(\varrho)|\big)^2 \ll \\ \ll X^{2\sigma-2-2\eta(1-\sigma)} & \sum_{\beta \geqq \sigma, |\gamma| \leqq \tau} |M(\varrho)|^2 \sum_{\beta \geqq \sigma, |\gamma| \leqq \tau} |N(\varrho)|^2 \ll \\ \ll X^{2\sigma-2-2\eta(1-\sigma)} & (MN)^{1-2\sigma}(MN+M\tau+N\tau+\tau^2)\log^2 X \ll \\ \ll X^{-20/43-\eta/3} & \big((MN)^{20/43}+M^{20/43}N^{-23/43}\tau+M^{-23/43}N^{20/43}\tau+(MN)^{-23/43}\tau^2\big) \ll 1 \end{aligned} \tag{4.4}$$

taking into account the propositions of Lemma 4. On the other hand for $1-(\log X)^{-4/5} \geqq \sigma \geqq 33/43$ we have from the Halász—Montgomery inequality and (4.3) the analogous estimate

$$\begin{aligned} H \ll X^{2\sigma-2-(\eta/2)(1-\sigma)}(MN)^{1-2\sigma}(MN+M\tau^{1/2+(43/20)(1-\sigma)}+ \\ + N\tau^{1/2+(43/20)(1-\sigma)}+\tau^{1+(86/20)(1-\sigma)}) \end{aligned} \tag{4.5}$$

where the first term is <1 in view of $MN \leqq x$ and all the other terms decrease monotonically in σ, since by our propositions

$$\frac{X^2}{(MN)^2\tau^{43/20}} \leqq \frac{X^{86/23}}{\tau^{2\,86/23+43/20}} < 1 \tag{4.6}$$

because $\tau > X^{4/9}$. Thus it is sufficient to check that the 2nd, 3rd and 4th term is less than 1 for $\sigma = 33/43$, i.e.

$$X^{-23/43-\eta/10}\big(M^{20/43}N^{-23/43}\tau+N^{20/43}M^{-23/43}\tau+(MN)^{-23/43}\tau^2\big) \ll 1 \tag{4.7}$$

which is the same as in (4.4).

5. Similarly to Heath-Brown and Iwaniec [3] we can now show asymptotic formulas for the following parts of $\sum_2$ (p, q, r are primes, $q \leqq r, pqr \in \mathcal{A}$)

$$\sum\nolimits_2^{(1)} = \sum_{\substack{Q<q\leqq 2Q \\ R<r\leqq 2R}} \left(\psi\left(\frac{x}{qr}\right)-\psi\left(\frac{x-y}{qr}\right)\right) \tag{5.1}$$

and

$$\sum\nolimits_2^{(2)} = \sum_{\substack{P<p\leqq 2P \\ R<r\leqq 2R}} \left(\psi\left(\frac{x}{pr}\right)-\psi\left(\frac{x-y}{pr}\right)\right) \tag{5.2}$$

which yield

$$\Sigma_2^{(1)} = y\Big(\sum_{q,r} \frac{1}{qr} + O(\log^{-3} x)\Big) \tag{5.3}$$

$$\Sigma_2^{(2)} = y\Big(\sum_{p,r} \frac{1}{pr} + O(\log^{-3} x)\Big) \tag{5.4}$$

provided that (according to Lemmas 3 and 4, respectively)

$$T^{18/5}X^{-1} \leqq QR \leqq XT^{-3/5}, \quad T^{13/5}X^{-1} \leqq R/Q \leqq XT^{-8/5} \tag{5.5}$$

and

(5.6)

$$T^{86/23}X^{-20/23} \leqq MN \leqq X, \quad MN^{-23/20} \leqq XT^{-43/20}, \quad NM^{-23/20} \leqq XT^{-43/20}$$

respectively, where $X = x^{1-\eta}$

The conditions

$$8RD \leqq Q^2 x, \quad 16Q^2R \leqq x, \quad x \leqq QRz, \quad 2Q \leqq R \tag{5.7}$$

and

$$64P^2R^3D \leqq x^3, \quad x \leqq P^2R, \quad 2P \leqq z, \quad x \leqq PR^2, \tag{5.8}$$

respectively, ensure that $(D/p)^{1/3} \leqq q < p < z$, $r \leqq q$ and so in these cases we obtain

$$\sum_{\substack{pqr\in\mathcal{A}\\ Q<q\leqq 2Q\\ R<r\leqq 2R}} 1 = y \sum_{\substack{Q<q\leqq 2Q\\ R<r\leqq 2R}} \frac{1}{qr\log\frac{x}{qr}} + O(y\log^{-4} x) \tag{5.9}$$

and

$$\sum_{\substack{pqr\in\mathcal{A}\\ P<p\leqq 2P\\ R<r\leqq 2R}} 1 = y \sum_{\substack{P<p\leqq 2P\\ R<r\leqq 2R}} \frac{1}{pr\log\frac{x}{pr}} + O(y\log^{-4} x). \tag{5.10}$$

Finally, the condition

$$P > xXT^{-18/5} \tag{5.11}$$

in the second case will ensure that disjoint sets of values of p, q and r are counted in the two cases. Summing over all admissible values we have (cf. [3])

$$\Sigma_2 > \frac{y}{\log x}\Big\{\iint_{\mathcal{M}_{21}\cup\mathcal{M}_{22}} \frac{d\alpha\, d\beta}{\alpha\beta(1-\alpha-\beta)} - c\eta\Big\} \tag{5.12}$$

where

$$\mathcal{M}_{21}=\Big\{(\alpha,\beta);\ \frac{18}{5}(1-\theta)-1\leqq\alpha+\beta\leqq 1-\frac{3}{5}(1-\theta),$$

$$\frac{13}{5}(1-\theta)-1\leqq\beta-\alpha\leqq 1-\frac{8}{5}(1-\theta),\ \beta-2\alpha\leqq 0.08-B,$$

$$\beta+2\alpha\leqq 1,\ 2-\frac{16}{5}(1-\theta)\leqq\alpha+\beta,\ \alpha\leqq\beta\Big\}= \tag{5.13}$$

$$=\Big\{(\alpha,\beta);\ \frac{18}{5}(1-\theta)-1\leqq\alpha+\beta\leqq 1-\frac{3}{5}(1-\theta),$$

$$\frac{13}{5}(1-\theta)-1\leqq\beta-\alpha\leqq 1-\frac{8}{5}(1-\theta),\ \beta-2\alpha\leqq 0{,}08-B,\ \beta+2\alpha\leqq 1\Big\}$$

and similarly to this

$$\mathcal{M}_{22}=\Big\{(\alpha,\beta);\ 2-\frac{18}{5}(1-\theta)\leqq\alpha\leqq\frac{16}{5}(1-\theta)-1,$$

$$\frac{86}{23}(1-\theta)-\frac{20}{23}\leqq\alpha+\beta,\ 2\alpha+3\beta\leqq 2{,}08-B\Big\}. \tag{5.14}$$

Using Lemmas 3 and 5, an analogous procedure leads to asymptotic formulas for

$$\sum_{\substack{Q<q\leqq 2Q\\ P<p\leqq 2P}}\left(\psi\left(\frac{x}{pq}\right)-\psi\left(\frac{x-y}{pq}\right)\right) \tag{5.15}$$

under two mutually exclusive sets of conditions on P, Q. This leads to

$$\sum\nolimits_5\geqq\frac{y}{\log x}\left(\iint_{\mathcal{M}_{51}\cup\mathcal{M}_{52}}\frac{d\alpha\,d\beta}{\alpha\beta(1-\alpha-\beta)}-c\eta\right) \tag{5.16}$$

where

$$\mathcal{M}_{51}=\Big\{(\alpha,\beta);\ 1-\frac{6}{5}(1-\theta)\leqq\alpha\leqq\frac{1}{2},$$

$$\alpha+\beta\leqq 1-\frac{3}{5}(1-\theta),\ \alpha-\beta\leqq 1-\frac{8}{5}(1-\theta)\Big\} \tag{5.17}$$

and

$$\mathcal{M}_{52}=\Big\{(\alpha,\beta);\ \alpha+\beta\leqq 1-\frac{4}{5}(1-\theta),\ \alpha-\beta\leqq 1-\frac{22}{15}(1-\theta),$$

$$\alpha\geqq 1-\frac{6}{5}(1-\theta),\ \alpha+3\beta\geqq 0.92-B\Big\}. \tag{5.18}$$

6. Denoting the corresponding integrals on $\mathcal{M}_{ij}$ in (5.12) and (5.16) by $c_{ij}(\theta)$, we get finally

$$\pi(x)-\pi(x-y)>\frac{y}{\log x}\big(c(\theta)-c\eta\big) \tag{6.1}$$

where

$$c(\theta) = c_1(\theta)+c_{21}(\theta)+c_{22}(\theta)-c_3(\theta)-c_4(\theta)+c_{51}(\theta)+c_{52}(\theta). \tag{6.2}$$

Since $c(\theta)$ is a continuous function of θ, our theorem will follow from

$$c\left(\frac{17}{31}\right) > \frac{1}{3000}. \tag{6.3}$$

This is really the case, since

$$\begin{aligned} c_1\left(\frac{17}{31}\right) &> 0.195839 \\ c_{21}\left(\frac{17}{31}\right) &> 0.14899 \\ c_{22}\left(\frac{17}{31}\right) &> 0.01699 \\ c_3\left(\frac{17}{31}\right) &< 0.052105 \\ c_4\left(\frac{17}{31}\right) &< 0.38602 \\ c_{51}\left(\frac{17}{31}\right) &> 0.06347 \\ c_{52}\left(\frac{17}{31}\right) &> 0.01317 \end{aligned} \tag{6.4}$$

and so

$$c\left(\frac{17}{31}\right) > 0.000334 > \frac{1}{3000}. \tag{6.5}$$

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ON A PROBLEM CONCERNING L^p MODULI OF SMOOTHNESS

V. TOTIK

The aim of this note is to prove the following

THEOREM. *Let (a, b) be a finite or infinite interval, $r \geqq 1$ an integer, $\alpha > 0$ and $1 \leqq p < \infty$, $f \in L^p(a, b)$. If*

$$\|\Delta_{t_n}^r f\|_{L^p(a,b-rt_n)} \leqq t_n^\alpha \tag{1}$$

for a sequence $t_n \to 0+0$ satisfying

$$\frac{t_n}{t_{n+1}} = O(1) \quad (n \to \infty) \tag{2}$$

then

$$\|\Delta_h^r f\|_{L^p(a,b-rh)} = O(h^\alpha) \quad (h \to 0+0). \tag{3}$$

Here

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r+i} \binom{r}{i} f(x+ih).$$

The analogous result in $C_{2\pi}$ was proved by DeVore [5] for $r=2$ and he raised the question if the same is true in L^p-norm. Freud [3] verified this in the case $p=2$ and Ditzian [2] for every $p \geqq 1$. Freud showed also that (2) is necessary for the implication (1)$\Rightarrow$(3) when $\alpha < r$ and the problem of the sufficiency of (2) for every $r \geqq 1$ was posed by Ditzian [2]. Boman [1] solved this problem in a very general setting. Since Boman's approach is rather abstract and heavily uses the translation invariance of $L^p(-\infty, \infty)$, it seems to be worthwhile to give a direct proof which applies also to finite interval.

PROOF. Let

$$\frac{t_n}{t_{n+1}} \leqq C, \qquad C \geqq 1. \tag{4}$$

First let us suppose that beside (1) f has an absolutely continuous $(r-1)$-th derivative and a.e. an r-th derivative belonging to $L^p(a, b)$. We give an estimate on $\|\Delta_h^r f\|_{L^p(a,b-rh)}$ in which the bound is independent of the posed smoothness assumption.

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By the formula (see [4, p. 107])

$$\Delta_{ih}^r f(x) = \sum_{v_r=0}^{i-1} \dots \sum_{v_1=0}^{i-1} \Delta_h^r f(x+v_1 h+\dots+v_r h)$$

it follows for every $i \geqq 1$ the estimate

$$\|\Delta_{ih}^r f\|_{L^p(a,b-irh)} \leqq i^r \|\Delta_h^r f\|_{L^p(a,b-rh)}. \tag{5}$$

Let

$$f_\delta(x) = \frac{1}{\delta^r} \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \int_0^\delta \dots \int f(x+i(u_1+\dots+u_r))\, du_1 \dots du_r \tag{6}$$

where $\int \dots \int$ denotes r-fold integration and let

$$\omega(f, \delta) = \sup_{0<h\leqq\delta} \|\Delta_h^r f\|_{L^p(a,b-h-rh)}.$$

Since the norm of an integral is not greater than the integral of the corresponding norms we get by the formulae

$$(f-f_\delta)(x) = \frac{(-1)^r}{\delta^r} \int_0^\delta \dots \int \Delta_{u_1+\dots+u_r}^r f(x)\, du_1 \dots du_r,$$

$$f_\delta^{(r)}(x) = \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \frac{1}{i^r} \frac{1}{\delta^r} \Delta_{i\delta}^r f(x)$$

and

$$\Delta_h^r f_\delta(x) = \int_0^\delta \dots \int f_\delta^{(r)}(x+u_1+\dots+u_r)\, du_1 \dots du_r$$

the estimates for $\delta < h/(r+1)r$

$$\|f-f_\delta\|_{L^p(a,b-h)} \leqq \|f-f_\delta\|_{L^p(a,b-r\delta-r^2\delta)} \leqq$$

$$\leqq \frac{1}{\delta^r} \int_0^\delta \dots \int \|\Delta_{u_1+\dots+u_r}^r f\|_{L^p(a,b-r\delta-r^2\delta)}\, du_1 \dots du_r \leqq \omega(f, r, \delta),$$

$$\|f_\delta^{(r)}\|_{L^p(a,b-r^2\delta)} \leqq 2^r \delta^{-r} \sum_{i=1}^{r} \|\Delta_{i\delta}^r f\|_{L^p(a,b-ir\delta)} \leqq \tag{7}$$

$$\leqq 2^r r^{r+1} \delta^{-r} \|\Delta_\delta^r f\|_{L^p(a,b-r\delta)}$$

and

$$\|\Delta_h^r f_\delta\|_{L^p(a,b-rh-r^2\delta)} \leqq \int_0^h \dots \int \|f_\delta^{(r)}\|_{L^p(a,b-r^2\delta)}\, du_1 \dots du_r \leqq$$

$$\leqq h^r \|f_\delta^{(r)}\|_{L^p(a,b-r^2\delta)} \leqq 2^r r^{r+1} \left(\frac{h}{\delta}\right)^r \|\Delta_\delta^r f\|_{L^p(a,b-r\delta)} \tag{8}$$

where we used also (5).

Now (7) and (8) yield for $\delta < h/r(r+1)$

$$\|\Delta_h^r f\|_{L^p(a,b-(r+1)h} \leqq \|\Delta_h^r(f-f_\delta)\|_{L^p(a,b-rh-h)} + \|\Delta_h^r f_\delta\|_{L^p(a,b-rh-r^2\delta)} \leqq$$

$$\leqq 2^r \|f-f_\delta\|_{L^p(a,b-h)} + 2^r r^{r+1}\left(\frac{h}{\delta}\right)^r \|\Delta_\delta^r f\|_{L^p(a,b-r\delta)} \leqq$$

$$\leqq 2^r \omega(f, r\delta) + 2^r r^{r+1}\left(\frac{h}{\delta}\right)^r \|\Delta_\delta^r f\|_{L^p(a,b-r\delta)}.$$

Putting here $\delta = t_n$ such that

$$\frac{h}{CA} \leqq t_n \leqq \frac{h}{A}, \qquad A = 2^{1+(r+1)/\alpha}(r+1)r$$

(such a t_n is guaranteed by (4) for every $h \leqq t_1$), we obtain the estimate

$$\|\Delta_h^r f\|_{L^p(a,b-(r+1)h)} \leqq 2^r \omega\left(f, \frac{h}{r2^{1+(r+1)/\alpha}}\right) + (2rCA)^{r+1} h^\alpha$$

and together with this also

$$\omega(f, h) \leqq 2^r \omega\left(f; \frac{h}{r2^{1+(r+1)/\alpha}}\right) + (2rCA)^{r+1} h^\alpha.$$

Iterating this k times it follows

$$\omega(f; h) \leqq 2^{rk} \omega\left(f; \frac{h}{(r2^{1+(r+1)/\alpha})^k}\right) +$$

$$+(2rCA)^{r+1} h^\alpha \bigl(1 + 2^r (r2^{1+(r+1)/\alpha})^{-\alpha} + 2^{2r}(r2^{1+(r+1)/\alpha})^{-2\alpha} + \tag{9}$$

$$+ \dots + 2^{(k-1)r}(r2^{1+(r+1)/\alpha})^{-(k-1)\alpha}\bigr) \leqq$$

$$\leqq 2^{rk} \omega\left(f; \frac{h}{(r2^{1+(r+1)/\alpha})^k}\right) + 2(2rCA)^{r+1} h^\alpha$$

because

$$2/(r2^{1+(r+1)/\alpha})^\alpha \leqq \frac{1}{2}.$$

Now we apply our assumption $f^{(r)} \in L^p(a, b)$ to derive

$$\omega(f; \delta) \leqq \sup_{\tau \leqq \delta} \left\| \int_0^\tau \dots \int f^{(r)}(. + u_1 + \dots + u_r)\, du_1 \dots du_r \right\|_{L^p(a,b-(r+1)\tau)} \leqq$$

$$\leqq \delta^r \|f^{(r)}\|_{L^p(a,b)}$$

by which

$$2^{rk} \omega\left(f; \frac{h}{(r2^{1+(r+1)/\alpha})^k}\right) \leqq 2^{rk}\left(\frac{h}{(r2^{1+(r+1)/\alpha})^k}\right)^r \|f^{(r)}\|_{L^p(a,b)} = o(1)$$

as $k \to \infty$, and hence (see (9))

$$\omega(f;\ h) \leqq 2(2rCA)^{r+1} h^{\alpha}$$

for every $h \leqq t_1$.

Since our starting condition (1) is symmetrical one can prove in exactly the same way the estimate

$$\|\Delta_h^r f\|_{L^p(a+h,\, b-rh)} \leqq 2(2rCA)^{r+1} h^{\alpha} \quad (h \leqq t_1)$$

and so we obtain

$$\|\Delta_h^r f\|_{L^p(a,\, b-rh)} \leqq Mh^{\alpha} \quad (h \leqq t_1)$$

with $M = 4(2rCA)^{r+1}$, i.e. the Theorem is proved under the assumption $f^{(r)} \in L^p(a, b)$.

In the general case all what we have to do is to apply a smoothing process to f. Let $\varepsilon > 0$ be fixed and for $\delta < \varepsilon/r$ let us consider the function

$$f_\delta^*(x) = \frac{1}{\delta^r} \int_0^\delta \cdots \int f(x+u_1+\ldots+u_r)\, du_1 \ldots du_r.$$

Clearly,

$$\Delta_h^r f_\delta^*(x) = \sum_{i=0}^{r} (-1)^{i+r} \binom{r}{i} \frac{1}{\delta^r} \int_0^\delta \cdots \int f(x+ih+u_1+\ldots+u_r)\, du_1 \ldots du_r = (\Delta_h^r f)_\delta^*(x)$$

and so

$$\|\Delta_{t_n}^r f_\delta^*\|_{L^p(a,\, b-\varepsilon-rt_n)} = \left\| \frac{1}{\delta^r} \int_0^\delta \cdots \int \Delta_{t_n}^r f(\cdot + u_1+\ldots+u_r)\, du_1 \ldots du_r \right\|_{L^p(a,\, b-\varepsilon-rt_n)} \leqq$$

$$\leqq \frac{1}{\delta^r} \int_0^\delta \cdots \int \|\Delta_{t_n}^r f\|_{L^p(a,\, b-rt_n)}\, du_1 \ldots du_r \leqq t_n^{\alpha}.$$

However, f_δ^* has already an absolutely continuous $(r-1)$-th derivative with $(f_\delta^*)^{(r)} \in L^p(a, b-\varepsilon)$, hence, according to what we have proved above

$$\|\Delta_h^r f_\delta^*\|_{L^p(a,\, b-\varepsilon-rh)} \leqq Mh^{\alpha} \quad (h \leqq t_1).$$

Letting here $\delta \to 0$ and observing that

$$\|f - f_\delta^*\|_{L^p(a,\, b-\varepsilon)} = o(1) \quad \text{as} \quad \delta \to 0$$

we get

$$\|\Delta_h^r f\|_{L^p(a,\, b-\varepsilon-rh)} \leqq Mh^{\alpha}$$

for $h \leqq t_1$, and since here $\varepsilon > 0$ is arbitrary, the Theorem follows.

Remark. Our proof works also for other norms (e.g. for the supremum norm) instead of the L^p-one.

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COMPACT PACKING OF CIRCLES

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Let the circle c be touched by the circles $c_1, \ldots, c_n$ so that no two circles overlap and c_i touches c_{i+1} ($i=1, \ldots, n$; $c_{n+1}=c_1$). We say that c is touched by a closed chain of circles. If in a packing of circles each circle is touched by a closed chain of circles then we call the packing *compact*.

We shall denote a domain and its area with the same symbol. We define the homogeneity of a set of circles $\{c_i\}$ by $\inf c_i/\sup c_i$.

We shall prove the following

THEOREM 1. *The lower density of a compact packing of circles of positive homogeneity is at least* $\pi/\sqrt{12}$.

PROOF. By the assumption that the circles have positive homogeneity, the circles can nowhere accumulate. It follows that the part of the plane not covered by the circles consists of triangular gaps bounded by three circles mutually touching one another. To each gap we construct the triangle spanned by the centres of the respective circles. It is easy to show that these triangles fill the plane without overlapping and without interstices.

Let Δ be a triangle considered above. Let c_1, c_2, c_3 be the respective circles. We shall show that

$$\frac{c_1\cap\Delta+c_2\cap\Delta+c_3\cap\Delta}{\Delta} \geqq \frac{\pi}{\sqrt{12}}. \tag{1}$$

Let O be the centre of the incircle of Δ (Fig. 1). Let T_1, T_2, T_3 be the points of tangency of c_2 and c_3, c_3 and c_1, and c_1 and c_2. We write $2\alpha_1=\sphericalangle T_2OT_3$, $2\alpha_2=\sphericalangle T_3OT_1$, $2\alpha_3=\sphericalangle T_1OT_2$. Without loss of generality, we may suppose that the inradius of Δ has unit length. Then we have

$$\Delta = \sum_{i=1}^{3} \tan\alpha_i, \quad c_i\cap\Delta = \left(\frac{\pi}{2}-\alpha_i\right)\tan^2\alpha_i, \quad i = 1, 2, 3.$$

Thus, with the notation

$$y(\alpha) = \left(\frac{\pi}{2}-\alpha\right)\tan^2\alpha-\frac{\pi}{\sqrt{12}}\tan\alpha,$$

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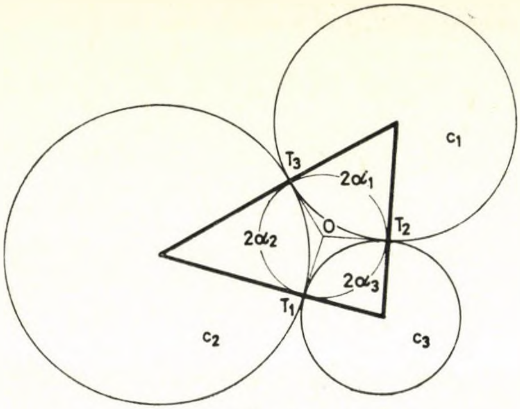

Fig. 1

we can write the inequality (1) in the form

$$y(\alpha_1)+y(\alpha_2)+y(\alpha_3) \geqq 0. \tag{2}$$

We claim that for $0<\alpha<\pi/2$ the function $y(\alpha)$ is convex. We have

$$y'(\alpha) = \frac{-\sin^2\alpha+(\pi-2\alpha)\tan\alpha-\dfrac{\pi}{\sqrt{12}}}{\cos^2\alpha},$$

$$y''(\alpha)\cos^4\alpha = \{-\sin 2\alpha-2\tan\alpha+(\pi-2\alpha)\cos^{-2}\alpha\}\cos^2\alpha+$$

$$+\sin 2\alpha\left\{-\sin^2\alpha+(\pi-2\alpha)\tan\alpha-\frac{\pi}{\sqrt{12}}\right\} =$$

$$= (\pi-2\alpha)(1+2\sin^2\alpha)-a\sin 2\alpha,$$

where $a=2+\pi/\sqrt{12}$. Writing $\alpha=(\pi-x)/2$, we obtain

$$y''\left(\frac{\pi-x}{2}\right)\sin^4\frac{x}{2} = x\left(1+2\cos^2\frac{x}{2}\right)-a\sin x = x(2+\cos x)-a\sin x \geqq$$

$$\geqq x(2+\cos x)-3\sin x = 3(2+\cos x)f(x)$$

where

$$f(x) = \frac{x}{3}-\frac{\sin x}{2+\cos x}.$$

Since $f(0)=0$ and

$$f'(x) = \frac{(\cos x-1)^2}{3(2+\cos x)^2} > 0, \qquad 0<x\leqq\pi$$

we have $f(x)\geqq 0$, $0\leqq x\leqq\pi$, which implies the convexity of $y(\alpha)$.

Using Jensen's inequality, we obtain the desired inequality

$$y(\alpha_1)+y(\alpha_2)+y(\alpha_3) \geqq 3y(\pi/3) = 0.$$

Let r be the supremum of the radii of the circles. Let $\sum_R$ denote a summation which extends to all circles of the packing contained in a circle $C(R)$ of radius R centred at a fixed point of the plane. The lower density d of the packing is defined by

$$d = \varliminf_{R\to\infty} \frac{1}{\pi R^2} \sum_R c_i.$$

We write (1) in the form

$$c_1 \cap \Delta + c_2 \cap \Delta + c_3 \cap \Delta \geqq \frac{\pi}{\sqrt{12}} \Delta$$

and sum up the corresponding inequalities for all triangles contained in $C(R-r)$. Obviously, the sum at the left-hand side is at most $\sum_R c_i$, and the sum at the right-hand side is at least equal to the area of the circle $C(R-3r)$. Therefore

$$\sum_R c_i \geqq \frac{\pi}{\sqrt{12}} \pi (R-3r)^2,$$

which, in accordance with the theorem, implies that $d \geqq \pi/\sqrt{12}$.

Besides the hexagonal packing of equal circles, the density $\pi/\sqrt{12}$ can be attained by a great variety of packing with incongruent circles (Fig. 2).

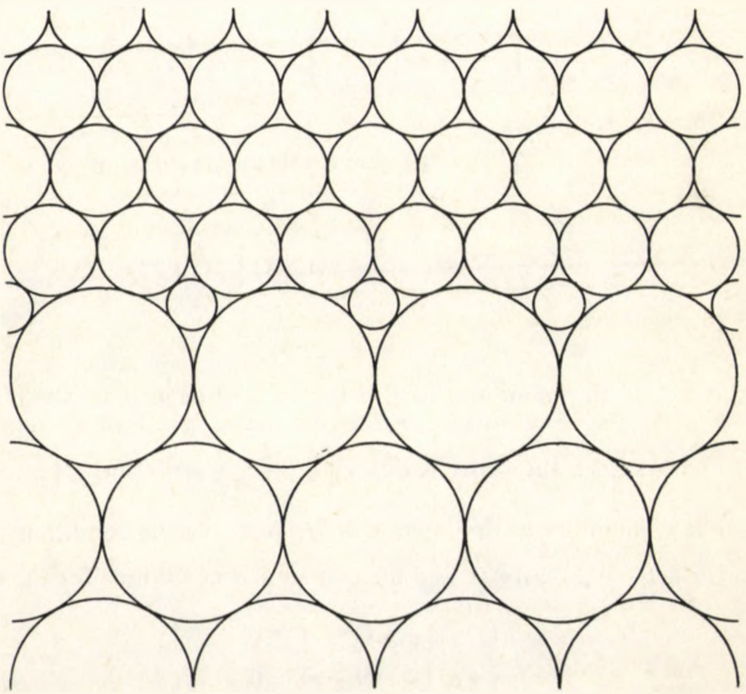

Fig. 2

In Poincaré's circle-model of the hyperbolic plane, consider the incircles of the faces of a regular trihedral tiling. In the Euclidean plane these circles constitute a compact packing of density 0. This example shows that dropping the condition made on the homogeneity we can construct a compact packing of circles with any density between 0 and 1.

Defining the compactness of a packing of convex discs and the homogeneity of a set of convex discs similarly as for circles, we have the following

THEOREM 2. *The lower density of a compact packing of homothetic centro-symmetric convex discs of positive homogeneity is at least* 3/4. *Equality is claimed only for affinely regular hexagons.*

PROOF. Let c_1, c_2, c_3 be homothetic centro-symmetric convex discs mutually touching one another. Let Δ be the triangle determined by the centres O_1, O_2, O_3 of c_1, c_2 and c_3. By the considerations used in the proof of Theorem 1, it suffices to show that

$$\frac{c_1\cap\Delta+c_2\cap\Delta+c_3\cap\Delta}{\Delta}\geqq\frac{3}{4}. \tag{3}$$

Since the quotients $c_i\cap\Delta/\Delta$ are invariant under affine transformations, we may suppose that Δ is an equilateral triangle of unit side-length. Let the boundaries of c_1 and c_2, c_2 and c_3, c_3 and c_1 intersect the boundary of Δ at T_3, T_1 and T_2. With the notations $O_1T_3=x$, $O_2T_1=y$, $O_3T_2=z$ we have

$$\frac{c_1}{c_2}=\left(\frac{x}{1-x}\right)^2,\quad \frac{c_2}{c_3}=\left(\frac{y}{1-y}\right)^2,\quad \frac{c_3}{c_1}=\left(\frac{z}{1-z}\right)^2.$$

Multiplying these equalities, we see that

$$g(x, y, z)=xyz-(1-x)(1-y)(1-z)=0. \tag{4}$$

Obviously

$$\frac{c_1\cap\Delta+c_2\cap\Delta+c_3\cap\Delta}{\Delta}\geqq x(1-z)+y(1-x)+z(1-y)=f(x, y, z).$$

Because of (4), we have

$$f(x, y, z)=1-2xyz.$$

We want to find the minimum of f in the cube Q defined by the inequalities $0\leqq x, y, z\leqq 1$ under the condition (4). At any boundary point of Q satisfying (4) f is equal to 1. Since, on the other hand, $g\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)=0$ and $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)=$ $=\frac{3}{4}<1$, there is a minimum in the interior of Q. We write the condition $f_x+\lambda g_x=$ $=-2yz+\lambda\{yz+(1-y)(1-z)\}=0$ and the two similar conditions for the minimum in the form

$$yz+\mu(1-y)(1-z)=0,$$

$$zx+\mu(1-z)(1-x)=0,$$

$$xy+\mu(1-x)(1-y)=0.$$

Multiplying these equalities, we obtain, by (4), that $\mu=-1$. Thus we have $yz=$ $=(1-y)(1-z)$, $xyz=x(1-y)(1-z)=(1-x)(1-y)(1-z)$, whence $x=1/2$. Similarly, we obtain that $y=1/2$ and $z=1/2$. This concludes the proof of (3).

Equality holds in (3) only if T_1, T_2 and T_3 are the midpoints of the sides of Δ, and the intersections $c_1\cap\Delta$, $c_2\cap\Delta$ and $c_3\cap\Delta$ are identical with the triangles $O_1T_2T_3$, $O_2T_3T_1$ and $O_3T_1T_2$. This implies that c_1, c_2 and c_3 are congruent affinely regular hexagons.

If in Theorem 2 we drop the condition of the central symmetry of the discs then the minimal density is conjectured to be 1/2. It is attained by a packing of triangles such that, roughly speaking, at almost each vertex three triangles meet.

E. Makai jr. asked the following question: Is it true that the density of a compact packing of homothetic convex discs with positive homogeneity cannot be less than the density of the thinnest six-neighbour lattice-packing of translates of one of the discs? An affirmative answer would imply besides Theorem 1 and 2 also the correctness of the above conjecture.

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DIE DÜNNSTE 2-FACHE DOPPELGITTERFÖRMIGE KREISÜBERDECKUNG DER EUKLIDISCHEN EBENE

ÁGOTA H. TEMESVÁRI

H sei ein Punktsystem in der euklidischen Ebene. Mit $\mathcal{L}_H$ bezeichnen wir die Anordnung von geschlossenen Einheitskreisen, deren Mittelpunkte mit dem Punktsystem H übereinstimmen. $\mathcal{L}_H$ wird eine k-fache Überdeckung genannt, wenn jeder Punkt der Ebene zu mindestens k der Kreisen gehört. Die Dichte von $\mathcal{L}_H$ bezeichnen wir mit $\Delta(\mathcal{L}_H)$ [3].

Im folgenden bezeichnen wir den Punkt X und den Ortsvektor $\overrightarrow{OX}$ mit demselben Buchstabe.

Mit Γ bezeichnen wir immer ein Punktgitter, d. h., die Punktmenge, deren Punkte durch die ganzzahligen linearen Kombinationen von den fixen, linear unabhängigen Ortsvektoren A und B bestimmt sind. A und B sind die Basisvektoren von Γ genannt.

$\mathcal{T}_X$ sei die Punktmenge, die aus der Punktmenge $\mathcal{T}$ durch die Verschiebung X stammt.

Mit Σ bezeichnen wir immer ein Doppelgitter, d. h., eine Punktmenge, für die es ein Punktgitter Γ und einen Ortsvektor X gibt, so daß $\Sigma=\Gamma\cup\Gamma_X$ ist. Es seien $D_2=\inf_H \Delta(\mathcal{L}_H)$, $D_2'=\inf_\Gamma \Delta(\mathcal{L}_\Gamma)$, $D_2''=\inf_\Sigma \Delta(\mathcal{L}_\Sigma)$, wo die Infima über sämtliche Punktmengen H, sämtliche Gitter Γ und sämtliche Doppelgitter Σ zu erstrecken sind, für die $\mathcal{L}_H$, $\mathcal{L}_\Gamma$, bzw. $\mathcal{L}_\Sigma$ eine zweifache Überdeckung ist.

Blundon [1] hat die Gleichheit $D_2'=2D_1$ bewiesen, wo D_1 die Dichte der dünnsten einfachen Kreisüberdeckung ist $\left(D_1=\frac{2\pi}{3\sqrt{3}}\right)$. Danzer [2] konstruierte eine 2-fache nicht gitterförmige Kreisüberdeckung, deren Dichte kleiner als D_2' ist.

Im folgenden beschäftigen wir uns mit den 2-fachen doppelgitterförmigen Überdeckungen von Einheitskreisen.

Im weiteren nehmen wir an, daß Γ eine normale Darstellung hat, d. h., die Ungleichungen

$$(1)\qquad |A|\leqq|B|\leqq|B-A| \quad\text{und}\quad \sphericalangle(AOB)\leqq\frac{\pi}{2}$$

für die Basisvektoren A und B von Γ gelten. Weil $\Sigma=\Gamma\cup\Gamma_X$ ein Doppelgitter ist, liegt wenigstens ein Gitterpunkt D von Γ_X im Parallelogramm $OA(A+B)B$. Wir können annehmen, daß D ein Punkt des geschlossenen Dreiecks OAB ist. Wir

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betrachten den Vektor D. Es ist klar, daß wir das Doppelgitter auch folgenderweise darstellen können:

(2) $$\Sigma = \Gamma \cup \Gamma_D, \quad D = kA + mB,$$

wo $0 \leqq k+m \leqq 1$, $k, m \geqq 0$ ist. Im folgenden nehmen wir an, daß (1) und (2) für die Angabe von Σ gelten (Abb. 1).

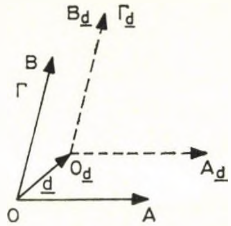

Abb. 1

Wir nennen Σ ein Doppelgitter vom Typ Σ_s, wenn Σ eine solche Angabe $\Gamma \cup \Gamma_X$ hat, bei der Γ das reguläre Dreiecksgitter ist und die Länge der Basisvektoren von Γ $\sqrt{3}$ ist. Eine Anordnung von Kreisen ist vom Typ $\mathcal{L}_{\Sigma_s}$, wenn das System der Kreismittelpunkte ein Doppelgitter vom Typ Σ_s bildet.

$\mathcal{L}_\Sigma$ sei eine 2-fache doppelgitterförmige Überdeckung von Einheitskreisen, wo die Bedingungen (1) und (2) für Σ gelten. So können wir die Überdeckung $\mathcal{L}_\Sigma$ in zwei Anordnungen $\mathcal{L}_\Gamma$ und $\mathcal{L}_{\Gamma_D}$ zerlegen. T sei der Inhalt des Grundparallelogramms von Γ. So ist die Dichte von $\mathcal{L}_\Sigma$ gleich $2\pi/T$. Auf diese Dichte bezieht sich der folgende

SATZ. *Die Dichte einer 2-fachen doppelgitterförmigen Überdeckung von Einheitskreisen ist* $\geqq 2D_1 \left(= \dfrac{4\pi}{3\sqrt{3}}\right)$. *Gleichheit tritt nur bei den Überdeckungen vom Typ* $\mathcal{L}_{\Sigma_s}$ *auf.*

Vor dem Beweis des Satzes sehen wir einige Hilfssätze ein. Wir führen die Bezeichnungen $|A|=a$, $|B|=b$, $|B-A|=c$ und $\sphericalangle(AOB)=\alpha$ (Abb. 1) ein.

HILFSSATZ 1. *Wir betrachten eine 2-fache doppelgitterförmige Überdeckung* $\mathcal{L}_\Sigma = \mathcal{L}_\Gamma \cup \mathcal{L}_{\Gamma_D}$. *Wenn die Anordnung* $\mathcal{L}_\Gamma$ *eine Überdeckung bildet, dann ist die Dichte von* $\mathcal{L}_\Sigma \geqq 2D_1$ *und Gleichheit tritt nur bei den Überdeckungen vom Typ* $\mathcal{L}_{\Sigma_s}$ *auf.*

Der Beweis ist klar, weil die Dichte von $\mathcal{L}_\Gamma \geqq D_1$ ist und Gleichheit nur bei dem regulären Dreiecksgitter Γ_s auftritt [3], [4].

HILFSSATZ 2. *Für die 2-fache doppelgitterförmige Überdeckung von Einheitskreisen* $\mathcal{L}_\Sigma$, *wo* $\Delta(\mathcal{L}_\Sigma) \leqq \dfrac{4\pi}{3\sqrt{3}}$ *ist, gelten notwendigerweise die folgenden:*

1. $b < 4$

2. $a \geqq \dfrac{3\sqrt{3}}{8}$

3. $a < 1$ für $b > 2$.

Beweis. 1. Wenn $b \geqq 4$ ist, dann hat der Basisvektor $\overrightarrow{OB}$ von Γ eine Strecke der Länge 2, die von den Kreisen der Anordnung $\mathscr{L}_\Gamma$ nicht überdeckt ist. Weil die Überdeckung $\mathscr{L}_\Sigma$ 2-fach und doppelgitterförmig ist, muß man die von $\mathscr{L}_\Gamma$ nicht überdeckten Ebenenteile mit der Verschiebung der von $\mathscr{L}_\Gamma$ mindestens 2-fach überdeckten Ebenenteile überdecken. Weil die Länge der gemeinsamen Sehne von zwei Kreisen in $\mathscr{L}_\Gamma$ kleiner als 2 ist, so ist die Überdeckung $\mathscr{L}_\Sigma$ im Fall $b \geqq 4$ nicht 2-fach.

2. Bei den Einheitskreisüberdeckungen vom Typ $\mathscr{L}_{\Sigma_s}$ ist $T=\frac{3\sqrt{3}}{2}$, deshalb müssen wir die Überdeckungen, für die $T<\frac{3\sqrt{3}}{2}$ ist, nicht untersuchen. In diesen Fällen ist nämlich die Dichte größer als $\frac{4\pi}{3\sqrt{3}}$. Wegen $b<4$ ist $T=ab\sin\alpha \leqq 4a$. Wenn $a<\frac{3\sqrt{3}}{8}$ ist, gilt $T \leqq 4a<\frac{3\sqrt{3}}{2}$. So gilt $a \geqq \frac{3\sqrt{3}}{8}$.

3. Im Fall $b>2$ gilt auch $c>2$ wegen (1). k und m seien ganze Zahlen. Wenn $a \geqq 1$ ist, dann berühren sich die Schnitte der um die Gitterpunkte kA und $(k+1)A$ geschlagenen Einheitskreise im Fall $a=1$ oder haben sie keinen gemeinsamen Punkt (Abb. 2). Aus $b>2$ und $c>2$ folgt, daß die Kreise mit den Mittelpunkten $mB+kA$ $(m \neq 0)$ und kA keinen gemeinsamen Punkt haben.

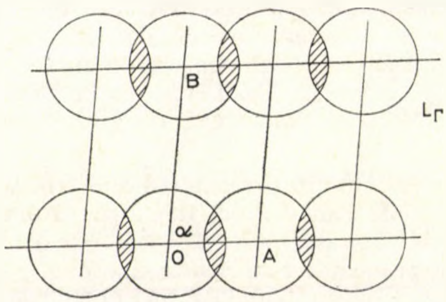

Abb. 2

Wir betrachten den von den Geraden OA und $B(A+B)$ begrenzten Streifen. Der von $\mathscr{L}_\Gamma$ nicht überdeckte Teil in diesem Streifen ist ein einfach zusammenhängendes Bereich, dessen Breite >0 ist. So können wir dieses Bereich durch Verschiebung der von $\mathscr{L}_\Gamma$ mindestens 2-fach überdeckten Ebenenteile nicht überdecken, d. h., die Überdeckung kann nicht 2-fach sein. So gilt $a<1$.

Hilfssatz 3. *$\mathscr{L}_\Sigma$ sei eine 2-fache doppelgitterförmige Überdeckung von Einheitskreisen. Wenn $\alpha=\pi/2$ ist, dann ist $\Delta(\mathscr{L}_\Sigma) \geqq 2D_1$ und Gleichheit tritt nur bei den Überdeckungen vom Typ $\mathscr{L}_{\Sigma_s}$ auf.*

Beweis. Es sei $\frac{3\sqrt{3}}{8} \leqq a<1$. Wenn $b \leqq 2$ ist, dann ist $T \leqq 2<\frac{3\sqrt{3}}{2}$, d. h., die Dichte der Überdeckung ist größer als $2D_1$. So müssen wir nur die Fälle $b>2$ untersuchen. Weil $a<1$ ist, schneiden sich die um die Punkte O und $2A$ geschlagenen Einheitskreislinien. Mit M und N bezeichnen wir die Schnittpunkte und m sei gleich $b-2$ (Abb. 3).

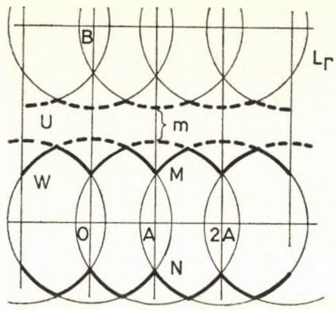

Abb. 3

Weil die Überdeckung 2-fach ist und Γ_D durch Verschiebung von Γ entsteht, gilt $MN \geqq m$ notwendigerweise. Wir betrachten nämlich die Kreise mit den Mittelpunkten kA (k ist eine ganze Zahl) in der Anordnung $\mathcal{L}_\Gamma$. $\mathcal{W}$ sei der von diesen Kreisen mindestens 2-fach überdeckte Ebenenteil. Wir nehmen die zu MN parallelen Strecken, die die Grenzpunkte von $\mathcal{W}$ verbinden. Unter diesen Strecken sind $(MN)_{kA}$ die kürzesten. Wir betrachten den von den Geraden OA und $B(A+B)$ begrenzten Stıeifen. $\mathcal{U}$ sei das Bereich in diesem Streifen, das von den Kreisen der Anordnung $\mathcal{L}_\Gamma$ nicht überdeckt ist. Auch in $\mathcal{U}$ nehmen wir die zu MN parallelen Strecken, die die Grenzpunkte von $\mathcal{U}$ verbinden. Die Länge der kürzesten unter diesen Strecken ist m. So im Fall $m > MN$ können wir mit $\mathcal{W}_D$ das ganze Bereich $\mathcal{U}$ nicht überdecken. So gilt $MN \geqq m$.

Die Basisvektoren des Gitters $\bar{\Gamma}$ seien $\overrightarrow{OA}$ und $\overrightarrow{O\bar{B}}$, wo $\overrightarrow{O\bar{B}} \perp \overrightarrow{OA}$ und $|\bar{B}| = MN+2$ gelten. Es ist offenbar, daß der Inhalt des Grundparallelogramms von $\bar{\Gamma}$ größer als T ist, wenn $MN > m$ ist. Es sei $D = \bar{B}/2$. Wir betrachten das Doppelgitter $\bar{\Sigma} = \bar{\Gamma} \cup \bar{\Gamma}_D$ und die entsprechende Einheitskreisanordnung $\mathcal{L}_{\bar{\Sigma}}$. Es ist leicht einzusehen, daß $\mathcal{L}_{\bar{\Sigma}}$ eine 2-fache Überdeckung ist. Es ist offenbar, daß wir im Fall $D \neq \bar{B}/2$ keine 2-fache Überdeckung bekommen und die Dichte von $\mathcal{L}_{\bar{\Sigma}}$ kleiner als die Dichte von $\mathcal{L}_\Sigma$ $(MN > m)$ ist. Mit einer einfachen Rechnung ergibt sich, daß

$$\bar{T}(a) = a(2+2\sqrt{1-a^2}), \qquad \frac{3\sqrt{3}}{8} \leqq a < 1$$

im Fall $\mathcal{L}_{\bar{\Sigma}}$ ist. Mit Hilfe der ersten Ableitung können wir uns davon überzeugen, daß der Inhalt $\bar{T}(a)$ im Fall $a = \frac{\sqrt{3}}{2}$ maximal ist. Im Fall $a = \frac{\sqrt{3}}{2}$ ist $b=3$, deshalb ist $\frac{3\sqrt{3}}{2}$ der maximale Inhalt, d. h., die minimale Dichte ist eben $2D_1$. Es ist offenbar, daß diese minimale Dichte nur bei einem einzigen Gitter auftritt. Weil $\left|A+\frac{B}{2}\right| = \sqrt{3}$ bei diesem Gitter ist, ist das Dreieck $O\left(A+\frac{B}{2}\right)(2A)$ regulär (Abb. 4). Das bedeutet,

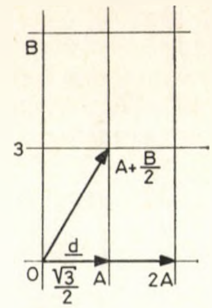

Abb. 4

daß das Doppelgitter mit vorigen extremalen Inhalt eigentlich vom Typ Σ_s ist. Das kann man leicht sehen, wenn wir das Doppelgitter auf eine andere Weise angeben. Es sei nämlich Γ_s mit den Basisvektoren $2A$ und $A+(B/2)$ gegeben und D sei der Vektor $\overrightarrow{OA}$. Das Doppelgitter $\Gamma_s \cup (\Gamma_s)_D$ ist eben unser extremales Doppelgitter.

2. Es sei $a \geqq 1$. Wegen des Hilfssatzes 2 ist $b \leqq 2$. Auf der Figur 5 kann man das Grundparallelogramm $OA(A+B)B$ und die um diese Gitterpunkte geschlagenen Einheitskreisen von $\mathscr{L}_\Gamma$ sehen ($\mathscr{L}_\Sigma = \mathscr{L}_\Gamma \cup \mathscr{L}_{\Gamma_D}$). Wenn $c \leqq 2$ ist, dann ist $\mathscr{L}_\Gamma$ eine einfache Überdeckung; mit diesem Fall beschäftigen wir uns wegen des Hilfssatzes 1 nicht. Es sei also $c > 2$. In diesem Fall gibt es ein Bereich $\mathscr{U}$ in $OA(A+B)B$, das von $\mathscr{L}_\Gamma$ nicht überdeckt ist. Der Schnitt der um O und A bzw. O und B geschlagenen Einheitskreise sei $\mathscr{W}_1$ bzw. $\mathscr{W}_2$. Offensichtlich gilt $\mathscr{W}_1 \cap \mathscr{W}_2 = \emptyset$. Hieraus folgt, daß das Bereich $\mathscr{U}$ durch ein verschobenes Exemplar von $\mathscr{W}_1$ oder $\mathscr{W}_2$ überdeckt ist. Wir beginnen mit dem ersten Fall. Weil die Überdeckung 2-fach ist, gilt $a - 1 \leqq \sqrt{1 - \left(\frac{b}{2}\right)^2}$. Jetzt geben wir eine 2-fache doppelgitterförmige Überdeckung $\mathscr{L}_{\Sigma'}$, deren Dichte nicht größer als die Dichte von $\mathscr{L}_\Sigma$ ist. Es seien $\overrightarrow{OA}$ und $\overrightarrow{OB}$ die Basisvektoren von Γ,

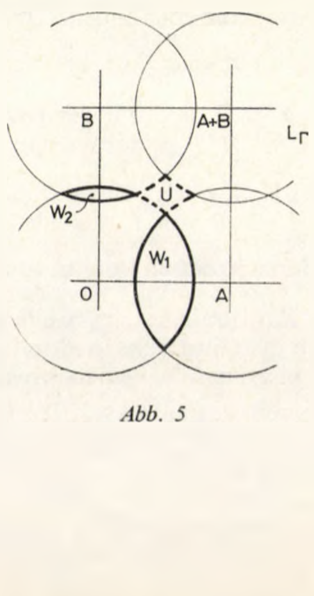

Abb. 5

wobei $\overrightarrow{OA} \perp \overrightarrow{OB}$ und $|\bar{A}| = \sqrt{1-\left(\frac{b}{2}\right)^2}+1$. Es sei $D=B/2$. Wir betrachten das Doppelgitter $\bar{\Sigma}=\bar{\Gamma}\cup\bar{\Gamma}_D$. Es ist leicht einzusehen, daß die Anordnung $\mathscr{L}_{\bar{\Sigma}}$ eine 2-fache Überdeckung ist. Man kann auch leicht sehen, daß wir im Fall $D\neq B/2$ keine 2-fache Überdeckung bekommen. Der Inhalt des Grundparallelogramms von $\bar{\Gamma}$ ist

$$\bar{T}(a) = \sqrt{-4a^4+8a^3}, \qquad 1 \leqq a \leqq 2,$$

woraus sich ergibt, daß wir den maximalen Inhalt bei $a=3/2$ bekommen. In diesem Fall ist $b=\sqrt{3}$ und der maximale Inhalt ist $\frac{3\sqrt{3}}{2}$. Das entsprechende Doppelgitter ist vom Typ Σ_s, so ist die Überdeckung vom Typ $\mathscr{L}_{\Sigma_s}$.

Betrachten wir jetzt den Fall, wo $\mathscr{U}$ durch ein verschobenes Exemplar von $\mathscr{W}_2$ überdeckt ist. In diesem Fall gilt $b-1 \leqq \sqrt{1-\left(\frac{a}{2}\right)^2}$. Hieraus ergibt sich, im Hinblick auf $a\leqq b$, $a\leqq 8/5$.

Die Basisvektoren des Gitters $\bar{\Gamma}$ seien $\overrightarrow{OA}$ und $\overrightarrow{OB}$, wo $\overrightarrow{OA}\perp\overrightarrow{OB}$ und $|\bar{B}| = \sqrt{1-\left(\frac{a}{2}\right)^2}+1$ gelten. Es sei $D=\frac{\bar{B}}{2}$. Bei dem Doppelgitter $\bar{\Sigma}=\bar{\Gamma}\cup\bar{\Gamma}_D$ ist der Inhalt des Grundparallelogramms von $\bar{\Gamma}$ größer als bei dem ursprünglichen Gitter Γ. Es ist leicht einzusehen, daß $\mathscr{L}_{\bar{\Sigma}}$ eine 2-fache Überdeckung ist und die Überdeckung im Fall $D\neq\bar{B}/2$ nicht 2-fach ist. Der Inhalt des Grundparallelogramms von $\bar{\Gamma}$ ist

$$\bar{T}(a) = a\left(\sqrt{1-\frac{a^2}{4}}+1\right), \qquad 1 \leqq a \leqq \frac{8}{5}.$$

Man kann sehr leicht einsehen, daß $\bar{T}(a)$ streng monoton wächst. So ist $\left(\frac{8}{5}\right)^2$ der maximale Inhalt. Wegen $\left(\frac{8}{5}\right)^2 < \frac{3\sqrt{3}}{2}$ ist die Dichte von $\mathscr{L}_{\bar{\Sigma}}$ größer als $2D_1$.

Beweis des Satzes. Wir betrachten eine 2-fache Überdeckung $\mathscr{L}_{\Sigma}$ mit der Dichte $\Delta(\mathscr{L}_{\Sigma}) \leqq \frac{4\pi}{3\sqrt{3}}$. Mit Rücksicht auf Hilfssatz 3 genügt es zu zeigen, daß sich zu $\mathscr{L}_{\Sigma}$ eine 2-fache Überdeckung $\mathscr{L}_{\bar{\Sigma}}$ mit $\bar{\alpha}=\pi/2$ und mit der Dichte $\Delta(\mathscr{L}_{\bar{\Sigma}}) \leqq \Delta(\mathscr{L}_{\Sigma})$ angeben läßt.

1. Wir beginnen mit dem Fall $a<1$. Es sei h die zu OA gehörige Höhe des Grundparallelogramms $OA(A+B)B$ von Γ. Es gilt $h>2$, sonst gilt $T\leqq 2$ und $\Delta(\mathscr{L}_{\Sigma}) \geqq \pi > \frac{4\pi}{3\sqrt{3}}$. $\mathscr{W}$ sei das Bereich, dessen Punkte von den Kreisen mit den Mittelpunkten kA (k ist eine ganze Zahl) mindestens zweifach überdeckt sind. M und N seien die Schnittpunkte der um die Gitterpunkte A und $(-A)$ geschlagenen Einheitskreislinien (Abb. 6). Es ist offenbar, daß $\mathscr{W}$ keinen Streifen der Breite größer als MN überdecken kann.

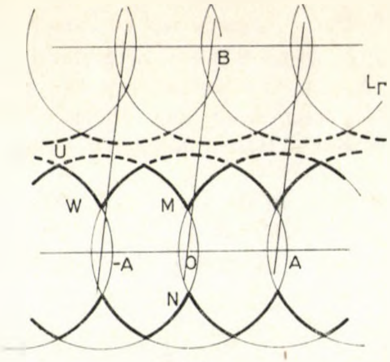

Abb. 6

Wir betrachten das Gitter $\bar{\Gamma}$, dessen Basisvektoren $\overrightarrow{OA}$ und $\overrightarrow{O\bar{B}}$ sind, wo $|\bar{B}|= =2+MN$ und $\sphericalangle(\bar{B}OA)=\pi/2$ gelten. Es gilt $\Delta(\mathscr{L}_{\bar{\Sigma}})\leqq\Delta(\mathscr{L}_{\Sigma})$, wo Gleichheit nur im Fall $\Sigma\equiv\bar{\Sigma}$ auftritt und $\mathscr{L}_{\bar{\Sigma}}$ offenbar eine 2-fache Überdeckung ist.

2. Der Fall $a\geqq 1$. Im Hinblick auf (1) und Hilfssatz 2 ergibt sich $a\leqq b\leqq 2$. Wegen des Hilfssatzes 1 können wir annehmen, daß $c>2$ ist. Es seien $\mathscr{W}_1, \mathscr{W}_4, \mathscr{W}_3, \mathscr{W}_2$ die Schnitte der um O und A; A und $A+B$; $A+B$ und B bzw. B und O geschlagenen Einheitskreise. E_i sei die im $OA(A+B)B$ liegende Ecke von $\mathscr{W}_i$ $(i=1, 2, 3, 4)$ (Abb. 7). Wir bezeichnen denjenigen Teil von $OA(A+B)B$, der von den Kreisen der Anordnung $\mathscr{L}_\Gamma$ nicht überdeckt ist, mit $\mathscr{U}$. $\mathscr{U}$ ist von den Kreisen der Anordnung $\mathscr{L}_{\Gamma_D}$ 2-fach überdeckt. Da die $\mathscr{W}_i$ disjunkt sind, ist $\mathscr{U}$ durch ein verschobenes Exemplar von $\mathscr{W}_i$ $(i=1, 2, 3, 4)$ überdeckt. Ohne Beschränkung der Allgemeinheit können wir annehmen, daß $i=1$ ist.

Das Bereich $\mathscr{W}_1$ und das Parallelogramm $E_1E_2E_3E_4$ sind zentralsymmetrisch und konvex, so, wenn wir $\mathscr{U}$ durch Verschiebung von $\mathscr{W}_1$ überdecken konnten, können wir $\mathscr{U}$ auch durch Verschiebung von $\mathscr{W}_1$ um $B/2$ überdecken.

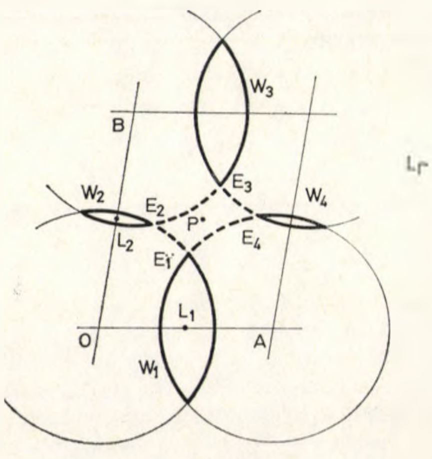

Abb. 7

Wir betrachten das Gitter $\bar{\Gamma}$, dessen Gittervektoren $\overrightarrow{OA}$ und $\overrightarrow{O\bar{B}}$ sind, für die $\overrightarrow{OA} \perp \overrightarrow{O\bar{B}}$ und $|\bar{B}| = |B|$ gelten. D sei $\bar{B}/2$. Wir betrachten das Doppelgitter $\bar{\Sigma} = \bar{\Gamma} \cup \bar{\Gamma}_D$. Es ist offenbar, daß $\bar{T} > T$ gilt. Wir sehen noch ein, daß die Anordnung $\mathcal{L}_{\Sigma}$ eine 2-fache Überdeckung ist. Es seien L_1, L_2 und P die Mittelpunkte der Strecken OA, OB und AB. Bei $\bar{\Gamma}$ wenden wir die Bezeichnungen der Figur 7, aber mit dem Zeichnen "–" an.

Die orthogonale Projektion auf AO führt den Punkt Y in Y^* über. Es ist leicht zu sehen, da $c > 2$ ist, daß die Reihenfolge der Bildpunkte von L_2, E_2, P genau L_2^*, E_2^*, P^* ist. Da $L_2 P \| OA$ und $L_2 P = \bar{L}_2 \bar{P} = a/2$ und $L_2^* E_2^* < \bar{L}_2^* \bar{E}_2^*$ ist, folgt

$$E_2^* E_4^* > \bar{E}_2^* \bar{E}_4^*. \tag{3}$$

Offensichtlich gilt $L_1 P \| OB$, $L_1 P = \bar{L}_1 \bar{P} = b/2$ und $L_1 E_1 = \bar{L}_1 \bar{E}_1$. Hieraus folgt wegen der Dreiecksungleichung

$$E_1 E_3 = 2E_1 P > 2(L_1 P - L_1 E_1) = 2(\bar{L}_1 \bar{P} - \bar{L}_1 \bar{E}_1) = 2\bar{E}_1 \bar{P} = \bar{E}_1 \bar{E}_3. \tag{4}$$

Aus (3) und (4) folgt, daß $\bar{\mathcal{U}}$ durch $(\mathcal{W}_1)_{B/2}$ überdeckt werden kann.

Damit haben wir den Beweis des Satzes beendet.

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SPECTRAL PROPERTIES OF VECTOR OPERATORS

T. MATOLCSI

1. Introduction

Usual quantum mechanical observables are self-adjoint operators, or better said, families of self-adjoint operators. For instance, position, a so-called vectorial observable, is considered as a family of three self-adjoint operators that are interpreted as the components of position relative to a basis of the physical space. If we want to get rid of bases and to look for a coordinate-free description, we face the problem, what mathematical objects represent quantum mechanical observables. The notion of vector operator is introduced to answer this question. Here we investigate only mathematical properties of vector operators and we do not enter into physical applications.

2. Preliminaries

In the sequel H and Z denote a complex Hilbert space and a finite dimensional complex vector space, respectively.

Inner products are denoted by the symbol $\langle , \rangle$ and are taken to be linear in the second variable.

$H\otimes Z$ is the algebraic tensor product of H and Z. It is well-known (see [1], Ch. II. 4) that if we equip Z with an inner product then $H\otimes Z$ turns into a Hilbert space with the inner product defined by

$$\langle h\otimes z,\ g\otimes y\rangle := \langle h, g\rangle\langle z, y\rangle \qquad (h, g\in H,\ z, y\in Z).$$

The corresponding topology on $H\otimes Z$ is independent of the particular inner product chosen on Z. That is why we consider $H\otimes Z$ as a topological vector space without specifying an inner product on Z.

If $z_1, \ldots, z_N$ is a basis of Z then every element of $H\otimes Z$ can be written in the form $\sum_{k=1}^{N} h_k\otimes z_k$.

Z^* stands for the dual of Z and the bilinear map of duality is denoted by $(|)$. We are given a continuous bilinear map

$$((|)) \colon Z^*\times(H\otimes Z) \to H,$$

defined by

$$((p|h\otimes z)) := (p|z)h \qquad (p\in Z^*,\ h\otimes z\in H\otimes Z),$$

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and a continuous sesquilinear map

$$\langle\langle , \rangle\rangle\colon H \times (H \otimes Z) \to Z$$

defined by

$$\langle\langle g, h \otimes z \rangle\rangle := \langle g, h \rangle z \qquad (g \in H,\ h \otimes z \in H \otimes Z).$$

We have the following relation:

$$\langle g, ((p|a)) \rangle = (p | \langle\langle g, a \rangle\rangle) \qquad (p \in Z^*,\ g \in H,\ a \in H \otimes Z).$$

If $p_1, \ldots, p_N$ is a basis of Z^* then the elements a and b of $H \otimes Z$ are equal if and only if $((p_k|a)) = ((p_k|b))$ $(k = 1, \ldots, N)$.

3. Basic facts about vector operators

Definition 1. A linear map defined in H and having values in $H \otimes Z$ is called a *Z valued vector operator* in H.

If A is a vector operator and $p \in Z^*$ then we define the linear map

$$((p|A))\colon H \supset \operatorname{Dom} A \to H, \quad h \mapsto ((p|Ah)).$$

Remarks. (i) A complex valued vector operator is a usual operator.

(ii) Since $H \otimes Z$ has a topology, we can speak about continuous and closed vector operators.

(iii) Let $z_1, \ldots, z_N$ be a basis of Z and let $p_1, \ldots, p_N$ be the corresponding dual basis of Z^*. Then we can consider $((p_k|A))$ $(k = 1, \ldots, N)$ as the *components* of the vector operator A relative to the given basis of Z. We have the equality

$$Ah = \sum_{k=1}^{N} [((p_k | Ah))] \otimes z_k \qquad (h \in \operatorname{Dom} A).$$

Consequently, if we are given a family $A_1, \ldots, A_N$ of operators with common domain D in H, then we can construct the vector operator

$$h \mapsto \sum_{k=1}^{N} (A_k h) \otimes z_k \qquad (h \in D)$$

whose components are precisely the given operators.

As a consequence, two Z valued vector operators are equal if and only if their components relative to any basis of Z coincide.

Examples. (i) If $u \in Z$ then $\otimes u\colon H \to H \otimes Z$, $h \mapsto h \otimes u$ is a continuous vector operator and $((p|\otimes u)) = (p|u)\,\mathrm{id}_H$.

(ii) Let V be a finite dimensional real vector space. Then $L^2(V) \otimes Z$ is identified, through the prescription $f \otimes z = (v \mapsto f(v)z)$, with the vector space of Z valued square integrable function classes. The *identity multiplication operator M* defined on

$$\operatorname{Dom} M := \{f \in L^2(V)\colon f\,\mathrm{id}_V \in L^2(V) \otimes V_{\mathbf{C}}\}$$

by

$$f \mapsto f\,\mathrm{id}_V := (v \mapsto f(v)v)$$

is a $V_{\mathbf{C}}$ valued vector operator in $L^2(V)$ where $V_{\mathbf{C}}$ stands for the complexification of V. If $r_1, \ldots, r_N$ is a basis in V then $((r_k|M))$ is contained in the operator of multiplication by the k-th coordinate.

If $f\colon V\to\mathbf{C}$ is differentiable, then $\mathrm{D}f(v)$, its derivative at $v\in V$, is a linear map $V\to\mathbf{C}$ which can be extended uniquely to a complex linear map $V_{\mathbf{C}}\to\mathbf{C}$; in other words, we can consider $\mathrm{D}f$ as a map $V\to(V_{\mathbf{C}})^*=(V^*)_{\mathbf{C}}=:V^*_{\mathbf{C}}$. Then the differentiation operator D defined on

$$\mathrm{Dom}\,\mathrm{D}:=\{f\in L^2(V)\colon f \text{ is differentiable, } \mathrm{D}f\in L^2(V)\otimes V^*_{\mathbf{C}}\}$$

is a $V^*_{\mathbf{C}}$ valued vector operator in $L^2(V)$. If $v_1, \ldots, v_N$ is a basis in $V=(V^*)^*$ then $((v_k|D))$ is contained in the k-th partial differentiation operator.

Definition 2. A bounded operator L is said to *commute* with the vector operator A if $AL\supset(L\otimes \mathrm{id}_Z)A$.

Proposition 1. *L commutes with A if and only if L commutes with $((p|A))$ for all $p\in Z^*$ which holds if and only if L commutes with $((p_k|A))$ $(k=1, \ldots, N)$ for an arbitrary basis $p_1, \ldots, p_N$ of Z^*.*

4. The spectrum of a vector operator

In the sequel A denotes a fixed densely defined vector operator.

Definition 3. A linear subspace D of $\mathrm{Dom}\,A$ is called *invariant* under A if $A(D)\subset D\otimes Z$.

Proposition 2. *D is invariant under A if and only if D is invariant under $((p|A))$ for all $p\in Z^*$ which holds if and only if D is invariant under $((p_k|A))$ $(k=1, \ldots, N)$ for an arbitrary basis $p_1, \ldots, p_N$ of Z^*.*

Definition 4. An element λ of Z is called an *eigenvalue* of A *if there is a nonzero* $h\in\mathrm{Dom}\,A$ such that $Ah=h\otimes\lambda$. The linear subspace $\{h\in\mathrm{Dom}\,A\colon Ah=h\otimes\lambda\}$ is the *eigenspace* of A corresponding to λ. The set of eigenvalues of A is denoted by $\mathrm{Eig}\,A$.

Definition 5. A linear subspace T of $H\otimes Z$ is called *bulky* if there is no proper closed linear subspace D of H such that $T\subset D\otimes Z$.

Proposition 3. *A linear subspace T of $H\otimes Z$ is bulky if and only if H is spanned by* $\bigcup_{p\in Z^*}\{((p|a))\colon a\in T\}$.

Definition 6. An element λ of Z is a *regular value* of A if
(i) $A-\otimes\lambda$ is injective,
(ii) $\mathrm{Ran}\,(A-\otimes\lambda)$ is bulky,
(iii) $(A-\otimes\lambda)^{-1}$ is continuous.
The set
$\mathrm{Sp}\,A:=\{\lambda\in Z\colon \lambda \text{ is not a regular value of } A\}$ is the *spectrum* of A.

Proposition 4. (i) $\mathrm{Eig}\, A \subset \mathrm{Sp}\, A$, *and for all* $p \in Z^*$
(ii) $(p|\mathrm{Eig}\, A) \subset \mathrm{Eig}\,((p|A))$,
(iii) $(p|\mathrm{Sp}\, A) \subset \mathrm{Sp}\,((p|A))$.

Proof. (i) and (ii) are evident. To prove (iii) suppose that $\lambda \in \mathrm{Sp}\, A$, $S(\lambda) := A - \otimes\lambda$ is injective, and distinguish the following two cases.

Firstly, assume that $\bigcup_{p \in Z^*} ((p|\mathrm{Ran}\, S(\lambda))$ does not span H. Then $\mathrm{Ran}\,((p|S(\lambda)) = ((p|\mathrm{Ran}\, S(\lambda)))$ cannot be dense in H, thus $(p|\lambda) \in \mathrm{Sp}\,((p|A))$ for all $p \in Z^*$.

Secondly, suppose that the inverse of $S(\lambda)$ is not continuous. Then there is an unbounded sequence h_n $(n \in \mathbf{N})$ in H such that $S(\lambda)h_n$ is bounded. Consequently, the sequence $((p|S(\lambda)h_n))$ is bounded, thus $((p|S(\lambda)))$ cannot have a continuous inverse (it may have no inverse at all), and $(p|\lambda) \in \mathrm{Sp}\,((p|A))$ $(p \in Z^*)$.

Proposition 5. *The spectrum of a vector operator is closed.*

Proof. To demonstrate this assertion let us equip Z with an inner product. Then for all $u \in Z$ the norm of the vector operator $\otimes u$ equals the norm of the vector u: $\|h \otimes u\| = \|u\| \|h\|$ for all $h \in H$. As a consequence, one can show as in usual operator theory that if B is a vector operator having a continuous inverse then $B - \otimes u$ has a continuous inverse for u in a convenient neighbourhood of the zero of Z. Furthermore, suppose that $\mathrm{Ran}\, B$ is bulky, i.e. for any $g \in H$ there are $p \in Z^*$ and $h \in \mathrm{Dom}\, B$ such that $\langle g, ((p|Bh))\rangle \neq 0$; then $\langle g, ((p|B - \otimes u))h\rangle = \langle g, ((p|Bh))\rangle - \langle g, h\rangle (p|u) \neq 0$ if u is small enough, hence $\mathrm{Ran}\,(B - \otimes u)$ is bulky. Substitute $A - \otimes\lambda$ for B with a regular value λ of A to have the desired result.

Proposition 6. *Let A be continuous. Equip Z with an inner product. Then the set $\{z \in Z\colon \|z\| > \|A\|\}$ is disjoint from* $\mathrm{Sp}\, A$.

Proof. If $\|z\| > \|A\|$ then $\|(A - \otimes z)h\| \geq \big|\|Ah\| - \|z\|\, \|h\|\big| \geq (\|z\| - \|A\|)\|h\|$ for all $h \in \mathrm{Dom}\, A$, hence $A - \otimes z$ has a continuous inverse. We have to show now that $\mathrm{Ran}\,(A - \otimes z)$ is bulky. Let $\dot{z}$ denote that element of Z^* for which $(\dot{z}|y) = \langle z, y\rangle$ $(y \in Z)$. Then $((\dot{z}|A)) = (\otimes z)^* A$, so $\|((\dot{z}|A))\| \leq \|z\| \|A\| < \|z\|^2$, and thus $\|z\|^2 = (\dot{z}|z)$ is not in the spectrum of $((\dot{z}|A))$ as it is well-known from usual operator theory. Consequently, $\mathrm{Ran}\,[((\dot{z}|A)) - (\dot{z}|z)\, \mathrm{id}_H] = ((\dot{z}|\mathrm{Ran}\,(A - \otimes z)))$ is dense in H; apply Proposition 3 to end the proof.

Proposition 7. *Let Y be a finite dimensional vector space containing Z as a linear subspace. Then $H \otimes Z \subset H \otimes Y$ and a Z valued vector operator is also a Y valued vector operator. The spectrum of A is independent of whether A is considered as Z valued or Y valued.*

Proof. We have to show that if $y \in Y$ and $y \notin Z$ then y is a regular value of A. Choose an inner product on Y and write $y = u + v$ such that u is in Z and $v \neq 0$ is orthogonal to Z. Then for all $h \in \mathrm{Dom}\, A$, $\|(A - \otimes y)h\|^2 = \|(A - \otimes u)h\|^2 + \|v\|^2 \|h\|^2 \geq \|v\|^2 \|h\|^2$, hence $A - \otimes y$ has a continuous inverse. Furthermore, using the notation introduced in Proposition 6, we have $((\dot{v}|(A - \otimes y)h)) = -\|v\|^2 h$ $(h \in \mathrm{Dom}\, A)$ which yields that $\mathrm{Ran}\,(A - \otimes y)$ is bulky.

Remarks. (i) If $Z = \mathbf{C}$, Definition 6 gives back the usual definition of the spectrum. If Z is one-dimensional, the spectrum of a Z valued vector operator has the usual properties.

(ii) To construct examples that the spectrum of a vector operator does not exhibit in general all the properties of the usual spectrum, we take two dimensional spaces. Let h_1, h_2 and z_1, z_2 be an orthonormal basis of H and a basis of Z, respectively, and let us consider vector operators of the form $H \to H \otimes Z$, $h \mapsto (A_1 h) \otimes z_1 + (A_2 h) \otimes z_2$.

— The vector operator given by $A_1 h_1 := h_1$, $A_2 h_2 := 0$, $A_2 h_1 := A_2 h_2 := h_1 + h_2$ has a void spectrum.

— The spectrum of the vector operator given by $A_1 h_1 := A_1 h_2 := h_1 + h_2$, $A_1 h_2 := := A_2 h_1 := 0$ contains zero, but not as an eigenvalue.

(iii) Observe that the norm of vector operators depends on the inner product on Z. It is interesting that even the set $\{z \in Z: \|z\| > \|A\|\}$ depends on it. To see this let H and Z be as in (ii) and let A_1 and A_2 be the projections onto the subspaces spanned by h_1 and h_2, respectively. Then the corresponding vector operator has one and the same norm whatever be the inner product on Z such that $\|z_1\| = \|z_2\| = 1$.

(iv) If $A_1, \ldots, A_N$ are operators defined on a common dense linear subspace in H, the spectrum of the $\mathbf{C}^N$ valued vector operator whose components relative to the standard basis are the given operators is some sort of *joint spectrum* for $A_1, \ldots, A_N$.

5. Spectral theorem for vector operators

If T is a Hausdorff topological space, $B(T)$ denotes the algebra of Borel subsets of T. If P is a projection valued measure defined on $B(T)$ and having values in the set of projections of H then for all $h, g \in H$, $E \mapsto P_{h,g}(E) := \langle h, P(E)g \rangle$ is a complex measure on $B(T)$.

An element t of T is called a *sharp value* of P if $P(\{t\}) \neq 0$. The set of sharp values of P is denoted by Sharp P.

The *support* of P is the set

$$\text{Supp}\, P := \{t \in T:\ P(G) \neq 0 \text{ for all open } G \text{ with } t \in G\}.$$

Definition 7. A Z valued vector operator A in H is called
(i) *partially normal* if
$((p|A))$ is closable and its closure is normal for all $p \in Z^*$,

$$\text{Dom}\, A = \bigcup_{p \in Z^*} \text{Dom}\, \overline{((p|A))};$$

(ii) *totally normal* if it is partially normal and $\overline{((p|A))}$ and $\overline{((q|A))}$ strongly commute for all $p, q \in Z^*$.

Proposition 8. (i) *A partially normal vector operator is densely defined and closed.*
(ii) *A continuous partially normal vector operator is totally normal.*

Proof. (i) is quite easy. To show (ii) observe the continuity of A implies that $\overline{((p|A))} = ((p|A))$. Take the bounded normal operators $((p+q|A)) = ((p|A)) + ((q|A))$ and $((p+iq|A))$ to obtain that $((p|A))$ commutes with $((q|A))^*$ which implies the commutativity of $((p|A))$ and $((q|A))$ $(p, q \in Z^*)$.

PROPOSITION 9. *Let A be a totally normal vector operator. Then there exists a unique projection valued measure R on $B(Z)$ such that*

$$\langle\langle h, Ag\rangle\rangle = \int_Z \mathrm{id}_Z dR_{h,g} \qquad (h\in H,\ g\in \mathrm{Dom}\, A).$$

PROOF. Let $p_1, \dots, p_N$ be a basis in Z^* and let R_k be the spectral resolution of the normal operator $\overline{((p_k|A))}$ $(k=1, \dots, N)$. Then $R_1, \dots, R_N$ are commuting projection valued measures, hence their product $\bigotimes_{k=1}^{N} R_k$ exists and is the unique projection valued measure on $B(\mathbf{C}^N)$ determined by $\left(\bigotimes_{k=1}^{N} R_k\right)\left(\bigtimes_{k=1}^{N} E_k\right)=\prod_{k=1}^{N} R_k(E_k)$. Let b denote the inverse of the linear bijection $Z\to\mathbf{C}^N$, $z\mapsto\{(p_k|z)\colon k=1, \dots, N\}$, and put $R:=\left(\bigotimes_{k=1}^{N} R_k\right)\circ b^{-1}$. Then for all $k=1, \dots, N$, $h\in H$ and $g\in\mathrm{Dom}\, A$

$$(p_k|\langle\langle h, Ag\rangle\rangle) = \langle h, ((p_k|A))g\rangle = \int_{\mathbf{C}} \mathrm{id}_{\mathbf{C}} d(R_k)_{h,g} =$$

$$= \int_{\mathbf{C}^N} \mathrm{pr}_k d\left(\bigotimes_{i=1}^{N} R_i\right)_{h,g} = \int_Z p_k\, dR_{h,g} = \left(p_k\Big| \int_Z \mathrm{id}_Z dR_{h,g}\right)$$

where $\mathrm{pr}_k\colon \mathbf{C}^N\to\mathbf{C}$ is the k-th canonical projection; we also used the relation $p_k\circ b= =\mathrm{pr}_k$ and the well-known integral transformation formula. The uniqueness of R follows from the uniqueness of the R_k's and from the equalities

$$R = \left[\bigotimes_{k=1}^{N} (R\circ p_k^{-1})\right]\circ b^{-1}, \qquad R_k = R\circ p_k^{-1}.$$

REMARK. We can define the integral of measurable functions $T\to Z$ with respect to projection valued measures on $B(T)$ as Z valued vector operators. It can be shown that all such vector operators are totally normal. In other words, only the totally normal vector operators have spectral resolutions, i.e. are integrals of id_z with respect to projection valued measures.

PROPOSITION 10. *A bounded operator L commutes with a totally normal vector operator A if and only if L commutes with the spectral resolution of A.*

The proof of the following assertion requires a number of notions and particular results from the theory of integration with respect to projection valued measures. Who is familiar with them, can argue similarly as in the case of usual normal operators (see [2]), needing only one new step, a consideration on bulky subspaces. We omit these details.

PROPOSITION 11. *Let A be a totally normal vector operator having R as its spectral resolution. Then*

$$\mathrm{Eig}\, A = \mathrm{Sharp}\, R, \qquad \mathrm{Sp}\, A = \mathrm{Supp}\, R.$$

DEFINITION 8. Let V be a finite dimensional real vector space. A $V_{\mathbf{C}}$ valued vector operator A in H is called

(i) *partially self-adjoint* if

$\overline{((r|A))}$ is closable and its closure is self-adjoint for all $r \in V^*$,

$$\operatorname{Dom} A = \bigcap_{r \in V^*} \operatorname{Dom} \overline{((r|V))};$$

(ii) *totally self-adjoint* if it is partially self-adjoint and $\overline{((r|A))}$ and $\overline{((s|A))}$ strongly commute for all $r, s \in V^*$.

REMARKS. (i) A partially self-adjoint vector operator is densely defined and closed.

(ii) A partially self-adjoint vector operator need not be partially normal. For instance, the first operator given in Remark (ii) at the end of Section 3, if $Z = V_{\mathbf{C}}$, $z_1, z_2 \in V$, is partially self-adjoint without being partially normal.

(iii) Taking a basis $r_1, \ldots, r_N$ in V^* (it is a basis in $V^*_{\mathbf{C}}$, too, with respect to the complex structure) and repeating the argument of the proof of Proposition 9, this time considering $((r_k|A))$ instead of $((p_k|A))$, we find that a totally self-adjoint vector operator is the integral of $\mathrm{id}_{V_{\mathbf{C}}}$ with respect to a projection valued measure whose support is in V. As a consequence, by the Remark to Proposition 9, a totally self-adjoint vector operator is totally normal, and its spectrum is contained in V.

EXAMPLES. (i) For $u \in z$, the vector operator $\otimes u$ is totally normal, its spectral resolution is the projection valued measure concentrated at u.

(ii) The identity multiplication operator in $L^2(V)$ is totally self-adjoint. Its spectral resolution is the projection valued measure that assigns to $E \in B(V)$ the operator of multiplication by the characteristic function of E (which is the projection onto $L^2(E) \subset L^2(V)$).

(iii) The differentiation operator in $L^2(V)$ is closable, its closure multiplied by the imaginary unit is totally self-adjoint. Its spectral resolution is the projection valued measure that assigns to $S \in B(V^*)$ the projection $F^{-1}K(S)F$ where $K(S)$ is the projection onto $L^2(S) \subset L^2(V^*)$ and $F\colon L^2(V) \to L^2(V^*)$ is the Fourier transformation defined by

$$(Ff)(r) := \int_V e^{i(r|v)} f(v)\, dv \qquad (f \in L^2(V) \cap L^1(V),\ r \in V^*)$$

with the translation invariant measure on $B(V^*)$ chosen in such a way that F be unitary.

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BEMERKUNG ZU EINER ARBEIT VON J. PINTZ

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1. Bekanntlich bewies Littlewood im Jahre 1914, daß die Funktion

$$\pi(x) - \operatorname{li} x = \sum_{p \leqq x} 1 - \int_0^x \frac{dt}{\log t} \qquad (x > 2) \tag{1.1}$$

unendlich viele Zeichenwechsel hat. Sein Beweis erlaubte es jedoch nicht, ein X_0 so zu bestimmen, daß $\pi(x) - \operatorname{li} x$ mindestens einen Zeichenwechsel im Intervall $[2, X_0]$ hat. Weiterhin war es nicht möglich, für die Anzahl $V(y)$ der Zeichenwechsel von $\pi(x) - \operatorname{li} x$ im Intervall $[2, y]$ eine Abschätzung nach unten anzugeben.

Skewes [10] zeigte 1955, daß

$$\pi(x) > \operatorname{li} x \quad \text{für mindestens ein} \quad x < \exp_4 7{,}705 \tag{1.2}$$

$$(\exp_1 x = \exp x, \ \exp_{n+1} x = \exp_n \exp x).$$

Dieses Ergebnis verbesserte Lehman [6] im Jahre 1966, indem er als obere Schranke für den ersten Zeichenwechsel $X_0 = 1{,}65 \cdot 10^{1165}$ berechnete.

Über die Anzahl $V(y)$ der Zeichenwechsel bewies Knapowski [4] im Jahre 1962

$$V(y) \geqq e^{-35} \log_4 y \quad \text{für} \quad y \geqq \exp_5 35$$
$$(\log_1 y = \log y, \ \log_{n+1} y = \log_n \log y). \tag{1.3}$$

Unter Verwendung der Beweismethode von Pintz [8] zeigen wir in diesem Aufsatz den folgenden

SATZ. *Für* $y \geqq \exp_4 3{,}57$ *gilt*

$$V(y) > \frac{1}{\exp_3 3{,}550} \frac{\sqrt{\log y}}{\log_2 y}. \tag{1.4}$$

2. Wir folgen bei den anschließenden Überlegungen der Beweismethode von Pintz [8] und geben die wesentlichen Änderungen an. Dabei werden wir die Bezeichnungsweise von Pintz [8] zugrundelegen.

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Zum Beweis unseres Satzes im Fall I zeigen wir Lemma 1 für $Z \geqq \exp_2 15$. Daraus ergibt sich

$$V(y) > 10^{-3} \frac{\sqrt{\log y}}{\log_2 y} \quad \text{für} \quad y \geqq \exp_2 16 \tag{2.1}$$

und damit der behauptete Satz im Fall I.

Die Überlegungen (6.12) bis (6.19) zur Reduktion des Problems können für $Z \geqq \exp_2 15$ (das bedeutet $A \geqq \exp_3 2{,}6$) beibehalten werden. Die Setzungen (7.2) bis (7.5) ändern wir folgendermaßen ab (anderenfalls läßt sich die Abschätzung (13.6) nicht beweisen):

Sei $\varrho_0 = \beta_0 + i\gamma_0$ eine Nullstelle mit maximalem Realteil unter denen, die (7.1) bei Pintz [8] erfüllen. Weiterhin sei nun sukzessive ϱ'_{n+1} eine Nullstelle mit maximalem Realteil β'_{n+1}, die

$$\begin{gathered} \gamma'_n \leqq \gamma'_{n+1} \leqq \gamma'_n + 2\lambda \\ \beta'_{n+1} \geqq \beta'_n + \frac{1}{\mu} \end{gathered} \tag{2.2}$$

erfüllt, falls eine solche Nullstelle existiert. Nach höchstens $[\mu/2]$ Schritten erhalten wir eine Nullstelle

$$\begin{gathered} \varrho'_N = \beta'_N + i\gamma'_N \stackrel{\text{def}}{=} \varrho_1 = \beta_1 + i\gamma_1 \quad \text{mit} \\ \beta_1 \geqq \frac{1}{2} + \frac{1}{\lambda} \\ 0 < \gamma_1 \leqq e^{2\lambda} \quad \text{für} \quad Z \geqq \exp_2 15. \end{gathered} \tag{2.3}$$

Dabei sind die Bereiche

$$|t| \leqq \lambda^5, \tag{2.4}$$

und

$$|t - \gamma_1| \leqq 2\lambda, \quad \sigma > \beta_1 + \frac{1}{\mu} \tag{2.5}$$

nullstellenfrei.

Als Abschätzung für $|U|$ nach oben erhalten wir im Fall A (vergleiche (12.9) bei Pintz [8]):

$$|U| \leqq \frac{e^{k\beta_1^2 + \mu\beta_1}}{e^{10M}} \quad \text{für} \quad Z \geqq \exp_2 15. \tag{2.6}$$

Dabei haben wir (11.1) bei Pintz [8] korrigiert durch

$$\int_1^\infty \frac{f(x)}{x^{s+1}}\,dx = \frac{1}{s}\left\{\int_2^s \left(\frac{\zeta'}{\zeta}(z) + \zeta(z)\right) dz + h\right\} \pm \frac{1}{s - \frac{1}{2}} - 1 \quad \text{für} \quad \sigma > 1 \tag{2.7}$$

mit $|h| \leqq 10$.

Bei der Berechnung der Konstanten in (12.5) bei Pintz [8] sind wir von der Weierstraßschen Produktdarstellung ausgegangen

$$(s-1)\zeta(s) = \frac{e^{bs}}{2\Gamma\left(\frac{s}{2}+1\right)} \prod_{\varrho} \left(1-\frac{s}{\varrho}\right) e^{s/\varrho} \quad \text{mit} \quad b = 0{,}549\ldots \tag{2.8}$$

und erhalten für (12.5)

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leqq 140 \frac{\log |t|}{\eta}, \tag{2.9}$$

wobei in (12.4)

$$2 \geqq \sigma \geqq \beta+\eta, \quad 2 \leqq |t| \leqq T \tag{2.10}$$

zu setzen ist.

Unter Verwendung von

$$\zeta(s) = \sum_{n \leqq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} (x-[x]) \frac{dx}{x^{s+1}}, \quad N \geqq 1, \quad \sigma > 0 \tag{2.11}$$

erhalten wir für (12.6)

$$|\zeta(s)| \leqq 5{,}5 \sqrt{|t|} \quad \text{für} \quad \sigma \geqq \frac{1}{2}, \quad |t| \geqq 10. \tag{2.12}$$

Unter Verwendung dieser Ergebnisse erhält man (2.6) analog zu Pintz [8].

Um $|U|$ nach unten abzuschätzen, verbessern wir (13.9) zu

$$1 \leqq n \leqq 320 \log L \tag{2.13}$$

und erhalten für (13.14)

$$|U| \geqq \frac{e^{k\beta_1^2+\mu\beta_1}}{e^{\delta M}} \quad \text{für} \quad Z \geqq \exp_2 15. \tag{2.14}$$

Dies steht im Widerspruch zu (2.6), womit Lemma 1 im Fall A bewiesen ist. Der Fall B läßt sich analog zu Pintz [8] für $Z \geqq \exp_2 15$ unter Verwendung ähnlicher Abschätzungen für $\zeta(s)$ und $\frac{\zeta'}{\zeta}(s)$ im kritischen Streifen wie im Fall A beweisen.

3. Zum Beweis des voranstehenden Satzes im Fall II können wir (18.5) ersetzen durch

$$\left|\Delta_1^*(x) - \Delta_4^*(x) + \frac{\Pi(x)-\pi(x)}{\frac{\sqrt{x}}{\log x}}\right| \leqq \frac{16}{\log x} \quad \text{für} \quad \exp 10^4 \leqq x \leqq e^{\lambda_0/2}. \tag{3.1}$$

Daher genügt es, anstelle von (19.2) und (19.3)

$$\max_{x\in J} \Delta_4^*(x) > 1+\frac{2}{10^3} \tag{3.2}$$

$$\min_{x\in J} \Delta_4^*(x) < -\left(1+\frac{2}{10^3}\right) \tag{3.3}$$

zu beweisen mit $J\subset[\exp 10^4, e^{\lambda_0/2}]$.

Unter Verwendung von

$$\left|\sum_{|\gamma|>T} \frac{x^\varrho}{\varrho}\right| \leqq 10^{-4}x^{1/2} \quad \text{für} \quad x\geqq \exp 10^4, \quad T\geqq x^2 \tag{3.4}$$

(vergleiche Skewes [10]) erhalten wir für (19.7) und (19.8)

$$\max_{a_1\leqq v\leqq a_2} G(v) > 1+\frac{1}{100} \tag{3.5}$$

$$\min_{a_1\leqq v\leqq a_2} G(v) < -\left(1+\frac{1}{100}\right) \tag{3.6}$$

mit $[e^{a_1}, e^{a_2}]\subset[\exp 10^4, e^{\lambda_0/2}]$.

Verwendet man die speziellen Ergebnisse

$$\sum_{\gamma>0} \frac{1}{\gamma^{3/2}} \leqq 0{,}372 \tag{3.7}$$

$$\sum_{\gamma>0} \frac{1}{\gamma^{15/8}} \leqq 0{,}053, \tag{3.8}$$

die man erhält, wenn man beim Beweis von $\sum_{\gamma>0}\frac{1}{\gamma^\delta}\leqq\frac{1}{2\pi}\frac{\delta}{(\delta-1)^2}$, $\delta>1$, die ersten 200 Nullstellen gesondert betrachtet (für die Werte der Nullstellen siehe Haselgrove und Miller [2]), so gilt für (20.7) bzw. (20.9)

$$|I_1(\omega)-I_2(\omega)| \leqq 1{,}12\frac{A}{\lambda_0} \tag{3.9}$$

bzw.

$$|I_2(\omega)-I_3(\omega)| \leqq 10^{-3} \tag{3.10}$$

für $10^4+1\leqq\omega\leqq\frac{\lambda_0}{2}-1$, $10^{-4}\lambda_0>A>e^{18}$.

Für $10^4+1\leqq\omega\leqq\dfrac{\lambda_0}{2\log A}$ und $10^{-4}\lambda_0>A>2e^{28}=:2T_1$ erhält man anstelle von (21.12) und (21.13)

$$J_\omega\left(\frac{1}{A}\right) > 0{,}226\log A-0{,}157 \tag{3.11}$$

und

$$J_{\omega}\left(-\frac{1}{A}\right) < 0{,}517 - 0{,}226 \log A. \tag{3.12}$$

Dabei haben wir verwendet, daß

$$N(T) > \frac{T}{7} \log T \quad \text{für} \quad T \geqq e^{28}. \tag{3.13}$$

Setzen wir in (22.4)

$$q = B \overset{\text{def}}{=} 100 \log^2 A, \tag{3.14}$$

so erhalten wir für (22.11)

$$\left| J_{\omega_\nu^{(l)}}(\omega_\nu^{(l)}) - J_{\omega_\nu^{(l)}}\left(\frac{(-1)^l}{A}\right) \right| < 0{,}087. \tag{3.15}$$

Zusammengefaßt ergibt sich dann für (22.13) und (22.14)

$$I_3(\omega_\nu^{(1)}) < -0{,}226 \log A + 0{,}604 \tag{3.16}$$

$$I_3(\omega_\nu^{(2)}) > 0{,}226 \log A - 0{,}244. \tag{3.17}$$

Damit können wir für (23.1)

$$A = \max\{e^{4{,}425(2\pi(1+1/100)+0{,}606)}, 2e^{28}\} = e^{30{,}762\ldots} \tag{3.18}$$

setzen.

Unter Berücksichtigung von

$$N(A) \leqq \frac{A}{2} \log A \tag{3.19}$$

setzen wir in (22.5)

$$c(A) = \frac{10^4 + 22}{\exp_3 3{,}550}. \tag{3.20}$$

Daraus ergibt sich die Behauptung unseres Satzes auch im Fall II.

4. Der Beweisgang im Fall II macht deutlich, daß es darauf ankommt, $I_3(\omega)$ möglichst groß nach oben bzw. nach unten abzuschätzen (vergleiche (3.16) und (3.17)). Außerdem zeigt (22 5) bei Pintz [8], daß A und damit die Anzahl der Nullstellen möglichst klein sein sollte, um möglichst gute Konstanten in der Aussage des Satzes zu erhalten. Diese Überlegungen legen es nahe, ein Ergebnis von Skewes [10] zu verwenden, das einen genügend großen Ausschlag für $I_3(\omega)$ garantiert, wobei $A = 500$. Für den Fall, daß die Riemannsche Vermutung wahr ist, wird dies in Dette, Meier und Pintz [1] durchgeführt.

5. Wir weisen darauf hin, daß die oben aufgeführten Überlegungen in unserer Dissertation (Über die Zeichenwechsel der Funktion $\pi(x) - \operatorname{li} x$, Bielefeld, 1982) ausführlich dargestellt sind.

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HOROCYCLES OF A DYNAMICAL SYSTEM ON THE PLANE

ZS. LANG

Introduction

The transformation of the plane $(x_1, y_1) = F(x_0, y_0)$ given by

$$x_1 = x_0 + \frac{1}{y_0}, \qquad y_1 = y_0 - x_0 - \frac{1}{y_0}$$

is connected with the Restricted Three Body Problem (see Henon [2]). F is an area preserving diffeomorphism, except along the x-axis where it is singular. Devaney showed recently that this mapping is topologically conjugate to the baker transformation. However, this result is purely topological and implies nothing about the ergodicity of F.

In this paper we construct contracting and expanding horocycles. They are important in studying the ergodic properties of dynamical systems (see [4]).

In § 1 we sketch the topological conjugacy to the baker transformation. Further we introduce the main notions and symbols.

In § 2 and § 3 we construct the horocycles by describing their tangents.

In § 4 we give the rate of expansion of the horocycles.

Our paper is based upon several ideas introduced by Devaney [1].

§ 1. Conjugacy to the baker transformation. Basic notions

F is defined on $\mathbf{R}^2 - \{y=0\}$. Its inverse is given by $(x_{-1}, y_{-1}) = F^{-1}(x_0, y_0)$ where

$$x_{-1} = x_0 - \frac{1}{x_0 + y_0}, \qquad y_{-1} = x_0 + y_0.$$

Hence F^{-1} is not defined on the line $y_0 = -x_0$.

Denote the point $F^j(x_0, y_0)$ by (x_j, y_j) for $j \in \mathbf{Z}$. Let $p \in \mathbf{R}^2$. If $F^k(p)$ is defined for all $k \in \mathbf{Z}$, then we may assign a sequence $s(p)$ to p, where

$$s(p) = (\dots, s_{-2}, s_{-1}, s_0;\; s_1, s_2, \dots)$$

$$s_j = \begin{cases} +1 & \text{if} \quad y_{-j}(p) > 0 \\ -1 & \text{if} \quad y_{-j}(p) < 0. \end{cases}$$

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Some F-orbits terminate when $y_j(p)=0$, $j\geqq 0$ or when $y_k(p)=-x_k(p)$, $k<0$. To these orbits we assign a sequence of the form

$$[0, s_{-j+1}, \dots, s_{-1}, s_0; s_1, s_2, \dots)$$

or

$$(\dots, s_{-2}, s_{-1}, s_0; s_1, \dots, s_{-k}; 0],$$

respectively. Under this identification F goes over to the shift transformation.

The set of sequences s of $+1$ and -1 may be mapped onto the open square $0\leqq |u|$, $|v|<1$ in the plane via the rule

$$u = \sum_{i=1}^{\infty} \frac{s_i}{2^i}, \quad v = \sum_{i=0}^{-\infty} \frac{s_i}{2^{1-i}}$$

when the shift transformation goes over to the baker transformation.

The mapping $p \leftrightarrow (u, v)$ is a topological conjugacy between the plane and the open square $0\leqq |u|$, $|v|<1$. We refer to [1] for complete details.

Now we explain several notions, terminology and symbols used in this paper. Define the sectors in the tangent bundle of $\mathbf{R}^2$

$$S^{+-} = \{(\xi, \eta) \mid \xi \geqq 0, \eta \leqq 0\}$$

$$S^{++} = \{(\xi, \eta) \mid \xi \geqq 0, \eta \geqq 0\}$$

$$S^{--} = \{(\xi, \eta) \mid \xi \leqq 0, \eta \leqq 0\}$$

$$S^{-+} = \{(\xi, \eta) \mid \xi \leqq 0, \eta \geqq 0\}.$$

So S^{++} consists of all tangent vectors to $\mathbf{R}^2$ which lie in the first quadrant. One checks easily that

$$dF(S^{+-}) \subset S^{+-}, \quad dF^{-1}(S^{++}) \subset S^{++}$$

$$dF(S^{-+}) \subset S^{-+}, \quad dF^{-1}(S^{--}) \subset S^{--}.$$

$S^u = S^{+-} \cup S^{-+}$ is called the unstable sector and $S^s = S^{++} \cup S^{--}$ is called the stable sector. We define an unstable curve to be a smooth curve whose tangents lie in the interior of S^u. Stable curves have tangents lying in the interior of S^s. It is clear that F maps unstable curves to unstable curves and F^{-1} maps stable curves to stable curves.

Let (ξ, η) be a tangent vector. We denote $dF^n(\xi, \eta)$ by (ξ_n, η_n) for $n\in\mathbf{Z}$. We introduce the norm

$$\|(\xi, \eta)\| = |\xi| + |\eta|$$

in the tangent bundle.

§ 2. Tangent vectors of the horocycles

Let us study the stable vectors all of whose dF-iterates are also stable vectors. We remark, that dF^{-1}-iterates of stable vectors are stable.

Let $(1,f)\in T_{(x,y)}$, $f>0$, $y\neq 0$. This vector lies in the first quadrant of $T_{(x,y)}$. The vector $dF(1,f)$ lies in the same quadrant if $y^2>f>\frac{y^2}{y^2+1}$. Similar conditions can be found by higher order dF-iterates.

PROPOSITION 1. *The condition-sets form a nested sequence of intervals.*

PROOF. Let f_n be the tangent of the vector $dF^n(1,f)$ for $n\geqq 0$. Hence

$$y_n^2 > f_n > \frac{y_n^2}{y_n^2+1}.$$

Suppose that we can transform this condition to $\tilde{b}>f_{i+1}>\tilde{a}$. Then we have

$$\frac{1}{\frac{1}{y_i^2}+\frac{1}{1+\tilde{b}}} > f_i > \frac{1}{\frac{1}{y_i^2}+\frac{1}{1+\tilde{a}}}.$$

DEFINITION. Let the n-th condition-interval be $\big(a_n(x,y), b_n(x,y)\big)$, for example, $a_0(x,y)=\frac{y^2}{y^2+1}$, $b_0(x,y)=y^2$. Consequently,

$$a_n(x,y) = \frac{1}{\frac{1}{y^2}+\frac{1}{1+a_{n-1}(x_1,y_1)}}, \quad b_n(x,y) = \frac{1}{\frac{1}{y^2}+\frac{1}{1+b_{n-1}(x_1,y_1)}}.$$

We remark that, if $y_n=0$, then $a_n=b_n$. Define $a_m=b_m=a_n=b_n$ for $m>n$ in this case. Thus the functions $a_n\colon \mathbf{R}^2\to\mathbf{R}$ and $b_n\colon \mathbf{R}^2\to\mathbf{R}$ are continuous on the whole $\mathbf{R}^2$. Let us denote $\lim a_n=a\leqq b=\lim b_n$.

LEMMA 2. $a=b$.

PROOF. We can assume that $y_n\neq 0$, $n=0,1,2,\ldots$. Suppose $0<a<b$. Let $\mathbf{s}_1=(\alpha,\gamma)$ and $\mathbf{s}_2=(\beta,\delta)$ lie in the first quadrant of $T_{(x,y)}$ and suppose that

$$a<\frac{\gamma}{\alpha}<\frac{\beta}{\delta}<b$$

and that

$$(\alpha-\beta,\gamma-\delta)=(\xi,\eta)=\mathbf{u}$$

lies in the second quadrant (see Fig. 1).

One immediately checks that the dF-iterates of $\mathbf{s}_1$ and $\mathbf{s}_2$ are stable vectors, the dF-iterates of $\mathbf{u}$ are unstable vectors.

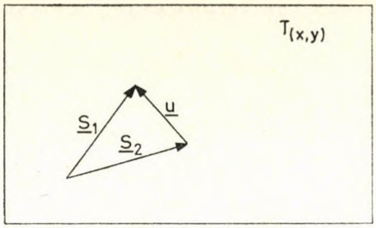

Fig. 1

The following equation is straightforward:

$$\begin{bmatrix} 1+\frac{1}{y_n^2} & \frac{1}{y_n^2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{n+1} \\ \gamma_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha_{n+1}+\frac{1}{y_n^2}(\alpha_{n+1}+\gamma_{n+1}) \\ \alpha_{n+1}+\gamma_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha_n \\ \gamma_n \end{bmatrix}.$$

There are analogous equations for (β, δ) and (ξ, η). Hence

$$0 \leqq \alpha_{n+1} \leqq \alpha_n, \quad 0 \leqq \beta_{n+1} \leqq \beta_n, \quad \xi_{n+1} \leqq \xi_n,$$

$$0 \leqq \gamma_{n+1} \leqq \gamma_n, \quad 0 \leqq \delta_{n+1} \leqq \delta_n.$$

Consequently, the following convergences hold:

$$\lim \alpha_n = \hat{\alpha}, \quad \lim \gamma_n = \hat{\gamma}, \quad \lim \beta_n = \hat{\beta}, \quad \lim \delta_n = \hat{\delta}, \quad \lim \xi_n = \hat{\xi}.$$

Furthermore, $\hat{\alpha}=\hat{\beta}=0$, because $\alpha_{n+1}=\gamma_n-\gamma_{n+1}$ and $\beta_{n+1}=\delta_n-\delta_{n+1}$. Hence $\hat{\xi}=\hat{\alpha}-\hat{\beta}=0$. But $\ldots\leqq\xi_{n+1}\leqq\xi_n\leqq\ldots\leqq\xi<0$, which gives a contradiction.

Denote $f=\lim a_n$.

THEOREM A. *f: $\mathbf{R}^2\to\mathbf{R}$ is continuous.*

PROOF. Let $(x, y)\in\mathbf{R}^2$, $\varepsilon>0$. Choose n such that $b_n(x, y)-a_n(x, y)<\varepsilon$, and $\delta>0$ such that if $|x-\bar{x}|+|y-\bar{y}|<\delta$, then $|a_n(x, y)-a_n(\bar{x}, \bar{y})|<\varepsilon$ and $|b_n(x, y)-b_n(\bar{x}, \bar{y})|<\varepsilon$. We have

$$a_n(x, y)-\varepsilon < a_n(\bar{x}, \bar{y}) \leqq f(\bar{x}, \bar{y}) \leqq b_n(\bar{x}, \bar{y}) < b_n(x, y)+\varepsilon$$

and

$$a_n(x, y) \leqq f(x, y) \leqq b_n(x, y).$$

Hence $|f(x, y)-f(\bar{x}, \bar{y})|<3\varepsilon$.

REMARK. (i) There is an equation for f:

$$f(x, y) = \frac{1}{\frac{1}{y^2}+\frac{1}{1+f(x_1, y_1)}},$$

(ii) $\frac{y^2}{y^2+1} < f(x, y) < y^2.$

In the following we describe the analogous properties for the unstable vectors. The proofs are similar to the preceding ones and hence are omitted.

We study the unstable vectors, whose dF^{-1}-iterates are also unstable.

Let $(1, g)\in T_{(x,y)}$, $x\neq -y$, $g<0$. This vector lies in the fourth quadrant of $T_{(x,y)}$. The vector $dF^{-1}(1, g)$ lies in the same quadrant if

$$-(x+y)^2-1 < g < -1.$$

The higher order dF^{-1}-iterates give us similar conditions for g.

PROPOSITION 3. *The condition sets form a nested sequence of intervals.*

Let the n-th condition interval be $\big(c_n(x, y), d_n(x, y)\big)$. Then

$$c_n(x, y) = -1+\frac{1}{-\frac{1}{y_{-1}^2}+\frac{1}{c_{n-1}(x_{-1}, y_{-1})}}, \quad d_n(x, y) = -1+\frac{1}{-\frac{1}{y_{-1}^2}+\frac{1}{d_{n-1}(x_{-1}, y_{-1})}}.$$

The functions c_n and d_n can be continuously extended to the whole plane. Furthermore, $\lim c_n = c \leqq d = \lim d_n$ holds.

LEMMA 4. $c=d$.

Let $g=\lim c_n$.

THEOREM B. *g is continuous and*

$$g(x, y) = -1+\frac{1}{-\frac{1}{y_{-1}^2}+\frac{1}{g(x_{-1}, y_{-1})}}.$$

REMARK. $-1-y_{-1}^2<g(x, y)<-1$.

§ 3. Description of the horocycles

Consider the differential equations

(1) $$\dot v(t) = f\big(t, v(t)\big) \qquad v(\tau) = \xi$$

and

(2) $$\dot u(t) = g\big(t, u(t)\big) \qquad u(\tau) = \xi.$$

These equations have solutions for all $(\tau, \xi)\in \mathbf{R}^2$, because f and g are continuous.

PROPOSITION 5. (i) *If v is a solution of* (1), *then* $v\equiv 0$ *or* $v(t)\neq 0$ *for all* t. (ii) *If u is a solution of* (2), *then* $u(t)\equiv -t$ *or* $u(t)\neq -t$ *for all* t.

PROOF. (i) Suppose $v(t_0)=0$. Then $F\circ(\mathrm{id}, v)$ is not a stable curve. (ii) is handled similarly.

Consider the baker transformation on the square $Q=(-1, 1)\times(-1, 1)$. On Q we can define the (u, v) coordinate system.

COROLLARY 6. *The line* $v=v_0$ *on* Q *defines on the plane a global solution of* (1). *Similarly, the line* $u=u_0$ *defines on the plane a global solution of* (2).

PROOF. On each point of the line $v=v_0$ there lies the image of a local solution of (1). These images wholly belong to the line $v=v_0$. The other case is similar.
In addition, it means also that (1) and (2) cannot have two local solutions.

COROLLARY 7. *The solutions of* (1) *and* (2) *are unique.*

DEFINITION. The global solutions of (1) are called contracting horocycles and the global solutions of (2) are called expanding horocycles.

§ 4. Hyperbolicity

For the remainder we give the rate of expansion of the horocycles. For this aim it is necessary to approximate F when y is large. But first of all we answer the question of why the horocycles are called contracting and expanding:

PROPOSITION 8. *Let* $\mathbf{s}\in T_{(x,y)}$ *be a tangent vector of a contracting horocycle. Respectively,* $\mathbf{u}\in T_{(x,y)}$ *is a tangent vector of an expanding horocycle. Then*

$$\|dF(\mathbf{s})\| = \frac{f(x,y)}{1+f(x,y)}\|\mathbf{s}\|$$

and

$$\|dF^{-1}(\mathbf{u})\| = \frac{\dfrac{1+g(x,y)}{(x+y)^2}-g(x,y)}{1-g(x,y)}\|\mathbf{u}\|.$$

Let $h=-1-g$. *Then*

$$\|dF^{-1}(\mathbf{u})\| < \frac{1+h(x,y)}{2+h(x,y)}\|\mathbf{u}\|.$$

The types of F-iterates of a point are represented in Fig. 2.
It is clear from the conjugacy to the baker transformation that all of F-iterates cannot remain in the upper (resp. in the lower) half plane (see [1]).

Let $\{z_j=(x_j,y_j)\,|\,j=0,\dots,N+1\}\subset\mathbf{R}^2$ be a set of points for which
(i) z_j lies in the upper half plane for $j=1,\dots,N$.
(ii) $F(z_j)=z_{j+1}$ for $j=0,\dots,N$.
(iii) z_0 and z_{N+1} lie in the lower half plane.
If there exists such z_j for which $0<y_j<3$ then the contraction is less than 9/10, for $f(z_j)<y_j^2$. Suppose $y_j\geqq 3$ for $j=1,\dots,N$. Let $k=\min\{j\,|\,1\leqq j\leqq N,\ x_j>0\}$, denote $(x,y)=(x_k,y_k)$.

LEMMA 9. *Consider the solutions p and q of the following initial value problems:*
(i) $p'(t)=-tp(t)$, $p(x)=y$.
(ii) $q'(t)=-tq(t)-2(t^2+t+1)$, $q(x)=y$.
Then for $N\geqq j\geqq k$ $p(x_j)>y_j>q(x_j)$.

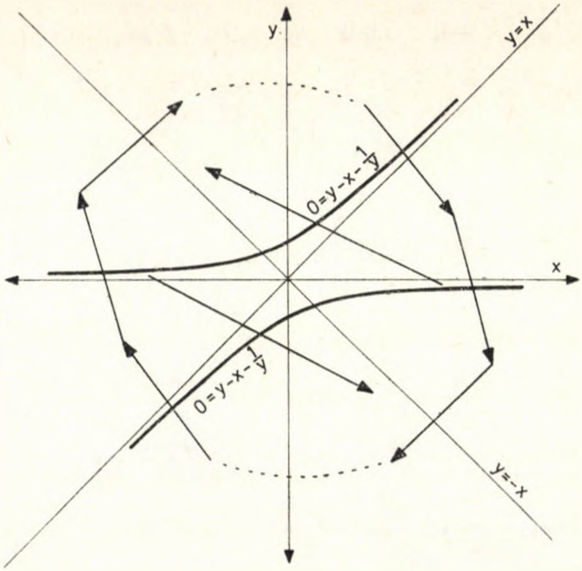

Fig. 2

PROOF. We prove only $p(x_j)>y_j$. The other case is similar. Let l consist of straight lines connecting z_j to z_{j+1} for $j=k, k+1, \ldots, N-1$. Compute the first tangent m of l:

$$m=\frac{y-x-\dfrac{1}{y}-y}{x+\dfrac{1}{y}-x}=-xy-1.$$

Hence there exists such $\delta>0$ that for $x<t<\delta$ the curve p lies above l. Suppose that p intersects l. Denote the first point of intersection by $(x_j+\lambda, v)=P$, where $0\leqq\lambda<1/y_j$. We have $v=y_j-\lambda(x_jy_j+1)$. Hence $p'(x_j+\lambda)=-(x_j+\lambda)\big(y_j-\lambda(x_jy_j+{}$ ${}+1)\big)>-x_jy_j-1$, which contradicts the assumption that P is the first point of intersection.

PROPOSITION 10. $(y+2)e^{-t^2/2}>p(t)>q(t)>(y+2)e^{-t^2/2}-2t-2$ *for* $t>0$.

The contraction of a tangent vector of a contracting horocycle on the upper half plane is

$$\varrho_1=\frac{f(x_1, y_1)}{1+f(x_1, y_1)}\cdot\ldots\cdot\frac{f(x_N, y_N)}{1+f(x_N, y_N)}<\frac{f(x_N, y_N)}{N+f(x_N, y_N)},$$

because $f(z_i)<1+f(z_{i+1})$. Now $x_N<\sqrt{2\log\dfrac{y+2}{2}}$, for $2<y_N<(y+2)e^{-\frac{x_N^2}{2}}$, further

$y_N < x_N + \frac{1}{y_N} < \sqrt{2\log\frac{y+2}{2}} + 1$. Thus we have $f(x_N, y_N) < \left(\sqrt{2\log\frac{y+2}{2}}+1\right)^2$.
For N we have

$$N = \sum_{j=1}^{N} 1 \geqq \sum_{j=k}^{N} y_j(x_j - x_{j-1}) > \int_0^{\sqrt{2\log\frac{y+2}{2}}} (y+2)e^{-t^2/2} - 2t - 2\,dt >$$

$$> 1{,}03(y+2) - \left(\sqrt{2\log\frac{y+2}{2}}+1\right)^2 + 1.$$

Hence

$$\varrho_1 < \frac{\left(\sqrt{2\log\frac{y+2}{2}}+1\right)^2}{1{,}03(y+2)+1}. \tag{3}$$

Similar estimates can be found for the expansion of expanding horocycles. Let $\{w_j = (x_j, y_j) | j = 0, \ldots, M+1\} \subset \mathbf{R}^2$ be a set of points for which

(i) $x_j + y_j > 0$ for $j = 1, \ldots, M$.

(ii) $F^{-1}(w_j) = w_{j-1}$, $j = 1, \ldots, M+1$.

(iii) $x_0 + y_0 < 0$ and $x_{M+1} + y_{M+1} < 0$.

Again we can suppose $x_j + y_j \geqq 3$. Let $k = \max\{j | 1 \leqq j \leqq M, x_j \leqq 0\}$, denote $(x, y) = (x_k, y_k)$. Then we have $-1/3 < -1/y \leqq x \leqq 0$.

LEMMA 11. *Consider the solutions of the following initial value problems:*

(i) $r'(t) = -tr(t) - t^2 + t - 1$, $r(x) = y$.

(ii) $s'(t) = -ts(t)$, $s(x) = y$.

Then for $1 \leqq j \leqq k$ $r(x_j) > y_j > s(x_j)$.

The proof of Lemma 11 is similar to that of Lemma 9.

To characterize the expansion define ϱ_{-1} as follows:

$$\varrho_{-1} = \frac{1+h(w_1)}{2+h(w_1)} \cdot \ldots \cdot \frac{1+h(w_M)}{2+h(w_M)}.$$

Then we have

$$\varrho_{-1} < \frac{1+h(w_1)}{M+1+h(w_1)},$$

for $h(x_i, y_i) < 1 + h(x_{i-1}, y_{i-1})$.

PROPOSITION 12.

$$ye^{-(t^2/2)} < s(t) < r(t) < (y+3)e^{-(t^2/2)} - t + 1 \quad \textit{for} \quad t < 0.$$

Now $h(w_1) < (x_1 + y_1)^2$. We have $x_1 > -\sqrt{2\log\frac{y+3}{2}}$, for

$$3 < x_1 + y_1 < (y+3)e^{-x_1^2/2} + 1.$$

Hence

$$x_1 + y_1 < -x_1 + \frac{1}{x_1+y_1} < \sqrt{2\log\frac{y+3}{2}} + \frac{1}{3}.$$

For M we have

$$M > \sum_{j=1}^{k} 1 = \sum_{j=1}^{k} (x_{j+1}+y_{j+1})(x_{j+1}-x_j) >$$

$$\int_0^{x_i} ye^{-t^2/2}+t\,dt > \int_0^{1,29} ye^{-t^2/2}-t\,dt > y-0,83.$$

Thus we have

(4) $$\varrho_{-1} < \frac{1+\left(\sqrt{2\log\frac{y+3}{2}}+\frac{1}{3}\right)^2}{y+0,17+\left(\sqrt{2\log\frac{y+3}{2}}+\frac{1}{3}\right)^2}.$$

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LATTICE-THEORETICAL CHARACTERIZATIONS OF INNER PRODUCT SPACES

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1. Introduction

1.1. *Motivation.* The importance of lattice-theoretical characterizations of Hilbert spaces is well-known, this interest coming not only from pure mathematics but also from quantum theory. For this reason there are plenty of results characterizing Hilbert spaces; see e.g. [4]—[6], [11]—[12] and the references therein.

On the other hand, also from the physical point of view there has been some interest to use indefinite inner product spaces instead of Hilbert spaces; see the introductions of [2]—[3] and their references.

This being said, it is a bit surprising that there are so few lattice-theoretical characterizations of "classical" indefinite inner product spaces; see [10]. The purpose of this paper is to try to fill this gap at least in case the scalars are the reals, complexes or real quaternions.

1.2. *Contents.* Because the terminology used in the theory of indefinite inner product spaces differs from that used in the lattice-theoretical approach we briefly introduce the necessary notions in Section 2.

In Section 3 we characterize various types of indefinite inner product spaces among normed spaces with the aid of a linear orthogonality relation. As corollaries we achieve also characterizations of (pre-)Hilbert spaces, e.g. the classical result of S. Kakutani and G. W. Mackey.

In Section 4 the basic space is supposed to be a decomposable (indefinite) inner product space. The results characterize Krein, Pontrjagin and Hilbert spaces under this assumption.

2. Preliminaries

2.1. *Basic assumptions and notation.* Throughout this paper the division ring $\mathbf{K}$ is the reals $\mathbf{R}$, the complexes $\mathbf{C}$ or the (real) quaternions $\mathbf{H}$, and * stands for the usual involution of $\mathbf{K}$, i.e. it is the identity of $\mathbf{R}$, the complex conjugation of $\mathbf{C}$ or the canonical conjugation of $\mathbf{H}$, resp. The norm on $\mathbf{K}$ is the mapping $|\cdot|: \mathbf{K}\to\mathbf{R}$, $a\to(a^*a)^{1/2}$.

The symbol E denotes always an infinite-dimensional (left) vector space over $\mathbf{K}$. Furthermore, L is the lattice of all subspaces of E, and E^* is the (algebraic) dual of E.

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2.2. *Linear orthogonality spaces.* A pair $(E, \perp)$, or simply E, is called a *linear orthogonality space* if $\perp$ is a binary relation on E such that

(i) $x \perp y$ iff $y \perp x$,

(ii) $\{x\}^\perp := \{y \in E \mid x \perp y\} \in L$ for all $x \in E$,

(iii) $x \perp y$ for all $y \in E$ implies $x = 0$.

For the rest of this subsection, let E be a linear orthogonality space. Basic properties of these spaces have been represented in [7] and [9]. Here we give only the necessary definitions.

For a subspace F of E we define $F^\perp := \{y \in E \mid y \perp x \text{ for all } x \in F\}$ and call it the *orthogonal* of F. If $F \cap F^\perp = \{0\}$, then F is said to be *non-degenerate* or *semisimple.* The set of all these is denoted by L_n. A subspace F is *definite* in case $x \in F$ and $x \perp x$ imply $x = 0$.

A subspace F of E is said to be *orthoclosed* if $F = F^{\perp\perp}$. The lattice of all orthoclosed subspaces is denoted by $L_{\perp\perp}$; the lattice operations are $F \wedge G := F \cap G$ and $F \vee G := (F+G)^{\perp\perp}$.

A *splitting* or *orthocomplemented* subspace F has the property $E = F + F^\perp$; the set of all these is L_s. The space E is said to be *Hilbertian* if $L_{\perp\perp} = L_s$.

A functional $f \in E^*$ with kernel $\ker(f) \in L_{\perp\perp}$ is called *orthocontinuous,* and they all form the *orthodual* $E^*_{\perp\perp}$ of E.

The closure operator ${}^{\perp\perp}$ has the *Mackey property* if $F \in L_{\perp\perp}$ and $x \in E$ imply $F + \langle x \rangle \in L_{\perp\perp}$; here $\langle \cdot\cdot \rangle$ means the subspace spanned by $\{\cdot\cdot\}$.

2.3. *Inner product spaces.* Let ν be an antiautomorphism of $\mathbf{K}$ and $(\cdot|\cdot)\colon E \times E \to \mathbf{K}$ a ν-sesquilinear form. The pair $(E, (\cdot|\cdot))$, or simply E, is called a *quadratic space* if $(\cdot|\cdot)$ is non-degenerate and such that $(x|y) = 0$ if and only if $(y|x) = 0$. For the theory of quadratic spaces, see [4] and [7]. By defining $x \perp y$ iff $(x|y) = 0$ we see that a quadratic space is a linear orthogonality space. Thus we have all the notions of the previous subsection at hand.

The form $(\cdot|\cdot)$ of a quadratic space is always *ε-hermitean* for some $\varepsilon \in \mathbf{K}$, i.e., $(y|x) = \varepsilon \cdot \nu(x|y)$ for all $x, y \in E$; see [4], Theorem I.1. If $\varepsilon = 1$ and $\nu = {}^*$, we call the form an *inner product* and the space an *inner product space* or a G-*space*; cf [3], where (in case $\mathbf{K} = \mathbf{C}$) these spaces are called non-degenerate G-spaces. For their importance and general theory, see [2] and [3].

Let E be a normed space. Denote by E^*_c the norm-dual of E and by L_c the lattice of all closed subspaces of E with the lattice operations $F \wedge G := F \cap G$ and $F \vee G := \overline{F+G}$. If in addition E is a G-space, the norm is said to be a *partial majorant* of the inner product in case every functional of the form $(\cdot|y)$ is continuous. A partial majorant is *admissible* if every $f \in E^*_c$ is of the form $(\cdot|y)$ for some $y \in E$. The norm is a *majorant* of the inner product if $(\cdot|\cdot)$ is jointly continuous.

A G-space is called a (B, G)-*space* if there is a Banach majorant of the inner product.

2.4. *Decomposable inner product spaces.* Let E be an inner product space. It is said to be *decomposable* if it can be represented in the form

$$E = E_+ \oplus E_-, \tag{2.1}$$

where $E_+ (E_-)$ is a positive definite (negative definite) subspace; here the symbol $\oplus$ denotes a direct and orthogonal sum. The *fundamental decomposition* (2.1) induces a positive definite *J-inner product* $(\cdot|\cdot)_J$:

$$(x|x)_J := (x_+|x_+)-(x_-|x_-), \tag{2.2}$$

where $x=x_++x_-$ with $x_\pm \in E_\pm$. Thus the space E with this J-inner product is a pre-Hilbert space and the corresponding *J-norm* is obviously a majorant of the inner product $(\cdot|\cdot)$.

If a decomposable inner product space E is complete with respect to the J-inner product (2.2), it is called a *J-space* or a *Krein space*. If in addition $k:=\min\{\dim E_+, \dim E_-\}$ is finite, the space is a *Pontrjagin space (with the rank of indefiniteness k)*. A *pre-Pontrjagin space* is a decomposable space with $\min\{\dim E_+, \dim E_-\}$ finite.

3. Inner product spaces among normed spaces

3.1. *Characterization of G-spaces.* In addition to the basic assumptions made in 2.1 we suppose in this section that E is normed. Our aim is to characterize various inner product spaces among normed spaces. We start with necessary conditions.

PROPOSITION 3.1. *Let the normed space E be also a G-space with inner product $(\cdot|\cdot)$, and define $x\perp y$ iff $(x|y)=0$. If the norm is an admissible partial majorant then $(E, \perp)$ is a linear orthogonality space such that*
(i) $L_{\perp\perp}=L_c$,
(ii) *there exists a 2-dimensional definite subspace.*

PROOF. Clearly $(E, \perp)$ is a linear orthogonality space. By the Fréchet—Riesz representation theorem, see [8], Theorem 3.1, every functional f in $E^*_{\perp\perp}$ is of the form $(\cdot|y)$ and conversely. By the assumption the functionals $(\cdot|y)$, $y\in E$, are precisely the elements of E^*_c. Thus $E^*_{\perp\perp}=E^*_c$.

Let F be an arbitrary closed subspace of E. We claim that it is orthoclosed. Otherwise there is an element x in $F^{\perp\perp}$ but not in F. Using the Hahn—Banach—Bohnenblust—Sobczyk—Soukhomlinoff theorem one finds a functional $f\in E^*_c$ with $F\subset\ker(f)$, $x\notin\ker(f)$. But as $E^*_c=E^*_{\perp\perp}$, we have $x\in F^{\perp\perp}\subset\ker(f)^{\perp\perp}=\ker(f)$, which is a contradiction. Thus $L_c\subset L_{\perp\perp}$. To prove the opposite inclusion it is enough to show that $F^\perp\in L_c$ for all subspaces F. This can be done in the same way as above by using [8], Corollary 3.4.

To prove (ii) note first that our spaces are assumed to be infinite-dimensional. So there exists an element x_0 with $(x_0|x_0)\neq 0$; otherwise

$$(x-(x|y)y|x-(x|y)y) = 0$$

for all $x, y\in E$, i.e., $(x|y)=0$ for all $x, y\in E$. Suppose for definiteness that $(x_0|x_0)>0$. If there exists $x_1\in\langle x_0\rangle^\perp$ with $(x_1|x_1)>0$, then $E_0:=\langle x_0, x_1\rangle$ is suitable. Suppose that $(x|x)\leq 0$ for all $x\in\langle x_0\rangle^\perp$. As above we can find $x_1\in\langle x_0\rangle^\perp$ with $(x_1|x_1)\neq 0$ and $x_2\in\langle x_1\rangle^\perp\cap\langle x_0\rangle^\perp$ with $(x_2|x_2)\neq 0$; note that $\langle x_0\rangle^\perp=\langle x_1\rangle^\perp\cap\langle x_0\rangle^\perp+\langle x_1\rangle$. In this case $E_0:=\langle x_1, x_2\rangle$ meets our requirements. □

The following result shows that the conditions of Proposition 3.1 are also sufficient for E to be a G-space.

THEOREM 3.2. *Let the normed space E be also a linear orthogonality space with the relation $\perp$. If the conditions* (i)—(ii) *of Proposition 3.1 hold true, there exists an inner product $(\cdot|\cdot)$ on E such that*

a) *E is a G-space;*

b) *$x \perp y$ iff $(x|y)=0$;*

c) *the norm is an admissible partial majorant of the inner product.*

PROOF. The condition (i) guarantees that the closure operator $F \to F^{\perp\perp}$ has the Mackey property. But a linear orthogonality space with the Mackey property is always a quadratic space; see [7], Theorem 4.8. Thus there exists an automorphism v of $\mathbf{K}$ and a non-degenerate v-sesquilinear form $(\cdot|\cdot)$ with property b).

By the assumption (ii) one can suppose that $(x_0|x_0)=1$ for an element $x_0 \in E$, which implies that v is an involution and the form $(\cdot|\cdot)$ is 1-hermitean; see [4], Theorem I.1. Let us consider the three division rings separately.

1) $\mathbf{K}=\mathbf{R}$. In this case v is obviously the identity and E is a G-space.

2) $\mathbf{K}=\mathbf{C}$. Setting $T(y):=(\cdot|y)$ for all $y \in E$ we obtain an injective, v-linear mapping $T\colon E \to E^*$. By using the property (i) it is not hard to see that the range of T is exactly E_c^*.

We claim that T maps closed hyperplanes of E onto closed hyperplanes of E_c^*. Indeed, if H is a closed hyperplane of E, then it is of the form $\langle x\rangle^\perp$ with $x \neq 0$. Thus the image $T(H)$ of H under T consists of all $f \in E_c^*$ which annihilate the subspace $\langle x\rangle$. As $H=\langle x\rangle^\perp$ is a hyperplane this implies that $T(H)$ is also a hyperplane. Furthermore, by using the definition of the norm of E_c^* the closedness of $T(H)$ is easily established.

A result of S. Kakutani and G. W. Mackey, see [6], Corollary to Lemma 2, guarantees now that the involution v is either the identity of $\mathbf{C}$ or complex conjugation.

Suppose that v is the identity. Let $\{x, y\}$ be a base of the 2-dimensional definite subspace; then

$$0 \neq (x+ay|x+ay) = (y|y)a^2+2(x|y)a+(x|x)$$

for all $a \in \mathbf{C}$, which contradicts the fundamental theorem of algebra. Thus v is complex conjugation and E is a G-space.

3) $\mathbf{K}=\mathbf{H}$. As the assumption (ii) implies that there exist $x \in E$ and $y \in \langle x\rangle^\perp$ with $(x|x), (y|y) \neq 0$, we can follow the reasoning represented in [11], pp. 62—64, which shows that v must be the canonical conjugation of $\mathbf{H}$. Thus E is a G-space.

To prove the property c) note that (i) implies the equality $E_{\perp\perp}^* = E_c^*$. This together with [8], Theorem 3.1, establishes c). □

3.2. *Characterization of Pontrjagin spaces.* It would be desirable to achieve an analogue to Theorem 3.2 for decomposable spaces and especially for Krein spaces, but we have not succeeded in this task. Nevertheless we can characterize (pre-) Pontrjagin spaces among normed spaces.

THEOREM 3.3. *Let the normed space E be also a linear orthogonality space. If*

(i) $L_{\perp\perp}=L_c$,

(ii) *there exists $x_0 \in E$ such that $x_0 \not\perp x_0$,*

(iii) $\max\{\dim F \mid F \subset F^\perp\} =: k < \infty$,

then there exists an inner product $(\cdot|\cdot)$ with the properties

a) *E is a pre-Pontrjagin space with k as the rank of indefiniteness,*
b) $x \perp y$ *iff* $(x|y)=0$,
c) *the norm is an admissible partial majorant.*

PROOF. As in the proof of Theorem 3.2 we find that there exist an involution v of $\mathbf{K}$ and a non-degenerate v-sesquilinear 1-hermitean form $(\cdot|\cdot)$ with property b). Furthermore, one can assume that $(x_0|x_0)=1$.

Let us show first that the involution v is canonical. For this we consider the three cases separately.

1) In case $\mathbf{K}$ is $\mathbf{R}$ everything is clear.

2) In case $\mathbf{K}$ is $\mathbf{C}$ the involution v must be either the identity or complex conjugation, see the proof of Theorem 3.2. Suppose the former. The subspace $\langle x_0\rangle$ is orthocomplemented and a simple calculation shows that there exists $x_1 \in \langle x_0\rangle^\perp$ with $(x_1|x_1)\neq 0$. Define $z_1:=ax_0+x_1$; then $\langle z_1\rangle \subset \langle z_1\rangle^\perp$ for a suitable $a\in\mathbf{C}$. By Lemma 1.5 in [4] the subspace $\langle z_1\rangle$ is contained in a finite-dimensional, orthocomplemented subspace E_1. One can find an element $x_2\in E_1^\perp$ with $(x_2|x_2)\neq 0$ and an element $x_3\in\langle x_2\rangle^\perp \cap E_1^\perp$ with $(x_3|x_3)\neq 0$. Applying the previous reasoning to x_2 and x_3 we can construct an element $z_2\in E_1^\perp$ with $(z_2|z_2)=0$. Thus there exists a subspace $E_2= =\langle z_1, z_2\rangle$ with $E_2\subset E_2^\perp$. As $\dim E=\infty$ this process can be continued infinitely contradicting the assumption (iii). Consequently, v must be complex conjugation.

3) Let $\mathbf{K}$ be $\mathbf{H}$. By [11], p. 62, the involution v is of the form $v(a)=qa^*q^{-1}$ with $q^2=\pm 1$. Suppose that $q^2=-1$. As in [11], pp. 63—64, we can find an element z_1 with $\langle z_1\rangle\subset\langle z_1\rangle^\perp$. Continuing in the same way as in part 2) one can construct subspaces F of arbitrary large dimensions with the property $F\subset F^\perp$, thus contradicting the assumption (iii). This means that $q^2=1$, which implies $q=\pm 1$. Consequently, v is the canonical conjugation.

Property c) is obvious and a) follows from [1], Lemma 2, which holds true also in case $\mathbf{K}=\mathbf{H}$. □

COROLLARY 3.4. *If in addition to the assumptions of Theorem 3.3 the space is complete, then E is a Pontrjagin space.*

PROOF. Combine Theorem 2.2 of [3] and Lemma 3 of [10] with the previous theorem. □

3.3. *Characterization of Hilbert spaces.* With the result of Subsection 3.1 it is easy to characterize (pre-)Hilbert spaces:

THEOREM 3.5. *Let the normed space E be also a linear orthogonality space with the properties*
(i) $L_{\perp\perp}=L_c$,
(ii) $x\perp x$ *implies* $x=0$.
Then there exists an inner product $(\cdot|\cdot)$ on E such that
a) *E is a pre-Hilbert space,*
b) $x\perp y$ *iff* $(x|y)=0$,
c) *the norm is an admissible partial majorant.*

PROOF. By Theorem 3.2 the existence of an inner product $(\cdot|\cdot)$ with the properties b) and c) is clear. Furthermore, one can assume that $(x_0|x_0)=1$ for an element $x_0 \in E$.

The assumption (ii) and the relation b) guarantee that $(\cdot|\cdot)$ is definite. Thus it is enough to prove that $(\cdot|\cdot)$ is positive. Clearly, it is positive on the subspace $\langle x_0 \rangle$. If there exists an element $x \in \langle x_0 \rangle^{\perp}$ with $(x|x)<0$, then

$$0 \neq (x+ax_0 | x+ax_0) = (x|x)+a^2$$

for all $a \in \mathbf{R}$; choose $a := \{-(x|x)\}^{1/2}$ to get a contradiction. Consequently, $\langle x_0 \rangle^{\perp}$ is positive and as $\langle x_0 \rangle$ is orthocomplemented also E is positive. □

COROLLARY 3.6. *Let the assumptions of Theorem 3.5 be fulfilled and, in addition, let E be complete. Then E is a Hilbert space with the natural norm equivalent to the norm of E.*

PROOF. By Theorem 3.5 E is a pre-Hilbert space. To prove the rest, one can proceed as in the proofs of Lemmata 7.7 and 7.8 in [11]. □

3.4. *Characterizations by orthocomplementation.* The classical lattice characterizations of Hilbert space use orthocomplemented lattices. Here we show that Theorem 3.2 allows us to prove several results of this kind. First, a variant of Theorem 3.2.

THEOREM 3.7. *Suppose that there exists a mapping $'$ on the lattice L_c of the normed space E with the properties*
(i) *$F \subset G$ implies $F' \supset G'$,*
(ii) *$F''=F$,*
(iii) *there exists a two-dimensional subspace $E_0 \in L_c$ such that $\langle x \rangle \cap \langle x \rangle' = \{0\}$ for all $x \in E_0$.*
Then there exists an inner product $(\cdot|\cdot)$ on E such that
a) *E is a G-space,*
b) *$F'=F^{\perp}$ for all $F \in L_c$,*
c) *the norm is an admissible partial majorant of the inner product.*

PROOF. Define a relation $\perp$ by setting $x \perp y$ iff $\langle x \rangle \subset \langle y \rangle'$. Then it is easy to see that $(E, \perp)$ is a linear orthogonality space and $\langle x \rangle^{\perp} = \langle x \rangle'$ for all $x \in E$. This implies that $L_{\perp\perp} \subset L_c$. On the other hand, by the assumption (ii)

$$F = F'' = \bigcap_{x \in F'} \langle x \rangle' = F'^{\perp} \in L_{\perp\perp}$$

for all $F \in L_c$. Thus $L_{\perp\perp} = L_c$. Furthermore, the assumption (iii) implies clearly the condition (ii) of Theorem 3.2. Now the result follows immediately from Theorem 3.2. □

COROLLARY 3.8. *If in addition to the assumptions of Theorem 3.7 the space E is complete, then E is a (B, G)-space.*

In case **K** is **R** or **C** this result is a special case of [10], Satz 2.

COROLLARY 3.9. *Suppose that the lattice L_c of the normed space E admits an orthocomplementation $'$. Then E is a pre-Hilbert space with the properties* b) *and* c) *of Theorem 3.7.*

Recall that an *orthocomplementation* $':L_c \to L_c$ is defined by the properties (i)—(ii) of Theorem 3.7 and

(iii)′ $F \cap F' = \{0\}$ for all $F \in L_c$.

COROLLARY 3.10. *If in addition to the assumptions of Corollary 3.9 the space E is complete, then E is a Hilbert space.*

This is the classical result of S. Kakutani and G. W. Mackey; see [6] or [11], Theorem 7.1.

There is also another way to guarantee that the induced inner product space is complete. Recall that a lattice L is called *orthomodular* if it has an orthocomplementation $'$ which satisfies the orthomodular identity

(iv) $G = F \vee (G \wedge F')$ for $F \subset G$.

First, a useful lemma, which extends a result of W. J. Wilbur; see [12], Theorem 4.1 and the comments after it.

