Studia

115930

Scientiarum Mathematicarum Hungarica

18/1983

VOLUME 18

NUMBER 1

1983

AKADÉMIAI KIADÓ, BUDAPEST

STUDIA SCIENTIARUM MATHEMATICARUM HUNGARICA

A QUARTERLY OF THE HUNGARIAN ACADEMY OF SCIENCES

Studia Scientiarum Mathematicarum Hungarica publishes original papers on mathematics mainly in English, but also in German, French and Russian.

Studia Scientiarum Mathematicarum Hungarica is published in yearly volumes of four issues (mostly double numbers published semiannually) by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to

J Merza
Managing Editor
H-1053 Budapest, Reáltanoda u. 13—15

Subscription information

Orders should be addressed to

KULTURA Foreign Trading Company
H-1389 Budapest, P. O. Box 149

or to its representatives abroad.

Studia Scientiarum Mathematicarum Hungarica

Editor-in-Chief

A. Hajnal

Volume 18

Akadémiai Kiadó, Budapest

1983

STUDIA SCIENTIARUM MATHEMATICARUM HUNGARICA

Volume 18

CONTENTS

84-782 — Szegedi Nyomda — F. v.: Dobó József igazgató

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 1—11

ON THE CHARACTERIZATION OF ADDITIVE AND MULTIPLICATIVE FUNCTIONS

KATALIN KOVÁCS

0. Summary of the results

Let f denote an additive arithmetical function and let

$$g(n) = \max\{f(n),\ f(n+r_1),\ f(n+r_2),\ \dots, f(n+r_k)\}$$

with arbitrary, but fixed r_i $(i=1, \dots, k)$. In [4] we proved the following results for $r_i=i$:

(i) If g is a strictly monotonic function on a set A having upper density one $\left(\overline{\lim\limits_{n\to\infty}}\left(\sum\limits_{\substack{a\in A\\ a\leqq n}} 1/n\right)=1\right)$, then $f(n)=c\log n$.

(ii) If $\lim\limits_{n\to\infty} g(n)=c$, then $c\geqq 0$ and $g=c$.

We now generalize these results by proving

THEOREM 1. *Let f denote an additive function. If g is a strictly monotonic function on a set having upper density one, then $f(n)=c\log n$.*

THEOREM 2. *Let f denote an additive function. If $\lim\limits_{n\to\infty} g(n)=c$ on a set having upper density one, then $c\geqq 0$ and $g=c$.*

Let f denote a multiplicative function. (i) and (ii) can be generalized also for multiplicative functions, and the proofs are similar to the additive case, except the end of the proofs.

THEOREM 3. *Let f denote a multiplicative function. If g is strictly monotonic on a set having upper density one, then $f(n)=n^k$.*

THEOREM 4. *Let f denote a multiplicative function. If $\lim\limits_{n\to\infty} g(n)=c\neq 0$ on a set having upper density one, then $c\geqq 1$ and $g=c$, except the case, when there exist an x_0 and an $i\in\{0, \dots, k\}$, for which $g(x_0)<c$ and $f(x_0+r_i)=-c$. This case is still unsolved and the problem seems to be difficult.*

Generalization to a "rare" set. Let f be an additive function and A a subsequence of the natural numbers $a_1<a_2<\dots<a_n\dots$. R. Freud [3] raised the following question: How rare can such a set A be so that if f is additive and monotonic on A, then $f(n)=$

1980 *Mathematics Subject Classification.* Primary 10A20.
Key words and phrases. Characterization of additive and multiplicative functions, rare sets, set having upper density one, function $c\log n$, monotonic functions.

$=c \log n$. He proved that

(0.1) $$a_n > h(n)$$

moreover

(0.2) $$a_{n+1}-a_n > h(n)$$

can be guaranteed with an arbitrary $h(n)$. (i) and (ii) can be generalized in this direction, too:

THEOREM 5. *Let f denote an additive function. For every $h(n)$ there exists a set A for which* (0.1) *holds, and if g is strictly monotonic on A, then $f(n)=c \log n$.*

THEOREM 6. *Let f denote an additive function. For every $h(n)$ there exists a set A for which* (0.2) *holds, and if $\lim_{n\to\infty} g(a_n)=c$, then $c\geqq 0$ and $g=c$.*

THEOREM 7. *The analogue of Theorem 5 is true for multiplicative functions.*

THEOREM 8. *The analogue of Theorem 6 is true for multiplicative functions: For every $h(n)$ there exists a set A for which* (0.2) *holds and if $\lim_{n\to\infty} g(a_n)=c\neq 0$, then $c\geqq 1$ and $g=c$ excluding the case mentioned in Theorem 4.*

In what follows p, p', P always denote primes and q a prime power.

1. Proof of Theorem 1

Let g be strictly monotonically increasing function on a set A having upper density one. We shall find a set A_1 also having upper density one, on which f is monotonic. Then we can use a theorem of I. Kátai [5] and B. J. Birch [1]: If f is additive and monotonic on a set having upper density one, then $f(n)=c \log n$. Let $A_1 := \{a | a-2r_k+r_i \in A,\ i=0, \dots, k;\ r_0=0\}$. Since

(1.1) $$f(a-r_k+r_i) \leqq g(a-2r_k+r_i) \qquad (i=0, \dots, k)$$

and g is strictly monotonically increasing, so

(1.2) $$g(a-r_k) = \max \{f(a-r_k), \dots, f(a)\} = f(a).$$

So to any $a_1 < a_2 \in A_1$

$$f(a_1) = g(a_1-r_k) < g(a_2-r_k) = f(a_2),$$

consequently, f is monotonic on A_1, if $a > r_k$. It is easy to show, that A_1 also has upper density one.

If g is strictly monotonically decreasing, the proof is similar. □

2. Proof of Theorem 2

(a) First we prove a lemma:

LEMMA. *If A is a set having upper density one, then for any fixed d there exist infinitely many numbers $s_i \in A$ for which $(d, s_i)=1$, s_i are pairwise relatively prime and $ds_i \in A$.*

PROOF of the lemma. Let us assume, that we have already constructed suitable $s_1, \ldots, s_{n-1}$. Let

$$S_n := \{s \in A \mid (s, ds_1 \ldots s_{n-1}) = 1,\ ds \in A\}.$$

If this set were empty, then for any s s or ds would be missing from A. So at least $\frac{1}{2} \cdot \frac{\varphi(ds_1 \ldots s_{n-1})}{ds_1 \ldots s_{n-1}} N$ elements would be missing from A till dN, because the same element b can play the role of s and ds, a contradiction. □

(b) Now let A be the set in question having upper density one. Since $\lim_{n\to\infty} g(a_n) = c$, to arbitrary $\varepsilon > 0$ there exists a number n_0 such that if $n \geqq n_0$, then

$$c - \varepsilon < g(a_n) < c + \varepsilon \tag{2.1}$$

and hence

$$f(a_n) < c + \varepsilon. \tag{2.2}$$

First we show that $f(q) \leqq 0$ for all $q = p^\alpha$ where $p > r_k$. Let T be the set where the function f assumes the values of g, i.e. $t \in T$, if for some $n \in \mathrm{N}$ $g(n) = f(t)$. The set T has positive upper density $\left(\geqq \frac{1}{r_k}\right)$, because $t_{i+1} - t_i \leqq r_k$. Let

$$T_q := \{t_i \in T \mid t_i,\ qt_i,\ t_{i+1},\ qt_{i+1} \text{ are all in } A\}.$$

T_q is an infinite set, otherwise for all $i \geqq i_0$, at least one of the elements t_i, t_{i+1}, qt_i, qt_{i+1} would not be in A, so at least $\frac{1}{4r_k}(N - i_0)$ elements would be missing from A till qN.

Consider a large $t_i \in T_q$. Since $t_{i+1} - t_i \leqq r_k$, therefore either $(q, t_i) = 1$ or $(q, t_{i+1}) = 1$ or both. If e.g. $(q, t_i) = 1$, then using (2.1) and (2.2), we have

$$c - \varepsilon < f(t_i)$$

and

$$f(qt_i) = f(q) + f(t_i) < c + \varepsilon.$$

Combining the inequalities we have

$$f(q) \leqq 2\varepsilon,$$

consequently

$$f(q) \leqq 0. \tag{2.3}$$

(c) We show now that f cannot be smaller than a fixed negative number on an infinite set of pairwise relatively prime integers. Let $p_1 < \ldots < p_l$ be the primes not greater than r_k. Consider

$$x \equiv 1 \pmod{p_1^{\alpha_1} \ldots p_l^{\alpha_l}}, \tag{2.4}$$

where $p_j^{\alpha_j - 1} \leqq r_k + 1 < p_j^{\alpha_j}$. If $p_j^{\beta_{ji}} \| x + r_i$ $i = 0, \ldots, k$ $(r_0 = 0)$, then $\beta_{ji} < \alpha_j$, otherwise $p_j^{\alpha_j} | x + r_i - (x-1) = r_i + 1$ would hold.

Assume now indirectly, that $w_1, w_2 \ldots$ are pairwise relatively prime and $f(w_t) < -\varepsilon$ for some fixed positive ε. We may assume that $(w_t, p_i) = 1$ holds, too, for all

i, t. (To achieve this, we have to delete at most l w_t-s.) Consider (2.4) and the system of congruences

$$x+r_i \equiv w_{im+1} \dots w_{im+m} \mod (w_{im+1} \dots w_{im+m})^2 \tag{2.5}$$

($i=0, 1, \dots, k$) for some large m. Using (2.3), we have

$$f(x+r_i) \leqq f\left(\prod_{\nu=1}^{m} w_{im+\nu}\right)+f\left(\prod_{j=1}^{l} p_j^{\beta_{ji}}\right) < -m\varepsilon+K, \tag{2.6}$$

which contradicts $\lim_{n\to\infty} g(a_n)=c$ for a large m, because infinitely many solutions of (2.4)—(2.5) are in A. So if we take any set of pairwise relatively prime w_t ($t=1, \dots$), then for any $\varepsilon>0$ there is a t_0 such that

$$-\varepsilon < f(w_t) \leqq 0 \quad \text{for} \quad t \geqq t_0. \tag{2.7}$$

This also proves that c cannot be negative.

(d) Now we prove that c is an upper bound for f. If there exists a number d such that

$$f(d) = c+\delta \quad \text{with} \quad \delta > 0, \tag{2.8}$$

then by the lemma in (a) there are infinitely many $s_i \in A$ for which $ds_i \in A$, $(d, s_i)=1$ and s_i are pairwise relatively prime.

Now we use (2.7) with $\varepsilon=\dfrac{\delta}{2}$: there exists an n_0 such that for $n \geqq n_0$

$$f(s_n) > -\frac{\delta}{2}.$$

So

$$f(ds_n) = f(d)+f(s_n) > c+\delta-\frac{\delta}{2} = c+\frac{\delta}{2}, \tag{2.9}$$

which contradicts $\lim_{n\to\infty} g(a_n)=c$.

(e) Finally we prove $g=c$. If there exists a number x_0 for which $g(x_0)=c_0<c$, then consider the solutions of the congruence

$$x \equiv x_0 \mod p_1 \dots p_l[x_0(x_0+r_1) \dots (x_0+r_k)]^2, \tag{2.10}$$

where the p_i's are the primes up to r_k. There are infinitely many solutions of (2.10) in A. Then also for these solutions

$$x+r_i = (x_0+r_i)[1+tp_1 \dots p_l x_0^2 \dots (x_0+r_i) \dots (x_0+r_k)^2]. \tag{2.11}$$

So using (2.3)

$$\begin{aligned} f(x+r_i) &= f(x_0+r_i)+f[1+tp_1 \dots p_l x_0^2 \dots (x_0+r_i) \dots (x_0+r_n)^2] \leqq \\ &\leqq f(x_0+r_i). \end{aligned} \tag{2.12}$$

Consequently, $g(x) \leqq g(x_0) < c$, a contradiction.

REMARK. We examine what kind of conditions can ensure $g=c>0$. In [4] we proved that the values of g are such values of the function $f(n)$, where n is a multiple of a number $v \leqq r_k$, if $c \neq 0$. So it is necessary, that $v | x+r_i$ should hold for any x with at least one $i \in \{0, ..., k\}$. This implies that $0, r_1, ..., r_k$ must contain a complete residue-system mod v.

On the other hand, if $0, r_1, ..., r_k$ contain a complete residue-system for some $v \leqq r_k$, then we can find a function f such that $g=c>0$. For example, let $v=p_1^{\alpha_1} \dots p_s^{\alpha_s}$ and let f be the following function:

$$f(p_i^{\beta_i}) = \begin{cases} 0, & \text{if } \beta_i < \alpha_i \text{ and } i=1, ..., s \\ \dfrac{c}{s} > 0, & \text{if } \beta_i \geqq \alpha_i \text{ and } i=1, ..., s \\ 0, & \text{if } i > s. \end{cases}$$

For this function $g=c>0$.

3. Proof of Theorem 3

(a) If g is strictly monotonically increasing on a set having upper density one, then — similarly to the proof of Theorem 1, we get that f is also strictly monotonically increasing on a set having upper density one. Similarly to the additive case, it is easy to show that f must be monotonically increasing everywhere.

(b) If g is strictly monotonically decreasing, then we can apply the same arguments, because f cannot be 0 or negative anywhere. If there existed a number s for which $f(s) \leqq 0$, then there exists a $t>s$ for which $st \in A$ and $(s, t)=1$ so $f(t)<0$, however, $f(st)=f(s)f(t)>0$, a contradiction. □

4. Proof of Theorem 4

Let A be a set having upper density one.

(a) c cannot be negative. If $c<0$, then f is negative from a number on in A. Let $f(a)<0$. Because A is a set having upper density one, there are infinitely many t for which $t \in A$, $at \in A$ and $(a, t)=1$. Then $f(a)<0$, $f(t)<0$ and $f(at)=f(a)f(t)>0$, a contradiction.

(b) Let $c>0$. The proof is similar to the proof of Theorem 2, but we must face some new difficulties. As in (2.3) we obtain

$$f(q) \leqq 1, \quad \text{if} \quad q=p^\alpha, \quad \text{where} \quad p>r_k. \tag{4.1}$$

We can prove in the same way that

$$f(P_1^{\alpha_1} P_2^{\alpha_2}) \leqq 1 \tag{4.2}$$

is valid, too, where $P_i > 2r_k$.

Before going to step (c) we have to show that $f(p^\alpha) < -1$ can occur at most finitely many times. In the opposite case, we could choose prime powers $P_1^{\alpha_1}, P_2^{\alpha_2}$

such that $P_i > 2r_k$ and $f(P_i^{\alpha_i}) < -1$ $(i=1, 2)$, so

$$f(P_1^{\alpha_1}P_2^{\alpha_2}) = f(P_1^{\alpha_1})f(P_2^{\alpha_2}) > 1,$$

which contradicts (4.2). (We obtained that there are at most $\Pi(2r_k)+1$ "bad" primes.)

(c) Now we can proceed as in 2(c) to obtain that for any $\varepsilon > 0$

$$|f(w_t)| < 1-\varepsilon \tag{4.3}$$

is impossible on an infinite set of pairwise relatively prime numbers w_t. Namely, similarly to (2.4)—(2.6) — using (4.1) and the result of (b) — we get

$$|f(x+r_i)| \leqq \left|f\left(\prod_{v=1}^{m} w_{im+v}\right)\right| \left|f\left(\prod_{j=1}^{s} P_j^{\beta_{ji}}\right)\right| \leqq K(1-\varepsilon)^m, \tag{4.4}$$

where P_j are the primes for which there exists a power γ_j that $|f(P_j^{\gamma_j})| > 1$. If $m \to \infty$, then $f(x+r_i) \to 0$, which contradicts $\lim\limits_{n\to\infty} g(a_n) = c$.

(d) Similarly to the proof of 2(d) we prove that c is an upper bound of f. The only extra difficulty is to guarantee that $f(s_i)$ is non-negative. Let $s_i \in S_i$, and $f(s_i) \geqq 0$, if $i=1, \ldots, n-1$. If for $s^* \in S_n$ $f(s^*) < 0$, then consider

$$S_n' := \{s \in A \mid (s, ds_1 \ldots s_{n-1}s^*) = 1,\ ds \in A,\ ss^* \in A,\ dss^* \in A\}.$$

As in the proof of the lemma we obtain that S_n' is non-empty. If $s' \in S'_n$ and $f(s') \geqq 0$, then let $s_n := s'$. If $s' \in S_n'$ and $f(s') < 0$, then $f(s^*s') > 0$, and let $s_n := s^*s'$. So $s_1, \ldots, s_n, \ldots$ is a set with the same properties as in the lemma and $f(s_i) \geqq 0$ for all $i=1, 2, \ldots$. So similarly to 2(d), using (4.3), we can guarantee

$$f(s_n) > \frac{c+\dfrac{\delta}{2}}{c+\delta}$$

too, and so with the same d as in 2(d)

$$f(ds_n) = f(d)f(s_n) > (c+\delta)\frac{c+\dfrac{\delta}{2}}{c+\delta} = c+\frac{\delta}{2}, \tag{4.5}$$

a contradiction.

(e) Finally we prove $g=c$ with the exception of some special cases. Assume that there exists a number x_0 such that

$$g(x_0) = c_0 < c.$$

(i) If there are only finitely many primes p for which $f(p^\alpha)$ is negative for some α, then adding these extra primes to the P_i's and using the analogue of (2.10)—(2.12), we get the contradiction in the same way.

(ii) If there are infinitely many primes p for which $f(p^\alpha)$ is negative for some α, then by (4.3), there are infinitely many primes p_i' such that $0 > f(p_i'^{\alpha_i}) \geqq -1+\varepsilon$.

This guarantees also that $f(n)$ cannot be smaller than $-c<0$. In the opposite case, if $f(a)=-c_1<-c$, then

$$f(ap_i'^{\delta_i})=f(a)f(p_i'^{\delta_i})=(-c_1)(-1+\varepsilon)>c,$$

if ε is small enough, which is a contradiction, because c is an upper bound of f.

If $f(x_0+r_i)>-c$ for $i=0, 1, \dots, k$, then for the solutions of the analogue of (2.10)—(2.12)

$$f(x+r_i)=f(x_0+r_i)f(1+t'P_1 \dots P_l),$$

since $-1\leqq f(1+t'P_1 \dots P_l)\leqq 1$ and

$$-c<-c_2\leqq f(x_0+r_i)<c_0,$$

consequently,

$$f(x+r_i)<\max\{c_2, c_0\}<c,$$

a contradiction. Thus the only uncleared case is when there exists a $j\in\{0, \dots, k\}$ for which $f(x_0+r_j)=-c$. This case is still unsolved. □

Remark. Similarly to the additive case (see Remark after Theorem 2), $c>1$ can occur only if there exists a v such that $0, r_1, \dots, r_k$ contain a complete residue-system mod v.

5. Proof of Theorem 5

Let g be strictly monotonically increasing on the set

$$A:=\{nt_n-2r_k+r_i,\ (n+1)t_n-2r_k+r_i;\ i=0, \dots, k;\ r_0=0:\ n=2, \dots;\ (t_n, n(n+1))=1\}.$$

The rarity (0.1) can be guaranteed with a suitable choice of the numbers t_n.

If $a=nt_n$ or $a=(n+1)t_n$, then similarly to the proof of Theorem 1 we get

$$f(a)=g(a-r_k).$$

So $g(nt_n-r_k)=f(nt_n)$ and

$$g((n+1)t_n-r_k)=f((n+1)t_n),$$

consequently,

$$f(nt_n)<f((n+1)t_n),$$

because g is monotonic on A. But f is additive, so $f(n)<f(n+1)$ for any n. A theorem of Erdős [2] implies now $f(n)=c\log n$.

If g is strictly monotonically decreasing, the proof is similar. □

6. Proof of Theorem 6

Let

$$A:=\{a_{ms},\ m(a_{ms}+r_i),\ m=2, \dots;\ p\nmid m \text{ for any prime } p\leqq r_k+1;\ i=0, \dots, k;\ s=1, 2, \dots;\ a_{ms}\equiv 1 \bmod m\}\cup$$

$\{p_1^{\alpha_1}\dots p_t^{\alpha_t}\cdot r$ for all primes $p_i\leqq r_k+1$ with all $\alpha_i\geqq 0$ and infinitely many primes $r\}\cup$

$\cup \{n+p_1 \dots p_t w[n(n+r_1)\dots(n+r_k)]^2$ for all n with infinitely many w for each of them$\}$.

The rarity (0.2) can be guaranteed with the suitable choice of the numbers a_{ms}, w and r.

Since $a_{ms} \in A$ $(s=1, 2, \dots)$, therefore

$$\lim_{s\to\infty} g(a_{ms}) = \lim f(a_{ms}+r_j) = c,$$

where $j=0, 1, \dots,$ or k. There exists a $j_0 \in \{0, \dots, k\}$ for which

$$c-\varepsilon < f(a_{ms}+r_{j_0}) < c+\varepsilon. \tag{6.1}$$

infinitely often. At the same time

$$f\big(m(a_{ms}+r_{j_0})\big) = f(m)+f(a_{ms}+r_{j_0}) < c+\varepsilon \tag{6.2}$$

is valid, since

$$(a_{ms}+r_j, m) = (mv+1+r_j, m) = (r_j+1, m) = 1.$$

Because of (6.1) and (6.2)

$$f(m) \leqq 2\varepsilon,$$

so

$$f(m) \leqq 0 \tag{6.3}$$

for all m having no prime-divisors smaller than r_k+1. Now we show that $f(w) < -\varepsilon$ cannot hold on an infinite set of relatively prime numbers. In the opposite case (as in 2(c)) we could guarantee $g(x_n) < -d$ with an arbitrary $d>0$ on the solutions x_n of the system of congruences (2.4)—(2.5). If there are infinitely many such numbers x_n in A, we are ready. If not, we can choose $y_n \in A$ such that $g(y_n) \leqq g(x_n) < -d$ the following way: Let

$$y_n = x_n + p_1 p_2 \dots p_t w[x_n(x_n+r_1)\dots(x_n+r_k)]^2. \tag{6.4}$$

By definition $y_n \in A$ for some w. So using (6.3)

$$f(y_n+r_i) = f(x_n+r_i)+f(1+p_1 \dots p_t w \dots) \leqq f(x_n+r_i) \leqq g(x_n), \tag{6.5}$$

consequently,

$$g(x_n) \leqq g(y_n) < -d,$$

a contradiction. So to any ε there exists a t_0 such that

$$-\varepsilon \leqq f(w_t) \leqq 0. \tag{6.6}$$

We have also proved that c cannot be negative. Now we are ready to prove that c is an upper bound of f. Assume that there exists a number z such that

$$g(z) = c+\delta \quad \text{with} \quad \delta > 0.$$

Let $z = p_1^{\alpha_1} \dots p_t^{\alpha_t} m$, where $f(m) \leqq 0$. So $f(p_1^{\alpha_1} \dots p_t^{\alpha}) \geqq c+\delta$ is true, too. Because of the definition of A and using (6.6) there are infinitely many primes r for which

$0 \geqq f(r) \geqq -\frac{\delta}{2}$, and $p_1^{\alpha_1} \ldots p_t^{\alpha_t} r \in A$. So

$$f(p_1^{\alpha_1} \ldots p_t^{\alpha_t} r) = f(p_1^{\alpha_1} \ldots p_t^{\alpha_t}) + f(r) \geqq (c+\delta) - \frac{\delta}{2} = c + \frac{\delta}{2}$$

with infinitely many r, a contradiction.

Using the fact that

$$n + p_1 \ldots p_t w[n(n+r_1) \ldots (n+r_k)]^2 \in A$$

with infinitely many w, we can prove similarly to part (e) in the proof of Theorem 2 (taking $n = x_0$) that g cannot be smaller than c. □

7. Proof of Theorem 7

(a) Let g be strictly monotonically increasing on the set $A := \{p_1, p_2, p_1 p_2$, where p_1, p_2 are primes$\} \cup \{t_n - 2r_k + r_i, \quad nt_n - 2r_k + r_i, \quad (n+1)t_n - 2r_k + r_i;$ $i = 0, \ldots, k\} = A_1 \cup A_2$.

Similarly to the proof of Theorem 5, we obtain

$$f(nt_n) < f((n+1)t_n). \tag{7.1}$$

Because $t_n - 2r_k + r_i \in A$, so

$$g(t_n - r_k) = f(t_n) \tag{7.2}$$

is valid, too. Since either $f(p_1) \geqq 0$ or $f(p_2) \geqq 0$ or $f(p_1 p_2) \geqq 0$, therefore $g(p_1) \geqq 0$ or $g(p_2) \geqq 0$ or $g(p_1 p_2) \geqq 0$, so $g \geqq 0$ from a number on. Hence $f(t_n) > 0$ from a number on in A (see (7.2)). So (7.1) implies $f(n+1) > f(n)$ for n large enough. Hence $f(n) = n^k$ [6].

(b) If g is strictly monotonically decreasing, we modify A_1 the following way:

$$\{p_{i1}, p_{i2}, p_{i1}p_{i2},\ i = 1, 2, \ldots, \text{ where } p_{ij} \text{ are primes}\}.$$

This guarantees that f is infinitely often positive on A. Hence g must be positive everywhere. Using this fact, we can argue as in (a). □

8. Proof of Theorem 8

Let $A := \{np_{in}^{(1)}, np_{in}^{(2)}, np_{in}^{(1)} p_{in}^{(2)},\ i = 1, 2, \ldots$ with infinitely many primes $p_{in}^{(1)}, p_{in}^{(2)}$ to any $n \in \mathbf{N}\} \cup \{a_{ms}, m(a_{ms} + r_i)$, where $a_{ms} \equiv 1 \pmod m$ and $(p, m) = 1$, if $p \leqq r_k + 1; \quad i = 0, \ldots, k\} \cup \{n + p_1 p_2 \ldots p_t w_{tn}[n(n+r_1) \ldots (n+r_k)]^2$, where $p_1 < p_2 < \ldots$ denote the series of the primes and we consider infinitely many t to all $n \in \mathbf{N}\}$.

The rarity (0.2) can be guaranteed with the suitable choice of the numbers a_{ms}, r, w_{tn}, $p_{in}^{(1)}$, $p_{in}^{(2)}$.

Because $p_{in}^{(1)}, p_{in}^{(2)}, p_{in}^{(1)}p_{in}^{(2)}$ are infinitely often in A and either $f(p_{in}^{(1)}) \geqq 0$ or $f(p_{in}^{(2)}) \geqq 0$ or $f(p_{in}^{(1)}p_{in}^{(2)}) \geqq 0$, so $c \geqq 0$. We excluded the case $c=0$, so $c>0$.

(a) Similarly to the proof of Theorem 6 we can prove that

$$f(n) \leqq 1 \tag{8.1}$$

if $p \nmid n$ for $p \leqq r_k+1$.

(b) In the next step we need $f(q)<-1$ only on the powers of finitely many primes. This follows easily:

If $p \nmid q_1 q_2$, where $p \leqq r_k+1$,

and

$$f(q_1) < -1 \quad \text{and} \quad f(q_2) < -1,$$

then

$$f(q_1 q_2) = f(q_1)f(q_2) > 1,$$

which contradicts to (8.1).

(c) Using (8.1) and the result of 8(b) we get a contradiction, if we assume that

$$|f(w_j)| < 1-\varepsilon$$

on an infinite set of pairwise relatively prime numbers. We can prove this by combining the relevant parts of 4(c) and 6.

(d) We prove that c is an upper bound of f. If there exists a number d such that

$$f(d) = c+\delta \qquad (\delta > 0),$$

then consider $dp_{id}^{(1)}, dp_{id}^{(2)}, dp_{id}^{(1)}p_{id}^{(2)} \in A$. We know that at least one of $f(p_{id}^{(1)}), f(p_{id}^{(2)})$, $f(p_{id}^{(1)}p_{id}^{(2)})$ is non-negative. On the other hand using the result of 8(c), e.g.

$$f(p_{id}^{(1)}) > 1-\varepsilon$$

for infinitely many $p_{id}^{(1)}$. So $f(dp_{id}^{(1)}) = f(d)f(p_{id}^{(1)}) > (c+\delta)(1-\varepsilon) > c+\frac{\delta}{2}$, if ε is small enough. Because $dp_{id}^{(1)} \in A$ infinitely often, this fact contradicts $\lim_{n\to\infty} g(a_n) = c$.

(e) The final part of the proof is similar to 4(e). The extra difficulty arises also now in the case, when there exist an x_0 and an $i \in \{0, \dots, k\}$ for which $g(x_0)<c$ and $f(x_0+r_i)=-c$. Except this case the proof runs in the same way using that $n+p_1p_2\dots p_t w_{tn}[n(n+r_1)\dots(n+r_k)]^2 \in A$ for all $n \in \mathbf{N}$ with infinitely many w_{tn}. $\square$

REFERENCES

[1] Birch, B. J., Multiplicative functions with non-decreasing normal order, *J. London Math. Soc.* **42** (1967), 149—151. *MR* **34** # 2535.
[2] Erdős, P., On the distribution function of additive functions, *Ann. of Math.* (2) **47** (1946), 1—20. *MR* 7—416.
[3] Freud, R., On additive functions which are monotone on a "rare" set, *Acta Math. Acad. Sci. Hungar.* **31** (1978), 151—156. *MR* **57** # 16240.
[4] Kovács, K., On the characterization of additive functions, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* (to appear).
[5] Kátai, I., A remark on number-theoretical functions, *Acta Arith.* **14** (1967/1968), 409—415. *MR* **38** # 130.
[6] Moser, L. and Lambek, J., On monotone multiplicative functions, *Proc. Amer. Math. Soc.* **4** (1953), 544—545. *MR* **15**—104.

(Received December 15, 1981)

BERZSENYI DÁNIEL GIMNÁZIUM
HUBA U. 7
H—1134 BUDAPEST
HUNGARY

83-3121 — Szegedi Nyomda — F. v.: Dobó József igazgató

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 13—19

RADICALS AND UNIQUENESS THEOREMS IN MULTIPLICATIVE LATTICES WITH CHAIN CONDITIONS

N. K. THAKARE and C. S. MANJAREKAR

1. Introduction

A major step was taken by Dilworth [2] in giving a concrete abstract formulation of the ideal theory of commutative rings. He succeeded in establishing the fundamental theorem on primary decompositions for Noether lattices, in particular. He also touched upon the notion of radicals of an element in a multiplicative lattice but did not elaborate it further. In this note we take up the concept of radicals and use it to prove uniqueness of primary decompositions in multiplicative lattices with ascending chain condition. For this we present an abstract formulation of results on radicals in commutative rings (cf. Zariski and Samuel [5]). Radicals were also studied by Murata [4].

A multiplicative lattice is a complete lattice L on which a multiplication is defined which is commutative, associative, and distributive over arbitrary joins, and the largest element of L acts as a multiplicative identity. An element $a\in L$ is called *compact* whenever $a\leqq \bigvee X, X\subseteq L$ implies the existence of a finite number of elements $x_1, x_2, \dots, x_n$ of X such that $a\leqq x_1\vee x_2\vee\dots\vee x_n$. It is well-known that a complete lattice L satisfies the ascending chain condition if and only if every element of L is compact; see Dilworth and Crawley [3]. In what follows L will stand for a multiplicative lattice every element of which is compact. An element p of L is said to be *prime* if $ab\leqq p$ implies $a\leqq p$ or $b\leqq p$ for $a, b\in L$. An element p of L is said to be *primary* if $ab\leqq p$ implies $a\leqq p$ or $b^n\leqq p$ for some integer n. The *radical* of an element a is $\bigvee\{x\in L|x^s\leqq a$ for some integer $s\}$ and is denoted by $\sqrt{a}$. If q is a primary element then $\sqrt{q}$ is the minimal prime containing q, and $\sqrt{q}$ is called the *prime associated with q*. An element a is said to have a *primary decomposition* if there exist primary elements $q_1, q_2, \dots, q_m$ such that

$$a = q_1\wedge q_2\wedge\dots\wedge q_m. \tag{0}$$

If this decomposition cannot be reduced further it is said to be *irredundant*. Clearly, $\sqrt{q_i}=p_i$ $(i=1, 2, \dots, m)$ are the associated primes arising from the irredundant primary decomposition (0). Sometimes the p_i's will be called associated primes with respect to the primary decomposition (0). The following properties of associated primes are well-known (see Dilworth [2]): (i) $\sqrt{q}^k\leqq q\leqq\sqrt{q}$ for some integer k. (ii) $ab\leqq q$ implies $a\leqq q$ or $b\leqq\sqrt{q}$.

1980 *Mathematics Subject Classification.* Primary 06E10, 06F10; Secondary 06A23, 06F99, 06B10, 06B99, 13A99, 13E99.

Key words and phrases. Multiplicative lattice, chain condition, radicals, irredundant primary decomposition, prime associated with an element, Noether lattice.

In Section 2 we characterize those elements of L that are their own radicals, besides obtaining necessary and sufficient conditions for a prime element p of L to be equal to some p_i which arises from the primary decomposition (0). In Section 3 we present uniqueness theorems for primary decompositions. Later these considerations are applied to zero divisors and nilpotent elements of L.

2. Radicals

Suppose that a in L has an irredundant primary decomposition given by (0). The proofs of the following two lemmas are immediate.

LEMMA 1. *If a prime element p contains a finite meet $\bigwedge_{i=1}^{m} q_i$ then it contains some q_i, if the q_i are primary then p contains the associated prime element p_i of one of them.*

LEMMA 2. *If a prime element $p=\bigwedge_{i=1}^{n} p_i$ where the p_i are prime then p contains one of them by Lemma 1 and thus is equal to it; the other p_j then contain this p_i.*

We state at the outset some properties of radicals the proofs of which can be easily worked out. We, however, prove the last property only.

Properties of radicals. For $a, b \in L$

$$(\mathrm{p}_1) \quad a \leqq \sqrt{a},$$

$$(\mathrm{p}_2) \quad a \leqq b \Rightarrow \sqrt{a} \leqq \sqrt{b},$$

$$(\mathrm{p}_3) \quad \sqrt{\sqrt{a}} = \sqrt{a},$$

$$(\mathrm{p}_4) \quad \sqrt{a \wedge b} = \sqrt{a} \wedge \sqrt{b} = \sqrt{ab},$$

$$(\mathrm{p}_5) \quad \sqrt{a \vee b} = \sqrt{\sqrt{a} \vee \sqrt{b}}.$$

For proving (p_5) take $y \in \{x \in L \mid x^s \leqq a \vee b$ for some integer $s\}$. Then by property (p_1) $y^s \leqq \sqrt{a} \vee \sqrt{b}$ and $y \in \{x \in L \mid x^s \leqq \sqrt{a} \vee \sqrt{b}\}$ and we have $\sqrt{a \vee b} \leqq \sqrt{\sqrt{a} \vee \sqrt{b}}$. For the reverse inequality let $y \in \{x \in L \mid x^s \leqq \sqrt{a} \vee \sqrt{b}$ for some integer $s\}$. Then by the definition of the radical $y^s \leqq [\vee \{x \in L \mid x^s \leqq a$ for some integer $s\}] \vee [\vee \{z \in L \mid z^l \leqq b$ for some integer $l\}]$. Since every element of L is compact, we have

$$y^s \leqq [x_1 \vee x_2 \vee \ldots \vee x_m] \vee [z_1 \vee z_2 \vee \ldots \vee z_n]$$

where $x_i^{s_i} \leqq a$ $i=1, 2, \ldots, m$ and $z_j^{l_j} \leqq b$ where $j=1, 2, \ldots, n$. Let $s_1+s_2+\ldots+\ldots+s_m+l_1+l_2+\ldots+l_n=k$. Using a well-known property (see Dilworth [2], (2.16)), we have

$$[x_1 \vee x_2 \vee \ldots \vee x_m \vee z_1 \vee z_2 \vee \ldots \vee z_n]^k \leqq x_1^{s_1} \vee x_2^{s_2} \vee \ldots \vee x_m^{s_m} \vee z_1^{l_1} \vee z_2^{l_2} \vee \ldots \vee z_n^{l_n} \leqq a \vee b.$$

Hence we infer that $y \in \{x \mid x^s \leqq a \vee b$ for some integer $s\}$. Then $\sqrt{\sqrt{a} \vee \sqrt{b}} \leqq \sqrt{a \vee b}$ and the proof is complete.

We see in our next result that the elementary properties above can be used to characterize those elements which are their own radicals.

THEOREM 3. *Let* $a\in L$ *admit an irredundant primary decomposition* (0). *For a to be equal to* $\sqrt{a}$, *it is necessary and sufficient that all the* q_i *are prime elements.*

PROOF. Suppose that each q_i in the representation (0) is prime. Let $y\in\{x|x^s\leqq a$ for some integer $s\}$. Then $y^s\leqq\bigwedge_{i=1}^{m}q_i\leqq q_i$ for all $i=1,2,\dots,m$ and since the q_i's are prime, we have $y\leqq q_i$ for each $i=1,2,\dots,m$. This permits us to have $\bigvee\{x\in L|x^s\leqq a$ for some integer $s\}\leqq a$. Property (p_1) gives now $a=\sqrt{a}$.

Conversely, suppose $a=\sqrt{a}$ where a has the representation (0). Let p_i be an associated prime of q_i, $i=1,2,\dots,m$. In view of the representation (0) and (p_4) we have $\sqrt{a}=\sqrt{q_1}\wedge\sqrt{q_2}\wedge\dots\wedge\sqrt{q_m}$. As $a=\sqrt{a}$, we obtain $a=\bigwedge_{i=1}^{m}p_i$. This representation of a as a meet of prime elements is irredundant, for so is (0). We have to show, that $q_i=p_i$ for each i. We always have $q_i\leqq p_i$ for each i. Let $y\leqq p_i$. As $\bigwedge_{i=1}^{m}p_i$ is an irredundant decomposition of a, there is a $z\leqq\bigwedge_{j\neq i}p_j$ such that $z\nleqq p_i$. Now $yz\leqq\leqq\bigwedge_{i=1}^{m}p_i=\bigwedge_{i=1}^{m}q_i\leqq q_i$ for each i. As each q_i is primary, we get $y\leqq q_i$. In particular, $p_i\leqq q_i$ and we are done. Q.e.d.

3. Uniqueness theorems

In Section 2 we discussed some properties of radicals. In the context of Noether lattices it is well-known that every element has an irredundant primary decomposition of type (0). Now we take up the question of uniqueness of representations of type (0) by showing that the associated primes p_i of q_i (for each $i=1,2,\dots,m$) are uniquely determined. We need to recall the concept of residuation. If a,b are elements of L, the residual of a by b, denoted by $a:b$, is defined to be the join of all $z\in L$ such that $zb\leqq a$. The associated prime elements of the primary elements occurring in an irredundant primary representation (0) of an element a are called the associated prime elements of a or simply prime elements of a. A minimal element in the family of associated prime elements of a is called an *isolated* prime element of a; a prime element of a which is not isolated is said to be *embedded*. If $a=\bigwedge_{i=1}^{m}q_i$ is an irredundant primary decomposition of a, the q_i are said to be the *primary components* of a and q_i is called *isolated* or *embedded* according as its associated prime element p_i is isolated or embedded. In this case we also say that q_i is primary for p_i.

The following theorem gives a characterization for a prime element of L to be equal to some associated prime p_i of q_i.

THEOREM 4. *Let* $a\in L$ *have irredundant primary decomposition* (0) *and let the* p_i *be the associated primes of* q_i. *For a prime element p of L to be equal to some* p_i, *it is necessary and sufficient that there exists an element* $c\in L$ *not contained in a and such that* $(a:c)$ *is primary for p (i.e. the associated prime of* $(a:c)$ *is p).*

PROOF. Suppose the p_i are as given in the hypothesis and let $p=p_i$ for some i. For this i there exists

$$c \leqq \bigwedge_{j\neq i} p_i \tag{1}$$

such that $c \not\leqq q_i$ since the primary decomposition (0) is irredundant. First we show that for such an element c, the element $(a:c)$ evidently contains q_i and is contained in p_i. Let $x\in\{y|yc\leqq a\}$; then $xc\leqq q_j$ for every j. But for our fixed i, $c\not\leqq q_i$, hence $x^s\leqq q_i$ for some integer s. Thus $x\in\{y|y^s\leqq q_i$ where s is some integer$\}$ which shows that $\bigvee\{y|yc\leqq a\}\leqq\bigvee\{y|y^s\leqq q_i$ for some integer $s\}$ i.e.

$$(a:c) \leqq p_i. \tag{2}$$

On the other hand select $y\leqq q_i$. From (1) we have $yc\leqq \bigwedge_{j=1}^{m} q_j\leqq a$, which implies $y\leqq(a:c)$ and hence

$$q_i \leqq (a:c). \tag{3}$$

From (2) and (3), we have

$$\sqrt{q_i} = p_i \leqq \sqrt{(a:c)} \leqq \sqrt{p_i} = p_i. \tag{4}$$

Take $yz\leqq(a:c)$ and suppose $z\not\leqq p_i$. Clearly, $yzc\leqq a\leqq q_i$ and thus $yc\leqq q_i$. Use of (1) gives $yc\leqq \bigwedge_{i=1}^{m} q_i=a$ yielding $y\leqq(a:c)$. From (4) we now conclude that $(a:c)$ is primary for p_i and hence primary for p. One way of the implication is thus complete.

Conversely, suppose that for some element c such that $c\not\leqq a$, the element $(a:c)$ is primary for a given prime element p. From (0) and Dilworth [[2], (2.4), p. 482] we have $(a:c)=\bigwedge_{i=1}^{m}(q_i:c)$. By the property ($p_4$) of radicals and by our assumption, we have

$$\sqrt{(a:c)} = \bigwedge_{i=1}^{m} \sqrt{(q_i:c)} = p. \tag{5}$$

From the assumption $c\not\leqq a$ and the irredundant decomposition (0) we obtain $c\not\leqq q_i$ for some i and $c\leqq q_i$ for the remaining ones.

Case (i). $c\leqq q_i$: We have by Dilworth [[2], (2.3.) p. 482] $\sqrt{(q_i:c)}=1$.

Case (ii). $c\not\leqq q_i$: Let $x\leqq q_i$. Then we can show that $x\leqq(q_i:c)$. Thus $q_i\leqq$ $\leqq(q_i:c)$ and $\sqrt{q_i}=p_i\leqq\sqrt{(q_i:c)}$.

For the reverse inequality suppose $x\leqq(q_i:c)$. Then $xc\leqq q_i$. But $c\not\leqq q_i$, and the primariness of q_i yields $x^s\leqq q_i$ for some integer s. That is, $x\leqq\sqrt{q_i}$ and $\sqrt{(q_i:c)}\leqq$ $\leqq p_i$. Thus $\sqrt{(q_i:c)}=p_i$. From (5) and the two cases above it follows that $p=\bigwedge_J p_j$ for some subset J of $\{1, 2, \dots, m\}$. Therefore by Lemma 2 we conclude that $p=p_j$ for some j, and we are done. Q.e.d.

As a consequence of the above theorem the primes p_i ($i=1, 2, \dots, m$) are now uniquely determined by a.

The next characterization is very simple.

THEOREM 5. *For a prime element p of L to contain a it is necessary and sufficient that p contains some p_i where a has an irredundant primary decomposition* (0) *and the p_i are the associated primes of the q_i's, respectively.*

PROOF. Suppose $p_i \leqq p$ for some i. Then $a \leqq p$. Conversely, $a \leqq p$ yields $q_i \leqq p$ for some i. But p_i is a minimal prime element containing q_i, which implies $p_i \leqq p$. Q.e.d.

Earlier we showed (Theorem 4) that the associated primes p_i arising from the irredundant primary decomposition of an element $a = \bigwedge_{i=1}^{m} q_i$ are uniquely determined. In our next result we show that even those q_i's can be uniquely determined which are isolated primary components of $a \in L$.

THEOREM 6. *Let $a \in L$ have an irredundant primary decomposition* (0) *with the p_i's as associated primes of the q_i's. The element*

$$q_i' = \bigvee \{x \in L \mid (a:x) \nleqq p_i\} \tag{6}$$

is an element of L which is contained in q_i. If q_i is an isolated primary component of a then $q_i = q_i'$.

PROOF. Take any element

$$y \in \{x \in L \mid (a:x) \nleqq p_i\} = \{x \in L \mid \bigvee \{y \mid yx \leqq a\} \nleqq p_i\}.$$

Then there is an element $\pi \in L$ such that $\pi x \leqq a$ and $\pi \nleqq p_i$. Then $\pi^n \nleqq q_i$ for all integers n. Now $\pi x \leqq q_i$, $\pi^n \nleqq q_i$ for every integer n and q_i is primary, which gives $x \leqq q_i$. Therefore $q_i' \leqq q_i$ and the first part is proved.

If p_i is a minimal associated prime of a, $p_j \nleqq p_i$ for $i \neq j$. Then there exists $b_j \leqq p_j$ such that $b_j \nleqq p_i$. Since every element of L is compact, we have

$$b_j \leqq p_j = \bigvee_{r=1}^{n} \{x_r \mid x_r^{s_r} \leqq q_j \text{ for some integer } s_r\}.$$

Put $s_1 + s_2 + \ldots + s_n = k(j)$. Then $b_j^{k(j)} \leqq (x_1 \vee x_2 \vee \ldots x_n)^{k(j)} \leqq q_j$. Clearly, $b = \prod_{j \neq i} b_j^{k(j)} \nleqq p_i$ as p_i is prime. However, $b \leqq \bigwedge_{j \neq i} q_j$. Next take any $y \leqq q_i$. Then $yb \leqq \bigwedge_{i=1}^{m} q_i = a$. Hence by (6) we can come to the conclusion that $y \leqq \bigvee \{x \in L \mid (a:x) \nleqq p_i\} = q_i'$. Thus $q_i \leqq q_i'$ and finally from the first part we get $q_i = q_i'$. Q.e.d.

We now go to relate the radical of a with the isolated primes of $a \in L$. In that direction, we have

THEOREM 7. *Let $a \in L$ have an irredundant primary decomposition* (0). *Then the radical of a is the meet of the isolated prime elements of a.*

PROOF. By (p_4) the radical of a is the meet of all associated primes p_i ($i = 1, 2, \ldots$ $\ldots, m$) of a, i.e., $\sqrt{a} = p_1 \wedge p_2 \wedge \ldots \wedge p_m$. If some p_k is not isolated then $p_k \geqq p_i$ for some p_i and hence we can delete such elements from the above representation and we are through. Q.e.d.

Recall that the nilpotent elements in a multiplicative lattice L are those elements $a \in L$ for which $a^n = 0$ for some integer n.

The following consequence is worth noting.

COROLLARY 8. *In a Noether lattice, the join of the set of nilpotent elements is the meet of the isolated primes of* 0.

PROOF. Let $0 = \bigwedge_{i=1}^{m} q_i$ be an irredundant primary decomposition of 0, with the p_i as associated primes of the q_i $(i = 1, 2, \ldots, m)$, respectively. By Theorem 7, $\sqrt{0} = p_i \wedge p_2 \wedge \ldots \wedge p_k$ where $p_1, p_2, \ldots, p_k$ are isolated primes. From this and the definition of the radical we have $\vee \{b \mid b \text{ is nilpotent}\} = p_1 \wedge p_2 \wedge \ldots \wedge p_k$. Q.e.d.

The primeness of the radical of a is characterized in the following

COROLLARY 9. *For $a \in L$, $\sqrt{a}$ is prime if and only if a has a single isolated prime element.*

PROOF. If a has a single isolated prime element p then we have $\sqrt{a} = p$.

Conversely, suppose $\sqrt{a} = p_1 \wedge p_2$ where p_1, p_2 are isolated primes of a. Then there are x, y in L such that $x \leqq p_1, x \nleqq p_2$ and $y \leqq p_2, y \nleqq p_1$. Then $xy \leqq \sqrt{a}$. But $\sqrt{a}$ is prime, which gives the contradiction that $x \leqq p_2$ and $y \leqq p_1$. Hence $\sqrt{a} \neq p_1 \wedge p_2$. Thus a cannot have more than one isolated primes. Q.e.d.

In the remaining part of the paper we shall let the multiplicative lattice L be a Noether lattice.

THEOREM 10. *Let a, b be any two elements of L such that $a \neq 1$. Then $a = (a:b)$ if and only if b is contained in no prime element of a.*

PROOF. Let $a = \bigwedge_{i=1}^{m} q_i$ be an irredundant primary decomposition of a and let $p_i = \sqrt{q_i}$. Suppose $b \nleqq p_i$, which permits us to write $b^n \nleqq p_i$ for any integer n. It is known that $(a:b)b \leqq a$ (see Dilworth [2]) and thus $(a:b)b \leqq q_i$. Primariness of q_i yields $(a:b) \leqq q_i$, i.e. $(a:b) \leqq \bigwedge_{i=1}^{m} q_i = a$, and we have $a = (a:b)$ as the reverse inequality is always valid (see Dilworth [2]).

Conversely, suppose $(a:b) = a$ and if possible, without loss of generality, suppose $b \leqq p_1$. Then $(q_1 : b^s) = 1$ for some integer s. From Dilworth [2] we can directly write $(a:b^s) = a$. Finally, by Dilworth [2], p. 482,

$$a = \bigwedge_{j \neq 1} (q_j : b^s) \geqq \bigwedge_{j \neq 1} q_j \geqq a,$$

i.e. $a = \bigwedge_{j \neq 1} q_j$, which is impossible, and the proof is complete. Q.e.d.

The above theorem can be restated in the following elegant form.

COROLLARY 11. *For an element b of L to be contained in some associated prime element of a in L it is necessary and sufficient that* $(a:b)\neq a$.

A direct application of Corollary 11 readily gives the uniqueness of the maximal associated prime elements of a.

COROLLARY 12. *For an element x of L to be contained in some associated prime of an element a of L, it is necessary and sufficient that there is an element* $y\not\leqq a$ *such that* $xy\leqq a$.

The concept of zero divisor in the context of an r-lattice was introduced by Anderson [1]. In our context we can restate it as follows.

Let L be a Noether lattice and $x\in L$. x is called a zero divisor if $(0:x)\neq 0$, i.e., there exists at least one $y\neq 0$ such that $xy=0$.

We now relate zero divisors with associate primes in the following result.

PROPOSITION 13. *In L the join of all zero divisors is contained in the join of all associated prime elements of* 0.

PROOF. If x is a zero divisor then $(0:x)\neq 0$. By applying Corollary 11 to $a=0$, we get $x\leqq p_i$ for some associated prime p_i of 0, where $\bigwedge_{i=1}^{m} q_i$ is an irredundant decomposition of 0, and we are done. Q.e.d.

REFERENCES

[1] ANDERSON, D. D., Abstract commutative ideal theory without chain condition, *Algebra Universalis* **6** (1976), 131—145. *MR* **54**#7332.
[2] DILWORTH, R. P., Abstract commutative ideal theory, *Pacific J. Math.* **12** (1962), 481—498. *MR* **26**#1333.
[3] DILWORTH, R. P. and CRAWLEY, P., Decomposition theory for lattices without chain conditions, *Trans. Amer. Math. Soc.* **96** (1960), 1—22. *MR* **22**#9461.
[4] MURATA, K., Primary decomposition of elements in compactly generated integral multiplicative lattices, *Osaka J. Math.* **7** (1970), 97—115. *MR* **43**#3178.
[5] ZARISKI, O. and SAMUEL, P., *Commutative algebra*, Vol. 1, Graduate Texts in Mathematics, No. 28, Springer-Verlag, New York—Heidelberg—Berlin, 1975. *MR* **52**#5641.

(Received November 26, 1981)

DEPARTMENT OF MATHEMATICS
JAIHIND COLLEGE OF
ARTS, SCIENCE AND COMMERCE
IND—424 002 DHULE (MAHARASHTRA)

L.B.S. COLLEGE
SATARA (MAHARASHTRA)
INDIA

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 21—41

СУММЫ МУЛЬТИПЛИКАТИВНЫХ ФУНКЦИЙ

Б. В. ЛЕВИН и Н. М. ТИМОФЕЕВ

Целью настоящей работы является получение оценок сумм мультипликативных функций в зависимости от их поведения в среднем на степенях простых чисел.

В дальнейшем $f(n)$—мультипликативная функция.

Определение 1. Условимся говорить, что $f\in V$, если

$$\sum_{p,\,r\geqq 2}\frac{|f(p^r)|}{p^r}\log^2 p^r=O(1)$$

и

$$\sum_{p}\frac{|f(p)|^2}{p^2}\log^2 p=O(1).$$

Введем следующие обозначения

$$\sigma(f,x)=\frac{1}{x}\sum_{p\leqq x}f(p)\log p,\quad M(f,x)=\sup_{u\leqq x}|\sigma(f,u)|.$$

Определение 2. Пусть $\mu(x)$ монотонно убывает. $f\in S(\mu)$ означает, что

$$\int_0^1|\sigma(f,x^u)-\sigma(f,x)|^2\,du\leqq\mu^2(x),$$

$$\sup_{1/3\leqq u\leqq 1}|\sigma(f,x^u)-\sigma(f,x)|\leqq\mu(x),$$

$$\frac{\log\log x}{\mu(x)\sqrt{\log x}}\leqq M(|f|,x)\leqq\sqrt{\log\log x}.$$

В дальнейшем изучаются только $f\in V\cap S(\mu)$.

В частности доказывается следующая

1980 *Mathematics Subject Classification.* Primary 10H25.

Key words and phrases. Multiplicative function, asymptotical behaviour of sums of the multiplicative functions.

Теорема 2. *Пусть* $f\in V\cap S(\mu)$, $f(n)\geqq 0$, *тогда*

$$m(f,x)=\frac{1}{x}\sum_{n\leqq x}f(n)=\frac{\exp\left[-\gamma\sigma(f,x)+\sum_{p\leqq x}f(p)/p\right]}{\Gamma(\sigma(f,x))\log x}\times$$

(A)

$$\times\left[c(f)+O\left(\mu(x)M^2(f,x)\cdot\Gamma(\sigma(f,x))+M(f,x)\sqrt{\mu(x)\Gamma(\sigma(f,x))}\right)\right],$$

где γ*—постоянная Эйлера,* $\Gamma(s)$*—гамма-функция Эйлера,*

$$c(f)=\prod_p\left(1+\sum_{r=1}^{\infty}f(p^r)/p^r\right)e^{-f(p)/p}.$$

В наших предположениях $c(f)$—сходится.

Теорема 2 содержит, в частности, теорему Е. Вирзинга [1]. Если положить, что $\sigma(f,x)\to\sigma\neq 0$, то найдется $\mu(x)$ такое, что $f\in S(\mu)$($M(f,x)$ и $\Gamma(\sigma(f,x))$ в этом случае ограничены) и (A) перепищется так:

$$m(f,x)=\frac{1}{\Gamma(\sigma(f,x))\log x}\exp\left[-\gamma\sigma(f,x)+\sum_{p\leqq x}\frac{f(p)}{p}\right]\cdot\left[c(f)+O\left(\sqrt{\mu(x)}\right)\right].$$

Если же $\sigma(f,x)\to 0$, то вместо

$$m(f,x)=o\left(\exp\left(\sum_{p\leqq x}\frac{f(p)-1}{p}\right)\right),$$

полученой в [1] из (A) получаем

$$m(f,x)=O\left(\exp\left(\sum_{p\leqq x}\frac{f(p)-1}{p}\right)\cdot(\mu(x)+\sigma(f,x))\right),$$

а в случае, если $f\in V\cap S(\mu)$, где $\mu(x)=o(\sigma(f,x))$ даже выделяется главный член

$$m(f,x)=\frac{1}{\Gamma(\sigma(f,x))\log x}\exp\left[-\gamma\sigma(f,x)+\sum_{p\leqq x}\frac{f(p)}{p}\right]\left[c(f)+O\left(\sqrt{\frac{\mu(x)}{\sigma(f,x)}}\right)\right].$$

Кроме того условие $f\in S(\mu)$, гораздо слабее существования конечного предела $\sigma(f,x)$ при $x\to\infty$ и допускает некоторый рост $\sigma(f,x)$. Например, если $\sigma(f,x)=$ $=(\log\log x)^{\alpha}$, $\alpha\leqq\frac{1}{2}$, то

$$m(f,x)=O\left(e^{-\gamma(\log\log x)^{\alpha}}\frac{(\log\log x)^{\alpha}}{\log x}e^{\sum_{p\leqq x}f(p)/p}\right),$$

а если $f\in V\cap S(\mu)$, при

$$\mu(x)=o\left(M^{-2}(f,x)\sigma(f,x)^{-\sigma(f,x)+1/2}e^{\sigma(f,x)}\right),$$

то происходит выделение главного члена

$$m(f,x)=\frac{\exp[-\gamma\sigma(f,x)+\sum_{p\le x}f(p)/p]}{\Gamma(\sigma(f,x))\log x}\times$$
$$\times\left(c(f)+O(M^2(f,x)\mu(x)\sigma(f,x)^{\sigma(f,x)-1/2}e^{-\sigma(f,x)})\right).$$

Соотношение (А) применимо и в случае, когда $\sigma(f,x)$ вообще не имеет предела. Условия $f\in S(\mu)$ означают по существу «медленное» изменение $\sigma(f,x)$.

В работе кроме того доказываются

Теорема 3. *Пусть $f(n)$—комплекснозначная мультипликативная функция $f\in \mathbf{V}\cap \mathbf{S}(\mu)$ и кроме того* $\operatorname{Re}\sigma(f,x)\ge 0$, $\operatorname{Im}\sigma(f,x)=O(1)$,

$$\sum_{y\le p\le z}\frac{|f(p)|}{p}\log p\le CM(|f|,x)\left(\log\frac{z}{y}+\mu_1(x)\right)$$

для всех $y,z\in\left[\frac{1}{4}\log x, x\right]$,

$$\sum_{\substack{\alpha\ge 2\\ p^\alpha\ge\frac{1}{4}\log x}}|f(p^\alpha)|/p^\alpha\le CM(|f|,x)\mu_1(x),$$

где $\mu_1(x)$ монотонно убывает, тогда

$$m(f,x)=\frac{1}{x}\sum_{n\le x}f(n)=\frac{1}{\Gamma(\sigma(f,x))\log x}\exp\left[-\gamma\sigma(f,x)+\sum_{p\le x}f(p)/p\right]\times$$
$$\times\left[c(f)+O\left(\exp\left[\sum_{p\le x}\frac{|f(p)|-\operatorname{Re}f(p)}{p}+\operatorname{Re}\frac{\gamma}{2}\sigma(f,x)\right]\times\right.\right.$$
$$\left.\left.\times\left[(M^{3/2}(|f|,x)\sqrt{\mu(x)}+\mu_1(x)M(|f|,x)\cdot e^{(\gamma/2)\operatorname{Re}\sigma(f,x)})|\Gamma(\sigma(f,x))|\right]\right.\right].$$

Вначале доказывается сформулированная ниже теорема 1, а затем с помощью асимптотического дифференцирования получается теорема 2 и 3.

Теорема 1. *Пусть $f(n)$—комплекснозначная мультипликативная функция, $f\in V\cap S(\mu)$,* $\operatorname{Re}\sigma(f,x)\ge 0$, $\operatorname{Im}\sigma(f,x)=O(1)$, *тогда*

$$l(x)=\frac{1}{x}\sum_{n\le x}f(n)\log n\log\frac{x}{n}=\frac{\exp\left(-\gamma\sigma(f,x)+\sum_{p\le x}\frac{f(p)}{p}\right)}{\Gamma(\sigma(f,x))}\times$$
$$\times\left(c(f)+O(e^{\sum_{p\le x}\frac{|f(p)|-\operatorname{Re}f(p)}{p}}\cdot\mu(x)M^2(|f|,x)|\Gamma(\sigma(f,x))|\right).$$

Отметим, что $f(n)$ может зависеть от x и от других параметров. Все ограничения на f предпологаются равномерными по всем параметрам и все результаты также равномерны по всем параметрам от которых зависит $f(n)$.

Зависимость от параметров дает возможность применять полученные результаты к суммам вида

$$\frac{1}{x}\sum_{n\leqq x}\exp\left(i\xi\,\frac{g(n)-A(x)}{B(x)}\right),$$

где $g(n)$—аддитивная функция, получать для них, при некоторых ограничениях на $g(n)$, асимптотические формулы с остатком и, следовательно, интегральные теоремы с остатком.

Метод доказательства аналитический:

1° Для оценки $m(f, x)$ рассмотрим вначале

$$l(x)=\frac{1}{x}\sum_{n\leqq x} f(n)\log n\log\frac{x}{n}. \tag{1}$$

Пусть $g(n)=g(n, y)$—мультипликативная функция такая, что

$$g(p^r, y)=\begin{cases} f(p^r), & \text{при} \quad r\geqq 2,\\ f(p), & \text{при} \quad p\leqq y, \quad r=1,\\ \sigma(f, x), & \text{при} \quad p>y, \quad r=1,\end{cases} \tag{2}$$

где $y\in\left[\frac{x}{2}, 2x\right]$, и

$$l(x, y)=\frac{1}{x}\sum_{n\leqq x} g(n, y)\log n\log\frac{x}{n}.$$

Очевидно, что $l(x, x)=l(x)$. Положив

$$F(s)=\sum_{n=1}^{\infty} g(n)/n^s, \qquad s=\sigma+it,$$

получаем при $\sigma>1$

$$l(x, y)=\frac{-1}{2\pi i}\int_{(\sigma)}\frac{x^{s-1}}{s^2}F'(s)\,ds. \tag{3}$$

2° Ясно, что поведение $F(s)$ зависит, в основном, от значений $g(n)$ в простых числах при некоторых довольно слабых ограничениях на $g(p^r)$. Покажем это. Пусть

$$\Phi(s)=\exp\left(\sum_p\frac{g(p)}{p^s}\right), \quad G(s)=\frac{F(s)}{\Phi(s)}=\sum_{n=1}^{\infty}\frac{a_n}{n^s}. \tag{4}$$

Лемма 1. *Пусть* $f\in V$, *то есть*

$$\sum_{p,\, r\geqq 2}\frac{|f(p^r)|}{p^r}\log^2 p^r=O(1),$$

(5)

$$\sum_p\frac{|f(p)|^2}{p^2}\log^2 p=O(1),$$

и

$$\sigma(f, x)=O(\sqrt{x/\log x}),$$

тогда $G(s)$, $G'(s)$, $G''(s)$ *регулярны при* $\sigma \geqq 1$ *и*

$$\sum_n \frac{|a_n|}{n} \log^2 n = O(1). \tag{6}$$

Доказательство. Фактически для доказательства леммы достаточно доказать (6). Представим $G(s)$ в виде $G(s) = G_1(s) \cdot G_2(s)$,

$$G_1(s) = \prod_{p \leqq M} \left(1 + \sum_{r=1}^{\infty} \frac{g(p^r)}{p^{rs}}\right) e^{-\frac{g(p)}{p^s}},$$

$$G_2(s) = \exp\left[\sum_{p>M} \log\left(1 + \sum_{r=1}^{\infty} \frac{g(p^r)}{p^{rs}}\right) - \frac{g(p)}{p^s}\right],$$

где M выбрано так, чтобы при $p > M$ выполнялось неравенство

$$\sum_{r=2}^{\infty} \frac{|f(p^r)|}{p^r} \log p^r + \frac{|g(p)|}{p} \log p < \frac{1}{4}. \tag{7}$$

Существование такого M вытекает из условий леммы. Тогда

$$G''(s) = \sum_n \frac{a_n \log^2 n}{n^s} = G_1'' G_2 + 2G_1' G_2' + G_2'' G_1. \tag{8}$$

Заметим, что если

$$\sum_{n=1}^{\infty} \frac{d_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

или

$$\sum_{n=1}^{\infty} \frac{d_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s} + \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

и

$$\sum_{n=1}^{\infty} |b_n|/n = O(1), \qquad \sum_{n=1}^{\infty} |c_n|/n = O(1),$$

то

$$\sum_n |d_n|/n = O(1).$$

Рассмотрим G_1''. Функция $G_1(s)$ представлена в виде конечного произведения рядов Дирихле

$$\sum_{n=1}^{\infty} \gamma_n/n^s = \left(1 + \sum_{r=1}^{\infty} f(p^r)/p^{rs}\right) e^{-f(p)/p^s}.$$

В силу (5)

$$\sum_{n=1}^{\infty} \frac{|\gamma_n| \log^2 n}{n} = O(1).$$

Отсюда на основании сделанного выше замечания,

$$\sum_{n=1}^{\infty} \frac{|g_1(n)| \log^2 n}{n} = O(1), \tag{9}$$

где $g_1(n)$—коэффициенты ряда Дирихле $G_1(s)$. Пусть

$$G_2(s)=\sum_{n=1}^{\infty}\frac{g_2(n)}{n^s}=\exp(A(s)),\quad A(s)=\sum_n\frac{\beta_n}{n^s}.$$

Ясно, что для доказательства неравенства

$$\sum_n |g_2(n)|\log^2 n/n=O(1) \tag{10}$$

достаточно показать, что имеет место оценка

$$\sum_n |\beta_n|\log^2 n/n=O(1). \tag{11}$$

Докажем (11). Так как

$$A''(s)=\sum_{p>M}\Biggl\{\sum_{r=2}^{\infty}\frac{g(p^r)\log p^r}{p^{rs}}+$$
$$+\sum_{k=2}^{\infty}\Biggl[(k-1)(-1)^k\Bigl(\sum_{r=1}^{\infty}\frac{g(p^r)}{p^{rs}}\Bigr)^{k-2}\Bigl(\sum_{r=1}^{\infty}\frac{g(p^r)}{p^{rs}}\log p^r\Bigr)^2+$$
$$+(-1)^k\Bigl(\sum_{r=1}^{\infty}\frac{g(p^r)}{p^{rs}}\Bigr)^{k-1}\Bigl(\sum_{r=1}^{\infty}\frac{g(p^r)}{p^{rs}}\log^2 p^r\Bigr)\Biggr]\Biggr\},$$

то из (5) и выбора M, получаем

$$\sum_n |\beta_n|\log^2 n/n=O(1).$$

На основании (8), (10) и (9) получаем доказательство леммы 1.

3° На основании представления (4) и (2)

$$l(x,y)=\frac{-1}{2\pi i}\int_{(\sigma)}\frac{x^{s-1}}{s^2}G'(s)\Phi(s)\,ds+\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^{s-1}}{s^2}G(s)\Phi(s)\sum_p\frac{g(p)\log p}{p^s}\,ds.$$

Полагая

$$H(s)=\frac{\zeta'}{\zeta}(s)+\sum_p\log p/p^s,$$

получаем

$$l(x,y)=\frac{-\sigma(f,x)}{2\pi i}\int_{(\sigma)}\frac{x^{s-1}}{s^2}F(s)\frac{\zeta'}{\zeta}(s)\,ds+$$
$$+\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^{s-1}}{s^2}F(s)\sum_{p\le y}\frac{f(p)-\sigma(f,x)}{p^s}\log p\,ds+ \tag{12}$$
$$+\frac{\sigma(f,x)}{2\pi i}\int_{(\sigma)}\frac{x^{s-1}}{s^2}\Phi(s)G(s)H(s)\,ds-\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^{s-1}}{s^2}G'(s)\Phi(s)\,ds.$$

Для оценки последних двух интегралов докажем лемму.

Лемма 2. *Пусть* $C_1(s)$ *и* $C_2(s)$ *ряды Дирихле с коэффициентами* $c_1(n)$ *и* $c_2(n)$, *где* $c_2(n)$ — *мультипликативная функция, причем*

$$\sum_{n} \frac{|c_1(n)| \log n}{n} = O(1), \qquad \sum_{r \geq 2,\, p \leq x} \frac{|c_2(p^r)| \log p^r}{p^r} = O(1). \tag{13}$$

Тогда

$$I = \frac{1}{2\pi i} \int_{(\sigma)} \frac{x^{s-1}}{s^2} C_1(s) C_2(s)\, ds = O\left(\frac{1}{\log x} M(|c_2|, x) \cdot \Pi(|c_2|, x)\right),$$

где

$$\Pi(|c_2|, x) = \prod_{p \leq x} \left(1 + \sum_{r=1}^{\infty} \frac{|c_2(p^r)|}{p^r}\right), \qquad M(|c_2|, x) = \max_{u \leq x} \sigma(|c_2|, u).$$

Доказательство. Из определения I имеем

$$I = \frac{1}{x} \sum_{n \leq x} \sum_{kl = n} c_1(k) c_2(l) \log x/n = \frac{1}{x} \int_1^x \sum_{k \leq n} c_1(k) \sum_{l \leq u/k} c_2(l) \frac{du}{u}.$$

Оценим внутреннюю сумму под знаком интеграла.

$$\sum_{n \leq v} |c_2(n)| \leq \sum_{n \leq \sqrt{3v}} |c_2(n)| + \frac{2}{\log 3v} \sum_{n \leq v} |c_2(n)| \log n \leq$$

$$\leq \sqrt{3v}\, \Pi(|c_2|, \sqrt{3v}) + \frac{2}{\log 3v} \sum_{n \leq v} |c_{2}, (n)| \sum_{p^r \leq \frac{v}{n}} |c_2(p^r)| \log p^r \leq \tag{13'}$$

$$\leq \sqrt{3v}\, \Pi(|c_2|, \sqrt{3v}) + \frac{2v M(|c_2|, v)}{\log 3v} \Pi(|c_2|, v) +$$

$$+ \frac{2v}{\log 3v} \sum_{n \leq v} \frac{|c_2(n)|}{n} \sum_{\substack{p^r \leq \frac{v}{n} \\ r \geq 2}} \frac{|c_2(p^r)|}{p^r} \log p^r = O\left(\frac{v M(|c_2|, v)}{\log 3v} \Pi(|c_2|, v)\right).$$

Подставляя эту оценку в выражение для I, находим

$$I = O\left(M(|c_2|, x) \Pi(|c_2|, x) \frac{1}{x} \int_1^x \sum_{k \leq u} \frac{|c_1(k)|}{k \log 3u/k}\, du\right) =$$

$$= O\left(M(|c_2|, x) \Pi(|c_2|, x) \frac{1}{x} \int_3^x \frac{1}{\log u} \left(\sum_{k \leq \sqrt{u}} \frac{|c_1(k)|}{k} + \sum_{k \leq u} \frac{|c_1(k)|}{k} \log k\right) du\right) =$$

$$= O\big(M(|c_2|, x) \Pi(|c_2|, x)/\log x\big).$$

Лемма доказана.

Выбирая в качестве $C_2(s)$ функцию $\Phi(s)$, а в качестве $C_1(s)$ функции $G(s)H(s)$ и $G'(s)$ соответсвенно и применяя лемму 2, к третьему и четвертому интегралу в (12) получаем, учитывая, что

$$M(|f|, x) \leqq \sqrt{\log\log x},$$

$$l(x, y) = \frac{-\sigma(f, x)}{2\pi i} \int\limits_{(\sigma)} \frac{x^{s-1}}{s^2} F(s) \cdot \frac{\zeta'}{\zeta}(s)\, ds + \tag{14}$$

$$+\frac{1}{2\pi i} \int\limits_{(\sigma)} \frac{x^{s-1}}{s^2} F(s) \sum_{p \leqq y} \frac{f(p)-\sigma(f, x)}{p^s} \log p\, ds + O\left\{\frac{\log\log x}{\log x} \exp\left(\sum_{p \leqq x} \frac{|g(p)|}{p}\right)\right\}.$$

Для обоснования применимости леммы 2 к оценке последних двух интегралов в (12) достаточно доказать, что для $f_1(n)$—коэффициентов ряда Дирихле $\Phi(s)$ выполнены условие (13). Действительно $f_1(p^r) = \frac{(g(p))^r}{r!}$ при $p \leqq x$, поэтому

$$\sum_{p \leqq x} \sum_{r=2}^{\infty} \frac{|f_1(p^r)|}{p^r} \log p^r \leqq \sum_{p \leqq \frac{x}{2}} \log p \sum_{r \geqq 2} \frac{|f(p)|^r}{(r-1)!\, p^r} + \sum_{p > \frac{x}{2}} \log p \sum_{r \geqq 2} \frac{M^r(|f|, x)}{(r-1)!\, p^r} = O(1)$$

на основании (5) и неравенства $M(|f|, x) \leqq \sqrt{\log\log x}$.

4°. Представим, теперь, $F(s)$ в виде

$$F(s) = \sum_{n \leqq \sqrt[3]{x}} \frac{g(n)}{n^s} + \sum_{n > \sqrt[3]{x}} \frac{g(n)}{n^s} = F_0(s) + [F(s) - F_0(s)].$$

Тогда второй интеграл в (14), представим в виде суммы двух интегралов, которые мы обозначим через I_1 и I_2.

Для I_1 имеем

$$|I_1| = \left|\frac{1}{x} \int\limits_1^x \sum_{\substack{k \leqq u \\ k \leqq \sqrt[3]{x}}} g(k) \sum_{p \leqq \frac{u}{k}} (g(p) - \sigma(f, x)) \log p \frac{du}{u}\right| \leqq$$

$$\leqq \frac{2M(|f|, x)}{x} \int\limits_1^{x/\log x} \sum_{k \leqq u} \frac{|g(k)|}{k}\, du + \frac{1}{x} \int\limits_{x/\log x}^{x} \sum_{k \leqq \sqrt[3]{x}} \frac{|f(k)|}{k} \left(\left|\sigma\left(f, \frac{u}{k}\right) - \sigma(f, x)\right| + \right.$$

$$\left. + O\left(\exp\left(-\sqrt{\log \frac{u}{k}}\right)\right)\right) du \ll \frac{M(|f|, x)}{\log x} \Pi(|g|, x) + \mu(x) \Pi(|f|, \sqrt[3]{x}) + \tag{15}$$

$$+ \frac{1}{\log x} \Pi(|f|, \sqrt[3]{x}) \ll \frac{\sqrt{\log\log x}}{\log x} \exp\left(\sum_{p \leqq x} \frac{|g(p)|}{p}\right) + \mu(x) \Pi(|f|, \sqrt[3]{x}).$$

Здесь и в дальнейшем используется, что $f \in S(\mu)$ и поэтому $M(|f|, x) \leqq \leqq \sqrt{\log\log x}$, $\mu(x) \geqq \sqrt{\log\log x / \log x}$.

Для оценки I_2 воспользуемся неравенством Шварца и следующим соображением: если

$$\Psi(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

то

$$\frac{1}{s}\Psi(s) = \int_1^{\infty} \sum_{n\le u} b(n)\frac{du}{u^{s+1}} = \int_0^{\infty} \sum_{n\le e^u} b(n)e^{-\sigma u}e^{-itu}\,du$$

и равенство Парсеваля дает

$$\frac{1}{2\pi}\int_{(\sigma)}\left|\frac{1}{s}\Psi(s)\right|^2|ds| = \int_0^{\infty}\Big|\sum_{n\le e^u} b(n)\Big|e^{-2\sigma u}\,du. \tag{16}$$

На основании неравенства Шварца

$$I_2 \le \sqrt{\frac{1}{2\pi}\int_{(\sigma)}\left|\frac{1}{s}\sum_{p\le y}\frac{f(p)-\sigma(f,x)}{p^s}\log p\right|^2|ds|\cdot\frac{1}{2\pi}\int_{(\sigma)}|F(s)-F_0(s)|^2\frac{|ds|}{|s|^2}} = \sqrt{I_3}\cdot\sqrt{I_4},$$

Здесь, как и всюду в дальнейшем, $\sigma = 1+1/\log x$.

Используем, теперь, (16). Имеем

$$I_3 = \frac{1}{2\pi}\int_{(\sigma)}\left|\frac{1}{s}\sum_{p\le y}\frac{f(p)-\sigma(f,x)}{p^s}\log p\right|^2|ds| =$$

$$= \int_0^{\log y}\Big|\sum_{p\le e^u}\big(f(p)-\sigma(f,x)\big)\log p\Big|^2 e^{-2\sigma u}\,du +$$

$$+\Big|\sum_{p\le y}\big(f(p)-\sigma(f,x)\big)\log p\Big|^2\int_{\log y}^{\infty} e^{-2\sigma u}\,du \ll$$

$$\ll \int_0^{\log y} e^{2u}|\sigma(f,e^u)-\sigma(f,x)|^2 e^{-2\sigma u}\,du + M^2(|f|,x)\int_0^{\log y} e^{-\sqrt{u}}\,du + \mu^2(x)\log x.$$

Таким образом,

$$I_3 = O\big(\mu^2(x)\log x\big). \tag{17}$$

Для оценки I_4 поступаем следующем образом. Согласно (16)

$$I_4 = \int_0^{\infty}\Big|\sum_{\sqrt[3]{x}\le n\le e^u} g(n)\Big|^2 e^{-2\sigma u}\,du \le \frac{9}{\log^2 x}\int_0^{\infty}\Big|\sum_{n\le e^u}|g(n)|\log n\Big|^2 e^{-2\sigma u}\,du =$$

$$= \frac{9}{2\pi\log^2 x}\int_{(\sigma)}\left|\frac{1}{s}\sum_{n=1}^{\infty}\frac{|g(n)|\log n}{n^s}\right|^2|ds|.$$

Представим ряд Дирихле

$$F_1(s) = \sum_{n=1}^{\infty}\frac{|g(n)|}{n^s}$$

в виде

$$F_1(s) = \Phi_1(s) G_3(s) = \exp\left(\sum_p |g(p)|/p\right) \cdot G_3(s).$$

Тогда согласно лемме 1, $G_3(s)$ и $G_3'(s)$ ограничены при $\sigma \geqq 1$ и так как

$$F_1' = \Phi_1' G_3 + \Phi_1 G_3',$$

то

$$I_4 \ll \frac{1}{\log^2 x} \exp\left(2 \sum_p \frac{|g(p)|}{p^\sigma}\right) \left(\int\limits_{(\sigma)} \left| \frac{1}{s} \sum_p \frac{|g(p)| \log p}{p^s} \right|^2 |ds| + 1 \right) \ll$$

$$\ll \frac{1}{\log^2 x} \exp\left(2 \sum_p \frac{|g(p)|}{p^\sigma}\right) \left(\int\limits_0^\infty \Big| \sum_{p \leqq e^u} |g(p)| \log p \Big|^2 e^{-2\sigma u}\, du + 1 \right) \ll$$

$$\ll \frac{M^2(|f|, x)}{\log x} \exp\left(2 \sum_p |g(p)|/p^\sigma\right).$$

Таким образом,

$$I_2 = O\Big(M(|f|, x) \mu(x) \exp\left(\sum_p |g(p)|/p^\sigma\right)\Big).$$

Следовательно, отсюда и из (14), (15) получаем

$$l(x, y) = -\frac{\sigma(f, x)}{2\pi i} \int\limits_{(\sigma)} \frac{x^{s-1}}{s^2} F(s) \frac{\zeta'}{\zeta}(s)\, ds +$$

(18)

$$+ O\Big(\mu(x) \Pi(|f|, \sqrt[3]{x}) + M(|f|, x) \mu(x) \exp\left(\sum_p |g(p)|/p^\sigma\right)\Big).$$

5° Исследуем теперь интеграл входящий в (18). Так как $\sigma = 1 + \dfrac{1}{\log x}$, то его можно представить в виде

(19)
$$\sigma(f, x) I_5 + O\left(\frac{|\sigma(f, x)|}{\log x} \exp\left(\sum_p |g(p)|/p^\sigma\right)\right),$$

где

$$I_5 = -\frac{1}{2\pi i} \int\limits_{D(\sigma)} \frac{x^{s-1}}{s^2} F(s) \frac{\zeta'}{\zeta}(s)\, ds,$$

$$D(\sigma) = \{s\colon \operatorname{Re} s = \sigma,\ |t| \leqq \log^2 x\}.$$

Проинтегрировав I_5 по частям и воспользовавшись тем, что при $|t| \leqq \log^2 x$, $t = \operatorname{Im} s$,

$$\int\limits_{-\log^2 x}^{t} \frac{x^{iu}}{(\sigma + iu)^2}\, du = \frac{x^{it}}{i(\sigma + it) \log x} + O\left(\frac{1}{|\sigma + it|^2 \log^2 x}\right),$$

получаем

$$I_5 = \frac{1}{2\pi i \log x} \int\limits_{D(\sigma)} \frac{x^{s-1}}{s^2} \left(G(s) \Phi(s) \frac{\zeta'}{\zeta}(s)\right)' ds +$$

$$+ O\left(\frac{1}{\log^2 x} \int\limits_{D(\sigma)} \frac{1}{|s|^2} \left| \left(G(s) \Phi(s) \frac{\zeta'}{\zeta}(s)\right)' \right| |ds| + \frac{1}{\log^4 x} \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right).$$

При выводе этой оценки, как и ниже, используется ограниченность $G(s)$ и $G'(s)$ при $\sigma \geqq 1$ и оценка $(\zeta'/\zeta)(\sigma)$. После несложных преобразований получим

$$
I_5 = \frac{1}{2\pi i \log x} \int\limits_{D(\sigma)} \frac{x^{s-1}}{s^2} G(s) \left(\Phi(s) \frac{\zeta'}{\zeta}(s)\right)' ds + O\left[\frac{1}{\log x} \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right] \times
$$

(20)

$$
\times \left(\int\limits_{D(\sigma)} \left| \left|\frac{\zeta'}{\zeta}(s)\right| + \frac{1}{\log x} \left|\sum_p \frac{g(p)\log p}{p^s}\right| \left|\frac{\zeta'}{\zeta}(s)\right| + \frac{1}{\log x} \left|\left(\frac{\zeta'}{\zeta}(s)\right)'\right| \right| \frac{|ds|}{|s|^2} \right).
$$

Так как

$$
\sum_p \frac{|g(p)| \log p}{p^\sigma} \leqq \int\limits_2^y \frac{1}{u} d \sum_{p \leqq u} |f(p)| \log p + M(|f|, x) \int\limits_y^\infty \frac{du}{u^\sigma} \ll M(|f|, x) \log x
$$

и

$$
\int\limits_{D(\sigma)} \left| \left(\frac{\zeta'}{\zeta}(s)\right)^{(\nu)} \right| \frac{|ds|}{|s|^2} = \begin{cases} O(\log\log x), & \text{при} \quad \nu = 0, \\ O(\log x), & \text{при} \quad \nu = 1, \end{cases}
$$

то на основании (20)

$$
I_5 = \frac{-1}{2\pi i \log x} \int\limits_{D(\sigma)} \frac{x^{s-1}}{s^2} F(s) \sum_p \frac{g(p) \log p}{p^s} \cdot \frac{\zeta'}{\zeta}(s)\, ds +
$$

(21)

$$
+ \frac{1}{2\pi i \log x} \int\limits_{D(\sigma)} \frac{x^{s-1}}{s^2} F(s) \left(\frac{\zeta'}{\zeta}(s)\right)' ds + O\left[\frac{(\log\log x)^2}{\log x} \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right].
$$

Первый и второй интегралы (21) обозначим I_6 и соответственно I_7.

$$
I_6 = \frac{\sigma(f, x)}{2\pi i \log x} \int\limits_{D(\sigma)} \frac{x^{s-1}}{s^2} F(s) \left(\frac{\zeta'}{\zeta}(s)\right)^2 ds +
$$

$$
+ O\left[\frac{1}{\log x} \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right) \left(\int\limits_{D(\sigma)} \frac{1}{|s|^2} \left| \sum_{p \leqq y} \frac{f(p) - \sigma(f, x)}{p^s} \log p \right| \left|\frac{\zeta'}{\zeta}(s)\right| |ds| + \right.\right.
$$

$$
\left.\left. + |\sigma(f, x)| \int\limits_{D(\sigma)} \frac{1}{|s|^2} \left|\frac{\zeta'}{\zeta}(s)\right| |H(s)| |ds| \right)\right].
$$

Для оценки первого остатка применяем неравенство Шварца и оценку (17), а для оценки второго вспомним что $H(s) = O(1)$ при $\sigma \geqq 1$ а оставившийся интеграл оценивается, как $O(\log\log x)$.

$$
I_6 = \frac{\sigma(f, x)}{2\pi i \log x} \int\limits_{D(\sigma)} \frac{x^{s-1}}{s^2} F(s) \left(\frac{\zeta'}{\zeta}(s)\right)^2 ds + O\left[\mu(x) \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right].
$$

В интеграле I_7 и в первом члене последнего выражения для I_6 воспользуемся соответственно соотношениями

$$
\left(\frac{\zeta'}{\zeta}(s)\right)' = \frac{1}{(s-1)^2} + O\left(\frac{|s|^\delta}{|s-1|}\right) \quad \text{и} \quad \left(\frac{\zeta'}{\zeta}(s)\right)^2 = \frac{1}{(s-1)^2} + O\left(\frac{|s|^\delta}{|s-1|}\right)
$$

справедливыми в области $D(\sigma)$ с любым $\delta > 0$. Тогда

$$I_5 = \frac{\sigma(f, x)+1}{2\pi i \log x} \int_{D(\sigma)} \frac{x^{s-1}}{s^2} F(s) \frac{ds}{(s-1)^2} +$$

$$(22) \quad + O\left[\mu(x) \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right) + \frac{M(|f|, x)}{\log x} \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right) \int_{D(\sigma)} \frac{|ds|}{|s|^{2-\delta}|s-1|}\right] =$$

$$= \frac{\sigma(f, x)+1}{2\pi i \log x} \int_{D(\sigma)} \frac{x^{s-1}}{s^2(s-1)^2} F(s)\, ds + O\left[\mu(x) \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right].$$

Последний интеграл равен

$$(23) \qquad \int_{T(\sigma)} \frac{x^{s-1}}{s^2(s-1)^2} F(s)\, ds + O\left[\frac{\log x}{k(x)} \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right],$$

где

$$T(\sigma) = \{s\colon \operatorname{Re} s = \sigma = 1 + 1/\log x, \ |\operatorname{Im} s| \leq k(x)/\log x\}.$$

Так как $G'(s) = O(1)$, то при $|t| \leq k(x)/\log x$

$$F(s) = G(s)\Phi(s) = G(\sigma)\Phi(s) + O\left[\frac{k(x)}{\log x} \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right].$$

Отсюда из (22) и (23) получаем

$$I_5 = \frac{\sigma(f, x)+1}{2\pi i \log x} G(\sigma) \int_{T(\sigma)} \frac{x^{s-1}}{s^2(s-1)^2} \Phi(s)\, ds +$$

$$(24) \quad + O\left[\exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\left(\mu(x) + \frac{1+|\sigma(f, x)|}{k(x)}\left(1 + \frac{k^2(x)}{\log x}\right)\right)\right].$$

И, следовательно, при $k(x)$ удовлетворяющем неравенству

$$(25) \qquad \frac{\mu(x)}{1+|\sigma(f, x)|} \leq k(x) \leq \sqrt{\log x}$$

из (18), (19) и (24) получаем

$$l(x, y) = \frac{\sigma(f, x)(\sigma(f, x)+1)}{2\pi i \log x} G(\sigma) \int_{T(\sigma)} \frac{x^{s-1}}{s^2(s-1)^2} \Phi(s)\, ds +$$

$$(26) \quad + O\left[\mu(x)\Pi(|f|, \sqrt[3]{x}) + \mu(x) M(|f|, x) \exp\left(\sum_p \frac{|g(p)|}{p^\sigma}\right)\right].$$

6°. Для изучение главного члена $l(x, y)$ надо изучить

$$I_8 = \frac{1}{2\pi i \log x} \int_{T(\sigma)} \frac{x^{s-1}}{s^2(s-1)^2} \frac{\Phi(s)}{\Phi(\sigma)}\, ds.$$

Вспоминая определение $T(\sigma)$ и делая замену переменных, получаем

(27) $$I_8=\frac{1}{2\pi i}\int\limits_{L(x)}\frac{e^z}{z^2}\,\frac{\Phi(1+z/\log x)}{\Phi(1+1/\log x)}\,dz+O\left(\frac{|\Phi(\sigma)|^{-1}}{\sqrt{\log x}}\exp\left(\sum_p\frac{|g(p)|}{p^\sigma}\right)\right),$$

где $L(x)=\{z\colon \operatorname{Re} z=1,\ |\operatorname{Im} z|\leqq k(x)\}$.

Исследуем поведение $B(1+z/\log x)=\dfrac{\Phi(1+z/\log x)}{\Phi(1+1/\log x)}$. Избавимся вначале от зависимости $B(s)$ от y. Имеем

$$B(s)=\exp\left[\sum_{p\leqq y}f(p)/p^s-\sum_{p\leqq y}\frac{f(p)}{p^\sigma}+\sigma(f,x)\sum_{p>y}\frac{1}{p^\sigma}\left(\frac{1}{p^{it}}-1\right)\right]=$$

$$=\exp\left[\sum_{p\leqq x}\frac{f(p)}{p^s}-\sum_{p\leqq x}\frac{f(p)}{p^\sigma}+\sigma(f,x)\sum_{p>x}\left(\frac{1}{p^s}-\frac{1}{p^\sigma}\right)\right]\times$$

$$\times\left\{1+O\left(\sum_{(x/2)\leqq p\leqq 2x}\frac{|f(p)|}{p}+|\sigma(f,x)|\sum_{(x/2)\leqq p\leqq 2x}\frac{1}{p}\right)\right\}.$$

Отсюда, учитывая что

$$\sum_{(x/2)\leqq p\leqq 2x}\frac{|f(p)|}{p}=\frac{\sigma(|f|,x2)}{\log 2x}-\frac{\sigma(|f|,x/2)}{\log x/2}+$$

$$+\int\limits_{x/2}^{2x}\sigma(|f|,u)\left(\frac{1}{\log u}+\frac{1}{\log^2 u}\right)\frac{du}{u}=O\left(\frac{M(|f|,x)}{\log x}\right),$$

получаем

$$B(s)=\exp\left(\sum_p\frac{g(p,x)}{p^s}-\sum_p\frac{g(p,x)}{p^\sigma}\right)\left(1+O\left(\frac{M(|f|,x)}{\log x}\right)\right).$$

Полагая $s=1+z/\log x$,

$$\varphi(s)=\sum_{p\leqq x}(f(p)-\sigma(f,x))/p^s,$$

и вспоминая определение $H(s)$, получаем

$$B(s)=\frac{\Phi(s)}{\Phi(\sigma)}=\left(\frac{\zeta(s)}{\zeta(\sigma)}\right)^{\sigma(f,x)}\exp\big(\varphi(s)-\varphi(\sigma)+\sigma(f,x)\big(H(s)-H(\sigma)\big)\big)\times$$

(28) $$\times\left(1+O\left(\frac{M(|f|,x)}{\log x}\right)\right)=\left(z^{-\sigma(f,x)}+O\left(\frac{M(|f|,x)}{\log x}\,|z|^{1-\operatorname{Re}\sigma(f,x)}\right)\right)\times$$

$$\times\exp\big(\varphi(s)-\varphi(\sigma)+\sigma(f,x)(H(s)-H(\sigma))\big),$$

так как по предположению $\operatorname{Im}\sigma(f,x)=O(1)$, $k(x)\leqq\sqrt{\log x}$, $M(|f|,x)\leqq$ $\leqq\sqrt{\log\log x}$.

Лемма 3. *При* $|\xi| \leqq k(x)$, $|\beta| \leqq k(x)$, $|\alpha| \leqq \log k(x)$ *равномерно по* α, ξ *и* β

$$\left|\varphi\left(1+\frac{\alpha+i\beta}{\log x}\right)-\varphi\left(1+\frac{1+i\xi}{\log x}\right)+\sigma(f, x)\left[H\left(1+\frac{\alpha+i\beta}{\log x}\right)-H\left(1+\frac{1+i\xi}{\log x}\right)\right]\right| =$$

$$= O\left(|\alpha-1+i(\beta-\xi)| \frac{1}{\sqrt{|\alpha|+1}} e^{|\alpha|} \mu(x)\right).$$

Доказательство. Так как $H'(s)=O(1)$, при $\operatorname{Re} s \geqq \frac{1}{2}$, то

$$\left|H\left(1+\frac{\alpha+i\beta}{\log x}\right)-H\left(1+\frac{1+i\xi}{\log x}\right)\right| \ll \frac{|\alpha-1+i(\beta-\xi)|}{\log x}. \tag{29}$$

С другой стороны, применяя суммирование по Абелю, получаем

$$\varphi(s) = s\int_2^x \frac{\sigma(f, u)-\sigma(f, x)}{u^s \log u}\,du + \int_2^x \frac{\sigma(f, u)-\sigma(f, x)}{u^s \log^2 u}\,du +$$

$$+\sigma(f, x)\int_2^x \frac{1}{u^{s+1}\log u}\left(s+\frac{1}{\log u}\right)\left(\sum_{p \leqq u} \log p - u\right) du.$$

Отсюда получаем

$$\left|\varphi\left(1+\frac{\alpha+i\beta}{\log x}\right)-\varphi\left(1+\frac{1+i\xi}{\log x}\right)\right| \ll \frac{|\alpha-1+i(\beta-\xi)|}{\log x}\int_2^x \frac{|\sigma(f, u)-\sigma(f, x)|}{u^{1+\alpha/\log x}}\,du +$$

$$+|\sigma(f, x)|\frac{|\alpha-1+i(\beta-\xi)|}{\log x} \ll$$

$$\ll |\alpha-1+i(\beta-\xi)|\left(\frac{e^{|\alpha|}}{\sqrt{|\alpha|+1}}\sqrt{\int_0^1 |\sigma(f, x^u)-\sigma(f, x)|^2\,du}+\frac{|\sigma(f, x)|}{\log x}\right) \ll$$

$$\ll |\alpha-1+i(\beta-\xi)| \frac{e^{|\alpha|}}{\sqrt{|\alpha|+1}} \mu(x).$$

Отсюда и из (29) следует утверждение леммы 3.