LEMMA 3.11. *Let* $(F, \perp)$ *be a linear orthogonality space over an arbitrary division ring. If the lattice* $L_{\perp\perp}$ *is orthomodular and if the closure operator* $G \to G^{\perp\perp}$ *has the Mackey property, then* F *is Hilbertian.*

PROOF. It is enough to show the inclusion $L_{\perp\perp} \subset L_s$. For this, let $G \in L_{\perp\perp}$ and $x \in F$ be arbitrary. The assumptions imply that

$$G + \langle x \rangle = (G + H)^{\perp\perp}$$

with $H := (G + \langle x \rangle) \cap G^{\perp}$. Using this it is easily established that $\dim H \leqq 1$. Consequently, $G + H$ is orthoclosed and thus $x \in G + H \subset G + G^{\perp}$. □

In case **K** is **R** or **C** the following result is included in Theorem 6.6 of [12]:

COROLLARY 3.12. *If the lattice* L_c *of the normed space E is orthomodular, then E is a Hilbert space.*

PROOF. Let $'$ denote the orthocomplementation of L_c in question. By Corollary 3.9 E is a pre-Hilbert space with $L_{\perp\perp} = L_c$. Thus $L_{\perp\perp}$ is orthomodular which, by Lemma 3.11, implies that E must be Hilbertian. But a pre-Hilbert space which is Hilbertian is necessarily complete; see [11], Lemma 7.42. □

4. Results for decomposable spaces

4.1. *Characterizations of Hilbert space.* In addition to the basic assumptions made in 2.1 we suppose in this section that E is a decomposable inner product space with the inner product $(\cdot|\cdot)$ and with the fundamental decomposition (2.1). Furthermore, the corresponding J-inner product $(\cdot|\cdot)_J$ is given by (2.2). All the topological notions are to be understood with respect to the J-norm.

The following result should be compared with Corollary 6 of [5], where the form $(\cdot|\cdot)$ is supposed to be positive definite and $\mathbf{K} \subset \mathbf{R}$. Note that in this section the term "(pre-)Hilbert space" includes also negative definite inner product spaces.

THEOREM 4.1. *For the decomposable inner product space E the following statements are equivalent:*

(i) *E is a Hilbert space,*

(ii) $L_s = L_c$,

(iii) $L_s = L_{\perp\perp}$,

(iv) *the lattice $L_{\perp\perp}$ is orthomodular.*

PROOF. 1) (i)$\Rightarrow$(ii) is clear because of the projection theorem.

2) (ii)$\Rightarrow$(iii): As $L_s \subset L_{\perp\perp}$ always, it is enough to prove $L_{\perp\perp} \subset L_c$. For this in turn we need only the inclusion $E^*_{\perp\perp} \subset E^*_c$, because it implies that $F^\perp \in L_c$ for all $F \in L$. So let $f \in E^*_{\perp\perp}$ be arbitrary. Then it is of the form $(\cdot\,|y)$; see [8], Theorem 3.1. A simple calculation using (2.2) and the Cauchy—Schwarz inequality shows that $|f(x)| \leqq M\|x\|$ for all $x \in E$; here $M := \|y_+ - y_-\|$ with $y = y_+ + y_-$ decomposed according to (2.2). Thus $f \in E^*_c$.

3) (iii)$\Rightarrow$(iv): As L_s is an orthomodular poset, see [9], Theorem 7, the assumption forces $L_{\perp\perp}$ to be an orthomodular lattice.

4) (iv)$\Rightarrow$(iii) is clear because of Lemma 3.11.

5) (iii)$\Rightarrow$(i): As a decomposable space, E has decomposition (2.1). Suppose that both E_+ and E_- are non-zero, and choose $e_\pm \in E_\pm \setminus \{0\}$. The definition $x := e_+ + ae_-$ with $a := \{-(e_-|e_-)^{-1}(e_+|e_+)\}^{1/2}$ gives a non-zero element with the property $\langle x\rangle \subset \subset \langle x\rangle^\perp$. On the other hand, $\langle x\rangle \in L_s$, which implies that $\langle x\rangle \cap \langle x\rangle^\perp = \{0\}$; a contradiction.

Thus E_- or E_+ must be zero; i.e., E is either a positive definite or a negative definite Hilbertian space. Lemma 7.42 in [11] yields now the desired result. □

In case of a definite inner product we have one more necessary and sufficient condition:

COROLLARY 4.2. *Let E be a pre-Hilbert space. Then the statements* (i)—(iv) *are equivalent to*

(v) $L_{\perp\perp} = L_c$.

PROOF. The implication (i)$\Rightarrow$(v) is well-known. For the converse note the following facts: E^*_c is a Banach space with the sup-norm; by (v), it consists of the functionals of the form $(\cdot\,|x)$ with $x \in E$, the norms of E and E^*_c are equivalent. Thus E must be complete. □

4.2. *Characterizations of Krein and Pontrjagin spaces.* If in Corollary 4.2 we consider a decomposable space instead of a pre-Hilbert space, we get a characterization of Krein spaces.

THEOREM 4.3. *The decomposable space E is a Krein space iff $L_{\perp\perp} = L_c$.*

PROOF. 1) Let E be a Krein space. The inclusion $L_{\perp\perp} \subset L_c$ has been shown to be true in part 2) of the proof of Theorem 4.1. As the J-norm topology is admissible, the norm-closure and orthoclosure are equal, see [2], Theorem III.6.1, which implies the inverse inclusion.

2) Suppose $L_{\perp\perp} = L_c$. It is quite easy to prove that $L_{\perp\perp}$ is the same as the lattice consisting of all subspaces which are orthoclosed with respect to the J-inner product (2.2). Thus the space E with the inner product $(\cdot\,|\,\cdot)_J$ satisfies the assumptions of Corollary 4.2. □

Pontrjagin spaces have the property that closed, non-degenerate subspaces are splitting. We prove now that this characterizes Pontrjagin spaces.

THEOREM 4.4. *The decomposable space E is a Pontrjagin space iff* $L_s = L_{\perp\perp} \cap L_n$.

PROOF. 1) Suppose that E is a Pontrjagin space. The inclusion $L_s \subset L_{\perp\perp} \cap L_n$ is always true. The converse inclusion follows from Theorem 4.3 and [2], Theorem IX.2.2.

2) Assume $L_s = L_{\perp\perp} \cap L_n$, and let (2.1) be a decomposition of E. Denote by $L_\cdot^{\pm}$ the set formed with respect to $E_\pm$ corresponding $L_\cdot$. The definiteness of the inner product on $E_\pm$ implies that $L_n^{\pm} = L^{\pm}$, and the assumption implies that $L_{\perp\perp}^{\pm} \subset L_s$. Consequently, $L_{\perp\perp}^{\pm} = L_s^{\pm}$. Thus E_+ and E_- are Hilbert spaces by Theorem 4.1, and E is a Krein space.

To complete the proof it is enough to show that $\dim E_+$ or $\dim E_-$ is finite. According to the assumption every closed definite subspace of E is splitting, which means that they all are uniformly definite, see [2], Theorem V.5.2. This is possible only if the rank of indefiniteness of E is finite, see [2], Theorem V.6.3. (The results of [2] referred to are proved there only for $\mathbf{K} = \mathbf{C}$, but they are obviously provable also for $\mathbf{K} = \mathbf{R}$ and $\mathbf{K} = \mathbf{H}$.) Thus E is a Pontrjagin space. $\square$

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ASYMPTOTIC RESULT CONCERNING EQUATION $x''|x'|^{n-1}+a(t)\overset{*}{x^n}=0$

EXTENSION OF A THEOREM BY ARMELLINI-TONELLI-SANSONE

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1. As it is well-known [1—2] at least one solution of the equation

$$x''+a(t)x=0, \quad a(t)\in C(I), \quad I=[x_0, \infty), \quad x_0\in \mathbf{R} \tag{1}$$

tends to zero as $t\to\infty$ provided $a(t)$ is non-decreasing and $\lim\limits_{t\to\infty} a(t)=\infty$ and every solution behaves so if $\log a(t)$ tends to infinity "regularly" as $t\to\infty$ [3]. The last theorem has extensions in two directions. According to the first one $a(t)$ can be increased by an additive term of bounded variation [4], the second one extends its validity to the nonlinear equation

$$x''+a(t)f(x)=0 \tag{2}$$

under suitable conditions [5].

2. In the present work the theorem will be extended to the ordinary second order half-linear differential equation

$$x''|x|^{n-1}+a(t)\overset{*}{x^n}=0, \quad t\in I, \quad n>0, \quad \overset{*}{x^n}=|x|^n \operatorname{sgn} x \tag{3}$$

which recently was thoroughly investigated by Á. Elbert [6], who has shown (inter alia) that every solution of (3) exists on I provided $a(t)\in C(I)$, and in a forthcoming paper also proved that the Sturmian comparison theorem concerning the magnitudes extends to (3).

First of all let us recall two notions.

a) Density of a sequence S of intervals (α_k, β_k) $k=1, 2, \dots$ having no point in common. If

$$0\leqq \alpha_1<\beta_1<\alpha_2<\beta_2<\dots, \quad \beta_k\to\infty \quad \text{as} \quad k\to\infty,$$

then

$$\varlimsup_{k\to\infty} \frac{\sum\limits_{i=1}^{k}(\beta_i-\alpha_i)}{\beta_k}=\delta(S)=\delta$$

defines the density in question (on R^+) and we put $S=S_\varepsilon$ provided $\delta\leqq\varepsilon$.

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b) The non-decreasing positive function $f(t)\in C(I)$ tends to infinity "intermittently" (or "quasi jumping") as $t\to\infty$ provided to *every* $\varepsilon>0$ *there is* an S_ε such that the increase of $f(t)$ on the complement of S_ε with respect to R^+ is finite, i.e.

$$\mathcal{S} = \sum_{k=1}^{\infty}[f(\alpha_k)-f(\beta_{k-1})] < \infty. \tag{4}$$

In the opposite case we say $f(t)\to\infty$ "regularly" as $t\to\infty$. In this case *there is an* $\varepsilon_0>0$ that on the complement of *every* S_{ε_0} the sum (4) is infinite.

Now we can formulate our result.

THEOREM. *If* $\log a(t)\to\infty$ *regularly as* $t\to\infty$, *then every solution of* (3) *tends to zero as* $t\to\infty$.

PROOF. For the sake of simplicity we suppose $a\in C_1(I)$. Consider the function

$$A(t) = |x|^{n+1}+\frac{|x'|^{n+1}}{a(t)}$$

where $x=x(t)$ is a non-trivial solution of (3). The function $A(t)$ is non-increasing, viz. taking (3) into account we have

$$A'(t) = -\frac{a'}{a^2}|x'|^{n+1}. \tag{5}$$

Consequently, the limit $A=\lim\limits_{t\to\infty} A(t)$ exists and $A\geqq 0$.

Contrary to our assertion suppose that there exists a solution $x(t)$ of (3) not tending to zero as $t\to\infty$. Then concerning to this solution $A>0$ what will lead to a contradiction. By (5) we have

$$\begin{aligned} A(t) = A(0)-\int_0^t \frac{a'}{a^2}|x'|^{n+1}\,dt &= A(0)-\int_0^t \frac{a'}{a}\left(A(\tau)-|x|^{n+1}\right)d\tau = \\ &= A(0)-\int_0^t \left(A(\tau)-|x|^{n+1}\right)\frac{da(\tau)}{a(\tau)}. \end{aligned} \tag{6}$$

Let $\varepsilon_0>0$ be a number such that for every sequence S_{ε_0} of intervals

$$\mathcal{S} = \sum_{i=1}^{k}[\log a(\alpha_{i+1})-\log a(\beta_i)] = \sum_{i=1}^{k}\log\frac{a(\alpha_{i+1})}{a(\beta_i)} \to\infty \tag{7}$$

as $k\to\infty$.

Later it will be proved that to every $\varepsilon_0>0$ a number $\eta>0$ can be chosen so that the density of the sequence S of all intervals where

$$A(t)-|x|^{n+1} \leqq \eta \tag{8}$$

is less than ε_0, i.e. $S=S_{\varepsilon_0}$. On the intervals (β_i, α_{i+1}) we have

$$A(t)-|x|^{n+1} > \eta, \tag{9}$$

therefore

$$A(\alpha_k) \leqq A(0) - \sum_{i=1}^{k-1} \int_{\beta_i}^{\alpha_{i+1}} (A(t)-|x|^{n+1}) \frac{da(t)}{a(t)} < A(0) - \eta \sum_{i=1}^{k} \log \frac{a(\alpha_{i+1})}{a(\beta_i)}$$

which implies by (7) that $A(\alpha_k)$ becomes negative for k large enough. This involves the contradiction $A<0$ and $A>0$ at the same time.

3. It remains the proof of the statement concerning η. Letting $A(t)=(B(t))^{n+1}$ we have

$$B(t)\downarrow, \quad \lim_{t\to\infty} B(t) = B = A^{1/(n+1)} > 0$$

and (8) can be written as follows

$$(B(t))^{n+1} - |x|^{n+1} \leqq \eta. \tag{8'}$$

Since $(B(t))^{n+1}-(B(t))^n|x| \leqq (B(t))^{n+1}-|x|^{n+1}$ holds, therefore (8′) involves

$$(B(t))^{n+1} - (B(t))^n |x| \leqq \eta$$

or

$$B(t) - |x| \leqq \frac{\eta}{B^n(t)} \leqq \frac{\eta}{B^n} = \frac{\eta}{\varrho}$$

where $\varrho = B^n = A^{n/(n+1)} > 0$. Therefore it is sufficient to show the existence of such a number η^* concerning which the density of all sequences S of intervals (α_k, β_k) where

$$B(t) - |x| \leqq \eta^* \tag{10}$$

is less than ε_0. Viz. if $k<n+1 \leqq k+1$, where k is an integer, then

$$(B(t))^{n+1} - |x|^{n+1} \leqq (B(t))^{k+1} - |x|^{k+1} \tag{10'}$$

provided $B(t) \geqq 1$. This can be assumed always, since if $B(t)<1$, then for $cx(t)$ — with a suitable $c=\text{const}>0$ — $cB(t) \geqq 1$. Inequality (10) multiplied by the (bounded) expression

$$(B(t))^k + (B(t))^{k-1}|x| + \ldots + B(t)|x|^{k-1} + |x|^k < K = \text{const}$$

gives

$$(B(t))^{n+1} - |x|^{n+1} \leqq K\eta^* = \bar{\eta}$$

and $\bar{\eta}$ satisfies our requirements.

Inequality (10′) can be proved by taking into consideration that the function

$$f(n) = (B(t))^{n+1} - |x|^{n+1}$$

has the derivative

$$f'(n) = (B(t))^{n+1} \log B(t) - |x|^{n+1} \log |x|$$

which is non-negative for $B(t) \geqq 1$.

By the definition of $B(t)$ and B

$$B + \eta^* > B(t) > B - \eta^*$$

for t large enough — say $t \geqq t_1$. Then for $t \geqq t_1$ satisfying (10) we have

$$B-\eta^*-|x| \leqq B(t)-|x| \leqq \eta^*$$

or

$$B\left(1-\frac{2\eta^*}{B}\right) \leqq |x(t)| \tag{11}$$

where $2\eta^*$ can be taken less than B.

By omitting terms in finite number from a sequence S of intervals we obtain a new sequence S_1 of the same density. Thus it is enough to prove: given $\varepsilon_0>0$ there exists a number $0<\sigma<1$ that the density of the sequence S' of intervals where

$$\sigma B \leqq |x(t)| \tag{12}$$

is less than ε_0. Indeed: conversely: (12) involves (11) with

$$\sigma = 1-\frac{2\eta^*}{B} \quad \text{or} \quad \eta^* = \frac{B}{2}(1-\sigma)$$

and this implies in turn

$$B(t)-\eta^*-|x(t)| \leqq B-|x(t)| \leqq 2\eta^*$$

i.e.

$$B(t)-|x(t)| \leqq 3\eta^* = \bar{\eta}.$$

With the notation $\sigma B=\mu$ relation (12) reads as

$$|x(t)| \geqq \mu. \tag{13}$$

Now we have to apply Sturm's comparison theorem to estimate the density of S' consisting of the intervals (α_i', β_i'), $i=1, 2, \dots$ where — besides (13) — we have

$$|x(\alpha_i')| = |x(\beta_i')| = \mu, \qquad i = 1, 2, \dots. \tag{14}$$

Now consider together with (3) the auxiliary comparison equation

$$y''|y'|^{n-1}+a(\alpha_i')\overset{*}{y^n} = 0 \tag{15}$$

which can be written also in the form

$$(|y'|^{n+1})'+a(\alpha_i')(|y|^{n+1})' = 0$$

and its solution $y(t)=y_i(t)$ with the initial conditions

$$y(\alpha_i') = |x(\alpha_i')|, \quad y'(\alpha_i') = |x'(\alpha_i')|.$$

Denote the first solution of equation $y(t)=\mu$ greater than α_i' by β_i'' and the first place on the left of α_i' where $y(t)=0$ by γ_i'. Then applying Sturm's theorem to equations (3), (15) and $x(t), y(t)$ we have

$$\beta_i'-\alpha_i' \leqq \beta_i''-\alpha_i' \quad \text{and} \quad \beta_i'-\beta_{i-1}' \geqq \alpha_i'-\gamma_i'.$$

Evaluate $\beta_i''-\alpha_i'$ and $\alpha_i'-\gamma_i'$. From the second form of equation (15)

$$|y'|^{n+1}+a(\alpha_i')|y|^{n+1}=K=\text{const}$$

where

$$K=|y'(\alpha_i')|^{n+1}+a(\alpha_i')|y(\alpha_i')|^{n+1}=|x'(\alpha_i')|^{n+1}+a(\alpha_i')|x(\alpha_i')|^{n+1}=a(\alpha_i')A(\alpha_i'),$$

whence

$$y'=\frac{dy}{dt}=[a(\alpha_i')A(\alpha_i')-a(\alpha_i')|y(t)|^{n+1}]^{1/(n+1)}.$$

Thus — being the arc (α_i', β_i'') of the curve of $y(t)$ symmetrical to its middle point $t=t_m$, $y_m=y(t_m)$ —

$$\beta_i''-\alpha_i'=\frac{2}{(a(\alpha_i'))^{1/(n+1)}}\int_{\mu}^{y_m}\frac{dy}{[A(\alpha_i')-y^{n+1}]^{1/(n+1)}}.$$

Carrying out the substitution $\lambda=y^{n+1}$ we have

$$\beta_i''-\alpha_i'=\frac{2}{(n+1)(a(\alpha_i'))^{1/(n+1)}}\int_{\mu^{n+1}}^{y_m^{n+1}}\frac{d\lambda}{\lambda^{n/(n+1)}(A(\alpha_i')-\lambda)^{1/(n+1)}}\leqq$$

$$\leqq\frac{2I}{(n+1)(a(\alpha_i'))^{1/(n+1)}\mu^n}$$

where

$$I=\int_{\mu^{n+1}}^{y_m^{n+1}}\frac{d\lambda}{(A(\alpha_i')-\lambda)^{1/(n+1)}}=\left[-\frac{n+1}{n}(A(\alpha_i')-\lambda)^{n/(n+1)}\right]_{\mu^{n+1}}^{y_m^{n+1}}=$$

$$=\frac{n+1}{n}\left[(A(\alpha_i')-\mu^{n+1})^{n/(n+1)}-(A(\alpha_i')-y_m^{n+1})^{n/(n+1)}\right],$$

but

$$A(\alpha_i')-\mu^{n+1}=A(\alpha_i')-\sigma^{n+1}B^{n+1}=A+v_i-\sigma^{n+1}A=A(1-\sigma^{n+1})+v_i,$$

where $v_i\geqq 0$ decreases and $\lim\limits_{i\to\infty} v_i=0$. Furthermore

$$A(\alpha_i')-y_m^{n+1}=0,$$

since $y'(t_m)=0$ and

$$\tilde{A}(t)=|y(t)|^{n+1}+\frac{|y'(t)|^{n+1}}{a(\alpha_i')}=\text{const}=\tilde{A}(\alpha_i')=A(\alpha_i')=\tilde{A}(t_m)=y_m^{n+1}.$$

Let $v>0$ be sufficiently small. Then there exists $k_0\in\mathbf{Z}$ such that $v_i\leqq v$ for $i\geqq k_0$, and

$$I=\frac{n+1}{n}[A(1-\sigma^{n+1})+v_i]^{n/(n+1)}\leqq\frac{n+1}{n}[A(1-\sigma^{n+1})+v]^{n/(n+1)}$$

and

$$\beta_i''-\alpha_i'\leqq\frac{2[A(1-\sigma^{n+1})+v]^{n/(n+1)}}{n(a(\alpha_i'))^{1/(n+1)}\mu^n}.$$

Similarly,

$$\alpha_i'-\gamma_i'=\frac{1}{(\alpha(\alpha_i'))^{1/(n+1)}}\int_0^\mu\frac{dy}{(A(\alpha_i')-y^{n+1})^{1/(n+1)}}\geqq$$

$$\geqq\frac{1}{(a(\alpha_i'))^{1/(n+1)}}\int_0^\mu\frac{dy}{(A(t_1)-y^{n+1})^{1/(n+1)}}.$$

Consequently, for $i\geqq k_0$

$$\frac{\beta_i'-\alpha_i'}{\beta_i'-\beta_{i-1}'}\leqq\frac{2[A(1-\sigma^{n+1})+v]^{n/(n+1)}}{n\mu^n\int_0^\mu\frac{dy}{(A(t_1)-y^{n+1})^{1/(n+1)}}}=G(\sigma,v),$$

whence for $k>k_0+1$

$$\sum_{i=k_0+1}^{k}(\beta_i'-\alpha_i')\leqq G(\sigma,v)\sum_{i=k_0+1}^{k}(\beta_i'-\beta_{i-1}')\leqq G(\sigma,v)(\beta_K'-\beta_{K_0}')<G(\sigma,v)(\beta_K'-\alpha_{K_0}'),$$

hence

$$\delta_k=\frac{\sum_{i=k_0+1}^{k}(\beta_i'-\alpha_i')}{\beta_K'-\alpha_{K_0}'}\leqq G(\sigma,v).$$

Here $0<\sigma<1$ and by increase of σ, μ increases, too, involving the increase of the denominator of $G(\sigma,v)$, thus: once chosen k_0 so large (v so small) that $G(1,v)<\varepsilon_0/2$, then $G(\sigma,v)$ will be less than ε_0 provided σ is near enough to 1, i.e. with this σ

$$\overline{\lim_{k\to\infty}}\,\delta_k\leqq\varepsilon_0$$

what was to be proved.

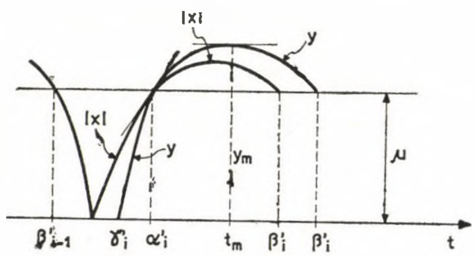

Fig. 1

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THE CHARACTERIZATION OF COMPLEX-VALUED ADDITIVE FUNCTIONS

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We examined the characterization of real-valued additive functions in [1]. We can generalize some results for complex-valued functions. Let f denote a complex-valued additive function and let $r_1, r_2, \dots, r_k$ be fixed positive integers and let us define

$$H_n = \left\{ f(n+r_i) \colon i \in \{1, \dots, k\} \,\middle|\, |f(n+r_i)| \text{ is maximal} \right\}.$$

Let $g(n)$ be an arbitrary but fixed element of H_n. It is possible, that H_n has more elements and so there are a lot of functions with the given property. Let us examine one of these functions, the function g, which is uniquely defined already.

We shall prove the following theorems:

THEOREM 1. *If $|g|$ is monotonically decreasing, then $|g| \equiv c$ with a constant $c \in \mathbf{R}$.*

THEOREM 2. *If $\lim_{n\to\infty} g(n) = c$, then $g \equiv c$ with a constant $c \in \mathbf{C}$.*

PROOF of Theorem 1. First we prove, that $|f(p_i^{\alpha_i})| > 0$ is only for finitely many p_i possible (where p_i denotes always prime numbers, here and later on). Namely, if $|f(p_i^{\alpha_i})| > 0$ $(i = 1, \dots, \infty)$ on the powers of infinitely many primes, then for any $\varepsilon' > 0$ there exists an angular domain with the midpoint origo and with the angle ε' $(\varphi_0 \leqq \arg z \leqq \varphi_0 + \varepsilon')$, which contains infinitely many $f(p_i^{\alpha_i})$ $(i = 1, \dots, \infty)$. If $\varepsilon' < \pi/2$ and $n \to \infty$, then $\left| f\left(\prod_{i=1}^{n} p_i^{\alpha_i} \right) \right|$ is monotonically increasing, which is a contradiction.

If there exists an x_0 with $g(x_0) = 0$ then clearly $g \equiv 0$. If there exists an x_0 with $|g(x_0)| = c \neq 0$, so we have $|f(x_1)| = c$, where $x_1 = x_0 - r_i$ for some i, $1 \leqq i \leqq k$. Since $f(p_j^{\alpha_j}) = 0$ if $p_j > P_0$, we have $|f(x_1 p_j^{\alpha_j})| = c$ for $p_j > \max(P_0, x_1)$. Thus there exists a sequence $b_n \to \infty$ with $|g(b_n)| \geqq c$. Taking into account that $|g|$ is monotonically decreasing, this is only possible, if $|g(n)| = c$ for $n \geqq n_0$. Since x_0 was arbitrary, this implies $g \equiv c$. □

REMARK 1. $|g|$ cannot be strictly monotonically decreasing, in wiew of the above proof.

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REMARK 2. For real-valued functions the assertion of the theorem gives $g\equiv c\in\mathbf{R}$. This is not true for complex-valued functions, as it is shown by the following counter-example.

Let

$$f(2^\alpha)=1$$

$$f(3^\alpha)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$f(p^\alpha)=0 \text{ in other cases.}$$

So if $k>1$ and for any $n\in\mathbf{N}$ there exists an $i\in\{1, \dots, k\}$, such that $2|n+r_i$ or $3|n+r_i$, which can easily be satisfied, then

$$H_n=\left\{f(2^\alpha) \quad \text{or} \quad f(3^\alpha) \quad \text{or} \quad f(2^\alpha 3^\beta)=\frac{1}{2}+\frac{\sqrt{3}}{2}i\right\}.$$

This yields a function g with $|g|=1$, but $g\not\equiv 1$.

PROOF of Theorem 2. (1) If $c=0$, $\lim_{n\to\infty} g(n)=0$ implies $\lim_{p_i^{\alpha_i}\to\infty} f(p_i^{\alpha_i})=0$. If there exists an $a\in\mathbf{N}$ such that $f(a)\neq 0$, then for any sufficiently large p_i

$$|f(ap_i^{\alpha_i})|=|f(a)+f(p_i^{\alpha_i})|\geqq|f(a)|-|f(p_i^{\alpha_i})|\geqq\frac{|f(a)|}{2},$$

which is a contradiction.

(2/i) In the case $c\neq 0$ we first prove that for any $\varepsilon>0$ there exists an i_0 such that if $i\geqq i_0$, then $|f(t_i)|<\varepsilon$ for all sequences $\{t_i\}_0^\infty$, where $(t_j, t_k)=1$ if $j\neq k$. In the opposite case for any $\varepsilon>0$ there exists an angular domain $\varphi_1\leqq\operatorname{arc} z\leqq\varphi_1+\varepsilon$, which contains infinitely many $f(t_{i_j})$ $(i_j\in\mathbf{N}, j=1, \dots, \infty)$. If $\varepsilon<\pi/2$ and $n\to\infty$, then $\left|f\left(\prod_{j=1}^{n} t_{i_j}\right)\right|\to\infty$, which is a contradiction.

(2/ii) We prove $g\equiv c$. Let $x_0\in\mathbf{N}$ be arbitrary. We shall construct such a sequence y_s, that $\lim_{s\to\infty} g(y_s)=g(x_0)$. The construction: Let us write any $s>k$ in the form $s=nk+i$, where $1\leqq i\leqq k$ and let us define

$$y_s:=1+r_k!X_i\prod_{j=1}^{nk} y_j,$$

where

$$X_i:=\frac{\sum_{m=1}^{k}(x_0+r_m)^2}{x_0+r_i} \qquad i=1, \dots, k,$$

further let $y_s=1$ for $s=1, \dots, k$.

We prove, that $(y_\nu, y_\mu)=1$, if $\nu<\mu$. If there exists an m with $\nu\leqq mk<\mu$, this is clear. Thus it is sufficient to prove, that

$$(1+r_k!X_u t, 1+r_k!X_v t)=1$$

for any $u<v$, $u, v\in\{1, \ldots, k\}$ and for any fixed $n\in\mathbf{N}$, where $t=\prod_{j=1}^{nk} y_j$. This is true, since

$$(1+r_k!X_u t,\ 1+r_k!X_v t)=(1+r_k!X_u t,\ r_k!(X_v-X_u)t)=$$

$$=(1+r_k!X_u t,\ X_v-X_u)=\left(1+r_k!X_u t,\ X_u\left(\frac{X_v}{X_u}-1\right)\right)=$$

$$=\left(1+r_k!X_u t,\ X_u\left(\frac{x_0+r_u}{x_0+r_v}-1\right)\right)=\left(1+r_k!X_u t,\ \frac{X_u}{x_0+r_v}(r_u-r_v)\right)=$$

$$=\left(1+r_k!X_u t,\ \frac{X_u}{x_0+r_v}\right)=\left(1,\ \frac{X_u}{x_0+r_v}\right)=1.$$

So to any $\varepsilon>0$ there exists an s_0 such that if $s\geqq s_0$, then $|f(y_s)|<\varepsilon$. Let

$$a_n:=x_0+r_k!\prod_{j=1}^{nk} y_j\prod_{m=1}^{k}(x_0+r_m)^2.$$

Then

$$f(a_n+r_i)=f(x_0+r_i)\left(1+r_k!\prod_{j=1}^{nk} y_j\frac{\prod_{m=1}^{k}(x_0+r_m)^2}{x_0+r_i}\right)=f(x_0+r_i)+f(y_{nk+i})$$

for any $n\in\mathbf{N}$ and $i\in\{1, \ldots, k\}$, since by the definition of y_s, we have clearly $(x_0+r_i,\ y_{nk+i})=1$.

Since $|f(y_{nk+i})|\to 0$, if $n\to\infty$, we obtain $f(a_n+r_i)\to f(x_0+r_i)$, if $n\to\infty$. Hence $g(a_n)\to g(x_0)$, and so $g(x_0)=c$. □

Problem 1. What can we assert if we assume the conditions of Theorems 1 and 2, resp., only on a set having upper density one?

Problem 2. Do Theorems 1 and 2 remain true if we assume the conditions for a suitable chosen, but "arbitrarily rare" set? (More precisely: Can we find to any $h(n)$ a set $\mathcal{A}=\{a_n\}_1^\infty$ with $a_n\in\mathbf{N}$, $a_n>h(n)$, such that assuming the conditions of Theorems 1 and 2 only on $\mathcal{A}$ instead of N, the consequences $|g(n)|\equiv c$ and $g(n)\equiv c$ remain true for all $n\in\mathbf{N}$?)

The answer for Problems 1 and 2 is positive if f is real-valued [2].

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CORRIGENDUM TO MY PAPER „ON THE ESTIMATION OF REGRESSION COEFFICIENT IN CASE OF AN AUTOREGRESSIVE NOISE PROCESS"[1]

I. HORVÁTH GAUDI

In formulas for A_C, B_C, A and B the summation goes from $t=1$. So the right formulas have the following form:

$$A_c = \sum_{t=1} (\cos \omega t - \alpha \cos \omega (t-1) - \beta \cos \omega (t-2))^2$$

$$B_c = \sum_{t=1} (y(t) - \alpha y(t-1) - \beta y(t-2))(\cos \omega t - \alpha \cos \omega (t-1) - \beta \cos \omega (t-2))$$

$$A = \frac{1}{r_{11}} \cos^2 (-\omega) + (\gamma \cos (-\omega) + \delta)^2 + A_c$$

$$B = \frac{1}{r_{11}} y(-1) \cos (-\omega) + [\gamma y(-1) + \delta y(0)][\gamma \cos (-\omega) + \delta] + B_c.$$

So the correct table are as follows:

TABLE I ($T=10$)

N	5	10	15	20	30	40	50	60
$\hat{\sigma}$	10.08	8.34	5.99	3.57	1.83	2.29	1.32	1.33
$\sigma_{M,c}$	2.77	1.96	1.60	1.38	1.13	0.98	0.88	0.80
σ_M	1.13	1.04	0.98	0.92	0.83	0.77	0.71	0.67

TABLE II ($N=40$)

T	40	30	20	18	16	14	12	10	8	5
$\hat{\sigma}$	4.06	2.99	9.34	11.19	12.31	9.83	3.76	2.29	1.32	0.73
$\sigma_{M,c}$	1.78	2.14	4.18	7.11	20.09	4.75	1.93	0.98	0.52	0.18
σ_M	0.88	0.92	1.02	1.05	1.08	1.08	0.99	0.77	0.49	0.18

TABLE III ($N=2T$)

T	60	40	36	32	28	24	20	16	12	10	8	6
$\hat{\sigma}$	1.34	1.64	2.43	2.09	3.66	3.31	9.34	12.47	7.26	3.57	1.22	21.5
$\sigma_{M,c}$	0.93	1.26	1.39	1.58	1.88	2.47	4.18	32.52	2.50	1.38	0.82	0.49
σ_M	0.68	0.79	0.82	0.86	0.90	0.96	1.02	1.08	1.05	0.92	0.71	0.47

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TABLE IV $(N=T)$

T	50	40	30	20	18	16	14	12	10	8	6	5	4	3
$\hat{\sigma}$	2.13	4.06	4.09	12.79	13.16	12.79	11.74	10.17	8.34	6.62	5.43	5.12	5.00	5.01
$\sigma_{M.c}$	1.49	1.78	2.42	5.92	10.63	45.98	8.06	3.53	1.96	1.16	0.69	0.52	0.386	0.290
σ_M	0.83	0.88	0.94	1.04	1.06	1.08	1.10	1.10	1.04	0.90	0.65	0.51	0.385	0.286

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TAUBERIAN THEOREMS FOR POWER SERIES OF TWO VARIABLES

L. ALPÁR

§ 1. Introduction

1.1. *The origin of the problems.* The following result due to Hardy and Littlewood ([4], Theorem 96, p. 155) plays an important part in the proof of certain Tauberian theorems for power series of one variable.

THEOREM HL_1. *If the series*

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu \tag{1.1}$$

with $a_\nu \geqq 0$ *is convergent for* $0 \leqq x < 1$, $f(x) \sim A(1-x)^{-1}$ (A=const.) *as* $x \to 1-0$ *and* $S_n = \sum_{\nu=0}^{n} a_\nu$, *then* $S_n/n \to A$ *as* $n \to \infty$.

With the help of this result the same authors have shown that if the partial sums S_n satisfy certain conditions then the A summability and the $(C, 1)$ summability of the series $\sum a_\nu$ are equivalent ([4], Theorems 92, 93, 94, p. 154).

THEOREM HL_2. *If the series* (1.1) *is convergent for* $0 \leqq x < 1$,

$$\lim_{x \to 1-0} f(x) = A$$

exists and is finite, and one of the conditions

(1.2) (i) $S_n = O(1)$; (ii) $S_n \geqq 0$; (iii) a_ν *is real and* $S_n \geqq -H$ (*or* $S_n \leqq H$)

is fulfilled, where $H > 0$ *is a constant, then*

$$\lim_{n \to \infty} C_n^1 = \lim_{n \to \infty} \frac{1}{n+1} \sum_{\nu=0}^{n} S_\nu = A.$$

Our aim is to generalize these theorems to power series of two variables and also to prove some related results. It will be evident from our reasonings that some of our propositions extend to the case of more than two variables.

REMARK 1. The following statements are well-known. We may suppose that the a_ν's are real in both theorems, otherwise we examine the real and the imaginary parts separately. Furthermore, the restriction $a_\nu \geqq 0$ can be replaced by $a_\nu \geqq -H$ ($H>0$), because the requirements of Theorem HL_1 are satisfied for the power series with

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coefficients $b_\nu = a_\nu + H$ and $c_\nu \equiv H$ and a subtraction yields the assertion. In Theorem HL_2 it suffices to consider the case $S_n \geqq 0$, since there are constants H such that $S_n \geqq 0$ for the power series of $H \pm f(x)$.

1.2. *Notation.* a) The partial sums S_n and their arithmetic means C_n^1 of a simple series are defined in a quite natural way. The situation is completely different in the case of double series. A double series is a network of terms a_{mn} $(m, n = 0, 1, 2, \dots)$ arranged like an infinite matrix whose "sum" is defined as the *unique* accumulation point at finite distance, if there exists, of the set of certain "partial sums" determined by some particular rule. Since this rule can be chosen variously, the limits obtained are not always the same. One such process is to form the rectangular partial sums

$$S_{mn} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} = \sum_{\mu, \nu=0}^{m, n} a_{\mu\nu}. \tag{1.3}$$

The double sequence of the partial sums $\{S_{mn}\}$ is said to be converge to the finite value A *in Pringsheim's sense* [9] as m and n tend to infinity *independently* of each other, if for any $\varepsilon > 0$ we can determine an $m_0 = m_0(\varepsilon)$ and an $n_0 = n_0(\varepsilon)$ such that

$$|A - S_{mn}| \leqq \varepsilon \quad \text{for} \quad m \geqq m_0, \quad n \geqq n_0.$$

In what follows convergence of a double series always means convergence in Pringsheim's sense, and we shall use the notations

$$\sum_{m, n=0}^{\infty} a_{mn}, \qquad \lim_{m, n \to \infty} S_{mn}.$$

b) The series $\sum a_{mn}$ is called $(C, 1, 1)$ summable, or the sequence $\{S_{mn}\}$ $(C, 1, 1)$ limitable, if the sequence of terms

$$C_{mn}^{1,1} = \frac{1}{(m+1)(n+1)} \sum_{\mu, \nu=0}^{m, n} S_{\mu\nu} = \frac{S_{mn}^{1,1}}{(m+1)(n+1)} \tag{1.4}$$

converges as $m, n \to \infty$.

c) One also defines the (C, ξ, η) summability of a double series for $\xi > 0$, $\eta > 0$ (see, e.g. [5], pp. 209—223). We call

(1.5)

$$S_{mn}^{\xi,\eta} = \sum_{\nu=0}^{n} S_{m, n-\nu}^{\xi, \eta-1} = \sum_{\mu=0}^{m} S_{m-\mu, n}^{\xi-1, \eta} = \sum_{\mu, \nu=0}^{m, n} S_{m-\mu, n-\nu}^{\xi-1, \eta-1}; \quad S_{mn}^{0,0} = S_{mn}; \quad S_{mn}^{-1,-1} = a_{mn}$$

the (m, n)th double sum of order (ξ, η). If $a_{00} = 1$ and $a_{mn} = 0$ for $m + n > 0$, i.e. $S_{mn} = 1$ for all m, n, then we write for $S_{mn}^{\xi,\eta}$

$$A_{mn}^{\xi,\eta} = A_m^\xi A_n^\eta = \binom{\xi+m}{m}\binom{\eta+n}{n} \sim m^\xi n^\eta / \Gamma(\xi+1)\Gamma(\eta+1), \quad m, n \to \infty. \tag{1.6}$$

The quotient $S_{mn}^{\xi,\eta} / A_{mn}^{\xi,\eta} = C_{mn}^{\xi,\eta}$ is the (m, n)th Cesàro means of order (ξ, η) of the series $\sum a_{mn}$. If the limit

$$\lim_{m, n \to \infty} C_{mn}^{\xi,\eta} = C^{\xi,\eta} \tag{1.7}$$

exists then the series $\sum a_{mn}$ is (C, ξ, η) summable or the sequence $\{S_{mn}\}$ is (C, ξ, η) limitable. One verifies that

$$S_{mn}^{\xi,\eta} = \sum_{\mu,\nu=0}^{m,n} A_{m-\mu,n-\nu}^{\xi-1,\eta-1} S_{\mu\nu} = \sum_{\mu,\nu=0}^{m,n} A_{m-\mu,n-\nu}^{\xi,\eta} a_{\mu\nu}. \tag{1.8}$$

d) The notation $\lim\limits_{x\to\alpha,\, y\to\beta} f(x, y)$ means that $x\to\alpha$, $y\to\beta$ independently.

We say that the series $\sum a_{mn}$ is A summable at the point $(1, 1)$ if the power series

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n \tag{1.9}$$

converges in the square

$$Q = \{(x, y): 0 \leqq x < 1, 0 \leqq y < 1\} \tag{1.10}$$

and there exists a finite limit

$$\lim_{x\to 1-0,\, y\to 1-0} f(x, y) = A \tag{1.11}$$

as the point $(x, y)\in Q$ approaches the point $(1,1)$ along an arbitrary continuous curve in Q.

e) We also consider some sets whose importance will soon turn out. Denote by L an open Jordan measurable set in the first quadrant of the plan $(x\geqq 0, y\geqq 0)$ with closure $\bar{L}$, boundary ∂L and measure $|L|>0$. This L may be connected or not, the sets $\partial L\cap Ox$ and $\partial L\cap Oy$ may be empty or even of positive linear Jordan measure. We derive from L other sets using two parameters $\varkappa$ and λ tending to infinity independently and two positive numbers ξ and η. If $(x, y)\in\bar{L}$ then

$$\begin{aligned} &\text{(i)} \quad (\varkappa x, \lambda y)\in \bar{L}_{\varkappa\lambda}, \quad \bar{L}_{11} = \bar{L}; \\ &\text{(ii)} \quad (x^{\xi}, y^{\eta})\in \bar{L}^{\xi,\eta}, \quad \bar{L}^{1,1} = \bar{L}; \\ &\text{(iii)} \quad [(\varkappa x)^{\xi}, (\lambda y)^{\eta}]\in \bar{L}_{\varkappa\lambda}^{\xi,\eta}, \quad \bar{L}_{11}^{\xi,\eta} = \bar{L}^{\xi,\eta}, \quad \bar{L}_{\varkappa\lambda}^{1,1} = \bar{L}_{\varkappa\lambda}. \end{aligned} \tag{1.12}$$

Clearly,

$$|L_{\varkappa\lambda}| = \varkappa\lambda|L|; \quad |L_{\varkappa,\lambda}^{\xi,\eta}| = \varkappa^{\xi}\lambda^{\eta}|L^{\xi,\eta}|. \tag{1.13}$$

f) Further we call

$$S_{\varkappa\lambda}(L) = \sum_{(m,n)\in L_{\varkappa\lambda}} a_{mn} \tag{1.14}$$

the $(\varkappa, \lambda)$th L partial sum of the series $\sum\limits_{m,n=0}^{\infty} a_{mn}$, or generally

$$S_{\varkappa\lambda}^{\xi,\eta}(L) = \sum_{(m,n)\in L_{\varkappa\lambda}} S_{mn}^{\xi-1,\eta-1} \tag{1.15}$$

the $(\varkappa, \lambda)$th L sum of order (ξ, η), where $S_{mn}^{\xi-1,\eta-1}$ is defined by (1.5). Also, using (1.6) and (1.13), we write

$$\frac{|L_{\varkappa\lambda}^{\xi,\eta}|}{\Gamma(\xi+1)\Gamma(\eta+1)} = \frac{\varkappa^{\xi}\lambda^{\eta}|L^{\xi,\eta}|}{\Gamma(\xi+1)\Gamma(\eta+1)} = A_{\varkappa\lambda}^{\xi,\eta}(L), \tag{1.16}$$

which is considered as a generalization of the double binomial coefficient $A_{mn}^{\xi,\eta}$.

Next we define

$$C_{\varkappa\lambda}^{\xi,\eta}(L) = \frac{S_{\varkappa\lambda}^{\xi,\eta}(L)}{A_{\varkappa\lambda}^{\xi,\eta}(L)} \tag{1.17}$$

as the $(\varkappa, \lambda)$th L Cesàro means of order (ξ, η). If the limit

$$\lim_{\varkappa, \lambda\to\infty} C_{\varkappa\lambda}^{\xi,\eta}(L) = C^{\xi,\eta}(L) \tag{1.18}$$

exists then the series $\sum a_{mn}$ is (C, ξ, η) L summable to the value $C^{\xi,\eta}(L)$. We shall see that under certain conditions $C^{\xi,\eta}(L) = C^{\xi,\eta}$ given by (1.7), that is, this limit does not depend on L.

1.3. *Results of Knopp and Turán.* Theorem HL_1 was generalized by Turán [12] and Theorem HL_2 by Knopp [8] to power series of two variables. Inspired by Turán's method we are going to prove more general statements under weaker conditions, containing these former results, too.

In Turán's paper $\xi = \eta = 1$, $\varkappa = \lambda$ and $\bar{L}_{\varkappa\lambda}$ is denoted by $\bar{L}_\lambda$ $\big((1.12)\big)$. He proved

THEOREM T. *Let the series*

$$f(x, y) = \sum_{m, n=0}^{\infty} a_{mn} x^m y^n$$

with $a_{mn} \geqq 0$ *be convergent in the square* Q *(see* (1.10)*) and assume that*

$$\lim f(x, y)(1-x)(1-y) = 1$$

as the point $(x, y) \in Q$ *approaches the point* (1,1) *along an arbitrary continuous curve in* Q. *Then*

$$\lim_{\lambda\to\infty} \lambda^{-2} \sum_{(m,n)\in \bar{L}_\lambda} a_{mn} = |L|. \tag{1.19}$$

The theorem of Knopp concerns bounded rectangular partial sums (cf. (1.3)) ([8], Satz 7, p. 586).

THEOREM K. *For bounded sequences* $\{S_{mn}\}$ *the* A *and the* $(C, 1, 1)$ *summabilities are equivalent.*

In other words, if the power series (1.9) converges in Q, if it is A summable at the point (1,1) (cf. (1.11)) and the sequence $\{S_{mn}\}$ is bounded, then $\lim\limits_{m, n\to\infty} C_{mn}^{1,1} = A$ (see (1.4)).

In proving these theorems, Knopp and Turán extended the classical idea of Karamata [6], [7] to power series of two variables. We do the same, but make use of other devices as well. In addition, Turán introduced partial sums of coefficients defined by means of the set $\bar{L}_\lambda$, which is an essential generalization of the usual rectangular partial sums and which led to new results [1], [2].

§ 2. Statement and proof of results

2.1. We begin with our main result which is a common generalization of Theorems HL_1 and T. The proofs of our other theorems are based on this one.

THEOREM 1. *Let the series*

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n \tag{2.1}$$

with $a_{mn} \geqq 0$ *be convergent in the square* Q *and assume that*

$$\lim f(x, y)(1-x)^{\xi}(1-y)^{\eta} = A \tag{2.2}$$

as the point $(x, y) \in Q$ *approaches the point* $(1,1)$ *along an arbitrary continuous curve in* Q, *where* A, ξ, η *are positive constants. Then*

$$\lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{(m,n) \in L_{\varkappa\lambda}} a_{mn} = \frac{A |L^{\xi,\eta}|}{\Gamma(\xi+1)\Gamma(\eta+1)} \tag{2.3}$$

or, by (1.13), (1.14), (1.16),

$$\lim_{\varkappa, \lambda \to \infty} |L^{\xi,\eta}_{\varkappa\lambda}|^{-1} \sum_{(m,n) \in L_{\varkappa\lambda}} a_{mn} = \frac{A}{\Gamma(\xi+1)\Gamma(\eta+1)} \tag{2.4}$$

and

$$\lim_{\varkappa, \lambda \to \infty} \frac{S_{\varkappa\lambda}(L)}{A^{\xi,\eta}_{\varkappa\lambda}(L)} = A. \tag{2.5}$$

PROOF. The fact that $x \to 1-0$, $y \to 1-0$ will be denoted by

$$x \sim e^{-1/r\varkappa}, \quad y \sim e^{-1/s\lambda} \tag{2.6}$$

where r and s are any finite positive parameters *fixed from case to case*, while $\varkappa$ and λ tend to infinity.

Thus we have, in virtue of (2.1), (2.2) and (2.6),

$$\sum_{m,n=0}^{\infty} a_{mn} x^m y^n \sim A/(1-x)^{\xi}(1-y)^{\eta} \sim A \varkappa^{\xi} \lambda^{\eta} r^{\xi} s^{\eta}$$

whence

$$\lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n = A r^{\xi} s^{\eta}. \tag{2.7}$$

Let $p \geqq 0$, $q \geqq 0$ be integers and replace x and y in (2.7) by x^{p+1} and y^{q+1}, respectively. Then we obtain

$$\begin{aligned} \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n x^{mp} y^{nq} &= \frac{A r^{\xi} s^{\eta}}{(p+1)^{\xi}(q+1)^{\eta}} = \\ &= \frac{A r^{\xi} s^{\eta}}{\Gamma(\xi)\Gamma(\eta)} \int_0^{\infty} \int_0^{\infty} e^{-u-v} e^{-pu-qv} u^{\xi-1} v^{\eta-1} \, du \, dv, \end{aligned} \tag{2.8}$$

where we applied the relation

$$\frac{1}{(p+1)^{\xi}} = \frac{1}{\Gamma(\xi)} \int_0^{\infty} e^{-(p+1)u} u^{\xi-1} du.$$

Consequently, if $g(x, y)$ is any polynomial and $(x, y) \in Q$, then we infer from (2.8)

$$\begin{aligned} \mathscr{F}^{r,s}[g] &= \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n g(x^m, y^n) = \\ &= \frac{A r^{\xi} s^{\eta}}{\Gamma(\xi)\Gamma(\eta)} \int_0^{\infty} \int_0^{\infty} e^{-u-v} g(e^{-u}, e^{-v}) u^{\xi-1} v^{\eta-1} du\, dv. \end{aligned} \tag{2.9}$$

We are going to show that $\mathscr{F}^{r,s}$ is a positive linear functional on $\bar{Q}$ in the normed linear space of real polynomials with maximum norm. For, by (2.9), it assigns a real number to any real polynomials; it is obviously positive additive and homogeneous, and, by (2.7), it is bounded:

$$|\mathscr{F}^{r,s}[g]| \leqq \Big(\lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n\Big) \max_{(x,y) \in \bar{Q}} |g(x, y)| = A r^{\xi} s^{\eta} \|g\|. \tag{2.10}$$

The sign of equality holds in (2.10) if $g \equiv 1$, hence

$$\|\mathscr{F}^{r,s}\| = A r^{\xi} s^{\eta}. \tag{2.11}$$

These polynomials form a subspace of the normed linear space of continuous functions in $\bar{Q}$ with maximum norm and so, by the Hahn—Banach theorem, $\mathscr{F}^{r,s}$ is extendable to this entire space without changing the norm. In our case this extension is unique by Weierstrass' approximation theorem. Accordingly, (2.9) remains valid if $g(x, y)$ is any continuous function in $\bar{Q}$.

Furthermore, the representation theorem of F. Riesz [10], [11] states that $\mathscr{F}^{r,s}$ can be extended to the class of functions which are limits (everywhere) of sequences of continuous increasing bounded functions. This larger class contains the function

$$w(x, y) = \begin{cases} 1/xy & \text{if} \quad e^{-1} \leqq x \leqq 1 \quad \text{and} \quad e^{-1} \leqq y \leqq 1 \\ 0 & \text{if} \quad x < e^{-1} \quad \text{or} \quad y < e^{-1}, \end{cases} \tag{2.12}$$

hence we can put $w(x, y)$ in place of $g(x, y)$ in (2.9):

$$\begin{aligned} \mathscr{F}^{r,s}[w] &= \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n w(x^m, y^n) = \\ &= \frac{A r^{\xi} s^{\eta}}{\Gamma(\xi)\Gamma(\eta)} \int_0^1 \int_0^1 u^{\xi-1} v^{\eta-1} du\, dv = \frac{A r^{\xi} s^{\eta}}{\Gamma(\xi+1)\Gamma(\eta+1)} = B r^{\xi} s^{\eta}. \end{aligned} \tag{2.13}$$

In virtue of (2.6) and (2.12), we have to consider the values $x^m \sim e^{-m/r\varkappa} \geqq e^{-1}$, $y^n \sim \sim e^{-n/s\lambda} \geqq e^{-1}$ only, hence $m \leqq r\varkappa$, $n \leqq s\lambda$ and (2.13) takes on the form

$$\lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{m \leqq r\varkappa} \sum_{n \leqq s\lambda} a_{mn} = B r^{\xi} s^{\eta}. \tag{2.14}$$

That is, if L is the rectangle with one vertex in the origin and two sides of lengths r and s on the axes Ox and Oy, respectively, then $r^\xi s^\eta = |L^{\xi,\eta}|$; on the other hand, $m \leqq r\varkappa$, $n \leqq s\lambda$ means that $(m, n) \in L_{\varkappa\lambda}$. Briefly, (2.14) proves (2.3) in this special case. Now if $0 \leqq a < b$, $0 \leqq c < d$ and L is the rectangle with vertices (a, c), (b, c), (b, d), (a, d), then we conclude easily from (2.14)

$$\text{(2.15)} \qquad \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{a\varkappa < m \leqq b\varkappa} \sum_{c\lambda < n \leqq d\lambda} a_{mn} = B(b^\xi - a^\xi)(d^\eta - c^\eta) = B|L^{\xi,\eta}|.$$

This means that Theorem 1 is proved if L is a rectangle with sides parallel to the axes and, consequently, when L is the union of a finite number of pairwise non-overlapping rectangles of this type. Hence if H is such a set, we may write, using (1.13) and (2.15),

$$\text{(2.16)} \qquad \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} \sum_{(m,n) \in H_{\varkappa\lambda}} a_{mn} = \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} S_{\varkappa\lambda}(H) = B|H^{\xi,\eta}|.$$

Finally, if L is a general Jordan measurable set, then for every $\varepsilon_0 > 0$ there exist an $\varepsilon = \varepsilon(\varepsilon_0)$, $\varepsilon \to 0$ as $\varepsilon_0 \to 0$, and two sets H and K which are unions of a finite number of pairwise non-overlapping rectangles of the mentioned kind such that

$$H \subset L \subset K; \quad |L| - \varepsilon_0 \leqq |H| \leqq |K| \leqq |L| + \varepsilon_0;$$

$$H^{\xi,\eta} \subset L^{\xi,\eta} \subset K^{\xi,\eta}; \quad |L^{\xi,\eta}| - \varepsilon \leqq |H^{\xi,\eta}| \leqq |K^{\xi,\eta}| \leqq |L^{\xi,\eta}| + \varepsilon.$$

Thus we have, by (2.16),

$$B(|L^{\xi,\eta}| - \varepsilon) \leqq \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} S_{\varkappa\lambda}(H) \leqq \varliminf_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} S_{\varkappa\lambda}(L) \leqq$$

$$\leqq \varlimsup_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} S_{\varkappa\lambda}(L) \leqq \lim_{\varkappa, \lambda \to \infty} \varkappa^{-\xi} \lambda^{-\eta} S_{\varkappa\lambda}(K) \leqq B(|L^{\xi,\eta}| + \varepsilon)$$

and $\varepsilon \to 0$ completes the proof of (2.3). The formulae (2.4) and (2.5) are simple modifications of (2.3), but we shall see that each of them has its own meaning.

REMARK 2. Turán's original result, see (1.19), is analogous to (2.3). However, if $\varkappa = \lambda$ then (2.6) is of the form $x \sim e^{-1/r\lambda}$, $y \sim e^{-1/s\lambda}$ and it is inevitable to admit that $0 < r$, $s < \infty$. This means that the passage to the limit $(x, y) \to (1,1)$ is carried out along a continuous curve C_{rs} which has a slope of tangent r/s at the point $(1,1)$, it touches there the curve $y = x^{r/s}$. It follows that at $(1,1)$ C_{rs} has a tangent non parallel to the axes. On the other hand, if $\varkappa \neq \lambda$ then $y \sim x^{r\varkappa/s\lambda}$ where again $0 < r$, $s < \infty$, but $\varkappa/\lambda$ may tend to any limit c where $0 \leqq c \leqq \infty$, and it is even possible that $\varkappa/\lambda$ does not have a limit as $\varkappa \to \infty$ and $\lambda \to \infty$ independently. Consequently, at $(1,1)$ C_{rs} can have a tangent parallel to one of the axes or it can happen that it has no tangent at all at this point.

REMARK 3. The relations (2.3) and (2.4) are not completely equivalent. (2.3) has a general meaning in itself already. Provided that $|L^{\xi,\eta}|$ does not change, (2.3) remains unaltered if we deform L anyhow, we may either decompose L in further components or unify as well as remove some of them arbitrarily. In (2.4) we can drop even the invariance of $|L^{\xi,\eta}| > 0$.

If $\xi=\eta=1$ then $A^{1,1}_{\varkappa\lambda}(L)=|L_{\varkappa\lambda}|$ (cf. (1.16), (1.20)) and (2.5) takes on the form

$$\lim_{\varkappa,\lambda\to\infty}\frac{S_{\varkappa\lambda}(L)}{A^{1,1}_{\varkappa\lambda}(L)}=\lim_{\varkappa,\lambda\to\infty}|L_{\varkappa\lambda}|^{-1}\sum_{(m,n)\in L_{\varkappa\lambda}}a_{mn}=A. \tag{2.17}$$

This shows that these L partial sums are uniformly distributed in the plane.

2.2. Bromwich and Hardy ([3], p. 173) proved that if $C^{1,1}_{mn}=O(1)$ (cf. (1.4)) and the sequence $\{C^{1,1}_{mn}\}$ converges then the series $\sum a_{mn}$ is A summable, that is, (1.11) holds. Therefore we do not investigate this problem. On the other hand, Knopp ([8], Satz 7, pp. 586—589) showed that (1.11), the A summability of the series $\sum a_{mn}$ at the point (1,1), implies the $(C, 1,1)$ summability of this series, if the sequence $\{S_{mn}\}$ (cf. (1.3)) is bounded. We are going to generalize this result of Knopp but for L partial sums and $(C, 1,1)$ L Cesàro limits (cf. (1.14)—(1.18)) with S_{mn} satisfying restrictions similar to that of (1.2).

THEOREM 2. *Let the series*

$$f(x, y)=\sum_{m,n=0}^{\infty}a_{mn}x^m y^n \tag{2.18}$$

be absolutely convergent in Q and assume the existence of a finite limit

$$\lim_{x\to1-0,\,y\to1-0}f(x, y)=A \tag{2.19}$$

as the point $(x, y)\in Q$ approaches the point (1, 1) *along an arbitrary continuous curve in Q. Furthermore suppose that one of the conditions below is fulfilled:*

$$\text{(i)}\quad S_{mn}=O(1);\quad \text{(ii)}\quad S_{mn}\geqq -H\quad (\textit{or}\quad S_{mn}\leqq H) \tag{2.20}$$

where $H\geqq0$ is a constant. Then

$$\lim_{\varkappa,\lambda\to\infty}|L_{\varkappa\lambda}|^{-1}\sum_{(m,n)\in L_{\varkappa\lambda}}S_{mn}=\lim_{\varkappa,\lambda\to\infty}\frac{S^{1,1}_{\varkappa\lambda}(L)}{A^{1,1}_{\varkappa\lambda}(L)}=C^{1,1}(L)=A. \tag{2.21}$$

PROOF. In our reasonings we need the assumption that the series (2.18) is absolutely convergent in Q, but this is not insured by (2.19) solely. Even the convergence of $\sum a_{mn}$ does not involve either the absolute or the ordinary convergence of (2.18) in the whole square Q (see [3], examples, p. 166). However, if $a_{mn}=O(1)$ then (2.18) is absolutely convergent in Q. Next we have, by (1.3),

$$a_{mn}=S_{mn}-S_{m-1,n}-S_{m,n-1}+S_{m-1,n-1} \tag{2.22}$$

and so, if $|S_{mn}|\leqq C$ (see (2.20) (i)) then $|a_{mn}|\leqq 4C$, that is, the condition of Knopp implies the absolute convergence of (2.18) in Q, but the converse is not true.

Since $(1-x)(1-y)\sum_{m,n=0}^{\infty}x^m y^n=1$ for $(x, y)\in Q$ and this series converges absolutely, we deduce from (2.22) by comparing the coefficients:

$$\begin{aligned} f(x, y)&=(1-x)(1-y)\Big(\sum_{m,n=0}^{\infty}x^m y^n\Big)\Big(\sum_{m,n=0}^{\infty}a_{mn}x^m y^n\Big)=\\ &=(1-x)(1-y)\sum_{m,n=0}^{\infty}S_{mn}x^m y^n \end{aligned} \tag{2.23}$$

where $\sum S_{mn}x^m y^n$, as a product of two absolutely convergent series, is itself absolutely convergent in Q. Now, if one of the conditions (2.20) is satisfied then there exists a constant $K>0$ such that $T_{mn}=S_{mn}+K\geqq 0$ and $A+K=A_0>0$. Thus we have, by (2.19) and (2.23),

$$\lim (1-x)(1-y) \sum_{m,n=0}^{\infty} T_{mn}x^m y^n = A_0. \tag{2.24}$$

Therefore all conditions of Theorem 1 hold for the series (2.24) with $\xi=\eta=1$. Hence (2.21) is a direct consequence of Theorem 1 with T_{mn} in place of S_{mn} and a subtraction gives the assertion originally stated.

In particular, if $r\varkappa=m+1$, $s\lambda=n+1$ and $\bar{L}_{\varkappa\lambda}$ is the rectangle with one vertex in the origin and two sides of lengths $m+1$ and $n+1$ on the axes Ox and Oy, respectively, then (2.21) is of the form

$$\frac{1}{(m+1)(n+1)} \sum_{\mu,\nu=0}^{m,n} S_{\mu\nu} = C_{mn}^{1,1} \to A \quad (m, n \to \infty).$$

This is the result of Knopp for $S_{mn}=O(1)$.

Remark 4. In spite of the resemblance of (2.17) and (2.21), these formulae have different meanings. For in Theorem 1 $f(x, y)$ is characterized by (2.2) and in Theorem 2 by (2.19).

2.3. After these considerations it seems to be obvious that A and (C, ξ, η) summabilities are connected by a relation like (2.21) if $\xi>1$ and $\eta>1$. We give a direct proof of this claim.

Theorem 3. *If the requirements of Theorem 2 are fulfilled and* $\xi>1, \eta>1$, *then*

$$C^{\xi,\eta}(L) = A. \tag{2.25}$$

Proof. Using (1.6) we have

$$(1-x)^{\xi}(1-y)^{\eta} \sum_{m,n=0}^{\infty} A_{mn}^{\xi-1,\eta-1} x^m y^n = 1$$

for $(x, y)\in Q$ and this identity holds as $(x, y)\to(1,1)$. Moreover, this series is absolutely convergent in Q and the coefficients $A_{mn}^{\xi-1,\eta-1}\geqq 1$ grow to infinity with m, n (see (1.6)). Thus the function represented by this series satisfies the conditions of Theorem 1. We conclude as previously, by (1.8),

$$f(x, y) = (1-x)^{\xi}(1-y)^{\eta}\Big(\sum_{m,n=0}^{\infty} A_{mn}^{\xi-1,\eta-1}x^m y^n\Big)\Big(\sum_{m,n=0}^{\infty} a_{mn}x^m y^n\Big) =$$

$$= (1-x)^{\xi}(1-y)^{\eta} \sum_{m,n=0}^{\infty} S_{mn}^{\xi-1,\eta-1} x^m y^n.$$

Again the last series is absolutely convergent in Q and, by (1.8),

$$S_{mn}^{\xi-1,\eta-1} = \sum_{\mu,\nu=0}^{m,n} A_{m-\mu,n-\nu}^{\xi-2,\eta-2} S_{\mu\nu}$$

where $\xi-2>-1$, $\eta-2>-1$ and so $A^{\xi-2,\eta-2}_{m-\mu,n-\nu}>0$. Hence if $|S_{mn}|\leqq C$ (cf. (2.20) (i)) then

$$|S^{\xi-1,\eta-1}_{mn}|\leqq C\sum_{\mu,\nu=0}^{m,n} A^{\xi-2,\eta-2}_{m-\mu,n-\nu}=CA^{\xi-1,\eta-1}_{mn},$$

while if $S_{mn}\geqq -H$ (cf. (2.22) (ii)) then $S^{\xi-1,\eta-1}_{mn}\geqq -HA^{\xi-1,\eta-1}_{mn}$. That is, in both cases there exists a constant $K>0$ such that $S^{\xi-1,\eta-1}_{mn}+KA^{\xi-1,\eta-1}_{mn}=T_{mn}\geqq 0$ and $A+K=A_0>0$. Thus we find (see (1.15))

$$\lim_{\varkappa,\lambda\to\infty}\sum_{(m,n)\in L}(S^{\xi-1,\eta-1}_{mn}+KA^{\xi-1,\eta-1}_{mn})/A^{\xi,\eta}_{\varkappa\lambda}(L)=$$

$$=\lim_{\varkappa,\lambda\to\infty}\frac{S^{\xi,\eta}_{\varkappa\lambda}(L)}{A^{\xi,\eta}_{\varkappa\lambda}(L)}+K=A_0=A+K$$

and (2.25) is proved.

Remark 5. If $0<\xi<1$, $0<\eta<1$ or $0<\xi<1$, $\eta\geqq 1$ then $A^{\xi-1,\eta-1}_{mn}>0$ but the latter tends to 0 for some values of m and n and $A^{\xi-2,\eta-2}_{m-\mu,n-\nu}$ may be positive or negative, therefore our former argumentation fails in these cases. Hence in Theorem 1 we can replace the condition $a_{mn}\geqq 0$ either by $a_{mn}=O(1)$ or by $a_{mn}\geqq -H$ for $\xi\geqq 1$ and $\eta\geqq 1$. If $0<\xi<1$ or $0<\eta<1$ we need some additional hypotheses to insure the validity of Theorem 1.

Remark 6. Theorems 1, 2 and 3 can be stated for power series of one variable, too. Here L is a Jordan-measurable set of the Ox axis ($x\geqq 0$) and to the point $x\in\bar{L}$ we assign the point $\lambda x\in\bar{L}_\lambda$. Then, if the conditions of Theorem HL_1 are satisfied we have not only the relation $S_n/n\sim S_n/A^1_n\to A$, but also

$$\lim_{\lambda\to\infty}\lambda^{-1}\sum_{\nu\in L_\lambda}a_\nu=A|L|,\quad \lim_{\lambda\to\infty}|L_\lambda|^{-1}\sum_{\nu\in L_\lambda}a_\nu=A.$$

In consequence of the quite natural definition of S_n these last relations have not been recognized. If the conditions of Theorem HL_2 are fulfilled, similar result can be obtained by substituting a_ν by S_ν.

2.4. Theorem 2 enables us to generalize the Cauchy formula. Let the sequence $\{S_\nu\}$ converge to a finite limit A, then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}S_\nu=A.$$

Theorem 4. *If the bounded sequence $\{S_{mn}\}$ converges to a finite limit:*

(2.26) $$\lim_{m,n\to\infty}S_{mn}=A$$

then

(2.27) $$\lim_{\varkappa,\lambda\to\infty}|L_{\varkappa\lambda}|^{-1}\sum_{(m,n)\in L_{\varkappa\lambda}}S_{mn}=A.$$

This last result may be formulated for simple series as well.

Proof. By means of (2.22) we determine the sequence $\{a_{mn}\}$ from $\{S_{mn}\}$. As we have seen in the proof of Theorem 2, the condition $|S_{mn}|\leqq C$ implies $|a_{mn}|\leqq 4C$ and the series (2.18) is absolutely convergent in Q. Thus using the theorem of Bromwich and Hardy ([3], p. 164), the Abel's continuity theorem for two variables, (2.19) follows from (2.26) and so we get (2.23) and (2.27), respectively.

Considering rectangular partial sums only, this result was obtained by Holzberger [5] already. The relation (2.27) is an L Cauchy formula.

2.5. For double sequences we can also prove an L Jensen theorem which is a generalization of Theorem 4. Jensen's theorem on simple series states the following: If the sequence $\{S_\nu\}$ converges to a finite limit A, and there exist a sequence $\{\alpha_\nu\}$ and a constant $\beta>0$ such that $\sum_{\nu=0}^{\infty}|\alpha_\nu|=\infty$ and

$$\Big|\sum_{\nu=0}^{n}\alpha_\nu\Big| \geqq \beta \sum_{\nu=0}^{n}|\alpha_\nu|$$

for all n, then

$$\lim_{n\to\infty}\frac{\sum_{\nu=0}^{n}\alpha_\nu S_\nu}{\sum_{\nu=0}^{n}\alpha_\nu}=A.$$

Theorem 5. *If the bounded sequence $\{S_{mn}\}$ converges to a finite limit A, and there exist a sequence $\{\alpha_{mn}\}$ and a constant $\beta>0$ such that*

$$\lim_{\varkappa,\lambda\to\infty}\sum_{(m,n)\in L_{\varkappa\lambda}}|\alpha_{mn}|=\infty, \tag{2.28}$$

$$|A_{\varkappa\lambda}|=\Big|\sum_{(m,n)\in L_{\varkappa\lambda}}\alpha_{mn}\Big|\geqq\beta\sum_{(m,n)\in L_{\varkappa\lambda}}|\alpha_{mn}|=\beta B_{\varkappa\lambda} \tag{2.29}$$

and

$$\begin{gathered}\lim_{\varkappa,\lambda\to\infty}B_{\varkappa\lambda}^{-1}\sum_{\substack{(m,n)\in L_{\varkappa\lambda}\\ m<m'}}|\alpha_{mn}|=0,\\ \lim_{\varkappa,\lambda\to\infty}B_{\varkappa\lambda}^{-1}\sum_{\substack{(m,n)\in L_{\varkappa\lambda}\\ n<n'}}|\alpha_{mn}|=0,\end{gathered} \tag{2.30}$$

where m' and n' are any natural numbers, then

$$\lim_{\varkappa,\lambda\to\infty}\frac{\sum\alpha_{mn}S_{mn}}{\sum_{(m,n)\in L_{\varkappa\lambda}}\alpha_{mn}}=\lim_{\varkappa,\lambda\to\infty}v_{\varkappa\lambda}=A. \tag{2.31}$$

Proof. Given $\varepsilon>0$, we can find an $m_0=m_0(\varepsilon)$ and an $n_0=n_0(\varepsilon)$ such that

$$|S_{mn}-A|\leqq\frac{\varepsilon}{4}\beta,\quad\text{for}\quad m>m_0,\quad n>n_0.$$

It follows that

$$|v_{\varkappa\lambda}-A|\leqq C_1+C_2+C_3+C_4$$

where

$$C_1 = \left|\frac{1}{A_{\varkappa\lambda}} \sum \alpha_{mn}(S_{mn}-A)\right|,$$

$$(m, n)\in \bar{L}_{\varkappa\lambda}, \quad m \leqq m_0, \quad n \leqq n_0.$$

C_2, C_3, C_4 are analogous expressions with $(m, n)\in \bar{L}_{\varkappa\lambda}$, but $m \leqq m_0$, $n > n_0$ for C_2; $m > m_0$, $n \leqq n_0$ for C_3; and $m > m_0$, $n > n_0$ for C_4. Since $S_{mn}=O(1)$, C_1, C_2, C_3 are less than $\varepsilon/4$ for $\varkappa$, λ large enough, by (2.28), (2.29), (2.30), and $C_4 \leqq \frac{\varepsilon}{4}\beta\frac{B_{\varkappa\lambda}}{|A_{\varkappa\lambda}|} \leqq$ $\leqq \varepsilon/4$. This proves that $|v_{\varkappa\lambda}-A| \leqq \varepsilon$, hence (2.31) is verified.

Holzberger [5] obtained also Jensen's theorem for rectangular partial sums.

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CONVERGENCE IN u-SECOND VARIATION AND RS_u INTEGRALS

A. G. DAS

1. Preliminaries and definitions

A. M. Russell in [7] obtained the definition of functions of bounded u-second variaton (BV_u function) alongwith certain properties of BV_u functions. Russell also obtained the definition of an integral (the RS_u integral) together with some important properties of the integral. The same concept of variaton has been introduced by Webb [8] and Huggins [2], [3] under the title bounded slope variation. Similar concept has also been introduced by Roberts and Varberg [6]. We retain the title of Russell as we pass to the RS_u integral as introduced by him. In [1] A. G. Das and B. K. Lahiri obtained some new results and also certain modifications of some results of [7]. A convergence theorem of RS_u integrals appears in [1] depending on the convergence of integrands. In the present paper the author presents a convergence theorem analogous to Arzelà's dominated convergence theorem. Convergence theorems of RS_u integrals depending on the convergence of integrators are also presented. For this purpose it is desirable to investigate the convergence in u-second variation.

The following definitions are known [7].

Let a', a, b, b' be fixed real numbers such that $a'<a<b<b'$. The real valued functions that occur are defined at least in $[a, b]$, $u(x)$ always being strictly increasing.

Definition 1. For $x, y\in[a', b']$, $x\neq y$

$$g_u(x, y) = \frac{g(x)-g(y)}{u(x)-u(y)}$$

is called the u-incrementary ratio of g.

Definition 2. If $x\in[a, b)$ and $\lim\limits_{h\to 0+} g_u(x, x+h)$ exists, we denote it by $g_u^+(x)$. A corresponding definition holds for $g_u^-(x)$, where $x\in(a, b]$. When $g_u^-(x)=g_u^+(x)$, we say g is u-differentiable at x and denote the common value by $g'_u(x)$.

Condition A. Suppose that $g_u^-(b)$ and $g_u^+(a)$ exist. The functions g, u are defined in $[a', a]$ and $[b, b']$ such that u is strictly increasing on $[a', b']$ and

$$g_u(x, y) = g_u^+(a) \quad \text{for all} \quad x, y\in[a', a]$$

and

$$g_u(x, y) = g_u^-(a) \quad \text{for all} \quad x, y\in[a', a].$$

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Definition 3. The total u-second variation of g on $[a, b]$ is defined by

$$V_u(g;\ a, b) = \sup_{\pi} \sum_{i=1}^{k-1} |g_u(x_{i+1}, x_i) - g_u(x_i, x_{i-1})|,$$

where the supremum is taken over all π: $a=x_0<x_1<\ldots<x_k=b$ subdivision of $[a, b]$. If $V_u(g;\ a, b)<\infty$, we say that g is of bounded u-second variation on $[a, b]$, and we write $g\in BV_u[a, b]$.

Definition 4. Let $\varepsilon>0$ be arbitrarily small. Then $\int_a^b f(x)\frac{d^2g(x)}{du(x)}$ is the number I, if it exists uniquely, and there is a number $\delta(\varepsilon)>0$ such that for all Δ: $a'\leqq x_{-1}<x_0=a<x_1<\ldots<x_k=b<x_{k+1}\leqq b'$ subdivision of $[a, b]$ and for $\xi_i\in[x_{i-1}, x_{i+1}]$, $i=1, 2, \ldots, k-1$, $\xi_0=a$, $\xi_k=b$

$$\left|I - \sum_{i=1}^{k} f(\xi_i)[g_u(x_{i+1}, x_i) - g_u(x_i, x_{i-1})]\right| < \varepsilon$$

whenever

$$\|\Delta\| = \max_{0\leqq i\leqq k+1}(x_i - x_{i-1}) < \delta(\varepsilon).$$

If the integral exists, we write $(f, g)\in RS_u[a, b]$.

Definition 5. A function g is u-convex on $[a, b]$ if for $a\leqq\alpha\leqq\xi\leqq\beta\leqq b$

$$g(\xi) \leqq \frac{u(\xi)-u(\alpha)}{u(\beta)-u(\alpha)}\, g(\beta) + \frac{u(\beta)-u(\xi)}{u(\beta)-u(\alpha)}\, g(\alpha).$$

We shall often, for the sake of simplicity, use the notation $d(g;\ x_{i-1}, x_i, x_{i+1})$ for the expression $[g_u(x_{i+1}, x_i) - g_u(x_i, x_{i-1})]$. For further definitions and notations which are not noted here see [7].

We note some results from [1] for ready references. Lemma 3, however, is not included in [1] whose proof is easy and omitted.

Theorem 1. *If f is continuous and g is u-convex on $[a, b]$ and g, u satisfy condition A, then $(f, g)\in RS_u[a, b]$.*

Theorem 2. *Let g be u-convex on $[a, b]$ and g, u satisfy condition A. If $(f, g)\in RS_u[a, b]$ and $g'_u(c)$ exist where $a<c<b$, then $(f, g)\in RS_u[a, c]$ and $(f, g)\in RS_u[c, b]$. Conversely, if $(f, g)\in RS_u[a, c]$, $(f, g)\in RS_u[c, b]$ and $g'_u(c)$ exists, then $(f, g)\in RS_u[a, b]$. In either case*

$$\int_a^b f(x)\frac{d^2g(x)}{du(x)} = \int_a^c f(x)\frac{d^2g(x)}{du(x)} + \int_c^b f(x)\frac{d^2g(x)}{du(x)}.$$

Theorem 3. *Let g be u-convex on $[a, b]$ and g, u satisfy condition A. Let $\{f_n(x)\}$ be a sequence of functions which converges uniformly to $f(x)$ on $[a, b]$. If for all n, $(f_n, g)\in RS_u[a, b]$, then $(f, g)\in RS_u[a, b]$ and*

$$\lim_{n\to\infty}\int_a^b f_n(x)\frac{d^2g(x)}{du(x)} = \int_a^b f(x)\frac{d^2g(x)}{du(x)}.$$

LEMMA 1. *If g is u-convex on $[a, b]$, then both $g_u^+(x), g_u^-(x)$ exist everywhere in (a, b). Further $g_u'(x)$ exists on $[a, b]$ except at most an enumerable set.*

LEMMA 2. *Let $a \leqq c < d \leqq b$ and $g_u'(c), g_u'(d)$ exist, then*

$$\int_c^d \frac{d^2 g(x)}{du(x)} = g_u'(d) - g_u'(c).$$

LEMMA 3. *If $g_u'(c)$ exists, where $a < c < b$, then*

$$V_u(g;\, a, b) = V_u(g;\, a, c) + V_u(g;\, c, b).$$

2. Convergence in u-second variation

Let $\{F_n(x)\}$ be a sequence of real functions defined in $[a, b]$ which is assumed, throughout the section, to be convergent and to converge to $F(x)$, say.

It is easily verified that $V_u(F;\, a, b) \leqq \liminf_{n\to\infty} V_u(F_n;\, a, b)$. We investigate the case of equality.

PROPERTY A_u. A sequence $\{F_n(x)\}$ is said to satisfy Property A_u on $[a, b]$ if a subdivision $\pi_0(\alpha_0, \alpha_1, \ldots, \alpha_\mu)$ of $[a, b]$ and a positive integer m exist such that

$$|d(F_n;\, x_0, x_1, x_2)| \geqq |d(F_m;\, x_0, x_1, x_2)|$$

when $n > m$ and for each set of 3 distinct points $x_r \in [\alpha_{i-1}, \alpha_{i+1}]$ $r = 0, 1, 2$ and $i = 1, \ldots, \mu - 1$.

Consider the sequence $\{F_n(x)\}$ defined by $F_n(x) = a_n x^{2p}$, $u(x) = x^2$, $|a_n| \geqq |a_m|$, $p \geqq 2$ is an integer. Clearly, $d(F_n;\, x_0, x_1, x_2) = a_n(x_2^2 - x_0^2)\Sigma x_0^{2\beta_0} x_1^{2\beta_1} x_2^{2\beta_2}$ where the summation is extended to all positive integers including zero which satisfy the relation $\beta_0 + \beta_1 + \beta_2 = p - 2$. The Property A_u is then immediate.

Let E denote the collection of all subdivisions π of $[a, b]$ and let

$$V_u(\varphi, \pi) = \sum_{i=1}^{k-1} |d(\varphi;\, x_{i-1}, x_i, x_{i+1})|.$$

We immediately obtain the following lemma.

LEMMA 2.1. $\lim_{n\to\infty} V_u(F_n;\, \pi) = V_u(F;\, \pi)$ *for every* $\pi \in E$.

LEMMA 2.2. *If K is a finite number and $V_u(F_n;\, a, b) \leqq K$ for all n, then $V_u(F;\, a, b) \leqq K$.*

PROOF. The proof follows easily using Lemma 2.1 or else directly from Definition 3.

LEMMA 2.3. *If the sequence $\{F_n(x)\}$ possesses property A_u on $[a, b]$ and if $V_u(F_n;\, a, b) > K$ for all n, K being a finite number, then a $\pi \in E$ exists such that $V_u(F_n;\, \pi) > K$ for all n.*

Proof. A subdivision $\pi_0(\alpha_0, \alpha_1, \ldots, \alpha_\mu)$ of $[a, b]$ and a positive integer m exist such that

$$|d(F_n;\ x_{i-1}, x_i, x_{i+1})| \geqq |d(F_m;\ x_{i-1}, x_i, x_{i+1})|$$

when $n>m$ and for each set of 3 distinct points $x_r \in [\alpha_{i-1}, \alpha_{i+1}]$, $r=i-1, i, i+1$ and $i=1, \ldots, \mu-1$.

If $\pi_1 \in E$ which contains all the points of subdivision of π_0, then, using Property A_u, it is easily seen that

$$V_u(F_n;\ \pi_1) \geqq V_u(F_m;\ \pi_1) \quad \text{for} \quad n>m. \tag{1}$$

Since $V_u(F_i;\ a, b) \geqq K$, $1 \leqq i \leqq m$, an element $\pi_2 \in E$ exists such that

$$V_u(F_i;\ \pi_2) > K \quad \text{for each} \quad i,\ 1 \leqq i \leqq m. \tag{2}$$

Let π be a subdivision in E consisting of all the points of subdivisions of π_1 and π_2. By Lemma 1.4 of [7] and the inequalities (1) and (2), we obtain

$$V_u(F_n;\ \pi) > K \quad \text{for all} \quad n.$$

This proves the lemma.

Lemma 2.4. *If $\{F_n(x)\}$ and all its subsequences possess Property A_u on $[a, b]$ and if $V_u(F;\ a, b) < K$, where K is a positive finite number, then $V_u(F_n;\ a, b) \leqq K$ for all n except possibly a finite number.*

Proof. Suppose, that the lemma is false. Then there exists a sequence of positive integers $\{n_i\}$ with $n_i \to \infty$ such that $V_u(F_{n_i};\ a, b) > K$. Using Lemma 2.3 and Lemma 2.1, we have $V_u(F;\ a, b) \geqq K$. The contradiction proves the lemma.

Theorem 2.1. *If $\{F_n(x)\}$ and all its subsequences possess Property A_u on $[a, b]$ and $V_u(F_n;\ a, b)$ is finite for each n, then*

$$\lim_{n\to\infty} V_u(F_n;\ a, b) = V_u(F;\ a, b).$$

Proof. Let $L = \overline{\lim}\, V_u(F_n;\ a, b)$ and $l = \underline{\lim}\, V_u(F_n;\ a, b)$. First suppose that there is a finite $K>0$ such that $V_u(F_n;\ a, b) < K$ for all n. Then $L<\infty$. There exists a sequence $\{n_i\}$ of positive integers such that $\lim\limits_{i\to\infty} V_u(F_{n_i};\ a, b) = L$. If $\varepsilon>0$ is arbitrary, an integer i_0 exists such that

$$L-\varepsilon < V_u(F_{n_i};\ a, b) < L+\varepsilon \quad \text{when} \quad i \geqq i_0.$$

Hence, by Lemma 2.2,

$$V_u(F;\ a, b) \leqq L+\varepsilon. \tag{3}$$

Again, by Lemma 2.3, an element $\pi \in E$ exists such that $V_u(F_{n_i};\ \pi) > L-\varepsilon$ for $i \geqq i_0$. Letting $i\to\infty$, we obtain, by Lemma 2.1,

$$V_u(F;\ \pi) \geqq L-\varepsilon \quad \text{and so} \quad V_u(F;\ a, b) \geqq L-\varepsilon.$$

Combining this with (3), we get $L-\varepsilon \leqq V_u(F;\ a, b) \leqq L+\varepsilon$ and so $V_u(F;\ a, b) = L$. Since $V_u(F;\ a, b) \leqq l$, it follows that $\lim\limits_{n\to\infty} V_u(F_n;\ a, b) = V_u(F;\ a, b)$.

Next, if there exists no such finite $K>0$ such that $V_u(F_n;\ a, b)<K$ for all n, then there is a sequence $\{n_i\}$ of positive integers such that $V_u(F_{n_i};\ a, b)\geqq K$. Obviously then $L=+\infty$. If possible, let $V_u(F;\ a, b)\leqq K$ for some finite $K>0$. Then, by Lemma 2.4, $V_u(F_n;\ a, b)\leqq K$ for all n, a contradiction. Hence $V_u(F;\ a, b)=+\infty$. Since $V_u(F;\ a, b)\leqq l$, we have

$$\lim_{n\to\infty} V_u(F_n;\ a, b) = V_u(F;\ a, b) = +\infty.$$

This proves the theorem.

NOTE 2.1. If g is u-convex in $[a, c]$ and u-concave in $[c, b]$ where $a<c<b$ and if $g'_u(x)$ exists everywhere in $[a, b]$, then

$$V_u(g;\ a, b) = |g'_u(a)-g'_u(c)|+|g'_u(c)-g'_u(b)|.$$

REMARK 2.1. For the validity of Theorem 2.1, the convergence of the sequence $\{F_n(x)\}$ or even the uniform convergence is not sufficient. This is shown by the following example. Let

$$F_n(x) = \frac{1-\cos nx^2}{n^2}, \quad u(x) = x^2, \quad 0 \leqq x \leqq \pi^{1/2}.$$

Then $\{F_n(x)\}$ converges uniformly to $F(x)=0$ in $[0, \pi^{1/2}]$. We observe that $F'_{n,u}(x)$ exists in $[0, \pi^{1/2}]$ and $F'_{n,u}(x)=\dfrac{\sin nx^2}{n}$, $0\leqq x\leqq\pi^{1/2}$. Also in view of Lemma 3 and the u-convex property of $F_n(x)$ in a sub-interval in which $F'_{n,u}(x)$ is increasing we have $V_u(F_n;\ 0, \pi^{1/2})=V(F'_{n,u};\ 0, \pi^{1/2})=2$ for each n. But $V_u(F;\ 0, \pi^{1/2})=0$ and so, $\lim\limits_{n\to\infty} V_u(F_n;\ 0, \pi^{1/2})\neq V_u(F;\ 0, \pi^{1/2})$.

3. Convergence in RS_u integral

We consider a $\Delta(x_{-1}, x_0, \dots, x_{k+1})$ subdivision of $[a, b]$ and make the following definitions:

$$M_i = \sup_{x_{i-1}\leqq x\leqq x_{i+1}} f(x), \qquad m_i = \inf_{x_{i-1}\leqq x\leqq x_{i+1}} f(x), \quad 1 \leqq i \leqq k-1;$$

$$M_0 = \sup_{x_0\leqq x\leqq x_1} f(x), \qquad m_0 = \inf_{x_0\leqq x\leqq x_1} f(x);$$

$$M_k = \sup_{x_{k-1}\leqq x\leqq x_k} f(x), \qquad m_k = \inf_{x_{k-1}\leqq x\leqq x_k} f(x);$$

$$S = \sum_{i=0}^{k} M_i\, d(g;\ x_{i-1}, x_i, x_{i+1}), \quad s = \sum_{i=0}^{k} m_i\, d(g;\ x_{i-1}, x_i, x_{i+1}).$$

As in Theorem 3.1 of [7], it is easily verified that the upper approximating sum S does not increase and the lower approximating sum s does not decrease with the insertion of an extra point of subdivision to (a, b). Furthermore, no lower sum can exceed any upper sum. We consider g to be u-convex in $[a, b]$ and g, u satisfy Condition A and define

$$\underline{\int_a^b} f(x)\frac{d^2g(x)}{du(x)}=\sup_\Delta s \quad\text{and}\quad \overline{\int_a^b} f(x)\frac{d^2g(x)}{du(x)}=\inf_\Delta S.$$

We then have

$$\underline{\int_a^b} f(x)\frac{d^2g(x)}{du(x)}\leqq \overline{\int_a^b} f(x)\frac{d^2g(x)}{du(x)},$$

the equality sign holds if and only if $(f,g)\in RS_u[a,b]$, and in that case

$$\int_a^b f(x)\frac{d^2g(x)}{du(x)}=\underline{\int_a^b} f(x)\frac{d^2g(x)}{du(x)}=\overline{\int_a^b} f(x)\frac{d^2g(x)}{du(x)}.$$

Following Luxemburg [4] it is not difficult to obtain Arzelà's dominated convergence theorem for RS_u integral:

THEOREM 3.1. *Let $g(x)$ be u-convex in $[a,b]$ and g,u satisfy Condition A. Let $\{f_n(x)\}$ be a sequence of functions which converges to $f(x)$ in $[a,b]$. If for all n, $(f_n,g)\in RS_u[a,b]$ and $(f,g)\in RS_u[a,b]$ and if there exists a constant $M>0$ satisfying $|f_n(x)|\leqq M$ for all $x\in[a,b]$ and for all n, then*

$$\lim_{n\to\infty}\int_a^b f_n(x)\frac{d^2g(x)}{du(x)}=\int_a^b f(x)\frac{d^2g(x)}{du(x)}.$$

To establish the proof of the theorem we note Theorem 2.1 of [7], Theorem 3 of § 1 and the obvious inequality $\int_a^b \varphi(x)\frac{d^2h(x)}{du(x)}\geqq 0$ for $\varphi(x)\geqq 0$, $h(x)$ u-convex on $[a,b]$ with h,u satisfying Condition A and $(\varphi,h)\in RS_u[a,b]$.

We prove the following lemma which will be useful to prove the remaining theorems.

LEMMA 3.1. *Let f be bounded and g,u satisfy Condition A. If $(f,g)\in RS_u[a,b]$, then*

$$\left|\int_a^b f(x)\frac{d^2g(x)}{du(x)}\right|\leqq M(f)V_u(g;\,a,\,b),$$

where $M(f)=\sup_{a\leqq x\leqq b}|f(x)|$.

PROOF. Let $\varepsilon>0$ be arbitrary. Consider a $\Delta(x_{-1},\dots,x_{k+1})$ subdivision of $[a,b]$. Now for $x_{i-1}\leqq\xi_i\leqq x_{i+1}$, $1\leqq i\leqq k-1$

$$\left|\sum_{i=0}^{k} f(\xi_i)\,d(g;\,x_{i-1},x_i,x_{i+1})\right|\leqq |f(a)|\,|g_u(x_1,a)-g_u^+(a)|+$$

$$+\left|\sum_{i=1}^{k-1} f(\xi_i)\,d(g;\,x_{i-1},x_i,x_{i+1})\right|+|f(b)|\,|g_u^-(b)-g_u(b,x_{k-1})|.$$

Since we are ultimately concerned with arbitrarily small norm of Δ we may take x_1 and x_{k-1} such that each of the first and the last terms of the right hand member can be made separately less then $\varepsilon/2$. Hence, by Definitions 3 and 4, it follows that

$$\left|\int_a^b f(x)\frac{d^2g(x)}{du(x)}\right| \leqq M(f)V_u(g;\ a, b)+\varepsilon.$$

As $\varepsilon>0$ is arbitrary, the lemma follows.

THEOREM 3.2. *Let f be bounded and $\{g_n(x)\}$ be a sequence of functions which converges to $g(x)$ in $[a, b]$ with $\{V_u(g_n;\ a, b)\}$ converging to $V_u(g;\ a, b)$. If for all n, g_n, u satisfy Condition A, $(f, g_n)\in RS_u[a, b]$ and $(f, g)\in RS_u[a, b]$, then*

$$\lim_{n\to\infty}\int_b^a f(x)\frac{d^2g_n(x)}{du(x)} = \int_a^b f(x)\frac{d^2g(x)}{du(x)}.$$

PROOF. Let $\varepsilon>0$ be arbitrary. Then correspondingly there exists a positive integer n_0 such that

(4) $$|V_u(g_n;\ a, b)-V_u(g;\ a, b)| < \frac{\varepsilon}{M(f)} \quad \text{when} \quad n \geqq n_0,$$

where

$$M(f) = \sup_{a\leqq x\leqq b} |f(x)|.$$

That $g(x)$ satisfy Condition A is immediate.
Now by Theorem 2.2 of [7]

$$\left|\int_a^b f(x)\frac{d^2g_n(x)}{du(x)}-\int_a^b f(x)\frac{d^2g(x)}{du(x)}\right| = \left|\int_a^b f(x)\frac{d^2[g_n(x)-g(x)]}{du(x)}\right|$$

$$\leqq M(f)\,V_u(g_n-g;\ a, b), \qquad \text{by Lemma 3.1}$$

$$<\varepsilon \qquad\qquad , \qquad \text{by (4).}$$

This proves the theorem.

The convergence of $\{g_n(x)\}$ in u-second variation to the function $g(x)$ is assured by Theorem 2.1, and then Theorem 3.2 takes the form:

THEOREM 3.3. *Let f be bounded and $\{g_n(x)\}$ be a sequence of functions which converges to $g(x)$ in $[a, b]$ and let $V_u(g_n;\ a, b)$ be finite for each n. Let each g_n, u satisfy Condition A and let $\{g_n(x)\}$ and all its subsequences possess Property A_u. If for all n, $(f, g_n)\in RS_u[a, b]$ and $(f, g)\in RS_u[a, b]$, then*

$$\lim_{n\to\infty}\int_a^b f(x)\frac{d^2g_n(x)}{du(x)} = \int_a^b f(x)\frac{d^2g(x)}{du(x)}.$$

Finally, we obtain a convergence formula similar to that of Stieltjes integral in Natanson [5], Theorem 3, p. 233.

THEOREM 3.4. *Let f be continuous and let $\{g_n(x)\}$ be a sequence which converges uniformly to $g(x)$ at every point of $[a, b]$ and let g_n, u satisfy Condition A. If there exists a constant $K>0$ such that $V_u(g_n; a, b)<K$ for all n, then*

$$\lim_{n\to\infty}\int_a^b f(x)\frac{d^2 g_n(x)}{du(x)} = \int_a^b f(x)\frac{d^2 g(x)}{du(x)}.$$

To prove the theorem we require the following lemma:

LEMMA 3.2. *Under the hypothesis on $\{g_n(x)\}$ in Theorem 3.4*

$$\lim_{n\to\infty} g^+_{n,u}(x) = g^+_u(x) \quad \textit{for all} \quad x\in[a, b)$$

and

$$\lim_{n\to\infty} g^-_{n,u}(x) = g^-_u(x) \quad \textit{for all} \quad x\in(a, b].$$

PROOF. By Lemma 2.2, $V_u(g; a, b)\leqq K$ and so each g_n and $g\in BV_u[a, b]$. Also g, u satisfy Condition A. The u-incrementary ratios of each g_n and g are bounded. The existence of either sided derivative is ensured by Lemma 3.3 of [8] or else by Theorem 2 of [3] and Lemma 1 of § 1. We prove the lemma for the right-hand derivative. The other case is analogous.

Let $\varepsilon>0$ be arbitrary and $a\leqq x\leqq b$. There exists $\delta_1=\delta_1(\varepsilon)>0$ such that

$$\left|\frac{g(x+h)-g(x)}{u(x+h)-u(x)} - g^+_u(x)\right| < \varepsilon/4$$

whenever $0<h<\delta_1$.

Since $\{g_n(x)\}$ converges uniformly to $g(x)$, there exists a positive integer n_0 such that for any $h>0$

$$\left|\frac{g_n(x+h)-g_n(x)}{u(x+h)-u(x)} - \frac{g(x+h)-g(x)}{u(x+h)-u(x)}\right| < \varepsilon/4$$

whenever $n\geqq n_0$. It, then, follows that

$$\left|\frac{g_n(x+h)-g_n(x)}{u(x+h)-u(x)} - g^+_u(x)\right| < \varepsilon/2 \tag{5}$$

whenever $0<h<\delta_1$ and $n\geqq n_0$.

Also for each n we can choose $\delta_2>0$, depending on ε and n, such that

$$\left|\frac{g_n(x+h)-g_n(x)}{u(x+h)-u(x)} - g^+_{n,u}(x)\right| < \varepsilon/2 \tag{6}$$

whenever $0<h<\delta_2$.

For each $n\geqq n_0$ choose δ_2 and then choose a fixed $h<\delta=\min(\delta_1, \delta_2)$. Then from (5) and (6), we obtain

$$|g^+_{n,u}(x)-g^+_u(x)| < \varepsilon \quad \text{whenever} \quad n\geqq n_0.$$

This proves the lemma.

PROOF of the theorem. By Theorem 1.1 of [7] and Theorem 1 of § 1, $(f, g) \in RS_u[a, b]$ and $(f, g_n) \in RS_u[a, b]$ for each n. By Theorem 1 of [2], there exists a subset E_1 of $[a, b]$, where $[a, b] - E_1$ is countable, such that g and each g_n possess u-derivative at each point of E_1. Let $\varepsilon > 0$ be arbitrary. There exist finite subintervals $[x_i, x_{i+1}]$, $i=0, 1, \ldots, m-1$, $x_0 = a$, $x_m = b$, $x_i \in E_1$, $1 \leq i \leq m-1$, of $[a, b]$ such that oscillation of $f(x)$ in each subinterval is less than $\varepsilon/3K$.

The equality now follows from [5] simply applying Theorem 2.1 of [7], Theorem 2, Lemma 2, Lemma 3.1, Lemma 3.2 and Lemma 3 of § 1 in appropriate steppings.

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MINIMAL GRAPHS OF DIAMETER TWO

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1. Introduction

Denote $H_2(n, k)$ the set of all undirected graphs of order n, diameter 2 and maximal degree k. Let $e(G)$ mean the number of edges of the graph G. In [2] the size of

$$F_2(n, k) = \min_{G \in H_2(n,k)} e(G)$$

was investigated. The exact values for $k=n-1, \ldots, n-4$ were determined there. In [3] the values of $F_2(n, k)$ for $\frac{n+1}{2} \leqq k \leqq n-5$ were studied and the following problem was stated (see page 235): "To determine the exact value of $F_2(n, k)$ for $k < \frac{n}{2}$, or at least the asymptotic value of $F_2(n, [cn])$ with $0 < c < \frac{1}{2}$."

For $\frac{3}{7} < c < \frac{1}{2}$ J. Pach and L. Surányi [4, 5] showed that we have $\lim_{n\to\infty} \frac{F_2(n, [cn])}{n} = 3$. Here we give the exact value, namely we prove

THEOREM. Let a positive $h < \frac{1}{14}$ be given. If

$$n > \frac{21}{h^2}, \quad \frac{3}{7} n \leqq k < \left(\frac{1}{2} - h\right) n, \tag{1}$$

then $F_2(n, k) = 3n - 12$.

2. Auxiliary results

For the remainder of the paper suppose G is a graph from $H_2(n, k)$, n, k fulfil (1) and

$$e(G) < 3n - 12. \tag{2}$$

Obviously, according to (1), G cannot contain any vertex of degree 1 or 2. One item of notation: $O(x)$ denotes the neighbourhood of x in G. We begin with establishing a number of properties of G.

PROPERTY 1 (P1). For every vertex $x \in G$ the sum of the degrees of vertices in $O(x)$ is at least $n-1$.

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PROOF. This is obvious since the diameter of G is 2.

PROPERTY 2 (P2). A vertex of degree 3 is adjacent only to vertices of degree at least $[2hn-1]$.

PROOF. This is an immediate consequence of P1.

Next denote R the set of all vertices of degree at least $[hn-1]$ in G; let $|R|=r$.

PROPERTY 3 (P3). $r<\frac{6}{h}+1$.

PROOF. If this were not the case, the total sum of degrees in G would be at least $[hn-1]\left(\frac{6}{h}+1\right)$, which contradicts (2).

Further, from P1 the following can be deduced:

PROPERTY 4 (P4). A vertex of degree 4 (5) is adjacent to at least 3 (2) vertices of R. A vertex of degree 6 is also adjacent to a vertex in R.

Now, let P be the set of all vertices of degree 3, Q the set of all vertices of degree 4, 5 or 6 and S the set of all vertices of degree 7, 8, ..., $[hn-2]$ in G. Let $|P|=p$, $|Q|=q$, $|S|=s$.

PROPERTY 5 (P5). $p>n-\frac{42}{h}$.

PROOF. By P2 and P4, the sum of degrees in G is at least $6p+7q+7s$. If $q+s\geqq 6r-24$, then we have $6p+7q+7s\geqq 6(p+q+r+s)-24=6n-24$, which contradicts (2). On the other hand, if $q+s<6r-24$, then by P3, $p=n-(q+r+s)>n-7r+24>n-\frac{42}{h}$. The proof is finished.

Denote T the set of all vertices adjacent to at least one vertex of degree 3 in G (by P2, $T\subset R$) and Z the set of all triples (from T) representing the neighbourhoods of vertices of degree 3. Let $|T|=t$.

PROPERTY 6 (P6). $t\geqq 7$.

PROOF. By P5 there exist at least $3n-\frac{126}{h}$ edges from P to T, and (1) implies our assertion.

We shall investigate the properties of Z, now.

PROPERTY 7 (P7). Any two triples of Z have a common element.

PROOF. G is of diameter 2.

DEFINITION. We say that the vertices $x_1, \ldots, x_j$ cover Z, if every triple of Z contains at least one x_i, and we say they 2-cover Z if every triple contains at least two x_i's.

PROPERTY 8 (P8). No couple of vertices covers Z and no 4-tuple 2-covers Z.

PROOF. This follows from (1) and P5.

PROPERTY 9 (P9). A couple of vertices cannot be contained in two (or more) triples of Z.

PROOF. We shall prove it indirectly. Suppose $a, b \in T$ occur in two triples of Z: abc, abd. Then by P8, a triple containing neither a nor b exist in Z, and by P7 it must be of the form

$$cde,\ e \neq a, b.$$

Thus by P7, Z may contain only the following further types of triples (the points represent elements of T distinct from a and b, but not necessarily distinct from c, d, e):

(3) $ac.$ $ad.$ $ae.$

(4) $bc.$ $db.$ $be.$

(5) $cd.$

(6) abe

Now we shall need two lemmas:

LEMMA 1. *In* (3) *or* (4) *a triple of the form ae. or be. exist, where the points are different from c and d.*

PROOF. If this were not the case, then the letters a, b, c, d would 2-cover Z (see P8).

LEMMA 2. *Not all triples of* (3) *contain c; not all triples of* (3) *contain d. The same is true for* (4).

PROOF. We prove only the first assertion: if all triples of (3) contain c, then the couple b, c covers Z (see P8).

We shall continue in the proof of property 9. We have to distinguish some cases:

Case 1. If in Z a cdf $(f \neq e)$ exists, then according to Lemma 1, the following 3 possibilities can occur only:

Case 1A. Z contains both triples aef, bef. Then by P7, in Z no seventh letter can occur, which contradicts P6.

Case 1B. Z contains no $be.$ (the point different from c, d), but $aef \in Z$. In this case by Lemma 2 and P7 there exists a triple bce or bcf and also a triple bde or bdf in (4). However, then in Z no seventh letter can occur, again.

Case 1C. Z contains no $ae.$, but $bef \in Z$. Here we can proceed as in case 1B.

Case 2. (5) is empty. Then by Lemma 1 in (3) or (4) there exists a triple z_1 containing b but not c, d. Since (5) is empty, there exists also a triple z_2 containing c but not d, e (in opposite case the symbols a, b, d, e would 2-cover Z) and (for similar reasons) a triple z_3 containing d but not c, e.

Case 2A. $z_1, z_2, z_3 \in (3)$. Let $z_1 = aex$, $z_2 = acy$, $z_3 = adw$. Obviously, if $bc.$ ($bd.$ or $be.$) exists, then

(7) $$x = w \ (x = y \quad \text{or} \quad y = w).$$

On the other hand, at least two distinct triples of types *bc.*, *bd.*, *be.* exist (if this were not the case, then 2 letters would cover Z), hence at least two of the equalities (7) hold and so we have $x=y=w$. Thus every triple of (4) contains x and we have only 6 letters in Z, again.

Case 2B. $z_1, z_2 \in (3)$, $z_3 \in (4)$. Let $z_1=aex$, $z_2=acy$, $z_3=bdw$. By P7, $x=y=w$ and (4) may contain only triples *bce*, *bcx*, *bex*. However, by Lemma 2, at least one of them must occur in (4); thus in (3) no seventh letter occurs.

The remaining cases can be considered similarly as 2A or 2B. The proof of P9 is finished.

PROPERTY 10 (P10). *T consists of 7 vertices. Z consists of 7 triples and no triple not belonging to Z has a common element with every triple of Z.*

PROOF. Let Z contain the triples *abc*, *ade* (see P7, P9). By P8, triples not containing a exist in Z. Obviously, any such triple must contain b or c and also d or e. Thus Z may contain only the following such triples: *bdf*, *beg*, *cdg*, *cef*. However, all mentioned triples can be 2-covered by b, c, d, e and hence the last possible triple *afg* occurs in Z, too. Thus the first assertion follows. Deleting a triple z of those 7 we get a system of triples 2-covered by 4 letters not contained in z; therefore by P8, Z consists of all mentioned triples. It is easy to check, that no triple beside of those of Z has a common element with every of them. The proof is finished.

REMARK. Incidentally we proved: If U is a system of non-disjoint triples formed from at least 7 symbols and U is not covered (2-covered) by any 2 (4) symbols, then in U any two triples have exactly one common element and any couple is contained in some triple (which can be checked above), i.e. U is a Steiner triple system.

PROPERTY 11 (P11). *In G there exist at least $3(n-7)$ edges with exactly one endpoint in T.*

PROOF. A vertex of degree 3 can be reached from a vertex of higher degree by a way of length $\leqq 2$ only through T. By P8, no two vertices cover Z, hence a vertex not belonging to T, must be adjacent to (at least) 3 vertices of T and the assertion follows.

3. Proof of the Theorem

We shall proceed indirectly: suppose a $G \in H_2(n, k)$ with (1) and (2) exists. Denote G_T the subgraph of G induced by the set T. According to (2) and P11,

$$e(G_T) \leqq 8.$$

Thus the sum of degrees in G_T is at most 16. However, in G_T no vertex of degree 1 exists (see P8) and since T has 7 elements (see P10), there exist at least 5 vertices of degree 2 in G_T.

Take vertex v_1 of degree 2 in G_T; let v_2 and v_3 be its neighbours. From v_1 a path of length $\leqq 2$ exists to every vertex of degree 3, hence the triple $z=v_1v_2v_3$ has a common vertex with every triple of Z. Therefore by P10, $z \in Z$ and hence z forms the neighbourhood of a vertex v_0 of degree 3 in G. From this v_0 we have to reach all vertices of T by a way of length $\leqq 2$ (in G) and thus the sum of degrees of vertices v_2 and v_3 in G_T is at least 6.

Hence for arbitrary vertex v_1 of degree 2 in G_T one of the following possibilities holds: a) v_1 is adjacent to 2 vertices of degree 3, b) v_1 is adjacent to a vertex of degree at least 4 in G_T. Obviously, if G_T has at most 8 edges, no of those possibilities can be fulfilled for 5 (or more) vertices of degree 2.

This contradiction proves, that $F_2(n, k) \geqq 3n-12$.

On the other hand, for any n, k fulfilling (1) we construct a graph $G_0 \in H_2(n, k)$, with $3n-12$ edges.

G_0 consists of:

1. seven vertices a, b, c, d, e, f, g and nine edges $ac, ag, bc, be, cd, df, ef, eg, fg$;
2. a group of vertices of degree 3 adjacent to vertices a, b and d (denote this group abd) and (using the same notation) the further 6 groups of vertices of degree 3: bce, cdf, deg, efa, fgb, gac.

It can be checked, that G_0 is of diameter 2. Next determine the cardinalities of those groups.

The groups abd contains $\left[\frac{6k-2n+2}{4}\right]$ vertices.

Denote $A=\left[\frac{n-k-5}{3}\right]$. The cardinalities of further groups are as follows: $|bce|=A+x_1$, $|cdf|=A+x_2$, $|deg|=A+x_3$, $|efa|=A+x_4$, $|fgb|=A+x_5$, $|gac|=$ $=A+x_6$, where

1. if $n-k \equiv 1 \pmod 4$, then $x_1 = \dots = x_6 = 0$;
2. if $n-k \equiv 2 \pmod 4$, then $x_1 = x_2 = 1$, $x_3 = \dots = x_6 = 0$;
3. if $n-k \equiv 3 \pmod 4$, then $x_1 = x_2 = x_4 = 1$, $x_3 = x_5 = x_6 = 0$;
4. if $n-k \equiv 0 \pmod 4$, then $x_1 = \dots = x_5 = 1$, $x_6 = 0$.

It can be shown that $e(G_0) = 3n-12$ and that $G_0 \in H_2(n, k)$ for n, k fulfilling (1). The proof of the theorem is finished.

REMARK. $G_0 \in H_2(n, k)$ for all $n \geqq 17$ and fulfilling (1).

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ON THE SOLUTIONS OF A HALF-LINEAR DIFFERENTIAL EQUATION

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Consider the differential equation

$$(1)\qquad y''|y'|^{n-1}+qy^{n*}=0,\quad y=y(x),\quad '=\frac{d}{dx},\quad y^{n*}=|y|^n\cdot\operatorname{sign} y,\quad n>0,$$

where $q=q(x)$ is a positive continuous function in the interval (a, b) $(-\infty<a<b\leqq\leqq\infty)$. Let $y(x)$ be a solution of the differential equation (1). Denote by $x_0, x_1, \ldots, \ldots, x_k$ and $x'_0, x'_1, \ldots, x'_k$ the roots of the equations $y(x)=0$ and $y'(x)=0$, respectively, such that $a\leqq x_0, x'_0$ and $(x_0<)x'_0<x_1<\ldots<b$, provided they exist at all.

DEFINITION. Let the function $q(x)$ belong to the class of functions $C_\nu[a, b]$ if it is continuous and $[q(x)]^\nu$ (ν is a real number) is concave in the interval (a, b).

Such a function is e.g. $q(x)=x^{1/\nu}$ for $a=0$ and $b=\infty$.

In what follows we shall give estimates on the location of roots and maxima of the solution $y(x)$ of the differential equation (1), and this will be done by means of the functional $\int_\xi^x q(\tau)^{1/(n+1)}\,d\tau$, provided that the function $q(x)$ belongs to the class $C_\nu[a, b]$ in the interval of integration. These estimates generalize to an arbitrary $n(>0)$ the results obtained for $n=1$ in [1] and [6].

It will be assumed throughout the sequel that $q(x)$ is twice continuously differentiable in the interval (a, b). From the point of view of our investigations this is not an essential restriction since any function $q\in C_\nu[a, b]$ can be arbitrarily closely approximated by a function $q_\varepsilon(x)$ which is not only twice continuously differentiable but even monotonically increasing or decreasing, respectively, in the interval (a, b) according as $q(x)$ is so, and for which we have $q_\varepsilon\in C_\nu[a, b]$ (see e.g. [1]).

Let us investigate the class $C_\nu[a, b]$. First we shall show that for any $q\in C_\nu[a, b]$ and any finite subinterval $(a, c]$ of (a, b) we have:

$$(2)\qquad \int_a^x q(\tau)^{1/(n+1)}\,d\tau<\infty\quad\text{if}\quad \nu\geqq 0\quad\text{or}\quad \nu<-\frac{1}{n+1},$$

$$(3)\qquad \int_a^x q(\tau)(\tau-a)^n\,d\tau<\infty\quad\text{if}\quad \nu\geqq 0\quad\text{or}\quad \nu<-\frac{1}{n+1},$$

$$(4)\qquad \int_a^x q(\tau)\,d\tau<\infty\quad\text{if}\quad \nu\geqq 0\quad\text{or}\quad \nu<-1,$$

where $a<x\leqq c$.

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Since $[q(x)]^\nu$ $(a<x<b)$ is a positive concave function, it holds

$$(5)\qquad q(x) \leqq \begin{cases} [(q(x)^\nu)'|_{x=c}(x-c)+q(c)^\nu]^{1/\nu} & \text{if} \quad \nu \geqq 0 \\ \left[\dfrac{q(c)^\nu}{c-a}(x-a)\right]^{1/\nu} & \text{if} \quad \nu < 0 \end{cases} \qquad (a < x \leqq c).$$

It is easy to check that (2), (3), (4) hold for any q of the form

$$q(x) = \big(c_1(x-a)+c_2\big)^{1/\nu} \quad (c_1 > 0,\ c_2 \geqq 0) \quad (a < x \leqq c),$$

whence, in view of (5), they hold for every function $q \in C_\nu[a, b]$ as well.

Put

$$(6)\qquad u(x) = \int_a^x q(\tau)^{1/(n+1)}\,d\tau \quad (a \leqq x < b)$$

and denote the inverse of $u(x)$ by $x_q(u)$. In what follows, this u will be considered as a new independent variable. Set

$$Y(u) = y\big(x(u)\big).$$

Then the differential equation (1) has the form

$$(7)\qquad \ddot{Y}|\dot{Y}|^{n-1}+s_q(u)\dot{Y}^{n*}+Y^{n*} = 0 \quad \big(0 < u \leqq u(b)\big),$$

where

$$\cdot = \frac{d}{du}$$

and

$$(8)\qquad s_q(u) = \frac{1}{n+1}\cdot\frac{q'\big(x_q(u)\big)}{q\big(x_q(u)\big)^{1+(1/(n+1))}} = \frac{1}{n+1}\frac{d}{du}\log q\big(x_q(u)\big) \quad \big(0 < u \leqq u(b)\big).$$

If it will not cause misunderstanding, we shall write $x(u)$ and $s(u)$ instead of $x_q(u)$ and $s_q(u)$, respectively, in the sequel.

The condition $q \in C_\nu[a, b]$ implies that

$$(9)\qquad (\nu-1)q'^2+qq'' \begin{cases} \leqq 0 & \text{if} \quad \nu \geqq 0 \\ \geqq 0 & \text{if} \quad \nu < 0 \end{cases} \qquad (a < x < b).$$

Differentiating the function $s(u)$ with respect to u we obtain

$$(10)\quad \dot{s}(u) = \frac{1}{n+1}\cdot\frac{q''\big(x(u)\big)q\big(x(u)\big)-\left(1+\dfrac{1}{n+1}\right)q'\big(x(u)\big)^2}{q\big(x(u)\big)^{2[1+(1/(n+1))]}} \qquad \big(0 < u < u(b)\big).$$

Substituting qq'' into (10) according to (9) we arrive at

$$(11)\qquad -\dot{s}(u) \begin{cases} \geqq \dfrac{1}{\alpha}s^2(u) & \text{if} \quad \nu \geqq 0 \\ \leqq \dfrac{1}{\alpha}s^2(u) & \text{if} \quad \nu < 0 \end{cases} \qquad \big(0 < u < u(b)\big)$$

where

$$\alpha = \frac{1}{1+\nu(n+1)}.$$

We shall deal with the cases

$$(12) \qquad \begin{matrix} 0 < \\ -\infty < \end{matrix} \left\} \alpha \left\{ \begin{matrix} \leqq 1 & \text{if} & \nu \geqq 0 \\ < 0 & \text{if} & \nu < -\dfrac{1}{n+1} \end{matrix} \right. \right. \qquad (0 < u < u(b))$$

and we shall show that

$$(13) \qquad s(u) \begin{cases} \leqq \alpha u^{-1} & \text{if} \quad \nu \geqq 0 \\ \geqq \alpha u^{-1} & \text{if} \quad \nu < -\dfrac{1}{n+1} \end{cases} \qquad (0 < u < u(b)).$$

The relation (13) is trivially fulfilled for $s \equiv 0$, therefore $s \not\equiv 0$ will be assumed in the sequel.

By (11), (12) we have

$$(14) \qquad \alpha \dot{s}(u) \leqq 0 \quad (0 < u < u(b)),$$

whence $\alpha s(u)$ $(0<u<u(b))$ is monotonically decreasing and therefore the following u^* is uniquely defined:

$$u^* = \begin{cases} 0 & \text{if} \quad |s(u)| > 0 \\ \sup \{u\colon 0 < u < u(b), \quad s(u) = 0\} & \text{otherwise.} \end{cases}$$

By the definition of u^*,

$$(15) \qquad \alpha s^{-1}(u) \begin{cases} \geqq 0 & \text{if} \quad 0 \leqq u < u^* \\ < 0 & \text{if} \quad u^* < u < u(b). \end{cases}$$

(15) implies that α and $s(u)$ are of different signs in the interval $(u^*, u(\mathrm{b}))$, hence (13) is trivially satisfied in this case.

Consider now the case $0<u<u^*$ and put

$$C(\xi) = \alpha s^{-1}(\xi) - \xi \quad (0 < \xi < u^*).$$

By (11) we have

$$(16) \qquad \frac{dC(\xi)}{d\xi} \geqq 0 \quad (0 < \xi < u^*)$$

and by (15)

$$C(\xi_1) \geqq -\xi_1 \quad (0 < \xi_1 < u^*),$$

hence

$$(17) \qquad C(\xi) \geqq -\xi_1 + \int_{\xi_1}^{\xi} \frac{dC(\xi)}{d\xi} d\xi \quad (0 < \xi_1 < \xi < u^*).$$

Since ξ_1 can be taken arbitrarily small, (17) yiedls in view of (16) that

$$(18) \qquad C(\xi) \geqq 0 \quad (0 < \xi < u^*).$$

Set

$$(19)\qquad \sigma_\xi(u)=\begin{cases}\alpha(u+C(\xi))^{-1} & \text{if}\quad s(\xi)\neq 0\\ 0 & \text{otherwise}\end{cases}\qquad (-C(\xi)<u<\infty,\ 0\leqq\xi<u^*).$$

Integrating (11) on the intervals (u, ξ) and (ξ, u) $(0<\xi<u^*)$, respectively, we obtain

$$(20)\qquad \alpha s(u)^{-1}\begin{cases}\leqq u-\xi+\alpha s(\xi)^{-1} & \text{if}\quad 0<u\leqq\xi\\ \geqq u-\xi+\alpha s(\xi)^{-1} & \text{if}\quad \xi\leqq u<u^*.\end{cases}$$

By (19), (20) we have

$$(21)\qquad s(u)\begin{cases}\geqq\sigma_\xi(u) & \text{if}\quad v\geqq 0\\ \leqq\sigma_\xi(u) & \text{if}\quad v<-\dfrac{1}{n+1}\end{cases}\qquad (0<\xi<u^*,\ 0<u<\xi)$$

and

$$(22)\qquad s(u)\begin{cases}\leqq\sigma_\xi(u) & \text{if}\quad v\geqq 0\\ \geqq\sigma_\xi(u) & \text{if}\quad v<-\dfrac{1}{n+1}\end{cases}\qquad (0<\xi<u^*,\ \xi<u<u^*).$$

In the special case $\xi=0$ (21) and (22) yield (13), which was to be proved.

Put $u_i=u(x_i)$, $u_i'=u(x_i')$ $(i=0, 1, 2, \ldots)$, and consider the solution $Y(u)$ of the differential equation (7). We have $Y(u_i)=0$ $(i=1, 2, \ldots)$. Next we show that

$$(23)\qquad \dot Y(u_i')=0\quad (i=0, 1, 2, \ldots).$$

We have

$$(24)\qquad \dot Y(u_i')=\lim_{x\to x_i+0} y'(x)\,q(x)^{-(1/(n+1))}\quad (i=0, 1, 2, \ldots).$$

Since $q(x)$ is a positive continuous function in (a, b), (24) implies (23) if $x_i'>a$ $(i=0, 1, 2, \ldots)$. Thus we have to deal only with the case $x_0'=a$. In what follows we shall restrict ourselves to the functions $q\in C_v[a, b]$. Therefore it can be supposed that q is monotonic in a right neighbourhood K_a of the point a. It suffices to consider the case when $\lim\limits_{x\to a+0} q(x)=0$. Then $q'(x)>0$ $(x\in K_a)$. From the differential equation (1) we obtain

$$((y')^{n+1})'=-(n+1)\,q y^{n*} y'\quad (a<x<b),$$

whence

$$(25)\qquad |y'(x)|^{n+1}=-\int_a^x (n+1)\,q(\tau)\,y^{n*}(\tau)\,y'(\tau)\,d\tau\quad (a<x<b).$$

We may assume that $|y(x)|$, $|y'(x)|<1$ $(x\in K_a)$. Since $q(x)$ is monotonically increasing for $x\in K_a$, we have

$$(26)\qquad \int_a^x q(\tau)\,y^{n*}(\tau)\,y'(\tau)\,d\tau\leqq(x-a)\,q(x)\quad (x\in K_a),$$

hence by (25) and (26)

$$\lim_{n\to a+0} |y'(x)|^{n+1}/q(x) = 0,$$

which yields (23) in view of (24).

Introduce now the following function $t(u)$:

$$t(u) = \frac{Y(u)}{\dot{Y}(u)} \quad u \neq u_i' \quad (i = 0, 1, 2, \ldots) \quad (0 < u < u(b)).$$

Differentiating here and taking into consideration (7) we get

$$\dot{t}(u) = 1 + s(u)t(u) + |t(u)|^{n+1} \quad u_i \leqq u < u_i' \text{ resp. } u_i' < u \leqq u_{i+1} \tag{27}$$

$$(i = 0, 1, 2, \ldots).$$

The function $q(x)$ is defined on the open interval (a, b), whereas we should like to have the solution $y(x)$ of the differential equation (1) defined on the left closed interval $[a, b)$. The possibility of this is investigated in the following.

THEOREM 1. *The differential equation* (1) *admits a unique solution* $y(x)$ *defined in a right neighbourhood of the point a and satisfying the initial conditions* $y(a)=A$, $y'(a)=B$ $(A^2+B^2>0)$, *if and only if there is a* $\xi\in(a, b)$ *such that*

$$\int_a^\xi q(\tau)|\psi(\tau)|^n\, d\tau < \min\left(\frac{1}{n}\frac{B_1^n}{\sqrt[n]{2}}, \frac{1}{n}\frac{B_1^n}{2}\right) \tag{28}$$

where

$$\xi - a \leqq \vartheta\frac{\max(|A+B|, A)}{B_1},$$

$$0 < \vartheta < \min\left(\frac{1}{2}, \frac{1}{\sqrt[n]{3}}\left(1 - \frac{1}{\sqrt[n]{2}}\right)\right),$$

$$\psi(x) = A + B(x-a),$$

$$B_1 = \begin{cases} |A| & \text{if } B = 0 \\ |B| & \text{otherwise.} \end{cases}$$

PROOF. **Sufficiency:** Put

$$S_0 = \{\varphi\colon \varphi\in C^0[a, \xi],\ \varphi\cdot\psi \geqq 0,\ 0 \leqq |\varphi| \leqq |\psi|\}.$$

Define the distance ϱ of two elements $\varphi_1, \varphi_2\in S_0$ by

$$\varrho(\varphi_1, \varphi_2) = \max_{a\leqq x\leqq\xi} |\varphi_1 - \varphi_2|/|\psi|. \tag{29}$$

It can be seen that the space $[S_0, \varrho]$ is complete.

Define an operator F on S_0 as follows:

$$F(\varphi)(x) = A+\int_a^x \left[B^{n^*}-n\int_a^v q(\tau)\varphi^{n^*}(\tau)\,d\tau\right]^{1/n^*} dv \quad x\in(a,\xi). \tag{30}$$

By (28) it can be seen that $F(\varphi)$ exists for all $\varphi\in S_0$.

In any case, the operator F produces a twice continuously differentiable function, and we have $F(\varphi)(a)=A$, $\frac{d}{dx}F(\varphi)(a)=B$. It is easy to see that

$$\varphi_1(x) \leqq \varphi_2(x)$$

for $\varphi_1, \varphi_2\in S_0$ implies

$$F(\varphi_1)(x) \geqq F(\varphi_2)(x).$$

Hence F reverses the order relations. In particular, we have

$$F(0)(x) = \psi(x),$$

thus

$$0 \leqq |\varphi| \leqq |F(0)| \tag{31}$$

for all $\varphi\in S_0$, and in view of the above remark we obtain that

$$|F(\psi)(x)| \leqq |F(\varphi)(x)| \leqq |\psi(x)|. \tag{32}$$

Denote by S_1 the following subset of S_0:

$$S_1 = \{\varphi\colon \varphi\in S_0,\ |F(\psi)| \leqq |\varphi|\}. \tag{33}$$

Then we have

$$F\colon S_1 \to S_1 \tag{34}$$

by (32) (moreover, $F\colon S_0\to S_1$).

If the differential equation (1) admits a solution $y(x)$ satisfying the required initial conditions then $y\in S_0$ and twice integrating the equation (1) we obtain that

$$y = F(y). \tag{35}$$

Hence the solution $y(x)$ — if it exists — is a fixed point of the operator F in S_0 and even in S_1 for $F\colon S_0\to S_1$.

Next we show for all $\varphi_1, \varphi_2\in S_1$:

$$\varrho(F(\varphi_1), F(\varphi_2)) \leqq \begin{cases} \varrho(\varphi_1, \varphi_2)/2^n & \text{if } 0 < n \leqq 1 \\ \varrho(\varphi_1, \varphi_2)/\sqrt[n]{2} & \text{if } n > 1 \end{cases} \quad (\varphi_1, \varphi_2\in S_1) \tag{36}$$

hence F is a contraction on the space $[S_1, \varrho]$.

By (28) we have for all $\varphi\in S_1$:

$$1/\sqrt[n]{2}\,|\psi| \leqq |\varphi| \leqq |\psi| \tag{37}$$

and

$$\frac{n}{2}\int_a^x q(\tau)\,|\psi(\tau)|^n\,d\tau \leqq |F(\varphi)'|^n \leqq (2B_1)^n. \tag{38}$$

We shall make use of the following simple inequality:

$$|v_1^{m^*}-v_2^{m^*}| \leqq \begin{cases} mk^{m-1}|v_1-v_2| & \text{if } 0<m\leqq 1,\\ mK^{m-1}|v_1-v_2| & \text{if } m>1,\end{cases} \tag{39}$$

where $0<k\leqq v_1, v_2\leqq K<\infty$ or $-\infty<K\leqq v_1, v_2\leqq -k<0$. By (37) and (39) we have for all $\varphi_1, \varphi_2\in S_1$

$$|\varphi_1^{n^*}-\varphi_2^{n^*}| \leqq \begin{cases} n2^{(1/n)-1}|\psi|^{n-1}|\varphi_1-\varphi_2| & \text{if } 0<n\leqq 1,\\ n|\psi|^{-1}|\varphi_1-\varphi_2| & \text{if } n>1,\end{cases}$$

whence by (29)

$$|\varphi_1^{n^*}-\varphi_2^{n^*}| \leqq \begin{cases} n2^{(1/n)-1}\varrho(\varphi_1,\varphi_2)|\psi|^n & \text{if } 0<n\leqq 1,\\ n\varrho(\varphi_1,\varphi_2)|\psi|^n & \text{if } n>1.\end{cases} \tag{40}$$

Now we estimate $|F(\varphi_1)-F(\varphi_2)|$. By (38) and (39) we have

$$|F(\varphi_1)-F(\varphi_2)| \leqq$$

$$\leqq \begin{cases} \int\limits_a^x (2B_1)^{1-n}\int\limits_a^v q(\tau)|\varphi_2^{n^*}(\tau)-\varphi_1^{n^*}(\tau)|\,d\tau\,dv & \text{if } 0<n\leqq 1\\ \int\limits_a^x \left(\frac{n}{2}\int\limits_a^v q(\tau)\psi^{n^*}(\tau)\,d\tau\right)^{(1/n)-1}\int\limits_a^v q(\tau)|\varphi_2^{n^*}(\tau)-\varphi_1^{n^*}(\tau)|\,d\tau\,dv & \text{if } n>1\end{cases} \qquad (\varphi_1, \varphi_2\in S_1)$$

hence by (40)

$$|F(\varphi_1)-F(\varphi_2)| \leqq$$

$$\leqq \begin{cases} \varrho(\varphi_1,\varphi_2)n\int\limits_a^x 2^{(1/n)-1}(2B_1)^{1-n}\int\limits_a^v q(\tau)|\psi(\tau)|^n\,d\tau\,dv & \text{if } 0<n\leqq 1\\ \varrho(\varphi_1,\varphi_2)\int\limits_a^x 2\left(\frac{n}{2}\int\limits_a^v q(\tau)|\psi(\tau)|^n\,d\tau\right)^{1/n}dv & \text{if } n>1\end{cases} \qquad (\varphi_1, \varphi_2\in S_1)$$

and by (28)

$$|F(\varphi_1)-F(\varphi_2)| \leqq \begin{cases} \dfrac{\varrho(\varphi_1,\varphi_2)}{2^n}B_1(x-a) & \text{if } 0<n\leqq 1\\ \dfrac{\varrho(\varphi_1,\varphi_2)}{\sqrt[n]{2}}B_1(x-a) & \text{if } n>1\end{cases} \qquad (\varphi_1, \varphi_2\in S_1). \tag{41}$$

Dividing here by $|\psi|$ and taking into consideration that $B_1(x-a)\leqq|\psi(x)|$ $(x\in[a,\xi])$, we obtain (36). Since $[S_1, \varrho]$ is a complete metric space and the operator $F\colon S_1\to S_1$ is a contraction by (36), we can apply the Picard—Banach fixed point theorem which says that F has one and only one fixed point in S_1. This proves the sufficiency of the condition in Theorem 1.

Necessity: Suppose that the differential equation (1) has a solution $y(x)$ satisfying the given initial conditions. It can be shown that $y(x)$ is concave in a right neighbourhood of the point $x=a$, so there is a ξ, $a<\xi<b$, such that

$$\min\left(\frac{1}{\sqrt[n]{2}}, \frac{1}{2}\right)|\psi| \leqq |y| \quad (a \leqq x \leqq \xi < b). \tag{42}$$

It can be assumed that

$$|y'| \leqq \sqrt[n]{2}\, B_1 \tag{43}$$

is also satisfied on the interval $a \leqq x \leqq \xi$. Integrating the differential equation (1) we obtain

$$y'^{n*} = B^{n*} - n \int_a^x q(\tau) y^{n*}(\tau)\, d\tau, \tag{44}$$

whence by (42) and (43) we get (28). This completes the proof of Theorem 1.

Next we prove a series of lemmas.

LEMMA 1. *If in every neighbourhood of a we have*

$$\int_a^x q(\tau)\, d\tau = \infty \quad (a < x < b) \tag{45}$$

then $x_0'(>a)$ *exists for any choice of* x_1 $(a<x_1<b)$.

PROOF. Consider the solution of the differential equation (1) satisfying the initial condition $y(x_1)=0$, $y'(x_1)=-1$. Assume that $x_0'(>a)$ does not exist. Then $y'(x)<0$ $(a<x<x_1)$, hence we have here $y(x)>0$.

Let $a<x^*<x_1$. We have

$$y(x) > y(x^*) > 0 \quad (a < x < x_1),$$

so by (45) one can find a ξ $(a<\xi \leqq x^*)$ such that

$$\int_\xi^{x_1} q(\tau) y^{n*}(\tau)\, d\tau = 1/n. \tag{46}$$

Integrating the differential equation (1) on the interval $[x, x_1]$ we obtain

$$y'^{n*}(x) = -1 + n \int_x^{x_1} q(\tau) y^{n*}(\tau)\, d\tau \quad (a < x < b). \tag{47}$$

Substitution of (46) into (47) yields

$$y'(\xi) = 0,$$

contrary to the indirect assumption. Thus Lemma 1 is proven.

LEMMA 2. *Consider the differential equations*

$$y_i' |y_i'|^{n-1} + q_i y_i^{n*} = 0 \quad (i = 1, 2), \tag{48}$$

where $q_i = q_i(x)$ $(i=1, 2)$ *are twice continuously differentiable functions on the interval*

(a, b) $(-\infty < a < b \leqq \infty)$. *Suppose the existence of solutions* $y_i(x)$ $(i=1, 2)$ *on the interval* $[a, b]$.

Considering u $\left(u=\int\limits_a^x q_i(\tau)^{1/(n+1)}\,d\tau \ \ (i=1,2)\right)$ *as an independent variable, the differential equations* (48) *take the following form:*

(49) $$\ddot{Y}_i\,|\dot{Y}_i|^{n-1}+s_{q_i}\dot{Y}_i^{n^*}+Y_i^{n^*}=0 \quad (i=1,2), \quad (0\leqq u<u(b)),$$

where

$$Y_i(u)=y_i\bigl(x_{q_i}(u)\bigr).$$

Denote by $u_{i,j}, u'_{i,j}$ $(i=1,2)$, $(j=0,1)$ *the roots of the equations* $Y_i(u)=0$ *and* $\dot{Y}_i(u)=0$ $(0\leqq u<u(b))$, *respectively, provided that they exist.*

Suppose

(50) $$s_{q_1}(u)\leqq s_{q_2}(u) \quad \bigl(0\leqq u\leqq u(b)\bigr).$$

Then: a) $u_{1,0}=u_{2,0}$ *implies*

(51) $$u'_{1,0}\geqq u'_{2,0},$$

b) $u'_{1,0}=u'_{2,0}$ *implies*

(52) $$u_{1,1}\leqq u_{2,1}.$$

PROOF. Put in Case a) $v_1=u_{1,0}(=u_{2,0})$, $v_2=\min(u'_{1,0}, u'_{2,0})$, in Case b) $v_1=u'_{1,0}$ $(=u'_{2,0})$, $v_2=\min(u_{1,1}, u_{2,1})$. Then

(53) $$\dot{Y}_i(u)\neq 0 \quad (v_1<u<v_2),\ (i=1,2).$$

Define

$$W(u)=y_1'\bigl(x_{q_1}(u)\bigr)y_2\bigl(x_{q_1}(u)\bigr)-y_1\bigl(x_{q_1}(u)\bigr)y_2'\bigl(x_{q_1}(u)\bigr) \quad \bigl(0\leqq u<u(b)\bigr).$$

It can be shown that in both cases a) and b)

(54) $$W(v_1)=0.$$

The function W can be written in the form

(55) $$W(u)=\bigl(\dot{Y}_1(u)Y_2(u)-Y_1(u)\dot{Y}_2(u)\bigr)\bar{q}_1(u)^{1/(n+1)} \quad \bigl(0\leqq u<u(b)\bigr),$$

where $\bar{q}_1(u)=q_1\bigl(x_{q_1}(u)\bigr)$.

Put

(56) $$t_i(u)=\frac{Y_i(u)}{\dot{Y}_i(u)} \qquad (v_1<u<v_2),\ (i=1,2).$$

By (53) $t_i(u)$ $(i=1,2)$ is continuous in (v_1, v_2). Differentiating W we obtain by (49) and (56)

(57) $$\dot{W}=[Y_1\dot{Y}_2(s_{q_1}-s_{q_2})+Y_1Y_2(|t_2|^{n-1}-|t_1|^{n-1})]\bar{q}_1^{1/(n+1)} \quad (v_1<u<v_2).$$

Define now

(58) $$M_1(t_1, t_2)=\begin{cases} \dfrac{t_1t_2^{n^*}-t_1^{n^*}t_2}{t_2-t_1} & \text{if } t_1\neq t_2 \\ (n-1)t_1^{n^*} & \text{if } t_1=t_2. \end{cases}$$

Clearly, the function M_1 is homogeneous and continuous in $(-\infty, \infty)\times(-\infty, \infty)$. Set

(59) $$M(u) = M_1(t_1(u),\ t_2(u)) \quad (v_1<u<v_2).$$

M is a continuous function of u in the interval (v_1, v_2).

By (57), (58), (59) we obtain the following first order differential equation for W:

$$\dot{W} = Y_1\dot{Y}_2(s_{q_2}-s_{q_1})+MW \quad (v_1 < u < v_2).$$

Solving this differential equation and taking (54) into consideration we get

(60) $$W = \int_{v_1}^{u} \varphi(v)e^{\int_v^u M(\tau)d\tau}\,dv \quad (v_1 < u < v_2),$$

where

$$\varphi(v) = Y_1\dot{Y}_2(s_{q_2}-s_{q_1}).$$

Since $\varphi(u)$ is of constant sign in (v_1, v_2), (60) yields

(61) $$W(u)\cdot \operatorname{sign}\varphi(u) \geqq 0 \quad (v_1 < u < v_2).$$

Consider now Case a). By (55) we have

$$\dot{Y}_1(u) = \frac{W(u)\bar{q}_1(u)^{-(1/(n+1))}+Y_1(u)\dot{Y}_2(u)}{Y_2(u)} \quad (v_1 < u < v_2),$$

so by (61)

$$|\dot{Y}_1(u)| > 0 \quad (v_1 < u < v_2),$$

whence (51) follows by the definition of v_2.