Пусть

$$\mu(x) k(x) \log k(x) = O(1). \tag{30}$$

Тогда из леммы 3 при $z \in L(x)$, $\xi=0$, учитывая, что в этом случае $\alpha=\operatorname{Re} z=1$, следует, что

$$\varphi(s)-\varphi(\sigma)+\sigma(f, x)\left(H(s)-H(\sigma)\right) = O(1).$$

Напомним, что $s=1+z/\log x$. Отсюда из (28), (27) и (25) следует, что

$$I_8=\frac{1}{2\pi i}\int_{L(x)} e^{z+\varphi(s)-\varphi(\sigma)+\sigma(f,x)(H(s)-H(\sigma))}\frac{dz}{z^{2+\sigma(f,x)}}+ \tag{31}$$
$$+O\left(\frac{|\Phi(\sigma)|^{-1}}{\sqrt{\log x}}\exp\left(\sum_p\frac{|g(p)|}{p^\sigma}\right)+\frac{M(|f|,x)}{\log x}\log\log x\right).$$

Для подсчета интеграла в (31) заметим, что он равен

$$I_9=\frac{1}{2\pi i}\int_\Gamma \frac{e^z}{z^{2+\sigma(f,x)}}e^{\varphi(s)-\varphi(\sigma)+\sigma(f,x)(H(s)-H(\sigma))}\,dz,$$

где

$$\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5\cup\Gamma_6\cup\Gamma_7,$$
$$\Gamma_1=\{z\colon \operatorname{Im} z=k(x),\ -\log k(x)\leqq \operatorname{Re} z\leqq 1\},$$
$$\Gamma_2=\{z\colon \operatorname{Re} z=-\log k(x),\ 0<\operatorname{Im} z\leqq k(x)\},$$
$$\Gamma_3=\{z\colon \operatorname{Im} z=0,\ -\log k(x)\leqq \operatorname{Re} z\leqq -1\},$$
$$\Gamma_4=\{z\colon |z|=1,\ -\pi<\arg z<\pi\},$$

$\Gamma_5, \Gamma_6, \Gamma_7$ симметричны относительно вещественной оси соответственно $\Gamma_3, \Gamma_2, \Gamma_1$. Направление на Γ задано по часовой стрелке. На контурах $\Gamma_0, \Gamma_2, \Gamma_6$ и Γ_7 на основании леммы 3 и выбора $k(x)$

$$\left|\varphi\left(1+\frac{\alpha+i\beta}{\log x}\right)-\varphi\left(1+\frac{1}{\log x}\right)+\sigma(f,x)\left(H\left(1+\frac{\alpha+i\beta}{\log x}\right)-H\left(1+\frac{1}{\log x}\right)\right)\right|\leqq$$
$$\leqq|\alpha-1+i\beta|\frac{e^{|\alpha|}}{\sqrt{|\alpha|+1}}\mu(x)=O(1).$$

Следовательно, учитывая, что $\operatorname{Im}\sigma(f,x)=O(1)$, $\operatorname{Re}\sigma(f,x)\geqq 0$, получаем

$$\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_6\cup\Gamma_7}\exp\left(z+\varphi(s)-\varphi(\sigma)+\sigma(f,x)(H(s)-H(\sigma))\right)\frac{dz}{z^{2+\sigma(f,x)}}=$$
$$=O\left(\int_{-\log k(x)}^{1} e^\alpha\left(\alpha^2+k^2(x)\right)^{-1-\frac{1}{2}\operatorname{Re}\sigma(f,x)}d\alpha+\right.$$
$$\left.+\int_0^{k(x)}\frac{1}{k(x)}\left(\log^2 k(x)+\beta^2\right)^{-1-\frac{1}{2}\operatorname{Re}\sigma(f,x)}d\beta\right)=O\left(\frac{1}{k(x)\log k(x)}\right).$$

Главную часть I_9 дают интегралы по $\Gamma_3, \Gamma_4, \Gamma_5$. Так как из леммы 3, при $\xi=0$, $|\beta|\leqq 1$ следует, что

$$\exp\left(\varphi\left(1+\frac{\alpha+i\beta}{\log x}\right)-\varphi\left(1+\frac{1}{\log x}\right)+\sigma(f,x)\left(H\left(1+\frac{\alpha+i\beta}{\log x}\right)-H\left(1+\frac{1}{\log x}\right)\right)\right)=$$
$$=1+O\left((|\alpha-1|+1)\frac{e^{|\alpha|}}{\sqrt{|\alpha|+1}}\mu(x)\right),$$

следовательно,

$$I_9 = \frac{1}{2\pi i} \int\limits_{\Gamma_2 \cup \Gamma_4 \cup \Gamma_5} e^z z^{-2-\sigma(f,x)} dz + O\left(\frac{1}{k(x)\log k(x)}\right) +$$

$$+ O\left(\mu(x) \int\limits_{-\log k(x)}^{1} \frac{\sqrt{|\alpha|+1}\, e^{\alpha} e^{|\alpha|}}{|\alpha|^{2+\operatorname{Re}\sigma(f,x)}} d\alpha + \mu(x)\right) =$$

$$= \frac{1}{2\pi i} \int\limits_{C} \frac{e^z}{z^{2+\sigma(f,x)}} dz + O\left(\frac{1}{k(x)\log k(x)} + \mu(x)\right),$$

где C-контур Ханкеля, $\mu(x)$ монотонно убывает, а

$$k(x) = \min\left(\frac{1}{\mu(x)\log 1/\mu(x)},\ \sqrt{\log x}\right)$$

удовлетворяет (25) и (30). При таком выборе $k(x)$ получаем

$$I_9 = \frac{1}{\Gamma(2+\sigma(f,x))} + O(\mu(x)).$$

На основании (26) пункта 6° получаем

$$I(x,y) = \frac{F(\sigma)}{\Gamma(\sigma(f,x))} + O\left(\mu(x)\Pi(|f|, \sqrt[3]{x})\right) + \tag{32}$$

$$+ O\left[\exp\left(\sum_p \frac{|g(p)|}{p^{\sigma}}\right)\left(M(|f|,x) + |\sigma(f,x)|(1+|\sigma(f,x)|)\right)\mu(x)\right].$$

7° Для того, чтобы придать единообразное выражение остаткам, покажем в начале, что

$$\sum_{p \le \sqrt[3]{x}} |f(p)|/p \leqq \sum_{p \le x} |f(p)|/p - \gamma\sigma(f,x) + O(1). \tag{33}$$

Действительно, суммируя по Абелю, получаем

$$\sum_{\sqrt[3]{x} \le p \le x} |f(p)|/p \geqq \int\limits_{\sqrt[3]{x}}^{x} \frac{\sigma(|f|,u)}{u\log u} du + O(1/\sqrt{\log x}) \geqq$$

$$\geqq |\sigma(f,x)|\log 3 + O\left(\int\limits_{\sqrt[3]{x}}^{x} |\sigma(f,u) - \sigma(f,x)| \frac{du}{u\log u} + \frac{1}{\sqrt{\log x}}\right),$$

что с учетом ограниченности $\mu(x)$ и неравенства $\log 3 > \gamma$ дает (33).

Далее, так как

$$\sum_p \frac{g(p)}{p^\sigma} - \sum_{p\leqq x} \frac{f(p)}{p} = \sum_{p\leqq x} \frac{f(p)}{p}\left(\frac{1}{p^{\sigma-1}}-1\right)+$$

$$+\sigma(f,x)\sum_{p>y}\frac{1}{p^\sigma}+O\Bigl(\sum_{(x/2)\leqq p\leqq 2x}|f(p)|/p\Bigr)=$$

$$=\int_2^x \sigma(f,u)\frac{1}{u\log u}\left(\frac{1}{u^{\sigma-1}}-1\right)du+\sigma(f,x)\int_y^\infty \frac{du}{u^\sigma \log u}+O\left(\frac{1}{\sqrt{\log x}}\right)=$$

$$=\int_0^1 \sigma(f,x^u)\frac{e^{-u}-1}{u}\,du+\sigma(f,x)\int_1^\infty \frac{e^{-u}}{u}\,du+O\left(\frac{1}{\sqrt{\log x}}\right),$$

то

$$\sum_p g(p)/p^\sigma - \sum_{p\leqq x}\frac{f(p)}{p} = -\gamma\sigma(f,x)+O(\mu(x)).$$

Вполне аналогично

$$\sum_p |g(p)|/p^\sigma - \sum_{p\leqq x}\frac{|f(p)|}{p} =$$

$$=\int_0^1 \sigma(|f|,x^u)\frac{e^{-u}-1}{u}\,du+|\sigma(f,x)|\int_1^\infty \frac{e^{-u}}{u}\,du+O(1)\leqq$$

$$\leqq -\gamma\,\mathrm{Re}\,\sigma(f,x)+O(1).$$

Эти соображения и условие $|\sigma(f,x)|\leqq M(|f|,x)$ позволяют переписать (32) в следующем виде

$$l(x,y)=G(\sigma)\exp\left(-\gamma\sigma(f,x)+\sum_{p\leqq x}\frac{f(p)}{p}\right)\times$$

(34)

$$\times\left(\frac{1}{\Gamma(\sigma(f,x))}+O\left(\exp\left(\sum_{p\leqq x}\frac{|f(p)|-\mathrm{Re}\,f(p)}{p}\right)\mu(x)M^2(|f|,x)\right)\right).$$

Так как $G'(s)=O(1)$ при $\mathrm{Re}\,s\geqq 1$, то

$$G(\sigma)=G\left(1+\frac{1}{\log x}\right)=G(1)+O(1/\log x)=$$

$$=\prod_{p\leqq y}\left(1+\sum_{r=1}^\infty f(p^r)/p^r\right)e^{-\frac{f(p)}{p}}\cdot\left(1+\sum_{\substack{n>y\\ \forall p|n\Rightarrow p>y}}\frac{a_n}{n}\right)+O\left(\frac{1}{\log x}\right)=$$

$$=c(f)+O(1/\log x)$$

на основании леммы 1. Следовательно,

$$
l(x, y) = \exp\left(-\gamma\sigma(f, x) + \sum_{p \le x} \frac{f(p)}{p}\right) \times
$$

(35)

$$
\times\left(\frac{c(f)}{\Gamma(\sigma(f, x))} + O\left(\exp\left(\sum_{p \le x} \frac{|f(p)| - \operatorname{Re} f(p)}{p}\right) \mu(x) M^2(|f|, x)\right)\right).
$$

8° Чтобы перейти от $l(x)$ к $m(x)$, подсчитаем в начале

$$
m_1(x) = \frac{1}{x} \sum_{n \le x} f(n) \log n.
$$

Рассмотрим разность $l(x(1+\varepsilon), x) - l(x, x)$, где $0 < \varepsilon < 1$. С одной стороны, она равна

$$
\frac{\log(1+\varepsilon)}{(1+\varepsilon)x} \sum_{n \le x} f(n) \log n - \frac{\varepsilon}{x(1+\varepsilon)} \sum_{n \le x} f(n) \log \frac{x}{n} \log n +
$$

$$
+ \frac{1}{x(1+\varepsilon)} \sum_{x < n \le x(1+\varepsilon)} f(n) \log \frac{(1+\varepsilon)x}{n} \log n =
$$

$$
= (\varepsilon + O(\varepsilon^2)) m_1(x) - (\varepsilon + O(\varepsilon^2)) l(x, x) + O\left(\frac{\varepsilon}{x} \sum_{x < n \le x(1+\varepsilon)} |f(n)| \log n\right).
$$

Для третьего слагаемого, при $y = \frac{1}{4} \log x$, имеем

$$
\sum_{x < n \le x(1+\varepsilon)} |f(n)| \log n \le \sum_{x < n \le x(1+\varepsilon)} |f(n)| \Big(\sum_{\substack{p^\alpha \| n \\ p^\alpha \le y(1+\varepsilon)}} \log p^\alpha + \sum_{\substack{p^\alpha \| n \\ p^\alpha > y(1+\varepsilon)}} \log p^\alpha \Big) \le
$$

$$
\le \frac{1}{2} \sum_{x < n \le x(1+\varepsilon)} |f(n)| \log n + \sum_{n \le \frac{x(1+\varepsilon)}{y(1+\varepsilon)}} |f(n)| \sum_{\frac{x}{n} \le p^\alpha \le \frac{x(1+\varepsilon)}{n}} |f(p^\alpha)| \log p^\alpha.
$$

Следовательно,

$$
\sum_{x < n \le x(1+\varepsilon)} |f(n)| \log n \le 2 \sum_{n \le \frac{x(1+\varepsilon)}{y}} |f(n)| \sum_{\frac{x}{n} < p \le \frac{x(1+\varepsilon)}{n}} \frac{x(1+\varepsilon)}{n} \frac{|f(p)| \log p}{p} +
$$

$$
+ 2x(1+\varepsilon) \sum_{n \le x} \frac{|f(n)|}{n} \sum_{\substack{p^\alpha > y \\ \alpha \ge 2}} \frac{|f(p^\alpha)|}{p^\alpha}.
$$

Пусть $\mu_1(x)$ такова, что

$$
\sum_{\substack{p^\alpha > y \\ \alpha \ge 2}} \frac{|f(p^\alpha)|}{p^\alpha} + \sum_{x_1 < p \le x_2} \frac{|f(p)|}{p} \log p \le CM(|f|, x) \left(\log \frac{x_2}{x_1} + \mu_1(x)\right)
$$

для всех $x_1, x_2 \in [y, x]$, $y = \frac{1}{4}\log x$. Тогда имеем

$$\frac{1}{x}\sum_{x<n\leqq x(1+\varepsilon)}|f(n)|\log n \ll \exp\left(\sum_{p\leqq x}\frac{|f(p)|}{p}\right)(\varepsilon+\mu_1(x))M(|f|, x).$$

Таким образом,

$$\begin{aligned}l(x(1+\varepsilon), x)-l(x, x) &= (\varepsilon+O(\varepsilon^2))m_1(x)- \\ &-(\varepsilon+O(\varepsilon^2))l(x, x)+O\left(\varepsilon\exp\left(\sum_{p\leqq x}\frac{|f(p)|}{p}\right)(\varepsilon+\mu_1(x))M(|f|, x)\right).\end{aligned}\tag{36}$$

С другой стороны, эту разность можно найти используя (35). Первый член (35) дает

$$\begin{aligned}&c(f)\exp\left(-\gamma\sigma(f, x)+\sum_{p\leqq x}\frac{f(p)}{p}\right)\cdot\frac{1}{\Gamma(\sigma(f, x))}\times \\ &\times\left\{\exp\left(-\gamma(\sigma(f, x(1+\varepsilon))-\sigma(f, x))+\sum_{x<p\leqq x(1+\varepsilon)}\frac{f(p)}{p}\right)\frac{\Gamma(\sigma(f, x))}{\Gamma(\sigma(f, x(1+\varepsilon)))}-1\right\} = \\ &=c(f)\exp\left(-\gamma\sigma(f, x)+\sum_{p\leqq x}\frac{f(p)}{p}\right)\frac{1}{\Gamma(\sigma(f, x))}\left(e^{O(\mu(x))}\frac{\Gamma(\sigma(f, x))}{\Gamma(\sigma(f, x(1+\varepsilon)))}-1\right) = \\ &=c(f)\exp\left(-\gamma\sigma(f, x)+\sum_{p\leqq x}\frac{f(p)}{p}\right)\times \\ &\times\left(-\frac{1}{\Gamma(\sigma(f, x))}+\frac{1}{\Gamma(\sigma(f, x(1+\varepsilon)))}+O\left(\frac{\mu(x)}{\Gamma(\sigma(f, x(1+\varepsilon)))}\right)\right).\end{aligned}$$

Отсюда так как $\operatorname{Re}\sigma(f, x)\geqq 0$, $\operatorname{Im}\sigma(f, x)=O(1)$, то

$$\begin{aligned}&\frac{1}{\Gamma(\sigma(f, x(1+\varepsilon)))}-\frac{1}{\Gamma(\sigma(f, x))} = \frac{\sigma(f, x(1+\varepsilon))}{\Gamma(1+\sigma(f, x(1+\varepsilon)))}-\frac{\sigma(f, x)}{\Gamma(\sigma(f, x)+1)} = \\ &= O\left(|\sigma(f, x(1+\varepsilon))-\sigma(f, x)|\cdot\max_{\substack{\Theta,\ \operatorname{Re}\Theta\geqq 1\\ \operatorname{Im}\Theta=O(1)}}\left(\left|\frac{1}{\Gamma(\Theta)}\right|+\left|\frac{\Theta-1}{\Gamma(\Theta)}\right|\left|\frac{\Gamma^1(\Theta)}{\Gamma(\Theta)}\right|\right)\right) = O(\mu(x)).\end{aligned}$$

Остальные члены в (35) при подстановке вместо x выражения $x(1+\varepsilon)$, в силу монотонности $\mu(x)$ и неравенства

$$\sum_{x<p\leqq x(1+\varepsilon)}|f(p)|/p = O\left(\frac{\sqrt{\log\log x}}{\log x}\right)$$

не увеличивают остаток. Отсюда следует, что

$$\begin{aligned}l(x(1+\varepsilon), x)-l(x, x) &= \exp\left(-\gamma\sigma(f, x)+\sum_{p\leqq x}\frac{f(p)}{p}\right)\times \\ &\times O\left(\exp\left(\sum_{p\leqq x}\frac{|f(p)|-\operatorname{Re}f(p)}{p}\right)M^2(|f|, x)\mu(x)\right).\end{aligned}\tag{37}$$

Из (36) и (37) получаем

$$m_1(x)=l(x,x)(1+O(\varepsilon))+O\left(\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)(\varepsilon+\mu_1(x))M(|f|,x)+\right.$$

$$\left.+e^{-\gamma\operatorname{Re}\sigma(f,x)}\exp\left(\sum_{p\le x}|f(p)|/p\right)\frac{M^2(|f|,x)\mu(x)}{\varepsilon}\right).$$

Выберем

$$\varepsilon=e^{-\gamma/2\operatorname{Re}\sigma(f,x)}\sqrt{M(|f|,x)\mu(x)}.$$

При таком ε получаем

$$m_1(x)=\exp\left(-\gamma\sigma(f,x)+\sum_{p\le x}\frac{f(p)}{p}\right)\times$$

$$\times\left(\frac{c(f)}{\Gamma(\sigma(f,x))}+O\left(\exp\left(\sum_{p\le x}\frac{|f(p)|-\operatorname{Re}f(p)}{p}\right)(M^{3/2}(|f|,x)\times\right.\right.$$

$$\left.\left.\times\sqrt{\mu(x)}\,e^{\frac{\gamma}{2}\operatorname{Re}\sigma(f,x)}+\mu_1(x)M(|f|,x)e^{\gamma\operatorname{Re}\sigma(f,x)})\right)\right).$$

Для перехода от $m_1(x)$ к $m(x)$ поступаем следующим образом

$$m(x)=\frac{1}{x\log x}\sum_{n\le x}f(n)\log n+\frac{1}{x\log x}\sum_{n\le x}f(n)\log\frac{x}{n}=$$

$$=\frac{m_1(x)}{\log x}+O\left(\frac{1}{\log^2 x}\sum_{n\le x/\log^2 x}|f(n)|/n+\frac{\log\log x}{x\log x}\sum_{n\le x}|f(n)|\right).$$

Используя неравенство (13′) с $|f(n)|$ вместо $|c_2(n)|$ получаем

$$m(x)=\frac{m_1(x)}{\log x}+O\left(\frac{\log\log x}{\log^2 x}M(|f|,x)\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)\right).$$

Следовательно,

$$m(x)=\frac{\exp\left(-\gamma\sigma(f,x)+\sum_{p\le x}f(p)/p\right)}{\log x}\times$$

$$\times\left(\frac{c(f)}{\Gamma(\sigma(f,x))}+O\left(\exp\left(\sum_{p\le x}\frac{|f(p)|-\operatorname{Re}f(p)}{p}\right)\times\right.\right.$$

$$\left.\left.\times(M^{3/2}(|f|,x)\sqrt{\mu(x)}\,e^{\gamma/2\operatorname{Re}\sigma(f,x)}+\mu_1(x)M(|f|,x)e^{\gamma\operatorname{Re}\sigma(f,x)})\right)\right).$$

В случае, если $f(n)\ge 0$, асимптотическое дифференцирование можно провести с меньшими потерями. При оценке $l(x,(1+\varepsilon),x)-l(x,x)$ в правой часть (35) поступаем точно как и раньше. А левая часть

$$(1+\varepsilon)l(x(1+\varepsilon),x)-l(x,x)=$$

$$=\frac{1}{x}\sum_{x<n\le x(1+\varepsilon)}f(n)\log n\log\frac{x(1+\varepsilon)}{n}+\frac{\log(1+\varepsilon)}{x}\sum_{n\le x}f(n)\log n\ge$$

$$\ge\log(1+\varepsilon)m_1(x).$$

Отсюда получим

$$m_1(x) \leqq l(x, x) + O\left(\frac{|l(x(1+\varepsilon), x) - l(x, x)|}{\varepsilon}\right) + O(\varepsilon |l(x, x)|).$$

Аналогично получим

$$l(x, x) + O\left(\frac{|l(x(1-\varepsilon), x) - l(x, x)|}{\varepsilon}\right) + O(\varepsilon |l(x, x)|) \leqq m_1(x).$$

Выбрав

$$\varepsilon = M(|f|, x)\sqrt{\Gamma(\sigma(f, x))}\sqrt{\mu(x)}\left(M^2(|f|, x)\mu(x)\Gamma(\sigma(f, x)) + 1\right)^{-1/2}$$

и переходя от $m_1(x)$ к $m(x)$ также как делалось выше, найдем, что в случае $f(n) \leqq 0$

$$m(x) = \frac{\exp\left(-\gamma(\sigma(f, x)) + \sum_{p \leqq x} f(p)/p\right)}{\log x} \times$$

$$\times\left(\frac{c(f)}{\Gamma(\sigma(f, x))} + O(M^2(f, x)\mu(x)) + O\left(M(|f|, x)\sqrt{\frac{\mu(x)}{\Gamma(\sigma(f, x))}}\right)\right).$$

Теорема 2 доказана.

ЛИТЕРАТУРА

[1] Wirsing, E., Das asymptotische Verhalten von Summen über multiplikative Funktionen, *Math. Ann.* **143** (1961), 75—102. *MR* **24** # A1241.

(Поступила 3 янбаря 1982)

ВЛАДИМИРСКИЙ ГОСУДАРСТВЕННЫЙ
ПЕДАГОГИЧЕСКИЙ ИНСТИТУТ
МАТЕМАТИЧЕСКИЙ ФАКУЛЬТЕТ
ПР. СТРОИТЕЛЕЙ Д. 11
SU—600024 ВЛАДИМИР
SOVIET UNION

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 43—56

ALMOST PRODUCT FINSLER STRUCTURES AND CONNECTIONS

GHEORGHE ATANASIU and FRANCISC C. KLEPP

Summary

The almost product Finsler structure is defined and the set of Finsler connections compatible with this structure (F-Finsler connections) is determined. A subset of F-Finsler connections, called F-Finsler connections is explicited and some invariant tensors of the group of transformations of such connections are determined.

Subgroups obtained by vanishing of some invariants, contain transformations of F-Miron connections and F-Gołąb connections, which are characterized by vanishing of Finsler tensors $\underset{1}{N}{}^i_{jk}$ and $\underset{2}{N}{}^i_{jk}$ given by (2.5) and (2.6), respectively. The class of F-Finsler connections having this property is much more comprehensive.

The almost product structures were introduced and studied by A. G. Walker (10), (11), K. Yano (12) and by many others. For example, in (8) and (9) the invariants for a sufficient vast category of compatible connections with the almost product structure are presented.

In this paper the almost product Finsler structure will be defined and we shall determine the set of all Finsler connections compatible with this structure (F-Finsler connections given by Theorem 3.1, formulae (3.4)). A subset of these connections, called $F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$-Finsler connections (formulae (3.8)) will be explicited and we will determine some of the invariants of the group of transformations among such connections, which have a common non-linear connection. For $F\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$- and for $F\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$-Finsler connections, respectively, these invariants are presented by Theorem 6.1 and 6.3, the generalized case is given by Theorem 5.1.

We remark that these groups of transformations contain the transformations of Miron connections in particulary (this notion was introduced by M. Hashiguchi in (2) and it was generalized by us in Definition 1.1), and the transformations of Gołąb connections, respectively (see (1) and Definition 1.2), when these connections are compatible with the almost product Finsler structure.

Moreover, the studied Finsler connections are characterized by vanishing of Finsler tensors $\underset{1}{N}{}^i_{jk}$ and $\underset{2}{N}{}^i_{jk}$ (given by (2.5) and (2.6)). These tensors have the well-known Nijenhuis tensor properties in the problem of the almost product structure integrability.

The notations and terminologies are those of M. Matsumoto (3), (4), with few modifications (6).

1980 *Mathematics Subject Classification.* Primary 53C60.

Key words and phrases. Almost product Finsler structures, Finsler connections, groups of Finsler connections and invariants.

§ 1. Preliminaries

By M. Matsumoto (3), (4) a Finsler connection $F\Gamma$ on a C^∞-differentiable manifold M_n is defined in three equivalent manners: as a pair (Γ^h, Γ^v), as a pair (Γ, N) or as a triad (Γ_v, N, Γ^v) where Γ and Γ^h (resp. Γ^v) are a connection and a horizontal (resp. vertical) connection in the Finsler bundle $F(M_n)$, N is a non-linear connection in the tangent bundle $T(M_n)$ and Γ_v is a V-connection in the linear frame bundle $L(M_n)$. If Γ^i_{jk}, C^i_{jk} are the coefficients of Γ, and N^i_k and F^i_{jk} are the respective ones of N and Γ_v, the Finsler connection $F\Gamma$ having $(N^i_k, F^i_{jk}, C^i_{jk})$ as the coefficients is denoted by $F\Gamma=(N, F, C)$.

For a Finsler tensor field $K^i_j(x, y)$, where y is the supporting element in x, the h- and v-covariant derivatives are defined by:

$$(1.1)\qquad \begin{cases} K^i_{j|k} = \delta K^i_j/\delta x^k - F^a_{jk}K^i_a + F^i_{ak}K^a_j \\ K^i_j|_k = \partial K^i_j/\partial y^k - C^a_{jk}K^i_a + C^i_{ak}K^a_j \end{cases}$$

where $\delta/\delta x^i = \partial/\partial x^i - N^a_i\partial/\partial y^a$.

The Ricci identities applying to K^i_j are:

$$(1.2)\qquad \begin{cases} K^i_{j|k|p} - K^i_{j|p|k} = -K^i_a R^a_{jkp} + K^a_j R^i_{akp} - K^i_j|_a R^a_{kp} - K^i_{j|a} T^a_{kp} \\ K^i_{j|k}|_p - K^i_j|_{k|p} = -K^i_a P^a_{jkp} + K^a_j P^i_{akp} - K^i_j|_a P^a_{kp} - K^i_{j|a} C^a_{kp} \\ K^i_j|_k|_p - K^i_j|_p|_k = -K^i_a S^a_{jkp} + K^a_j S^i_{akp} - K^i_j|_a S^a_{kp} \end{cases}$$

where five torsion tensor fields R^1, P^1, S^1, T, C and three curvature tensor fields R^2, P^2, S^2 appear, and their components are given by:

$$(1.3)\qquad \begin{cases} R^i_{jk} = \delta N^i_j/\delta x^k - \delta N^i_k/\delta x^j; \; P^i_{jk} = \partial N^i_j/\partial y^k - F^i_{kj} \\ T^i_{jk} = F^i_{jk} - F^i_{kj}; \; S^i_{jk} = C^i_{jk} - C^i_{kj}; \end{cases}$$

$$(1.4)\qquad \begin{cases} R^i_{jkp} = \delta F^i_{jk}/\delta x^p - \delta F^i_{jp}/\delta x^k + F^a_{jk}F^i_{ap} - F^a_{jp}F^i_{ak} + C^i_{ja}R^a_{kp} \\ P^i_{jkp} = \partial F^i_{jk}/\delta y^p - C^i_{jp|k} + C^i_{ja}P^a_{kp} \\ S^i_{jkp} = \partial C^i_{jk}/\partial y^p - \partial C^i_{jp}/\delta y^k + C^a_{jk}C^i_{ap} - C^a_{jp}C^i_{ak}. \end{cases}$$

If $F\Gamma=(N, F, C)$ and $F\bar\Gamma=(\bar N, \bar F, \bar C)$ are two Finsler connections on M_n, then a unique triad of Finsler tensor fields A, B, D is determined by (3), (4):

$$(1.5)\qquad \bar N^i_j = N^i_j + A^i_j; \quad \bar F^i_{jk} = F^i_{jk} - C^i_{ja}A^a_k + B^i_{jk}; \quad \bar C^i_{jk} = C^i_{jk} + D^i_{jk}.$$

Reciprocally, let $F\Gamma$ be a Finsler connection and A, B, D a triad of Finsler tensor fields, then $F\bar\Gamma$ given by (1.5) is a Finsler connection. The map $F\Gamma \to F\bar\Gamma$ defined by (1.5) is called a *transformation of Finsler connections*. The group of these transformations was discussed at large in (5) and (6).

Transformations of Finsler connections, which preserve the nonlinear connection N are given by:

$$(1.5')\qquad \bar N^i_j = N^i_j; \quad \bar F^i_{jk} = F^i_{jk} + B^i_{jk}; \quad \bar C^i_{jk} = C^i_{jk} + D^i_{jk}$$

and will be denoted by $F\Gamma(N) \to F\bar\Gamma(N)$.

We also notice (6):

$$(1.6)\qquad K^i_{jkp} = R^i_{jkp} - C^i_{ja}R^a_{kp}; \quad \mathfrak{P}^i_{jkp} = P^i_{jkp} - P^i_{jpk} - C^i_{ja}t^a_{kp}$$

where t^i_{jk} is the *torsion* tensor of the non-linear connection N (4);

(1.7) $$t^i_{jk} = \partial N^i_j/\partial y^k - \partial N^i_k/\partial y^j.$$

So we have (6):

PROPOSITION 1.1. *By a transformation* (1.5′) *of Finsler connections* $F\Gamma(N) \to \to F\bar{\Gamma}(N)$, *the tensor fields* K^i_{jkp}, $\mathfrak{P}^i_{jkp}$, S^i_{jkp} *are transformed as follows:*

(1.8)
$$\begin{aligned}
\bar{K}^i_{jkp} &= K^i_{jkp} + B^i_{ja} T^a_{kp} + B^i_{jk|p} - B^i_{jp|k} + B^a_{jk} B^i_{ap} - B^a_{jk} B^i_{ak} \\
\bar{\mathfrak{P}}^i_{jkp} &= \mathfrak{P}^i_{jkp} + B^i_{ja} S^a_{kp} + D^i_{ja} T^a_{kp} + B^i_{jk}|_p - B^i_{jp}|_k + \\
&\quad + B^a_{jk} D^i_{ap} - B^a_{jp} D^i_{ak} + D^a_{jk} B^i_{ap} - D^a_{jp} B^i_{ak} + D^i_{jk|p} - D^i_{jp|k} \\
\bar{S}^i_{jkp} &= S^i_{jkp} + D^i_{ja} S^a_{kp} + D^i_{jk}|_p - D^i_{ap}|_k + D^a_{jk} D^i_{ap} - D^a_{jp} D^i_{ak}.
\end{aligned}$$

We introduce the following definitions:

DEFINITION 1.1. A Finsler connection $F\Gamma(N) = (N, F, C)$ is called *Miron connection* if:

(1.9) $$F^i_{jk} - F^i_{kj} = \delta^i_j p_k - \delta^i_k p_j; \quad C^i_{jk} - C^i_{kj} = \delta^i_j q_k - \delta^i_k q_j$$

where p_k and q_k are arbitrary Finsler covector fields.

This definition generalizes the definition given by M. Hashiguchi on Miron connections in (2), where Miron connections $M\Gamma(p_k, q_k)$ were metrical Finsler connections.

DEFINITION 1.2. A Finsler connection $F\Gamma(N) = (N, F, C)$ is called *Gołąb connection* if:

(1.10) $$F^i_{jk} - F^i_{kj} = t^i_j p_k - t^i_k p_j; \quad C^i_{jk} - C^i_{kj} = t^i_j q_k - t^i_k q_j$$

where p_k, q_k and t^i_j are arbitrary Finsler tensor fields.

§ 2. *F*-Finsler connections

In this section we shall define the almost product Finsler structure and compatible Finsler connections with this structure (*F*-Finsler connections), evidencing some of the properties of these connections.

DEFINITION 2.1. A Finsler tensor field $F^i_j(x, y)$ is called *almost product Finsler structure* on M_n, if:

(2.1) $$F^a_j F^k_a = \delta^k_j,$$

(2.2) $$\mathrm{Det}\,[F^i_j(x, y)] \neq 0 \quad \forall y \neq 0.$$

DEFINITION 2.2. A Finsler connection $F = (N, F, C)$ is called *almost product*, or compatible with almost product Finsler structure, or we say it is *an F-Finsler connection*, if and only if:

(2.3) $$F^i_{j|k} = 0; \quad F^i_j|_k = 0$$

So we have:

THEOREM 2.1. *If* $F\overset{\circ}{\Gamma}=(\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ *is an arbitrary fixed Finsler connection on* M_n, *then the following Finsler connection:*

$$\overset{k}{N}{}^i_j = \overset{\circ}{N}{}^i_j; \quad \overset{k}{F}{}^i_{jk} = \overset{\circ}{F}{}^i_{jk} - \frac{1}{2} F^a_j F^i_{a\overset{\circ}{|}k}; \quad \overset{k}{C}{}^i_{jk} = \overset{\circ}{C}{}^i_{jk} - \frac{1}{2} F^a_j F^i_a \overset{\circ}{|}_k \tag{2.4}$$

is an F-Finsler connection, where $\overset{\circ}{|}$ *and* $\overset{\circ}{|}$ *denote the h- and v-covariant derivatives relative to* $F\overset{\circ}{\Gamma}$.

Indeed, a direct substitution in (2.1) shows that $K\overset{\circ}{\Gamma}=(\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ given by (2.4) satisfies (2.3).

DEFINITION 2.3. We shall call the Finsler connection $K\overset{\circ}{\Gamma}=(\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ given by (2.4) the almost product Kawaguchi connection derived from $F\overset{\circ}{\Gamma}$.

We introduce the following Finsler tensors for an almost product Finsler structure:

$$\underset{1}{N}{}^i_{jk} = F^a_j(\delta_a F^i_k - \delta_k F^i_a) - F^a_k(\delta_a F^i_j - \delta_j F^i_a), \tag{2.5}$$

$$\underset{2}{N}{}^i_{jk} = F^a_j(\partial_a F^i_k - \partial_k F^i_a) - F^a_k(\partial_a F^i_j - \partial_j F^i_a). \tag{2.6}$$

Relations (1.1), (1.3), (2.3), (2.5) and (2.6) give

THEOREM 2.2. *For any F-Finsler connection the Finsler tensors* $\underset{1}{N}{}^i_{jk}$ *and* $\underset{2}{N}{}^i_{jk}$ *can be expressed as follows:*

$$\underset{1}{N}{}^i_{jk} = T^i_{jk} + F^a_j F^b_k T^i_{ab} - T^a_{jb} F^b_k F^i_a - F^a_j T^b_{ak} F^i_b, \tag{2.5'}$$

$$\underset{2}{N}{}^i_{jk} = S^i_{jk} + F^a_j F^b_k S^i_{ab} - S^a_{jb} F^b_k F^i_a - F^a_j S^b_{ak} F^i_b. \tag{2.6'}$$

We may consider the Finsler tensor fields:

$$\underset{1}{P}{}^{ks}_{ij} = \frac{1}{2}(\delta^k_i \delta^s_j + F^k_i F^s_j); \quad \underset{2}{P}{}^{ks}_{ij} = \frac{1}{2}(\delta^k_i \delta^s_j - F^k_i F^s_j), \tag{2.7}$$

and we shall call them the K. Yano—Obata operators (7), (12) of the almost product Finsler structure $F^i_j(x, y)$.

From their properties we mention:

$$\underset{1}{P} + \underset{2}{P} = \delta\delta; \quad \underset{1}{P}\cdot\underset{1}{P} = \underset{1}{P}; \quad \underset{2}{P}\cdot\underset{2}{P} = \underset{2}{P}; \quad \underset{1}{P}\cdot\underset{2}{P} = 0; \quad \underset{2}{P}\cdot\underset{1}{P} = 0 \tag{2.8}$$

Using properties (2.8) we have immediately:

LEMMA 2.3. *A tensorial equation*

$$\underset{1}{P}\cdot X = U \quad \textit{(respectively } \underset{2}{P}\cdot X = U\textit{)}$$

with X as unknown, has solution if and only if

$$\underset{2}{P}\cdot U = 0 \quad \textit{(respectively } \underset{1}{P}\cdot U = 0\textit{).}$$

In this case the general solution is given by

$$X = \underset{2}{P} \cdot Y + U \quad (\text{respectively } X = \underset{1}{P} \cdot Y + U)$$

where Y is an arbitrary Finsler tensor of the same type as X.

Here $(\underset{1}{P} \cdot \underset{1}{P})^{ks}_{ij} = \underset{1}{P}{}^{as}_{ib} \underset{1}{P}{}^{kb}_{aj}$ and the contraction of a Finsler tensor of type (1, 2) by $\underset{1}{P}$ has the local expression $(\underset{1}{P} \cdot U)^i_{jk} = \underset{1}{P}{}^{ai}_{jb} U^b_{ak}$. From $\underset{2}{P}U = 0$ we have $\underset{1}{P}U = U$, since $\underset{1}{P} + \underset{2}{P} = \delta\delta$ and $(\underset{1}{P} + \underset{2}{P})U = \delta\delta U = U$.

It follows that $X = \underset{2}{P}Y + U$ is a solution of the equation: $\underset{1}{P}X = U$.

From (2.7) and (2.3) we have:

THEOREM 2.4. *For any F-Finsler connection, the K. Yano—Obata operators of the almost product Finsler structure, are h- and v-covariant constants, i.e.*

$$\underset{1}{P}{}^{ks}_{ij|h} = 0; \quad \underset{1}{P}{}^{ks}_{ij}|_h = 0; \quad \underset{2}{P}{}^{ks}_{ij|h} = 0; \quad \underset{2}{P}{}^{ks}_{ij}|_h = 0. \tag{2.9}$$

Now, if $F\Gamma$ is an F-Finsler connection, then the Ricci identities (1.2) applied to $F^i_j(x, y)$ give:

$$F^i_a R^a_{jkh} - F^a_j R^i_{akh} = 0 \qquad (R^i_{jkh} = F^i_a F^b_j R^a_{bkh}) \tag{2.10}$$

and by analogy the relation (2.10) in P^i_{jkh} and S^i_{jkh}.

Both last result and Theorem 2.4 by (2.7) lead us to

THEOREM 2.5. *The curvature tensor fields R^i_{jkh}, P^i_{jkh} and S^i_{jkh} of an F-Finsler connection have the property that the Finsler tensor fields $\underset{2}{P}{}^{ai}_{jb} R^b_{akh}$, $\underset{2}{P}{}^{ai}_{jb} P^b_{akh}$ and $\underset{2}{P}{}^{ai}_{jb} S^b_{akh}$ and their h- and v-covariant derivatives of any order vanish.*

§ 3. The set of *F*-Finsler connections

We shall determine all almost product Finsler connections by a well-known method based on Lemma 2.3. For this purpose, if $\mathring{F\Gamma} = (\mathring{N}, \mathring{F}, \mathring{C})$ is a Finsler connection, then according to (1.5), the triad $F\Gamma = (N, F, C)$ given by:

$$\begin{aligned} N^i_j &= \mathring{N}^i_j - X^i_j, \\ F^i_{jk} &= \mathring{F}^i_{jk} + \mathring{C}^i_{ja} X^a_k + B^i_{jk}, \\ C^i_{jk} &= \mathring{C}^i_{jk} + D^i_{jk} \end{aligned} \tag{3.1}$$

where X, B, D are arbitrary, is a Finsler connection, too, and reciprocally.

If $F\Gamma$ given by (3.1) is an F-Finsler connection, for the Finsler tensors X, B, D we obtain the following system of equations:

$$B^i_{jk} - F^a_j F^i_b B^b_{ak} = F^i_a (F^a_j \mathring{|}_k + F^a_j \mathring{|}_b X^b_k),$$

$$D^i_{jk} - F^a_j F^i_b D^b_{ak} = F^i_a F^a_j \mathring{|}_k$$

which, by (2.7), can be expressed in the form:

$$(3.2)\qquad \begin{aligned} \underset{2}{P}{}^{ai}_{jb} B^b_{ak} &= \frac{1}{2} F^i_a (F^a_{j\,\overset{\circ}{|}k} + F^a_j \overset{\circ}{|}_b X^b_k),\\ \underset{2}{P}{}^{ai}_{jb} D^b_{ak} &= \frac{1}{2} F^i_a F^a_j \overset{\circ}{|}_k. \end{aligned}$$

A direct calculus shows us that Lemma 2.3 is fulfilled and then the general solution of that system is given by:

$$(3.3)\qquad \begin{aligned} B^i_{jk} &= \frac{1}{2} F^i_a (F^a_{j\,\overset{\circ}{|}k} + F^a_j \overset{\circ}{|}_b X^c_k) + \underset{1}{P}{}^{ai}_{jb} Y^b_{ak}\\ D^i_{jk} &= \frac{1}{2} F^i_a F^a_j \overset{\circ}{|}_k + \underset{1}{P}{}^{ai}_{jb} Z^b_{ak} \end{aligned}$$

where Y^i_{jk} and Z^i_{jk} are arbitrary Finsler tensor fields.

Using the above affirmations we have

THEOREM 3.1. *The set of F-Finsler connections* $F\Gamma = (N, F, C)$ *is given by*

$$(3.4)\qquad \begin{aligned} N^i_j &= \overset{\circ}{N}{}^i_j - X^i_j,\\ F^i_{jk} &= \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{ja} X^a_k - \frac{1}{2} F^a_j (F^i_{a\,\overset{\circ}{|}k} + F^i_a \overset{\circ}{|}_b X^b_k) + \underset{1}{P}{}^{ai}_{jb} Y^b_{ak},\\ C^i_{jk} &= \overset{\circ}{C}{}^i_{jk} - \frac{1}{2} F^a_j F^i_a \overset{\circ}{|}_k + \underset{1}{P}{}^{ai}_{jb} Z^b_{ak}, \end{aligned}$$

where $\overset{\circ}{F\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ is a fixed Finsler connection on M_n, $\overset{\circ}{|}$ and $\overset{\circ}{|}$ denote the h- and v-covariant derivatives relative to $\overset{\circ}{F\Gamma}$ and X^i_j, Y^i_{jk}, Z^i_{jk} are arbitrary Finsler tensors.

OBSERVATION 3.1. The almost product Kawaguchi connection $\overset{\circ}{K\Gamma} = (\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ given by (2.4) is obtained from (3.4) for $X=0$, $Y=0$, $Z=0$.

COROLLARY 3.2. *If* $\overset{\circ}{F\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ *is an F-Finsler connection, then the set of all F-Finsler connections* $F\Gamma = (N, F, C)$ *is given by*

$$(3.5)\qquad \begin{aligned} N^i_j &= \overset{\circ}{N}{}^i_j - X^i_j,\\ F^i_{jk} &= \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{ja} X^a_k + \underset{1}{P}{}^{ai}_{jb} Y^b_{ak},\\ C^i_{jk} &= \overset{\circ}{C}{}^i_{jk} + \underset{1}{P}{}^{ai}_{jb} Z^b_{ak} \end{aligned}$$

where X^i_j, Y^i_{jk}, Z^i_{jk} *are arbitrary Finsler tensor fields.*

If we denote by $F\Gamma(\overset{\circ}{N})$ Finsler connections which preserve the non-linear connection $\overset{\circ}{N}$, then we have:

COROLLARY 3.3. *The set of all F-Finsler connections $F\Gamma(N)$ with the same non-linear connection N, is given by*

$$
\begin{aligned}
N^i_j &= \overset{\bullet}{N}{}^i_j,\\
F^i_{jk} &= \overset{\bullet}{F}{}^i_{jk} - \frac{1}{2} F^a_j F^i_{a|k} + \underset{1}{P}{}^{ai}_{jb} Y^b_{ak}, \qquad (3.6)\\
C^i_{jk} &= \overset{\bullet}{C}{}^i_{jk} - \frac{1}{2} F^a_j F^i_a|_k + \underset{1}{P}{}^{ai}_{jb} Z^b_{ak}
\end{aligned}
$$

where Y^i_{jk} and Z^i_{jk} are arbitrary Finsler tensor fields.

COROLLARY 3.4. *If $F\overset{\circ}{\Gamma} = (\overset{\bullet}{N}, \overset{\bullet}{F}, \overset{\bullet}{C})$ is an F-Finsler connection, then the set of all F-Finsler connections $F\Gamma(\overset{\bullet}{N})$ is written as follows:*

$$
\begin{aligned}
N^i_j &= \overset{\bullet}{N}{}^i_j,\\
F^i_{jk} &= \overset{\bullet}{F}{}^i_{jk} + \underset{1}{P}{}^{ai}_{jb} Y^b_{ak}, \qquad (3.7)\\
C^i_{jk} &= \overset{\bullet}{C}{}^i_{jk} + \underset{1}{P}{}^{ai}_{jb} Z^b_{ak}
\end{aligned}
$$

where Y^i_{jk} and Z^i_{jk} are arbitrary Finsler tensor fields.