Consider Case b). Also by (55)

$$Y_2(u) = \frac{W(u)\bar{q}_1(u)^{-(1/(n+1))}+Y_1(u)\dot{Y}_2(u)}{\dot{Y}_1(u)} \quad (v_1 < u < v_2),$$

so by (61)

$$|Y_2(u)| > 0 \quad (v_1 < u < v_2),$$

which implies (52).

For $q\equiv 1$ the differential equation (7) reduces to

(62) $$\ddot{Y}|\dot{Y}|^{n-1}+Y^{n*} = 0,$$

which admits as solutions satisfying the initial conditions $Y(0)=0$, $\dot{Y}(0)=1$ and $Y(0)=1$, $\dot{Y}(0)=0$ the functions

(63) $$Y(u) = S_n(u) \quad\text{and}\quad Y(u) = S_n\left(u\pm\frac{\hat{\pi}}{2}\right),$$

respectively. These solutions are periodical generalized trigonometric functions of period

(64) $$\hat{\pi} = \hat{\pi}(n) = 2\,\frac{\dfrac{\pi}{n+1}}{\sin\dfrac{\pi}{n+1}},$$

and the distance of neighbouring roots and maxima for them is $\hat{\pi}/2$ (see Á. Elbert [2]).

In the case $q(x)=Cx^{1/\nu}$ $(C>0)$, $(\nu\geqq 0$ or $\nu<-1/(n+1))$, $(a=0,\ b=\infty)$ let $z_1(x)$ and $z_2(x)$ be the solutions of the differential equation (1) satisfying the initial conditions $z_1(a)=0$, $z_1'(a)=1$ $\left(\nu\geqq 0,\ \nu<-\frac{1}{n+1}\right)$ and $z_2(a)=1$, $z_2'(a)=0$ $(\nu\geqq 0,$ $\nu<-1)$, respectively. These solutions exist by Theorem 1. In this case the differential equation (7) is

$$\ddot{Y}|\dot{Y}|^{n-1}+\alpha u^{-1}\dot{Y}^{n*}+Y^{n*}=0 \quad (0<u<\infty) \tag{65}$$

where α has been defined already. Put $Z_1(u)=z_1(x(u))$, $Z_2(u)=z_2(x(u))$. Now we have $Z_1(0)=0$, $Z_2(0)=1$, $\dot{Z}_2(0)=0$. Denote by $j_\nu'(n)$ and $j_\nu(n)$ the first positive roots of the equations $\dot{Z}_1(u)=0$ and $Z_2(u)=0$, respectively. For $n=1$, $Z_1(u)$ and $Z_2(u)$ are just the Bessel functions of the first kind $J_\mu(u)$ and $J_{-\mu}(u)$, respectively, where $\mu=\frac{\nu}{1+2\nu}$ $(\alpha=1-2\mu)$. One can see that $j_0'(1)=j_{\mu-1}$, $j_0(1)=j_{-\mu}$, where $j_{\mu-1}$ and $j_{-\mu}$ stand for the first positive roots of the Bessel functions $J_{\mu-1}$ and $J_{-\mu}$, respectively.

LEMMA 3. *For* $q\in C_\nu[a,b]$ $\left(\nu\geqq 0 \text{ or } \nu<-\frac{1}{n+1}\right)$ *we have*

$$\int_{x_0}^{x_0'} q(\tau)^{1/(n+1)}\,d\tau \begin{cases}\leqq \hat{\pi}/2 & \text{if } q'\geqq 0\\ \geqq \hat{\pi}/2 & \text{if } q'\leqq 0\end{cases} \quad (a<x<b). \tag{67}$$

PROOF. By (8) we have

$$s(u)\begin{cases}\geqq 0 & \text{if } q'\geqq 0\\ \leqq 0 & \text{if } q'\leqq 0\end{cases} \quad (a<x<b). \tag{68}$$

Applying Lemma 2 to the differential equation (7) and taking into consideration (68), we obtain (67).

LEMMA 4. *For* $q\in C_\nu[a,b]$ *we have*

a)

$$\int_{x_0}^{x_0'} q(\tau)^{1/(n+1)}\,d\tau \begin{cases}\geqq j_\nu(n) & \text{if } \nu\geqq 0\\ \leqq j_\nu'(n) & \text{if } \nu<-\frac{1}{n+1};\end{cases} \tag{69}$$

b)

$$\int_{x_0'}^{x_1} q(\tau)^{1/(n+1)}\,d\tau \begin{cases}\leqq j_\nu(n) & \text{if } \nu\geqq 0\\ \geqq j_\nu(n) & \text{if } \nu<-1\\ >0 & \text{otherwise } \left(-1\leqq \nu<-\frac{1}{n+1}\right).\end{cases} \tag{70}$$

PROOF. a) Applying Lemma 2 to the solutions $Y(u)$, $Z_1(u)$ of the differential equations (7) and (65), respectively, (13) yields (67).

b) The cases $\nu \geqq 0$ and $\nu < -1$ follow from Lemma 2 in the same way as in a). Consider the case $-1 \leqq \nu < -\frac{1}{n+1}$. Then there is a function $q \in C_\nu[a, b]$ such that $\int_a^x q(\tau)d\tau = \infty$ for any $a < x < b$ (e.g. $q = (x-a)^{1/\nu}$). In view of (2) an x^* $a < x^* < b$) can be chosen such that

$$\int_a^{x^*} q(\tau)^{1/(n+1)}\, d\tau < \varepsilon, \tag{71}$$

(where $\varepsilon > 0$ is an arbitrarily fixed positive number. Let $a < x_1 < x$. By Lemma 1 there exists an x_0' $(a < x_0' < x_1)$, hence (78) implies

$$\int_{x_0'}^{x_1} q(\tau)^{1/(n+1)}\, d\tau < \varepsilon,$$

which proves Lemma 4.

Lemma 5. *For* $q \in C_\nu[a, b]$ *we have*

$$\int_{x_0'}^{x_1'} q(\tau)^{1/(n+1)}\, d\tau \begin{cases} \geqq \hat{\pi} & \textit{if} \quad \nu \geqq 0 \\ \leqq \hat{\pi} & \textit{if} \quad \nu \leqq -\dfrac{2}{n+1}. \end{cases} \tag{72}$$

(In the case $n=1$ this lemma was proved by J. H. E. Cohn [4]).

Proof. Firstly we exhibit that it suffices to restrict oneself to the case when q is a monotonic function. One can assume q to be monotonic on the intervals $(a, x^*]$ and $[x^*, b)$, where $x_1 \leqq x^* < b$. Since $q \in C_\nu[a, b]$, (14) implies that q is in case $\nu \geqq 0$ first monotonically increasing and then decreasing, in case $\nu < -\frac{1}{n+1}$ first monotonically decreasing and then increasing.

Put

$$\bar{q}(x) = \begin{cases} q(x) & \text{if} \quad a < x < x^* \\ q(x^*) & \text{if} \quad x^* < x < b \end{cases} \quad (x_1 \leqq x^* < b).$$

We have

$$s_{\bar{q}}(u) \begin{cases} \geqq s(u) & \text{if} \quad \nu \geqq 0 \\ \leqq s(u) & \text{if} \quad \nu < -\dfrac{1}{n+1} \end{cases} \quad (0 < u < u(b)) \tag{73}$$

where $s(u) = s_q(u)$.

Consider the differential equation

$$\ddot{\bar{Y}}|\dot{\bar{Y}}|^{n-1} + s_{\bar{q}}\dot{\bar{Y}}^{n*} + \bar{Y}^{n*} = 0 \quad (0 < u < u(b)). \tag{74}$$

Suppose that $\bar{Y}(u)$ is a solution of (74) satisfying the inital conditions $\bar{Y}(u_1) = Y(u_1)$ $(=0)$, $\dot{\bar{Y}}(u_1) = \dot{Y}(u_1)$. Denote by $\bar{u}_i$ and $\bar{u}_i'$ $(i = 0, 1, \ldots)$ the roots of the equations $\bar{Y}(u) = 0$ and $\dot{\bar{Y}}(u) = 0$, respectively $(\bar{u}_1 = u_1)$. Comparing the solutions Y and $\bar{Y}$

of the differential equations (7) and (74), respectively, from (73) we infer by Lemma 2 that

$$(75)\qquad \bar{u}_1'-\bar{u}_0' \begin{cases} \leqq u_1'-u_0' & \text{if} \quad v \geqq 0 \\ \geqq u_1'-u_0' & \text{if} \quad v < -\dfrac{1}{n+1}. \end{cases}$$

By (75) it suffices to prove (72) for the monotonic function $\bar{q}$.

In the case $q \equiv \text{const}$ $(a<x<b)$ (72) is trivially true, therefore we can assume that $q \not\equiv \text{const}$. Furthermore we can suppose

$$(76)\qquad q' \begin{cases} \geqq 0 & \text{if} \quad v \geqq 0 \\ \leqq 0 & \text{if} \quad v \leqq -\dfrac{2}{n+1} \end{cases} \qquad (a<x<b).$$

Consider the differential equation

$$(77)\qquad \ddot{\bar{Y}}|\dot{\bar{Y}}|^{n-1}+\sigma_{u_1}\dot{\bar{Y}}^n+\bar{Y}^n=0 \quad (-C_1(u_1)<u<\infty)$$

where $C(u_1)$ and σ_{u_1} are defined in (18) and (19), respectively. By (21) and (22) we have

$$(78)\qquad s(u) \begin{cases} \geqq \sigma_{u_1} & \text{if} \quad v \geqq 0 \\ \leqq \sigma_{u_1} & \text{if} \quad v < -\dfrac{1}{n+1}, \end{cases} \qquad (0<u<u_1)$$

$$(79)\qquad s(u) \begin{cases} \leqq \sigma_{u_1} & \text{if} \quad v \geqq 0 \\ \geqq \sigma_{u_1} & \text{if} \quad v < -\dfrac{1}{n+1}. \end{cases} \qquad (u_1 \leqq u < u(b))$$

Let $\bar{Y}(u)$ be a solution of the differential equation (77) satisfying $\bar{Y}(u_1)=Y(u_1)$, $\dot{\bar{Y}}(u_1)=\dot{Y}(u_1)$. Denote by $\bar{u}_i$ and $\bar{u}_i'$ $(i=0, 1, \ldots)$ the roots of the equations $\bar{Y}(u)=0$ and $\dot{\bar{Y}}(u)=0$, respectively. We shall show that $\bar{u}_0'$ exists, i.e.

$$(80)\qquad \bar{u}_0' \geqq -C(u_1).$$

In the case $v \geqq 0$ we apply Lemma 2 to the solutions $Y(u)$ and $\bar{Y}(u)$ of the differential equations (7) and (77), respectively, and by (78) we obtain $\bar{u}_0' \geqq u_0'$, which implies (80).

Consider the case $v<-1/(n+1)$. Let $\bar{\bar{Y}}(u)$ be a solution of (77) satisfying $\bar{\bar{Y}}(-C(u_1)/2)=1$, $\dot{\bar{\bar{Y}}}(-C(u_1)/2)=0$. Denote by $\bar{\bar{u}}_1$ the first root of the equation $\bar{\bar{Y}}(u)=0$ $(-C(u_1)/2<u<\infty)$. Taking into consideration that $\sigma_{u_1}(-C(u_1)/2+u) \leqq \sigma_{u_1}(u)$ $(-C(u_1)/2<u<\infty)$, by Lemma 2 we obtain

$$\bar{\bar{u}}_1-C(u_1)/2<u_1-u_0',$$

which implies (80) for the solutions $\bar{\bar{Y}}$ and $\bar{Y}$ separate one another's zeros and maxima or minima, respectively (see Á. Elbert [2]).

In view of Lemma 2, from (78) and (79) we obtain that

$$u_1-\bar{u}_0' \begin{cases} \leqq u_1-u_0' & \text{if} \quad v \geqq 0 \\ \geqq u_1-u_0' & \text{if} \quad v < -\dfrac{1}{n+1} \end{cases}$$

and

$$\bar{u}_1'-u_1 \begin{cases} \leqq u_1'-u_1 & \text{if} \quad v \geqq 0 \\ \geqq u_1'-u_1 & \text{if} \quad v < -\dfrac{1}{n+1}, \end{cases}$$

hence

$$\bar{u}_1'-\bar{u}_0' \begin{cases} \leqq u_1'-u_0' & \text{if} \quad v \geqq 0 \\ \geqq u_1'-u_0' & \text{if} \quad v < -\dfrac{1}{n+1}. \end{cases} \tag{81}$$

By (81) it suffices to prove that

$$\bar{u}_1'-\bar{u}_0' \begin{cases} \geqq \hat{n} & \text{if} \quad v \geqq 0 \\ \leqq \hat{n} & \text{if} \quad v < -\dfrac{2}{n+1}. \end{cases} \tag{82}$$

This is what will be done in what follows. We shall suppose that

$$\sigma_{u_1} = \alpha u^{-1} \quad (0 < u < \infty) \tag{83}$$

(this can be achieved by a simple transformation).

Putting $\bar{t}=\bar{Y}/\dot{\bar{Y}}$, we have, similarly to (27),

$$\dot{\bar{t}} = 1+\alpha\bar{u}^{-1}\bar{t}+|\bar{t}|^{n+1} \quad (\dot{\bar{Y}} \neq 0). \tag{84}$$

Consider the inverse functions $\bar{u}(t)$ to $\bar{t}(u)$ in the interval $(\bar{u}_0', \bar{u}_1')$ and introduce the following notations:

$$\begin{aligned} v_1(t) &= \bar{u}_1-u(-t) \\ v_2(t) &= \bar{u}(t)-\bar{u}_1. \end{aligned} \quad (0 \leqq t < \infty). \tag{85}$$

From (84) and (85) we get

$$\frac{dv_1(t)}{dt} = \frac{1}{1-\alpha t(\bar{u}_1-v_1)^{-1}+t^{n+1}}, \tag{86}$$

$$\frac{dv_2(t)}{dt} = \frac{1}{1+\alpha t(v_2+\bar{u}_1)^{-1}+t^{n+1}}, \tag{87}$$

$$(0 \leqq t < \infty).$$

Define a function $F(t)$ as follows:

$$F(t) = v_1(t)+v_2(t)-2\int\limits_0^t \frac{d\tau}{1+\tau^{n+1}} \quad (0 \leqq t < \infty). \tag{88}$$

As is easy to see, $F(0)=0$. We are going to show that

(89)
$$\lim_{t\to\infty} F(t)\begin{cases}\geqq 0 & \text{if} \quad v \geqq 0\\ \leqq 0 & \text{if} \quad v \leqq -\dfrac{2}{n+1}.\end{cases}$$

This in turn implies (82) for $\int\limits_0^\infty \frac{d\tau}{1+\tau^{n+1}}=\hat{\pi}/2$.

By (84), (85), (86) we have

(90)
$$\frac{dF(t)}{dt}=$$
$$=\alpha t\frac{2\alpha t+(1+t^{n+1})\big(v_1(t)+v_2(t)\big)}{\big[(1+t^{n+1})^2\big(\bar{u}_1+v_1(t)\big)\big(v_2(t)-\bar{u}_1\big)+(1+t^{n+1})\big(v_1(t)+v_2(t)\big)-\alpha^2t^2\big](1+t^{n+1})}.$$

On the right-hand side of (90), the denominator is always positive, therefore it suffices to look at the sign of the numerator.

If $v\geqq 0$ then $\alpha>0$, hence by (90) we have $\frac{dF}{dt}>0$ $(0\leqq t<\infty)$, which establishes (89) in this case.

Let now $v\leqq -2/(n+1)$ and assume that (89) is not valid. Then there exists a subinterval $[t^*, t^{**})$ $(0\leqq t^*<t^{**}<\infty)$ of $[0,\infty)$ such that

(91)
$$F(t)\begin{cases}>0 & \text{if} \quad t^*<t<t^{**}\\ =0 & \text{if} \quad t=t^*.\end{cases}$$

As is easy to see,

(92)
$$\int\limits_0^t \frac{d\tau}{1+\tau^{n+1}}>\frac{t}{1+t^{n+1}} \quad (0<t<\infty).$$

By (91) and (92) we have

(93)
$$v_1(t)+v_2(t)>\frac{2t}{1+t^{n+1}} \quad (t^*<t<t^{**}).$$

Our condition $v\leqq -2/(n+1)$ implies $-1\leqq\alpha<0$, hence by (92) and (93):

$$2\alpha t+(1+t^{n+1})\big(v_1(t)+v_2(t)\big)>2t(\alpha+1)>0 \quad (t^*<t<t^{**}),$$

so $\frac{dF(t)}{dt}<0$ $(t^*<t<t^{**})$, and therefore

$$F(t)=\int\limits_{t^*}^t\frac{dF(\tau)}{d\tau}d\tau<0 \quad (t^*<t<t^{**}),$$

contrary to the indirect assumption. Thus Lemma 5 is proven.

LEMMA 6. *For* $q \in C_\nu[a, b]$ *we have*

(94) $$\int_{x_0}^{x_1} q(\tau)^{1/(n+1)}\, d\tau \begin{cases} \leqq \hat{\pi} & \text{if} \quad \nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right) \\ \geqq \hat{\pi} & \text{if} \quad \nu < -\dfrac{1}{n+1}. \end{cases}$$

(In the case $n=1$ this assertion is known as Makai's Lemma).

PROOF. We shall proceed in a similar way as in Lemma 5. Just as was done there, it can be shown that it suffices to deal with the case of monotonic functions $q(x)$.

Consider the differential equation

(95) $$\ddot{\bar{Y}}|\dot{\bar{Y}}|^{n-1} + \sigma_{u_0'}\dot{\bar{Y}}^{n^*} + \bar{Y}^{n^*} = 0 \quad (-C(u_0') < u < \infty).$$

By (21) and (22) we have

(96) $$s(u) \begin{cases} \geqq \sigma_{u_0'} & \text{if} \quad \nu \geqq 0 \\ \leqq \sigma_{u_0'} & \text{if} \quad \nu < -\dfrac{1}{n+1}, \end{cases} \qquad (0 < u < u_0')$$

(97) $$s(u) \begin{cases} \leqq \sigma_{u_0'} & \text{if} \quad \nu \geqq 0 \\ \geqq \sigma_{u_0'} & \text{if} \quad \nu < -\dfrac{1}{n+1}. \end{cases} \qquad (u_0' \leqq u < u(b))$$

Let $\bar{Y}$ be a solution of the differential equation (96) satisfying $\bar{Y}(u_0') = Y(u_0')$, $\dot{\bar{Y}}(u_0') = \dot{Y}(u_0') = 0$. Denote by u_i and $\bar{u}_i$ $(i=0, 1, \ldots)$ the roots of the equations $\bar{Y}=0$ and $\dot{\bar{Y}}=0$ $(-C(u_0') \leqq u < \infty)$, respectively. One can show that $\bar{u}_0 (\geqq -C(u_0'))$ exists.

Applying Lemma 2, from (96) and (97) we obtain

(98) $$\bar{u}_1 - \bar{u}_0 \begin{cases} \geqq u_1 - u_0 & \text{if} \quad \nu \geqq 0 \\ \leqq u_1 - u_0 & \text{if} \quad \nu < -\dfrac{1}{n+1}, \end{cases}$$

hence it suffices to exhibit

(99) $$\bar{u}_1 - \bar{u}_0 \begin{cases} \leqq \hat{\pi} & \text{if} \quad \nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right) \\ \geqq \hat{\pi} & \text{if} \quad \nu < -\dfrac{1}{n+1}. \end{cases}$$

We can assume that

$$\sigma_{u_0'} = \alpha u^{-1} \quad (0 < u < \infty).$$

Then, putting $\bar{t} = \bar{Y}/\dot{\bar{Y}}$, from (95) we obtain (84). Consider the inverse function $\bar{u}(t)$ to $\bar{t}(u)$ in intervals $[\bar{u}_0, u_0')$, $(u_0', u_1]$. Set

(100) $$\begin{aligned} v_1(t) &= u_0' - \bar{u}(t) \\ v_2(t) &= u(-t) - u_0. \end{aligned} \qquad (0 \leqq t < \infty)$$

By (84) and (100) we have

(101) $$\frac{dv_1(t)}{dt} = \frac{-1}{1+\alpha t(u_0'-v_1(t))^{-1}+t^{n+1}}$$

$$(0 \leqq t < \infty)$$

(102) $$\frac{dv_2(t)}{dt} = \frac{-1}{1-\alpha t(u_0'+v_2(t))^{-1}+t^{n+1}}.$$

Define

(103) $$F(t) = v_1(t)+v_2(t)-2\int_0^\infty \frac{d\tau}{1+\tau^{n+1}} \quad (0 \leqq t < \infty).$$

One can show that $\lim_{t\to\infty} F(t)=0$. We are going to prove

(104) $$F(0)\begin{cases} \leqq 0 & \text{if} \quad \nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right) \\ \geqq 0 & \text{if} \quad \nu < -\dfrac{1}{n+1}, \end{cases}$$

which in turn implies (99).

By (104), (101), (102) we have

(105)
$$\frac{dF(t)}{dt} = \alpha t \frac{-2\alpha t+(1+t^{n+1})(v_1(t)+v_2(t))}{[(1+t^{n+1})^2(u_0'-v_1(t))(u_0'+v_2(t))+(1+t^{n+1})(v_1(t)+v_2(t))-\alpha^2 t^2](1+t^{n+1})}$$
$$(0 \leqq t < \infty).$$

The denominator on the right-hand side of (105) is always positive, hence it suffices to consider the sign of the numerator.

If $\nu < -1/(n+1)$ then $\alpha < 0$, hence from (105) we get $\dfrac{dF}{dt} < 0$ $(0 \leqq t < \infty)$, thus

$$F(0) = -\int_0^\infty \frac{dF(\tau)}{d\tau} > 0,$$

and we are done.

Consider the case $\nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right)$. Suppose that (104) does not hold, then there is a t^* $(0 \leqq t^* < \infty)$ such that

(106) $$F(t)\begin{cases} > 0 & \text{if} \quad 0 \leqq t < t^* \\ = 0 & \text{if} \quad t = t^*. \end{cases}$$

By (103) and (106) we have

(107) $$v_1(t)+v_2(t) > 2\int_t^\infty \frac{d\tau}{1+\tau^{n+1}} \quad (0 \leqq t < t^*).$$

As is easy to see,

(108) $$\int_t^\infty \frac{d\tau}{1+\tau^{n+1}} > \frac{t}{n(1+t^{n+1})} \quad (0 \leqq t < \infty),$$

whence by (107):

$$v_1(t)+v_2(t) > \frac{2t}{1+t^{n+1}} \quad (0 \leqq t < t^*). \tag{109}$$

Since $0<\alpha\leqq 1/n$ in view of $v>\max\left(0, \frac{n-1}{n+1}\right)$, from (109) we get

$$-2\alpha t+(1+t^{n+1})\big(v_1(t)+v_2(t)\big) > 2t\left(\frac{1}{n}-\alpha\right) > 0 \quad (0 \leqq t < t^*), \tag{110}$$

whence $\frac{dF(t)}{dt}>0$ $(0\leqq t<t^*)$ and therefore

$$F(0) = -\int_0^t \frac{dF(\tau)}{d\tau}\,d\tau < 0,$$

contrary to the indirect assumption. This completes the proof of Lemma 6.

THEOREM 2. *If the function* $q\in C_v[a, b]$ *is monotonically increasing in the interval* (a, b) *then we have:*

a)

$$\left.\begin{array}{r} j'_v(n)+(n+1/2)\hat{\pi} \leqq \\ n\hat{\pi} \leqq \end{array}\right| \int_{x_0}^{x_n} q(\tau)^{1/(n+1)}\,d\tau \left\{\begin{array}{lll} \leqq n\hat{\pi} & \text{if} & v \geqq \max\left(0, \frac{n-1}{n+1}\right) \\ \leqq j'_n(n)+(n+1/2)\hat{\pi} & \text{if} & v \leqq -\frac{2}{n+1}, \end{array}\right.$$

b)

$$\left.\begin{array}{r} (n+1/2)\hat{\pi} \leqq \\ j'_v(n)+(n-1)\hat{\pi} < \end{array}\right| \int_{x_0}^{x'_n} q(\tau)^{1/(n+1)}\,d\tau \left\{\begin{array}{lll} \leqq j'_v(n)+n\hat{\pi} & \text{if} & v \geqq \max\left(0, \frac{n-1}{n+1}\right) \\ \leqq (n+1/2)\hat{\pi} & \text{if} & v \leqq -\frac{2}{n+1}, \end{array}\right.$$

c)

$$\left.\begin{array}{r} (n-1/2)\hat{\pi} \leqq \\ j_v(n)+(n-1)\hat{\pi} \leqq \\ (n-1)\hat{\pi} < \end{array}\right\} \int_{x'_0}^{x_{\cdot}} q(\tau)^{1/(n+1)}\,d\tau \left\{\begin{array}{lll} \leqq j_v(n)+(n-1)\hat{\pi} & \text{if} & v \geqq \max\left(0, \frac{n-1}{n+1}\right) \\ \leqq (n-1/2)\hat{\pi} & \text{if} & v < -1 \\ \leqq (n-1/2)\hat{\pi} & \text{if} & v \leqq -\frac{2}{n+1}, \end{array}\right.$$

d)

$$\left.\begin{array}{r} n\hat{\pi} \leqq \\ j_v(n)+(n-1/2)\hat{\pi} \leqq \\ (n-1)\hat{\pi} < \end{array}\right\} \int_{x'_0}^{x'_n} q(\tau)^{1/(n+1)}\,d\tau \left\{\begin{array}{lll} \leqq j_v(n)+(n-1/2)\hat{\pi} & \text{if} & v \geqq \max\left(0, \frac{n-1}{n+1}\right) \\ \leqq n\hat{\pi} & \text{if} & v < -1 \\ \leqq n\hat{\pi} & \text{if} & v \leqq -\frac{2}{n+1}. \end{array}\right.$$

PROOF. a) Consider the case $v\geqq\max\left(0, \frac{n-1}{n+1}\right)$. The right-hand side of the relation follows from Lemma 6. In order to prove the left hand side, decompose the interval $[x_0, x_n]$ into $[x_0, x_n]=[x_0, x_0']\cup[x_0', x_{n-1}']\cup[x_{n-1}', x_4]$ and apply Lemmas 3, 5, 4. The case $\nu \leqq -\dfrac{2}{n+1}$ can be treated in a similar way.

b) Decompose the interval $[x_0, x_n']$ into $[x_0, x_n']=[x_0, x_0']\cup[x_0', x_n']$ and apply Lemmas 3, 5 in case $\nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right)$ and Lemmas 4, 5 in case $\nu < -\dfrac{2}{n+1}$.

The cases c) and d) can be settled similarly to the previous ones.

THEOREM 3. *For* $q\in C_\nu[a, b]$ *we have:*

a)

$$\left.\begin{array}{r} 2j_n'(n)+(n-1)\hat{\pi} \leqq \\ n\hat{\pi} \leqq \end{array}\right| \int_{x_0}^{x_n} q(\tau)^{1/(n+1)}\, d\tau \left\{\begin{array}{lll} \leqq n\hat{\pi} & if & \nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right) \\ \leqq 2j_\nu'(n)+(n-1)\hat{\pi} & if & \nu \leqq -\dfrac{2}{n+1}, \end{array}\right.$$

b)

$$\left.\begin{array}{r} j_\nu'(n)+n\hat{\pi} \leqq \\ j_\nu(n)+n\hat{\pi} \leqq \\ n\hat{\pi} < \end{array}\right\} \int_{x_0}^{x_n'} q(\tau)^{1/(n+1)}\, d\tau \left\{\begin{array}{lll} \leqq (n+1/2)\hat{\pi} & if & \nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right) \\ \leqq j_\nu'(n)+n\hat{\pi} & if & \nu < -1 \\ \leqq j_\nu'(n)+n\hat{\pi} & if & \nu < -\dfrac{2}{n+1}, \end{array}\right.$$

c)

$$\left.\begin{array}{r} n\hat{\pi} \leqq \\ 2j_\nu(n)+(n-1)\hat{\pi} \leqq \\ (n-1)\hat{\pi} < \end{array}\right\} \int_{x_0'}^{x_n'} q(\tau)^{1/(n+1)}\, d\tau \left\{\begin{array}{lll} \leqq 2j_\nu(n)+(n-1)\hat{\pi} & if & \nu \geqq \max\left(0, \dfrac{n-1}{n+1}\right) \\ \leqq n\hat{\pi} & if & \nu < -1 \\ \leqq n\hat{\pi} & if & \nu < -\dfrac{2}{n+1}. \end{array}\right.$$

PROOF. As was in Theorem 2, the proof is done by applying Lemmas 3, 4, 5, 6.

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ОПЕРАЦИЯ ФАКТОРИЗАЦИИ НА ГИПЕРГРАФАХ

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Резюме

В статье рассматривается операция, являющаяся обобщением факторизации конечного множества под действием введенного на нем отношения эквивалентности. Показано, что введение этой операции резко упрощает решение ряда задач теории перечисления гиперграфов, теории функций Мёбиуса.

Используется определение гиперграфа из [1]. Таким образом, запрещены кратные и пустые ребра, но могут иметься вложенные ребра и ребра, содержащие одну вершину.

Операция факторизации на гиперграфах является обобщением понятия факторизации конечного множества под действием отношения эквивалентности. При такой операции сохраняется внутренняя структура гиперграфа, но меняется число его вершин, некоторые из них образуют естественные классы эквивалентности, дающие по представителю в факторизованный гиперграф.

Впервые, насколько известно автору, операция факторизации в неявной форме была применена при попытке перечислить всевозможные топологии на конечном множестве в работе [2, p. 1092, (I)].

Определение 1. Минимальной окрестностью вершины $x \in X$ гиперграфа $G(X, V_X)$ будем называть пересечение всех ребер, содержащих x.

Определение 2. Две вершины гиперграфа $G(X, V_X)$ будем называть эквивалентными, если их минимальные окрестности совпадают.

Пусть дан произвольный гиперграф $G(X, V_X)$.

Определение 3. Назовем факторгиперграфом $\mathscr{F}act\, G(X, V_X)$ гиперграфа $G(X, V_X)$ гиперграф, получаемый заменой каждого класса эквивалетности в смысле определения 2 единственной вершиной. Множество вершин в $\mathscr{F}act\, G(X, V_X)$ является индуцированным ребром тогда и только тогда, когда полный прообраз этого множества при операции $\mathscr{F}act$ является ребром $G(X, V_X)$.

Пусть дан класс H гиперграфов. Тогда операция $\mathscr{F}act$ факторизации индуцирует операцию $H \to \mathscr{F}act\, H$ на множестве классов. Пусть $H(n, r)$-число гиперграфов из H с n (помеченными) вершинами и ребрами;

$$H(n) \overset{\mathrm{def}}{=\!=} \sum_r H(n, r).$$

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Основная лемма. *Предположим что* $G(X, V_X) \in H$ *тогда и только тогда когда* $\mathscr{F}act\; G(X, V_X) \in \mathscr{F}act\; H$. *Тогда имеет место пара обратимых соотношений (inverse relations):*

$$H(n, r) = \sum_{k=0}^{n} S(n, k)(\mathscr{F}act\, H)(k, r); \quad n \geqq 0, \tag{1}$$

$$(\mathscr{F}act\, H)(n, r) = \sum_{k=0}^{n} s(n, k) H(k, r); \quad n \geqq 0, \tag{2}$$

с последовательностями Стирлинга второго и первого родов соответственно.

Доказательство. Рассмотрим конечное множество X. Распределим вершины X в блоки, и будем трактовать последние в духе «обощенных» вершин, на множестве которых индуцирована линейная лексикографическая пометка. (1) следует применением правил произведения и суммы; (2) следует обращением из (1) с учетом «квазиортогональности», ибо

$$\{S(n, k)\}_{\substack{n \geqq 0 \\ k \geqq 0}} \quad \text{и} \quad \{s(n, k)\}_{\substack{n \geqq 0 \\ k \geqq 0}}$$

суть взаимно обратные функции алгебры инцидентности множества неотрицательных целых чисел с обычным порядком «больше либо равно» [3; p. 344], [4; ch. IV, § 64, pp. 182—183 (1)—(2)].

Замечание 1. (1) и (2) доставляют каноническую комбинаторную интерпретацию пары обратимых соотношений, наиболее часто встречающейся в исчислении комбинаций (перечислении) [5; гл. 2, стр. 94—95, № 21].

Замечание 2. Соотношение (2) в неявной операционной форме приведено в [6; p. 4, лемма 3.1; corollary 3.2] — достаточно сравнить с [1; лемма на стр. 76]; см также [7; § 4, p. 119, (4.1)—(4.2)].

Следствие 1. *Пусть MC (minimal covers) есть класс минимальных покрытий* [8]. *Каждое ребро которых содержит по меньшей мере одну вершину степени (кратности) 1. Тогда*

$$(\mathscr{F}act\, MC)(k, r) = \binom{k}{r} (2^r - r - 1)_{k-r}. \tag{3}$$

Действительно:

1. Выделим r вершин из числа k и образуем с их помощью r одновершинных ребер.

2. Покроем каждую из оставшихся $k-r$ вершин попарно различными неупорядоченными наборами (всего их имеется $2^r - r - 1$) ребер из не менее чем двух ребер. Это дает второй множитель.

Из (1) и (3) имеем:

$$MC(n, r) = \sum_{k=0}^{n} S(n, k) \binom{k}{r} (2^r - r - 1)_{k-r}, \quad n \geqq 0. \tag{4}$$

Соотношение (4) аннонсировано в [9; стр. 1067, (6)] подробности см. в [8; стр. 89—90, теорема 5].

Следствие 2. *Пусть* $\mathscr{B}$*-класс белловских разбиений, трактуемых как гиперграфы* [1; стр. 71]. *Число таких разбиений на множестве* X; $|X| = n$, *есть* *n-е число Белла* $B(n) \overset{\text{def}}{=} \sum_{k=0}^{n} S(n, k)$; $n \geqq 0$. *Рассмотрим класс* $\mathcal{T}r$ *тривиальных разбиений, все ребра которых имеют мощность* 1.

Очевидно, что $\mathcal{T}r(n) \equiv 1$; $n \geqq 0$. *С другой стороны,* $\mathcal{T}r = \mathcal{F}ac^t\, \mathcal{B}$, *и* (2) *дает нам тождество* [1; стр. 75, (10)]:

$$\sum_{k=0}^{n} s(n, k) B(k) = 1; \quad n \geqq 0. \tag{5}$$

Следствие 3. *Пусть* $\mathcal{P}$*-класс неупорядоченных пар независимых разбиений конечных множеств* (1). *Заметим, что* $\mathcal{P}$ *сть факторкласс класса* $\mathcal{B}^{\langle 2 \rangle}$ *произвольных неупорядоченных пар разбиений, причем каждая пара трактуется как гиперграф с отождествлением кратных ребер. Имеет место*

$$B^{\langle 2 \rangle}(k) = \binom{B(k)+1}{2} = \frac{B(k)[B(k)+1]}{2}. \tag{6}$$

Действительно, в нашем распоряжении $B(k)$ типов разбиений, и мы выбираем 2 разбиения, причем повторение типов возможно.

Из (2) и (6) имеем:

$$P(n) = \sum_{k=0}^{n} s(n, k) \frac{B(k)[B(k)+1]}{2}; \quad n \geqq 0, \tag{7}$$

откуда, используя (5), выводим основной результат из (1):

$$P(n) = \frac{1}{2} + \sum_{k=0}^{n} s(n, k) \frac{[B(k)]^2}{2}; \quad n \geqq 0, \tag{8}$$

который ранее был получен путем громоздкого анализа двудольного соответствия [10; p. 103, п. 3] между блоками пары разбиений.

Следствие 4. *Рассмотрим решетку* $B(X)$ *разбиений конечного множества* X, *трактуемых как гиперграфы* [1]. *Пусть число блоков разбиения* α *есть* k $(1 \leqq k \leqq |X|)$. *Тогда сегмент* $[\alpha, I]$ *в смысле Рота* [3] *изоморфен сегменту* [0, 1] *в решетке разбиений множества из* k *элементов. Легко видеть, что изоморфизм устанавливается фактор—операцией, а роль вершин образа играют блоки* α.

Положим $\mu_0(0, I) = 0 = \mu_0$; $\mu_n(0, I) = \mu_n = \mu(0, I)$ в $B(X)$; $|X| = n \geqq 1$, и, применяя обычную рекуррентность с «левым свободным концом» для функции μ Мёбиуса:

$$\sum_{x:\, 0 \leqq x \leqq I} \mu(x, I) = \delta(0, I),$$

имеем:

$$\sum_{k=0}^{n} S(n, k) \mu_k = \delta_{n,1}; \quad \text{где} \quad n = |X| \geqq 0, \tag{9}$$

откуда обращением получаем:

$$(10) \qquad \mu_n = \sum_{k=0}^{n} s(n, k)\delta_{k,1} = s(n, 1) = (-1)^{n-1}(n-1)!; \quad n \geqq 1.$$

Последнее равенство можно найти, например, в [4; ch. IV, § 52, p. 147, 1].

Замечание 3. Этим путем Марсель—Поль Шютценберже пытался вычислить μ на $B(X)$ в своей диссертации [11; p. 25], но ряд неаккуратностей привел его к ошибке. Затем, уже совершенно точно, этот результат установили Р. Фрухт и Дж. К. Рота в двух совместных работах 63 и 65 годов [12; p. 113, (12)]; [13; p. 9, (20)] после чего с более общих позиций этот результат репродуцирован Рота [3; pp. 359—360]. В настоящее время существует немало различных доказательств этого замечательного предложения, имеющего многочисленные применения в математике и физике.

Замечание 4. При подстановке явного выражения μ из (10) в (9) получаем хорошо известное тождество, которое приводится в обширном числе источников, как-то: [14; p. 263, (36)]; [4, p. 189, § 67, (18)]; [5, гл. V, стр. 188—189, № 5a)]; [15]; [16; п. 4, p. 296, 5-е тождество]; [17; II, p. 254]; [18; p. 68, № 44]; [19; стр. 181, (5—135)] и пр. с ошибками и без надлежащей комбинаторной интерпретации. Элементарное доказательство этого тождества будет дано в Дополнении 1.

Замечание 5. Можно интерпретировать пару (9)—(10) аналогично (1)—(2), вводя «идеальные» классы гиперграфов Отметим, что $\delta_{n,1}$ «перечисляет» класс, состоящий из единственного одноточечного гиперграфа.

Введем частичный порядок на множестве $G(X)$ гиперграфов с множеством вершин X, обобщающий частичный порядок на решетке $B(X)$ разбиений конечного X.

Определение 4 (Ф. В. Широков). Будем говорить, что $G_1(X, V_X^1) \geqq G_2(X, V_X^2)$, если каждое ребро $A \in V_X^2$ покрыто по меньшей мере одним ребром $B \in V_X^1$, т. е. $A \subseteqq B$.

Задачи (Ф. В. Широков).

1. Вычислить функцию Мёбиуса на системе $MC(X)$ минимальных покрытий множества X; $|X| \geqq 1$.

2. Вычислить функцию Мёбиуса системы минимальных покрытий из ровно r ребер множества X. Отметим, что в отличие от случая разбиений эта система не является антицепью.

Решение первой задачи автору настоящей работы не известно. Решение второй задачи дано В. А. Сигнаевским в 72 году в период сотрудничества с Ф. В. Широковым в области теории покрытий и приводится в Дополнении II с любезного разрешения Ф. В. Широкова и В. А. Сигнаевского.

Дополнение 1

Дадим элементарное доказательство тождества

$$\sum_{k=1}^{n} S(n, k)(-1)^{k-1}(k-1)! = \delta_{n,1}; \quad n \geqq 1, \tag{11}$$

с помощью которого считается μ на $B(X)$. Используя общеизвестное представление общего члена $\{S(n, k)\}_{\substack{n \geqq 0 \\ k \geqq 0}}$

$$S(n, k) = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} l^n; \quad n \geqq k \geqq 0; \quad 0^0 \overset{\text{def}}{=} 1, \tag{12}$$

перепишем (11) в виде

$$\sum_{r=1}^{n} (-1)^{r-1} \frac{1}{r} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} i^n = \delta_{n,1}; \quad n \geqq 1. \tag{13}$$

В таком виде (11) фигурирует в [15], [18] и [19] (уточнения страниц и формул см. выше).

Рассмотрим два случая:

а) $n=1$. Этот случай рассматривается непосредственно.

в) $n>1$. Используя тот факт, что

$$\frac{i}{r}\binom{r}{i} = \binom{r-1}{i-1} = \binom{r}{i} - \binom{r-1}{i}; \quad r \geqq 1; \quad i \geqq 1$$

перепишем левую часть (13) в виде

$$\sum_{r=1}^{n} \left(\sum_{i=0}^{r} \binom{r}{i} (-1)^{i-1} i^{n-1} - \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^{i-1} i^{n-1} \right) =$$

$$= (a_1 - a_0) + (a_2 - a_1) + \ldots + (a_n - a_{n-1}) = a_n \quad (\text{т.к. } a_0 = 0),$$

где

$$a_r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{i-1} i^{n-1}; \quad r \geqq 0; \; n \geqq 2.$$

Таким образом, при $n>1$ левая часть (13) с точностью до знака равна разности порядка n от полинома x^{n-1} степени $n-1$, т.е. равна нулю. Случай в) проверен, и тождество (11) полностью доказано.

Дополнение 2

Пусть у нас имеются два r-рёберных минимальных покрытия $\alpha=[A_1, \ldots, \ldots, A_r]$ и $\beta=[B_1, \ldots, B_r]$ одного и того же конечного множества и пусть $\alpha \leqq \beta$ в смысле определения 4. Это значит, что для любого A_i найдется B_j такое, что $A_i \subseteqq B_j$.

Под кратностью точки (степенью вершины) будем понимать число ребер фиксированного покрытия, содержащих эту вершину.

Теорема (В. А. Сигнаевский).

$$\mu(\alpha, \beta) = (-1)^{\text{сумма кратностей вершин } \beta \text{ — сумма кратностей вершин } \alpha}.$$

Доказательство[1] разбивается на ряд предложений.

I. Каждое $B_j \in \beta$ содержит целиком по меньшей мере одно $A_i \in \alpha$. Действительно, в противном случаей найдется в α семейство $A_{i_1}, \ldots, A_{i_s}$, покрывающее B_j, и соответствующее семейство $B_{i_1}, \ldots, B_{i_s}$, где $A_{i_k} \subseteq B_{i_k}$, полностью покроет B_j, что противоречит минимальности β.

II *(Основное предложение)*. Каждое B_j содержит в точности одно A_i. Доказательство проведем индукцией по r. Проверка случаев $r=1$ и $r=2$ проводится непосредственно. Перед индуктивным переходом докажем 2 вспомогательных утверждения.

а) Пусть B_j содержит A_i и A_k, тогда существует A_s, нокрытое как минимум двумя ребрами из β. Действительно, в силу 1 имеются две возможности:

1° Либо существует ребро B_t, $t \neq j$, содержащее A_i или A_k.

2° Либо всем ребрам B_t, $t \neq j$, соответствуют A_p $(p \neq i, k)$, но здесь работает предложение 1 и принцип Дирихле, поскольку имеется $r-1$ значений параметра t и $r-2$ значений параметра p.

в) Пусть $A_l \subseteq B_j \cap B_k$. Тогда найдутся такие A_s и A_t, что:

$$A_s \subseteq B_j, \quad A_s \cap (B_j \setminus B_k) \neq \emptyset;$$

$$A_t \subseteq B_k, \quad A_t \cap (B_k \setminus B_j) \neq \emptyset.$$

Отметим, что A_s и A_t различны, ибо $(B_j \setminus B_k) \cap (B_k \setminus B_j) = \emptyset$. Рассуждаем так же, как и при доказательстве предложения 1. Если нет искомого A_s, то найдется набор $A_{i_1}, \ldots, A_{i_q}$, покрывающий $B_j \setminus B_k$, причем $A_{i_k} \nsubseteq B_j$ $(k=1, \ldots, q)$. Но тогда соответсвующий набор $B_{i_1}, \ldots, B_{i_q}$ покроет $B_j \setminus B_k$, а набор $B_{i_1}, \ldots, \ldots, B_{i_q}, B_k$ покроет B_j, что противоречит минимальности β, ибо B_{i_p} отличны от B_j при всех $p=1, \ldots, q$.

с) Доказательство предложения II. Пусть оно выполнено для всех значений вплоть до $r-1$ и не выполнено для r. Тогда найдутся A_l, B_j и B_k, A_s и A_t, указанные в предыдущем пункте.

Удалим из α ребро A_l, а из β вершины $\{A_l\}$. Покажем, что оставшиеся покрытия α' и β', индуцированные исходными α и β, суть минимальные.

Действительно α'-минимально, ибо однократные точки ребер лежат вне удаленного A_l, β'-минимально, так как вершины A_l по меньшей мере двукратны (покрыты B_j и B_k) в β.

После удаления $\{A_l\}$ объединим в β' ребра $\{B_j \setminus A_l\}$ и $\{B_k \setminus A_l\}$ в одно ребро $\{(B_j \cup B_k) \setminus A_l\}$. Получим так же минимальное покрытие β''. Ребро $\{(B_j \cup B_k) \setminus A_l\} \in \beta''$ содержит целиком два различных ребра $\{A_s \setminus A_l\}$ и $\{A_t \setminus A_l\}$

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[1] Настоящее доказательство было дано автором после того, как В. А. Сигнаевзкий сообщил автору формулировку теоремы.

в силу пункта в). Кроме того, $\beta'' \geqq \alpha'$, α' и β'' имеют одно и то же число ребер. Это приводит к противоречию с предположением индукции. *Итак, мы доказали предложение* II.

В дальнейшем соответствующие друг другу ребра минимальных покрытий α и β будем метить одним индексом.

III. Пусть A_i и B_i соответствующие ребра в α и β; $A_i \subsetneqq B_i$.

Тогда множество однократных точек B_i содержится в A_i. Пусть, напротив, y-однократная точка B_i, лежащая в $B_i \setminus A_i$. Тогда найдется ребро $A_j \in \alpha$ содержащее эту точку, и соответствующее ему $B_j \supseteqq A_j \ni y$, отличное от B_i в силу II, что приводит к противоречию предположение об однократности y в β.

IV. Пусть $A_i \subsetneqq B_i$. Тогда множество однократных точек ребра B_i содержится в множестве однократных точек ребра A_i. Пусть, напротив, x-однократная точка B_i, которая является кратной (т. е. кратность которой не меньше 2) в A_i. Тогда найдется отличное от A_i ребро $A_j \ni x$, и отличное от B_i ребро $B_j \supseteqq A_j \ni x$, что приводит к противоречию.

V. Теперь, после предварительной подготовки, можно перейти к выводу формулы Сигнаевского.

Рассмотрим точку i. Если она однократна в β, то в силу IV она заведомо однократна в α. Если же i покрыта в α ребрами $A_{i_1}, \ldots, A_{i_s}$; $s \geqq 2$, то она покрыта в β соответствующими ребрами: $B_{i_1}, \ldots, B_{i_s}$, где $B_{i_k} \supseteqq A_{i_k}$; $k=1, \ldots, s$. Но из числа $r-s$ ребер B_p $(p \neq i_k,\ 1 \leqq k \leqq s)$, покрывающих другие A_t $(t \neq i_k;\ 1 \leqq k \leqq s)$, могут так же найтись покрывающие вершину i. Пусть имеется $n-p$ кратных вершин $(p \geqq r$, p-число однократных вершин) в β.

Тогда сегмент $[\alpha, \beta]$ распадается в прямое произведение $n-p$ сегментов (некоторые из них могут вырождаться в точки), каждый из которых изоморфен единичному кубу размерности, равной избытку кратности β над α в соответствующей кратной вершине.

Отсюда, в силу формулы для μ на множестве всех подмножеств конечного множества [3; p. 345] и в силу теоремы о произведении [3; p. 345, proposition 5], следует формула Сигнаевского для $\mu(\alpha, \beta)$, *что и требовалось.*

В заключение автор выражает свою искреннюю признательность коллегам: Ф. В. Широкову за сообщение определения 4 и связанных с этим опрделением задач 1 и 2, и В. А. Сигнаевскому за сообщение результата теоремы из Дополнения 2.

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ON THE PROPERTIES OF $k^{\sim}$-COERCIVE LINEAR PARTIAL DIFFERENTIAL OPERATORS

JOUKO TERVO

1. Introduction

In this article we shall generalize the notion of $2t$-coercive linear partial differential operators (see [3], [6] and [7]). We work in certain subspaces of tempered distributions having a Banach space structure. For the properties of these spaces we refer to [4], pp. 33—62.

In the third part of this work we consider the global regularity of the solutions u for the distributional equation

$$L(D)u = f, \tag{1.1}$$

where $L(D)$ is a $k^{\sim}$-coercive (see 3.1) partial differential operator with constant coefficients. By applying Sobolev's lemma to the results in Section 3.3 we obtain the following particular result: Every solution $u\in\mathscr{H}_{p,-\infty}$ of (1.1) with $f\in\mathscr{H}_{p,\infty}$ lies in $C^{\infty}(\mathbf{R}^n)$. We point out that in this result it is essential that the distributional solution u is originally an element of the space $\mathscr{H}_{p,-\infty}$, which is in some sense a global subspace of distribution space.

In the fourth part we consider a $k^{\sim}$-coercive operator with variable coefficients in a bounded set $G\subset\mathbf{R}^n$. The main interest lies in the semi-Fredholm properties of $L^{\sim}_{p,k,G}$.

2. Preliminaries

2.1. Let G be an open subset in $\mathbf{R}^n$. For the definition of spaces $D(G)$, $S(\mathbf{R}^n)$, $D'(G)$, $S'(\mathbf{R}^n)$ and $E'(G)$ we refer to [4]. Furthermore, let K be a totality of all temperate weight functions such as in [4]. Denote by F the Fourier transform $S'(\mathbf{R}^n)\to\to S'(\mathbf{R}^n)$.

Define a norm $\|\cdot\|_{p,k}$: $C_0^{\infty}(G)\to\mathbf{R}$ by the requirement

$$\|\varphi\|_{p,k} = \Big(\lambda_n \int_{\mathbf{R}^n} |(F\varphi)(\xi)k(\xi)|^p\, d\xi\Big)^{1/p}, \tag{2.1}$$

where $1\leq p<\infty$, $k\in K$ and $\lambda_n=(2\pi)^{-n}$. Let $\mathscr{H}^{\sim}_{p,k}(G)$ be the completion of $C_0^{\infty}(G)$ with respect to the norm (2.1). Then the mapping L: $\mathscr{H}^{\sim}_{p,k}(G)\to S'(\mathbf{R}^n)$ defined by

$$L(E)(\psi) = \lim_{n\to\infty} \varphi_n(\psi) := \lim_{n\to\infty} \int_{\mathbf{R}^n} \varphi_n(x)\psi(x)\,dx, \quad \psi\in C_0^{\infty}(\mathbf{R}^n) \tag{2.2}$$

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is a linear injection, where $\{\varphi_n\}$ is a representative of E. Let $\mathcal{H}_{p,k}(G)$ be a subspace of $S'(\mathbf{R}^n)$ such that $\mathcal{H}_{p,k}(G)=L(\tilde{\mathcal{H}}_{p,k}(G))$ equipped with the topology induced by the norm $\|T\|_{p,k}=\|L^{-1}(T)\|_{p,k}=\lim_{n\to\infty}\|\varphi_n\|_{p,k}$, where $\{\varphi_n\}$ is a representative of $L^{-1}(T)$.

The following characterization of the space $\mathcal{H}_{p,k}:=\mathcal{H}_{p,k}(R^n)$ is obvious.

THEOREM 2.1. *A distribution $T\in S'(\mathbf{R}^n)$ lies in $\mathcal{H}_{p,k}$ iff FT is a function and*

$$N_{p,k}(T) := \Big((2\pi)^{-n}\int_{\mathbf{R}^n}|(FT)(\xi)k(\xi)|^p\,d\xi\Big)^{1/p} < \infty. \tag{2.3}$$

Moreover $N_{p,k}(T)=\|T\|_{p,k}$.

2.2. Let $L(x, D)$ be a partial differential operator of the form

$$L(x, D) = \sum_{|\sigma|\le r} a_\sigma(x)D^\sigma \tag{2.4}$$

with $a_\sigma\in C^\infty(G)$. Furthermore, let

$$L'(x, D) = \sum_{|\sigma|\le r} (-D)^\sigma\big(a_\sigma(x)(\cdot)\big) \tag{2.5}$$

be the formal transpose operator of $L(x, D)$. For arbitrary $k\in K$ and $1\le p<\infty$ we introduce a linear operator $L_{p,k,G}$: $\mathcal{H}_{p,k}(G)\to\mathcal{H}_{p,k}(G)$ by the requirement

$$\begin{cases} D(L_{p,k,G}) = C_0^\infty(G) \\ L_{p,k,G}\varphi = L(x, D)\varphi, \quad \varphi\in D(L_{p,k,G}). \end{cases} \tag{2.6}$$

Then $L_{p,k,G}$ is closeable; let $\tilde{L}_{p,k,G}$ be its smallest closed extension. We write $\tilde{L}_{p,k,\mathbf{R}^n}=\tilde{L}_{p,k}$.

We define an operator $L'^{\#}_{p,k}$: $\mathcal{H}_{p,k}\to\mathcal{H}_{p,k}$ such that

$$\begin{cases} D(L'^{\#}_{p,k}) = \{u\in\mathcal{H}_{p,k}| \text{ for which there exists } f\in\mathcal{H}_{p,k}; \\ \qquad\qquad u(L'_{p,k}\varphi) = f(\varphi) \quad \text{for all} \quad \varphi\in C_0^\infty(\mathbf{R}^n)\} \\ L'^{\#}_{p,k}u = f. \end{cases} \tag{2.7}$$

The $L'^{\#}_{p,k}$ is a closed operator and moreover $\tilde{L}_{p,k}\subset L'^{\#}_{p,k}$.

For the operators with constant coefficients we have the following theorem (see [2] and [6]).

THEOREM 2.2. *Let $L(D)$ be an operator with constant coefficients. Then*

$$L'^{\#}_{p,k} = \tilde{L}_{p,k}. \tag{2.8}$$

PROOF. Let $\psi\in C_0^\infty(\mathbf{R}^n)$ such that $0\le\psi$, $\operatorname{supp}\psi\subset\bar{B}(0, 1)$ and $(F\psi)(0)=\int_{\mathbf{R}^n}\psi(x)dx=1$. Furthermore, let $\psi_j\in C_0^\infty(\mathbf{R}^n)$ be such that $\psi_j(x)=j^n\psi(jx)$, $j\in\mathbf{N}$. Then for arbitrary $u\in D(L'^{\#}_{p,k})$ the convolution $u*\psi_j$ lies in $D(\tilde{L}_{p,k})\cap C^\infty(\mathbf{R}^n)$. In addition $u*\psi_j\to u$ in $\mathcal{H}_{p,k}$ and $\tilde{L}_{p,k}(u*\psi_j)=L(D)(u*\psi_j)=(L'^{\#}_{p,k}u)*\psi_j\to L'^{\#}_{p,k}u$ in $\mathcal{H}_{p,k}$. This means $L'^{\#}_{p,k}\subset\tilde{L}_{p,k}$, as required. □

3. On the global regularity of the solutions of the distributional equation $L(D)u=f$

3.1. We consider the algebraic characterization of the following inequality (that is, the $k^{\sim}$-coercivity of $L(D)$)

$$\|L(D)\varphi\|_{p,k} \geqq C_1\|\varphi\|_{p,kk^{\sim}} - C_2\|\varphi\|_{p,k}, \quad \varphi\in C_0^\infty(\mathbf{R}^n), \tag{3.1}$$

where $k, k^{\sim}\in K$ and $1\leqq p<\infty$. In the case when $p=2$, $k=1$ and $k\cong k_{2t}$ we refer to [6] and [7]. Assume first that G is the whole space $\mathbf{R}^n$. Then we have

THEOREM 3.1. *Let $L(D)$ be an operator with constant coefficients. Then there exist constants $C_1>0$ and $C_2\geqq 0$ such that* (3.1) *holds for every $\varphi\in C_0^\infty(\mathbf{R}^n)$ iff there exists a constant $C>0$;*

$$(|L(\xi)|+1) \geqq Ck^{\sim}(\xi) \quad \textit{for every} \quad \xi\in\mathbf{R}^n. \tag{3.2}$$

PROOF. Suppose that (3.1) is true. Let $\psi\in C_0^\infty(\mathbf{R}^n)$ be such as in the proof of Theorem 2.2, and let $\psi_j\in C_0^\infty(\mathbf{R}^n)$ be such that

$$\psi_j(x) = j^{-n+n/p}\psi(x/j). \tag{3.3}$$

The function $\phi_j\colon \mathbf{R}^n\to\mathbf{C}$ defined by

$$\phi_j(x) = (\psi_j(x)e^{i(\xi,x)})/k(\xi) \tag{3.4}$$

is an element in $C_0^\infty(\mathbf{R}^n)$ for every $\xi\in\mathbf{R}^n$. We have for every $j\in\mathbf{N}$

$$\begin{aligned}\|\phi_j\|_{p,k} &= \Big(\lambda_n \int_{R^n} |j^{n/p}(F\psi)(j(\eta-\xi))k(\eta)/k(\xi)|^p\,d\eta\Big)^{1/p} = \\ &= \Big(\lambda_n \int_{R^n} |(F\psi)(\tau)k(\tau/j+\xi)/k(\xi)|^p\,d\tau\Big)^{1/p} \leqq \\ &\leqq \Big(\lambda_n \int_{R^n} |(F\psi)(\tau)M_k(\tau/j)|^p\,d\tau\Big)^{1/p} \leqq C\|\psi\|_{p,k_N},\end{aligned} \tag{3.5}$$

where $M_k\in K$ such that $k(\xi+\eta)\leqq M_k(\xi)k(\eta)$ and $N\in\mathbf{N}$ such that $M_k(\xi)\leqq \leqq C(1+|\xi|^2)^{N/2}=:Ck_N(\xi)$. Moreover by Leibniz's rule

$$\begin{aligned}\|L(D)\phi_j\|_{p,k} &\leqq \sum_\alpha \frac{1}{\alpha!}|L^{(\alpha)}(\xi)|\,\|(D^\alpha\psi_j)e^{i(\xi,x)}/k(\xi)\|_{p,k} \\ &\leqq \sum_\alpha \frac{1}{\alpha!}|L^{(\alpha)}(\xi)|\frac{1}{j^{|\alpha|}}C\|D^\alpha\psi\|_{p,k_N}.\end{aligned} \tag{3.6}$$

In addition we have the estimate

$$\begin{aligned}\|\phi_j\|_{p,kk^{\sim}} &= \Big(\lambda_n \int_{R^n} |(F\psi)(\eta)(kk^{\sim})(\eta/j+\xi)/k(\xi)|^p\,d\eta\Big)^{1/p} \geqq \\ &\geqq \Big(\lambda_n \int_{R^n} |(F\psi)(\eta)(1/M_{kk^{\sim}}(-\eta/j))|^p\,d\eta\Big)^{1/p} k^{\sim}(\xi) \geqq \\ &\geqq (1/C')\|\psi\|_{p,1/k_{N'}}\cdot k^{\sim}(\xi).\end{aligned} \tag{3.7}$$

Taking into account the assumption (3.1) we get by the inequalities (3.5)—(3.7)

$$
(3.8) \qquad
\begin{aligned}
&C_1(1/C')\|\psi\|_{p,1/k_{N'}}k^{\sim}(\xi) \leqq \\
&\leqq \sum_{\alpha} \frac{1}{\alpha!} C\|D^{\alpha}\psi\|_{p,k_N} \frac{1}{j^{|\alpha|}} |L^{(\alpha)}(\xi)| + CC_2\|\psi\|_{p,k_N}
\end{aligned}
$$

for every $j\in\mathbf{N}$. Letting $j\to\infty$, the inequality (3.8) implies the assertion.

Conversely, it is easy to see that (3.2) implies (3.1). □

If $k^{\sim}\in K$ such that $k^{\sim}(\xi)\to\infty$ for $|\xi|\to\infty$ the inequality (3.2) can be given in the following equivalent form: There exist constants $E>0$ and $R\geqq 0$ such that

$$
(3.9) \qquad |L(\xi)| \geqq Ek^{\sim}(\xi) \quad \text{for} \quad |\xi| \geqq R.
$$

3.2. In this subsection we give a necessary algebraic condition for the validity of the inequality (3.1), when G is an open set in $\mathbf{R}^n$. When $G\subset\mathbf{R}^n$ is bounded this condition is sufficient to imply (3.1).

THEOREM 3.2. *Suppose that* (3.1) *is valid for all* $\varphi\in C_0^{\infty}(G)$. *Then there exists a constant* $C>0$ *such that*

$$
(3.10) \qquad L^{\sim}(\xi) \geqq Ck^{\sim}(\xi) \quad \textit{for all} \quad \xi\in\mathbf{R}^n.
$$

(*The definition of* $L^{\sim}(\xi)$ *see in* [4].)

PROOF. For all $\varphi\in C_0^{\infty}(G)$, $\|L(D)\varphi\|_{p,k}\leqq\|\varphi\|_{p,kL^{\sim}}$ and $\|\varphi\|_{p,k}\leqq M\|\varphi\|_{p,kL^{\sim}}$ (the last inequality follows from the fact that there exists $\gamma>0$; $L^{\sim}(\xi)\geqq\gamma$ for all $\xi\in\mathbf{R}^n$). Thus by the inequality (3.1) we have

$$
(3.11) \qquad \|\varphi\|_{p,kk^{\sim}} \leqq C'\|\varphi\|_{p,kL^{\sim}} \quad \text{for all} \quad \varphi\in C_0^{\infty}(G).
$$

Let $\psi\in C_0^{\infty}(G)$; $\psi\neq 0$ and let $\varphi\in C_0^{\infty}(G)$ such that $\varphi(x)=\psi(x)e^{i\langle\xi,x\rangle}$. Then as in the proof of Theorem 3.1 we see by (3.11) that there exist $C>0$, $C'>0$ and $N, N'\in\mathbf{N}$ such that

$$
C'(kk^{\sim})(\xi)\|\psi\|_{p,1/k_{N'}} \leqq Ck(\xi)L^{\sim}(\xi)\|\psi\|_{p,k_N}.
$$

This completes the proof. □

Denote by $\mathscr{L}\varphi$ the Fourier—Laplace transform of $\varphi\in C_0^{\infty}(G)$. We shall first show the following lemma

LEMMA 3.3. *Assume that G is an open bounded subset of* $\mathbf{R}^n$. *Then there exists a constant* $C>0$ *such that*

$$
(3.12) \qquad \|L^{(\alpha)}(D)\varphi\|_{p,k} \leqq C\|L(D)\varphi\|_{p,k} \quad \textit{for all} \quad \varphi\in C_0^{\infty}(G).
$$

PROOF. Let $\phi\in C_0^\infty(\mathbf{R}^n)$ be such that $\phi(x)=1$, $x\in\bar{G}$. Then for all $\varphi\in C_0^\infty(G)$ and $\sigma\in S:=\{\sigma=\xi+i\eta\in\mathbf{C}^n||\sigma|\leqq 1\}$

$$\lambda_n\int_{\mathbf{R}^n}|(\mathcal{L}\varphi)(\tau+\sigma)k(\tau)|^p\,d\tau\leqq$$

$$\leqq\lambda_n M_k(-\xi)^p\int|F(e^{(\eta,x)}\phi\varphi)(\tau+\xi)k(\tau+\xi)|^p\,d\tau= \tag{3.13}$$

$$=M_k(-\xi)^p\|e^{(\eta,x)}\phi\varphi\|_{p,k}^p\leqq M_k(-\xi)^p\|e^{(\eta,x)}\phi\|_{1,M_k}^p\|\varphi\|_{p,k}^p.$$

Furthermore for every $\tau\in\mathbf{R}^n$

$$F(e^{(\eta,x)}\phi)(\tau)=(\mathcal{L}\phi)(\tau+i\eta).$$

Due to the Paley—Wiener Theorem for every $q\in\mathbf{N}$ there exists $C>0$ such that

$$|(\mathcal{L}\phi)(\tau+i\eta)|\leqq C(1+|\tau|^2+|\eta|^2)^{-q}e^{A|\eta|},$$

where $A\in\mathbf{R}$; supp $\phi\subset\bar{B}(0,A)$ (see [4], p. 21). Since $|\eta|\leqq 1$, we obtain with some $C>0$

$$|F(e^{(\eta,x)}\phi)(\tau)|\leqq C(1+|\tau|^2)^{-q}. \tag{3.14}$$

Hence it is easy to see that $\|e^{(\eta,x)}\phi\|_{1,M_k}\leqq M$ for all $|\eta|\leqq 1$, with some constant $M>0$. According to (3.13) we have

$$\lambda_n\int_{\mathbf{R}^n}|(\mathcal{L}\varphi)(\tau+\sigma)k(\tau)|^p\,d\tau\leqq\sup_{|\xi|\leqq 1}M_k(-\xi)^pM^p\|\varphi\|_{p,k}^p \tag{3.15}$$

Define functions H, R and θ: $\mathbf{C}^n\to\mathbf{C}$ such that

$$H(\sigma)=(\mathcal{L}\varphi)(\tau+\sigma),\quad R(\sigma)=L(\tau+\sigma)$$

and

$$\theta(\sigma)=\begin{cases}1, & |\sigma|\leqq 1\\ 0, & |\sigma|>1.\end{cases}$$

Then we obtain for every $|\alpha|\leqq r$

$$|H(0)(D^\alpha R)(0)|\int|\sigma^\alpha|\theta(\sigma)\,d\sigma\leqq(\bar{r}!/(\bar{r}-\alpha)!)\int|H(\sigma)R(\sigma)|\theta(\sigma)\,d\sigma, \tag{3.16}$$

where $\bar{r}\in\mathbf{N}^n$; $\bar{r}=(r,\ldots,r)$ (see for example [8], p. 186). In other words

$$|(F\varphi)(\tau)L^{(\alpha)}(\tau)|E_\alpha\leqq\bar{r}!/(\bar{r}-\alpha)!\int_{|\sigma|\leqq 1}\left|\mathcal{L}(L(D)\varphi)(\tau+\sigma)\right|d\sigma, \tag{3.17}$$

where

$$E_\alpha=\int|\sigma^\alpha|\theta(\sigma)\,d\sigma>0.$$

Hence by Fubini's Theorem with some constant $C>0$

$$E_\alpha^p\|L^{(\alpha)}(D)\varphi\|_{p,k}^p\leqq C\lambda_n\int_{\mathbf{R}^n}\Big(\int_{|\sigma|\leqq 1}\left|\mathcal{L}(L(D)\varphi)(\tau+\sigma)k(\tau)\right|^p d\sigma\Big)\,d\tau \tag{3.17}$$

$$=C\lambda_n\int_{|\sigma|\leqq 1}\Big(\int_{\mathbf{R}^n}\left|\mathcal{L}(L(D)\varphi)(\tau+\sigma)k(\tau)\right|^p d\tau\Big)\,d\sigma.$$

Applying the inequality (3.15) we obtain (3.12). □

Lemma 3.3 gives the following theorem.

THEOREM 3.4. *Assume that* (3.10) *is true. Then there exist constants* $C_1>0$ *and* $C_2\geqq 0$ *such that* (3.1) *is valid for all* $\varphi\in C_0^\infty(G)$, *where* G *is an open bounded subset of* $\mathbf{R}^n$.

PROOF. The inequality (3.10) implies that

$$\|\varphi\|_{p,kk^\sim} \leqq (1/C)\|\varphi\|_{p,kL^\sim} \quad \text{for all} \quad \varphi\in C_0^\infty(G). \tag{3.18}$$

Since for every $\varphi\in C_0^\infty(G)$,

$$\|\varphi\|_{p,kL^\sim}^p \leqq C' \sum_\alpha \|L^{(\alpha)}(D)\varphi\|_{p,k}^p,$$

Lemma 3.3 proves our assertion. □

3.3. Let us define

$$\mathcal{H}_{p,\infty} = \bigcap_{k\in K} \mathcal{H}_{p,k}, \quad \mathcal{H}_{p,-\infty} = \bigcup_{k\in K} \mathcal{H}_{p,k}. \tag{3.19}$$

We prove a global regularity result for the distributional solutions u of the equation

$$L(D)u = f, \quad u\in\mathcal{H}_{p,-\infty}, \quad f\in\mathcal{H}_{p,k}. \tag{3.20}$$

THEOREM 3.5. *Suppose that the inequality* (3.1) *is valid for all* $\varphi\in C_0^\infty(\mathbf{R}^n)$ *with* $k\in K$ *such that* $k^\sim(\xi)\to\infty$ *for* $|\xi|\to\infty$. *Then every solution of the equation* (3.20) *lies in* $\mathcal{H}_{p,kk^\sim}$.

PROOF. The validity of (3.20) implies the relation

$$L(\xi)(Fu)(\xi) = (Ff)(\xi) \quad \text{a.e.} \quad \xi\in\mathbf{R}^n \tag{3.22}$$

Since by Theorem 3.1 $|L(\xi)|+1\geqq Ck^\sim(\xi)$ and since $k^\sim(\xi)\to\infty$ for $|\xi|\to\infty$, we have with some $\varrho\geqq 0$ and $d>0$

$$d(kk^\sim)(\xi)|(Fu)(\xi)| \leqq k(\xi)|(Fu)(\xi)L(\xi)| = k(\xi)|(Ff)(\xi)| \tag{3.23}$$

a.e. $\xi\in\mathbf{R}^n$ such that $|\xi|\geqq\varrho$. Hence one can see that u lies in $\mathcal{H}_{p,kk^\sim}$ (because u lies in some $\mathcal{H}_{p,k}$). □

On semi-Fredholm properties of the operator $L^\sim_{p,k,G}$

4.1. In the sequel we assume that G is an open bounded subset in $\mathbf{R}^n$. Furthermore we assume that $k^\sim\in K$; $k^\sim(\xi)\to\infty$ for $|\xi|\to\infty$. Then the imbedding $\lambda\colon \mathcal{H}_{p,kk^\sim}(G)\to\mathcal{H}_{p,k}(G)$ is compact (see [4], pp. 38—39; note that $\mathcal{H}_{p,k}(G)\subset \subset E'(G)\cap\mathcal{H}_{p,k}$).

We assume that there exist two constants $C_1>0$ and $C_2\geqq 0$ such that for all $\varphi\in C_0^\infty(G)$

$$\|L(x,D)\varphi\|_{p,k} \geqq C_1\|\varphi\|_{p,kk^\sim} - C_2\|\varphi\|_{p,k} \tag{4.1}$$

(where $L(x, D)$ is a differential operator (2.4)).

Let $\mathscr{H}_{p,k}(G)^+$ be the dual space of $\mathscr{H}_{p,k}(G)$ and let $L^+_{p,k,G}\colon \mathscr{H}_{p,k}(G)^+ \to \mathscr{H}_{p,k}(G)^+$ be the dual operator of $L_{p,k,G}$. We shall prove for arbitrary $1<p<\infty$ the following result.

THEOREM 4.1. *Assume that the differential operator* (2.4) *satisfies the condition* (4.1) *for all* $\varphi\in C_0^\infty(G)$, *where G is an open bounded subset of* $\mathbf{R}^n$. *Then* $L^{\sim}_{p,k,G}$ *is a semi-Fredholm operator such that*

(4.2) $$\dim N(L^{\sim}_{p,k,G})<\infty.$$

Furthermore

(4.3) $$N(L^{\sim}_{p,k,G})^\perp = R(L^+_{p,k,G})$$

and

(4.4) $$R(L^{\sim}_{p,k,G}) = N(L^+_{p,k,G})^\perp.$$

PROOF. By the assumption (4.1) we have for all $u\in D(L^{\sim}_{p,k,G})$

(4.5) $$C_1\|u\|_{p,k\tilde{k}} \leqq \|L^{\sim}_{p,k,G}u\|_{p,k}+C_2\|u\|_{p,k}.$$

Hence it is easy to see that $\dim N(L^{\sim}_{p,k,G})<\infty$.

We now have to prove that $R(L^{\sim}_{p,k,G})\subset\mathscr{H}_{p,k}(G)$ is closed. We do it by showing that $L^{\sim}_{p,k,G}(B)\subset\mathscr{H}_{p,k}(G)$ is closed whenever B is closed bounded subset of $D(L^{\sim}_{p,k,G})\subset\mathscr{H}_{p,k}(G)$ ([1], pp. 99—100). Let $\{f_n\}\subset L^{\sim}_{p,k,G}(B)$ be a sequence such that $\|f_n-f\|_{p,k}\to 0$ for some $f\in\mathscr{H}_{p,k}(G)$, and let $u_n\in B$ be such that $L^{\sim}_{p,k,G}u_n=f_n$. Because $\{f_n\}$ is convergent and B is bounded there are constants $M_1>0$ and $M_2>0$;

$$\|u_n\|_{p,k}\leqq M_1,$$

$$\|f_n\|_{p,k}\leqq M_2.$$

Thus by the inequality (4.5)

(4.6) $$\|u_n\|_{p,k\tilde{k}}\leqq (M_2+C_2M_1)/C_1 \quad \text{for all} \quad n\in\mathbf{N}.$$

This implies that there exists a subsequence $\{u_{n_j}\}\subset\{u_n\}$ and $u\in\mathscr{H}_{p,k}(G)$; $\|u_{n_j}-u\|_{p,k}\to 0$, $j\to\infty$. As B is closed then $u\in B$. Because $\|u_{n_j}-u\|_{p,k}\to 0$ and $\|L^{\sim}_{p,k,G}u_{n_j}-f\|_{p,k}\to 0$ it holds that $u\in D(L^{\sim}_{p,k,G})\cap B$ and $L^{\sim}_{p,k,G}u=f\in L^{\sim}_{p,k,G}(B)$.

Since $L^{\sim}_{p,k,G}$ is a semi-Fredholm operator we have that

(4.7) $$R(L^{\sim}_{p,k,G}) = N(L^{\sim +}_{p,k,G})^\perp \quad\text{and}\quad N(L^{\sim}_{p,k,G})^\perp = R(L^{\sim +}_{p,k,G}).$$

Since $L_p(R^n)$ is reflexive, by Millman's Theorem ([8], pp. 126—128) the space $\mathscr{H}_{p,k}$ is reflexive for $1<p<\infty$. Therefore $\mathscr{H}_{p,k}(G)$ regarded as a closed subspace of $\mathscr{H}_{p,k}$ is also reflexive. This implies that $L^{\sim}_{p,k,G}=(L^+_{p,k,G})^+$, where $(L^+_{p,k,G})^+$ is the dual operator of $L^+_{p,k,G}$ ([5], p. 168).

Hence the relations (4.3) and (4.4) follow from (4.7). □

4.2. Let U be an open ball in $\mathbf{R}^n$ and let $H_{p,k}(U)$ be a subspace of $D'(U)$ such that for all $u\in H_{p,k}(U)$ there exists $f_u\in\mathscr{H}_{p,k}$;

(4.8) $$u(\varphi)=f_u(\varphi), \quad \text{for all} \quad \varphi\in C_0^\infty(U).$$

We equip the space $H_{p,k}(U)$ with the topology induced by the norm

$$|||u|||_{p,k} = \inf_{\psi \in C_0^\infty(\mathbf{R}^n \setminus \bar{U})} \|f_u - \psi\|_{p,k}.$$

Assume that $p \in \mathbf{R}$; $1 < p < \infty$. Then for every $F \in \mathcal{H}_{p',1/\check{k}}(U)^+$ (where $p' \in \mathbf{R}$; $1/p + 1/p' = 1$ and $\check{k}(\xi) = k(-\xi)$) there exists $f \in H_{p,k}(U)$ such that

$$F(\varphi) = f(\varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(U)$$

and on the contrary.

The operator $L'^{\#}_{p,k,U}\colon H_{p,k}(U) \to H_{p,k}(U)$ is defined such as the operator $L'^{\#}_{p,k}\colon \mathcal{H}_{p,k} \to \mathcal{H}_{p,k}$. Assume that there exist constants $D_1 > 0$ and $D_2 \geqq 0$;

$$\|L'(x, D)\varphi\|_{p',1/\check{k}k^\sim} \geqq D_1 \|\varphi\|_{p',1/\check{k}} - D_2 \|\varphi\|_{p',1/\check{k}k^\sim} \tag{4.9}$$

for all $\varphi \in C_0^\infty(G)$ where $k^\sim(\xi) \to \infty$ for $|\xi| \to \infty$. Let $G^\sim$ be an open subset of G such that

$$N(L'^{\sim}_{p',1/\check{k}k^\sim,G}) \cap E'(\{x\}) = \{0\} \quad \text{for all} \quad x \in G^\sim, \tag{4.10}$$

where $E'(\{x\})$ is a subspace of $E'(\mathbf{R}^n)$ such that for each $u \in E'(\{x\})$, supp $u \subset \{x\}$. If dim $N(L'^{\sim}_{p',1/\check{k}k^\sim,G}) < \infty$, $G^\sim = G - M$, where M is a finite subset of G. One must note that (4.10) is always valid for all $x \in G$, if $N(L'^{\sim}_{p',1/\check{k}k^\sim,G}) \subset L^1_{\mathrm{loc}}$. In addition (4.10) is true for all $x \in G$ for every non-trivial operator with constant coefficients. We prove the following

Corollary 4.2. *Assume that the inequality* (4.9) *is valid. Then for every* $x \in G^\sim$ *there exists an open ball* $U_x = B(x, \varrho) \subset G^\sim$ *such that*

$$R(L'^{\#}_{p,k,U_x}) = H_{p,k}(U_x). \tag{4.11}$$

Proof. By Theorem 4.1 $R(L'^{\sim}_{p',1/\check{k}k^\sim,U})$ is closed and $\dim N(L'^{\sim}_{p',1/\check{k}k^\sim,U}) < \infty$ for all open $U \subset G$. Assume that $U_{x,\varepsilon} := B(x, \varepsilon) \subset G^\sim$. If $N(L'^{\sim}_{p',1/\check{k}k^\sim,U_{x,\varepsilon}}) \neq \{0\}$, there exists $\varepsilon' < \varepsilon$ such that $\dim N(L'^{\sim}_{p',1/\check{k}k^\sim,U_{x,\varepsilon'}}) < \dim N(L'^{\sim}_{p',1/\check{k}k^\sim,U_{x,\varepsilon}}) < \infty$ (because $N(L'^{\sim}_{p',1/\check{k}k^\sim,U_{x,\varepsilon}}) \subset E'(\bar{U}_{x,\varepsilon})$). Hence $N(L'^{\sim}_{p',1/\check{k}k^\sim,U_{x,\varrho}}) = \{0\}$, with some $\varrho > 0$. We set $U_{x,\varrho} = U_x$.

Let f be in $H_{p,k}(U_x)$ and let F be in $\mathcal{H}_{p',1/\check{k}}(U_x)^+$ such that $F(\varphi) = f(\varphi)$ for all $\varphi \in C_0^\infty(U_x)$. Since $N(L'^{\sim}_{p',1/\check{k}k^\sim,U_x}) = \{0\}$,

$$R(L'^{+}_{p',1/\check{k}k^\sim,U_x}) = \mathcal{H}_{p,1/\check{k}k^\sim}(U_x)^+.$$

Let W be in $\mathcal{H}_{p',1/\check{k}k^\sim}(U_x)^+$ such that

$$L'^{+}_{p',1/\check{k}k^\sim,U_x} W = F. \tag{4.12}$$

Then there exists $w \in H_{p,kk^\sim}(U_x)$; $W(\varphi) = w(\varphi)$ for all $\varphi \in C_0^\infty(U_x)$. In addition according to the relation (4.12) $w(L'_{p,k,U_x}\varphi) = f(\varphi)$ for all $\varphi \in C_0^\infty(U_x)$ and then $f = L'^{\#}_{p,k,U_x} w \in R(L'^{\#}_{p,k,U_x}\varphi)$. $\square$

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A SIMPLE STRATEGY FOR THE RAMSEY-GAME

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0. Introduction

Recently the following game has been investigated widely: two players I and II alternatively choose (previously unchosen) finite subsets of a given set S at a transfinite sequence of moves. I chooses at limit steps. I wins if he can produce a large subset every finite subset of which is chosen by him. The motivation from partition calculus is clear. In [1] the following problem is asked: if $|S|$ is measurable and a normal measure on S is given can I produce a set of measure one? It was not even clear whether any large cardinal axiom can guarantee an infinite winning set for I. Recently, the problem was answered affirmatively by Zsigmond Nagy. His proof used the notion of sequoia, and indeed it was later discovered that it yields much stronger results namely that $\varkappa\to(\alpha)^{<\omega}$ implies that I can produce a set of order type α (α limit) (see [2]). It seems to be worthwhile to give a direct strategy for the game.

1. The strategy

THEOREM (Zs. Nagy). *Assume $\varkappa>\omega$ is measurable and U is a normal ultrafilter on it. Then* I *wins $R(\varkappa, <\omega, U)$, i.e.* I *has a winning strategy in the following game:* I *and* II *alternatively choose (previously unchosen) finite subsets of $\varkappa$,* I *chooses at limit steps and having completed $\varkappa$ steps,* I *wins if and only if there is a set $X\in U$, with all $[X]^{<\omega}$ chosen by* I.

PROOF. We describe the strategy of I. To start, he chooses $\emptyset$. In the α'th step I picks $\{\gamma_n, \dots, \gamma_0\}$ where the following conditions hold: $\gamma_0>\gamma_1>\dots>\gamma_n$, $\alpha=\omega^{\gamma_0}+\dots$ $\dots+\omega^{\gamma_n}$, and for every $s\in[\gamma_{n-1}]^{<\omega}$ it is true that if II has chosen $s\cup\{\gamma_{n-1}, \dots, \gamma_0\}$ before the α'th step then $s\subseteq\gamma_n$. If these conditions are not fulfilled, I does not choose anything at all. It is clear that the strategy is correct, i.e. the subsets picked by I are always untouched by II. We are going to prove if $\{\gamma_{n-1}, \dots, \gamma_0\}$ is picked by I and if γ_{n-1} is regular, then there is a closed unbounded $C_{\gamma_{n-1}, \dots, \gamma_0}\subseteq\gamma_{n-1}$ such that for $\gamma\in C_{\gamma_{n-1}, \dots, \gamma_0}$, $\{\gamma, \gamma_{n-1}, \dots, \gamma_0\}$ is chosen by I.

For $n=0$ this reduces to: there is a closed unbounded $C\subseteq\varkappa$ such that for $\gamma\in C$ it is true that if s is touched by II earlier than the γ'th move, then $s\subseteq\gamma$. This can easily be proved by Skolem-functions.

If $n>1$ and $\{\gamma_{n-1}, \dots, \gamma_0\}$ is a move for I and γ_{n-1} is a regular uncountable cardinal, there is a closed unbounded $C_{\gamma_{n-1}, \dots, \gamma_0}\subseteq\gamma_{n-1}$ for which the following is

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true: if $\gamma\in C_{\gamma_{n-1},\ldots,\gamma_0}$, $s\subseteq\gamma_{n-1}$ and $s\cup\{\gamma_{n-1},\ldots,\gamma_0\}$ is chosen by II between the $\omega^{\gamma_0}+\ldots+\omega^{\gamma_{n-1}}$-th and $\omega^{\gamma_0}+\ldots+\omega^{\gamma_{n-1}}+\omega^{\gamma}$-th moves then $s\subseteq\gamma$. This can be proved by the standard Skolem-closure. If s is as given, $s\cup\{\gamma_{n-1},\ldots,\gamma_0\}$ cannot be chosen earlier as I has chosen $\{\gamma_{n-1},\ldots,\gamma_0\}$ and so $s\cup\{\gamma_{n-1}\}\subseteq\gamma_{n-1}$ should hold.

Next we shall prove that for every $n<\omega$ there is a set $X_n\in U$ with $[X_n]^n$ picked by I.[1] For $n=1$ this is clear, as every closed unbounded set is in U. Assume that our statement is true for n, X_n witnesses this fact and every element of X_n is uncountable, regular. By normality, there is a closed, unbounded C such that for a $Y_n\in U$ the following holds: if $x\in[Y_n]^n$ then $C_x=C\cap\min x$. Let us define $X_{n+1}=C\cap Y_n\cap X_n$. If $\{\gamma_n,\ldots,\gamma_0\}\in X_{n+1}$ with $\gamma_n\in C_{\gamma_{n-1},\ldots,\gamma_0}$, $\{\gamma_n,\ldots,\gamma_0\}$ is chosen by I and the proof is finished.

2. Remark

From the proof of the theorem the following problem type arises: if $\varkappa$ is a cardinal and for every $\gamma_n<\ldots<\gamma_0$ a closed unbounded $C_{\gamma_n,\ldots,\gamma_0}\subseteq\gamma_n$ is given whether a "large" homogeneous $X\subseteq\varkappa$ exists, i.e. for every $\gamma_{n+1}<\ldots<\gamma_0$ sequence from X, $\gamma_{n+1}\in C_{\gamma_n,\ldots,\gamma_0}$ must hold. These questions will be treated elsewhere.

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[1] Clearly this gives the result.

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SOME RESULTS IN PARTIAL EXCHANGEABILITY

A. I. DALE

1. Introduction

The notion of independence is of long-standing and has been painstakingly researched in statistics. However, the usefulness of this concept is somewhat attenuated in Bayesian statistics, where the possibility of "learning through experience" is of prime importance. Should independence be discarded, the next simplest thing is to continue to regard the order of the events as irrelevant. In this case the events are said to be *exchangeable*[1] (i.e. symmetric, at least as far as all probabilistic properties are concerned, as regards to order).

In 1938 de Finetti broached the idea of *partial exchangeability* ("équivalence partielle"), a concept between which and exchangeability a meaningful distinction may be drawn (for further details see de Finetti [9], p. 227). Until recently, however, little had been written on the subject, its recrudescence perhaps being stimulated by de Finetti's [8] (in which work details of pertinent writings may be found). The scant attention this subject received in the decades following its first airing is perhaps not altogether surprising, for de Finetti himself write ([7], p. 11)

> Toutes les conclusions et les formules valables pour le cas de l'équivalence s'étendent aisément au cas actuel des événements que l'on pourrait qualifier de *partiellement équivalents*, et définir par la même condition de symétrie, en spécifiant toutefois que les événements se divisent en un certain nombre de types 1, 2, ..., g, et que ce sont seulement les événements de même type qui s'avèrent comme "interchangeables" par rapport à tout problème de probabilité.

Since, however, there certainly exist cases in which partial exchangeability rather than exchangeability of events seems the appropriate thing to consider (see Section 2 below), it seems worthwhile to examine some analogues, in the setting of partial exchangeability, of results known to hold for exchangeable events, and it is to this end that this paper is written. More specifically, after a short section on partial exchangeability and de Finetti's Theorem, we present, in the third section, a finite version of this latter result. In the fourth section partially exchangeable random variables are considered, while in the fifth a Poisson limit theorem is presented.

A recent paper by Link [21] has also been devoted to this topic, some exceedingly deep and general results being obtained by approaching partial exchangeability via

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[1] On the origin of the term (and various alternatives to it) see de Finetti [8] p. 211, and Fréchet [10], p. 72.

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the theory of abstract convex sets (cf. Hewitt and Savage [17]). Our aim here is rather to emphasize the probabilistic approach to the subject, since it is felt that such an approach, while not leading to results of the same depth as Link's, might nevertheless be of interest to probabilists.

2. Partial exchangeability

Consider firstly a sequence of tosses of a single coin (i.e. a *simple sample* in the sense of Good [12], p. 12, or a *single sample* in the sense of Girschik et al. [11], p. 19). If this coin is of irregular appearance, we might be somewhat hesitant to say much more about the probability of the sequence $E_1, E_2, \ldots$ of events that this probability depends only on the number of events and not on the actual n-tuple observed (for example, the probability of getting six heads in fifteen tosses of a coin is independent of the places in the sequence at which the six heads are observed) — i.e. an assumption of *exchangeability*[2]. If, now, we assume the same game to be played with g different coins, we may well suppose that each coin will generate (by appropriate tossing) a sequence of exchangeable events, but neither wish nor indeed be able to say anything more about the *overall* sequence of events — that is, this latter sequence is *partially exchangeable.* (For further details on partial exchangeability see the paper by Bruno [1], reprinted as Chapter 10 in de Finetti [8], and de Finetti [9], Chapters 10 & 11.)

De Finetti [7] has shown that, given n such events of which n_i are of type i, where $i\in\{1, 2, \ldots, g\}$ and $\sum_1^g n_i=n$, there exists a (unique) g-dimensional distribution function Φ such that the probability $\omega_{r_1,\ldots,r_g}^{(n_1,\ldots,n_g)}$ that, for each i, r_i "favourable" results will be obtained from n_i, is given by

$$\omega_{r_1,\ldots,r_g}^{(n_1,\ldots,n_g)} = \int_G \prod_{i=1}^{g} \binom{n_i}{r_i} \xi_i^{r_i}(1-\xi_i)^{n_i-r_i}\, d\Phi \tag{1}$$

where G is the g-dimensional product space $[0, 1]\times\ldots\times[0, 1]$.

3. A finite version of de Finetti's Theorem

Recalling that all exchangeable processes which end after a finite number of steps are mixtures of the hypergeometric processes, while all those which can be continued indefinitely are mixtures of Bernoulli processes (see de Finetti [9], p. 217), and bearing in mind the intimate connection between exchangeable and partially exchangeable events, we shall not be surprised to find similar mixtures arising when we turn our attention to the latter class of events.

Let $(\Omega, \mathscr{A}, \mathsf{P})$ be a probability space on which a g-fold partially exchangeable sequence $\{A_{ij}\colon i\in\{1, 2, \ldots, g\}, j\in\mathrm{N}\}$ of events is defined. The condition of partial exchangeability can be formulated in terms of the probabilities (the so-called *de*

[2] Some remarks on the question of priority are given in Appendix 1.

Finetti constants)

$$\omega_{i_1,\dots,i_g} = \mathsf{P}[A_{1,r_1},\dots,A_{1,r_{i_1}},\dots,A_{g,r_1},\dots,A_{g,r_{i_g}}]$$

by requiring that each such probability should depend only on the g-tuple $(i_1,\dots,i_g)$. (Here, for each $j\in\{1,2,\dots,g\}$, the $r_1,\dots,r_{i_j}$ are all different. Moreover, the r's attached to any A are not necessarily the same as those attached to any other: we have chosen this notation merely for convenience.)

Defining the partial differences

$$\Delta_j\omega_{i_1,\dots,i_g} = \omega_{i_1,\dots,i_j,\dots,i_g}-\omega_{i_1,\dots,i_j+1,\dots,i_g}$$

$$\Delta_j\Delta_k\omega_{i_1,\dots,i_g} = \omega_{i_1,\dots,i_g}-\omega_{i_1,\dots,i_j+1,\dots,i_k+1,\dots,i_g}-\omega_{i_1,\dots,i_{k+1},\dots,i_g}+\omega_{i_1,\dots,i_{j+1},\dots,i_{k+1},\dots,i_g},$$

with $\Delta_j^{h_j}\Delta_k^{h_k}\omega_{i_1,\dots,i_g}$ defined similarly,

we find that

$$\text{(2)}\qquad \begin{aligned}\Delta_1^{h_1}\dots\Delta_g^{h_g}\omega_{i_1,\dots,i_g} = \mathsf{P}[&A_{1,r_1}\dots A_{1,i_{r_1}}\bar{A}_{1,r_{i_1}+1}\dots\bar{A}_{1,r_{i_1}+h_1}\dots\\ &\dots A_{g,r_1}\dots A_{g,r_{i_g}}\bar{A}_{g,r_{i_g}+1}\dots\bar{A}_{g,r_{i_g}+h_g}]\end{aligned}$$

the bar denoting complementation with respect to Ω. Setting $\omega_{0,\dots,0}=1$, we find that

$$\text{(3)}\qquad \sum_{i_1=0}^{n_1}\dots\sum_{i_g=0}^{n_g}\prod_{j=1}^{g}\binom{n_j}{i_j}\Delta_1^{i_1}\dots\Delta_g^{i_g}\omega_{n_1-i_1,\dots,n_g-i_g} = 1.$$

We might note, in passing, that (2) and (3) imply that the $\prod_{j=1}^{g}(1+n_j)$ points in the sequence $\{\omega_{0,\dots,0},\dots,\omega_{n_1,\dots,n_g}\}$ can be associated with a sequence of g-fold partially exchangeable events. To see this, let Ω be a probability space of $2^{n_1}\times\dots\times 2^{n_g}$ points labelled

$$\text{(4)}\qquad \varepsilon_{1,1},\dots,\varepsilon_{1,n_1},\dots,\varepsilon_{g,1},\dots,\varepsilon_{g,n_g},$$

each ε being either 0 or 1. The attaching of the weight

$$\Delta_1^{i_1}\dots\Delta_g^{i_g}\omega_{n_1-i_1,\dots,n_g-i_g}$$

to (4), where i_j is the number of elements of $\{\varepsilon_{j,1},\dots,\varepsilon_{j,n_j}\}$ which are zero, gives the required result.

Notice next that

$$\text{(5)}\qquad \sum_{i_1=0}^{n_1-m_1}\dots\sum_{i_g=0}^{n_g-m_g}\prod_{j=1}^{g}\binom{n_j-m_j}{i_j}\Delta_1^{i_1}\dots\Delta_g^{i_g}\omega_{n_1-i_1,\dots,n_g-i_g} = \omega_{m_1,\dots,m_g}.$$

Letting

$$\text{(6)}\qquad \omega_{i_1,\dots,i_g}^{(n_1,\dots,n_g)} = \prod_{j=1}^{g}\binom{n_j}{i_j}\Delta_1^{n_1-i_1}\dots\Delta_g^{n_g-i_g}\omega_{i_1,\dots,i_g},$$

we see that, in the light of (2) and (3), this defines a probability distribution. Moreover, from (5),

$$\omega_{m_1,\ldots,m_g} = \sum_{k_1=m_1}^{n_1} \cdots \sum_{k_g=m_g}^{n_g} \prod_{j=1}^{g} \binom{n_j-m_j}{k_j-m_j} \Delta_1^{k_1-m_1} \ldots \Delta_g^{k_g-m_g} \omega_{n_1-(k_1-m_1),\ldots,n_g-(k_g-m_g)} =$$

$$(7) \qquad = \sum_{k_1=m_1}^{n_1} \cdots \sum_{k_g=m_g}^{n_g} \prod_{j=1}^{g} \binom{n_j-m_j}{k_j-m_j} \omega^{(n_1,\ldots,n_g)}_{n_1-k_1+m_1,\ldots,n_g-k_g+m_g} \Big/ \prod_{j=1}^{g} \binom{n_j}{k_j-m_j} =$$

$$= \sum_{s_1=m_1}^{n_1} \cdots \sum_{s_g=m_g}^{n_g} \prod_{j=1}^{q} [(s_j)_{m_j}/(n_j)_{m_j}] \omega^{(n_1,\ldots,n_g)}_{s_1,\ldots,s_g},$$

where $(s_j)_{m_j}=s_j(s_j-1)\ldots(s_j-m_j+1)$. This is the required result, the finite result corresponding to (1) above with $n_j=r_j$. (Note that $\omega^{(n_1,\ldots,n_g)}_{n_1,\ldots,n_g}=\omega_{n_1,\ldots,n_g}$.) The similar finite theorem for exchangeable events may be found in de Finetti [6]: more recent useful references include Diaconis [2], Heath and Sudderth [16] — a result for exchangeable *random variables* rather than *events* — and Kendall [20].

A limiting form of the above result is easily obtainable. Equation (2) above states that the multiple sequence $\{\omega_{i_1,\ldots,i_g}\}$ is *completely monotonic* (in the sense of Hildebrandt and Schoenberg [18]). It follows, by Theorem 1 of this latter paper, that there exists a function F such that

$$(8) \qquad \omega_{m_1,\ldots,m_g} = \int_0^1 \cdots \int_0^1 p_1^{m_1} \ldots p_g^{m_g}\, d_1 \ldots d_g F(p_1, \ldots, p_g),$$

where F is monotonic (the term being interpreted as in Hildebrandt and Schoenberg [18]). This F is unique (in the sense that any two such F's can differ only in a denumerable number of hyperplanes), and is strictly unique if continuous (cf. Good [12], p. 23, and Hildebrandt and Schoenberg [18] § 3). Putting all the m_i's equal to zero, we see that F is in fact a distribution function.

From (8) it follows that

$$\Delta_1 \omega_{m_1,\ldots,m_g} = \omega_{m_1,\ldots,m_g} - \omega_{m_1+1,\ldots,m_g} =$$

$$= \int_0^1 \cdots \int_0^1 p_1^{m_1}(1-p_1) \prod_{j=2}^{g} p_j^{m_j}\, d_1 \ldots d_g F(p_1, \ldots, p_g)$$

and, more generally,

$$(9) \qquad \Delta_1^{n_1-m_1} \ldots \Delta_g^{n_g-m_g} \omega_{m_1,\ldots,m_g} = \int_0^1 \cdots \int_0^1 \prod_{j=1}^{g} p_j^{m_j}(1-p_j)^{n_j-m_j}\, d_1 \ldots d_g F(p_1, \ldots, p_g).$$

On combining this latter result with (6) we obtain (1), as required.

4. Partially exchangeable random variables

DEFINITION. A sequence $\{X_n\}$ of random variables on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ is *g-fold partially exchangeable* if $\{X_n\}$ is divisible into a g-fold infinite sequence $\{X_{ij}\colon i\in\{1, 2, \ldots, g\},\ j\in\mathbf{N}\}$ such that, for all $n_1, \ldots, n_g\in\mathbf{N}$ and all $x_{ij}\in\mathbf{R}$

$(i \in \{1, 2, \ldots, g\}, j \in \{1, 2, \ldots, n_i\})$

$$\mathsf{P}[\bigcap_{i=1}^{g} \bigcap_{j=1}^{n_i} \{X_{i,\varrho_i(j)} < x_{ij}\}] = \mathsf{P}[\bigcap_{i=1}^{g} \bigcap_{j=1}^{n_i} \{X_{ij} < x_{ij}\}]$$

where ϱ_i is any "finite" permutation of **N** onto **N**.

The proof of de Finetti's Theorem to be presented here is analogous to that of Heath and Sudderth [16] for exchangeable random variables. This proof does not rely as heavily on the consideration of moment sequences as does that in the preceding section. We shall restrict our attention to two-valued random variables: the extension to sequences of random variables taking on more than two values is, at least conceptually, not difficult.

Let $\{X_n\}$ be a g-fold partially exchangeable sequence of random variables (in the sense of the preceding definition) taking on values in $\{0, 1\}$, and let

(10) $$\omega_{i_1, \ldots, i_g} = \mathsf{P}[X_{11} = 1, \ldots, X_{1i_1} = 1, X_{1,i_1+1} = 0, \ldots, X_{1,n_1} = 0, \ldots, X_{g1} = \\ = 1, \ldots, X_{g,i_g} = 1, X_{g,i_g+1} = 0, \ldots, X_{g,n_g} = 0],$$

(11) $$\omega_{i_1,\ldots,i_g}^{(m_1,\ldots,m_g)} = \mathsf{P}[\sum_{j=1}^{m_1} X_{1j} = i_1, \ldots, \sum_{j=1}^{m_g} X_{gj} = i_g]$$

where, for each $j \in \{1, 2, \ldots, g\}$, $i_j \leq n_j \leq m_j$.

LEMMA 1.

$$\omega_{i_1,\ldots,i_g} = \sum_{r_1=0}^{m_1} \cdots \sum_{r_g=0}^{m_g} \prod_{j=1}^{g} \binom{m_j - n_j}{r_j - i_j} \Big/ \binom{m_j}{r_j} \omega_{r_1,\ldots,r_g}^{(m_1,\ldots,m_g)} =$$

$$= \sum_{r_1=0}^{m_1} \cdots \sum_{r_g=0}^{m_g} \prod_{j=1}^{g} (r_j)_{i_j} (m_j - r_j)_{n_j - i_j} / (m_j)_{n_j} \omega_{r_1,\ldots,r_g}^{(m_1,\ldots,m_g)}.$$

PROOF. Consider g urns of $m_1, \ldots, m_g$ balls, respectively, of which $r_1, \ldots, r_g$ are red (score 1). Let n_j be the size of a sample of balls drawn from the j-th urn. Then

$$\mathsf{P}[i_1 \text{ red in 1st sample}, \ldots, i_g \text{ red in } g\text{-}th \text{ sample } | r_1, \ldots, r_g] =$$

$$= \prod_{j=1}^{g} \binom{r_j}{i_j} \binom{m_j - r_j}{n_j - i_j} \Big/ \binom{m_j}{n_j} = \prod_{j=1}^{g} \binom{n_j}{i_j} \binom{m_j - n_j}{r_j - i_j} \Big/ \binom{m_j}{r_j}.$$

Thus

$$\mathsf{P}[\text{the first } i_j \text{ in the } j\text{-th sample are red, for each } j | r_1, \ldots, r_g] =$$

$$= \prod_{j=1}^{g} \binom{m_j - n_j}{r_j - i_j} \Big/ \binom{m_j}{r_j}.$$

Now, for each j, r_j may be any one of the values $\{0, 1, \ldots, m_j\}$, and hence

$$\omega_{i_1,\ldots,i_g} = \sum_{r_1=0}^{m_1} \cdots \sum_{r_g=0}^{m_g} \prod_{j=1}^{g} \binom{m_j - n_j}{r_j - i_j} \Big/ \binom{m_j}{r_j} \mathsf{P}[\sum_{j=1}^{m_1} X_{1j} = r_1, \ldots, \sum_{j=1}^{m_g} X_{gj} = r_g] =$$

$$= \sum_{r_1=0}^{m_1} \cdots \sum_{r_g=0}^{m_g} \prod_{j=1}^{g} \binom{m_j - n_j}{r_j - i_j} \Big/ \binom{m_j}{r_j} \omega_{r_1,\ldots,r_g}^{(m_1,\ldots,m_g)}$$

This proves the first result stated in the Lemma; the second can be obtained from this by a simple manipulation of the binomial coefficients.

Recalling (10) and (11) above, we have, from this lemma,

$$\omega_{i_1,\dots,i_g}^{(n_1,\dots,n_g)} = \sum_{r_1=0}^{m_1}\dots\sum_{r_g=0}^{m_g}\sum_{j=1}^{g}\binom{n_j}{i_j}\binom{m_j-n_j}{r_j-i_j}\Big/\binom{m_j}{r_j}\,\omega_{r_1,\dots,r_g}^{(m_1,\dots,m_g)}. \tag{12}$$

Now let $F_{m_1,\dots,m_g}$ be the distribution function concentrated on $\left\{\left(\frac{r_1}{m_1},\dots,\frac{r_g}{m_g}\right):\right.$ $\left. 0\leqq r_j\leqq m_j,\ j\in\{1,2,\dots,g\}\right\}$ with saltus $\omega_{r_1,\dots,r_g}^{(m_1,\dots,m_g)}$ at $\left(\frac{r_1}{m_1},\dots,\frac{r_g}{m_g}\right)$. Then (12) can be written

$$\omega_{i_1,\dots,i_g}^{(n_1,\dots,n_g)} = \int_0^1\dots\int_0^1\prod_{j=1}^{g}\binom{n_j}{i_j}(m_j\theta_j)_{i_j}(m_j(1-\theta_j))_{n_j-i_j}(m_j)_{n_j}d_1\dots d_g F_{m_1,\dots,m_g}(\theta_1,\dots,\theta_g).$$

Application of Helly's Theorem (Feller [5], § VIII. 6) yields the existence of a subsequence of $\{F_{m_1,\dots,m_g}\}$ which converges in distribution to a function F, say. By the uniform convergence of the integrand in the above expression as $m_j\to\infty$, $j\in\{1,2,\dots,g\}$, we find finally that

$$\omega_{i_1,\dots,i_g}^{(n_1,\dots,n_g)} = \int_0^1\dots\int_0^1\prod_{j=1}^{g}\binom{n_j}{i_j}\theta_j^{i_j}(1-\theta_j)^{n_j-i_j}d_1\dots d_g F(\theta_1,\dots,\theta_g). \tag{13}$$

To get an idea of the difference between the expressions in (12) and (13), notice that

$$\begin{aligned}
&\Bigg|\sum_{r_1=0}^{m_1}\dots\sum_{r_g=0}^{m_g}\prod_{j=1}^{g}\binom{n_j}{i_j}\binom{m_j-n_j}{r_j-i_j}\Big/\binom{m_j}{r_j}\,\omega_{r_1,\dots,r_g}^{(m_1,\dots,m_g)} -\\
&\qquad -\int_0^1\dots\int_0^1\prod_{j=0}^{g}\binom{n_j}{i_j}\theta_j^{i_j}(1-\theta_j)^{n_j-i_j}d_1\dots d_g F(\theta_1,\dots,\theta_g)\Bigg| \leqq\\
&\leqq \sum_{r_1}\dots\sum_{r_g}\omega_{r_1,\dots,r_g}^{(m_1,\dots,m_g)}\left|\prod_{j=1}^{g}\binom{m_j}{i_j}\binom{m_j-n_j}{r_j-i_j}\Big/\binom{m_j}{r_j}-\prod_{j=1}^{g}\binom{n_j}{i_j}\left(\frac{r_j}{m_j}\right)^{i_j}\left(1-\frac{r_j}{m_j}\right)^{n_j-i_j}\right| \leqq\\
&\leqq \sum_{r_1}\dots\sum_{r_g}\omega_{r_1,\dots,r_g}^{(m_1,\dots,m_g)}\sum_{j=1}^{g}\left|\binom{n_j}{i_j}\binom{m_j-n_j}{r_j-i_j}\Big/\binom{m_j}{r_j}-\binom{n_j}{i_j}\left(\frac{r_j}{m_j}\right)^{i_j}\left(1-\frac{r_j}{m_j}\right)^{n_j-i_j}\right| \leqq\\
&\leqq \sum_{r_1}\dots\sum_{r_g}\omega_{r_1,\dots,r_g}^{(m_1,\dots,m_g)}\sum_{j=1}^{g}2n_j/m_j \leqq\\
&\leqq 2\sum_{j=1}^{g}i_j/n_j,
\end{aligned} \tag{14}$$

where the transition from the third-to the second-last line is effected by Theorem (4) of Diaconis and Freedman [3]. The inequality (14) provides a measure of the accuracy of the approximation of the true (finite) state of affairs (as given in (12)) by the infinite form of de Finetti's Theorem (13).

5. A Poisson limit theorem

By postulating the existence of the limits of various functions of certain de Finetti constants it is possible to prove that a certain limit law is a product of Poisson distributions. To prove this result, however, we require the following generalization of a lemma of Feller [5] § VII. 1, the proof of which may be found in Appendix 2.

LEMMA 2. *Let* $\{u_i(\cdot): i=1, 2, \ldots, k\}$ *be a sequence of real-valued continuous functions with* $|u_i(\cdot)|\leqq 1$ *for each i. Consider a family of k-dimensional distribution functions* $F_{n,\underline{\theta}}$ *with mean* $\underline{\theta}=(\theta_1\theta_2\ldots\theta_k)'$ *and with* $\mathsf{Var}\, X_i=\sigma_n^2(\theta_i)$, $i=1, 2, \ldots, k$. *Finally, let*

$$\mathsf{E}_{n,\theta}(u_1, \ldots, u_k) = \int_{\mathbf{R}^k} \prod_1^k u_i(x_i)\, d_1\ldots d_k F_{n,\underline{\theta}}(x_1, \ldots, x_k).$$

If, for each i, $\sigma_n^2(\theta_i)\to 0$ *as* $n\to\infty$, *then* $\mathsf{E}_{n,\underline{\theta}}(u_1, \ldots, u_k)\to \prod_1^k u_i(\theta_i)$.

THEOREM 1. *For each* $\nu\in\mathbf{N}$, *let* $(\Omega_\nu, \mathscr{A}_\nu, \mathsf{P}_\nu)$ *be a probability space on which an infinite sequence* $\{A_{ij}^{(\nu)}: i=1, 2, \ldots, g;\ j\in\mathbf{N}\}$ *of g-fold partially exchangeable events with de Finetti constants* $\omega_{m_1,\ldots,m_g}^{(\nu)}$, *is defined. Let* $X_i^{(\nu)}$ *be the number of the events*

$$A_{i_1}^{(\nu)}, A_{i_2}^{(\nu)}, \ldots, A_{i_\nu}^{(\nu)}$$

that occur, $i\in\{1, 2, \ldots\} g\}$. *If for each permutation* $\varrho(1, 0, \ldots, 0)$ *of* $(1, 0, \ldots, 0)$ *and* $\varrho(2, 0, \ldots, 0)$ *of* $(2, 0, \ldots, 0)$

$$\nu\omega_{\varrho(1,0,\ldots,0)}^{(\nu)} \to \mu_{\varrho(1,0,\ldots,0)}$$

(15) *and*

$$\nu^2\omega_{\varrho(2,0,\ldots,0)}^{(\nu)} \to \mu_{\varrho(2,0,\ldots,0)}^2$$

as $\nu\to\infty$, *then*

$$\lim_{\nu\to\infty} \mathsf{P}_\nu[X_1^{(\nu)} = s_1, \ldots, X_g^{(\nu)} = s_g] = \prod_{j=1}^g e^{-\mu_j}\mu_j^{s_j}/s_j! \tag{16}$$

where $\mu_1\equiv\mu_{(1,0,\ldots,0)}$, $\mu_2=\mu_{(0,1,0,\ldots,0)}$, *etc.*

PROOF. From equation (13) above it follows that

$$\mathsf{P}_\nu[X_1^{(\nu)} = s_1, \ldots, X_g^{(\nu)} = s_g] = \int_0^1\cdots\int_0^1 \prod_{j=1}^g \binom{\nu}{s_j}\theta_j^{s_j}(1-\theta_j)^{\nu-s_j}\, d_1\ldots d_g F_\nu(\theta_1, \ldots, \theta_g).$$

For $|\xi_i|\leqq 1$, $i\in\{1, 2, \ldots, g\}$,

$$\begin{aligned}\mathsf{E}_\nu\Big[\prod_{j=1}^g \xi_j^{X_j^{(\nu)}}\Big] &= \sum_{s_1}\cdots\sum_{s_g}\xi_1^{s_1}\ldots\xi_g^{s_g}\,\mathsf{P}_\nu[X_1^{(\nu)} = s_1, \ldots, X_g^{(\nu)} = s_g] =\\ &= \int_0^1\cdots\int_0^1 \sum_{s_1}\cdots\sum_{s_g}\prod_{j=1}^g\binom{\nu}{s_j}(\xi_j\theta_j)^{s_j}(1-\theta_j)^{\nu-s_j}\, d_1\ldots d_g F_\nu(\theta_1, \ldots, \theta_g) =\\ &= \int_0^1\cdots\int_0^1 \prod_{j=1}^g\Big[\sum_{s_j}\binom{\nu}{s_j}(\xi_j\theta_j)^{s_j}(1-\theta_j)^{\nu-s_j}\Big]\, d_1\ldots d_g F_\nu(\theta_1, \ldots, \theta_g) =\\ &= \int_0^1\cdots\int_0^1 \prod_{j=1}^g[\xi_j\theta_j+(1-\theta_j)]^\nu\, d_1\ldots d_g F_\nu(\theta_1, \ldots, \theta_g).\end{aligned}$$

Now let $\eta_j = \nu\theta_j$ and let G_ν denote the transformed distribution. Then

$$(17)\quad \mathsf{E}_\nu\Big[\prod_{j=1}^{g} \xi_j^{X_j^{(\nu)}}\Big] = \int_0^\nu \dots \int_0^\nu \prod_{j=1}^{g} [\xi_j\eta_j/\nu + (1-\eta_j/\nu)]^\nu \, d_1 \dots d_g G_\nu(\eta_1, \dots, \eta_g) =$$

$$= \int_0^\nu \dots \int_0^\nu \prod_{j=1}^{g} [1-(1-\xi_j)\eta_j/\nu]^\nu \, d_1 \dots d_g G_\nu(\eta_1, \dots, \eta_g).$$

Now, for $0 \leqq u_i \leqq \nu$, $i \in \{1, 2, \dots, g\}$, we have (using a result from Whittaker and Watson [23], p. 242)

$$0 \leqq \Big|\prod_{j=1}^{g} e^{-u_i} - (1-u_i/\nu)^\nu\Big| \leqq \sum_{j=1}^{g} |e^{-u_i} - (1-u_i/\nu)^\nu| \leqq$$

$$\leqq \sum_{j=1}^{g} u_i^2 e^{-u_i}/\nu \leqq$$

$$\leqq 4e^{-2} g/\nu.$$

It follows, then, that if the integrand in (17) is replaced by $\prod_{j=1}^{g} \exp[-(1-\xi_j)\eta_j]$, where $0 \leqq \xi_j \leqq 1$, the error in the integral will be at most $4e^{-2}g/\nu$.

Notice next that if $\underline{\eta}$ has the distribution G_ν, then, for $i \in \{1, 2, \dots, g\}$

$$\mathsf{E}_\nu \eta_i = \nu E_\nu X_i^{(\nu)} = \nu\omega_{0,\dots,1,\dots,0}^{(\nu)}$$

$$\mathsf{Var}_\nu(\eta_i) = \nu^2[\omega_{0,\dots,2,\dots,0}^{(\nu)} - (\omega_{0,\dots,1,\dots,0}^{(\nu)})^2]$$

and hence, by (15), $\mathsf{Var}_\nu(\eta_i) \to 0$ as $\nu \to \infty$. It thus follows from the preceding Lemma that

$$\lim_{\nu\to\infty} \int_0^\nu \dots \int_0^\nu \prod_{j=1}^{g} \exp[-(1-\xi_j)\eta_j] \, d_1 \dots d_g G_\nu(\eta_1, \dots, \eta_g) = \prod_{j=1}^{g} \exp[-(1-\xi_j)\mu_j].$$

That is

$$\lim_{\nu\to\infty} \mathsf{E}_\nu\Big[\prod_{j=1}^{g} \xi_j^{X_j^{(\nu)}}\Big] = \prod_{j=1}^{g} \exp[-(1-\xi_j)\mu_j].$$

It then follows from the continuity theorem for probability generating functions (Feller [4], § XI. 6) that

$$\mathsf{P}_\nu[X_1^{(\nu)} = s_1, \dots, X_g^{(\nu)} = s_g] \xrightarrow[\infty]{\nu} \prod_{j=1}^{g} e^{-\mu_j}\mu_j^{s_j}/s_j!,$$

as asserted.

Appendix 1

The attribution of priority is usually a difficult matter. In their important paper [17] of 1955 Hewitt and Savage state (p. 470)

Jules Haag seems to have been the first author to discuss symmetric sequences of random variables (see [13]). This paper deals only with 2-valued random variables. It hints at, but does not rigorously state or prove, the representation theorem for this case.

(the reference "see [13]" appears as [15] in my list of references). However, as Good [12], p. 13, has pointed out, Haag "was perhaps anticipated by W. E. Johnson". Since, however, Haag's paper and Johnson's Logic, Part III, were both published in 1924, there seems little reason, on the basis of date of publication, to regard one or other as the first to investigate these matters.

As regards the pertinent work on exchangeability by Johnson and by Haag, let us note firstly that Johnson's contribution was limited to some few pages in the Appendix on Eduction in Part III of his Logic. There Johnson introduced his *Permutation-Postulate*, one which is readily seen to be that of the definition of the exchangeability of events. This postulate, appearing as it does in an appendix to what is probably a relatively little used work nowadays, has perhaps for that reason not received its due recognition: moreover it appears in a book on logic, and as such might well not be readily available to the probabilist.

Haag, on the other hand, introduced the idea of probabilities which are completely *symmetric* with respect to the events $E_1, E_2, \ldots, E_n$, but are not independent (see his papers [13], [14] and [15]). Denoting by z_p^q the probability that, of m events chosen from this class, the first p are favourable and the next $q=m-p$ are unfavourable, Haag [15] derived various general formulae expressing relationships between z_p^q, $x_p(\equiv z_p^0)$ and $y_p(\equiv z_0^q)$; and he also proved that, in the case of an infinite number of events,

$$z_p^q = \int_0^1 f(x)x^p(1-x)^q\,dx,$$

where "$f(x)dx$ est la probabilité pour que la *fréquence* des événements favorables soit comprise entre x et $x+dx$" [15], p. 664. While it may be true, as Hewitt and Savage have suggested, that Haag's result lacked rigour both in statement and in proof, yet no small credit should, I believe, be attributed to him for his pioneering effort.

It was, of course, only in de Finetti's work [6] of 1937 that the importance of exchangeability in subjective probability become realized, and it was in this latter setting that the representation theorem received its first complete statement and proof.

Appendix 2

Proof of Lemma 2.

Notice firstly that, from our assumption that the u_i are uniformly bounded by 1,

$$\left|\prod_1^k u_i(x_i) - \prod_1^k u_i(\theta_i)\right| \leq \sum_1^k |u_i(x_i)-u_i(\theta_i)|.$$

Thus

$$\begin{aligned}(1)\qquad \mathscr{I} &\equiv \int \left|\prod_1^k u_i(x_i) - \prod_1^k u_i(\theta_i)\right| d_1\ldots d_k F_{n,\underline{\theta}}(x_1,\ldots,x_k) \leq \\ &\leq \sum_1^k \int |u_i(x_i)-u_i(\theta_i)|\, d_1\ldots d_k F_{n,\underline{\theta}}(x_1,\ldots,x_k) \equiv \sum_1^k \mathscr{I}_i,\end{aligned}$$

say where all integrals, unless otherwise specified, are over $\mathbf{R}^k$. Now, since each u is bounded and continuous,

(i) there exists a sequence $\{M_i\}_1^k$, of constants such that $|u_i(x_i)-u_i(\theta_i)|<M_i$ on the whole of the range of u_i, and

(ii) for each $i\in\{1, 2, \dots, k\}$, given $\varepsilon_i>0$, there exists $\delta_i\equiv\delta(\varepsilon_i)>0$ such that

$$|x_i-\theta_i| < \delta_i \Rightarrow |u_i(x_i)-u_i(\theta_i)| < \varepsilon_i.$$

Let, then, $\{\varepsilon_i\}_1^k$ be a given sequence of positive constants, and let a corresponding sequence $\{\delta_i\}_1^k$ be chosen so that (ii) above holds. Then, for $j\in\{1, 2, \dots, k\}$,

$$\begin{aligned}\mathscr{I}_j &= \int |u_j(x_j)-u_j(\theta_j)|\, d_1\dots d_k F_{n,\underline{\theta}}(x_1, \dots, x_k) = \\ &= \int_A |u_j(x_j)-u_j(\theta_j)|\, d_1\dots d_k F_{n,\underline{\theta}}(x_1, \dots, x_k) + \\ &\qquad + \int_B |u_j(x_j)-u_j(\theta_j)|\, d_1\dots d_k F_{n,\underline{\theta}}(x_1, \dots, x_k) \equiv \\ &\equiv \mathscr{I}_j(A)+\mathscr{I}_j(B),\end{aligned} \tag{2}$$

say, where

$$A = \{(x_1, \dots, x_k)\colon\ |x_j-\theta_j| < \delta_j,\ \ -\infty < x_i < \infty \quad \text{for} \quad i \neq j\}$$

$$B = \{(x_1, \dots, x_k)\colon\ |x_j-\theta_j| \geqq \delta_j,\ \ -\infty < x_i < \infty \quad \text{for} \quad i \neq j\}.$$

Now, on A, $|u_j(x_j)-u_j(\theta_j)|<\varepsilon_j$ (by (ii)), and hence

$$\mathscr{I}_j(A) \leqq \varepsilon_j \int_A d_1\dots d_g F_{n,\underline{\theta}}(x_1, \dots, x_k) \leqq \varepsilon_j, \tag{3}$$

since $F_{n,\underline{\theta}}$ is a distribution function.

Let B be decomposed into the union of the two mutually exclusive events

$$\begin{aligned} B_1 &= \{(x_1, \dots, x_k)\colon\ |x_j-\theta_j| \geqq \delta_j,\ |x_i-\theta_i| < \delta_i \text{ for at least one } i \neq j\}. \\ B_2 &= \{(x_1, \dots, x_k)\colon\ |x_j-\theta_j| \geqq \delta_j,\ |x_i-\theta_i| \geqq \delta_i \text{ for all } i \neq j\}\end{aligned} \tag{4}$$

and let $\mathscr{I}_j(B)=\mathscr{I}_j(B_1)+\mathscr{I}_j(B_2)$. Now on B_1, $|u_j(x_j)-u_j(\theta_j)|<M_j$ (by (i) above). Thus

$$\begin{aligned}\mathscr{I}_j(B_1) &= \int_B |u_j(x_j)-u_j(\theta_j)|\, d_1\dots d_k F_{n,\underline{\theta}}(x_1, \dots, x_k) \leqq \\ &\leqq M_j \mathsf{P}[|X_j-\theta_j| \geqq \delta_j,\ |X_i-\theta_i| < \delta_i \text{ for at least one } i \neq j] \leqq \\ &\leqq M_j \mathsf{P}[|X_j-\theta_j| \geqq \delta_j] \leqq \\ &\leqq M_j \sigma_n^2(\theta_j)/\delta_j^2\end{aligned} \tag{5}$$

by Čebyšev's inequality.

Turning our attention to the event B_2, we see that

$$\mathscr{I}_j(B_2) = \int_B |u_j(x_j)-u_j(\theta_j)|\, d_1\dots d_k F_{n,\underline{\theta}}(x_1, \dots, x_k) \leqq$$

$$\leqq M_j \mathsf{P}[|X_i-\theta_i| \geqq \delta_i \text{ for all } i] \leqq \tag{6}$$

$$\leqq M_j \sum_1^k [\sigma_n^2(\theta_i)/\delta_i^2]$$

by an inequality of Tong [22], § 7.2.

Combining the above results (2)—(6) we see that

$$\mathscr{I}_j \leqq \varepsilon_j + M_j\big[\sigma_n^2(\theta)/\delta_j^2 + \sum_1^k \sigma_n^2(\theta_i)/\delta_i^2\big].$$

Hence, from (1),

$$\mathscr{I} \leqq \sum_{j=1}^k \big\{\varepsilon_j + M_j\big[\sigma_n^2(\theta_j)/\delta_j^2 + \sum_1^k \sigma_n^2(\theta_i)/\delta_i^2\big]\big\} \leqq$$

$$\leqq \sum_{j=1}^k \varepsilon_j + \sum_{j=1}^k M_j \sigma_n^2(\theta_j)/\delta_j^2 + \Big(\sum_{j=1}^k M_j\Big)\Big(\sum_1^k \sigma_n^2(\theta_i)/\delta_i^2\Big).$$

From the definition of $\mathsf{E}_{n,\underline{\theta}}(u_1, \dots, u_k)$ it follows that

$$\Big|\mathsf{E}_{n,\underline{\theta}}(u_1, \dots, u_k) - \prod_{i=1}^k u_i(\theta_i)\Big| \leqq \int \Big|\prod_1^k u_i(x_i) - \prod_1^k u_i(\theta_i)\Big|\, d_1\dots d_g F_{n,\underline{\theta}}(x_1, \dots, x_k) \leqq$$

$$\leqq \sum_1^k \varepsilon_j + \sum_1^k M_j \sigma_n^2(\theta_j)/\delta_j^2 + \Big(\sum_1^k M_j\Big)\Big(\sum_1^k \sigma_n^2(\theta_i)/\delta_i^2\Big).$$

Since the variances $\sigma_n^2(\theta_i) \to 0$ as $n \to \infty$ and since the ε_i are arbitrary, the stated result is now immediate.

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A SECOND NOTE ON HAJNAL—MÁTÉ GRAPHS

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In this paper graphs on ω_1, the set of countable ordinals, are considered. If $\mathcal{G}$ is a graph of this kind, let us define $\Gamma(\alpha)$ as the set of ordinals smaller than α and adjacent to it. $\mathcal{G}$ is called a Hajnal—Máté graph if $\Gamma(\alpha)$ is either finite or cofinal in α with ordertype ω. In [1] a Hajnal—Máté graph with chromatic number $\aleph_1$ was constructed under $\Diamond^*$, a principle deduced from the exiom of constructibility. In [2] we showed that a trianglefree Hajnal—Máté graph with chromatic number $\aleph_1$ also exists, we used only $\Diamond$. Here we show that this result can be extended to graphs not containing circuits which have only one increasing and one decreasing part.

THEOREM ($\Diamond^*$). *There is a Hajnal—Máté graph $\mathcal{G}$ with chromatic number $\aleph_1$ and without circuits of the following type $\{x_0, x_1, \dots, x_{n-1}, x_0\}$ where $x_0 < x_1 < \dots$ $\dots < x_r > x_{r+1} > \dots > x_{n-1} > x_0$.*

PROOF. Choose a decomposition $\omega_1 = \bigcup_{\tau<\omega_1} X_\tau$ with $\tau < \min X_\tau$, X_τ stationary. Assume that $S_\alpha \subseteq P(\alpha)$, $|S_\alpha| \leqq \aleph_0$ witnesses the $\Diamond^*$-property.

If γ, $\xi < \omega_1$, say that ξ is γ-covered if there is an increasing path $\{x_0, x_1, \dots, x_n\}$ with $x_0 \leqq \gamma$, $x_n = \xi$. Clearly, ξ is γ-covered, if $\xi \leqq \gamma$. We define $\Gamma(\alpha) \subseteq \alpha$ inductively, so assume that $\Gamma(\beta)$ is defined for $\beta < \alpha$. Choose a sequence $\{y_n: n < \omega\}$ cofinal in α. We call $A \subseteq \alpha$ covered if there is a $\gamma < \alpha$ such that every element of A is γ-covered. Otherwise, A is uncovered. Enumerate the set of the uncovered subsets $A \in S_\alpha$ as $\{A_0, A_1, \dots\}$, and choose $x_0, x_1, \dots$ with $x_0 > \tau$ if $\alpha \in X_\tau$, $x_n \in A_n$ and x_{n+1} is not x_n-covered (x_0 is not τ-covered), $x_n \geqq y_n$. Put $\Gamma(\alpha) = \{x_0, x_1, \dots\}$ (it may be finite or even empty).

First we prove that no cycle mentioned in the theorem exists in the graph. If $\{a_0, a_1, \dots, a_{n-1}, a_0\}$ is a circuit and a_r is its maximal point, and, for definiteness, $a_{r-1} < a_{r+1} < a_r$ then a_{r+1} is a_{r-1}-covered and both are in $\Gamma(a_r)$, a contradiction.

Next we prove that our graph is not ω-chromatic. Assume $f: \omega_1 \to \omega$ is a good colouring. $H_n = f^{-1}(\{n\})$. Call $n < \omega$ small, if there is a $\gamma_n < \omega_1$ such that every element of H_n is γ_n-covered, otherwise n is large. Put $K = \{n: n \text{ small}\}$, $\tau = \sup\{\gamma_n: n \in K\}$.

If n is large, for every $\gamma < \omega_1$ there is a $\xi \in H_n$ which is not γ-covered, and by closure, it is easy to show that $H_n \cap \alpha$ is an uncovered subset of α for a closed unbounded set of α's.

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By property $\diamondsuit^*$ $H_n \cap \alpha \in S_\alpha$ for a closed unbounded set of α's and, as X_τ is stationary we can choose an α with $\alpha \in X_\tau$, $H_n \cap \alpha \in S_\alpha$ and uncovered if n is large. We do not claim that there are large numbers. $f(\alpha)$ is undefined as α is not τ-covered by construction, so $f(\alpha)$ is surely not small. $f(\alpha)$ can not be large as α is connected to a point in H_n for every large n.

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ON THE TANGENCY OF MULTIFUNCTIONS

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Abstract

We present a definition of tangency of multifunctions and relate it with various notions of differentiability of multifunctions. We define a characteristic number on each multifunction with the property that it remains the same for two tangent multifunctions. The sign of this number is proved to give a sufficient condition for the asymptotic stability of multivalued differential equations and ordinary differential equations which cannot be locally linearized. At last we obtain some perturbation results.

0. The purpose of this work is to define the notion of α-tangent multifunctions at the origin, where the multifunctions vanish. We relate α-tangency with various definitions of differentiability of multifunctions ([2], [4], [5]). Furthermore, to each multifunction F we associate a characteristic number $\chi_\alpha(F)$, which does not change on α-tangent multifunctions. This characteristic number is a multivalued and Hilbert space version of the characteristic exponents of nonlinear single-valued functions in a Banach space, which were introduced in [3]. Also we show that the negativeness of $\chi_\alpha(F)$ implies the asymptotic stability of the multivalued differential equation $x' \in F(x)$. Such a result is applied to ordinary differential equations which cannot be locally linearized. Finally we obtain some perturbation results.

1. Let H be a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$. We denote by $c(H)$ the collection of all compact convex non-empty subsets of H. One can endow $c(H)$ with the following metric, usually called Hausdorff distance,

$$\delta(A, B) = \inf\{\lambda > 0\colon\ A \subset B+\lambda S,\ B \subset A+\lambda S\},$$

where S is the unit ball around 0 in H.

Given two upper semi-continuous multifunctions $F, G\colon H \to c(H)$, such that $F(0)=G(0)=0$, and a number $\alpha > 0$, we say that F and G are *α-tangent* (at the origin) if

$$\lim_{x\to 0} \frac{\delta(F(x), G(x))}{|x|^\alpha} = 0.$$

Clearly α-tangency is a relation of equivalence, because of the properties of the Hausdorff distance (e.g. [2]).

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Now, given $x \in H$, $A \in c(H)$, the following function is well defined ([1])

$$\sigma(x, A) = \sup_{y \in A}(x, y).$$

$\sigma(x, A)$ is called the support function of the convex set A.

If F: $H \to c(H)$ is upper semi-continuous, $F(0)=0$, and $\alpha > 0$, we define the following (extended real) number

$$\chi_\alpha(F) = \limsup_{x \to \infty} \frac{\sigma(x, F(x))}{|x|^{\alpha+1}}.$$

Obviously $\chi_\alpha(F)$ is finite, if F is quasi-bounded of order α at the origin, i.e. $\limsup_{x \to 0} \frac{1}{|x|^\alpha} \sup_{y \in F(x)} |y| < \infty$.

It is a fundamental property of the number χ_α that it only depends on the equivalence class of α-tangent multifunctions. This is seen by the following proposition.

Proposition 1. *Let* F, G: $H \to c(H)$ *upper semi-continuous,* $F(0)=G(0)=0$, *and* $\alpha > 0$. *If F and G are α-tangent, then* $\chi_\alpha(F) = \chi_\alpha(G)$.

Proof. For any $\varepsilon > 0$ there exists a neighborhood of the origin where

$$F(x) \subset B(x) + \varepsilon |x|^\alpha S$$

$$G(x) \subset F(x) + \varepsilon |x|^\alpha S.$$

Therefore we get

$$\sigma(x, F(x)) \leqq \sigma(x, G(x)) + \varepsilon |x|^{\alpha+1}$$

$$\sigma(x, G(x)) \leqq \sigma(x, F(x)) + \varepsilon |x|^{\alpha+1}.$$

Thus it is implied

$$|\chi_\alpha(F) - \chi_\alpha(G)| \leqq \varepsilon$$

and the proof is completed, as ε is arbitrary.

Proposition 2. *Let* F, G: $H \to c(H)$ *upper semi-continuous,* $F(0)=G(0)=0$, $\alpha \geqq 1$, $J(x) = x|x|^{\alpha-1}$, $\lambda \geqq 0$ *and* $k \in \mathbf{R}$. *Then we have*

$$\chi_\alpha(\lambda F) = \lambda \chi_\alpha(F)$$

$$\chi_\alpha(F+G) \leqq \chi_\alpha(F) + \chi_\alpha(G)$$

$$\chi_\alpha(F+kJ) = \chi_\alpha(F) + k.$$

Proof. It is a direct consequence of the following relations:

$$\sigma(x, \lambda F(x)) = \lambda \sigma(x, F(x)),$$

$$\sigma(x, F(x) + G(x)) = \sigma(x, F(x)) + \sigma(x, G(x)),$$

$$\sigma(x, F(x) + kJ(x)) = \sigma(x, F(x)) + k|x|^{\alpha+1}.$$

We say that the multifunction G: $H \to c(H)$ is α-order homogeneous ($\alpha > 0$), if $G(\lambda x) = \lambda^\alpha G(x)$, $\lambda \geqq 0$, $x \in H$.

PROPOSITION 3. *If G is α-order homogeneous, then* $\chi_\alpha(G)=\sup_{|x|=1}\sigma(x, G(x))$.

PROOF. It follows from the definition of limsup and the homogeneity of $G(x)$ and $\sigma(x, G(x))$.

2. In order to fix the ideas we consider in this section that H is finite dimensional, e.g. $H=\mathbf{R}^n$. In the sequel we assume $\alpha\geqq 1$.

A multifunction F: $\mathbf{R}^n\to c(\mathbf{R}^n)$ is called *α-order Lipschitzian* at the origin, if there exist constants $L\geqq 0$ and $\delta>0$ such that for all $y\in F(x)$, $|x|\leqq\delta$, we have

$$|y| \leqq L|x|^\alpha.$$

An upper semi-continuous α-order homogeneous multifunction Φ: $\mathbf{R}^n\to c(\mathbf{R}^n)$ is called *α-order upper differential* of the α-order Lipschitzian multifunction F, if there exists $\delta>0$ such that for all $|x|<\delta$

$$F(x)\subset\Phi(x).$$

Note that, if F is α-order Lipschitzian, an α-order upper differential always exists; it is $\Phi(x)=\{y\in\mathbf{R}^n\colon |y|\leqq L|x|^\alpha\}$.

We define the *α-order differential* at the origin of the α-order Lipschitzian multifunction F by

$$D_0^\alpha F(x) = \bigcap\{\Phi(x)\colon \Phi \text{ is an } \alpha\text{-order differential of } F\}.$$

The above definition of α-order differentiability of multifunctions generalizes the definition of Lasota and Strauss [4] concerning the existence of multivalued (first order, i.e. $\alpha=1$) differentials of non-differentiable Lipschitzian at 0 single-valued functions (cf. the first order generalization of De Blasi in [2]).

The following result shows that $D_0^\alpha F$ is *well-behaved.*

PROPOSITION 4. *If $D_0^\alpha F$ is the α-order differential of F at* 0, *then*

(i) *the range of $D_0^\alpha F$ is $c(\mathbf{R}^n)$,*

(ii) *the multifunction $D_0^\alpha F$: $\mathbf{R}^n\to c(\mathbf{R}^n)$ is upper semi-continuous, α-order homogeneous and*

(iii) *there exists a sequence $\{\Phi_n\}$ of α-order upper differentials such that $\Phi_{n+1}(x)\subset\subset\Phi_n(x)$, $x\in\mathbf{R}^n$, $n=1, 2, \ldots$, and $D_0^\alpha F(x)=\bigcap_{n=1}^{\infty}\Phi_n(x)$.*

PROOF. (i) The only fact which requires a proof is that $D_0^\alpha F(x)\neq\emptyset$ for $x\neq 0$ (the case $x=0$ is trivial). For sufficiently large integer n we have for all $y_n\in F(x/n)$ that

$$\frac{|y_n|}{|x/n|^\alpha}\leqq L.$$

Thus there exists a limit point z of $y_n/|x/n|^\alpha$ as $n\to\infty$. Let $n_k\to\infty$ such that $y_{n_k}/|x/n_k|^\alpha\to z$. Let Φ any α-order upper differential of F. Then for n_k sufficiently large

$$\frac{y_{n_k}}{|x/n_k|^\alpha}\in\frac{F(x/n_k)}{|x/n_k|^\alpha}\subset\frac{\Phi(x/n_k)}{|x/n_k|^\alpha}=\Phi\left(\frac{x}{|x|}\right).$$

Letting $n_k \to \infty$, we obtain $z \in \Phi(x/|x|)$. Thus $|x|^\alpha z \in \Phi(x)$ for every α-order upper differential Φ, i.e. $|x|^\alpha z \in D_0^\alpha F(x)$.