In (3.4) we take the Finsler tensor fields Y and Z in the form

$$
\begin{aligned}
Y^i_{jk} &= (\lambda\delta^i_j + \mu F^i_j)\sigma_k\\
Z^i_{jk} &= (\alpha\delta^i_j + \beta F^i_j)\tau_k
\end{aligned}
$$

and with (2.7) we have:

THEOREM 3.5. *If $F\overset{\bullet}{\Gamma}$ is a fixed F-Finsler connection on M_n, $\lambda, \mu, \alpha, \beta \in R$ and σ_k, τ_k, X^i_j are arbitrary Finsler tensors, then the following Finsler connections are F-Finsler connections:*

$$
(3.8)\qquad \begin{cases}
N^i_j = \overset{\bullet}{N}{}^i_j - X^i_j,\\
F^i_{jk} = \overset{\bullet}{F}{}^i_{jk} + \overset{\bullet}{C}{}^i_{ja} X^a_k + (\lambda\delta^i_j + \mu F^i_j)\sigma_k,\\
C^i_{jk} = \overset{\bullet}{C}{}^i_{jk} + (\alpha\delta^i_j + \beta F^i_j)\tau_k.
\end{cases}
$$

DEFINITION 3.1. F-Finsler connections given by (3.8) will be called $F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$-Finsler connections.

COROLLARY 3.6. *If $F\overset{\bullet}{\Gamma}$ is a fixed F-Finsler connection, with the non-linear connection $\overset{\bullet}{N}$, then $F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$-Finsler connections $F\Gamma(\overset{\bullet}{N})$ are given by:*

$$
\begin{aligned}
N^i_j &= \overset{\bullet}{N}{}^i_j,\\
F^i_{jk} &= \overset{\bullet}{F}{}^i_{jk} + (\lambda\delta^i_j + \mu F^i_j)\sigma_k, \qquad (3.9)\\
C^i_{jk} &= \overset{\bullet}{C}{}^i_{jk} + (\alpha\delta^i_j + \beta F^i_j)\tau_k.
\end{aligned}
$$

Particularly, $F\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$-Finsler connections $F\Gamma(\overset{\circ}{N})$ are given by

$$(3.10)\qquad \begin{aligned} N^i_j &= \overset{\circ}{N}{}^i_j,\\ F^i_{jk} &= \overset{\circ}{F}{}^i_{jk}+\delta^i_j\sigma_k,\\ C^i_{jk} &= \overset{\circ}{C}{}^i_{jk}+\delta^i_j\tau_k \end{aligned}$$

and $F\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$-Finsler connections $F\Gamma(\overset{\circ}{N})$ are given by

$$(3.11)\qquad \begin{aligned} N^i_j &= \overset{\circ}{N}{}^i_j,\\ F^i_{jk} &= \overset{\circ}{F}{}^i_{jk}+F^i_j\sigma_k,\\ C^i_{jk} &= \overset{\circ}{C}{}^i_{jk}+F^i_j\tau_k \end{aligned}$$

where σ_k and τ_k are arbitrary Finsler covector fields.

§ 4. The group of transformations of F-Finsler connections

If $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ is an F-Finsler connection with $F\overset{\circ}{\Gamma}$ fixed like in (3.4), then according to (1.5) and Theorem 3.1, we affirm:

THEOREM 4.1. *Let $F\Gamma$ and $F\bar{\Gamma}$ be two F-Finsler connections corresponding to arbitrary Finsler tensors X, Y, Z and $\bar{X}, \bar{Y}, \bar{Z}$, respectively. Then, the map $F\Gamma \to F\bar{\Gamma}$ is given by:*

$$(4.1)\qquad \begin{aligned} \bar{N}^i_j &= N^i_j-\tilde{X}^i_j,\\ \bar{F}^i_{jk} &= F^i_{jk}+C^i_{ja}\tilde{X}^a_k+\underset{1}{P}{}^{ai}_{jb}\tilde{Y}^b_{ak},\\ \bar{C}^i_{jk} &= C^i_{jk}+\underset{1}{P}{}^{ai}_{jb}\tilde{Z}^b_{ak} \end{aligned}$$

where we denoted $X-\bar{X}=\tilde{X}$; $Y-\bar{Y}=\tilde{Y}$; $Z-\bar{Z}=\tilde{Z}$.

The set G of transformations $F\Gamma \to F\bar{\Gamma}$ of F-Finsler connections, together with the mapping product is a group. The structure of this group G can be studied in the same way as the group of transformations of metric Finsler connections (6).

We want to pay attention to some particular cases.

Let us consider transformations of F-Finsler connections $F\Gamma(N) \to F\bar{\Gamma}(N)$, having the same-non-linear connection N. According to (3.6) these are:

$$(4.2)\qquad \begin{aligned} \bar{N}^i_j &= N^i_j,\\ \bar{F}^i_{jk} &= F^i_{jk}+\underset{1}{P}{}^{ai}_{jb}\tilde{Y}^b_{ak},\\ \bar{C}^i_{jk} &= C^i_{jk}+\underset{1}{P}{}^{ai}_{jb}\tilde{Z}^b_{ak} \end{aligned}$$

where $\tilde{X}^i_{jk}$ and $\tilde{Z}^i_{jk}$ are arbitrary Finsler tensor fields.

The above considerations allow us to state

THEOREM 4.2. *The set of the transformations of F-Finsler connections $F\Gamma(N) \to F\bar{\Gamma}(N)$, given by* (4.2) *is a group G_{ap} relative to the product of transformations and* *this group is isomorphic with the additive group of Finsler tensor pairs in the form* $(\underset{1}{P}{}^{ai}_{jb}Y^b_{ak},\ \underset{1}{P}{}^{ai}_{jb}Z^b_{ak})$.

COROLLARY 4.3. *The group* $G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$ *of transformations of* $F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$*-Finsler connections* $F\Gamma(N)\to F\bar{\Gamma}(N)$ *having the same non-linear connection* N *and the same constants* $\lambda, \mu, \alpha, \beta\in R$, *respectively, is given by*

$$\bar{N}^i_j = N^i_j,$$

$$\bar{F}^i_{jk} = F^i_{jk}+(\lambda\delta^i_j+\mu F^i_j)\,p_k, \tag{4.3}$$

$$\bar{C}^i_{jk} = C^i_{jk}+(\alpha\delta^i_j+\beta F^i_j)\,q_k$$

where p_k *and* q_k *are arbitrary Finsler covector fields.*

According to (1.3), (1.4), (1.6), Proposition 1.1 and the above Corollary, a direct calculation lead us to

THEOREM 4.4. *By a transformation* (4.3) *of* $F\begin{pmatrix}\lambda\mu\\ \alpha\beta\end{pmatrix}$*-Finsler connections* $F\Gamma(N)\to$ $\to F\bar{\Gamma}(N)$, *the Finsler tensor fields* $T^i_{jk}, S^i_{jk}, K^i_{jkh}; \mathfrak{P}^i_{jkh}; S^i_{jkh}$ *are transformed as follows:*

$$\begin{cases}\bar{T}^i_{jk} = T^i_{jk}+(\lambda\delta^i_j+\mu F^i_j)\,p_k-(\lambda\delta^i_k+\mu F^i_k)\,p_j\\ \bar{S}^i_{jk} = S^i_{jk}+(\alpha\delta^i_j+\beta F^i_j)\,q_k-(\alpha\delta^i_k+\beta F^i_k)\,q_j\end{cases} \tag{4.4}$$

$$\begin{cases}\bar{K}^i_{jkh} = K^i_{jkh}+(\alpha\delta^i_j+\beta F^i_j)\,q_{kh}\\ \bar{\mathfrak{P}}^i_{jkh} = \mathfrak{P}^i_{jkh}+(\lambda\delta^i_j+\mu F^i_j)\,u_{kh}+(\alpha\delta^i_j+\beta F^i_j)\,v_{kh}\\ \bar{S}^i_{jkh} = S^i_{jkh}+(\alpha\delta^i_j+\beta F^i_j)\,q_{kh}\end{cases} \tag{4.5}$$

where we denoted:

$$\begin{aligned} p_{kh} &= p_{k|h}-p_{h|k}+p_aT^a_{kh}; & q_{kh} &= q_k|_h-q_h|_k+q_aS^a_{kh}\\ u_{kh} &= p_k|_h-p_h|_k+p_aS^a_{kh}; & v_{kh} &= q_{k|h}-q_{h|k}+q_aT^a_{kh}.\end{aligned} \tag{4.6}$$

§ 5. Invariants in transformations of $F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$-Finsler connections

We decide upon to find some of invariant tensors of the group $G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$ of transformations of $F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$-Finsler connections given by (4.3).

THEOREM 5.1. *If* $\lambda^2-\mu^2\neq 0$; $\lambda n+\mu f\neq 0$; $\lambda f+\mu n\neq 0$ *and* $\alpha^2-\beta^2\neq 0$; $\alpha n+\beta f\neq 0$; $\alpha f+\beta n\neq 0$, *where* $f=F^a_a$, *then the following Finsler tensors are invariants of the group*

$G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$:

$$
(5.1)\quad \begin{cases} t^i_{jk}, \qquad\qquad R^i_{jk}, \\ \mathcal{T}^i_{jk}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = T^i_{jk} - \dfrac{1}{(\lambda^2-\mu^2)(n-1)}[(\lambda\delta^i_j+\mu F^i_j)(\lambda T_k+\mu F^a_b T^b_{ka}) - \\ \qquad -(\lambda\delta^i_k+\mu F^i_k)(\lambda T_j+\mu F^a_b T^b_{ja})], \\ \mathcal{S}^i_{jk}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = S^i_{jk} - \dfrac{1}{(\alpha^2-\beta^2)(n-1)}[(\alpha\delta^i_j+\beta F^i_j)(\alpha S_k+\beta F^a_b S^q_{ka}) - \\ \qquad -(\alpha\delta^i_k+\beta F^i_k)(\alpha S_j+\beta F^a_b S^b_{ja})], \end{cases}
$$

$$
(5.2)\quad \begin{cases} \underset{1}{\mathcal{K}}{}^i_{jkh}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = K^i_{jkh} - \dfrac{1}{\lambda n+\mu f}(\lambda\delta^i_j+\mu F^i_j)K^a_{akh} \\ \underset{1}{\mathfrak{P}}{}^i_{jkh}\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix} = \mathfrak{P}^i_{jkh} - \dfrac{1}{\alpha n+\beta f}(\alpha\delta^i_j+\beta F^i_j)\mathfrak{P}^a_{akh} \\ \underset{1}{\mathcal{S}}{}^i_{jkh}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = S^i_{jkh} - \dfrac{1}{\alpha n+\beta f}(\alpha\delta^i_j+\beta F^i_j)S^a_{akh}, \end{cases}
$$

$$
(5.3)\quad \begin{cases} \underset{2}{\mathcal{K}}{}^i_{jkh}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = K^i_{jkh} - \dfrac{1}{\lambda f+\mu n}(\lambda\delta^i_j+\mu F^i_j)F^a_b K^b_{akh} \\ \underset{2}{\mathfrak{P}}{}^i_{jkh}\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix} = \mathfrak{P}^i_{jkh} - \dfrac{1}{\alpha f+\beta n}(\alpha\delta^i_j+\beta F^i_j)F^a_b\mathfrak{P}^b_{akh} \\ \underset{2}{\mathcal{S}}{}^i_{jkh}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = S^i_{jkh} - \dfrac{1}{\alpha f+\beta n}(\alpha\delta^i_j+\beta F^i_j)F^a_b S^b_{akh}, \end{cases}
$$

$$
(5.4)\quad \begin{cases} \underset{3}{\mathcal{K}}{}^i_{jkh}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = K^i_{jkh} - \dfrac{1}{\lambda^2-\mu^2}(\lambda\delta^i_j+\mu F^i_j)(\lambda K^a_{hka}-\mu F^a_b K^b_{hka}) \\ \underset{3}{\mathfrak{P}}{}^i_{jkh}\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix} = \mathfrak{P}^i_{jkh} - \dfrac{1}{\alpha^2-\beta^2}(\alpha\delta^i_j+\beta F^i_j)(\alpha\mathfrak{P}^a_{hka}-\beta F^a_b\mathfrak{P}^b_{hka}) \\ \underset{3}{\mathcal{S}}{}^i_{jkh}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = S^i_{jkh} - \dfrac{1}{\alpha^2-\beta^2}(\alpha\delta^i_j+\beta F^i_j)(\alpha S^a_{hka}-\beta F^a_b S^b_{hka}), \end{cases}
$$

where $n>1$ *and* $T_k=T^a_{ak}$, $S_k=S^a_{ak}$.

PROOF. Because transformations of the group $G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$ preserve the non-linear connection N it follows that t^i_{jk} and R^i_{jk} are invariants of this group.

Going on we shall indicate a way of getting the invariant $\underset{3}{\mathfrak{P}}{}^i_{jkh}\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix}$ for the other invariants analogous ways were used.

According to (4.4), for $\lambda=\alpha$ and $\mu=\beta$, we have:

$$
(5.5)\qquad \overline{\mathfrak{P}}^i_{jkh} = \mathfrak{P}^i_{jkh} + (\alpha\delta^i_j+\beta F^i_j)r_{kh}
$$

where $r_{kh}=u_{kh}+v_{kh}$, u_{kh} and v_{kh} are given by (4.6).

From (5.5) we have:

$$F_a^i \overset{*}{\mathfrak{P}}{}^a_{jkh} = F_a^i \mathfrak{P}^a_{jkh} + (\alpha F_j^i + \beta\delta_j^i) r_{kh}. \tag{5.6}$$

By contraction $i=h$ in (5.5) and by summing, we obtain:

$$\overset{*}{\mathfrak{P}}{}^a_{jka} = \mathfrak{P}^a_{jka} + \alpha r_{kj} + \beta F_j^a r_{ka} \tag{5.7}$$

and by the same contraction in (5.6) and by summing we obtain:

$$F_b^i \overset{*}{\mathfrak{P}}{}^b_{jka} = F_b^a \mathfrak{P}^b_{jka} + \alpha F_j^a r_{ka} + \beta r_{kj}. \tag{5.8}$$

If we multiply (5.7) by α and (5.8) by β, respectively, and the equalities such obtained are subtracted, then we can express the tensor r_{kj} in the form:

$$r_{kj} = \frac{1}{\alpha^2-\beta^2}[\alpha(\overset{*}{\mathfrak{P}}{}^a_{jka} - \mathfrak{P}^a_{jka}) - \beta F_b^a(\overset{*}{\mathfrak{P}}{}^b_{jka} - \mathfrak{P}^b_{jka})]. \tag{5.9}$$

Substituting r_{kj} given by (5.9) in (5.5) and separating the marked quantities by the other terms we obtain:

$$\underset{3}{\overset{*}{\mathfrak{P}}}{}^i_{jkh}\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix} = \underset{3}{\mathfrak{P}}{}^i_{jkh}\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix}$$

where $\underset{3}{\mathfrak{P}}{}^i_{jkh}\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix}$ is given by (5.4). Q.e.d.

We remark that if we denote by ${}^*G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$ the subgroup of $G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$ given by

$${}^*G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = \left\{t \in G_{ap}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}\middle| \mathscr{T}^i_{jk}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = 0;\ \mathscr{S}^i_{jk}\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix} = 0\right\}$$

it consists of transformations of ${}^*F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$-Finsler connections which have the following properties:

$$F^i_{j|k} = 0; \quad F^i_j|_k = 0; \quad N^i_j = \overset{*}{N}{}^i_j \quad \text{(fixed)} \tag{5.10}$$

$$\begin{cases} F^i_{jk} - F^i_{kj} = (\lambda\delta^i_j + \mu t^i_j)p_k - (\lambda\delta^i_k + \mu t^i_k)q_j \\ C^i_{jk} - C^i_{kj} = (\alpha\delta^i_j + \beta t^i_j)q_k - (\alpha\delta^i_k + \beta t^i_k)p_j \end{cases} \tag{5.11}$$

where

$$t^i_j = F^i_j;$$

$$p_k = \frac{1}{(\lambda^2-\mu^2)(n-1)}(\lambda T_k + \mu F_b^a T^b_{ka})$$

$$q_k = \frac{1}{(\alpha^2-\beta^2)(n-1)}(\alpha S_k + \beta F_b^a S^b_{ka}).$$

Then, using Theorem 2.2, a direct calculus lead us to

THEOREM 5.2. *For any* $^*F\begin{pmatrix}\lambda & \mu\\ \alpha & \beta\end{pmatrix}$*-Finsler connection we have:*

$$^*\underset{1}{N}{}^i_{jk}=0;\quad ^*\underset{2}{N}{}^i_{jk}=0. \tag{5.12}$$

REMARK 5.1. The class of F-Finsler connections having the property that $\underset{1}{N}{}^i_{jk}=0$; $\underset{2}{N}{}^i_{jk}=0$ is much more comprehensive. A future note will present it.

REMARK 5.2. Invariants of exceptional cases in Theorem 5.1 can be determined for each case, using the above method, or for allowed cases, particularizing invariants determined earlier. For instance we obtain the following invariants:

$$\mathscr{T}^i_{jk}\begin{pmatrix}\lambda & \lambda\\ \alpha & \alpha\end{pmatrix}\equiv\mathscr{T}^i_{jk}\begin{pmatrix}-\lambda & -\lambda\\ -\alpha & -\alpha\end{pmatrix}=T^i_{jk}-F^i_aT^a_{jk}$$

$$\mathscr{T}^i_{jk}\begin{pmatrix}\lambda & -\lambda\\ \alpha & -\alpha\end{pmatrix}\equiv\mathscr{T}^i_{jk}\begin{pmatrix}-\lambda & \lambda\\ -\alpha & \alpha\end{pmatrix}=T^i_{jk}+F^i_aT^a_{jk}$$

$$\underset{3}{\mathscr{K}}{}^a_{jka}\begin{pmatrix}\lambda & \lambda\\ \alpha & \alpha\end{pmatrix}=\underset{3}{\mathscr{K}}{}^a_{jka}\begin{pmatrix}-\lambda & -\lambda\\ -\alpha & -\alpha\end{pmatrix}=K^a_{jka}-F^a_bK^b_{jka}.$$

§ 6. Particular cases

In this final section, we shall study two important particular cases:

Case 1. If $\lambda=1$, $\mu=0$ and $\alpha=1$, $\beta=0$, according to (4.3) we obtain the group $G_{ap}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ of transformations of $F\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$-Finsler connections, given by (3.10). So we have:

THEOREM 6.1. *If* $f=F^a_a\neq 0$ *and* $n>1$, *then the following Finsler tensors are invariants of the group* $G_{ap}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$:

$$\begin{cases} t^i_{jk}, \qquad R^i_{jk} \\ \mathscr{T}^i_{jk}\begin{pmatrix}10\\10\end{pmatrix}=T^i_{jk}-\dfrac{1}{n-1}(\delta^i_jT_k-\delta^i_kT_j); \\ \mathscr{S}^i_{jk}\begin{pmatrix}10\\10\end{pmatrix}=S^i_{jk}-\dfrac{1}{n-1}(\delta^i_jS_k-\delta^i_jS_j)\end{cases} \tag{6.1}$$

$$\begin{cases} \underset{1}{\mathscr{K}}{}^i_{jkh}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}=K^i_{jkh}-\dfrac{1}{n}\delta^i_jK^a_{akh}; \\ \underset{2}{\mathscr{K}}{}^i_{jkh}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}=K^i_{jkh}-\dfrac{1}{f}\delta^i_jF^a_bK^b_{akh} \\ \underset{3}{\mathscr{K}}{}^i_{jkh}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}=K^i_{jkh}-\delta^i_jK^a_{hka}\end{cases} \tag{6.2}$$

and by analogy the relations (6.2) *in* $\mathfrak{P}^i_{jkh}$ *and* S^i_{jkh}.

REMARK 6.1. The subgroup ${}^M G_{ap}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ of $G_{ap}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ given by:

$$ {}^M G_{ap}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = \left\{ t\in G_{ap}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} \middle| \mathscr{T}^i_{jk}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = 0 \right\} $$

consists of transformations of *F-Miron connections.*

Indeed, connections of ${}^M G_{ap}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ have the following properties:

$$ F^i_{j|k} = 0, \quad F^i_j|_k = 0, \quad N^i_j = \overset{*}{N}{}^i_j \quad \text{(fixed)} \tag{6.3} $$

$$ F^i_{jk} - F^i_{kj} = \delta^i_j p_k - \delta^i_k p_j; \quad C^i_{jk} - C^i_{kj} = \delta^i_j q_k - \delta^i_k q_j \tag{6.4} $$

where: $p_k = T_k/(n-1)$, $q_k = S_k/(n-1)$, $T_k = T^a_{ak}$, $S_k = S^a_{ak}$.

A direct calculus, or using Theorem 5.2 gives:

THEOREM 6.2. *For any F-Miron connection we have:*

$$ {}^M\underset{1}{N}{}^i_{jk} = 0, \quad {}^M\underset{2}{N}{}^i_{jk} = 0. \tag{6.5} $$

Case 2. If $\lambda=0, \mu=1$ and $\alpha=0$, $\beta=1$, according to (4.3) we obtain the group $G_{ap}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$ of transformations of $F\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$-Finsler connections, given by (3.11). So we have:

THEOREM 6.3. *If* $f = F^a_a \neq 0$ *and* $n>1$, *then the following Finsler tensors are invariants of the group* $G_{ap}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$:

$$ \left\{ \begin{aligned} & t^i_{jk}, \qquad\qquad R^i_{jk}; \\ & \mathscr{T}^i_{jk}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = T^i_{jk} - \frac{1}{n-1} F^a_b (F^i_j T^b_{ak} - F^i_k T^b_{aj}) \\ & \mathscr{S}^i_{jk}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = S^i_{jk} - \frac{1}{n-1} F^a_b (F^i_j S^b_{ak} - F^i_k S^b_{aj}) \end{aligned} \right. \tag{6.6} $$

$$ \left\{ \begin{aligned} & \underset{1}{\mathscr{K}}{}^i_{jkh}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = K^i_{jkh} - \frac{1}{f} F^i_j K^a_{akh}, \\ & \underset{2}{\mathscr{K}}{}^i_{jkh}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = K^i_{jkh} - \frac{1}{n} F^i_j F^a_b K^b_{akh}, \\ & \underset{3}{\mathscr{K}}{}^i_{jkh}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = K^i_{jkh} - F^i_j F^a_b K^b_{hka} \end{aligned} \right. \tag{6.7} $$

and by analogy the relation (6.7) *in* $\mathfrak{P}^i_{jkh}$ *and* S^i_{jkh}.

REMARK 6.2. The subgroup ${}^G G_{ap}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$ of $G_{ap}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$ given by

$$ {}^G G_{ap}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = \left\{ t\in G_{ap}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \middle| \mathscr{T}^i_{jk}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = 0;\ \mathscr{S}^i_{jk}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = 0 \right\} $$

consists of transformations of *F-Gołąb connections.*

Indeed, connections of ${}^{G}G_{ap}\begin{Bmatrix}0 & 1\\ 0 & 1\end{Bmatrix}$ have the following properties:

(6.8) $$F^i_{j|k} = 0, \quad F^i_j|_k = 0, \quad N^i_j = \dot{N}^i_j \quad \text{(fixed)}$$

(6.9) $$F^i_{jk} - F^i_{kj} = t^i_j p_k - t^i_k p_j, \quad C^i_{jk} - C^i_{kj} = t^i_j q_k - t^i_k q_j$$

where:

$$t^i_j = F^i_j; \; p_k = F^a_b T^b_{ak}/(n-1); \; q_k = F^a_b S^b_{ak}/(n-1).$$

Finally, by a direct calculus, or using Theorem 5.2 we have:

THEOREM 6.4. *For any Gołąb connection we have:*

(6.10) $${}^{G}\underset{1}{N}{}^i_{jk} = 0, \quad {}^{G}\underset{2}{N}{}^i_{jk} = 0.$$

REFERENCES

[1] GOŁĄB, S., On semi-symmetric and quarter-symmetric linear connections, *Tensor (N. S.)* **29** (1975), 249—254. *MR* **52** # 4156.

[2] HASHIGUCHI, M., Wagner connections and Miron connections of Finsler spaces, *Rev. Roumaine Math. Pures Appl.* **25** (1980), 1387—1390. *MR* **82k**: 53042.

[3] MATSUMOTO, M., *The theory of Finsler connections,* Publications of the Study Group of Geometry, Vol. 5, Department of Mathematics, Okayama University, Okayama, 1970. *MR* **42** # 2409.

[4] MATSUMOTO, M., *Foundations of Finsler geometry and special Finsler spaces,* VEB Deutscher Verlag Wiss., Berlin (to appear).

[5] MIRON, R., On transformation groups of Finsler connections, *Tensor N. S.* (to appear).

[6] MIRON, R. and HASHIGUCHI, M., Metrical Finsler connections, *Rep. Fac. Sci. Kagoshima Univ.* **12** (1979), 21—35. *MR* **81e**: 53018.

[7] OBATA, M., Affine connections on manifolds with almost complex, quaternion or Hermitian structure, *Jap. J. Math.* **26** (1956), 43—77. *MR* **20** # 17962.

[8] VAMANU, E., Connections and almost-product structure, *Bul. Inst. Politehn. Iaşi Sect. I* **24 (28)** (1978), 39—45. *MR* **80k**: 53059.

[9] VAMANU, E. and ATANASIU, GH., Sur les transformations de connexions presque produit, *Bul. Univ. Braşov Ser. C* (1980) (to appear).

[10] WALKER, A. G., Connexions for parallel distributions in the large, *Quart. J. Math. Oxford Ser.* (2) **6** (1955), 301—308. *MR* **19**—312; Connexions for parallel distributions in the large II, *Quart. J. Math. Oxford Ser.* (2) **9** (1958), 221—231. *MR* **20** # 6135.

[11] WALKER, A. G., Almost product structures, *Proc. Sympos. Pure Math.*, Vol. III, American Mathematical Society, Providence, R. I., 1961, 94—100. *MR* **23** # A1314.

[12] YANO, K., *Differential geometry on complex and almost complex spaces,* International Series of Monographs in Pure and Applied Mathematics, Vol. 49, A Pergamon Press Book, The Macmillan Co., New York, 1965. *MR* **32** # 4635.

(Received January 7, 1982)

DEPARTMENT OF GEOMETRY
FACULTY OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF BRAŞOV
STR. KARL MARX 50
R—2200 BRAŞOV

DEPARTMENT OF MATHEMATICS
POLYTECHNICAL INSTITUTE "TRAIAN VUIA"
R—1900 TIMIŞOARA
ROMANIA

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 57—67

IRREGULARITIES IN THE DISTRIBUTION OF PRIME IDEALS I

SZ. GY. RÉVÉSZ

To my father, György E. Révész, on his fiftieth birthday

1. Let us denote, as usual, $\Psi(x)=\sum_{n\leq x}\Lambda(n)$ where $\Lambda(n)$ is von Mangoldt's function, i.e. $\log p$ for the powers of the prime p and zero for numbers having more than one prime divisor. Then the remainder term in the prime number theorem is $\Delta(x)=\Psi(x)-x$. It is known for a long time, that the order of magnitude of $\Delta(x)$ is closely connected with the non-trivial zeros of $\zeta(s)$, the Riemann zeta function, e.g.

$$\Delta(x)=-\sum_{\varrho}\frac{x^{\varrho}}{\varrho}+O(\log x). \tag{1.1}$$

From this formula it is easy to see an old result of Phragmén, which states, that if θ is the least upper bound for the real parts of the ζ-roots, then

$$\Delta(x)=\Omega(x^{\theta-\varepsilon}). \tag{1.2}$$

However, this is a completely ineffective result. Littlewood [3] pointed out in 1937: "If we give a particular ϱ_0 with $\beta_0>\frac{1}{2}$ then there are no known ways to give an explicit X depending on ϱ_0 and ε, such that

$$|\Delta(x)|>X^{\beta_0-\varepsilon} \tag{1.3}$$

for some x in $[0, X]$." The first effective result in this field is due to P. Turán [11] who could prove the following

Theorem (P. Turán). *If $\varrho_0=\beta_0+i\gamma_0$, $\beta_0\geqq\frac{1}{2}$ is an arbitrary non-trivial zero of $\zeta(s)$, then for*

$$T>\max\{c_0, c_1(\varrho_0)\} \tag{1.4}$$

one has

$$\max_{1\leqq x\leqq T}|\Delta(x)|>\frac{T^{\beta_0}}{|\varrho_0|^{\frac{10\log T}{\log\log T}}}\exp\left(-c_2\frac{\log T\log\log\log T}{\log\log T}\right) \tag{1.5}$$

with effectively calculable c_i—s.

1980 *Mathematics Subject Classification.* Primary 12A70; Secondary 10H22.
Key words and phrases. Zeros of the Dedekind zeta-functions, oscillation of the remainder term in the prime ideal formula.

But (1.1) suggests that the oscillation of $\Delta(x)$ is of order $\frac{x^{\beta_0}}{|\varrho_0|}$, which is larger than the right side of (1.5). The complete answer in this respect was given by J. Pintz [5], who proved the undermentioned

THEOREM (J. Pintz). *If* $0<\varepsilon<\frac{1}{50}$ *and* $\zeta(\varrho_0)=0$ *for* $\varrho_0=\beta_0+i\gamma_0$, $\beta_0=\frac{1}{2}+\delta>\frac{1}{2}$ *then for* $\gamma_0>\frac{c_3}{\varepsilon^2}$ *and*

(1.6) $$H^{\varepsilon^2/4}>\max(\gamma_0, c_4)$$

we have in the interval

(1.7) $$[H, H^{100\log\gamma_0}]$$

an x *and an* x', *for which*

(1.8) $$\Delta(x)>(1-\varepsilon)\frac{x^{\beta_0}}{|\varrho_0|}, \qquad \Delta(x')<-(1-\varepsilon)\frac{(x')^{\beta_0}}{|\varrho_0|}.$$

Here the c_j*-s are effective constants, too.*

2. As the situation seems to be very similar in the theory of the remainder term in the prime-ideal theorem, the question for similar results is natural.

Denote by $K=K(\vartheta)$ an algebraic number field with degree n and discriminant Δ over $\mathbf{Q}$. P denotes always a prime ideal, $\mathrm{N}P$ the norm of P in K. As usual, let

$$\Psi_K(x)=\sum_{m\leq x}G(m), \qquad G(m)=\sum_{\substack{P,k\\ \mathrm{N}P^k=m}}\log\mathrm{N}P$$

and let $\Delta_K(x)=\Psi_K(x)-x$ be the remainder term in the prime-ideal theorem. Let $\zeta_K(s)$ be the Dedekind zeta function of K, and so in $\sigma>1$

(2.1) $$-\frac{\zeta_K'}{\zeta_K}(s)=\sum_{m=1}^{\infty}\frac{G(m)}{m^s}.$$

By $\varrho=\beta+i\gamma$ we shall denote always non-trivial roots of $\zeta_K(s)$.

It is easy to obtain results like (1.2), but these will be again ineffective. So the natural aim would be to get effective results, in which the constants depends also effectively on the parameters of K. Such a result was first attained by W. Staś [8], who proved

THEOREM (W. Staś). *If* $0<\eta\leqq 2$, $\zeta_K(\varrho_0)=0$, $\varrho_0=\beta_0+i\gamma_0$ *and* $\beta_0\geqq\frac{1}{2}$, *then for*

(2.2) $$T>\max(c_5, c_\eta, c_K, c_{\varrho_0})$$

we have

(2.3) $$\max_{1\leqq x\leqq T}|\Delta_K(x)|>T^{\beta_0}\exp\left\{-(\eta+1)\frac{\log T\log\log\log T}{\log\log T}\right\}.$$

Here c_5, c_6 are explicitly calculable effective absolute constants, and

$$c_\eta = \exp\exp\exp\left(\frac{128}{\eta}\right), \tag{2.4}$$

$$c_K = \max\left\{\exp(c_6^n|\Delta|^{3/2}),\ \exp\exp\left(\frac{3}{2}n+2\right)^n\right\}, \tag{2.5}$$

$$c_{\varrho_0} = \exp\exp(|\varrho_0|^{1/\eta}+|\varrho_0|^{28/\eta}). \tag{2.6}$$

3. In the present paper we show, that the expectable oscillation can be proved. With the above notations, and explicit absolute constants c, C we assert:

THEOREM. *If $\varrho_0=\beta_0+i\gamma_0$, $\zeta_K(\varrho_0)=0$, $\beta_0\geqq\frac{1}{2}$ and $\gamma_0\geqq 0$ then for every*

$$Y > C\max\{(\gamma_0+2), n, \log|\Delta|\} \tag{3.1}$$

there exists an

$$x\in[Y, Y^{c(n\log(\gamma_0+10)+\log|\Delta|)}] \tag{3.2}$$

for which

$$|\Delta_K(x)| > \frac{x^{\beta_0}}{\sqrt{\gamma_0^2+50}}. \tag{3.3}$$

We remark, that (3.1) is much nicer, than the condition (2.2) with the constants (2.4), (2.5) and (2.6), and the oscillation is the expected one. The constant 50 in (3.3) is not important. In the examination of $\max_{1\leqq x\leqq T}|\Delta_K(x)|$, our result is not as good as (2.3), in view of the localization (3.2). In the classical case $K=\mathbf{Q}$, when $\zeta_K(s)$ is the Riemann zeta function and $\Delta_K(x)$ is the remainder term of the prime number formula the above version of Pintz's theorem was proved by himself in a lecture held in 1980 at the Eötvös Loránd University (Budapest).

In the special case, when $\beta_0+i\gamma_0$ is a zero of ζ_K having minimal distance from the real axis, we get the classical Landau-type estimate with a uniform localization. To obtain this, we remark, that by the symmetry of the ζ_K-roots to $\sigma=\frac{1}{2}$, we can take $\beta_0\geqq\frac{1}{2}$, and that if $N_K(T)$ denotes the number of ζ_K-roots having imaginary part not exceeding $T>2$ in absolute value, by classical methods it can be easily seen that[1]

$$N_K(T) = \frac{n}{\pi}T\log T+\frac{\log|\Delta|-n-n\log 2\pi}{\pi}T+O\big((n+\log|\Delta|)\log T\big). \tag{3.4}$$

---

[1] It is well-known that the remainder term is $\frac{1}{\pi}\operatorname{Im}\int_2^{\frac{1}{2}+iT}\frac{\zeta_K'}{\zeta_K}(s)\,ds$, from which $\int_2^{2+iT}$ is $O(n)$ by (2.1) and the estimate of $G(m)$ in the proof of Lemma 1. An application of Littlewood's lemma (see e.g. [10] § 9.9) gives that for some T' with $|T-T'|\leqq 1$ the horizontal integral is $O((n+\log|\Delta|)\log T)$. Taking into account $N_K(T)-N_K(T')=O((n+\log|\Delta|)\log T)$ (c.f. Lemma 4) we have (3.4).

Note added in proof. Professor J. Kaczorowsky called my attention that an alternative proof can be found in [7].

From the fact, that for a sufficiently large constant the main term exceeds the remainder, we conclude, that all the ζ_K functions have at least one zero $\beta_0+i\gamma_0$ for which $|\gamma_0|$ does not exceed an absolute constant and for which $\beta_0\geqq\frac{1}{2}$. Applying our Theorem to this zero, we get

COROLLARY. *There exist positive, absolute, effective constants c_0, c', c'' that for every algebraic number field K and any*

$$Y > c' \max\{n, \log|\Delta|\}, \tag{3.5}$$

we can find an

$$x\in[Y, Y^{c''(n+\log|\Delta|)}] \tag{3.6}$$

for which

$$|\Delta_K(x)| > c_0\sqrt{x}. \tag{3.7}$$

I am deeply indebted to J. Pintz for drawing my attention to the field, and for many helpful remarks during the time of my work.

4. In the course of proof $c_1, c_2, \ldots$ denote always positive, absolute, effectively calculable constants.

LEMMA 1.

$$|\Delta_K(x)| \leqq \frac{n}{\log 2} x \log^2 x. \tag{4.1}$$

PROOF. While $0\leqq G(m)\leqq\frac{n}{\log 2}\log^2 m$ ([8], Lemma 2) this is trivial.

LEMMA 2. *For* $-\infty<t<\infty$

$$\left|\frac{1}{\zeta_K\left(\frac{3}{2}+it\right)}\right| < \left(\zeta\left(\frac{3}{2}\right)\right)^n. \tag{4.2}$$

PROOF. This is Corollary 2 and Corollary 3 in p. 295 of [4] with $d=\frac{1}{2}$.

LEMMA 3. *For* $-1\leqq\sigma\leqq 4$, $-\infty<t<\infty$ *one has*

$$|(s-1)\zeta_K(s)| \leqq c_1^n|\Delta|^{3/2}(|t|+1)^{(3/2)n+2}. \tag{4.3}$$

PROOF. This is Lemma 3 in [9].

LEMMA 4. *If* $T\geqq 0$, *then*

$$\sum_{|\gamma-T|\leqq 1} 1 \leqq c_2 n\log(T+2)+c_3\log|\Delta|. \tag{4.4}$$

PROOF. We use Jensen's inequality for $r=\sqrt{2}$, $R=\frac{5}{2}$ around $s_0=\frac{3}{2}+iT$.

$$\sum_{|\gamma-T|\leq 1} 1 \leq 2 \sum_{\substack{\varrho \\ \beta\geq 1/2 \\ |\gamma-T|\leq 1}} 1 \leq 2 \sum_{\substack{\varrho \\ |\varrho-s_0|\leq\sqrt{2}}} 1 \leq$$

(4.5)

$$\leq \frac{2}{\log\frac{R}{r}} \log \max_{|s-s_0|\leq R} \left|\frac{\zeta_K(s)(s-1)}{\zeta_K(s_0)(s_0-1)}\right| \leq c_4 \log\left(\frac{M}{\zeta_K\left(\frac{3}{2}+iT\right)\cdot\frac{1}{2}}\right),$$

where

(4.6) $$M = \max_{\substack{|\sigma-(3/2)|\leq(5/2) \\ |t|\leq T+(5/2)}} |\zeta_K(s)(s-1)| < c_5^n|\Delta|^{3/2}(T+2)^{(3/2)n+2},$$

in view of Lemma 3, and so by Lemma 2 we get

(4.7) $$\sum_{|\gamma-T|\leq 1} 1 < c_4 \log\left(c_6^n|\Delta|^{3/2}(T+2)^{(3/2)n+2}\right) < c_2\, n \log(T+2)+c_3 \log|\Delta|.$$

Lemma 4 trivially implies

LEMMA 5. *Let l be a natural number. Then*

(4.8) $$\sum_{|\gamma-T|\leq l} 1 \leq c_7\, l\big(n \log(|T|+l+2)+\log|\Delta|\big).$$

LEMMA 6. *When* $-\infty<t<\infty$, *we have for* $s=-\frac{1}{2}+it$

(4.9) $$\left|\frac{1}{\zeta_K\left(-\frac{1}{2}+it\right)}\right| \leq c_8^n \frac{1}{|\Delta|\,|s|^n},$$

where

(4.10) $$c_8 = 2\pi\zeta\left(\frac{3}{2}\right).$$

PROOF. If F denotes the canonical polynomial of ϑ over $\mathbf{Q}$, we denote the number of its real zeros by r_1, and the number of the complex conjugate pairs of roots by r_2, so $r_1+2r_2=n$. Following Landau, we denote the constant in the functional equation (see e.g. [2], Satz 154, Satz 155) by A, i.e.

(4.11) $$A = \sqrt{|\Delta|}\, 2^{-r_2}\pi^{-n/2}.$$

For the proof of the Lemma we write

$$\frac{1}{\zeta_K(s)} = \frac{\zeta_K(1-s)}{\zeta_K(s)} \frac{1}{\zeta_K(1-s)}.$$

Now let $s=-\frac{1}{2}+it$, then

$$\frac{1}{|\zeta_K(1-s)|} < \left(\zeta\left(\frac{3}{2}\right)\right)^n,$$

and for $f(s)=\frac{\zeta_K(1-s)}{\zeta_K(s)}$ we have by the functional equation ([2], Satz 155, 2)

$$f(s) = A^{2s-1}\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right)^{r_1}\left(\frac{\Gamma(s)}{\Gamma(1-s)}\right)^{r_2} =$$

(4.12)

$$= A^{2s-1}\left(\frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right)^{r_1}\frac{1}{\left(\frac{s}{2}\right)^{r_1}}\left(\frac{\Gamma(s+2)}{\Gamma(1-s)}\right)^{r_2}\frac{1}{(s(s+1))^{r_2}},$$

$$(4.13)\quad |f(s)| = A^{-2}\left|\frac{\Gamma\left(\frac{3}{4}+i\frac{t}{2}\right)}{\Gamma\left(\frac{3}{4}-i\frac{t}{2}\right)}\right|^{r_1}\left|\frac{\Gamma\left(\frac{3}{2}+it\right)}{\Gamma\left(\frac{3}{2}-it\right)}\right|^{r_2}\frac{2^{r_1}}{|s|^{r_1+r_2}|s+1|^{r_2}} = A^{-2}2^{r_1}\frac{1}{|s|^{r_1+2r_2}}.$$

In view of (4.11) and $r_1+2r_2=n$, we get from this the lemma.

LEMMA 7. *For* $-\infty<t_0<\infty$, $s_0=-\frac{1}{2}+it_0$

$$(4.14)\qquad \left|\frac{\zeta_K'}{\zeta_K}(s_0)+\frac{1}{s_0-1}\right| < c_9\, n\log(|t_0|+2)+3\log|\Delta|.$$

PROOF. Let $g(s)=\zeta_K(s)(s-1)$, $r=\frac{1}{3}$. Then

$$(4.15)\qquad \frac{1}{|g(s_0)|} \leqq \frac{1}{|\zeta_K(s_0)|} \leqq \frac{1}{|\Delta|}c_8^n\frac{1}{|s_0|^n}$$

by Lemma 6, and

$$(4.16)\qquad \max_{|s-s_0|\leqq r}|g(s)| \leqq c_1^n|\Delta|^{3/2}\left(|t_0|+\frac{4}{3}\right)^{(3/2)n+2}$$

by Lemma 3.

Since $g(s)$ is regular and nonvanishing in the whole circle $|s-s_0|\leqq r$, we can apply Satz 4.3, Anhang [6]

$$(4.17)\qquad \left|\frac{g'}{g}(s_0)\right| \leqq \frac{2}{r}\log\max_{|s-s_0|\leqq r}\left|\frac{g(s)}{g(s_0)}\right|,$$

and so considering our estimates (4.15) and (4.16)

$$\left|\frac{\zeta_K'}{\zeta_K}(s_0)+\frac{1}{s_0-1}\right| \leqq 6\log\left|(c_1c_8)^n\sqrt{|\Delta|}\left(\frac{|t_0|+\frac{4}{3}}{|s_0|}\right)^n\left(|t_0|+\frac{4}{3}\right)^{(1/2)n+2}\right| \leqq$$

(4.18)

$$\leqq c_9\, n\log(|t_0|+2)+3\log|\Delta|.$$

The main tool in the proof of the theorem will be Turán's powersum method. Here we shall use Cassels' powersum theorem [1] which we formulate as

LEMMA 8. *For arbitrary complex numbers* $z_1, z_2, \ldots, z_N$ *we have*

$$\max_{1 \leqq \nu \leqq 2N-1} \left| \frac{\sum_{j=1}^{N} z_j^\nu}{z_1^\nu} \right| \geqq 1. \tag{4.19}$$

Substituting here $z_j = e^{\alpha_j a}$ we get from (4.19)

LEMMA 9. *For arbitrary complex numbers* $\alpha_1, \alpha_2, \ldots, \alpha_N$ *and for any* $a > 0$

$$\max_{a \leqq t \leqq (2N-1)a} \left| \frac{\sum_{j=1}^{N} e^{\alpha_j t}}{e^{\alpha_1 t}} \right| \geqq 1. \tag{4.20}$$

5. PROOF of the theorem. We use the "kernel-function" e^{ks^2+Hs} with real $k > 1$, for which the well-known integral formula

$$\frac{1}{2\pi i} \int_{(b)} e^{ks^2+Hs}\, ds = \frac{1}{2\sqrt{\pi k}} \exp\left(-\frac{H^2}{4k}\right) \tag{5.1}$$

holds for every real k and complex H. If we define for $\sigma > 1$ the convergent integral

$$\begin{aligned} D(s) &= \int_1^\infty \Delta_K(x) \frac{d}{dx}(x^{-s})\, dx = -s \int_1^\infty \frac{\Psi_K(x) - x}{x^{s+1}}\, dx = \\ &= \frac{\zeta_K'}{\zeta_K}(s) + \frac{s}{s-1} = \frac{\zeta_K'}{\zeta_K}(s) + \frac{1}{s-1} + 1, \end{aligned} \tag{5.2}$$

then we can see from the latter form that D is a meromorf function.