The proof of (ii) and (iii) is similar to the one in [4] and so it is omitted.

The next result relates α-order differentiability with α-tangency.

PROPOSITION 5. *Let $F\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ α-order Lipschitzian at the origin. If there exists an α-order homogeneous upper semi-continuous multifunction $G\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ such that F and G are α-tangent at the origin, then $G = D_0^\alpha F$, i.e. F and $D_0^\alpha F$ are α-tangent at the origin.*

PROOF. First we remark that G, as defined above, is the multivalued differential of F at 0 in the sense of De Blasi [2] (if $\alpha=1$) and of [5] (if $\alpha>1$). Thus the conclusion of the proposition follows from a direct extension of the proof of Theorem 4.8 of [2] for any α.

Let us remark that the existence of G in Proposition 5 is an indispensable assumption. Indeed the (single-valued) function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^\alpha \sin(1/x)$, $x \neq 0$, $f(0)=0$, is α-order Lipschitzian at 0, $D_0^\alpha F(x) = x^\alpha S$, but f and $D_0^\alpha f$ are not α-tangent at 0.

3. Now we are going to give some applications of the above ideas to the stability of multivalued differential equations.

PROPOSITION 6. *Let $F\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ upper semi-continuous, $F(0)=0$, and $\alpha \geqq 1$. If $\chi_\alpha(F) < 0$, then the zero solution of the multivalued differential equation*

$$x' \in F(x) \tag{1}$$

is asymptotically stable.

PROOF. Let $\varepsilon > 0$ be such that $k = \chi_\alpha(F) + \varepsilon < 0$ and $\delta > 0$ be such that $\sigma(x, F(x)) \leqq k|x|^{\alpha+1}$ whenever $|x| < \delta$. We consider any solution $x(t)$ of (1) (in its right maximal interval of existence $[0, T)$) such that $|x(0)| < \delta$. If there exists $0 < t_1 \leqq T$ such that $|x(t)| < \delta$, $t \in [0, t_1)$, $|x(t_1)| = \delta$, then we would have in $[0, t_1)$

$$\frac{d}{dt}|x(t)|^2 = 2(x(t), x'(t)) \leqq 2\sigma(x(t), F(x(t))) \leqq 2k|x(t)|^{\alpha+1},$$

i.e. by integrating the above differential inequality in $[0, t_1)$ we get

$$\delta = |x(t_1)| \leqq |x(0)| e^{kt_1} < \delta, \quad \text{if} \quad \alpha = 1,$$

$$\delta = |x(t_1)| \leqq \frac{|x(0)|}{[1 - k(\alpha-1)|x(0)|^{\alpha-1} t_1]^{1/(\alpha-1)}} < \delta, \quad \text{if} \quad \alpha > 1,$$

a contradiction. Therefore for all $t \in [0, T)$ we have $|x(t)| > \delta$. A standard argument shows that $T = \infty$. So, since for all $t \geqq 0$

$$|x(t)| \leqq |x(0)| e^{kt}, \quad \text{if} \quad \alpha = 1$$

$$|x(t)| \leqq \frac{|x(0)|}{[1 - k(\alpha-1)|x(0)|^{\alpha-1} t]^{1/(\alpha-1)}}, \quad \text{if} \quad \alpha > 1,$$

it follows that the zero solution of (1) is asymptotically stable.

Combining the above proposition with Propositions 1 and 3 we get the following result.

COROLLARY 7. *Let* $F\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ *upper semi-continuous,* $F(0)=0$, $\alpha \geqq 1$. *Suppose that there exists an* α*-order homogeneous upper semi-continuous multifunction* $G\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ *such that* F *and* G *are* α*-tangent at the origin. If for any* x, $|x|=1$, *we have* $\sigma(x, G(x))<0$, *then the zero solution of* (1) *is asymptotically stable.*

Now we give an example of a scalar single-valued differential equation with right-hand side not linearized at 0. For any $\alpha \geqq 1$ we consider the function $f_\alpha\colon \mathbf{R}\to\mathbf{R}$ defined by $f_\alpha(x)=x^\alpha\left(\theta\sin\frac{1}{x}-1\right)$, $x\neq 0$, $f_\alpha(0)=0$, for some fixed $\theta\in(0, 1)$. Clearly f_1 is not differentiable at 0 and for any $\alpha>1$ the differential of f_α at 0 vanishes. An easy computation shows that if α is an odd integer, then $\chi_\alpha(f_\alpha)<0$, which implies that the zero solution of the differential equation

$$x' = x^\alpha\left(\theta\sin\frac{1}{x}-1\right)$$

is asymptotically stable.

Finally we obtain some perturbation results applying the previous propositions.

PROPOSITION 8. *Let* $F\colon \mathbf{R}^n \to c(\mathbf{R}^n)$ *upper semi-continuous,* $F(0)=0$, $\alpha\geqq 1$. *If* $\chi_\alpha(F)<0$ *and* $\varepsilon>0$ *sufficiently small, then the zero solution of the perturbed multi-valued differential equation*

$$x'\in F(x)+\varepsilon|x|^\alpha S \tag{2}$$

is asymptotically stable.

PROOF. Let $G(x)=F(x)+\varepsilon|x|^\alpha S$. Then we have

$$\chi_\alpha(G) = \limsup_{x\to 0}\frac{\sigma(x, F(x))+\sigma(x, \varepsilon|x|^\alpha S)}{|x|^{\alpha+1}} \leqq \chi_\alpha(F)+\varepsilon.$$

Since $\chi_\alpha(F)<0$, we can take $\varepsilon>0$ such that $\chi_\alpha(G)=\chi_\alpha(F)+\varepsilon<0$. Then the conclusion follows from Proposition 6.

Combining Propositions 3 and 8 we get the following result.

COROLLARY 9. *Let* $F\colon \mathbf{R}^n\to c(\mathbf{R}^n)$ α*-order homogeneous upper semi-continuous,* $\alpha\geqq 1$. *If for any* x, $|x|=1$, *we have* $\sigma(x, F(x))<0$ *and* $\varepsilon>0$ *sufficiently small, then the zero solution of* (2) *is asymptotically stable.*

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ASYMPTOTIC SOLUTION OF A LOCALLY-TURÁN PROBLEM

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1. Introduction

Let $S_n=\{a_1, ..., a_n\}$ be an n-element set, $C^k(S_n)=\{S\subseteq S_n: |S|=k\}$ the family of its k-element subsets, in other words, the complete k-graph with vertex set S_n. S_i denotes an arbitrary i-element subset of S_n. The paper considers k-graphs $G^k\subseteq C^k(S_n)$ satisfying the following condition:

$$\forall S_p \subseteq S_n\colon \exists S_q \subseteq S_p\colon \forall S_k \subseteq S_q\colon S_k\in G^k.$$

We write $G^k\sim LT_{n,p,q,k}$ (locally Turán) iff G^k has the above property.

Let $T(n, p, q, k)$ denote the minimum number of edges of G^k satisfying $G^k\sim LT_{n,p,p,k}$. Turán [1] determined $T(n, p, 2, 2)$ and posed the problem of finding $T(n, p, q, q)$ (see [2]). As a special case, he conjectured [3]:

$$T(2n, 5, 3, 3) = 2\binom{n}{3}. \tag{1}$$

This is still unsettled. Concerning the LT-property see [4], [5], and [6].

In the present paper we investigate the asymptotic behaviour of $T(n, p, q, k)$. It is easy to see (like for $T(n, p, q, q)$ in [7]) that $T(n, p, q, k)/\binom{n}{k}$ is a monotonically increasing function of n, thus the limit

$$\lim_{n\to\infty} T(n, p, q, k)/\binom{n}{k} = t(p, q, k)$$

exists. It is clear that the validity of (1) would result in $t(5, 3, 3)=1/4$. The main result of the paper is the following

THEOREM.

$$\lim_{r\to\infty} t(2r+1, r+1, 3) = 1/4. \tag{2}$$

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Key words and phrases. Turán's problem, subsets, degree, limit.

2. Proof of the theorem

Let us first observe that the graph $H^3=C^3(A)\cup C^3(B)$, where $A\cup B=S_n$, $A\cap B=\emptyset$, $|A|=\left\lfloor\frac{n}{2}\right\rfloor$ ($\lfloor x\rfloor$ denotes the largest integer $\leqq x$), satisfies $H^3\sim LT_{n,2r+1,r+1,3}$ for any $r\geqq 2$. Therefore $t(2r+1, r+1, 3)\leqq 1/4$. We have to prove that the limit (2) exists and is not smaller than 1/4. The proof of this is based on three lemmas.

LEMMA 1. *If G^3 is a 3-graph satisfying*

$$G^3\sim LT_{n,2r+1,r+1,3},$$

$$|G^3|<\binom{n}{3}\left(\frac{1}{4}-\varepsilon\right),$$

where $\varepsilon>0$, $n\geqq 28r+16$, then there exist a natural number m and a 3-graph G_* such that for $n\geqq m\geqq 14r+8$,

$$G_*\sim LT_{m,2r+1,r+1,3},$$

$$|G_*|<\binom{n}{3}\left(\frac{1}{4}-\varepsilon\right),$$

and the degree of every vertex $x\in S_m$ satisfies

$$d(x)<\frac{2}{7}\binom{m}{2}.$$

PROOF. Suppose the contrary. Then there is a vertex $x_0\in S_n$ with $d(x_0)\geqq\frac{2}{7}\binom{n}{2}$. Let $G_0=G^3$ and suppose that

$$G_i\sim LT_{n-i,2r+1,r+1,3},\tag{3}$$

$$|G_i|<\binom{n-i}{3}\left(\frac{1}{4}-\varepsilon\right),\tag{4}$$

and

$$d(x_i)\geqq\frac{7}{2}\binom{n-i}{2}\tag{5}$$

hold for some 3-graph G_i on $n-i$ vertices $\left(0\leqq i<\left\lceil\frac{n}{2}\right\rceil\right.$; $\lceil x\rceil$ denotes the smallest integer $\geqq x$). We define G_{i+1} by

$$G_{i+1}=\{e\in G_i:\ x_i\notin e\}$$

and prove that G_{i+1} satisfies the same conditions (3)—(5).

$$G_{i+1}\sim LT_{n-i-1,2r+1,r+1,3}$$

is obvious. On the other hand,

$$|G_{i+1}|=|G_i|-d(x_i)<\binom{n-i}{3}\left(\frac{1}{4}-\varepsilon\right)-\frac{2}{7}\binom{n-i}{2}\leqq\binom{n-i-1}{3}\left(\frac{1}{4}-\varepsilon\right)$$

holds by (4) and (5). The existence of $x_{i+1}\in S_{n-i-1}$ and $d(x_{i+1})\geqq\frac{2}{7}\binom{n-i}{2}$ follows by the indirect assumption. It follows that (3)—(5) hold for $i=\left\lceil\frac{n}{2}\right\rceil$ and for the 3-graph $G_{\lceil n/2\rceil}$. Now we can obtain a lower estimate of the size of $G^3=G_0$:

$$|G^3|\geqq\sum_{i=0}^{\lceil\frac{n}{2}\rceil}\frac{2}{7}\binom{n-i}{2}=\frac{2}{7}\binom{n+1}{3}-\frac{2}{7}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{3}>\frac{1}{4}\binom{n}{3}.$$

This inequality contradicts the assumption of the lemma. Consequently, there are an $m\geqq\left\lfloor\frac{n}{2}\right\rfloor\geqq 14r+8$ and a $G_*\sim LT_{m,2r+1,r+1,3}$ satisfying the conditions of the lemma.

LEMMA 2. *If* $G_*\sim LT_{m,2r+1,r+1,3}$, $m\geqq 14r+8$, *and*

$$d(x)<\frac{2}{7}\binom{m}{2}$$

for every vertex x, *and* $2\leqq l\leqq r$, *then*

(6) $$G\sim LT_{m,2l+1,l+1,3}.$$

PROOF. If the lemma is not true then there is a maximal $l<r$ not satisfying (6). That is

(7) $$\exists S_{2l+1}\subseteqq S_m\colon\ \forall S_{l+1}\subset S_{2l+1}\colon\ \exists S_3\subseteqq S_{l+1}\colon\ S_3\notin G_*.$$

Let D be the above S_{2l+1}. Let further x, y be arbitrary elements of S_m-D. As $G_*\sim LT_{m,2l+3,l+2,3}$, we know that

$$\exists S_{l+2}\subseteqq(D\cup\{x,y\})\colon\ \forall S_3\subseteqq S_{l+2}\colon\ S_3\in G_*.$$

(7) implies that $\{x,y\}\subseteqq S_{l+2}$, therefore there are at least l different $z\in D$ satisfying $\{x,y,z\}\in G_*$. As x and y were chosen arbitrarily, G_* contains at least $l\binom{m-2l-1}{2}$ edges e such that $e\cap D=\emptyset$. Consequently,

$$\exists z\in D\colon\ d(z)\geqq\frac{l}{2l+1}\binom{m-2l-1}{2}.$$

However, this latter quantity is greater than $\frac{2}{7}\binom{m}{2}$ if $m\geqq 14l+8$

(use $\min\,(l/(2l+1))=2/5$).

This contradiction proves the lemma.

LEMMA 3. *Let* G_* *be the 3-graph of the preceding lemmas, and define* $G_*(A)=$ $=G_*\cap C^3(A)$ *where* A *is a set of vertices of* G_* *such that* $|A|\leqq 2r+2$. *Then*

(8) $$|G_*(A)|\geqq\binom{\lfloor|A|/2\rfloor}{3}+\binom{\lceil|A|/2\rceil}{3}.$$

PROOF. We use induction on $|A|$. For $|A| \leqq 4$ (8) is trivial. Now we suppose that (8) is true for $|A|-1$ and prove it for $|A|$. We distinguish two cases:

a) $|A|=2l$. Then $|G_*(A-a)| \geqq \binom{l}{3}+\binom{l-1}{3}$ holds for any $a \in A$. Therefore

$$|G_*(A)| = \frac{1}{2l-3} \sum_{a \in A} |G_*(A-a)| \geqq \frac{2l}{2l-3}\left(\binom{l}{3}+\binom{l-1}{3}\right) = 2\binom{l}{3}.$$

b) $|A|=2l+1$. $G_*(A) \sim LT_{2l+1, 2l+1, l+1, 3}$ by Lemma 2, that is

$$\exists B \subset A,\ |B| = l+1:\ \forall S_3 \subset B:\ S_3 \in G_*(A).$$

If $b \in B$ then b is an element of at least $\binom{l}{2}$ edges of $G_*(A)$. On the other hand, $|G_*(A-b)| \geqq 2\binom{l}{3}$ by the induction hypothesis. Consequently,

$$|G_*(A)| \geqq \binom{l}{2}+2\binom{l}{3} = \binom{l+1}{3}+\binom{l}{3}.$$

The proof is complete.

Now let us go back to the proof of the theorem. Fix an $\varepsilon > 0$ and choose $r_0(\varepsilon)$ so that

$$2\binom{r+1}{3}\Big/\binom{2r+2}{3} > \frac{1}{4}-\varepsilon \tag{9}$$

whenever $r \geqq r_0(\varepsilon)$. If the theorem is not true then

$$t(2r+1, r+1, 3) < 1/4-\varepsilon \tag{10}$$

for infinitely many r, that is, we can find an r satisfying both (9) and (10). By the definition of $t(2r+1, r+1, 3)$ and the monotonicity of $T(n, 2r+1, r+1, 3)\Big/\binom{n}{3}$, there is a 3-graph $G^3 \sim T(n, 2r+1, r+1, 3)$ for $n \geqq 28r+16$ such that

$$|G^3| < \binom{n}{3}\left(\frac{1}{4}-\varepsilon\right). \tag{11}$$

By Lemma 1 there is a 3-graph G_* satisfying Lemmas 1, 2 and 3. Therefore

$$|G_*(A)| \geqq 2\binom{r+1}{3}$$

holds for any $2r+2$-element subset of the vertex set S_m of G_*. Hence

$$|G_*| = \frac{1}{\binom{m-3}{2r-1}} \sum_{S_{2r+1}\subset S_m} |G_*(S_{2r+2})| \geqq$$

$$\geqq \left(\binom{m}{2r+2}\Big/\binom{m-3}{2r-1}\right) 2\binom{r+1}{3} = \binom{m}{3} \frac{2\binom{r+1}{3}}{\binom{2r+2}{3}} > \left(\frac{1}{4}-\varepsilon\right)\binom{m}{3}$$

follows by (9). This inequality contradicts the statement of Lemma 1. The theorem is proved.

3. A short survey

We list the known asymptotic results ([4], [5] and [6]):

$$t(p, q, 2) = 1\Big/\left[\frac{p-1}{q-1}\right];$$

$$t(p, q, k) = 1 \quad \text{if} \quad p \leqq \left[\frac{k}{k-1}(q-1)\right].$$

[4], [8] and [9] contain asymptotic results of the opposite type: $p=n-p'$, $q=n-q'$, $k=n-k'$, where p', q', k' are fixed and n tends to infinity.

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CLASSES OF CONGRUENCE LATTICES OF FILTRAL VARIETIES

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Abstract

For a variety **V** of algebras, let Con (**V**) denote the class $\{\mathrm{Con}(\mathfrak{A}) \mid \mathfrak{A} \in \mathbf{V}\}$, where Con ($\mathfrak{A}$) is the congruence lattice of the algebra $\mathfrak{A}$. In this paper we describe Con (**V**) for filtral varieties. It turns out that for a nontrivial filtral variety **V**, Con (**V**) is the class of all ideal lattices of generalized Boolean lattices or the class of all ideal lattices of Boolean lattices.

We also describe the congruence lattice of an algebra in a filtral variety in terms of an equivalence relation on filters of a power set.

1. Introduction

Let **B** and **GB** denote the class of Boolean lattices and generalized Boolean lattices, respectively. For a lattice or join-semilattice L with 0, let $I(L)$ denote the ideal lattice of L. For a class **K** of lattices, let $I(\mathbf{K})$ denote the class of all $I(L)$, $L \in \mathbf{K}$.

Filtral varieties were introduced (under a different name) by R. Magari [8]. To define filtral varieties we need the concept of a filtral congruence.

Let $\mathfrak{A}_i$, $i \in X$, be simple algebras and let $\mathfrak{A}$ be a subdirect product of $\mathfrak{A}_i$, $i \in X$. Let F be a filter (dual ideal) of $P(X)$ (the power set of X). For $f, g \in A$, define the *equalizer* of f and g:

$$E(f, g) = \{i \mid i \in X, f(i) = g(i)\}$$

Then

$$f \equiv g(\theta_F) \quad \text{iff} \quad E(f, g) \in F$$

defines a congruence relation on $\mathfrak{A}$.

A variety **V** is *filtral* iff **V** is *semisimple* (that is, all subdirectly irreducible algebras are simple) and whenever an algebra $\mathfrak{A} \in \mathbf{V}$ is respresented as a subdirect product of simple algebras $\mathfrak{A}_i$, $i \in X$, for every congruence θ of $\mathfrak{A}$, there is a filter F of $P(I)$, such that $\theta = \theta_F$.

Various characterizations of filtral varieties were given in [1], [2], [3] and [6].

A variety **V** is called *congruence distributive* iff Con ($\mathfrak{A}$) is distributive for all $\mathfrak{A} \in \mathbf{V}$. It was proved in [1] and in [3], that a filtral variety is congruence distributive.

Let Comp ($\mathfrak{A}$) denote the join-semilattice of compact congruences of $\mathfrak{A}$. It is well-known (see, e.g., [5]) that $I(\mathrm{Comp}(\mathfrak{A}))$ is isomorphic to Con ($\mathfrak{A}$).

One of the most important properties of filtral varieties was found in [3]: every $\theta \in \mathrm{Comp}(\mathfrak{A})$ ha sa complement in Con ($\mathfrak{A}$). This is used in proving

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THEOREM 1. *Let* **V** *be a filtral variety. Then* Con $(\mathbf{V}) \subseteq I(\mathbf{GB})$.

In this paper we describe which subclasses of $I(\mathbf{GB})$ can be represented as Con $(\mathbf{V})$ (up to isomorphism). It is somewhat surprising that there are only three such subclasses.

Let us call a filtral variety **V** *trivial,* if **V** consists of one-element algebras.

A variety **V** is *Iota Compact* (IC for short) iff for all $\mathfrak{A} \in \mathbf{V}$, the largest congruence, ι, of $\mathfrak{A}$ is compact.

Now we can state our main result:

THEOREM 2. *Let* **V** *be a nontrivial filtral variety. If* **V** *is an* IC *variety, then (up to isomorphism)* Con $(\mathbf{V}) = I(\mathbf{B})$; *otherwise,* Con $(\mathbf{V}) = I(\mathbf{GB})$.

This shows that from the point of view of Con $(\mathbf{V})$, there are only three types of filtral varieties.

Finally, in a filtral variety, we look at the representation of the congruence lattice by equivalences on filters.

Let **V** be a filtral variety, $\mathfrak{A} \in \mathbf{V}$. Let $\mathfrak{A}$ be represented as a subdirect product of the simple algebras $\mathfrak{A}_i$, $i \in X$. Then, by definition, every $\theta \in$ Con $(\mathbf{V})$ can be represented as θ_F for a suitable filter F of $P(X)$.

Obviously, if $\mathfrak{A}$ is not the full direct product, then for distinct filters F and G, we may have $\theta_F = \theta_G$. Define the equivalence relation $\equiv_{\mathfrak{A}}$ on the lattice of filters, $F(P(X))$, of $P(X)$ by

$$F \equiv_{\mathfrak{A}} G \quad \text{iff} \quad \theta_F = \theta_G.$$

Clearly, Con $(\mathfrak{A})$ is isomoprhic to $F(P(X))/\equiv_{\mathfrak{A}}$. The question is: which equivalences $\equiv$ on $F(P(X))$ can be represented as $\equiv_{\mathfrak{A}}$ for a suitable algebra $\mathfrak{A}$ in a filtral variety.

To state our characterization, it is more convenient to use ideals rather than filters. For $f, g \in A$, define the *distinguisher* of f and g:

$$D(f, g) = \{i \mid i \in X,\ f(i) \neq g(i)\}$$

Let F be a filter of $P(X)$. Define $I_F = \{X - Y \mid Y \in F\}$. Then I_F is an ideal of $P(X)$ and every ideal of $P(X)$ is of the form I_F for a suitable filter F of $P(X)$. Obviously, $E(f, g) \in F$ iff $D(f, g) \in I_F$. Hence we can describe θ_F as θ_{I_F}: $f \equiv g(\theta_{I_F})$ iff $D(f, g) \in I_F$. Moreover, the equivalence on $F(P(X))$ copies over to an equivalence on $I(P(X))$: $F \equiv_{\mathfrak{A}} G$ iff $I_F \equiv_{\mathfrak{A}} I_G$.

Thus we may consider $\equiv_{\mathfrak{A}}$ an equivalence relation on $I(P(X))$. We denote by $\varrho_{\mathfrak{A}}$ the natural map: $I \to \theta_I$, where $I \in I(P(X))$, $\theta_I \in$ Con $(\mathfrak{A})$.

This equivalence $\equiv_{\mathfrak{A}}$ on $I(P(X))$ is characterized in our last result:

THEOREM 3. *Let X be a set, let $\sim$ be an equivalence relation of ideals of $P(X)$, and let* **V** *be a nontrivial filtral variety. There exists in* **V** *a subdirect product $\mathfrak{A}$ of simple algebras $\mathfrak{A}_i$, $i \in X$, with the property that $\sim$ is the kernel of $\varrho_{\mathfrak{A}}$ iff the following conditions are satisfied:*

(i) *For any ideal I of $P(X)$, the $\sim$ class containing I contains a smallest ideal denoted by $\varphi(I)$.*

(ii) *φ is idempotent and preserves arbitrary intersections.*

(iii) *The generating elements of the principal ideals of the form* $\varphi(I)$ *form a generalized Boolean sublattice* B *of* $P(X)$. *If* $\mathbf{V}$ *is* IC, *then* B *is a Boolean lattice.*

(iv) *For every ideal* I *of* $P(X)$, $\varphi(I)$ *is generated by* $\varphi(I)\cap B$.

While Theorems 1 and 2 can be viewed as an abstract characterization of Con $(\mathbf{V})$, Theorem 3 is a concrete characterization.

For basic concepts and notation, the reader is referred to [4] and [5].

2. Proof of Theorem 1

Let $\mathbf{V}$ be a filtral variety and let $\mathfrak{A}\in\mathbf{V}$. By Theorem 4.13 in [3], every principal congruence $\theta(a, b)$ of $\mathfrak{A}$ has a complement in Con $(\mathfrak{A})$. This implies immediately that every $\theta\in$ Comp $(\mathfrak{A})$ has a complement.

Now let $\alpha, \beta\in$ Comp $(\mathfrak{A})$, $\alpha\leqq\beta$, and let α' be the complement of α.

CLAIM 1. $\alpha'\wedge\beta$ *is the relative complement of* α *in* $[\omega, \beta]$ *and* $\alpha'\wedge\beta\in$ Comp $(\mathfrak{A})$.

PROOF. Since Con $(\mathfrak{A})$ is distributive, the first part of the claim is obvious. Let $\{\beta_y | y\in Y\}$ be the set of all compact congruences β_y with $\beta_y\leqq\alpha'\wedge\beta$. Then

$$\alpha'\wedge\beta=\bigvee(\beta_y\mid y\in Y).$$

Hence,

$$\beta=\alpha\vee(\alpha'\wedge\beta)=\alpha\vee\bigvee(\beta_y\mid y\in Y).$$

By the compactness of β, there is a finite subset Y_1 of Y such that

$$\beta=\alpha\vee\bigvee(\beta_y\mid y\in Y_1).$$

Since, obviously,

$$\alpha\wedge\bigvee(\beta_y\mid y\in Y_1)=\omega,$$

we conclude that $\bigvee(\beta_y | y\in Y_1)$ is also a relative complement of α in $[\omega, \beta]$, hence, by distributivity,

$$\alpha'\wedge\beta=\bigvee(\beta_y\mid y\in Y_1),$$

verifying the claim.

CLAIM 2. *If* $\alpha, \beta\in$ Comp $(\mathfrak{A})$, *then* $\alpha\wedge\beta\in$ Comp $(\mathfrak{A})$.

PROOF. Let α_1 (resp. β_1) be the relative complement of α (resp. β) in $[\omega, \alpha\vee\beta]$. Obviously, $\alpha_1\leqq\beta$ and $\beta_1\leqq\alpha$. By the previous claim, α_1 and $\beta_1\in$ Comp $(\mathfrak{A})$. Thus $\gamma=\alpha_1\vee\beta_1\in$ Comp $(\mathfrak{A})$ and $\gamma\leqq\alpha\vee\beta$. Again, by Claim 1, γ has a relative complement γ_1 in $[\omega, \alpha\vee\beta]$ and $\gamma_1\in$ Comp $(\mathfrak{A})$. Using De Morgan's Law, it is clear that $\gamma_1=\alpha\wedge\beta$.

By Claims 1 and 2, Comp $(\mathfrak{A})$ is a sublattice of Con $(\mathfrak{A})$ and Comp $(\mathfrak{A})$ is a generalized Boolean lattice. Since $I($Comp $(\mathfrak{A}))\cong$ Con $(\mathfrak{A})$, Theorem 1 follows.

3. The first construction

CLAIM 3. *Let* $\mathbf{V}$ *be a nontrivial filtral variety. For every Boolean lattice* B, *there exists a* $\mathfrak{B}\in\mathbf{V}$ *such that* Con $(\mathfrak{B})$ *is isomorphic to* $I(B)$.

PROOF. Let $\mathfrak{A}$ be a simple algebra in $\mathbf{V}$, $|A|>1$. Let us regard B as a field of sets of some set I, that is, B is a sublattice of the power set $P(I)$, and the 0 and 1 of B are $\emptyset$ and I, respectively.

Now we construct the set C as follows: C consists of all functions $f\colon I\to A$ satisfying

(1) $|f(I)|$ is finite.

(2) For $a\in A$, $f^{-1}(a)\in B$.

Obviously, $C\subseteq A^I$. Now let $f, g\in C$ and let $+$ be a binary operation. We verify that $f+g$ (as an element of $\mathfrak{A}^I$) is also in C. Indeed,

$$(f+g)(I) = \{f(i)+g(i) \mid i\in I\}$$

is finite since $f(I)$ and $g(I)$ are finite. Now for $a\in A$

$$(f+g)^{-1}(a) = \{i \mid f(i)+g(i)=a\} = \cup(\{i \mid f(i) = b,\ g(i) = c\} \mid b+c = a) =$$
$$= \cup(f^{-1}(b)\cap g^{-1}(c) \mid b+c = a,\ b\in f(I),\ c\in g(I))$$

Since, by (1), there are only finitely many pairs $\langle b, c\rangle\in A^2$ with $b\in f(I)$, $c\in g(I)$, $b+c=a$, we conclude that $(f+g)^{-1}(a)\in B$. The proof for an arbitrary operation is similar. Thus C defines a subalgebra $\mathfrak{B}$ of $\mathfrak{A}^I$; in fact, $\mathfrak{B}$ is a subdirect power of $\mathfrak{A}$.

Now, for $X\in B$ we define a congruence relation θ_X of $\mathfrak{B}$ by

$$f\equiv g(\theta_X) \quad \text{iff} \quad f(i) = g(i) \quad \text{for all} \quad i\notin X.$$

θ_X is obviously a congruence relation. We claim that θ_X is principal.

Let $a, b\in A$, $a\neq b$. Define $p, q\in C$ by

(3) $p(i)=a$ and $q(i)=b$ for $i\in X$

(4) $p(i)=a$ and $q(i)=a$ for $i\notin X$.

By the definition of θ_X, $p\equiv q(\theta_X)$, thus $\theta(p, q)\leqq\theta_X$. To prove $\theta_X\leqq\theta(p, q)$, let $f\equiv g(\theta_X)$. Then

$$E(f, g) \supseteq I-X = E(p, q).$$

It was observed in [2], that in a filtral variety this implies that $\theta(f, g)\leqq\theta(p, q)$, concluding the proof of $\theta_X=\theta(p, q)$.

Thus $\varphi\colon X\to\theta_X$ embeds B into Comp ($\mathfrak{B}$). To conclude the proof of Claim 3, we have to show that φ is onto. It is sufficient to prove that every principal congruence is of the form θ_X.

Let $\theta=\theta(p, q)$ and define $X=D(p, q)$. Since $p, q\in C$, X is a finite union of sets of the form

$$\{i \mid p(i) = u \quad \text{and} \quad q(i) = v\} = p^{-1}(u)\cap q^{-1}(v)$$

hence $X\in B$. We verify $\theta=\theta_X$ as above.

4. The second construction

We start with

CLAIM 4. *Let* $\mathbf{V}$ *be a filtral variety. Then* Con ($\mathbf{V}$)$\subseteq I(\mathbf{B})$ *iff no* $\mathfrak{A}\in\mathbf{V}$ *with* $|A|>1$ *has a one-element subalgebra.*

PROOF. Let us assume that no $\mathfrak{A}\in\mathbf{V}$ with $|A|>1$ has a one-element subalgebra.

By the result of J. Kollár [7], **V** is an IC variety. By Theorem 1, $\mathrm{Con}(\mathbf{V}) \subseteq I(\mathbf{GB})$. The IC members of $I(\mathbf{GB})$ are exactly the ones in $I(B)$. Hence, $\mathrm{Con}(\mathbf{V}) \subseteq I(\mathbf{B})$.

Conversely, if $\mathrm{Con}(\mathbf{V}) \subseteq I(\mathbf{B})$, then **V** is an IC variety, hence, again by Kollár's result [7], no $\mathfrak{A} \in \mathbf{V}$ with $|A| > 1$ has a one-element subalgebra.

Now we can do our second construction.

CLAIM 5. *Let* **V** *be a filtral variety failing* IC. *For every generalized Boolean lattice B there exists a $\mathfrak{B} \in \mathbf{V}$ satisfying* $\mathrm{Con}(\mathfrak{B}) \cong I(B)$.

PROOF. By Claim 4, there is an $\mathfrak{A} \in \mathbf{V}$ with a one-element subalgebra $\{0\}$ and $|A| > 1$. We can assume that B is represented in $P(I)$ (that is, B is a sublattice of $P(I)$, $\emptyset$ is the zero of B, and $\bigcup (X \mid X \in B) = I$).

We can further assume that $\mathfrak{A}$ is simple. Indeed, if $\mathfrak{A}$ is not simple, then there is a subdirectly irreducible algebra $\mathfrak{A}'$ with $|A'| > 1$ such that $\mathfrak{A}$ has a homomorphism onto $\mathfrak{A}'$. Since **V** is semisimple, $\mathfrak{A}'$ is simple and has a one-element subalgebra (the image of $\{0\}$).

Now we define $C \subseteq A^I$ by the rules: $f \in C$ iff

(1) $|f(I)|$ is finite

(2′) For $a \in A$, $a \neq 0$, $f^{-1}(a) \in B$.

Note that (1) is the same as (1) in § 3, while (2′) is (2) of § 3 modified.

If $f, g \in C$, then we show that $f+g \in C$ as in § 3. The only new case to consider is a set of the form

$$f^{-1}(0) \cap g^{-1}(c), \quad 0+c = a, \quad c \neq 0.$$

Now $f^{-1}(0) = I - f^{-1}(f(A - \{0\})) = I - \bigcup (f^{-1}(x) \mid x \neq 0)$. Since this union is finite, $f^{-1}(0) = I - X$ for some $X \in B$. Hence

$$g^{-1}(c) = (g^{-1}(c) \cap X) \cup (f^{-1}(0) \cap g^{-1}(c)).$$

Since $g^{-1}(c)$ and $g^{-1}(c) \cap X \in B$, this equation implies that $f^{-1}(0) \cap g^{-1}(c) \in B$.

For $X \in B$, we again define $\theta_X \in \mathrm{Con}(\mathfrak{B})$. To verify that θ_X is principal, now define

(3) $p(i) = a$ and $q(i) = b$ for $i \in X$

(4′) $p(i) = 0$ and $q(i) = 0$ for $i \notin X$.

The rest of the proof is identical.

5. Proof of Theorem 2

Let **V** be a filtral variety. By Theorem 1, $\mathrm{Con}(\mathbf{V}) \subseteq I(\mathbf{GB})$. By Claim 3, $I(\mathbf{B}) \subseteq \mathrm{Con}(\mathbf{V})$.

Now let **V** be an IC variety. Then ι is compact in $\mathrm{Con}(\mathfrak{A})$ for $\mathfrak{A} \in \mathbf{V}$. Since for a generalized Boolean lattice B, the unit of $I(B)$ is compact iff B is Boolean, we conclude that $\mathrm{Con}(\mathbf{V}) \subseteq I(B)$, hence $\mathrm{Con}(\mathbf{V}) = I(\mathbf{B})$. Conversely, if $\mathrm{Con}(\mathbf{V}) = I(\mathbf{B})$, then **V** is obviously IC.

Finally, assume that **V** does not have IC. Then, by Claim 5, $\mathrm{Con}(\mathbf{V}) \supseteq I(\mathbf{GB})$, hence $\mathrm{Con}(\mathbf{V}) = I(\mathbf{GB})$. This completes the proof of Theorem 2.

6. Proof of Theorem 3

To prove necessity of the condition, we define φ by

$$\varphi(I) = \{D(f, g) \mid D(f, g) \in I\}.$$

Now, conditions (i) and (ii) are obvious. The principal ideals of the form $\varphi(I)$ are, clearly, the ones which are generated by a single $D(f, g)$, hence, they form a (generalized) Boolean sublattice B of $P(X)$. Condition (iv) follows immediately from the definition of $\varphi(I)$.

Next, we show that conditions (i)—(iv) are sufficient. Indeed, in this case the algebra constructed in Theorem 2, with B defined in condition (iii), will give, clearly, the desired equivalence relation.

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SMALL VALUES OF INDEFINITE QUADRATIC FORMS AND POLYNOMIALS IN MANY VARIABLES

R. J. COOK

1. Introduction

A well-known result, due to Birch, Davenport and Ridout ([1], [7] and [8]), states that if $Q(x)$ is an indefinite quadratic form in $n \geqq 21$ variables then for any $\varepsilon > 0$ there is a non-zero integer vector x with

$$|Q(x)| < \varepsilon. \tag{1}$$

More recently, Schmidt [10] has shown that if $F(x)$ is a form of odd degree k then for any $E > 0$ there is an integer vector x with

$$0 < |x| \leqq X \quad \text{and} \quad |F(x)| < |F| X^{-E},$$

where $|x| = \max |x_i|$ and $|F|$ is the largest absolute value of the coefficients of $F(x)$, provided only that n and X are large enough as functions of k and E. Using the diagonalization procedure of Birch and Davenport [1] we obtain a similar result for quadratic forms in many variables. We say that an indefinite quadratic form $Q(x)$ in n variables is of type $(r, n-r)$ if, when Q is expressed as a sum of squares of n real linear forms with positive and negative signs, there are r positive signs and $n-r$ negative signs. We use Vinogradov's $\ll$-notation where the implicit constants may depend on Q or F as well as n and ε.

THEOREM 1. *Let $Q(x)$ be of type $(r, n-r)$ where*

$$1 \leqq \min(r, n-r) \leqq 4 \quad \textit{and} \quad n \geqq 21, \tag{2}$$

then for any $\varepsilon > 0$ and $X > X_0(\varepsilon, n)$ there is a non-zero integer vector x such that

$$|x| \leqq X \quad \textit{and} \quad |Q(x)| \ll X^{-\frac{1}{2}+\frac{25}{2n+10}+\varepsilon}. \tag{3}$$

We see that the exponent of X in (3) is negative for $n \geqq 21$, and tends to the limit $-\frac{1}{2}+\varepsilon$ as $n \to \infty$. The condition on the type of Q can be dispensed with at the cost of extra variables. If $Q(x)$ is an indefinite quadratic form in n variables, of type $(r, n-r)$, then for any $r' < r$ $Q(x)$ represents a form of type $(r', n-r)$; the latter being a form in $r'+n-r$ variables (see [1]). Replacing Q by $-Q$, if necessary, we may suppose that $\min(r, n-r) = r$, then $r \leqq [n/2]$. Now Q represents a form of type

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$(4, n-r)$ in $n-r+4$ variables, assuming that $r>4$. Thus for $n \geqq 33$ we have $n-r+4 \geqq 21$ and

$$2(n-r+4) \leqq 2n-2[n/2]+8 = n+8+\theta \tag{4}$$

where

$$\theta = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \tag{5}$$

We use $X_0(\varepsilon, n)$, $n_0(\varepsilon)$ as suitable boundary points, not necessarily the same at each occurence.

THEOREM 2. *Let $Q(x)$ be a non-singular indefinite quadratic form in $n \geqq 33$ variables. Then for any $\varepsilon>0$ and $X>X_0(\varepsilon, n)$ there is a non-zero integer vector x such that*

$$|x| \leqq X \quad \text{and} \quad |Q(x)| \ll X^{-\frac{1}{2}+\frac{25}{n+18+\theta}+\varepsilon} \tag{6}$$

where θ is defined by (5).

Let

$$F(x) = f(x)+2^{1/2}(x_1^2+\ldots+x_n^2)$$

where f has integer coefficients. Then for $0<|x| \leqq X$ we have

$$|F(x)| \geqq \|2^{1/2}(x_1^2+\ldots+x_n^2)\| \gg X^{-2}, \tag{7}$$

where $\|\theta\|$ denotes the distance from θ to the nearest integer. A recent result of Schlickewei [9] on additive Diophantine inequalities can be used to obtain a result for the fractional parts of quadratic polynomials in many variables.

THEOREM 3. *Let $F(x)$ be a quadratic polynomial in n variables having no constant term. For any $\varepsilon>0$ there exist $n_0(\varepsilon)$, $X_0(\varepsilon, n)$ such that if $n \geqq n_0(\varepsilon)$ and $X \geqq X_0(\varepsilon, n)$ then there exists a non-zero integer vector x such that*

$$|x| \leqq X \quad \textit{and} \quad \|F(x)\| < X^{-2+\varepsilon}. \tag{8}$$

This improves on our previous results for quadratic forms [4] and quadratic polynomials [5]. From (7) we see that apart from the arbitrary ε the exponent is best possible.

2. Preliminary lemmas

Davenport [6] showed that if $Q(x)$ is an indefinite quadratic form of type $(r, n-r)$ then there is a non-singular linear transformation $y = Tx$ which takes Q into a quadratic form $Q'(y)$ with the following properties

$$Q'(y_1, \ldots, y_r, 0, \ldots, 0) > 0 \tag{9}$$

for all integers $y_1, \ldots, y_r$, not all zero, and

$$Q'(0, \ldots, 0, y_{r+1}, \ldots, y_n) < 0 \tag{10}$$

for all integers $y_{r+1}, \ldots, y_n$, not all zero. Since T is non-singular there exist numbers

c, C such that

$$c|y| \leqq |x| \leqq C|y|$$

so there is no loss of generality in proving Theorem 1, and also Theorem 3, under the additional assumption that Q also satisfies (9) and (10).

Our first lemma, due to Birch and Davenport [2], shows that a diagonal indefinite quadratic form in 5 variables takes small values.

LEMMA 1. *For any* $\delta > 0$ *there exists* $C(\delta)$ *with the following property. For any real* $\lambda_1, \ldots, \lambda_5$, *not all of the same sign and all of absolute value* 1 *at least, there exist integers* $x_1, \ldots, x_5$ *which satisfy*

$$|\lambda_1 x_1^2 + \ldots + \lambda_5 x_5^2| < 1 \tag{11}$$

and

$$0 < \sum_{i=1}^{5} |\lambda_i x_i^2| < C(\delta)|\lambda_1 \ldots \lambda_5|^{1+\delta}. \tag{12}$$

We apply Lemma 1 to the diagonal quadratic form

$$\lambda_1 Y x_1^2 + \ldots + \lambda_5 Y x_5^2 \quad (Y > 1)$$

and a straightforward calculation gives the following result.

LEMMA 2. *For any* $\tau > 0$ *there exists* $C(\tau)$ *with the following property. For any real* $\lambda_1, \ldots, \lambda_5$, *not all of the same sign and real numbers* $X_1, \ldots, X_5$, Y, *all at least* 1, *satisfying*

$$Y(Y^5 \Pi)^\tau < C(\tau) X_i^{1/2} |\lambda_i \Pi^{-1}|^{1/4} \quad \textit{for} \quad 1 \leqq i \leqq 5, \tag{13}$$

where $\Pi = |\lambda_1 \ldots \lambda_5|$, *there exist integers* $x_1, \ldots, x_5$, *not all zero, satisfying*

$$0 \leqq x_i \leqq X_i \quad \textit{for} \quad i = 1, \ldots, 5 \tag{14}$$

and

$$|\lambda_1 x_1^2 + \ldots + \lambda_5 x_5^2| < Y^{-1}. \tag{15}$$

To reduce quadratic forms and polynomials to almost diagonal shape we use the following lemma which is essentially due to Birch and Davenport [1], with minor modifications which can be left to the reader.

LEMMA 3. *Suppose that* $m < n$, *and let* $L_1(x), \ldots, L_m(x)$ *be* m *real linear forms in* n *variables* $x_1, \ldots, x_n$, *say*

$$L_i(x) = \sum_{j=1}^{n} \gamma_{ij} x_j \quad (1 \leqq i \leqq m). \tag{16}$$

Then, for any $P \geqq 2$, *there exists a non-zero integer vector* x *such that*

$$|x| \leqq P^m \quad \textit{and} \quad |L_i| \leqq C P^{m-n} \sum_{j=1}^{n} |\gamma_{ij}| \quad (1 \leqq i \leqq m), \tag{17}$$

where C *is an absolute constant.*

For the proof of Theorem 3 we need the following result on additive quadratic forms, due to Schlickewei [9].

LEMMA 4. *For any* $\eta>0$ *there exists* $s_0(\eta)$ *such that for any* $s\geqq s_0$, $N\geqq 1$ *and real* $\theta_1, \ldots, \theta_s$ *the inequalities*

(18) $$|x| \leqq N \quad \textit{and} \quad \|\theta_1 x_1^2+\ldots+\theta_s x_s^2\| < N^{-2+\eta}$$

have a non-zero solution.

Finally, in order to reduce a quadratic polynomial to a polynomial that is almost an additive form mod 1 we need the following version of Dirichlet's box principle which is Theorem VI of Chapter 1 of Cassels [3].

LEMMA 5. *Let* $L_1(x), \ldots, L_m(x)$ *be m real linear forms in n variables* $x_1, \ldots, x_n$. *Then for any* $P>1$ *there exist integers* $x_1, \ldots, x_n$, *not all zero, such that*

(19) $$|x| \leqq P \quad \textit{and} \quad \|L_i(x)\| < P^{-n/m} \quad \textit{for} \quad 1 \leqq i \leqq m.$$

3. Proof of Theorem 1

Let

(20) $$Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i x_j \quad (\alpha_{ij} = \alpha_{ji})$$

and with $Q(x)$ we associate the bilinear form

(21) $$B(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i y_j.$$

Replacing Q by $-Q$, if necessary, we may suppose that $r\leqq 4$ and then our assumption that $Q(x)$ satisfies (9) implies that $\alpha_{11}>0$. We shall use a suitably chosen linear transformation

(22) $$x = u_1 z^1+\ldots+u_5 z^5$$

to show that Q represents an almost diagonal quadratic form in 5 variables.

We take $z^1=(1, 0, \ldots, 0)$ and, having chosen $z^1, \ldots, z^{j-1}$, we choose z^j by applying Lemma 3 with $m=j-1$ and $L_i(x)=B(z^i, x)$ for $i=1, \ldots, j-1$. In this way we obtain non-zero integer vectors $z^1, \ldots, z^5$ such that

(23) $$|z^j| \leqq P^{j-1} \quad \text{for} \quad j = 1, \ldots, 5$$

and

(24) $$|B(z^i, z^j)| \ll P^{i+j-n-2} \quad (i \neq j)$$

where $P>2$ is to be chosen later.

Under the linear transformation (22)

(25) $$Q(x) = \varphi(u_1, \ldots, u_5) = \sum_{i=1}^{5} \sum_{j=1}^{5} \beta_{ij} u_i u_j$$

say, where $\beta_{ij}=B(z^i, z^j)$. We consider the values taken by φ for

(26) $$|u_i| \leqq \frac{1}{5} X P^{1-i} \quad 1 \leqq i \leqq 5$$

so that $|x| \leqq X$. Now $\beta_{11}=\alpha_{11}>0$ and since Q represents φ, φ is of type (r', s') where $r' \leqq r \leqq 4$. Thus $\beta_{11}, \ldots, \beta_{55}$ are not all of the same sign and

$$\Pi = |\beta_{11} \ldots \beta_{55}| \ll P^{20}.$$

Taking

$$Y = X^{(1/2)-\varepsilon} P^{-5} \tag{27}$$

and choosing $\tau>0$ sufficiently small we see that for $i=1, \ldots, 5$

$$Y(Y^5 \Pi^\tau) \ll X^{1/2} P^{(1-i)/2} |\beta_{ii} \Pi^{-1}|^{1/4} \ll X^{1/2} \Pi^{-1/4} \tag{28}$$

provided that P is a fixed power of X. Applying Lemma 2, we see that there are integers $u_1, \ldots, u_5$, not all zero, with

$$|\beta_{11} u_1^2 + \ldots + \beta_{55} u_5^2| < Y^{-1}. \tag{29}$$

The off-diagonal terms of $\varphi(u)$ contribute

$$\ll P^{i+j-n-2} X P^{1-i} X P^{1-j} = X^2 P^{-n}. \tag{30}$$

Taking

$$P = X^{5/2(n+5)},$$

so that $X^2 P^{-n} = X^{-1/2} P^5$, we have

$$|Q(x)| = |\varphi(u_1, \ldots, u_5)| \ll X^{-\frac{1}{2}+\varepsilon} P^5 + X^2 P^{-n} \ll X^{-\frac{1}{2}+\frac{25}{2(n+5)}+\varepsilon} \tag{31}$$

so it now only remains to check that $x \neq 0$.

If any $|\beta_{ii}| < Y^{-1}$ then $x=z^i$ gives a non-zero solution of (31), so now we may suppose that

$$Y^{-1} < |\beta_{ii}| \ll P^{2(i-1)}, \quad 1 \leqq i \leqq 5. \tag{32}$$

Therefore the contribution of the main diagonal to $\det \varphi$ is greater than Y^{-5} in absolute value. The contribution coming from a product of terms all of which lie off the main diagonal is

$$\ll P^{\sum_{i=1}^{5} i + \sum_{j=1}^{5} j - 5n - 10} = P^{20-5n} = o(Y^{-5}).$$

The contribution of a product with just one term β_{ii} on the main diagonal is

$$\ll \beta_{ii} P^{30-2i-4n-8} \ll P^{20-4n} = o(Y^{-5}).$$

The final contributions come from products with three terms on the main diagonal and are

$$\ll \beta_{ii}\beta_{jj}\beta_{kk} P^{2(l+m-n-2)} \ll \beta_{ii}\beta_{jj}\beta_{kk} P^{14-2n} = o(\beta_{ii}\beta_{jj}\beta_{kk} Y^{-2}),$$

where $i, j, \ldots, m$ is a permutation of $1, 2, \ldots, 5$. Thus

$$\det \varphi = \beta_{11} \ldots \beta_{55}(1+o(1)) \neq 0, \tag{33}$$

the rank of the substitution (22) is 5 and so $x \neq 0$.

4. Proof of Theorem 3

Let $s=s_0(\eta)$ be the value arising in Lemma 4, and consider the linear transformation

$$x = u_1 z^1 + \ldots + u_s z^s, \tag{34}$$

where $z^1, \ldots, z^s$ will be chosen suitably. Then

$$F(x) = Q(x)+L(x) = \varphi(u)+\Lambda(u) \tag{35}$$

where

$$\varphi(u) = \sum_{i=1}^{s}\sum_{j=1}^{s} \beta_{ij} u_i u_j, \ \beta_{ij} = B(z^i, z^i) \tag{36}$$

and

$$\Lambda(u) = \sum_{j=1}^{s} \mu_j u_j, \quad \mu_j = L(z^j). \tag{37}$$

We choose z^j by applying Lemma 5 with $m=1$ and $L_1(x)=L(x)$. We obtain a non-zero integer vector z^j such that

$$|z^1| \leqq P \quad \textit{and} \quad \|L(z^1)\| < P^{-n}. \tag{38}$$

Having chosen $z^1, \ldots, z^{j-1}$ we choose z^j by applying Lemma 5 with $m=j$, P^j in place of P, $L_i(x)=B(z^i, x)$ for $i=1, 2, \ldots, j-1$ and $L_j(x)=L(x)$. We obtain a non-zero integer vector z^j such that

$$|z^j| \leqq P^j, \max_{1<i\leqq j-1} \|B(z^i, z^j)\| < P^{-n} \quad \text{and} \quad \|L(z^j)\| < P^{-n}, \tag{39}$$

and continue the process as far as z^s.

Let

$$\varphi_0(u) = \beta_{11} u_1^2 + \ldots + \beta_{ss} u_s^2$$

and $\varphi_1(u)=\varphi(u)-\varphi_0(u)$. Then

$$\begin{aligned} \|F(x)\| &\leqq \|\varphi_0(u)\| + \|\varphi_1(u)\| + \|\Lambda(u)\| \leqq \\ &\leqq \|\varphi_0(u)\| + \sum_{i=1}^{s}\sum_{j=1}^{s} \|\beta_{ij}\| \, |u_i| \, |u_j| + \sum_{j=1}^{s} \|\mu_j\| \, |u_j|. \end{aligned} \tag{40}$$

We apply Lemma 5 to the diagonal form $\varphi_0(u)$ and for $U>1$ choose a nonzero integer vector u such that

$$|u| \leqq U \quad \text{and} \quad \|\varphi_0(u)\| < U^{-2+\eta}. \tag{41}$$

Then

$$|x| < sUP^{s-1} \tag{42}$$

and

$$\|F(x)\| < U^{-2+\eta} + s^2 P^{-n} U + sP^{-n} U. \tag{43}$$

We take $P=X^p$ and $U=X^u$ where

$$p = \frac{1}{(s-1)\sqrt{n}} \quad \text{and} \quad u = 1-(s-1)p-\eta. \tag{44}$$

Then

$$|x| < sX^{u+(s-1)p} = (sX^{-\eta})X < X \tag{45}$$

and

$$\begin{aligned} |F(x)| &< X^{-2+2(s-1)p+3\eta}+2s^2X^{2-2(s-1)p-2\eta-\sqrt{n}/(s-1)} < \\ &< X^{-2+4\eta}+X^{-2} < X^{-2+\varepsilon} \end{aligned} \tag{46}$$

provided that we take $\eta=\varepsilon/5$, $n>n_0(s,\varepsilon)$ and $X>X_0(s,\varepsilon)$, which completes the proof when $x\neq 0$.

If $x=0$ then $z^1, \ldots, z^s$ are linearly independent over **Q** so for some $j\leqq s$ and integers $\alpha_1, \ldots, \alpha_j$

$$\alpha_1 z^1+\ldots+\alpha_j z^j = 0 \tag{47}$$

where $z^1, \ldots, z^{j-1}$ are linearly independent over **Q**. Hence

$$z^j = \beta_1 z^1+\ldots+\beta_{j-1}z^{j-1} \tag{48}$$

where $\beta_i=\alpha_i/\alpha_j$. These equations (48) are n linear equations on the β_i, with coefficients which are the coordinates of the vectors z^r. By construction, some $j-1$ of them are linearly independent, and so have determinant $\Delta\neq 0$. Applying Cramer's rule to this independent subset of $j-1$ equations we obtain

$$\beta_i = \Delta_i/\Delta \quad \text{for} \quad i = 1, \ldots, j-1$$

where the Δ's are determinants of $(j-1)\times(j-1)$ integer matrices. Thus we may take $\alpha_j=\Delta$ and $\alpha_i=-\Delta_i$ for $i=1, \ldots, j-1$ in (47). Since the elements of the k-th column of Δ have absolute value at most P^{k-1} and Δ is non-singular

$$0 < |\alpha_j| = |\Delta| \leqq (j-1)P^{j(j-1)/2} \leqq (s-1)P^{s(s-1)/2} \tag{49}$$

and similarly

$$|\alpha_i| = |\Delta_i| \leqq (s-1)P^{s(s+1)/2}. \tag{50}$$

Then $x=\alpha_j z^j\neq 0$ satisfies

$$|x| \leqq (s-1)P^{s(s-1)/2+(s-1)} < X \tag{51}$$

provided that $n>n_0(s)$ and $X>X_0(s)$. Further

$$\begin{aligned} \|F(x)\| &= \|B(\alpha_j z^j, \alpha_j z^j)+L(\alpha_j z^j)\| \leqq \Big\|\alpha_j \sum_{i=1}^{j-1}(-\alpha_i)B(z^i, z^j)\Big\|+\|\alpha_j L(z^j)\| \leqq \\ &\leqq (s-1)^3P^{s(s-1)/2+s(s+1)/2-n}+(s-1)P^{s(s-1)/2-n} < X^{-2} \end{aligned} \tag{52}$$

provided that $n>n_0(s)$ and $X>X_0(s,n)$, and this completes the proof of Theorem 3.

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A REMARK TO A PAPER OF J. ACZÉL AND J. K. CHUNG

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In their paper [1] J. Aczél and J. K. Chung have proved among other results that, if the functions f_i $(i=1, 2, \ldots, n)$ are locally Lebesgue integrable and the functions p_k and q_k $(k=1, 2, \ldots, n)$ are L-independent, moreover the functional equation

$$(1)\qquad \sum_{i=1}^{m} f_i(x+\lambda_i y) = \sum_{k=1}^{m} p_k(x) q_k(y), \quad x\in]A, B[, \quad y\in]C, D[$$

is satisfied, where $0\neq\lambda_i\neq\lambda_j$ for $i\neq j$, then the functions f_i, p_k and q_k are in C^∞. L-independence means, e.g. for the q_k's, that

$$\sum_{k=1}^{m} c_k q_k(y) = 0 \quad \text{for almost every} \quad y\in]C, D[$$

implies that $c_1=c_2=\ldots=c_n=0$.

In this short note we observe that L-independence and local Lebesgue integrability may be replaced by linear independence and by Lebesgue measurability, respectively.

In order to prove this we observe (as in [1]) that

$$(2)\qquad p_k(x) = \sum_{i,j} a_{i,j,k} f_i(x+\lambda_i y_j) \quad \text{if} \quad x\in]A, B[$$

for a suitably chosen sequence $C<y_1<y_2<\ldots<y_n<D$ because of the linear independence of the q_k. Similarly,

$$(3)\qquad q_k(y) = \sum_{i,j} b_{i,j,k,} f_i(x_j+\lambda_i y), \quad y\in]C, D[$$

where $A<x_1<x_2<\ldots<x_n<B$. Hence the p_k and q_k are Lebesgue measurable too. Now, with the substitution $t=x+\lambda_i y$, we have that

$$(4)\qquad f_i(t) = \sum_{k=1}^{m} p_k(t-\lambda_i y) q_k(y) - \sum_{j\neq i} f_j\big(t+(\lambda_j-\lambda_i)y\big)$$

whenever $C<y<D$ and $A+\lambda_i y<t<B+\lambda_i y$. Hence, using Theorem 3.3 of [2], we have that f_i is continuous. So, by (2) and (3), the functions p_k and q_k are continuous

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too. Similarly, as in [1], choosing C^* between C and D and integrating, we obtain that

$$\sum_{i=1}^{m} \int_{C^*}^{t} f_i(x+\lambda_i y)\,dy = \sum_{k=1}^{m} p_k(x) \int_{C^*}^{t} q_k(y)\,dy.$$

If we introduce into each integral on the left hand side individually new variables $k=x+\lambda_i y$ we have that

$$\sum_{i=1}^{m} \frac{1}{\lambda_i} \int_{x+\lambda_i C^*}^{x+\lambda_i t} f_i(k)\,dk = \sum_{k=1}^{n} p_k(x) Q_k(t) \tag{5}$$

where

$$Q_k(t) = \int_{C^*}^{t} q_k(y)\,dy.$$

The functions Q_k are linearly independent for else there would exist constants c_k not all 0 such that

$$\sum_{k=1}^{n} c_k Q_k(t) \equiv 0$$

that is,

$$\int_{C^*}^{t} \Big(\sum_{k=1}^{n} c_k \cdot q_k(y)\,dy \equiv 0 \quad \text{for all} \quad t \in]C, D[,$$

which is impossible because the q_k are linearly independent and continuous. Hence the p_k are linear combinations of the continuously differentiable functions

$$x \mapsto \int_{x+\lambda_i C^*}^{x+\lambda_i t} f_i(s)\,ds$$

and so continously differentiable. The same holds for the q_k. By (4) and by Theorem 5.2 of [2], the f_i are continuously differentiable too. Using now (4) and Theorem 7.2 of [2], we have that the f_i are twice continuously differentiable and, by (2) and (3), so are the p_k and q_k too. By repeating this argument we get the result that f_i, p_k and q_k are in C^∞.

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ÜBER DIE DICHTE MEHRFACHER GITTERFÖRMIGER KREISANORDNUNGEN IN DER EBENE

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1. Sei G ein Gitter in der Ebene, $k\in\mathbf{N}$ und $K(\mathbf{x})$ bzw. $\mathrm{B}(\mathbf{x})$ die offene bzw. abgeschlossene Einheitskreisscheibe mit Mittelpunkt $\mathbf{x}$.

G liefert eine k-Packung (der Einheitskreisscheibe) genau dann, wenn jeder Punkt der Ebene in höchstens k Kreisscheiben der Form $K(\mathbf{g})$, $\mathbf{g}\in G$ liegt.

Von einer k-Überdeckung spricht man, wenn jeder Punkt in mindestens k Kreisscheiben $\mathrm{B}(\mathbf{g})$, $\mathbf{g}\in G$ liegt.

Die Dichte einer solchen Kreisanordnung wird durch $d(G):=\frac{\pi}{\Delta(G)}$ gegeben, wobei $\Delta(G)$ die Determinante des Gitters G bezeichnet. Sei schließlich

$d_k:=\sup\,\{\mathrm{d}(G)|G \text{ liefert k-Packung}\}$,

$D_k:=\inf\,\{\mathrm{d}(G)|G \text{ liefert k-Überdeckung}\}$.

Dann besteht das Problem darin, d_k, D_k zu bestimmen oder wenigstens abzuschätzen. Die genauen Werte sind für einige „kleine" k bekannt (s. z. B. [2]); ferner gilt über das asymptotische Verhalten von d_k und D_k: Es gibt Konstanten $c_i>0$, so daß

$$k-c_1k^{2/5}\leqq d_k\leqq k-c_2k^{1/4}$$

$$k+c_3k^{1/4}\leqq D_k\leqq k+c_4k^{2/5}.\quad ([1])$$

Ich möchte in dieser Arbeit zeigen, daß der Exponent 1/4 in beiden Abschätzungen nicht mehr zu verbessern ist.

SATZ. *Es gibt* $c_5, c_6>0$ *(konstant) und unendliche Teilmengen* $N, N'\subseteqq\mathbf{N}$, *so daß*

$$d_k\geqq k-c_5k^{1/4}\quad \textit{für alle}\quad k\in N$$

$$D_k\leqq k+c_6k^{1/4}\quad \textit{für alle}\quad k\in N'.$$

Zum Beweis wird das Gitter G_n mit den Basisvektoren $(a;0)$ und $(0;h)$, $a=\frac{\pi}{2n}$ und $h=\frac{2}{n}$ für alle $n\in\mathbf{N}$ auf seine Lagerungsvielfachheit untersucht. Dabei ergibt sich, daß die Packungsvielfachheit $k_n\leqq n^2+O(\sqrt{n})$ ist, während die Überdeckungsvielfachheit $k_n'\geqq n^2+O(\sqrt{n})$ ist. Die Dichte $\mathrm{d}(G_n)$ ist offenbar $=\frac{\pi}{ah}=n^2$. Ich

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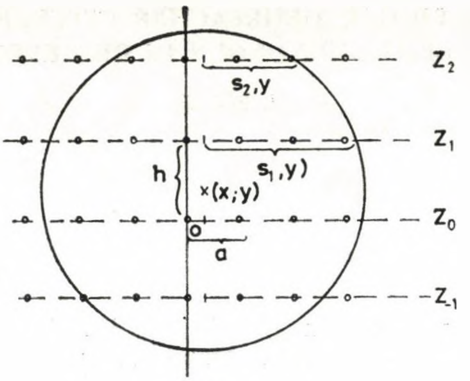

Abb. 1

beschränke den Beweis zunächst auf k-Packungen, die Aussage für k-Überdeckungen ergibt sich dann sehr leicht.

2. Sei $G=G_n$ wie oben definiert und für alle Punkte $(x;y)$ $v(x;y)=$ $=\operatorname{card}\big(K(x;y)\cap G\big)$ die Anzahl der Gitterpunkte in der offenen Einheitskreisscheibe um $(x;y)$, oder auch die Anzahl der offenen Einheitskreisscheiben $K(\mathbf{g})$, $\mathbf{g}\in G$, in denen $(x;y)$ liegt. Dann gilt:

$$v(x;\,y)=\sum_{|jh-y|\leq 1}\left(\left[\frac{1}{a}(s_j(y)+x)\right]+\left[\frac{1}{a}(s_j(y)-x)\right]+1\right)-R(x;\,y), \tag{1}$$

wobei $s_j(y):=\sqrt{1-(jh-y)^2}$ (für $|jh-y|\leq 1$) und $R(x;y)=$ Anzahl der Gitterpunkte auf dem Rand von $K(x;y)$ ist. Diese Gleichung ergibt sich folgendermaßen: Die Punkte von G liegen auf den Geraden $Z_j:=\{(u;jh)|u\in\mathbf{R}\}$, $j\in\mathbf{Z}$, und $K(x;y)$ schneidet Z_j in einer Strecke der Länge $2s_j(y)$, auf der bei $(0;jh)$ ein Gitterpunkt liegt. Die Anzahl der Gitterpunkte „rechts" bzw. „links" von $(0;jh)$ ist $\left[\frac{1}{a}(s_j(y)+x)\right]$ bzw. $\left[\frac{1}{a}(s_j(y)-x)\right]$, wobei eventuelle Randpunkte mitgezählt werden, außerdem muß man natürlich noch $(0;jh)\in G$ berücksichtigen.

(1) läßt sich etwas umformen, wenn man die durch $\psi(u):=u-[u]-\frac{1}{2}$ definierte Funktion benutzt:

$$\begin{aligned} v(x,y)=\frac{2}{a}\sum_{|jh-y|\leq 1}s_j(y)+\sum_{|jh-y|\leq 1}\bigg(\psi\left(\frac{1}{a}(s_j(y)+x\right)+\\ +\psi\left(\frac{1}{a}(s_j(y)-x)\right)\bigg)-R(x,y). \end{aligned} \tag{2}$$

Es bleiben also noch die beiden Summen abzuschätzen, wobei die erste relativ leicht zu behandeln ist, während die zweite einigermaßen langwierig wird.

3. Lemma 1. *Für* $a=\frac{\pi}{2n}$ *und* $h=\frac{2}{n}$ *gilt:*

$$\frac{2}{a}\sum_{|jh-y|\le 1}\sqrt{1-(jh-y)^2}=n^2+O(\sqrt{n}).$$

Beweis. Mit Hilfe einer bekannten Summenformel (s. z. B. [3], S. 113) bekommt man

$$\frac{2}{a}\sum_{|jh-y|\le 1}\sqrt{1-(jh-y)^2}=\frac{2}{a}\sum_{r=-\infty}^{+\infty}\int_{-(1-y)/h}^{(1+y)/h}\sqrt{1-(wh-y)^2}\cos 2\pi rw\,dw=$$

$$=\frac{4}{ah}\sum_{r=-\infty}^{+\infty}\cos\frac{2\pi ry}{h}\int_0^1\sqrt{1-w^2}\cos\frac{2\pi rw}{h}\,dw=$$

$$=n^2+\frac{8}{ah}\sum_{r=1}^{\infty}\cos\frac{2\pi ry}{h}\int_0^1\sqrt{1-w^2}\cos\frac{2\pi rw}{h}\,dw=$$

$$=n^2+\frac{2}{a}\sum_{r=1}^{\infty}\frac{1}{r}J_1\left(\frac{2\pi r}{h}\right)\cos\frac{2\pi ry}{h}=n^2+\frac{1}{a}\,0\left(\sqrt{h}\sum_{r=1}^{a}\frac{1}{r^{3/2}}\right)=$$

$$=n^2+O(\sqrt{n}).$$

Für die benutzten Eigenschaften der Bessel-Funktion J_1 vergleiche [5] S. 366 und S. 368.

4. In diesem Abschnitt seien stets $m, r\in\mathbf{N}$ und $m_1:=\frac{m}{h}=\frac{1}{2}mn$, $r_1:=\frac{r}{a}=$ $=\frac{2}{\pi}rn$. Außerdem sei $e(z):=e^{2\pi iz}$. Ich beweise zunächst einige Hilfssätze, die für die Abschätzung der gesuchten Summe benötigt werden.

Lemma 2.

$$\left|\frac{2}{h}\int_0^1\frac{z}{\sqrt{1-z^2}}e(r_1z)\,dz\right|<6\sqrt{\frac{n}{r}}.$$

Beweis. Sei $\varepsilon:=\frac{1}{r_1}$. Dann gilt

$$\left|\int_{1-\varepsilon}^1\frac{z}{\sqrt{1-z^2}}e(r_1z)\,dz\right|<\int_{1-\varepsilon}^1\frac{1}{\sqrt{1-z}}\,dz=\frac{2}{\sqrt{r_1}}$$

und nach dem 2. Mittelwertsatz der Integralrechnung

$$\left|\int_0^{1-\varepsilon}\frac{z}{\sqrt{1-z^2}}e(r_1z)\,dz\right|\le\frac{1-\varepsilon}{\sqrt{2\varepsilon-\varepsilon^2}}\cdot\frac{2}{r_1}<\frac{\sqrt{2}}{\sqrt{r_1}},\quad\text{also}$$

$$\left|\frac{2}{h}\int_0^1\frac{z}{\sqrt{1-z^2}}e(r_1z)\,dz\right|<\frac{8}{h}\frac{1}{\sqrt{r_1}}=4n\frac{\sqrt{\pi}}{\sqrt{2nr}}<6\sqrt{\frac{n}{r}}.$$

LEMMA 3.

$$\left|\int_0^1 e\left(-m_1\sqrt{1-w^2}+r_1 w\right)dw\right| < \frac{3}{r^{1/2}m^{1/4}n^{3/4}}.$$

BEWEIS. Für $F(w)=-m_1\sqrt{1-w^2}+r_1 w$ ergibt sich:

$$F'(w) = r_1+\frac{m_1 w}{\sqrt{1-w^2}} \geqq r_1,$$

und

$$F''(w) = m_1(1-w^2)^{-3/2} \geqq m_1.$$

Nun ist

$$T := \int_0^1 e\left(F(w)\right)dw = \frac{1}{2\pi i}\int_0^1 \frac{1}{F'(w)}\left(e(F(w))\right)' dw.$$

Aus $|F'(w)|\geqq r_1$ folgt nach dem 2. Mittelwertsatz

$$|T| \leqq \frac{1}{2\pi}\frac{1}{r_1}\cdot 2 = \frac{1}{\pi r_1} < \frac{1}{rn}.$$

Andererseits gilt:

$$|T| \leqq \left|\int_0^{1/\sqrt{m_1}} e\left(F(w)\right)dw\right| + \left|\int_{1/\sqrt{m_1}}^1 e\left(F(w)\right)dw\right|,$$

und

$$\left|\int_0^{1/\sqrt{m_1}} e\left(F(w)\right)dw\right| \leqq \frac{1}{\sqrt{m_1}}.$$

Für $\frac{1}{\sqrt{m_1}}\leqq w\leqq 1$ hat man $F'(w)\geqq r_1+\frac{1}{\sqrt{m_1}}\cdot m_1>\sqrt{m_1}$. Damit wird wieder nach dem 2. Mittelwertsatz

$$\left|\int_{1/\sqrt{m_1}}^1 e\left(F(w)\right)dw\right| \leqq \frac{2}{\sqrt{m_1}},$$

und daher

$$|T| \leqq \frac{3}{\sqrt{m_1}} < \frac{5}{\sqrt{mn}}.$$

Aus $|T|<\frac{1}{rn}$ und $|T|\leqq\frac{5}{\sqrt{mn}}$ ergibt sich $|T|^2<\frac{5}{rm^{1/2}n^{3/2}}$, also

$$|T| < \frac{3}{r^{1/2}m^{1/4}n^{3/4}}.$$

LEMMA 4. *Sei* $B:=\sqrt{m_1^2+r_1^2}$ *und* $m_1\geqq r_1$.
Dann gilt

$$\left|\int_{r_1/B}^{m_1/B} e\left(B\sqrt{1-w^2}\right)dw\right| \leqq \frac{3\cdot m^{5/4}}{r^{1/2}n^{3/4}(r^2+m^2)^{3/4}}.$$

Beweis (analog zu Lemma 3). Sei

$$G(w) := B\sqrt{1-w^2} \quad \text{für} \quad \frac{r_1}{B} \leqq w \leqq \frac{m_1}{B}.$$

Dann gilt:

$$|G'(w)| \geqq |G'(r_1/B)| = \frac{r_1 B}{m_1}$$

und

$$|T| = \left| \int_{r_1/B}^{m_1/B} e(G(w))\, dw \right| = \frac{1}{2\pi} \left| \int_{r_1/B}^{m_1/B} \frac{1}{G'(w)} (e(G(w)))'\, dw \right| \leqq \frac{m_1}{\pi r_1 B} < \frac{m}{rB}.$$

Ferner gilt

$$|G''(w)| \geqq \left| G''\left(\frac{r_1}{B}\right) \right| = \frac{B^4}{m_1^3} =: K,$$

und für

$$\frac{r_1}{B} + \frac{1}{\sqrt{K}} \leqq w \leqq \frac{m_1}{B}:$$

$$|G'(w)| = \left| G'\left(\frac{r_1}{B}\right) \right| + \left(w - \frac{r_1}{B}\right) \left| G''\left(\frac{r_1}{B} + \vartheta \left(w - \frac{r_1}{B}\right)\right) \right| > \sqrt{K}.$$

Damit bekommt man:

$$|T| \leqq \left| \int_{r_1/B}^{(r_1/B)+\frac{1}{\sqrt{K}}} e(G(w))\, dw \right| + \frac{1}{2\pi} \left| \int_{(r_1/B)+\frac{1}{\sqrt{K}}}^{m_1/B} \frac{1}{G'(w)} (e(G(w)))'\, dw \right| \leqq$$

$$\leqq \frac{1}{\sqrt{K}} + \frac{1}{\pi} \frac{1}{\sqrt{K}} < \frac{2}{\sqrt{K}},$$

also

$$|T| < 2 \frac{m_1^{3/2}}{B^2} < \frac{m^{3/2} n^{3/2}}{B^2}.$$

Genau wie oben ergibt sich

$$|T|^2 < \frac{m^{5/2} n^{3/2}}{rB^3} \Rightarrow |T| < \frac{m^{5/4} n^{3/4}}{r^{1/2} B^{3/2}} < 3 \frac{m^{5/4}}{r^{1/2} n^{3/4} (r^2 + m^2)^{3/4}}.$$

Um nun $\sum\limits_{|jh-y| \leqq 1} \psi\left(\frac{1}{a}(s_j(y) \pm x)\right)$ abzuschätzen, benutze ich den „Pfeifferschen Kunstgriff" (s. z. B. [4], S. 47, u. S. 448). $f(w)$ bezeichne $\frac{1}{a}(\sqrt{1-(wh-y)^2} \pm x)$.

Lemma 5.

$$\left| \sum_{|jh-y| \leqq 1} \psi(f(j)) \right| \leqq 1260\sqrt{n}.$$

Beweis. Der Beweis ist in wesentlichen Zügen dem des „Hilfssatzes 4" in [4], S. 47 nachkonstruiert. Für meine spezielle Funktion ergibt sich durch schärfere

Abschätzungen im einzelnen ein schärferes Ergebnis. Der „Hilfssatz 4" stammt von van der Corput (s. auch [4], S. 448).

Da $\psi(z) = -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2\pi mz)}{m}$ in jedem abgeschlossenen Intervall, das keine ganze Zahl enthält, gleichmäßig konvergiert, gilt für $t>0$

$$t\int_0^{\pm 1/t}\psi(z+w)\,dw = -\frac{t}{2\pi i}\sum_{m=1}^{\infty}\frac{1}{m}\int_0^{\pm 1/t}\big(e(m(z+w)) - e(-m(z+w))\big)\,dw =$$

$$= \sum_{m=-\infty}^{\infty} p_m e(mz) \quad \text{mit} \quad p_0 = 0 \quad \text{und}$$

$$p_m = -\frac{t}{2\pi i m}\int_0^{\pm 1/t} e(mw)\,dw \quad \text{für} \quad m \neq 0.$$

Daher ist für $m \neq 0$: $|p_m| \leq \min\left(\frac{1}{|m|}, \frac{t}{m^2}\right)$ und

$$t\left|\sum_{|jh-y|\leq 1}\int_0^{\pm 1/t}\psi(f(j)+w)\,dw\right| \leq 2\sum_{r=1}^{\infty}|S_r|\min\left(\frac{1}{r}, \frac{t}{r^2}\right), \tag{3}$$

wobei

$$S_r := \sum_{|jh-y|\leq 1}{}' e(rf(j)).$$

$T_\pm$ bezeichne

$$t\sum_{|jh-y|\leq 1}\int_0^{\pm 1/t}\psi(f(j)+w)\,dw.$$

Da für $z_1 \leq z_2$ stets $\psi(z_2) - \psi(z_1) \leq z_2 - z_1$ ist, gilt

$$t\sum_{|jh-y|\leq 1}\int_0^{1/t}\big(\psi(f(j)+w) - \psi(f(j))\big)\,dw \leq t\sum_{|jh-y|\leq 1}\int_0^{1/t} w\,dw < \frac{1}{2t}(n+2),$$

also

$$\sum_{|jh-y|\leq 1}\psi(f(j)) \geq T_+ - \frac{n+2}{2t}.$$

Ebenso ergibt sich

$$\sum_{|jh-y|\leq 1}\psi(f(j)) \leq \frac{n+2}{2t} + T_-.$$

Ich setze nun $t := \sqrt{n}$ und zeige noch, daß

$$\sum_{r=1}^{\infty}|S_r|\min\left(\frac{1}{r}, \frac{\sqrt{n}}{r^2}\right) = O(\sqrt{n})$$

ist (ab hier weiche ich vom Beweis des „Hilfssatzes 4" ab). Dazu ist zunächst S_r abzuschätzen.

Die Eulersche Summenformel liefert:

(4)
$$S_r = \sum_{|jh-y|\le 1} e(rf(j)) = $$
$$= \int_{-(1-y)/h}^{(1+y)/h} e(rf(w))\,dw + c_1 + 2\pi i r \int_{-(1-y)/h}^{(1+y)/h} \psi(w) f'(w) e(rf(w))\,dw,$$

wobei $|c_1| \le 2$.

Durch naheliegende Umformung ergibt sich für das erste Integral:

$$T_1 := \int_{-(1-y)/h}^{(1+y)/h} e(rf(w))\,dw = \frac{2}{h} e(\pm r_1 x) \int_0^1 e(r_1\sqrt{1-u^2})\,du =$$

$$= \frac{2}{h} e(\pm r_1 x) \int_0^1 \frac{z}{\sqrt{1-z^2}} e(r_1 z)\,dz, \quad \text{also nach Lemma 2:}$$

$$|T_1| < 6\sqrt{\frac{n}{r}}.$$

Das zweite Integral liefert

$$T_2 := r \int_{-(1-y)/h}^{(1+y)/h} \psi(w) f'(w)\ e(rf(w))\,dw =$$

$$= -\frac{hr}{a} \int_{-(1-y)/h}^{(1+y)/h} \psi(w) \frac{wh-y}{\sqrt{1-(wh-y)^2}} e(\pm r_1 x + r_1\sqrt{1-(wh-y)^2})\,dw =$$

$$= \frac{r_1}{\pi} e(\pm r_1 x) \sum_{m=1}^{\infty} \frac{1}{m} \int_{-1}^{+1} \sin\left(2\pi m \frac{u+y}{h}\right) \frac{u}{\sqrt{1-u^2}} e(r_1\sqrt{1-u^2})\,du,$$

wenn man die bekannte Fourier—Reihe für ψ benutzt. Die Vertauschung von $\sum$ und $\int$ ist durch die gleichmäßige Konvergenz auf jedem kompakten Intervall, das keine ganze Zahl enthält, gerechtfertigt.

$$T_2 = \frac{2r_1}{\pi} e(\pm r_1 x) \sum_{m=1}^{\infty} \frac{1}{m} \cos(2\pi m_1 y) \int_0^1 \sin(2\pi m_1 u) \frac{u}{\sqrt{1-u^2}} e(r_1\sqrt{1-u^2})\,du =$$

(5)
$$= \frac{2r_1}{\pi} e(\pm r_1 x) \sum_{m=1}^{\infty} \frac{1}{m} \cos(2\pi m_1 y) \int_0^1 \sin(2\pi m_1\sqrt{1-u^2}) e(r_1 u)\,du =$$

$$= -\frac{ir_1}{\pi} e(\pm r_1 x) \sum_{m=1}^{\infty} \frac{1}{m} \cos(2\pi m_1 y)(X_m - Y_m).$$

Dabei ist

$$X_m = \int_0^1 e(r_1 u + m_1\sqrt{1-u^2})\,du$$

$$Y_m = \int_0^1 e(r_1 u - m_1\sqrt{1-u^2})\,du.$$

Für Y_m gilt nach Lemma 3:

(6) $$|Y_m| < \frac{3}{r^{1/2} m^{1/4} n^{3/4}},$$

X_m ist noch zu behandeln.

Setzen wir wie oben $B:=\sqrt{r_1^2+m_1^2}$, außerdem $m_1=B\sin\lambda$, $r_1=B\cos\lambda$ $\left(0<\lambda<\frac{\pi}{2}\right)$ und $u=\sin\varphi$, so erhalten wir

$$X_m = \int_0^{\pi/2} \cos(\varphi)e(B\sin(\varphi+\lambda))\,d\varphi = \int_\lambda^{\pi/2+\lambda} \cos(\varphi-\lambda)e(B\sin\varphi)\,d\varphi =$$

(7) $$= \cos\lambda\cdot U_m + \sin\lambda\cdot V_m$$

mit

$$U_m := \int_\lambda^{\pi/2+\lambda} \cos\varphi e(B\sin\varphi)\,d\varphi = \frac{1}{2\pi i B} e(B\sin\varphi)|_\lambda^{\pi/2+\lambda},$$

also

(8) $$|\cos(\lambda)\cdot U_m| < \frac{r_1}{B^2} < \frac{3r}{n(r^2+m^2)},$$

und

$$V_m := \int_\lambda^{\pi/2+\lambda} \sin\varphi e(B\sin\varphi)\,d\varphi = \int_{\sin\lambda}^{\cos\lambda} \frac{w}{\sqrt{1-w^2}} e(Bw)\,dw.$$

Ist nun $m_1\leqq r_1$ so liefert der 2. Mittelwertsatz:

$$|V_m| \leqq \frac{r_1/B}{m_1/B}\cdot\frac{2}{B}, \quad \text{d. h.}$$

(9) $$|\sin(\lambda)V_m| < \frac{2r_1}{B^2} < \frac{6r}{n(r^2+m^2)}.$$

Für $m_1>r_1$ dagegen ergibt sich:

$$-V_m = \int_{r_1/B}^{m_1/B} \frac{w}{\sqrt{1-w^2}} e(Bw)\,dw,$$

und daher nach Lemma 4

(10) $$|\sin(\lambda)V_m| < 3\frac{m^{9/4}}{r^{1/2}n^{3/4}(r^2+m^2)^{5/4}}.$$

Aus (4), (5) und (6) folgt nun

(11) $$|S_r| < 6\sqrt{\frac{n}{r}}+2+2r_1\sum_{m=1}^{\infty}\frac{1}{m}|X_m|+2r_1\sum_{m=1}^{\infty}\frac{3}{r^{1/2}m^{5/4}n^{3/4}}.$$

Nach (7), (8), (9) und (10) gilt:

$$\sum_{m=1}^{\infty}\frac{1}{m}|X_m| \leqq \frac{3r}{n}\sum_{m=1}^{\infty}\frac{1}{(r^2+m^2)m}+\sum_{m\leqq\alpha r}\frac{6r}{mn(r^2+m^2)}+$$

$$+\sum_{m>\alpha r}\frac{3m^{5/4}}{r^{1/2}n^{3/4}(r^2+m^2)^{5/4}} \quad (\alpha = 4/\pi).$$

Damit ergibt sich für die Teilsummen

$$\frac{3r}{n}\sum_{m=1}^{\infty}\frac{1}{m(r^2+m^2)} < \frac{3}{rn}+\frac{3}{rn}\int_1^{\infty}\frac{dw}{w(1+w^2/r^2)} < \frac{3\log r}{rn}+\frac{9}{rn}$$

$$\sum_{m\leqq\alpha r}\frac{6r}{mn(r^2+m^2)} < \frac{6}{rn}+\frac{6}{rn}\int_1^{\infty}\frac{dw}{w(1+w^2/r^2)} < \frac{18}{rn}+\frac{6\log r}{rn}$$

$$\frac{3}{r^{1/2}n^{3/4}}\sum_{m>\alpha r}\frac{m^{5/4}}{(r^2+m^2)^{5/4}} < \frac{3}{r^{1/2}n^{3/4}}\left(\frac{(\alpha r)^{5/4}}{r^{5/2}(1+\alpha^2)^{5/4}}+\frac{1}{r^{5/2}}\int_{\alpha r}^{\infty}\frac{w^{5/4}}{(1+w^2/r^2)^{5/4}}\right) <$$

$$< \frac{3}{r^{7/4}n^{3/4}}+\frac{15}{r^{3/4}n^{3/4}}.$$

Daher ist

$$|S_r| < 18\log r+56+6\sqrt{\frac{n}{r}}+30r^{1/2}n^{1/4}+30r^{1/4}n^{1/4}+6\frac{n^{1/4}}{r^{3/4}} \leqq$$

$$\leqq 18\log r+56+6\sqrt{\frac{n}{r}}+60r^{1/2}n^{1/4}+6\frac{n^{1/4}}{r^{3/4}},$$

und schließlich (s. (3))

$$|T_\pm| \leqq 2\sum_{r=1}^{\infty}|S_r|\min\left(\frac{1}{r},\frac{\sqrt{n}}{r^2}\right) <$$

$$< 36\sqrt{n}\sum_{r=1}^{\infty}\frac{\log r}{r^2}+112\sqrt{n}\sum_{r=1}^{\infty}\frac{1}{r^2}+12\sqrt{n}\sum_{r=1}^{\infty}\frac{1}{r^{3/2}}+$$

$$+12\sum_{r=1}^{\infty}\frac{n^{1/4}}{r^{7/4}}+120n^{1/4}\sum_{r=1}^{\infty}\min\left(\frac{1}{\sqrt{r}},\frac{\sqrt{n}}{r^{3/2}}\right).$$

Für die letzte Summe ergibt sich

$$\sum_{r=1}^{\infty}\min\left(\frac{1}{r^{1/2}},\frac{\sqrt{n}}{r^{3/2}}\right) \leqq \sum_{r\leqq\sqrt{n}}\frac{1}{n^{1/2}}+\sqrt{n}\sum_{r>\sqrt{n}}\frac{1}{r^{3/2}} <$$

$$< 1+\int_1^{\sqrt{n}}w^{-1/2}\,dw+\sqrt{n}\left(\frac{1}{n^{3/4}}+\int_{\sqrt{n}}^{\infty}w^{-3/2}\,dw\right) < 2n^{1/4}+2n^{1/4},$$

so daß

$$|T_\pm| < 1256\sqrt{n}.$$

Schließlich erhält man

$$\Big|\sum_{|jh-y|\leqq 1} \psi(f(j))\Big| \leqq |T_\pm| + \frac{n+2}{2\sqrt{n}} < 1260\sqrt{n}.$$

Mit (2) ergibt sich nun: es gibt ein $c_7>0$, das von n unabhängig ist, mit

$$v(x;\ y) < n^2+c_7\sqrt{n} \quad \text{für alle} \quad (x;\ y),$$

und daher auch

(12) $$k_n := \max_{x,y} v(x;\ y) < n^2+c_7\sqrt{n}.$$

Daher ist

$$\frac{d_{k_n}}{k_n} \geqq \frac{d(G_n)}{k_n} > \frac{n^2}{n^2+c_7\sqrt{n}} > 1-\frac{c_8}{n^{3/2}}.$$

Aus (12) folgt $k_n<c_9 n^2$ oder $n>c_{10}k_n^{1/2}$, also

$$\frac{d_{k_n}}{k_n} > 1-\frac{c_{11}}{k_n^{3/4}}$$

(alle $c_i>0$ und von n unabhängig). Wählt man noch $N:=\{k_n|n\in\mathbf{N}\}$, so ist der Satz für k-Packungen bewiesen.

Sei nun $w(x;y):=\operatorname{card}(B(x;y)\cap G)$ die Anzahl der abgeschlossenen Kreisscheiben $B(\mathrm{g})$, $\mathrm{g}\in G$, die $(x;y)$ überdecken. Dann gilt offenbar: $w(x;y)= =v(x;y)+R(x,y)$ ($R(x,y)$ die Anzahl der $B(\mathrm{g})$, auf deren Rand $(x;y)$ liegt), so daß mit (2) und Lemma 5:

$$w(x;\ y) = n^2+O(\sqrt{n}).$$

Sei $k_n':=\min\limits_{x,y} w(x;y)$, so gilt $k_n'\geqq n^2-c_{12}\sqrt{n}$ (c_{12} von n unabhängig), und daher

$$\frac{D_{k_n}'}{k_n'} \leqq \frac{d(G_n)}{k_n'} \leqq \frac{n^2}{n^2-c_{12}\sqrt{n}} \leqq 1+\frac{c_{13}}{n^{3/2}} \leqq 1+\frac{c_{14}}{k^{3/4}}.$$

Mit $N':=\{k_n'|n\in\mathbf{N}\}$ ergibt das den zwiten Teil des Satzes.

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DIE DÜNNSTE GITTERFÖRMIGE 5-FACHE KREISÜBERDECKUNG DER EBENE

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Eine Menge von abgeschlossenen Kreisen in der Ebene bildet eine k-fache Überdeckung, wenn jeder Punkt der Ebene zu mindestens k Kreisen gehört. Eine Überdeckung von kongruenten Kreisen ist gitterförmig, wenn die Kreismittelpunkte ein ebenes Punktgitter Γ bilden.

Es sei D_k das Infinum der Dichten aller gitterförmigen k-fachen Überdeckungen. Über die Größen D_k sind die folgenden Ergebnisse bekannt:

$$D_1 = \frac{2\pi}{\sqrt{27}} \qquad \text{Kershner [6]}$$

$$D_2 = 2 \cdot D_1$$

$$D_3 = \mu D_1, \ \mu = 2{,}841\ldots \qquad \text{Blundon [1]}$$

$$D_4 = \frac{25}{4\sqrt{3}} \cdot D_1$$

$$D_5 \leqq \frac{48}{49}\sqrt{21} \cdot D_1 = \frac{32\pi}{7\sqrt{7}} \qquad \text{Danzer [3]}$$

Bolle [2] zeigte, daß es eine Konstante $c_2 > 0$ gibt, so daß

$$\frac{D_k}{k} \leqq 1 + \frac{c_2}{\sqrt{k}}$$

gilt.

In dieser Arbeit bestimmen wir D_5. Wir zeigen, daß die von Danzer konstruierte gitterförmige 5-fache Überdeckung die dünnste ist. Das geht aus dem folgenden Satz hervor.

SATZ. *Die Dichte einer gitterförmigen 5-fachen Überdeckung von Einheitskreisen ist größer oder gleich* $\frac{32\pi}{7\sqrt{7}}$ *und Gleichheit tritt nur dann auf, wenn das Gitter durch*

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zwei Vektoren der Längen $\frac{\sqrt{7}}{4}$ *und* $\sqrt{\frac{7}{8}}$ *erzeugt wird, die einen Winkel der Größe* $\arccos \frac{1}{\sqrt{8}}$ *einschließen.*[1]

Der Beweis des Satzes beruht auf einigen Hilfssätzen. Vor der Formulierung dieser Hilfssätze führen wir einige Bezeichnungen ein. Es seien $\overrightarrow{OA}$ und $\overrightarrow{OB}$ die Basisvektoren des Gitters Γ. Mit X bzw. $|X|$ bezeichnen wir den Ortsvektor $\overrightarrow{OX}$ bzw. die Länge von $\overrightarrow{OX}$. Wir schlagen Kreise vom Radius r um die Gitterpunkte von Γ. Diese Kreisanordnung wird mit $L(\Gamma, r)$ bezeichnet.

Ein Gitter Γ ist von normaler Darstellung, wenn die folgenden Ungleichungen für seine Basisvektoren A und B gelten:

$$(1) \qquad |A| \leqq |B| \leqq |B-A| \quad \sphericalangle(AOB) \leqq \frac{\pi}{2}.$$

Es seien $|A|=a$, $|B|=b$, $|B-A|=c$, $\frac{a}{b}=x$ und $\sphericalangle(AOB)=\alpha$. Mit diesen Bezeichnungen kann man (1) folgenderweise aufschreiben:

$$(2) \qquad 0 < x \leqq 1, \quad 0 \leqq \cos\alpha \leqq \frac{x}{2}.$$

Auf der Abb. 1 haben wir die Menge der Punkte im rechtwinkligen Koordinatensystem x, $y=\cos\alpha$ dargestellt, die die Ungleichungen (2) befriedigen. Diese Menge ist das rechtwinklige Dreieck OPQ mit Ausnahme von O, wo $OP=1$, $PQ=\frac{1}{2}$ und $\overrightarrow{OP} \perp \overrightarrow{PQ}$ sind.

Zu jedem Gitter von normaler Darstellung gehört also ein Punkt im Dreieck OPQ. Und umgekehrt, zu jedem Punkt $(x, \cos\alpha) \neq (0, 0)$ des Dreiecks OPQ gehört ein Gitter von normaler Darstellung, das abgesehen von einer Ähnlichkeit eindeutig ist. Jetzt zerlegen wir das Dreieck OPQ in Teilmengen. Es seien

$$\overline{H}_1 := \left\{(x, \cos\alpha) \,\middle|\, \sqrt{\frac{1}{2}} \leqq x \leqq 1,\ 0 \leqq \cos\alpha \leqq \frac{1}{2x} - \frac{x}{2}\right\},$$

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[1] Nach der Fertigstellung dieser Arbeit habe ich erfahren, daß dieses Ergebnis auch von Subak [8] erzielt worden ist. In seiner Dissertation hat er auch D_6 bestimmt. Weiter enthält die Dissertation von Haas [7] die Lösung des Problems für $k=7$. (Die Ergebnisse von Subak und Haas wurden nicht veröffentlicht.) Die hier folgende Ableitung der Gleichheit $D_5 = \frac{32\pi}{7\sqrt{7}}$ ist von den Beweisen von Subak und Haas weitgehend verschieden. In einer weiteren Arbeit ist es mir gelungen unter Anwendung der hierigen Ideen eine allgemeine Methode für die Bestimmung von D_k zu geben. Durch diese Methode wird das Problem der Bestimmung von D_k auf die Bestimmung der Minima von endlich vielen stetigen Funktionen in einem Veränderlichen zurückgeführt. Dadurch ist es gelungen auch D_8 außer D_6 und D_7 zu bestimmen.

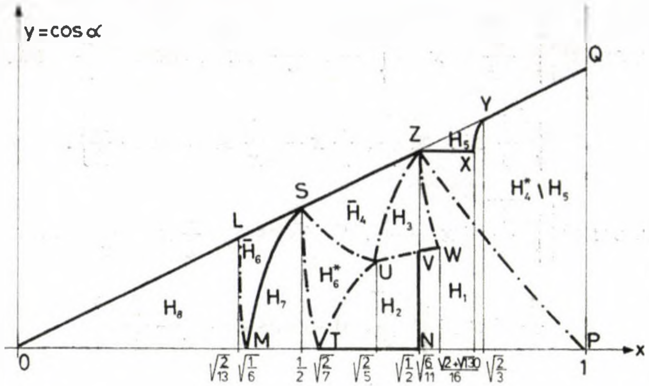

Abb. 1

$$H_2 := \left\{(x, \cos\alpha) \,\middle|\, \sqrt{\frac{2}{7}} \leqq x \leqq \sqrt{\frac{2}{5}},\ 0 \leqq \cos\alpha \leqq \frac{7}{8}x - \frac{1}{4x} \quad \text{oder} \quad \sqrt{\frac{2}{5}} \leqq x \leqq \sqrt{\frac{1}{2}},\ 0 \leqq \cos\alpha \leqq \frac{x}{4}\right\},$$

$$H_3 := \left\{(x, \cos\alpha) \,\middle|\, \sqrt{\frac{2}{5}} \leqq x \leqq \sqrt{\frac{1}{2}},\ \frac{x}{4} \leqq \cos\alpha \leqq \frac{3}{2}x - \frac{1}{2x} \quad \text{oder} \quad \sqrt{\frac{1}{2}} \leqq x \leqq \sqrt{\frac{6}{11}},\ \frac{x}{4} \leqq \cos\alpha \leqq \frac{3}{2x} - \frac{5}{2}x\right\},$$

$$\bar{H}_4 := \left\{(x, \cos\alpha) \,\middle|\, \frac{1}{2} \leqq x \leqq \sqrt{\frac{2}{5}},\ \frac{1}{6x} - \frac{1}{6}x \leqq \cos\alpha \leqq \frac{x}{2} \quad \text{oder} \quad \sqrt{\frac{2}{5}} \leqq x \leqq \sqrt{\frac{1}{2}},\ \frac{3}{2}x - \frac{1}{2x} \leqq \cos\alpha \leqq \frac{x}{2}\right\},$$

$$H_4^* := \left\{(x, \cos\alpha) \,\middle|\, \sqrt{\frac{1}{2}} \leqq x \leqq 1,\ \frac{1}{2x} - \frac{1}{2}x \leqq \cos\alpha \leqq \frac{x}{2}\right\},$$

$$H_5 := \left\{(x, \cos\alpha) \,\middle|\, \sqrt{\frac{1}{2}} \leqq x \leqq \frac{\sqrt{2} + \sqrt{130}}{16},\ \frac{1}{2}\sqrt{\frac{1}{2}} \leqq \cos\alpha \leqq \frac{x}{2} \quad \text{oder} \quad \frac{\sqrt{2} + \sqrt{130}}{16} \leqq x \leqq \sqrt{\frac{2}{3}},\ 2x - \frac{1}{x} \leqq \cos\alpha \leqq \frac{x}{2}\right\},$$

$$H_6^* := \left\{(x, \cos\alpha) \,\middle|\, \frac{1}{2} \leqq x \leqq \sqrt{\frac{2}{7}},\ \frac{1}{x} - \frac{7}{2}x \leqq \cos\alpha \leqq \frac{1}{6x} - \frac{1}{6}x \quad \text{oder} \quad \sqrt{\frac{2}{7}} \leqq x \leqq \sqrt{\frac{2}{5}},\ \frac{7}{8}x - \frac{1}{4x} \leqq \cos\alpha \leqq \frac{1}{6x} - \frac{1}{6}x\right\},$$

$$\overline{H}_6 := \left\{(x, \cos\alpha) \,\middle|\, \sqrt{\frac{2}{13}} \leqq x \leqq \sqrt{\frac{1}{6}},\ \frac{1}{x} - 6x \leqq \cos\alpha \leqq \frac{x}{2} \quad \text{oder} \right.$$

$$\left. \sqrt{\frac{1}{6}} \leqq x \leqq \frac{1}{2},\ \frac{3}{2}x - \frac{1}{4x} \leqq \cos\alpha \leqq \frac{x}{2} \right\},$$

$$H_7 := \left\{(x, \cos\alpha) \,\middle|\, \sqrt{\frac{1}{6}} \leqq x \leqq \frac{1}{2},\ 0 \leqq \cos\alpha \leqq \frac{3}{2}x - \frac{1}{4x} \quad \text{oder} \right.$$

$$\left. \frac{1}{2} \leqq x \leqq \sqrt{\frac{2}{7}},\ 0 \leqq \cos\alpha \leqq \frac{1}{x} - \frac{7}{2}x \right\},$$

$$H_8 := \left\{(x, \cos\alpha) \,\middle|\, 0 < x \leqq \sqrt{\frac{2}{13}},\ 0 \leqq \cos\alpha \leqq \frac{x}{2} \quad \text{oder} \right.$$

$$\left. \sqrt{\frac{2}{13}} \leqq x \leqq \sqrt{\frac{1}{6}},\ 0 \leqq \cos\alpha \leqq \frac{1}{x} - 6x \right\}.$$

Es ist leicht einzusehen, daß die Teilmengen $H_1 = \overline{H}_1 \setminus (\overline{H}_1 \cap H_3)$ $(PNVWZ)$, H_2 $(NTUV)$, H_3 (UWZ), $H_4 = \overline{H}_4 \cup (H_4^* \setminus H_5)$ $(XYQPZUS)$, $H_5 (XYZ)$, $H_6 = \overline{H}_6 \cup H_6^*$ $(LMSTU)$, H_7 (MTS), H_8 (OLM) keinen gemeinsamen inneren Punkt haben und jeder Punkt der Dreiecks OPQ (außer 0) zu irgendeiner von den Mengen H_i ($i = 1, \ldots, 8$) gehört.

Mit $k[XYZ]$ bzw. $\hat{k}[XYZ]$ bezeichnen wir den abgeschlossenen Kreis bzw. die Kreislinie, die von den nicht kollinearen Punkten X, Y und Z bestimmt sind. Die Formel

$$r = \frac{xyz}{4T} \tag{3}$$

gibt den Umkreisradius r des Dreiecks mit den Seitenlängen x, y und z und mit dem Inhalt T an.

Es seien $k_1 := k[OB(3A)]$, $k_2 := k[O(2A)(A+2B)]$, $k_3 := k[O(2A)(2B)]$, $k_4 := k[O(A+B)(3A-B)]$, $k_5 := k[O(3A)(2A+B)]$, $k_6 := k[O(2A+B)(3A-B)]$, $k_7 := k[O(4A)(A+B)]$ und $k_8 := k[O(5A)(2A+B)]$. Mit Δ_i $(1 \leqq i \leqq 8)$ bezeichnen wir das Gitterdreieck, durch das der Kreis k_i $(1 \leqq i \leqq 8)$ oben bestimmt wurde. Der Radius des Kreises k_i wird mit r_i bezeichnet. Auf Grund von (3) können wir diese Radien wie folgt aufschreiben:

$$r_1^2 = \frac{9a^2 + b^2 - 6ab\cos\alpha}{4\sin^2\alpha}, \tag{4}$$

$$r_2^2 = \frac{(a^2 + 4b^2)^2 - 16a^2b^2\cos^2\alpha}{16b^2\sin^2\alpha}, \tag{5}$$

$$r_3^2 = \frac{a^2 + b^2 - 2ab\cos\alpha}{\sin^2\alpha}, \tag{6}$$

(7) $$r_4^2 = \frac{((a^2+b^2)^2-4a^2b^2\cos^2\alpha)(9a^2+b^2-6ab\cos\alpha)}{16a^2b^2\sin^2\alpha},$$

(8) $$r_5^2 = \frac{(a^2+b^2-2ab\cos\alpha)(4a^2+b^2+4ab\cos\alpha)}{4b^2\sin^2\alpha},$$

(9) $$r_6^2 = \frac{(a^2+4b^2-4ab\cos\alpha)(4a^2+b^2+4ab\cos\alpha)(9a^2+b^2-6ab\cos\alpha)}{100a^2b^2\sin^2\alpha},$$

(10) $$r_7^2 = \frac{(a^2+b^2+2ab\cos\alpha)(9a^2+b^2-6ab\cos\alpha)}{4b^2\sin^2\alpha},$$

(11) $$r_8^2 = \frac{(4a^2+b^2+4ab\cos\alpha)(9a^2+b^2-6ab\cos\alpha)}{4b^2\sin^2\alpha}.$$

Bemerkung 1. Für die Basisvektoren des im Satz erwähnten Gitters gelten

(12) $$|A| = \frac{\sqrt{7}}{4}, \quad |B| = |B-A| = \sqrt{\frac{7}{8}}.$$

Es ist leicht einzusehen, daß beim Gitter (12) die Kreise k_1, k_3, k_4 und k_5 (Abb. 2)

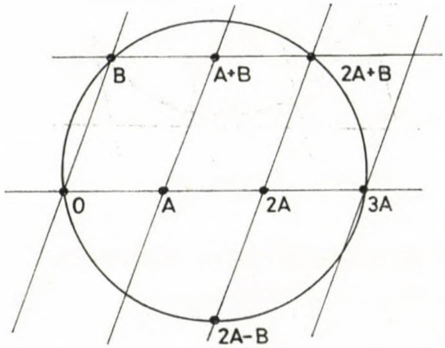

Abb. 2

Einheitskreise sind. Danzer [3] zeigte, daß das Gitter (12) eine 5-fache Überdeckung von Einheitskreisen erzeugt. Die Dichte dieser Überdeckung ist $\frac{32\pi}{7\sqrt{7}}$.

Bemerkung 2. Für die zu (12) ähnlichen Gitter gelten

(13) $$x = \sqrt{\frac{1}{2}} \quad \text{und} \quad \cos\alpha = \frac{1}{2}\sqrt{\frac{1}{2}},$$

d. h., der Punkt $Z\left(\sqrt{\frac{1}{2}}, \frac{1}{2}\sqrt{\frac{1}{2}}\right)$ (Abb. 1) entspricht den Gittern (13) im rechtwinkligen Dreieck OPQ. Γ sei ein Gitter, für das (13) gilt. Dann sind die dem Gitter Γ entsprechenden Kreisradien r_1, r_3, r_4, r_5 gleich. Auf Grund der Bemerkung 1. ist die Anordnung $L(\Gamma, r_i)$ $(i=1, 3, 4, 5)$ eine 5-fache Überdeckung und $L(\Gamma, r_i)$ hat die Dichte $\frac{32\pi}{7\sqrt{7}}$.

Zum Beweis des Satzes verwenden wir einige Hilfssätze.

HILFSSATZ 1. *Wir betrachten ein Gitter von normaler Darstellung, bei dem* $(x, \cos\alpha)\in H_i$ $(i=1, \ldots, 8)$ *gilt. Dann liegen höchstens* 4 *Gitterpunkte im Inneren des Kreises* k_i *und das Dreieck* Δ_i *ist nicht stumpfwinklig.*

BEWEIS. Nun untersuchen wir den Fall $(x, \cos\alpha)\in H_1$. In diesem Fall gilt $2A+B\in k_1$ wegen (1). $A, 2A, A+B\in k_1$ gelten offenbar. $3A+B\notin k_1\setminus\widehat{k_1}$ ist und $3A+B\in\widehat{k_1}$ gilt nur im Fall $\alpha=\frac{\pi}{2}$ wegen (1).

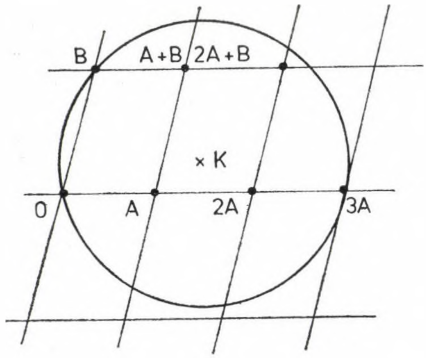

Abb. 3

Mit K bezeichnen wir den Mittelpunkt des Kreises k_1 (Abb. 3). $2A-B\notin k_1\setminus\widehat{k_1}$ ist, wenn

(14) $$|(2A-B)-K|^2 \geqq |K|^2$$

gilt. Aus (14) ergibt sich

(15) $$4|A|^2+|B|^2-4A\cdot B-4A\cdot K+2B\cdot K \geqq 0.$$

Weil $O, 3A$ und $B\in\widehat{k_1}$ sind, gelten $|K|^2=|B-K|^2=|3A-K|^2$, d. h., sind

(16) $$|B|^2 = 2B\cdot K \quad \text{und} \quad 3|A|^2 = 2A\cdot K.$$

Aus (15) und (16) bekommen wir die Ungleichung

$$|B|^2-|A|^2 \geqq 2A\cdot B.$$

Wegen $A\cdot B=|A||B|\cos\alpha$ und $x=\frac{a}{b}$ ist

(17) $$\cos\alpha \leqq \frac{1}{2x}-\frac{x}{2}$$

mit (14) äquivalent. (17) gilt aber nach unserer Voraussetzung, deshalb gilt

$2A-B\notin k_1\setminus\widehat{k_1}$. Daraus ergibt sich $sA-B\notin k_1$, wo s eine ganze Zahl ist. Es ist offenbar, daß k_1 keinen weiteren Gitterpunkt enthält.

$\sphericalangle\left(BO(3A)\right)\leqq\frac{\pi}{2}$, $\sphericalangle\left(O(3A)B\right)<\frac{\pi}{2}$ gelten offenbar. $\sphericalangle\left(OB(3A)\right)<\frac{\pi}{2}$ ist dann und nur dann, wenn $\left|B-\frac{3}{2}A\right|>\left|\frac{3}{2}A\right|$ gilt. Daraus ergibt sich $\cos\alpha<\frac{1}{3x}$, die wegen $\frac{1}{3x}>\frac{1}{2x}-\frac{x}{2}$ $\left(x^2\geqq\frac{1}{2}\right)$ gilt.

Ebenso kann man in den weiteren Fällen unsere Behauptung beweisen, deshalb legen wir die nicht ausführlich dar. Wir geben nur die 4 Gitterpunkte an, die im Inneren von k_i liegen können:

$$
\begin{array}{ll}
(x,\cos\alpha)\in H_1 & A,\ 2A,\ A+B,\ 2A+B\in k_1,\\
(x,\cos\alpha)\in H_2 & A,\ \ B,\ A+B,\ 2A+B\in k_2,\\
(x,\cos\alpha)\in H_3 & A,\ \ B,\ A+B,\ 2A+B\in k_3,\\
(x,\cos\alpha)\in \overline{H}_4 & A,\ 2A,\ \ \ 3A,\ 2A-B\in k_4,\\
(x,\cos\alpha)\in H_4^*\setminus H_5 & A,\ 2A,\ A-B,\ 2A-B\in k_4,\\
(x,\cos\alpha)\in H_5 & A,\ 2A,\ A+B,\ 2A-B\in k_5,\\
(x,\cos\alpha)\in H_6^* & A,\ 2A,\ \ \ 3A,\ 2A-B\in k_6,\\
(x,\cos\alpha)\in \overline{H}_6 & A,\ 2A,\ \ \ 3A,\ \ \ 4A\in k_6,\\
(x,\cos\alpha)\in H_7 & A,\ 2A,\ \ \ 3A,\ 2A+B\in k_7,\\
(x,\cos\alpha)\in H_8 & A,\ 2A,\ \ \ 3A,\ \ \ 4A\in k_8.
\end{array}
$$

Wir betrachten ein Gitter Γ von normaler Darstellung. Mit g bezeichnen wir die folgende Transformation von Γ. Wir halten den Basisvektor A fest und bewegen den Endpunkt des Basisvektors B auf der zu OA parallelen Gerade derart, daß $|B|$ zunimmt. Wir wenden die Transformation g höchstens bis der Lage $|B|=|B-A|$ $\left(\cos\alpha=\frac{x}{2}\right)$ an.

Hilfssatz 2. *Wir betrachten ein Gitter Γ von normaler Darstellung, für das $(x,\cos\alpha)\in H_i$, $i=1, 3, 4, 6, 7$ oder 8 gilt und wir wenden die Transformation g auf das Gitter Γ an. Dann nimmt der Radius r_i $(i=1, 3, 4, 6, 7, 8)$ streng ab.*

Beweis. Im Fall $(x,\cos\alpha)\in H_i$, $i=1, 3, 7, 8$ ist das Gitterdreieck Δ_i nach dem Hilfssatz 1 nicht stumpfwinklig. Es ist leicht einzusehen, daß Δ_i höchstens in bestimmten Randpunkten von H_i rechtwinklig sein kann. Folglich nimmt der entsprechende Kreisradius r_i offensichtlich ab.

Gilt $(x,\cos\alpha)\in H_4$, dann ist das Dreieck $\Delta_4=O(A+B)(3A-B)$ nicht stumpfwinklig. B_1 und B_2 seien zwei beliebige Lagen von B während der Anwendung von g, so daß $|B_1|<|B_2|$ gilt (Abb. 4). Es gilt auch $\sphericalangle\left(O(A+B_1)(2A)\right)>\sphericalangle\left(O(A+B_2)(2A)\right)$. Es ist offenbar, daß $|B_1-A|>|B_2-A|$ und $\sphericalangle\left((A+B_2)(2A)O\right)>\sphericalangle\left((A+B_1)(2A)O\right)$ sind. Wir drehen das Dreieck $O(A+B_2)(3A-B_2)$ um $2A$ mit dem Winkel

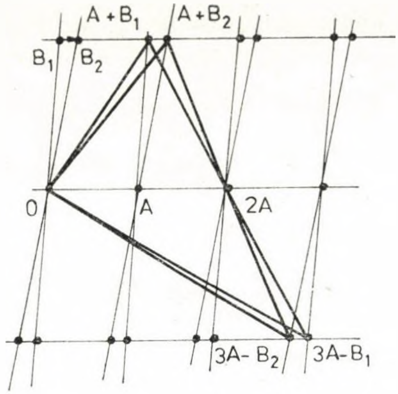

Abb. 4

$(A+B_2)(2A)(A+B_1)$. Aus den obigen folgt, daß das so erhaltene Dreieck im Inneren von $k[O(A+B_1)(3A-B_1)]$ liegt. So ist der Radius von $k[O(A+B_2)(3A-B_2)]$ kleiner als der Radius von $k[O(A+B_1)(3A-B_1)]$, d. h., r_4 nimmt streng ab.

Ebenso kann man die Monotonie von r_6 im Fall $(x, \cos\alpha)\in H_6$ beweisen. Damit haben wir den Beweis des Hilfssatzes beendet.

Die Transformation g_1 von Γ ist folgenderweise definiert. Wir halten den Basisvektor A fest und drehen den Basisvektor B um O, so daß α zunimmt. Wir verwenden diese Transformation nur im Fall $\alpha<\frac{\pi}{2}$.

HILFSSATZ 3. *Es sei Γ ein Gitter von normaler Darstellung, für das $(x, \cos\alpha)\in H_2$ oder $(x, \cos\alpha)\in H_5$ gilt. Dann nimmt r_2 bzw. r_5 während der Anwendung von g_1 streng ab.*

BEWEIS. Es ist leicht einzusehen, daß Δ_2 bzw. Δ_5 spitzwinklig ist, wenn der dem Gitter entsprechende Punkt $(x, \cos\alpha)$ ein innerer Punkt von H_2 bzw. H_5 ist. Es seien $\sphericalangle(B_1OA)=\alpha_1$ und $\sphericalangle(B_2OA)=\alpha_2$, wo B_1 und B_2 zwei verschiedene Lagen von B während der Transformation g_1 sind. Es sei $\alpha_2>\alpha_1$ und $(x_i, \cos\alpha_i)\in H_2$ $(i=1, 2)$ (Abb. 5). In diesem Fall ist $\sphericalangle\big(O(A+2B_i)(2A)\big)<\frac{\pi}{2}$. Während g_1 ist A fix und $|B_1|=|B_2|$ gilt. Deshalb sind $A(A+2B_1)$ und $A(A+2B_2)$ gleiche Strecken. Weil $\alpha_2>\alpha_1$ ist, gilt $A+2B_2\in k[O(2A)(A+2B_1)]$, d. h., nimmt r_2 streng ab.

Ebenso kann man beweisen, daß r_5 im Fall $(x, \cos\alpha)\in H_5$ streng abnimmt-

DER BEWEIS DES SATZES. Wir betrachten eine beliebige gitterförmige 5-fache Kreisüberdeckung $L(\Gamma, r)$, wo Γ von normaler Darstellung ist. Wir nehmen solche Kreise, die von nicht *kollinearen* Gitterpunkten bestimmt sind und höchstens 4 Gitterpunkte in ihren Inneren enthalten. Weil die Überdeckung 5-fach ist, müssen wir auch die Mittelpunkte der vorigen Kreise mindestens 5-fach überdecken. Das bedeutet, daß die Radien dieser Kreise $\leqq r$ sind. Auf Grund des Hilfssatzes 1 gilt $r\geqq r_i$, wenn $(x, \cos\alpha)\in H_i$ $(i=1, \dots, 8)$ ist.

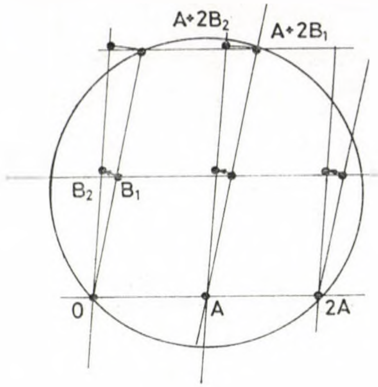

Abb. 5

Die Dichte von $L(\Gamma, r)$ ist $\frac{r^2\pi}{T(\Gamma)}$, wo $T(\Gamma)$ der Inhalt des Grundparallelogramms von Γ ist. Für die Basisvektoren von Γ gilt (2). Entsprechend den Werten x und $\cos\alpha$ können wir die Dichte von $L(\Gamma, r)$ mit einem von den Quotienten $\frac{r_i^2\pi}{T(\Gamma)}$ $(i=1, \dots, 8)$ von unten schätzen. Wir zeigen, daß

$$\frac{r_i^2}{T(\Gamma)} \geqq \frac{32}{7\sqrt{7}}$$

bei dem entsprechenden Index i gilt und die Gleichheit nur bei den in der Bemerkung 2 erklärten Kreisüberdeckungen auftritt.

Mit Hilfe (4)—(11) und $T(\Gamma)=ab\sin\alpha$ können wir die Quotienten $\frac{r_i^2}{T(\Gamma)}$, $i=1, \dots, 8$ als Funktion von x und α aufschreiben. Die Definitionsbereiche der Funktionen $\frac{r_i^2}{T(\Gamma)}$, H_i $(i=1, \dots, 8)$ kann man auf der Abb. 1 sehen.

1. Wir betrachten eine 5-fache Überdeckung $L(\Gamma, r)$, wo $(x, \cos\alpha)\in H_2$ für das Gitter Γ gilt. In diesem Fall ist

$$\frac{r^2}{T(\Gamma)} \geqq \frac{r_2^2}{T(\Gamma)}.$$

Wir wenden die im Hilfssatz 3. gegebene Transformation g_1 an. Während dieser Transformation bleibt x konstant, $\cos\alpha$ nimmt ab, der Inhalt des Grundparallelogramms nimmt zu und nach dem Hilfssatz 3. nimmt r_2 streng ab. Das bedeutet, daß die Funktion $\frac{r_2^2}{T(\Gamma)}$, H_2 während g_1 streng abnimmt. Deshalb nimmt unsere Funktion ihr Minimum im Fall $\cos\alpha=0$ auf. Im Fall $\cos\alpha=0$ können wir unsere

Funktion folgenderweise aufschreiben:

$$(18) \qquad \frac{(x^2+4)^2}{16x} \qquad \sqrt{\frac{2}{7}} \leqq x \leqq \sqrt{\frac{1}{2}}.$$

Die erste Ableitung von (18) ist

$$\frac{(x^2+4)}{16x^2}(3x^2-4) < 0.$$

So gilt

$$\frac{r^2}{T(\Gamma)} \geqq \frac{r_2^2}{T(\Gamma)} \geqq \frac{4{,}5^2}{16\sqrt{\frac{1}{2}}} > \frac{32}{7\sqrt{7}}.$$

2. Jetzt untersuchen wir solche 5-fache Überdeckungen $L(\Gamma, r)$, bei denen $(x, \cos\alpha)\in H_5$ gilt. In diesem Fall gilt

$$\frac{r^2}{T(\Gamma)} \geqq \frac{r_5^2}{T(\Gamma)}$$

und wir können die Transformation g_1 anwenden (s. Hilfssätze 1 und 3). Während der Anwendung von g_1 nimmt $\frac{r_5^2}{T(\Gamma)}$ streng ab. Auch $\cos\alpha$ nimmt ab und x ist konstant. Deshalb nimmt die Funktion $\frac{r_5^2}{T(\Gamma)}$ auf H_5 ihr Minimum bei $\cos\alpha=\frac{1}{2}\sqrt{\frac{1}{2}}$ oder $\cos\alpha=2x-\frac{1}{x}$ auf.

Aus der Gleichheit $\frac{r_4^2}{T(\Gamma)}=\frac{r_5^2}{T(\Gamma)}$ ((7) und (8)) ergibt sich

$$(19) \qquad \cos\alpha_0 = \frac{-x^2-1+2\cdot\sqrt{-5x^4+5x^2+1}}{6x}.$$

Im Fall $\sqrt{\frac{1}{2}} \leqq x \leqq 1$ nimmt (19) streng ab und die Koordinaten von Z und P befriedigen (19). Man kann leicht sehen, daß $\frac{r_4^2}{T(\Gamma)}<\frac{r_5^2}{T(\Gamma)}$ im Fall $\cos\alpha>\cos\alpha_0$ ist. Deshalb gilt $\frac{r_5^2}{T(\Gamma)}>\frac{r_4^2}{T(\Gamma)}$, wenn $\cos\alpha=\frac{1}{2}\sqrt{\frac{1}{2}}$ oder $\cos\alpha=2x-\frac{1}{x}$ ist. Diese Fälle werden aber später untersucht (3.6. und 3.7.).

3. Endlich betrachten wir die 5-fachen Überdeckungen $L(\Gamma, r)$, bei denen $(x, \cos\alpha)\in H_i$ $(i=1, 3, 4, 6, 7, 8)$ gilt. Aus dem Hilfssatz 1 folgt, daß $\frac{r^2}{T(\Gamma)} \geqq$ $\geqq \frac{r_i^2}{T(\Gamma)}$ $(i=1, 3, 4, 6, 7, 8)$ ist. Nun wenden wir die Transformation g auf Γ an. So können wir $\frac{r_i^2}{T(\Gamma)}$ vermindern (Hilfssatz 2). Während der Anwendung der Trans-

formation g nimmt x ab und nimmt $\cos\alpha$ zu. (Wir bemerken, daß die den Gittern entsprechenden Punkte im Koordinatensystem $x, y=\cos\alpha$ während der Anwendung von g auf einer Ellipse bewegen.) Mit Hilfe von g können wir die Grenze der entsprechenden Menge H_i erreichen. Wenn wir während der weiteren Anwendung von g die Punkte irgendeiner Menge H_j ($j=1, 3, 4, 6, 7, 8$) bekommen, dann gilt $\frac{r_i^2}{T(\Gamma)}=$ $=\frac{r_j^2}{T(\Gamma)}$ im Grenzpunkt G_{ij} mit den folgenden Ausnahmen. Diese sind $H_1 \cap H_3$ im Fall $\cos\alpha=\frac{x}{4}$ und $H_7 \cap \overline{H}_6$ im Fall $\cos\alpha=\frac{3}{2}x-\frac{1}{4x}$. Sonst kann man die Gleichheit $\frac{r_i^2}{T(\Gamma)}=\frac{r_j^2}{T(\Gamma)}$ z. B. mit Hilfe von (4), (6), (7), (9), (10), (11) einsehen. Gilt $\frac{r_i^2}{T(\Gamma)}=\frac{r_j^2}{T(\Gamma)}$ im Grenzpunkt G_{ij}, so können wir mit der Anwendung von g und der Funktion $\frac{r_j^2}{T(\Gamma)}$, H_j weiter vermindern. Es ist leicht einzusehen, daß wir mit Hilfe der Anwendung von g einen von den folgenden Fällen erreichen. (Diese Fälle sind auf der Abb. 1 ununterbrochen dick bezeichnet.)

3.1. den Punkt $Z\left(\sqrt{\frac{1}{2}}, \frac{1}{2}\sqrt{\frac{1}{2}}\right)$, d. h., das Gitter (13).

3.2. die Funktion $\frac{r_1^2}{T(\Gamma)}$, H_1, wenn $\cos\alpha=\frac{x}{4}$ (VW) ist.

3.3. die Funktion $\frac{r_1^2}{T(\Gamma)}$, H_1, wenn $x=\sqrt{\frac{1}{2}}$ (NWV) ist.

3.4. die Funktion $\frac{r_4^2}{T(\Gamma)}$, $\overline{H}_4$, wenn $\cos\alpha=\frac{x}{2}$ (SZ) ist.

3.5. die Funktion $\frac{r_4^2}{T(\Gamma)}$, $H_4^* \setminus H_5$, wenn $\cos\alpha=\frac{x}{2}$ (YQ) ist.

3.6. die Funktion $\frac{r_4^2}{T(\Gamma)}$, $H_4^* \setminus H_5$, wenn $\cos\alpha=\frac{1}{2}\sqrt{\frac{1}{2}}$ (ZX) ist.

3.7. die Funktion $\frac{r_4^2}{T(\Gamma)}$, $H_4^* \setminus H_5$, wenn $\cos\alpha=2x-\frac{1}{x}$ (XY) ist.

3.8. die Funktion $\frac{r_7^2}{T(\Gamma)}$, H_7, wenn $\cos\alpha=\frac{3}{2}x-\frac{1}{4x}$ (MS) ist.

3.9. die Funktion $\frac{r_6^2}{T(\Gamma)}$, $\overline{H}_6$, wenn $\cos\alpha=\frac{x}{2}$ (LS) ist.

3.10. die Funktion $\frac{r_8^2}{T(\Gamma)}$, H_8, wenn $\cos\alpha=\frac{x}{2}$ (OL) ist.

In den Fällen 3.2. und 3.3. müssen wir das Minimum der Funktion

$$\frac{r_1^2}{T(\Gamma)} = \frac{9x^2+1-6x\cos\alpha}{4x\sin^3\alpha}, \quad H_1$$

mit der Bedingung $\cos\alpha=\frac{x}{4}$ bzw. $x=\sqrt{\frac{1}{2}}$ finden. Bei $\cos\alpha=\frac{x}{4}$ bzw. $x=\sqrt{\frac{1}{2}}$ bekommen wir die Funktion

$$\text{(20)} \qquad \frac{8(15x^2+2)}{x\sqrt{16-x^2}^3}, \qquad \sqrt{\frac{1}{2}} \leqq x \leqq \sqrt{\frac{6}{11}}$$

bzw.

$$\text{(21)} \qquad \frac{5{,}5-3\sqrt{2}\cos\alpha}{2\sqrt{2}\sin^3\alpha}, \qquad 0 \leqq \cos\alpha \leqq \frac{1}{4}\sqrt{\frac{1}{2}}.$$

Mit Hilfe der ersten Ableitung können wir zeigen, daß (20) und (21) ihr Minimum an der Stelle $x=\sqrt{\frac{1}{2}}$, $\cos\alpha=\frac{1}{4}\sqrt{\frac{1}{2}}$ aufnehmen. An dieser Stelle ist aber $\frac{r_1^2}{T(\Gamma)}$ größer als $\frac{32}{7\sqrt{7}}$.

In den Fällen 3.4—3.7. brauchen wir die Funktion

$$\text{(22)} \qquad \frac{r_4^2}{T(\Gamma)} = \frac{\left((x^2+1)^2-4x^2\cos^2\alpha\right)(9x^2+1-6x\cos\alpha)}{16x^3\sin^3\alpha}, \quad H_4$$

zu untersuchen. Bei 3.4. und 3.5. ist $\cos\alpha=\frac{x}{2}$, deshalb müssen wir

$$\text{(23)} \qquad \frac{12x^4+8x^2+1}{2x^3\sqrt{4-x^2}^3}, \qquad \frac{1}{2} \leqq x \leqq \sqrt{\frac{1}{2}} \quad \text{und} \quad \sqrt{\frac{2}{3}} \leqq x \leqq 1$$

untersuchen. Mit Hilfe der ersten Ableitung können wir uns davon überzeugen, daß (23) im Fall $\frac{1}{2}\leqq x\leqq\sqrt{\frac{1}{2}}$ streng abnimmt und bei $\sqrt{\frac{2}{3}}\leqq x\leqq 1$ zunimmt. Bei $x=\sqrt{\frac{2}{3}}$ nimmt die Funktion (23) einen größeren Wert als $\frac{32}{7\sqrt{7}}$ auf, deshalb erreicht unsere Funktion ihr Minimum an der Stelle $x=\sqrt{\frac{1}{2}}$, d. h., für die Gitter (13).

Im Fall 3.4. $\left(\cos\alpha=\frac{1}{2}\sqrt{\frac{1}{2}}\right)$ bekommen wir aus (22) die Funktion

$$\text{(24)} \qquad \frac{(2x^4+3x^2+2)(9\sqrt{2}x^2-3x+\sqrt{2})}{14\sqrt{7}x^3}, \qquad \sqrt{\frac{1}{2}} \leqq x \leqq \frac{\sqrt{2}+\sqrt{130}}{16}.$$

Es ist leicht einzusehen, daß (24) zunimmt, d. h., (24) ihr Minimum an der Stelle $x=\sqrt{\frac{1}{2}}$ aufnimmt.

Im Fall 3.7. $\left(\cos\alpha=2x-\frac{1}{x}\right)$ müssen wir die Funktion

$$\frac{3(-5x^4+6x^2-1)(7-3x^2)}{16\sqrt{-4x+5x^2-1}^3}, \quad \frac{\sqrt{2}+\sqrt{130}}{16} \leqq x \leqq \sqrt{\frac{2}{3}} \tag{25}$$

untersuchen. Mit Hilfe der ersten Ableitung können wir beweisen, daß (25) streng abnimmt. Ihrer Minimumwert ist aber größer als $\frac{32}{7\sqrt{7}}$.

Im Fall 3.8. $\left(\cos\alpha=\frac{3}{2}x-\frac{1}{4x}\right)$ können wir die Funktion $\frac{r_7^2}{T(\Gamma)}$ auf Grund (10) folgenderweise aufschreiben:

$$\frac{20x^2(8x^2+1)}{(\sqrt{-36x^4+28x^2-1})^3}, \quad \sqrt{\frac{1}{6}} \leqq x \leqq \frac{1}{2}. \tag{26}$$

Man kann leicht einsehen, daß (26) zunimmt. An der Stelle $x=\sqrt{\frac{1}{6}}$ ist aber (26) größer als $\frac{32}{7\sqrt{7}}$.

Im Fall 3.9. bzw. 3.10. müssen wir die Funktion $\frac{r_6^2}{T(\Gamma)}$ bzw. $\frac{r_8^2}{T(\Gamma)}$ unter der Bedingung $\cos\alpha=\frac{x}{2}$ untersuchen. Auf Grund von (9) bzw. (11) bekommen wir die Funktionen

$$\frac{2(6x^2+1)^2}{25x^3\sqrt{4-x^2}}, \quad \sqrt{\frac{2}{13}} \leqq x \leqq \frac{1}{2} \tag{27}$$

bzw.

$$\frac{2(6x^2+1)^2}{x\sqrt{4-x^2}^3}, \quad 0 < x \leqq \sqrt{\frac{2}{13}}. \tag{28}$$

Mit Hilfe der ersten Ableitung können wir uns davon überzeugen, dass der Minimum wert von (27) bzw. (28) größer als $\frac{32}{7\sqrt{7}}$ ist.

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EXTREMALE GROEMERPACKUNGEN

GERD WEGNER

Wir verwenden die Bezeichnungen aus [1]. Wir wollen eine Groemerpackung $\mathcal{G}$ aus n Kreisen *extremal* nennen, wenn $p_0(n)=p(\mathcal{G})$ gilt. Wie in [1] dargelegt, ist für eine extremale Groemerpackung $F(\operatorname{conv}\mathcal{G})=F(P(\mathcal{G}))=F_0(n)$ und die Frage nach der minimalen Fläche, die die konvexe Hülle einer Packung aus n Einheitskreisen haben kann, für solche n geklärt, für welche extremale Groemerpackungen existieren.

Bei Groemerpackungen aus mindestens zwei Kreisen ist $p(\mathcal{G})$ die Anzahl der peripheren Kreise. Für die Randsequenz $p_1, p_2, p_3, p_4, p_5, p_6$ gilt $p_{i+1}+p_{i+2}=$ $=p_{i+3}+p_{i+4}$ (Indices modulo 6) und damit erhält man folgende, rein zahlentheoretische Charakterisierung: Eine extremale Groemerpackung aus n Kreisen existiert genau dann, wenn es zu n natürliche Zahlen p_1, p_2, p_3, p_4 gibt mit

$$\begin{aligned}(1)\qquad n &= (p_1+p_2-1)(p_3+p_4-1)-\binom{p_1}{2}-\binom{p_2}{2}\\ p_0(n) &= p_1+2p_2+2p_3+p_4-6.\end{aligned}$$

Mit Hilfe der Einschränkung der p_i in [1] 4.4 — die sich für großes n leicht verschärfen läßt — ist für eine gegebene natürliche Zahl n leicht entscheidbar, ob es zu n eine Extremalpackung gibt oder nicht. Die Zahlen n, zu denen keine Extremalpackung existiert, seien *Ausnahmezahlen* genannt. Diese Bezeichnung rechtfertigt sich durch die dünne Verteilung dieser Zahlen. So gibt es unterhalb 1000 nur 24 Ausnahmezahlen: 121, 163, 211, 235, 265, 292, 325, 355, 391, 424, 463, 499, 541, 580, 625, 667, 706, 715, 760, 802, 811, 859, 904, 913, 955, 964. Eine theoretische Charakterisierung der Ausnahmezahlen anzugeben, scheint jedoch schwierig zu sein. Die bis $n=10^6$ fortgeführten Rechnungen stützen die folgende

VERMUTUNG 1. Ausnahmezahlen sind genau diejenigen Zahlen n, bei denen in der Darstellung

$$n=1+6\binom{a}{2}+ab+c \quad \text{mit} \quad 0\leqq b<6, \quad 0\leqq c<a$$

(vgl. 4.3 in [1]) die Parameter a, b, c eine der beiden folgenden Bedingungen erfüllen:

a) $b=2$ und $a-c\equiv -6m \bmod 9^{m+1}$ mit $m\in\mathbf{N}_0$;

b) $b=5$ und $a-c\equiv 4\cdot 9^m \bmod 9^{m+1}$ mit $m\in\mathbf{N}_0$.

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Zu dieser Vermutung wollen wir nun Teilergebnisse angeben. Natürlich sind die Zahlen $n=1+6\binom{a}{2}$ keine Ausnahmezahlen; zu diesen gehören die Groemerpackungen mit regulärer Sechsecksgestalt. Dabei erhält man die Packung mit $1+6\binom{a+1}{2}$ Kreisen, indem man um diejenige mit $1+6\binom{a}{2}$ Kreisen eine neue Kreisschicht herumlegt. Dieser Prozeß des „Ränderns" läßt sich auch auf andere Groemerpackungen anwenden: Rändert man eine extremale Groemerpackung aus $n=1+6\binom{a}{2}+ab+c$ Kreisen (nun seien b, c nicht beide 0), so erhält man eine Groemerpackung aus $1+6\binom{a+1}{2}+(a+1)b+(c+1)$ Kreisen und diese ist wieder extremal. Dasselbe gilt für den umgekehrten Prozeß des „Schälens" (Entfernung aller peripheren Kreise), wenn nicht $b=0 \wedge c=1$ bzw. $b\neq 0 \wedge c=0$ ist. Somit gilt:

LEMMA. *Entweder alle Zahlen einer Serie*

$$1+6\binom{a+k}{2}+(a+k)b+\begin{matrix} k+1 & \text{für} & b=0, \\ k & \text{für} & b\neq 0, \end{matrix} \quad k\in\mathbf{N}_0$$

sind Ausnahmezahlen oder keine.

Es genügt also, die „Pilotzahlen" $n=1+6\binom{a}{2}+ab+c$ dieser Serien mit $b=0\wedge c=1$ oder $b\neq 0\wedge c=0$ zu untersuchen. — Als Konsequenz des Lemmas hat man beispielsweise, daß Zahlen n mit $c=a-1$ keine Ausnahmezahlen sind. Die zugehörigen extremalen Groemerpackungen ergeben sich durch c-faches [bzw. $(c-1)$-faches] Rändern der entsprechenden extremalen Groemerpackungen zu $n=2, 3, 4, 5, 6$ [bzw. 8].

SATZ 1. *Die Pilotzahlen*

$$n=1+6\binom{a}{2}+2a \quad \text{mit} \quad a\equiv 0 \bmod 9$$

und

$$n=1+6\binom{a}{2}+5a \quad \text{mit} \quad a\equiv 6 \bmod 9$$

sind Ausnahmezahlen.