Now we define

$$U = \frac{1}{2\pi i} \int_{(2)} D(s + i\gamma_0)\, e^{ks^2+\mu s}\, ds. \tag{5.3}$$

We can evaluate this integral in two ways; by passing the path of integration to $\sigma = -\frac{1}{2}$ and estimating the obtained sum of residues by the powersum method, or by using the definition of D and interchanging the order of integration. We will see, that our kernel function in this way relates the behaviour of $\Delta_K(x)$ around e^μ and the residues of the $\frac{\zeta_K'}{\zeta_K}$ function near to ϱ_0. Let $k = \frac{\mu}{16}$ where μ will be chosen later with the property $\mu \geqq a = 8 \log Y \geqq 40$.

$$\begin{aligned} U &= \frac{1}{2\pi i} \int_{(2)} \left(\frac{\zeta_K'}{\zeta_K}(s + i\gamma_0) + \frac{s + i\gamma_0}{s + i\gamma_0 - 1} \right) e^{ks^2+\mu s}\, ds = \\ &= \frac{1}{2\pi i} \int_{(-1/2)} \left(\frac{\zeta_K'}{\zeta_K}(s + i\gamma_0) + \frac{s + i\gamma_0}{s + i\gamma_0 - 1} \right) e^{ks^2+\mu s}\, ds + \sum_\varrho e^{k(\varrho - i\gamma_0)^2 + \mu(\varrho - i\gamma_0)}. \end{aligned} \tag{5.4}$$

Denoting the sum of residues by $W(\mu, \varrho_0)=W$, we have by $\mu>k>1$ and by Lemma 7

$$|W-U| \leqq \frac{1}{2\pi} \int_{-\infty}^{\infty} \{c_9\, n \log(|t+\gamma_0|+2)+3\log|\Delta|+1\}\, e^{k\left(\frac{1}{4}-t^2\right)-\frac{\mu}{2}}\, dt \leqq$$

$$\leqq e^{\frac{k}{4}-\frac{\mu}{2}}\left\{n\log(\gamma_0+2)\, c_{10} \int_{-\infty}^{\infty} (1+\log(|t|+2))\, e^{-t^2}\, dt+3\log|\Delta|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2}\, dt\right\}<$$

(5.5) $$< e^{-\frac{\mu}{3}}\{c_{11}\, n\log(\gamma_0+2)+3\log|\Delta|\}<1,$$

if

(5.6) $$\mu>3\log[c_{11}\, n\log(\gamma_0+2)+3\log|\Delta|].$$

Let

(5.7) $$V\overset{\text{def}}{=}\sum_{|\gamma-\gamma_0|<8} e^{\left(\frac{1}{16}(\varrho-i\gamma_0)^2+(\varrho-i\gamma_0)\right)\mu}.$$

The number of summands in V is less than $c_{12}(n\log(\gamma_0+10)+\log|\Delta|)$, according to Lemma 5. Using Lemma 4 we have

$$|W-V| \leqq \sum_{|\gamma-\gamma_0|\geqq 8} e^{k(\beta^2-(\gamma-\gamma_0)^2)+\mu\beta} < e^{\mu+k}\sum_{\nu=8}^{\infty} e^{-k\nu^2}\sum_{\nu\leqq|\gamma-\gamma_0|<\nu+1} 1<$$

(5.8) $$< e^{\mu+k}\sum_{\nu=8}^{\infty} 2\left(c_2\, n\log(\nu+\gamma_0+2)+c_3\log|\Delta|\right)e^{-k\nu^2}<$$

$$< e^{\mu+k-64k}\, c_{13}\left(n\log(\gamma_0+2)+\log|\Delta|\right)<1,$$

if

(5.9) $$\mu>\log\left[c_{13}\left(n\log(\gamma_0+2)+\log|\Delta|\right)\right].$$

Now we use Lemma 9 for V $\Big(a>0,\ \alpha_j=\frac{1}{16}(\varrho-i\gamma_0)^2+(\varrho-i\gamma_0),\ 1\leqq N<c_{12}(n\log(\gamma_0+$ $+10)+\log|\Delta|),\ \alpha_1=\frac{1}{16}\beta_0^2+\beta_0$ (from the summand for $\varrho_0)\Big)$.

We get the existence of a

(5.10) $$\mu\in\left[a,\ 2c_{12}\left(n\log(\gamma_0+10)+\log|\Delta|\right)a\right],$$

for which

(5.11) $$|V|\geqq e^{k\beta_0^2+\mu\beta_0}.$$

On the other hand, interchanging the order of integrations and derivation we have

$$U=\frac{1}{2\pi i}\int_{(2)}\left(\int_1^{\infty}\Delta_K(x)\frac{d}{dx}(x^{-s-i\gamma_0})\, dx\right)e^{ks^2+\mu s}\, ds=$$

$$=\int_1^{\infty}\Delta_K(x)\frac{d}{dx}\left[x^{-i\gamma_0}\frac{1}{2\pi i}\int_{(2)} e^{ks^2+(\mu-\log x)s}\, ds\right]dx=$$

(5.12)

$$= \int_1^\infty \Delta_K(x) \frac{d}{dx}\left[x^{-i\gamma_0} \frac{1}{2\sqrt{\pi k}} \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\right] dx =$$

$$= \frac{1}{2\sqrt{\pi k}} \int_1^\infty \frac{\Delta_K(x)}{x} x^{-i\gamma_0} \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\left(-i\gamma_0 - \frac{2(\log x-\mu)}{4k}\right) dx.$$

Now let $U = U_1 + U_2 + U_3$ where

$$(5.13)\quad U_1 = \int_1^{e^{\mu-14k}} = \int_1^{e^{(1/8)\mu}}, \quad U_2 = \int_{e^{\mu-14k}}^{e^{\mu+14k}} = \int_{e^{(1/8)\mu}}^{e^{(15/8)\mu}}, \quad U_3 = \int_{e^{\mu+14k}}^{\infty} = \int_{e^{(15/8)\mu}}^{\infty}.$$

Using Lemma 1 and $\mu \geqq 40$, we gain for U_1 and U_3

$$|U_1| < \int_1^{e^{\mu-14k}} \frac{n}{\log 2} \log^2 x \exp(-49k)\left(\gamma_0 + \frac{\mu}{2k}\right) dx <$$

(5.14)

$$< \frac{n}{\log 2}(\gamma_0+8)\frac{\mu^2}{64} \exp\left(-\frac{49}{16}\mu\right) e^{(1/8)\mu} < 1,$$

if

$$(5.15)\qquad \mu > \log(n(\gamma_0+2)),$$

and

$$|U_2| < \frac{1}{2\sqrt{\pi k}} \int_{e^{(15/8)\mu}}^{\infty} \frac{n}{\log 2} \log^2 x \exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\left(\gamma_0 + \frac{\log x - \mu}{2k}\right) dx =$$

$$= \frac{1}{2\sqrt{\pi k}} \frac{n}{\log 2} \int_{(7/8)\mu}^{\infty} (\mu+y)^2 \left(\gamma_0 + \frac{y}{2k}\right) \exp\left(-\frac{y^2}{4k}\right) e^{\mu+y}\, dy <$$

$$< \frac{e^{\mu+k}}{2\sqrt{\pi k}} \frac{n}{\log 2} \int_{(7/8)\mu}^{\infty} \left(\frac{15}{7} y\right)^2 (\gamma_0+2) \frac{y}{2k} \exp\left(-\frac{y^2}{4k} + y - k\right) dy =$$

$$(5.16)\qquad = \frac{e^{\mu+k} n(\gamma_0+2)}{2\log 2\sqrt{\pi k}} \left(\frac{15}{7}\right)^2 \frac{1}{2k} \int_{14k}^{\infty} y^3 \exp\left[-\left(\frac{y}{2\sqrt{k}} - \sqrt{k}\right)^2\right] dy <$$

$$< \frac{e^{\mu+k} n(\gamma_0+2)}{2\log 2\sqrt{\pi k}} \left(\frac{15}{7}\right)^2 \frac{1}{2k} \int_{14k}^{\infty} \left[2\sqrt{k}\frac{7}{6}\left(\frac{y}{2\sqrt{k}} - \sqrt{k}\right)\right]^3 \exp\left[-\left(\frac{y}{2\sqrt{k}} - \sqrt{k}\right)^2\right] dy =$$

$$= e^{\mu+k} \frac{n(\gamma_0+2)}{\sqrt{\pi}} \frac{175}{12\log 2} \int_{6\sqrt{k}}^{\infty} t^3 \exp(-t^2)\, 2\sqrt{k}\, dt =$$

$$= e^{17k} n(\gamma_0+2)\sqrt{\mu} \frac{175}{48\sqrt{\pi}\log 2} \left\{[t^2(-e^{-t^2})]_{6\sqrt{k}}^{\infty} - \int_{6\sqrt{k}}^{\infty} 2t(-e^{-t^2})\, dt\right\} <$$

$$< e^{17k} n(\gamma_0+2) 3\sqrt{\mu}(36k+1)e^{-36k} < n(\gamma_0+2) 7\mu^{3/2} e^{-(19/16)\mu} < 1,$$

if

$$\mu > \log(n(\gamma_0+2)). \tag{5.17}$$

Now from (5.5), (5.8), (5.11), (5.13), (5.14) and (5.16) we have

$$|U_2| \geqq e^{k\beta_0^2+\mu\beta_0} - 4 \tag{5.18}$$

with a suitable μ satisfying (5.10) and the further conditions (5.6), (5.9), (5.15) and (5.17), for which it is enough if

$$a > 3\log\{c_{14}(n(\gamma_0+2)+\log|\Delta|)\}. \tag{5.19}$$

Let us suppose, that with some A

$$|\Delta_K(x)| < Ax^{\beta_0} \tag{5.20}$$

for all x in $[e^{\mu/8}, e^{(15/8)\mu}]$. Then for this A we get

$$|U_2| \leqq A\frac{1}{2\sqrt{\pi k}}\int_{e^{\mu-14k}}^{e^{\mu+14k}} x^{\beta_0}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)|7+i\gamma_0|\frac{dx}{x} =$$

$$= A\frac{1}{2\sqrt{\pi k}}\sqrt{49+\gamma_0^2}\int_{-14k}^{14k}\exp\left(-\frac{y^2}{4k}\right)e^{\beta_0(y+\mu)}\,dy < \tag{5.21}$$

$$< A\sqrt{49+\gamma_0^2}\,e^{\beta_0\mu+\beta_0^2k}\frac{1}{2\sqrt{\pi k}}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{y}{2\sqrt{k}}-\beta_0\sqrt{k}\right)^2\right]dy =$$

$$= Ae^{\beta_0\mu+\beta_0^2k}\sqrt{49+\gamma_0^2}.$$

By (5.18) and (5.21) we have for any A satisfying (5.20)

$$A > \frac{1}{\sqrt{49+\gamma_0^2}}\left(1-\frac{4}{e^{\mu\beta_0+k\beta_0^2}}\right). \tag{5.22}$$

Now, as $\sqrt{1-x}<1-\frac{x}{3}$ for $0\leqq x\leqq 0.1$, and as $\mu>a$ and $\beta_0\geqq\frac{1}{2}$, we have with suitable c_{15} and c_{16} for any

$$a > 4\log(\gamma_0+2)+c_{15} \tag{5.23}$$

for $\gamma_0>c_{16}$

$$\sqrt{\frac{49+\gamma_0^2}{50+\gamma_0^2}} < 1-\frac{1}{3(50+\gamma_0^2)} < 1-\frac{4}{(\gamma_0+2)^{33/16}} < 1-\frac{4}{e^{\mu(33/64)}} < 1-\frac{4}{e^{\mu\beta_0+k\beta_0^2}},$$

for $\gamma_0<c_{16}$

$$\sqrt{\frac{49+\gamma_0^2}{50+\gamma_0^2}} < 1-\frac{1}{3(50+c_{16}^2)} < 1-\frac{4}{e^{(1/2)c_{15}}} < 1-\frac{4}{e^{\mu\beta_0+k\beta_0^2}},$$

that is, by (5.22)

$$A > \frac{1}{\sqrt{50+\gamma_0^2}}.$$

So we have an $x \in [e^{(1/8)\mu}, e^{(15/8)\mu}]$ with the chosen μ for which (3.3) holds. Finally, considering (5.10) and the restrictions (5.19) and (5.23) for a, for every

$$Y = e^{a/8} > \max \left\{e^{3/8 \log (c_{14}(n(\gamma_0+2)+\log|\Delta|))}, e^{(1/2)\log(\gamma_0+2)+(1/8)c_{15}}\right\},$$

so for every

$$Y > C \max \{\log|\Delta|, n, (\gamma_0+2)\}$$

we get the existence of an x,

$$x \in [e^{\mu/8} e^{(15/8)\mu}] \subset [Y, Y^{c(n\log(\gamma_0+10)+\log|\Delta|)}]$$

for which (3.3) holds. Q.e.d.

REFERENCES

[1] CASSELS, J. W. S., On the sums of powers of complex numbers, *Acta Math. Acad. Sci. Hungar.* **7** (1956), 283—289. *MR* **19**—397.

[2] LANDAU, E., *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Teubner, Leipzig—Berlin, 1918.

[3] LITTLEWOOD, J. E., Mathematical notes XII. An inequality for a sum of cosines, *J. London Math. Soc.* **12** (1937), 217—221. *Zbl* **16**, 395.

[4] NARKIEWICZ, W., *Elementary and analytic theory of algebraic numbers*, Monografie Matematyczne, Tom 57, PWN-Polish Scientific Publishers, Warszawa, 1974. *MR* **50** # 268.

[5] PINTZ, J., On the remainder term of the prime number formula. I. On a problem of Littlewood, *Acta Arith.* **36** (1980), 341—365.

[6] PRACHAR, K., *Primzahlverteilung*, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1957. *MR* **19**—393.

[7] STAŚ, W. and WIERTELAK, K., A comparison of certain remainders connected with prime ideals in ideal classes mod f, *Functiones et Approximatio* **4** (1976), 97—107. *MR* **56** # 281.

[8] STAŚ, W., Über eine Anwendung der Methode von Turán auf die Theorie des Restgliedes im Primidealsatz, *Acta Arith.* **5** (1959), 179—195. *MR* **21** # 6355.

[9] STAŚ, W. and WIERTELAK, K., Some estimates in the theory of Dedekind zeta-functions, *Acta Arith.* **23** (1973), 127—135. *MR* **49** # 2668.

[10] TITCHMARSH, E. C., *The theory of the Riemann zeta-function*, Oxford at the Clarendon Press, Oxford, 1951. *MR* **13**—741.

[11] TURÁN, P., On the remainder-term of the prime number formula I, *Acta Math. Acad. Sci. Hungar.* **1** (1950), 48—63. *MR* **13**—208.

(Received January 13, 1982)

SOMLÓI ÚT 62
H—1118 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 69—73

PACKING AND COVERING WITH r-CONVEX DISCS

L. FEJES TÓTH

We define an *r-convex* domain as the intersection of circular discs of radius r. We shall denote a domain and its area with the same symbol. Let $c_1, \ldots, c_n$ be convex domains with average area $a=(c_1+\ldots+c_n)/n$. Let R be a region. The quotient na/R is said to be the density of the domains relative to R. We say that two domains cross each other if removing their intersection causes both domains to fall into disjoint components.

We shall prove the following theorems.

THEOREM 1. *If a finite number of r-convex discs with average area a and average perimeter p are packed into a convex polygon h with at most six sides then the density d of the discs relative to h satisfies the inequality*

$$\frac{1}{d} \geqq 1+6r^2\,\frac{\tan\dfrac{p}{12r}-\dfrac{p}{12r}}{a}. \tag{1}$$

THEOREM 2. *If a finite number of r-convex discs with average area a and average perimeter p cover a convex polygon h with at most six sides so that no two discs cross then the density D of the discs relative to h satisfies the inequality*

$$\frac{1}{D} \leqq 1-3r^2\,\frac{\dfrac{p}{6r}-\sin\dfrac{p}{6r}}{a}. \tag{2}$$

For congruent circles (1) and (2) yield the known exact bounds $d\leqq\pi/\sqrt{12}$ and $D\geqq 2\pi/\sqrt{27}$. The question how sharp the inequalities (1) and (2) are in other cases will be discussed later.

Using the inequality $\tan x-x>x^3/3$, $0<x<\pi/2$, we obtain from (1)

$$\frac{1}{d} > 1+\frac{p^3}{864ra}.$$

This corollary of Theorem 1 has been proved in [3].

To the proof of (1) and (2) we shall need the following

1980 *Mathematics Subject Classification.* Primary 52A45.
Key words and phrases. Packing and covering, convexity.

LEMMA. *In the interval* $(0, b)$ *let the function* $f(x)$ *have a positive second derivative:* $f''(x)>0$. *Then, in the angular region* $0<y<bx$, *the function* $z(x, y)=xf(y/x)$ *is convex. In the case when* $f''(x)<0$ *the function* $z(x, y)$ *is concave.*

For we have $z_{xx}=\frac{y^2}{x^3}f''\left(\frac{y}{x}\right)>0$ or <0 according as $f''>0$ or $f''<0$, and

$$z_{xx}z_{yy}-z_{xy}^2 = \frac{y^2}{x^3}f''\left(\frac{y}{x}\right)\frac{1}{x}f''\left(\frac{y}{x}\right)-\left\{-\frac{y}{x^2}f''\left(\frac{y}{x}\right)\right\}^2 = 0.$$

Let the r-convex discs referred to in Theorem 1 and 2 be $c_1, \dots, c_n$. We may suppose that $r=1$. We also may assume that each disc c is the intersection of finitely many unit circles. Let the perimeter of c_i be p_i. Obviously, $p_i\leqq 2\pi$. Let a_k^i be a k-gon of minimal area circumscribed about c_i. By a simple argument one can show (see [3]) that

$$a_k^i-c_i \geqq k\tan\frac{p_i}{2k}-\frac{p_i}{2}.$$

Supposing that the discs $c_1, \dots, c_n$ are packed into the convex polygon h with at most six sides, we use the known construction [2, 4, 5] of blowing up the discs to non-overlapping convex polygons $q_1, \dots, q_n$ of number of sides $k_1, \dots, k_n$ such that $c_i\subset q_i\subset h$ and

(3) $$k_1+\dots+k_n \leqq 6n.$$

Then we have

$$h \geqq \sum_{i=1}^{n} q_i \geqq \sum_{i=1}^{n} a_{k_i}^i \geqq \sum_{i=1}^{n}\left(c_i+k_i\tan\frac{p_i}{2k_i}-\frac{p_i}{2}\right) = na-\frac{1}{2}np+\sum_{i=1}^{n}k_i\tan\frac{p_i}{2k_i}.$$

But by our Lemma the function $k\tan\frac{p}{2k}$ is convex for $k\geqq 3$, $0<p\leqq 2\pi$, and by

$$\frac{\partial}{\partial k}\left(k\tan\frac{p}{2k}\right) = -\frac{1}{2}\cos^{-2}\frac{p}{2k}\left(\frac{p}{k}-\sin\frac{p}{k}\right) < 0,$$

it is a decreasing function of k. So we have, by Jensen's inequality and (3),

$$h \geqq na-\frac{1}{2}np+n6\tan\frac{p}{2\cdot 6}$$

which is equivalent with (1).

Turning to the proof of (2), let $b_k^i=b_k$ be a k-gon of maximal area inscribed into $c_i=c$. Let A and B be consecutive vertices of b_k, l the length of the arc AB of the boundary of c, and s the segment of c cut off by the line AB. If the open arc AB does not contain any vertex then

$$s = \frac{l}{2}-\frac{1}{2}\sin l.$$

Otherwise

$$s > \frac{l}{2}-\frac{1}{2}\sin l.$$

To see this we suppose that the arc AB contains some vertices of c which we denote in their proper order with $V_1, \ldots, V_{m-1}$. We write $A=V_0$, $B=V_m$, and consider the centro-symmetric arc-sided polygon $V=V_0V_1\ldots V_{2m}$ of area $2s$ and perimeter $2l$ bounded by the unit circular-arcs $V_0V_1, \ldots, V_{m-1}V_m$ and their images reflected in the midpoint of the straight segment $V_0V_m=AB$. The assumption that AB is a side of a k-gon of maximal area inscribed into c implies that V is convex.

Let V_jV_{j+m} be a greatest diagonal of V. Let V_i be any vertex of V other than V_j and V_{j+m}. Obviously, $\sphericalangle V_jV_iV_{j+m} \geqq 90°$. We consider the four segments of V outside the quadrangle $V_jV_iV_{j+m}V_{i+m}$ as rigid discs fixed to each other at the vertices of $V_jV_iV_{j+m}V_{i+m}$ by joints. We move this gadget so as to increase the angle $\sphericalangle V_jV_iV_{j+m}$ until the arcs $V_{i-1}V_i$ and V_iV_{i+1} will form one circular arc, obtaining a new arc-sided polygon of number of sides $2m-2$, perimeter $2l$, and area less than $V=2s$. Repeating successively this construction we obtain in $m-1$ steps a digon of area $l-\sin l$.

Summing up the inequalities $s \geqq \frac{l}{2}-\frac{1}{2}\sin l$ for all the k segments of c outside b_k and using Jensen's inequality we obtain

$$c_i-b_k^i \geqq \frac{1}{2}p_i-\frac{1}{2}k\sin\frac{p_i}{k}.$$

Supposing that the discs $c_1, \ldots, c_n$ fulfil the conditions of Theorem 2, we contract them [1, 4] to convex polygons $q_1, \ldots, q_n$ which fill h without overlapping and without interstices. Then the number of sides $k_1, \ldots, k_n$ of $q_1, \ldots, q_n$ automatically satisfy (3). Using the above inequality for b_k^i, we have

$$h=\sum_{i=1}^{n} q_i \leqq \sum_{i=1}^{n} b_k^i \leqq \sum_{i=1}^{n}\left(c_i-\frac{1}{2}p_i+\frac{1}{2}k_i\sin\frac{p_i}{k_i}\right)=na-\frac{1}{2}np+\frac{1}{2}\sum_{i=1}^{n}k_i\sin\frac{p_i}{k_i}.$$

Since, by the above lemma, the function $k\sin\frac{p}{k}$ is concave for $k\geqq 3$, $0<p<2\pi$, and since, by

$$\frac{\partial}{\partial k}\left(k\sin\frac{p}{k}\right)=\sin\frac{p}{k}-\frac{p}{k}\cos\frac{p}{k}>0, \qquad 0<\frac{p}{k}\leqq\pi$$

it is an increasing function of k, we have by Jensen's inequality and (3)

$$h \leqq na-\frac{1}{2}np+\frac{1}{2}n6\sin\frac{p}{6}$$

which is equivalent with (2).

In order to get an idea how sharp the inequalities (1) and (2) are, we give them a new interpretation. Let the r-convex discs $c_1, \ldots, c_n$ have average area a and average perimeter p. Let congruent replicas of $c_1, \ldots, c_n$ be distributed in the whole plane so that the number-density (see [5]) of the copies of c_i exists and has for all values of $i=1, \ldots, n$ the same value, say, δ. (If c_i and c_j are congruent, we imagine them to be differently coloured so as to distinguish the copies of c_i from those of c_j.) Theorem 1 implies that if the copies of $c_1, \ldots, c_n$ form a packing then the density $d=na\delta$ of the packing satisfies the inequality (1). Similarly, Theorem 2 implies that if the copies of

$c_1, \ldots, c_n$ cover the plane without crossing then the density $D=na\delta$ of the covering satisfies the inequality (2). In what follows we shall refer to (1) and (2) in this interpretation.

In (1) equality can be attained only for equal circles. We still consider the case when c_1 is a unit circle and c_2 a regular arc-sided quadrangle of inradius $\sqrt{2}-1$ bounded by unit circular arcs. An equal number of copies of these discs can be packed (Fig. 1) with density 0.938 while (1) yields only the bound $d<0.941$.

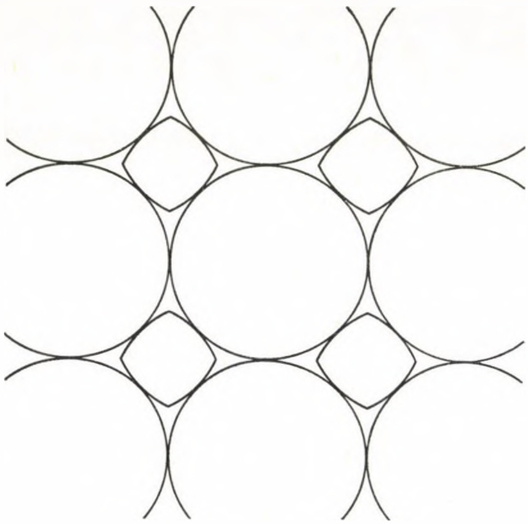


Fig. 1

On the other hand, in (2) equality can be attained in infinitely many cases. Let $q_1, \ldots, q_n$ be equilateral convex polygons of total number of sides $6n$ whose congruent copies are distributed in the same proportion so as to form a tiling of the plane. Attach to each side of these polygons congruent segments of a circle of radius r so as to obtain r-convex arc-sided polygons $c_1, \ldots, c_n$. The density D of the obtained covering is $D=\dfrac{c_1+\ldots+c_n}{q_1+\ldots+q_n}$. If s is a segment attached to a side of a polygon q_i then $c_1+\ldots+c_n=q_1+\ldots+q_n+6ns$. Since the central angle of the arc of s is equal to $\dfrac{np}{r}:6n=\dfrac{p}{6r}$, we have

$$s=\frac{r^2}{2}\left(\frac{p}{6r}-\sin\frac{p}{6r}\right).$$

Thus

$$\frac{1}{D}=\frac{q_1+\ldots+q_n}{c_1+\ldots+c_n}=\frac{c_1+\ldots+c_n-6ns}{c_1+\ldots+c_n}=1-6\frac{s}{a}$$

which is equal to the right side of (2).

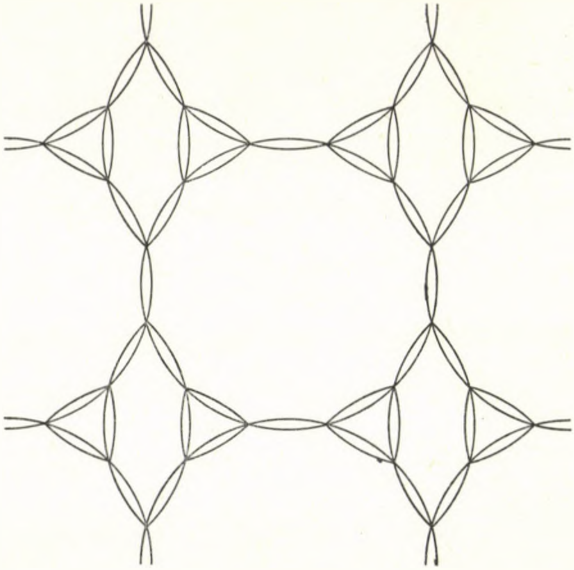


Fig. 2

Examples are given by the regular and Archimedean trihedral tilings (6, 6, 6), (3, 12, 12), (4, 8, 8) and (4, 6, 12). For instance (4, 6, 12) consists of an equal number of copies of three squares, two regular hexagons and a regular dodecagon. These tilings can continuously be distorted so as to yield further examples. But there are also polygons $q_1, \ldots, q_n$ satisfying the above conditions which generate tilings topologically different from those mentioned above. In the covering exhibited in Fig. 2 the corresponding set of polygons consists of two triangles, one hexagon and one dodecagon. The arc-sided hexagons and arc-sided dodecagons are degenerated into digons and circles.

REFERENCES

[1] Bambah, R. P. and Rogers, C. A., Covering the plane with convex sets, *J. London Math. Soc.* **27** (1952), 304—314. *MR* **13**—971.

[2] Cassels, J. W. S., *An introduction to the geometry of numbers*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. 99, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1959. *MR* **28**#1175.

[3] Fejes Tóth, L., Packing of r-convex discs, *Studia Sci. Math. Hungar.* **17** (1982), 449—452.

[4] Fejes Tóth, L., Some packing and covering theorems, *Acta Sci. Math. (Szeged)* **12** (1950), Pars A, 62—67. *MR* **12**—352.

[5] Fejes Tóth, L., *Lagerungen in der Ebene, auf der Kugel und im Raum*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. 65, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1953. *MR* **15**—248; Zweite verbesserte und erweiterte Auflage, Springer-Verlag, Berlin—New York, 1972. *MR* **50**#5603.

(Received January 13, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZET
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 75—77

THE GROWTH OF COEFFICIENTS OF RATIONAL FUNCTIONS WITH INTEGRAL COEFFICIENTS WHICH APPROXIMATE $|x|$

L. B. O. FERGUSON and J. SZABADOS

In the design of digital filters there arises the problem of approximating a given function by means of a rational function. Since the rational function will eventually be realized on a digital computer its coefficients can only be implemented with a certain accuracy (often to one part in $2^8=256$). After an appropriate scaling, the coefficients of the rational functions are integers. For more details see Ferguson [1] or Oppenheim and Shafer [3].

For this reason we are interested in bounds on the size of the coefficients which arise in approximation by rational functions. As a first step in this direction we submit the following two results on the approximation of the function $f(x)=|x|$. Throughout the history of approximation theory the study of this function has led to particularly significant results.

THEOREM 1. *There exist rational functions $r_n(x)$ of degree n with integer coefficients of order $O(\exp n^{3/2})$ such that*

$$\||x|-r_n(x)\|_{[-1,1]} = O\left(\exp\left(-\frac{1}{2}\sqrt{n}\right)\right).$$

PROOF. A slight modification of the famous Newman's rational function (see [2]) will make it of integer coefficient. Let

$$a = \exp n^{-1/2}, \quad p_n(x) = \prod_{k=0}^{n-1} ([a^k]x+1).$$

Then

$$r_n(x) = x\frac{p_n(x)-p_n(-x)}{p_n(x)+p_n(-x)}$$

will have the required properties. Namely, if $0\leqq x\leqq [a^n]^{-1}$ then $p_n(x)$ and $p_n(-x)$ are nonnegative and therefore $0\leqq r_n(x)\leqq x\leqq [a^n]^{-1}$. Let now $[a^{j+1}]^{-1}\leqq x\leqq [a^n]^{-1}$, $0\leqq j\leqq n-1$, then by $\frac{1-u}{1+u}\leqq e^{-2u}$ $(u\geqq 0)$ and $a-1<2/\sqrt{n}$ we get

$$\left|\frac{p_n(-x)}{p_n(x)}\right| = \prod_{k=0}^{j}\frac{1-x[a^k]}{1+x[a^k]}\prod_{k=j+1}^{n-1}\frac{x[a^k]-1}{x[a^k]+1} \leqq \prod_{k=0}^{j}\frac{[a^n]-[a^k]}{[a^n]+[a^k]}\prod_{k=j+1}^{n-1}\frac{[a^k]-[a^j]}{[a^k]+[a^j]} \leqq$$

$$\leqq \exp\left(-\frac{2}{[a^n]}\sum_{k=0}^{j}[a^k]-2[a^j]\sum_{k=j+1}^{n-1}\frac{1}{[a^k]}\right) \leqq$$

$$\leqq \exp\left(2\frac{j+1}{[a^n]}-2\frac{a^{j+1}-1}{a^n(a-1)}-2\frac{[a^j]}{a^j}\,\frac{1-a^{-n+j+1}}{a-1}\right) \leqq \exp\left(2-2\frac{a^{n-j}[a^j]-1}{a^n(a-1)}\right) \leqq$$

$$\leqq \exp\left(3-\frac{1}{a-1}\right) \leqq \exp\left(3-\frac{\sqrt{n}}{2}\right).$$

1980 *Mathematics Subject Classification.* Primary 41A20; Secondary 41A25.
Key words and phrases. Rational functions, integral coefficients, degree of approximation.

Thus

$$\big||x|-r_n(x)\big| = \frac{2x}{\left|\frac{p_n(x)}{p_n(-x)}+1\right|} \leqq \frac{2}{\left|\frac{p_n(x)}{p_n(-x)}\right|-1} \leqq \frac{2}{\exp\left(-3+\frac{\sqrt{n}}{2}\right)-1} =$$

$$= O\left(\exp\left(-\frac{\sqrt{n}}{2}\right)\right) \qquad ([a^n]^{-1} \leqq x \leqq 1).$$

The coefficient of x^k in $p_n(x)$ is evidently less than

$$\binom{n}{k} a^{k(2n-k-1)/2} \leqq \binom{n}{k} a^{n^2/2} = \binom{n}{k} \exp\left(\frac{1}{2}n^{3/2}\right) = O(\exp n^{3/2})$$

for all k, $0 \leqq k \leqq n$. Q.e.d.

THEOREM 2. *If $|x|$ is uniformly approximated on $[-1, 1]$ by rational functions of integer coefficients with an error ε, $0<\varepsilon<1/4$, then at least one of the coefficients is greater than $1/(5\sqrt{\varepsilon})$ in absolute value.*

PROOF. Let

$$r(x) = \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{n} b_k x^k} \qquad (a_k, b_k \text{ integers, } b_0 \neq 0)$$

be such that $\| |x| - r(x) \| < \varepsilon$. Consider

$$R(x) = \frac{r(x)+r(-x)}{2} = \frac{1}{2}\left(\frac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{n} b_k x^k} + \frac{\sum_{k=0}^{n} a_k (-1)^k x^k}{\sum_{k=0}^{n} b_k (-1)^k x^k}\right) =$$

$$= \frac{a_0 b_0 + (a_0 b_2 - a_1 b_1 + a_2 b_0)x^2 + \dots + (-1)^n a_n b_n x^{2n}}{b_0^2 + (2b_0 b_2 - b_1^2)x^2 + \dots + (-1)^n b_n^2 x^{2n}} = \frac{\sum_{k=0}^{n} A_k x^{2k}}{\sum_{k=0}^{n} B_k x^{2k}},$$

where

$$A_k = \sum_{j=\max(0,2k-n)}^{\min(2k,n)} (-1)^j a_j b_{2k-j},$$

(1)

$$B_k = \sum_{j=\max(0,2k-n)}^{\min(2k,n)} (-1)^j b_j b_{2k-j} \qquad (k = 0, 1, \dots, n).$$

Then obviously

(2) $$\big\| |x| - R(x) \big\| < \varepsilon.$$

Denoting $c_n = \max_{0 \leqq k \leqq n} (|a_k|, |b_k|)$, we have from (1)

$$|A_k| \leqq (2k+1)c_n^2, \quad |B_k| \leqq (2k+1)c_n^2 \qquad (k = 1, 0, ..., n). \tag{3}$$

We now distinguish two cases.

Case 1. $a_0 \neq 0$. Then $|r(0)| = |a_0/b_0| < \varepsilon$, and hence

$$c_n \geqq |b_0| > |a_0|/\varepsilon \geqq 1/\varepsilon > 1/\sqrt{20\varepsilon}.$$

Case 2. $a_0 = 0$. Then $A_0 = 0$, $|B_0| \geqq 1$ and by (3)

$$\left|\sum_{k=0}^{n} A_k (2\varepsilon)^{2k}\right| \leqq 4c_n^2\varepsilon^2 \sum_{k=1}^{\infty} (2k+1)(2\varepsilon)^{2k-2} \leqq 16c_n^2\varepsilon^2 \sum_{k=1}^{\infty} \frac{2k+1}{4^k} \leqq 20c_n^2\varepsilon^2,$$

$$\left|\sum_{k=0}^{n} B_k (2\varepsilon)^{2k}\right| \geqq 1 - 4c_n^2\varepsilon^2 \sum_{k=1}^{\infty} (2k+1)(2\varepsilon)^{2k-2} \geqq 1 - 20c_n^2\varepsilon^2.$$

Now if $1 - 20c_n^2\varepsilon^2 \leqq 0$ then

$$c_n \geqq \frac{1}{\varepsilon\sqrt{20}} > 1/(5\sqrt{\varepsilon})$$

and the statement is proved. Otherwise by (2)

$$\varepsilon \leqq R(2\varepsilon) = \frac{\sum_{k=0}^{n} A_k (2\varepsilon)^{2k}}{\sum_{k=0}^{n} B_k (2\varepsilon)^{2k}} \leqq \frac{20c_n^2\varepsilon^2}{1 - 20c_n^2\varepsilon^2},$$

i.e., $1 \leqq 20c_n^2(\varepsilon + \varepsilon^2) < 25c_n^2\varepsilon$. Q.e.d.

REFERENCES

[1] Ferguson, Le Baron O., Approximation by polynomials with integral coefficients and digital filter design, *Numerical Methods of Approximation Theory*, Vol. 5 (Conf. Math. Res. Inst., Oberwolfach, 1979), Internat. Ser. Numer. Math., 52, Birkhäuser-Verlag, Basel, 1980, 111—116. *MR* **81j**: 41008.

[2] Newman, D. J., Rational approximation to $|x|$, *Michigan Math. J.* **11** (1964), 11—14. *MR* **30** # 1344.

[3] Oppenheim, A. V. and Shafer, R. W., *Digital signal processing*, Prentice-Hall, Englewood Cliffs, N. J., 1975.

(Received February 1, 1982)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
RIVERSIDE, CA 92502
U.S.A.

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 79—141

CATEGORICAL ALGEBRAIC PROPERTIES. A COMPENDIUM ON AMALGAMATION, CONGRUENCE EXTENSION, EPIMORPHISMS, RESIDUAL SMALLNESS, AND INJECTIVITY

E. W. KISS, L. MÁRKI, P. PRÖHLE and W. THOLEN

Introduction

The existence of amalgamated products, injectives, non-surjective epimorphisms or large subdirectly irreducible objects in particular categories are closely related and frequently discussed questions in algebra and topology. Their solution becomes in many cases easier from a more general, say universal algebraic or categorical, point of view. For instance, one can often make use of some general theorems which are not always well-known for the specialist of a concrete field. The aim of this paper is to give a survey on concrete and general results in this area. The first part gives the definitions, some fundamental general theorems and key references concerning them. In the second part we present a table which lists for many familiar categories whether they enjoy the amalgamation property, the intersection property of amalgamations (hence also the strong amalgamation property), the congruence extension property; whether they have surjective epimorphisms, enough absolute retracts, cogenerating sets, injective hulls; whether they are residually small. The article ends with a classified bibliography which is intended to include all known results concerning these topics by May 1983. By the word 'classified' we mean that for each item we indicate which of the properties discussed in our survey are treated there. Of course, the authors do realize that their intention of completeness of references may not have been achieved, and they apologize to those colleagues whose work may be missing here.

The present survey grew out of a preprint with a similar intention of the last named author: *Amalgamations in categories,* Seminarberichte, Fachbereich Mathematik, Fernuniversität Hagen, **5** (1979), 121—151, and its principles have been agreed upon during a visit of his in Budapest in December 1979, sponsored by the J. Bolyai Mathematical Society within the frameworks of an agreement with the Deutsche Mathematiker-Vereinigung. The authors are indebted to many colleagues for helpful suggestions, especially to K. Głazek, H.-J. Hoehnke, J. R. Isbell, T. Katriňák, F. E. J. Linton, L. N. Shevrin, L. A. Skornjakov, and above all, to G. M. Bergman, whose contribution would have justified him to become a co-author.

1. General results

§0. Introductory remarks. In this first part of our paper we sum up the most important universal algebraic or categorical results concerning our topic, and we also present some general methods and ideas of proofs which facilitate settling problems of these kinds in concrete classes. We do not deal with the model-theoretic (or algebraic-logical) aspects of these problems, but papers treating them are included in the bibliography. Especially amalgamation and related properties have a rich literature of this kind, giving e.g. syntactic characterizations for them, and showing that some of them are equivalent to definability or interpolation properties. Without claiming completeness even among key references on this topic, we refer to Andréka and Sain [81], Bacsich [74], [75a], Bacsich and Rowlands Hughes [74], Comer [69], Pigozzi [71], Preller [69].

The first five sections deal with classes of algebras. In §1 we define the notions which will be investigated in the sequel, and give the basic connections among them. §§2—5 treat residual smallness, congruence extension, amalgamation, and injectivity, respectively. §6 describes how the preceding considerations carry over to abstract categories.

In order to give due credit to the people who invented the notions we are going to investigate, we shall begin by giving exact references of the first occurrences of these concepts (as far as we know). Amalgamation was first considered for groups by Schreier [27], the general form of the property had its first appearance in Fraïssé [54], whereas the strong amalgamation property was introduced by Jónsson [56]. The term 'intersection property of amalgamations' in the present meaning was first used by Ringel [72], but the name turned up in Dwinger [70] and meant what we call strong amalgamation property, whereas the property itself had been investigated before, first probably by Isbell [66a]. Semigroup theorists call the same notion 'special amalgamation property'; however, the latter name has a different meaning in Grätzer and Lakser [71]. The classical form of the property 'epimorphisms are surjective' goes back to Isbell [57]. The congruence extension property was introduced by Grätzer and Lakser [71], and the transferability property appeared first in Banaschewski [70] and then, under this name, in Bacsich [72c]. The existence of a cogenerating set was first considered by Grothendieck [57], that of enough absolute retracts by Węglorz [66], whereas the notion of residual smallness is due to Taylor [72]. Having enough injectives was first established by Baer [40] for abelian groups, the general notion appeared in Buchsbaum [55]. The existence of injective hulls was proved for modules by Eckmann and Schopf [53], the general form of this property goes back to Mitchell [65].

We use notations and terminology of Grätzer [79], but make no distinction between an algebra and its underlying set, and write the mappings on the left. The sign $\square$ after a statement means that it can be proved by the reader with no difficulty.

§1. Generalities. For the following arguments we fix a class $\mathscr{K}$ of algebras with $\mathrm{HS}\mathscr{K} \subseteq \mathscr{K}$ and suppose that all algebras — unless otherwise stated — are in $\mathscr{K}$. We must emphasize that the closedness under HS is not necessary, but it makes life easier. However, the reasonings can be carried over to rather general arbitrary categories, see §6.

Before getting down to the investigations of our properties separately let us put their interrelations in their proper light. What follows is belonging to the folklore, the main references are: Banaschewski [70], Taylor [72], Grätzer, Lakser [71, 72a].

Let us start with some definitions. Our first concept appears in several places in mathematics. An algebra Q (not necessarily in $\mathscr{K}$) is said to be *injective* over $\mathscr{K}$ if whenever a diagram

$$\begin{tikzcd} A \arrow[r, tail, "\alpha"] \arrow[d, "\tau"'] & B \arrow[dl, dashed, "\varphi"] \\ Q & \end{tikzcd}$$

is given with an injective homomorphism α then there exists a φ such that $\varphi\alpha=\gamma$. $\mathscr{K}$ is said to have *enough injectives* (EI) if each object can be embedded into an injective one (in and over $\mathscr{K}$).

In order to characterize this concept let us generalize it. We say that a subalgebra A of B ($A \leqq B$ in notation) is a *retract* of B if it is the image of an idempotent endomorphism of B. R is called *absolute retract* in $\mathscr{K}$ if it is a retract in each of its extensions (in $\mathscr{K}$).

PROPOSITION 1.1. *The injectives in $\mathscr{K}$ are absolute retracts.* □

If we want to get EI via absolute retracts, we have to face two problems: 1) embedding each object into an absolute retract, and 2) finding conditions which ensure the injectivity of the latter.

A class $\mathscr{K}$ is said to have EAR *(enough absolute retracts)* if each of its objects can be embedded into an absolute retract in $\mathscr{K}$. This concept has as yet been investigated only for varieties, where it is equivalent to having only a set of (non-isomorphic) subdirectly irreducible algebras. Classes with this latter property are named *residually small* (RS). Results on RS varieties are summed up in §2 below.

In order to facilitate finding absolute retracts, let us make some general observations. For the first question call a subdirectly irreducible (for short: SI) algebra S *maximal subdirectly irreducible* in $\mathscr{K}$ if it cannot be properly embedded into any SI algebra in $\mathscr{K}$.

PROPOSITION 1.2. *The maximal SI algebras are absolute retracts.*

This statement is clear by virtue of the following definition and claims. An extension $A \leqq B$ is called *essential* if each non-0 congruence of B restricts to a non-0 one of A.

PROPOSITION 1.3. (a) *Essential extensions of SI algebras are SI.* □

(b) *If $A \leqq B$ then among the congruences θ on B with $\theta \restriction A=0$ there is a maximal one, θ_0, and the extension $A \leqq B/\theta_0$ is essential.* □

PROPOSITION 1.4. *An algebra has a proper essential extension iff it is not an absolute retract.* □

Now let us turn to the question: when are absolute retracts injective? We say that the *injections are transferable* in $\mathscr{K}$ or $\mathscr{K}$ has the *transferability property* (TP) if each diagram

(1)
$$
\begin{tikzcd}
A \arrow[r, tail, "\beta"] \arrow[d, "\gamma"'] & B \\
C &
\end{tikzcd}
$$

can be completed to a commutative square as follows:

(2)
$$\begin{tikzcd} A \arrow[r, tail, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d, "\gamma'"] \\ C \arrow[r, tail, "\beta'"'] & D \end{tikzcd}$$

PROPOSITION 1.5. *EI implies TP. Conversely, if $\mathscr{K}$ has TP then the absolute retracts are injective.* □

TP is a join of two properties which are very important in themselves. Namely, if γ in (1) is supposed to be surjective, then we get the *congruence extension property* (CEP), which can be formulated in a more algebraic way: we say that an algebra A has CEP if all congruences on its subalgebras can be extended to A, and by the definition above, a class $\mathscr{K}$ has CEP if each of its objects has CEP.

For the other concept let us call a quintuple $(A;\ \beta,\ B;\ \gamma,\ C)$ an *amalgam* if $\beta\colon A \rightarrowtail B$ and $\gamma\colon A \rightarrowtail C$ are injective homomorphisms (the mappings are sometimes omitted; here A is not empty since it is an algebra). We say that this amalgam can be *completed* if β, γ admit a commutative square of the form

(3)
$$\begin{tikzcd} A \arrow[r, tail, "\beta"] \arrow[d, tail, "\gamma"'] & B \arrow[d, tail, "\gamma'"] \\ C \arrow[r, tail, "\beta'"'] & D \end{tikzcd}$$

$\mathscr{K}$ is said to have the *amalgamation property* (AP) if each amalgam can be completed in $\mathscr{K}$. We mention that some difficulties can arise with respect to the empty subalgebra. For this problem see Lakser [73] and, concerning injectivity, Day [72].

PROPOSITION 1.6. *CEP and AP imply TP. Conversely, TP implies CEP and, if $\mathscr{K}$ has finite products, also AP.*

To explain the last assertion we mention that if γ in (1) is injective then TP does not give AP since γ' in (2) is not necessarily 1-1. So we must apply TP "on the other side" and then consider a product (cf. Proposition 4.1).

COROLLARY 1.7. *If $\mathscr{K}$ has finite products then EI is equivalent to AP and CEP and EAR.* □

AP has a frequently investigated stronger version. We say that an amalgam $(A;\ \beta,\ B;\ \gamma,\ C)$ can be *embedded* if it can be completed as in (3) so that

$$\operatorname{Im}\beta' \cap \operatorname{Im}\gamma' = \operatorname{Im}\gamma'\beta.$$

If each amalgam can be embedded then $\mathscr{K}$ has the *strong amalgamation property* (SAP).

To capture the extent to which SAP is stronger than AP we make an observation.

PROPOSITION 1.8. *The following two conditions are equivalent.*
(a) *If an amalgam can be completed, it can be embedded as well.*
(b) *Each amalgam of the form* $(A; \beta, B; \beta, B)$ *can be embedded.* □

(Note that an amalgam of the form $(A; \beta, B; \beta, B)$ can always be completed by B.)

Condition (a) is called the *intersection property of amalgamation* (IPA) and (b) is sometimes named the *special amalgamation property.*

PROPOSITION 1.9. $\mathscr{K}$ *has SAP iff it has AP and IPA.* □

Our last property to deal with is the following. Consider $\mathscr{K}$ as a category, then one can define epimorphisms in the usual sense: they are the morphisms α with the property that for all β and γ, $\beta\alpha=\gamma\alpha$ implies $\beta=\gamma$. We say that *epimorphisms are surjective* in $\mathscr{K}$ (ES) if the epimorphisms are onto mappings.

In many cases the validity of ES can be decided positively:

PROPOSITION 1.10. *IPA implies ES.* □

We are going to give a refined version of this statement in §4 as well as an example of a variety showing that the converse of 1.10 fails. However, we can prove the following:

PROPOSITION 1.11. *In a variety having AP the properties ES and IPA are equivalent.*

We shall prove this proposition at the end of §4, but let us remark here that the statement holds in more general categories, too (see §6).

We close with a practical remark.

PROPOSITION 1.12. *Suppose that in* $\mathscr{K}$ *each algebra can be embedded into a simple one, and that* $\mathscr{K}$ *is closed under finite products. Then*
(a) $\mathscr{K}$ *does not have CEP,*
(b) $\mathscr{K}$ *has only trivial absolute retracts.* □

Note that a weaker condition (instead of the existence of finite products) would also do. If $\mathscr{K}$ has the stronger property that each — in some way naturally defined — "partial" algebra (e.g. an amalgam) can be embedded into a simple algebra in $\mathscr{K}$ then SAP holds as well.

This property occurs several times. The embedding is usually carried out by constructing a larger partial algebra which kills the congruences of the original one, and then one repeats the procedure countably many times. This method works e.g. for non-unary similarity types and quasigroups.

§2. Residual smallness. This is a frequently discussed question in universal algebra, or, to be more precise, in the theory of varieties. The following is a basic result here.

THEOREM 2.1 (Taylor [72]). *The following are equivalent for a variety* V*:*
(i) V *has only a set of (isomorphism types of) SI algebras.*
(ii) $V=ISP(\mathscr{K})$ *for some subset* $\mathscr{K}\subseteq V$.

(iii) *Each algebra in V has only a set of (pairwise nonequivalent) essential extensions.*

(iv) *Each algebra in V can be embedded into an absolute retract in V.*

Sets $\mathcal{K}$ as in (ii) are called *cogenerating sets* for V; property ECS means the Existence of a Cogenerating Set.

This theorem is not at all easy to prove, in fact there are two difficult parts in it. The proof of (iv)⇒(i) goes through the theory of equationally compact algebras. The reader may get acquainted with it by reading the paper of Banaschewski and Nelson [72]. Note that (i)⇒(iv) is not trivial: the way which seems promising, namely, to construct the required absolute retracts as products of maximal SI algebras, cannot be followed because in spite of the trivial fact that products of injectives are injective, the product of two absolute retracts need not be an absolute retract even in a CD (congruence distributive) variety (Taylor [73]).

For proving (i)⇒(iii) we make use of another concept which occurs in several places. A formula $\varphi(x, y, z, u)$ in the first order language of V is called *congruence formula if*

(i) φ is positive,

(ii) $\forall y, z(\exists x \;\; \varphi(x, x, y, z) \Rightarrow y=z)$.

It is clear that $A \models \varphi(a, b, c, d)$ implies that $c \equiv d \;\; \theta(a, b)$, where $\theta(a, b)$ denotes the smallest congruence collapsing a and b. Conversely, Mal'cev's lemma implies that if $c \equiv d \;\; \theta(a, b)$ in A, then $A \models \varphi(a, b, c, d)$ for some congruence formula $\varphi(x, y, z, u)$ of the form

$$\exists z_0, \dots, z_n \left(z_0 = z \wedge z_n = u \wedge \bigwedge_{i=0}^{n-1} \exists x_0, \dots, x_m (z_i = \tau_i \wedge z_{i+1} = \tau_i(\sigma)) \right)$$

where each τ_i is a term with variables among $x, y, z, u, x_0, \dots, x_m$ and σ is the substitution switching x and y. Formulas of this form are called *Mal'cev schemes* and they are said to be *restricted* if the τ_i depend only on x, y, z and u.

The congruence formulas describe how the congruences spread, and though they cannot be handled in most of the cases, sometimes they prove very useful. This is so in our case as well, as we have:

THEOREM 2.2 (Taylor [72]). *A variety V is RS iff for each congruence formula φ in the language of V there exists a finite number n such that*

$$V \models \forall y, z[\exists x_1, \dots, x_n (\bigwedge_{1 \leq i < j \leq n} \varphi(x_i, x_j, y, z)) \Rightarrow y = z].$$

In fact this statement provides a deep insight into the behaviour of RS varieties. It is worth mentioning that its proof is based on a Ramsey-type theorem of combinacorial set theory due to Erdős and Radó. This way of proof yields even some numerital results on the size of subdirectly irreducible algebras.

THEOREM 2.3 (Taylor [72]). *Let $\varkappa = \aleph_0 +$ (the number of operations of V). Then if V is RS then each SI algebra in V has power $\leq 2^{\varkappa}$ and their number is $\leq 2^{2^{\varkappa}}$. It is also true that in V each essential extension of any $A \in V$ has at most $2^{|A|+\varkappa}$ elements.*

Concerning the number and the size of subdirectly irreducible algebras in a variety, further information can be found in McKenzie—Shelah [74], Baldwin—Berman [75], Baldwin [80].

There is also another problem to investigate. In most cases we are given a set (class) $\mathcal{K}$ of algebras and we have to say something about the SI algebras in the variety V generated by $\mathcal{K}$. If V is CD then they are in $HSU_p(\mathcal{K})$ (Jónsson [67]), but generally the problem is very hard. Anyhow, in many familiar varieties we find arbitrarily large SI algebras as soon as an infinite one can be constructed. This explains the importance of the following fundamental observation.

THEOREM 2.4 (Quackenbush [71]). *If a locally finite variety contains an infinite SI algebra then the size of its finite SI algebras is not bounded.*

The converse problem had a great influence on investigations concerning RS varieties and is still unsolved.

PROBLEM (Quackenbush [71]). Suppose that V is generated by a finite algebra and that V has infinitely many finite SI algebras. Is it true that V has an infinite SI algebra?

McKenzie [81, 82] described all the varieties of rings and semigroups which are RS, in the semigroup case up to groups. These results show that if the finite algebra in the above problem is a ring or a semigroup then the answer is yes. The same holds for all finite algebras generating a congruence modular (CM) variety because of the following deep result.

THEOREM 2.5 (Freese—McKenzie [81]). *The following are equivalent for a finite algebra A generating a CM variety V:*
(i) *V is RS.*
(ii) *Each SI in V has power $\leqq (l+1)!m$ where $m=|A|$ and $l=m^{m^{m+1}}$.*
(iii) *Each subalgebra of A satisfies the commutator identity $[x, x]\wedge y \leqq [x, y]$.*

Here $[x, y]$ denotes the commutator of congruences x and y. For groups, this notion coincides with mutual commutator subgroup, whereas in rings we have $[I, J]=IJ+JI$ (replacing congruences by normal subgroups and ideals, respectively). Commutators can be defined in any CM variety (for a very readable account see H. P. Gumm, An easy way to the commutator in modular varieties, *Arch. Math. (Basel)* **34** (1980), 220—228) and they can be handled essentially as in groups and rings. Considering commutators generally yields much information in the investigation of CM varieties, e.g. it is condition (iii) which makes Theorem 2.5 so effective.

Among others the well-known fact that a finite group generates an RS variety iff its Sylow subgroups are abelian, is a consequence of Theorem 2.5.

There is still another type of algebras for which the answer to the Quackenbush problem is known to be "yes". We say that a variety V has *definable principal congruences* (DPC) if there exists a formula $\varphi(x, y, z, u)$ in the first order language of V such that for each $a, b, c, d \in A \in V$ we have

$$c \equiv d\ \theta(a, b) \quad \text{iff} \quad A \models \varphi(a, b, c, d).$$

It can be easily seen that in this case φ is equivalent to a finite disjunction of Mal'cev schemes (which is a congruence formula). We have

THEOREM 2.6 (Baldwin—Berman [75]). *Let V be an RS variety with DPC. Then there exists a natural number N such that each SI algebra in V has $\leqq N$ elements.*

Note that locally finite CEP varieties have DPC (Baldwin—Berman [75]). Theorem 2.6 follows clearly from Theorem 2.2.

On the other hand two interesting 'almost counterexamples' to the Quackenbush problem can be found in Baldwin—Berman [75], and Baldwin [80].

Similar investigations of simple algebras instead of SI ones have been carried out in Magari, R., Una dimostrazione del fatto che ogni varietà ammette algebre semplici, *Ann. Univ. Ferrara, Sez. VII* **14** (1969), 1—4, Lampe, W. A. — Taylor, W., Simple algebras in varieties (preprint), McKenzie—Shelah [74], Freese—McKenzie [81]. In fact, in many cases the negation of RS is proved by finding arbitrarily large simple algebras.

§3. Congruence extension. It is not easy to obtain information about the CEP in general. Many results exist, however, dealing with CD (and recently CM) varieties. The CEP varieties of groups, rings, semigroups, and monoids, respectively, have also been described (see Biró—Kiss—Pálfy [82]).

Most works are based on the following observation.

PROPOSITION 3.1 (Day [71], Grätzer—Lakser [72b]). *Suppose $B \leqq A$ and each principal congruence on each C with $B \leqq C \leqq A$ can be extended to A. Then each congruence on B can be extended to A.*

The CEP is hereditary for subalgebras but not for homomorphic images (Fried [78]). It is preserved by direct limits but neither inverse limits (Biró—Kiss—Pálfy [82]) nor direct products. However, the situation changes in particular classes.

PROPOSITION 3.2 (a) (Kiss [81]). *In a congruence permutable (CP) variety homomorphisms preserve CEP.* □

(b) *In CD varieties finite products preserve CEP.* □

(c) (Kiss [a]). *In a CM variety a finite product is CEP provided that the square of each factor is CEP.*

We can use Proposition 3.1 to characterize CEP by means of Mal'cev schemes.

PROPOSITION 3.3. *A variety V has CEP iff for each Mal'cev scheme φ there exists a restricted one ψ such that $V \models \varphi \Rightarrow \psi$.* □

Now we have two aims: to find properties of CEP varieties and to provide sufficient conditions for a variety to have CEP. It may be surprising that there are natural 'non-artificial' examples for varieties having a single restricted Mal'cev scheme for all congruences — call them URCS varieties after Fried—Grätzer—Quackenbush [80b].

PROPOSITION 3.4 (Fried—Pixley [79]). *Discriminator and dual discriminator varieties are URCS, and so have CEP by Proposition 3.3.*

Thus Werner [78] gives us a wide range of examples of CEP varieties. Unfortunately one cannot get but CD examples in this way (Fried—Kiss [a]). It seems to be very hard to find conditions under which a general variety has CEP. Much work has been done in this direction in CD varieties (Quackenbush [74a], Davey [77], Kollár

[80]). All these results can be generalized to CM varieties and are summed up in the following results. We work within a fixed CM variety V.

PROPOSITION 3.5 (Kiss [a]). *$A\times A$ has CEP iff*
(a) *A has CEP,*
(b) *A satisfies the commutator identity*

$$[x, y] = x\wedge y\wedge[1, 1],$$

(c) *for each $B\leqq A$ and congruences θ, ψ on A,*

$$[\theta, \psi]\upharpoonright B = [\theta\upharpoonright B, \psi\upharpoonright B].$$

THEOREM 3.6 (Kiss [a]). *V has CEP iff $U_pSi(V)$ has CEP and the square of each SI has CEP. ($Si(\mathscr{K})$ stands for the class of SI members in $\mathscr{K}$.)*

THEOREM 3.7 (Kiss [a]). *Suppose that the free algebra on four generators in V is finite. Then V has CEP iff the square of each SI has CEP.*

THEOREM 3.8 (Kiss [a]). *If V_1 and V_2 are two subvarieties of V with CEP then so is their join.*

These results give one the feeling that CEP for $\mathrm{Si}(V)$ is not sufficient to imply CEP for all of a variety V. Indeed, Day [73] has given a counterexample; though his example is not CM, the methods of commutator theory enable one to produce modular examples, too.

§4. Amalgamation and surjectivity. These areas are among the neglected fields of universal algebra. There seems to exist no general theory or result which would provide deeper information.* There are only some easy technical observations which apply in certain cases.

On the other hand there are strong theorems on concrete structures (especially lattices and semigroups).

Both the algebraic and model theoretic aspects of AP are summed up in the very readable dissertation of Zeitler [76]. Here we restrict ourselves to listing two of the most fundamental facts.

PROPOSITION 4.1 (Grätzer—Lakser [71]). *The amalgam $(A; \beta, B; \gamma, C)$ over a variety V can be completed in V iff for each $b_1\neq b_2\in B$ there exist a $D\in V$ and homomorphisms $\gamma_1: B\to D$, $\beta_1: C\to D$ such that $\gamma_1\beta=\beta_1\gamma$ and $\gamma_1(b_1)\neq\gamma_1(b_2)$ and the same holds for C.* □

(Indeed, the amalgam can be completed by the product of these D-s.)

PROPOSITION 4.2 (Quackenbush [74b]). *Let V be a CEP variety containing only simple SI algebras. The amalgam $(A; \beta, B; \gamma, C)$ can be completed in V iff for any maximal congruence θ of B there exists a maximal congruence ψ of C with the same restriction Φ to A such that $(A/\Phi; B/\theta; C/\psi)$ can be completed by a simple algebra of V, and the same condition for B and C interchanged.* □

* Added in proof. For recent progress see C. H. Bergman [81].

COROLLARY 4.3 (Quackenbush [74b]). *A variety generated by a quasi-primal algebra A has the AP iff each proper inner automorphism of A extends to an automorphism (that is, A is demi-semi-primal).*

(This corollary is easy to prove by using basic facts about quasiprimal algebras (see e.g. Werner [78]).)

AP can be related to the solvability of algebraic equations (Hule [76], [78], [79]), and to free products (Grätzer—Lakser [71]) as well. Moreover, one can define in an arbitrary $\mathscr{K}$ the *free product* of algebras B, C *with amalgamated subalgebra* A to be an algebra $D\in\mathscr{K}$ together with embeddings $\gamma': B \rightarrowtail D$, $\beta': C \rightarrowtail D$ which coincide on A, with the property that for each D_1 and homomorphisms $\gamma_1: B\to D_1$, $\beta_1: C\to D_1$ coinciding on A there exists a unique homomorphism $\delta: D\to D_1$ with $\delta\beta'=\beta_1$ and $\delta\gamma'=\gamma_1$.

$$\begin{tikzcd}
A \arrow[r, tail, "\beta"] \arrow[d, tail, "\gamma"'] & B \arrow[d, tail, "\gamma'"'] \arrow[ddr, "\gamma_1"] & \\
C \arrow[r, tail, "\beta'"] \arrow[drr, "\beta_1"'] & D \arrow[dr, dashed, "\delta"] & \\
& & D_1
\end{tikzcd}$$

PROPOSITION 4.4. *In varieties with AP the free products with amalgamated subalgebra exist.* □

For a general idea of settling some specific structures note that the passage from semigroups to semigroup rings and from rings to their multiplicative semigroups makes it possible to carry over several results concerning AP, SAP, IPA, and ES from semigroups to rings and vice versa. One also often investigates in concrete cases which algebras A have the property that all amalgams $(A; \beta, B; \gamma, C)$ can be completed (embedded).

Finally we mention that many papers deal with the problem of amalgamating several (concrete) structures with all intersections prescribed. For a general investigation see Lanckau [69, 70] and Iskander [65].

There is only one purely algebraic paper which deals with the surjectivity problem in the most general setting. Let us recall its main result.

First of all it is clearly sufficient to investigate whether embeddings of proper subalgebras can be epimorphisms. If $A\leq B$ then define the *dominion* $\mathrm{Dom}_B(A)$ of A in B to be the set of all elements b of B with the property that for each pair of homomorphisms $\alpha, \beta: B\to C$ into some $C\in\mathscr{K}$ $\alpha\restriction A=\beta\restriction A$ implies $\alpha(b)=\beta(b)$. Clearly, $\mathrm{Dom}_B(A)=B$ iff the embedding of B is an epimorphism; such subalgebras are called *dense*. An algebra A is called *saturated* if for each $B\gneqq A$ we have $\mathrm{Dom}_B(A)\lneqq B$ and it is *absolutely closed* if $\mathrm{Dom}_B(A)=A$ for each $B\geqq A$.

PROPOSITION 4.5. *A variety V has IPA iff each $A \in V$ is absolutely closed. V has ES iff each $A \in V$ is saturated.* $\square$

One has the following characterization of the dominion in classes admitting coproducts (e.g. varieties; we do not assume that the coordinate mappings are injective).

ZIG-ZAG THEOREM 4.6 (Isbell [66a]). *Suppose $\mathscr{K}$ admits coproducts and $A \leq B \in \mathscr{K}$. Let $B * B$ be the coproduct of two copies of B with coordinate mappings $\varrho_1, \varrho_2 : B \to B * B$. The following are equivalent for a $d \in B$:*

(i) $d \in \mathrm{Dom}_B(A)$.

(ii) *There exists a finite sequence $w_0 = \varrho_1(d), \ldots, w_n = \varrho_2(d)$ in $B * B$ such that for each $0 \leq i < n$ the element (w_i, w_{i+1}) lies in the subalgebra of $(B * B) \times (B * B)$ generated by all elements of three forms (x, x); $(\varrho_1(a), \varrho_2(a))$; $(\varrho_2(a), \varrho_1(a))$ $(a \in A)$.*

(iii) *$(\varrho_1(d), \varrho_2(d))$ is in the congruence of $B * B$ generated by the pairs $(\varrho_1(a), \varrho_2(a))$ $(a \in A)$.* $\square$

COROLLARY 4.7 (Isbell [66a]). *Dom_B is a closure operator on the subalgebras of each $B \in \mathscr{K}$. No object of $\mathscr{K}$ is the domain of a proper class of inequivalent epimorphisms (since the cardinalities of the dominions are bounded). If $\mathscr{K}$ admits infinite coproducts as well, then each object can be embedded into an absolutely closed algebra.*

Now we prove Proposition 1.11. Suppose V is a variety having AP and ES, and $A \leq B \in V$. It is enough to show, by Proposition 4.5, that for each $c \in B \setminus A$ there exist homomorphisms α, β into some $C \in V$ coinciding on A but not at c. Consider the subalgebra B' of B generated by c and A. Because of ES there exist homomorphisms α' and β' from B' to some $C' \in V$ coinciding on A but not on B' and hence not at c. Replacing C' by $B' \times C'$ we may suppose that α' and β' are injective. Now completing the amalgam $(B'; \mathrm{id}, B; \alpha', C')$ we can "extend" α' to B. Extending β' similarly we are ready with the proof.

Finally, we give an example of a variety V which has ES but not IPA. Let $n \geq 4$ and consider the variety V_n of all semigroups with zero satisfying $S^n = 0$. We claim that V_n does the job. First we prove ES. In fact, if $A \lneqq B \in V_n$ then it is easy to see that $A \cup B^2 \subsetneqq B$. Therefore the natural mapping $B \to B/A \cup B^2$ coincides with the 0 mapping on A but not on B.

Now we show that IPA is not satisfied. We use Proposition 4.5. Consider the free semigroup S on the generators x, y, z, and let B be the subsemigroup generated by xz, zy, z. Set $\bar{S} = S/S^4$ and $\bar{B} = B \cup S^4/S^4$. Then $\bar{S} \in V_n$ and for the images $\bar{x}, \bar{y}, \bar{z}$ of x, y, z we have $\overline{xz}, \overline{zy}, \bar{z} \in \bar{B}$ but $\overline{xzy} \notin \bar{B}$ (since $xzy \notin B \cup S^4$). But $\overline{xzy} \in \mathrm{Dom}_{\bar{S}}(\bar{B})$ is shown by the Zig-Zag Theorem.

We mention that the same computations show that the variety of rings satisfying $R^n = 0$ also has ES but not IPA if $n \geq 4$.

§5. Injectivity. What more can we say about this concept? Whenever we have to decide whether a class has EI, we can just test the conditions having been settled in the preceding paragraphs. Nevertheless, it is possible that if a generating subset of a variety is given with some property on injectivity, then this "goes up" despite the fact that this is not necessarily true e.g. for amalgamation. Also, we may be interested in the structure of injective algebras in order to see whether or not there are enough of them.

Universal algebraic theorems of this kind have been proved only for varieties being very close to CD ones so far. Before presenting them we have to introduce an "intermediate" concept. We say that I is a *weak injective* (over an HS-closed $\mathscr{K}$) if each "injectivity diagram" with epic γ

$$\begin{tikzcd} A \arrow[r, tail, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[dl, dashed, "\Phi"] \\ I & \end{tikzcd}$$

can be completed. We have

PROPOSITION 5.1 (Grätzer—Lakser [72a]). *Each injective is a weak injective and each weak injective is an absolute retract. Conversely, if $\mathscr{K}$ has CEP then absolute retracts are weak injectives.* □

We have already mentioned in §2 that products of injectives are injective but this is not true for absolute retracts. Let us call a subalgebra $A \leqq \prod_{i\in I} A_i$ a *subdirect retract* of the family $\{A_i\colon i\in I\}$ if it is a retract of $\prod_{i\in I} A_i$ and all its projections are onto the A_i-s.

PROPOSITION 5.2 (Grätzer—Lakser [72a]). *A retract of an injective is an injective. A subdirect retract of weak injectives is a weak injective.* □

There have been very nice initiatives to describe injectives as Boolean extensions in Day [72], Quackenbush [74a] and Davey [76], and of course there were similar results in concrete classes. All these results can be put under a common roof as was shown by Davey and Werner [79]. In order to formulate their main theorem we remind the reader of some definitions.

For a finite algebra A and a Boolean algebra B the *bounded Boolean power*, $A[B]^*$, is defined as the algebra of continuous functions from the Boolean space of prime ideals of B into the discretely topologized algebra A.

A first order formula $\alpha(x, y)$ which is a $\exists\forall$ conjunct of equations is said to be a *simplicity formula* for a class $\mathscr{K}$ if for each $a, b \in C \in \mathscr{K}$, $C \models \alpha(a, b)$ iff $\theta(a, b)$ is trivial (that is, it is the least or the greatest congruence on C). Finally we say that $\mathscr{K}$ has *factorizable congruences* if for all n and all $A_0, \dots, A_n \in \mathscr{K}$ the natural map from $\mathrm{Con}(A_0)\times\dots\times\mathrm{Con}(A_n)$ to $\mathrm{Con}(A_0\times\dots\times A_n)$ is onto.

THEOREM 5.3 (Davey—Werner [79]). *Let V be a variety, let $\mathscr{K}$ be a finite set of finite algebras from V, and assume that:*

(a) $\mathrm{Si}(V) \subseteq \mathrm{IS}(\mathscr{K})$,

(b) *there exists a simplicity formula for $\mathscr{K}$,*

(c) *$\mathscr{K}$ has factorizable congruences.*

Then I is a (weak) injective in V iff I is isomorphic to $A_0[B_0]^*\times\dots\times A_n[B_n]^*$, where for all $j\leqq n$ $A_j\in H(\mathscr{K})\cap\mathrm{Si}(V)$, A_j is a (weak) injective over V, B_j is a complete Boolean algebra, and the algebras A_j are pairwise nonisomorphic.

In Davey—Werner [79] there is also a complete discussion concerning the applications of this theorem for proving known and new concrete and general results.

$A[B]^*$ is always a subdirect retract of copies of A (Davey [77]), hence the "if" part is clear. If V is CD then (c) is satisfied.

A frequent choice of $\mathscr{K}$ is the set of maximal SI algebras of V. If $\mathscr{K}$ consists of simple algebras then (b) holds, and we have the same in some particular classes of lattice-ordered algebras.

The last question is: how to find (weak) injectives over V in $H(\mathscr{K})\cap \mathrm{Si}(V)$? This leads us to the field of "going up" theorems. We present the most general ones.

THEOREM 5.4 (Davey [77]). *A SI member of a CD variety V is a weak injective over V iff it is a weak injective over $U_p\mathrm{Si}(V)$.*

THEOREM 5.5 (Kollár [80]). *Let V be a CD variety generated by finitely many finite algebras and set $\mathscr{K}=\mathrm{HSSi}(V)$. Then V has enough injectives iff*

(i) *each maximal SI is injective over $\mathscr{K}$,*

(ii) *every retract of any maximal SI is the direct product of SI algebras which are injective over $\mathscr{K}$.*

Note that (ii) cannot be omitted (Kollár [80]).

We mention a result settling injective hulls. Fixing an HS-closed $\mathscr{K}$ we say that $Q\geqq A$ is an *injective hull* of A if Q is injective and this extension is essential. We have:

THEOREM 5.6 (Banaschewski [70], Bacsich [72b]). *The injective hull of A is unique up to isomorphism over A. If $\mathscr{K}$ has EI then each $A\in\mathscr{K}$ has an injective hull. In this case each maximal essential extension and each minimal injective extension is the injective hull.* □

The theory of *equationally compact* algebras, that is, algebras in which whenever each finite subsystem of a system of equations can be solved, then the whole system admits a solution, is related to injectivity as well. Indeed, if we consider a special class of monomorphisms (the so-called pure embeddings) then the resulting injectivity concept is exactly equational compactness. For an exposition of this topic see Appendix 6 (written by G. H. Wenzel) in Grätzer [79].

We mention that all members of a variety V are injective iff they are all equationally compact (Mamedov [78]).

§6. Categorical generalizations. In what follows we show how to carry over the preceding considerations from classes of algebras to abstract categories. We fix a category $\mathscr{K}$ with small Hom-sets and two classes of morphisms $\mathscr{E}$ and $\mathscr{M}$ containing all $\mathscr{K}$-isomorphisms and being closed under them, such that the following holds:

(I) Every $\alpha\in\mathscr{K}$ allows a factorization $\alpha=\mu\varepsilon$ with $\mu\in\mathscr{M}$ and $\varepsilon\in\mathscr{E}$.

(II) Every $\varepsilon\in\mathscr{E}$ is a $\mathscr{K}$-epimorphism and every $\mu\in\mathscr{M}$ is a $\mathscr{K}$-monomorphism.

(III) For every commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, two heads, "\varepsilon"] \arrow[d, "\alpha"'] & \cdot \arrow[d, "\beta"] \\ \cdot \arrow[r, tail, "\mu"'] & \cdot \end{tikzcd}$$

with $\varepsilon\in\mathscr{E}$ and $\mu\in\mathscr{M}$ there is a (necessarily unique) diagonal morphism δ with $\delta\varepsilon=\alpha$ (and $\mu\delta=\beta$).

(IV) For every object A, there is only a set of nonisomorphic $\mathscr{M}$-morphisms with codomain A and only a set of nonisomorphic $\mathscr{E}$-morphisms with domain A.

From (I)—(III) one gets the following properties: $\mathscr{E}$ and $\mathscr{M}$ are closed under composition; $\mathscr{E}\cap\mathscr{M}$ is the class of all isomorphisms; $\mathscr{E}$ and $\mathscr{M}$ are uniquely determined by each other; $\mathscr{E}$ is right cancellable (i.e., $\varepsilon\alpha\in\mathscr{E}$ only if $\varepsilon\in\mathscr{E}$) and, dually, $\mathscr{M}$ is left cancellable; $\mathscr{E}$ contains all extremal epimorphisms and, dually, $\mathscr{M}$ contains all extremal monomorphisms. (An epimorphism ε is *extremal* if it does not factorize over a proper monomorphism, i.e. $\varepsilon=\mu\beta$ with a monomorphism μ only if μ is an isomorphism; extremal monomorphism is dual.) □

For a class $\mathscr{K}$ of algebras the natural choice for $\mathscr{E}$, $\mathscr{M}$ is $\mathscr{E}=\{$surjective homomorphisms$\}$ and $\mathscr{M}=\{$injective homomorphisms$\}$. If $\mathscr{K}$ is closed under H or S (and isomorphisms), conditions (I)—(IV) hold. Categorically spoken, $\mathscr{E}$ is just the class of extremal epimorphisms and $\mathscr{M}$ is the class of all monomorphisms of $\mathscr{K}$. Note that, up to categorical equivalence, here the $\mathscr{E}$-morphisms with fixed domain A describe the congruences on A.

Replacing injective homomorphisms by $\mathscr{M}$-morphisms and congruences by $\mathscr{E}$-morphisms we are now able to generalize the notions introduced before. Properties (I)—(IV) are not needed in full for the next propositions but, for simplicity, it is convenient to have them present throughout this section. We begin with the properties transferability (TP), congruence extension (CEP), and amalgamation (AP), which now depend on the choice of $\mathscr{E}$ and $\mathscr{M}$.

$\mathscr{K}$ satisfies (TP) ((CEP); (AP) resp.), if each span

(4)
$$\begin{tikzcd} {} \arrow[r, "\mu"] \arrow[d, "\alpha"'] & {} \\ {} & \end{tikzcd}$$

with $\mu\in\mathscr{M}$ ($\mu\in\mathscr{M}$ and $\alpha\in\mathscr{E}$; $\mu\in\mathscr{M}$ and $\alpha\in\mathscr{M}$ resp.) can be completed to a commutative diagram

(5)
$$\begin{tikzcd} {} \arrow[r, "\mu"] \arrow[d, "\alpha"'] & {} \arrow[d, "\alpha'"] \\ {} \arrow[r, "\mu'"'] & {} \end{tikzcd}$$

with $\mu'\in\mathscr{M}$ ($\mu'\in\mathscr{M}$; $\mu'\in\mathscr{M}$ and $\alpha'\in\mathscr{M}$ resp.). Proposition 1.6 *remains true* in this general setting, i.e. for $\mathscr{K}$ having finite products one has

$$(\mathrm{TP})\Leftrightarrow(\mathrm{CEP})\wedge(\mathrm{AP}).\quad\square$$

If $\mathscr{K}$ has pushouts, it can be assumed without loss of generality that (5) is a pushout. So (AP) means the existence of free products with amalgamated

$\mathscr{M}$-subobject (cf. Proposition 4.4). Cf. Banaschewski [70], Dwinger [70], and Bacsich [72c].

The Strong Amalgamation Property (SAP) means that, for every span (4) with $\mu\in\mathscr{M}$ and $\alpha\in\mathscr{M}$, there is a pullback diagram (5) with $\mu'\in\mathscr{M}$ and $\alpha'\in\mathscr{M}$. If $\mathscr{K}$ has pushouts, (5) can be assumed to be also a pushout. The Intersection Property of Amalgamations (IPA) means that, for every span (4) with $\mu\in\mathscr{M}$ and $\alpha\in\mathscr{M}$ for which there is a commutative diagram (5) with $\mu'\in\mathscr{M}$ and $\alpha'\in\mathscr{M}$, there is even a pullback diagram (5) with $\mu'\in\mathscr{M}$ and $\alpha'\in\mathscr{M}$. Proposition 1.9 *remains true*, i.e.

$$(\mathrm{SAP}) \Leftrightarrow (\mathrm{AP})\wedge(\mathrm{IPA}). \quad \square$$

Also, Proposition 1.8 *remains true*. Moreover, one has the following characterization of (IPA) (cf. Kelly [69], Ringel [72], Tholen [82a]):

PROPOSITION 6.1. *A category $\mathscr{K}$ with pushouts satisfies (IPA) (with respect to $\mathscr{M}$), iff $\mathscr{M}$ consists of all regular monomorphisms.*

(A monomorphism $\mu\colon A\to B$ is regular, if every $\alpha\colon C\to B$ satisfying the equation $\xi\alpha=\eta\alpha$ whenever $\xi\mu=\eta\mu$ holds, factorizes as $\alpha=\mu\beta$; the class of regular monomorphisms coincides with that of equalizers in such a category.)

Since regular monomorphisms are in particular extremal, and since all extremal monomorphisms belong to $\mathscr{M}$, (IPA) implies that $\mathscr{M}$ is just the class of all extremal monomorphisms. Then $\mathscr{E}$ has to be the class of all epimorphisms. We denote the latter property by (ES), since, for a class of algebras (as at the beginning) with the natural factorization system, this means that epimorphisms are surjective. But note that in general (ES) depends, like (AP), (EI), etc., on the choice of $(\mathscr{E},\mathscr{M})$.

Even without assuming the existence of pushouts one has

COROLLARY 6.2. *(IPA)$\Rightarrow$(ES).* $\square$

Note that Proposition 6.1 and Corollary 6.2 reformulate Proposition 4.5. (An object A in $\mathscr{K}$ is *saturated, absolutely closed* resp. if every monomorphism with domain A is extremal, regular resp.). Note furthermore that the converse implication in 6.2 does not hold, even for varieties (§4). Categories satisfying (ES) but not (IPA) contain extremal monomorphisms which are not regular. Now we get the following sharpening of Proposition 1.9:

PROPOSITION 6.3 (Ringel [72]). *For $\mathscr{K}$ having pushouts of monomorphisms and equalizers one has*

$$(\mathit{SAP}) \Leftrightarrow (\mathit{AP})\wedge(\mathit{ES}). \quad \square$$

Let us now consider the conditions (EAR), (EI), and (EIH). An object Q in $\mathscr{K}$ is an ($\mathscr{M}$-) *absolute retract*, if any $\mu\in\mathscr{M}$ with domain Q is a split monomorphism (i.e., has a left inverse). Q is ($\mathscr{M}$-) *injective*, if for all $\mu\colon A\to B$ in $\mathscr{M}$ and $\alpha\colon A\to Q$ in $\mathscr{K}$ there is a $\beta\colon B\to Q$ with $\beta\mu=\alpha$. Every injective object is an absolute retract, and the converse proposition holds under (TP) (cf. Proposition 1.5). A morphism $\mu\colon A\to B$ in $\mathscr{M}$ is called ($\mathscr{M}$-) *essential*, if for any $\gamma\colon B\to C$ one has $\gamma\mu\in\mathscr{M}$ only if $\gamma\in\mathscr{M}$. This is the same as to say that for every nonisomorphic $\varepsilon\colon B\to C$ in $\mathscr{E}$ one has $\varepsilon\mu\notin\mathscr{M}$. An essential morphism into an injective object is called an *injective hull* of its domain and is, up to isomorphisms, uniquely determined. $\mathscr{K}$ is said to satisfy (EAR) ((EI); (EIH) resp.) if for every object A there is an $\mathscr{M}$-morphism

$\mu: A \to Q$ with Q being an absolute retract (Q being injective; Q being injective and μ being essential resp.). Trivially one has

$$\text{(EIH)} \Rightarrow \text{(EI)} \Leftrightarrow \text{(EAR)} \wedge \text{(TP)}.$$

(EI)$\Rightarrow$(EIH) holds if the category is, in a sense, finitary. However, this is false in the infinite case, as shown by the category of compact spaces. (Finitariness can be expressed by different conditions and is, implicitly, contained in conditions (V) and (VI) below.)

In order to analyze property (EAR) in more detail we put a further condition on our factorization system $(\mathcal{E}, \mathcal{M})$:

(V) For every well-ordered chain $(\alpha_{ij}: A_i \to A_j)_{0 \leq i \leq j < m}$ with $\alpha_{ii}=1$, $\alpha_{jk}\alpha_{ij}=\alpha_{ik}$ for $i \leqq j \leqq k$, and all α_{0i} in $\mathcal{M}$, one has an "upper bound" $(\alpha_i: A_i \to A)_{0 \leq i < m}$ with $\alpha_j \alpha_{ij} = \alpha_i$ for $i \leqq j$ and α_0 in $\mathcal{M}$.

Using (V) one proves the categorical generalization of Proposition 1.3 (b) which is condition (E3) in Banaschewski [70]: For every $\mu \in \mathcal{M}$ there is an $\varepsilon \in \mathcal{E}$ such that $\varepsilon\mu$ is an essential $\mathcal{M}$-morphism (cf. Tholen [81]). Furthermore, in the presence of the harmless conditions (I)—(IV), condition (V) is strong enough to imply the following important result:

Proposition 6.4 (Banaschewski [71]). *If, for every object A in $\mathcal{K}$, there is only a set of nonisomorphic essential $\mathcal{M}$-morphisms with domain A, then $\mathcal{K}$ has property (EAR).*

One can even show that every object admits an essential $\mathcal{M}$-morphism into an absolute retract. Therefore, having (TP) one is able to construct injective hulls by 6.4.

The reverse implication in Proposition 6.4 does not hold in general; it does, if $\mathcal{K}$ fulfils the following weakening of condition (AP):

(ap) For every span (4) with $\mu \in \mathcal{M}$ and $\alpha \in \mathcal{M}$ one has a commutative diagram (5) with $\mu'\alpha \in \mathcal{M}$ (not necessarily $\mu' \in \mathcal{M}$).

(ap) follows from (EI) even if $\mathcal{K}$ does not have (finite) products. Therefore, in the presence of conditions (I)—(V), from Proposition 6.4 one gets the following generalization of Theorem 5.6:

Corollary 6.5. *(EIH)$\Leftrightarrow$(EI).*

Finally we want to consider the properties (ECS) and (RS) for abstract categories. A (small) set $\mathcal{C}$ of objects of $\mathcal{K}$ is called an *($\mathcal{M}$-) cogenerating set* of $\mathcal{K}$, if all direct products of objects in $\mathcal{C}$ exist in $\mathcal{K}$ and if every object A in $\mathcal{K}$ admits some $\mathcal{M}$-morphism

$$A \to \prod_{i \in I} C_i$$

into a product of objects in $\mathcal{C}$; the latter is the same as to say that the canonical morphism

$$A \to \prod_{C \in \mathcal{C}} C^{\mathcal{K}(A,C)}$$

belongs to $\mathcal{M}$. If $\mathcal{M}$ is the class of all monomorphisms and if $\mathcal{K}$ has products, then $\mathcal{C}$ is a cogenerating set iff for every pair $\alpha, \beta\colon A \to B$ of different $\mathcal{K}$-morphisms one can find a morphism $\gamma\colon B \to C$ with $\gamma\alpha \neq \gamma\beta$ and $C \in \mathcal{C}$. The importance of the existence of a cogenerating set (ECS) in $\mathcal{K}$ was pointed out a long time ago: by the Special Adjoint Functor Theorem (cf. Freyd [64]), a functor from a category with (ECS) and certain limits has a left adjoint iff it preserves these limits.

The following theorem compares (ECS) with (EAR); it shows that Theorem 2.1 can be almost completely proved for abstract categories with (ap):

THEOREM 6.6. *Let $\mathcal{K}$ fulfil the property* (ap) *(see above). One then has the implication*

$$(ECS) \Rightarrow (EAR),$$

and (EAR) is equivalent to each of the following conditions:

(i) *Every object admits only a set of nonisomorphic $\mathcal{M}$-essential extensions.*

(ii) *For every object A there is an $\mathcal{M}$-morphism $\mu\colon A \to Q$ such that every $\mathcal{M}$-morphism $\nu\colon A \to B$ admits a morphism β with $\beta\nu = \mu$.*

The idea how to prove (ECS)$\Rightarrow$(ii) is in Barr [75]. For (EAR)$\Leftrightarrow$(i) see Proposition 6.4, and for (i)$\Leftrightarrow$(ii) cf. Tholen [81]. Note that (ii) implies (ap).

COROLLARY 6.7. *$(ECS) \wedge (AP) \Rightarrow (EAR)$.*

An object S in $\mathcal{K}$ is called ($\mathcal{M}$-) *subdirectly irreducible,* iff for every $\mathcal{M}$-morphism $\mu\colon S \to \prod_{i \in I} A_i$ into a direct product there is at least one index $i \in I$ with $\pi_i \mu \in \mathcal{M}$, π_i being a canonical projection. $\mathcal{K}$ is ($\mathcal{M}$-) *residually small,* iff $\mathcal{K}$ contains only a set of nonisomorphic subdirectly irreducible objects. One has (without any condition on $\mathcal{K}$):

LEMMA 6.8. *$(ECS) \Rightarrow (RS)$.* $\square$

To get further results on the relationships between (ECS), (RS), and (EAR) we shall restrict ourselves throughout the rest of this section to the case

$\mathcal{M}$ = all monomorphisms

and impose a sixth condition (VI) which sharpens condition (V) if $\mathcal{K}$ has colimits of well-ordered chains.

(VI) $\mathcal{K}$ has products and a generating set $\mathcal{G}$ (this is dual to (ECS)) such that, for every $G \in \mathcal{G}$, for every pair of different morphisms $\xi, \eta\colon G \to A_0$, and for every well-ordered chain $(\alpha_{ij}\colon A_i \to A_j)_{0 \leq i \leq j < m}$ with $\alpha_{ii} = 1$, $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ for $i \leq j \leq k$, and $\alpha_{0i}\xi \neq \alpha_{0i}\eta$ for all i, there is an upper bound $(\alpha_i\colon A_i \to A)_{0 \leq i < m}$ with $\alpha_j \alpha_{ij} = \alpha_i$ for $i \leq j$ and $\alpha_0 \xi \neq \alpha_0 \eta$.

Conditions (I)—(VI) *are still satisfied by every quasi-variety of (finitary) universal algebras* and, more generally, by every $\aleph_0$-presentable category in the sense of Gabriel and Ulmer [71].

In every category satisfying conditions (I)–(VI) one has Birkhoff's Subdirect Representation Theorem:

PROPOSITION 6.9 (Tholen [81], [82b]). *Every object A admits a monomorphism* $\mu\colon A\to\prod_{i\in I} S_i$ *into a product of subdirectly irreducible objects such that all morphisms* $\pi_i\mu$ *are extremal epimorphisms* (π_i *are the projections*).

COROLLARY 6.10. $(ECS)\Leftrightarrow(RS)$.

The connection between (RS) and (EAR) is given by

LEMMA 6.11 (Tholen [81]). *Condition* (i) *of Theorem 6.6 implies* (RS).

We therefore get:

THEOREM 6.12. *If* $\mathcal{K}$ *fulfils* (ap), *then* (ECS), (RS), *and* (EAR) *are pairwise equivalent.*

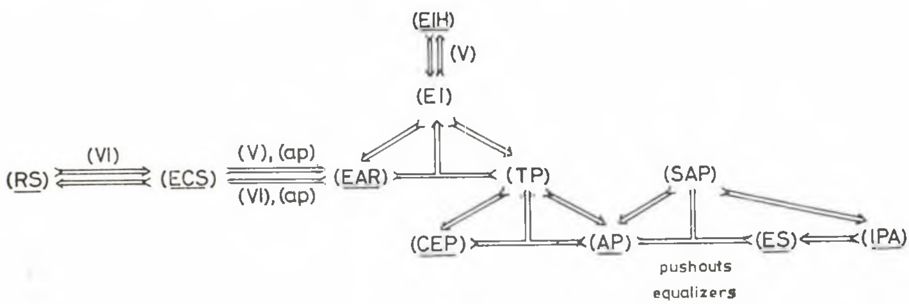
COROLLARY 6.13. $(ECS)\wedge(TP)\Leftrightarrow(RS)\wedge(TP)\Leftrightarrow(EIH)$.

2. Table of results

In what follows we list a number of categories of algebras and other structures, and indicate whether or not they possess the properties discussed above. In most cases we name only the objects of the categories, morphisms being the obvious ones (e.g. homomorphisms in categories of algebras); however, morphisms are specified in cases which might be ambiguous. For the readers' convenience we first repeat the abbreviations we use and the most important logical connections:

AP	Amalgamation Property	TP	Transferability Property
IPA	Intersection Property of Amalgamations	ECS	Existence of a Cogenerating Set
SAP	Strong Amalgamation Property	RS	Residual Smallness
ES	Epimorphisms are Surjective	EAR	Enough Absolute Retracts
CEP	Congruence Extension Property	EI	Enough Injectives
		EIH	Existence of Injective Hulls

For a category with direct products satisfying the (standard) assumptions (I)—(IV) (see §6) one has the implications below. Some of them require the additional assumptions (V) or (VI) which are still satisfied by every quasi-variety of universal algebras. Two implications require the additional assumption (ap) which is a weakening of (AP) (see the remark after Proposition 6.5); but (ap) is not needed in case of a variety. Another implication requires the existence of pushouts and equalizers which is also given in every quasi-variety.



The table contains only the underlined properties since the others can be obtained from them, even if the additional conditions (V), (VI), etc. are not satisfied.

We remind the reader that *all these properties depend on the choice of the factorization system* $(\mathscr{E}, \mathscr{M})$. If not otherwise stated (except for some cases where the choice is clear from the foregoing categories in the table) we choose $\mathscr{M}$=all monomorphisms and, consequently, $\mathscr{E}$=extremal epimorphisms. In most cases $\mathscr{E}$ is contained in the class of morphisms having underlying surjective mappings; then, if (ES) holds, epimorphisms are really surjective. But in general, (ES) just means that $\mathscr{E}$ is the class of all epimorphisms. So it may happen that (ES) holds even though epimorphisms are not surjective (Hausdorff spaces with $\mathscr{E}$=dense maps, for example), or that (ES) does not hold even though epimorphisms are surjective (topological spaces with $\mathscr{E}$=quotient maps). Those entries which may cause misunderstandings of (ES) are marked by †.

In many entries of the table we give the first reference (up to our knowledge) in which it is determined whether or not the category in question has the given property. In some cases, if the answer is easy, we refer to the first paper containing non-trivial results about the respective property. If the sign ∘ appears instead of a reference, this means we could not find any explicit reference, have checked the property ourselves, and felt that some of the readers might need a hint at the proof. These hints are put together and follow the table. Finally, if neither a reference nor the sign ∘ appears, this means that the given result is either well-known or easy to verify (maybe using some general theorems figuring in the previous part). For some varieties of algebras there is also a description of all its subvarieties possessing the given property; this is indicated by an $*$ above the answer in the given entry, and an appropriate reference is also included. In some cases we do not know whether a category listed below enjoys some of the properties in question. In such a case either the entry is left blank, or a question mark is put there. The latter means that we think the problem is difficult.