BEWEIS. Setzen wir $x:=p_1+p_2-1$, $y:=p_3+p_4-1$, $r:=p_1$ und $s:=p_4$, so geht (1) über in

$$n=xy-\binom{r}{2}-\binom{s}{2} \tag{2}$$

$$p_0(n)=2x+2y-r-s-2.$$

Im Falle $n=1+6\binom{a}{2}+2a$ sind also natürliche Zahlen x, y, r, s gesucht mit

$$1+6\binom{a}{2}+2a=xy-\binom{r}{2}-\binom{s}{2} \tag{3}$$

$$6a-3=2(x+y)-(r+s)-2$$

und damit haben diese Zahlen wegen $a \equiv 0$ mod 9 die folgenden Kongruenzen zu erfüllen:

$$xy - \binom{r}{2} - \binom{s}{2} \equiv 1 \bmod 9 \tag{4}$$

$$2(x+y)-(r+s) \equiv -1 \bmod 9.$$

Nun nimmt $\binom{m}{2}$ modulo 9 nur vier Werte an, nämlich

$$\binom{m}{2} \equiv 0 \bmod 9 \quad \text{für} \quad m \equiv 0,1 \bmod 9$$

$$\binom{m}{2} \equiv 1 \bmod 9 \quad \text{für} \quad m \equiv 2 \bmod 3$$

$$\binom{m}{2} \equiv 3 \bmod 9 \quad \text{für} \quad m \equiv 3,7 \bmod 9$$

$$\binom{m}{2} \equiv 6 \bmod 9 \quad \text{für} \quad m \equiv 4,6 \bmod 9.$$

Damit läßt sich durch Fallunterscheidung leicht zeigen, daß unter Beachtung von $2(x+y)-(r+s) \equiv -1$ mod 9 bei jeglicher Wahl von x, y, r, s stets $xy - \binom{r}{2} - \binom{s}{2} \not\equiv 1$ mod 9 ausfällt, d. h. (4) und somit auch (3) sind nicht lösbar. Entsprechend führt der Fall $n = 1 + 6\binom{a}{2} + 5a$ mit $a \equiv 6$ mod 9 auf das unlösbare Kongruenzensystem

$$xy - \binom{r}{2} - \binom{s}{2} \equiv 4 \bmod 9$$

$$2(x+y)-(r+s) \equiv 2 \bmod 9.$$

Außer der trivialen Serie $n = 1 + 6\binom{a}{2}$ lassen sich weitere Serien von Zahlen angeben, die keine Ausnahmezahlen sind, und zwar in Abhängigkeit davon, daß sich a durch gewisse binäre quadratische Formen darstellen läßt. Wir stellen diese Ergebnisse, die unmittelbar verifiziert werden können, im folgenden Satz tabellarisch zusammen.

Satz 2: *Für die Pilotzahlen* $n = 1 + 6\binom{a}{2} + ab + \delta_{0,b}$ *mit den in den ersten beiden Spalten der nachfolgenden Tabelle angegebenen Werten von a und b, wobei u, v nichtnegative ganze Zahlen sind, gibt es extremale Groemerpackungen und zwar bilden* $x = 2a + u$, $y = 2a - u$ *und die in den letzten beiden Spalten angegebenen Werte von r und s eine Lösung von (2).*

a	b	r	s
$5+u^2+v^2-3v$	0	$a-v+3$	$a+v$
$1+u^2+v^2$	1	$a-v+1$	$a+v-1$
$1+u^2+v^2-v$	2	$a-v+1$	$a+v$
$1+u^2+v^2$	3	$a+v$	$a-v$
$2+u^2+v^2-v$	4	$a+v-1$	$a-v$
$3+u^2+v^2$	5	$a-v-1$	$a+v-1$

Die Randsequenz der Groemerpackung ergibt sich dann zu

$$r,\ x-r+1,\ y-s+1,\ s,\ x-s+1,\ y-r+1.$$

Hier war nicht beabsichtigt, möglichst viele solche Serien anzugeben, sondern für jeden Wert von b wenigstens eine. Eine Abdeckung aller Möglichkeiten durch endlich viele solche Serien, bei denen a dargestellt wird durch binäre quadratische Formen, ist ohnedies nicht zu erreichen.

Für Ausnahmezahlen n bleibt die Frage nach der minimalen Fläche der konvexen Hülle einer Packung aus n Einheitskreisen zunächst völlig offen. Vermutlich wird auch in diesen Fällen der Minimalwert von Groemerpackungen geliefert und zwar hier von solchen mit $p_0(n)+1$ peripheren Kreisen:

VERMUTUNG 2. Ist $\mathcal{G}$ eine Packung aus n Einheitskreisen und n eine Ausnahmezahl, so gilt

$$F(\operatorname{conv}\mathcal{G}) \geqq F_0(n)+2-\sqrt{3}$$

und das Gleichheitszeichen tritt für jedes n ein und zwar genau für geeignete Groemerpackungen.

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A NOTE ON THE ARGUESIAN LATTICE IDENTITY

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Abstract

In a series of (sometimes joint) papers, Jónsson (et al.) introduced the Arguesian lattice identity, and proved it was equivalent to (the lattice theoretical formulation of) Desargues' implication. In this note we present two new equivalent formulations of the Arguesian law together with a simplified, complete proof of the aforementioned earlier results.

Let $(L; +, \cdot)$ be a lattice. A *triangle* in L is an element of L^3. For two triangles in L, $\mathbf{a}=(a_0, a_1, a_2)$ and $\mathbf{b}=(b_0, b_1, b_2)$ we define auxiliary polynomials $p_i=p_i(\mathbf{a}, \mathbf{b})=(a_j+b_j)\cdot(a_k+b_k)$, $p=p_ip_j(=p_ip_k=p_jp_k)$, and $c_i=c_i(\mathbf{a}, \mathbf{b})=(a_j+a_k)\cdot\cdot(b_j+b_k)$. Two triangles, $\mathbf{a}$ and $\mathbf{b}$ in L, are called *centrally perspective* if $p_2(\mathbf{a}, \mathbf{b})\leqq\leqq a_2+b_2$ and are called *axially perspective* if $c_2(\mathbf{a}, \mathbf{b})\leqq c_0(\mathbf{a}, \mathbf{b})+c_1(\mathbf{a}, \mathbf{b})$. We abbreviate these concepts as $CP(\mathbf{a}, \mathbf{b})$ and $AP(\mathbf{a}, \mathbf{b})$ respectively. *Desargues' implication* is the Horn sentence $CP(\mathbf{a}, \mathbf{b})\Rightarrow AP(\mathbf{a}, \mathbf{b})$.

THEOREM. *In the theory of lattices the following are equivalent.*

(1) *Desargues' Implication*

(2) $p(\mathbf{a}, \mathbf{b})\leqq a_0(a_1+c_2(c_0+c_1))+b_0(b_1+c_2(c_0+c_1))$

(3) $p(\mathbf{a}, \mathbf{b})\leqq a_0+b_0(b_1+c_2(c_0+c_1))$

(4) $p(\mathbf{a}, \mathbf{b})\leqq a_0+b_1+c_2(c_0+c_1)$

(5) $(a_0+c_1)(b_0(a_0+p_0)+b_1)\leqq c_0+c_1+b_1(a_0+a_1)$.

PROOF. We first note that all of the above statements imply modularity. For (1), consider the triangles $\mathbf{a}=(xz, z, xz)$ and $\mathbf{b}=(xz, y, y)$. For the rest use $\mathbf{a}=(xyz, x, x)$ and $\mathbf{b}=(y+z, yz, yz)$. Secondly, we have trivially that (2)$\Rightarrow$(3) and (3)$\Rightarrow$(4).

(3)$\Rightarrow$(2): Using the $\mathbf{a}-\mathbf{b}$ symmetry of $p(\mathbf{a}, \mathbf{b})$, (3) implies

$$\begin{aligned} p &\leqq [a_0+b_0(b_1+c_2(c_0+c_1))][b_0+a_0(a_1+c_2(c_0+c_1))] \\ &= a_0(a_1+c_2(c_0+c_1))+b_0(b_1+c_2(c_0+c_1))+a_0b_0, \quad \text{by mod} \\ &= a_0(a_1+c_2(c_0+c_1))+b_0(b_1+c_2(c_0+c_1)), \quad \text{by mod} \end{aligned}$$

and the fact that $a_0b_0\leqq c_2(c_0+c_1)$.

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(5)⇔(3): Making heavy use of modularity we have

$$\begin{aligned} p \leqq a_0+b_0(b_1+c_2(c_0+c_1)) \quad \text{iff} \quad & a_0+p \leqq a_0+b_0(b_1+c_2(c_0+c_1)) \\ \text{iff} \quad & b_0(a_0+p_0) \leqq a_0+b_0(b_1+c_2(c_0+c_1)) \\ \text{iff} \quad & b_0(a_0+p_0) \leqq b_0[a_0b_0+b_1+c_2(c_0+c_1)] = \\ & = b_0[b_1+c_2(c_0+c_1)] = \\ & = b_0[b_1+(a_0+a_1)(c_0+c_1)] \\ \text{iff} \quad & b_1+b_0(a_0+p_0) \leqq b_1+(a_0+a_1)(c_0+c_1) \end{aligned}$$

and by meeting with a_0+a_1, iff $(a_0+a_1)(b_0(a_0+p_0)+b_1) \leqq c_0+c_1+b_1(a_0+a_1)$.

(1)⇒(5): By modularity, for any triangles, **a** and **b**, the modified triangles $\mathbf{a}' = (a_0, a_1, a_2+a_0(a_1+b_1))$ and $b' = (b_0(a_0+p_0), b_1, b_2)$ are centrally perspective. By (1), $AP(\mathrm{a}', \mathrm{b}')$, namely:

$$\begin{aligned} (a_0+a_1)(b_0(a_0+p_0)+b_1) \leqq\ & (a_0+a_2)(b_0(a_0+p_0)+b_2)+ \\ & +(a_1+a_2+b_1(a_0+a_1))(b_1+b_2) \leqq \\ \leqq\ & c_1+c_0+b_1(a_0+a_1). \end{aligned}$$

(4)⇒(1): Let **a** and **b** be centrally perspective triangles. By substituting into (4) we obtain

$$(a_0+b_0)(a_1+b_1) \leqq a_0+b_1+c_2(c_0+c_1).$$

By joining with a_0+b_1 and then meeting with c_2 we obtain:

$$\begin{aligned} c_2 &= c_2(a_0+b_0+b_1)(a_0+a_1+b_1) \leqq \\ &\leqq c_2(a_0+b_1+c_2(c_0+c_1)) \quad \text{by (4)} \\ &= c_2(c_0+c_1+c_2(a_0+b_1)) \leqq \\ &\leqq c_0+c_1+a_0(b_0+b_1)+b_1(a_0+a_1) = \\ &= (b_1+b_2)(a_1+a_2+b_1(a_0+a_1))+(a_0+a_2)(b_0+b_2+a_0(b_0+b_1)) = \\ &= (b_1+b_2)(a_1+a_2+a_0(a_1+b_1))+(a_0+a_2)(b_0+b_2+b_1(a_0+b_0)) \leqq \\ &\leqq (b_1+b_2)(a_1+a_2+a_0(a_2+b_2))+(a_0+a_2)(b_0+b_2+b_1(a_2+b_2)), \end{aligned}$$

by $CP(\mathbf{a}, \mathbf{b})$

$$\begin{aligned} &= (b_1+b_2)(a_1+a_2+b_2(a_0+a_2))+(a_0+a_2)(b_0+b_2+a_2(b_1+b_2)) = \\ &= c_0+b_2(a_0+a_2)+c_1+a_2(b_1+b_2) = \\ &= c_0+c_1. \end{aligned}$$

This completes the proof.

Using the Desargues' Implication, Jónsson ([3]) showed that Arguesian lattices (i.e. lattices satisfying any of the above) formed a self dual variety of lattices. None of the above equations make that result transparent. Since distributive lattices $((x+y)(x+z)(y+z)=xy+xz+yz)$ and modular lattices $((y+z)(x+yz)=$ $=yz+x(y+z))$ are self-dual variaeties *and* are defined by a self-dual equation $p=p^{dual}$, one might ask if such an equation exists for Arguesian lattices.

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ON POLYADIC GROUPS WHICH ARE TERM-DERIVED FROM GROUPS

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0. Introduction

W. Dörnte introduced in [4] the notion of *n-group* (called also a *polyadic group* [16]; for the definition see also [7] and [5]), which is a natural generalization of the notion of group. We recall that $(G; f)$ is an $(n+1)$-group if $f\colon G^{n+1}\to G$ is associative and for every $k\in\{0, 1, \ldots, n\}$ (and fixed $x_i\in G$; $i\neq k$) the mapping

$$z\mapsto f(x_0, x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_n)$$

is bijective. One can observe that if

$$f(x_0, x_1, \ldots, x_n) = x_0\circ x_1\circ\ldots\circ x_n \tag{0}$$

in a group $(G; \circ)$, then $(G; f)$ is an $(n+1)$-group. This polyadic group is said to be *derived* from the group $(G; \circ)$. In [4] a criterion was proved in order that a polyadic group were derived from a group. The following more general situation has also been considered:

$$f(x_0, x_1, \ldots, x_n) = \varphi_0(x_0)\circ\varphi_1(x_1)\circ\ldots\circ\varphi_n(x_n)\circ d, \tag{1}$$

where φ_i maps G into itself and $d\in G$.

Timm proved [18] that the operation f defined by (1), where

$$\varphi_0(x) = x, \quad \varphi_i(e) = e \quad (i = 0, 1, \ldots, n), \tag{a}$$

and e is the neutral element (identity) of the group $(G; \circ)$, is an $(n+1)$-group operation over $(G; \circ)$ if and only if

(b) φ_1 is an automorphism of the group $(G; \circ)$,

(c) $\varphi_i=\varphi_1^i$ $(i=0, 1, \ldots, n-1)$,

(d) $\varphi_n(x)=d\circ x\circ d^{-1}$,

(e) $\varphi_1(d)=d$.

Let $\varphi=\varphi_1$. Then an $(n+1)$-group $(G; f)$ (or the operation f), where f is of the form (1) with (a)$-$(e), is said to be *(φ, d)-derived* from the group $(G; \circ)$ (see [6]).

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Post and Hosszú proved that every polyadic group is (φ, d)-derived from a group ([16], p. 245, [9], see also [19]).

In this paper we shall consider special (φ, d)-derived operations called here *β-derived* (namely, if $d=e$ and $\varphi(x)=x^\beta$, where β is an integer; of course, 1-derived means derived in the previous sense). It will be shown that such operations coincide with term operations on the group $(G; \circ)$ of the form

$$f(x_0, x_1, \ldots, x_n) = x_0^{\alpha_0}\circ x_1^{\alpha_1}\circ \ldots \circ x_n^{\alpha_n}, \tag{2}$$

which are $(n+1)$-group operations. Note, for example, that in the group $(\mathbf{Z}_8; +)$ the operation $f(x, y, z)=x+3y+z$ is a 3-group operation.

By $(G; \circ)$ we shall always mean a group (2-group). By $\mathbf{F}_n(G; \circ)$ we shall denote the set of all term operations on $(G; \circ)$ of the form (2) which are $(n+1)$-group operations. Of course, $\mathbf{F}_n(G; \circ)$ is a subset of the set $\mathbf{T}(G; \circ)$ of all term operations on $(G; \circ)$. If $f\in \mathbf{T}(G; \circ)$ and $(G; f)$ is a polyadic group, then $(G; f)$ (or the operation f) is called *term-derived* from the group $(G; \circ)$.

It is convenient to use the following abbreviated notation:

$$g(x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_{i+s+1}, \ldots, x_m) = g(x_0, x_1, \ldots, x_i, \overset{(s)}{x}, x_{i+s+1}, \ldots, x_m)$$

whenever $x_{i+1}=\ldots=x_{i+s}=x$ (and $\overset{(s)}{x}$ is the empty symbol for $s=0$). Denote also:

$$g_{(r)}(x_0, \ldots, x_m, x_{m+1}, \ldots, x_{2m}, \ldots, x_{(r-1)m+1}, \ldots, x_{rm}) =$$
$$= g(g(\ldots g(g(x_0, \ldots, x_m), x_{m+1}, \ldots, x_{2m}), \ldots), x_{(r-1)m+1}, \ldots, x_{rm})$$

for an arbitrary m-ary operation g.

1. β-derived operation

By using the results of Post, Hosszú and Timm we shall prove the following

PROPOSITION 1. *Let $(G; \circ)$ be an arbitrary group and assume that f is of the form (2). Then f is an $(n+1)$-group operation if and only if the following identities hold:*

$$x^{\alpha_0} = x = x^{\alpha_n}, \tag{3}$$

$$x^{\alpha_i} = x^{\beta^i}, \tag{4}$$

$$(x\circ y)^\beta = x^\beta\circ y^\beta \tag{5}$$

for some integer $\beta\neq 0$. Moreover, the mapping $x\mapsto x^\beta$ is an automorphism of $(G; \circ)$, and for every $x\in G$ the element $x^{\beta-1}$ belongs to the center of $(G; \circ)$.

PROOF. Let $f\in \mathbf{F}_n(G; \circ)$. Then for every $x\in G$ there exists a skew element $\bar{x}\in G$ such that

$$f(x, x, \ldots, x, \bar{x}, y) = y \tag{6}$$

and

$$f(y, x, \ldots, x, \bar{x}, x) = y \tag{7}$$

for every $y \in G$ (see [4]). So we have

$$x^{\alpha_0} \circ x^{\alpha_1+\ldots+\alpha_{n-2}} \circ \bar{x}^{\alpha_{n-1}} \circ y^{\alpha_n} = y, \tag{6'}$$

$$y^{\alpha_0} \circ x^{\alpha_1+\ldots+\alpha_{n-2}} \circ \bar{x}^{\alpha_{n-1}} \circ x^{\alpha_n} = y. \tag{7'}$$

By putting $y=e$, we conclude that $x^{\alpha_1+\ldots+\alpha_{n-2}} \circ \bar{x}^{\alpha_{n-1}}$ is the inverse to x^{α_0} and to x^{α_n}. Therefore, from (6′) and (7′) we get (3) and also (a) for $\varphi_0(x)=x^{\alpha_0}$. Put $\varphi_i(x)=x^{\alpha_i}$, $\alpha_1=\beta$ and $d=e$. Then f is of the form (1) with (a), and by Timm's result the mappings φ_i fulfil the conditions (b)—(e). Therefore we obtain the identities (4) and (5). It is easy to check that $x^{\beta-1}$ belongs to the center of $(G; \circ)$.

Now, let the identities (3)—(5) hold in the group $(G; \circ)$. Define $\varphi_i(x)=x^{\beta^i}$. By (5), φ_1 is an endomorphism of $(G; \circ)$. Taking into account (3) and (4) we have $(\varphi_1)^n(x)=x$ and φ_1 is an automorphism of $(G; \circ)$. Therefore, the operation f is of the form (1) with $d=e$, and the conditions (a)—(e) are satisfied. Hence $(G;f)$ is an $(n+1)$-group (cf. [18], [9] and [16], p. 245), which completes the proof.

According to the definitions above we get immediately

Corollary 1. *$f \in F_n(G; \circ)$ iff $(G;f)$ is β-derived from $(G; \circ)$ for a suitable β, i.e.*

$$f(x_0, x_1, \ldots, x_n) = x_0 \circ x_1^{\beta} \circ \ldots \circ x_{n-1}^{\beta^{n-1}} \circ x_n. \tag{1'}$$

From [7] and from the formula

$$\bar{x} = x^{-(\beta+\beta^2+\ldots+\beta^{n-1})} \tag{8}$$

we infer easily

Corollary 2. *If $(G;f)$ is an $(n+1)$-group β-derived from the group $(G; \circ)$, then $(G;f,{}^-)$ is a reduct of $(G; \circ)$ (i.e. every term operation of $(G;f)$ is a term operation of $(G; \circ)$).*

We observe that the conditions (3) and (4) describe all term operations which are $(n+1)$-group operations over an abelian group $(G; \circ)$. In particular, we have

Corollary 3 (see also [8]). *The operation $x \circ y$ is the only term which is a (binary) group operation in an arbitrary abelian group $(G; \circ)$.*

From Corollary 3 we can infer (as J. T. Baldwin has remarked) also Lemma 5.1 of [2].

The next corollary is a generalization of a result of Prüfer and Certaine for $n=2$ ([17] and [3]); in this case we obtain a heap (or a flock) with operation $x \circ y^{-1} \circ z$ (see also [1], [12], [14], [15] and [20]). Firstly, observe that if $(G;f)$ is an $(n+1)$-group (-1)-derived from a group $(G; \circ)$, then n is even or $x^{-1}=x$, because we have $x^{(-1)^n}=x$.

Corollary 4. *Let $(G; \circ)$ be a group and*

$$f(x_0, x_1, \ldots, x_n) = x_0 \circ x_1^{-1} \circ \ldots \circ x_{n-1}^{-1} \circ x_n, \tag{9}$$

where n is even. Then $(G;f)$ is an $(n+1)$-group iff $(G; \circ)$ is abelian.

Indeed, if n is even and $\beta=-1$, then the identities (3) and (4) hold. Moreover, in this case identity (5) is equivalent to commutativity of the multiplication.

In a similar manner (by putting $\beta=2$) we get

COROLLARY 5. *Let* $(G; \circ)$ *be a group of exponent* 2^n-1, *and*

$$f(x_0, x_1, \ldots, x_n) = x_0 \circ x_1^{2} \circ x_2^{4} \circ \ldots \circ x_{n-1}^{2^{n-1}} \circ x_n. \tag{10}$$

Then $(G;f)$ *is an* $(n+1)$*-group iff* $(G; \circ)$ *is abelian.*

It is easy to verify

COROLLARY 6. *Let* $n>1$. $\mathbf{F}_n(G; \circ)=\{x_0 \circ x_1 \circ \ldots \circ x_n\}$ *iff the group* $(G; \circ)$ *is Boolean, or without exponent and either n is odd or* $(G; \circ)$ *is non-abelian.*

In the case $n=1$ the part "if" is true, and for a free group we have $\mathbf{F}_1(G; \circ)= =\{x_0 \circ x_1\}$, which follows also from a result of H. Neumann [13]. We have also

COROLLARY 7. $\mathbf{F}_n(G; \circ) \subset \{x_0 \circ x_1 \circ \ldots \circ x_n,\ \ x_0 \circ x_1^{-1} \circ x_2 \circ \ldots \circ x_n\}$ *if* $(G; \circ)$ *is an abelian group without exponent.*

2. Criteria of β-derivability

In this section we shall give some necessary and sufficient conditions for an $(n+1)$-group to be β-derived from a group. Observe that if an $(n+1)$-group is β-derived from a group with $\beta<-1$, then the group $(G; \circ)$ is of finite exponent (which divides β^n-1 or $-\beta^n+1$) and there exists $\gamma>0$ such that $x^{-1}=x^{\gamma}$. So $(G;f)$ is also $(-\beta\gamma)$-derived from $(G; \circ)$, where $-\beta\gamma>0$. Therefore, without loss of generality, we may consider only the cases: $\beta>0$ and $\beta=-1$.

For the next theorem we need the following

LEMMA 1. *Let* $\beta>0$, *and assume that there exists* $e \in G$ *such that*

$$f(e, e, \ldots, e) = e \tag{11}$$

and

$$f_{(\beta)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta}, e) = x. \tag{12}$$

Then the identities

$$f_{(\beta^i)}(\overset{(n-i)}{e}, x, \underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta^i-1}, \overset{(i)}{e}) = x \tag{13}$$

hold for every $i=0, 1, \ldots, n$.

PROOF. Of course, for $i=0$ we get

$$f(\overset{(n)}{e}, x) = x \tag{14}$$

and for $i=1$ we have (12). Observe that (14) is equivalent to (11) in any $(n+1)$-group ([4]). Assume (13) is satisfied for $i=k$. We shall prove that (13) holds for

$i=k+1$. By the assumption and by (14) we have

$$f_{(\beta^{k+1})}(\overset{(n-k-1)}{e}, \underbrace{x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}}_{\beta^{k+1}-1}, x, \overset{(k+1)}{e}) =$$

$$= f(\overset{(n)}{e}, f_{(\beta^{k+1})}(\overset{(n-k-1)}{e}, x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}, x, \overset{(k+1)}{e})) =$$

$$= f_{(\beta^k(\beta-1)+1)}(\overset{(n-1)}{e}, f_{(\beta^k)}(\overset{(n-k)}{e}, \underbrace{x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}}_{\beta^k-1}, x, \overset{(k)}{e}),$$

$$\overset{(n-k-1)}{e}, \underbrace{x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}}_{\beta^k(\beta-1)-1}, x, \overset{(k+1)}{e}) =$$

$$= f_{(\beta^k(\beta-1)+1)}(\overset{(n-1)}{e}, x, \overset{(n-k-1)}{e}, \underbrace{x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}}_{\beta^k(\beta-1)-1}, x, \overset{(k+1)}{e}) =$$

$$= f_{(\beta^k(\beta-2)+2)}(\overset{(n-1)}{e}, x, \overset{(n-1)}{e}, f_{(\beta^k)}(\overset{(n-k)}{e}, x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}, x, \overset{(k)}{e}),$$

$$\overset{(n-k-1)}{e}, x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}, x, \overset{(k+1)}{e}) =$$

$$= f_{(\beta^k(\beta-2)+2)}(\overset{(n-1)}{e}, x, \overset{(n-1)}{e}, x, \overset{(n-k-1)}{e}, x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}, x, \overset{(k+1)}{e}) = \ldots =$$

$$= f_{(\beta^k+\beta-1)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta-2}, \overset{(n-1)}{e}, f_{(\beta^k)}(\overset{(n-k)}{e}, x, \overset{(n-1)}{e}, \ldots, x, \overset{(k)}{e}),$$

$$\overset{(n-k-1)}{e}, \underbrace{x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}}_{\beta^k-1}, x, \overset{(k+1)}{e}) =$$

$$= f_{(\beta^k+\beta-1)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta-1}, \overset{(n-k-1)}{e}, \underbrace{x, \overset{(n-1)}{e}, \ldots, x, \overset{(n-1)}{e}}_{\beta^k-1}, x, \overset{(k+1)}{e}) =$$

$$= f_{(\beta)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta-1}, \overset{(n-1)}{e}, f_{(\beta^k)}(\overset{(n-k)}{e}, x, \overset{(n-1)}{e}, \ldots, x, \overset{(k)}{e}), e) =$$

$$= f_{(\beta)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta}, e) = x,$$

which completes the proof.

The following theorem is a generalization of Dőrnte's criterion ([4], p. 7, see also [16], p. 231, and [10], p. 54).

THEOREM 1. *$(G;f)$ is an $(n+1)$-group which is β-derived, with $\beta>0$, from some group $(G;\circ)$ if and only if there exists an element $e\in G$ such that* (11) *and* (12) *hold.*

Moreover,

$$x \circ y = f(x, \overset{(n-1)}{e}, y). \tag{15}$$

PROOF. Let $(G;f)$ be an $(n+1)$-group β-derived from a group $(G;\circ)$, and $\beta>0$. Further, let e be the neutral element of $(G;\circ)$. Then (11) and (15) are obvious, and we have

$$f_{(\beta)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta}, e) = f_{(\beta-1)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta-1}, f(\overset{(n-1)}{e}, x, e)) =$$

$$= f_{(\beta-1)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta-1}, x^{\beta^{n-1}}) = f_{(\beta-2)}(\underbrace{\overset{(n-1)}{e}, x, \ldots, \overset{(n-1)}{e}, x}_{\beta-2}, x^{\beta^{n-1}} \circ x^{\beta^{n-1}}) =$$

$$= \ldots = \underbrace{x^{\beta^{n-1}} \circ \ldots \circ x^{\beta^{n-1}}}_{\beta} = x^{\beta^n} = x.$$

We get the last equality by (3) and (4). Therefore the identity (12) is also satisfied.

Conversely, let the $(n+1)$-group fulfil (11) and (12). Consider the binary operation defined by (15). By Lemma 1 of [9] the groupoid $(G;\circ)$ is a group. Taking into account (14) and the associativity of the $(n+1)$-ary operation f, we get

$$x_0 \circ x_1^{\beta} \circ \ldots \circ x_{n-1}^{\beta^{n-1}} \circ x_n =$$

$$= f_{(\gamma)}(x_0, \underbrace{\overset{(n-1)}{e}, x_1, \ldots, \overset{(n-1)}{e}, x_1}_{\beta}, \underbrace{\overset{(n-1)}{e}, x_2, \ldots, \overset{(n-1)}{e}, x_2}_{\beta^2}, \ldots$$

$$\ldots, \underbrace{\overset{(n-1)}{e}, x_{n-1}, \ldots, \overset{(n-1)}{e}, x_{n-1}}_{\beta^{n-1}}, \overset{(n-1)}{e}, x_n),$$

where $\gamma = 1+\beta+\beta^2+\ldots+\beta^{n-1}$. Hence, by our Lemma 1, we have

$$x_0 \circ x_1^{\beta} \circ \ldots \circ x_{n-1}^{\beta^{n-1}} \circ x_n =$$

$$= f(x_0, f_{(\beta)}(\overset{(n-1)}{e}, x_1, \ldots, \overset{(n-1)}{e}, x_1, e), f_{(\beta^2)}(\overset{(n-2)}{e}, x_2, \overset{(n-1)}{e}, x_2, \ldots$$

$$\ldots, \overset{(n-1)}{e}, x_2, \overset{(2)}{e}), \ldots, f_{(\beta^{n-1})}(e, x_{n-1}, \overset{(n-1)}{e}, x_{n-1}, \ldots, \overset{(n-1)}{e}, x_{n-1}, \overset{(n-1)}{e}), x_n) =$$

$$= f(x_0, \ldots, x_n).$$

Therefore $(G;f)$ is an $(n+1)$-group β-derived from the group $(G;\circ)$, which completes the proof.

For the case $n=2$ Theorem 1 takes a simpler form.

COROLLARY 8. *A 3-group $(G;f)$ is β-derived, with $\beta>0$, from a group $(G;\circ)$ iff there exists $e \in G$ such that $\bar{e}=e$ and*

$$f_{(\beta)}(e, x, \ldots, e, x, e) = x. \tag{12'}$$

Indeed, it is enough to observe

$$x_0 \circ x_1 \circ x_2 = f(x_0, f_{(\beta)}(e, x_1, \ldots, e, x_1, e), x_2) = f(x_0, x_1, x_2).$$

Theorem 1 gives a new description of a certain variety of groups.

COROLLARY 9. *The class of all $(n+1)$-groups $(G;f)$ β-derived, with fixed $\beta>0$, from a group $(G;\circ)$ is polynomially equivalent to the variety of algebras $(G;f,{}^{-},e)$ (of the type $(n+1,1,0)$, where $(G;f,{}^{-})$ is an $(n+1)$-group equationally defined as in* [7] *or* [5], *and $e\in G$ is a fixed element satisfying* (11) *and* (12)), *and to the variety of all groups satisfying* (5) *and $x^{\beta^n}=x$.*

For $\beta=-1$ we obtain

THEOREM 2. *Let n be even, and let $(G;f)$ be an $(n+1)$-group. Then $(G;f)$ is (-1)-derived from some group if and only if the following identities hold in G:*

$$f(x,x,\dots,x)=x, \tag{16}$$

$$f(x_0,\dots,x_i,y,y,x_{i+3},\dots,x_n)=f(x_0,\dots,x_i,z,z,x_{i+3},\dots,x_n) \tag{17}$$

(for all $i\leqq n-2$). In this case $(G;f)$ is (-1)-derived from the group $(G;\circ)$ where

$$x\circ y=f(x,\overset{(n-1)}{c},y) \tag{18}$$

for an arbitrary $c\in G$, and the inverse operation is given by

$$x^{-1}=f(c,\overset{(n-1)}{x},c). \tag{19}$$

PROOF. Let an $(n+1)$-group $(G;f)$ satisfy the conditions (16) and (17). It is easy to verify that $(G;\circ)$ defined by (18) is a group with neutral element c, and the formula (19) defines the inverse operation $x\mapsto x^{-1}$. Indeed, we have

$$(x\circ y)\circ z=f\big(f(x,\overset{(n-1)}{c},y),\overset{(n-1)}{c},z\big)=x\circ(y\circ z).$$

Since by (16) c is self-skew (i.e. $\bar{c}=c$), so $x\circ c=f(x,\overset{(n-1)}{c},c)=x$. And finally we obtain

$$x\circ x^{-1}=f\big(x,\overset{(n-1)}{c},f(c,\overset{(n-1)}{x},c)\big)=f\big(x,f(\overset{(n)}{c},x),\overset{(n-2)}{x},c\big)=f(\overset{(n)}{x},c)=c.$$

Taking into account (16)—(19) we get

$$\begin{aligned}
&x_0\circ x_1^{-1}\circ x_2\circ\ldots\circ x_{n-1}^{-1}\circ x_n=\\
&=f_{(n+n/2)}(x_0,\overset{(n)}{c},\overset{(n-1)}{x_1},\overset{(n)}{c},x_2,\dots,\overset{(n-1)}{x_{n-1}},\overset{(n)}{c},x_n)=\\
&=f_{(n/2)}(x_0,\overset{(n-1)}{x_1},x_2,\dots,\overset{(n-1)}{x_{n-1}},x_n)=\\
&=f_{(n/2-1)}\big(x_0,x_1,f(\overset{(n-2)}{x_1},x_2,\overset{(2)}{x_3}),\dots,\overset{(n-1)}{x_{n-1}},x_n\big)=\\
&=f_{(n/2-1)}\big(x_0,x_1,f(\overset{(n-2)}{x_2},x_2,\overset{(2)}{x_2}),\dots,\overset{(n-1)}{x_{n-1}},x_n\big)=\\
&=f_{(n/2-1)}(x_0,x_1,x_2,\overset{(n-3)}{x_3},\dots,\overset{(n-1)}{x_{n-1}},x_n)=\ldots=\\
&=f\big(x_0,x_1,\dots,x_{n-3},f(\overset{(2)}{x_{n-3}},x_{n-2},\overset{(n-2)}{x_{n-1}}),x_{n-1},x_n\big)=\\
&=f\big(x_0,x_1,\dots,x_{n-3},f(\overset{(2)}{x_{n-2}},x_{n-2},\overset{(n-2)}{x_{n-2}}),x_{n-1},x_n\big)=f(x_0,\dots,x_n).
\end{aligned}$$

Therefore $(G;f)$ is (-1)-derived.

Conversely, if an $(n+1)$-group $(G;f)$ is (-1)-derived from some group $(G;\circ)$, then the formulas (16) and (17) are obvious, and $(G;f)$ is also (-1)-derived from each group $(G;\circ_c)$, where $x\circ_c y=x\circ c\circ y$ and $c\in G$ is an arbitrary element. Thus the proof of Theorem 2 is complete.

Now we have (using the notion of polynomial in the sense of [11]) immediately (see also [3] and [12] for $n=2$):

COROLLARY 10. *All $(n+1)$-groups (-1)-derived from some group (n is even) form a variety which is polynomially equivalent to the variety of all abelian groups.*

Finally we prove

COROLLARY 11. *Let $(G;f)$ be an $(n+1)$-group and n be odd. Then the following conditions are equivalent:*

(A) *$(G;f)$ is β-derived from a Boolean group $(G;\circ)$ (i.e. from a group with exponent 2),*

(B) *$(G;f)$ is simultaneously 1-derived and (-1)-derived from some group $(G;\circ)$'*

(C) *the following identities hold in G:*

(20) $$f(x, \ldots, x, y, y) = f(z, \ldots, z, t, t),$$

(21) $$f_{(n+1)}(\overset{(n-1)}{x}, \overset{(2)}{y}, \ldots, \overset{(n-1)}{x}, \overset{(2)}{y}, z, \overset{(n-1)}{x}, \overset{(2)}{y}) = z,$$

(D) *the equalities*

(22) $$f(x, \ldots, x, y, y) = c$$

(23) $$f(\overset{(n-1)}{c}, x, c) = x$$

hold for every $x, y\in G$ and for some $c\in G$.

PROOF. Firstly we prove that (A)$\Leftrightarrow$(B). Indeed, if $(G;f)$ is β-derived from a Boolean group $(G;\circ)$, then of course it is (-1)-derived and 1-derived from the group $(G;\circ)$, because $x=x^\beta=x^{-1}$, so (A)$\Rightarrow$(B). Conversely, if $(G;f)$ is β-derived, for $\beta=1, -1$, from some group $(G;\circ)$, then

$$x_0\circ x_1\circ x_2\circ\ldots\circ x_{n-1}\circ x_n = f(x_0, x_1, \ldots, x_n) = x_0\circ x_1^{-1}\circ x_2\circ\ldots\circ x_{n-1}^{-1}\circ x_n.$$

Hence by putting $x_i=e$ for $i\neq 1$ we get $x_1=x_1^{-1}$ for an arbitrary $x_1\in G$. Therefore $(G;\circ)$ is Boolean.

The implication (A)$\Rightarrow$(C) is obvious. Now if (C) holds, then $f(x, \ldots, x, y, y)$ does not depend on x and y, so it has some constant value c, and we get $f(c, \ldots, c)=c$. By putting $x=y=c$ in (21) we obtain (23). Thus (C)$\Rightarrow$(D).

Finally, from (23) we have (11) and (12) for $e=c$ and $\beta=1$, hence $(G;f)$ is 1-derived from some group $(G;\circ)$. Now, by (22), we have

$$x^{n-1}\circ y^2 = f(x, \ldots, x, y, y) = c.$$

Taking into account that c is the neutral element of $(G;\circ)$, we get $y^2=c$ for every $y\in G$, and so $y=y^{-1}$ in $(G;\circ)$, which completes the proof of Corollary 11. Observe that (A)$\Leftrightarrow$(B) also for even n.

Therefore we easily get

Corollary 12. *All $(n+1)$-groups which are simultaneously 1-derived and (-1)-derived from some group $(G; \circ)$ form a variety which is polynomially equivalent to the variety of all Boolean groups.*

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ON BASES FOR NORMAL IDENTITIES

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Abstract

Given a set E of identities of type τ: $T \to \mathbf{N}^+$ where $\mathbf{N}^+$ denotes the set of positive integers. K_E denotes the equational class of algebras of type τ defined by E. Let p, q be polynomial symbols of type τ. An identity $p \equiv q$ is called normal if either it is of the form $x \equiv x$ (where x is a variable) or none of the symbols p, q are variables. Denote by $N(\tau)$ the set of all normal identities of type τ. $C(E)$ denotes the set of all consequences of E, $N(E) = C(E) \cap N(\tau)$.

In this note we deal with the problem of indicating an axiomatic for $N(E)$, for a given set E of identities.

We give syntactic proof of the fact that $C(E)$ is finitely based (i.e. has finite axiomatic) if and only if $N(E)$ is finitely based; as well as the variety K_E has the finite basis property if and only if $K_{N(E)}$ has (if we assume that T is finite).

§ 1. Our nomenclature and notation are basically those of [1], [9].

The notion of a term which is "trivial" in a variety K was introduced in [8]. An identity is called to be "trivializing" if it is of the form $x \equiv y$ (where x, y are different variables) or $x_k \equiv p(x_1, ..., x_n)$ where p is a polynomial symbol which is not a variable. The first type of identity we shall call "almost contradictory", the next one "an absorbtion law". Following the nomenclature introduced in [2], in this paper we shall use the name "normal" instead of "non-trivializing" (cf. [3], [4], [5]).

§ 2. Given a set E of identities of type τ. Let $r(x, ..., x) \equiv x$ be an absorption law which belongs to $C(E)$. In the sequel we shall write $r(x)$ instead of $r(x, ..., x)$. Consider the set E^r of identities including all normal identities from E together with all identities of the form $r(x_k) \equiv p(x_1, ..., x_n)$, where the identity $x_k \equiv p(x_1, ..., x_n)$ or $p(x_1, ..., x_n) \equiv x_k$ belongs to $E - N(\tau)$:

$E^r = E \cap N(\tau) \cup \{r(x_k) \equiv p(x_1, ..., x_n)\colon\ x_k \equiv p(x_1, ..., x_n)$ or $p(x_1, ..., x_n) \equiv x_k$ is an identity of $E - N(\tau)\}$.

For $t \in T$ with $n = \tau(t)$ consider the following axioms:

$$(t_1) \qquad r(f_t(x_1, ..., x_n)) \equiv f_t(x_1, ..., x_n);$$

$$(t_2) \qquad f_t(x_1, ..., x_n) \equiv f_t(r(x_1), ..., r(x_n)).$$

Let $\mathcal{N} = \{(t_1), (t_2)\colon\ t \in T\}$.

Remark. A consequence of the axiom (t_1) is:

$$(t_3) \qquad r(r(x)) = r(x).$$

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Firstly we give a syntactic proof of a theorem on bases for $N(E)$, which generalizes Theorem 2 of [5]:

THEOREM 1. *Given a consistent set E of identities of type τ. Let $x\equiv r(x, \ldots, x)$ be an absorbtion law from $C(E)-N(\tau)$. Consider $p\equiv q\in N(\tau)$. Then $E\vdash p\equiv q$ if and only if $E^r\cup\mathcal{N}\vdash p\equiv q$.*

PROOF. Sufficiency is obvious. To prove necessity we shall show firstly that any normal identity which is derived from E by the superposition rule, is a consequence of $E^r\cup\mathcal{N}$. Let $p_i\equiv q_i\in E$ for $i=1, \ldots, n$ and $p=f_t(p_1, \ldots, p_n)$, $q=f_t(q_1, \ldots, q_n)$. Denote by $r(p_i)$ the polynomial symbol $r(x, \ldots, x)(p_i, \ldots, p_i)$ (and similarly for q_i). If $p_i\equiv q_i$ is not normal, then it is of the form:

$$x_k \equiv q_i(x_1, \ldots, x_m) \quad \text{or} \quad p_i(x_1, \ldots, x_m)\equiv x_k \quad \text{for some} \quad 1 \leqq k \leqq m.$$

Assume $x_k\equiv q_i(x_1, \ldots, x_m)\in E-N(\tau)$. Then $r(x_k)\equiv q_i(x_1, \ldots, x_m)$ belongs to E^r. By an easy induction on the rank of q_i we deduce:

$$\mathcal{N}\vdash q_i(x_1, \ldots, x_m) \equiv q_i(r(x_1), \ldots, r(x_m)),$$

$$\mathcal{N}\vdash q_i(r(x_1), \ldots, r(x_m)) \equiv r(q_i(x_1, \ldots, x_m))$$

and thus:

$$E^r\cup\mathcal{N}\vdash r(x_k) = r(q_i(x_1, \ldots, x_m)).$$

Thus $E^r\cup\mathcal{N}\vdash r(p_i)\equiv r(q_i)$, for $i=1, \ldots, n$. So

$$E^r\cup\mathcal{N}\vdash f_t(r(p_1), \ldots, r(p_n)) \equiv f_t(r(q_1), \ldots, r(q_n)),$$

$$\mathcal{N}\vdash f_t(q_1, \ldots, q_n) \equiv f_t(r(q_1), \ldots, r(q_n))$$

and finally: $E^r\cup\mathcal{N}\vdash p\equiv q$. Analogously one can show that any normal identity which is obtained from E by the substitution rule can be derived from $E^r\cup\mathcal{N}$.

Let $\mathrm{Sb}(E)$ denote the smallest set including E and closed under the substitution rule. Assume that p and q are not variables. Assume $p\neq q$ (i.e. p and q are different terms). If $E\vdash p\equiv q$ then $\mathrm{Sb}(E)\vdash p\equiv q$ which implies (see [7], [9]) that there exists a derivation $p_1, \ldots, p_s$ such that $p_1=p$, $p_s=q$ and for each $i<s$, there exists an identity $\alpha_i\equiv\beta_i\in\mathrm{Sb}(E)$ such that α_i (or β_i) is a subterm of p_i and p_{i+1} results from p_i by replacing the subterm α_i by β_i (resp. α_i). Let i be the smallest number such that p_i is a variable. Thus $p_{i-1}\equiv p_i\in\mathrm{Sb}(E)-N(\tau)$ and $p_i\equiv p_{i+1}\in\mathrm{Sb}(E)-N(\tau)$ (or is equal to $x\equiv x$). But then $p_{i-1}\equiv r(p_i)$, $r(p_i)\equiv p_{i+1}\in N(\tau)$ (or we can omit p_{i+1} in the sequence). By the first part of the proof we conclude that

$$E^r\cup\mathcal{N}\vdash r(p_{i-1}) \equiv r(p_i), \quad r(p_i) \equiv r(p_{i+1})$$

and $E^r\cup\mathcal{N}\vdash r(p_{i-1})\equiv p_{i-1}$, $r(p_{i+1})\equiv p_{i+1}$, thus $E^r\cup\mathcal{N}\vdash p_{i-1}\equiv r(p_i)$, $r(p_i)\equiv$ $\equiv p_{i+1}$. If p_{i+1} is not a variable, we consider p_{i-1}, $r(p_i)$, p_{i+1} instead of the subsequence p_{i-1}, p_i, p_{i+1}. Otherwise we omit p_i in the derivation. By induction on i, we can exchange each occurence of a variable in the proof of $p\equiv q$.

We can now assume that $p_1, \ldots, p_s$ is a derivation of $p\equiv q$, such that p_i is different from a variable, for each $i\leqq s$. If $\alpha_i\equiv\beta_i\in\mathrm{Sb}(E)-N(\tau)$ then assume that: $\alpha_i=x_k$ and $\beta_i=f_t(\gamma_1, \ldots, \gamma_n)$ for some $t\in T$. By assumption, p_i is not a variable, thus: $\mathcal{N}\vdash p_i(x_1, \ldots, x_m)\equiv p_i(x_1, \ldots, r(x_k), \ldots, x_m)$ by (t_2), (t_3) and an easy induction on the rank of p_i. Similarly for p_{i+1}. Instead of p_i, p_{i+1} consider $p_i, p_i(x_1, \ldots, \ldots, r(x_k), \ldots, x_m)$, $p_{i+1}(x_1, \ldots, r(x_k), \ldots, x_m), p_{i+1}$ and instead of $\alpha_i \equiv \beta_i$ consider $\alpha_i(x_1, \ldots, r(x_k), \ldots, x_m) \equiv \beta_i(x_1, \ldots, r(x_k), \ldots, x_m) \in \mathrm{Sb}(E) \cap N(\tau)$.

Applying this procedure for each $i<n$ such that $\alpha_i \equiv \beta_i \in \mathrm{Sb}(E) - N(\tau)$ we obtain a proof of $p \equiv q$ from $E^r \cup \mathcal{N}$.

REMARK. If E is not consistent (i.e. $C(E)$ contains an almost contradictory identity $x \equiv y$), consider the set $E' = \{r(x) \equiv r(y)\} \cup \mathcal{N}$. Then for a normal identity $p \equiv q$ of type τ, we obtain: $E' \vdash p \equiv q$. If $T = \emptyset$ then the empty set is a base for $N(E)$.

THEOREM 2. *Assume that E is a set of identities of type τ and e is an identity from the set $C(E) - N(\tau)$. Then $C(E) = C(N(E) \cup \{e\})$.*

PROOF. The inclusion $\supseteq$ is obvious. To show the converse, let us assume that e is an identity of the form $x_k \equiv p(x_1, \ldots, x_n)$, where $p(x_1, \ldots, x_n)$ is a term of type τ. If p is a variable (different from x_k) then the inclusion $\subseteq$ obviously holds. Let us assume that p is not a variable and $x_j \equiv q(x_1, \ldots, x_n)$ is an identity from the set $C(E) - N(\tau)$. If $k = j$ then $p \equiv q$ belongs to $N(E)$ and then $N(E) \cup \{x_k \equiv p\} \vdash x_k \equiv q$. If $k<j$ then let $p^*(x_1, \ldots, x_n)$ denotes the polynomial symbol $p(x_1, \ldots, x_{k-1}, x_j, x_{k+1}, \ldots, x_n)$, obtained from p by replacing x_k by x_j. Then $x_k \equiv p \vdash x_j \equiv p^*$. Now, if q is not a variable, then $p^* \equiv q$ belongs to $N(E)$ and thus $x_k \equiv p$, $x_j \equiv p^*$, $p^* \equiv q$, $x_j \equiv q$ is a proof of $x_j \equiv q$ from the set $N(E) \cup \{e\}$. Otherwise, i.e. when q is a variable y and $y \neq x_j$, then $p(z, \ldots, z) \equiv z$, $p(x_1, \ldots, x_n) \equiv p(z, \ldots, z)$ belong to $E(K)$ for any variable z, different from $x_1, \ldots, x_k, \ldots, x_n$. Thus $x_k \equiv p(x_1, \ldots, x_n)$, $p(x_1, \ldots, x_n) \equiv p(z, \ldots, z)$, $p(z, \ldots, z) \equiv z$, $x_k \equiv z$, $x_j \equiv y$ is a proof of $x_j \equiv q$ from the set $N(E) \cup \{e\}$.

We say that $C(E)$ is finitely based if there exists a finite set E_0 of identities such that $C(E) = C(E_0)$ (see [7], [9]).

Applying Theorem 2, we conclude:

COROLLARY. *If card (T) is finite then $C(E)$ is finitely based if and only if $N(E)$ is finitely based.*

§ 3. Given an algebra $\mathfrak{A}$ of type $\tau\colon T \to \mathbf{N}^+$, $E(\mathfrak{A})$ denotes the set of all identities satisfied in $\mathfrak{A}$, $N(\mathfrak{A}) = E(\mathfrak{A}) \cap N(\tau)$.

Recall, that a variety K is said to have the finite basis property if for any finite algebra $\mathfrak{A} \in K$ the set $E(\mathfrak{A})$ is finitely based (see [6]).

Our next theorem shows that the operator N lifts varieties with the finite basis property into varieties with the same property, if we deal with algebras of finite type.

THEOREM 3. *Given a set E of identities of type $\tau\colon T \to \mathbf{N}^+$, with T finite. Then the variety K_E has the finite basis property if and only if $K_{N(E)}$ has the finite basis property.*

PROOF. Sufficiency follows from the inclusion $K_E \subseteq K_{N(E)}$. If $T = \emptyset$ then the theorem is obvious. Assume that $T \neq \emptyset$ and $\mathfrak{A} = (A, F)$ is a finite algebra from $K_{N(E)}$, where $F = \{f_t\colon t \in T\}$. Applying the theorem of [4] we can assume that r is a mapping from A into A such that $rr(a) = r(a)$ for $a \in A$ and $\mathfrak{B} = (r(A), F)$ is a subalgebra of $\mathfrak{A}$ and $\mathfrak{B} \in K_E$. Moreover, the following equations holds: if $t \in T$,

$\tau(t)=n$ and $a_1, \ldots, a_n \in A$ then

(1) $$f_t(a_1, \ldots, a_n) = f_t(r(a_1), \ldots, r(a_n)),$$

(2) $$r(f_t(a_1, \ldots, a_n)) = f_t(a_1, \ldots, a_n).$$

By induction on the rank of p we can show that for any polynomial symbol p of type τ which is not a variable, the following identities are satisfied in $\mathfrak{A}$:

(1′) $$p(x_1, \ldots, x_n) \equiv p(r(x_1), \ldots, r(x_n)),$$

(2′) $$r(p(x_1, \ldots, x_n)) \equiv p(x_1, \ldots, x_n).$$

We shall show, that one of the conditions below is satisfied in $\mathfrak{A}$:

(i) r is the identity mapping in A (so $E(\mathfrak{A})=E(\mathfrak{B})$);

or

(ii) $E(\mathfrak{A})=N(\mathfrak{B})$.

To prove this let us assume that there exists an element $a \in A$ such that $a \neq r(a)$. Firstly we show that $E(\mathfrak{A}) \subseteq N(\tau)$. Assume the opposite, i.e. let $x \equiv p(x, \ldots, x) \in E(\mathfrak{A}) - N(\tau)$. If p is a variable (different from x) then $\mathfrak{A}$ is trivial and (i) holds. If p is not a variable, then $a=p(a, \ldots, a)$ but the identities $x \equiv p(x, \ldots, x) \equiv p(r(x), \ldots, r(x)) \equiv r(p(x, \ldots, x))$ hold in $\mathfrak{A}$; thus $a=p(a, \ldots, a)=r(p(a, \ldots, a))=r(a)$, a contradiction. On the other hand, for any normal identity $p \equiv q \in N(\mathfrak{B})$ which is not of the form $x=x$ we obtain: $p(x_1, \ldots, x_n) \equiv p(r(x_1), \ldots, r(x_n)) \equiv q(r(x_1), \ldots, r(x_n)) \equiv q(x_1, \ldots, x_n)$ in $\mathfrak{A}$ and thus $p \equiv q \in E(\mathfrak{A})$; i.e. $E(\mathfrak{A})=N(\mathfrak{B})$.

Finally, in the case (i) we conclude that $\mathfrak{A} \in K_E$ so $E(\mathfrak{A})$ is finitely based by assumption. In the case (ii) we have: $E(\mathfrak{A})=N(\mathfrak{B})$, but $\mathfrak{B}$ is finite and $\mathfrak{B} \in K_E$ so $E(\mathfrak{B})$ is finitely based. By our Theorem 1, $E(\mathfrak{A})$ is finitely based.

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APPLICATIONS OF UNIVERSAL ALGEBRA TO COMPUTER SCIENCE

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Abstract

We show how concepts from universal algebra, notably those of free and "Herbrand" algebra, and the notions of (quasi-) varieties of algebras can be applied to formalize and prove properties of programs.

1. Introduction

In this paper we show some applications of universal algebra to theoretical computer science, and more precisely to one branch of it called algebraic semantics [GU]. Universal algebra provides computer scientists nice concepts to organize their thought patterns. Some of the problems in computer science can be expressed within the framework of universal algebra which helps in providing partial answers for them. This in turn usually implies new questions or a different formulation of problems, leading to new problems in universal algebra, etc... and results in a two way communication channel between the two disciplines. Through this paper we shall try to make explicit this duality between computer science and universal algebra concepts.

Algebraic semantics' main goal has been to provide a clean and sound semantics of programming languages by splitting as much as possible the syntactic and semantic parts of a program. Using universal algebra or category theory tools, one can then describe abstractly, i.e. independent of any interpretation, the syntactic properties of a program; after what one is well equipped for, given any concrete interpretation, translating the abstract or syntactic properties, via that interpretation, into concrete or semantic properties of the real program. Moreover, this can be done stepwise, introducing at each step the exactly needed amount of semantic knowledge. Algebraic semantics thus makes easier and more natural the concepts of modularization and abstraction, essential in software development.

In algebraic semantics, interpretations are nothing but certain algebraic systems in the sense of Mal'cev [MA], or equivalently Σ-structures or models in the sense of Grätzer [GR]. The main tool in characterizing the syntax of a program is to have it compute symbolically in a free interpretation, i.e. in an absolutely free algebra of terms: the result of all possible symbolic computations is then represented by an infinite tree (i.e. an infinite term) which characterizes the behaviour of the program

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with respect to all interpretations; Properties of programs are thus described by properties of the associated infinite tree. One of the main problems one has to deal with in proving any program property, is to prove equivalences of programs: this approach shows that two programs are equivalent w.r.t. all possible interpretations iff their associated infinite trees are equal.

However, equivalence w.r.t. all possible interpretations is by far too exacting to be of any real use. In practice one has to take into account some of the constraints or properties verified by the interpretations one is interested in; this extra information should even be modularized according to needs. We thus have to look for alternate syntactic objects and structures for finding and describing the set of all possible symbolic computations under some given constraints. The constraints will be described by a class $\mathscr{C}$ of interpretations, and we will look for a generic — or free, or Herbrand — interpretation H for any given class $\mathscr{C}$, together with effective ways of describing:

1) the free interpretation H

2) the equivalence w.r.t. H; this equivalence will in turn characterize the equivalence w.r.t. the class $\mathscr{C}$. This H corresponds roughly to what is usually called a free $\mathscr{C}$-algebra, or an algebra free relative to the class $\mathscr{C}$ of algebraic systems. The description of this free interpretation H, its elements and its equivalence can then be fruitfully applied to prove various kinds of program properties, tranformations, simplifications, etc...

In algebraic semantics, one can study [GU] a few types of classes $\mathscr{C}$ of interpretations of interest: (in)equational classes (defined by a set of (in)equations), algebraic classes (where, intuitively, any (in)equation between programs can be proved by computation induction), first-order definable classes. According to the various types of classes, we have different methods for describing the interpretation free with respect to that class and its properties which we shall sketch. Finally, in order to illustrate the differences and similarities between those various types of classes, we will single out one of the numerous applications of the characterization of the free interpretation and compare the results obtained in each case; this will be an application to logics of programs: how to deduce from the free interpretation a complete proof system for deriving all valid (in)equations w.r.t. some class $\mathscr{C}$?

The main concepts of universal algebra used in our approach are:

— free (and related) algebraic systems in various classes,

— classes of algebraic systems,

— equational, quasi-equational (or inequational, or relational) and other kinds of classes of algebraic systems.

This paper is primarily a survey written with an intuition minded bias: we give numerous examples and informal explanations, but refer the reader to the literature and mainly [GU] where most of the results here given are proved. For those kinds of applications of universal algebra in computer science which are not surveyed here (e.g. algebraic logic in computer science both the category theoretic and the cylindric algebraic versions, ultraproducts and related constructions in program verification etc.) the reader is referred to the survey series Parts I—V [AN2]. The present paper is organized as follows: Section 2 contains the preliminaries and notations on algebraic semantics; Section 3 introduces the "class of interpretations" approach, and illustrates problems and questions inherent to it; in Section 4 we study equational classes in a more detailed way.

2. Program schemes and semantics: basic results

We will briefly outline in this section the basic results of algebraic semantics. The fundamental idea of algebraic semantics is:

1) using universal algebra tools, characterize a program by a mathematical object which is an element of a universal algebra [GU, GU1, N]. An alternate approach uses category theory instead of universal algebra [ADJ, BG1, BG2, E, M1];

2) use that mathematical object to obtain sound proofs of properties of the program.

2.1. *Basic results*

In order to fix the notation, we will sketch the universal algebra approach in this section. For more details see [D, GU].

Let $F=\{f, g, h, \ldots\}$ (resp. $\Phi=\{G, H, K, \ldots\}$) be a ranked alphabet of base function symbols (resp. of function variables); the rank of a symbol s is denoted by $r(s)$ and F_n(resp. Φ_n) denotes the set of base function symbols (resp. function variables) of rank n. Ω, representing the "undefined", is a special rank 0 symbol in F; F is a signature, or similarity type in terms of universal algebra. Let $V=\{u, v, w, x, y, z, \ldots\}$ be a set of variable symbols (intended to represent parameters or positions) of rank 0.

An *ordered F-algebra* [ADJ], or F-magma [N], or algebraic structure [AN, C, GR], M is an ordered set $(D_M, \leqq_M)$ together with for each f in F_n, a total and monotone mapping f_M: $D_M^n \to D_M$, and such that Ω_M is the least element of D_M. Ordered F-algebras are actually a special kind of algebraic systems of similarity type, or signature, $F\cup\{\leqq\}$.

The class of all ordered F-algebras forms a quasi-variety (i.e. a class axiomatizable by quasi-atomic formulae, i.e. by universally quantified Horn formulae) in the sense of Mal'cev [MA], cf. also [SA]. See also [GR] p. 339, and [AN] (Section 4), [GU, GU2], for quasi equational logic. Since quasi-varieties are epireflective in the category of all algebras with the same signature, all the nice properties of similarity classes of algebraic systems are inherited by ordered F-algebras, too.

An F-magma M is said to be Δ-complete (resp. ω-complete) iff all directed subsets (resp. countable chains) of D_M have a l.u.b. in D_M and the f_M's are continuous, i.e. preserve l.u.b.s of directed sets (resp. countable chains). In the sequel we will consider mainly Δ-complete magmas (the theory is exactly similar for ω-complete ones) and we will call them complete to shorten notations. Whenever we consider a different notion of completeness, this will be mentioned explicitly.

Define the category of (Δ-)complete F-algebras, or F-magmas by: objects are (Δ-)complete F-magmas; morphisms are the continuous homomorphisms, that is, in addition to being an $F\cup\{\leqq\}$ homomorphism they have to preserve lubs of directed sets, too. We shall define $M^\infty(F, V)$ as being the algebraic system freely generated by V in the category of Δ-complete F-algebras. A more precise definition comes later. The category of ω-complete F-algebras can be defined similarly.

It might be of interest to note that it was proved in [PA], cf. also [LP] that the category of ω-complete F-algebras is reducible to a variety of partial algebras in the sense of [AN]. This shows the strong relationship of algebraic semantics with well investigated concepts of universal algebra.

A (complete) $F\cup V$-magma I is said to be *free over generators* V iff for any (complete) $F\cup V$-magma M there exists a unique morphism $\varphi\colon I\to M$ making the following diagram commutative:

$$\begin{tikzcd} V \arrow[r, "i_I"] \arrow[dr, "i_M"'] & I \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

where, for any $F\cup V$-magma J, $i_J(v)=v_J$.

I is also called the free (resp. free complete) F-magma generated by V.

The free and free complete F-magmas exist and can be constructed as follows: the free F-magma generated by V is the set of finite, well formed (with respect to ranks) trees (i.e. terms) on the alphabet $F\cup V$. It is ordered by the least ordering $<$ such that: (a) Ω is the least element, (b) the magma operations are monotone. The free F-magma generated by V is denoted by $M(F, V)$. The free complete F-magma generated by V is the ideal completion of $M(F, V)$ [B] and is denoted by $M^\infty(F, V)$: it can be viewed as the set of all finite and infinite trees (terms) on $F\cup V$; the ordering $<$ extends the ordering on $M(F, V)$ and can be intuitively described by: $T<T'$ for any trees (terms) T, T' iff T' can be deduced from T by replacing Ω's by trees (terms) different from Ω. In the [ADJ] terminology, $M(F, V)$ is denoted by $FT_F(V)$ and $M^\infty(F, V)$ by $CT_F(V)$. See [GU] for more details.

A *recursive program scheme*, in short RPS, on F is a pair (S, t), where S is a system of n equations:

$$S\colon\ G_i(v_1, \dots, v_{n_i}) = t_i \quad i = 1, \dots, n \tag{1}$$

where for $i=1, \dots, n$, $G_i\in\Phi_{n_i}$, $t_i\in M(F\cup\Phi, \{v_1, \dots, v_{n_i}\})$ and t is a tree in $M(F\cup\Phi, V)$.

It is associated with a schematic tree rewriting system (or context free tree grammar [BO, EN, GU, R]), defined by: $G_i(v_1, \dots, v_{n_i})\to t_i+\Omega$, for $i=1, \dots, n$, and which is also denoted by S. Let $L(S, t)=\{t'\mid t'\in M(F, V),\ t \overset{*}{\underset{S}{\Rightarrow}} t'\}$ be the tree language generated by S with axiom t.

It is well-known [GU] that $L(S, t)$ is a directed subset of $M^\infty(F, V)$; let $T(S, t)=\operatorname{lub} L(S, t)$.

A recursive program scheme is *iterative* iff $\Phi=\Phi_0$, i.e. iff all function variables have rank 0. Iterative schemes have been considered in [E, COU, G, GI, NE, PP, T, etc...] and they have been named iterations, regular schemes, rational schemes, etc...

An *interpretation* I of F is a complete F-algebra; a valuated interpretation is an interpretation of $F\cup V$; equivalently it is a pair (I, v) consisting of an interpretation I together with a valuation $v\colon V\to D_I$.

Since any interpretation I is complete, the function computed by an RPS (S, t) with respect to I can be defined as the lub. of the finite computations of (S, t). More formally, suppose t is in $M(F\cup\Phi, \{v_1, \dots, v_n\})$; for any $x_1, \dots, x_n$ in D_I let $v_{x_1\dots x_n}$ be the valuation $V\to D_I$ defined by: for $i=1, \dots, n$, $v_{x_1\dots x_n}(v_i)=x_i$, where $V=\{v_1, \dots, v_n\}$; since $M^\infty(F, V)$ is the free complete F-algebra on generators V, $v_{x_1\dots x_n}$ has a unique extension $v^\infty_{x_1\dots x_n}\colon M^\infty(F, V)\to D_I$; define the function T_I computed by an infinite tree T in $M^\infty(F, V)$ w.r.t. I by: for all $x_1, \dots, x_n$ in D_I,

$T_I(x_1, \dots, x_n)=v^{\infty}_{x_1 \dots x_n}(T)$; define now the function computed by scheme (S, t) w.r.t. I by $(S, t)_I=T(S, t)_I$.

The adequacy of this definition is expressed by the following

THEOREM 1. *Let (S, t) and (S', t') be two RPSs:*
(i) $T(S, t)_I \leqq_I T(S', t')_I$ *for all* I *iff* $T(S, t) < T(S', t')$;
(ii) *for all* I, $T(S, t)_I = \text{lub}\,\{\theta_I / \theta < T(S, t)\}$.

(ii) expresses the fact that the function computed by (S, t) w.r.t. I is defined as a lub of finite computations, by successive approximations; and (i) says that the infinite tree $T(S, t)$ characterizes the behaviour of (S, t) w.r.t. all interpretations, which was goal # 1 stated at the beginning of this section. Introducing some terminology let us lay down the next

DEFINITION 1. An interpretation H is said to be a *Herbrand interpretation* iff: for any T, T' in $M^{\infty}(F, V)$:

$$T_I \leqq_I T'_I \quad \text{for all } I \text{ iff } \quad T_H \leqq_H T'_H.$$

Then Theorem 1 expresses the fact that $M^{\infty}(F, V)$, together with the identity valuation $v(v)=v$, for all v in V, is a Herbrand interpretation.

Notice that, since a program scheme (S, t) is characterized by the associated infinite tree $T(S, t)$, we may w.l.g. study infinite trees instead of program schemes, although not every infinite tree is associated with a program scheme. However, the facts that:

(1) there are much more infinite trees than trees associated with program schemes
(2) it might then be too exacting to require that all directed sets, even those which are not associated with any program scheme, have lubs (in order to ensure completeness)

led some authors to introduce different notions of interpretations. To this end, they define algebras which, though they are imcomplete in the above sense, contain enough lubs to express the functions computed by the programs one is interested in. Let us briefly outline some of these approaches.

We need first recall some terminology from denotational semantics. Note first that if A is any F-algebra (not necessarily ordered or complete), we can define as above a derived operation $t_A(x_1, \dots, x_n)=v_{x_1 \dots x_n}(t)$ for any t in $M(F, V)$ (but not necessarily any t in $M^{\infty}(F, V)$). Now, let A be an F-algebra and S be a recursive program scheme defined by a system of equations (1), let $\mathscr{D}=(D_A^{n_1} \to D_A) \times \dots \times (D_A^{n_n} \to D_A)$ be the set of n-tuples of mappings $D_A^{r_i} \to D_A$, for $i=1, \dots, n$. Any $\vec{g}=(g_1, \dots, g_n)$ in $\mathscr{D}$ defines an $F \cup \Phi$-algebra $\mathrm{A}(\vec{g})$ by: $f_{A(\vec{g})}=f_A$ for f in F and $G_{i_{A(\vec{g})}}=g_i$ for G_i in Φ. Hence, we can associate to each recursive program scheme S a mapping S_A: $\mathscr{D} \to \mathscr{D}$ defined by $S_A(\vec{g})=(t_{1_{A(\vec{g})}}, \dots, t_{n_{A(\vec{g})}})$.

Now, an F-algebra A is said to be *iterative* iff, for any ideal (or proper) iteration S, S_A has a unique fixpoint in $\mathscr{D}$ [E, GI, NE, T].

An ordered F-algebra A is said to be

— *regular* iff, for any iteration S, S_A has a least fixpoint which is defined by lub $\{S_A^n(\Omega_A, \dots, \Omega_A) | n \in N\}$ [G, GPP, PP, T].

— *1-rational* iff, for any recursive program scheme S, S_A has a least fixpoint which is defined by lub $\{S_A^n(\Omega_A, \ldots, \Omega_A)|n\in N\}$ [G].

By allowing for higher type schemes, Gallier also defines n-rational algebras which we will not consider here in order to keep the notations simple.

A subset of D_A of the form $\{S_A^n(\Omega_A, \ldots, \Omega_A)|n\in N\}$ for some iteration (resp. RPS) S, is called a *regular* (resp. an *algebraic*) *subset* of A. Regular subsets are also called *iterations*.

The following is then clear [BG2, G]:

THEOREM 2. *Any complete F-algebra (and any interpretation) is regular and rational.* $M^\infty(F, V)$ *is iterative.*

2.2. *Applications*

By characterizing the semantic behaviour of a program scheme by a syntactic object, its associated infinite tree, Theorem 1 provides us with the corner-stone of algebraic semantics. Let us illustrate this by two very simple, though interesting, applications.

EXAMPLE 1. One can simplify a program scheme by deleting all useless branches. Consider for instance Morris' program: $G(u, v)=$ *if* $u=0$ *then* 0 *else* $G(u-1, G(u, v))$. The underlined occurrence of $G(u, v)$ is clearly a useless loop; this can be recognized very easily by looking at the corresponding program scheme: S: $G(u, v)= =g(u, G(p(u), G(u, v)))$. Its associated infinite tree is:

T(S,G) = 

$T(S, G)$ can also be generated more straightforwardly by S': $G'(u)=g(u, G(p(u)))$. Hence, $T(S, G)=T(S', G')$, and by Theorem 1, $T(S, G)_I=T(S', G')_I$ for all I, and (S', G') is equivalent to (S, G) which we denote by $(S', G')\equiv(S, G)$. Now, there is an easy algorithm realizing the transformation from (S, G) into (S', G'): it suffices to delete the useless variable v from (S, G), and this can be done in a standard way in language theory [GU]. □

Let $\mathcal{I}$ denote the class of all interpretations; then, even though $\mathcal{I}$ is clearly not first-order, Theorem 1 can be viewed as a Birkhoff-like completeness theorem for deriving all valid (in)equations in $\mathcal{I}$. Note first that, in the formalism of logics, $T_I \leqq_I T'_I$ for all I is denoted by $\mathcal{I} \models T \leqq T'$. We then can introduce a set Ax of axioms and deduction rules for inequational logic such that $T < T'$ iff $\mathrm{Ax} \vdash T \leqq T'$. Hence Theorem 1 can be restated as $\mathcal{I} \models T \leqq T'$ iff $\mathrm{Ax} \vdash T \leqq T'$. The set Ax of axioms is defined by the axiom scheme:

(2) $\vdash \Omega \leqq t$ for any t in $M^\infty(F, V)$.

Consider then the following set of deduction rules:

(3) $\vdash t \leqq t$ for any t in $M(F, V)$ (reflexivity);

(4) $t \leqq t'$ and $t' \leqq t'' \vdash t \leqq t''$ for any t, t', t'' in $M(F, V)$ (transitivity);

(5) $t_i \leqq t_i'$ for $i=1, \ldots, r \vdash f(t_1, \ldots, t_r) \leqq f(t_1', \ldots, t_r')$ for any t_i, t_i' in $M(F, V)$, $i=1, \ldots, r$, and f in F_r (monotonicity);

(6) $\forall i \in N \exists j \in N\ t_i \leqq t_j' \vdash \mathrm{lub}\,(t_i) \leqq \mathrm{l.u.b.}\,(t_j')$, for any t_i, t_j' in $M(F, V)$ (algebraicity and continuity).

For t and t' finite trees in $M(F, V)$, let $\vdash t \leqq t'$ iff $t \leqq t'$ is deducible from axiom Ax using deduction rules (3)—(5); then clearly: $t \leqq t'$ iff $\vdash t \leqq t'$.

For T and T' (possibly infinite) trees in $M^\infty(F, V)$, it can be shown that it is necessary and sufficient to add the induction rule (6); let $\overset{\infty}{\vdash} T \leqq T'$ iff $T \leqq T'$ is deducible from axiom Ax using deduction rules (3)—(6).

Theorem 1 can thus be translated into two completeness theorems for finite and infinite trees:

for t, t' in $M(F, V)$, $\mathscr{I} \models t \leqq t'$ iff $\mathrm{Ax} \vdash t \leqq t'$

for T, T' in $M^\infty(F, V)$, $\mathscr{I} \models T \leqq T'$ iff $\mathrm{Ax} \overset{\infty}{\vdash} T \leqq T'$.

This makes clear how algebraic semantics can be applied to yield results in more model theoretic or logic minded approaches as in [BL, BT, G]. This connection between algebraic semantics and logics will be further investigated in the subsequent sections.

Remark finally that, since $\mathscr{I} \models T = T'$ iff $\mathscr{I} \models T \leqq T'$ and $\mathscr{I} \models T' \leqq T$, the above also provides us with a complete proof system for equational logic; however, this proof system does not immediately translate into a proof system using the deduction rules of equational logic [B, BT] and we will see a sample of the difficulties involved in Section 4.

3. Classes of interpretations

We showed in the previous section that the characterization of a program scheme (S, t) by an infinite tree, which is the function computed by (S, t) in a Herbrand interpretation, can be rewarding. However, this approach is usually far too general: in practice, one never considers all interpretations $\mathscr{I}$, but rather subclasses of $\mathscr{I}$, where, e.g. some operation is associative and commutative, or some base function symbol is always interpreted as an "*if* ... *then* ... *else* ...". Hence, if one wants to come any closer to real programs, one has to take into account some constraints on interpretations. For instance, equivalence with respect to all interpretations is far too exacting as is illustrated by the following

Example 2. Let S and S' be program schemes defined by

S: $G(v) = g(v, g(v, G(v)))$

S': $G'(v) = g(v, \Omega)$.

Then $(S, G(v))$ has no useless branch and is strictly larger than $(s', G'(v))$ since the former generates an infinite tree whereas the latter only generates $g(v, \Omega)$. However, if we restrict our attention to interpretations where g_I has the following form: $g_I(x, y) =$ *if* $p(x)$ *then* x *else* y, with $p(x)$ some predicate on D_I, then $(S, G(v))$ can be simplified into $(S', G'(v))$. More formally, let $\mathscr{C}$ be the subclass of $\mathscr{I}$ defined by $\mathscr{C} = \{I/g_I(x, g_I(x, y)) = g_I(x, y)$ for any x, y in $D_I\}$; then, for any I in $\mathscr{C}$: $T(S, G(v))_I = T(S', G'(v))_I$. □

Formally, let us define:

Definition 2. A *class* $\mathscr{C}$ of interpretations is a subclass of $\mathscr{I}$. For $\mathscr{C}$ a class of interpretations and T, T' in $M^\infty(F, V)$, define

$$T \leqq_{\mathscr{C}} T' \quad \text{iff for any } I \text{ in } \mathscr{C}: T_I \leqq_I T'_I$$

$$T \equiv_{\mathscr{C}} T' \quad \text{iff for any } I \text{ in } \mathscr{C}: T_I = T'_I.$$

For $\mathscr{C}$ and $\mathscr{C}'$ two classes of interpretations, define

$$\mathscr{C} < \mathscr{C}' \quad \text{iff} \quad \leqq_{\mathscr{C}'} \subseteq \leqq_{\mathscr{C}}$$

$$\mathscr{C} \ll \mathscr{C}' \quad \text{iff} \quad \equiv_{\mathscr{C}'} \subseteq \equiv_{\mathscr{C}}$$

$$\mathscr{C} \sim \mathscr{C}' \quad \text{iff} \quad \leqq_{\mathscr{C}'} = \leqq_{\mathscr{C}}$$

$$\mathscr{C} \approx \mathscr{C}' \quad \text{iff} \quad \equiv_{\mathscr{C}'} = \equiv_{\mathscr{C}}. \quad \square$$

Note that, clearly, $\mathscr{C} < \mathscr{C}' \Rightarrow \mathscr{C} \ll \mathscr{C}'$ and $\mathscr{C} \sim \mathscr{C}' \Rightarrow \mathscr{C} \approx \mathscr{C}'$, counterexamples for reverse inclusions will be shown later (see Theorem 8 below and [GM]). Note also that $\mathscr{C} \subset \mathscr{C}' \Rightarrow \mathscr{C} < \mathscr{C}'$.

We will now try to generalize the approach of Section 2 to classes of interpretations: namely, given a class $\mathscr{C}$ of interpretations, we first have to characterize the behaviour of a program scheme w.r.t. all interpretations in $\mathscr{C}$ by the function it computes in some Herbrand interpretation. We will show now that this is much less straightforward than in the previous completely free case: we have to accept a trade-off between a better modelling of reality versus an increased complexity of proofs and results. We need one more definition:

Definition 3. Let $\mathscr{C}$ be a class of interpretations and I an interpretation; I is said to be *$\mathscr{C}$-Herbrand* iff $\mathscr{C} \sim \{I\}$. A valuated interpretation (I, v) is said to be *$\mathscr{C}$-free* (over generators V) iff for any I' in $\mathscr{C}$ and valuation v': $V \to D_{I'}$ there exists a unique morphism φ: $I \to I'$ such that $\varphi(v(v)) = v'(v)$ for any v in V (i.e. the restriction of φ to $v(V)$ coincides with v').

Remark 1. a) We are implicitly considering classes of non-valuated interpretations (or functional interpretations). Valuated interpretations can be treated similarly [GU].

b) Note that our definitions of $\mathscr{C}$-free and $\mathscr{C}$-Herbrand are slightly different from the classical notions of universal or free object in universal algebra or category theory [AN, C, GR, MA]: we do not require that the $\mathscr{C}$-free or $\mathscr{C}$-Herbrand interpretation belong to the class $\mathscr{C}$. As a result, it may indeed be the case that none of them belongs to $\mathscr{C}$; for instance, $M^\infty(F, V)$ is $\mathscr{C}$-free for any $\mathscr{C}$. We will see later various examples of $\mathscr{C}$-Herbrand interpretations I which are not in $\mathscr{C}$ (§ 4.2). Note, however, that, whenever a $\mathscr{C}$-free interpretation belongs to $\mathscr{C}$, then (i) it is also $\mathscr{C}$-

Herbrand, (ii) it is the unique $\mathscr{C}$-free interpretation belonging to $\mathscr{C}$ (up to isomorphism).

c) We can also ensure uniqueness of the free interpretation by postulating the following stronger definition [AN].

DEFINITION 3.1. Let $\subseteq$ be the preordering defined on valuated interpretations by: $(I, v)\subseteq(I', v')$ iff there exists a unique morphism φ: $D_I \to D_{I'}$ such that: for any v in V $\varphi(v(v))=v'(v)$. Then (I_0, v_0) is said to be *strongly $\mathscr{C}$-free* iff (i) it is $\mathscr{C}$-free, (ii) for any $\mathscr{C}$-free interpretation (I, v), $(I, v)\subseteq(I_0, v_0)$ i.e. (I_0, v_0) is the largest $\mathscr{C}$-free interpretation w.r.t. the preordering $\subseteq$.

d) Note finally that the notion of Herbrand interpretation can be extended for arbitrary atomic formulae.

DEFINITION 3.2. An interpretation I is said to be *strongly $\mathscr{C}$-Herbrand* iff for any atomic formulae Φ: $I\models\Phi\leftrightarrow\mathscr{C}\models\Phi$. □

This concept, even though it was never given a name, has been investigated in universal algebra (see e.g. [AN]).

Constructing $\mathscr{C}$-free and $\mathscr{C}$-Herbrand interpretations.

For completeness sake, let us state the following

PROPOSITION 1. For any class $\mathscr{C}$ of interpretations, there exists a $\mathscr{C}$-free and a $\mathscr{C}$-Herbrand interpretation.

However, the proof of this result is highly non-constructive: it consists in taking some suitable subclass of the infinite product of all interpretations in $\mathscr{C}$. Hence this result is of no help and we have to find alternate characterizations of the $\mathscr{C}$-free and $\mathscr{C}$-Herbrand interpretation. Let us give first some terminology about preorderings.

DEFINITION 4. A *preordering* on an ordered F-magma M is a reflexive and transitive relation π containing $\leqq_M$ and which is compatible with the F-algebra structure of M, namely: for any f in F_r, d_i, d_i' in D_M, for $i=1, \ldots, r$, $d_i\pi d_i'$ imply $f_M(d_1, \ldots, d_r)\pi f_M(d_1', \ldots, d_r')$. π is said to be *continuous* iff for any directed set E in D_M, and any d' in D_M, $e\pi d'$ for any e in E, imply lub $\{e/e$ in $E\}\pi d'$. When $M=M^\infty(F, V)$ with the syntactic ordering $<\pi$ is said to be

— *substitution-closed* if or any t, t' in $M(F, \{v_1, \ldots, v_n\})$ and any $T_1, \ldots, T_n$ in $M^\infty(F, V)$, $t\pi t'$ implies $t(T_1/v_1, \ldots, T_n/v_n)\pi t'(T_1/v_1, \ldots, T_n/v_n)$. Equivalently, one may require that for any endomorphism h: $M^\infty(F, V)$, $t\pi t'$ implies $h(t)\pi h(t')$.

— *algebraic* iff for any T, T' in $M^\infty(F, V)$, $T\pi T'$ implies: for any $t<T$, t in $M(F, V)$, there exists some $t'<T'$, t' in $M(F, V)$, such that $t\pi t'$.

For a preordering π on $M^\infty(F, V)$ let $\mathscr{C}_\pi=\{I/\pi\subseteq\equiv_{\{I\}}\}$. We can now state

PROPOSITION 2 [GU]. *$\mathscr{C}\to\leqq_{\mathscr{C}}$ and $\pi\to\mathscr{C}_\pi$ define a Galois isomorphism between classes of interpretations and continuous, substitution-closed preorderings on $M^\infty(F, V)$.*

One might then expect to use the above Galois bijection in order to get somehow more tractable characterizations of the $\mathscr{C}_\pi$-free and Herbrand interpretations. This goal can be only partially fulfilled, as will be shown by the sequel.

For any ordered F-magma M, and preordering π on M, let:
— M^∞ denote the ideal completion of M [B, CR, GU]
— M/π denote the F-magma obtained by factoring M through the equivalence $\bar{\pi}=\pi\cap\pi^{-1}$ associated with π, and ordering the factor algebra by $\pi/\bar{\pi}$. The equivalence class of an element t of M modulo $\bar{\pi}$ will be denoted by $[t]_\pi$.

Let π be a continuous and substitution closed preordering on $M^\infty(F, V)$. We now try to find $\mathscr{C}_\pi$-free and/or $\mathscr{C}_\pi$-Herbrand interpretations. The most natural choice would be $I=M^\infty(F, V)/\pi$.

However, I is usually neither $\mathscr{C}_\pi$-free nor $\mathscr{C}_\pi$-Herbrand. The most immediate reason for that is that I, being usually not complete, is not large enough and does not contain enough lubs. However, there are more subtle causes for which I cannot be $\mathscr{C}_\pi$-free or Herbrand: even when I is complete, lubs might be there just by chance and might not be the right lubs. Hence:

1) the operations on I may be non continuous
2) even when the operations on I are continuous, I may not be $\mathscr{C}_\pi$-free because the unique mapping $\varphi\colon I\to I'$ into an I' in $\mathscr{C}_\pi$ may be not continuous, hence φ will not be a morphism.

This will be made clearer by the following example:

Example 2. We will show here continuous, substitution-closed and algebraic preorderings π such that $M^\infty(F, V)/\pi$ is neither $\mathscr{C}_\pi$-Herbrand nor $\mathscr{C}_\pi$-free.

Let $F=\{\Omega, a, b, c, h\}$ $r(\Omega)=r(a)=r(b)=r(c)=0$ $r(h)=1$. Define π as being the preordering generated by: $a\pi b$, $a\pi h(a)$, $b\pi h(b)$, $c\pi h(c)$, $h(c)\pi c$ $h^n(a)\pi b$, for all n, and $b\pi c$. Then $I=M^\infty(F)/\pi$ is complete: lub $\{h^n(a)_I/n\in N\}=b_I$ and lub $\{h^n(b)_I/n\in N\}=c_I$, and $h^r(c)\bar{\pi}c$ for any n in N.

However h_I is not continuous since $h_I(b_I)=[h(b)]_\pi=h_I(\operatorname{lub}_n \{h^n(a)_I\}\neq$ $\neq\operatorname{lub}_n \{h^{n+1}(a)_I\}=[b]_\pi=b_I$. Hence I is not even an interpretation.

Now, even when the operations are continuous, I may still be not $\mathscr{C}_\pi$-free. Let π' be the (continuous, algebraic, substitution-closed) preordering generated by π together with the relation $h(b)\pi'b$. Then $I'=M^\infty(F)/\pi'$ is clearly a complete F-magma, with continuous operations. But $b_{I'}$ happens to be the lub of the $h^n(a)_{I'}s$ by mere chance, and this results in I' being not $\mathscr{C}_{\pi'}$-free. For instance, the unique $\varphi\colon I'\to I''=(M(F)/\pi)^\infty$ is clearly not continuous.

This example shows that neither $M^\infty(F, V)/\pi$ nor $(M^\infty(F, V)/\pi)^\infty$ can be $\mathscr{C}_\pi$-free or $\mathscr{C}_\pi$-Herbrand in general, even for a very smooth choice of π. We nevertheless can state

Proposition 3. *Let π be a preordering on $M^\infty(F, V)$; then $(M(F, V)/\pi)^\infty$ is $\mathscr{C}_\pi$-free.* □

Example 3. Note that, except when π is algebraic (cf. Theorem 3) $(M(F, V)/\pi)^\infty$ is usually not $\mathscr{C}_\pi$-Herbrand. Let $F=\{\Omega, h_1, h_2\}$ with $r(h_1)=r(h_2)=1$. Let $T_i=\operatorname{lub}_n \{h_i^n(\Omega)\}$, for $i=1, 2$, and π be defined by: $T_1\pi T_2$. Then, clearly $M(F)/\pi=$ $=M(F)$ and $(M(F)/\pi)^\infty=M^\infty(F)$ is not $\mathscr{C}_\pi$-Herbrand.

The previous example amply illustrate that the standard completion by ideals method cannot give us a $\mathscr{C}$-Herbrand interpretation. However, for any continuous and substitution-closed preorder π, one can construct a $\mathscr{C}_\pi$-free and $\mathscr{C}_\pi$-Herbrand interpretation by a more refined completion method: one has to perform a "continuous" completion preserving lubs of algebraic subsets of $M^{\infty}(F, V)/\pi$; by transfinite induction, one than gets the required $\mathscr{C}_\pi$-Herbrand interpretation [CR, GU, M1] thus yielding

PROPOSITION 4. *For any continuous and substitution-closed preordering, one can construct a $\mathscr{C}_\pi$-free and $\mathscr{C}_\pi$-Herbrand interpretation.*

This construction by transfinite induction is, however, only very slightly more effective than the bare existential result of Proposition 1. In particular, this construction can lead neither to a nice characterization of the function computed by an RPS w.r.t. a class $\mathscr{C}$, nor to the faintest hope of getting a complete proof system for deducing valid inequations $t \leqq_{\mathscr{C}} t'$, nor even to some characterization of $\leqq_{\mathscr{C}}$. Hence, in order to get more manageable results, we will have to consider somehow more specific classes of interpretations.

Note that most of the problems in constructing a $\mathscr{C}$-Herbrand and/or $\mathscr{C}$-free interpretation stemmed from completeness and continuity. Hence considering *classes of X-interpretations,* where X is intended to be replaced by iterative, regular or 1-rational, one would expect to cancel some of these problems, since completeness and continuity are replaced by weaker conditions that only those lubs one effectively wants to compute should exist and be preserved by operations. This is indeed partly the case. See [G, GU2, GPP] for more details.

4. Equational and relational classes of interpretations

4.1. *Relational classes*

DEFINITION 5. A class of interpretations is said to be *relational* iff it is of the form $\mathscr{C}_R = \{I/R \subseteq \leqq_{(I)}\}$ for some binary relation $R \subseteq M(F, V) \times M(F, V)$. If R is an equivalence relation, $\mathscr{C}_R$ is said to be *equational.*

Relational classes are called algebraic varietal in [M2], varieties in [BL], and semi-varieties in [G], who considers classes of rational algebras, defined by some relation R possibly involving infinite trees. Relational classes are those classes definable by a set of inequations between finite terms; they are the most tractable classes of interpretations: for a relational class $\mathscr{C}$, we will get an easy characterization of $\leqq_{\mathscr{C}}$ restricted to finite trees, show that $\mathscr{C}$ is algebraic, hence obtain a complete deduction system within inequational logic to prove all valid inequations in $\mathscr{C}$ (cf. Sections 2.2 and 3.2).

Let us state first a Birkhoff theorem characterizing relational classes in terms of closure operations [M2].

THEOREM 3. *$\mathscr{C}$ is relational iff it is closed under products, continuous subalgebras and factor algebras, and ideal completions.*

A slightly simpler version of this theorem is proved in [G] for semi-varieties of rational algebras and in [BL] for varieties of ordered algebras. In this connection cf. also [SA].

For a relational class $\mathscr{C}_R$ define $<_R$ as the least substitution-closed preordering containing R on $M(F, V)$. Then, clearly, $\mathscr{C}_R = \mathscr{C}_{<_R}$. Finally, let us abbreviate $\leqq_{\mathscr{C}_R}$ by $\leqq_R$ and $\equiv_{\mathscr{C}_R}$ by $=_R$.

We can now state the main theorem of this section [GU].

THEOREM 4. *Let R be a binary relation on $M(F, V)$ and $H_R = (M(F, V)/<_R)^\infty$, then:*

(i) *H_R is $\mathscr{C}_R$-free and $\mathscr{C}_R$-Herbrand;*
(ii) *$\leqq^f_R = <_R$ and $\leqq_R = <^a_R$;*
(iii) *$\mathscr{C}_R$ is an algebraic class.* □

The importance of this theorem can be illustrated by its consequences. We simply state two of them.

COROLLARY 1. *A class $\mathscr{C}$ is algebraic iff it is equivalent to some relational class $\mathscr{C}_R$, i.e. iff $\mathscr{C} \sim \mathscr{C}_R$ (cf. Definition 2).* □

Recall that, as noted in Section 2.2,

$t \leqq_R t'$ is equivalent to $\mathscr{C}_R \models t \leqq t'$;

$t <^a_R t'$ amounts to saying that $t \leqq t'$ can be deduced from $(R \cup <)$ using the deduction rules (3)—(5) of inequational logic, completed by induction rule (6) and rule (7) expressing substitution closure: $t \leqq t' \vdash \varphi(t) \leqq \varphi(t')$ for any t, t' in $M(F, V)$ and endomorphism φ: $M^\infty(F, V) \to M^\infty(F, V)$.

Hence the statement $\leqq_R = <^a_R$ in (ii) of the above theorem can be restated as the following completeness theorem:

COROLLARY 2 (completeness theorem). *Let* Ax_R *be the axiom system defined by: $\vdash t \leqq t'$ for any t, t' in $M(F, V)$ such that (t, t') is in R, and $\vdash \Omega \leqq T$ for any T in $M^\infty(F, V)$.*

Then, for any t, t' in $M^\infty(F, V)$, $\mathscr{C}_R \models t \leqq t'$ iff $\mathrm{Ax}_R \overset{\infty}{\vdash} t \leqq t'$ *using the deduction rules* (3)—(7). □

One can obtain similar results when considering varieties (or relational classes) of regular [T], rational [G] or recursive [BG1] algebras, see [GU2]. In the case of varieties of iterative algebras though, the existence and construction of a free or Herbrand interpretation for a variety of iterative algebras is more problematic [CO, GPP, PP].

Theorem 4 has many other applications which we merely list here for lack of space:

— proofs of program properties (with or without induction) and program equivalences [G, GU, BK].
— program simplifications and transformations w.r.t. classes of interpretations and correctness proofs of such transformations [CO, EG, GU, K].

4.2. *Application to "if ... then ... else ..."*

We will however detail some more the study of classes of interpretations where a given base function symbol is interpreted as a test — i.e. an "*if ... then ... else ...*". This will give some clues about how relational and equational classes can help in studying non relational ones.

We will start by recalling a result from [BT]. Let $B=\{\Omega, tt, ff, g\}$ where $r(tt)=$ $=r(ff)=0$ and $r(g)=3$. tt and ff are intended to be interpreted as the constants "true" and "false" in any (not necessarily ordered) B-algebra I; we will thus say that g_I is a *test* iff it satisfies:

$$(8) \qquad \text{for any} \quad x, y, z, \quad \text{in} \quad D_I\, g_I(x, y, z) = \begin{cases} y & \text{if} \quad x=tt_I \\ z & \text{if} \quad x=ff_I \\ \Omega_I & \text{otherwise.} \end{cases}$$

Let $\mathscr{K}=\{I/I$ is a B-algebra satisfying (8)$\}$. It is well-known that $\mathscr{K}$, being not closed under products, is neither equational nor relational. However, the following completeness theorem has been shown in [BT]:

THEOREM 5. *For any t, t' in $M(B, V)$, $\mathscr{K}\models t\equiv t'$ iff $A\mid_{\overline{E}} t\equiv t'$, where — A is the following set of equational axioms, in which $g(x, y, z)$ has been abbreviated into $[x, y, z]$:*

$$A \quad \begin{array}{ll} [tt, x, y] \equiv x & [x, x, y] \equiv [x, tt, y] \\ [ff, x, y] \equiv y & [x, y, x] \equiv [x, y, ff] \\ [\Omega, x, y] \equiv \Omega & [x, \Omega, \Omega] \equiv \Omega \\ [x, [x, y, z], w] \equiv [x, y, w] & [x, y, [x, z, w]] \equiv [x, y, w] \\ \multicolumn{2}{l}{[x, [y, z, u], [y, v, w]] \equiv [y, [x, z, v], [x, u, w]]} \\ \multicolumn{2}{l}{[[x, y, z], u, v] \equiv [x, [y, u, v], [z, u, v]];} \end{array}$$

$-A\mid_{\overline{E}} t\equiv t'$ means: $t\equiv t'$ is deducible from A using the deduction rules of equational logic [B, BT], *or, equivalently, $t\equiv t'$ is in the least compatible and substitution closed congruence containing A, which we also denote by $\equiv_A$.* □

Intuitively, and somehow incorrectly, this means that $H=M(B, V)/\equiv_A$ is both "$\mathscr{K}$ and $\mathscr{A}$-Herbrand", where $\mathscr{A}=\{I/I$ is a B-algebra satisfying $A\}$. Note that H is not in $\mathscr{K}$ since $\mathscr{K}$ is not equational; hence $\mathscr{K}\subsetneq\mathscr{A}$ and $\mathscr{K}\approx\mathscr{A}$, and this is an example of a $\mathscr{K}$-Herbrand algebra which does not belong to $\mathscr{K}$.

Note that Theorem 5 was obtained assuming two restrictive hypotheses, namely that one deals with unordered algebras, and that the unique operation allowed in the signature is the test g. We can now extend this result to complete F-algebras whose signature F contains base function symbols F' other than g, provided those new symbols in F' represent strict operations: this is a not too stringent restriction since, in practice (see for instance abstract data types), the only base function symbol which is assumed to be non strict, and needs to be such, is the test g.

Let now $F=B\cup F'$, for some set F' of base function symbols, and let C be the following set of axioms:

for any f in F, $n\geqq 1$, and $i=1, \dots, n$:

$$C \begin{cases} f(x_1, \dots, x_{i-1}, \Omega, x_{i+1}, \dots, x_n) \equiv \Omega \\ f(x_1, \dots, x_{i-1}, [u, v, w], x_{i+1}, \dots, x_n) \equiv \\ \equiv [u, f(x_1, \dots, x_{i-1}, v, x_{i+1}, \dots, x_n), f(x_1, \dots, x_{i-1}, w, x_{i+1}, \dots, x_n)]. \end{cases}$$

Let $A' = A \cup C$ and define:
$\mathscr{A}' = \{I/I$ is an F-algebra satisfying $A'\} = \{I/I \models A'\}$
$\mathscr{A}'_0 = \mathscr{A}' \cap \mathscr{I}\{I/I$ is a complete F-algebra satisfying $A'\}$
$\mathscr{A}'_d = \mathscr{A}'_0 \cap \mathscr{D} = \{I/I$ is a discrete interpretation of F satisfying $A'\}$.
And define similarly
$\mathscr{K}^s = \mathscr{K} \cap \{I/I$ is an F-algebra where, for any f in F'_n, $n \geq 1$, f_I is strict$\}$
$= \mathscr{K} \cap \mathscr{A}'$ (recall an operation f_I is strict iff it yields the result Ω_I whenever one of its arguments is Ω_I) $\mathscr{K}^s_0 = \mathscr{K}^s \cap \mathscr{I}$ and $\mathscr{K}^s_d = \mathscr{K}^s_0 \cap \mathscr{D}$.

Then, extending in an obvious way the relations $\equiv_{\mathscr{C}}$ and $\approx$ of Definition 2 to classes of algebras which are not necessarily complete, one obtains the following [GM]:

THEOREM 6.

(i) $$\mathscr{A}' \approx \mathscr{A}'_0 \approx \mathscr{A}'_d \approx \mathscr{K}^s \approx \mathscr{K}^s_0 \approx \mathscr{K}^s_d;$$

(ii) $$\mathscr{K}^s_0 \sim \mathscr{K}^s_d, \mathscr{A}'_0 \sim \mathscr{A}'_d;$$

(iii) $$\mathscr{K}^s_0 \lneqq \mathscr{A}'_0, \mathscr{K}^s_d \lneqq \mathscr{A}'_d. \quad \square$$

This theorem has several consequences. It first yields a nice characterization of the equivalences $\equiv_{\mathscr{C}}$, for $\mathscr{C}$ in $\{\mathscr{K}^s, \mathscr{K}^s_0, \mathscr{K}^s_d, \mathscr{A}'_0, \mathscr{A}'_d\}$: by (i) those equivalences coincide with $\equiv_{\mathscr{A}'}$, which is by Birkhoff's theorem, the congruence $\equiv_{A'}$ generated by A'. Note that, here, $\equiv_{\mathscr{A}'_0}$ also coincides with $\equiv_{A'}$, whereas in general, $\equiv_{\mathscr{C}_R} = \,<_R \cap <_R^{-1} \supsetneqq \equiv_R$ (note that in the present subsection we are dealing with finite trees only).

Hence we can state

COROLLARY 1. *For $\mathscr{C}$ in $\{\mathscr{A}'_0, \mathscr{A}'_d, \mathscr{K}^s, \mathscr{K}^s_0, \mathscr{K}^s_d\}$, $\equiv_{\mathscr{C}}$ is the compatible substitution-closed congruence generated by A'.*

COROLLARY 2. *$\mathscr{K}^s_d$ is not $\mathscr{D}$-equational* [GU].

This stems from: $\mathscr{K}^s_d \approx \mathscr{A}'_d$ but $(\mathscr{K}^s_d \not\sim \mathscr{A}'_d)$: by the first equivalence, the only possible congruence is $\equiv_{A'}$, which is excluded by the second inequivalence. $\square$

Similar results hold slightly modifying B in order to obtain an equality test; let $B' = \{\Omega, g'\}$, $r(g') = 4$, and let $\mathscr{K}' = \{I/I$ is a B'-algebra satisfying (9)$\}$, where for any x, y, u, v in D_I

$$g'_I(x, y, u, v) = \begin{cases} \Omega_I & \text{if} \quad x = \Omega_I \quad \text{or} \quad y = \Omega_I \\ u & \text{if} \quad x = y \neq \Omega_I \\ v & \text{if} \quad \Omega_I \neq x \neq y \neq \Omega_I. \end{cases} \tag{9}$$

Then there exists an axiom system $\bar{A}$ [BT] such that for any t, t' in $M(B', V)$: $\mathscr{K}' \models t \equiv t'$ iff $\bar{A} \mid\!\!\frac{}{E}\, t \equiv t'$. Letting $\bar{A}' = \bar{A} \cup \bar{C}$, where $\bar{C} = C \cup \{(a), (b)\}$, with, for

f in F'_n, $n \geqq 1$, and $i=1, \ldots, n$:

(a)

$$g(u, v, f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n), y) \equiv g(u, v, f(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n), y)$$

(b)

$$g(u, v, g(f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n), y, z, z'), w) \equiv$$
$$\equiv g(u, v, g(f(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n), y, z, z'), w).$$

The proofs and results of the previous case easily go through (see [GM] for more details).

Other classes of interpretations, e.g. algebraic, first-order, ..., meaningful from the computer science standpoint, can also be fruitfully investigated, see [GU, GU2].

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SOME REMARKS ON THE TWO DISCRIMINATORS

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1. Introduction

A variety V is a discriminator variety if there is a ternary term $t(x, y, z)$ of V such that for subdirectly irreducible (SI) $S \in V$ and $x, y, z \in S$,

$$\begin{aligned} t(x, y, z) &= z \quad \text{if} \quad x = y, \\ &= x \quad \text{if} \quad x \neq y. \end{aligned} \tag{1.1}$$

V is a dual discriminator variety if there is a ternary term $d(x, y, z)$ of V such that for SI $S \in V$ and $x, y, z \in S$,

$$\begin{aligned} d(x, y, z) &= x \quad \text{if} \quad x = y, \\ &= z \quad \text{if} \quad x \neq y. \end{aligned} \tag{1.2}$$

In the decade since their introduction in [9] discriminator varieties, and more recently dual discriminator varieties, introduced in [5], have played key roles in several areas of universal algebra, e.g.: in understanding the structure, spectra, and decidability of certain varieties. (For examples see [3], [8], [11].)

Discriminator varieties and dual discriminator varieties have several elementary properties which we briefly recall. First, both types of varieties, are, by (1.1) and (1.2), both semi-simple (non-trivial SI algebras are simple) and sub semi-simple (non-trivial subalgebras of simple algebras are simple). Next, a discriminator variety is necessarily arithmetical (i.e.: both CD-congruence distributive and CP-congruence permutable), since a discriminator term t must clearly satisfy the Mal'cev equations

$$t(x, x, z) = t(z, x, x) = t(z, x, z) = z \tag{1.3}$$

characterizing arithmetricity ([9]). On the other hand a dual discriminator term d obviously satisfies the ternary majority equations

$$d(x, x, z) = d(z, x, x) = d(x, z, x) = x \tag{1.4}$$

which imply that the variety is CD (and in the strongest way in which a variety can be CD — see [7]). Since the lattice median term

$$m(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

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induces the dual discriminator on the two element lattice, the variety of distributive lattices is a dual discriminator variety but, lacking CP, is not a discriminator variety. However if t is a discriminator term and if $f(x, y, z)$ is any ternary function satisfying Mal'cev's identities for CP,

$$f(x, x, z) = f(z, x, x) = z,$$

(e.g. if f is t), then $f(x, t(x, y, z), z)$ is the dual discriminator. Hence every discriminator variety is a dual discriminator variety but not conversely, and the difference is precisely congruence permutability:

LEMMA 1-1. *A dual discriminator variety is a discriminator variety iff it is CP.*

In spite of the wide gulf — congruence permutability — separating these two kinds of varieties, the definitions (1.1) and (1.2) suggest that they are, none-the-less closely related and, more precisely, that properties of discriminator varieties might generally be obtainable from more general properties of dual discriminator varieties by imposing congruence permutability. This has, in fact, already turned out to be the case in [12] where "Stone" duality for discriminator varieties becomes a special case of the more general "Priestly" duality for dual discriminator varieties. The purpose of the present note is to continue this theme by giving two characterizations of dual discriminator varieties and obtaining, as easy corollaries, corresponding characterizations of discriminator varieties by imposing CP. In Section 2 we shall do this for general varieties and in Section 3 examine the special case of locally finite varieties which are semi-simple.

2. General varieties

An algebra A has the PCC property (principal congruences are complemented) if each principal congruence $\theta(x, y) \in \mathrm{Con}(A)$ has a complement $\theta' \in \mathrm{Con}(A)$. A has the PCI (principal congruence intersection) property if whenever a principal congruence $\theta(a, b) \in \mathrm{Con}(A)$ has a complement $\theta' \in \mathrm{Con}(A)$, then for all $x \in A$.

$$[a]\theta(a, b) \cap [x]\theta' \neq \emptyset,$$

i.e.: the congruence class of $\theta(a, b)$ containing a intersects every θ' congruence class. A variety has the PCC or PCI property if each algebra in the variety has the corresponding property.

The PCC property is discussed by Fried and Kiss in [4] where it is shown that V is a discriminator variety iff V is arithmetical and has the PCC property. We shall obtain this result as Corollary D below. The PCI property is clearly implied by congruence permutability. That it is strictly weaker follows from the fact that every dual discriminator variety has the PCI property. To see this recall (from [5], page 91, proof of Theorem 3.8) that if V is a dual discriminator variety then for $a, b \in A \in V$, the relation

$$\gamma(a, b) = \{(u, v) \colon d(a, b, u) = d(a, b, v)\}$$

is a congruence relation (called co-principal) and is the complement of the principal congruence $\theta(a, b)$; in particular

$$a = d(a, a, x)\theta(a, b)\, d(a, b, x)\gamma(a, b)x$$

for all $x \in A$. Hence V is PCI. Using this observation we can formulate our characterization of dual discriminator varieties as follows:

THEOREM DD. *For a variety V the following are equivalent:*
1) *V is a dual discriminator variety.*
2) *V has the properties:*
   i) *CD*, ii) *PCC*, iii) *PCI*.

PROOF. 1)→2) is clear from the remarks above. To prove 2)→1) let $F = F_V(x, y, z, v_0, v_1, \ldots)$ be the free algebra of V with the denumerable set $\{x, y, z, v_0, v_1, \ldots\}$ of free generators. By ii) the principal congruence $\theta(x, y)$ has a complement θ' and by iii) there is a term, say $t(x, y, z, v_0, \ldots, v_{m-1})$ in F such that

$$x\theta(x, y)t(x, y, z, v_0, \ldots, v_{m-1})\theta' z.$$

On the subalgebra F' of F generated by $\{x, z, v_0, \ldots\}$, $\theta(x, y) = \omega$ and hence $x = t(x, x, v_0, \ldots, v_{m-1})$ in F'. Since $\{x, z, v_0, \ldots\}$ are free generators of F' it follows that

$$x = t(x, x, z, v_0, \ldots, v_{m-1}) \tag{2.1}$$

is an equation of V.

To complete the proof it will suffice to show that for any SI S in V and $a, b, c \in S$ with $a \neq b$,

$$t(a, b, c, c, \ldots, c) = c, \tag{2.2}$$

for (2.1) and (2.2) together will show that the term $t_1(x, y, z) = t(x, y, z, z, \ldots, z)$ is the dual discriminator on S. To accomplish this first notice that ii) (PCC) implies that the meet of all proper congruences of any $A \in V$ is ω and hence that V is semisimple. Thus we need only consider simple S. Second, observe that we need only establish (2.2) for at most countably generated S, for the class K of at most countably generated simple members of V generates V and, by Jónssons Lemma ([7]), all simple members of V are in $\mathrm{HSP}_u(K)$. Thus if we establish that the sentence (1.2) asserting that $t_1(x, y, z)$ is the dual discriminator is true for all members of K it follows that it will be true for all members of $\mathrm{SP}_u(K) = \mathrm{HSP}_u(K)$.

Hence let S be at most countably generated, by say $g_0, g_1, \ldots$, and simple with $a, b, c \in S$, $a \neq b$. Define a homomorphism $\varphi\colon F \to S$ by

$$\varphi(x) = a, \quad \varphi(y) = b, \quad \varphi(c) = c, \quad \varphi(v_i) = c,$$

for $i = 0, \ldots, m-1$ and $\varphi(v_i) = g_{i-m}$ for $i = m, m+1, \ldots$. Since φ is surjective, $\ker\varphi$ is maximal in $\mathrm{Con}(F)$. We claim that $\theta' \leqq \ker\varphi$. If this were not so then $\theta' \vee \ker\varphi = \iota$. But $\theta(x, y) \vee \theta' = \iota$ and $\theta(x, y) \wedge \theta' = \omega$ so that congruence distributivity (i) would imply $\ker\varphi = \iota$, a contradiction. Hence $\theta' \leqq \ker\varphi$. Thus it follows that $\varphi(t(x, y, z, v_0, \ldots, v_{m-1})) = \varphi(z)$, i.e.: $t(a, b, c, c, \ldots, c) = t_1(a, b, c) = c$, completing the proof.

COROLLARY D (Fried, Kiss [4]). *For a variety V the following are equivalent:*
1) *V is a discriminator variety*
2) *V has the properties:*
   i) *V is arithmetical,* ii) *PCC.*

The corollary is immediate from the remarks preceeding the Theorem.

3. Locally finite semi-simple varieties

Recall that a variety is locally finite if its finitely generated members are finite. For locally finite semisimple varieties we can obtain sharp characterizations of dual discriminator and discriminator varieties. To do this we need to first review (from [5], [10]) the concepts of rectangular and p-rectangular subalgebras. Let A_1, A_2 be algebras and S a subalgebra of $A_1 \times A_2$. S is rectangular if $(x, y), (x, v), (u, v) \in S \Rightarrow (u, y) \in S$; briefly, if three vertices of a rectangle in $A_1 \times A_2$ are in S the fourth vertex is also in S. A variety V is CP iff for all $A_1, A_2 \in V$ each subalgebra of $A_1 \times A_2$ is rectangular ([10]). S is p-rectangular if S has the properties:

$$(x, y_1), (x, y_2), (u, v) \in S \quad \text{and} \quad y_1 \neq y_2 \Rightarrow (x, v) \in S, \tag{3.1}$$

$$(x_1, y), (x_2, y), (u, v) \in S \quad \text{and} \quad x_1 \neq x_2 \Rightarrow (u, y) \in S. \tag{3.2}$$

Geometrically this means that if S contains two distinct points of a vertical (or horizontal) line in $A_1 \times A_2$ then the horizontal (or vertical) projection of any other element of S onto this line is also in S. In the present setting the significance of these concepts is given by the following result from [10].

THEOREM 3.1. *Let A be a finite algebra.*

a) *The dual discriminator is a term function of A iff each $f\colon A^n \to A$ under which each p-rectangular subalgebra of $A \times A$ is closed, is a term function of A.*

b) *The discriminator is a term function of A iff each $f\colon A^n \to A$ under which each subalgebra of $A \times A$ which is both rectangular and p-rectangular is closed, is a term function of A.*

With this background we can state our result. ($F_V(3)$ is the 3 generated free algebra in V.)

THEOREM DD′. *Let V be a variety having the following properties:*

a) *V has a majority term, i.e.: a term $m(x, y, z)$ satisfying (1.4),*

b) $|F_V(3)| < \omega$,

c) *V is semi-simple,*

d) *For each pair of finite simple algebras $S_1, S_2 \in V$, each subdirect product in $S_1 \times S_2$ is p-rectangular.*

Then the following are equivalent:

i) *V is a dual discriminator variety.*

ii) *V is sub semi-simple (i.e.: non trivial subalgebras of simple algebras in V are simple)*

iii) *For $n \leq 3$ every n-generated non-trivial subalgebra of any simple algebra of V is simple.*

The implication i)→ii) is clear and the equivalence of ii) and iii) is always true (any non-trivial algebra is simple if its subalgebras of no more than 3 generators are simple). Hence to complete the proof we need only demonstrate iii)→i). We defer this until Section 4 and in the present section show how to obtain the following corollary from the Theorem.

COROLLARY D′. *Let V be a variety having the following properties:*

a) *V is arithmetical,*

b) $|F_V(3)|<\omega$

c) *V is semi-simple.*

Then the following are equivalent:

i) *V is a discriminator variety.*

ii) *V is sub semi-simple*

iii) *For $n\leqq 3$ every n-generated non-trivial subalgebra of any simple algebra of V is simple.*

PROOF. Since V is CP and hence has the PCI property it will be enough, by Lemma 1.1, to establish the following Lemma.

LEMMA 3.2. *The PCI property in a congruence modular variety implies condition d) of Theorem DD′.*

PROOF of Lemma 3.2. Let S be a subdirect product in $S_1\times S_2$, S_i simple and suppose $(x, y_1), (x, y_2), (u, v)\in S$, $y_1\neq y_2$. Then $\theta=\theta((x, y_1), (x, y_2))\leqq\pi_1$, the kernel of the first projection. Hence, from the simplicity of the S_i, π_1 and π_2 are complementary proper maximal congruences. But by modularity, $\theta=\theta\vee(\pi_1\wedge\pi_2)=$ $=\pi_1\wedge(\theta\vee\pi_2)$. Again by maximality $\theta\vee\pi_2=\iota$ or π_2 and if $\theta\vee\pi_2=\pi_2$ then $\theta\leqq\pi_2$ so $\theta=\omega$ contradicting $y_1\neq y_2$. Hence $\theta\vee\pi_2=\iota$ so $\theta=\pi_1$, i.e.: π_1 is principal. But then by the PCI property,

$$[(x, y_1)]\pi_1\cap[(u, v)]\pi_2\neq\emptyset,$$

which means that $(x, v)\in S$ so (3.1) is satisfied. Symmetrically we obtain (3.2) so S is p-rectangular.

Hence Corollary D′ is a simple consequence of Theorem DD′.

4. Completion of the proof of Theorem DD′

We assume that V has properties a)—d) and iii) as stated in the Theorem. By b) and c) the free algebra $F_V(3)$ may be identified with a subdirect product S in $S_1\times\ldots\times S_n$ where the S_i are finite simple members of V, S is freely generated by $x=(x_1, \ldots, x_n)$, $y=(y_1, \ldots, y_n)$, $z=(z_1, \ldots, z_n)$, and for each $i=1, \ldots, n$, S_i is generated by $\{x_i, y_i, z_i\}$. For all $i<j$ let S_{ij} be the projection of S into $S_i\times S_j$. Then, in particular, the pairs (x_i, x_j), (y_i, y_j), (z_i, z_j) are in S_{ij}. We first show that for all $1\leqq i<j\leqq n$ there is a term function d_{ij} of three variables such that $d_{ij}(x_k, y_k, z_k)=$ $=d(x_k, y_k, z_k)$ for $k=i$ or j, and where d is the dual discriminator function. We show this by considering the following possible cases:

1) If $x_i=y_i$ and $x_j=y_j$ the projection on the first coordinate is the sought d_{ij}.

2) If $x_i\neq y_i$ and $x_j\neq y_j$ take d_{ij} to be projection on the third coordinate.

3) If $x_i=y_i$ and $x_j\neq y_j$ then by d) and (3.1) there is a term function d_{ij} of three variables such that

$$d_{ij}((x_i, x_j), (y_i, y_j), (z_i, z_j)) = (x_i, z_j),$$

i.e.:

$$d_{ij}(x_i, y_i, z_i) = x_i \quad\text{and}\quad d_{ij}(x_j, y_j, z_j) = z_j,$$

which means $d_{ij}(x_k, y_k, z_k)=d(x_k, y_k, z_k)$ for $k=i$ or j, as required.

4) If $x_i \neq y_i$ and $x_j = y_j$ then we obtain the d_{ij} as in 3) using (3.2).

Now for $i=1, \ldots, n$ define the functions d_i: $\{x, y, z\}^3 \to \{x, y, z\} \subset S$ by

$$d_i(x, y, z) = x \quad \text{if} \quad x_i = y_i,$$
$$= z \quad \text{if} \quad x_i \neq y_i,$$

and let π_i be the kernels of the projections of S onto S_i. Then in the free algebra S, the system of n congruences

$$p(x, y, z) \equiv d_i(x, y, z)(\pi_i), \quad i = 1, \ldots, n, \tag{4.1}$$

has the $d_{ij}(x, y, z)$ as pairwise solutions. Hence by the existence of a majority term (a), it follows from a result of A. Huhn ([6], see also [1]) that the system (4.1) has a single simultaneous solution, i.e.: there is a term $t(x, y, z)$ such that the induced term function satisfies

$$t(x_k, y_k, z_k) = d(x_k, y_k, z_k), \quad k = 1, \ldots, n.$$

Finally let A be any SI (hence simple) member of V and choose $a, b, c \in A$. Let $\pi = \ker h$ were h is the homomorphism of S into A defined by $h(x)=a$, $h(y)=b$, $h(z)=c$. The subalgebra B of A generated by a, b, c is, by iii), simple or trivial so π is either maximal or ι in Con (S). If π is maximal then since a) implies that V is CD, π is some π_k for $k=1, \ldots, n$, so that $B \cong S_k$ and hence $t(a, b, c) = d(a, b, c)$. If B is trivial there is nothing to prove. Hence the term $t(x, y, z)$ induces the dual discriminator on each SI in V and this completes the proof.

5. Special results

For varieties generated by very small algebras some special remarks can be made. First, S. Burris [2] has observed that if $|A| \leq 4$ and $V(A)$ is semi-simple and arithmetical, then $V(A)$ is automatically sub semi-simple so that $V(A)$ is a discriminator variety. (This is easily verified by considering cases.) Burris also shows that 4 is the least integer for which this statement is true.

No corresponding special result holds for dual discriminator varieties. It, is, of course, obvious that if $|A| = 2$ and A has a majority term function, then with no further requirements, $V(A)$ is a dual discriminator variety. On the other hand for $|A| = 3$ we can construct an algebra A such that conditions a), b), c) and ii) of Theorem DD′ hold for $V(A)$ but condition d) fails so $V(A)$ is not a dual discriminator variety. This observation and the following illustrative example were kindly supplied by the referee. Let A have the set $\{0, 1, 2\}$ as universe and take for operations min, max, f, g where

$$f(0) = 1, \quad f(1) = f(2) = 2,$$

$$g(2) = 1, \quad g(1) = g(0) = 0.$$

Then A is simple and $A \times A - \{0, 2\}$ is a subalgebra of $A \times A$ which is not p-rectangular. Hence d) (and thus i)) fails for $V(A)$ while all other conditions of Theorem DD′ hold.

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COMPLETE RULES OF INFERENCE FOR UNIVERSAL SENTENCES

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Introduction

(i) Background

In 1935, G. Birkhoff [1] gave complete rules of inference for identities (equational logic). Three decades later, A. Selman [7] gave complete rules for two other fragments of first-order logic without relation symbols; one of this fragments was strict universal Horn sentences. Each of the above three fragments is a set of universal sentences. Subsequently, the present author also obtained complete rules for universal Horn sentences (as mentioned in McNulty [5]). In each case that we mention, the different rules of inference are easily shown to be equivalent. Selman applied a proof-theoretic technique of L. Henkin [3] to restrict the completeness theorem of first-order logic to each fragment. Instead of using such a technique, we follow Birkhoff by constructing suitable models.

In this paper, we give complete rules of inference for certain classes of universal sentences, including the classes of universal Horn sentences and positive universal sentences. For their completeness results, Birkhoff and Selman gave distinct proofs; even Selman's two proofs are only related by analogy. We were motivated by this disparity to find a unified approach. In fact, all our completeness results follow in a uniform way from the case of all universal sentences.

G. McNulty [5] has also given a completeness proof for universal Horn sentences. H. Andréka and I. Németi [0] proved the same result. Besides Birkhoff's result, Andréka and Németi only knew of Selman's result. Moreover, for their result on universal Horn sentences, they give a Birkhoff-style proof and allow partial operations.

(ii) Notation

Since universal sentences can be assumed to be in prenex form, we suppress all universal quantifiers. Each universal sentence can be written as a finite conjunction of expressions of the form

$$(*) \qquad \sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_m \rightarrow \tau_1 \vee \tau_2 \vee \ldots \vee \tau_n$$

where $m+n>0$ and $\sigma_1, \sigma_2, \ldots, \sigma_m, \tau_1, \tau_2, \ldots, \tau_n$ are atomic formulas. An atomic formula looks like $p \doteq q$ for polynomials (terms) p and q, or like $\mathbf{R}(p_1, \ldots, p_n)$ where $\mathbf{R}$ is an n-ary relation symbol and $p_1, \ldots, p_n$ are polynomials.

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We shall only consider universal sentences of the form $(*)$ because rules of inference are easily extended to conjunctions. Moreover, we do not consider the slight extension that allows arbitrary sentences as hypotheses (existentially quantified variables are replaced with Skolem functions). Henceforth, the term "sentence" means a universal sentence of the form $(*)$. (Although the Gentzen form in Chapter 15 of Kleene [4] looks similar, it does not require the formulas on each side of $(*)$ to be atomic.)

In fact, we shall represent each sentence as

$$S \rightarrow T,$$

where S and T are *finite sets* of atomic formulas. We permit both S and T to be empty. The sentence $\emptyset \rightarrow \emptyset$, denoted by $\mathbf{F}$, represents falsity, a sentence without a model. Our notation emphasizes that we are dealing with ordered pairs of finite sets of atomic formulas when we consider sentences syntactically. One reads $S \rightarrow T$ as "the conjunction of S implies the disjunction of T". Since an empty conjunction is "true" and an empty disjunction is "false", $S \rightarrow \emptyset$ says that the conjunction of S does not hold, and $\emptyset \rightarrow T$ says that the disjunction of T holds.

There is a fixed similarity type ξ that lists all the operation symbols and the relation symbols, together with their arities. Since all operation and relation symbols are understood to be of this type, ξ is only rarely mentioned explicitly. We shall always denote the set of variables by $\mathbf{X}$ and the set of constants (nullary operation symbols in ξ) by $\mathbf{C}$. We do allow $\mathbf{X}$ to be empty, but unless ξ consists entirely of nullary relation symbols, $\mathbf{X} \cup \mathbf{C}$ is nonempty.

Let us introduce some notational conventions. Σ is a set of sentences of type ξ, and $\mathbf{X}$ contains each variable that occurs in Σ. In the rules that we shall give, $\mathbf{f}$ denotes any operation symbol and $\mathbf{R}$ denotes any relation symbol (both of nonzero arity n). In addition, p, q and r (with or without subscripts) are polynomials; σ, τ and ϱ are atomic formulas; and R, S and T are finite sets of atomic formulas. Writing the sentence φ as $\varphi(x_1, \ldots, x_m)$ means that the distinct variables $x_1, \ldots, x_m$ may appear in φ; moreover, $\varphi(p_1, \ldots, p_m)$ denotes the sentence obtained from φ by replacing each occurrence the variable x_i by the polynomial p_i $(1 \leqq i \leqq m)$.

(iii) Some rules of inference

We shall always assume that $0 \leqq \alpha \leqq \omega$ and $1 \leqq \beta \leqq \omega$. We write $\mathscr{U}_{\alpha,\beta}$ for the set of sentences $S \rightarrow T$ with both S and T finite, $|S| \leqq \alpha$, and $|T| \leqq \beta$. In particular, $\mathscr{U}_{\omega,\omega}$ is the set of all sentences. For any such α and β, we now define nine rules of inference for $\mathscr{U}_{\alpha,\beta}$. All sentences that appear, including those of Σ, are assumed to be in $\mathscr{U}_{\alpha,\beta}$, and both (U7) and (U8) are omitted if $\alpha = 0$. We also assume that $|T| < \beta$ in (U2), (U3), and (U4).

(U0) $\Sigma \vdash \varphi$ whenever φ is in Σ.

(U1) $\Sigma \vdash \emptyset \rightarrow \{p \doteq p\}$ for every polynomial p.

(U2) From $\Sigma \vdash S \rightarrow T \cup \{p \doteq q\}$, infer $\Sigma \vdash S \rightarrow T \cup \{q \doteq p\}$.

(U3) From $\Sigma \vdash S \rightarrow T \cup \{p \doteq q\}$ and $\Sigma \vdash S \rightarrow T \cup \{q \doteq r\}$, infer $\Sigma \vdash S \rightarrow T \cup \{p \doteq r\}$.

$$\text{(U4)} \quad \begin{cases} \text{From} & \Sigma \vdash S \to T \cup \{p_i \doteq q_i\} \quad \text{for} \quad 1 \leqq i \leqq n, \\ \text{infer} & \Sigma \vdash S \to T \cup \{\mathbf{f}(p_1, \ldots, p_n) \doteq \mathbf{f}(q_1, \ldots, q_n)\}. \\ \text{From} & \Sigma \vdash S \to T \cup \{p_i \doteq q_i\} \quad \text{for} \quad 1 \leqq i \leqq n \\ \text{and} & \Sigma \vdash S \to T \cup \{\mathbf{R}(p_1, \ldots, p_n)\}, \\ \text{infer} & \Sigma \vdash S \to T \cup \{\mathbf{R}(q_1, \ldots, q_n)\}. \end{cases}$$

(U5) From $\Sigma \vdash \varphi(x_1, \ldots, x_m)$, infer $\Sigma \vdash \varphi(p_1, \ldots, p_m)$.

(U6) From $\Sigma \vdash S \to T$, infer $\Sigma \vdash S' \to T'$ for any $S' \supseteq S$ and $T' \supseteq T$.

(U7) $\Sigma \vdash \{\sigma\} \to \{\sigma\}$ for every atomic formula σ.

(U8) If $\Sigma \vdash S \to R_i$ for $1 \leqq i \leqq k \leqq \alpha$ with k finite, each R is nonempty, and for any choice of $\varrho_i \in R_i$, $\Sigma \vdash \{\varrho_1, \ldots, \varrho_k\} \to T$, then infer $\Sigma \vdash S \to T$.

(iv) The main result

Let $\mathscr{L}$ be a set of formulas of first-order logic (for some fixed similarity type). A rule of inference for $\mathscr{L}$ has all its hypotheses and its conclusion in $\mathscr{L}$. A set of rules of inference for $\mathscr{L}$ is (semantically) *complete* if the formulas that can be proved from any subset Σ of $\mathscr{L}$ using these rules are exactly the first-order consequences of Σ that are in $\mathscr{L}$. (The semantic notion of first-order consequence is intended.) It should be obvious that any sentence proved via our rules is also a first-order consequence. Our main result is the

THEOREM. *If* $\alpha = \omega$, *or* $\beta = 1$, *or* $\beta = \omega$, *then the rules* (U0) *to* (U8) *form a complete set of rules of inference of* $\mathscr{U}_{\alpha,\beta}$. *Moreover, for any other values of* α *and* β *(i.e., both* α *and* β *are finite and* $\beta \geqq 2$*), these rules are not complete for any type with at least* $3\alpha + 2\beta + 2$ *constants.*

In particular, the $\langle \alpha, \beta \rangle = \langle 0, \omega \rangle$ case of the theorem means that the rules (U0) to (U8) are complete rules of inference for positive universal sentences (with each positive universal sentence replaced by the set of its conjuncts, each of which is a disjunction). We have mentioned an explicit number of constants in the theorem to emphasize how easy it is to get incompleteness when α and β are finite and $\beta \geqq 2$.

Let $\mathscr{U}^*_{\alpha,\beta}$ denote the subset of $\mathscr{U}_{\alpha,\beta}$ consisting of the sentences $S \to T$ with T nonempty. If $\Sigma \subseteq \mathscr{U}^*_{\alpha,\beta}$, then our rules will never introduce a sentence of the form $S \to \emptyset$. Therefore, the theorem implies that the $\langle \alpha, \beta \rangle$-rules are complete rules of inference for $\mathscr{U}^*_{\alpha,\beta}$ whenever $\alpha = \omega$, or $\beta = 1$, or $\beta = \omega$. Examples in Section 1 will demonstrate the incompleteness of our rules for the remaining values of $\langle \alpha, \beta \rangle$. These examples also show that these rules, when applied to $\mathscr{U}^*_{\alpha,\beta}$, are incomplete for the same values of $\langle \alpha, \beta \rangle$. For these values of $\langle \alpha, \beta \rangle$, we *conjecture* that no finite set of rules of inference for $\mathscr{U}_{\alpha,\beta}$ or $\mathscr{U}^*_{\alpha,\beta}$ is complete.

(v) Consequences and modifications

If $\mathbf{X}$ is infinite, then we can assume that $m = 1$ in (U5). Thus, if $\mathbf{X}$ is infinite, and there are no relation symbols, then we obtain Birkhoff's rules for $\mathscr{U}^*_{0,1}$ (as given, for example, in Grätzer [2]) from our theorem. (Of course, the formulation of our rules was motivated by those of Birkhoff.) When there are no relation symbols,

A. Selman [7] found complete sets of rules of inference for $\mathscr{U}^*_{1,1}$ and $\mathscr{U}^*_{\omega,1}$. Observe that the completeness theorem of A. Robinson [6] is not a completeness theorem for $\mathscr{U}_{\omega,\omega}$ in our sense because his rules allow conjunctions and disjunctions in formulas on both sides of the implication sign (even when the initial formulas are in $\mathscr{U}_{\omega,\omega}$).

The examples of Section 1 also serve as illustrations of our rules. In particular, the strength of (U8) will be demonstrated. In rule (U8), we shall call each of the sets $\{\varrho_1, \ldots, \varrho_k\}$ a *selection* (for the sets of atomic formulas R_i, $1 \leqq i \leqq k$).

There are some minor modifications to our rules that yield equivalent sets of rules. If **X** is empty, then (U5) can be omitted. If **X** is nonempty, then (U1) can be replaced by

(U1)′ $$\Sigma \vdash \emptyset \rightarrow \{x \doteq x\} \quad \text{for some} \quad x \in \mathbf{X}.$$

If $\alpha \geqq 1$, then we can replace (U2) by

(U2)′ $$\Sigma \vdash \{p \doteq q\} \rightarrow \{q \doteq p\}.$$

If $\alpha \geqq 2$, then we can replace (U3) by

(U3)′ $$\Sigma \vdash \{p \doteq q, q \doteq r\} \rightarrow \{p \doteq r\}.$$

If $\alpha = \omega$, then we can replace (U4) by

(U4)′ $$\begin{cases} \Sigma \vdash \{p_1 \doteq q_1, \ldots, p_n \doteq q_n\} \rightarrow \{\mathbf{f}(p_1, \ldots, p_n) \doteq \mathbf{f}(q_1, \ldots, q_n)\} \\ \Sigma \vdash \{p_1 \doteq q_1, \ldots, p_n \doteq q_n, \mathbf{R}(p_1, \ldots, p_n)\} \rightarrow \{\mathbf{R}(q_1, \ldots, q_n)\}. \end{cases}$$

If α is finite, then we can require $k = \alpha$ in (U8) because we could repeat $S \rightarrow R_1$ often enough; since the new selections contain the old ones, (U6) can be applied.

Sections 2 and 3 reduce to the $\alpha = \omega$ case. If α is infinite, then Section 4 allows us to assume that β is also infinite. Completeness is proved in Section 5.

An obvious consequence of the theorem is that our notion of proof depends only superficially on the type ξ and the set **X** of variables when $\alpha = \omega$, $\beta = 1$, or $\beta = \omega$. More precisely, if there is an $\langle \alpha, \beta \rangle$-proof of the sentence φ from the set Σ of sentences for some choice of the type and set of variables, then there is also an $\langle \alpha, \beta \rangle$-proof of φ from Σ when ξ (respectively, **X**) consists only of those symbols (respectively, variables) that appear in Σ or φ. (In fact, we show in Section 1 that this dependence is superficial for all α and β.)

1. Examples of incompleteness

For $\Sigma \subseteq \mathscr{U}_{\alpha,\beta}$, the *$\langle \alpha, \beta \rangle$-closure* of Σ, denoted by $\mathrm{cl}_{\alpha,\beta}(\Sigma)$, is the set of all sentences that can be proved from Σ using the $\langle \alpha, \beta \rangle$-rules. A set of $\langle \alpha, \beta \rangle$-sentences is *$\langle \alpha, \beta \rangle$-closed* if it is the $\langle \alpha, \beta \rangle$-closure of some set, and so in particular, of itself. Writing $\Sigma \vdash \varphi$ means that there is an $\langle \alpha, \beta \rangle$-proof of the sentence φ from Σ. The *length* of a proof is the total number of applications of all rules except (U0).

Let us indicate why an $\langle \alpha, \beta \rangle$-proof of $\Sigma \vdash \varphi$ can be assumed to use only the symbols and variables that occur in Σ or φ. Let ξ and **X** denote this minimum type and set of variables. First, delete any sentences in the proof in which a relation symbol not in ξ appears on the left side, and remove all atomic formulas involving an extraneous relation symbol from all the sets on the right sides of the remaining sentences.

What remains is still an $\langle\alpha, \beta\rangle$-proof of φ from Σ. We can now assume that $\mathbf{X}\cup\mathbf{C}$ is nonempty; choose some $x\in\mathbf{X}\cup\mathbf{C}$. Replace each constant not in $\mathbf{C}$ by x, and "interpret" each extraneous operation symbol of nonzero arity as the first projection. The resulting sequence is now a proof of $\Sigma\vdash\varphi$ of type ξ, but extraneous variables may still occur. Since the only operation symbols are constants and $\mathbf{X}$ is empty in all the examples of this section, we first make these assumptions. In this case, every variable is replaced by each constant in $\mathbf{C}$. (For example, a sentence with two extraneous variables is replaced by 36 sentences if there are 6 constants.) For the general case, we remove one extraneous variable y at a time. We substitute finitely many polynomials (of type ξ and not involving the variable y) for y at its introduction (which will be an application of one of (U1), (U5), (U6), or (U7)), and in all subsequent steps, for later use in the modified proof. (One analyzes subsequent applications of (U5) in the original proof.)

Each set Σ of sentences that we shall define in this section is finite and has only constant operation symbols. Consequently, $\mathrm{cl}_{\alpha,\beta}(\Sigma)$ is finite for all α and β. Thus, for any definite values of α and β, any claim that we make about Σ could, in principle, be verified by a direct calculation of $\mathrm{cl}_{\alpha,\beta}(\Sigma)$.

The first lemma expresses an old idea: if constants are systematically replaced in the original hypotheses and in each step of a proof, then the resulting sequence is also a proof. We omit the simple inductive proof of this lemma. The second lemma eliminates (U2) from $\langle\alpha, \beta\rangle$-proofs.

Lemma 1.1. *Let $c_1, \ldots, c_m$ be distinct constants that do not appear in the set $\Sigma(x_1, \ldots, x_m)$ of sentences, and let $d_1, \ldots, d_m$ be polynomials in which no variables appear (closed terms). Let α and β be arbitrary. If $\Sigma(c_1, \ldots, c_m)\vdash\varphi(c_1, \ldots, c_m)$, then $\Sigma(d_1, \ldots, d_m)\vdash\varphi(d_1, \ldots, d_m)$. Moreover, the $\langle\alpha, \beta\rangle$-proof of $\varphi(d_1, \ldots, d_m)$ from $\Sigma(d_1, \ldots, d_m)$ has the same length as the $\langle\alpha, \beta\rangle$-proof of $\varphi(c_1, \ldots, c_m)$ from $\Sigma(c_1, \ldots, c_m)$.*

Lemma 1.2. *Let Σ be a set of sentences that is closed under* (U2). *For any $\langle\alpha, \beta\rangle$-proof of a sentence φ from Σ, there is an $\langle\alpha, \beta\rangle$-proof of φ from Σ that does not use* (U2), *and is no longer than the original proof.*

Proof. Let φ' be obtained by zero or more applications of (U2) to the sentence φ. If φ is obtained by a single application of any rule except (U2) to a (U2)-closed set Σ, one shows that φ' can also be obtained by a single application of the same rule to Σ.

Let $\alpha=0$ and β be finite with $\beta\geqq 2$. We define a subset Σ of $\mathscr{U}_{0,\beta}$ and a sentence $\varphi\in\mathscr{U}_{0,\beta}$ such that φ can be proved from Σ by the $\langle 0, \beta+1\rangle$-rules, but not by the $\langle 0, \beta\rangle$-rules. Let $Q=\{\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_{\beta-2}\}$, with $Q=\emptyset$ if $\beta=2$. Σ consists of the following three sentences:

1. $\emptyset \to \{a \doteq b,\ e_2 \doteq d\}\cup Q,$

2. $\emptyset \to \{a \doteq e_1,\ c \doteq e_2\}\cup Q,$

3. $\emptyset \to \{e_1 \doteq b,\ c \doteq d\}\cup Q,$

and φ is $\emptyset\to T$, where $T=\{a\doteq b,\ c\doteq d\}\cup Q$. Since $\langle\alpha, \beta\rangle$-proofs depend only superficially on the type and variables, we can assume that the type consists of the symbols appearing in Σ, and that the set $\mathbf{X}$ of variables is empty. An example with only constant symbols is obtained by introducing $2\beta-4$ new constants, and replacing each relation symbol by an equation involving two constants. The modified example is the incompleteness example of the theorem for $\alpha=0$. However, since the modifications entail only minor changes in the argument, we continue with the original example.

It is easy to find $\langle 0, \beta+1\rangle$-proofs of φ from Σ. (For the theorem, we only need the obvious fact that φ is a first order consequence of Σ.) We require a preliminary analysis to verify the other condition. We define $C_1=\{a, b, e_1\}$ and $C_2=\{c, d, e_2\}$; thus, the set $\mathbf{C}$ of constants is the union of C_1 and C_2. We shall only consider structures whose underlying set is obtained from $\mathbf{C}$ by identifying elements. Any such structure satisfies Σ iff C_1 or C_2 is collapsed to a single point, or some $\mathbf{R}_i$ is true. Let s, t, u, v be in $\mathbf{C}$ with $s\neq t$ and $u\neq v$, and consider the sentence

$$\emptyset \to \{s \doteq t,\ u \doteq v\}\cup Q,$$

with Q as defined above. It follows that:

(i) If the right side of this sentence is replaced by a proper subset, the new sentence is not a first-order consequence of Σ.

(ii) This sentence is a first-order consequence of Σ iff each of $\{s, t\}$ and $\{u, v\}$ is contained in C_1 or C_2.

Let Σ_i be the (U2)-closure of sentence i of Σ for $1\leqq i\leqq 3$. We define $\Sigma'=\Sigma_1\cup\Sigma_2\cup\Sigma_3$, a (U2)-closed set. There are six more such 4-element sets: $\Sigma_4, \Sigma_5, \ldots, \Sigma_9$, each corresponding to a pair of 2-element subsets of C_1 and C_2. Obviously, any permutation of $\mathbf{C}$ induces a permutation of the nine objects $\Sigma_1, \Sigma_2, \ldots, \ldots, \Sigma_9$. Suppose that the permutation λ of $\mathbf{C}$ permutes the three objects $\Sigma_1, \Sigma_2, \Sigma_3$. If, for some sentence ψ, $\Sigma'\vdash\psi$, then Lemma 1.1 implies that there is a proof of $\lambda(\psi)$ from Σ' of the same length. Two such permutations are $\lambda_1=(be_1)(cd)$ and $\lambda_2=(ab)(de_2)$.

Suppose there is a $\langle 0, \beta\rangle$-proof of φ from Σ. In order to reach a contradiction, let us consider a shortest $\langle 0, \beta\rangle$-proof of φ from Σ'. Since φ is not in Σ', the minimum length is nonzero, and (U0) is not applied in the last step. We now consider which rule could have been applied in the last step. By the form of φ, (U1) is not possible. By Lemma 1.2, we can assume that (U2) does not occur in the proof. (U4) is excluded because there are no symbols of positive arity. Since $\mathbf{X}$ is empty, (U5) does not apply. By (i), rule (U6) was not used.

For (U3), assume that, for some $p\in\mathbf{C}$, both $\emptyset\to\{a\doteq p, c\doteq d\}\cup Q$ and $\emptyset\to\{p\doteq b, c\doteq d\}\cup Q$ have shorter proofs from Σ'. We would have a shorter proof of φ if p were a or b. By (ii), we conclude that p is e_1. Applying λ_1 to $\emptyset\to\{a\doteq e_1, c\doteq d\}\cup Q$, we conclude by Lemmas 1.1 and 1.2 that φ has a shorter proof, a contradiction. (For the other possible application of (U3), the permutation λ_2 is used.) We have verified that Σ and φ have the properties we claimed.

Let us further investigate the above example for $\beta=2$. The theorem says that there is an $\langle\omega, 2\rangle$-proof of φ from Σ. As an illustration, we give a $\langle 1, 2\rangle$-proof of φ from Σ. (When we apply one of our rules, we give the line numbers of the hypotheses, with zero indicating that a hypothesis is in Σ.)

	Rule	Lines
1. $\{a \doteq e_1\} \to \{a \doteq e_1\}$	(U7)	—
2. $\{a \doteq e_1\} \to \{a \doteq e_1, c \doteq d\}$	(U6)	1
3. $\{a \doteq e_1\} \to \{e_1 \doteq b, c \doteq d\}$	(U6)	0
4. $\{a \doteq e_1\} \to \{a \doteq b, c \doteq d\}$	(U3)	2, 3
5. $\{c \doteq e_2\} \to \{c \doteq e_2\}$	(U7)	—
6. $\{c \doteq e_2\} \to \{a \doteq b, c \doteq e_2\}$	(U6)	5
7. $\{c \doteq e_2\} \to \{a \doteq b, e_2 \doteq d\}$	(U6)	0
8. $\{c \doteq e_2\} \to \{a \doteq b, c \doteq d\}$	(U3)	6, 7
9. $\emptyset \to \{a \doteq b, c \doteq d\}$	(U8)	0, 4, 8

Before we describe our examples for nonzero α, let us make an observation. If we replace the empty set by $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ on the left side of each sentence in the above Σ and φ, then we do *not* obtain an incompleteness example for $\alpha=3$. Since both

$$\{a \doteq e_1,\ e_1 \doteq b,\ e_2 \doteq d\} \to \{a \doteq b,\ c \doteq d\}$$

and

$$\{e_1 \doteq b,\ c \doteq e_2,\ e_2 \doteq d\} \to \{a \doteq b,\ c \doteq d\}$$

have $\langle 3, 2\rangle$-proofs, the new version of φ now follows from an application of (U8) to the three new sentences of Σ.

Suppose α and β are finite with $\alpha \geqq 1$ and $\beta \geqq 2$. Let $S=\{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_\alpha\}$, $Q=\{\mathbf{R}_3, \mathbf{R}_4, \ldots, \mathbf{R}_\beta\}$, with $Q=\emptyset$ if $\beta=2$, and let $T=\{\mathbf{R}_1, \mathbf{R}_2\} \cup Q$. The set Σ consists of the following $\alpha+2$ sentences $(1 \leqq i \leqq \alpha+1)$:

$$S \to \{a_{i-1} \doteq a_i,\ \mathbf{R}_1\} \cup Q, \quad i \text{ odd},$$

$$S \to \{a_{i-1} \doteq a_i,\ \mathbf{R}_2\} \cup Q, \quad i \text{ even}$$

$$\{a_0 \doteq a_{\alpha+1}\} \to T.$$

We define φ to be $S \to T$. (As in the $\alpha=0$ case, additional constants can be introduced to eliminate the relation symbols.) Clearly, φ is a first-order consequence of Σ. (Applying (U8) to every sentence of Σ but the last yields an $\langle \alpha+1, \beta\rangle$-proof of φ from Σ. There is also an obvious $\langle \alpha, \beta+1\rangle$-proof.)

For S as above, suppose that the $\langle \alpha, \beta\rangle$-sentence $S \to R$ is a first-order consequence of Σ, $S \cap R=\emptyset$, and R does not contain $u \doteq u$ for any $u \in \mathbf{C}$. Using structures in which each relation symbol in S is true, it is easy to show that:

(i) $Q \subseteq R$ and R contains $\mathbf{R}_1$ or $\mathbf{R}_2$.

(ii) If $R=\{u \doteq v, \mathbf{R}_j\} \cup Q$ with $u, v \in \mathbf{C}$ and $j \in \{1, 2\}$, then $\{u, v\}=\{a_{i-1}, a_i\}$, and i and j have the same parity.

(iii) R is not a proper subset of T.

Suppose there is an $\langle \alpha, \beta\rangle$-proof of φ from Σ. We consider the last step in a shortest $\langle \alpha, \beta\rangle$-proof of φ from Σ. It is easily seen that none of (U0), (U6), or (U7) was applied. Since $\mathbf{X}=\emptyset$ and there is no equation in T, rule (U8) was used in the last step. Suppose that (U8) was applied to $S \to R_m$, $1 \leqq m \leqq k \leqq \alpha$. If $R_m=T$ for some m, then we have a shorter proof of φ, a contradiction. Using (iii), we conclude that no R_m is a subset of T. We shall write $u \sim v$ to denote either $u \doteq v$ or $v \doteq u$. By (i) and (ii), there is a selection of the form

$$P = \{a_{i-1} \sim a_i \mid i \in I\} \cup \{\mathbf{P}_j \mid j \in J\} \cup \{u \doteq u \mid u \in H\}$$

where $I \subseteq \{1, 2, ..., \alpha+1\}$, $J \subseteq \{1, 2, ..., \alpha\}$, $H \subseteq \mathbf{C}$, and

$$|I|+|J|+|H| \leqq k \leqq \alpha.$$

By the assumptions of (U8), $P \to T$ has a shorter proof from Σ. If $|J|=\alpha$, then $P \to T$ would be φ, a contradiction. Therefore, $|J|<\alpha$. Choose $i \in \{1, 2, ..., \alpha+1\}-I$ and $j \in \{1, 2, ..., \alpha\}-J$. We define a 2-element model of Σ in which $P \to T$ fails. In this structure, $\mathbf{P}_j$ is false and every other relation symbol in S is true. Also, every relation symbol in T is false. We identify the elements of $\mathbf{C}$ as follows:

$$a_0 = a_1 = \ldots = a_{i-1} \quad \text{and} \quad a_i = a_{i+1} = \ldots = a_{\alpha+1}.$$

Consequently, $P \to T$ is not a first-order consequence of Σ. With this contradiction, the proof of the second statement of the theorem is complete.

2. From $\beta=1$ to $\beta=\omega$

If $\alpha=\omega$, $\beta=1$, or $\beta=\omega$, then we shall show that

$$\mathrm{cl}_{\alpha,\beta}(\Sigma) = \mathscr{U}_{\alpha,\beta} \cap \mathrm{cl}_{\omega,\omega}(\Sigma).$$

We now give the first of many steps required to prove this "upward compatibility" result.

PROPOSITION 2.1. *If* $\Sigma \subseteq \mathscr{U}_{\alpha,1}$, *then*

$$\mathrm{cl}_{\alpha,1}(\Sigma) = \mathscr{U}_{\alpha,1} \cap \mathrm{cl}_{\alpha,\omega}(\Sigma).$$

In fact, if Σ is $\langle\alpha, 1\rangle$-closed, then $\mathrm{cl}_{\alpha,\omega}(\Sigma)$ *is the set of all $\langle\alpha, \omega\rangle$-sentences $S \to T$ for which $S \to T'$ is in Σ for some $T' \subseteq T$.*

PROOF. It obviously suffices to prove the second statement. Let Σ be $\langle\alpha, 1\rangle$-closed, and define Σ_1 to be the set of all $\langle\alpha, \omega\rangle$-sentences $S \to T$ for which $S \to T'$ is in Σ for some $T' \subseteq T$. Since Σ_1 is contained in $\mathrm{cl}_{\alpha,\omega}(\Sigma)$, it only remains to show that Σ_1 is $\langle\alpha, \omega\rangle$-closed. We shall apply each $\langle\alpha, \omega\rangle$-rule to sentences in Σ_1. If we apply (U0), (U1) or (U7), then the conclusion is in Σ', a subset of Σ_1. If (U3) is applied to $S \to T \cup \{p \doteq q\}$ and $S \to T \cup \{q \doteq r\}$, and $S \to T$ is not in Σ_1, then both $S \to \{p \doteq q\}$ and $S \to \{q \doteq r\}$ must be in Σ; hence, $S \to \{p \doteq r\}$ is in Σ so that the conclusion of (U3) is in Σ_1. We omit the similar proofs for (U2) and (U4). Suppose that $1 \leqq k \leqq \alpha$ and (U8) is applied to $S \to R_i$, $1 \leqq i \leqq k$. If $S \to \emptyset$ is in Σ, then the conclusion is in Σ_1 by (U6). We can now assume that, for each i, there is $\varrho_i \in R_i$ such that $S \to \{\varrho_i\}$ is in Σ. Since $\{\varrho_1, ..., \varrho_k\} \to T$ is in Σ_1 by the assumptions of (U8), there is $T' \subseteq T$ such that $\{\varrho_1, ..., \varrho_k\} \to T'$ is in Σ. By applying (U8) to Σ, we conclude that $S \to T'$ is in Σ. Hence, $S \to T$ is in Σ_1. Since the conclusion is obvious if (U5) or (U6) is applied, the proof of the proposition is complete.

3. Upward compatibility from $\langle\alpha, \omega\rangle$ to $\langle\omega, \omega\rangle$

To each variable $x \in \mathbf{X}$, we associate a new constant $\dot{x}$. We extend the type from ξ to $\xi^{\#}$ by extending the set of constants from $\mathbf{C}$ to $\mathbf{C}^{\#} = \mathbf{C} \cup \{\dot{x} | x \in \mathbf{X}\}$. We shall apply the superscript $\#$ to polynomials, atomic formulas, sets of atomic formulas, sentences, and sets of sentences (all of type ξ) to denote the corresponding object of type $\xi^{\#}$. If Σ is $\langle\alpha, \beta\rangle$-closed, then so is $\Sigma^{\#}$. (The converse of this statement does *not* hold.) For sentences of type $\xi^{\#}$, we shall always assume that the set of variables is empty.

Clearly, $\Sigma^{\#} \vdash \varphi^{\#}$ means that there is a proof of φ from Σ in which (U5) is not used. By converting to the type $\xi^{\#}$, we can apply results that require the set of variables to be empty. Observe that Lemma 3.3 below would be invalid if $\mathbf{X} = \{x\}$ were allowed. (Let $\Sigma = \emptyset$ and $R = \{x \doteq a, x \doteq b\}$. The sentence $\emptyset \to \{a \doteq b\}$ would be in the right side, but not in the left.)

By Proposition 2.1, the results of this section for infinite β are also valid when $\beta = 1$. Because of rule (U8), the cases when α is zero and nonzero have significant differences. Specifically, Lemma 3.1 only applies to nonzero α, and Lemma 3.2 is vacuously true for $\alpha = 0$. Moreover, in Lemma 3.3, we require a separate argument for the $\alpha = 0$ case.

Although the formulation of the following result suggests rule (U8), the differences should be noted.

LEMMA 3.1. *Let* $\mathbf{X} = \emptyset$, *let* $\Sigma \subseteq \mathscr{U}_{\alpha,\omega}$, *let S and T be finite sets of atomic formulas, and let* $1 \leqq k \leqq \alpha$. *For* $1 \leqq i \leqq k$, *let* R_i *be a nonempty finite set of atomic formulas such that* $S \to R_i$ *is in* $\mathrm{cl}_{\alpha,\omega}(\Sigma)$. *If the sentence* $\{\varrho_1, \ldots, \varrho_k\} \to T$ *is in* $\mathrm{cl}_{\alpha,\omega}(\Sigma \cup \{\emptyset \to \{\sigma\} | \sigma \in S\})$ *for any choice of* $\varrho_i \in R_i$, *then* $\mathrm{cl}_{\alpha,\omega}(\Sigma)$ *contains* $S \to T$.

PROOF. We can assume that Σ is $\langle\alpha, \omega\rangle$-closed. Let $\Sigma_1 = \Sigma \cup \{\emptyset \to \{\sigma\} | \sigma \in S\}$. If P is a selection for $R_1, \ldots, R_k$, we call the sentence $Q \to T'$ the *witness* of $P \to T$ if it has the shortest proof from Σ_1 for $Q \subseteq P$ and $T' \subseteq T$. (The witness need not be unique, but its proof length is.) If each witness is in Σ, then $S \to T$ is in Σ by (U6) and (U8). If $\emptyset \to \{\sigma\}$ is a witness where $\sigma \in S$, then $S \to T$ is in Σ by (U7) and (U6). Thus, we are done if each proof has zero length. We prove by induction, primarily on the maximum length of the proof of a witness, that $S \to T$ is in Σ. The induction is secondarily on the number of times the maximum length occurs (i.e., on the number of such selections). Since Σ is closed under (U1) and (U7), neither of these rules is used in any proof. We can also exclude (U5) because $\mathbf{X} = \emptyset$. For $1 \leqq i \leqq k$, choose $\varrho_i \in R_i$ so that the proof of the witness $Q \to T'$ of $\{\varrho_1, \ldots, \varrho_k\} \to T$ has maximum length. In the last step of this proof, (U2), (U3), (U4), or (U8) was used. (The induction assumptions were formulated to exclude (U6) from the final step.) For (U3), suppose that $Q \to T_1 \cup \{p \doteq q\}$ and $Q \to T_1 \cup \{q \doteq r\}$ both have shorter proofs from Σ_1. By induction, Σ contains both $S \to T \cup \{p \doteq q\}$ and $S \to T \cup \{q \doteq r\}$. (In the previous argument, two different sets played the role of the original set T.) Applying (U3), we conclude that Σ contains $S \to T \cup \{p \doteq r\}$; since $p \doteq r \in T$, the last sentence is $S \to T$, completing the proof if (U3) was last applied.

We omit the similar proofs for (U2) and (U4). We can now assume that (U8) was last applied. Let $1 \leqq m \leqq \alpha$, and suppose that $Q \to P_j$ and $\{\pi_1, \ldots, \pi_m\} \to T'$ have shorter proofs from Σ_1 whenever $1 \leqq j \leqq m$ and $\pi_j \in P_j$. For each j, the induction hypotheses apply if T is replaced by $T \cup P_j$. Therefore, Σ contains $S \to T \cup P_j$

for each j. The induction hypotheses now apply with the original set T, and $R_1, \ldots, \ldots, R_k$ replaced by $T \cup P_1, \ldots, T \cup P_m$. (If a selection for these last sets contains $\tau \in T$, then its witness can be taken to be $\{\tau\} \to \{\tau\}$, which has a proof from Σ_1 of zero length. Any other selection P is of the form $\{\pi_1, \ldots, \pi_m\}$ with $\pi_j \in P_j$; the witness of $P \to T$ does not have a longer proof than $P \to T'$ does.) We conclude that $S \to T$ is in Σ, completing the proof of the lemma.

LEMMA 3.2. *Let* $\mathbf{X} = \emptyset$, *let* $\Sigma \subseteq \mathscr{U}_{\alpha,\omega}$, *and let S be a finite set of atomic formulas with* $|S| \leqq \alpha$. *If* $\emptyset \to T$ *is in* $\mathrm{cl}_{\alpha,\omega}(\Sigma \cup \{\emptyset \to \{\sigma\} | \sigma \in S\})$, *then* $S \to T$ *is in* $\mathrm{cl}_{\alpha,\omega}(\Sigma)$.

PROOF. We can assume that Σ is $\langle \alpha, \omega \rangle$-closed. By induction on the length of the proof of $\emptyset \to T$ from $\Sigma_1 = \Sigma \cup \{\emptyset \to \{\sigma\} | \sigma \in S\}$, we show that $S \to T$ is in Σ. This is obvious if $\emptyset \to T$ is in Σ. Thus, the conclusion follows if (U0) or (U1) is last applied to prove $\emptyset \to T$. (U5) is excluded since $\mathbf{X}$ is empty, and (U7) is excluded by the form of the sentence $\emptyset \to T$. We omit the straightforward cases (U2), (U3), (U4), and (U6). Suppose that $\emptyset \to R_i$ $(1 \leqq i \leqq k \leqq \alpha)$, and for any choice of $\varrho_i \in R_i$, $\{\varrho_1, \ldots, \varrho_k\} \to T$ have shorter proofs from Σ_1. By induction, $S \to R_i$ $(1 \leqq i \leqq k)$ are in Σ. By Lemma 3.1, we conclude that $S \to T$ is in Σ, completing the proof of the lemma.

LEMMA 3.3. *Let* $\mathbf{X} = \emptyset$. *For* $\Sigma \subseteq \mathscr{U}_{\alpha,\omega}$ *and a nonempty finite set R of atomic formulas,*

$$\mathrm{cl}_{\alpha,\omega}(\Sigma \cup \{\emptyset \to R\}) = \bigcap_{\varrho \in R} \mathrm{cl}_{\alpha,\omega}(\Sigma \cup \{\emptyset \to \{\varrho\}\}).$$

PROOF. Since the sentence $\emptyset \to R$ is in the right side, the left side is contained in the right. We can assume that Σ is $\langle \alpha, \omega \rangle$-closed. Let $\Sigma' = \mathrm{cl}_{\alpha,\omega}(\Sigma \cup \{\emptyset \to R\})$ and let $R = \{\varrho_1, \ldots, \varrho_n\}$. For $1 \leqq i \leqq n$, let $\Sigma_i = \Sigma \cup \{\emptyset \to \{\varrho_i\}\}$, and let S be a finite set of atomic formulas with $|S| \leqq \alpha$. We first assume that $\alpha \geqq 1$. Let

$$\Sigma(S) = \mathrm{cl}_{\alpha,\omega}(\Sigma \cup \{\emptyset \to \{\sigma\} \mid \sigma \in S\}),$$

and define $\Sigma'(S)$ and $\Sigma_i(S)$ similarly. Suppose that $\Sigma_i \vdash S \to T$ whenever $1 \leqq i \leqq n$. By (U8), it follows that $\emptyset \to T$ is in $\Sigma_i(S)$ for all i. By Lemma 3.2, $\{\varrho_i\} \to T$ is in $\Sigma(S)$ for all i. Therefore, we can apply (U8) to $\emptyset \to R$ to conclude that $\emptyset \to T$ is in $\Sigma'(S)$. By Lemma 3.2, $S \to T$ is in Σ', completing the proof for nonzero α.

Now let $\alpha = 0$, and retain the meaning of Σ', n, ϱ_i, and Σ_i. Assume that $\Sigma_i \vdash \emptyset \to T_i$ for $1 \leqq i \leqq n$, and let $T = T_1 \cup \ldots \cup T_n$. We prove by induction on the sum of lengths of the n proofs $\Sigma_i \vdash \emptyset \to T_i$ that Σ' contains $\emptyset \to T$. If Σ contains $\emptyset \to T_i$ for any i, the conclusion is obvious. For any i, it follows that Σ' contains $\emptyset \to T$ whenever (U0) or (U1) is last applied in the proof of $\emptyset \to T_i$ from Σ. Since $\mathbf{X}$ is empty, (U5) does not apply. We can assume that (U2), (U3), (U4), or (U6) was last applied in the proof $\Sigma_1 \vdash \emptyset \to T_1$. Since the first three cases are similar, we consider only (U3). Let $Q = T_2 \cup \ldots \cup T_n$. If $\emptyset \to T' \cup \{p \doteq q\}$ and $\emptyset \to T' \cup \{q \doteq r\}$ both have shorter proofs from Σ_1, then Σ' contains both $\emptyset \to T' \cup \{p \doteq q\} \cup Q$ and $\emptyset \to T' \cup \{q \doteq r\} \cup Q$. Applying (U3), we conclude that $\emptyset \to T$ is in Σ'. Since the final case, that (U6) was last applied, is not difficult, the lemma has been demonstrated.

PROPOSITION 3.4. *If α is finite and* $\Sigma \subseteq \mathscr{U}_{\alpha,\omega}$, *then*

$$\mathrm{cl}_{\alpha,\omega}(\Sigma) = \mathscr{U}_{\alpha,\omega} \cap \mathrm{cl}_{\alpha+1,\omega}(\Sigma).$$

In fact, if Σ *is* $\langle\alpha, \omega\rangle$*-closed, then* $\mathrm{cl}_{\alpha+1,\omega}(\Sigma)$ *is the set of all* $\langle\alpha+1, \omega\rangle$*-sentences* $S\to T$ *for which* $\mathrm{cl}_{\alpha,\omega}(\Sigma^* \cup \{\emptyset\to\{\sigma^*\}|\sigma\in S\})$ *contains* $\emptyset\to T^*$.

PROOF. Let Σ be $\langle\alpha, \omega\rangle$-closed. For a set S of atomic formulas, $\Sigma^*(S)$ denotes

$$\mathrm{cl}_{\alpha,\omega}(\Sigma^* \cup \{\emptyset \to \{\sigma^*\} \mid \sigma\in S\}).$$

Let Σ_1 be the set of all $\langle\alpha+1, \omega\rangle$-sentences $S\to T$ for which $\emptyset\to T^*$ is in $\Sigma^*(S)$. If $S\to T$ is in Σ_1 and $|S|\leqq\alpha$, then $(S\to T)^*$ is in Σ^* by Lemma 3.2; hence, $S\to T$ is in Σ. Thus, it suffices to prove the second statement. By (U8), $\Sigma\subseteqq\Sigma_1$. It follows by Lemma 3.2 (with α replaced by $\alpha+1$) that $(\Sigma_1)^*\subseteqq\mathrm{cl}_{\alpha+1,\omega}(\Sigma^*)$. Since $\Sigma\subseteqq\Sigma_1\subseteqq\mathrm{cl}_{\alpha+1,\omega}(\Sigma)$, it only remains to show that Σ_1 is $\langle\alpha+1, \omega\rangle$-closed. Since $\Sigma^*(S)$ is $\langle 0, \omega\rangle$-closed for any S, Σ_1 is obviously closed under (U1) to (U4), and applications of (U6) in which the right side is increased. Let $S\to T$ be in Σ_1 and let $x_1, x_2, \ldots, x_m$ be all the variables that appear in S or T. Since $\emptyset\to T(\hat{x}_1, \ldots, \hat{x}_m)$ is in $\Sigma^*(S(\hat{x}_1, \ldots, \hat{x}_m))$, it follows by Lemma 1.1 (each $\hat{x}_i$ is replaced by $(p_i)^*$) that $\emptyset\to(T(p_1, \ldots, p_m))^*$ is in $\Sigma^*(S(p_1, \ldots, p_m))$. (We used that Σ is (U5)-closed.) Consequently, Σ_1 is closed under (U5). Σ_1 is (U6)-closed because the only remaining application is an increase of the left side. Σ_1 obviously contains each sentence $\{\sigma\}\to\{\sigma\}$ of (U7). Finally, for (U8), suppose that Σ_1 contains $S\to R_i$ $(1\leqq i\leqq k\leqq\alpha+1)$, and $\{\varrho_1, \ldots, \varrho_k\}\to T$ for any choice of $\varrho_i\in R_i$. Fix $\varrho_1\in R_1$. By Lemma 3.2, $\Sigma^*(S\cup\{\varrho_1\})$ contains $\emptyset\to(R_i)^*$ and $(\{\varrho_2, \ldots, \varrho_k\}\to T)^*$ whenever $2\leqq i\leqq k$ and $\varrho_i\in R_i$. It follows by (U8) that $\emptyset\to T^*$ is also in $\Sigma^*(S\cup\{\varrho_1\})$. Because the choice of $\varrho_1\in R_1$ was arbitrary, it follows by Lemma 3.3 that $\emptyset\to T^*$ is in $\mathrm{cl}_{\alpha,\omega}(\Sigma^*(S)\cup\{\emptyset\to(R_1)^*\})$. Since $\emptyset\to(R_1)^*$ is in $\Sigma^*(S)$ by the definition of Σ_1, we conclude that $\emptyset\to T^*$ is in $\Sigma^*(S)$. Thus, $S\to T$ is in Σ_1, completing the proof of the proposition.

COROLLARY 3.5. *If* $\Sigma\subseteqq\mathcal{U}_{\alpha,\omega}$, *then*

$$\mathrm{cl}_{\alpha,\omega}(\Sigma) = \mathcal{U}_{\alpha,\omega}\cap\mathrm{cl}_{\omega,\omega}(\Sigma).$$

In fact, if Σ *is* $\langle\alpha, \omega\rangle$*-closed, then* $\mathrm{cl}_{\omega,\omega}(\Sigma)$ *is the set of all sentences* $S\to T$ *for which* $\mathrm{cl}_{\alpha,\omega}(\Sigma^*\cup\{\emptyset\to\{\sigma^*\}|\sigma\in S\})$ *contains* $\emptyset\to T^*$.

PROOF. We can assume that α is finite. For any integer $n\geqq\alpha$, repeated applications of the proposition yield upward compatibility from $\langle\alpha, \omega\rangle$ to $\langle n, \omega\rangle$, and a description of $\mathrm{cl}_{n,\omega}(\Sigma)$. Since $\mathrm{cl}_{\omega,\omega}(\Sigma)$ is the union of $\mathrm{cl}_{n,\omega}(\Sigma)$ for all $n\geqq\alpha$, the result follows.

4. Upward compatibility from $\langle\omega, \beta\rangle$ to $\langle\omega, \omega\rangle$

Although there is an analogy between the results of this section and the previous one, the proofs in this section do contain some new ideas.

LEMMA 4.1. *Let* $\mathbf{X}=\emptyset$, *let* $\Sigma\subseteqq\mathcal{U}_{\omega,\beta}$, *let* S *and* T *be finite sets of atomic formulas, and let* I *be a nonempty finite set. For* $i\in I$, *let* R_i *be a nonempty finite set of atomic formulas. If* $S\to R_i$ *is in* $\mathrm{cl}_{\omega,\beta}(\Sigma\cup\{\{\tau\}\to\emptyset|\tau\in T\})$ *for each* $i\in I$, *and the sentence* $\{\varrho_i|i\in I\}\to T$ *is in* $\mathrm{cl}_{\omega,\beta}(\Sigma)$ *for any choice of* $\varrho_i\in R_i$, *then* $\mathrm{cl}_{\omega,\beta}(\Sigma)$ *contains* $S\to T$.

PROOF. We can assume that Σ is $\langle\omega, \beta\rangle$-closed. Let $\Sigma_1=\Sigma\cup\{\{\tau\}\to\emptyset|\tau\in T\}$. For $i\in I$, we call the sentence $S_i\to R_i'$ the *witness* of $S\to R_i$ if it has the shortest proof from Σ_1 for $S_i\subseteq S$ and $R_i'\subseteq R_i$. If each witness is in Σ, or $\{\tau\}\to\emptyset$ is a witness for some $\tau\in T$, then $S\to T$ is in Σ by (U6) and (U8). Similarly as in Lemma 3.1, we prove that Σ contains $S\to T$ by induction on the maximum proof lengths of witnesses, and on how often the maximum occurs. Choose $\theta\in I$ such that the proof of the witness of $S\to R_\theta$ has maximum length, and let $I'=I-\{\theta\}$. The rules (U1), (U5), (U6), and (U7) were not used in the last step of the proof of $S_\theta\to R_\theta'$. We omit the cases when (U2) or (U4) was last applied. Suppose that (U3) was applied to $S_\theta\to Q\cup\{p\doteq q\}$ and $S_\theta\to Q\cup\{q\doteq r\}$ in the last step. Let $I_1=I'\cup\{u, v\}$ for two new elements u and v. We define a new system $S\to R_i$, $i\in I_1$, by retaining the original meaning of R_i for $i\in I'$, and defining $R_u=Q\cup\{p\doteq q\}$ and $R_v=Q\cup\{q\doteq r\}$. Clearly, the witnesses of the new system have shorter proofs. Let $P=\{\varrho_i|i\in I_1\}$ be a selection for R_i, $i\in I_1$. If ϱ_u or ϱ_v is in Q, then P contains a selection for the original sets R_i, $i\in I$; thus, $P\to T$ is in Σ by (U6). We can now assume that ϱ_u is $p\doteq q$ and ϱ_v is $q\doteq r$. Since Σ contains

$$\{p \doteq q, q \doteq r\} \to \{p \doteq r\},$$

$P\to\{p\doteq r\}$ is in Σ. Let $P'=\{\varrho_i|i\in I'\}$. Since $P'\cup\{p\doteq r\}$ is a selection for R_i, $i\in I$, Σ contains $P'\cup\{p\doteq r\}\to T$. Since $P\to\{\varrho_i\}$ is in Σ whenever $i\in I'$, it follows by (U8) that $P\to T$ is in Σ. Therefore, by induction, $S\to T$ is in Σ.

We now assume that (U8) was last applied. Let J be a nonempty finite set disjoint from I, and suppose that $S_\theta\to P_j$ and $Q\to R_\theta'$ have shorter proofs whenever $j\in J$ and Q is a selection for P_j, $j\in J$. Fix such a selection Q, let $\{\varrho_i|i\in I'\}$ be a selection for R_i, $i\in I'$, and define $Q'=Q\cup P'$. We consider the system $Q'\to\{\varrho_i\}$, $i\in I'$ together with $Q'\to R_\theta$. Since the witnesses for all sentences but the last have proofs of zero length, it follows easily that the induction hypotheses apply. Thus, we conclude that $Q'\to T$ is in Σ. Let $I_2=I'\cup J$. We define a new system $S\to R_i$, $i\in I_2$, by retaining the meaning of R_i for $i\in I'$, and defining R_j to be P_j for $j\in J$. Since any selection for R_i, $i\in I_2$, is of the form Q' considered above, it follows by induction that $S\to T$ is in Σ, completing the proof of the lemma.

LEMMA 4.2. *Let* $\mathbf{X}=\emptyset$, *let* $\Sigma\subseteq\mathscr{U}_{\omega,\beta}$, *and let T be a finite set of atomic formulas with* $|T|\leqq\beta$. *If* $S\to\emptyset$ *is in* $\mathrm{cl}_{\omega,\beta}(\Sigma\cup\{\{\tau\}\to\emptyset|\tau\in T\})$, *then* $S\to T$ *is in* $\mathrm{cl}_{\omega,\beta}(\Sigma)$.

PROOF. We can assume that Σ is $\langle\omega, \beta\rangle$-closed. By induction on the length of the proof of $S\to\emptyset$ from $\Sigma_1=\Sigma\cup\{\{\tau\}\to\emptyset|\tau\in T\}$, we show that $S\to T$ is in Σ. The conclusion is obvious if (U0) is last applied to prove $S\to\emptyset$. (U5) is excluded since $\mathbf{X}$ is empty. (U1), (U2), (U3), and (U7) are excluded by the form of the sentence $S\to\emptyset$. We omit the easy case (U6). Suppose that $S\to R_i$ $(1\leqq i\leqq k<\omega)$, and for any choice of $\varrho_i\in R_i$, $\{\varrho_1, \ldots, \varrho_k\}\to\emptyset$ have shorter proofs from Σ_1. By induction, $\{\varrho_1, \ldots, \varrho_k\}\to\emptyset$ are in Σ. By Lemma 4.1, $S\to T$ is in Σ, completing the proof.

LEMMA 4.3. *Let* $\mathbf{X}=\emptyset$. *For* $\Sigma\subseteq\mathscr{U}_{\omega,\beta}$ *and a nonempty finite set R of atomic formulas,*

$$\mathrm{cl}_{\omega,\beta}(\Sigma\cup\{R\to\emptyset\}) \equiv \bigcap_{\varrho\in R} \mathrm{cl}_{\omega,\beta}(\Sigma\cup\{\{\varrho\}\to\emptyset\,|\,\varrho\in R\}).$$

Lemma 4.3 follows immediately from the proof of Proposition 4.4. It is included to preserve the analogy with the previous section.

PROPOSITION 4.4. *If β is finite and $\Sigma \subseteq \mathcal{U}_{\omega,\beta}$, then*

$$\mathrm{cl}_{\omega,\beta}(\Sigma) = \mathcal{U}_{\omega,\beta} \cap \mathrm{cl}_{\omega,\beta+1}(\Sigma).$$

In fact, if Σ is $\langle\omega, \beta\rangle$-closed, then $\mathrm{cl}_{\omega,\beta+1}(\Sigma)$ is the set of all $\langle\omega, \beta+1\rangle$-sentences $S \to T$ for which $\mathrm{cl}_{\omega,\beta}(\Sigma^* \cup \{\{\tau^*\} \to \emptyset \mid \tau \in T\})$ contains $S^* \to \emptyset$.

PROOF. Let Σ be $\langle\omega, \beta\rangle$-closed. For a set T of atomic formulas, $\Sigma^*[T]$ denotes

$$\mathrm{cl}_{\omega,\beta}(\Sigma^* \cup \{\{\tau^*\} \to \emptyset \mid \tau \in T\}).$$

Let Σ_1 be the set of all $\langle\omega, \beta+1\rangle$-sentences $S \to T$ for which $S^* \to \emptyset$ is in $\Sigma^*[T]$. If $S \to T$ is in Σ_1 and $|T| \leqq \beta$, then $(S \to T)^*$ is in Σ^* by Lemma 4.2; hence, $S \to T$ is in Σ. Thus, it suffices to prove the second statement. By (U8), $\Sigma \subseteq \Sigma_1$. It follows by Lemma 4.2 (with β replaced by $\beta+1$) that $(\Sigma_1)^* \subseteq \mathrm{cl}_{\omega,\beta+1}(\Sigma^*)$. Since $\Sigma \subseteq \Sigma_1 \subseteq \mathrm{cl}_{\omega,\beta+1}(\Sigma)$, it only remains to show that Σ_1 is $\langle\omega, \beta+1\rangle$-closed. Since the proof is analogous to that of Proposition 3.4, we shall only consider applications of (U3) and (U8). Suppose that both $S \to T \cup \{p \doteq q\}$ and $S \to T \cup \{q \doteq r\}$ are in Σ_1 with $|T| \leqq \beta$. By Lemma 4.2, $(S \to \{p \doteq q\})^*$ and $(S \to \{q \doteq r\})^*$ are in $\Sigma^*[T]$. Applying (U3), $(S \to \{p \doteq r\})^*$ is in $\Sigma^*[T]$, and therefore, $S^* \to \emptyset$ is in $\Sigma^*[T \cup \{p \doteq r\}]$. Thus, Σ_1 is (U3)-closed.

For (U8), assume that Σ_1 contains $S \to R_i$ $(1 \leqq i \leqq k < \omega)$, and $\{\varrho_1, \dots, \varrho_k\} \to T$ whenever $\varrho_i \in R_i$. For each i, fix $\varrho_i \in R_i$. If $|R_i| = \beta+1$, let $R_i' = R_i - \{\varrho_i\}$; otherwise, $R_i' = R_i$. Clearly, $1 \leqq |R_i'| \leqq \beta$. By Lemma 4.2, $(S \to R_i')^*$ is in $\Sigma^*[\{\varrho_i\}]$ for each i. Therefore, by (U8), $S^* \to \emptyset$ is in $\Sigma^*[T \cup \{\varrho_1, \dots, \varrho_k\}]$. To show that $S^* \to \emptyset$ is in $\Sigma^*[T]$, we shall require k applications of (U8) supplemented with symmetry arguments. Since one case reveals the argument, we assume that $k=3$. By Lemma 4.2, $\Sigma^*[T \cup \{\varrho_2, \varrho_3\}]$ contains $(S \to \{\varrho_1\})^*$, $(S \to R_2')^*$, and $(S \to R_3')^*$. Applying (U8), $S^* \to \emptyset$ is in $\Sigma^*[T \cup \{\varrho_2, \varrho_3\}]$. By symmetry, $S^* \to \emptyset$ is also in $\Sigma^*[T \cup \{\varrho_1, \varrho_3\}]$. By Lemma 4.2, $\Sigma^*[T \cup \{\varrho_3\}]$ contains $(S \to \{\varrho_1\})^*$, $(S \to \{\varrho_2\})^*$, and $(S \to R_3')^*$; thus, $S^* \to \emptyset$ is in $\Sigma^*[T \cup \{\varrho_3\}]$ by (U8). By symmetry, $S^* \to \emptyset$ is in $\Sigma^*[T \cup \{\varrho_i\}]$ for each i. By Lemma 4.2, $\Sigma^*[T]$ contains $(S \to \{\varrho_i\})^*$ for each i; hence, by (U8), $\Sigma^*[T]$ contains $S^* \to \emptyset$. Thus, $S \to T$ is in Σ_1, completing the proof.

COROLLARY 4.5. *If $\Sigma \subseteq \mathcal{U}_{\omega,\beta}$, then*

$$\mathrm{cl}_{\omega,\beta}(\Sigma) = \mathcal{U}_{\omega,\beta} \cap \mathrm{cl}_{\omega,\omega}(\Sigma).$$

In fact, if Σ is $\langle\omega, \beta\rangle$-closed, then $\mathrm{cl}_{\omega,\omega}(\Sigma)$ is the set of all sentences $S \to T$ for which $\mathrm{cl}_{\omega,\beta}(\Sigma^* \cup \{\{\tau^*\} \to \emptyset \mid \tau \in T\})$ contains $S^* \to \emptyset$.

PROOF. See the proof of Corollary 3.5.

5. Completeness

We shall omit both α and β when $\alpha = \beta = \omega$. Recall that $\mathbf{F}$, called falsity, denotes the sentence $\emptyset \to \emptyset$.

LEMMA 5.1. *Assume that $\mathbf{X} = \emptyset$ and that $\alpha = \omega$, $\beta = 1$, or $\beta = \omega$. If $\mathrm{cl}_{\alpha,\beta}(\Sigma)$ contains $S \to T$ but does not contain $\mathbf{F}$, then $\mathbf{F} \notin \mathrm{cl}(\Sigma \cup \{\{\sigma\} \to \emptyset\})$ for some $\sigma \in S$ or $\mathbf{F} \notin \mathrm{cl}(\Sigma \cup \{\emptyset \to \{\tau\}\})$ for some $\tau \in T$.*

PROOF. By Proposition 2.1, Corollary 3.5, and Corollary 4.5, we can assume that $\alpha=\beta=\omega$. Also, we can assume that Σ is closed. In order to reach a contradiction, we suppose that the hypotheses hold but that none of the conclusions do. By Lemma 4.2, Σ contains $\emptyset\to\{\sigma\}$ for each $\sigma\in S$. If S is nonempty, we conclude that Σ contains $\emptyset\to T$ by (U8). By a similar argument using Lemma 3.2, we conclude that Σ contains $\emptyset\to\emptyset$, a contradiction.

LEMMA 5.2. *Assume that* $\mathbf{X}=\emptyset$ *and that* $\alpha=\omega$, $\beta=1$, *or* $\beta=\omega$. *If* $|S|\leqq\alpha$ *and* $|T|\leqq\beta$ *but* $S\to T$ *is not in* $\mathrm{cl}_{\alpha,\beta}(\Sigma)$, *then* $\mathbf{F}\notin \mathrm{cl}(\Sigma\cup\{\emptyset\to\{\sigma\}|\sigma\in S\}\cup\{\{\tau\}\to\emptyset|\tau\in T\})$.

PROOF. As above, we can assume that $\alpha=\beta=\omega$. If $\emptyset\to\emptyset$ were in the last set, then by Lemma 3.2, $S\to\emptyset$ is in $\mathrm{cl}\,(\Sigma\cup\{\{\tau\}\to\emptyset|\tau\in T\}$. Hence, by Lemma 4.2, $S\to T$ is in $\mathrm{cl}\,(\Sigma)$, contrary to assumption.

Since the completeness results of the theorem now follow by standard methods, we shall not give all the details. Firstly, the syntactic and semantic notions of consistency are shown to be equivalent. The nontrivial direction is given by

PROPOSITION 5.3. *Let* $\alpha=\omega$, $\beta=1$, *or* $\beta=\omega$. *If there is no* $\langle\alpha,\beta\rangle$-*proof of* $\mathbf{F}$ *from* Σ, *then there is a model of* Σ.

PROOF. As above, we can assume that $\alpha=\beta=\omega$. We can also assume that Σ is closed, and by converting to the type ξ^*, that the set of variables is empty. By hypothesis, Σ does not contain $\mathbf{F}$. For each atomic formula σ, $\{\sigma\}\to\{\sigma\}$ is in Σ. Hence, by Lemma 5.1, either $\{\sigma\}\to\emptyset$ or $\emptyset\to\{\sigma\}$ can be added to Σ so that the closure of this new set does not contain $\mathbf{F}$. By iteration, we obtain a closed set Σ' of sentences that contains Σ, does not contain $\mathbf{F}$, and contains $\{\sigma\}\to\emptyset$ or $\emptyset\to\{\sigma\}$ for each atomic formula σ. Let $\mathfrak{A}$ be the structure whose underlying set A is the set of all polynomials without variables modulo Σ' and an atomic formula σ is true in $\mathfrak{A}$ iff $\emptyset\to\{\sigma\}$ is in Σ'. (Two polynomials p and q are identified in A iff $\emptyset\to\{p\doteq q\}$ is in Σ'.) Let $S\to T$ be in Σ. Suppose that no $\sigma\in S$ is false in $\mathfrak{A}$ and that no $\tau\in T$ is true in $\mathfrak{A}$. This means that Σ' contains $\emptyset\to\{\sigma\}$ and $\{\tau\}\to\emptyset$ whenever $\sigma\in S$ and $\tau\in T$. Since Σ' also contains $S\to T$, an application of (U8) shows that $\emptyset\to T$ is in Σ', and a second application shows that $\emptyset\to\emptyset$ is in Σ'. This contradiction implies that $\mathfrak{A}$ is a model of Σ, completing the proof.

PROPOSITION 5.4. *Let* $\alpha=\omega$, $\beta=1$, *or* $\beta=\omega$. *If there is no* $\langle\alpha,\beta\rangle$-*proof of* $S\to T$ *from* Σ, *then* $S\to T$ *is not a first-order consequence of* Σ.

PROOF. As above, we can assume that $\alpha=\beta=\omega$, that Σ is closed, and that the set of variables is empty. By Lemmas 5.2 and 5.3, there is a structure that satisfies

$$\Sigma\cup\{\emptyset\to\{\sigma\}\,|\,\sigma\in S\}\cup\{\{\tau\}\to\emptyset\,|\,\tau\in T\}.$$

Since $S\to T$ fails in this structure, the proof of the proposition is complete.

The last proposition finishes the proof of the theorem. For $\langle\alpha,\beta\rangle=\langle\omega,1\rangle$ (universal Horn sentences), we used Propositions 2.1, 3.2, and 4.2 from previous sections. Observe that the generality of the last two results is not required in this case. (Versions for $\alpha=\beta=\omega$ would suffice.)

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PROBLEMS

RAISED AT THE PROBLEM SESSION OF THE VISEGRÁD CONFERENCE

A conference on universal algebra was held on May 30—June 6, 1982 in the resort house of the Eötvös University (Budapest) in Visegrád, Hungary. Some of the contributions by the 38 participants are published in this issue. The problems raised at the problem session (on June 3, 1982) were the following.

HAJNAL ANDRÉKA and ISTVÁN NÉMETI

1. Is $ZF \models (\exists n \in \omega)\ P^n = P^{n+1}$ true? That is: Is $(\exists n \in \omega)\ P^n = P^{n+1}$ true without the axiom of choice? Here P is defined to be such that $P = IP$ (otherwise $P \neq PP$ would be the case).

2. Are all epimorphisms surjective in the variety CA_α of cylindric algebras of infinite dimension α?

REMARK. I. Sain proved that they are surjective if $\alpha = 1$, and H. Andréka, S. Comer, I. Németi proved that they are not surjective if $1 < \alpha < \omega$. Related results: Andréka, Comer, and Németi showed that for representable CA_α-s, α finite, the picture is the same as in CA_α.

3. Does there exist a finitely generated pseudosimple algebra which is not simple?

HINT. Such an algebra, if there is any, must have infinite similarity type.

ALAN DAY

1. Is there some sort of minimal list of minimal non-Arguesian lattices (of finite length)?

2. Is a self dual finitely based variety of lattices determined by self dual equations?

KAZIMIERZ GŁAZEK

1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be n-groups and $\mathrm{gl}(\mathfrak{A})$ and $\mathrm{gl}(\mathfrak{B})$ their global algebras, respectively. Is the implication

$$\mathrm{gl}(\mathfrak{A}) \cong \mathrm{gl}(\mathfrak{B}) \Rightarrow \mathfrak{A} \cong \mathfrak{B}$$

valid?

2. Let $+$ be a commutative group (or quasigroup, or inverse semigroup) operation. Consider the ternary operations $f_1(x, y, z) = x+y+z$, $f_2(x, y, z) = x+y-z$ $f_3(x, y, z) = x-y+z$, $\dots, f_8(x, y, z) = -x-y-z$ (or more generally: $g(x, y, z) =$

$\alpha x+\beta y+\gamma z$, where α, β, γ are group automorphisms). Give a finite basis for the identities in $*$ when $* \in K \subseteq \{f_1, \ldots, f_8\}$.

3. Let H and G be two groups such that $H \lhd G$ and G/H is isomorphic to the multiplicative group of the rationals. Does there exist a field F of characteristic 0 such that G is isomorphic to the group of all weak automorphisms of F and $H \cong \cong \mathrm{Aut}\, F$?

4. Investigate algebras $\mathfrak{A}$ with the following properties:

(i) $\mathfrak{A}$ has no proper subalgebra, and

(ii) every function $f\colon A^n \to A$ which is preserved by any weak automorphism τ of $\mathfrak{A}$ (i.e., $f=\tau \circ f \circ \tau^{-1}$) is a term of $\mathfrak{A}$. (Proposed name: "super-demi primal algebras".)

5. (with J. Dudek) Which algebras are n-groupoids for some natural number n? (I.e., $\mathfrak{A}=(A; F)$ is termally (polynomially) equivalent to $(A; g)$ where $g\colon A^n \to A$.)

6. (with J. Dudek) Investigate the parameter $n(\mathfrak{A})= \min \{n \mid \exists g\colon A^n \to A$ such that $(A; g)$ is polynomially equivalent to $\mathfrak{A}=(A; F)\}$. (The same question with "termally" instead of "polynomially".)

7. Which algebras are Aut-derived from groupoids, i.e., termally (or polynomially) equivalent to the algebra $(A; \{\circ\} \cup \mathrm{Aut}\,(A; \circ))$, where $\circ$ is a binary operation on A?

MATTHEW I. GOULD

1. Given a finite group G of even order, does there exist a *finite* algebra $\mathfrak{A}$ such that $G \cong \mathrm{Aut}\,(\mathfrak{A} \times \mathfrak{A})$? It is equivalent to ask for a finite free algebra on two generators such that $G \cong \mathrm{Aut}\, \mathfrak{F}$. If G retracts onto a two element subgroup, the answer is affirmative. Thus, the smallest group for which the problem is open is $\mathbf{Z}_4$.

DAVID KELLY (with R. Padmanabhan)

1. Let $\mathscr{K}$ be a variety of groupoids such that every algebra in $\mathscr{K}$ is cancellative. Show that every algebra in $\mathscr{K}$ is (the reduct of) a quasigroup.

2. Let the type be $\langle 2 \rangle$ (i.e., groupoids), and let $\Sigma \cup \{\sigma\}$ be a set of identities of type $\langle 2 \rangle$. Let $\vdash_Q$ denote the consequence in the language of quasigroups and $\vdash_C$ that in the language of cancellative groupoids. Find a counterexample to the statement: "If $\Sigma \vdash_Q \sigma$, then $\Sigma \vdash_C \sigma$".

PETER KÖHLER

1. Let $\mathfrak{A}$ be a finite (unary) algebra. Is there a *natural* way to find a finite set Ω and a permutation group G on Ω such that

$$\mathrm{Con}\, \mathfrak{A} \cong \mathrm{Con}\, (\Omega;\ G)?$$

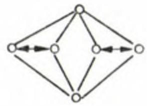
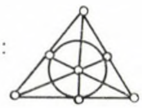
PÉTER P. PÁLFY

1. Characterize those monoids operating on a finite set which occur as the monoid of unary polynomials of essentially unary algebras only.

ROBERT W. QUACKENBUSH

1. Give a nice categorical characterization of varieties of affine algebras (as abelian categories are a nice characterization of varieties of modules).

2. Let $\mathscr{V}$ be the variety generated by the orthomodular lattice L or by the Steiner quasigroup Q.

L :  Q : 

It is not boolean representable; find an easily described subalgebra of L^{ω} (Q^{ω}) which is easily seen not to be boolean representable.

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.NTERSECTIONS OF IMBEDDED SUBGROUPS IN ABELIAN p-GROUPS

WILLIAM J. KEANE

Characterizing the intersections of subgroups of abelian p-groups with various purity properties has been a common problem. For example, Charles solved the problem for divisible subgroups in [2], Rangaswamy for neat subgroups in [7], and Megibben for pure subgroups in [5]. Recently, Moore, in [6], has topologically generalized the notion of purity. If l: $\mathbf{Z}^+ \to \mathbf{Z}^+$ is a strictly increasing function, a subgroup K of the p-group G is l-imbedded, written $K <_l G$, if $K \cap p^{l(n)} G \subset p^n K$, for $n \in \mathbf{Z}^+$. Equivalently, K is imbedded if its p-adic topology coincides with the relative one inherited from G. In this note, we solve the above problem for imbedded subgroups, both with and without the assumption of a fixed function l. Throughout, "group" means abelian p-group, and the notation and terminology is that of [3].

DEFINITION. If G is a group and n a non-negative integer, then

$$G_n = (p^n G)[p] = \{x \in p^n G \mid px = 0\}.$$

To determine which subgroups are intersections of l-imbedded subgroups for a fixed function l, we need the following lemmas.

LEMMA 1. *Let A be a nonzero subgroup of G and K a subgroup of G such that $A \cap K = 0$. Then K is the intersection of all A-high subgroups of G containing K.*

PROOF. Let H be an A-high subgroup containing K, and suppose $x \in H - K$. It will suffice to construct an A-high subgroup L containing K but not x, and we may assume $px \in K$. Choose a nonzero $y \in A[p]$, and let $L' = \langle K, x+y \rangle$. Then $L' \cap A = 0$, so if L is A-high containing L', $x \notin L$.

LEMMA 2. *Let K be a p^n-pure subgroup of G (i.e., $K \cap p^m G = p^m K$ for $m \leqq n$) which contains $p^{l(n+1)-1} G$. Then $K <_l G$.*

PROOF. Clearly, $K \cap p^{l(m)} G \subset p^m K$, for $m \leqq n$, so assume $m > n$, and let $x \in K \cap p^{l(m)} G$. Then $x = p^{l(m)-l(n+1)+1} y$, where $y \in p^{l(n+1)-1} G \subset K \cap p^n G = p^n K$. Hence $x \in p^{l(m)-l(n+1)+n+1} K \subset p^{m-(n+1)+n+1} K = p^m K$.

THEOREM 1. *A subgroup A of G is the intersection of l-imbedded subgroups if and only if, for each non-negative integer n, $G_n \subset A$ implies $p^{l(n+1)-1} G \subset A$.*

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Key words and phrases. Imbedded subgroups, p-adic topology, closed subgroups, high subgroups.

Proof. Since every l-imbedded subgroup must satisfy the condition (see [4]), the necessity is clear. For sufficiency, if $G_n \not\subset A$ for all $n \in \mathbf{Z}^+$, A is the intersection of pure and hence l-imbedded subgroups, by the Theorem in [5]. Now if $G_0 = G[p] \subset A$, that is if A is essential in G, then $p^{l(1)-1}G \subset A$, and by Lemma 2, A is in fact l-imbedded in G. Suppose then that $G_n \subset A$, $G_{n-1} \not\subset A$, and let $g \in G_{n-1} - A$. Then by Lemma 1 A is the intersection of all $\langle g \rangle$-high subgroups K containing A, each of which then contains $p^{l(n+1)-1}G$. By Lemma 2 it is enough to show that each K is p^n-pure in G. Now K is neat in G, so suppose $K \cap p^m G = p^m K$ for $m < n$, and let $x \in K$, $x = p^{m+1}y$, $y \in G$. Then $x = pz$, for $z \in K$, so $z - p^m y \in G[p]$, and $z - p^m y = g' + y'$, $g' \in \langle g \rangle[p]$, $y' \in K[p]$. Then $z - y' \in K \cap p^m G = p^m K$. Hence $x = p(z - y') \in p^{m+1}K$.

We now remove the restriction of a fixed function l.

Theorem 2. *A subgroup A of G is the intersection of imbedded subgroups of G if and only if, for each non-negative integer n, $G_n \subset A$ implies A is closed.*

Proof. If K is l-imbedded and $G_n \subset K$, then K also contains $p^{l(n+1)-1}G$, so G/K is bounded and hence K is closed. Thus if $G_n \subset A$, and A is the intersection of imbedded subgroups, A is also closed. Conversely, we again need only consider the case when $G_n \subset A$ and A is closed. Now for each non-negative integer k, $A + p^k G <_{l_k} G$, where $l_k(i) = i + k$. But $\bar{A} = A = \bigcap_k (A + p^k G)$.

We note that by the proof of Theorem 2, if $A \subset G$ is the intersection of imbedded subgroups but not of pure subgroups, then the imbedded subgroups may be chosen to be a countable family.

The first corollary to Theorem 2 is an immediate consequence, but we give a slightly different proof using the next lemma, which is of interest in its own right.

Lemma 3. *If $A <_l G$, then A is pure in $\bar{A}$.*

Proof. Let $x \in A$, $x = p^n y$, for $y \in \bar{A}$. Then $y = p^{l(n)-n}g + x'$, for some $g \in G$, $x' \in A$, so we have $x = p^{l(n)}g + p^n x' = p^n x'' + p^n x'$, for some $x'' \in A$, since $p^{l(n)}g \in A <_l G$. Thus $x \in p^n A$.

Corollary 1. *If A is essential in G, then A is the intersection of imbedded subgroups if and only if A is closed.*

Proof. An essential imbedded subgroup of G is essential and pure in its closure, and hence closed. The proof is then similar to that of the theorem.

Recently (see [1]), some attention has been given to finite intersections of subgroups. We conclude by showing that in one important case, this question can be easily resolved for imbedded subgroups.

Corollary 2. *If a subgroup A of G is the intersection of a finite number of imbedded subgroups, but is not the intersection of pure subgroups, then A is imbedded.*

Proof. If $A = \bigcap_{i=1}^{m} K_i$, we can find a function l such that $K_i <_l G$, for $i = 1, \ldots, m$. Now since $G_n \subset A$, for some n, by Theorem 1, $A \supset p^{l(n+1)-1}G$, and is thus imbedded for a sufficiently large imbedding function.

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SUR LES INTERSECTIONS DES SURFACES ALEATOIRES AVEC DES HYPERPLANS

MARIO WSCHEBOR

Au Professeur István Vincze pour son 70-ème anniversaire

0. Introduction

Cet article contient deux résultats liés à la formule de Rice d-dimensionelle: le premier est de caractère local, le deuxième est une formule pour les moments d'ordre supérieur de la mesure des intersections avec des hyperplans horizontales. Nous allons utiliser les mêmes notations que dans [10]. Si O est un ensemble ouvert dans R^d et B un Borélien, le «perimètre de B relativement à O» (fini ou infini) est défini par:

$$Q_O(B) = \sup\left\{\int_B \operatorname{div}(u)\,dt:\ u\in\left(C_k^\infty(O)\right)^d,\ \|u(t)\| \leqq 1\ \forall t\in R^d\right\}$$

où $\left(C_k^\infty(O)\right)^d$ note les fonctions C^∞ à valeurs dans R^d et support compacte contenu dans O, $\|.\|$ la norme euclidienne; la mesure de Lebesgue dans R^d va être representée indifféremment par (dt) ou $d\mu_d(t)$. Pour les propriétés et méthodes d'estimation de $Q_O(B)$ voir [7]. Si $\{X(t):\ t\in R^d\}$ est un processus aléatoire à d paramètres réels, on va noter A_u, B_u, C_u respectivement les ensembles aléatoires

$$\{t:\ X(t) < u\},\quad \{t:\ X(t) > u\},\quad \{t:\ X(t) = u\}.$$

Nous supposons par la suite que, avec probabilité égale à 1, les trajectoires du processus sont continûment différentiables et qu'il y a une densité jointe

$$p_{t_1,\dots,t_k;\,t_1',\dots,t_h'}(x_1,\dots,x_k;\ \dot{x}_1,\dots,\dot{x}_h)$$

de $X(t_1),\dots,X(t_k)$; grad $(X(t_1')),\dots,$ grad $(X(t_h'))$ pour $t_i\neq t_j$, $t_i'\neq t_j'$ si $i\neq j$.

Le rapport entre, d'une part, les formules globales concernant $Q_T(A_u)$ et $Q_T(B_u)$ — T étant un ouvert borné dans R^d — et leurs interprétations en termes de C_u, et, d'autre part, la presque sûre non-existence d'extrema locaux sur la barrière de niveau égale à u, est connu. Plus précisément, pour que $\partial A_u = \partial B_u = C_u$ (∂B dénote la frontière essentielle du Borélien B, i.e., $\partial B = \bigcap\{\tilde{\partial}(B\triangle N):\ \mu_d(N)=0\}$ où $\tilde{\partial}$ indique la frontière ordinaire), il faut et il suffit que p.s. il n'y ait pas d'extrema locaux du processus sur la barrière u. (Voir, pour $d=1$, [4] et références citées et [3], [11]. Pour $d>1$, [2] et [10]). Le Théorème 2 de [10] permet d'assurer cette conclusion quand les hypothèses suivantes sont vérifiées:

1) La densité jointe $p_{t;t}(x,\dot{x})$ est une fonction bornée pour x dans un voisinage de u et $(t,\dot{x})$ dans un compacte de $R^d\times R^d$.

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2) Le champ aléatoire grad $(X(t))$ satisfait, en probabilité, une condition de Hölder d'ordre plus grand que $\alpha_d=(d-1)/(d+1)$, c'est-à-dire, que pour tout $\varepsilon>0$ et tout ensemble compacte K dans R^d, il existe $\alpha>\alpha_d$ et une constante positive γ telles que

$$\mathsf{P}\Big(\sup_{\substack{\|t'-t''\|<\delta\\ t',t''\in K}} \|\mathrm{grad}\,(X(t')-X(t''))\| > \gamma\delta^\alpha\Big) < \varepsilon$$

pour tout $\delta>0$.

Dans le paragraphe 1 nous montrons que cette condition 2) sur la régularité des trajectoires est précise, dans le sens que l'on peut construire un processus $\{X(t)\colon t\in R^d\}$ $(d>1)$, qui vérifie 1) etd ont grad$(X(t))$ satisfait une condition de Hölder d'ordre α_d pour toutes les trajectoires, et d'autre part $\mathsf{P}\big(\min_{t\in R^d} X(t)=0\big)=1$, c'est-à-dire qu'il ne vérifie pas la conclusion du théorème.

Le paragraphe 2 contient des formules pour les moments d'ordre supérieur de $Q_T(A_u)$ et $Q_T(B_u)$. Pour $d=1$, ces formules sont connues depuis longtemps, et ont été démontrées sous certaines conditions dans les cas gaussien [1], et dans le cas général [12]. Nous avons inclu une démonstration qui contient une bonne partie des méthodes dont on a besoin après, pour le cas $d>1$. L'enoncé sur la finitude des moments d'ordre supérieur dans le cas $d=1$ contenu dans le Théorème 2, peut-être amélioré si on ajoute l'hypothèse que le processus est gaussien [1], ou gaussien et stationnaire [5], [6], [8]. Ce Théorème peut être utilisé à son tour pour vérifier les hypothèses du Théorème 4 qui concerne le cas $d>1$.

1. Exemple sur l'existence d'extrema locaux d'une surface aléatoire sur une barrière donnée

Théorème 1. *Il existe un processus à d paramètres* $\{X(t)\colon t\in R^d\}$ *tel que :*

1) $p_{t;t}(x,\dot{x})$ *est une fonction bornée pour x dans un voisinage de* $u=0$ *et* $(t,\dot{x})$ *dans un compacte de* $R^d\times R^d$.

2) grad $(X(t))$ *vérifie une condition de Hölder d'ordre* $\alpha_d=(d-1)/(d+1)$ *pour toutes les trajectoires.*

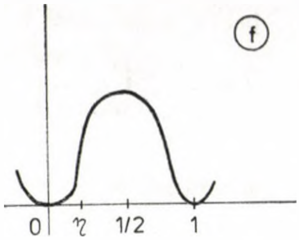
3) $\mathsf{P}\big(\min_{t\in R^d} X(t)=0\big)=1$.

Démonstration. Nous donnons une construction explicite simple. Considérons le processus:

$$X(t)=X(t_1,\dots,t_d)=\sum_{i=1}^d f(\zeta_i t_i+\xi_i) \tag{1}$$

où $\xi_1,\dots,\xi_d,\zeta_1,\dots,\zeta_d$ sont $2d$ variables aléatoires indépendantes, uniformément distribuées sur $(0,1)$, et f est une fonction de R^1 dans R^1, périodique de période égale à 1, qui sur l'intervalle $]0,1[$ est positive, infiniment dérivable, pair par rapport à 1/2 et coïncide avec la fonction $x^{1+\alpha_d}$ sur un intervalle de la forme $[0,\eta]$, $0<\eta<1/2$.

Il est évident que toutes les trajectoires du processus défini par (1) ont gradient hölderien d'ordre α_d et que $\min_{t\in R^d} X(t)=0$.



Voyons donc, qu'il vérifie la condition 1).

On prouve d'abord, sans grande difficulté, que le processus est stationnaire. En effet, si $h^j=(h_1^j, \ldots, h_d^j)$ et $Y^j=X(t+h^j)$ $(j=1, \ldots, n)$, la distribution jointe de $Y^1, \ldots, Y^n$ est indépendante de t, ce que l'on voit en conditionnant sur $\zeta_1, \ldots, \zeta_d$ et en tenant compte du fait que f est de période égale à 1 et que la distribution conditionnelle de $\xi_1', \ldots, \xi_d'$ $(\xi_i'=\mathrm{fr}(\xi_i+\zeta_i t_i)$, où $\mathrm{fr}(x)$ note la partie fractionnaire du nombre réel x), étant données $\zeta_1, \ldots, \zeta_d$, est aussi celle de d variables indépendantes avec distribution uniforme sur $(0, 1)$.

Donc, la densité $p_{t,t}(x, \dot{x})$ est indépendante de t. Posons $a=1+\alpha_d$.

Pour

$$0 < x < x+\Delta x < f(\eta) = \eta^a$$

$$x_1^a+x_2^a+\ldots+x_{d-1}^a \leqq x$$

$$\dot{x}_i > 0 \qquad (i = 1, \ldots, d)$$

la probabilité conditionnelle :

$$\mathsf{P}\Big(X(0)\in(x, x+\Delta x), \frac{\partial X}{\partial t_i}\Big|_0 \in(\dot{x}_i, \dot{x}_i+\Delta \dot{x}_i) \quad (i = 1, \ldots, d)/$$

$$\xi_1 = x_1, \ldots, \xi_{d-1} = x_{d-1}\Big)\Big)$$

(2)
$$= \mathsf{P}\Big(f(\xi_d)\in\Big(x-\sum_1^{d-1} x_i^a, x-\sum_1^{d-1} x_i^a+\Delta x\Big),$$

$$\zeta_i f'(x_i)\in(\dot{x}_i, \dot{x}_i+\Delta\dot{x}_i) \quad (i = 1, \ldots, d-1),$$

$$\zeta_d f'(\xi_d)\in(\dot{x}_d, \dot{x}_d+\Delta\dot{x}_d)/\xi_1 = x_1, \ldots, \xi_{d-1} = x_{d-1}\Big)$$

si $\Delta\dot{x}, \ldots, \Delta\dot{x}_d$ sont suffisamment petits.

Évidemment, la condition peut être supprimée dans le probabilité conditionnelle (2).

Quand on a la distribution de $f(\xi)$, ξ étant uniformément distribuée sur $(0, 1)$, on a pour t suffisemment petit et $f^{-1}(t)<1/2$:

$$\mathsf{P}(f(\xi) \leqq t) = 2\mathsf{P}(\xi \leqq f^{-1}(t)) = 2f^{-1}(t) = 2t^{1/a}.$$

Donc, la densité $g(t)$ de $f(\xi)$ est égale (pour t petit) à $\dfrac{2}{a}\, t^{(1/a)-1}$, et en substituant dans (2), pour $\Delta x, \Delta \dot{x}_1, \dots, \Delta \dot{x}_d$ petits, nous avons la majoration

$$\text{(3)} \qquad (\text{const})\, \Delta x \Delta \dot{x}_1, \dots, \Delta \dot{x}_d\, g\Big(x - \sum_1^{d-1} x_i^a\Big) \left(\prod_{i=1}^{d-1} \frac{1}{a x_i^{a-1}} \right) \frac{1}{a\big(x - \sum_1^{d-1} x_i^a\big)^{(a-1)/a}}$$

de la probabilité (2). Ici, nous avons utilisé le fait que la densité jointe de $\zeta_1, \dots, \zeta_d$ est partout majorée par 1.

Écrivons maintenant, pour x petit et $\dot{x}_i \neq 0$ $(i = 1, \dots, d)$:

$$\text{(4)} \qquad p_{0;0}(x, \dot{x}) = p_{0;0}(x, \dot{x}_1, \dots, \dot{x}_d) = \underset{\substack{x_1^a + \dots + x_{d-1}^a \leqq x \\ x_i \geqq 0\ (i=1,\dots,d-1)}}{\overset{d-1}{\int \dots \int}} p_{0;0}(x;\ \dot{x}_1, \dots, \dot{x}_d / \xi_1 = x_1, \dots, \xi_{d-1} = x_{d-1})\, dx_1, \dots, dx_{d-1}.$$

Il faut remarquer que nous pouvons avoir $f(x_1) + \dots + f(x_d) = x$ avec chacune des variables $x_1, \dots, x_d$ près de **0** ou de **1**, mais que l'un ou l'autre sont univoquement déterminés par les signes de $\dot{x}_1, \dots, \dot{x}_d$.

En substituant le majoration (3) dans l'égalité (4) on obtient :

$$p_{0;0}(x, \dot{x}) \leqq (\text{const}) \underset{\substack{x_1^a + \dots + x_{d-1}^a \leqq x \\ x_i \geqq 0\ (i=1,\dots,d-1)}}{\overset{d-1}{\int \dots \int}} \Big(x - \sum_{i=1}^{d-1} x_i^a\Big)^{(1/a)-1} \prod_{i=1}^{d-1} \frac{1}{x_i^{a-1}} \frac{dx_1, \dots, dx_{d-1}}{\big(x - \sum_1^{d-1} x_i^a\big)^{(a-1)/a}}$$

et en faisant :

$$x_i^a = x\sigma_i, \quad dx_i = x^{1/a} \frac{1}{a} \sigma_i^{(1/a)-1} d\sigma_i \qquad (i = 1, \dots, d-1):$$

$$p_{0;0}(x, \dot{x}) \leqq$$

$$\leqq (\text{const}) \underset{\substack{\sigma_1 + \dots + \sigma_{d-1} \leqq 1 \\ \sigma_i \geqq 0\ (i=1,\dots,d-1)}}{\overset{d-1}{\int \dots \int}} \frac{1}{\big(1 - \sum_1^{d-1} \sigma_i\big)^{2-(2/a)}} \frac{d\sigma_1 \dots d\sigma_{d-1}}{(\sigma_1 \dots \sigma_{d-1})^{2-(2/a)}} \frac{1}{x^{((a-1)/a)(d+1)-(d-1)/a}} =$$

$= (\text{const})$,

puisque $a = 1 + (d-1)/(d+1)$. Ceci montre que si x est suffisemment petit, $p_{0;0}(x, \dot{x})$ est bornée, donc, que le processus $\{X(t) : t \in R^d\}$ défini par (1) satisfait aux besoins de notre construction.

2. Moments d'ordre supérieur de $Q_T(A_u)$ et $Q_T(B_u)$

Le but de cette section est de donner des formules pour les moments d'ordre supérieur à 1 des variables aléatoires $Q_T(A_u)$ et $Q_T(B_u)$. Le premier moment a été calculé sous diverses conditions (voir [9], [10] et références citées). Les formules qu'on obtient pour $Q_T(A_u)$ et $Q_T(B_u)$ sont les mêmes; donc, nous allons nous restreindre à la considération de $Q_T(A_u)$.

Les outils pour les démonstrations sont les mêmes que ceux déjà utilisés dans [9] et [10], c'est-à-dire, des estimations pour les perimètres. Nous avons séparé l'ennoncé pour $d=1$ de celui pour $d>1$, étant donné qu'il y a des différences significatives entre les deux cas.

Introduisons les notations:

$$I_K(x_1, \ldots, x_K) = \int_{T^K}\!\!\ldots\int dt_1, \ldots, dt_K \times$$

$$\times \int_{R^{dK}}\!\!\ldots\int \|\dot{x}_1\| \ldots \|\dot{x}_K\| p_{t_1, \ldots, t_K; t_1, \ldots, t_K}(x_1, \ldots, x_K;\ \dot{x}_1, \ldots, \dot{x}_K)\, d\dot{x}_1, \ldots, d\dot{x}_K$$

$$A_{t_1, \ldots, t_K}(x_1, \ldots, x_K) =$$

$$= \int_{R^{dK}}\!\!\ldots\int \|\dot{x}_1\| \ldots \|\dot{x}_K\| p_{t_1, \ldots, t_K; t_1, \ldots, t_K}(x_1, \ldots, x_K;\ \dot{x}_1, \ldots, \dot{x}_K)\, d\dot{x}_1, \ldots, d\dot{x}_K$$

où nous rappellons que T est un ouvert borné dans R^d.

Quand nous aurons besoin de considérer les coordonnées de $t_i \in R^d$, nous mettrons t_{ij} $(j=1, \ldots, d)$ pour la j-ième coordonnée de t_i.

Finalement

$$E = E_K(T) = \{(t_1, \ldots, t_K):\ t_i \in T,\ t_i \neq t_j \text{ pour } i \neq j\} \subset T^K$$

et, pour $\delta > 0$:

$$E_\delta = E_{K,\delta}(T) = \{(t_1, \ldots, t_K):\ t_i \in T,\ \|t_i - t_j\| \geqq \delta \text{ pour } i \neq j\}.$$

Il est clair que $E_\delta \uparrow E$ quand $\delta \neq 0$.

Hypothèse $H_{1,K}$.

Nous dirons que le processus $\{X(t):\ t \in R^d\}$ satisfait l'hypothèse $H_{1,K}$ si :

(i) la densité $p_{s_1, \ldots, s_K; t_1, \ldots, t_l}(x_1, \ldots, x_K;\ \dot{x}_1, \ldots, \dot{x}_l)$ est une fonction continue de $(x_1, \ldots, x_K)$ au point $(u, \ldots, u)$ quand les autres variables restent fixes et de $(s_1, \ldots, s_K)$ quand les autres variables restent fixes $\big((s_1, \ldots, s_K) \in E_K(T),\ (t_1, \ldots, t_l) \in E_l(T)\big)$.

(ii) $$\int_{R^{3d}} \|\dot{x}_1\|^{K-1} \|\dot{x}_2 - \dot{x}_3\| p_{s_1, \ldots, s_K; t_1, t_2, t_3}(x_1, \ldots, x_K;\ \dot{x}_1, \dot{x}_2, \dot{x}_3)\, d\dot{x}_1\, d\dot{x}_2\, d\dot{x}_3 =$$

$$= \mathsf{E}(\|\mathrm{grad}\,(X(t_1))\|^{K-1} \times$$

$$\times \|\mathrm{grad}\,(X(t_2) - X(t_3))\| / X(s_1) = x_1, \ldots, X(s_K) = x_K) p_{s_1, \ldots, s_K}(x_1, \ldots, x_K)$$

tend vers zéro quand $\|t_2 - t_3\| \to 0$, uniformément pour $(s_1, \ldots, s_K)$ dans un compacte de $E_K(T)$, $x_1, \ldots, x_K$ dans un voisinage de u et (t_1, t_2) dans un compacte de $E_2(T)$, et (t_1, t_2, t_3) dans un compacte.

(iii) $A_{t_1, \ldots, t_K}(x_1, \ldots, x_K)$ est continue comme fonction de ses arguments au point $(t_1, \ldots, t_K;\ u, \ldots, u)\big((t_1, \ldots, t_K) \in E\big)$.

Hypothèse $H_{2,K}$.

Considérons le processus à un paramètre

$$Z_j(\tau) = \frac{\partial X}{\partial t_j}(t_1, \ldots, t_{j-1}, \tau, t_{j+1}, \ldots, t_d).$$

Nous allons employer la notation

$$Q_T^{Z,u} = Q_T(\{t\colon Z(t) < u\})$$

puisque on va calculer des perimètres associés à plusieurs processus stochastiques Z et barrières u.

Soit

$$g_q^{(j)}(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_d) = \mathsf{E}\left([Q_{(a,b)}^{Z_j,0}]^q\right)$$

où (a, b) est un intervalle dans R^1.

Nous dirons que $\{X(t)\colon t\in R^d\}$ vérifie $H_{2,K}$, si pour chaque intervalle borné (a, b) dans R^1 et chaque $j=1, \ldots, d$, la fonction $g_{2(K-1)}^{(j)}$ est une fonction bornée de ses arguments.

Dans les théorèmes ci-dessous où ces hypothèses $H_{1,K}$ et $H_{2,K}$ vont être utilisées, elles peuvent être substituées par d'autres conditions analogues sans changer l'essentiel des résultats et des démonstrations, mais nous ne mettrons pas l'accent sur ce point ici. A titre d'exemple on vérifie sans grande difficulté qu' un processus Gaussien à gradient continu satisfait $H_{1,K}$ si les distribution jointes de $X(t_1), \ldots, X(t_K)$; grad $(X(t_1'))$, ..., grad $(X(t_l'))$ ne dégénèrent pas pour $(t_1, \ldots, t_K)\in E_K(T)$, $(t_1', \ldots, \ldots, t_l')\in E_l(T)$.

Le théorème suivant donne une condition suffisante pour que $I_K(u, \ldots, u)$ soit fini, dans le cas $d=1$. Il peut s'appliquer à la vérification de $H_{2,K}$, dans le cas $d>1$.

Théorème 2. *Soit $d=1$ et K entier positif. Supposons que $\{X(t)\colon t\in[a, b]\}$ est p.s. $(K+1)$-fois continûment différentiable et que la densité jointe $q_t(x^{(0)}, x^{(1)}, \ldots, \ldots, x^{(K)})$ de $X(t)$, $X^{(1)}(t), \ldots, X^{(K)}(t)$, est bornée par la constante L pour $x^{(0)}$ dans un voisinage de la barrière u.*

Soit

$$\mathfrak{X} = \sup\{\|X^{(h)}\|_\infty\colon 1 \leq h \leq K+1\}$$

$\left(\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|\right)$. *S'il existe $M>(K+1)^2$ tel que*

(a) $$\mathfrak{M}_M = E(\mathfrak{X}^M) < \infty,$$

alors

$$\mathsf{E}\left((N_u(a, b))^K\right) \leq CL(1+\mathfrak{M}_M C')$$

où $N_u(a, b)= \# \{t\colon X(t)=u,\ a<t<b\}$ C, C' sont des constantes qui dépendent de K, de $b-a$ et de M.

Remarques. Si le processus $\{X(t)\colon t\in R^1\}$ est gaussien et p.s. $(K+1)$-fois continûment différentiable, le Théorème de Landau—Shepp—Fernique assure la vérification de (a) pour tout M.

Si le processus est gaussien à d paramètres et p.s. $(K+1)$-fois continûment différentiable, la non-dégénerescence des distributions qui figure dans l'énoncé assure la vérification de l'hypothèse appellée $H_{2,K}$.

Démonstration du Théorème 2. Pour simplifier le calcul on va supposer $(a, b)= =(0, 1)$ et poser $N_u(a, b)=N_u$. Il s'agit de majorer

$$\mathsf{E}(N_u^K) = \sum_{q=1}^{\infty} \mathsf{P}(N_u^K \geq q).$$

Observons d'abord que

$$\{N_u^K \geqq q\} \subset \left\{\exists t \in C_u \text{ tel que } |X^{(h)}(t)| \leqq \frac{\mathfrak{X}}{m_q} \quad \forall h = 1, \ldots, K\right\} \tag{5}$$

où $m_q = [q^{1/K}/(K+1)] \vee 1$ ($[x]$ dénote le plus grand entier qui n'est pas plus grand que x).

Si $1 \leqq q^{1/K} \leqq K+1$ (5) est immédiate.

Si $q^{1/K} > K+1$, $N_u^K \geqq q$ entraîne l'existence d'un intervalle de longueur $1/m_q$ contenant au moins $(K+1)$ points de C_u, donc, l'inclusion (5) découle du Théorème de la valeur moyenne.

Posons encore

$$B_j = \{j \leqq \mathfrak{X} < j+1\} \quad \text{et}$$

$$A_{j,q} = \left\{\exists t \in C_u \text{ tel que } |X^{(h)}(t)| \leqq \frac{j+1}{m_q} \quad \forall h = 1, \ldots, X\right\}.$$

D'après l'inégalité de Hölder, si $\alpha, \beta \geqq 1$, $1/\alpha + 1/\beta = 1$, on a :

$$\mathsf{P}(\{N_u^K \geqq q\} \cap B_j) \leqq \mathsf{P}(A_{j,q} \cap B_j) \leqq [\mathsf{P}(A_{j,q})]^{1/\alpha} [\mathsf{P}(B_j)]^{1/\beta}. \tag{6}$$

Pour majorer $\mathsf{P}(A_{j,q})$ nous employons la méthode du Lemme 2 de [10].

Soit, pour $\varepsilon > 0$:

$$C_{u,\varepsilon} = \{t \colon X(t) = u, \ |X^{(h)}(t)| < \varepsilon \ \forall h = 1, \ldots, K\}$$

et

$$U_\varepsilon = \{t \colon |X^{(h)}(t)| < \varepsilon \ \forall h = 1, \ldots, K\}.$$

L'hypothèse entraîne la p.s. non-existence d'extrema locaux sur la barrière u, donc, que $\partial A_u = C_u$, d'où :

$$\text{p.s. } Q_{(0,1) \cap U_\varepsilon}(A_u) = \#(\partial A_u \cap (0,1) \cap U_\varepsilon) = \#(C_u \cap (0,1) \cap U_\varepsilon) = \#(C_{u,\varepsilon}).$$

Maintenant, si $\{f_m\}$ est une suite de fonctions réelles, C^∞, noncroissantes, $f_m(x) = 0$ pour $x \geqq u$, $f_m(x) = 1$ pour $x \leqq u - 1/m$, on a bien (voir [7], [10]):

$$\mathsf{E}(\#(C_{u,\varepsilon})) = \mathsf{E}(Q_{(0,1) \cap U_\varepsilon}(A_u)) \leqq$$

$$\leqq \mathsf{E}\left[\liminf_{m \to \infty} \int_{(0,1) \cap U_\varepsilon} |f_m'(X(t))| \, |X^{(1)}(t)| \, dt\right] =$$

$$\leqq \liminf_{m \to \infty} \mathsf{E}\left[\int_0^1 |f_m'(X(t))| \, |X^{(1)}(t)| \prod_{h=1}^K \chi_{\{|X^{(h)}(t)| < \varepsilon\}} \, dt\right] =$$

$$= \liminf_{m \to \infty} \int_0^1 \int_{R^{K+1}} |f_m'(x^{(0)})| \, |x^{(1)}| \prod_{h=1}^K \chi_{\{|x^{(h)}| < \varepsilon\}} q_t(x^{(0)}, x^{(1)}, \ldots, x^{(K)}) \, dx^{(0)} \ldots dx^{(K)} \, dt \leqq$$

$$\leqq L\varepsilon^{K+1}.$$

Or

$$A_{j,q} = \{\# C_{u,(j+1)/m_q} \geqq 1\}$$

c'est-à-dire que :

$$\mathsf{P}(A_{j,q}) \leqq \mathsf{E}(\# C_{u,(j+1)/m_q}) \leqq L\left(\frac{j+1}{m_q}\right)^{K+1}.$$

D'autre part, nous utilisons la majoration

$$\mathsf{P}(B_j) \leqq \frac{1}{j^M}\mathsf{E}(\mathfrak{X}^M) = \frac{\mathfrak{M}_M}{j^M}$$

pour $j \geqq 1$.

En substituant dans (6), nous obtenons ($q \geqq 1$) :

$$\mathsf{P}(N_u^K \geqq q) \leqq \sum_{j=0}^{\infty} \mathsf{P}(A_{j,q} \cap B_j) \leqq L\left\{\left(\frac{1}{m_q}\right)^{(K+1)/\alpha} + \sum_{j=1}^{\infty}\left(\frac{j+1}{m_q}\right)^{(K+1)/\alpha} \frac{\mathfrak{M}_M}{j^{M/\beta}}\right\}$$

d'où:

$$\sum_{q=1}^{\infty} \mathsf{P}(N_u^K \geqq q) \leqq L\left[\sum_{q=1}^{\infty} \frac{1}{m_q^{(K+1)/\alpha}}\right]\left[1 + \mathfrak{M}_M \sum_{j=1}^{\infty} \frac{(j+1)^{(K+1)/\alpha}}{j^{M/\beta}}\right].$$

La démonstration sera achevée si l'on choisi α tel que

$$\frac{K+1}{K\alpha} > 1 \quad \text{et} \quad \frac{M}{\beta} - \frac{K+1}{\alpha} > 1,$$

ce qui est possible, compte tenue de $M > (K+1)^2$.

THÉORÈME 3 (Cas $d=1$). *Sous les hypothèses $H_{1,K}$ on a:*

$$\mathsf{E}(V_K^{Q_T(A_u)}) = I_K(u, \ldots, u) \tag{7}$$

où $V_K^m = m(m-1)\ldots(m-K+1)$ *et* $I_K(u, \ldots, u)$ *peut être fini ou* $+\infty$.

REMARQUE. Dans le cas $d=1$ on a $Q_T(B) = \#(\partial B \cap T)$ pour n'importe quel Borélien $B \subset R^1$ (B dénote la frontière essentielle de B). En plus, si p.s. on n'a pas d'extrema locaux sur la barrière u ce qui découle de $H_{1,K}$, alors $Q_T(A_u) = N_u(T) = \#\{t\colon X(t) = u,\ t \in T\}$.

THÉORÈME 4 (Cas $d>1$). *Sous les hypothèses $H_{1,K}$ et $H_{2,K}$ on a*

$$\mathsf{E}((Q_T(A_u))^K) = I_K(u, \ldots, u). \tag{8}$$

DÉMONSTRATION du Théorème 3. Considérons l'ensemble

$$M_K(u) = (\partial A_u \cap T) \overset{K}{\times \cdots \times} (\partial A_u \cap T).$$

Il est clair que

$$V_K^{Q_T(A_u)} = \#(M_K(u) \cap E).$$

Pour démontrer la formule (7) il suffit de prouver que:

$$\mathsf{E}(\#\{M_K(u) \cap J\}) = \int_J \cdots \int A_{t_1, \ldots, t_K}(u, \ldots, u)\, dt_1, \ldots, dt_K \tag{9}$$

pour tout rectangle ouvert $J = I_1 \times \ldots \times I_k$ avec adhérence contenue dans E, $I_i \subset T$

$(i=1, \dots, K)$. Ceci est une conséquence du fait que la mesure de Lebesgue de E^C dans R^{dK}, est nulle.

Or, pour un tel rectangle $J=I_1\times\dots\times I_K$ on a :

$$\text{(10)}\quad \begin{aligned} &\mathsf{E}(\#\{M_K(u)\cap J\}) = \mathsf{E}\Big(\prod_{i=1}^{K}(\#\{\partial A_u\cap I_i\})\Big) = \mathsf{E}\Big(\prod_{i=1}^{K} Q_{I_i}(A_u)\Big) \leqq \\ &\leqq \liminf_{m_1\to\infty}\dots\liminf_{m_K\to\infty}\mathsf{E}\Big\{\int_{I_1\times\dots\times I_K}\!\!\dots\int \prod_{i=1}^{K}\|\operatorname{grad}(f_{m_i}(X(t_i)))\|\,dt_1\dots dt_K\Big\}\end{aligned}$$

avec $\{f_m\}$ choisie comme dans la démonstration du Théorème 2 (c.f. le lemme 1 (i) et la démonstration du Théorème 1 de [10]). Donc,

$$\mathsf{E}(\#\{M_K(u)\cap J\}) \leqq \liminf_{m_1\to\infty}\dots\liminf_{m_K\to\infty}\int_J\!\dots\int dt_1\dots dt_K\,\mathsf{E}\Big\{\prod_{i=1}^{K}|f'_{m_i}(X(t_i))|\,\|\operatorname{grad}(X(t_i))\|\Big\} =$$

$$= \liminf_{m_1\to\infty}\dots\liminf_{m_K\to\infty}\int_J\!\dots\int dt_1\dots dt_K \int_{R^K}\prod_{i=1}^{K}|f'_{m_i}(x_i)|\,A_{t_1\dots t_K}(x_1,\dots,x_K)\,dx_1\dots dx_K =$$

$$= \int_J\!\dots\int A_{t_1\dots t_K}(u,\dots,u)\,dt_1\dots dt_K$$

compte tenue de l'hypothèse (iii) faite sur $A_{t_1,\dots,t_K}(x_1,\dots,x_K)$.

Pour avoir l'inégalité inverse et donc prouver (9), nous appellons au lemme 1 (ii) énoncé dans [10]. Ψ_ε dénote une approximation C^∞ de l'unité, Ψ_ε: $R^d\to R^+$ avec support contenu dans la boule de rayon $\varepsilon(\varepsilon>0)$ et $g_\varepsilon=\Psi_\varepsilon * g$ pour chaque fonction $g\in L^1_{\mathrm{loc}}$. On obtient pour $0<\varepsilon<\delta$:

$$\text{(11)}\quad \begin{aligned} &\mathsf{E}(\#\{M_K(u)\cap J\}) \geqq \mathsf{E}\Big(\prod_{i=1}^{K}\int_{(I_i)_{-\delta}}|\operatorname{grad}(\chi_{A_u})_\varepsilon(t_i)|\,dt_i\Big) = \\ &= \lim_{m\to\infty}\int_{(I_1)_{-\delta}\times\dots\times(I_K)_{-\delta}}\!\!\dots\int \mathsf{E}\Big\{\prod_{i=1}^{K}\Big|\int_{R^1}\Psi_\varepsilon(t_i-s_i)f'_m(X(s_i))\operatorname{grad}(X(s_i))\,ds_i\Big|\Big\}\,dt_1\dots dt_n.\end{aligned}$$

Pour minorer l'espérance qui figure dans (11) on peut utiliser l'inégalité suivante, dont la démonstration est immédiate :

$$\text{(12)}\quad \prod_{i=1}^{K} a_i \geqq \prod_{i=1}^{K} b_i - \sum_{j=1}^{K} b_1\dots b_{j-1}c_j a_{j+1}\dots a_K$$

a_i, b_i, c_i $(i=1, \dots, K)$ étant des nombres non-négatifs tels que

$$a_i \geqq b_i - c_i \qquad (i = 1, \dots, K).$$

Nous l'appliquons avec :

$$a_i = \Big|\int_{R^1}\Psi_\varepsilon(t_i-s_i)f'_m(X(s_i))\operatorname{grad}(X(s_i))\,ds_i\Big|$$

$$b_i = \int_{R^1}\Psi_\varepsilon(t_i-s_i)|f'_m(X(s_i))|\,|\operatorname{grad}(X(t_i))|\,ds_i$$

$$c_i = \int_{R^1}\Psi_\varepsilon(t_i-s_i)\,|f'_m(X(s_i))|\,|\operatorname{grad}(X(s_i))-\operatorname{grad}(X(t_i))|\,ds_i.$$

On a:

$$\mathsf{E}\Big(\prod_{i=1}^{K} b_i\Big) = \int_{R^K} ds_1 \dots ds_K \prod_{i=1}^{K} \Psi_\varepsilon(t_i - s_i) \int_{R^K \times R^K} \prod_{i=1}^{K} \big(|f_m'(x_i)|\,|\dot{x}_i|\big) \cdot$$

$$\cdot\, P_{s_1,\dots,s_K; t_1\dots t_K}(x_1, \dots, x_K;\ \dot{x}_1, \dots, \dot{x}_K)\, dx_1 \dots dx_K\, d\dot{x}_1 \dots d\dot{x}_K$$

et, en faisant $m \to \infty$ et $\varepsilon \to 0$ (dans cet ordre), le lemme de Fatou donne :

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \lim_{m \to \infty} \int_{(I_1)_{-\delta} \times \dots \times (I_K)_{-\delta}} \!\!\dots\! \int \mathsf{E}\Big(\prod_{i=1}^{K} b_i\Big)\, dt_1 \dots dt_K \geqq \int_J \!\dots\! \int A_{t_1 \dots t_K}(u, \dots, u)\, dt_1 \dots dt_K.$$

Pour chacun des termes de la somme de (12) nous avons la majoration:

$$\mathsf{E}(b_1 \dots b_{j-1} c_j a_{j+1} \dots a_K) \leqq$$

$$\leqq \int_{R^K} ds_1 \dots ds_K \prod_{i=1}^{K} \Psi_\varepsilon(t_i - s_i) \int_{R^K \times R^{K+1}} |f_m'(x_1)| \dots |f_m'(x_K)|\, |\dot{x}_1| \dots \cdot$$

$$\cdot \dots |\dot{x}_{j-1}|\, |\dot{x}_j - \dot{y}_j|\, |\dot{x}_{j+1}| \dots |\dot{x}_K|\, p_{s_1, \dots, s_K;\, t_1, \dots, t_{j-1}, s_j, t_j, s_{j+1}, \dots, s_K}$$

$$(x_1, \dots, x_K;\ \dot{x}_1, \dots, \dot{x}_{j-1}, \dot{x}_j, \dot{y}_j, \dot{x}_{j+1}, \dots, \dot{x}_K)\, dx_1 \dots dx_K\, d\dot{x}_1, \dots, d\dot{x}_K, d\dot{y}_j.$$

Nous faisons d'abord $m \to \infty$ et obtenons :

$$\mathsf{E}(b_1 \dots b_{j-1} c_j a_{j+1} \dots a_K) \leqq \int_{R^K} ds_1 \dots ds_K \prod_{i=1}^{K} \Psi_\varepsilon(t_i - s_i) \int_{R^{K+1}} \Big(\prod_{i \neq j} |\dot{x}_i|\Big) |\dot{x}_j - \dot{y}_j|$$

(13)

$$p_{s_1, \dots, s_K;\, t_1 \dots t_{j-1}, s_j, t_j, s_{j+1}, \dots, s_K}(u, \dots, u;\ \dot{x}_1, \dots, \dot{x}_{j-1}, \dot{x}_j, \dot{y}_j, \dot{x}_{j+1}, \dots, x_K)\, d\dot{x}_1 \dots d\dot{x}_K\, d\dot{y}_j.$$

Le 2ème membre de (13) tend vers zéro quand $\varepsilon \to 0$, ce que peut se prouver à partir de l'inégalité

$$a_1 \dots a_m \leqq \sum_{i=1}^{m} a_i^m \qquad (a_1, \dots, a_m \geqq 0)$$

la condition (ii) et le fait que $t_i \in (I_i)_{-\delta}$, $s_i \in I_i$ $(i = 1, \dots, K)$. L'uniformité dans (ii) permet de conclure aussi que

(14) $$\lim_{\varepsilon \to 0} \lim_{m \to \infty} \int_{(I_1)_{-\delta} \times \dots \times (I_K)_{-\delta}} \!\!\dots\! \int \mathsf{E}(b_1 \dots b_{j-1} c_j a_{j+1} \dots a_K)\, dt_1 \dots dt_K = 0$$

d'où

$$\mathsf{E}(\#\{M_K(u) \cap J\}) \geqq \int_J \!\dots\! \int A_{t_1 \dots t_K}(u, \dots, u)\, dt_1 \dots dt_K.$$

Ceci prouve (9) et, par conséquent, le théorème 3.

DÉMONSTRATION du Théorème 4 ($d>1$). Nous avons, de la même façon que dans [10] et le théorème 3:

$$\mathsf{E}([Q_T(A_u)]^K) \leqq \liminf_{m_K\to\infty}\ldots\liminf_{m_1\to\infty} \mathsf{E}\left\{\int_{T^K}\ldots\int \prod_{i=1}^{K} \left\|\operatorname{grad}(f_{m_i}(X(t_i)))\right\| dt_1\ldots dt_K\right\} =$$

(15) $$= \liminf_{m_K\to\infty}\ldots\liminf_{m_1\to\infty} \mathsf{E}\left\{\int_{T^K}\ldots\int \prod_{i=1}^{K} (|f'_{m_i}(X(t_i))|\,\|\operatorname{grad}(X(t_i))\|)\, dt_1\ldots dt_K\right\}$$

$$= \liminf_{m_K\to\infty}\ldots\liminf_{m_1\to\infty} \left[\mathsf{E}\left\{\int_{E_\delta}\ldots\int\right\}+\mathsf{E}\left\{\int_{E_\delta^C}\ldots\int\right\}\right].$$

Le passage à la limite dans le premier terme de (15) ne pose pas de nouveaux problèmes par rapport au cas $d=1$, sauf pour des modifications évidentes, et on obtient comme limite

$$\int_{E_\delta}\ldots\int A_{t_1,\ldots,t_K}(u,\ldots,u)\,dt_1\ldots dt_K.$$

Il s'agit donc de prouver que le deuxième terme de (15) est arbitrairement petit si $\delta>0$ est sufisemment petit. Ceci prouvera que

(16) $$\mathsf{E}([Q_T(A_u)]^K) \leqq I_K(u,\ldots,u)$$

après faire $\delta\to 0$.

En fait, ceci finit la démonstration de (8), puisque l'inégalité inverse à (16) peut être déduite de façon entièrement analogue à ce qu'on a fait pour le théorème 3, en changeant seulement des petits détails. Dénotons par R_δ le deuxième terme de (15). En vue de la définition de E_δ^C, nous avons :

(17)

$$R_\delta \leqq \binom{K}{2} \liminf_{m_K\to\infty}\ldots\liminf_{m_1\to\infty} \mathsf{E}\left\{\int_{T^K\cap\{\|t_1-t_2\|<\delta\}}\ldots\int \prod_{i=1}^{K} (|f'_{m_i}(X(t_i))|\,\|\operatorname{grad}(X(t_i))\|)\, dt_1\ldots dt_K\right\}.$$

Il suffit de prouver que le deuxième membre tend vers zéro avec δ quand T est un hypercube. Nous allons supposer, pour simplifier la notation que $T=(0,1)^d$. Introduisons les notations :

$$\xi_m = \int_T |f'_m(X(t))|\,\|\operatorname{grad}(X(t))\|\,dt$$

et

$$Z_{j,h}(\tau) = \left.\frac{\partial X}{\partial t_j}\right|_{(t_{h,1},t_{h,2},\ldots,t_{h,j-1},\tau,t_{h,j+1},\ldots,t_{h,d})}$$

(c.f. l'hypothèse $H_{2,K}$).

Une première observation c'est qu'on a :

$$\xi_m \leqq \sum_{j=1}^{d}\int_T |f'_m(X(t))|\left|\frac{\partial X}{\partial t_j}(t)\right|dt = \sum_{j=1}^{d}\int_{(0,1)^{d-1}}\ldots\int \prod_{i\neq j} dt_i \int_0^1 |f'_m(X(t))|\left|\frac{\partial X}{\partial t_j}(t)\right|dt_j.$$

Or, l'intégrale intérieure est majorée par

$$Q_{(0,1)}^{Z_j,0}+1.$$

En effet, si $Q_{(0,1)}^{Z_j,0} = +\infty$ il n'y a rien à prouver. Dans le cas contraire rappellons que $Q_{(0,1)}^{Z_j,0}$ est le nombre des points $\tau \in (0, 1)$ tels que $\frac{\partial X}{\partial t_j}$ change de signe dans tout voisinage de τ ($\frac{\partial X}{\partial t_j}$ étant considerée comme fonction de la j-ième coordonnée, avec les autres fixées). Mais, d'autre part, si Z_j ne change pas de signe sur l'intervalle (α, β), compte tenu du fait que $f_m' \leqq 0$, on a:

$$\int_\alpha^\beta |f_m'(X(t_1, \ldots, t_d)| \left|\frac{\partial X}{\partial t_j}(t_1, \ldots, t_d)\right| dt_j =$$

$$= \left|\int_\alpha^\beta f_m'(X(t_1, \ldots, t_d)) \frac{\partial X}{\partial t_j}(t_1, \ldots, t_d)\, dt_j\right| =$$

$$= |f_m(X(t_1, \ldots, t_{j-1}, \beta, t_{j+1}, \ldots, t_d) - f_m(X(t_1, \ldots, t_{j-1}, \alpha, t_{j+1}, \ldots, t_d))| \leqq 1.$$

Ceci permet de'affirmer que

$$\int_0^1 |f_m'(X(t))| \left|\frac{\partial X}{\partial t_j}(t)\right| dt_j \leqq Q_{(0,1)}^{Z_j,0} + 1 \tag{18}$$

et aussi, donc :

$$\xi_m \leqq \sum_{j=1}^d \int \underset{(0,1)^{d-1}}{\cdots} \int \prod_{i \neq j} dt_i (Q_{(0,1)}^{Z_j,0} + 1) = \xi. \tag{19}$$

L'inégalité de Hölder plus l'hypothèse $H_{2,K}$ entraînent maintenant que $\xi_m \in L^K(\Omega)$ et que $\mathsf{E}(\xi_m^K)$ est bornée indépendemment de m. Donc, $\mathsf{E}((Q_T(A_u))^K) < \infty$.

Nous avons aussi :

$$\iint\limits_{\substack{\|t_1 - t_2\| < \delta \\ 0 < t_{1j},\, t_{2j} < 1 \quad (j = 1, \ldots, d)}} |f_{m_1}'(X(t_1))|\, |f_{m_2}'(X(t_2))|\, \|\text{grad}\,(X(t_1))\|\, \|\text{grad}\,(X(t_2))\|\, dt_1\, dt_2 \leqq$$

$$\leqq \sum_{j,j'=1}^d \iint\limits_{\substack{\|t_1 - t_2\| < \delta \\ 0 < t_{1,j},\, t_{2,j} < 1 \quad (j = 1, \ldots, d).}} |f_{m_1}'(X(t_1))|\, |f_{m_2}'(X(t_2))| \left|\frac{\partial X}{\partial t_j}(t_1)\right| \left|\frac{\partial X}{\partial t_{j'}}(t_2)\right| dt_1\, dt_2$$

Notons chaque terme de cette somme par $\eta_{j,j'}$. Nous avons les majorations suivantes:

Si $j = j'$,

$$\eta_{jj} \leqq \int_0^1 \cdots \int_0^1 \prod_{h \neq j} dt_{1h} \int \underset{\substack{|t_{2h} - t_{1h}| < \delta \\ h \neq j}}{\cdots} \int \prod_{h \neq j} dt_{2h} \int_0^1 \int_0^1 |f_{m_1}'(X(t_1))|\, |f_{m_2}'(X(t_2))| \left|\frac{\partial X}{\partial t_j}(t_1)\right| \times$$

$$\times \left|\frac{\partial X}{\partial t_j}(t_2)\right| dt_{1j}\, dt_{2j} \leqq$$

$$\leqq \int_0^1 \cdots \int_0^1 \prod_{h \neq j} dt_{1h} \int \underset{\substack{|t_{2h} - t_{1h}| < \delta \\ h \neq j}}{\cdots} \int \prod_{h \neq j} dt_{2h} (Q_{(0,1)}^{Z_j,t_1,0} + 1)(Q_{(0,1)}^{Z_j,t_2,0} + 1).$$

Soit C une borne supérieur de $\mathsf{E}\big[(Q^{Z_{j'},0}_{(0,1)}+1)^{2(K-1)}\big]$ pour tout j, qui existe à cause de $H_{2,K}$. En appliquant une autre fois l'inégalité de Hölder, nous obtenons :

(20) $$\mathsf{E}(\eta_{jj}^{K-1}) \leqq C\delta^{(d-1)(K-1)}.$$

Considérons maintenant les termes avec $j\neq j'$:

$$\eta_{jj'} \leqq \int_0^1 \dots \int_0^1 \prod_{h\neq j,j'} dt_{1h} \int_0^1 \dots \int_0^1 \prod_{h'\neq j,j'} dt_{2h'} \int_0^1 dt_{2j} \int_0^1 dt_{2j'} \int_{t_{2j'}-\delta}^{t_{2j'}+\delta} dt_{1j'} \int_{t_{2j}-\delta}^{t_{2j}+\delta} dt_{1j}\cdot$$

$$\cdot |f'_{m_1}(X(t_1))|\,|f'_{m_2}(X(t_2))| \left|\frac{\partial X}{\partial t_j}(t_1)\right| \left|\frac{\partial X}{\partial t_{j'}}(t_2)\right| =$$

$$= \int_0^1 \dots \int_0^1 \prod_{h\neq j,j'} dt_{1h} \int_0^1 \dots \int_0^1 \prod_{h'\neq j,j'} dt_{2h'} \int_0^1 dt_{2j} \int_0^1 dt_{2j'}\cdot$$

$$\cdot \int_{t_{2j'}-\delta}^{t_{2j'}+\delta} |f'_{m_2}(X(t_2))| \left|\frac{\partial X}{\partial t_{j'}}(t_2)\right| dt_{1j'} (Q^{Z_{j,1},0}_{(t_{2j}-\delta,t_{2j}+\delta)}+1)$$

par une majoration analogue à celle qui conduit à (18).

Le terme qui provienne du « 1 » est évidemment borné par

$$2\delta\xi_{m_2}.$$

Quand à l'autre terme, puisque

$$Q^{Z_{j,1},0}_{(t_{2j}-\delta,t_{2j}+\delta)}$$

est fonction de $t_{1j}, \dots, t_{1,j-1}, t_{1,j+1}, \dots, t_{1d}, t_{2j}$ *et non* de t_{2j}, il est borné par

$$\int_0^1 \dots \int_0^1 \prod_{h\neq j,j'} dt_{1h} \int_0^1 \dots \int_0^1 \prod_{h'\neq j,j'} dt_{2h'} \int_0^1 dt_{2j} \int_0^1 dt_{1j'} Q^{Z_{j,1},0}_{(t_{2j}-\delta,t_{2j}+\delta)} (Q^{Z_{j',2},0}_{(0,1)}+1).$$

Donc,

(21) $$\eta_{jj'} \leqq 2\delta\xi + \int_0^1 \dots \int_0^1 \prod_{h\neq j} dt_{1h} \int_0^1 \dots \int_0^1 \prod_{h'\neq j'} dt_{2h'} Q^{Z_{j,1},0}_{(t_{2j}-\delta,t_{2j}+\delta)} (Q^{Z_{j',2},0}_{(0,1)}+1).$$

Dénotons par $\tilde\eta_{j,j'}$ le deuxième terme du second membre de (21) et remplaçons (20) et (21) dans (17). On a:

$$\mathsf{E}\Big(\prod_{i=3}^K \xi_{m_i} \int_{T^2\cap\{\|t_2-t_1\|<\delta\}} |f'_{m_1}(X(t_1))|\,|f'_{m_2}(X(t_2))|\,\|\mathrm{grad}\,(X(t_1))\|\,\|\mathrm{grad}\,(X(t_2))\|\,dt_1\,dt_2\Big) \leqq$$

$$\leqq \mathsf{E}\big(\xi^{K-2}\{\sum_{j=1}^d \eta_{jj} + 2\delta\, d^2\xi + \sum_{j\neq j'} \tilde\eta_{jj'}\}\big) \leqq$$

$$\leqq 2d^2\delta\,\mathsf{E}(\xi^{K-1}) + \mathsf{E}\big(\xi^{K-2}\{\sum_{j=1}^d \eta_{jj} + \sum_{j\neq j'} \tilde\eta_{jj'}\}\big) \leqq$$

$$\leqq 2d^2\delta \mathsf{E}(\xi^{K-1})+[\mathsf{E}(\xi^{K-1})]^{(K-2)/(K-1)}\Big\{\sum_{j=1}^{d}[\mathsf{E}(\eta_{jj}^{K-1})]^{1/(K-1)}+\sum_{j\neq j'}[\mathsf{E}(\tilde\eta_{j,j'}^{K-1})]^{1/(K-1)}\Big\}\leqq$$

$$\leqq 2d^2\delta \mathsf{E}(\xi^{K-1})+[\mathsf{E}(\xi^{K-1})]^{(K-2)/(K-1)}\Big\{dC^{1/(K-1)}\delta^{d-1}+\sum_{j\neq j'}[\mathsf{E}(\tilde\eta_{j,j'}^{K-1})]^{1/(K-1)}\Big\}.$$

Puisque cette borne ne dépend pas de $m_1, \dots, m_K$, la démonstration sera achevée si

$$\mathsf{E}(\tilde\eta_{j,j'}^{K-1})=o(1) \qquad (\delta\to 0)$$

pour chaque couple $j, j', j\neq j'$. Ceci résulte de

$$\mathsf{E}(\tilde\eta_{j,j'}^{K-1})\leqq C^{1/2}\int_0^1\dots\int_0^1\prod_{h\neq j}dt_{1h}\int_0^1\dots\int_0^1\prod_{h'\neq j'}dt_{2h'}\Big\{\mathsf{E}[(Q^{Z_{j,1},0}_{(t_{2j}-\delta,t_{2j}+\delta)})^{2(K-1)}]\Big\}^{1/2}.$$

L'intégrand est majoré par $C^{1/2}$ et tend vers zéro quand $\delta\to 0$ pour $t_{1j}, \dots, t_{1,j-1}, t_{1,j+1}, \dots, t_{1,d}, t_{2,j}$ fixés. Le Théorème de Lebesgue donne alors la conclusion.

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ON THE GLOBAL ASYMPTOTIC STABILITY OF THE ZERO SOLUTION OF THE EQUATION $\ddot{x}+g(t, x, \dot{x})\dot{x}+f(x) = 0$

J. KARSAI

1. Introduction

In this paper we consider the differential equation

$$\ddot{x}=g(t, x, \dot{x})\dot{x}+f(x) = 0, \tag{1}$$

where functions $f\colon \mathbf{R}\to\mathbf{R}$, $g\colon \mathbf{R}_+\times\mathbf{R}^2\to\mathbf{R}_+$ are continuous, $xf(x)>0$ $(x\neq 0)$, and

$$F(x) := 2\int_0^x f \to \infty \quad (x\to\infty).$$

Equation (1) is the model of an oscillator of one degree of freedom; where $-f(x)$ is the elastic force, $-g(t, x, \dot{x})\dot{x}$ is viscous friction.

For the function g we assume the estimate

$$0 \leqq a(t)\varphi(x, y) \leqq g(t, x, y) \leqq b(t)\psi(x, y) \tag{2}$$

for all $x, y\in\mathbf{R}$ and $t>t_0$ (for some t_0), where a, b, φ, ψ are continuous, and if $y\neq 0$, then $\varphi(x, y)>0$.

The zero solution of (1) is said to be *stable* if for every $\varepsilon>0$, $t_0\geqq 0$ there exist a $\delta(\varepsilon, t_0)>0$ such that $|x(t_0)|+|\dot{x}(t_0)|<\delta$ implies $|x(t)|+|\dot{x}(t)|<\varepsilon$ for all $t\geqq t_0$. The zero solution of (1) is *globally asymptotically stable (g. a. s.)* if it is stable and every solution of (1) tends to zero as t goes to infinity.

In this paper we give sufficient conditions for the zero solution of equation (1) to be g. a. s. As (2) shows, theorems will include appropriate lower and upper estimation for $g(t, x, y)$. If g becomes "too large" or "too small" as $t\to\infty$ then the zero solution generally is not g. a. s.. For instance $x(t)=1+1/t$ is a solution of the equation

$$\ddot{x}+\left(t^2+\frac{2}{t}+t\right)\dot{x}+x = 0.$$

On the other hand, all solutions of the equation

$$\ddot{x}+\frac{1}{t^2}\dot{x}+x = 0$$

are oscillatory and do not tend to zero (see Section 2).

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Numerous papers are concerned with finding weaker and weaker estimates for g. Zvi Arnstein and E. F. Infante [1], L. H. Thruston and J. S. W. Wong [7] investigated such cases of equation (1) in which $g(t,x,y)>\varphi(x,y)$. L. Hatvani [3], R. J. Ballieu and K. Peiffer allows the general estimate (2).

Now we cite two of Peiffer's and Ballieu's results [2].

THEOREM A. *Suppose that* $a(t)\equiv 1$, $b(t)$ *is nondecreasing on* $[t_0,\infty]$. *If* $\int^{\infty} 1/b=\infty$, *then the zero solution of* (1) *is g. a. s..*

THEOREM B. *Suppose that* $b(t)\equiv 1$, $a(t)$ *is nonincreasing on* $[t_0,\infty]$. *If* $\int^{\infty} a=\infty$, *then the zero solution of* (1) *is g. a. s..*

Consider the equation

$$\ddot{x}+g(t)\dot{x}+f(x)=0. \tag{1'}$$

By Theorems A and B the zero solution of (1') is globally asymptotically stable if either

$$0<a_0\leqq g(t)\leqq b(t)\quad (t\geqq 0),\qquad \int_0^{\infty}\frac{1}{b}=\infty,$$

or

$$0<a(t)\leqq g(t)\leqq b_0\quad (t\geqq 0),\qquad \int_0^{\infty} a=\infty$$

is satisfied with nondecreasing b and nonincreasing a.

There arises the following problems: Can the zero solution of (1) be g. a. s. also if $b(t)$ is unbounded and $a(t)$ takes arbitarily small values? Do conditions $\int^{\infty} a=\infty$, $\int^{\infty} 1/b=\infty$ imply g. a. s. of the zero solution of (1); in other words, is there a common generalization of Theorems A and B? In this paper we give such a generalization in case $\varphi(x,y)>0$. In addition, we improve conditions of Theorem B provided that $\varphi(x,y)>0$ if $y\neq 0$.

2. Preliminary lemmas

The function

$$E(t)=\dot{x}^2(t)+F(x(t))$$

will be used as a Ljapunov function. The derivative of $E(t)$ with respect to equation (1) reads

$$\dot{E}(t)=-2g(t,x(t),\dot{x}(t))\dot{x}^2(t).$$

Consequently every solution of (1) is defined on $[t_0,\infty)$, $\lim_{t\to\infty} E(t)=\lambda$ exists and is finite. So $x(t)$, $\dot{x}(t)$ are bounded on $[t_0,\infty)$, and by Ljapunov's theorem [8] the zero solution of (1) is stable.

The asymptotic behaviour of oscillatory and nonoscillatory solutions will be investigated separately, which is made possible by the following lemma.

LEMMA 1 ([3]). *Let $x(t)$ be a solution of equation* (1). *If t_1 and t_2 are consecutive zeros of $\dot{x}(t)$, then there exists a $\bar{t}\in(t_1, t_2)$ such that $x(\bar{t})=0$.*

By this lemma solutions are either oscillatory or monotone for sufficiently large values of t. The following lemma is concerned with the asymptotic behaviour of the monotone solutions.

LEMMA 2 ([2]). *Assume that function g admits estimate* (2).

a) *If $b(t)$ is nondecreasing on $[t_0, \infty)$ and $\int^{\infty} 1/b=\infty$, then for every monotone solution of* (1) $\lim_{t\to\infty} (x(t), \dot{x}(t))=0$.

b) *If $\varphi(x, y)>0$ $(x, y\in\mathbf{R})$ and $\int^{\infty} 1/a<\infty$, then there exists a monotone solution $x(t)$ of* (1) *such that $\lim_{t\to\infty} \dot{x}(t)=0$, and $\lim_{t\to\infty} x(t)\neq 0$.*

This lemma shows that the monotone solutions and their derivatives tend to zero as $t\to\infty$ provided that g is "not too large in average". However, the g. a. s. of the zero solution is influenced also by the lower estimate of g. Indeed, if $x(t)$ is a solution of (1), then x, $\dot{x}$ are bounded, hence

$$\dot{E}(t) = -2g(t, x(t), \dot{x}(t))\dot{x}^2(t) \geqq -Kb(t)E(t).$$

So, if $\int^{\infty} b<\infty$, then $\lim_{t\to\infty} E(t)>0$, consequently $x(t)\not\to 0$ $(t\to\infty)$. Therefore, it is reasonable to assume $\int^{\infty} a=\infty$. However, as it was shown in [5] by an example, this condition even with bounded a is not sufficient for the zero solution of $\ddot{x}+a(t)\dot{x}+x=0$ to be a g. a. s. (for a sufficient condition see Cor. 2 of this paper).

LEMMA 3. *Let $x(t)$ be an oscillatory solution of* (1) *for which $\lim_{t\to\infty} E(t)=\lambda>0$. Let $0<\varepsilon_1<\varepsilon_2<\lambda$. Then there exist $\delta_2>\delta_1>0$, such that if $t_2>t_1$, $F(x(t_1))=\varepsilon_1$, $F(x(t_2))=\varepsilon_2$ and $\varepsilon_1<F(x(t))<\varepsilon_2$ on (t_1, t_2), then $\delta_2>t_2-t_1>\delta_1$.*

PROOF. Obviously,

$$\lambda-\varepsilon_2 < \dot{x}^2(t) = E(t)-F(x(t)) < E(t_0) \quad (t_1 \leqq t \leqq t_2).$$

We can assume, that $\dot{x}(t)>0$ on $[t_1, t_2]$. By integration we have

$$\sqrt{\lambda-\varepsilon_2}(t_2-t_1) < x(t_2)-x(t_1) < \sqrt{E(t_0)}(t_2-t_1).$$

Now, the existence of δ_1, δ_2 follows from the continuity of $F(x)$ and boundedness of $x(t)$.

3. Results

THEOREM 1. *Suppose that* $\varphi(x, y)>0$ $(x, y\in\mathbf{R})$, *and there exists a nonincreasing, nonnegative and differentiable function* α *on* $[t_0, \infty)$, *having the following properties:*

$$\int_{t_0}^{\infty} \alpha = \infty; \tag{0}$$

(i) $\alpha(t)b(t)$ *is bounded on* $[t_0, \infty)$;

(ii) *there exists a positive number* σ, $0<\sigma<1$, *such that for every* k, l $\left(0<k<\sup\limits_{\mathbf{R}\setminus\{0\}} F(x)/xf(x),\ l\geq \inf\limits_{x,y\in\mathbf{R}} \varphi(x, y)\right)$

$$\overline{\lim_{t\to\infty}} \left\{\left(\int_{t_0}^{t} \alpha\right)^{-1} \int_{t_0}^{t} \left[la(\tau)\left(\int_{t_0}^{\tau} \alpha\right)-(1+k)\alpha(\tau)\right]^{-} d\tau\right\} = \mu(k) < 1-\sigma.^{1)}$$

Then the zero solution of equation (1) *is g. a. s..*

For example, for the equation

$$\ddot{x}+\left(\frac{1}{t\log t}+t\sin^2 t\right)\dot{x}+xe^x = 0$$

the conditions of the above theorem are satisfied by $\alpha(t)=1/(t\log t)$, however neither Theorem A nor Theorem B can be applied. In general, by choice of $\alpha(t)=\min\left(a(t), 1/b(t)\right)$ a common generalization of Theorems *A* and *B* can be derived.

COROLLARY 1. *Suppose that* $\varphi(x, y)>0$ $(x, y\in R)$. *If* $a(t)$ *is nonincreasing and* $b(t)$ *is nondecreasing on* $[t_0, \infty)$ *and* $\int^{\infty} a=\int^{\infty} 1/b=\infty$, *then the zero solution of equation* (1) *is g. a. s..*

It may happen in practice, that function g does not meet conditions of Corollary 1. For example, consider equation (1′) with $g(t)=\sin^2 t/t$. Such cases were treated by L. Hatvani [3, 4]. By the help of Theorem 1 there may be investigated also such equations which are beyond the scope of method in [4] as the following example shows.

Let $\varepsilon>0$ be given, and define $g(t)$ in equation (1′) as follows:

$$g(t) = \begin{cases} 1 & t\in[2n-1+(n-1)\varepsilon,\ 2n+n\varepsilon] \\ 0 & t\in\left[2n+n\varepsilon+\dfrac{1}{n+3},\ 2n+1+n\varepsilon-\dfrac{1}{n+3}\right]. \end{cases} \tag{3}$$

On the left parts of $\mathbf{R}_+$ let $g(t)$ be defined linearly such that g be continuous. Moreover let $f(x)=x$. With $\alpha(t)=1/t$ Condition (i) of Theorem 1 is satisfied trivially. Condition (ii) holds because of the following estimate (since $f(x)=x$, it

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[1] For $a\in\mathbf{R}$ we denote by $[a]^+$ and $[a]^-$ the positive and negative part of a, respectively, i.e. $[a]^+=\max(0, a)$, $[a]^-=\max(0, -a)$.

must be claimed only for $k=1$):

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t}\frac{1}{\tau}d\tau\Big)^{-1}\int_{t_0}^{t}\Big[g(\tau)\Big(\int_{t_0}^{\tau}\frac{1}{s}ds\Big)-\frac{2}{\tau}\Big]^{-}d\tau<$$

$$<\lim_{t\to\infty}\Big(\int_{t_0}^{t}\frac{1}{\tau}d\tau\Big)^{-1}2\int_{H_t}\frac{1}{\tau}d\tau<\frac{2}{2+\varepsilon}$$

$$\Big(H_t=\bigcup_{2n+n\varepsilon<t}(2n+n\varepsilon,2n+1+n\varepsilon)\Big),\quad \text{as}$$

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t}\frac{1}{\tau}d\tau\Big)^{-1}\int_{H_t}\frac{1}{\tau}d\tau<\frac{1}{2+\varepsilon}.$$

By repeating the above procedure (with $1/b(t)$ instead of $1/t$) the following corollary can be proved.

COROLLARY 2. *Suppose that* $\varphi(x,y)>0$ $(x,y\in R)$ *and there exists a nondecreasing function* $b(t)$, *majorizing* $g(t)$ *in equation* (1'), *such that*

$$\int^{\infty}\frac{1}{b}=\infty;\ \lim_{t\to\infty}\Big(\int_{t_0}^{t}\frac{1}{b}\Big)^{-1}\int_{H_t}\frac{1}{b}<\frac{1}{1+k}$$

hold with $H_t:=\{\in[t_0,t]\colon g(\tau)<1/b(\tau)\}$, $0<k<\sup_{R\setminus\{0\}}F(x)/xf(x)$. *Then the zero solution of* (1') *is g. a. s..*

The following theorem is concerned with the case, when $\varphi(x,y)>0$ is required only for $y\neq0$.

THEOREM 2. *Suppose that function g admits estimate* (2) *with* $b(t)\equiv1$. *If there exists a positive nonincreasing differentiable function* $\alpha(t)$ *for which* $\int^{\infty}\alpha=\infty$ *and condition* (ii) *in Theorem 1 is satisfied, then the zero solution of equation* (1) *is g. a. s..*

Similarly to Corollary 2 we can prove

COROLLARY 3. *Suppose that* $b(t)\equiv1$. *Let a function* $\alpha(t)$ *be positive, nonincreasing, differentiable on* $[t_0,\infty)$ *and* $\int^{\infty}\alpha=\infty$. *If the inequality*

$$\varlimsup_{t\to\infty}\Big(\int_{t_0}^{t}\alpha\Big)^{-1}\int_{H_t}\alpha<\frac{1}{1+k}$$

holds for every $0<k<\sup_{R\setminus\{0\}}\dfrac{F(x)}{xf(x)}$, *where* $H_t:=\{\tau\in[t_0,t]\colon a(\tau)<\alpha(\tau)\}$, *then the zero solution of equation* (1) *is g. a. s..*

For example, while results in [3, 4] cannot be applied to equation

$$\ddot{x}+g(t)\dot{x}^3+x=0$$

where $g(t)$ is defined by (3), conditions of Corollary 3 are satisfied with $\alpha(t)=1/t$.

4. Proofs

PROOF of Theorem 1. By Lemma 2 it will be sufficient to show that $\lim_{t\to\infty} E(t)=$ $=\lambda=0$ for every oscillatory solution of equation (1).

Suppose the contrary. Let $x(t)$ be an oscillatory solution for which $\lambda>0$. F. J. Scott [6] observed the simple fact, that on account of boundedness of $x(t)$ for any given positive ε there is a v $(0<v<\sup_{R\setminus\{0\}} F(x)/xf(x))$, such that

$$F(x(t))-vx(t)f(x(t)) < \varepsilon \quad \text{on} \quad [t_0, \infty).$$

Therefore we have

$$E(t) = \dot{x}^2(t)+F(x(t)) = (1+v)\dot{x}^2(t)-vg(t, x(t), \dot{x}(t))x(t)\dot{x}(t)-$$
$$-v(x\dot{x})^{\cdot}(t)+F(x(t))-vx(t)f(x(t)) < (1+v)\dot{x}^2(t)-$$
$$-vg(t, x(t), \dot{x}(t))x(t)\dot{x}(t)-v(x\dot{x})^{\cdot}(t)+\varepsilon.$$

For the derivative of function $W(t):=E(t)\int_{t_0}^{t}\alpha$ the following estimate is found:

$$\dot{W}(t) \leqq -2g(t, x(t), \dot{x}(t))\dot{x}^2(t)\int_{t_0}^{t}\alpha+ \tag{4}$$
$$+(1+v)\alpha(t)\dot{x}^2(t)-v\alpha(t)(x\dot{x})^{\cdot}(t)-vg(t, x(t), \dot{x}(t))\alpha(t)x(t)\dot{x}(t)+\varepsilon\alpha(t).$$

Let $\{t_n\}$ be the sequence of zeros of $x(t)$ $(t_n\nearrow\infty$ if $n\to\infty$, $0<E(t_1)-\lambda<\varepsilon)$. By integration of (4) on $[t_1, t_n]$ we get:

$$W(t_n) < W(t_1)+\varepsilon\int_{t_1}^{t_n}\alpha-v\int_{t_1}^{t_n}\alpha(x\dot{x})^{\cdot}+$$
$$\int_{t_1}^{t_n}\dot{x}^2(\tau)\Big[g(\tau, x(\tau), \dot{x}(\tau))\Big(\int_{t_0}^{\tau}\alpha\Big)-(1+v)\alpha(\tau)\Big]^{-}d\tau- \tag{5}$$
$$-\int_{t_1}^{t_n}\dot{x}(\tau)f(\tau, x(\tau), \dot{x}(\tau))\Big(\int_{t_0}^{\tau}\alpha\Big)\Big(v\Big(\int_{t_0}^{\tau}\alpha\Big)^{-1}\alpha(\tau)x(\tau)+\dot{x}(\tau)\Big)d\tau=$$
$$=W(t_1)+\varepsilon\int_{t_1}^{t_n}\alpha-v\int_{t_1}^{t_n}\alpha(x\dot{x})^{\cdot}+I(t_1, t_n)+J(t_1, t_n).$$

By choice of t_1 and (2) we have

$$I(t_1, t_n) < (\lambda+\varepsilon)\int_{t_1}^{t_n}\Big[la(\tau)\Big(\int_{t_0}^{\tau}\alpha\Big)-(1+v)\alpha(\tau)\Big]^{-}d\tau,$$

where $l=\inf_{t>t_0}\varphi(x(t), \dot{x}(t))>0$ because of boundedness of $(x(t), \dot{x}(t))$. Integrating by parts, we obtain

$$-v\int_{1}^{t_n}\alpha(x\dot{x})^{\cdot} = v\int_{t_1}^{t_n}\alpha^{\cdot}x\dot{x} < vKLV,$$

where

$$K = \sup_{[t_0,\infty)} |x(t)|, \quad L = \sup_{[t_0,\infty)} |\dot{x}(t)|, \quad V = \int_{t_0}^{\infty} |\alpha^{\cdot}|.$$

If we show, that

$$[J(t_1, t_n)]^+ = o\Big(\int_{t_1}^{t_n} \alpha\Big) \quad (n \to \infty),$$

then dividing (5) by $\int_{t_1}^{t_n} \alpha$, and limiting $n \to \infty$ we get the following inequality:

$$\lambda < \varepsilon + \mu(v)(\lambda+\varepsilon) < \varepsilon + (1-\sigma)(\lambda+\varepsilon).$$

If ε is sufficiently small, we get a contradiction.

It remains to prove $[J(t_1, t_n)]^+ = o\Big(\int_{t_1}^{t_n} \alpha\Big)$ $(n \to \infty)$. Let $\{\sigma_n\}$ be the sequence of zeros of $\dot{x}(t)$ $(\sigma_n < t_n < \sigma_{n+1})$. By Lemma 1 we have $x(t)\dot{x}(t) \leqq 0$ on $[\sigma_n, t_n]$ and $x(t)\dot{x}(t) \geqq 0$ on $[t_n, \sigma_{n+1}]$. Hence

$$J(t_1, t_n) < \sum_{k=1}^{n} J(\sigma_k, t_k).$$

Since $\alpha(t)$, $E(t)$ are bounded and $\dot{x}(\sigma_k) = 0$. $\dot{x}^2(t_k) \geqq \lambda$, $\int^{\infty} \alpha = \infty$, there exist a sequence $\{\tau_k\}$ $(\sigma_k < \tau_k < t_k)$ and a number δ $(0 < \delta < \lambda)$, such that

$$-\dot{x}(t)\Big(v\Big(\int_{t_0}^{t} \alpha\Big)^{-1} x(t)\alpha(t) + \dot{x}(t)\Big) < 0 \quad t \in [\tau_k, t_k],$$

and

$$\min_{t \in [\sigma_k, \tau_k]} \big|f(x(t))\big| = \delta.$$

Hence

$$J(t_1, t_n) < \sum_{k=1}^{n} J(\sigma_k, \tau_k).$$

Now we examine the behaviour of integrals $J(\sigma_k, \tau_k)$ $(k = 1, 2, \ldots)$. We can assume, that $\dot{x}(t) \geqq 0$, $x(t) < 0$ on $[\sigma_k, \tau_k]$. The opposite case may be handled in a similar way.

By integration the following equation is obtained for $\dot{x}(t)$ on $[\sigma_k, t_k]$:

$$\dot{x}(t) = -\exp\Big\{-\int_{\sigma_k}^{t} g(s, x(s), \dot{x}(s))\,ds\Big\} \int_{\sigma_k}^{t} f(x(\tau)) \exp\Big\{\int_{\sigma_k}^{\tau} g(s, x(s), \dot{x}(s))\,ds\Big\}\,d\tau. \tag{6}$$

Let M, N be defined so that $M > b(t)\alpha(t)$ and $N > \psi(x(t), \dot{x}(t))$ on $[t_1, \infty)$. Then for $t \in [\sigma_k, \tau_k]$

$$\dot{x}(t) > \delta \exp\Big\{\int_{\sigma_k}^{t} \frac{MN}{\alpha}\Big\} \int_{\sigma_k}^{t} \exp\Big\{\int_{\sigma_k}^{\tau} \frac{MN}{\alpha}\Big\}\,d\tau \geqq \frac{\delta}{MN}\,\alpha(t)\Big(1 - \exp\Big\{-\int_{\sigma_k}^{t} \frac{MN}{\alpha}\Big\}\Big).$$

Now we can majorize $J(\sigma_k, \tau_k)$:

$$J(\sigma_k,\tau_k)=\frac{\delta}{MN}\int_{\sigma_k}^{\tau_k}\dot x(\tau)g(\tau,x(\tau),\dot x(\tau))\alpha(\tau)\Big(\int_{t_0}^{\tau}\alpha\Big)\times \tag{7}$$
$$\times\left(\frac{vKMN}{\delta}\Big(\int_{t_0}^{\tau}\alpha\Big)^{-1}-\left\{1-\exp\left\{-\int_{\sigma_k}^{\tau}\frac{MN}{\alpha}\right\}\right\}\right)d\tau.$$

By the choice of τ_k Lemma 3 ensures the existence of $\delta_1>0$ such that

$$1-\exp\left\{-\int_{\sigma_k}^{\tau_k}\frac{M}{\alpha}\right\}>\delta_1 \quad (k=1,\ 2\ldots)$$

Hence, if k is large enough, then there exists a $\tau_k'\in(\sigma_k,\tau_k)$, such that

$$\frac{vKMN}{\delta}\Big(\int_{t_0}^{\tau_k'}\alpha\Big)^{-1}=1-\exp\left\{-\int_{\sigma_k}^{\tau_k'}\frac{MN}{\alpha}\right\},$$

and

$$1-\exp\left\{-\int_{\sigma_k}^{\tau}\frac{MN}{\alpha}\right\}>\frac{vKMN}{\delta}\Big(\int_{t_0}^{\tau}\alpha\Big)^{-1}$$

if $\tau_k'<\tau<\tau_k$. It follows from the definition of τ_k', that

$$\int_{\sigma_k}^{\tau_k'}\frac{1}{\alpha}<R\Big(\int_{t_0}^{\tau_k'}\alpha\Big)^{-1}\quad\text{and so}\quad \tau_k'-\sigma_k<R\alpha(\sigma_k)\Big(\int_{t_0}^{\sigma_k}\alpha\Big)^{-1} \tag{8}$$

with a suitable number R.

We majorize $J(\sigma_k, \tau_k)$ integrating only on $[\sigma_k, \tau_k']$:

$$J(\sigma_k,\tau_k)<vKLN\int_{\sigma_k}^{\tau_k'}b(\tau)\alpha(\tau)\,d\tau<vKLMN(\tau_k'-\sigma_k).$$

Substituting this estimate into (5), by the aid of (8) we obtain

$$J(t_1,t_n)<vKLMN\sum_{k=1}^{n}(\tau_k'-\sigma_k)\leqq KLMNR\sum_{k=1}^{n}\alpha(\sigma_k)\Big(\int_{t_0}^{\sigma_k}\alpha\Big)^{-1}$$

where the right-hand side is $o\Big(\int_{t_0}^{t_n}\alpha\Big)$ $(n\to\infty)$. The theorem has been proved.

PROOF of Theorem 2. The proof of Theorem 1 will be refined. First we show that for arbitary given number $\varepsilon>0$ there exists a set $\Omega\subset[t_1,\infty)$ and a number $\varepsilon_1>0$ such that

$$\overline{\lim_{n\to\infty}}\Big(\int_{\Omega\cap[t_1,t_n]}2\lambda(1+v)\alpha\Big)\Big(\int_{t_1}^{t_n}\alpha\Big)^{-1}<\varepsilon \tag{9}$$

and $\varphi(x(t),\dot x(t))>\varepsilon_1$ on $[t_1,\infty)\setminus\Omega$. Indeed, because $b(t)\equiv1$, the sequence

$\{\sigma_{k+1}-\sigma_k\}$ is bounded (see [2], proof of Theorem 5). Hence there exists a set of form $\Omega=\bigcup_{k=1}^{\infty}[\tau_k^1, \tau_k^2]$ with property (9) $(t_{k-1}<\tau_k^1<\sigma_k<\tau_k^2<t_k)$. In fact, this choice is possible so that $|\tau_k^i-\sigma_k|>\delta>0$ and $|f(x(\tau_k^i))|>\delta$ $(i=1,2;\ k=1,2,\ldots)$ for some number δ. Then by (6) there exists a number $\delta_1>0$, such that $|\dot{x}(\tau_k^i)|>\delta_1$ $(i=1,2;$ $k=1,2,\ldots)$. From properties of $\varphi(x,y)$ the existence of ε_1 follows.

Now consider the term $I(t_1, t_n)$ in estimate of $W(t)$:

$$I(t_1,t_n)<(\lambda+\varepsilon)\int_{t_1}^{t_n}\Big[a(\tau)\varphi(x(\tau),\dot{x}(\tau))\Big(\int_{t_0}^{t}\alpha\Big)-$$

$$-(1+\nu)\alpha(\tau)\Big]^-\,d\tau<(\lambda+\varepsilon)(1+\nu)\int_{\Omega\cap[t_1,t_n]}\alpha+$$

$$+(\lambda+\varepsilon)\int_{t_1}^{t_n}\Big[a(\tau)\varepsilon_1\Big(\int_{t_0}^{\tau}\alpha\Big)-(1+\nu)\alpha(\tau)\Big]^-\,d\tau.$$

The proof can be continued in a similar way as in the proof of Theorem 1.

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AN EMBEDDING THEOREM FOR ORDERED TOPOLOGICAL SPACES

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1. According to an earlier theorem due to L. Nachbin it is well-known that the uniformizable ordered (compact ordered) spaces are exactly the (closed) subspaces of the naturally ordered Tychonoff cubes ([5], p. 104). Recently T. H. Choe and Y. S. Park [1] showed that under certain conditions an ordered topological space can be embedded into a suitable topological join-semilattice.

In this paper we generalize the classical theorem that any T_0 topological space of weight $\leqq \mathfrak{m}$ is isomorphic to a subspace of $\mathbf{F}^{\mathfrak{m}}$ (where $\mathbf{F}=\{0, 1\}$ with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$) by constructing so-called *convex Alexandroff cubes*. Such a cube is a complete distributive atomic topological lattice (with a non-continuous complementation). Each convex T_0-ordered topological space is embedded into a convex Alexandroff cube. In this way a compactification of such spaces can be obtained.

2. A set X equipped with a topology and a partial order is called an *ordered topological space*. If X is a lattice, and its order issues from the lattice operations $\vee$ and $\wedge$ by the definition $x\leqq y\Leftrightarrow x\vee y=y\Leftrightarrow x\wedge y=x$, moreover both $\vee$ and $\wedge$ are continuous mappings of $X\times X$ onto X, then X will be called a *topological lattice*. A subset $E\subset X$ is said to be *decreasing (increasing)* iff $x\in E$, $y<x$ $(x\leqq y)$ imply $y\in E$. The decreasing (increasing) open sets form a topology on X called the *lower (upper) topology* of X. If this topology agrees with the initial one, i.e. in X each open set is decreasing (increasing) then we say that X is *decreasing (increasing)*. X is *convex* ([5], p. 100) iff it has a subbase consisting of decreasing and increasing sets (that is, the topology of X is the supremum of its lower and upper topology). X is said to be T_0*-ordered* iff, for any $x, y\in X$, $x\not\leqq y$, there exists either an increasing open $V\subset X$ such that $x\in V$, $y\notin V$, or a decreasing open $W\subset X$ such that $y\in W$, $x\notin W$. (This is a new separation axiom for ordered topological spaces introduced first for preordered syntopogenous spaces in [3].) X is called *upper (lower)* T_1*-ordered* iff $x, y\in X$, $x\not\leqq y$ imply $y\in W$, $x\notin W$ $(x\in V, y\notin V)$ for some decreasing open $W\subset X$ (increasing open $V\subset X$). (Note that this definition is equivalent to that of McCartan [4].)