The categories of lattices, modular lattices, and distributive lattices, respectively, are named with the supplement 'bounded or not'. In fact, it is easy to verify that the answers must be the same in the two cases. (In categories consisting of bounded lattices, morphisms are 0-1-preserving homomorphisms.)

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
sets	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
M-sets (M a monoid) = any variety of unary algebras	Yes$_\circ$	Yes	Yes	Yes	Yes	Yes	Yes	Yes$_\circ$ Berthiaume [67]
any similarity type of non-unary algebras	Yes	Yes	Yes	No	No	No	No	No
any functor category [$\mathscr{D}$, $\mathscr{S}et$] ($\mathscr{D}$ small)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Kacov [76]
any Grothendieck topos	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Ebrahimi [82]
any elementary topos	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
abelian groups	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Baer [40]
R-modules (R a unital ring) = any subvariety of such	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Eckmann—Schopf [53]
any additive functor category Add [$\mathscr{C}$, $\mathscr{A}b$] ($\mathscr{C}$ small, additive)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Weidenfeld [70]
any Grothendieck category (= abelian Ab5-categ.) with generator	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Grothendieck [57]
semigroups	No Kimura [57]	No	No	No* Biró—Kiss—Pálfy [82]	No	No* McKenzie [81]	No	No* Biró—Kiss—Pálfy [82]
finite semigroups	No Kimura [57]	No	No Howie—Isbell [67]	No	Yes	Yes	No$_\circ$	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
commutative semigroups	No Kimura [57]	No	No* Higgins [83]	No* Biró—Kiss—Pálfy [82]	No	No* McKenzie [81]	No	No* Biró—Kiss—Pálfy [82]
regular semigroups	No Hall [78a]	No Scheiblich [76]		No	No	No	No$_{\circ}$	No
orthodox semigroups	No Hall [78a]	No Scheiblich [76]		No	No	No	No$_{\circ}$	No
inverse semigroups	Yes* Hall [75] Hall [78c]	Yes Howie—Isbell [67]	Yes	No	No$_{\circ}$	No	No$_{\circ}$	No
finite inverse semigroups	No Hall [75]	Yes Hall [78b]	Yes	No	Yes	Yes	No$_{\circ}$	No
commutative inverse semigroups	Yes* Imaoka [76a] Hall [78c]	Yes Imaoka [76a]	Yes	Yes$_{\circ}$	Yes	Yes	Yes	Yes Schein [76]
unions of groups	No Hall [78a]	No Scheiblich [76]		No	No	No	No$_{\circ}$	No
semilattices of groups (=Clifford semigroups)	No Hall [75]	Yes$_{\circ}$	Yes	No	No	No	No$_{\circ}$	No
bands	No* Hall [78a]	No Scheiblich [76]	?	No* Biró—Kiss—Pálfy [82]	No	No* McKenzie [81]	No	No
normal bands	Yes* Imaoka [76b] Biró—Kiss—Pálfy [82]	Yes Imaoka [76b] Scheiblich [76]	Yes	Yes* Biró—Kiss—Pálfy [82]	Yes	Yes	Yes	Yes* Biró—Kiss—Pálfy [82]
semilattices	Yes	Yes Horn—Kimura [71]	Yes	Yes	Yes	Yes	Yes	Yes Bruns—Lakser [70]
compact Lawson semilattices	No Hofmann—Mislove [76]	Yes	Yes Hofmann—Mislove [75]	No Stralka [77]				No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
left cancellative semigroups	No Howie [63]	No	No	No	No	No	No$_o$	No
commutative cancellative semigroups	Yes$_o$	No	No$_o$	Yes$_o$	Yes	Yes	Yes	Yes$_o$
monoids (with 1-preserving homomorphisms)	No$_o$	No	No	No$^*_o$	No	No	No	No$^*_o$
commutative monoids	No$_o$	No	No	No	No	No$_o$	No	No$^*_o$ Banaschewski [70]
commutative cancellative monoids	Yes$_o$	No	No$_o$	Yes$_o$	Yes	Yes	Yes	Yes Georgescu [71b]
small categories	No Trnková [65]	No	No Isbell [68b]	No	No	No	No	No
quasi-groups	Yes Ježek—Kepka [79]	Yes Ježek—Kepka [79]	Yes	No	No	No	No	No
commutative quasi-groups	Yes* Ježek—Kepka [77]	Yes* Ježek-Kepka [77]	Yes*	No	Yes	Yes Ježek—Kepka [77]	Yes	No
medial quasi-groups	Yes* Ježek—Kepka [77]	Yes* Ježek-Kepka [77]	Yes*	No	Yes	Yes Ježek—Kepka [77]	Yes	No
Steiner quasi-groups	Yes$_o$	Yes$_o$	Yes	No* Quacken-bush [76]				No Quackenbush [76]
loops	Yes$_o$	Yes$_o$	Yes	No	No	No	No	No
Steiner loops	Yes$_o$	Yes$_o$	Yes	No* Quacken-bush [76]				No Quackenbush [76]

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
groups	Yes Schreier [27]	Yes Schreier [27]	Yes	No* Biró—Kiss— Pálfy [82]	No	No	No	No
finite groups	Yes B. H. Neumann [54]	Yes B. H. Neumann [54]	Yes	No	Yes	Yes	No$_o$	No
solvable groups	No B. H. Neumann [60b]	Yes Hilton [72]	Yes	No	No	No$_o$	No$_o$	No
nilpotent groups	No Wiegold [59]	No$_o$	Yes Enjalbert [78]	No	No	No$_o$	No$_o$	No
torsion groups	No B. H. Neumann [60a]	Yes Hilton [72]	Yes	No	No	No$_o$	No$_o$	No
p-groups	No B. H. Neumann [60b]	Yes Hilton [72]	Yes	No	No	No$_o$	No$_o$	No
not necessarily associative rings	Yes Dididze [57]	Yes Dididze [57]	Yes	No	No	No	No	No
associative *R*-algebras (*R* a commutative unital ring)	No	No	No	No* E. W. Kiss [a]	No	No* McKenzie [82]	No	No
commutative associative *R*-algebras (*R* a commutative unital ring)	No	No	No	No* E. W. Kiss [a]	No	No* McKenzie [82]	No	No
Lie algebras (over a field)	Yes$_o$	Yes Reid [70]	Yes	No* E. W. Kiss [a]	No Pareigis— Sweedler [70]	No	No	No
regular rings	Yes$_o$	Yes Stenström [75]	Yes Gardner [75]	No	No	No	No$_o$	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
commutative regular rings	Yes Cornish [77]	No	No$_{o}$	Yes Cornish [77]	No$_{o}$	No	No$_{o}$	No
integral domains	Yes Cornish [77]	No	No	No	No	No	No	No
Ore domains	Yes Felgner [a]	No	No$_{o}$	No	No	No	No	No
division rings	Yes Cohn [71]	Yes Cohn [71]	Yes	Yes	No	No	No	No
fields	Yes Jónsson [65]	No	No Burgess [65]	Yes	No	No	No	No
near-rings$_{o}$	No	No	No	No	No	No	No	No
preordered sets (with monotone mappings)	Yes	No	No†	Yes	Yes	Yes	Yes	Yes
partially ordered sets ($\mathscr{M}$=embeddings ($x \leq y \Leftrightarrow fx \leq fy$), $\mathscr{E}$=surj. morphisms)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Bacsich [72b]
lattices (bounded or not)	Yes Jónsson [60]	Yes Grätzer [78]	Yes	No	No	No	No	No
modular lattices (bounded or not)	No Grätzer—Jónsson—Lakser [73]	No	No Freese [79]	No	No	No	No	No
distributive lattices (bounded or not)	Yes Pierce [68]	No	No$_{o}$	Yes	Yes	Yes	Yes	Yes Banaschewski—Bruns [68]
complete lattices (with 0-1-preserving homomorphisms)	Yes$_{o}$	Yes$_{o}$	Yes	No	No$_{o}$	No$_{o}$	No$_{o}$	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
algebraic lattices (with 0-1-preserving homomorphisms)	Yes$_o$	Yes$_o$	Yes	No	No$_o$	No$_o$	No$_o$	No
Stone algebras	Yes			Yes	Yes	Yes	Yes	Yes* Balbes—Grätzer [71] A. Day [72]
double Stone algebras	Yes*			Yes	Yes	Yes* Katriňák [74a]	Yes	Yes* Katriňák [77b]
distributive p-algebras	Yes* Grätzer—Lakser [71]			Yes Grätzer—Lakser [71]	No	No* Lee [70], Lakser [71]	No	No* A. Day [72] Grätzer—Lakser [72a]
distributive double p-algebras				Yes Katriňák [74b]	No	No Katriňák [80]	No	No
distributive Ockham algebras	Yes	No	No$_o$	Yes Berman [77]	Yes	Yes Berman [77]	Yes	Yes Goldberg [81]
de Morgan algebras	Yes			Yes	Yes	Yes	Yes	Yes Cignoli [75]
Kleene algebras	Yes			Yes	Yes	Yes	Yes	Yes Cignoli [75]
Heyting algebras	Yes A. Day [a]	Yes A. Day [a]	Yes	Yes A. Day [a]	No	No	No	No* A. Day [a]
Boolean algebras	Yes Dwinger—Yaqub [63]	Yes Dwinger—Yaqub [63]	Yes	Yes Sikorski [48]	Yes	Yes	Yes	Yes Halmos [61]
m-complete Boolean algebras	Yes Lagrange [74]	Yes Lagrange [74]	Yes	Yes	No	No Monk [67]	No	No Monk [67]

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
cylindric algebras (of fixed dimension α)	Yes if $\alpha=1$ No if $\alpha>1$ Comer [69] Pigozzi [71]	Yes if $\alpha=1$ No if $1<\alpha<\omega$	Yes if $\alpha=1$ Sain [a] No if $1<\alpha<\omega$ Andréka—Comer—Németi [a]	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
cylindric algebras of fixed characteristic $\neq 0$ (and of fixed dimension)	Yes Comer [68] Pigozzi [71]	Yes if $\alpha<\omega$	Yes if $\alpha<\omega$ Comer [a]	Yes	Yes	Yes Henkin—Monk—Tarski—Andréka—Németi [81]	Yes	Yes
locally finite cylindric algebras (of fixed infinite dimension)	Yes Daigneault [64b]	Yes Daigneault [64b]	Yes	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
representable cylindric algebras (of fixed dimension α)	Yes if $\alpha=1$ No if $\alpha>1$ Comer [69] Pigozzi [71]	Yes if $\alpha=1$ No if $1<\alpha<\omega$	Yes if $\alpha=1$ No if $1<\alpha<\omega$ Andréka—Comer—Németi [a]	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
cylindric-relativised set algebras (of fixed dimension)	Yes Németi [a]	Yes Németi [a]	Yes	Yes	No	No Henkin—Monk—Tarski—Andréka—Németi [81]	No	No
weakly associative lattices	Yes Fried—Grätzer [76]	Yes Fried—Grätzer [76]	Yes	No* Fried [74b]	No	No	No	No

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
the variety generated by the weakly associative lattices with unique bound property	Yes Fried—Grätzer—Quackenbush [80a]	No	No$_o$	Yes Fried [74b]	No	No	No	No
topological spaces ($\mathcal{M}$ = inj. cont. maps, $\mathcal{E}$ = quotient maps)	Yes	No	No†	Yes	Yes	Yes	Yes	Yes
topological spaces ($\mathcal{M}$ = embeddings, $\mathcal{E}$ = surjective maps)	Yes	Yes	Yes	Yes	Yes	Yes		No Wyler [77]
T_o-spaces ($\mathcal{M}$ = embeddings, $\mathcal{E}$ = surjective maps)	Yes	No	No Baron [68]	Yes	Yes	Yes	Yes	No Banaschewski [77b]
T_1-spaces ($\mathcal{M}$ = embeddings, $\mathcal{E}$ = surjective maps)	Yes	Yes	Yes	Yes	No Mrówka [82]	Yes Vinárek [82]	No	No R.-E. Hoffmann [81]
Hausdorff spaces ($\mathcal{M}$ = embeddings, $\mathcal{E}$ = surjective maps)	No	No	No	No Kelly [69]	No			No
Hausdorff spaces ($\mathcal{M}$ = closed embeddings, $\mathcal{E}$ = dense maps)	Yes	Yes	Yes†	No Kelly [69]	No			No
completely regular Hausdorff spaces ($\mathcal{M}$ = embeddings, $\mathcal{E}$ = surjective maps)	Yes	No	No	Yes	Yes	Yes		
completely regular Hausdorff spaces ($\mathcal{M}$ = closed embeddings, $\mathcal{E}$ = dense maps)	Yes	Yes	Yes†	Yes	No Herrlich [67]		Yes	

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
compact Hausdorff spaces ($\mathscr{M}$ = closed embeddings = inj. maps., $\mathscr{E}$ = dense maps = surj. maps)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No Banaschewski [70]
compact 0-dim. Hausdorff spaces ($\mathscr{M}$ = closed embeddings = inj. maps, $\mathscr{E}$ = dense maps = surj. maps)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No$_o$
topological groups (not necessarily T_2)	Yes	No	No†	No	No	No	No	No
Hausdorff groups	No Tholen [82a]	No	No Nummela [78]	No	No			No
compact (Hausdorff) groups	No Bergman [d]	Yes Poguntke [73]	Yes Reid [70], Poguntke [70]	No	Yes$_o$	Yes$_o$	No$_o$	No
Hausdorff abelian groups	No Tholen [82a]	No	No					No
compact abelian groups	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No Banaschewski [70]
topological vector spaces (over a Hausdorff top. field)	Yes	No	No†	Yes	Yes	Yes	Yes	Yes
Hausdorff topological vector spaces	No	No	No					
locally convex spaces	Yes	No	No†	Yes	Yes	Yes	Yes	Yes

CATEGORY	AP	IPA	ES	CEP	ECS	RS	EAR	EIH
Hausdorff locally convex spaces	No	No	No		Yes	Yes		
metric spaces (with contractions) ($\mathscr{M}$ = isometric embeddings, $\mathscr{E}$ = surj. morphisms)	Yes	No	No	Yes	Yes	Yes	Yes	Yes Isbell [64a]
compact metric spaces	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes Isbell [64a]
normed vector spaces (with linear contractions)	Yes	No	No	Yes	Yes	Yes	Yes	Yes Nachbin [50]
Banach spaces	Yes	No	No	Yes	Yes	Yes	Yes	Yes Phillips [40]
commutative C*-algebras	Yes	Yes	Yes Reid [70]	Yes	Yes	Yes	Yes	Yes Gleason [58]

Hints for proving some of the results in the table

M-sets: AP, EIH

Notice that algebras are assumed to be non-empty. The two answers in question are false if this assumption is cancelled and we have a variety of unary algebras with two constant operations which may or may not coincide (see Higgs [71] and Lakser [73]).

Inverse semigroups, Unions of groups, Semilattices of groups, Commutative regular rings: EAR, ECS

Given a class $\mathscr{K}$ of algebras, if new operations can be introduced in these algebras so that their homomorphisms remain the same and $\mathscr{K}$ becomes a variety, then RS$\Leftrightarrow$EAR in $\mathscr{K}$ (in fact, between $\mathscr{K}$ and its enriched copy there is an isomorphism which commutes with the underlying set functor).

Commutative inverse semigroups: CEP

Make use of the fact that these semigroups are strong semilattices of abelian groups.

Semilattices of groups: IPA

These semigroups are strong semilattices of groups; first we amalgamate the corresponding components, then we extend these larger amalgams (by using the structure homomorphisms) to the 'skeleton' obtained by amalgamating the two semilattices.

Left cancellative semigroups: EAR

This class is a quasi-variety containing arbitrarily large subdirectly irreducible algebras, so it does not have EAR by Taylor [72].

Commutative cancellative semigroups

Embed the semigroups under consideration into their quotient groups.

Monoids, Commutative monoids, Commutative cancellative monoids

The results in question follow from those on the corresponding classes of semigroups by choosing an appropriate semigroup and adjoining an external unit to it.

Steiner quasi-groups, Loops, Steiner loops: SAP

Use the result that in these categories every partial algebra can be extended to a complete one (see Bruck [58]).

Finite groups: EAR

Notice that every finite group can be embedded in a suitable finite alternating group.

Solvable groups, Nilpotent groups, Torsion groups, p-groups: RS

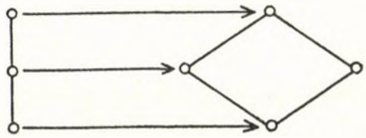
The central product of arbitrarily many copies of a fixed non-abelian group of order p^3 is subdirectly irreducible.

Near-rings

The ring constructions work.

Distributive lattices, distributive Ockham algebras, HSP*(weakly ass. lattices with UBP):* ES

Consider the embedding



Complete lattices, Algebraic lattices

Notice that every lattice can be embedded into a partition lattice, which is a simple algebraic lattice.

(HINTS BASED ON SUGGESTIONS OF G. M. BERGMAN)

Regular rings: EAR

The 2×2 matrix ring over R is an essential extension of R.

Regular, orthodox, inverse, finite inverse, finite semigroups: EAR

The same as above, using the 2×2 Rees matrix semigroup with the identity as sandwich matrix. We start by adjoining an external zero and an external identity to the given semigroup.

Solvable, nilpotent, torsion, and p-groups: EAR

For a group G and $n(=1, 2, \dots$ or $\infty)$ let $\bar{G}$ be the semidirect product of Z_n and G^n with generator x of Z_n acting on G^n by the shifting operator. Embed G as the diagonal; observe that x centralizes this subgroup but the commutator of x and $(e, a, \dots, a^{n-1})$ (or $(\dots, a^{-1}, e, a, \dots)$) is $(\dots, a, a, \dots)$ for each $a\in G$, $|a|\,|\,n$, thus G is not a retract in $\bar{G}$. Now for

torsion resp. p-groups: if $e\neq a\in G$ and $n=|a|$ then $\bar{G}$ is a torsion resp. p-group;

solvable groups: Let A be a non-trivial abelian normal subgroup of G, $n=\infty$, and consider the subgroup of $\bar{G}$ generated by Z_∞ and the elements $(\dots, ga^{-1}, g, ga, \dots)$ with $g\in G$, $a\in A$. This is solvable but G is not a retract of it;

nilpotent groups: the previous construction works with $A=Z(G)$.

Nilpotent groups: IPA

Let B be the group on $Q\times Q\times Q$ with rule of multiplication $(a, b, c)\times \times(a', b', c')=(a+a', b+b', c+c'+a'b)$ and $A<B$ be the subgroup consisting of the elements with entries in Z. Then $(0, 0, a)\in \mathrm{Dom}_B(A)$ for all $a\in Q$. In order to show this, write $(a, b, c)\in B$ as $p(a)q(b)r(c)$, where $p(a)=(a, 0, 0)$ etc., p, q, r are

homomorphisms of the additive group Q into B, and $q(a)p(b)=p(b)q(a)r(ab)$, further, make use of the fact that for each pair α, β of homomorphisms of Q to a nilpotent group, if $\alpha(Z)$ centralizes $\beta(Q)$ then so does $\alpha(Q)$.

Lie algebras: SAP

If we have an amalgam of A, B, C, $A \subseteq B$, $A \subseteq C$, then consider the universal enveloping algebras $k(A)$, $k(B)$, $k(C)$, and notice that $k(B)$ and $k(C)$ as modules are free over $k(A)$. Now the coproduct of $k(B)$ and $k(C)$ with amalgamation of $k(A)$ regarded as a Lie algebra can be seen to do the job.

Regular rings: AP

First observe that every regular ring is a subdirect product of regular rings containing prime fields, and that a product of regular rings is regular. This allows us to reduce to the case where our rings are algebras over a fixed *field* k. Second, note that every k-algebra is embeddable in a regular k-algebra, namely a full algebra of endomorphisms of a vector space. Hence it suffices to show that every amalgam of regular algebras over k can be completed with a not necessarily regular k-algebra; equivalently, that given regular k-algebras $A \subseteq B, A \subseteq C$, the algebras B and C embed in their coproduct over A. This can be deduced from a description of the coproduct due to P. M. Cohn [59]. Namely, it is shown that as left A-module, this coproduct is the direct limit of the chain $B \to C \otimes_A B \to B \otimes_A C \otimes_A B \to \ldots$. Now since A is regular, tensoring with inclusions $A \subseteq B$ and $A \subseteq C$ gives 1-1 maps, so B embeds in the direct limit; likewise C does, as required.

Commutative regular rings: ES

If $A < B$ are fields of characteristic $p > 0$, and B is generated by an element x over A with $x^p \in A$, then $x \in \mathrm{Dom}_B(A)$, thus A is dense in B.

Ore domains: ES

Let B be the subring of $Q[x]$ generated by x and $A < B$ be generated by x^2 and x^3. Then we again have $x \in \mathrm{Dom}_B(A)$, thus $\mathrm{Dom}_B(A) = B$.

Compact 0-dimensional Hausdorff spaces: EIH

Since the essential embeddings here are clearly surjective, we have to show (by Stone duality) that not every Boolean algebra is projective. But the natural homomorphism of $P(\omega)$ to its factor modulo the ideal of finite subsets is easily seen not to be right invertible.

Compact groups: ECS, RS

The Peter—Weyl Theorem says that the Hilbert space of L^2 functions on a compact group is a Hilbert-space direct sum of finite-dimensional subspaces closed under the action of the group by translation. Since the action of the group on the full space separates group-elements, so do the actions on these finite-dimensional spaces, taken together. But a norm-preserving action on an n-dimensional Hilbert space is equivalent to a homomorphism into the unitary group $U(n)$, which is compact, so the unitary groups form a cogenerating set.

Compact groups: EAR

First note that every $U(n)$ can be embedded in a $PSU(n')$, $n' \geqq 4$. These are simple groups, hence a retract of a direct product of copies of these groups will be the direct product of a subfamily thereof. Now since $PSU(i) \subseteq PSU(i+1)$, we see that every compact group G can be embedded, on the one hand, in a direct product of groups $PSU(i)$ where i takes on even values, and on the other hand, in a direct product where i takes on odd values. Hence if G is a retract of both of these products, it must itself be a direct product of both sorts; contradiction.

3. Bibliography

The following list of papers is intended to be (though it is certainly not) complete concerning items which might be useful in deciding whether a given category (of structures) enjoys one of the properties discussed in the preceding text and table. E.g., it contains papers determining rings over which injectivity of modules is equivalent with some weaker property. Furthermore, if not all objects of a category have a given property, it might be interesting to know which of them do; papers going in this line are also included here, e.g. investigations of semigroup amalgams which can be completed or embedded. In order to facilitate the work of a reader who wants to find information on a given topic, we also include classification codes of the enlisted papers. These codes are the following:

AP	Amalgamation Property	EIH	Existence of Injective Hulls
CEP	Congruence Extension Property	ES	Epimorphisms are Surjective
EAR	Enough Absolute Retracts	IPA	Intersection Property of Amalgamations
ECS	Existence of Cogenerating Sets	RS	Residual Smallness
EI	Enough Injectives	SAP	Strong Amalgamation Property

On the other hand, we restricted ourselves to consider only the validity of these properties and not to go further. E.g., the reader will find here neither the papers dealing with the structure of amalgamated free products of groups nor those investigating injective modules in order to solve ring theoretical problems, nor a comprehensive literature on equational compactness though, as we have seen in the expository part, the latter is in connection with injectivity.

Abramovič, Ju. A.
71 Injective hulls of normed lattices, *Dokl. Akad. Nauk SSSR* **197** (1971), 743—745 (in Russian). *MR* **44** # 7257 EIH
Agapitov, K. V.
81 Epimorphisms of generalized uniserial rings and bicentralizers of torsion modules over bounded hereditary noetherian prime rings, *Abelian Groups and Modules*, ed. by L. A. Skornyakov, Tomsk University, Tomsk, 1981, 3—19 (in Russian). ES
Ahsan, J. and Deeba, E. Y.
81 Injective BCK-algebras, *Math. Japon.* **26** (1981), 547—551. *MR* **82m**: 06011 EI
Allouch, D.-A.
74 Monoïdes de fractions et enveloppe injective d'un monoïde, *C. R. Acad. Sci. Paris Sér. A* **279** (1974), 833—836. *MR* **51** # 589 EIH
Amir, D. and Arbel, B.
73 On injections and surjections of continuous function spaces, *Israel J. Math.* **15** (1973), 301—310. *MR* **53** # 1239 ES
Anderson, F. W.
78 Simple injective modules, *Math. Scand.* **43** (1978), 204—210. *MR* **80g**: 16027 EI
Anderson, M. and Conrad, P.
81 Epicomplete l-groups, *Algebra Universalis* **12** (1981), 224—241. *MR* **82d**: 06017 ES
Andréka, H., Comer, S. D. and Németi, I.
a Epimorphisms of finite dimensional cylindric algebras need not be surjective (manuscript). ES

ANDRÉKA, H., GERGELY, T. and NÉMETI, I.
77 On universal algebraic constructions of logics, *Studia Logica* **36** (1977), 9—47. *MR* **58** # 21588 AP SAP ES RS

ANDRÉKA, H. and SAIN, I.
81 Connections between algebraic logic and initial algebra semantics of CF languages, *Mathematical Logic in Computer Science* (Proc. Conf. Salgótarján, 1978), Colloq. Math. Soc. J. Bolyai, Vol. 26, North-Holland, Amsterdam, 1981, 25—83. ES

ARAGANE, K. and FUJIWARA, T.
79 On a problem of Lausch and Nöbauer, *Math. Japon.* **24** (1979/80), 121—122. *MR* **80g**: 08003 AP IPA

ARDUINI, P.
69 Monomorphisms and epimorphisms in abstract categories, *Rend. Sem. Mat. Univ. Padova* **42** (1969), 135—166. *MR* **43** # 6279 ES

BACSICH, P. D.
72a Compact injectives and non-standard analysis, *Conference in Mathematical Logic* — London '70 (Proc. Conf., Bedford Coll., London, 1970), Lecture Notes in Math., Vol. 255, Springer, Berlin, 1972, 10—28. *MR* **49** # 2875 EI
72b Injective hulls as completions, *Glasgow Math. J.* **13** (1972), 17—23. *MR* **47** # 3488 EI EIH
72c Injectivity in model theory, *Colloq. Math.* **25** (1972), 165—176. *MR* **48** # 3840 AP CEP EI
73 An epi-reflector for universal theories, *Canad. Math. Bull.* **16** (1973), 167—171. *MR* **49** # 370 AP EIH
74 Model theory of epimorphisms, *Canad. Math. Bull.* **17** (1974), 471—477. *MR* **51** # 10081 ES
75a Amalgamation properties and interpolation theorems for equational theories, *Algebra Universalis* **5** (1975), 45—55. *MR* **52** # 2873 AP CEP TP
75b The strong amalgamation property, *Colloq. Math.* **33** (1975), 13—23. *MR* **53** # 116 AP SAP IPA

BACSICH, P. D. and ROWLANDS HUGHES, D.
74 Syntactic characterisation of amalgamation, convexity and related properties, *J. Symbolic Logic* **39** (1974), 433—451. *MR* **52** # 71 AP SAP CEP

BAER, R.
40 Abelian groups that are direct summands of every containing abelian group, *Bull. Amer. Math. Soc.* **46** (1940), 800—806. *MR* **2**—126 EIH
46 Absolute retracts in group theory, *Bull. Amer. Math. Soc.* **52** (1946), 501—506. *MR* **8**—14 EAR

BALBES, R.
67 Projective and injective distributive lattices, *Pacific J. Math.* **21** (1967), 405—420. *MR* **35** # 2802 EI

BALBES, R. and DWINGER, PH.
71 Coproducts of Boolean algebras and chains with applications to Post algebras, *Colloq. Math.* **24** (1971/72), 15—25. *MR* **46** # 3388 EI

BALBES, R. and GRÄTZER, G.
71 Injective and projective Stone algebras, *Duke Math. J.* **38** (1971), 339—347. *MR* **43** # 7378 EI

BALBES, R. and HORN, A.
70 Injective and projective Heyting algebras, *Trans. Amer. Math. Soc.* **148** (1970), 549—559. *MR* **41** # 1607 EI

BALDWIN, J. T.
73 A sufficient condition for a variety to have the amalgamation property, *Colloq. Math.* **28** (1973), 181—183. *MR* **48** # 5957 AP
80 The number of subdirectly irreducible algebras in a variety, II, *Algebra Universalis* **11** (1980), 1—6. *MR* **82c**: 08008 RS

BALDWIN, J. T. and BERMAN, J. D.
75 The number of subdirectly irreducible algebras in a variety, *Algebra Universalis* **5** (1975), 379—389. *MR* **52** # 13578 RS

BALLARD, J. W.
78 Injective modules for restricted enveloping algebras, *Math. Z.* **163** (1978), 57—63. *MR* **80b**: 20053 EI
BANASCHEWSKI, B.
70 Injectivity and essential extensions in equational classes of algebras, *Proc. Conf. Universal Algebra* (Queen's Univ., Kingston, Ont., October, 1969), Queen's Papers in Pure and Applied Math., Vol. 25, Queen's Univ., Kingston, Ontario, 1970, 131—147. *MR* **41** # 3354 TP EI EIH
71 Projective covers in categories of topological spaces and topological algebras, *General Topology and its Relations to Modern Analysis and Algebra* (Proc. Conf., Kanpur, 1968), Academia, Praha, 1971, 63—91. *MR* **44** # 1616 EI EIH
72 On profinite universal algebras, *General Topology and its Relations to Modern Analysis and Algebra, III* (Proc. Third Prague Topological Sympos., 1971), Academia, Praha, 1972, 51—62. *MR* **50** # 4448 ES
73 Equational compactness in universal algebra, Lecture Notes, McMaster Univ., Hamilton, 1973. RS EI EIH
77a Sheaves of Banach spaces, *Quaestiones Math.* **2** (1977/78), 1—22. *MR* **58** # 11073 EIH
77b Essential extensions of T_0-spaces, *General Topology and Appl.* **7** (1977), 233—246. *MR* **56** # 16557 EI EIH
79 Injective Banach sheaves, *Applications of Sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham,* Durham, 1977), Lecture Notes in Math., Vol. 753, Springer, Berlin—Heidelberg—New York, 1979, 101—112. *MR* **81j**: 46109 EI EIH
80a Extension of invariant linear functionals: Hahn—Banach in the topos of *M*-sets, *J. Pure Appl. Algebra* **17** (1980), 227—248. *MR* **82c**: 46081 EI EIH
80b Injective sheaves of abelian groups: a counterexample, *Canad. J. Math.* **32** (1980), 1518—1521. *MR* **82f**: 18015a EI
81 When are divisible groups injective?, *Quaestiones Math.* **4** (1981), 285—307. EI
a Essential extensions and injectivity for metric spaces and Banach spaces (unpublished note). EIH
BANASCHEWSKI, B. and BRUNS, G.
67 Categorial characterization of the MacNeille completion, *Arch. Math. (Basel)* **18** (1967), 369—377. *MR* **36** # 5036 EI EIH ES
68 Injective hulls in the category of distributive lattices, *J. Reine Angew. Math.* **232** (1968), 102—109. *MR* **38** # 5888 EIH
BANASCHEWSKI, B. and NELSON, E.
72 Equational compactness in equational classes of algebras, *Algebra Universalis* **2** (1972), 152—165. *MR* **46** # 7125 EI RS
77 On the non-existence of injective near-ring modules, *Canad. Math. Bull.* **20** (1977), 17—23. *MR* **57** # 12612 EI RS
BARANOVIČ, T. M. and BURGIN, M. S.
75 Linear Ω-algebras, *Uspehi Mat. Nauk* **30** (1975), no. 4 (184), 61—106 (in Russian). *MR* **54** # 5088 AP
BARON, S.
68 Note on epi in $\mathscr{T}_0$, *Canad. Math. Bull.* **11** (1968), 503—504. *MR* **38** # 3315 ES
BARR, M.
73 Non-abelian torsion theories, *Canad. J. Math.* **25** (1973), 1224—1237. *MR* **50** # 13188 EI
75 The existence of injective effacements, *Canad. Math. Bull.* **18** (1975), 1—6. *MR* **54** # 5311 ECS EIH
BEACHY, J. A.
72 A generalization of injectivity, *Pacific J. Math.* **41** (1972), 313—327. *MR* **47** # 1872 EI
BEAZER, R.
75a Post-like algebras and injective Stone algebras, *Algebra Universalis* **5** (1975), 16—23. *MR* **52** # 2988 EI
75b A note on injective double Stone algebras, *Algebra Universalis* **5** (1975), 239—241. *MR* **52** # 10527 EI

78 Some remarks on distributive double p-algebras, *Algebra Universalis* **8** (1978), 9—14. *MR* **56** # 8456 CEP

80a Subdirectly irreducible double Heyting algebras, *Algebra Universalis* **10** (1980), 220—224. *MR* **81d**: 06015 RS

80b Subdirectly irreducibles for various pseudocomplemented algebras, *Algebra Universalis* **10** (1980), 225—231. *MR* **81g**: 06005 RS

BELESOV, A. U.

76 Injective and equationally compact modules, *Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat.* **1976**, no. 1, 85—86 (in Russian). *MR* **56** # 15410 EI

BELJAEV, V. JA.

82 On algebraic closure and amalgamation of semigroups, *Semigroup Forum* **25** (1982), 223—267. AP

BERAN, L.

73 On a construction of amalgamation, I, *Acta Univ. Carolinae — Math. Phys.* **14** (1973), no. 2, 31—39. *MR* **50** # 9717 AP

BERGMAN, C. H.

81 Concerning the amalgamation bases of congruence distributive equational classes, Ph. D. Thesis, Univ. California, Berkeley, Calif., 1981. AP CEP RS

BERGMAN, G. M.

74 Modules over coproducts of rings, *Trans. Amer. Math. Soc.* **200** (1974), 1—32. *MR* **50** # 9970. SAP

78 The diamond lemma for ring theory, *Adv. in Math.* **29** (1978), 178—218. *MR* **81b**: 16001 AP SAP

a Epimorphisms of Lie-algebras (unpublished note). ES IPA

b Notes on epimorphisms of rings (unpublished note). ES

c A free epic subalgebra of a free associative algebra (unpublished note). ES

d Compact groups do not have the amalgamation property (manuscript). AP

BERGMAN, G. M. and DICKS, W.

75 On universal derivations, *J. Algebra* **36** (1975), 193—211. *MR* **52** # 8196 ES

BERMAN, J.

77 Distributive lattices with an additional unary operation, *Aequationes Math.* **16** (1977), 165—171. *MR* **58** # 418 CEP RS

81 Interval lattices and the amalgamation property, *Algebra Universalis* **12** (1981), 360—375. *MR* **82k**: 06007 AP

BERTHIAUME, P.

67 The injective envelope of S-sets, *Canad. Math. Bull.* **10** (1967), 261—273. *MR* **35** # 4321 EIH

BIRÓ, B., KISS, E. W. and PÁLFY, P. P.

82 On the congruence extension property, *Universal Algebra* (Proc. Conf. Esztergom, 1977), Colloq. Math. Soc. J. Bolyai, Vol. 29, North-Holland, Amsterdam, 1982, 129—151. CEP EI

BLACKADAR, B. E.

78 Weak expectations and injectivity in operator algebras, *Proc. Amer. Math. Soc.* **68** (1978), 49—53. EI

BÖRGER, R., THOLEN, W., WISCHNEWSKY, M. B. and WOLFF, H.

81 Compact and hypercomplete categories, *J. Pure Appl. Algebra* **21** (1981), 129—144. ECS

BOTNARU, D. V.

80 Injective objects with respect to varieties, *Algebraic structures, Mat. Issled.*, No. 56, 28—32 (in Russian). *MR* **82b**: 18006 EI

82 On the connection between adjoint pairs of subcategories and bicategory structures, *Izv. Vysš. Učebn. Zaved. Matematika* **1982**, no. 4, 9—12 (in Russian). EI

BOURBAKI, N.

70 *Éléments de mathématique. Algèbre*, Chapitres 1—3, Hermann, Paris, 1970. *MR* **43** # 2 SAP

BRAY, E. E., BYRD, K. A. and BERNHARDT, R. L.

71 The injective envelope of the upper triangular matrix ring, *Amer. Math. Monthly* **78** (1971), 883—886. *MR* **44** # 5338 EIH

BRUCK, R. H.
58 *A survey of binary systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 20, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1958. *MR* **20** # 76 SAP
BRUNS, G. and LAKSER, H.
70 Injective hulls of semilattices, *Canad. Math. Bull.* **13** (1970), 115—118. *MR* **43** # 137 EI EIH
BUCHSBAUM, D. A.
55 Exact categories and duality, *Trans. Amer. Math. Soc.* **80** (1955), 1—34. *MR* **17**—579 EI
BULASZEWSKA, A. and KREMPA, J.
75 On epimorphisms in the category of all associative rings, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **23** (1975), 1153—1159. *MR* **53** # 536 ES
BULMAN-FLEMING, S.
75 A note on injective semilattices, *Algebra Universalis* **5** (1975), 305—306. *MR* **52** # 10517 EI
BULMAN-FLEMING, S. and McDOWELL, K.
79 The category of mono-unary algebras, *Algebra Universalis* **9** (1979), 339—345. *MR* **80i**: 08005 EI EIH
BUNU, I. D.
74 Injectivity in modules, *Mat. Issled.* **9** (1974), no. 4 (34), 23—41 (in Russian). *MR* **52** # 5738 EI EIH
76a On the concept of injectivity in a category of modules, *Rings and modules, Mat. Issled.* **38** (1976), 71—85 (in Russian). *MR* **55** # 10524 EI EIH
76b (ϱ, σ)-injectivity in modules, *Rings and modules, Mat. Issled.* **38** (1976), 86—96 (in Russian). *MR* **55** # 10523 EI EIH
BURGESS, W.
65 The meaning of mono and epi in some familiar categories, *Canad. Math. Bull.* **8** (1965), 759—769. *MR* **33** # 161 ES
81 The injective hull of S-sets, S a semilattice of groups, *Semigroup Forum* **23** (1981), 241—246. EIH
BURGIN, M. S.
67 Imbedding an amalgam of groups with a certain property into a group with the same property, *Mat. Sb. (N. S.)* **74 (116)** (1967), 147—160 (in Russian). *MR* **36** # 1547 AP SAP
BUŞNEAG, D.
80 Injective objects in the Hilbert algebras category, *Math. Sem. Notes Kobe Univ.* **8** (1980), 433—442. *MR* **82f**: 06017 EI
BYRD, K. A.
72a Rings whose quasi-injective modules are injective, *Proc. Amer. Math. Soc.* **33** (1972), 235—240. *MR* **46** # 9112 EI
72b When are quasi-injectives injective? *Canad. Math. Bull.* **15** (1972), 599—600. *MR* **47** # 277 EI
CALENKO, M. Š.
69 Representations of concrete categories in the category of sets, *Mat. Zametki* **6** (1969), 125—127 (in Russian). *MR* **40** # 1444 ES
CALVO, B.
75 Dominions de foncteurs relatifs, I—II, *Comm. Algebra* **3** (1975), 95—127, 129—150. *MR* **51** # 5709 ES
CARSON, A. B.
73 The model completion of the theory of commutative regular rings, *J. Algebra* **27** (1973), 136—146. *MR* **52** # 7893 AP
CARTWRIGHT, D. I.
75 Extension of positive operators between Banach lattices, *Mem. Amer. Math. Soc.* **3** (1975), no. 164. *MR* **52** # 3913 EI
CĂZĂNESCU, V. E.
78 Catégories d'algèbres universelles, I—II, *Mathematica (Cluj)* **20 (43)** (1978), 5—20 and 133—141. *MR* **80g**: 18006 and **80k**: 18008 ES

CHAN, W.-M., HIGGS, D. A. and LENIHAN, B.
74 Compactness as a type of injectivity, *General Topology and Appl.* **4** (1974), 103—107. *MR* **49** # 9809 EI

CHEATHAM, T. J. and ENOCHS, E.
72 The epimorphic images of a Dedekind domain, *Proc. Amer. Math. Soc.* **35** (1972), 37—42. *MR* **46** # 1784 ES

CHERLIN, G. L.
76a *Model theoretic algebra—selected topics,* Lecture Notes in Mathematics, Vol. 521, Springer-Verlag, Berlin—New York, 1976. *MR* **58** # 7455 AP
76b Amalgamation bases for commutative rings without nilpotent elements, *Israel J. Math.* **25** (1976), 87—96. *MR* **56** # 11779 AP

CHOE, T. H.
73 Injective compact distributive lattices, *Proc. Amer. Math. Soc.* **37** (1973), 241—245. *MR* **46** # 8923 EI
77 Injective and projective zero-dimensional compact universal algebras, *Algebra Universalis* **7** (1977), 137—142. *MR* **55** # 10354 EI

CHOI, M.-D. and EFFROS, E. G.
77a Injectivity and operator spaces, *J. Functional Analysis* **24** (1977), 156—209. *MR* **55** # 3814 EI
77b Nuclear C^*-algebras and injectivity: the general case, *Indiana Univ. Math. J.* **26** (1977), 443—446. *MR* **55** # 3799 EI

CIGNOLI, R.
75 Injective De Morgan and Kleene algebras, *Proc. Amer. Math. Soc.* **47** (1975), 269—278. *MR* **50** # 9727 EI

CLARK, D. M. and KRAUSS, P. H.
77 Varieties generated by para primal algebras, *Algebra Universalis* **7** (1977), 93—114. *MR* **55** # 2707 CEP

CLARKE, G. T.
80 On completely regular semigroup varieties and the amalgamation property, *Semigroups* (Proc. Conf., Monash Univ., Clayton, 1979), Academic Press, New York, 1980, 159—165. *MR* **82b**: 20083 AP SAP
81 Commutative semigroup varieties with the amalgamation property, *J. Austral. Math. Soc. Ser. A* **30** (1980/81), 278—283. *MR* **82f**: 20084 AP SAP
83 Semigroup varieties with the amalgamation property, *J. Algebra* **80** (1983), 60—72. AP SAP

CLIFFORD, A. H. and PRESTON, G. B.
67 *The algebraic theory of semigroups,* Vol. II, Mathematical Surveys, No. 7, American Mathematical Society, Providence, R. I., 1967. *MR* **36** # 1558 SAP

COHEN, H. B.
64 Injective envelopes of Banach spaces, *Bull. Amer. Math. Soc.* **70** (1964), 723—726. *MR* **32** # 1536 EIH

COHN, P. M.
59 On the free product of associative rings, *Math. Z.* **71** (1959), 380—398. *MR* **21** # 5648 SAP
64 Free ideal rings, *J. Algebra* **1** (1964), 47—69. *MR* **28** # 5095 SAP
71 The embedding of firs in skew fields, *Proc. London Math. Soc.* (3) **23** (1971), 193—213. *MR* **45** # 6866 SAP

COMER, S. D.
67 Some representation theorems and the amalgamation property in algebraic logic, Ph. D. Thesis, Univ. Colorado. AP EI
68 Galois theory and the amalgamation property in finite dimensional cylindric algebras, *Notices Amer. Math. Soc.* **15** (1968), 103. AP EI
69 Classes without the amalgamation property, *Pacific J. Math.* **28** (1969), 309—318. *MR* **39** # 3980 AP
a Galois theory for cylindric algebras and its applications (preprint). ES

CONRAD, P. F.
73 The hulls of representable *l*-groups and *f*-rings, *J. Austral. Math. Soc.* **16** (1973), 385—415. *MR* **49** # 8913 EIH

CORNISH, W. H.
76 The variety of commutative Rickart rings has no non-trivial injectives, *Nanta Math.* **9** (1976), 84—85. *MR* **55** # 10440 — EI
77 Amalgamating commutative regular rings, *Comment. Math. Univ. Carolinae* **18** (1977), 423—436. *MR* **57** # 211 — CEP ES AP EIH EI RS

CORNISH, W. H. and FOWLER, P. R.
77 Coproducts of de Morgan algebras, *Bull. Austral. Math. Soc.* **16** (1977), 1—13. *MR* **55** # 7871 — CEP AP

CRAWLEY, J. W.
75 Some categorical results for semigroup categories, *Notices Amer. Math. Soc.* **22** (1975), A—723. — ES
78 Epimorphisms and H-coextensions, *Arch. Math. (Basel)* **30** (1978), 449—457. *MR* **80g**: 18009 — ES
80 Epimorphisms and semilattices of semigroups, *Semigroup Forum* **21** (1980), 95—112. *MR* **81k**: 20084 — ES

CROWN, G. D.
70 Projectives and injectives in the category of complete lattices with residuated mappings, *Math. Ann.* **187** (1970), 295—299. *MR* **42** # 5868 — EI

CUSICK, D.
73 Torsion submodule and injective hull, *Proc. West Virginia Acad. Sci.* **45** (1973), 347—351. *MR* **50** # 7254 — EIH

DAIGNEAULT, A.
64a Freedom in polyadic algebras and two theorems of Beth and Craig, *Michigan Math. J.* **11** (1964), 129—135. *MR* **29** # 3408 — SAP
64b On automorphisms of polyadic algebras, *Trans. Amer. Math. Soc.* **112** (1964), 84—130. *MR* **29** # 45 — AP
69 Injective envelopes, *Amer. Math. Monthly* **76** (1969), 766—774. *MR* **40** # 5690 — EIH

DAVEY, B. A.
76 Dualities for equational classes of Brouwerian algebras and Heyting algebras, *Trans. Amer. Math. Soc.* **221** (1976), 119—146. *MR* **54** # 192 — EI
77 Weak injectivity and congruence extension in congruence-distributive equational classes, *Canad. J. Math.* **29** (1977), 449—459. *MR* **56** # 217 — CEP EI
78 Subdirectly irreducible distributive double p-algebras, *Algebra Universalis* **8** (1978), 73—88. *MR* **56** # 8457 — RS

DAVEY, B. A. and WERNER, H.
79 Injectivity and Boolean powers, *Math. Z.* **166** (1979), 205—223. *MR* **80b**: 08004 — CEP EI

DAY, A.
70 Injectives in non-distributive equational classes of lattices are trivial, *Arch. Math. (Basel)* **21** (1970), 113—115. *MR* **43** # 122 — EI
71 A note on the congruence extension property, *Algebra Universalis* **1** (1971/72), 234—235. *MR* **45** # 3288 — CEP
72 Injectivity in equational classes of algebras, *Canad. J. Math.* **24** (1972), 209—220. *MR* **45** # 145 — EI CEP AP
73 The congruence extension property and subdirectly irreducible algebras — an example, *Algebra Universalis* **3** (1973), 229—237. *MR* **50** # 2027 — CEP
a Varieties of Heyting algebras, II (Amalgamation and injectivity) (unpublished note). — AP SAP CEP RS EI

DAY, B. J.
74 On adjoint functor factorization, *Category Seminar* (Proc. Sem. Sydney, 1972/73), Lecture Notes in Math., Vol. 420, Springer, Berlin—Heidelberg—New York, 1974, 1—19. *MR* **53** # 579 — ECS
78 On injective profinite algebras, *Bull. Austral. Math. Soc.* **19** (1978), 329—331. *MR* **80g**: 08012 — EI EIH
81 Injective uniform structures, *Algebra Universalis* **13** (1981), 167—170. — EI

DEMLOVÁ, M. and KOUBEK, V.
82 Completion of partial lattices, *Universal Algebra* (Proc. Conf. Esztergom, — AP SAP

1977), Colloq. Math. Soc. J. Bolyai, Vol. 29, North-Holland, Amsterdam, 1982, 173—193. ES
DIDIDZE, C. E.
57 Non-associative free sums of algebras with an amalgamated subalgebra, *Mat. Sb. N. S.* **43 (85)** (1957), 379—396 (in Russian). *MR* **20** # 3198 SAP
68 Free sums of Ω-algebras with amalgamated Ω-subalgebra, *Soobšč. Akad. Nauk Gruz. SSR* = Sakharth. SSR Mecn. Akad. Moambe **50** (1968), 531—534 (in Russian). *MR* **38** # 196 SAP
DIKHTJAR, M. B.
77 Epimorphisms and monomorphisms in the category of partial *P*-operatives, *Ordered sets and lattices*, **4**, 24—27. Saratov Univ. Press (in Russian). ES
DOBBS, D. E.
77 On characterizing injective sheaves, *Canad J. Math.* **29** (1977), 1031—1039; Corrigendum, *ibid.* **32** (1980), 1522. *MR* **56** # 5685 EI
DOROFEEVA, M. P.
73 Injective and flat polygons over hereditary monoids, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **28** (1973), no. 1, 47—51 (in Russian). *MR* **48** # 8670 EI
DÖRGE, K. and JUNG, H. A.
70 Abschließungen in Klassen strukturierter Mengen, *Math. Ann.* **187** (1970), 332—338. *MR* **42** # 167 EIH
DRBOHLAV, K.
63 A note on epimorphisms in algebraic categories, *Comment. Math. Univ. Carolinae* **4** (1963), 81—85. *MR* **32** # 2454 ES
DWINGER, PH.
64 Amalgamation of Boolean spaces, *Nieuw Arch. Wisk.* (3) **12** (1964), 25—31. *MR* **29** # 4027 SAP
70 The amalgamation problem from a categorial point of view, *Proc. Conf. Universal Algebra* (Queen's Univ., Kingston, Ont., October, 1969), Queen's Papers in Pure and Applied Math., Vol. 25, Queen's Univ., Kingston, Ontario, 1970, 190—210. *MR* **41** # 7268 AP SAP
DWINGER, PH. and YAQUB, F. M.
63 Generalized free products of Boolean algebras with an amalgamated subalgebra, *Nederl. Akad. Wetensch. Proc. Ser. A* **66** = *Indag. Math.* **25** (1963), 225—231. *MR* **26** # 6094 SAP
DZIOBIAK, W.
81 On infinite subdirectly irreducible algebras in locally finite equational classes, *Algebra Universalis* **13** (1981), 393—394. *MR* **82m**: 08009 RS
EBRAHIMI, M. M.
82 Algebra in a Grothendieck topos: injectivity in quasi-equational classes, *J. Pure Appl. Algebra* **26**, 269—280. ECS EI EIH
ECKMANN, B. and SCHOPF, A.
53 Über injektive Moduln, *Arch. Math.* **4** (1953), 75—78. *MR* **15**—5 EIH
EILENBERG, S. and MOORE, J. C.
65 Foundations of relative homological algebra, *Mem. Amer. Math. Soc.* No. **55** (1965), 39 pp. *MR* **31** # 2294 EI ES
EKLOF, P. C.
74 Algebraic closure operators and strong amalgamation bases, *Algebra Universalis* **4** (1974), 89—98. *MR* **52** # 5521 AP SAP IPA
EKLOF, P. and SABBAGH, G.
70 Model-completions and modules, *Ann. Math. Logic.* **2** (1970/71), 251—295. *MR* **43** # 3105 AP EI
ENJALBERT, P.
78 Epimorphismes: quelques résultats algébriques et logiques, *C. R. Acad. Sci. Paris Sér. A* **287** (1978), 175—177. ES
FAITH, C.
73 When are proper cyclics injective? *Pacific J. Math.* **45** (1973), 97—112. *MR* **47** # 8610 EI

FAJTLOWICZ, S.