Let $\{X_a\colon a\in A\neq\emptyset\}$ be a family of ordered topological spaces, and consider the order $\leqq'$ on the product of the sets X_a by postulating $(x_a)\leqq'(y_a)$ iff $x_a\leqq_a y_a$ for every $a\in A$. Endowing the topological Cartesian product of the X_a's with this order, the *product* $\bigtimes_{a\in A} X_a$ of the ordered topological spaces can be obtained. It is easy to

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check that if the spaces X_a are decreasing, increasing, convex, T_0-, upper or lower T_1-ordered and topological lattices, then so is their product, too.

We shall say that X *is embedded* into the ordered topological space Z iff there exists a mapping h of X into Z, which is a homeomorphism and at the same time an order isomorphism onto a topological and ordered subspace Y of Z.

3. Let us consider the natural order $\leqq$ on the set $\{0, 1\}$, and denote by $\mathbf{F}$ ($\mathbf{F}'$) this ordered space with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$ ($\{\emptyset, \{1\}, \{0, 1\}\}$). Then the ordered topological space $\mathbf{F}$ is decreasing upper T_1-ordered, and dually, $\mathbf{F}'$ is increasing lower T_1-ordered, consequently the products $\mathbf{F}^{\mathfrak{m}}$ and $\mathbf{F}'^{\mathfrak{m}}$ also have the corresponding property for any cardinal number $\mathfrak{m}$. $\mathbf{F}^{\mathfrak{m}}(\mathbf{F}'^{\mathfrak{m}})$ wil be called the *decreasing (increasing) Alexandroff cube of weight* $\mathfrak{m}$. Since both $\mathbf{F}^{\mathfrak{m}}$ and $\mathbf{F}'^{\mathfrak{n}}$ are convex T_0-ordered for any cardinal numbers $\mathfrak{m}, \mathfrak{n}$, the ordered product space $\mathbf{F}^{\mathfrak{m}}\times\mathbf{F}'^{\mathfrak{n}}$ is also convex T_0-ordered, thus it will be said to be the *convex Alexandroff cube of type* $(\mathfrak{m}, \mathfrak{n})$. (The topological weight of this cube is obviously $\mathfrak{m}+\mathfrak{n}$.) It is trivial that under the operations $\wedge=\min$ and $\vee=\max$ both $\mathbf{F}$ and $\mathbf{F}'$ are topological lattices with the same natural order $\leqq$, therefore each of the decreasing, increasing and convex Alexandroff cubes is a topological lattice. It is also well-known that these lattices are complete, atomic and Boolean, but the complementation is not continuous with respect to the topologies considered above.

4. Our embedding theorem is the following one:

THEOREM. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be arbitrary cardinal numbers. In order that the ordered topological space* X *be isomorphic to a topological ordered subspace of the convex Alexandroff cube of type* $(\mathfrak{m}, \mathfrak{n})$, *it is necessary and sufficient that there exist two systems* $\mathcal{D}$ *and* $\mathcal{I}$ *of subsets of* X *with the following properties:*

(1) $\mathcal{D}\cup\mathcal{I}$ *is an open subbase in* X.

(2) *For* $x, y\in X$, $x\leqq y$ *iff* $y\in D\in\mathcal{D}$ *implies* $x\in D$ *and* $x\in I\in\mathcal{I}$ *implies* $y\in I$.

(3) $|\mathcal{D}|\leqq\mathfrak{m}$ *and* $|\mathcal{I}|\leqq\mathfrak{n}$.

PROOF. **Necessity:** Let A and B be disjoint sets of indices such that $|A|=\mathfrak{m}$, $|B|=\mathfrak{n}$, $\mathbf{F}^{\mathfrak{m}}=\bigtimes_{a\in A}\mathbf{F}_a$, $\mathbf{F}'^{\mathfrak{n}}=\bigtimes_{b\in B}\mathbf{F}'_b$, where $\mathbf{F}_a=\mathbf{F}$, $\mathbf{F}'_b=\mathbf{F}'$ for any $a\in A$, $b\in B$. Suppose that $h\colon X\to Y\subset\mathbf{F}^{\mathfrak{m}}\times\mathbf{F}'^{\mathfrak{n}}$ is an isomorphism. Define, for $a_0\in A$, $b_0\in B$, the sets $D^*_{a_0}=(\bigtimes_{a\in A}P_a)\times\{0, 1\}^{\mathfrak{n}}$ and $I^*_{b_0}=\{0, 1\}^{\mathfrak{m}}\times(\bigtimes_{b\in B}Q_b)$, where $P_{a_0}=\{0\}$, $Q_{b_0}=\{1\}$, $P_a=Q_b=\{0, 1\}$ for $a_0\neq a\in A$ and $b_0\neq b\in B$. Let $A^*=\{a\in A\colon D^*_a\cap Y\neq\emptyset\}$, $B^*=\{b\in B\colon I^*_b\cap Y\neq\emptyset\}$, finally put $\mathcal{D}=\{D_a=h^{-1}(D^*_a\cap Y)\colon a\in A^*\}$ and $\mathcal{I}$ $\mathcal{I}=\{I_b=h^{-1}(I^*_b\cap Y)\colon b\in B^*\}$.

Since $\{D^*_a, I^*_b\colon a\in A, b\in B\}$ is a subbase for the open sets of $\mathbf{F}^{\mathfrak{m}}\times\mathbf{F}'^{\mathfrak{n}}$, the system $\mathcal{D}\cup\mathcal{I}$ is also a subbase in X. It is clear that $|\mathcal{D}|\leqq|A|=\mathfrak{m}$ and $|\mathcal{I}|\leqq|B|=\mathfrak{n}$, thus (1) and (3) are satisfied. In order to verify (2) put $x\leqq y$ for some $x, y\in X$. If $y\in D_a$, $a\in A^*$, then $h(x)_a\leqq h(y)_a=0$ implies $h(x)_a=0$, that is $h(x)\in D^*_a$ and $x\in D_a$. Similarly, $x\in I_b$, $b\in B^*$ give $1=h(x)_b\leqq h(y)_b$, therefore $h(y)_b=1$, $h(y)\in I^*_b$ and $y\in I_b$. Conversely, suppose $x\nleqq y$. Then $h(x)\nleqq h(y)$, i.e. $h(x)_c\nleqq h(y)_c$ for at least one index $c\in A\cup B$. This means $h(y)_c=0$, $h(x)_c=1$. If $c\in A$, then $h(y)\in D^*_c$, hence $c\in A^*$, at the same time $h(x)\notin D^*_c$, so that $y\in D_c$, $x\notin D_c$. If $c\in B$, then $h(x)\in I^*_c$, therefore $c\in B^*$, simultaneously $h(y)\notin I^*_c$, thus $x\in I_c$, $y\notin I_c$.

Sufficiency: Suppose $\mathscr{D}=\{D_a\colon a\in A^*\}$ and $\mathscr{I}=\{I_b\colon b\in B^*\}$. $|A^*|\leqq\mathfrak{m}$, $|B^*|\leqq\mathfrak{n}$, thus there are sets A, B such that $A^*\subset A$, $B^*\subset B$, $|A|=\mathfrak{m}$, $|B|=\mathfrak{n}$. We can assume $A\cap B=\emptyset$. Let us define a mapping h of X into $\mathbf{F}^{\mathfrak{m}}\times\mathbf{F}'^{\mathfrak{n}}$ as follows:

For given indices $a\in A^*$, $b\in B^*$ and a point $x\in X$, suppose

(i) $$h_a(x)=\begin{cases}0\Leftrightarrow x\in D_a\\ 1\Leftrightarrow x\notin D_a,\end{cases}$$

(ii) $$h_b(x)=\begin{cases}0\Leftrightarrow x\notin I_b\\ 1\Leftrightarrow x\in I_b,\end{cases}$$

moreover assume $h_c(x)=0$ for any $c\in(A-A^*)\cup(B-B^*)$. In this way a mapping h_c of X into the c-th component of the product is obtained. Let the c-th component of the value of h taken on $x\in X$ be determined by

(iii) $$h(x)_c=h_c(x).$$

Let $h(X)=Y\subset\mathbf{F}^{\mathfrak{m}}\times\mathbf{F}'^{\mathfrak{n}}$. It is easy to show that $x\leqq y$ is equivalent to $h(x)\leqq h(y)$ by (2), consequently h is an order isomorphism. In order to verify that h is a homeomorphism of X onto Y it is sufficient to see that h_c is continuous for any $c\in A\cup B$, and that if $V\subset X$ is open, $x\in V$, then there exists an open W in $\mathbf{F}^{\mathfrak{m}}\times\mathbf{F}'^{\mathfrak{n}}$ such that $x\in h^{-1}(W)\subset V$ (see [2], (7.1.8) and (2.6.4)(b)). The first condition is an immediate consequence of (1) and the definition of the mappings h_c $(c\in A\cup B)$. If V is open in X, $x\in V$, then there are $A_0\subset A^*$, $B_0\subset B^*$, $|A_0|<\aleph_0$, $|B_0|<\aleph_0$ such that $x\in(\bigcap_{a\in A_0}D_a)\cap(\bigcap_{b\in B_0}I_b)\subset V$. Put

$$W=(\mathop{\times}_{a\in A}(P_a)\times(\mathop{\times}_{b\in B}Q_b),$$

where $P_a=\{0\}$ for $a\in A_0$, $Q_b=\{1\}$ for $b\in B_0$, $P_a=Q_b=\{0,1\}$ for $a\in A-A_0$ and $b\in B-B_0$. Then W is open in $\mathbf{F}^{\mathfrak{m}}\times\mathbf{F}'^{\mathfrak{n}}$ such that $h^{-1}(W)=(\bigcap_{a\in A_0}D_a)\cap(\bigcap_{b\in B_0}I_b)$. ∎

5. Let us mention some simple consequences of the embedding theorem.

Corollary 1. *Any decreasing upper T_1-ordered (increasing lower T_1-ordered) topological space can be embedded into a decreasing (increasing) Alexandroff cube.*

Corollary 2. *Any convex T_0-ordered topological space can be embedded into a convex Alexandroff cube.*

Proof of 1 and 2. Let $\mathscr{D}(\mathscr{I})$ be an arbitrary base for the lower (upper) topology of X, $\mathfrak{m}=|\mathscr{D}|$, $\mathfrak{n}=|\mathscr{I}|$. In addition, if X is decreasing upper T_1-ordered (increasing lower T_1-ordered) then one can choose $\mathscr{I}=\emptyset$ $(\mathscr{D}=\emptyset)$, i.e. $\mathfrak{n}=0$ $(\mathfrak{m}=0)$. In each one of the three cases (decreasing, increasing and convex) $\mathscr{D}$ and $\mathscr{I}$ satisfy the conditions of the theorem, thus X can be embedded into the convex Alexandroff cube of type $(\mathfrak{m},\mathfrak{n})$, and in particular, for $\mathfrak{n}=0$ $(\mathfrak{m}=0)$ this cube agrees with the decreasing (increasing) Alexandroff cube of weight $\mathfrak{m}$ (and $\mathfrak{n}$ respectively). ∎

Note that the T_0-spaces are convex T_0-ordered topological spaces with the discrete order ($x\leqq y$ iff $x=y$). Since the convex Alexandroff cube of type $(\mathfrak{m},\mathfrak{n})$ is homeomorphic to the cube $\mathbf{F}^{\mathfrak{m}+\mathfrak{n}}$, our Theorem contains Alexandroff's theorem on the embedding of T_0-spaces. ∎

Corollary 3. *Every decreasing upper T_1-ordered, increasing lower T_1-ordered, or convex T_0-ordered topological space is a dense subspace of a compact partially ordered topological space having the same property.*

Proof. Let Z denote the corresponding decreasing, increasing or convex Alexandroff cube in which the ordered topological space X in question is embedded. Without loss of generality X can be identified with its isomorphic image Y in Z. Since Z is evidently compact, the closure of Y in Z is also compact and, as a subspace of Z, has the convexity and separation properties of Z. ∎

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NORM FORM EQUATIONS WITH SEVERAL DOMINATING VARIABLES AND EXPLICIT LOWER BOUNDS FOR INHOMOGENEOUS LINEAR FORMS WITH ALGEBRAIC COEFFICIENTS

I. GAÁL

1. Effective bounds for the solutions of norm form equations with several dominating variables

The purpose of the present paper is to give a common generalization of some results of Sprindžuk [33] and of Győry and Papp [15] concerning norm form equations.

Let $L \subset K$ be algebraic number fields with rings of integers $\mathbf{Z}_L$ and $\mathbf{Z}_K$. Let μ denote a non-zero element of L, and let $\alpha_1=1, \alpha_2, \ldots, \alpha_k$ be elements of K linearly independent over L, such that $K=L(\alpha_1, \ldots, \alpha_k)$.

Suppose $n=[K: L] \geqq 3$. Let us consider the solutions $(x_1, \ldots, x_k) \in \mathbf{Z}_L^k$ of the norm form equation

$$N_{K|L}(x_1+\alpha_2 x_2+\ldots+\alpha_k x_k) = \mu. \tag{1}$$

This equation plays an essential role in the theory of diophantine equations and in its applications (see e.g. Borevich and Shafarevich [3], Baker [2], Pethő [21], Győry [10], Schmidt [27]). When $k=n$, the problem of the resolution of (1) is essentially solved (cf. [3]). Hence we shall restrict ourselves to the case $k<n$. When $k=2$, equation (1) is called Thue's equation. Then, by a well-known theorem of Thue ([35], case $L=\mathbf{Q}$) and Siegel ([28], case of arbitrary L), (1) has only a finite number of solutions. Moreover, Baker [1], [2] gave an effective upper bound for all solutions of (1) which made possible to determine all the solutions. In the case $k=2$ a number of generalizations and sharpenings of these theorems of Baker have been established (see e.g. Coates [4], Sprindžuk [30], [32], [33], Feldman [5], Stark [34], Kotov [16], Kotov and Sprindžuk [20], Győry [6], [9], [11], [12], [13]).

Sprindžuk [33] studied an inhomogen generalization of Thue's equation. He gave an effective upper bound for the solutions x_1, x_2, λ of the equation

$$N_{K|\mathbf{Q}}(x_1+\alpha x_2+\lambda) = \mu, \tag{2}$$

where $K=\mathbf{Q}(\alpha)$, $[K: \mathbf{Q}]=n \geqq 3$, $\alpha \in \mathbf{Z}_K$, $0 \neq \mu \in \mathbf{Z}$, $x_1, x_2 \in \mathbf{Z}$ and $\lambda \in \mathbf{Z}_K$ is a non-dominating variable for which[1] $\overline{|\lambda|} \leqq X^{1-\zeta}$, $X=\max(|x_1|, |x_2|)$ with a given small positive number ζ. Sprindzuk's result containes Baker's well-known theorem as a special case.

Let now $k \geqq 2$ be an arbitrary integer. In the case $L=\mathbf{Q}$ Schmidt [25] obtained a general criterion for (1) to have only finitely many solutions in $\mathbf{Z}$. For a generaliz-

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[1] Using standard notation, $\overline{|\alpha|}$ will denote the size of the algebraic number α, i.e. the maximum absolute value of the conjugates of α.

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ation see Schlickewei [24]. These results are ineffective generalizations of Thue's theorem mentioned above. In the case of an arbitrary algebraic number field L Győry and Papp [14], [15], Győry [6], [9], [10], [11], [12], [13], and Kotov [17], [18], [19] obtained, under certain general assumptions concerning $\alpha_1, \dots, \alpha_k$, effective bounds for the solutions of (1) and of certain generalizations of (1). In Győry and Papp [15] the bound has been derived subject to the condition that

$$[L(\alpha_i)\colon L] = n_i \geqq 3 \quad (i = 2, \dots, k) \quad \text{and} \quad n_2 \dots n_k = n. \tag{3}$$

Apart from the form of the bound, the mentioned result of [15] also includes Baker's famous theorem [1], [2] as a special case.

A natural common generalization of the above two equations is the equation

$$N_{K|L}(x_1+\alpha_2 x_2+\dots+\alpha_k x_k+\lambda) = \mu \tag{4}$$

in $x_1, \dots, x_k, \lambda$, where $K, L, \alpha_1, \alpha_2, \dots, \alpha_k, \mu, x_1, \dots, x_k$ satisfy the same assumptions as in equation (1), for $\alpha_2, \dots, \alpha_k$ (3) holds, $\lambda \in \mathbf{Z}_K$ is an algebraic number with $\overline{|\lambda|} < X_0^{1-\zeta}$, $X_0 = \max_{2 \leqq i \leqq k} \overline{|x_i|}$, and ζ is a given small positive number. Our aim is to give an effectively computable upper bound for the sizes $X = \max_{1 \leqq i \leqq k} \overline{|x_i|}$ of all solutions $(x_1, \dots, x_k) \in \mathbf{Z}_L^k$ of equation (4).

To formulate our Theorem 1 we shall need some further notations. Let $[L\colon \mathbf{Q}] = l$ and $N = ln$. Suppose[2] $H(\alpha_i) \leqq H$ $(i = 2, \dots, k)$ and $H(\mu) \leqq m$. Denote by R_K and r the regulator and the unit rank of K. Under the above conditions our main result is as follows.

THEOREM 1. *There are effectively computable constants T_1 and T_2, depending only on l, n, R_K and r, such that if $(x_1, \dots, x_k) \in \mathbf{Z}_L^k$ and $\lambda \in \mathbf{Z}_K$ satisfy equation* (4) *and $\overline{|\lambda|} < X_0^{1-\zeta}$ with $X_0 = \max_{2 \leqq i \leqq k} \overline{|x_i|}$ and $0 < \zeta < 1$ then*

$$\max_{1 \leqq i \leqq k} \overline{|x_i|} < [8H^k(2m)^{l/n}]^{(T_2/\zeta)\log(T_1/\zeta)}. \tag{5}$$

In the special case $\lambda = 0$ our Theorem 1 gives Theorem 1 of [15] with another estimate.

COROLLARY 1.1. *If $K, L, \alpha_1, \alpha_2, \dots, \alpha_k$ and μ satisfy the above assumptions, then we have* (5) *with $\zeta = 1/2$ for all solutions $(x_1, \dots, x_k) \in \mathbf{Z}_L^k$ of* (1).

For certain improvements and (homogeneous) generalizations of Corollary 1.1 see Győry [11], [12], [13] and Kotov [17], [18], [19].

In the special case when $L = \mathbf{Q}$ and $k = 2$, from our Theorem 1 we get a modified version of Theorem 1 of [33].

COROLLARY 1.2. *Suppose that in* (2) $[K\colon \mathbf{Q}] = n \geqq 3$ *and that $H(\alpha) \leqq H$. There are effectively computable constants T_1' and T_2', depending only on n, R_K and r, such that if $x_1, x_2 \in \mathbf{Z}$ and $\lambda \in \mathbf{Z}_K$ satisfy equation* (2) *and $\overline{|\lambda|} < |x_2|^{1-\zeta}$ with some ζ, $0 < \zeta < 1$, then*

$$\max(|x_1|, |x_2|) < [8H^2(2|\mu|)^{1/n}]^{(T_2'/\zeta)\log(T_1'/\zeta)}.$$

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[2] As usual, $H(\alpha)$ denotes the height of an algebraic number α, that is the maximum absolute value of the coefficients of its defining polynomial over $\mathbf{Z}$.

2. Effective lower bounds for inhomogeneous linear forms with algebraic coefficients

Suppose $\alpha_1=1, \alpha_2, \ldots, \alpha_k$ are algebraic numbers, linearly independent over $\mathbf{Q}$. Let $K=\mathbf{Q}(\alpha_2, \ldots, \alpha_k)$ and $[K: \mathbf{Q}]=n$. By a generalization of a well-known theorem of Liouville there exists an effectively computable constant $c=c(\alpha_2, \ldots, \alpha_k)>0$ such that

$$|x_1+x_2\alpha_2+\ldots+x_k\alpha_k| > cX^{-((n-\sigma)/\sigma)},\ X = \max_{1\le i\le k} |x_i|$$

for all $(x_1, \ldots, x_k)\in\mathbf{Z}^k\setminus\{0\}$ where $\sigma=1$ or 2 according as K is real or not. By the Thue—Siegel—Roth—Schmidt theorem

$$(6)\qquad |x_1+x_2\alpha_2+\ldots+x_k\alpha_k| > c'X^{-\varkappa}$$

for any $\varkappa>(k-\sigma)/\sigma$. In (6) the exponent is best possible, but the constant $c'=c'(\varkappa, \alpha_2, \ldots, \alpha_k)>0$ cannot be effectively computed. (For further references see Schmidt [26], [27], Baker [2] and Győry [8]. Historical survey can be found in Schmidt [26].)

Norm form equations are in close connection with approximation of linear forms with algebraic coefficients. The explicit bounds obtained for the solutions of Thue's equation and its generalizations made possible to give explicit lower bounds for linear forms with algebraic coefficients, improving the above Liouville inequality (cf. e.g. [1], [4], [5], [33], [20], [19], [8], [15]).

Sprindžuk's result [33] for equation (2) makes possible to give effectively computable constants d and d' such that (with the notation of equation (2))

$$(7)\qquad |x_1+\alpha x_2+\lambda| > dX^{(-n+\sigma+d')/\sigma}\quad (X=\max(|x_1|, |x_2|))$$

for all $(x_1, x_2)\in\mathbf{Z}^2$, $\lambda\in\mathbf{Z}_K$ if $\overline{|\lambda|}<X^{1-\zeta}$ and $x_1+\alpha x_2+\lambda\neq 0$. (Here $\sigma=1$ or 2 according as K is real or not.) In fact Sprindžuk [33] obtained from his theorem another consequence as well, representing $\lambda\in\mathbf{Z}_K$ in an integer basis of K.

Let again $L\subset K$ be algebraic number fields with the parameters given in our Theorem 1. Denote by R_L the regulator of L. Let $\alpha_1, \ldots, \alpha_k$ be algebraic numbers in K with heights $\le H$ such that $[L(\alpha_i): L]=n_i\ge 3$ $(i=1, \ldots, k)$ and $n_1, \ldots, n_k=n$.

Suppose there are s real and $2t$ complex conjugate fields to K over $\mathbf{Q}$. Let Ω denote the set of Archimedean valuations $|.|_v$ of K, where v is one of the natural numbers $1, 2, \ldots, s+t$. For $\beta\in K$ put $\|\beta\|_v=|\beta|_v^{n_v}$ where $n_v=[K_v: \mathbf{Q}_v]$. Under the above conditions Győry and Papp [15] proved that there are effectively computable constants ϱ and τ, and there exists a unit ε in L such that

$$(8)\qquad \prod_{v\in\Gamma} \|(\varepsilon x_0)+(\varepsilon x_1)\alpha_1+\ldots+(\varepsilon x_k)\alpha_k\|_v > \varrho X^{-N+s'+2t'+\tau},\quad X=\max_{0\le i\le k}\overline{|\varepsilon x_i|}$$

for all $\{0\}\neq(x_0, x_1, \ldots, x_k)\in\mathbf{Z}_L^{k+1}$ where Γ is any subset of Ω and s' and t' denote the number of real and complex valuations in Γ. For certain generalizations, when, among other things, Γ includes both Archimedean and non-Archimedean valuations see Győry [8] and Kotov [19].

Our Theorem 1 enables us to give a common generalization of the above approximation results of Sprindžuk and of Győry and Papp. To formulate our Theorem 2 we need the same assumptions on $K, L, \alpha_1, \alpha_2, \ldots, \alpha_k$ as above (in (8)). Given any

$\lambda \in \mathbf{Z}_K$ and $(x_0, x_1, \dots, x_k) \in \mathbf{Z}_L^{k+1}$ let

(9) $$N_{K|L}(x_0+x_1\alpha_1+\dots+x_k\alpha_k+\lambda) = \mu.$$

By Lemma 3 of Győry [7] there exists a unit $\varepsilon=\varepsilon(x_0, x_1, \dots, x_k, \lambda)$ in L such that for $\mu'=\mu\varepsilon^n$

(10) $$\overline{|\mu'|} \leqq |N_{L|\mathbf{Q}}(\mu')|^{1/l} \exp(6nl^{3l-1}R_L).$$

Under the above conditions we have the following theorem.

THEOREM 2. *Let $\Gamma \subseteq \Omega$. Denote by s' and t' the number of real and complex valuations in Γ. Let $(x_0, x_1, \dots, x_k) \in \mathbf{Z}_L^{k+1}$ and $\lambda \in \mathbf{Z}_K$. Suppose that with the above $\varepsilon=\varepsilon(x_0, x_1, \dots, x_k, \lambda)$ λ satisfies $\overline{|\varepsilon\lambda|}<(\max_{1\leqq i\leqq k} \overline{|\varepsilon x_i|})^{1-\zeta}$ where $0<\zeta<1$ is a given real number. If $x_0+x_1\alpha_1+\dots+x_k\alpha_k+\lambda\neq 0$ then*

(11) $$\prod_{v\in\Gamma} \|(\varepsilon x_0)+(\varepsilon x_1)\alpha_1+\dots+(\varepsilon x_k)\alpha_k+(\varepsilon\lambda)\|_v > \varrho_1 X^{-N+s'+2t'+\tau_1}$$

where $X=\max_{0\leqq i\leqq k} \overline{|\varepsilon x_i|}$ and

$$\varrho_1=[2\cdot 8^{n/l} H^{(n/l)(k+1+kl)} \exp(6nl^{3l} R_L)]^{-1}(2+k+kH)^{-N+s'+2t'},$$

$$\tau_1 = \frac{n}{l}\left(\frac{T_2}{\zeta}\log\frac{T_1}{\zeta}\right)^{-1}.$$

(Here T_1 and T_2 denote the effectively computable constants of Theorem 1.)

We remark that ϱ_1 and τ_1 do not depend on ε. In the special case $L=\mathbf{Q}$ Theorem 2 gives the following result.

COROLLARY 2.1. *Let $\alpha_1, \dots, \alpha_k$ be algebraic numbers with height $\leqq H$. Suppose $[\mathbf{Q}(\alpha_i): \mathbf{Q}]=n_i\geqq 3$ $(i=1, \dots, k)$ and $n_1\dots n_k=n$. Let $(x_1, \dots, x_k)\in\mathbf{Z}^k$ and $\lambda\in\mathbf{Z}_K$. Suppose that $\overline{|\lambda|}<X_0^{1-\zeta}$ where $X_0=\max_{1\leqq i\leqq k}|x_i|$ and $0<\zeta<1$ is a given real number. If*[3] *$\|x_1\alpha_1+\dots+x_k\alpha_k+\lambda\|>0$ then*

(12) $$\|x_1\alpha_1+\dots+x_k\alpha_k+\lambda\| > \varrho_2 X_0^{(-n+\sigma+\tau_2)/\sigma}$$

where $\sigma=1$ or 2 according as K is real or not,

$$\varrho_2^\sigma = (2\cdot 8^n H^{n(2k+1)} e^{6n})^{-1}(2+k+kH)^{-n+\sigma}(2k(1+H))^{-n+\sigma+\tau_2}$$

and

$$\tau_2 = n\left(\frac{T_2'}{\zeta}\log\frac{T_1'}{\zeta}\right)^{-1}.$$

(Here T_1' and T_2' are the effectively computable constants of Corollary 1.2).

Let again L be an algebraic number field as above. Let ϑ be an algebraic number with height $\leqq H$ and with degree $n\geqq 3$ over L. Let $K=L(\vartheta)$, and let s, $2t$ and Ω be as in Theorem 2. Let $\alpha\in L$ and denote by a the leading coefficient of the minimal

[3] $\|\alpha\|$ denotes the distance from α to the nearest integer.

defining polynomial of α over $\mathbf{Z}$. Further, let $\lambda \in \mathbf{Z}_K$, and let

(13) $$N_{K|L}(a\vartheta - a\alpha + a\lambda) = \mu.$$

By Lemma 3 of Győry [7] there exists a unit $\varepsilon = \varepsilon(\alpha, \lambda)$ in L such that for $\mu' = \mu\varepsilon^n$ (10) holds. Under the above assumptions, from our Theorem 1 we shall deduce the following:

THEOREM 3. *Let $\Gamma \subseteq \Omega$. Denote by s' and t' the number of real and complex valuations in Γ. Let $\alpha \in L$, $\lambda \in \mathbf{Z}_K$ and let a, $\varepsilon = \varepsilon(\alpha, \lambda)$ be as above. Suppose λ satisfies $\overline{|\varepsilon a \lambda|} < (\overline{|\varepsilon a|})^{1-\zeta}$ with $\varepsilon = \varepsilon(\alpha, \lambda)$, where $0 < \zeta < 1$ is a given real number. If $\vartheta - \alpha + \lambda \neq 0$ then*

(14) $$\prod_{v \in \Gamma} \|\vartheta - \alpha + \lambda\|_v > \varrho_3 (H(\alpha))^{-(1+(1/l))ln + (s' + 2t' + \tau_3)/l}$$

and if α is an algebraic integer, then

(15) $$\prod_{v \in \Gamma} \|\vartheta - \alpha + \lambda\|_v > \varrho_3 (H(\alpha))^{(1+(1/l))(-ln + s' + 2t') + (\tau_3/l)}$$

where

$$\varrho_3 = (2 \cdot 8^{n/l} H^{((2+l)n)/ln} \exp(6nl^{3l} R_L))^{-1} (4H)^{-ln + s' + 2t'} \cdot 2^{ln - s' - 2t' - \tau_3}$$

and

$$\tau_3 = \frac{n}{l} \left(\frac{T_2}{\zeta} \log \frac{T_1}{\zeta} \right)^{-1}.$$

(Here T_1 and T_2 denote the same effectively computable constants as in Theorem 1.)

In the case $\lambda = 0$ our Theorem 3 becomes Theorem 3 of Győry and Papp [15], with constants of other form.

3. Proofs

Proof of Theorem 1. To prove our theorem we shall combine the arguments of the proofs of the main results of [33] and [15].

If $X_0 = 0$, (5) obviously holds. So we may suppose that $X_0 > 0$. Further we may assume that $X_0 = \overline{|x_k|}$ and

(16) $$X_0 > \exp\left\{ \frac{8}{\zeta} (N^2 k \log(4H(k+2))) + c_1 r R_K + 4 \log(2m) \right\}$$

where $c_1 = \left(\frac{6rN^2}{\log N} \right)^r$. Let us consider an isomorphism $K \to K'$ into $\mathbf{C}$ for which $x_k \to x_k'$ and $|x_k'| = \overline{|x_k|}$. Let us denote by $x_1', \ldots, x_k', \lambda', \mu', \alpha_2', \ldots, \alpha_k', L'$ the conjugates of $x_1, \ldots, x_k, \lambda, \mu, \alpha_2, \ldots, \alpha_k, L$ respectively under this isomorphism. For $x_1', \ldots, x_k', \lambda'$ we have

(17) $$N_{K'|L'}(x_1' + \alpha_2' x_2' + \ldots + \alpha_k' x_k' + \lambda') = \mu'.$$

Let $\alpha_i' = \alpha_{i,1}', \ldots, \alpha_{i,n_i}'$ denote the conjugates of α_i' over L'. Put

(18) $$\beta'_{j_2 \ldots j_k} = x_1' + \alpha'_{2,j_2} x_2' + \ldots + \alpha'_{k,j_k} x_k' + \lambda'_{j_2 \ldots j_k},$$

$(j_i=1, \dots, n_i;\ i=2, \dots, k)$, where $\lambda'_{j_2 \dots j_k}$ is the conjugate of λ' under the isomorphism for which $\alpha'_2 \to \alpha'_{2, j_2}, \dots, \alpha'_k \to \alpha'_{k, j_k}$. By assumption $K'=L'(\alpha'_2, \dots, \alpha'_k)$ and $[K' : L'] = n_2 \dots n_k$, so equation (17) can be written in the form

$$\prod_{j_2=1}^{n_2} \dots \prod_{j_k=1}^{n_k} \beta'_{j_2 \dots j_k} = \mu'. \tag{19}$$

Suppose the product $\prod_{j_k=1}^{n_k} |\beta'_{j_2 \dots j_k}|$ attains its minimum for $j'_2, \dots, j'_{k-1}$ and assume that

$$\beta'_{j_k} = \beta'_{j'_2 \dots j'_{k-1} j_k} \quad (j_k = 1, \dots, n_k).$$

Then (19) implies

$$\prod_{j_k=1}^{n_k} |\beta'_{j_k}| \leqq |\mu'|^{n_k/n} \leqq (2m)^{n_k/n}. \tag{20}$$

If $a_k > 0$ denotes the leading coefficient of the minimal defining polynomial of α'_k over $\mathbf{Z}$, then $a_k(\alpha'_{k,i} - \alpha'_{k,j})$ is a non-zero algebraic integer for any $i \neq j$ with $1 \leqq i, j \leqq n_k$ and

$$a_k |\alpha'_{k,i} - \alpha'_{k,j}| \leqq 2 \overline{|a_k \alpha'_k|} \leqq 4H, \tag{21}$$

whence

$$|\alpha'_{k,i} - \alpha'_{k,j}| \geqq (4H)^{-ln_k(ln_k - 1)}. \tag{22}$$

Let $|\beta'_q| = \min_{1 \leqq j \leqq n_k} |\beta'_j|$. Then for any j $(1 \leqq j \leqq n_k)$

$$|\beta'_j| \geqq |\beta'_q - \beta'_j| - |\beta'_q|$$

and from this we have

$$|\beta'_j| \geqq \frac{1}{2} |\beta'_q - \beta'_j| = \frac{1}{2} |\alpha'_{k,q} x'_k - \alpha'_{k,j} x'_k + \lambda'_q - \lambda'_j|, \tag{23}$$

where $\lambda'_{j_k} = \lambda'_{j'_2 \dots j'_{k-1} j_k}$ $(1 \leqq j_k \leqq n_k)$. Inequality (22) implies

$$X_0 (4H)^{-ln_k(ln_k-1)} \leqq X_0 |\alpha'_{k,q} - \alpha'_{k,j}| = |\alpha'_{k,q} x'_k - \alpha'_{k,j} x'_k| \leqq$$

$$\leqq |\alpha'_{k,q} x'_k - \alpha'_{k,j} x'_k + \lambda'_q - \lambda'_j| + |\lambda'_q - \lambda'_j| \leqq |\alpha'_{k,q} x'_k - \alpha'_{k,j} x'_k + \lambda'_q - \lambda'_j| + 2X_0^{1-\zeta},$$

whence

$$X_0 (4H)^{-ln_k(ln_k-1)} - 2X_0^{1-\zeta} \leqq |\alpha'_{k,q} x'_k - \alpha'_{k,j} x'_k + \lambda'_q - \lambda'_j|. \tag{24}$$

It follows from (23) and (24) together with (16) that

$$|\beta'_j| \geqq \frac{1}{2} X_0^{1-\zeta} \left(X_0^{\zeta} (4H)^{-ln_k(ln_k-1)} - 2 \right) \geqq \frac{1}{2} X_0^{1-\zeta} X_0^{\zeta/2} = \frac{1}{2} X_0^{(1-(\zeta/2))}. \tag{25}$$

From (20) and (25) we obtain

$$|\beta'_q| \leqq (2m)^{n_k/n} \left(\frac{1}{2} X_0^{1-(\zeta/2)} \right)^{1-n_k} = 2^{n_k-1} (2m)^{n_k/n} X_0^{(1-(\zeta/2))(1-n_k)}. \tag{26}$$

For any $\beta'_{j_2\ldots j_k}$ (using also (16))

$$|\beta'_{j_2\ldots j_k}| \leqq |\beta'_{j_2\ldots j_k}-\beta'_q|+|\beta'_q| =$$

$$= |(\alpha'_{2,j_2}-\alpha'_{2,j'_2})x'_2+\ldots+(\alpha'_{k-1,j_{k-1}}-\alpha'_{k-1,j'_{k-1}})x'_{k-1}+(\alpha'_{k,j_k}-\alpha'_{k,q})x'_k+$$

$$+\lambda'_{j_2\ldots j_k}-\lambda'_q|+|\beta'_q| \leqq 4H(k-1)X_0+2X_0^{1-\zeta}+2^{n_k-1}(2m)^{n_k/n}X_0^{(1-(\zeta/2))(1-n_k)} \leqq$$

$$\leqq 4H(k-1)X_0+2X_0+2^{n_k-1}(2m)^{n_k/n} \leqq 4H(k-1)X_0+3X_0 \leqq 4H(k+2)X_0$$

that is

$$|\beta'_{j_2\ldots j_k}| \leqq 4H(k+2)X_0. \tag{27}$$

By assumptions $[K\colon \mathbf{Q}]=N$. Let us suppose that there are s real and $2t$ complex conjugate fields to K and that they are chosen in the usual manner: $\alpha^{(j)}$ is real for $j=1, \ldots, s$ and $\alpha^{(j+t)}=\overline{\alpha^{(j)}}$ for $j=s+1, \ldots, s+t$ (for any element α of K). Let

$$e_j = \begin{cases} 1 & \text{for } \quad j = 1, \ldots, s \\ 2 & \text{for } \quad j = s+1, \ldots, s+t. \end{cases}$$

From the explicit form of a theorem of Siegel [29] and Stark [34] (see also Győry [7] Lemma 2) follows that there are independent units $\eta_1, \ldots, \eta_r$ in K such that

$$\prod_{i=1}^{r} \max(\log \overline{|\eta_i|}, 1) < c_1 R_K \tag{28}$$

and the absolute values of the elements of the inverse matrix of $(e_j \log |\eta_i^{(j)}|)_{1\leqq i,j\leqq r}$ do not exceed $c_2=\frac{6r!N^2}{\log N}$. Let $\beta_{1\ldots 1}=\beta$ and $|N_{K|\mathbf{Q}}(\beta)|=M$. Then there exist rational integers $b_1, \ldots, b_r$. (cf. [7], Lemma 3) such that

$$\gamma = \beta\eta_1^{b_1}\ldots\eta_r^{b_r} \tag{29}$$

and

$$\left|\log|M^{-1/N}\gamma^{(j)}|\right| < \frac{c_1 r}{2} R_K, \quad j = 1, \ldots, N. \tag{30}$$

Since $M=|N_{L|\mathbf{Q}}(\mu)|$, we have

$$|\log M| \leqq l\log(2m)+kN\log H.$$

From this inequality together with (30), (27) and (16) we get

$$b_1 e_j \log|\eta_1^{(j)}|+\ldots+b_r e_j \log|\eta_r^{(j)}| = e_j \log|\gamma^{(j)}/\beta^{(j)}| =$$

$$= e_j \log|M^{-1/N}\gamma^{(j)}|+\frac{e_j}{N}\log M-e_j\log|\beta^{(j)}| \leqq$$

$$\leqq c_1 r R_K+4l\log(2m)+4kN\log H+2n\log(4H(k+2)X_0) =$$

$$= c_1 r R_K+4l\log(2m)+4kN\log H+2n\log(4H(k+2))+2n\log X_0 \leqq$$

$$\leqq 3n\log X_0 \quad (1 \leqq j \leqq r).$$

This gives an upper bound for $|b_i|$:

(31) $$\max_{1\le i\le r}|b_i|\leqq 3rnc_2\log X_0.$$

Consider now the identity

(32) $$(\alpha'_{k,h}-\alpha'_{k,q})(\beta'_g-\lambda'_g)+(\alpha'_{k,g}-\alpha'_{k,h})(\beta'_q-\lambda'_q)-(\alpha'_{k,g}-\alpha'_{k,q})(\beta'_h-\lambda'_h)=0$$

where $q\neq h\neq g\neq q$; $1\leqq g, h\leqq n_k$. From (25) and (32) together with $|\alpha'_{k,i}-\alpha'_{k,j}|\leqq$ $\leqq 4H$ we have

$$\left|\alpha'_{k,h}-\alpha'_{k,q}+(\alpha'_{k,g}-\alpha'_{k,h})\frac{\beta'_q}{\beta'_g}-(\alpha'_{k,g}-\alpha'_{k,q})\frac{\beta'_h}{\beta'_g}\right|=$$

$$=\frac{1}{|\beta'_g|}|(\alpha'_{k,h}-\alpha'_{k,q})\beta'_g+(\alpha'_{k,g}-\alpha'_{k,h})\beta'_q-(\alpha'_{k,g}-\alpha'_{k,q})\beta'_h|=$$

$$=\frac{1}{|\beta'_g|}|(\alpha'_{k,h}-\alpha'_{k,q})\lambda'_g+(\alpha'_{k,g}-\alpha'_{k,h})\lambda'_q-(\alpha'_{k,g}-\alpha'_{k,q})\lambda'_h|\leqq$$

$$\leqq\frac{1}{|\beta'_g|}12H\overline{|\lambda|}\leqq\left(\frac{1}{2}X_0^{(1-(\zeta/2))}\right)^{-1}12HX_0^{1-\zeta}=24HX_0^{-2/\zeta}.$$

From the above estimate we get

$$\left|\alpha'_{k,h}-\alpha'_{k,q}-(\alpha'_{k,g}-\alpha'_{k,q})\frac{\beta'_h}{\beta'_g}\right|-\left|(\alpha'_{k,g}-\alpha'_{k,h})\frac{\beta'_q}{\beta'_g}\right|\leqq 24HX_0^{-\zeta/2}$$

whence

$$\left|\frac{(\alpha'_{k,g}-\alpha'_{k,q})\beta'_h}{(\alpha'_{k,h}-\alpha'_{k,q})\beta'_g}-1\right|\leqq\left|\frac{(\alpha'_{k,g}-\alpha'_{k,h})\beta'_q}{(\alpha'_{k,h}-\alpha'_{k,q})\beta'_g}\right|+\frac{24HX_0^{-\zeta/2}}{|\alpha'_{k,h}-\alpha'_{k,q}|}.$$

By (22) we have

$$|\alpha'_{k,i}-\alpha'_{k,j}|^{-1}\leqq(4H)^{ln_k(ln_k-1)}$$

with $i\neq j$, $1\leqq i,j\leqq n_k$. So we get

$$\left|\frac{(\alpha'_{k,g}-\alpha'_{k,q})\beta'_h}{(\alpha'_{k,h}-\alpha'_{k,q})\beta'_g}-1\right|\leqq\left|\frac{(\alpha'_{k,g}-\alpha'_{k,h})\beta'_q}{(\alpha'_{k,h}-\alpha'_{k,q})\beta'_g}\right|+24H(4H)^{ln_k(ln_k-1)}-X_0^{-\zeta/2}.$$

Using (16), from our estimates (21), (22), (25) and (26) we have

$$\left|\frac{(\alpha'_{k,g}-\alpha'_{k,q})\beta'_h}{(\alpha'_{k,h}-\alpha'_{k,q})\beta'_g}-1\right|\leqq$$

$$\leqq\frac{4H}{(4H)^{-ln_k(ln_k-1)}}\frac{2^{n_k-1}(2m)^{n_k/n}X_0^{(1-(\zeta/2))(1-n_k)}}{\frac{1}{2}X_0^{(1-(\zeta/2))}}+24H(4H)^{ln_k(ln_k-1)}X_0^{-\zeta/2}=$$

$$=(4H)^{ln_k(ln_k-1)+1}(2m)^{n_k/n}2^{n_k}X_0^{-(1-(\zeta/2))n_k}+24H(4H)^{ln_k(ln_k-1)}X_0^{-\zeta/2}\leqq$$

$$\leqq X_0X_0^{-(1-(\zeta/2))n_k}+X_0^{\zeta/4}X_0^{-\zeta/2}\leqq X_0^{1-3(1-(\zeta/2))}+X_0^{-\zeta/4}\leqq$$

$$\leqq X_0^{-1/2}+X_0^{-\zeta/4}\leqq 2X_0^{-\zeta/4}<X_0^{-\zeta/8}$$

that is

(33)
$$\left|\frac{(\alpha'_{k,g}-\alpha'_{k,q})\beta'_h}{(\alpha'_{k,h}-\alpha'_{k,q})\beta'_g}-1\right| < X_0^{-\zeta/8}.$$

Let

$$\alpha_i = \begin{cases} \eta'_{i,g}/\eta'_{i,h} & \text{for} \quad i = 1, \dots, r, \\ \dfrac{(\alpha'_{k,g}-\alpha'_{k,q})\gamma'_h}{(\alpha'_{k,h}-\alpha'_{k,q})\gamma'_g} & \text{for} \quad i = r+1, \end{cases}$$

where $\eta'_{i,g}, \gamma'_g$ (resp. $\eta'_{i,h}, \gamma'_h$) denote the conjugates of η_i, γ corresponding to β'_g (resp. to β'_h). With this notation (33) can be written in the form

$$0 < |\alpha_1^{b_1}\dots\alpha_r^{b_r}\alpha_{r+1}-1| < X_0^{-\zeta/8}.$$

This implies

(34) $$0 < |b_0 \log\alpha_0 + b_1\log\alpha_1 + \dots + b_r\log\alpha_r - \log\alpha_{r+1}^{-1}| < e^{(-\zeta/8)\log X_0} = e^{-\delta B},$$

where $\alpha_0 = -1$, log denotes the principal value of the logarithm and b_0 is a rational integer with

(35) $$|b_0| \leqq |b_1| + \dots + |b_r|$$

and $B = 3r^2nc_2 \log X_0$, $\delta = (\zeta/8)(3r^2nc_2)^{-1}$. Then by (31) we have

(36) $$\max_{0\leqq i\leqq r} |b_i| \leqq B.$$

Let $A_i = \max\left(H(\alpha_i), e^e\right)$ for $i=0, \dots, r$. Since

$$H(\alpha_i) \leqq (2\overline{|\eta'_{i,g}/\eta'_{i,h}|})^{N(n-1)} \leqq (2\overline{|\eta_i|}^N)^{N(n-1)},$$

we have

$$\log A_i < 2(n-1)N^2 \max(\log\overline{|\eta_i|}, 1)$$

and this together with (28) implies

(37) $$\Omega' = \log A_0 \log A_1, \dots, \log A_r < c_1 c_3 R_K,$$

where $c_3 = e[2(n-1)N'^2]^r$. Put $T = c_4\Omega' \log\Omega^1$ with $c_4 = (25(r+3)N)^{10(r+3)}$ and

$$A = [8H^k(2m)^{l/n}e^{8c_1 r R_K}]^{N^3(n-1)}.$$

Further, let $a = a_2 \dots a_k$ where a_i denotes the leading coefficient of the minimal defining polynomial of α'_i over $\mathbf{Z}$ $(i=2, \dots, k)$. Then $A_i < A$ for $i=0, \dots, r$ and since $a\cdot\gamma$ is an algebraic integer, from (21) and (30) we have

$$H(\alpha_{r+1}) \leqq \left(\overline{|a_k(\alpha'_{k,g}-\alpha'_{k,q})a\gamma'_h|} + \overline{|a_k(\alpha'_{k,h}-\alpha'_{k,q})a\gamma'_g|}\right)^{N(n-1)(n-2)} < A.$$

Obviously $\delta < c_4^{-1/2}T$. So we may apply a theorem of van der Poorten and Loxton (see Theorem 3 in [22] and [23]). By this theorem we get from (34)

$$B < \delta^{-1}T\log(\delta^{-1}T)\log A.$$

This yields and estimate for X_0:

(38) $$\log X_0 < \frac{8}{\zeta} T\log(\delta^{-1}T)\log A.$$

If x_1' is an arbitrary conjugate of x_1, by (27) we obtain

$$|x_1'| = |\beta'_{j_2\dots j_k} - (\alpha'_{2,j_2}x_2' + \dots + \alpha'_{k,j_k}x_k' + \lambda'_{j_2\dots j_k}| \leqq$$
$$\leqq 4H(k+2)X_0 + (k-1)2HX_0 + X_0^{1-\zeta} < 8H(k+1)X_0.$$

The above estimate together with (38) and (16) gives

$$\log X < 2\log X_0 < \frac{16}{\zeta} T \log(\delta^{-1}T)\log A. \tag{39}$$

By (37) $T < c_4c_1c_3R_K\log(c_1c_3R_K) = T_0$. Put

$$T_1 = 24r^2nc_2T_0 \quad \text{and} \quad T_0' = 16T_0 8c_1 rR_K N^2(n-2).$$

By (39) we obtain

$$\log X < \frac{T_0'}{\zeta}\log\left(\frac{T_1}{\zeta}\right)\log\left[(8H^k(2m)^{l/n})^{N^2(n-2)}\right] = \tag{40}$$
$$= \frac{T_2}{\zeta}\log\left(\frac{T_1}{\zeta}\right)\log\left[8H^k(2m^{l/n})\right]$$

where $T_2 = T_0'N^2(n-2)$. Then T_1 and T_2 are effectively computable constants depending only on l, n, R_K and r. From (40) our assertion (5) follows.

Proof of Theorem 2. We shall follow the proof of Theorem 2 of [15].

Denote by a the product of the leading coefficients of the minimal defining polynomials of $\alpha_1, \dots, \alpha_k$ over $\mathbf{Z}$. Then $a^n\mu \in \mathbf{Z}_L$. By (10) we have

$$H(\mu') \leqq a^n(2\overline{|\mu'|})^l \leqq H^{kn}|N_{L|\mathbf{Q}}(\mu')|\exp(6nl^{3l}R_L). \tag{41}$$

From equation (9) we get with $\varepsilon = \varepsilon(x_0, x_1, \dots, x_k, \lambda)$

$$N_{K|L}(\varepsilon x_0 + \varepsilon x_1\alpha_1 + \dots + \varepsilon x_k\alpha_k + \varepsilon\lambda) = \mu'. \tag{42}$$

By Theorem 1 we have

$$X = \max_{0\leqq i\leqq k}\overline{|x_i\varepsilon|} < \left[8H^{k+1}(2H(\mu'))^{l/n}\right]^{(T_2/\zeta)\log(T_1/\zeta)}. \tag{43}$$

From (41) and (43) we obtain

$$X < \left[8H^{k+1}(2H^{kn}|N_{L|\mathbf{Q}}(\mu')|\exp(6nl^{3l}R_L))^{l/n}\right]^{(T_2/\zeta)\log(T_1/\zeta)} =$$
$$= \left[2^{l/n}8H^{k+1+kl}|N_{L|\mathbf{Q}}(\mu')|^{l/n}(\exp(6nl^{3l}R_L))^{l/n}\right]^{(T_2/\zeta)\log(T_1/\zeta)}$$

whence

$$X^{((T_2/\zeta)\log(T_1/\zeta))^{-1}}\left[2^{l/n}8\cdot H^{k+1+kl}\exp(6nl^{3l}R_L))^{l/n}\right]^{-1} < |N_{L|\mathbf{Q}}(\mu')|^{l/n}$$

that is

$$|N_{L|\mathbf{Q}}(\mu')| > \varrho_1' X^{\tau_1} \tag{44}$$

with

$$\varrho_1' = \left[2\cdot 8^{n/l}H^{(n/l)(k+1+kl)}\exp(6nl^{3l}R_L)\right]^{-1}.$$

By (42) we have

$$\prod_{v\in\Omega}\|\varepsilon x_0 + \varepsilon x_1\alpha_1 + \dots + \varepsilon x_k\alpha_k + \varepsilon\lambda\|_v = |N_{L|\mathbf{Q}}(\mu')|. \tag{45}$$

For each $v \in \Omega$
$$\|\varepsilon x_0+\varepsilon x_1\alpha_1+\ldots+\varepsilon x_k\alpha_k+\varepsilon\lambda\|_v \leqq (2+k+kH)^{n_v} X^{n_v}$$
so from (45) and (44) we get
$$\prod_{v\in\Gamma} \|\varepsilon x_0+\varepsilon x_1\alpha_1+\ldots+\varepsilon x_k\alpha_k+\varepsilon\lambda\|_v > \frac{|N_{L|\mathbf{Q}}(\mu')|}{((2+k+kH)X)^{N-s'-2t'}} \geqq$$
$$\geqq \varrho_1'(2+k+kH)^{-N+s'+2t'} X^{-N+s'+2t'+\tau_1} = \varrho_1 X^{-N+s'+2t'+\tau_1}$$
and this is our assertion (11).

Proof of Corollary 2.1. Denote by $-y$ the nearest integer to $x_1\alpha_1+\ldots+x_k\alpha_k+\lambda$. If $\|x_1\alpha_1+\ldots+x_k\alpha_k+\lambda\| \geqq 1$ then (12) obviously holds; if <1, then

(46)
$$|y| \leqq 1+k(1+H)X_0+X_0.$$

Applying our Theorem 2 to $y+x_1\alpha_1+\ldots+x_k\alpha_k+\lambda$ we get
$$|y+x_1\alpha_1+\ldots+x_k\alpha_k+\lambda|^\sigma >$$
$$> (2\cdot 8^n H^{n(2k+1)n} e^{6n})^{-1}(2+k+kH)^{-n+\sigma} X^{-n+\sigma+n[(T_2'/\zeta)\log(T_1'/\zeta)]^{-1}},$$
where $X=\max(|y|, |x_1|, \ldots, |x_k|)$.
By (46) we have
$$X \leqq 1+k(1+H)X_0+X_0 \leqq 2k(1+H)X_0,$$
and from this we get
$$|y+x_1\alpha_1+\ldots+x_k\alpha_k+\lambda|^\sigma > \varrho_2^\sigma X_0^{-n+\sigma+\tau_2}$$
from which (12) follows.

Proof of Theorem 3. In our proof we shall use the arguments of the proof of Theorem 3 of [15].

Let a_1 denote the leading coefficient of the minimal defining polynomial of ϑ over $\mathbf{Z}$. Then $a_1^n\mu \in \mathbf{Z}_L$. By (10) we have

(47)
$$H(\mu') \leqq a_1^n(2\,\overline{|\mu'|})^l \leqq H^n |N_{L|\mathbf{Q}}(\mu')| \exp(6nl^{3l}R_L).$$

From (13) we get

(48)
$$N_{K|L}(\varepsilon a\vartheta-\varepsilon a\alpha+\varepsilon a\lambda) = \mu'.$$

Applying our Theorem 1 to (48) we have
$$X_1 = \max(\overline{|\varepsilon a|}, \overline{|\varepsilon a\alpha|}) < [8H^2(2H(\mu'))^{l/n}]^{(T_2/\zeta)\log(T_1/\zeta)}.$$
This together with (47) implies
$$X_1 < [8H^2(2H^n|N_{L|\mathbf{Q}}(\mu')|\exp(6nl^{3l}R_L))^{l/n}]^{(T_2/\zeta)\log(T_1/\zeta)} =$$
$$= [2^{l/n}8\cdot H^{2+l}|N_{L|\mathbf{Q}}(\mu')|^{l/n}(\exp(6nl^{3l}R_L))^{l/n}]^{(T_2/\zeta)\log(T_1/\zeta)},$$
whence
$$X_1^{((T_2/\zeta)\log(T_1/\zeta))}[2^{n/l}\cdot 8\cdot H^{2+l}(\exp(6nl^{3l}R_L))^{l/n}]^{-1} < |N_{L|\mathbf{Q}}(\mu')|^{l/n}$$
from which we get

(49)
$$|N_{L|\mathbf{Q}}(\mu')| > \varrho_3' X_1^{\tau_3}$$

with $\varrho_3' = (2\cdot 8^{n/l} H^{((2+l)n)/l} \exp(6nl^{3l} R_L))^{-1}$. By equation (48) we have

(50) $$\prod_{v\in\Omega} \|a\vartheta - a\alpha + a\lambda\|_v = |N_{L|\mathbf{Q}}(\mu')|.$$

Further, we have

$$\|\alpha\vartheta - a\alpha + a\lambda\|_v \leqq (H(\alpha)(4H+X_1))^{n_v} < (H(\alpha)4HX_1)^{n_v}$$

for each $v\in\Omega$, so from (50) and (49) we obtain

(51) $$\prod_{v\in\Gamma} \|\vartheta - \alpha + \lambda\|_v > \frac{|N_{L|\mathbf{Q}}(\mu')|\, a^{-s'-2t'}}{(4H\cdot H(\alpha)X_1)^{ln-s'-2t'}} \geqq$$

$$\geqq \varrho_3'(4H)^{-ln+s'+2t'}(H(\alpha))^{-ln+s'+2t'} a^{-s'-2t'} X_1^{-ln+s'+2t'+\tau_3}.$$

Since

$$H(\alpha) = H\left(\frac{\varepsilon a\alpha}{\varepsilon a}\right) \leqq (\overline{|\varepsilon a|} + \overline{|\varepsilon a\alpha|})^l \leqq 2^l X^l,$$

thus we have

$$X_1 \geqq \frac{1}{2}(H(\alpha))^{1/2}.$$

So from (51) we get

$$\prod_{v\in\Gamma} \|\vartheta - \alpha + \lambda\|_v > \varrho_3'(4H)^{-ln+s'+2t'}(H(\alpha))^{-ln+s'+2t'} a^{-s'-2t'} \left[\frac{1}{2}(H(\alpha))^{1/l}\right]^{-ln+s'+2t'+\tau_3} =$$

$$= \varrho_3 (H(\alpha))^{(1+(1/l)(-ln+s'+2t')+(\tau_3/l))}\cdot Q^{-s'-2t'}.$$

If α is algebraic integer, then $a=1$ and from this follows (15). If α is not algebraic integer then by $a \leqq H(\alpha)$ we get (14).

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WEAK HOMOMORPHISMS IN SOME CLASSES OF ALGEBRAS

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1. Introduction

By an algebra there is meant a couple $(A; F)$ where A is a set and F is an (eventually well-ordered) set of operations on A. Given two algebras $\mathcal{A}=(A; F)$ and $\mathcal{B}=(B; G)$, a mapping $\varphi\colon A\to B$ is called a semi-weak homomorphism if for each operation $f\in F$ (say n-ary) there is a term function (polynomial in the sense of [13]) g of $\mathcal{B}$ such that

$$(1)\qquad \varphi f(a_1, \ldots, a_n) = g(\varphi a_1, \ldots, \varphi a_n) \quad \text{for all} \quad a_1, \ldots, a_n\in A.^1$$

If, in addition, for each operation $g\in G$ (say n-ary) there is a term function f of $\mathcal{A}$ such that (1) holds, φ is said to be a weak homomorphism. Substituting in these definitions "polynomial function" (in the sense of [21] — "algebraic function" in the sense of [13])) for "term function" we get the notions of semi-pseudo-weak homomorphism and pseudo-weak homomorphism respectively. A bijective weak homomorphism or a bijective pseudo-weak homomorphism is called a weak isomorphism or a pseudo-weak isomorphism respectively.

The notion of weak homomorphism was suggested by E. Marczewski and A. Goetz (see [22] and [11]). The notion of pseudo-weak isomorphism in the class of distributive lattices was studied by J. Jakubík [15] (under the name "W-isomorphism"). Various authors described weak homomorphisms or weak isomorphisms in specific classes of algebras (see e.g. [4]—[8], [11], [25]). In the present paper this is done for semi-weak, pseudo-weak and semi-pseudo-weak homomorphisms in some classes of semigroups, groups, lattices and median algebras.[2] It turns out that in some of these classes some of the mentioned weaker forms of homomorphism coincide mutually or with the usual homomorphism. Theorem 4.2 shows that in the class of bounded lattices pseudo-weak homomorphisms and weak homomorphisms do not coincide. The following example shows a semi-weak homomorphism which is no weak homomorphism. Consider the groupoids $\mathcal{Z}=(\mathbf{Z}; \circ)$ and $\mathcal{Z}'=(\mathbf{Z}; \cdot)$ where $\mathbf{Z}$ is the set of all integers, $x\cdot y$ is the usual product and $x\circ y= =x^2\cdot y^2$. The mapping $\mathrm{id}_{\mathbf{Z}}\colon \mathcal{Z}\to\mathcal{Z}'$ is not even a pseudo-weak homomorphism.

Given algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}\times\mathcal{B}$ will denote their direct product. $\tilde{\ell}$ will denote the dual of the lattice ℓ.

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[1] Under assumption of the Axiom of Choice the notion of the semi-weak homomorphism is equivalent to Fajtlowicz's notion of morphism [3].

[2] Some results of the paragraph 4 of the present note were published without proof earlier [18]. The author is grateful to K. Głazek for critical comments.

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2. Some general propositions

In this section $\mathscr{A}$ and $\mathscr{B}$ will denote algebras $(A; F)$ and $(B; G)$ respectively.

2.1. *The composition of semi-pseudo-weak homomorphisms (or pseudo-weak or semi-weak homomorphisms) is a semi-pseudo-weak homomorphism (or pseudo-weak or semi-weak homomorphism respectively).*

The proof is straightforward. (For an analogous assertion concerning weak homomorphisms see [11].)

2.2. *Any surjective semi-pseudo-weak homomorphism $\varphi\colon \mathscr{A}\to\mathscr{B}$ can be expressed in the form $\varphi=\psi\circ\nu$ where ν is a (usual) homomorphism and ψ a bijective semi-pseudo-weak homomorphism. If φ is a pseudo-weak or semi-weak homomorphism then so is ψ.* (For an analogous proposition concerning weak homomorphisms see [9], p. 655 and [10], p. 223.)

Proof. It can be easily checked that $\Theta=\operatorname{Ker}\varphi$ is a congruence relation in $\mathscr{A}$. If $\nu\colon \mathscr{A}\to\mathscr{A}/_\Theta$ is the canonical homomorphism then $\varphi=\psi\circ\nu$ where $\psi\colon \mathscr{A}/_\Theta\to\mathscr{B}$ is a bijection. It can be readily shown that ψ is a semi-pseudo-weak homomorphism.

2.3. *Any bijective semi-pseudo-weak homomorphism $\psi\colon \mathscr{A}\to\mathscr{B}$ can be expressed in the form $\psi=\iota\circ\chi$ where $\chi\colon (A; F)\to(B; F)$[3] is a (usual) isomorphism and $\iota=\mathrm{id}_B\colon (B; F)\to(B; G)$ is a semi-pseudo-weak homomorphism. If ψ is a pseudo-weak or semi-weak or weak homomorphism then so is χ.*

Proof. Define the set F of operations in B as follows. Given a fundamental operation f of $\mathscr{A}$ (say n-ary) and $b_1, \dots, b_n\in B$, set $f(b_1, \dots, b_n)=\psi f(\psi^{-1}b_1, \dots, \dots, \psi^{-1}b_n)$. Then $\chi\colon (A; F)\to(B; F)$, where $\chi a=\psi a$ for each $a\in A$, is an isomorphism. The rest is straightforward.

2.4. *Any surjective semi-pseudo-weak homomorphism $\varphi\colon \mathscr{A}\to\mathscr{B}$ can be expressed in the form $\varphi=\iota\circ\mu$ where $\mu\colon (A; F)\to(B; F)$ is a (usual) homomorphism and $\iota=\mathrm{id}_B\colon (B; F)\to(B; G)$[4] is a semi-pseudo-weak homomorphism.*

Proof. Using 2.1, 2.2 and 2.3 we get $\varphi=\iota\circ\chi\circ\nu$ where $\chi\circ\nu$ is a homomorphism. (See Figure 1; the usual homomorphisms are marked by circled arrows.)

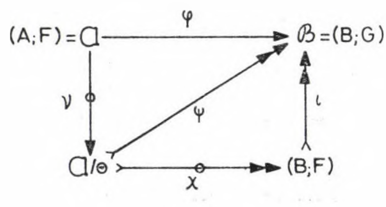

Fig. 1

[3] We denote the set of operations in B by the same symbol F. This is justified because of the isomorphism χ.

[4] See 2.3.

2.5. Remark. According to 2.4 the investigation of weaker kinds of homomorphisms under consideration can be limited to (usual homomorphisms and to) corresponding kinds of homomorphisms of the form $\iota=\mathrm{id}_B\colon (B; F)\to(B; G)$. In this case "$\iota$ is a semi-pseudo-weak homomorphism" means that for each operation $f\in F$ a polynomial function g of the algebra $(B; G)$ exists such that for any $b_1, \ldots, b_n\in B$ (if f is n-ary), $f(b_1, \ldots, b_n)=g(b_1, \ldots, b_n)$. Similarly other cases can be formulated.

3. Semigroups

3.1. Lemma. *Let $\mathcal{S}=(S;\ \cdot)$ and $\mathcal{S}'=(S;\ \circ)$ be semigroups with units e and u respectively, and let $\mathcal{S}$ satisfy the identity $x^2\cdot y=y\cdot x^2$. If there is a term function $f(x, y)$ in $\mathcal{S}$ such that $x\circ y=f(x, y)$ for each $x, y\in S$ then either the operations $\cdot$ and $\circ$ coincide or they are opposites $(x\circ y=y\cdot x)$.*

Remark. This lemma is a generalization of a theorem by A. Goetz [11, Theorem 1] for groups. The present proof is a modification of that in [11].

Proof. According to supposition,

(2) $x\circ y=x^{m_1}\cdot y^{n_1}\cdot x^{m_2}\cdot y^{n_2}\ldots x^{m_k}\cdot y^{n_k}$, m_i, n_i non-negative integers.

It can be easily shown that $x\circ y$ can be expressed in the form

(3) $x\circ y=x^m\cdot(y\cdot x)^n\cdot y^p$, m, n, p non-negative integers.

(The terms with even exponents can be translated to the beginning or to the end. If, say, m_i is odd, translate x^{m_i-1} at the beginning.) The following consideration shows that n can be chosen to be 0 or 1.

First $u^{m+n}=u\circ e=e=e\circ u=u^{n+p}$ hence $u=u\circ u=u^{m+n}\cdot u^{n+p}=e$. Moreover $x^{m+n}=x\circ e=x=e\circ x=x^{n+p}$. If $p=0$ then $x=x^n$, hence $x\circ y=x^m\cdot y\cdot x$. Suppose $p>0$.

If $n\geqq 3$ then $x\circ y=y\cdot x\cdot y\cdot x^{m+1}\cdot(y\cdot x)^{n-2}\cdot y^p=y\cdot x\cdot y\cdot x^m\cdot(x\cdot y)^{n-1}\cdot y^{p-1}=$ $=x\cdot y\cdot x\cdot y^2\cdot x\cdot y\cdot x^m\cdot(x\cdot y)^{n-3}\cdot y^{p-1}=x^{m+3}\cdot(x\cdot y)^{n-3}\cdot y^{p+3}$. $n\geqq 4$ then gives $x\circ y=x^{m+4}\cdot(y\cdot x)^{n-4}\cdot y^{p+4}$. Repeating this we get in (3) $0\leqq n<4$.

$n=3$ gives $x\circ y=x^{m+3}\cdot y^{p+3}=x\cdot y$. If $n=2$ then $x^{m+2}=x=x^{p+2}$. If m is even then x is commutative and $x\circ y=x^{m+2}\cdot y^{p+2}=x\cdot y$. Otherwise $x\circ y=$ $=y\cdot x\cdot y\cdot x^{m+1}\cdot y^p=x^{m+1}\cdot y\cdot x\cdot y^{p+1}$. Hence in (3) n can be taken 0 or 1.

If $n=0$ then $x=x^m, y=y^p$ and $x\circ y=x\cdot y$. The case $n=1$ gives $x^{m+1}=x=$ $=x^{p+1}$. If m is odd then x is commutative and we get $x\circ y=x\cdot y$. Otherwise $x\circ y=y\cdot x^{m+1}\cdot y^p=y\cdot x\cdot y^p$ and we get $x\circ y=y\cdot x$ if p is even and $x\circ y=x\cdot y$ if p is odd ($y=y^{p+1}$ is commutative). This proves the lemma.

3.2. Theorem. *Let $\mathcal{S}=(S;\ \circ)$ and $\mathcal{S}'=(S';\ \cdot)$ be semigroups with units and let $\mathcal{S}'$ satisfy the identity $x^2\cdot y=y\cdot x^2$. Then any surjective semi-weak homomorphism $\varphi\colon \mathcal{S}\to\mathcal{S}'$ is either an usual homomorphism or an anti-homomorphism (i.e. $\varphi(x\circ y)=\varphi y\cdot\varphi x$). In particular φ is a weak homomorphism.*

Proof. It suffices to use 2.4 and 3.1.

3.3. Remark. Theorem 3.2 would be false if the condition that $\mathcal{S}$ has a unit was omitted. E.g., if $(S;\ \cdot)$ is an arbitrary semigroup (satisfying the identity $x^2\cdot y=$

$=y\cdot x^2$) and $(S;\circ)$ is a semigroup with $x\circ y=x$, the surjective semi-weak homomorphism $\mathrm{id}_S\colon (S;\circ)\to(S;\cdot)$ fails in general to satisfy the conclusion of the theorem.

3.4. THEOREM. *Let $\mathcal{S}=(S;\circ)$ and $\mathcal{S}'=(S;\cdot)$ be semigroups with units u and e respectively, and let $\mathcal{S}'$ be commutative. The mapping $\mathrm{id}_S\colon \mathcal{S}\to\mathcal{S}'$ is a semi-pseudo-weak homomorphism if and only if the inverse u^{-1} of u in $\mathcal{S}'$ exists and $x\circ y=x\cdot u^{-1}\cdot y$. If this is the case then $\mathcal{S}$ is also commutative and the inverse $e^{\dot{-}1}$ of e in $\mathcal{S}$ exists and $x\cdot y=x\circ e^{\dot{-}1}\circ y$. In particular id_S is a pseudo-weak isomorphism of $\mathcal{S}$ and $\mathcal{S}'$.*

PROOF. Let id_S be a semi-pseudo-weak homomorphism. Then $x\circ y$ can be expressed in the form $x\circ y=a\cdot x^m\cdot y^n$, $a\in S$, m,n non-negative integers. Then $a\cdot x^m\cdot u^n=x\circ u=x=a\cdot u^m\cdot x^n$, hence $x\circ y=a\cdot(a\cdot x^m\cdot u^n)^m\cdot(a\cdot u^m\cdot y^n)^n=a^{m+n+1}\cdot x^{m^2}\cdot y^{n^2}u^{2mn}$ and $x\cdot y=a\cdot x^m\cdot u^n\cdot a\cdot u^m\cdot y^n=a^2\cdot(a\cdot x^m\cdot u^n)^m\cdot(a\cdot u^m\cdot y^n)^n\cdot u^{m+n}=a^{m+n+2}\cdot x^{m^2}\cdot y^{n^2}\cdot u^{2mn+m+n}$. Combining this with $u=u\circ u=a\cdot u^{m+n}$ we get $x\cdot y=(x\circ y)\cdot u$, in particular $e=e\cdot e=(e\circ e)\cdot u$. Hence $e\circ e=u^{-1}$ is the inverse of u in $\mathcal{S}'$ and $x\circ y=x\cdot u^{-1}\cdot y$. Obviously if the last relation holds, id_S is a semi-pseudo-weak homomorphism. Moreover $(u\cdot u)\circ e=u\cdot u\cdot u^{-1}\cdot e=u$, hence $u\cdot u=e^{\dot{-}1}$ is the inverse of e in $\mathcal{S}$. Further $(x\circ y)\circ(u\cdot u)=x\cdot y\cdot u^{-1}\cdot u\cdot u\cdot u^{-1}=x\cdot y$, and consequently $x\cdot y=x\circ e^{\dot{-}1}\circ y$.

3.5. COROLLARY. *Any semi-pseudo-weak homomorphism of commutative semigroups with units is a pseudo-weak homomorphism.*

3.6. COROLLARY. *Given two groups $\mathcal{G}=(G;\circ)$ and $\mathcal{G}'=(G;\cdot)$ where $\mathcal{G}'$ is commutative, the mapping $\mathrm{id}_G\colon\mathcal{G}\to\mathcal{G}'$ is a semi-pseudo-weak homomorphism (and a pseudo-weak homomorphism) if and only if $x\circ y=x\cdot u^{-1}\cdot y$ for some $u\in G$. In this case $\mathcal{G}$ is commutative, too.*

3.7. REMARK. The operation $xu^{-1}y$ in a group was investigated by H. Prüfer [24] and R. Baer [1] in connection with the notion "Schar" (heap). Some connections between the group operation and the operation $xu^{-1}y$ were established in [2]. Associative operations $x\circ y=f(x,y)$ in a free group, where $f(x,y)$ is a polynomial, were described in [23].

4. Lattices

4.1. LEMMA. *Let $\mathcal{A}=(A;\wedge,\vee)$, $\mathcal{B}=(B;\wedge,\vee)$ be lattices, $\mathcal{A}\times\mathcal{B}=(A\times B;\wedge,\vee)=\mathcal{L}$, $\mathcal{A}\times\tilde{\mathcal{B}}=(A\times B;\cap,\cup)=\mathcal{L}'$ and let $x\cap y=g(x,y)$, $x\cup y=h(x,y)$ where $g(x,y)$ and $h(x,y)$ are polynomial functions in $\mathcal{L}$. Then one of the following cases occurs.*

(i) $\mathcal{L}'=\mathcal{L}$,

(ii) $\mathcal{L}'=\tilde{\mathcal{L}}$,

(iii) both $\mathcal{L}$ and $\mathcal{L}'$ are bounded.

If $g(x,y)$ is a term function then one of the cases (i) *and* (ii) *occurs.*

PROOF. The order relations of $\mathcal{L}$ and $\mathcal{L}'$ will be denoted by $\leqq$ and $\subseteq$ respectively. The cases card $A=1$ and card $B=1$ give $\mathcal{L}'=\tilde{\mathcal{L}}$ and $\mathcal{L}'=\mathcal{L}$, respectively. We shall suppose card $A>1$ and card $B>1$.

1. First let $g(x, y)$ be a term function. Then it is equal to one of the following functions: $x, y, x \wedge y, x \vee y$. The first two cases give $x = g(x, y) = g(y, x) = y$ — a contradiction. In the third case $x \cap y = x \wedge y$ and $\mathscr{L}, \mathscr{L}'$ are identical; in the last case $\mathscr{L}'$ is the dual of $\mathscr{L}$.

2. Suppose $g(x, y)$ is not a term function. Then there is a term function $f(x_1, \ldots, x_n, x_{n+1}, x_{n+2})$ and elements $a^1, \ldots, a^n$ of $\mathscr{L}$ such that $g(x, y) = f(a^1, \ldots, a^n, x, y)$. First we shall show that $\mathscr{L}'$ has a least element.

a) Consider first the case $n=1$ (set $a^1=a$). Then $a \cap y = f(a, a, y)$ is equal to one of the functions $a, y, a \wedge y, a \vee y$. In the case $a \cap y = a$, a is the least element of $\mathscr{L}'$ as asserted. Any element $t \in A \times B$ has the form (t_1, t_2), $t_1 \in A$, $t_2 \in B$. The case $f(a, a, y) = a \wedge y$ gives $(a_1 \wedge y_1, a_2 \vee y_2) = a \cap y = a \wedge y = (a_1 \wedge y_1, \ a_2 \wedge y_2)$ hence card $B=1$. Analogously the last case gives card $A=1$. It remains the case $f(a, a, y) = y$. Then a is the greatest element of $\mathscr{L}'$, a_1 is the greatest element of $\mathscr{A}$ and a_2 the least element of $\mathscr{B}$ hence a is a neutral element[5] of $\mathscr{L}$. Therefore $f(a, x, y)$ can be expressed as a join of some of the elements[6]

$$a, x, y, a \wedge x, a \wedge y, x \wedge y, a \wedge x \wedge y.$$

If $f(a, x, y) = a \vee D(a, x, y)$ $(D(a, x, y)$ is a join of some of the above elements — eventually empty$)$ then $y = a \vee D(a, a, y)$ hence a_1 is the least element of $\mathscr{A}$ and $A = \{a_1\}$ — a contradiction. If $f(a, x, y) = x \vee D_1(a, x, y)$ then $y = f(a, a, y) = a \vee D_1(a, a, y)$ which is the previous case. The same situation occurs in the case $f(a, x, y) = y \vee D_2(a, x, y)$ $(f(a, x, y) = f(a, y, x))$. It follows that $f(a, x, y)$ is a union of some of the terms $a \wedge x$, $a \wedge y$, $x \wedge y$, $a \wedge x \wedge y$. Then $y = f(a, a, y) = a \wedge D_3(a, y)$ hence a_2 is the greatest element of $\mathscr{B}$ which follows $B = \{a_2\}$ — a contradiction. Summarizing, in the case a) $\mathscr{L}'$ has a least element.

b) If n is arbitrary, set $a^1 \cap \ldots \cap a^n = b$. Then $a_1^i \geqq b_1$,

$$a_2^i \leqq b_2, f(a_1^1, \ldots, a_1^n, x_1, y_1) \geqq f(b_1, \ldots, b_1, x_1, y_1),$$
$$f(a_2^1, \ldots, a_2^n, x_2, y_2) \leqq f(b_2, \ldots, b_2, x_2, y_2),$$

hence

$$x \cap y = (f(a_1^1, \ldots, a_1^n, x_1, y_1), f(a_2^1, \ldots, a_2^n, x_2, y_2)) \supseteqq$$
$$\supseteqq (f(b_1, \ldots, b_1, x_1, y_1), f(b_2, \ldots, b_2, x_2, y_2)) = f(b, \ldots, b, x, y)$$

and

$$b \cap y \supseteqq f(b, \ldots, b, b, y) \in \{b, y, b \wedge y, b \vee y\}.$$

In the case $b \cap y \supseteqq b$, b is the least element of $\mathscr{L}'$ as asserted. The case $b \cap y \supseteqq y$ gives $b \cap y = y$, $b = b \cap a^i = a^i$, hence $a^i = a^j$ which yields the case a). $b \cap y \supseteqq b \wedge y$ yields $b_2 \vee y_2 \leqq b_2 \wedge y_2$, hence card $B=1$, a contradiction. Analogously $b \cap y \supseteqq b \vee y$ yields $b_1 \wedge y_1 \geqq b_1 \vee y_1$ hence card $A=1$. It follows that $\mathscr{L}'$ has a least element.

Using the relation $x \cup y = h(x, y)$ the dual reasoning gives that $\mathscr{L}'$ has a greatest element hence it is bounded. By an easy reasoning we get that $\mathscr{L}$ is bounded, too, which completes the proof.

4.2. Theorem. *Let $\mathscr{L}$, $\mathscr{L}'$ be lattices and let φ: $\mathscr{L} \to \mathscr{L}'$ be a bijection.*

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[5] See e.g. [14, p. 138].

[6] The sublattice generated by the elements a, x, y is distributive (see [14, p. 140]).

a) *φ is a semi-weak homomorphism if and only if one of the following two cases occurs.*

(i) *φ is a (usual) isomorphism,*

(ii) *φ is a dual isomorphism.*

b) *φ is a semi-pseudo-weak homomorphism if and only if one of the following three cases occurs:* (i), (ii) *and*

(iii) *both $\mathcal{L}$ and $\mathcal{L}'$ are bounded and there are lattices $\mathcal{A}$ and $\mathcal{B}$ and isomorphisms $\psi\colon \mathcal{L}\to\mathcal{A}\times\mathcal{B}$, $\chi\colon \mathcal{L}'\to\mathcal{A}\times\tilde{\mathcal{B}}$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \mathcal{L}' \arrow[d, "\chi"] \\ \mathcal{A}\times\mathcal{B} \arrow[r, "\mathrm{id}_{A\times B}"'] & \mathcal{A}\times\tilde{\mathcal{B}} \end{tikzcd}$$

Fig. 2

Moreover the condition (iii) *is equivalent with the following one.*

(iv) *The operations $\cap$, $\cup$ of $\mathcal{L}$ and $\wedge$, $\vee$ of $\mathcal{L}'$ are connected by the rule (denote $\varphi x = x'$ and $u=\varphi^{-1}(o')$, $v=\varphi^{-1}(i')$ where o' and i' are the least and the greatest elements of $\mathcal{L}'$, respectively)*

$$(4)\quad x'\wedge y' = ((x\cap y)\cup(y\cap u)\cup(u\cap x))',\quad x'\vee y' = ((x\cap y)\cup(y\cap v)\cup(v\cap x))'.$$

Proof. The equivalence of (iii) and (iv) was proved in [17]. According to 2.3 it suffices to consider the case $\mathcal{L}=(L;\cap,\cup)$, $\mathcal{L}'=(L;\wedge,\vee)$ and $\varphi=\mathrm{id}_L$. The "if" part is obvious. Suppose φ is a semi-pseudo-weak homomorphism. Then there are polynomial functions $g(x,y)$ and $h(x,y)$ in $\mathcal{L}'$ such that $x\cap y=g(x,y)$ and $x\cup y=h(x,y)$. If $\leqq$ denotes the order relation in $\mathcal{L}'$, $x\leqq u$ and $y\leqq v$ imply $x\cap y\leqq u\cap v$ and $x\cup y\leqq u\cup v$. Hence the assertions 2.6, 2.8 [16] and Theorem 1 [18] are applicable and we get that there are lattices $\mathcal{A}$, $\mathcal{B}$ and a bijection $\psi\colon L\to A\times B$ such that ψ is an isomorphism of $\mathcal{L}$ to $\mathcal{A}\times\mathcal{B}$ and an isomorphism of $\mathcal{L}'$ to $\mathcal{A}\times\tilde{\mathcal{B}}$. According to 4.1 one of the conditions (i) and (ii) in the case a) and one of the conditions (i), (ii), (iii) in the case b) is fulfilled.

4.3. Corollary. *Any bijective semi-pseudo-weak homomorphism of lattices is a pseudo-weak isomorphism, and any bijective semi-weak homomorphism of lattices is a weak isomorphism.*

4.4. Corollary. *For non-bounded lattices the notions "pseudo-weak homomorphism" and "weak homomorphism" coincide.*

4.5. Corollary. *The only surjective weak homomorphisms of lattices are the usual homomorphisms and the dual homomorphisms. In the class of non-bounded lattices the same holds for pseudo-weak homomorphisms.*

4.6. Remark. The first assertion of Corollary 4.5 for specific classes of lattices with some unary operations was proved e.g. in [25] and [8]. Theorem 4.2 b) for pseudo-weak homomorphisms in the class of distributive lattices (with a different formulation of the rule (4)) was proved by J. Jakubík [15]. The pseudo-weak isomorphisms of Boolean algebras were considered by A. Goetz [12].

5. Modular median algebras

By a modular median algebra we mean an algebra with one ternary operation (xyz) satisfying the identities

(5) $$(xyy) = y, \quad ((xyz)tz) = (xz(tzy)).$$

These algebras are closely related to modular lattices. Namely, the operation $(xyz) = (x \wedge (y \vee z)) \vee (y \wedge z)$ (due to J. Hashimoto) derived from a modular lattice satisfies (5). Moreover in a modular lattice with the least element o and the greatest element i,

(6) $$(oxi) = x.$$

Conversely, any modular median algebra with elements, o, i satisfying (6) gives rise to a bounded modular lattice with the operations $x \wedge y = (xoy)$, $x \vee y = (xiy)$ (see [20]).

The identities (5) imply (see [20])

(7) $$(xyz) = (xzy).$$

Further we shall use the following (yet unpublished) result of J. Hedlíková.

The free modular median algebra with three generators x, y, z consists of the six elements

(8) $$x, y, z, (xyz), (yzx), (zxy).$$

5.1. *If $f(x, y, z)$ is a term function in a modular median algebra $(M; (\quad))$ such that (M, f) is a modular median algebra then $f(x, y, z) = (xyz)$.*

Proof. Following the above result of J. Hedlíková, $f(x, y, z)$ is equal to a function given by one of the terms (8). In the case $f(x, y, z) = x$ we get $\operatorname{card} M = 1$ because of $f(x, y, y) = y$, hence the assertion is true. The same result will be obtained in the cases $f(x, y, z) = y$ and $f(x, y, z) = z$ because of (7). The case $f(x, y, z) = (yzx)$ gives $f(x, y, z) = (yxz) = f(z, y, x) = f(z, x, y) = (xyz)$. The same result appears if $f(x, y, z) = (zxy)$. From this the assertion follows.

Combining 5.1 with 2.4 we get

5.2. Theorem. *The only surjective semi-weak homomorphisms of modular median algebras are the usual homomorphisms.*

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ON APPROXIMATION OF THE SOLUTIONS OF STRONGLY NONLINEAR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

L. SIMON

0. Introduction

In [1] the following elliptic equation has been considered:

$$Au(x)+g(x, u(x)) = f(x), \quad x\in\Omega \tag{0.1}$$

where Ω is a possibly unbounded domain in $\mathbf{R}^n$,

$$Au(x) = \sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, D^\beta u, \ldots), \; |\beta| \leqq m$$

and the terms $A_\alpha(x, \zeta)$ are required to have polynomial growth in ζ, however, in the term $g(x, u)$ no such growth restriction is imposed but it is supposed that g (essentially) satisfy the sign condition $g(x, u)u \geqq 0$. The existence of solutions of boundary value problems for (0.1) has also been proved there.

In the present paper it will be shown that the solutions of boundary value problems for (0.1) in unbounded Ω can be approximated by the solutions of boundary value problems, considered in large bounded domains Ω_ϱ where

$$\Omega_\varrho \supset \Omega\cap B_\varrho, \quad B_\varrho = \{x\in\mathbf{R}^n\colon |x| < \varrho\}.$$

Such approximation theorems has been proved in [2] and [3] for other boundary value problems for nonlinear elliptic equations.

1. Preliminaries

Let $\Omega\subset\mathbf{R}^n$ be an unbounded domain, $p>1$ and m a nonnegative integer. Denote by $W_p^m(\Omega)$ the usual Sobolev space of real valued functions u whose distributional derivatives of order $\leqq m$ belong to $L^p(\Omega)$. The norm on $W_p^m(\Omega)$ is defined by

$$\|u\|_{W_p^m(\Omega)} = \Big\{ \sum_{|\alpha|\leqq m} \int_\Omega |D^\alpha u|^p \Big\}^{1/p}$$

where $\alpha=(\alpha_1, \ldots, \alpha_n)$, $D^\alpha = D_1^{\alpha_1}\ldots D_n^{\alpha_n}$, $D_j=\partial x_j$.

The expression $W_{p,0}^m(\Omega)$ will denote the closure in $\|\cdot\|_{W_p^m(\Omega)}$ of $C_0^\infty(\Omega)$, the infinitely differentiable functions with compact support contained in Ω.

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Let N be the number of multiindices α, satisfying the condition $|\alpha|\leqq m$. For $\xi=(\xi_0, \dots, \xi_\beta, \dots)\in\mathbf{R}^N$ write $\xi=(\eta, \zeta)$ where $\eta=(\xi_0, \dots, \xi_\gamma, \dots)$ such that $|\gamma|\leqq \leqq m-1$. Suppose that

I. Functions A_α: $\Omega\times\mathbf{R}^N\to\mathbf{R}$ satisfy the Carathéodory conditions, i.e. they are measurable in x for each fixed $\xi=(\xi_0, \dots, \xi_\beta, \dots)\in\mathbf{R}^N$ and continuous in ξ for almost all $x\in\Omega$.

II. There exist a constant $c_1>0$ and a function $K_1\in L^q(\Omega)$ where $\frac{1}{p}+\frac{1}{q}=1$ such that

$$|A_\alpha(x, \xi)| \leqq c_1|\xi|^{p-1}+K_1(x)$$

for all $|\alpha|\leqq m$, a.e. in Ω and all $\xi\in\mathbf{R}^N$.

III. For all (η, ζ), $(\eta, \zeta')\in\mathbf{R}^N$ with $\zeta\neq\zeta'$ and a.e. in Ω

$$\sum_{|\alpha|\leqq m} [A_\alpha(x, \eta, \zeta)-A_\alpha(x, \eta, \zeta')](\zeta_\alpha-\zeta'_\alpha) > 0.$$

IV. There exist a constant c_2 and a function $K_2\in L^1(\Omega)$ such that for a.e. in Ω and all $\xi\in\mathbf{R}^N$

$$\sum_{|\alpha|\leqq m} A_\alpha(x, \xi)\xi_\alpha \geqq c_2|\xi|^p-K_2(x).$$

V. There exist constants $c_3>0$ and $\varrho_0>0$ such that for a.e. in Ω and all $\xi=(\eta, \zeta)\in\mathbf{R}^N$, $\xi'=(\eta', \zeta')\in\mathbf{R}^N$

$$\sum_{|\alpha|\leqq m} [A_\alpha(x, \xi)-A_\alpha(x, \xi')](\xi_\alpha-\xi'_\alpha) \geqq -c_3|\eta-\eta'|^p\psi(x)$$

where

$$\psi(x) = \begin{cases} 1 & \text{if} \quad |x|\leqq \varrho_0 \\ 0 & \text{if} \quad |x|> \varrho_0 \end{cases}$$

VI. Functions p, r: $\Omega\times\mathbf{R}\to\mathbf{R}$ satisfy the Carathéodory conditions (i.e. $p(x, t)$, $r(x, t)$ are measurable in x for each $t\in R$ and continuous in t for almost all $x\in\Omega$) and

$$p(x, t)t \geqq 0, \quad |r(x, t)| \leqq h(x), \quad h\in L^q(\Omega)\cap L^1(\Omega)$$

for all $t\in\mathbf{R}$ and a.e. in Ω.

VII. Let

$$g = p+r \quad \text{and} \quad \tilde{g}_s(x) = \sup_{|t|\leqq s} |g(x, t)|.$$

Suppose that for any $0\leqq s<\infty$

$$\tilde{g}_s\in L^1(\Omega).$$

VIII. Let V be a closed subspace of $W_p^m(\Omega)$ with the property that for any $u\in V$ there exist a constant $c>0$ and a sequence of functions $w_j\in V\cap L^\infty(\Omega)$ such that $(w_j)\to u$ in V and $|w_j(x)|\leqq c|u(x)|$ a.e. in Ω.

In [1] it is proved that conditions I—IV, VI—VIII imply the existence of variational solutions of boundary value problems for (0.1), more exactly: for any $f\in V^*$

(i.e. for any continuous linear functional f on V) there exists $u \in V$ such that

$$g(\cdot, u) \quad \text{and} \quad ug(\cdot, u) \in L^1(\Omega), \tag{1.1}$$

$$\sum_{|\alpha| \leq m} \int_\Omega A_\alpha(x, u, \ldots, D^\beta u, \ldots) D^\alpha v \, dx + \int_\Omega g(x, u) v \, dx = \langle f, v \rangle \tag{1.2}$$

for all $v \in V \cap L^\infty(\Omega)$ and for $v = u$. It is also shown that sequences (w_j) in condition VIII can be found in the interesting cases $V = W^m_{p,0}(\Omega)$ and $V = W^m_p(\Omega)$.

Now for $\varrho \geq \varrho_0$ let $\Omega_\varrho \subset \Omega$ be any bounded domain and V_ϱ be the function space

$$V_\varrho = \{u|_{\Omega_\varrho} : u \in V\}$$

with the norm of $W^m_p(\Omega_\varrho)$, satisfying the following assumptions:

a) $\Omega_\varrho \supset \Omega \cap B_\varrho$.

b) There exists a bounded linear operator $L_\varrho : V_\varrho \to V$ such that

$$L_\varrho u|_{\Omega_\varrho} = u \quad \text{a.e.}$$

and

$$\|L_\varrho\| \leq c \quad \text{for all} \quad \varrho \geq \varrho_0.$$

REMARK 1. From assumption b) it follows that V_ϱ is a closed subspace of $W^m_p(\Omega_\varrho)$.

REMARK 2. Condition VIII implies that for any $u_\varrho \in V_\varrho$ there exist a constant $c_\varrho > 0$ and a sequence of functions $w_{j,\varrho} \in V_\varrho \cap L^\infty(\Omega_\varrho)$ such that $(w_{j,\varrho}) \to u_\varrho$ in V_ϱ (i.e. with respect to the norm of $W^m_p(\Omega_\varrho)$) and $|w_{j,\varrho}(x)| \leq c_\varrho |u_\varrho(x)|$ a.e. in Ω_ϱ.

Indeed, according to the definition of V_ϱ there is a function $u \in V$ such that $u_\varrho = u|_{\Omega_\varrho}$. By assumption VIII there exist a constant $c > 0$ and $w_j \in V \cap L^\infty(\Omega)$ such that $(w_j) \to u$ in V and $|w_j(x)| \leq c|u(x)|$ a.e. in Ω. Thus,

$$w_j|_{\Omega_\varrho} \in V_\varrho \cap L^\infty(\Omega_\varrho),$$

$$(w_j|_{\Omega_\varrho}) \to u_\varrho \quad \text{in} \quad V_\varrho$$

and

$$|w_j|_{\Omega_\varrho}(x)| \leq c|u_\varrho(x)| \quad \text{a.e. in } \Omega_\varrho.$$

REMARK 3. Let $V = W^n_p(\Omega)$ or $V = W^n_{p,0}(\Omega)$. If $\partial\Omega$ (i.e. the boundary of Ω) is bounded then assumption b) is fulfilled for sufficiently smooth $\partial\Omega_\varrho \setminus \partial\Omega$ (it is sufficient to suppose that $\partial\Omega_\varrho \setminus \partial\Omega$ belong to C^m, see e.g. [5]).

If $\partial\Omega$ is not bounded then by use of [5] it is easy to formulate assumptions on $\partial\Omega_\varrho$ which imply condition b).

LEMMA 1. *Let assumptions I, II, IV, VI be fulfilled. Then there exists a constant c_4 such that for any $\varrho \geq \varrho_0$, $u \in V_\varrho$ the estimation*

$$\begin{aligned} &\sum_{|\alpha| \leq m} \int_{\Omega_\varrho} A_\alpha(x, u, \ldots, D^\beta u, \ldots) D^\alpha u \, dx + \int_{\Omega_\varrho} g(x, u) u \, dx \geq \\ &\geq c_2 \|u\|^p_{V_\varrho} - c_4 - \|h\|_{L^q(\Omega_\varrho)} \|u\|_{L^p(\Omega_\varrho)} \end{aligned} \tag{1.3}$$

holds where c_2 denotes the constant in condition IV.

PROOF. By assumptions I, II, IV the first term in the left is finite and for any $u \in V_\varrho$ it can be estimated as follows:

$$\sum_{|\alpha| \leq m} \int_{\Omega_\varrho} A_\alpha(x, u, \dots, D^\beta u, \dots) D^\alpha u \, dx \geq c_2 \|u\|_{V_\varrho}^p - \int_{\Omega_\varrho} K_2(x) \, dx \geq \tag{1.4}$$

$$\geq c_2 \|u\|_{V_\varrho}^p - \int_\Omega K_2(x) \, dx.$$

Moreover, condition VI implies that

$$\int_{\Omega_\varrho} g(x, u) u \, dx \geq \int_{\Omega_\varrho} r(x, u) u \, dx \geq - \int_{\Omega_\varrho} |h \cdot u| \, dx \geq - \|h\|_{L^q(\Omega_\varrho)} \cdot \|u\|_{L^p(\Omega_\varrho)}.$$

Thus estimation (1.3) follows from inequality (1.4).

Set

$$g_\varrho(x, u) = \begin{cases} g(x, u) & \text{if} \quad x \in \Omega_\varrho \\ 0 & \text{if} \quad x \in \Omega \setminus \Omega_\varrho. \end{cases}$$

LEMMA 2. *Suppose that conditions VI, VII are satisfied and the sequence* (u_j) *tends to* u *weakly in* V *such that estimation*

$$\int_\Omega |g_{\varrho_j}(x, u_j) u_j| \, dx \leq c_5 \tag{1.5}$$

holds with a constant c_5, $\lim_{j \to \infty} \varrho_j = +\infty$. *Then*

$$g(\cdot, u) u \in L^1(\Omega)$$

and there exists a subsequence (u_j') *of* (u_j) *such that* $\lim_{j \to \infty} (u_j') = u$ *a.e., and* $g_{\varrho_j}(\cdot, u_j')$ *tends to* $g(\cdot, u)$ *with respect to the norm of* $L^1(\Omega)$.

PROOF. As (u_j) tends to u weakly in V, there is a subsequence (u_j') such that $(u_j') \to u$ a.e. in Ω (see e.g. [4]). Therefore by asumption I

$$g_{\varrho_j}(\cdot, u_j') \to g(\cdot, u) \quad \text{a.e. in} \quad \Omega.$$

Thus Fatou's lemma implies

$$\int_\Omega |g(x, u) u| \, dx \geq \liminf_{j \to \infty} \int_\Omega |g_{\varrho_j}(x, u_j') u_j'| \, dx \leq c_5.$$

$$g(\cdot, u) u \in L^1(\Omega).$$

Inequality

$$|g_{\varrho_j}(x, u)| \leq \sup_{|t| \leq \delta^{-1}} |g_{\varrho_j}(x, t)| + \delta |g_{\varrho_j}(x, u) u| \leq \sup_{|t| \leq \delta^{-1}} |g(x, t)| + \delta |g_{\varrho_j}(x, u) u|,$$

assumption VII. and (1.5) imply that for any measurable set $E \subset \Omega$

$$\int_E |g_{\varrho_j}(x, u_j')| \, dx \leq \int_E \bar{g}_{\delta^{-1}}(x) \, dx + \delta \cdot c_5, \tag{1.6}$$

Given $\varepsilon > 0$, let $\delta = \varepsilon/(2c_5)$. Then in case meas$(E)$ is sufficiently small we have

$\int\limits_E |g_{\varrho_j}(x, u_j')|\, dx < \varepsilon$ and there is a set $A_\varepsilon \subset \Omega$ of finite measure with $\int\limits_{\Omega \setminus A_\varepsilon} |g(x, u_j')|\, dx < \varepsilon$. Thus by Vitali's theorem we have that $g_{\varrho_j}(\cdot, u_j')$ tends to $g(\cdot, u)$ in $L^1(\Omega)$.

2. The approximation theorem

Suppose that conditions I—VIII and a), b) are fulfilled and let $f \in V^*$ be given in the form

$$\langle f, v\rangle = \sum_{|\alpha| \leqq m} \int\limits_\Omega f_\alpha D^\alpha v\, dx,$$

where $f_\alpha \in L^q(\Omega)$. Consider the following boundary value problem in Ω_ϱ. We seek for a function $u_\varrho \in V_\varrho$ satisfying the conditions

$$g(\cdot, u_\varrho) \in L^1(\Omega_\varrho), \quad g(\cdot, u_\varrho) u_\varrho \in L^1(\Omega_\varrho) \tag{2.1}$$

and

$$\begin{aligned} &\sum_{|\alpha| \leqq m} \int\limits_{\Omega_\varrho} A_\alpha(x, u_r, \ldots, D^\beta u_\varrho, \ldots) D^\alpha v_\varrho\, dx + \\ &+ \int\limits_{\Omega_\varrho} g(x, u_\varrho) v_\varrho\, dx = \sum_{|\alpha| \leqq m} \int\limits_{\Omega_\varrho} f_\alpha D^\alpha v_\varrho\, dx \end{aligned} \tag{2.2}$$

for all $v_\varrho \in V_\varrho \cap L^\infty(\Omega_\varrho)$ and for $v_\varrho = u_\varrho$.

REMARK 4. $\sum\limits_{|\alpha| \leqq m} \int\limits_{\Omega_\varrho} f_\alpha (D^\alpha v_\varrho\, dx$ defines a continuous linear functional on V_ϱ.

THEOREM. *Suppose that conditions I—VIII and* a), b) *are satisfied. Then for any* $\varrho \geqq \varrho_0$ *the problem* (2.1), (2.2) *has at least one solution* u_ϱ.

Furthermore, let $\lim\limits_{j\to\infty} \varrho_j = +\infty$, $\varrho_j \geqq \varrho_0$ *and* u_{ϱ_j} *be a solution of* (2.1), (2.2) *for* $\varrho = \varrho_j$. *Then* (ϱ_j) *contains a subsequence* (ϱ_j') *such that* $(L_{\varrho_j'} u_{\varrho_j'})$ *tends to a solution* u^* *of* (1.1), (1.2) *weakly in* V.

If the solution u *of* (1.1), (1.2) *is unique then* $(L_{\varrho_j} u_{\varrho_j})$ *tends to* u *weakly in* V.

PROOF. From conditions I—IV, VI—VIII it follows that all conditions of the existence theorem in [1] are fulfilled for the boundary value problem (2.1), (2.2). (See Remarks 1, 2 and 4). Thus the problem (2.1), (2.2) has at least one solution $u_\varrho \in V_\varrho$ for arbitrary $f_\alpha \in L^q(\Omega)$.

Now consider a sequence (u_{ϱ_j}) of solutions of (2.1), (2.2) for $\varrho = \varrho_j$ with $\varrho_j \geqq \varrho_0$, $\lim\limits_{j\to\infty} \varrho_j = +\infty$. By (1.3), (2.2) we have the inequality

$$c_2 \|u_{\varrho_j}\|_{V_{\varrho_j}}^p - c_4 - \|h\|_{L^q(\Omega_{\varrho_j})} \cdot \|u_{\varrho_j}\|_{L^p(\Omega_{\varrho_j})} \leqq \sum_{|\alpha| \leqq m} \|f_\alpha\|_{L^q(\Omega_{\varrho_j})} \cdot \|D^\alpha u_{\varrho_j}\|_{L^p(\Omega_{\varrho_j})} \leqq$$

$$\leqq \Big(\sum_{|\alpha| \leqq m} \|f_\alpha\|_{L^q(\Omega)}\Big) \cdot \|u_{\varrho_j}\|_{V_{\varrho_j}}.$$

Thus the sequence (u_{ϱ_j}) is bounded in V_{ϱ_j} and by the assumption b) the sequence $(L_{\varrho_j} u_{\varrho_j})$ is bounded in V.

The assumptions I—II imply that the formulas

$$\langle T(u), v\rangle = \sum_{|\alpha|\le m} \int_\Omega A_\alpha(x, u, \dots, D^\beta u, \dots) D^\alpha v\, dx, \tag{2.3}$$

$$\langle T_{\varrho_j}(u), v\rangle = \sum_{|\alpha|\le m} \int_{\Omega_{\varrho_j}} A_\alpha(x, u, \dots, D^\beta u, \dots) D^\alpha v\, dx \tag{2.4}$$

define bounded linear operators $T\colon V\to V^*$, $T_{\varrho_j}\colon V\to V^*$ such that $\|T_{\varrho_j}\|\le c'$ (where c' does not depend on j). Therefore the sequence $(T_{\varrho_j}(L_{\varrho_j}u_{\varrho_j}))$ is bounded in V^*. Since V is a reflexive Banach space there exist a subsequence (ϱ_j') of (ϱ_j) and $u^*\in V$, $y\in V^*$ such that

$$\lim_{j\to\infty} L_{\varrho_j'} u_{\varrho_j'} = u^* \quad \text{weakly in } V \tag{2.5}$$

and

$$\lim_{j\to\infty} T_{\varrho_j'}(L_{\varrho_j'} u_{\varrho_j'}) = y \quad \text{weakly in } V^*. \tag{2.6}$$

As $u_{\varrho_j'}$ is a solution of (2.1), (2.2) with $\varrho=\varrho_j$ thus by (2.4)

$$\langle T_{\varrho_j'}(L_{\varrho_j'} u_{\varrho_j'}), L_{\varrho_j'} u_{\varrho_j'}\rangle + \int_{\Omega_{\varrho_j'}} g(x, u_{\varrho_j'}) u_{\varrho_j'}\, dx = \sum_{|\alpha|\le m} \int_{\Omega_{\varrho_j'}} f_\alpha D^\alpha u_{\varrho_j'}\, dx. \tag{2.7}$$

Hence by use of assumption VI we find

$$\begin{aligned} \int_{\Omega_{\varrho_j'}} |g(x, u_{\varrho_j'}) u_{\varrho_j'}|\, dx &\le \int_{\Omega_{\varrho_j'}} p(x, u_{\varrho_j'}) u_{\varrho_j'}\, dx + \int_{\Omega_{\varrho_j'}} |r(x, u_{\varrho_j'}) u_{\varrho_j'}|\, dx \le \\ &\le \int_{\Omega_{\varrho_j'}} g(x, u_{\varrho_j'}) u_{\varrho_j'}\, dx + 2 \int_{\Omega_{\varrho_j'}} |h u_{\varrho_j'}|\, dx = \\ &= \sum_{|\alpha|\le m} \int_{\Omega_{\varrho_j'}} f_\alpha D^\alpha u_{\varrho_j'}\, dx - \langle T_{\varrho_j'}(L_{\varrho_j'} u_{\varrho_j'}), L_{\varrho_j'} u_{\varrho_j'}\rangle + 2 \int_{\Omega_{\varrho_j'}} |h u_{\varrho_j'}|\, dx \le \\ &\le \sum_{|\alpha|\le m} \|f_\alpha\|_{L^q(\Omega)} \|u_{\varrho_j'}\|_{V_{\varrho_j'}} + \|T_{\varrho_j'}(L_{\varrho_j'} u_{\varrho_j'})\|_{V^*} \|L_{\varrho_j'} u_{\varrho_j'}\|_V + 2\|h\|_{L^q(\Omega)} \|L_{\varrho_j'} u_{\varrho_j'}\|_{L^p(\Omega)}. \end{aligned} \tag{2.8}$$

Therefore from Lemma 2 it follows that

$$g(\cdot, u^*) u^* \in L^1(\Omega) \tag{2.9}$$

and there exists a subsequence (ϱ_j'') of (ϱ_j') such that

$$\lim_{j\to\infty} L_{\varrho_j''} u_{\varrho_j''} = u^* \quad \text{a.e. in } \Omega, \tag{2.10}$$

$$\lim_{j\to\infty} \|g_{\varrho_j''}(\cdot, u_{\varrho_j''}) - g(\cdot, u^*)\|_{L^1(\Omega)} = 0. \tag{2.11}$$

Since $u_{\varrho_j''}$ is a solution of (2.1), (2.2) with $\varrho=\varrho_j''$ thus for any fixed $v\in V\cap L^\infty(\Omega)$ we have

$$\langle T_{\varrho_j''}(L_{\varrho_j''} u_{\varrho_j''}), v\rangle + \int_\Omega g_{\varrho_j''}(x, u_{\varrho_j''})\, v\, dx = \sum_{|\alpha|\le m} \int_{\Omega_{\varrho_j''}} f_\alpha D^\alpha v\, dx,$$

because by the definition of $V_{\varrho_j''}$ for any $v \in V \cap L^\infty(\Omega)$ we have $v|_{\Omega_{\varrho_j''}} \in V_{\varrho_j''} \cap L^\infty(\Omega_{\varrho_j''})$. Hence by (2.6) and (2.11) as $j \to \infty$ we obtain that for all $v \in V \cap L^\infty(\Omega)$

$$\langle y, v\rangle + \int_\Omega g(x, u^*) v\, dx = \sum_{|\alpha| \le m} \int_\Omega f_\alpha D^\alpha v\, dx. \tag{2.12}$$

We shall show that

$$y = T(u^*). \tag{2.13}$$

First we prove that

$$\limsup_{j \to \infty} \langle T_{\varrho_j''}(L_{\varrho_j''} u_{\varrho_j''}), L_{\varrho_j''} u_{\varrho_j''} - u^*\rangle \le 0. \tag{2.14}$$

Equality (2.7) implies

$$\begin{aligned} &\langle T_{\varrho_j''}(L_{\varrho_j''} u_{\varrho_j''}), L_{\varrho_j''} u_{\varrho_j''} - u^*\rangle = \\ &= \sum_{|\alpha| \le m} \int_{\Omega_{\varrho_j''}} f_\alpha D^\alpha u_{\varrho_j''}\, dx - \langle T_{\varrho_j''}(L_{\varrho_j''} u_{\varrho_j''}), u^*\rangle - \int_{\Omega_{\varrho_j''}} g(x, u_{\varrho_j''}) u_{\varrho_j''}\, dx. \end{aligned} \tag{2.15}$$

By (2.10) and assumption VI

$$\lim_{j \to \infty} p(\cdot, u_{\varrho_j''}) u_{\varrho_j''} = p(\cdot, u^*) u^* \quad \text{a.e.},$$

and

$$p(x, u_{\varrho_j''}) u_{\varrho_j''} \ge 0,$$

thus from Fatou's lemma and (2.8) we obtain

$$\int_\Omega p(x, u^*) u^*\, dx \le \liminf_{j \to \infty} \int_{\Omega_{\varrho_j''}} p(x, u_{\varrho_j''}) u_{\varrho_j''}\, dx. \tag{2.16}$$

Furthermore,

$$\lim_{j \to \infty} r(\cdot, u_{\varrho_j''}) u_{\varrho_j''} = r(\cdot, u^*) u^* \quad \text{a.e.}$$

and by assumption VI

$$|r(x, u_{\varrho_j''}) u_{\varrho_j''}| \le h |u_{\varrho_j''}|.$$

Therefore for arbitrary measurable set E

$$\int_E |r(x, u_{\varrho_j''}) u_{\varrho_j''}|\, dx \le \Big\{\int_E |h|^q dx\Big\}^{1/q} \cdot \|u_{\varrho_j''}\|_{L^p(\Omega)}$$

and from the Vitali convergence theorem we obtain

$$\lim_{j \to \infty} \int_{\Omega_{\varrho_j''}} r(x, u_{\varrho_j''}) u_{\varrho_j''}\, dx = \int_\Omega r(x, u^*) u^*\, dx. \tag{2.17}$$

Equalities (2.16), (2.17) imply

$$\int_\Omega g(x, u^*) u^*\, dx \le \liminf_{j \to \infty} \int_{\Omega_{\varrho_j''}} g(x, u_{\varrho_j''}) u_{\varrho_j''}\, dx,$$

whence

$$\limsup_{j \to \infty} \Big[- \int_{\Omega_{\varrho_j''}} g(x, u_{\varrho_j''}) u_{\varrho_j''}\, dx\Big] \le - \int_\Omega g(x, u^*) u^*\, dx. \tag{2.18}$$

Furthermore,

$$\sum_{|\alpha|\leq m}\int_{\Omega_{\varrho_j''}} f_\alpha D^\alpha u_{\varrho_j''}\,dx = \sum_{|\alpha|\leq m}\int_{\Omega} f_\alpha D^\alpha L_{\varrho_j''} u_{\varrho_j''}\,dx - \sum_{|\alpha|\leq m}\int_{\Omega\setminus\Omega_{\varrho_j''}} f_\alpha D^\alpha L_{\varrho_j''} u_{\varrho_j''}\,dx,$$

thus by (2.5) and the boundedness of $\|L_{\varrho_j''}u_{\varrho_j''}\|_V$ we have

$$\lim_{j\to\infty}\sum_{|\alpha|\leq m}\int_{\Omega_{\varrho_j''}} f_\alpha D^\alpha u_{\varrho_j''}\,dx = \langle f, u^*\rangle. \tag{2.19}$$

From (2.6), (2.15), (2.18), (2.19) it follows that

$$\limsup_{j\to\infty}\langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''}), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle \leq \langle f-y, u^*\rangle - \int_\Omega g(x, u^*)u^*\,dx.$$

By assumption VIII there exist a constant $c>0$ and a sequence of functions $w_k\in V\cap L^\infty(\Omega)$ such that

$$\lim_{k\to\infty}\|w_k-u^*\| = 0, \quad \lim_{k\to\infty} w_k = u^* \quad \text{a.e. in } \Omega \tag{2.20}$$

and

$$|w_k(x)| \leq c|u^*(x)| \quad \text{a.e. in } \Omega. \tag{2.21}$$

From equality (2.12) we obtain

$$\langle y, w_k\rangle + \int_\Omega g(x, u^*)w_k\,dx = \langle f, w_k\rangle.$$

Consequently,

(2.22)

$$\limsup_{j\to\infty}\langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''}), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle \leq \langle f-y, u^*-w_k\rangle + \int_\Omega g(x, u^*)(w_k-u^*)\,dx.$$

It is clear that

$$\lim_{k\to\infty}\langle f-y, u^*-w_k\rangle = 0. \tag{2.23}$$

Furthermore, by Lebesgue's dominated convergence theorem and (2.9), (2.20), (2.21) we find

$$\lim_{k\to\infty}\int_\Omega g(x, u^*)(w_k-u^*)\,dx = 0.$$

Thus (2.22), (2.23) implies (2.14).

Now we shall show that

$$\lim_{j\to\infty}\langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''}), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle = 0. \tag{2.24}$$

Assumption V implies that

$$\langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''}) - T_{\varrho_j''}(u^*), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle \geq -c\|u_{\varrho_j''}-u^*\|^p_{W_p^{m-1}(\Omega_{\varrho_0})}$$

with a constant $c>0$. This inequality can be written in the form

$$(2.25)\qquad \begin{aligned}\langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''}), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle \geqq \langle T_{\varrho_j''}(u^*)-T(u^*), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle +\\ +\langle T(u^*), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle - c\|u_{\varrho_j''}-u^*\|^p_{W_p^{m-1}(\Omega_{\varrho_0})}.\end{aligned}$$

From the boundedness of $\|L_{\varrho_j''}u_{\varrho_j''}\|_V$ and Hölder's inequality we obtain

$$(2.26)\qquad \lim_{j\to\infty}\langle T_{\varrho_j''}(u^*)-T(u^*), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle = 0.$$

Moreover, by (2.5) we have

$$(2.27)\qquad \lim_{j\to\infty}\langle T(u^*), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle = 0.$$

Finally as the imbedding of $W_p^m(\Omega_{\varrho_0})$ into $W_p^{m-1}(\Omega_{\varrho_0})$ is compact thus there exists a subsequence $(\bar{\varrho}_j'')$ of (ϱ_j'') such that

$$\lim_{j\to\infty}\|u_{\bar{\varrho}_j''}-u^*\|_{W_p^{m-1}(\Omega_{\varrho_0})} = 0.$$

Therefore (2.25)—(2.27), (2.14) imply (2.24).

Now we shall show that for any $v\in V$

$$(2.28)\qquad \langle T(u^*), u^*-v\rangle \leqq \langle y, u^*-v\rangle.$$

This inequality implies (2.13). Consider the element

$$w=(1-t)u^*+tv,\quad t>0.$$

By assumption V

$$\langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''})-T_{\varrho_j''}(w), L_{\varrho_j''}u_{\varrho_j''}-w\rangle \geqq -c\|u_{\varrho_j''}-w\|^p_{W_p^{m-1}(\Omega_{\varrho_0})},$$

which can be written in the form

$$\begin{aligned}&\langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''}), L_{\varrho_j''}u_{\varrho_j''}-u^*\rangle + \langle T_{\varrho_j''}(L_{\varrho_j''}u_{\varrho_j''}), t(u^*-v)\rangle -\\ &-\langle T[u^*+t(v-u^*)], L_{\varrho_j''}u_{\varrho_j''}-u^*+t(u^*-v)\rangle +\\ &+\langle T[u^*+t(v-u^*)]-T_{\varrho_j''}[u^*+t(v-u^*)], L_{\varrho_j''}u_{\varrho_j''}-u^*+t(u^*-v)\rangle \geqq\\ &\geqq -c\|u_{\varrho_j''}-u^*+t(u^*-v)\|^p_{W_p^{m-1}(\Omega_{\varrho_0})}.\end{aligned}$$

Hence by use of (2.5), (2.6), (2.24), Hölder's inequality and the compactness of the imbedding of $W_p^m(\Omega_{\varrho_0})$ into $W_p^{m-1}(\Omega_{\varrho_0})$ we find

$$\langle y, t(u^*-v)\rangle - \langle T[u^*+t(v-u^*)], t(u^*-v)\rangle \geqq -c\|t(u^*-v)\|^p_{W_p^{m-1}(\Omega_{\varrho_0})}.$$

Thus

$$\langle y, u^*-v\rangle - \langle T[u^*+t(v-u^*)], u^*-v\rangle \geqq -ct^{p-1}\|u^*-v\|^p_{W_p^{m-1}(\Omega_{\varrho_0})}.$$

By assumptions I—II

$$\lim_{t\to+0}\langle T[u^*+t(v-u^*)], u^*-v\rangle = \langle T(u^*), u^*-v\rangle$$

and, taking into account $p>1$, we obtain (2.28).

Thus (2.12), (2.13) imply

$$\langle T(u^*), v\rangle + \int_\Omega g(x, u^*)v\,dx = \langle f, v\rangle \tag{2.29}$$

for all $v \in V \cap L^\infty(\Omega)$.

Applying this equality to $v = w_k$, by (2.20), (2.21), (2.9) as $k \to \infty$ we find that (2.29) is valid also for $v = u^*$.

If the solution u of (1.1), (1.2) is unique but $(L_{\varrho_j} u_{\varrho_j})$ does not tend to u weakly in V then by the above argument we get easily a contradiction.

REMARK 5. Since for bounded domains $\tilde{\Omega}$ the imbedding of $W_p^m(\tilde{\Omega})$ into $W_p^{m-1}(\tilde{\Omega})$ is compact thus from the above theorem it follows that there is a subsequence $(\tilde{\varrho}_j)$ of (ϱ_j) such that for any bounded $\tilde{\Omega}$

$$\lim_{j\to\infty} \|u_{\tilde{\varrho}_j} - u^*\|_{W_p^{m-1}(\tilde{\Omega})} = 0.$$

Moreover, if the solution u of (1.1), (1.2) is unique, we have

$$\lim_{j\to\infty} \|u_{\varrho_j} - u^*\|_{W_p^{m-1}(\tilde{\Omega})} = 0.$$

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SELECTIVE ALGEBRAS AND COMPATIBLE VARIETIES

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1. Introduction

In this paper, the notion of a selective algebra is introduced and applied to characterize equational theories which have models over every variety. Another characterization was already proposed by Isbell [9]; the use of selective algebras makes it possible to prove or refute this property for several concrete equational theories.

We shall use the standard terminology of universal algebra [7]. A *non-trivial* set or algebra always has at least two elements. The set consisting of the first k non-negative integers will be denoted by $\mathbf{k}$.

Let P and $M_p (p\in P)$ be arbitrary non-empty sets and k a natural number. We define a k-ary operation f on $S:=\prod_{p\in P} M_p$ in the following way. We consider two mappings $f_1\colon P\to\mathbf{k}$ and $f_2\colon P\to P$, such that, for all $p\in P$, $M_{f_2(p)}\subseteq M_p$ and $M_{f_2(p)}$ is non-trivial if M_p is non-trivial. Let $\sigma_0, \ldots, \sigma_{k-1}\in S$. Put

$$f(\sigma_0, \ldots, \sigma_{k-1})(p) = \sigma_{f_1(p)}(f_2(p)), \tag{1}$$

for every $p\in P$. In words, in order to get the p-component of the result, first we select the $f_1(p)^{\text{th}}$ operand, and then the $f_2(p)$-component of it. Operations f obtained in this way will be called *selective operations*. The mappings f_1 and f_2 will be referred to as the *first* and *second selectors* of f. We say that $\langle S; F\rangle$ is a *selective algebra* if each $f\in F$ is a selective operation on S. If $M_p=M$ for every $p\in P$ (i.e. $S=M^P$), we call $\langle S; F\rangle$ a *regular selective algebra.*

Special kinds of selective algebras have been in use for a long time. A selective algebra $\langle S; F\rangle$ with $P=\mathbf{k}$, f k-ary, and $f_1(p)=f_2(p)=p$ for each $p\in P$ is a *k-dimensional diagonal algebra* (Płonka [13]) which often appears in the study of free spectra of varieties (see, e.g. [10]). Diagonal algebras of a given dimension form a variety in which regularity in the above sense means freeness. *Rectangular bands, left* and *right zero semigroups* are examples of diagonal algebras, hence also of selective algebras. A further example is the *k-dimensional die,* introduced by Fajtlowicz [4]; such an object is a free k-dimensional diagonal algebra whose structure is enriched by a further unary selective operation c with $c_2(i)\equiv i-1 \pmod k$ for every $i\in\mathbf{k}$. Regular selective groupoids with two-element P and non-trivial cyclic selectors were

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characterized by Evans [3] by means of identities; a more general result for regular selective groupoids was obtained by Saade [15]. Regular selective algebras with k-element P and with all possible selective operations appear at Taylor [16] as members of the k^{th} *power-variety of sets.*

Regular selective algebras are a special case of the *wreath algebras* introduced and applied to the study of completeness properties of finite algebras by Rosenberg [14]. Take a selective operation f on M^P and a mapping π of P into the symmetric group over M. Define the operation $w_{f,\pi}$ on M^P by $w_{f,\pi}(\sigma_0, \ldots, \sigma_{k-1})(p) = \pi(p)\big(f(\sigma_0, \ldots, \sigma_{k-1})(p)\big)$. The operations arising in this way are the wreath operations; and wreath algebras are the ones with wreath basic operations.

Now we make some observations we will need in the sequel.

Polynomials of selective algebras are selective operations.

Indeed, each projection on a product set $S = \prod_{p\in P} M_p$ is a selective operation having the first selector constant and the second selector the identical map. Further, if f and $g^0, \ldots, g^{k-1}$ are n-ary, resp. k-ary, selective operations on S, then, for any $\sigma_0, \ldots, \sigma_{k-1} \in S$ and $p \in P$

$$(2) \qquad f\big(g^0(\sigma_0, \ldots, \sigma_{k-1}), \ldots, g^{n-1}(\sigma_0, \ldots, \sigma_{k-1})\big)(p) = \sigma_{g_1^{f_1(p)}(f_2(p))}\big(g_2^{f_1(p)}(f(p))\big),$$

as, in view of (1), both sides are equal to $g^{f_1(p)}(\sigma_0, \ldots, \sigma_{k-1})\big(f_2(p)\big)$. Note also $M_{g_2^{f_1(p)}(f_2(p))} \subseteq M_p$; thus we see that $f(g^0, \ldots, g^{n-1}) = h$ is a selective operation on S with selectors $h_1 \colon p \mapsto g_1^{f_1(p)}\big(f_2(p)\big)$ and $h_2 \colon p \mapsto g_2^{f_1(p)}\big(f_2(p)\big)$.

This consideration also shows that we can attribute a well-determined pair of selectors to every polynomial symbol h of a selective algebra $\mathbf{S}$, which are also selectors of the polynomial induced by h in $\mathbf{S}$.

For a product set $S = \prod_{p\in P} M_p$, the *support* of S is the set $Q = \{p \in P \colon |M_p| > 1\}$.

An n-ary selective operation f on $\mathbf{S}$ *depends essentially on its i^{th} variable* $(i \in \mathbf{n})$ *if and only if the image of the support of S under f_1 contains i.* This follows directly from the definition.

LEMMA. *Two selective operations f and g of the same arity on S are equal iff their first selectors as well as their second selectors coincide on the support of S.*

The easy proof may be omitted. We note only that $f(\sigma_0, \ldots, \sigma_{n-1})(p) = a$ for all $p \in P$ such that $M_{f_2(p)} = \{a\}$, and also that for $p \in P \setminus Q$ we have $M_{f_2(p)} = M_p$ (because $M_{f_2(p)} \subseteq M_p$). Thus, without loss of generality we may assume that $f_2 \restriction (P \setminus Q) = \mathrm{id}_{P \setminus Q}$.

2. Compatibility of varieties

An *n-ary operation over an algebra* $\mathbf{A}$ is a homomorphism $h \colon \mathbf{A}^n \to \mathbf{A}$. For the algebra $\mathbf{A} = \langle A; \emptyset \rangle$ (i.e., a set) this is the common notion of the operation. Expressing it in other way, f is an operation over $\mathbf{A}$ iff f commutes with all operations of $\mathbf{A}$, i.e. belongs to the centralizer of $\mathbf{A}$ ([1], p. 127; cf. [12], [11]).

$\mathbf{B} = \langle A; H \rangle$ is an *algebra over* $\mathbf{A}$ if every $h \in H$ is an operation over $\mathbf{A}$. We can thus speak of algebras of a given type over $\mathbf{A}$, and of algebras over $\mathbf{A}$ which are models of a given equational theory, i.e. belong to a given variety.

Following Isbell [9], for two varieties $\mathscr{V}$ and $\mathscr{W}$, we say that $\mathscr{V}$ is compatible with $\mathscr{W}$ if there exists an algebra $A\in\mathscr{V}$ over a nontrivial $B\in\mathscr{W}$. For operations f, g the relation f commutes with g is symmetric, hence compatibility of varieties is symmetric, too. We say that a variety $\mathscr{V}$ is *ubiquitous* if $\mathscr{V}$ is compatible with every variety. Isbell proved ([9], Theorem 1.1) that every variety compatible with the variety of Boolean algebras is ubiquitous. The next proposition slightly extends this result, and throws light on the relationship of ubiquity and selective algebras.

PROPOSITION. *For a variety $\mathscr{V}$ the following are equivalent:*
(I) *$\mathscr{V}$ is compatible with a variety generated by a primal algebra.*
(II) *$\mathscr{V}$ contains a nontrivial regular selective algebra.*
(III) *$\mathscr{V}$ contains a nontrivial selective algebra.*
(IV) *$\mathscr{V}$ is ubiquitous.*

PROOF. Our proposition is implied by the following four claims:

CLAIM 1. *Let a variety $\mathscr{W}$ be generated by a primal algebra M. If* **B** *is a nontrivial algebra over an algebra* $\mathbf{A}\in\mathscr{W}$, *then* **B** *is a dense subalgebra of a regular selective algebra on a power of M.* ($A\subseteq M^P$ *is dense* if $A|P'=M^{P'}$ *for every finite* $P'\subseteq P$).

CLAIM 2. *If some dense subalgebra of a regular selective algebra* **S** *belongs to the variety $\mathscr{V}$, then* **S** *belongs to $\mathscr{V}$.*

CLAIM 3. *If a variety $\mathscr{V}$ contains a nontrivial selective algebra then for an arbitrary nontrivial set M, the variety $\mathscr{V}$ contains a regular selective algebra on some power of M.*

CLAIM 4. *For an arbitrary algebra* **K**, *every selective operation on a power K^P commutes with every operation of K^P.*

Indeed, (II)→(III) and (IV)→(I) are obvious; (I)→(II) follows from Claims 1 and 2; and (III)→((II)→)(IV) follows from Claims 3 and 4. Hence it remains to prove the Claims.

1. Let $\mathbf{B}=\langle A; F\rangle$ be an algebra over $\mathbf{A}\in\mathscr{W}$. As **M** is primal, **A** is isomorphic to a subdirect power of **M**. (Concerning primal algebras, consult [7], pp. 177—180, 401—403.) Hence the maximal congruences of **A** are exactly those having $|M|$ distinct congruence classes. We can represent **A** as a subdirect product of all factoralgebras modulo maximal congruences, which is the same as a subdirect product of copies of **M** indexed by the set P of all maximal congruences of **A**. Thus, **A** is, up to isomorphism, a subalgebra of $\mathbf{M}^P$, and the primality of **M** implies that A is dense.

Consider an n-ary operation $f\in F$, i.e. a homomorphism $f\colon \mathbf{A}^n\to\mathbf{A}$. Let $\pi\in P$ and, for $\langle\alpha_0, \dots, \alpha_{n-1}\rangle, \langle\alpha'_0, \dots, \alpha'_{n-1}\rangle\in\mathbf{A}^n$, put $\langle\alpha_0, \dots, \alpha_{n-1}\rangle\sim\langle\alpha'_0, \dots, \alpha'_{n-1}\rangle$ if the π-components of $f(\alpha_0, \dots, \alpha_{n-1})$ and $f(\alpha'_0, \dots, \alpha'_{n-1})$ coincide. Then $\sim$ is a maximal congruence of A^n. As the algebras in $\mathscr{W}$ may be considered as lattices with additional operations, the congruences of A^n are factorizable [5]. Thus, $\sim=\iota_A\times\dots\times\pi'\times\iota_A\times\dots$, where $\pi'\in P$ and π' is the $k_\pi{}^{\text{th}}$ factor. This shows that the π-component of $f(\alpha_0, \dots, \dots, \alpha_{n-1})$ is a bijective function of the π'-component of $\alpha_{k_\pi-1}$. As f is a homomorphism, this function is an automorphism of **M**, hence identical, because **M** is primal. We obtained that f is the restriction to **A** of a selective operation f' on M^P with $f_1'(\pi)=k_{\pi-1}$ and $f_2(\pi)=\pi'$ for every $\pi\in P$. Hence **A** is a dense subalgebra of a regular selective algebra on M^P, as asserted. (Note that this consideration may also be formulated using the Stone-Hu duality for primal algebra theory [8]).

2. Let $\mathbf{S}=\langle M^P; F\rangle$ be a regular selective algebra, and $\mathbf{D}$ a dense subalgebra of $\mathbf{S}$. We have to prove that the identities of $\mathbf{D}$ are satisfied in $\mathbf{S}$, too. This means that distinct (n-ary) polynomials h, h' of $\mathbf{S}$ can be distinguished by suitable $\delta_0, \ldots, \delta_{n-1}\in \mathbf{D}$. We can suppose that $\mathbf{S}$ is non-trivial, hence the support of $\mathbf{S}$ is P. Now, by the Lemma, $h\neq h'$ on $\mathbf{S}$ means that at least one of $h_1\neq h_1'$ and $h_2\neq h_2'$ is valid. First suppose that $h_1\neq h_1'$ and let $p\in P$ be such that $h_1(p)\neq h_1'(p)$. Take distinct elements m_1, m_2 from M. As $\mathbf{D}$ is dense, there exist $\delta, \delta'\in D$ with $\delta(p)=m_1$, $\delta'(p)=m_2$. Let $\delta_{h_1(p)}=\delta$, $\delta_{h_1'(p)}=\delta'$ and choose all the remaining $\delta_i\in D$ $(i\in \mathbf{n};\ i\neq h_1(p),\ h_1'(p))$ arbitrarily. Then

$$(3)\qquad h(\delta_0, \ldots, \delta_{n-1})(p) = m_1 \neq m_2 = h'(\delta_0, \ldots, \delta_{n-1})(p).$$

Assume $h_1=h_1'$; then there is a $p\in P$ with $h_2(p)\neq h_2'(p)$. As $\mathbf{D}$ is dense, there exists $\delta\in D$ with $\delta(h_2(p))=m_1\neq m_2=\delta(h_2'(p))$. Let $\delta_{h_1(p)}=\delta$ and choose the other δ_i's arbitrarily. Under these assumptions again (3) holds. Thus, h is distinct from h' on D, as stated.

3. Let $\mathbf{S}=\langle S; F\rangle$ be a non-trivial selective algebra. For an arbitrary non-trivial set M we present a regular selective algebra on some power of M which is the same type as $\mathbf{S}$ and satisfies all the identities of $\mathbf{S}$.

S has the form $\prod_{p\in P} M_p$ with non-empty support $Q\subseteq P$. Take an operation f of $\mathbf{S}$. Restrict f_1, f_2 to Q, thus obtaining f_1', f_2'. Let f' be the selective operation on M^Q determined by selectors f_1', f_2'. Now, $\mathbf{S}'=\langle M^Q; f'\colon f\in F\rangle$ is the regular selective algebra in question. Indeed, if g and h are polynomial symbols of $\mathbf{S}$, and $\mathbf{S}$ satisfies $g=h$, then, by the Lemma, $\mathbf{S}'$ satisfies $g'=h'$, where g', h' are the corresponding polynomial symbols of $\mathbf{S}'$.

4. Let $\mathbf{S}=\langle K^P; F\rangle$ be a selective algebra and take an n-ary $f\in F$. We have to show that f is a homomorphism of $(\mathbf{K}^P)^n$ into $\mathbf{K}^P$. Let g be an m-ary operation of $\mathbf{K}^P$. Choose m elements from $(\mathbf{K}^P)^n$ arbitrarily: $\langle \mu_0^i, \ldots, \mu_{n-1}^i\rangle$ $(i=0, \ldots, m-1)$. Then

$$\begin{aligned} f(\langle g(\mu_0^0, \ldots, \mu_0^{m-1}), \ldots, g(\mu_{n-1}^0, \ldots, \mu_{n-1}^{m-1})\rangle)(p) &= g(\mu_{f_1(p)}^0, \ldots, \mu_{f_1(p)}^{m-1})(f_2(p)) = \\ &= g(\mu_{f_1(p)}^0(f_2(p)), \ldots, \mu_{f_1(p)}^{m-1}(f_2(p))) = \\ &= g(f\langle \mu_0^0, \ldots, \mu_{n-1}^0\rangle(p), \ldots, f\langle \mu_0^{m-1}, \ldots, \mu_{n-1}^{m-1}\rangle(p)) = \\ &= g(f\langle \mu_0^0, \ldots, \mu_{n-1}^0\rangle, \ldots, f\langle \mu_0^{m-1}, \ldots, \mu_{n-1}^{m-1}\rangle)(p) \end{aligned}$$

holds for each $p\in P$, i.e. f commutes with g, as required, and the Proposition is proved.

3. Applications

The fact that ubiquitous varieties can be characterized by the presence of algebras with a quite transparent structure allows us to decide on several varieties whether they are ubiquitous.

No congruence modular variety is ubiquitous.

We prove this by showing that there is no non-trivial regular selective algebra in a congruence modular variety. Let $\mathscr{V}$ be congruence modular. By the Mal'cev type theorem of Day [2], there exist quaternary polynomial symbols $d^0, \ldots, d^n$ $(n \geqq 1)$ such that for $i=0, \ldots, n-1$ the following identities hold in $\mathscr{V}$:

$$d^i(x, y, y, x) = x, \tag{4}$$

$$\begin{cases} d^0(x, y, z, u) = x, \\ d^i(x, y, y, u) = d^{i+1}(x, y, y, u) & \text{for } i \text{ odd} \\ d^i(x, x, u, u) = d^{i+1}(x, x, u, u) & \text{for } i \text{ even} \\ d^n(x, y, z, u) = u. \end{cases} \tag{5}$$

Assume that there exists a regular selective algebra $\mathbf{S}=\langle M^P; F\rangle$ in $\mathscr{V}$. Set $e^i(x, y)=d^i(x, y, y, x)$ $(i=0, \ldots, n-1)$. Then for arbitrary $\sigma_0, \sigma_1 \in \mathbf{S}$ and for every $p \in P$

$$e^i(\sigma_0, \sigma_1)(p) = \sigma_0(p).$$

Applying (1), it follows $\sigma_0(p)=\sigma_{e_1^i(p)}(e_2^i(p))$, and the right side equals $\sigma_0(e_2^i(p))$ if $d_1^i(p) \in \{0, 3\}$ while it equals $\sigma_1(e_2^i(p))$ if $d_1^i(p) \in \{1, 2\}$. As we can choose σ_0 and σ_1 with $\sigma_0(p) \neq \sigma_1(e_2^i(p))$, the second case cannot occur, i.e., $d_1^i(p) \in \{0, 3\}$ for each i and p. This means that no d^i depends essentially upon its second and third variables. Hence, by (5), $\mathbf{S}$ satisfies $x=u$, thus $\mathbf{S}$ is trivial, a contradiction.

As a consequence, no varieties of quasigroups, groups, rings, or lattices are ubiquitous. As for semigroups, an easy argument shows that a variety of semigroups is ubiquitous if and only if it contains a non-trivial rectangular band.

Varieties $\mathscr{A}_{m,n}$ (with natural numbers m and n) having n-ary operations $g^0, \ldots, g^{m-1}$ and m-ary operations $h^0, \ldots, h^{n-1}$ which satisfy for each meaningful i

$$\begin{aligned} &h^i(g^0(x_0, \ldots, x_{n-1}), \ldots, g^{m-1}(x_0, \ldots, x_{n-1})) = x_i, \\ &g^i(h^0(x_0, \ldots, x_{m-1}), \ldots, h^{n-1}(x_0, \ldots, x_{m-1})) = x_i \end{aligned} \tag{6}$$

were first studied by Goetz and Ryll-Nardzewski [6]. They have the notable property that a free algebra in $\mathscr{A}_{m,n}$ with an m-element free generating set has also an n-element free generating set. Hence, for $m \neq n$, these varieties do not contain non-trivial finite algebras. Here we prove:

The varieties $\mathscr{A}_{m,n}$ are ubiquitous.

By the Proposition, we have to produce a selective algebra $\mathbf{S}$ with operations g^i $(i=0, \ldots, m-1)$, h^i $(i=0, \ldots, n-1)$ satisfying (6). Take a non-trivial set M. We shall define $\mathbf{S}$ on the set $M^{\mathbf{N}}$ where $\mathbf{N}=\{1, 2, \ldots\}$. Write $i \operatorname{div} j$ for the quotient of the Euclidean division of i by j, and $i \bmod j$ for the remainder of that. Define g^i and h^j by their selectors as follows:

$$g_1^i(k) = k \bmod n, \quad g_2^i(k) = m(k \operatorname{div} n)+i,$$

$$h_1^j(k) = k \bmod m, \quad h_2^j(k) = n(k \operatorname{div} m)+j,$$

for every $k \in \mathbf{N}$. Then, for arbitrary $\sigma_0, \ldots, \sigma_{n-1} \in M^{\mathbf{N}}$ it holds

$$h^i\big(g^0(\sigma_0, \ldots, \sigma_{n-1}), \ldots, g^{m-1}(\sigma_0, \ldots, \sigma_{n-1})\big)(k) =$$
$$= g^{k \bmod m}(\sigma_0, \ldots, \sigma_{n-1})\big(n(k \operatorname{div} m)+i\big) =$$
$$= \sigma_{(n(k \operatorname{div} m)+i) \bmod k}\big(m\,((n(k \operatorname{div} m)+i) \operatorname{div} n)+k \bmod m\big) = \sigma_i(k).$$

The identities in the second line of (6) can be verified in the same way. Thus, $S \in \mathscr{A}_{m,n}$, as required.

For a variety, to contain free algebras which have m-element and also n-element free generating sets ($m, n \in \mathbf{N}$; $m \neq n$) is a strong Mal'cev property ([7], p. 400), characterized by the identities (6). Hence we can conclude that the fulfilment of a Mal'cev condition does not exclude ubiquity. Using selective algebras, it is easy to establish that several other syntactical properties of varieties, e.g. equational completeness, definability by regular identities, and definability by linear identities are independent from ubiquity as well.

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ON OPTIMAL QUADRATURE FORMULAE

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1. Introduction. In 1950 P. Turán [7] observed that for preassigned nodes

$$-1 < x_n < x_{n-1} < \ldots < x_2 < x_1 < 1 \tag{1.1}$$

we can construct a quadrature formula (q.f.)

$$\int_{-1}^{1} f(x)\,dx = \sum_{p=0}^{2} \sum_{v=1}^{n} c_v^{(p)} f^{(p)}(x_v) + Rf \tag{1.2}$$

such that $Rf=0$ if f is any polynomial of degree $\leqq 3n-1$. Turán now asks: can we place the nodes x_v such that $Rf=0$ if f is any polynomial of degree $4n-1$? The answer to this interesting question is given by the following

THEOREM A (P. Turán). *Among the quadrature formula* (1.2) *valid for all polynomials $f(x)$ of degree $\leqq 3n-1$ there is exactly one choice* of $(x_1, x_2, \ldots, x_n)$ *for which the formula* (1.2) *is valid for all polynomials of degree $\leqq 4n-1$. The* $(x_1, x_2, \ldots, x_n)$ *system consists of the n real distinct zeros in the interior of* $[-1, 1]$ *of that polynomial $\pi_{n,4}(x) = x^n + \ldots$ which minimizes the integral*

$$I_4(\pi_n) = \int_{-1}^{1} (\pi_n(x))^4\,dx. \tag{1.3}$$

Turán also proved that for any weight function, $w(x)$ there is a unique quadrature formula

$$\int_{-1}^{1} f(x)w(x)\,dx = \sum_{j=1}^{n} [\lambda_j^{(0)} f(x_j) + \lambda_j^{(1)} f'(x_j) + \ldots + \lambda_j^{(2k-2)} f^{(2k-2)}(x_j)] \tag{1.4}$$

valid for $f \in P_{2kn-1}$, $k=1, 2, \ldots$. This formula is obtained by choosing the nodes $x_1, x_2, \ldots, x_n$ to be the zeros of the unique polynomial which minimizes $\int_{-1}^{1} (\pi_n(x))^{2k} w(x)\,dx$ among all $\pi_n(x)$ with leading coefficient 1, and integrating the Hermite interpolating polynomial of degree at most $(2k-1)n-1$ which agrees with f and its $2(k-1)$ derivatives at $x_1, x_2, \ldots, x_n$.

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In 1971 Micchelli and Rivlin [2] made an important contribution concerning optimal q.f. corresponding to the weight function $w(x)=(1-x^2)^{-1/2}$. In order to state their main results let us recall the notion of the divided differences of a function. Let ξ_i^r is shorthand for $\xi_i, \xi_i, \ldots, \xi_i$ (r times). If g has a continuous k^{th} derivative on $[-1, 1]$ and $x_1, x_2, \ldots, x_s$ are distinct points of that interval then $g(x_1^{m_1}, x_2^{m_2}, \ldots \ldots, x_s^{m_s})$, where $m_j \leq k+1$, $j=1, 2, \ldots, s$, the divided difference of g with respect to the x_i with multiplicity m_i, $i=1, 2, \ldots, s$, is the leading coefficient of the unique $p \in P_l$, $l=\sum_{i=1}^{s} m_i - 1$ which satisfies $p^{(j)}(x_i)=g^{(j)}(x_i)$, $i=1, 2, \ldots, s$; $j=0, \ldots, m_i-1$.

Now the main results of Micchelli and Rivlin can be formulated as follows.

THEOREM B (Micchelli and Rivlin). *The q.f.*

$$\int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{n}\Big[\sum_{j=1}^{n} f(\xi_j) + \sum_{j=1}^{k-1} \alpha_j f'(\xi_1^{2j}, \ldots, \xi_1^{2j})\Big] \tag{1.5}$$

and the q.f.

$$\int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{n}\Big[\sum_{j=0}^{n}{}'' f(\eta_j) + \sum_{j=1}^{k-1} \alpha_j (-1)^j f'(\eta_0^j, \eta_1^{2j}, \ldots, \eta_{n-1}^{2j}, \eta_n^j)\Big] \tag{1.6}$$

are both valid if $f \in P_{2kn-1}$, provided

$$\alpha_j = \frac{(-1)^j \begin{pmatrix} -\frac{1}{2} \\ j \end{pmatrix}}{2j 4^{(n-1)j}}, \quad j = 1, 2, \ldots \tag{1.7}$$

$$\xi_i = \cos\frac{(2i-1)\pi}{2n}, \quad i = 1, 2, \ldots, n; \quad \eta_i = \cos\frac{i\pi}{n}, \quad i = 0, 1, \ldots, n. \tag{1.8}$$

Further, the double stoke on the summation sign in (1.6) indicates that the first and the last terms are to be halved.

NOTE. In the special case $k=2$ (1.5) can be written as

(1.9)
$$\int_{-1}^{1} f(x)(1-x^2)^{-1/2}\, dx = \frac{\pi}{n}\Big[\sum_{i=1}^{n} f(\xi_i) - \frac{1}{4n^2}\sum_{i=1}^{n} \xi_i f'(\xi_i) + \frac{1}{4n^2}\sum_{i=1}^{n}(1-\xi_i^2) f''(\xi_i)\Big]$$

This formula was also proved in [2] (formula 18).

P. Turán gave a series of lectures held at the University of Montreal in 1975 where he raised 89 open problems on approximation theory. These problems first appeared in Mat. Lapok (in Hungarian) and later in Journal of Approximation Theory [8] (English translation). Remarking on his work [8] he mentioned that the case $w(x)=(1-x^2)^{-1/2}$ is particularly interesting (see formula (1.4)), since it is used in methods of Runge—Kutta type. In this case Turán's problem can be formulated as follows:

PROBLEM XXVI (P. Turán). *Let* $w(x)=(1-x^2)^{-1/2}$ *in* (1.4). *Give an explicit formula for* $\lambda_j^{(i)}$ $i=0, 1, \dots, 2k-2$ *and determine its asymptotic behavior as* $n\to\infty$.

NOTE. For $k=2$ the solution of the problem is given by (1.9).

The object of this paper is to give explicit solution of the above problem for $k=3$. We will also state the corresponding results explicitly ($k=2$) when the nodes are given by the zeros of extended Chebyshev polynomials of the second kind. [See Theorem B (1.6)]. More precisely we prove the following Theorems.

THEOREM 1. *Let*

$$x_k = \cos\frac{(2k-1)\pi}{2n} \quad k = 1, 2, \dots, n \tag{1.10}$$

be the zeros of Chebyshev polynomials of the first kind. Then the q.f.

$$\int_{-1}^{1} f(t)(1-t^2)^{-1/2}\,dt = \sum_{p=0}^{4}\sum_{\nu=1}^{n} f^{(p)}(x_\nu)\lambda_\nu^{(p)} \tag{1.11}$$

where

$$\lambda_\nu^{(0)} = \frac{\pi}{n}, \quad \lambda_\nu^{(1)} = \frac{-\pi x_\nu(20n^2-1)}{64n^5}$$

$$\lambda_\nu^{(2)} = \frac{\pi}{64n^5}[3+(20n^2-7)(1-x^2)], \tag{1.12}$$

$$\nu = 1, 2, \dots, n$$

$$\lambda_\nu^{(3)} = \frac{-6\pi x_\nu}{64n^5}(1-x_\nu^2), \quad \lambda_\nu^{(4)} = \frac{\pi}{64n^5}(1-x_\nu^2)^2$$

is valid for all polynomials $f(t)$ of degree $\leqq 6n-1$.

THEOREM 2. *The q.f.*

$$\int_{-1}^{1} f(t)(1-t^2)^{-1/2}\,dt = \sum_{\nu=0}^{n}\lambda_\nu^{(0)}f(\eta_\nu)+\sum_{\nu=0}^{n}\lambda_\nu^{(1)}f'(\eta_\nu)+\sum_{\nu=1}^{n-1}\lambda_\nu^{(2)}f''(\eta_\nu)$$

where

$$\lambda_0^{(0)} = \lambda_n^{(0)} = \frac{\pi}{2n}, \quad \lambda_\nu^{(0)} = \frac{\pi}{n}$$

$$\lambda_0^{(1)} = -\lambda_n^{(1)} = \frac{-\pi}{8n^3}, \quad \lambda_\nu^{(1)} = \frac{-\pi t_\nu}{4n^3}$$

$$\lambda_\nu^{(2)} = \frac{\pi(1-t_\nu^2)}{4n^3} \quad \nu = 1, 2, \dots, n-1,$$

is valid for all polynomials $f(t)$ *of degree* $\leqq 4n-1$. *Here* n_ν *are defined by* (1.8).

Let $w(x)=(1-x^2)^{-1/2}$ in (1.4). Turán [7] was interested to know whether $\lambda_j^{(i)}$ are nonnegative? For $k=2$ it was shown by Micchelli and Rivlin [2] that $\lambda_j^{(0)}>0$, $\lambda_j^{(1)}<0$, $\lambda_j^{(2)}>0$ $j=1, 2, \dots, n$. Actually in this case they computed explicitly (see (1.9)). Later Micchelli [3] proved similar theorem in a more general case. We refer Theorem 3 page 429 for details. In 1976 Samuel Karlin and Allan Pinkus [1] wrote two interesting papers related to Turán's work. Here they have extended the findings of Turán and Popoviciu [4] to extended complete Chebyshev system. For interested readers we may also refer an interesting book on Quadrature formulae [9].

2. Preliminaries. Let us denote by

$$x_k = \cos\theta_k = \cos\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n \tag{2.1}$$

the zeros of Chebyshev polynomial of the first kind

$$T_n(x) = \cos n\theta, \; x = \cos\theta. \tag{2.2}$$

Following L. Fejér the fundamental functions of Lagrange interpolation based on (2.1) is given by

$$l_k(x) = \frac{T_n(x)}{(x-x_k)T'(x_k)} = \frac{1}{n} + \frac{2}{n}\sum_{r=1}^{n-1} T_r(x)T_r(x_k). \tag{2.3}$$

Clearly

$$l_k(x_i) = \delta_{ik} \tag{2.4}$$

where δ_{ik} is the Kronecker delta. Similarly we also represent fundamental functions of Hermite Interpolation in the form

$$r_k(x) = \left(\frac{1-xx_k}{1-x_k^2}\right) l_k^2(x) = \frac{1}{n} + \frac{1}{n^2}\sum_{r=1}^{2n-1}(2n-r)T_r(x)T_r(x_k), \tag{2.5}$$

$$\varrho_k(x) = (x-x_k)l_k^2(x) = \frac{\sin\theta_k}{n^2}\sum_{r=1}^{2n-1}\sin r\theta_k T_r(x). \tag{2.6}$$

They satisfy the following conditions:

$$r_k(x_i) = \delta_{ik}, \; r_k'(x_i) = 0, \quad i = 1, 2, \ldots, n, \tag{2.7}$$

$$\varrho_k(x_i) = 0, \; \varrho_k'(x_i) = \delta_{ik}, \quad i = 1, 2, \ldots, n. \tag{2.8}$$

The following orthogonal property of Chebyshev polynomials plays an important role.

$$\int_{-1}^{1} T_i(t)T_j(t)(1-t^2)^{-1/2}\,dt = 0 \quad i \neq j \tag{2.9}$$

$$= \frac{\pi}{2} \quad i = j \neq 0$$

$$= \pi \quad i = j = 0.$$

3. Some identities. Here we will derive some identities with the help of Chebyshev-Gauss q.f. and integration by parts. It is well known that

$$\int_{-1}^{1} f(t)(1-t^2)^{-1/2}\,dt = \frac{\pi}{n}\sum_{i=1}^{n} f(x_i) \tag{3.1}$$

is exact provided f is a polynomial of degree $\leqq 2n-1$. Let $f=(1-x^2)r_k''(x)$, $f=xr_k'(x)$ in (3.1) where $r_k(x)$ is defined by (2.5). We have

$$\frac{\pi}{n}\sum_{i=1}^{n}(1-x_i^2)r_k''(x_i) = \int_{-1}^{1} r_k''(x)(1-x^2)^{1/2}\,dx,$$

and

$$0 = \frac{\pi}{n}\sum_{i=1}^{n}x_i r_k'(x_i) = \int_{-1}^{1} xr_k'(x)(1-x^2)^{-1/2}\,dx.$$

On integrating by parts we have

$$\int_{-1}^{1} r_k''(x)(1-x^2)^{1/2}\,dx = \int_{-1}^{1} xr_k'(x)(1-x^2)^{-1/2}\,dx.$$

Therefore

$$\sum_{i=1}^{n}(1-x_i^2)r_k''(x_i) = \sum_{i=1}^{n}x_i r_k'(x_i) = 0. \tag{3.2}$$

Similarly

$$\sum_{i=1}^{n}(1-x_i^2)\varrho_k''(x_i) = x_k, \quad k = 1, 2, \ldots, n. \tag{3.3}$$

Next, we will prove that

$$\sum_{i=1}^{n}\{(1-x_i^2)^2\varrho_k^{(\mathrm{IV})}(x_i)-6x_i(1-x_i^2)\varrho_k'''(x_i)+3\varrho_k''(x_i)\} = 6x_k \tag{3.4}$$

and

$$\sum_{i=1}^{n}\{(1-x_i^2)^2 r_k^{(\mathrm{IV})}(x_i)-6x_i(1-x_i^2)r_k'''(x_i)+3r_k''(x_i)\} = 0. \tag{3.5}$$

Proof of (3.4) and (3.5) are similar, so we only give details for (3.4). For this purpose we first note that

$$f(x) = (1-x^2)^2\varrho_k^{(\mathrm{IV})}(x)-6x(1-x^2)\varrho_k'''(x)+3\varrho_k''(x) \tag{3.6}$$

is indeed a polynomial of degree $\leqq 2n-1$. Therefore on applying (3.1) with $f(x)$ as given by (3.6) and make use of (2.8), (3.3) and integration by parts we obtain

$$\frac{\pi}{n}\sum_{i=1}^{n}f(x_i) = \int_{-1}^{1}(1-t^2)^{3/2}\varrho_k^{(\mathrm{IV})}(t)\,dt$$

$$-6\int_{-1}^{1}t(1-t^2)^{1/2}\varrho_k'''(t)\,dt+3\int_{-1}^{1}\varrho_k''(t)(1-t^2)^{-1/2}\,dt =$$

$$=3\int_{-1}^{1}(1-t^2)^{1/2}t\varrho_k'''(t)\,dt-6\int_{-1}^{1}t(1-t^2)^{1/2}\varrho_k'''(t)\,dt+3\int_{-1}^{1}\varrho_k''(t)(1-t^2)^{-1/2}\,dt =$$

$$=-3\int_{-1}^{1}t(1-t^2)^{1/2}\varrho_k'''(t)\,dt+3\int_{-1}^{1}\varrho_k''(t)(1-t^2)^{-1/2}\,dt =$$

$$=3\int_{-1}^{1}(1-2t^2)\varrho_k''(t)(1-t^2)^{-1/2}\,dt+3\int_{-1}^{1}\varrho_k''(t)(1-t^2)^{-1/2}\,dt=$$

$$=6\int_{-1}^{1}(1-t^2)^{1/2}\varrho_k''(t)\,dt=6\int_{-1}^{1}t(1-t^2)^{-1/2}\varrho_k'(t)\,dt=6\frac{\pi}{n}\sum_{i=1}^{n}x_i\varrho_k'(x_i)=6x_k\frac{\pi}{n}.$$

From this (3.4) follows at once.

4. Proof of Theorem 1. From the work of P. Turán [7] it follows that there exists $\lambda_\nu^{(p)}$ such that

$$\int_{-1}^{1}f(x)(1-x^2)^{-1/2}\,dx=\sum_{p=0}^{4}\sum_{\nu=1}^{n}\lambda_\nu^{(p)}f^{(p)}(x_\nu) \tag{4.1}$$

is valid for all polynomials $f(x)$ of degree $\leqq 6n-1$. Let

$$f_k(x)=\frac{T_n^4(x)l_k(x)}{24(T'(x_k))^4} \tag{4.2}$$

where $l_k(x)$ is defined by (2.3). On applying (4.1) with $f_k(x)$ as given by (4.2) we obtain

$$\lambda_k^{(4)}=\int_{-1}^{1}\frac{T^4(x)l_k(x)}{24(T_n'(x_k))^4}(1-x^2)^{-1/2}\,dx \tag{4.3}$$

Since

$$(T_n'(x_k))^2=n^2(1-x_k^2)^{-1},\quad k=1,2,\dots,n,$$

and

$$T_n^4(x)=\frac{3}{8}+\frac{1}{2}T_{2n}(x)+\frac{1}{8}T_{4n}(x),$$

therefore (4.3) becomes

$$\lambda_k^{(4)}=\frac{(1-x_k^2)^2}{24n^4}\int_{-1}^{1}\left[\frac{3}{8}+\frac{1}{2}T_{2n}(x)+\frac{1}{8}T_{4n}(x)\right]l_k(x)(1-x^2)^{-1/2}\,dx.$$

On using (2.9) and (2.3) we have

$$\lambda_k^{(4)}=\frac{(1-x_k^2)^2}{24n^4}\,\frac{3}{8}\int_{-1}^{1}l_k(x)(1-x^2)^{-1/2}\,dx=\frac{\pi(1-x_k^2)^2}{64n^5}. \tag{4.4}$$

Next, we will prove that

$$\lambda_k^{(3)}=\frac{-6x_k(1-x_k^2)\pi}{64n^5},\quad k=1,2,\dots,n. \tag{4.5}$$

For this purpose we set

$$g_k(x)=\frac{T_n^3(x)r_k(x)}{6(T_n'(x_k))^3}-\frac{1}{4}\frac{T_n''(x_k)}{T_n'(x_k)^5}l_k(x)T^4(x) \tag{4.6}$$

and note that

$$\begin{aligned} g_k^{(p)}(x_i)&=0, && p=0,1,2,4 \\ &=\delta_{ik}, && p=3 \end{aligned}\qquad i=1,2,\dots,n. \tag{4.7}$$

Further

$$T_n^3(x) = \frac{3}{4} T_n(x) + \frac{1}{4} T_{3n}(x). \tag{4.8}$$

Therefore from (2.5), (2.9) and (4.8) we obtain

$$\int_{-1}^{1} T_n^3(x) r_k(x)(1-x^2)^{-1/2}\, dx = 0, \quad k = 1, 2, \ldots, n. \tag{4.9}$$

Now, we use (4.6), (4.9), (4.3) and (4.4) and we obtain

$$\begin{aligned} \int_{-1}^{1} g_k(x)(1-x^2)^{-1/2}\, dx &= \frac{-x_k}{4(1-x_k^2)(T_n'(x_k))^4} \int_{-1}^{1} \frac{l_k(x) T_n^4(x)\, dx}{(1-x^2)^{1/2}} \\ &= \frac{-6x_k}{(1-x_k^2)} \lambda_k^{(4)} = \frac{-6x_k(1-x_k^2)\pi}{64n^5}. \end{aligned} \tag{4.10}$$

Now, on applying (4.1) with $g_k(x)$ as given by (4.6) and make use of (4.7) and (4.10) we obtain

$$\lambda_k^{(3)} = \frac{-6x_k(1-x_k^2)\pi}{64n^5}.$$

This proves (4.5). Next we will prove that

$$\lambda_k^{(2)} = \frac{\pi}{64n^5}[3 + (20n^2-7)(1-x_k^2)], \quad k = 1, 2, \ldots, n. \tag{4.11}$$

For this purpose, we will first prove

$$\int_{-1}^{1} \frac{T_n^2(x) r_k(x)}{2(T_n'(x_k))^2} (1-x^2)^{-1/2}\, dx = \frac{\pi(1-x_k^2)}{4n^3}. \tag{4.12}$$

This follows at once from (2.5), (2.9) and

$$T_n^2(x) = \frac{1+T_{2n}(x)}{2}.$$

Now we set in (4.1)

$$h_k(x) = \frac{T_n^2(x) r_k(x)}{2(T_n'(x_k))^2}$$

and note that $h_k(x) \in \pi_{4n-1}$. Since (4.1) is exact for polynomials of degree $\leqq 6n-1$

we have

$$\frac{\pi(1-x_k^2)}{4n^3} = \int_{-1}^{1} \frac{T_n^2(x)r_k(x)}{2(T_n'(x_k))^2}(1-x^2)^{-1/2}dx$$

$$= \lambda_k^{(2)} + \frac{3T_n''(x_k)}{T_n'(x_k)}\lambda_k^{(3)} + \frac{6(T_n''(x_k))^2 + 8T_n'(x_k)T_n'''(x_k)}{2(T_n'(x_k))^2}\lambda_k^{(4)} +$$

$$+ \frac{6\sum_{i=1}^{n} 2(T_n'(x_i))^2 r_k''(x_i)\lambda_i^{(4)}}{2(T_n'(x_k))^2}.$$

On using

$$T_n''(x_k) = \frac{x_k T_n'(x_k)}{(1-x_k^2)},$$

$$T_n'''(x_k) = \left(\frac{3x_k^2}{(1-x_k^2)^2} - \frac{(n^2-1)}{1-x_k^2}\right)T_n'(x_k).$$

and (4.4), (4.5) we obtain

$$\lambda_k^{(2)} = \frac{\pi}{4n^3}(1-x_k^2) + \frac{18x_k^2\pi}{64n^5} - \frac{\pi[15x_k^2 - 4(n^2-1)(1-x_k^2)]}{64n^5} -$$

$$- \frac{6(1-x_k^2)\pi}{64n^5}\sum_{i=1}^{n}(1-x_i^2)r_k''(x_i).$$

Now on using (3.2) we obtain (4.11).

Next, we turn to prove

$$\lambda_k^{(1)} = \frac{-\pi x_k}{64n^5}(20n^2-1). \tag{4.13}$$

For this purpose we set $f(x)=\varrho_k(x)$ in (4.1). We obtain

$$\int_{-1}^{1} \varrho_k(x)(1-x^2)^{-1/2}dx = \lambda_k^{(2)} + \sum_{i=1}^{n}\lambda_i^{(2)}\varrho_k''(x_i) + \tag{4.14}$$

$$+ \sum_{i=1}^{n}\lambda_i^{(3)}\varrho_k'''(x_i) + \sum_{i=1}^{n}\lambda_i^{(4)}\varrho_k^{(\mathrm{IV})}(x_i).$$

From (2.6) and (2.9) we have

$$\int_{-1}^{1} \varrho_k(x)(1-x^2)^{-1/2}dx = 0, \quad k = 1, 2, \dots, n. \tag{4.15}$$

On using (4.4), (4.5), (4.4) and (4.11) we obtain

$$0 = \lambda_k^{(1)} + \frac{\pi}{64n^5}\sum_{i=1}^{n}[3+(20n^2-7)(1-x_i^2)]\varrho_2''(x_i) -$$

$$- \frac{6\pi}{64n^5}\sum_{i=1}^{n}x_i(1-x_i^2)\varrho_k'''(x_i) + \frac{\pi}{64n^5}\sum_{i=1}^{n}(1-x_i^2)^2\varrho_k^{(\mathrm{IV})}(x_i).$$

On using (3.3) and (3.4) we obtain

$$\lambda_k^{(1)} = \frac{-\pi}{64n^5}\sum_{i=1}^{n}[(1-x_i^2)\varrho_k^{(IV)}(x_i)-6x_i(1-x_i^2)\varrho_k'''(x_i)+3\varrho_k''(x_i)]-$$

$$-\frac{\pi}{64n^5}(20n^2-7)\sum_{i=1}^{n}(1-x_i^2)\varrho_k''(x_i) =$$

$$= \frac{-6\pi x_k}{64n^5}-\frac{\pi(20n^2-7)}{64n^5}x_k =$$

$$= \frac{-\pi x_k}{64n^5}(20n^2-1), \qquad k=1,2,\dots,n.$$

Now, we will prove

$$\lambda_k^{(0)} = \frac{\pi}{n} \quad k=1,2,\dots,n. \tag{4.16}$$

For this purpose, we set $f(x)=r_k(x)$ in (4.1). We obtain on using (2.5),

$$\begin{aligned}\frac{\pi}{n} &= \int_{-1}^{1} r_k(x)(1-x^2)^{-1/2}\,dx = \\ &= \lambda_k^{(0)}+\sum_{i=1}^{n}\lambda_i^{(2)}r_k''(x_i)+\sum_{i=1}^{n}\lambda_i^{(3)}r_k'''(x_i)+\sum_{i=1}^{n}\lambda_i^{(4)}r_k^{(IV)}(x_i).\end{aligned} \tag{2.9}$$

On using (3.2), (3.5), (4.4), (4.5) and (4.11) we have

$$\frac{\pi}{n} = \lambda_k^{(0)}+\frac{\pi}{64n^5}\sum_{i=1}^{n}[3+(20n^2-7)(1-x_i^2)]r_k''(x_i)-$$

$$-\frac{6\pi}{64n^5}\sum_{i=1}^{n}x_i(1-x_i^2)r_k'''(x_i)+\frac{\pi}{64n^5}\sum_{i=1}^{n}(1-x_i^2)^2 r_i^{(IV)}(x_i).$$

Therefore

$$\lambda_k^{(0)} = \frac{\pi}{n}-\frac{\pi}{64n^5}\sum_{i=1}^{n}\{(1-x_i^2)r_k^{(IV)}(x_i)-6x_i(1-x_i^2)r_k'''(x_i)+3r_k''(x_i)\}-$$

$$-\frac{\pi}{64n^5}(20n^2-7)\sum_{i=1}^{n}(1-x_i^2)r_k''(x_i) = \frac{\pi}{n}, \quad k=1,2,\dots,n.$$

This completes the proof of Theorem 1. Proof of Theorem 2 is very similar to the proof of Theorem 1. We omit the details.

After the paper was written, in April 1981 the author came to know the work of Dr. R. D. Riess entitled "Gauss—Turán Quadratures of Chebyshev Type and Error Formulae" published in *Computing*, Vol. **15**, 173—179 (1975). He also obtained in this work same statement as Theorem 1 but the proof of Theorem 1 given there is very different than that, obtained in this work.

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ASYMPTOTIC BEHAVIOUR OF AUTONOMOUS HALF-LINEAR DIFFERENTIAL SYSTEMS ON THE PLANE

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0. Introduction. We consider the system of the half-linear differential equations of the form

$$(0.1)\qquad \begin{aligned} y' &= ay + bz^{1/n^*} \\ z' &= cy^{n^*} + dz \end{aligned}$$

for the functions $y=y(t)$, $z=z(t)$ where the coefficients a, b, c, d are constants, the number n is real, positive and the function u^{n^*} means $|u|^n \cdot \operatorname{sgn} u$ for $u \in \mathbf{R}$. The system (0.1) with non-constant coefficients was studied in [1] where only the qualitative properties of the solutions were investigated.

The system (0.1) is closely related to the half-linear second order differential equation with constant coefficients

$$(0.2)\qquad (x'^{n^*})' + px'^{n^*} + qx^{n^*} = 0,$$

because the substitutions

$$y = x, \quad z = x'^{n^*}$$

transform the equation (0.2) into the system

$$(0.1')\qquad \begin{aligned} y' &= z^{1/n^*} \\ z' &= -qy^{n^*} - pz. \end{aligned}$$

Let us remark here that the differential equation

$$(0.3)\qquad (x'^{n^*})' + \frac{n\gamma}{t^{n+1}} x^{n^*} = 0 \quad \text{for} \quad t > 0$$

with constant $\gamma \neq 0$ can be transformed by the substitutions $s = \log t$ and $X(s) = x(t)$ into the differential equation with constant coefficients

$$(0.4)\qquad (X'^{n^*})' - nX'^{n^*} + n\gamma X^{n^*} = 0,$$

which is also of the form of (0.2). The differential equation (0.3) has occured already

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in [2] in connection with an oscillation criterion and it turned out that the value

$$\gamma_0 = \frac{n^n}{(n+1)^{n+1}} \tag{0.5}$$

plays an essential role because the solutions of (0.3) are oscillatory when $t\to\infty$ if $\gamma>\gamma_0$ and nonoscillatory if $\gamma\leqq\gamma_0$.

The paper is devided into several parts. After the introduction in the first part we define an equivalence relation among the systems of the form (0.1). We shall see that besides the trivial classes there are two special classes and two one-parameter families of classes. The asymptotic behaviour will be determined for the trivial and the two special classes in the second part, while the one-parameter families of classes together with the differential equation (0.3), (0.4) will be characterized in the third part.

1. Classification. In what follows we classify the systems of the form (0.1). We say that the system

$$\begin{aligned} \bar y' &= \bar a \bar y + \bar b \bar z^{1/n^*} \\ \bar z' &= \bar c \bar y^{n^*} + \bar d \bar z \end{aligned} \tag{1.1}$$

with

$$\bar y = \bar y(\tau), \quad \bar z = \bar z(\tau), \quad \tau = \beta t$$

is equivalent to the system (0.1) if and only if there exist constants A, α, β such that $A\neq 0$, $\beta>0$ and the substitutions

$$\begin{aligned} y(t) &= e^{\alpha t}\bar y(\tau) \\ z(t) &= A^{n^*} e^{n\alpha t}\bar z(\tau) \end{aligned} \tag{1.2}$$

transform the system (0.1) into (1.1). A simple calculation provides that

$$\begin{aligned} \bar a &= \frac{a-\alpha}{\beta} \\ \bar b &= \frac{bA}{\beta} \\ \bar c &= \frac{c}{\beta}A^{n^*} \\ \bar d &= \frac{d-n\alpha}{\beta}. \end{aligned} \tag{1.3}$$

It is clear that the relations (1.2) or (1.3) define an equivalence relation for classification of the four parametric system (0.1).

On the other hand it is clear by the relations (1.3) that if $b=0$ (or $c=0$) then also $\bar b=0$ (or $\bar c=0$). Hence we may say that a system with $b=0$ or $c=0$ of the form (0.1) belongs to one of the trivial classes. We shall see that such trivial systems can be solved explicitly.

More interesting are the cases which are not trivial, i.e. for which $bc \neq 0$. By (1.3) we have then $\bar{b}\bar{c} \neq 0$, or more precisely

$$\operatorname{sgn} bc = \operatorname{sgn} \bar{b}\bar{c}$$

for the elements of the same class. A second similar relation is

$$\operatorname{sgn}(d-na) = \operatorname{sgn}(\bar{d}-n\bar{a}).$$

Now let us choose the values

$$\alpha = a$$
$$\beta = bA,$$

then the system equivalent to (0.1) has the form

$$\bar{y}' = \bar{z}^{1/n^*}$$

$$\bar{z}' = \frac{c}{b|A|^{n+1}} \bar{y}^{n^*} + \frac{d-na}{bA} \bar{z}, \tag{1.4}$$

which is the same as (0.1′). By our assumption $\beta > 0$ hence $\operatorname{sgn} A = \operatorname{sgn} b$.

According to the value of $d-na$ there are two main cases.

Case 1. $d-na=0$. Let $|A|=|c/bn|^{1/(n+1)}$, then we have two subcases depending on the sign of the ratio c/b:

Case 1a.

$$y' = z^{1/n^*}$$
$$z' = -ny^{n^*}.$$

Case 1b.

$$y' = z^{1/n^*}$$
$$z' = ny^{n^*}.$$

Case 2. $d-na \neq 0$. Let $A=|d-na|/(bn)$ then we should distinguish two subcases corresponding to the sign of the expression $d-na$.

Case 2a.

$$y' = z^{1/n^*}$$
$$z' = -n\gamma y^{n^*} + nz, \tag{1.5}$$

where

$$\gamma = -\frac{cb^{n^*}n^n}{|d-na|^{n+1}}, \quad \gamma \neq 0.$$

Case 2b.

$$y' = z^{1/n^*}$$
$$z' = -n\gamma y^{n^*} - nz. \tag{1.6}$$

For the sake of the completeness we add to the above cases the neglected (trivial) classes representing them by the coefficient matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of one element of the class as follows

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

2. In this part we shall consider the trivial systems and the equivalence classes (1a) and (1b).

Let us consider a trivial system with $b=0$. By (0.1) we have

$$y = Ce^{at}$$

$$z = \begin{cases} cC^{n*}te^{dt} + De^{dt} & \text{if } \ d = na, \\ \dfrac{cC^{*}}{na-d}e^{nat} + De^{dt} & \text{if } \ d \neq na, \end{cases}$$

where C and D are arbitrary constants.

In the case $c=0$ we can use the above formulas with the permutations

$$\begin{pmatrix} y & z \\ z & y \end{pmatrix}, \quad \begin{pmatrix} n & 1/n \\ 1/n & n \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}.$$

Concerning the study of the equivalence classes we remark first that it is sufficient to characterize only the solutions of *one* system representing the equivalence class because the solutions of another system of the same class are connected by the relations (1.2).

Case 1a. Let $(y(t), z(t))$ be a solution of the system (1a). Then the function $y(t)$ satisfies the second order differential equation

$$(x'^{n*})' + nx^{n*} = 0$$

or

$$x''|x'|^{n-1} + x^{n*} = 0. \tag{2.1}$$

In [1] the generalized sine function $S=S_n(t)$ was introduced as the solution of (2.1) with the initial conditions $S(0)=0$, $S'(0)=1$. Hence the general solutions of (2.1) are $x=C\cdot S_n(t-t_0)$, where C and t_0 are parameters. The equivalent statement for the system is formulated in the next theorem.

THEOREM 2.1. *The system of Case 1a has the solutions*

$$y = C \cdot S_n(t-t_0)$$
$$z = C^{n^*} \cdot S_n'^{n^*}(t-t_0),$$

which are bounded oscillatory periodic functions with the period of $2\hat{\pi}$ where

$$\hat{\pi} = 2\frac{\frac{\pi}{n+1}}{\sin\frac{\pi}{n+1}}.$$

Case 1b. Now the function $y(t)$ satisfies the differential equation

$$x''|x'|^{n-1} - x^{n^*} = 0. \tag{2.2}$$

Multiplying by x' and integrating over $[0, t]$ we have

$$|x'|^{n+1} - |x|^{n+1} = |x'(0)|^{n+1} - |x(0)|^{n+1} = C. \tag{2.3}$$

If $C=0$ then $x'=\pm x$ thus $x_1=e^t$ and $x_2=e^{-t}$ are solutions of (2.2).

Let $E=E_n(t)$ be the solution of (2.2) with the initial conditions $E(0)=0$, $E'(0)=1$ and similarly $F=F_n(t)$ with the initial conditions $F(0)=1$, $F'(0)=0$. Let us observe that the function E corresponds to $C=1$ while F to $C=-1$ in (2.3). Moreover for $n=1$, i.e. if the differential equation (2.2) is linear, we have $E_1(t)=\operatorname{sh} t$ and $F_1(t)=\operatorname{ch} t$.

Due to (2.3) the function E satisfies also the relation

$$E' = \sqrt[n+1]{1+|E|^{n+1}},$$

hence $E'>1$ for $t>0$. Consequently

$$\int_0^t \frac{E'}{\sqrt[n+1]{1+E^{n+1}}} = \int_0^{E(t)} \frac{ds}{\sqrt[n+1]{1+s^{n+1}}} = t. \tag{2.4}$$

In order to compare the function $E(t)$ with e^t let the function $f(s)$ be defined by

$$f(s) = \begin{cases} 1 & \text{for} \quad 0 \leqq s \leqq 1 \\ \dfrac{1}{s} & \text{for} \quad s \geqq 1. \end{cases}$$

Then by (2.4) we obtain

$$t - \log E(t) = 1 + \int_0^E \frac{ds}{\sqrt[n+1]{1+s^{n+1}}} - \int_0^E f(s)\,ds \quad \text{for} \quad E > 1.$$

Hence

$$\log \delta_n = \lim_{t\to\infty} [t - \log E(t)] = 1 - \int_0^\infty \left[f(s) - \frac{1}{\sqrt[n+1]{1+s^{n+1}}} \right] ds. \tag{2.5}$$

The integral on the right hand side can be interpreted as the area of the domain on the plane (s, y) given by the inequalities

$$\frac{1}{\sqrt[n+1]{1+s^{n+1}}} \leqq y \leqq f(s) \quad \text{for} \quad 0 \leqq s < \infty.$$

Taking y as independent variable we find for the integral in (2.5)

$$\log \delta_n = 1 - \int_0^1 \frac{1-\sqrt[n+1]{1-y^{n+1}}}{y}\,dy = 1 - \frac{1}{n+1}\int_0^1 \frac{1-u^{1/(n+1)}}{1-u}\,du. \tag{2.6}$$

Since $0<1-\sqrt[n+1]{1-y^{n+1}}<y$ for $0<y<1$ we have $0<\log\delta_n<1$, i.e. $1<\delta_n<e$. On the other hand the integral in (2.6) can be expressed by the aid of the function $\Psi(z)=d\log\Gamma(z)/dz$ as

$$\Psi(z) = -\tilde{\gamma} + \int_0^1 \frac{1-t^{z-1}}{1-t}\,dt \quad \text{for} \quad \operatorname{Re} z > 0,$$

where $\tilde{\gamma}$ is the Euler—Mascheroni constant and $\Gamma(z)$ denotes the gamma function (see [3]). Making use of this relation we obtain

$$\log \delta_n = 1 - \frac{1}{n+1}\left[\tilde{\gamma} + \Psi\left(\frac{n+2}{n+1}\right)\right]. \tag{2.7}$$

Finally, the relation (2.5) can be rewritten as

$$\lim_{t\to\infty} \frac{e^t}{E_n(t)} = \delta_n \quad \text{where} \quad 1 < \delta_n < e. \tag{2.5'}$$

A similar relation is expected also for the function $F_n(t)$. First we need a lemma.

LEMMA. *Let $I_1(R)$, $I_2(R)$, I_3 be integrals defined by*

$$I_1(R) = \int_0^R \frac{d\xi}{\sqrt[n+1]{1+\xi^{n+1}}}, \quad R>0$$

$$I_2(R) = \int_1^R \frac{d\xi}{\sqrt[n+1]{\xi^{n+1}-1}}, \quad R>1$$

$$I_3 = \int_0^1 \frac{d\xi}{\sqrt[n+1]{1-\xi^{n+1}}}.$$

Then

$$\lim_{R\to\infty} [I_1(R) - I_2(R)] = \frac{\pi}{n+1}\operatorname{ctg}\frac{\pi}{n+1}$$

and

$$I_3 = \frac{\frac{\pi}{n+1}}{\sin \frac{\pi}{n+1}}.$$

REMARK. The integral I_3 has played role already in [1] and its value is known as $\hbar/2$. The evaluation of I_3 takes place here in natural way and that is the reason why it is displayed again.

PROOF. Let us consider the function $H(z) = 1/\sqrt[n+1]{1+z^{n+1}}$ on the complex plane $\mathbf{C}$. Then $H(z)$ is holomorphic on the angular domain

$$0 \leqq \theta < \frac{\pi}{n+1} \quad \text{where} \quad z = re^{i\theta},$$

and it has singularity only on the boundary of this domain at $\omega = \exp\left(i\pi/(n+1)\right)$ and, in general, at $z=\infty$. For any $\varepsilon>0$ and $R>1+\varepsilon$ we define a closed curve $c_{\varepsilon,R}$ by the following components:

$$c_1 = \{x+0i;\ 0 \leqq x \leqq R\}$$

$$c_2 = \left\{Re^{i\theta};\ 0 \leqq \theta \leqq \frac{\pi}{n+1}\right\}$$

$$c_3 = \{\xi\omega;\ 1+\varepsilon \leqq \xi \leqq R\}$$

$$c_4 = \{\omega(1+\varepsilon e^{-i\theta});\ 0 \leqq \theta \leqq \pi\}$$

$$c_5 = \{\xi\omega;\ 0 \leqq \xi \leqq 1-\varepsilon\}.$$

Then we have

(2.8)
$$0 = \oint_{c_{\varepsilon,R}} H(z)\,dz = \sum_{k=1}^{5} \int_{c_k} H(z)\,dz = \sum_{k=1}^{5} \mathrm{J}_k,$$

where

$$\mathrm{J}_1 = \int_0^R \frac{dx}{\sqrt[n+1]{1+x^{n+1}}} = I_1(R)$$

$$\mathrm{J}_2 = i \int_0^{\pi/(n+1)} \frac{d\theta}{\sqrt[n+1]{1+\dfrac{e^{-i(n+1)\theta}}{R^{n+1}}}}$$

$$\mathrm{J}_3 = -\int_{1+\varepsilon}^{R} \frac{d\xi}{\sqrt[n+1]{\xi^{n+1}-1}}$$

$$\mathrm{J}_4 = O(\varepsilon^{n/(n+1)})$$

$$\mathrm{J}_5 = -\omega \int_0^{1-\varepsilon} \frac{d\xi}{\sqrt[n+1]{1-\xi^{n+1}}}.$$

Letting $\varepsilon \to +0$ and $R \to \infty$ in (2.8) we obtain

$$\lim_{R\to\infty}[I_1(R)-I_2(R)]+\frac{\pi}{n+1}i-\omega I_3=0,$$

which implies the statements of our Lemma.

Now we want to obtain a relation for $F_n(t)$ similar to (2.5′). Since F fulfils the differential equation

$$|F'|^{n+1}-|F|^{n+1}=-1,$$

hence

$$\frac{F'}{\sqrt[n+1]{F^{n+1}-1}}=1 \quad \text{for} \quad t>0.$$

An integration yields

$$\int_1^{F(t)}\frac{d\xi}{\sqrt[n+1]{\xi^{n+1}-1}}=t \quad \text{for} \quad t>0.$$

This relation implies that $\lim_{t\to\infty}F(t)=\infty$. On the other hand

(2.9)

$$\lim_{t\to\infty}[t-\log F(t)]=\lim_{t\to\infty}\int_1^{F(t)}\left[\frac{1}{\sqrt[n+1]{\xi^{n+1}-1}}-f(\xi)\right]d\xi=\int_1^{\infty}\left[\frac{1}{\sqrt[n+1]{\xi^{n+1}-1}}-\frac{1}{\xi}\right]d\xi$$

because the improper integral on the right-hand side exists. Let $\Delta=\Delta_n$ be introduced by

$$\log\Delta_n=\int_1^{\infty}\left[\frac{1}{\sqrt[n+1]{\xi^{n+1}-1}}-\frac{1}{\xi}\right]d\xi. \tag{2.10}$$

It is clear that $\Delta_n>1$. The relation (2.9) can be rewritten as

$$\lim_{t\to\infty}\frac{e^t}{F_n(t)}=\Delta_n \quad \text{with} \quad \Delta_n>1. \tag{2.11}$$

Now we want to establish a connection between Δ and δ. By (2.5), (2.10) and taking into consideration the definition of the function $f(\xi)$ we have

$$\log\frac{\Delta}{\delta}=\lim_{R\to\infty}\left[\int_1^R\left(\frac{1}{\sqrt[n+1]{\xi^{n+1}-1}}-\frac{1}{\xi}\right)d\xi-1+\int_0^R\left(f(\xi)-\frac{1}{\sqrt[n+1]{1+\xi^{n+1}}}\right)d\xi\right]=$$

$$=\lim_{R\to\infty}[I_2(R)-I_1(R)],$$

where the functions $I_1(R)$, $I_2(R)$ were introduced in the Lemma. Then by the Lemma we get the wanted relation as

$$\log\frac{\delta_n}{\Delta_n}=\frac{\pi}{n+1}\operatorname{ctg}\frac{\pi}{n+1}. \tag{2.12}$$

We may observe here that this relation implies in the linear case $(n=1)$ that $\delta_1=\Delta_1$. In fact we have $\delta_1=2=\Delta_1$.

By (2.7) the value of δ_n can be considered to be known, consequently by the relation (2.12) the value of Δ_n is also known.

Finally there are interesting functional relations between the functions $E_n(t)$, $F_n(t)$ as follows

$$(2.13)\qquad \begin{aligned} E_n'(t) &= \{F_{1/n}(nt)\}^{1/n} \\ F_n'(t) &= \{E_{1/n}(nt)\}^{1/n^*}. \end{aligned}$$

To prove these relations one must show that the functions on both sides of the equality satisfy the same differential equation and fulfil the same initial conditions.

The relations (2.13) provides another connections between the values of δ_n and Δ_n, too. Indeed, by (2.3) and (2.5′) we have

$$(2.14)\qquad \lim_{t\to\infty}\frac{E_n'(t)}{e^t} = \lim_{t\to\infty}\frac{E_n(t)}{e^t} = \frac{1}{\delta_n}.$$

On the other hand this and (2.13), (2.11) imply

$$\frac{1}{\delta_n^n} = \lim_{t\to\infty}\frac{E_n'^n(t)}{e^{nt}} = \lim_{t\to\infty}\frac{F_{1/n}(nt)}{e^{nt}} = \frac{1}{\Delta_{1/n}},$$

hence

$$(2.15)\qquad \Delta_{1/n} = \delta_n^n,$$

and similarly

$$(2.16)\qquad \delta_{1/n} = \Delta_n^n.$$

We remark that the last two relations are equivalent because substituting n by $1/n$ we get each from the other. By relations (2.12), (2.15) (or (2.16)) it is sufficient to know one of the values of Δ_n, δ_n, $\Delta_{1/n}$, $\delta_{1/n}$, and then all the other values can be obtained easily.

As in the linear case where the function sh t is odd and the function ch t is even, the functions $E_n(t)$, $F_n(t)$ behave in similar way:

$$(2.17)\qquad \begin{aligned} E_n(-t) &= -E_n(t) \\ F_n(-t) &= F_n(t). \end{aligned}$$

To prove this statement it is sufficient to show that the functions on the both side of the equality are solutions of the differential equation (2.2) and satisfy same initial conditions at $t=0$. Then the uniqueness of the initial value problem (see [1]) proves (2.17).

Now we know all the solutions of the differential equation (2.2). We display them in the next theorem.

THEOREM 2.2. *The solutions of the differential equation* (2.2) *on* $-\infty<t<\infty$ *are the following:*

i) $C_1 e^t$,
ii) $C_2 e^{-t}$,
iii) $C_3 E_n(t+t_0)$,
iv) $C_4 F_n(t+t_0)$,

where C_i *and* t_0 *are real parameters.*

PROOF. It is sufficient to show that to every initial condition

$$x(0)=x_0,\ x'(0)=x_0' \quad \text{with} \quad |x_0|^{n+1}+|x_0'|^{n+1}>0$$

there is only one possibility among the cases i)—iv) to that the solution belongs and then the parameter C_i (and t_0, if necessary) can be uniquely determined.

By (2.3) $C=|x_0'|^{n+1}-|x_0|^{n+1}$. If $C=0$ then only the cases i) and ii) are possible. If $x_0x_0'>0$ then the solution is $x(t)=x_0e^t$ and if $x_0x_0'<0$ then $x(t)=x_0e^{-t}$.

Suppose $C\neq 0$. If $C>0$ then by the definition of the function $E_n(t)$ the third possibility holds. Let $C_3=\operatorname{sgn} x_0' \sqrt[n+1]{C}$. Since the function $E_n(t)$ is strictly increasing (which follows from the fact that the function $E'(t)$ is continuous, $|E'|^{n+1}=1+|E|^{n+1}\geqq 1$ and $E'(0)=1$) there is a value $t_0\in(-\infty,\infty)$ such that $C_3E_n(t_0)=x_0$. We should still check the relation $C_3E_n'(t_0)=x_0'$. By the definition of C_1, C_3 and $E_n(t)$ we have

$$|x_0'|^{n+1}=C+|x_0|^{n+1}=C+C\cdot|E|^{n+1}=C\cdot(1+|E|^{n+1})=C\cdot|E'|^{n+1},$$

hence $x_0'=\pm C_3E'(t_0)$. But $E'(t_0)>0$ and $\operatorname{sgn} C_3=\operatorname{sgn} x_0'$, therefore the sign $+$ is the correct one.

If $C<0$ then let $C_4=\operatorname{sgn} x_0\cdot \sqrt[n+1]{-C}$ and t_0 be the solution of the equation $C_4F'(t_0)=x_0'$ and, as in case iii), the function $C_4F_n(t+t_0)$ is the wanted solution. Hereby the proof is complete.

Finally we pass over to investigate the asymptotic behaviour of the solutions in the Case 1b. The results are annunciated by the next theorem.

THEOREM 2.3. *The solution of the Case 1b behave as follows:*

i) *there is a one-parameter family of solutions with* $y(t)=Ce^{-t}$, $z(t)=-C^{n^*}e^{-nt}$;

ii) *there is another one-parameter family of solutions of the form* $y(t)=Ce^t$, $z(t)=C^{n^*}e^{nt}$;

iii) *all the other solutions form a two-parameter family and satisfy the following asymptotic relations:*

$$\lim_{t\to\infty}\frac{y(t)}{e^t}=C$$

$$\lim_{t\to\infty}\frac{z(t)}{e^{nt}}=C^{n^*}$$

with the same C in both relations, where C is some constant depending on the solution.

PROOF. The cases i) and ii) are trivial. To prove statement iii) we recall that $y(t)$ is a solution of the differential equation (2.2) and $z(t)=\{y'(t)\}^{n^*}$. According to Theorem 2.2 either $x(t)=C_3E_n(t+t_0)$ or $x(t)=C_4F_n(t+t_0)$. Hence by (2.5′) or (2.11) $\lim_{t\to\infty} y(t)/\exp(t)$ is either C_3/δ_n or C_4/Δ_n. Consequently $\lim_{t\to\infty} z(t)/\exp(nt)$ is either C_3^{n*}/δ_n^n or C_4^{n*}/Δ_n^n, i.e. C is either C_3/δ_n or C_4/Δ_n as it was stated.

3. In this part we shall study the differential equation (0.4) and the obtained results will be applied to characterize the asymptotic behaviour of the solutions of the systems (2a) and (2b).

The solutions of (0.4). The main tool for investigation of the solutions $x(t)$ of (0.4) is the *generalized Prüfer transformation* in which the generalized polar coordinates φ, ϱ play an essential role (see in [1]):

$$x(t) = \varrho(t)S(\varphi(t)) \qquad (3.1)$$
$$x'(t) = \varrho(t)S'(\varphi(t)).$$

It is known from [1] that the functions $\varphi(t)$, $\varrho(t)$ satisfy the differential equations

$$\varphi' = \Phi(\varphi) \qquad (3.2)$$

$$\varrho' = \varrho R(\varphi), \qquad (3.3)$$

where

$$\Phi = |S'(\varphi)|^{n+1} - S(\varphi)\cdot S'^{n*}(\varphi) + \gamma|S(\varphi)|^{n+1} \qquad (3.4)$$
$$R = S'(\varphi)[(1-\gamma)S^{n*}(\varphi) + S'^{n*}(\varphi)].$$

The functions Φ, R are continuous and periodic with the period of $\hat{\pi}$. On the other hand there is an interesting relation between them:

$$\Phi' + (n+1)R = n, \qquad (3.5)$$

which can be verified by direct calculation taking into account that S is a solution of the system in Case 1a and $|S|^{n+1}+|S'|^{n+1}=1$. Hence Φ is Lipschitzian and the solutions of (3.2), (3.3) are uniquely determined by initial conditions.

The properties of the solutions of (0.4) are determined by the sign of Φ, and we show that there are only three possibilities:

Subcase A. The function Φ has two zeros on $(0, \hat{\pi})$, say φ_1, φ_2.

Subcase B. The function Φ has only one zero φ_0, which is double.

Subcase C. The function Φ is positive.

Since $S(0)=0$ and $S'(0)=1$ we have $\Phi(0)=1$ and by the periodicity $\Phi(k\hat{\pi})= =1>0$, $k=0, \pm 1, \ldots$. On the other hand let $G(\alpha)$ be defined by

$$G(\alpha) = |\alpha|^{n+1} - \alpha^{n*} + \gamma,$$

then

$$G\left(\frac{S'(\varphi)}{S(\varphi)}\right) = \frac{\Phi(\varphi)}{|S(\varphi)|^{n+1}} \quad \text{for} \quad \varphi \neq 0, \pm\hat{\pi}, \pm 2\hat{\pi}, \ldots. \qquad (3.6)$$

Since G has its minimum at $\alpha_0=n/(n+1)$ and $G(\alpha_0)=\gamma-\gamma_0$, the subcases A, B, C correspond to the relations

$$\gamma<\gamma_0,\quad \gamma=\gamma_0,\quad \gamma>\gamma_0.$$

Subcase A. Now $G(\alpha_0)<0$ and $G(\alpha)$ is decreasing on $(-\infty, \alpha_0]$ and increasing on $[\alpha_0, \infty)$ hence by (3.6) there are values φ_1, φ_2 with

$$\Phi(\varphi)\begin{cases}>0 & \text{for} \quad 0\leqq\varphi<\varphi_1\\ <0 & \text{for} \quad \varphi_1<\varphi<\varphi_2\\ >0 & \text{for} \quad \varphi_2<\varphi\leqq\hat{\pi},\end{cases} \tag{3.7}$$

and $\Phi(\varphi_i)=0,\ i=1,2.$

Let α_i $(i=1,2)$ be defined by

$$\alpha_i=\frac{S'(\varphi_i)}{S(\varphi_i)} \quad i=1,2, \tag{3.8}$$

and φ_0 by

$$\frac{S'(\varphi_0)}{S(\varphi_0)}=\alpha_0 \quad 0<\varphi_0<\hat{\pi}. \tag{3.9}$$

The value φ_0 is uniquely determined because the function $S'(\varphi)/S(\varphi)$ is strictly decreasing from $+\infty$ to $-\infty$ as φ varies from 0 to $\hat{\pi}$. Then it is clear that $\varphi_1<\varphi_0<\varphi_2$ and

$$\alpha_2<\alpha_0<\alpha_1. \tag{3.10}$$

Moreover

$$R(\varphi_i)=\alpha_i \quad \text{for} \quad i=1,2. \tag{3.11}$$

Namely by (3.8) and (3.4) we can rewrite (3.11) as

$$1=S(\varphi_i)[(1-\gamma)S^{n*}(\varphi_i)+S'^{n*}(\varphi_i)]$$

and then making use of the relation $|S(\varphi)|^{n+1}+|S'(\varphi)|^{n+1}=1$ on the left hand side we are led to the equation $\Phi(\varphi_i)=0$, which proves the equality (3.11).

By the definition of the φ_i's the functions $\varphi(t)\equiv\varphi_i$ $(i=1,2)$ are solutions of (3.2). Now let us consider the other solutions. Since Φ is periodic with the period $\hat{\pi}$ we have in general two different situations: either $\bar{\varphi}(t_0)\in(\varphi_1, \varphi_2)$ or $\bar{\bar{\varphi}}(t_0)\in(\varphi_2-\hat{\pi}, \varphi_1)$. Let the corresponding solutions be denoted by $\bar{\varphi}(t)$, $\bar{\bar{\varphi}}(t)$, resp. The uniqueness of the solutions of (3.2) implies that $\varphi_1<\bar{\varphi}(t)<\varphi_2$ and $\varphi_2-\hat{\pi}<\bar{\bar{\varphi}}(t)<\varphi_1$ for all t. The differential equation (3.2) can be written in integral form

$$\int_{\varphi(t_0)}^{\varphi(t)}\frac{d\psi}{\Phi(\psi)}=t-t_0. \tag{3.12}$$

Hence by (3.7) the following limit relations hold

$$(3.13)\qquad \begin{gathered}\lim_{t\to\infty}\bar\varphi(t)=\lim_{t\to\infty}\bar{\bar\varphi}(t)=\varphi_1\\ \lim_{t\to-\infty}\bar\varphi(t)=\varphi_2\\ \lim_{t\to-\infty}\bar{\bar\varphi}(t)=\varphi_2-\hbar.\end{gathered}$$

Let us determine the asymptotic behaviour of the solutions $\bar\varphi(t)$, $\bar\varrho(t)$ as t tends to ∞, where $\bar\varrho$ is the solution of (3.3) with $\varphi=\bar\varphi$. By (3.13) we should investigate (3.12) when $\varphi(t)\sim\varphi_1$. Since $\Phi(\varphi_1)=0$ we need the value of $\Phi'(\varphi_1)$: by (3.5) and (3.11) we have

$$\Phi'(\varphi_1)=(n+1)\left[\frac{n}{n+1}-R(\varphi_1)\right]=(n+1)[\alpha_0-\alpha_1].$$

According to the inequalities in (3.10) $\Phi'(\varphi_1)<0$. Since

$$\Phi(\bar\varphi)=\Phi'(\varphi_1)\cdot(\bar\varphi-\varphi_1)+O\big((\bar\varphi-\varphi_1)^2\big)\quad\text{as}\quad \bar\varphi\to\varphi_1$$

we obtain from (3.12)

$$(3.14)\qquad \bar\varphi(t)=\varphi_1+\big(\bar C+o(1)\big)e^{\Phi'(\varphi_1)t}\quad\text{as}\quad t\to\infty,$$

where $\bar C$ is some positive constant due to the inequality $\bar\varphi(t)>\varphi_1$. In (3.3) we substitute

$$R(\bar\varphi)=R(\varphi_1)+O(\bar\varphi-\varphi_1)\quad\text{as}\quad t\to\infty,$$

hence a quadrature provides for $\bar\varrho$ by (3.11), (3.14)

$$(3.15)\qquad \bar\varrho(t)=\big(\bar C_1+o(1)\big)e^{\alpha_1 t}\quad\text{as}\quad t\to\infty.$$

Similar statement is true for $\bar{\bar\varphi}(t)$ and $\bar{\bar\varrho}(t)$, too.

The method used above works also when $t\to-\infty$ and we have the following results

$$\begin{gathered}\bar\varphi(t)=\varphi_2+\big(\bar{\bar C}+o(1)\big)e^{\Phi'(\varphi_2)t}\\ \bar\varrho(t)=\big(\bar{\bar C}_1+o(1)\big)e^{\alpha_2 t}\quad\text{as}\quad t\to-\infty,\end{gathered}$$

where $\Phi'(\varphi_2)=(n+1)[\alpha_0-\alpha_2]>0$ and $\bar{\bar C}$, $\bar{\bar C}_1$ are constant. The corresponding statement for $\bar{\bar\varphi}$ and $\bar{\bar\varrho}$ reads as

$$\begin{gathered}\bar{\bar\varphi}(t)=\varphi_2-\hbar+\big(\bar{\bar{\bar C}}+o(1)\big)e^{\Phi(\varphi_2)t}\\ \bar{\bar\varrho}(t)=\big(\bar{\bar{\bar C}}_1+o(1)\big)e^{\alpha_2 t}\quad\text{as}\quad t\to-\infty.\end{gathered}$$

Subcase B. Now $\gamma=\gamma_0$ and the function Φ is positive on $[0,\hbar]$ with the exception $\varphi=\varphi_0$, where $\Phi(\varphi_0)=0$. Since also $\Phi'(\varphi_0)=0$ we have to show that $\Phi''(\varphi_0)>0$. For $\alpha_0=n/(n+1)$ the definition of φ_0 by (3.9) yields $S(\varphi_0)=(n+1)\varkappa$, $S'(\varphi_0)=n\varkappa$,

$$\varkappa^{n+1}=\frac{1}{n^{n+1}+(n+1)^{n+1}}.$$

We compute first $R'(\varphi_0)$. From (3.3) we obtain

$$R' = (n+1)S''S'^{n*}+(1-\gamma_0)[S''S'^{n*}+nS'^2|S|^{n-1}],$$

hence by (2.1), (0.5)

$$R'(\varphi_0) = -(n\gamma_0+1)^2\varkappa^{n+1}(n+1)^{2n}n^{1-n} < 0,$$

then from (3.5)

$$\Phi''(\varphi_0) = -(n+1)R'(\varphi_0) > 0. \tag{3.16}$$

Thus the function Φ can be written in the form

$$\Phi(\varphi) = \frac{1}{2}\Phi''(\varphi_0)(\varphi-\varphi_0)^2+O((\varphi-\varphi_0)^3) \quad \text{as} \quad \varphi \to \varphi_0. \tag{3.17}$$

It is clear that the function $\varphi(t)\equiv\varphi_0$ is a solution of (3.2). Suppose $\varphi(t_0)\in(\varphi_0-\hat{\pi}, \varphi_0)$. Then by (3.12) we get

$$\lim_{t\to\infty}\varphi(t) = \varphi_0$$
$$\lim_{t\to-\infty}\varphi(t) = \varphi_0-\hat{\pi}. \tag{3.18}$$

Let us study the case when $t\to\infty$. Then we can apply (3.17) to obtain the next relation

$$\frac{1}{\Phi(\varphi)} = \frac{1}{\frac{1}{2}\Phi''(\varphi_0)(\varphi-\varphi_0)^2}+O\left(\frac{1}{\varphi-\varphi_0}\right) \quad \text{as} \quad \varphi \to \varphi_0,$$

hence by (3.12)

$$\frac{1}{\frac{1}{2}\Phi''(\varphi_0)}\,\frac{1}{\varphi_0-\varphi}+O(\log(\varphi_0-\varphi)) = t-t_0 \quad \text{as} \quad t\to\infty,$$

consequently

$$\frac{1}{\varphi_0-\varphi} = \frac{1}{2}\Phi''(\varphi_0)t+O(\log t) \quad \text{as} \quad t\to\infty,$$

or

$$\varphi = \varphi_0-\frac{1}{\frac{1}{2}\Phi''(\varphi_0)t}+O\left(\frac{\log t}{t^2}\right) \quad \text{as} \quad t\to\infty. \tag{3.19}$$

On the other hand

$$R(\varphi) = R(\varphi_0)+R'(\varphi_0)(\varphi-\varphi_0)+O((\varphi-\varphi_0)^2) \quad \text{as} \quad \varphi \to \varphi_0,$$

hence by (3.16), (3.19) we have for ϱ'/ϱ by (3.3)

$$\frac{\varrho'}{\varrho} = R(\varphi) = \alpha_0+\frac{2}{n+1}\,\frac{1}{t}+O\left(\frac{\log t}{t^2}\right) \quad \text{as} \quad t\to\infty.$$

Therefore by integration and letting $t \to \infty$ we obtain the limit

$$\lim_{t\to\infty}\left(\log \varrho(t) - \alpha_0 t - \frac{2}{n+1}\log t\right) = \log \varrho_+,$$

i.e. there exists a finite value ϱ_+ with

$$\lim_{t\to\infty}\frac{\varrho(t)}{e^{tn/(n+1)}t^{2/(n+1)}} = \varrho_+. \tag{3.20}$$

By similar considerations we have also the asymptotic relation

$$\lim_{t\to-\infty}\frac{\varrho(t)}{e^{tn/(n+1)}|t|^{2/(n+1)}} = \varrho_-. \tag{3.21}$$

Subcase C. Now the function $\Phi(\varphi)$ is positive for all φ hence by (3.12) for every solution $\varphi(t)$ to (3.2) we have

$$\lim_{t\to\pm\infty}\varphi(t) = \pm\infty.$$

Let

$$\tau = \int_0^{2\hat{\pi}}\frac{d\varphi}{\Phi(\varphi)} = 2\int_0^{\hat{\pi}}\frac{d\varphi}{\Phi(\varphi)}. \tag{3.22}$$

Then by (3.12) and by $\Phi(\varphi+\hat{\pi}) = \Phi(\varphi)$ the relation

$$\varphi(t+\tau) = \varphi(t) + 2\hat{\pi} \tag{3.23}$$

holds. The value of τ in (3.22) can be evaluated without the knowledge of the functions $S(\varphi)$, $S'(\varphi)$ in spite of their presence in the formula of Φ by (3.4), because the substitution $\sigma = S(\varphi)/S'(\varphi)$ transforms the integral (3.22) as follows

$$\tau = 2\int_{-\infty}^{\infty}\frac{d\sigma}{1-\sigma+\gamma|\sigma|^{n+1}}. \tag{3.24}$$

Let us determine the value $\varrho(t+\tau)/\varrho(t)$. By (3.2), (3.3) and (3.23) we have

$$\log\frac{\varrho(t+\tau)}{\varrho(t)} = \int_t^{t+\tau} R(\varphi(t))\,dt = \int_{\varphi(t)}^{\varphi(t+\tau)}\frac{R(\varphi)}{\Phi(\varphi)}\,d\varphi = \int_0^{2\pi}\frac{R(\varphi)}{\Phi(\varphi)}\,d\varphi.$$

If we apply the relations (3.5), (3.22) we get

$$\log\frac{\varrho(t+\tau)}{\varrho(t)} = \int_0^{2\hat{\pi}}\frac{\dfrac{n}{n+1} - \dfrac{1}{n+1}\Phi'(\varphi)}{\Phi(\varphi)}\,d\varphi = \frac{n}{n+1}\tau. \tag{3.25}$$

A consequence of (3.25) is that the function $\varrho(t)e^{-(n/n+1)t}$ is periodic with the period τ, because

$$\frac{\varrho(t+\tau)e^{-(n/(n+1))(t+\tau)}}{\varrho(t)e^{-(n/(n+1))t}} = \frac{\varrho(t+\tau)}{\varrho(t)}e^{-(n/(n+1))\tau} = 1. \tag{3.26}$$

We summarize the results concerning the behaviour of the solutions of (0.4).

THEOREM 3.1. *The asymptotic behaviour of the solutions of the differential equation* (0.4) *can be characterized as follows:*

in Case A: if $\gamma<\gamma_0$, $\gamma\neq 0$. *There are two zeros* α_1, α_2 *of the function*

$$G(\alpha) = |\alpha|^{n+1}-\alpha^{n^*}+\gamma$$

with $\alpha_1>n/(n+1)>\alpha_2$ *and the functions* $x_1(t)=C_1e^{\alpha_1 t}$, $x_2(t)=C_2e^{\alpha_2 t}$ *are solutions of* (0.4). *For every other solution the limits*

$$\lim_{t\to\infty} x(t)e^{-\alpha_1 t} = x_+, \quad \lim_{t\to\infty} x'(t)e^{-\alpha_1 t} = x'_+$$

exist as finite numbers and $x'_+=\alpha_1\cdot x_+$, *and similar statement is true for* $t\to-\infty$ *with* $\alpha=\alpha_2$: *there exist the limits*

$$\lim_{t\to-\infty} x(t)e^{-\alpha_2 t} = x_-, \quad \lim_{t\to-\infty} x'(t)e^{-\alpha_2 t} = x'_-,$$

where $x'_-=\alpha_2\cdot x_-$;

in Case B: if $\gamma=\gamma_0$. *Then* $\alpha_0=n/(n+1)$ *and* $x(t)=Ce^{\alpha_0 t}$ *is a solution of* (0.4). *For every other solution there exist the limits*

$$\lim_{t\to\infty} x(t)e^{-\alpha_0 t}\frac{1}{t^{2/(n+1)}} = x_+$$

$$\lim_{t\to\infty} x'(t)e^{-\alpha_0 t}\frac{1}{t^{2/(n+1)}} = x'_+,$$

where $x'_+=\alpha_0\cdot x_+$. *Same limit relations are true if* $t\to-\infty$;

in Case C: if $\gamma>\gamma_0$. *Then the solutions* $x(t)$ *are oscillatory, more precisely the functicns*

$$x(t)e^{-(n/(n+1))t}, \quad x'(t)e^{-(n/(n+1))t}$$

are periodic oscillatory functions with the period τ *given by* (3.24).

PROOF. By (3.1) we should consider the behaviour of the functions $\varphi(t)$, $\varrho(t)$ as $t\to\infty$. We saw in Case A that the functions $\varphi(t)\equiv\varphi_i$ $(i=1, 2)$, or more general, $\varphi(t)=\varphi_i+k\hat{\pi}$ $(i=1, 2, k=0, \pm 1, \ldots)$ are solutions of (3.2). On the other hand $R(\varphi_i+k\hat{\pi})=\alpha_i$ and by (3.3) $\varrho(t)=\varrho_0\exp(\alpha_i t)$ $(i=1, 2)$. Hence

$$x(t) = \varrho_0 e^{\alpha_i t}S(\varphi_i+k\hat{\pi}), \quad i = 1, 2, \quad k = 0, \pm 1, \ldots,$$

i.e. they are the special solutions which are of exponential form. Concerning the other situations we have by (3.14)

$$\lim_{t\to\infty}\varphi(t) = \varphi_1 \quad \text{or in general} \quad \lim_{t\to\infty}\varphi(t) = \varphi_1+k\hat{\pi}.$$

Since $R(\varphi)$ is periodic with the period of $\hat{\pi}$ the relation (3.15) holds again, hence by (3.1), (3.15)

$$\lim_{t\to\infty} x(t)e^{-\alpha_1 t} = \bar{C}_1\cdot S(\varphi_1+k\hat{\pi}) = x_+$$

$$\lim_{t\to\infty} x'(t)e^{-\alpha_1 t} = \bar{C}_1\cdot S'(\varphi_1+k\hat{\pi}) = x'_+.$$

But $S(\varphi)$ and $S'(\varphi)$ are periodic with the period of $2\hat{\pi}$ and $S(\varphi+\hat{\pi})=-S(\varphi)$, $S'(\varphi+\hat{\pi})=-S(\varphi)$ (see [1]), hence by (3.8)

$$x_+\alpha_1 = \bar{C}_1\cdot S(\varphi_1+k\hat{\pi})\cdot\frac{S'(\varphi_1)}{S(\varphi_1)} = (-1)^k\bar{C}_1\cdot S'(\varphi_1) = \bar{C}_1\cdot S'(\varphi_1+k\hat{\pi}) = x'_+,$$

as we stated. The proofs of the second part of Case A and the Case B is similar.

In Case C the function $\varphi(t)$ tends monotonically to $+\infty$ hence by (3.1) the solutions $x(t)$ are oscillatory.

By (3.23) and (3.26) the functions $\varphi(t)$ and $\varrho(t)\cdot\exp\left(-\frac{n}{n+1}t\right)$ are periodic with the period τ given by (3.22) which completes the proof of Theorem 3.1.

Case 2a. Now we shall consider the system (1.5).

THEOREM 3.2. *The solutions* $(y(t), z(t))$ *of system* (1.5) *have the asymptotic behaviour as* $|t|\to\infty$:

in Case A, when $\gamma<\gamma_0$, $\gamma\neq 0$: *there are two one-parameter family of solutions of the form*

$$y(t) = Ce^{\alpha_i t}, \quad z(t) = C^{n^*}\alpha_i^{n^*}e^{n\alpha_i t}$$

for $i=1,2$ *where* α_1, α_2 *are the same as above, C is arbitrary constant, while all the other solutions satisfy the relations*

$$\lim_{t\to\infty} y(t)e^{-\alpha_1 t} = y_+, \quad \lim_{t\to\infty} z(t)e^{-n\alpha_1 t} = z_+$$

with $z_+=\alpha_1^{n^*}y_+^{n^*}$ *and*

$$\lim_{t\to-\infty} y(t)e^{-\alpha_1 t} = y_-, \quad \lim_{t\to-\infty} z(t)e^{-n\alpha_2 t} = z_-$$

with $z_-=\alpha_2^{n^*}\cdot y_+^{n^*}$;

in Case B, when $\gamma=\gamma_0$: *there is a one-parameter family of solutions*

$$y(t) = Ce^{n/(n+1)t}, \quad z(t) = C^{n^*}\left(\frac{n}{n+1}\right)^n e^{n^2/(n+1)t}$$

with constant $C\neq 0$, *and all the other solutions satisfy the limit relations*

$$\lim_{t\to\pm\infty} y(t)\frac{e^{-n/(n+1)t}}{|t|^{2/(n+1)}} = y_\pm, \quad \lim_{t\to\pm\infty} z(t)\frac{e^{-n_2/(n+1)t}}{|t|^{2n/(n+1)}} = z_\pm,$$

where $z_+=\alpha_0^n y_+^{n^*}$, $z_-=\alpha_0^n y_+^{n^*}$;

in Case C, when $\gamma>\gamma_0$: *the solutions* $(y(t), z(t))$ *are oscillatory, more precisely the functions*

$$y(t)e^{-n/(n+1)t}, \quad z(t)e^{-n^2/(n+1)t}$$

are bounded periodic functions with period τ *given by* (3.22).

PROOF. The statements follow directly from Theorem 3.1 because the component $y(t)$ satisfies the differential equation (0.4) and on the other hand from the fact that $z(t)=\{y'(t)\}^{n^*}$.

Case 2b. This case can be reduced to Case 2a. Let $(y(t), z(t))$ be a solution of (1.6). Let $\bar{y}(t), \bar{z}(t)$ be introduced by

$$\bar{y}(t) = y(-t), \quad \bar{z}(t) = -z(-t).$$

Hence by (1.6)

$$\bar{y}'(t) = -y'(-t) = -\{z(-t)\}^{1/n^*} = \{\bar{z}(t)\}^{1/n^*}$$

$$\bar{z}'(t) = '(-t) = -n\gamma\{y(-t)\}^{n^*} - nz(-t) = -n\gamma\{\bar{y}(t)\}^{n^*} + n\bar{z}(t),$$

i.e. the functions $\bar{y}, \bar{z}$ form a solution to (1.5). Hence the asymptotic behaviour of the solutions of (1.6) as $t\to\infty$ is the same as the one of the solutions of (1.5) as $t\to\infty$, and similar statement is true for $y(t)$, $z(t)$ as $t\to\infty$.

The differential equation (0.3). Since this differential equation plays important role it seems to be useful to formulate its asymptotic behaviour, too.

THEOREM 3.3. *Let $x(t)$ be a solution of* (0.3). *Then the following possibilities can occur:*

Case A: $\gamma<\gamma_0$, $\gamma\neq 0$. Either $x(t)=C_i t^{\alpha_i}$, $i=1, 2$ or

$$\lim_{t\to\infty} x(t)t^{-\alpha_1} = x_+, \quad \lim_{t\to+0} x(t)-t^{-\alpha_2} = x_0,$$

where C_1, C_2, x_0, x_+ are constants, depending on the solution $x(t)$;

Case B: $\gamma=\gamma_0$. Either $x(t)=Ct^{n/(n+1)}$ or

$$\lim_{t\to\infty} \frac{x(t)}{t^{n/(n+1)}(\log t)^{2/(n+1)}} x_+, \quad \lim_{t\to+0} \frac{x(t)}{t^{n/(n+1)}|\log t|^{2/(n+1)}} = x_0,$$

where C, x_+, x_0 are constants, depending on the solution $x(t)$;

Case C: $\gamma>\gamma_0$. The solutions are oscillatory in both cases: when $t\to\infty$ and $t\to+0$. Moreover the function $\bar{x}(s)=x(e^s)\exp\left(\frac{-n}{n+1}s\right)$ is periodic with the period τ given by (3.22).

PROOF. The statements above follows from Theorem 3.1 by observing that if $\bar{x}(s)$ is a solution to (0.4) then $x(t)=\bar{x}(\log t)$ is a solution of (0.3).

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DEFORMATIONS OF NILPOTENT KAC—MOODY ALGEBRAS

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The main goal of this article is the calculation of the one and two-dimensional cohomology of maximal nilpotent subalgebras of affine Kac-Moody type Lie algebras. This calculation allows us to classify the exterior derivations and deformations of the indicated algebras.

The article consists of two sections: The first section contains basic definitions and statements of the results, while the second one contains the proofs.[1]

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§ 1. Definitions and the statements of the results

1. Let $A=\|a_{ij}\|$ be an integer $n\times n$ matrix with $a_{11}=\ldots=a_{nn}=2$ and $a_{ij}\leqq 0$ for $i\neq j$. Suppose that A is symmetrisable, i.e. there exist positive numbers $\varrho_1, \ldots, \varrho_n$ such that the matrix $\|\varrho_i a_{ij}\|=\varrho A$ is symmetric. From now on $\varrho_1, \ldots, \varrho_n$ denote the minimal positive integers with the property above. Define the *Kac—Moody Lie algebra* $\mathfrak{g}^A$ *with the Cartan matrix* A as a complex Lie algebra with the generators $e_1, \ldots, e_n$, $f_1, \ldots, f_n$, $h_1, \ldots, h_n$ and the relations

$$[e_i, f_j]=\delta_{ij}h_j, \quad [h_i, h_j]=0,$$
$$[h_i, e_j]=a_{ij}e_j, \quad [h_i, f_j]=-a_{ij}f_j,$$
$$[\underbrace{e_i, [e_i, \ldots, [e_i,}_{-a_{ij}+1} e_j]\ldots]]=0, \quad [\underbrace{f_i, [f_i, \ldots, [f_i,}_{-a_{ij}+1} f_j]\ldots]]=0 \quad (i\neq j).$$

Define in $\mathfrak{g}^A$ a (multi-) gradation by

$$\deg h=(\underbrace{0, \ldots, 0}_{n}), \quad \deg e_i=(\underbrace{0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0}_{n}), \quad \deg f_i=(\underbrace{0, \ldots, 0, \overset{(i)}{-1}, 0, \ldots, 0}_{n}).$$

Here n is called the rank of $\mathfrak{g}^A$.

Suppose that A is nondecomposable, i.e. it can not become of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ under any simultaneus permutation of rows and columns.

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[1] For another proof of a part of these results see in [7].

[2] The work was done during my stay in Moscow.

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The Weyl group $W=W^A$ of $\mathfrak{g}^A$ is defined as the subgroup of $GL(n, \mathbf{Z})$, generated by the matrices $\sigma_i=E-A_i$, where E is the unity and in A_i the ith row coincides with the ith row of A, while the other rows are zeros. (The elements of W may be considered as transformations of the "weight lattice" $\mathbf{Z}^n$, which grades $\mathfrak{g}^A$.)

Remind some facts about the Kac—Moody Lie algebras (see [1], [2], [3]).

(i) $\mathfrak{g}^A=\mathfrak{n}_+(A)+\mathfrak{h}+\mathfrak{n}_-(A)$, where $\mathfrak{n}_+(A)$ and $\mathfrak{n}_-(A)$ are subalgebras of $\mathfrak{g}^A$, generated by $e_1, ..., e_n$ and $f_1, ..., f_n$ respectively, while $\mathfrak{h}$ is n-dimensional (commutative) subalgebra, spanned by $h_1, ..., h_n$.

(ii) The defining relation system for the generators $e_1, ..., e_n$ of $\mathfrak{n}_+(A)$ consists of

$$\underbrace{[e_i, [e_i, ..., [e_i}_{-a_{ij}+1}, e_j]...]] = 0.$$

The similar relations are true for $\mathfrak{n}_-(A)$.

It is natural to divide the Kac—Moody Lie algebras into three classes: algebras with positive definite matrix ϱA, algebras with nonnegative definite matrices of rank $n-1$ and the remaining algebras.

(iii) The class of algebras $\mathfrak{g}^A$ with positive definite matrices ϱA coincides with the class of simple finite-dimensional complex Lie algebras.

In this paper we restrict ourselves to the so called *affine algebras* of the second type. The nondecomposable matrices corresponding to these algebras are listed in Tables 1 and 2.

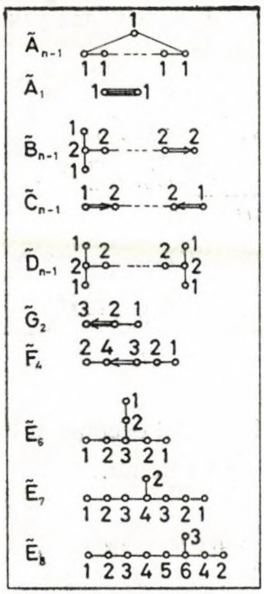


Table 1

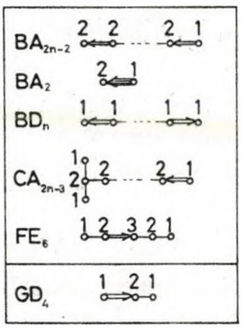


Table 2

The vertices in Tables 1—2 correspond to the rows of A. The ith vertice is joined with the jth one by $a_{ij}a_{ji}$ edges; if $|a_{ij}|>|a_{ji}|$, these edges have an arrow, pointing towards the ith vertice. Numerical marks are the coefficients of linear dependence between the corresponding columns of the Cartan matrix A. Fix for these numbers the notation $\omega_1, \ldots, \omega_n$.

(iv) Let A be a positive definite Cartan matrix, corresponding to certain Dynkin diagram and $\tilde{A}$ be the Cartan matrix of the extended Dynkin diagram from Table 1. Then $\mathfrak{g}^{\tilde{A}}$ is the central extension of the *current algebra* $\mathfrak{g}^A \otimes \mathbf{C}[t, t^{-1}]$.

By this the canonical generators $e_1, \ldots, e_n$ of $\mathfrak{g}^{\tilde{A}}$ correspond to the products $e_1 \otimes 1, \ldots, e_{n-1} \otimes 1$, $f \otimes t$, where $e_1, \ldots, e_{n-1}$ are canonical generators of $\mathfrak{g}^A$ and f is the root vector of $\mathfrak{g}^A$, corresponding to the negative root of maximal length. Moreover, for $(m_1, \ldots, m_n) \neq (0, \ldots, 0)$

$$\mathfrak{g}^{\tilde{A}}_{(m_1, \ldots, m_n)} = \mathfrak{g}^{A}_{(m_1 - m_n\alpha_1, \ldots, m_{n-1} - m_n\alpha_{n-1})} \otimes t^{m_n}$$

where $(\alpha_1, \ldots, \alpha_{n-1})$ is the weight of f.

We notice also, that $\mathfrak{n}_+(\tilde{A}) = (\mathfrak{n}_+(A) \otimes 1) \oplus (\bigoplus_{m>0} (\mathfrak{g}^A \otimes t^m))$ and similar is true for $\mathfrak{n}_-(A)$.

Algebras, corresponding to matrices from Table 2 are defined by means of finite order exterior automorphisms of finite-dimensional simple algebras. Namely, if $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ is such an automorphism and l is its order, than we define $\mathfrak{g}_\varphi$ as the subalgebra $\bigoplus_{\lambda=-\infty}^{\infty} \mathfrak{G}(\lambda) \otimes t^\lambda$ of $\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$, where $\mathfrak{g}(\lambda)$ is the root subspace of the automorphism φ, corresponding to the eigenvalue $e^{2\pi i\lambda/l}$.

(v) The algebras from Table 2 are central extensions of the algebras $\mathfrak{g}_\varphi$. Namely, the first 5 cases correspond to two-order automorphisms, while the last one to three-order automorphism.

The homology of $\mathfrak{n}_+(A)$ with trivial coefficients is known [4], [5]. Let

$$Q_A(x_1, \ldots, x_n) = -\frac{1}{2} \sum \varrho_i a_{ij} x_i x_j + \sum \varrho_i x_i.$$

(vi) If $Q_A(m_1, \ldots, m_n) \neq 0$, then

$$H_k^{(m_1, \ldots, m_n)}(\mathfrak{n}_+(A)) = 0$$

for arbitrary k. If $Q_A(m_1, \ldots, m_n) = 0$, then there is a unique $k(m_1, \ldots, m_n)$, for which

$$H_k^{(m_1, \ldots, m_n)}(\mathfrak{n}_+(A)) = \begin{cases} \mathbf{C} & \text{for } k = k(m_1, \ldots, m_n), \\ 0 & \text{for the others.} \end{cases}$$

For the practical computation of the number $k(m_1, \ldots, m_n)$ it is convenient to use the transformations $s_i\colon \mathbf{Z}^n \to \mathbf{Z}^n$, defined by

$$s_i(m) = \sigma_i(m) + (0, \ldots, 0, \overset{(i)}{\varrho_i}, \ldots, 0).$$

(The transformations s_i also define an action of W in $\mathbf{Z}^n$.) It is easy to show, that $Q_A \circ s_i = Q_A$ and that an arbitrary sequence $(m_1, \ldots, m_n)$ with $Q_A(m_1, \ldots, m_n) = 0$

may be obtained from $(0, \ldots, 0)$ by means of finite number of transformations s_i. The minimal number of these transformations is $k(m_1, \ldots, m_n)$.

In particular,

$$H_0(\mathfrak{n}_+(A)) = H_0^{(0,\ldots,0)}(\mathfrak{n}_+(A)) = \mathbf{C},$$

$$H_1(\mathfrak{n}_+(A)) = H_1^{(1,0,\ldots,0)}(\mathfrak{n}_+(A)) \oplus \ldots \oplus H_0^{(0,\ldots,0,1)}(\mathfrak{n}_+(A)) = \mathbf{C}^n.$$

2. Let A be a Cartan matrix from Tables 1, 2. The main result of this paper is the computation of one- and two-dimensional cohomologies of $\mathfrak{n}_+(A)$ with coefficients in the adjoint representation. Remind that the computation of one-dimensional cohomology is equivalent to the classification of exterior derivations, and it is that language, in which we formulate here the result. The calculation of two-dimensional cohomology allows us to classify the deformations of the considered algebras.

THEOREM 1. *The next derivations form a basis in the space of exterior derivations of* $\mathfrak{n}_+(A)$:

$$\bar{h}_i\colon\ g \to [h_i, g], \quad i = 1, \ldots, n-1;$$

$$\tau_i\colon\ t^{il+1}\frac{d}{dt}, \quad i = 0, 1, 2, \ldots.$$

Here l and t have the same sense as in (iv) and (v) of subsection 1.

We describe now some concrete deformations of $\mathfrak{n}_+(A)$.

1°. Let $\alpha \in H^1(\mathfrak{n}_+(A);\ \mathfrak{n}_+(A))$, $\beta \in H^1(\mathfrak{n}_+(A))$. The element α corresponds to the right extension

$$0 \to \mathfrak{n}_+(A) \to \tilde{\mathfrak{n}}_+(A) \to \mathbf{C} \to 0$$

(the elements of $H^1(\mathfrak{n}_+(A);\ \mathfrak{n}_+(A))$ may be interpreted not only as exterior derivations, but also as right extensions — see [5]), β to a functional $\varphi\colon \mathfrak{n}_+(A) \to \mathbf{C}$. For $t \in \mathbf{C}$ denote η_t the embedding $\mathfrak{n}_+(A) \to \tilde{\mathfrak{n}}_+(A) \cong \mathfrak{n}_+(A) \oplus \mathbf{C}$ defined by $\eta_t(g) = (g, t\varphi(g))$. It may be easily checked that $\eta_t(\mathfrak{n}_+(A))$ is a subalgebra of $\tilde{\mathfrak{n}}_+(A)$, that this subalgebra is connected with $\mathfrak{n}_+(A)$ by a natural linear isomorphism, and that for $t=0$ this isomorphism is compatible with the bracket operation. Thus we have a deformation of $\mathfrak{n}_+(A)$. The corresponding infinitesimal deformation is evidently the product

$$\alpha\beta \in H^2(\mathfrak{n}_+(A);\ \mathfrak{n}_+(A)).$$

(By all means, this construction may be applied to an arbitrary Lie algebra.)

2°. Let $1 \leq i \leq n$. The algebra $\mathfrak{n}_+(A)$ deforms inside $\mathfrak{g}^A$. The deformed algebra is spanned by the spaces $\mathfrak{g}^A_{(m_1, \ldots, m_n)}$ with $(m_1, \ldots, m_n) \neq (0, \ldots, 0, \overset{(i)}{1}, \ldots, 0)$ and by the vector $e_i + tf_i$, where t is a parameter. (Informally speaking, e_i deforms into $e_i + tf_i$, while the other additive generators of $\mathfrak{n}_+(A)$ do not change.)

The number of such deformations is equal to the rank of $\mathfrak{g}^A$.

3°. Let $1 \leq i, j \leq n$; consider the entry $a_{ij} = -1$ and if $a_{ij} = a_{ji}$, then $i<j$. The algebra $\mathfrak{n}_+(A)$ deforms again inside $\mathfrak{g}^A$. The deformed algebra is generated by the spaces $\mathfrak{g}^A_{(m_1, \ldots, m_n)}$ with

$$(m_1, \ldots, m_n) \neq (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0), \quad (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0, \overset{(j)}{1}, 0, \ldots, 0)$$

and the vectors e_i+tf_j and $[e_i, e_j]-th_j$. (Informally speaking, e_i and $[e_i, e_j]$ deform into e_i+tf_j and $[e_i, e_j]-th_j$, while the other additive generators of $\mathfrak{n}_+(A)$ are not deformed.)

The number of this type deformations is equal to the number of nonzero pairs (a_{ij}, a_{ji}) with $i\neq j$; this number we denote below by p.

Remark, that the equality $a_{ij}=-1$ is necessary for the verification of the fact that the deformed algebras are closed under the bracket and that with the only exception of the case $\tilde{A}_1$, at least one of two nontrivial nondiagonal entries of the Cartan matrix a_{ij}, a_{ji} is equal to -1. This specific property of $\tilde{A}_1$ comples us to consider the case $\mathfrak{n}_+(\tilde{A}_1)$ separately.

THEOREM 2. *Suppose that* $A\neq\tilde{A}_1$. *Then*

(i) *All the homogeneous infinitesimal deformations of* $\mathfrak{n}_+(A)$ *may be extended to its real deformations.*

(ii) *The space of infinitesimal deformations,* $H^2(\mathfrak{n}_+(A); \mathfrak{n}_+(A))$ *is spanned by deformations, corresponding to the above types* 1°, 2°, 3°. *In other words, the mapping*

$$\psi\colon \left[H^1(\mathfrak{n}_+(A);\ \mathfrak{n}_-(A))\otimes H^1(\mathfrak{n}_+(A))\right]\oplus\mathbf{C}^n\oplus\mathbf{C}^p \to H^2(\mathfrak{n}_+(A);\ \mathfrak{n}_+(A))$$

defined by the infinitesimal deformations listed above is epimorphism.

(iii) *The kernel of the mapping* ψ *is contained in*

$$H^1(\mathfrak{n}_+(A);\ \mathfrak{n}_+(A))\otimes H^1(\mathfrak{n}_+(A))$$

and its dimension is n. It is spanned by the elements $\varkappa_1, \dots, \varkappa_n$ *defined as follows. Let* $1\leq i\leq n$. *Choose the numbers* $\beta_1, \dots, \beta_{n-1}$ *so that* $\sum_1^{n-1}\beta_j a_{kj}=1$ *for* $k\neq i$ *(such numbers can be found, because the rank of the Cartan matrix with one column removed equals to* $n-1$*). Then*

$$\varkappa_i = \bar{h}_i\otimes\bar{e}_i, \qquad i = 1, \dots, n-1,$$

$$\varkappa_n = \Big(\sum_1^{n-1}\beta_j\bar{h}_j\Big)\otimes\bar{e}_i,$$

where $\bar{e}_i$ *is the class of the cocycle from* $\mathbf{C}^1(\mathfrak{n}_+(A))$, *assigning 1 to* e_i *and* 0 *to other* e_i*'s, while the* $\bar{h}_j$*-s were introduced in Theorem 1.*

Now turn to the case $A=\tilde{A}_1$. In this case the Cartan matrix is $\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix}$, and this excludes the possibility of applying the construction 3°. Mention also that it is not true for this case that all infinitesimal deformations may be extended to real deformations.

THEOREM 3. *(i) Infinitesimal deformations, corresponding to deformations of type* 1°, 2° *span in* $H^2(\mathfrak{n}_+(\tilde{A}_1); \mathfrak{n}_+(\tilde{A}_1))$ *a codimension 2 subspace. The complementary subspace is spanned by elements from* $H^2_{(-1,-2)}$ *and* $H^2_{(-2,-1)}$ *respectively. These elements can not be extended to the deformation of* $\mathfrak{n}_+(\tilde{A}_1)$. *(Cocycles representing these two classes are given in p. 2 §2 (section 2))*

(ii) *The kernel of the mapping*

$$[H^1(\mathfrak{n}_+(\tilde{A}_1);\ \mathfrak{n}_+(\tilde{A}_1))\otimes H^1(\mathfrak{n}_+(\tilde{A}_1))]\oplus \mathbf{C}^2 \to H^2(\mathfrak{n}_+(\tilde{A}_1);\ \mathfrak{n}_+(\tilde{A}_1))$$

may be described just as the kernel of ψ *in the part (iii) of Theorem* 2.

§ 2. Proofs

1. Let $\mathfrak{g}=\bigoplus_{i>0}\mathfrak{g}_i$ be a nilpotent graded Lie algebra and $B=\oplus B_j$ be a graded $\mathfrak{g}$-module. The space $C_k^{(m)}(\mathfrak{g};B)$ is spanned by "monomials", i.e. by the chains

$$g_1\wedge\ldots\wedge g_k\otimes b,\quad \text{where}\quad g_s\in\mathfrak{g}_{i_s},\quad b\in B_j,\ i_1+\ldots+i_k+j=m.$$

Denote by $F_pC_k^{(m)}(\mathfrak{g};B)$ the subspace of $C_k^{(m)}(\mathfrak{g};B)$, generated by monomials with $i_1+\ldots+i_k\leqq p$. Evidently, $\{F_p\}$ is a decreasing filtration in $C_*^{(m)}(\mathfrak{g};B)$. The spectral sequence corresponding to this filtration we will call *Feigin—Fuchs spectral sequence* and denote it by $\mathscr{E}(\mathfrak{g},B,m)$. Here $E_{p,q}^0=C_{p+q}^{(p)}(\mathfrak{g};B_{m-p})$, where B_{m-p} is considered as trivial $\mathfrak{g}$-module and $d_{p,q}^0$ is the differential

$$d_{p+q}\colon C_{p+q}^{(p)}(\mathfrak{g};B_{m-p})\to C_{p+q-1}^{(p)}(\mathfrak{g};B_{m-p})$$

hence

$$E_{p,q}^1=H_{p+q}^{(p)}(\mathfrak{g};B_{m-p})=H_{p+q}^{(p)}(\mathfrak{g})\otimes B_{m-p}.$$

For the algebra L_1 of polynomial vector fields on the line with trivial 1-jets in the point 0 this spectral sequence was considered in [6]. In the cases interesting for us the algebra $\mathfrak{g}$ has multigradation $\mathfrak{g}=\bigoplus_{(i_1,\ldots,i_k)>(0,\ldots,0)}\mathfrak{g}_{(i_1,\ldots,i_k)}$. In this case the spectral sequence $\mathscr{E}(\mathfrak{g},B,m)$ decomposes into the sum of spectral sequences $\mathscr{E}(\mathfrak{g},B,m_1,\ldots,m_k)$, $m_1+\ldots+m_k=m$. The initial term of the last spectral sequence is given by the formula

$$E_{p,q}^1=\bigoplus_{p_1+\ldots+p_k=p} H_{p+q}^{(p_1,\ldots,p_k)}(\mathfrak{g})\otimes B_{m_1-p_1,\ldots,m_k-p_k}.$$

We apply the above spectral sequence to the computation of the one- and two-dimensional homology of the algebra $\mathfrak{n}_+(A)$ with coefficients in the coadjoint representation $\mathfrak{n}_+(A)'$. (This is equivalent to the computation of the cohomology of $\mathfrak{n}_+(A)$ with coefficients in the adjoint representation.) For each of the matrices from Tables 1, 2 the terms and differentials of the spectral sequence $\mathscr{E}(\mathfrak{n}_+(A),\mathfrak{n}_+(A)',m)$ may be explicitly determined, and this leads to the calculation of the indicated homology. All computations are similar, and we shall give details only for the cases $\tilde{A}_{n-1}$ and BA_2.

2. Let us begin with $\tilde{A}_1$. There is a convenient explicit description of the quotient algebra of $\mathfrak{g}^{\tilde{A}_1}$ by its (one-dimensional) centre. Namely, it contains an additive basis ε_i $(i\in\mathbf{Z})$ such that

$$[\varepsilon_i,\varepsilon_j]=\alpha_{ij}\varepsilon_{i+j},\quad\text{where}\quad \alpha_{ij}\begin{cases}=-1,0,1,\\ \equiv(j-i)\bmod 3.\end{cases}$$

(In this notation $\varepsilon_1,\varepsilon_2,\varepsilon_{-1},\varepsilon_{-2}$ correspond to e_1,e_2,f_1,f_2, defined in §1.) (Bi-)

gradation in this basis is given by

$$\deg \varepsilon_{3m} = (m, m), \quad \deg \varepsilon_{3m-1} = (m, m+1), \quad \deg \varepsilon_{3m+1} = (m, m-1).$$

The subspace $\mathfrak{n}_+(\tilde{A}_1)$ of $\mathfrak{g}^{\tilde{A}_1}$ is spanned by ε_i, where $i>0$.

According to (vi) in §1, for $k>0$

$$H_k(\mathfrak{n}_+(\tilde{A}_1)) = H_k^{((k(k-1))/2, (k(k+1))/2)} \oplus H_k^{((k(k+1))/2, (k(k-1))/2)} = \mathbf{C} \oplus \mathbf{C}$$

(see Fig. 1) moreover, nontrivial elements of the spaces

$$H_k^{((k(k-1))/2, (k(k+1))/2)}, \; H_k^{((k(k+1))/2, (k(k-1))/2)}$$

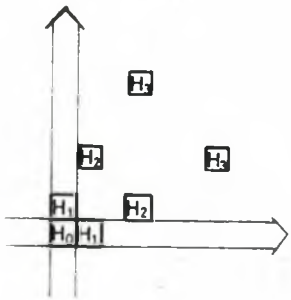


Fig. 1

are represented by cycles $\varepsilon_1 \wedge \varepsilon_4 \wedge \ldots \wedge \varepsilon_{3k-2}$, $\varepsilon_2 \wedge \ldots \wedge \varepsilon_5 \wedge \ldots \wedge \varepsilon_{3k-1}$ (see [5]). Since

$$\dim (\mathfrak{n}_+(\tilde{A}_1))_{(m_1, m_2)} = \begin{cases} 1 & \text{if} \quad |m_2 - m_1| \leqq 1, \; m_2 + m_1 > 0, \\ 0 & \text{in all other cases} \end{cases}$$

(see Fig. 2), in the spectral sequence

Fig. 2

$$\mathscr{E}(m_1, m_2) = \mathscr{E}(\mathfrak{n}_+(\tilde{A}_1), \mathfrak{n}_+(\tilde{A}_1)', m_1, m_2)$$

$$\dim E_k^1 = \begin{cases} 2 & \text{if} \quad k = 1, \quad m_1 = m_2 \leqq 0, \\ 1 & \text{if} \quad k-1 \leqq |m_2 - m_1| \leqq k+1, \quad m_1 + m_2 < k^2, \\ 0 & \text{in all other cases.} \end{cases}$$

(See Fig. 3; the circles and points show the degrees of the homology with trivial coefficients and the degrees of the nontrivial spaces E_k^1, respectively.)

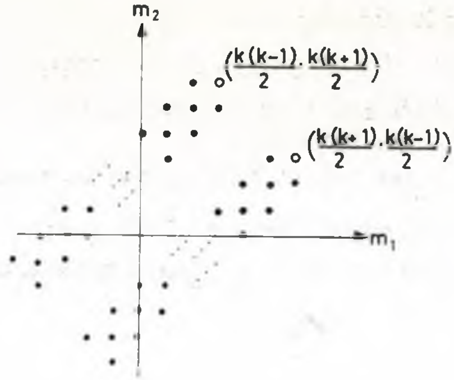


Fig. 3

So, the term E^1 of the spectral sequence $\mathscr{E}(m_1, m_2)$ is constructed in the following way. Let $l=|m_2-m_1|$ and $m=\min(m_1, m_2)$. If $l>0$, then the dimensions of the spaces E_k^1 are given by the table

$k=$ ...	$l-2$	$l-1$	l	$l+1$	$l+2$	...	
...	0	1	1	1	0	...	for $m \leqq \frac{l^2-3l}{2}$,
...	0	0	1	1	0	...	for $\frac{l^2-3l}{2} < m < \frac{l^2-l}{2}$,
...	0	0	0	1	0	...	for $\frac{l^2-l}{2} \leqq m \leqq \frac{l^2+l}{2}$,
...	0	0	0	0	0	...	for $\frac{l^2+l}{2} < m$,

and if $l=0$, then by the table

$k=0$	1	2	...	
1	2	0	...	for $m<0$
0	2	0	...	for $m=0$
0	0	0	...	for $m>0$.

LEMMA. *The non-trivial differentials d_k^1 are the following ones:*

$$d_l^1\colon E_l^1 \to E_{l-1}^1, \quad \text{if} \quad l \neq 0, \quad m \leqq \frac{l^2-3l}{2},$$

$$d_1^1\colon E_1^1 \to E_0^1, \quad \text{if} \quad l = 0, \quad m < 0;$$

the differentials d_k^r with $r>1$ are all trivial.

From this lemma it follows

PROPOSITION.
$$H_0(\mathfrak{n}_+(\tilde{A}_1);\ \mathfrak{n}_+(\tilde{A}_1)')=0;$$

$$\dim H_1^{(m_1,m_2)}(\mathfrak{n}_+(\tilde{A}_1);\ \mathfrak{n}_+(\tilde{A}_1)')=\begin{cases}2 & \textit{if} \quad m_1=m_2=0,\\ 1 & \textit{if} \quad m_1=m_2<0,\\ 0 & \textit{in the other cases;}\end{cases}$$

if $k>1$, *then*
$$\dim H_k^{(m_1,m_2)}(\mathfrak{n}_+(\tilde{A}_1);\ \mathfrak{n}_+(\tilde{A}_1)')=$$
$$=\begin{cases}1 & \textit{if} \quad |m_1-m_2|=k-1,\ m_1+m_2<k^2-1\\ & \textit{and if} \quad |m_1-m_2|=k,\ (k-1)^2<m_1+m_2\leqq k^2-2,\\ 0 & \textit{in the other cases.}\end{cases}$$

(See Fig. 4, on which there are shown the weights of one- and two-dimensional homologies.)

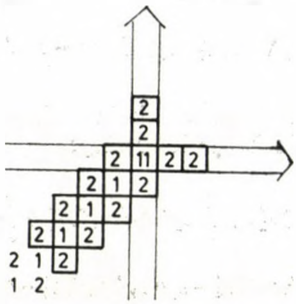


Fig. 4

Lemma may be proved by performing a straight but not particularly short calculation. Since we are interested only in homology of dimension one and two, we give the proof here only for the cases $k\leqq 2$. We have to show that the differentials

(i) d_1^1 for $m_2=m_1<0$,

(ii) d_1^1 for $|m_2-m_1|=1$, $\min(m_1,m_2)\leqq -1$,

(iii) d_2^1 for $|m_2-m_1|=2$, $\min(m_1,m_2)\leqq -1$,

(iv) d_3^1 for $|m_2-m_1|=3$, $\min(m_1,m_2)\leqq 0$

are non-trivial and

(v) d_3^1 for $m_1=2,\ m_2=0$ and $m_1=0,\ m_2=2$

is trivial. Since the roles of m_1 and m_2 are symmetric, we may consider only the case $m_1 \leqq m_2$. The differential d_k^1 in the spectral sequence $\mathscr{E}(m_1, m_2)$ is non-trivial if there exists a chain

$$c \in C_k^{(m_1, m_2)}(\mathfrak{n}_+(\tilde{A}_1), \mathfrak{n}_+(\tilde{A}_1)') \quad \text{such that}$$

$$c = \varepsilon_1 \wedge \ldots \wedge \varepsilon_{3k-2} \otimes \varepsilon_1' + \ldots$$

$$\partial c = \mu \varepsilon_1 \wedge \ldots \wedge \varepsilon_{3k-5} \otimes \varepsilon_j' + \ldots$$

where $\mu \neq 0$ and dots in the general case stand for terms of smaller filtration. We find such chains for the cases (i)—(iv), putting $m = -m_2$.

(i) $\quad c = \varepsilon_1 \otimes \varepsilon_{3m+1}'; \ \partial c = \varepsilon_{3m}',$

(ii) $\quad c = \varepsilon_1 \otimes \varepsilon_{3m}'; \ \partial c = -\varepsilon_{3m-1}',$

(iii) $\quad c = \varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_{3m}' - \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_{3m-1}'; \ \partial c = 2\varepsilon_1 \otimes \varepsilon_{3m-4}',$

(iv) $$c = \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon_{3m}' - \frac{1}{2} \varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 \otimes \varepsilon_{3m-1}' - \frac{1}{2} \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6 \wedge \otimes \varepsilon_{3m-1}' + \\ + \frac{3}{2} \varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 \otimes \varepsilon_{3m-4}'; \ \partial c = -3\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_{3m-7}'.$$

The differential d_k^1 is trivial, if there is a chain c of the above form, for which $\partial c = 0$. For the case (v) such a chain is the following:

(v) $$c = \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon_{10}' + \frac{1}{2} \varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 \otimes \varepsilon_9' + \frac{1}{2} \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6 \otimes \varepsilon_9' - \\ - \frac{1}{2} \varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_6 \otimes \varepsilon_8' - \varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 \otimes \varepsilon_6' - \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 \otimes \varepsilon_5'.$$

Now we describe cycles, representing bases in $H_k(\mathfrak{n}_+(A_1); \mathfrak{n}_+(A_1)')$ for $k = 1, 2$.

In $C_1^{(0,0)}$: $\varepsilon_1 \otimes \varepsilon_1', \ \varepsilon_2 \otimes \varepsilon_2'$.

In $C_1^{(m,m)}$, $m < 0$: $\varepsilon_1 \otimes \varepsilon_{-3m+1}' + \varepsilon_2 \otimes \varepsilon_{-3m+2}'$.

In $C_2^{(0,2)}$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_3' - \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_2'$.

In $C_2^{(1,2)}$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_1'$.

In $C_2^{(m,m+1)}$, $m \leqq 0$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_{-3m+4}' + \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_{-3m+3}' + \varepsilon_1 \wedge \varepsilon_2 \otimes \varepsilon_{-3m+2}'$.

Cycles in $C_2^{(2,0)}, C_2^{(m+1,m)}$ are given similarly, by substituting $\varepsilon_1 \leftrightarrow \varepsilon_2$, $\varepsilon_4 \leftrightarrow \varepsilon_5, \ldots$.

Since $\dim H^k_{(m_1, m_2)} = \dim H_k^{(-m_1, -m_2)}$, the cohomology needed for us is completely computed. It is easy to see that the above result agrees with the corresponding parts of Theorems 1, 3.

Cocycles, representing basis elements of the cohomology spaces are indicated in the next table.

weight	cocycle
$(0, -2)$	$(\varepsilon_1, \varepsilon_{3j}) \mapsto \varepsilon_{3j-1}$, $(\varepsilon_1, \varepsilon_{3j+1}) \mapsto -\varepsilon_{3j}$ for $j>0$, the rest $\mapsto 0$.
$(-2, 0)$	$(\varepsilon_2, \varepsilon_{3j}) \mapsto \varepsilon_{3j-2}$, $(\varepsilon_2, \varepsilon_{3j+2}) \mapsto -\varepsilon_{3j}$ for $j>0$, the rest $\mapsto 0$.
$(m, m-1)$, $m \geqq 0$	$(\varepsilon_1, \varepsilon_j) \mapsto j\varepsilon_{j+3m}$ for $j \neq 1$, the rest $\mapsto 0$.
$(m-1, m)$, $m \geqq 0$	$(\varepsilon_2, \varepsilon_j) \mapsto j\varepsilon_{j+3m}$ for $j \neq 1$, the rest $\mapsto 0$.
$(-1, -2)$	$(\varepsilon_1, \varepsilon_4) \mapsto 9\varepsilon_1$, $(\varepsilon_1, \varepsilon_j) \mapsto j\varepsilon_{j-3}$ for $j \geqq 5$, $(\varepsilon_3, \varepsilon_{3j}) \mapsto 2\varepsilon_{3j-1}$, $(\varepsilon_3, \varepsilon_{3j-2}) \mapsto -2\varepsilon_{3j-3}$ for $j \geqq 2$, $(\varepsilon_4, \varepsilon_{3j-1}) \mapsto 5\varepsilon_{3j-1}$, $(\varepsilon_4, \varepsilon_{3j+4}) \mapsto -5\varepsilon_{3j+4}$ for $j \geqq 1$, the rest $\mapsto 0$.
$(-2, -1)$	$(\varepsilon_2, \varepsilon_5) \mapsto 9\varepsilon_2$, $(\varepsilon_2, \varepsilon_j) \mapsto j\varepsilon_{j-3}$ for $j = 4, 6, 7, \ldots$, $(\varepsilon_3, \varepsilon_{3j}) \mapsto \varepsilon_{3j-2}$, $(\varepsilon_3, \varepsilon_{3j+2}) \mapsto -\varepsilon_{3j}$ for $j \geqq 1$, $(\varepsilon_5, \varepsilon_{3j-2}) \mapsto 4\varepsilon_{3j-2}$, $(\varepsilon_5, \varepsilon_{3j+5}) \mapsto -4\varepsilon_{3j+5}$ for $j \geqq 1$, the rest $\mapsto 0$.

We can easily verify that the indicated cochains are really cocycles and they do not vanish on the above cycles.

It remained to show that infinitesimal deformations, determined by two-dimensional cocycles of weight $(0, -2)$, $(-2, 0)$ and $(m+1, m)$, $(m, m+1)$ with $m \geqq -1$ can be extended to real deformations, while infinitesimal deformations of weight $(-1, -2)$, $(-2, -1)$ can not. The extensions in question are explicitly given in p. 2 §1. On the other hand, the cocycles of weight $(-1, -2)$, $(-2, -1)$ have nontrivial squares; for instance the first of them takes the value 135 at the cycle

$$\varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon_4' + \frac{1}{2}(\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 + \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6) \otimes \varepsilon_3' - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_6 \otimes \varepsilon_2'.$$

3. Let us now consider the case BA_2. The corresponding Cartan matrix is $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. The quotient algebra of $\mathfrak{g}^{BA_2}$ by its centre has explicite description. Namely, it contains an additive basis ε_i $(i \in \mathbf{Z})$ with $[\varepsilon_i, \varepsilon_j] = \alpha_{ij}\varepsilon_{i+j}$, where α_{ij} depends only on i, j mod 8, $\alpha_{i,j} + \alpha_{i',j'} = 0$ if $i+i'$ and $j+j'$ are multiples of 8, and for $0 \leqq i, j \leqq 7$ it is given in the following table:

$i \bmod 8$ \ $j \bmod 8$	1	2	3	4	5	6	7
0	1	−2	−1	0	1	2	−1
1		1	−1	3	−2	0	1
2			0	0	1	−1	
3				−3	−1		

Gradation is given by formulas

$$\deg \varepsilon_{8m} = (2m, 4m), \quad \deg \varepsilon_{8m+1} = (2m, 4m+1), \quad \deg \varepsilon_{8m+2} = (2m+1, 4m),$$

$$\deg \varepsilon_{8m+3} = (2m+1, 4m+1), \quad \deg \varepsilon_{8m+4} = (2m+1, 4m+2),$$

$$\deg \varepsilon_{8m+5} = (2m+1, 4m+3)$$

$$\deg \varepsilon_{8m+6} = (2m+1, 4m+4), \quad \deg \varepsilon_{8m+7} = (2m+2, 4m+3).$$

The subalgebra $\mathfrak{n}_+(BA_2)$ is spanned by ε_i with $i>0$.

By (vi) from §1 for $k>0$

$$H_{2k-1}(\mathfrak{n}_+(BA_2)) = H_{2k-1}^{((3k^2-k)/2,\, 3k^2-4k+1)} \oplus H_{2k-1}^{((3k^2-5k+2)/2,\, 3k^2-2k)} = \mathbf{C} \oplus \mathbf{C},$$

$$H_{2k}(\mathfrak{n}_+(BA_2)) = H_{2k}^{((3k^2+k)/2,\, 3k^2-2k)} \oplus H_{2k}^{((3k^2-k)/2,\, 3k^2+2k)} = \mathbf{C} + \mathbf{C}$$

(see Fig. 5) and nontrivial elements of the homology in question are represented with the cycles

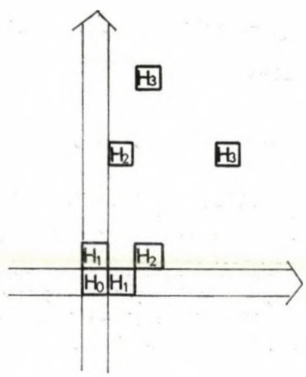


Fig. 5

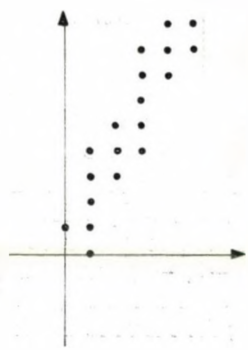


Fig. 6

$$(\varepsilon_2 \wedge \varepsilon_{10} \wedge \ldots \wedge \varepsilon_{8k-6}) \wedge (\varepsilon_3 \wedge \varepsilon_7 \wedge \ldots \wedge \varepsilon_{4k-5}), \quad (\varepsilon_6 \wedge \varepsilon_{14} \wedge \ldots \wedge \varepsilon_{8k-10}) \wedge (\varepsilon_1 \wedge \varepsilon_5 \wedge \ldots \wedge \varepsilon_{4k-3}),$$

$$(\varepsilon_2 \wedge \varepsilon_{10} \wedge \ldots \wedge \varepsilon_{8k-6}) \wedge (\varepsilon_3 \wedge \varepsilon_7 \wedge \ldots \wedge \varepsilon_{4k-1}), \quad (\varepsilon_6 \wedge \varepsilon_{14} \wedge \ldots \wedge \varepsilon_{8k-2}) \wedge (\varepsilon_1 \wedge \varepsilon_5 \wedge \ldots \wedge \varepsilon_{4k-3}).$$

The dimensions of the spaces $\mathfrak{n}_+(BA_2)_{(m_1, m_2)}$ equal to 0 and 1; the points (m_1, m_2) corresponding to spaces of dimension 1 are shown on Fig. 6.

In this way we can determine the dimensions of the spaces, forming the initial terms of the spectral sequences $\mathscr{E}_{(m_1,m_2)} = \mathscr{E}(\mathfrak{n}_+(BA_2), \mathfrak{n}_+(BA_2)', m_1, m_2)$. We

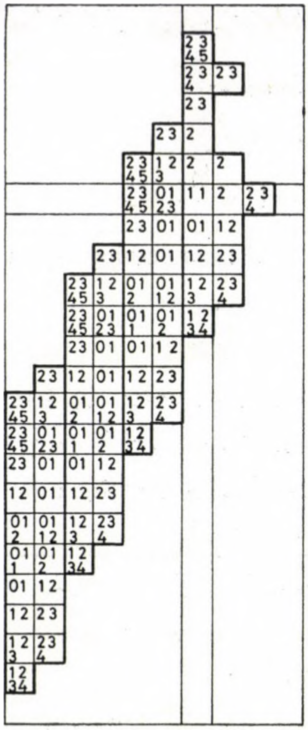


Fig. 7

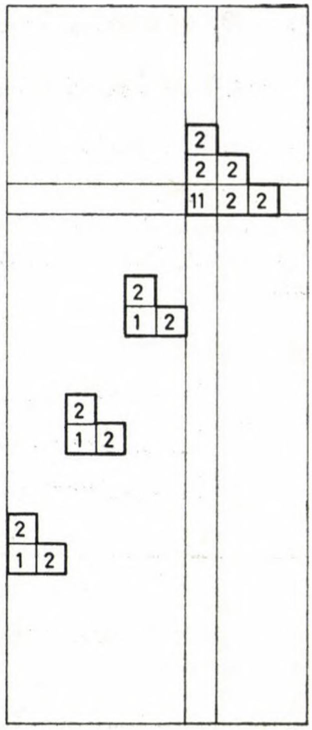


Fig. 8

restrict ourselves to (m_1, m_2) such that the space $\bigoplus_{k=0}^{2} E_k^1$ is nontrivial. These (m_1, m_2) are represented by small circles on Fig. 7. On this figure the cell (m_1, m_2) contains as many k's as the dimension of E_k^1 (for instance, in the spectral sequence $\mathcal{E}(-1, -3)$ the dimensions of E_k^1 are 1, 2, 1, 0, 0, ...). We remark, that the left half-plane on Fig. 7 is periodic with period 2 on the abscissa axis and with period 4 on the ordinate axis. The action of the differentials in these spectral sequences may be calculated in the same way as in p. 2. The result of the computations is shown on Fig. 8: the number of the 1's and 2's in the cell (m_1, m_2) equals to the dimension of $H_1^{(m_1, m_2)}$ and $H_2^{(m_1, m_2)}$, respectively.

Now we describe the cycles, which represent the basis in $H_k(BA_2, BA_2')$, $k=1, 2$.

In $C_1^{(0,0)}$: $\varepsilon_1 \otimes \varepsilon_1'$, $\varepsilon_2 \otimes \varepsilon_2'$.

In $C_1^{(2m,4m)}$, $m<0$: $2\varepsilon_1 \otimes \varepsilon'_{-8m+1} + \varepsilon_2 \otimes \varepsilon'_{-8m+2}$.

In $C_2^{(0,2)}$: $\varepsilon_1 \wedge \varepsilon_6 \otimes \varepsilon_5' + \frac{2}{3} \varepsilon_1 \wedge \varepsilon_5 \otimes \varepsilon_4' + \frac{2}{9} \varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_3' - \frac{2}{9} \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_2'$.

In $C_2^{(1,1)}$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon_2'$.

In $C_2^{(2,0)}$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon_1'$.

In $C_2^{(2m,4m+1)}$, $m\leqq 0$: $\varepsilon_1\wedge\varepsilon_6\otimes\varepsilon'_{-8m+6}-\varepsilon_1\wedge\varepsilon_5\otimes\varepsilon'_{-8m+5}+\varepsilon_1\wedge\varepsilon_4\otimes\varepsilon'_{-8m+4}-$

$$-\varepsilon_1\wedge\varepsilon_3\otimes\varepsilon'_{-8m+3}+\varepsilon_1\wedge\varepsilon_2\otimes\varepsilon'_{-8m+2}.$$

In $C_2^{(2m+1,4m)}$, $m\leqq 0$: $\varepsilon_2\wedge\varepsilon_3\otimes\varepsilon'_{-8m+3}-\varepsilon_2\wedge\varepsilon_1\otimes\varepsilon'_{-8m+1}$.

So, the cohomology needed for us is computed. It is easy to see, that the above result agrees with Theorems 1, 2.

Cocycles, representing basis elements of the cohomology spaces are indicated in the next table.

weight	cocycle
$(0,-2)$	$(\varepsilon_1,\varepsilon_{8j})\mapsto-\varepsilon_{8j-1},\ (\varepsilon_1,\varepsilon_{8j+1})\mapsto\varepsilon_{8j}\quad (j\geqq 1)$ $\left.\begin{array}{l}(\varepsilon_1,\varepsilon_{8j+3})\mapsto-2\varepsilon_{8j+2},\ (\varepsilon_1,\varepsilon_{8j+4})\mapsto 3\varepsilon_{8j+3}\\ (\varepsilon_1,\varepsilon_{8j+5})\mapsto-\varepsilon_{8j+4},\ (\varepsilon_1,\varepsilon_{8j+6})\mapsto\varepsilon_{8j+5}\end{array}\right\}\ j\geqq 0$ the rest $\mapsto 0$
$(-1,-1)$	$\left.\begin{array}{c}(\varepsilon_1,\varepsilon_{8j-1})\mapsto\varepsilon_{8j-3},\ (\varepsilon_1,\varepsilon_{8j})\mapsto-2\varepsilon_{8j-2}\\ (\varepsilon_1,\varepsilon_{8j+2})\mapsto\varepsilon_{8j},\ (\varepsilon_1,\varepsilon_{8j+3})\mapsto-\varepsilon_{8j+1}\\ (\varepsilon_3,\varepsilon_{j+1})\mapsto\varepsilon_{8j+1}\end{array}\right\}\ j\geqq 1$ $\left.\begin{array}{l}(\varepsilon_3,\varepsilon_{8j+2})\mapsto-2\varepsilon_{8j+2},\ (\varepsilon_3,\varepsilon_{8j+5})\mapsto\varepsilon_{8j+5}\\ (\varepsilon_3,\varepsilon_{8j+6})\mapsto 2\varepsilon_{8j+6},\ (\varepsilon_3,\varepsilon_{8j+7})\mapsto-\varepsilon_{8j+7}\end{array}\right\}\ j\geqq 0$ $(\varepsilon_1,\varepsilon_3)\mapsto-2\varepsilon_1$, the rest $\mapsto 0$
$(-2,0)$	$(\varepsilon_2,\varepsilon_{8j})\mapsto 2\varepsilon_{8j-2},\ (\varepsilon_2,\varepsilon_{8j+2})\mapsto-\varepsilon_{8j}\quad (j\geqq 1)$ $(\varepsilon_2,\varepsilon_{8j+3})\mapsto\varepsilon_{8j+1},\ (\varepsilon_2,\varepsilon_{8j+7})\mapsto-\varepsilon_{8j+5}\quad (j\geqq 0)$ the rest $\mapsto 0$
$(2m,4m-1)$ $m\geqq 0$	$(\varepsilon_1,\varepsilon_j)\mapsto j\varepsilon_{j+8m}$ for $j\neq 1$ the rest $\mapsto 0$
$(2m-1,4m)$ $m\geqq 0$	$(\varepsilon_2,\varepsilon_j)\mapsto j\varepsilon_{j+8m}$ for $j\neq 2$ the rest $\mapsto 0$

4. Now consider the case $\tilde{A}_{n-1}$ with $n\geqq 3$. The case $n=3$ is somewhat different from the general case (the main difference, from our point of view, is in the structure of the three-dimensional homology with trivial coefficients). Nevertheless, the final formula is the same, and the differences in the proofs are not essential. Therefore from now on we shall ignore the specific case $n=3$, nondirectly assuming that $n\geqq 4$.

The Cartan matrix of $\mathfrak{g}^{\tilde{A}_{n-1}}$ is:

$$\begin{pmatrix} 2 & -1 & 0 \dots & 0 & -1 \\ -1 & 2 & -1 \dots & 0 & 0 \\ 0 & -1 & 2 \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & 2 & -1 \\ -1 & 0 & 0 \dots & -1 & 2 \end{pmatrix}.$$

By (vi) in §1

$$\dim H_*^{(m_1, \dots, m_n)}(\mathfrak{n}_+(\tilde{A}_{n-1})) = \begin{cases} 1, & \text{if } \ P(m_1, \dots, m_n) = 0, \\ 0 & \text{in the other cases,} \end{cases}$$

where $P(m_1, \dots, m_n) = m_1^2 + \dots + m_n^2 - (m_1 m_2 + \dots + m_{n-1} m_n + m_n m_1) - (m_1 + \dots + m_n)$. In more details, if $k = 1, 2, 3$ then the space $H_k^{(m_1, \dots, m_n)}(\mathfrak{n}_+(\tilde{A}_{n-1}))$ has dimension 1 for the following sequences $(m_1, \dots, m_n)$:

$$k = 0\colon\ (0, \dots, 0); \quad k = 1\colon\ (1, 0, \dots, 0);$$

$$k = 2\colon\quad (2, 1, 0, \dots, 0),\ (1, \underbrace{0, \dots, 0}_{\geq 0}, 1, \underbrace{0, \dots, 0}_{\geq 0});$$

$$\begin{aligned} k = 3\colon\quad & (2, 2, 0, \dots, 0),\ (2, 1, 2, 0, \dots, 0), \\ & (3, 2, 1, 0, \dots, 0),\ (1, 3, 1, 0, \dots, 0), \\ & (2, 1, \underbrace{0, \dots, 0}_{\geq 0}, 1, \underbrace{0, \dots, 0}_{\geq 0}),\ (1, \underbrace{0, \dots, 0}_{\geq 0}, 1, \underbrace{0, \dots, 0}_{\geq 0}, 1, \underbrace{0, \dots, 0}_{\geq 0}), \end{aligned}$$

and also for the cases, obtained from these by cyclic permutation and reflection; for the remaining $(m_1, \dots, m_n)$ the named homology is 0.

Next we give cycles which represent generators of the above homology (ε_{ij} here and below stand for the matrix with 1 in the section of ith row and jth column and 0 elsewhere).

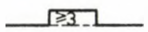
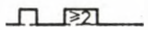
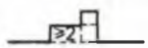
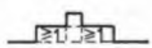
$$\begin{array}{ll} 1, & \varepsilon_{12}, \\ \varepsilon_{12} \wedge \varepsilon_{i,i+1}, & \varepsilon_{12} \wedge \varepsilon_{13} \wedge \varepsilon_{23}, \\ \varepsilon_{12} \wedge \varepsilon_{13} \wedge \varepsilon_{23}, & \varepsilon_{12} \wedge \varepsilon_{14} \wedge \varepsilon_{34}, \\ \varepsilon_{12} \wedge \varepsilon_{13} \wedge \varepsilon_{14}, & \varepsilon_{13} \wedge \varepsilon_{23} \wedge \varepsilon_{24}, \\ \varepsilon_{12} \wedge \varepsilon_{13} \wedge \varepsilon_{i,i+1}, & \varepsilon_{12} \wedge \varepsilon_{i,i+1} \wedge \varepsilon_{j,j+1}, \end{array}$$

where $\varepsilon_{n,n+1} = \varepsilon_{n,1} t$ by definition. Similarly, if as the result of cyclic permutation, we find the first index to be larger than the second one, we have to multiply ε by t.

Now we can determine the dimensions of the space which form the initial terms of the spectral sequences

$$\mathscr{E}(m_1, \dots, m_n) = \mathscr{E}(\mathfrak{n}_+(\tilde{A}_{n-1}), \mathfrak{n}_+(\tilde{A}_{n-1})', m_1, \dots, m_n).$$

We restrict ourselves to such $m_1, \dots, m_n$, that $\bigoplus\limits_{k=0}^{2} E_k^1$ are nontrivial. The dimensions of E_k^1 for these sequences are presented in Table 3.

$(m_1, \dots, m_n)$	dim E_0^1	dim E_1^1	dim E_2^1	dim E_3^1	
	n-1	n	0	0	*
	0	n	0	0	(m=0)*
	1	n-1	n-1	0	*
	0	0	n-1	0	(m=0)*
	1	2	2	1	
	0	0	2	1	(m=0)*
	1	2	1	0	
	0	2	n-1	n-2	
	0	1	2	1	
	0	1	2	1	
	0	0	2	1	(m=0)
	0	1	n-1	n-1	
	0	1	2	1	
	0	1	2	1	
	0	0	2	≥2	

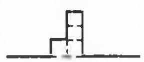
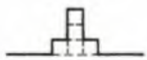
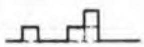
$(m_1, \dots, m_n)$	dim E_0^1	dim E_1^1	dim E_2^1	dim E_3^1
	0	0	1	≥1
	0	0	1	≥1
	0	0	2	2
	0	0	1	≥1
	0	0	1	1
	0	0	2	≥2
	0	0	1	≥1
	0	0	1	≥1
	0	0	1	≥1
	0	0	1	≥1
	0	0	3	≥3

Table 3

In this table the sequence $(m_1, \dots, m_n)$ is presented as a graph: the thick broken line is the graph of the step function with equally long steps and $m_1, \dots, m_n$ sequence of values. The left end of the line corresponds to the level $-m$ $(m_1 = -m)$. Whenever $m=0$ it is written at the end of the row. All calculations and dimensions are the same for those $(m_1, \dots, m_n)$ which can be obtained by reflection and cyclic permutation from those ones in the table.

It is easy to compute the differentials of the spectral sequences and it turns out, that homologies with dimension 1, 2 occur only in the cases which are marked in the table by stars. We calculate the differentials in these cases.

1°. $$(m_1, \dots, m_n) = (-m, \dots, -m).$$

In this case E_0^1 is trivial for $m=0$, and for $m>0$ it is spanned by the classes of the chains

$$\alpha_i = \big((\varepsilon_{i,i} - \varepsilon_{i+1,i+1})t^m\big)',$$

and E_1^1 is always spanned by classes of the chains

$$\beta_i = \varepsilon_{i,i+1} \otimes (\varepsilon_{i,i+1} t^m)', \quad i = 1, \dots, n-1, \quad \beta_n = \varepsilon_{n,1} t \otimes (\varepsilon_{n,1} t^{m+1})'.$$

Evidently, $d\beta_i=\alpha_i$ for $i=1, \dots, n-1$ and $d\beta_n=-\alpha_1-\dots-\alpha_{n-1}$. So,

$$\dim H_1^{(-m, \dots, -m)} = \begin{cases} 1 & \text{for} \quad m>0 \\ n & \text{for} \quad m=0, \end{cases} \qquad H_2^{(-m, \dots, -m)}=0.$$

One-dimensional cohomologies for $m>0$ are spanned by the class of the chain $\beta_1+\dots+\beta_n$, and for $m=0$ by classes of the chains $\beta_1, \dots, \beta_n$.

2°. $(m_1, \dots, m_n) = (\underbrace{-m, \dots, -m}_{i-1}, -m+1, -m, \dots, -m), \quad 1 \leqq i \leqq n.$

In this case E_0^1 is trivial for $m=0$, and for $m>0$ it is spanned by the class of the chain

$$\alpha = (\varepsilon_{i+1,i}t^m)';$$

E_1^1 is trivial for $m=0$, and for $m>0$ it is spanned by the classes of the chains

$$\beta_j = \varepsilon_{i,i+1}\otimes((\varepsilon_{j,j}-\varepsilon_{j+1,j+1})t^m)', \quad j=1, \dots, n-1;$$

E_2^1 is always spanned by the classes of the chains

$$\gamma_j = \varepsilon_{i,i+1}\wedge\varepsilon_{j,j+1}\otimes(\varepsilon_{j,j+1}t^m)', \quad j=1, \dots, i-2, i+2, \dots, n-1$$

$$\gamma_n = \varepsilon_{i,i+1}\wedge\varepsilon_{n,1}t\otimes(\varepsilon_{n,1}t^{m+1})',$$

$$\gamma_{i-1} = \varepsilon_{i-1,i+1}\wedge\varepsilon_{i,i+1}\otimes(\varepsilon_{i-1,i+1}t^m)',$$

$$\gamma_{i+1} = \varepsilon_{i,i+1}\wedge\varepsilon_{i,i+2}\otimes(\varepsilon_{i,i+2}t^m)'$$

(γ_i is absent). The differential $d=d^1$ acts by

$$d\beta_j = \begin{cases} -2\alpha & \text{for} \quad j=i \\ \alpha & \text{for} \quad j=i\pm 1, \\ 0 & \text{in the other cases;} \end{cases}$$

$$d\gamma_j = \begin{cases} \beta_j & \text{for} \quad j\neq i, \quad i\pm 1, \\ -\beta_1-\dots-\beta_{n-1} & \text{for} \quad j=n, \\ -2\beta_{i-1}-\beta_i & \text{for} \quad j=i-1, \\ \beta_i+2\beta_{i+1} & \text{for} \quad j=i+1. \end{cases}$$

So,

$$H_1^{(-m, \dots, -m+1, \dots, -m)} = 0,$$

$$\dim H_2^{(-m, \dots, -m+1, \dots, -m)} = \begin{cases} 1 & \text{for} \quad m>0 \\ n-1 & \text{for} \quad m=0. \end{cases}$$

The two-dimensional homologies for $m>0$ are spanned by the class of thy cycle $\gamma_1+\dots+\gamma_{i-2}+\frac{1}{2}\gamma_{i-1}+\frac{1}{2}\gamma_{i+1}+\gamma_{i+2}+\dots+\gamma_n$, while for $m=0$ by the classes of the cycles $\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n$.

3°. $(m_1, \dots, m_n) = (\underbrace{0, \dots, 0}_{i-1}, 1, 1, 0, \dots, 0), \quad 1 \leqq i \leqq n-1.$

In this case $E_0^1 = E_1^1 = 0$, E_2^1 is spanned by the classes of the chains

$$\gamma_1 = \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \otimes (\varepsilon_{i,i+1})', \quad \gamma_2 = \varepsilon_{i,i+2} \wedge \varepsilon_{i+1,i+2} \otimes (\varepsilon_{i+1,i+2})',$$

E_3^1 is spanned by the class of

$$\delta = \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \wedge \varepsilon_{i+1,i+2} \otimes (\varepsilon_{i,i+2})';$$

the differential acts by $d\delta = \gamma_1 - \gamma_2$. That means,

$$H_1^{(0,\dots,0,1,1,0,\dots,0)} = 0,$$

$$\dim H_2^{(0,\dots,0,1,1,0,\dots,0)} = 1.$$

The two-dimensional homologies are spanned by the class of γ_1 (or γ_2).

The case $(m_1, \dots, m_n) = (1, 0, \dots, 0, 1)$ is similar to the above one.

4°. $$(m_1, \dots, m_n) = (\underbrace{0, \dots, 0}_{i-1}, 2, 0, \dots, 0), \quad 1 \leqq i \leqq n.$$

In this case $E_0^1 = E_1^1 = 0$, E_2^1 is spanned by the classes of

$$\gamma_i = \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \otimes (\varepsilon_{i+1,i+2})', \quad \gamma_2 = \varepsilon_{i-1,i+1} \wedge \varepsilon_{i,i+1} \otimes (\varepsilon_{i-1,i})',$$

E_3^1 is spanned by the class of the chain

$$\delta = \varepsilon_{i-1,i+1} \wedge \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \otimes (\varepsilon_{i-1,i+2})';$$

the differential acts by $d\delta = \gamma_1 - \gamma_2$. That means,

$$H_1^{(0,\dots,0,2,0,\dots,0)} = 0,$$

$$\dim H_2^{(0,\dots,0,2,0,\dots,0)} = 1.$$

The two-dimensional homologies are spanned by the class of γ_1 (or γ_2).

As usually, we have isomorphism between the cohomology and homology. As it is clear from the list of deformations given before Theorem 1 in §1, all classes of two-dimensional cohomologies are represented by deformations of $\mathfrak{n}_+(\tilde{A}_{n-1})$.

5. The general case of an affine algebra $\mathfrak{g}^A$ for $A \neq \tilde{A}_1$, is quite similar to the above case. We restrict ourselves to formulate the final result.

$$\dim H_1^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A);\ \mathfrak{n}_+(A)') =$$

$$= \begin{cases} n & \text{for} \quad (m_1, \dots, m_n) = (0, \dots, 0), \\ 1 & \text{for} \quad (m_1, \dots, m_n) = (-ml\omega_1, \dots, -ml\omega_n), \quad m > 0, \\ 0 & \text{in all other cases,} \end{cases}$$

where $\omega_1, \dots, \omega_n$ are the coefficients of linear dependence between the columns of the Cartan matrix, while l equals to 1 for the current algebras (Table 1) and for the matrices from Table 2 is indicated in (v), §1.

$$\dim H_2^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A);\ \mathfrak{n}_+(A)') =$$

$$=\begin{cases} n-1 & \text{for}\quad (m_1,\dots,m_n)=(\underbrace{0,\dots,0}_{i-1},1,0,\dots,0),\quad 1\leqq i\leqq n \\ 1 & \text{for}\quad (m_1,\dots,m_n)=(-ml\omega_1,\dots,-ml\omega_{i-1},-ml\omega_i+1,-ml\omega_{i+1},\dots,-ml\omega_n) \\ & \qquad 1\leqq i\leqq n,\ m>0 \\ 1 & \text{for}\quad (m_1,\dots,m_n)=(\underbrace{0,\dots,0}_{i-1},2,0,\dots,0),\quad 1\leqq i\leqq n \\ 1 & \text{for}\quad (m_1,\dots,m_n)=(\underbrace{\underbrace{0,\dots,0}_{i-1},1,0,\dots,0}_{j-1},1,0,\dots,0) \\ & \qquad 1\leqq i<j\leqq n,\ a_{ij}\neq 0 \\ 0 & \text{in all other cases.}\end{cases}$$

As in the previous case, $H_k^{(m_1,\dots,m_n)}=H^k_{(-m_1,\dots,-m_n)}$, and all the two-dimensional cohomologies are represented as deformations.

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