82 On polynomially distributive lattices, *Universal Algebra* (Proc. Conf. Esztergom, 1977), Colloq. Math. Soc. J. Bolyai, Vol. 29, North-Holland, Amsterdam, 1982, 267—269. AP

FAKIR, S.

71 On relative injective envelopes, *J. Pure Appl. Algebra* **1** (1971), 377—383. *MR* **46** # 233 EIH

72 Catégories localement cohérentes et objets α-injectifs, *C. R. Acad. Sci. Paris Sér. A—B* **274** (1972), A1392—A1395. *MR* **45** # 5188 EI

75 Objets algébriquement clos et injectifs dans les catégories localement présentables, *Bull. Soc. Math. France*, Mémoire no. 42, Supplément au 103, no. 2. EI

FAY, T. H.

75 A note on Hausdorff groups, *Bull. Austral. Math. Soc.* **13** (1975), 117—119. *MR* **52** # 8316 ES

FELGNER, U.

a Die Amalgamierungseigenschaft der Ore-Bereiche (unpublished manuscript). AP

FELLER, E. H. and GANTOS, R. L.

69a Completely injective semigroups with central idempotents, *Glasgow Math. J.* **10** (1969), 16—20. *MR* **40** # 256 EI

69b Completely injective semigroups, *Pacific J. Math.* **31** (1969), 359—366. *MR* **41** # 1904 EI

69c Indecomposable and injective S-systems with zero, *Math. Nachr.* **41** (1969), 37—48. *MR* **40** # 7374 EI

FERRAND, D.

68a Monomorphismes de schémas noethériens, *Séminaire d'algèbre commutative P. Samuel*, 1967/68, Les épimorphismes d'anneaux, Secrétariat math., Paris, 1968. ES

68b Épimorphismes d'anneaux à source noethérienne et monomorphismes de schémas, *C. R. Acad. Sci. Paris Sér. A—B* **266** (1968), A319—A321. *MR* **39** # 197e ES

FINDLAY, G. D.

70 Flat epimorphic extensions of rings, *Math. Z.* **118** (1970), 281—288. *MR* **43** # 6270 ES

FITZPATRICK, P. and WIEGOLD, J.

81 Embedding group amalgams, *Bull. Austral. Math. Soc.* **24** (1981), 373—379. SAP

FLACHSMEYER, J.

76 Envelopes in the category of Kakutani-M-spaces, *Categorical topology* (Proc. Conf., Mannheim, 1975), Lecture Notes in Math., Vol. 540, Springer, Berlin—Heidelberg—New York, 1976, 243—253. *MR* **55** # 8743 EIH

FLEISCHER, I.

68 A new construction of the injective hull, *Canad. Math. Bull.* **11** (1968), 19—21. *MR* **37** # 5254 EIH

75 On extending congruences from partial algebras, *Fund. Math.* **88** (1975), 11—16. *MR* **52** # 3017 CEP

76 Amalgamation for semilattices, *Algebra Universalis* **6** (1976), 411—412. *MR* **55** # 211 SAP

80 Relative pseudocomplemented semilattices amalgamate strongly, *Algebra Universalis* **11** (1980), 130—132. *MR* **81m**: 06008 SAP

FLEISCHER, V.

77 Projective and injective varieties of abelian Ω-algebras, *Contributions to Universal Algebra* (Proc. Conf. Szeged, 1975), Colloq. Math. Soc. J. Bolyai, Vol. 17, North-Holland, Amsterdam, 1977, 113—132. *MR* **57** # 5863 EI

FOFANOVA, T. S.

72 Injectivity of polygons over Boolean algebras, *Sibirsk. Mat. Ž.* **13** (1972), 452—458 (in Russian). *MR* **45** # 5043 EI EIH

78 Injective polygons over chains, *Math. Slovaca* **28** (1978), 21—32 (in Russian). *MR* **80h**: 06003 EI

FONTANA, M.
70 Epimorfismi e monomorfismi nella teoria delle categorie concrete, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **49** (1970), 5—10. *MR* **45** # 3284 ES

FORREST, W. K.
77 Model theory for universal classes with the amalgamation property: a study in the foundations of model theory and algebra, *Ann. Math. Logic* **11** (1977), 263—366. *MR* **58** # 202 AP

FOSSUM, R. M.
71 Injective modules over Krull orders, *Math. Scand.* **28** (1971), 233—246. *MR* **46** # 7289 EI

FOUNTAIN, J. B.
74 Completely right injective semigroups, *Proc. London Math. Soc.* (3) **28** (1974), 28—44. *MR* **48** # 8664 EI

FRAÏSSÉ, R.
54 Sur l'extension aux relations de quelques propriétés des ordres, *Ann. Sci. École Norm. Sup.* (3) **71** (1954), 363—388. *MR* **16**—1006 AP

FREESE, R.
79 The variety of modular lattices is not generated by its finite members, *Trans. Amer. Math. Soc.* **255** (1979), 277—300. *MR* **81g**: 06003 ES

FREESE, R. and MCKENZIE, R.
81 Residually small varieties with modular congruence lattices, *Trans. Amer. Math. Soc.* **264** (1981), 419—430. RS

FREYD, P.
64 *Abelian categories. An introduction to the theory of functors,* Harper's Series in Modern Mathematics, Harper and Row, Publishers, New York, 1964. *MR* **29** # 3517 ES ECS EIH

FRIED, E.
74a Subdirect irreducible weakly associative lattices with congruence extension property, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **17** (1974), 59—68. *MR* **52** # 7980 CEP
74b Weakly associative lattices with congruence extension property, *Algebra Universalis* **4** (1974), 151—162. *MR* **51** # 12634 CEP
78 A note on congruence extension property, *Acta Sci. Math. (Szeged)* **40** (1978), 261—263. *MR* **80a**: 08005 CEP

FRIED, E. and GRÄTZER, G.
73 A nonassociative extension of the class of distributive lattices, *Pacific J. Math.* **49** (1973), 59—78. *MR* **49** # 4884 AP CEP RS EI EIH
76 Partial and free weakly associative lattices, *Houston J. Math.* **2** (1976), 501—512. *MR* **55** # 2680 SAP

FRIED, E., GRÄTZER, G. and LAKSER, H.
71 Amalgamation and weak injectives in the equational class of modular lattices M_n, *Notices Amer. Math. Soc.* **18** (1971), A—624. AP EI

FRIED, E., GRÄTZER, G. and QUACKENBUSH, R. W.
80a The equational class generated by weakly associative lattices with the unique bound property, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **22/23** (1979/80), 205—211. *MR* **82a**: 06008 AP CEP
80b Uniform congruence schemes, *Algebra Universalis* **10** (1980), 176—188. *MR* **81m**: 08013 CEP

FRIED, E. and KISS, E. W.
a Connection between congruence lattices and polynomial properties, *Algebra Universalis* (to appear). CEP

FRIED, E. and PIXLEY, A. F.
79 The dual discriminator function in universal algebra, *Acta Sci. Math. (Szeged)* **41** (1979), 83—100. *MR* **80g**: 08007 CEP

FUCHS, E.
75 Projective and injective complete lattices, *Arch. Math. (Brno)* **11** (1975), 217—239. *MR* **54** # 5065 EI

Gabriel, P. and Ulmer, F.
71 *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics, Vol. 221, Springer, Berlin—Heidelberg—New York, 1971. *MR* **48** # 6205 ECS EI
Gagen, T. M. and Weichsel, P. M.
72 Dense subgroups of finite groups, *J. Austral. Math. Soc.* **13** (1972), 385—390. *MR* **47** # 5107 ES
García, O. C.
74 Injectivity in categories of groups and algebras, *An. Inst. Mat. Univ. Nac. Autónoma México* **14** (1974), 95—115. *MR* **57** # 16421 EI
García, O. C. and Larrión, F.
82 Injectivity in varieties of groups, *Algebra Universalis* **14** (1982), 280—286. EI
Gardner, B. J.
75 Epimorphisms of regular rings, *Comment. Math. Univ. Carolinae* **16** (1975), 151—160. *MR* **52** # 5725 ES
79 A note on ring epimorphisms and polynomial identities, *Comment. Math. Univ. Carolinae* **20** (1979), 293—307. *MR* **80e**: 16013 ES
81 Absolute retracts, relatively injective simple objects and Brown—McCoy radicals, *Papers in Algebra, Analysis and Statistics* (Hobart, 1981), 277—283. Contemp. Math., 9, Amer. Math. Soc., Providence. EI
82 Injectives for ring monomorphisms with accessible images, *Comm. in Algebra* **10** (1982), 673—694. EI
Garling, D.
71 Some remarks on injective envelopes, *Proc. Amer. Math. Soc.* **27** (1971), 503—505. *MR* **43** # 2470 EIH
Georgescu, G.
70 On an injective immersion for Boolean algebras, *Stud. Cerc. Mat.* **22** (1970), 1335—1342 (in Romanian). *MR* **51** # 10190 EI
71a Some remarks on the adjoint functors, *Rev. Roumaine Math. Pures Appl.* **16** (1971), 179—194. *MR* **46** # 230 EI
71b Injective envelopes in the category of regular commutative monoids, *Stud. Cerc. Mat.* **23** (1971), 1049—1055 (in Romanian). *MR* **49** # 7333 EIH
73 Injective and projective objects in various concrete categories, *Stud. Cerc. Mat.* **25** (1973), 505—532 (in Romanian). *MR* **51** # 5712 EI EIH
Georgescu, G. and Vraciu, C.
69 Sur les épimorphismes centrés d'algèbres de Łukasiewicz, *C. R. Acad. Sci. Paris Sér. A—B* **269** (1969), A4—A6. *MR* **42** # 7488 ES
Gerhard, J. A.
71 Subdirectly irreducible idempotent semigroups, *Pacific J. Math.* **39** (1971), 669—676. *MR* **47** # 1980 RS
74 Injectives in equational classes of idempotent semigroups, *Semigroup Forum* **9** (1974/75), 36—53. *MR* **52** # 535 EI EIH
Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W. and Scott, D. S.
80 *A compendium of continuous lattices*, Springer-Verlag, Berlin—Heidelberg—New York, 1980. *MR* **82h**: 06005 EI
Gilmour, C. R. A.
75 Special morphisms for zero-set spaces, *Bull. Austral. Math. Soc.* **13** (1975), 57—68. *MR* **51** # 13978 ES IPA
Gleason, A. M.
58 Projective topological spaces, *Illinois J. Math.* **2** (1958), 482—489. *MR* **22** # 12509 EI
Gluskin, L. M.
56 Extension of homomorphisms of semigroups, *Učen. Zap. Harkov. Ped. Inst.* **18** (1956), no. 1, 33—39 (in Russian). CEP
79 Extensions of commutative semigroups, *Izv. Vysš. Učebn. Zaved. Matematika* **1979**, no. 11, 3—12. Correction, *ibid.*, **1982**, no. 7, 88 (in Russian). *MR* **81i**: 20081 EI
Goldberg, M. S.
81 Distributive Ockham algebras: free algebras and injectivity, *Bull. Austral. Math. Soc.* **24** (1981), 161—203. EI AP CEP

GOLDMAN, O. and SAH, C. H.
69 Locally compact rings of special type, *J. Algebra* **11** (1969), 363—454. *MR* **38** # 4530 EI EIH
GOLOVIN, O. N. and BRONŠTEĬN, M. A.
73 An axiomatic classification of exact operations, *Selected questions of algebra and logic* (A collection dedicated to the memory of A. I. Mal'cev), Izdat. Nauka, Novosibirsk, 1973, 40—96 (in Russian). *MR* **49** # 5189 AP
GOLUBOV, E. A. and SAPIR, M. V.
82 Varieties of finitely approximable semigroups, *Izv. Vysš. Učebn. Zaved. Matematika*, **1982**, no. 11, 21—29 (in Russian). RS
GONSHOR, H.
68 Injective hulls of C^* algebras, *Trans. Amer. Math. Soc.* **131** (1968), 315—322. *MR* **37** # 4631 EIH
70 Injective hulls of C^* algebras, II, *Proc. Amer. Math. Soc.* **24** (1970), 486—491. *MR* **44** # 4525 EIH
GRÄTZER, G.
66 Equational classes of lattices, *Duke Math. J.* **33** (1966), 613—622. *MR* **33** # 7278 RS
71 *Lattice theory. First concepts and distributive lattices,* W. H. Freeman and Co., San Francisco, Calif., 1971. *MR* **48** # 184 CEP AP EI EIH RS
78 *General lattice theory,* Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 52, Birkhäuser-Verlag, Basel—Stuttgart, 1978. *MR* **80c**: 06001a AP SAP
79 *Universal algebra,* Second edition, Springer-Verlag, New York—Heidelberg, 1979. *MR* **80g**: 08001 CEP AP EI SAP EIH
GRÄTZER, G., JÓNSSON, B. and LAKSER, H.
73 The amalgamation property in equational classes of modular lattices, *Pacific J. Math.* **45** (1973), 507—524. *MR* **51** # 3014 AP
GRÄTZER, G. and LAKSER, H.
71 The structure of pseudocomplemented distributive lattices, II. Congruence extension and amalgamation, *Trans. Amer. Math. Soc.* **156** (1971), 343—358. *MR* **43** # 124 CEP AP
72a The structure of pseudocomplemented distributive lattices, III. Injective and absolute subretracts, *Trans. Amer. Math. Soc.* **196** (1972), 475—487. *MR* **46** # 8926 EI
72b Two observations on the congruence extension property, *Proc. Amer. Math. Soc.* **35** (1972), 63—64. *MR* **45** # 6729 CEP
GRAVEN, A. W. M.
79 Injective and projective Banach modules, *Nederl. Akad. Wetensch. Indag. Math.* **41** (1979), 253—272. *MR* **81e**: 46063 EI
GRÉGOIRE, J. P. and SCHREIDEN, A.
71 Algèbre — généralisation de la notion de produit amalgamé et application à des catégories concrètes, *Bull. Soc. Roy. Sci. Liège* **40** (1971), 543—554. *MR* **46** # 9135 IPA
GRILLET, P.-A. and PETRICH, M.
70 Free products of semigroups amalgamating an ideal, *J. London Math. Soc.* (2) **2** (1970), 389—392. *MR* **43** # 2132 SAP
GROTHENDIECK, A.
57 Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* (2) **9** (1957), 119—221. *MR* **21** # 1328 EIH ECS
GRUSON, L.
66 Catégories d'espaces de Banach ultramétriques, *Bull. Soc. Math. France* **94** (1966), 287—299. *MR* **36** # 1957 EI EIH
GRUSON, L. and VAN DER PUT, M.
74 Banach spaces, *Bull. Soc. Math. France Mémoire No.* 39—40, Soc. Math. France, Paris, 1974, 55—100. *MR* **51** # 1426 EI EIH
GRUŠKO, P. JA.
80 Injective, projective, and group objects in the category of geometric structures, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **1980**, no. 4, 7—10 (in Russian). *MR* **82a**: 53036 EI

HACQUE, M.
70 Sur l'existence de l'épimorphisme injectif plat à gauche maximum, *C. R. Acad. Sci. Paris Sér. A—B* **271** (1970), A1218—A1221. *MR* **43** # 289 ES
HAJEK, D. W. and MYSIOR, A.
79 On non-simplicity of topological categories, *Categorical Topology* (Proc. Internat. Conf., Free Univ. Berlin, Berlin, 1978), Lecture Notes in Math., Vol. 719, Springer, Berlin—Heidelberg—New York, 1979, 84—93. *MR* **81j**: 18009 ECS
HAJEK, D. W. and WILSON, R. G.
73 The non-simplicity of certain categories of topological spaces, *Math. Z.* **131** (1973), 357—359. *MR* **47** # 9535 ECS
HAINES, D. C.
73 Injective objects in the category of *p*-rings, *Proc. Amer. Math. Soc.* **42** (1973), 57—60. *MR* **48** # 3837 EI
HALL, T. E.
75 Free products with amalgamation of inverse semigroups, *J. Algebra* **34** (1975), 375—385. *MR* **52** # 3401 AP SAP
78a Representation extension and amalgamation for semigroups, *Quart. J. Math. Oxford Ser.* (2) **29** (1978), 309—334. *MR* **80a**: 20084 AP SAP
78b Amalgamation and inverse and regular semigroups, *Trans. Amer. Math. Soc.* **246** (1978), 395—406. *MR* **80d**: 20053 AP SAP
78c Inverse semigroup varieties with the amalgamation property, *Semigroup Forum* **16** (1978), 37—51. *MR* **58** # 11174 AP SAP CEP
79 Inverse and regular semigroups and amalgamation: A brief survey, *Proceedings of a Symposium on Regular Semigroups* (Northern Illinois Univ., De Kalb, Ill., 1979), Northern Illinois Univ., De Kalb, Ill., 1979, 49—79. *MR* **82j**: 20120 AP SAP IPA
HALL, T. E. and JONES, P. R.
83 Epis are onto for finite regular semigroups, *Proc. Edinburgh Math. Soc.* **26** (1983), 151—162. ES
HALMOS, P.
61 Injective and projective Boolean algebras, *Lattice Theory* (Proc. Sympos. Pure Math., Vol. II), American Mathematical Society, Providence, R. I., 1961, 114—122. *MR* **25** # 1121 EI
HAMANA, M.
78 Injective envelopes of Banach modules, *Tôhoku Math. J.* (2) **30** (1978), 439—453. *MR* **80c**: 46070 EIH
79a Injective envelopes of C^*-algebras, *J. Math. Soc. Japan* **31** (1979), 181—197. *MR* **80g**: 46048 EIH
79b Injective envelopes of operator systems, *Publ. Res. Inst. Math. Sci.* **15** (1979), 773—785. *MR* **81h**: 46071 EIH
HANNA, A.
73 On injective modules, *Amer. Math. Monthly* **80** (1973), 297—298. *MR* **47** # 5051 EI
HARDIE, K. A.
76 Projectivity and injectivity relative to a functor, *Math. Colloq. Univ. Cape Town* **10** (1975/76), 68—80. *MR* **54** # 12863 EI
HARTING, R.
81 A remark on injectivity of sheaves of abelian groups, *Arch. Math. (Basel)* **36** (1981), 230—233. *MR* **82j**: 20101 EI
81 Injective Abelian groups in an elementary topos, Habilitationsschrift, Univ. Düsseldorf, 1981. EI
82 Internal coproduct, *Comm. Algebra* **10** (1982), 1173—1237. EI
HARTLEY, B.
77 Injective modules over group rings, *Quart. J. Math. Oxford* Ser. (2) **28** (1977), 1—29. *MR* **55** # 8088 EI
HAYDON, R.
77 Injective Banach lattices, *Math. Z.* **156** (1977), 19—47. *MR* **57** # 13438 EI
78 Sur les espaces de Banach réticulés injectifs, *Séminaire Choquet*, 16^e année EI

(1976/77), *Initiation à l'analyse,* Exp. No. 14, Secrétariat Math., Paris, 1978. *MR* **58** # 17777

HEAD, T. and ANDERSON, J. A.

80 On injective and flat commutative regular semigroups, *Semigroup Forum* **21** (1980), 283—284. *MR* **82b**: 20095 EI

HENKIN, L., MONK, J. D., TARSKI, A., ANDRÉKA, H. and NÉMETI, I.

81 *Cylindric Set Algebras,* Lecture Notes in Math., Vol. 883, Springer, Berlin—Heidelberg—New York, 1981. AP RS ES

HERRLICH, H.

65 Wann sind alle stetigen Abbildungen in Y konstant? *Math. Z.* **90** (1965), 152—154. *MR* **32** # 3029 ECS

67 Fortsetzbarkeit stetiger Abbildungen und Kompaktheitsgrad topologischer Räume, *Math. Z.* **96** (1967), 64—72. *MR* **34** # 8370 ECS

HERRLICH, H. and STRECKER, G. E.

73 *Category theory: an introduction,* Allyn and Bacon Series in Advanced Mathematics, Allyn and Bacon Inc., Boston, Mass., 1973. *MR* **50** # 2284 ECS

HICKMAN, R.

79 Congruence extensions for semilattices with distributivity, *Algebra Universalis* **9** (1979), 179—198. *MR* **80c**: 06007 CEP

HIGGINS, P. M.

81 Epis are onto for generalised inverse semigroups, *Semigroup Forum* **23** (1981), 255—259. ES

83 The varieties of commutative semigroups for which epis are onto, *Proc. Roy. Soc. Edinburgh* **94A** (1983), 1—7. ES

HIGGS, D.

71 Remarks on residually small varieties, *Algebra Universalis* **1** (1971/72), 383—385. *MR* **46** # 8939 RS EI AP

HIGMAN, G.

64 Amalgams of p-groups, *J. Algebra* **1** (1964), 301—305. *MR* **29** # 4799 SAP

HILLER, H. L.

77 Injectives in arrow categories, *Portugaliae Math.* **36** (1977), 239—246. EI

HILTON, P. J.

72 Category theory, Notes from a NSF Short Course, Colgate Univ., Hamilton ES

HOFFMANN, B.

80 An injective characterization of Peano spaces, *Topology Appl.* **11** (1980), 37—46. *MR* **80m**: 54049 EI

HOFFMANN, R.-E.

81 Essential extensions of T_1-spaces, *Canad. Math. Bull.* **24** (1981), 237—240. *MR* **82j**: 18009 EIH

HOFMANN, K. H. and MISLOVE, M. W.

75 Epics of compact Lawson semilattices are surjective, *Arch. Math. (Basel)* **26** (1975), 337—345. *MR* **51** # 12636 ES

76 Amalgamation in categories with concrete duals, *Algebra Universalis* **6** (1976), 327—347. *MR* **56** # 5676 AP IPA ES

HONG, S. S.

73 On $\mathfrak{k}$-compactlike spaces and reflective subcategories, *General Topology and Appl.* **3** (1973), 319—330. *MR* **48** # 12450 ECS

HORN, A. and KIMURA, N.

71 The category of semilattices, *Algebra Universalis* **1** (1971/72), 26—38. *MR* **47** # 6568 ES EI EIH

HOWIE, J. M.

62 Embedding theorems with amalgamation for semigroups, *Proc. London Math. Soc.* (3) **12** (1962), 511—534. *MR* **25** # 2139 SAP

63a An embedding theorem with amalgamation for cancellative semigroups, *Proc. Glasgow Math. Assoc.* **6** (1963), 19—26. *MR* **27** # 221 SAP

63b Subsemigroups of amalgamated free products of semigroups, *Proc. London Math. Soc.* (3) **13** (1963), 672—686. *MR* **27** # 3724 SAP

64 The embedding of semigroup amalgams, *Quart. J. Math. Oxford. Ser.* (2) **15** (1964), 55—68 and 277. *MR* **28** # 3107 SAP

68 Commutative semigroup amalgams, *J. Austral. Math. Soc.* **8** (1968), 609—630. *MR* **37** # 6388 AP ES

75 Semigroups amalgams whose cores are inverse semigroups, *Quart. J. Math. Oxford Ser.* (2) **26** (1975), 23—45. *MR* **52** # 10922 SAP

76 *An introduction to semigroup theory,* L. M. S. Monographs, No. 7, Academic Press, London—New York, 1976. *MR* **57** # 6235 SAP AP ES

a Epimorphisms and amalgamations: a survey of recent progress, *Semigroups. Structure and universal algebraic problems* (Proc. Conf. Szeged, 1981), Colloq. Math. Soc. J. Bolyai, Vol. 39, North-Holland, Amsterdam (to appear). AP SAP ES IPA

Howie, J. M. and Isbell, J. R.

67 Epimorphisms and dominions, II, *J. Algebra* **6** (1967), 7—21. *MR* **35** # 105b ES

Huet, P.

72 Un example d'épimorphisme d'anneaux non surjectif, *Bull. Sci. Math.* (2) **96** (1972), 151—153. *MR* **52** # 13807 ES

Hule, H.

76 Über die Eindeutigkeit der Lösungen algebraischer Gleichungssysteme, *J. Reine Angew. Math.* **282** (1976), 157—161. *MR* **53** # 13080 AP IPA

78 Relations between the amalgamation property and algebraic equations, *J. Austral. Math. Soc. Ser. A* **22** (1978), 257—263. *MR* **58** # 5454 AP IPA

79 Solutionally complete varieties, *J. Austral. Math. Soc. Ser. A* **28** (1979), 82—86. *MR* **81i**: 08007 AP IPA

Hule, H. and Müller, W. B.

76 On the compatibility of algebraic equations with extensions, *J. Austral. Math. Soc. Ser. A* **21** (1976), 381—383. *MR* **53** # 13076 AP

Hung, H. H.

73 The amalgamation property for *G*-metric spaces, *Proc. Amer. Math. Soc.* **37** (1973), 53—58. *MR* **47** # 2560 AP

Huq, S. A.

72a A note on monics and epics in varieties of categories, *Fund. Math.* **76** (1972), 117—119. *MR* **46** # 3587 ES

72b A note on generators in categories, *Cahiers Topologie Géom. Différentielle* **13** (1972), 393—395, 443. *MR* **48** # 367 ECS

78 Discussion of the condition "monics are equalizers" implies "epis are surjective" in categories, *Comment. Math. Univ. St. Paul.* **27** (1978/79), 5—9. *MR* **80b**: 18002 IPA ES

Hušek, M.

69 The class of *k*-compact spaces is simple, *Math. Z.* **110** (1969), 123—126. *MR* **39** # 6260 ECS

72 Perfect images of *E*-compact spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **20** (1972), 41—45. *MR* **46** # 6289 ECS

Imaoka, T.

76a Free products with amalgamation of commutative inverse semigroups, *J. Austral. Math. Soc. Ser. A* **22** (1976), 246—251. *MR* **55** # 12838 SAP

76b Free products with amalgamation of bands, *Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci.* **10** (1976), 7—17. *MR* **55** # 3118 SAP

77a Free products with amalgamation of orthodox semigroups, *Seminar on semigroup theory* (Res. Inst. Math. Sci., Kyoto Univ., Kyoto 1977), *Sûri-kaisekikenkyûsho Kyôkyûroku* No. 292 (1977), 40—55. *MR* **57** # 12733 SAP

77b Free products with amalgamation of orthodox semigroups, II, *Proc. First Sympos. on Semigroups* (Shimane Univ., Matsue, 1977), Shimane University, Matsue, 1977, 65—71. *MR* **57** # 12734 SAP

78a The embedding of generalized inverse semigroup amalgams, *Mem. Fac. Sci. Shimane Univ.* **12** (1978), 1—4. *MR* **80c**: 20083b SAP

78b The embedding of generalized inverse semigroup amalgams, *Proceedings 2nd Symposium on Semigroups* (Tokyo Gakugei Univ., Tokyo, 1978), Tokyo Gakugei Univ., Tokyo, 1978, 41—44. *MR* **80c**: 20083a SAP

79 The embedding of regular semigroup amalgams, *Proceedings of a Symposium on Regular Semigroups* (Northern Illinois Univ., De Kalb, Ill, 1979), Northern Illinois Univ., De Kalb, Ill, 1979, 92—100. *MR* **82j**: 20120 (See also, SAP

in a shortened form, in *Proceedings of the 3rd Symposium on Semigroups* (Inter-Univ. Sem. House of Kansai, Kobe, 1979), Osaka Univ., Osaka, 1980, 7—10. *MR* **81i**: 20082)

INGRAO, B.
72a Quelques propriétés de la catégories des anneaux absolument plats réduits. Formule des annulateurs, *C. R. Acad. Sci. Paris Sér. A—B* **274** (1972), A1597—A1600. *MR* **46** # 202 SAP
72b Quelques propriétés de la catégorie des anneaux absolument plats réduits. Surjectivité des épimorphismes, *C. R. Acad. Sci. Paris Sér. A—B* **274** (1972), A1753—A1756. *MR* **46** # 203 ES

ISBELL, J. R.
57 Some remarks concerning categories and subspaces, *Canad. J. Math.* **9** (1957), 563—577. *MR* **20** # 923 ES
64a Six theorems about injective metric spaces, *Comment. Math. Helv.* **39** (1964), 65—76. *MR* **32** # 431 EI
64b Injective envelopes of Banach spaces are rigidly attached, *Bull. Amer. Math. Soc.* **70** (1964), 727—729. *MR* **32** # 1537 EIH
66a Epimorphisms and dominions, *Proc. Conf. Categorical Algebra* (La Jolla, Calif. 1965), Springer, New York, 1966, 232—246. *MR* **35** # 105a ES
66b Structure of categories, *Bull. Amer. Math. Soc.* **72** (1966), 619—655. *MR* **34** # 5896 ECS EAR
68a Small subcategories and completeness, *Math. Systems Theory* **2** (1968), 27—50. *MR* **37** # 269 EAR ECS
68b Epimorphisms and dominions, III, *Amer. J. Math.* **90** (1968), 1025—1030. *MR* **38** # 5877 ES
69a Three remarks on injective envelopes of Banach spaces, *J. Math. Anal. Appl.* **27** (1969), 516—518. *MR* **40** # 4739 EIH
69b Epimorphisms and dominions, IV, *J. London Math. Soc.* (2) **1** (1969), 265—273. *MR* **41** # 1774 ES
72 Beatific semigroups, *J. Algebra* **23** (1972), 228—238. *MR* **46** # 5497 EI EIH
73 Epimorphisms and dominions, V, *Algebra Universalis* **3** (1973), 318—320. *MR* **50** # 2029 ES

ISÉKI, K.
75 Epimorphisms of categories of various topological spaces, *Math. Sem. Notes Kobe Univ.* **3** (1975), 189—191. *MR* **53** # 14389 ES

ISKANDER, A. A.
65 Universal algebras with relations and amalgams, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **1965**, no. 4, 22—28 (in Russian). *MR* **33** # 86 AP

ISKANDER, A. A. and YAQUB, F. M.
71 Commutative rings with the congruence extension property, *Notices Amer. Math. Soc.* **18** (1971), 804. CEP

JEŽEK, J.
76 EDZ-varieties: The Schreier property and epimorphisms onto, *Comment. Math. Univ. Carolinae* **17** (1976), 281—290. *MR* **53** # 13077 AP IPA SAP ES
77 Varieties of algebras with equationally definable zeros, *Czechoslovak Math. J.* **27** (**102**) (1977), 394—414. *MR* **56** # 11872 AP SAP ES RS EI

JEŽEK, J. and KEPKA, T.
77 Varieties of abelian quasigroups, *Czechoslovak Math. J.* **27** (**102**) (1977), 473—503. *MR* **56** # 8740 SAP RS
79 Varieties of quasigroups determined by short strictly balanced identities, *Czechoslovak Math. J.* **29** (104) (1979), 84—96. *MR* **80b**: 20092 SAP CEP RS

JOHN, R.
77 A note on implicational subcategories, *Contributions to Universal Algebra* (Proc. Conf. Szeged, 1975), Colloq. Math. Soc. J. Bolyai, Vol. 17, North-Holland, Amsterdam, 1977, 213—222. *MR* **58** # 28119 EI

JOHNSON, JR. C. S.
73 A characterization of injective *S*-systems over a semilattice, *J. Natur. Sci. Math.* **13** (1973), 239—241. *MR* **52** # 10519 EI

JOHNSON, JR., C. S. and MCMORRIS, F. R.

72a Injective hulls of certain S-systems over a semilattice, *Proc. Amer. Math. Soc.* **32** (1972), 371—375. *MR* **44** # 6875 EIH

72b Completely cyclic injective semilattices, *Proc. Amer. Math. Soc.* **36** (1972), 385—388. *MR* **46** # 9213 EI

JOHNSON, J. S.

70 Amalgamation of polyadic algebras, *Trans. Amer. Math. Soc.* **149** (1970), 627—652. *MR* **44** # 1548 AP

JOHNSTONE, P. T.

81 Injective toposes, *Continuous Lattices* (Proc. Conf. Bremen, 1979), Lecture Notes in Math., Vol. 871, Springer, Berlin—Heidelberg—New York, 1981, 284—297. EI

JOHNSTONE, P. T. and JOYAL, A.

82 Continuous categories and exponentiable toposes, *J. Pure Appl. Algebra* **25** (1982), 255—296. EI

JÓNSSON, B.

56 Universal relational systems, *Math. Scand.* **4** (1956), 193—208. *MR* **20** # 3091 SAP

60 Homogeneous universal relational systems, *Math. Scand.* **8** (1960), 137—142. *MR* **23** # A2328 AP

62 Algebraic extensions of relational systems, *Math. Scand.* **11** (1962), 179—205. *MR* **27** # 4777 AP SAP

65 Extensions of relational structures, *Theory of Models* (Proc. Internat. Sympos. Berkeley, 1963), North-Holland, Amsterdam, 1965, 146—157. *MR* **34** # 2463 AP

67 Algebras whose congruence lattices are distributive, *Math. Scand.* **21** (1967), 110—121. *MR* **38** # 5689 RS

82 Varieties of lattices: Some open problems, *Universal Algebra* (Proc. Conf. Esztergom, 1977), Colloq. Math. Soc. J. Bolyai, 29, North-Holland, Amsterdam, 421—436 (to appear). AP

KACOV, E. B.

76 Flat functors, *Mat. Zametki* **19** (1976), 577—586 (in Russian). *MR* **53** # 5700 EI EIH

KAIJSER, S.

76 Some remarks on injective Banach algebras, *Spaces of Analytic Functions* (Sem. Functional Anal. and Function Theory, Kristiansand, 1975), Lecture Notes in Math., Vol. 512, Springer, Berlin, 1976, 84—95, *MR* **58** # 17879 EI

KALMAN, J. A.

71 Subdirect decomposition of distributive quasilattices, *Fund. Math.* **71** (1971), 161—163. *MR* **45** # 5040 RS

KATRIŇÁK, T.

72 Subdirectly irreducible modular p-algebras, *Algebra Universalis* **2** (1972), 166—173. *MR* **46** # 8921 RS

73 Primitive Klassen von modularen S-Algebren, *J. Reine Angew. Math.* **261** (1973), 55—70. *MR* **52** # 13542 RS

74a Injective double Stone algebras, *Algebra Universalis* **4** (1974), 259—267. *MR* **51** # 10187 EI

74b Congruence extension property for distributive double p-algebras, *Algebra Universalis* **4** (1974), 273—276. *MR* **50** # 6953 CEP

77a Essential extensions of Stone algebras, *Algebra Universalis* **7** (1977), 1—3. *MR* **55** # 224 EIH

77b Essential extensions and injective hulls of double Stone algebras, *Algebra Universalis* **7** (1977), 5—23. *MR* **55** # 2695 EIH

79 Subdirectly irreducible p-algebras, *Algebra Universalis* **9** (1979), 116—126. *MR* **80f**: 06004 RS

80 Subdirectly irreducible distributive double p-algebras, *Algebra Universalis* **10** (1980), 195—219. *MR* **81g**: 06006 RS

KATZ, E. and MORRIS, S. A.

83 Free products of k_ω-topological groups with normal amalgamation, *Topology Appl.* **15** (1983), 189—196. SAP

83 On the Hausdorffness of free products of topological groups with normal amalgamation, *Math. Chron.* **12** (1983), 113—117. AP
a Free products of topological groups with amalgamation, I—II (preprint). AP

KAWAHARA, Y.
75 On the class of regular epimorphisms, *Comm. Algebra* **3** (1975), 851—857. *MR* **52** # 10836 ES

KELLY, G. M.
69 Monomorphisms, epimorphisms, and pull-backs, *J. Austral. Math. Soc.* **9** (1969), 124—142. *MR* **39** # 1515 IPA TP ECS

KEPKA, T.
77 Epimorphisms in some groupoid varieties, *Comment. Math. Univ. Carolinae* **18** (1977), 265—279. *MR* **55** # 12600 SAP ES
81 Strong amalgamation property in some groupoid varieties, *Finite Algebra and Multiple-Valued Logic* (Proc. Conf. Szeged, 1979), Colloq. Math. Soc. J. Bolyai, Vol. 28, North-Holland, Amsterdam, 1981, 373—385. SAP

KERTÉSZ, A.
62 Eine kennzeichnende Eigenschaft der injektiven Moduln, *Wiss. Z. Martin-Luther-Univ. Halle—Wittenberg Math.-Natur. Reihe* **11** (1962), 737—739. *MR* **29** # 5860 AP EI

KHAN, M. S. and MORRIS, S. A.
82 Free products of topological groups with central amalgamation, I—II, *Trans. Amer. Math. Soc.* **273** (1982), 405—416, 417—432. AP

KHAN, N. M.
82 Epimorphisms, dominions, and varieties of semigroups, *Semigroup Forum* **25** (1982), 331—337. ES

KIMURA, N.
57 On semigroups, Thesis, Tulane University, New Orleans. SAP

KISS, E.
74 Über eine Kategorie von Halbmoduln, *Studia Univ. Babeş—Bolyai Ser. Math.-Mech.* **19** (1974), no. 2, 11—15. *MR* **51** # 5680 ES

KISS, E. W.
81 Each Hamiltonian variety has the congruence extension property, *Algebra Universalis* **12** (1981), 395—398. *MR* **82h**: 08002 CEP
a Injectivity and related concepts in modular varieties, I. The congruence extension property (manuscript). CEP

KLEIN, A.
79 Injectives and simple objects, *J. Pure Appl. Algebra* **15** (1979), 243—245. *MR* **80**g: 18005 EI

KLEINER, I.
66 Free and injective Lie modules, *Canad. Math. Bull.* **9** (1966), 29—42. *MR* **33** # 4111 EI EIH

KLEISLI, H. and WU, Y. C.
64 On injective sheaves, *Canad. Math. Bull.* **7** (1964), 415—423. *MR* **29** # 5882 EI

KNAUER, U.
69 Special morphisms and amalgamations in commutative semigroups, *Materials of the* 10th *Colloq. on Algebra*, vol. 2, p. 117, Novosibirsk (in Russian). SAP

KNIGHT, J.
70 On epimorphisms of non-commutative rings, *Proc. Cambridge Philos. Soc.* **68** (1970), 589—600. *MR* **42** # 4595 ES

KNÖRR, R.
75 Ringepimorphismen und Morita-projektive Moduln über kommutativen Dedekind-Ringen, *Comment. Math. Helv.* **50** (1975), 267—275. *MR* **52** # 13795 ES

KOLLÁR, J.
80 Injectivity and congruence extension property in congruence distributive equational classes, *Algebra Universalis* **10** (1980), 21—26. *MR* **81c**: 08014 CEP EI

KOPPELBERG, S.
73 Injective hulls of chains, *Arch. Math. (Basel)* **24** (1973), 225—229. *MR* **48** # 194 EIH

KORNIENKO, V. S.
77 Flat polygons over distributive lattices, *Ordered sets and lattices*, **4**, Izdat. Saratov. Univ., Saratov, 1977, 69—85 (in Russian). *MR* **58** # 5435 EI
KOVÁCS, L. G. and NEWMAN, M. F.
82 Injectives in varieties of groups, *Algebra Universalis* **14** (1982), 398—400. EI
KRAUSS, P.
72 Extending congruence relations, *Proc. Amer. Math. Soc.* **31** (1972), 517—520. *MR* **44** # 2688 CEP
73 On the embedding and amalgamation properties, *Notices Amer. Math. Soc.* **20** (1973), 283. AP
a The structure of filtral varieties, II. Weakly injective and injective members (preprint). EI
KUROŠ, A. G.
67 *Theory of groups*, Third augmented edition, Izdat. Nauka, Moscow, 1967. *MR* **40** # 2740 SAP
KUROŠ, A. G., LIVŠIC, A. KH. and ŠUL'GEĬFER, E. G.
60 Foundations of the theory of categories, *Uspehi Mat. Nauk* **15** (1960), no. 6 (96), 3—52 (in Russian). *MR* **22** # 9526; *Russian Math. Surveys* **15** (1960), no. 6, 1—46. *MR* **23** # A1688. In German with CALENKO, M. S., *Zur Theorie der Kategorien*, Math. Forschungsberichte, 15, VEB Deutscher Verlag der Wiss., Berlin, 1963. *MR* **32** # 135 ES
LACEY, H. E. and COHEN, H. B.
69 On injective envelopes of Banach spaces, *J. Functional Analysis* **4** (1969), 11—30. *MR* **39** # 4644 EIH
LAGRANGE, R.
74 Amalgamation and epimorphisms in m-complete Boolean algebras, *Algebra Universalis* **4** (1974), 277—279. *MR* **51** # 300 SAP ES
LAI, H.-F.
80 On Kleene algebras, *Algebra Universalis* **11** (1980), 117—126. *MR* **82a**: 06016 ES EI
LAKSER, H.
70 Injective hulls of Stone algebras, *Proc. Amer. Math. Soc.* **24** (1970), 524—529. *MR* **41** # 119 EI EIH
71 The structure of pseudocomplemented distributive lattices, I. Subdirect decomposition, *Trans. Amer. Math. Soc.* **156** (1971), 335—342. *MR* **43** # 123 RS
73 Injective completeness of varieties of unary algebras: A remark on a paper of Higgs, *Algebra Universalis* **3** (1973), 129—130. *MR* **49** # 189 AP EI
LALLEMENT, G.
75 Amalgamated products of semigroups: The embedding problem, *Trans. Amer. Math. Soc.* **206** (1975), 375—394. *MR* **51** # 759 SAP
77 Sur les produits amalgamés de monoïdes, *Séminaire d'Algèbre Paul Dubreil, 29ème année* (Paris, 1975/76), Lecture Notes in Math., Vol. 586, Springer, Berlin, 1977, 25—33. *MR* **58** # 28232 SAP
LAMARTIN, W. F.
76 Epics in the category of T_2 k-groups need not have dense range, *Colloq. Math.* **36** (1976), 37—41. *MR* **55** # 556 ES
77 On the foundations of k-group theory, *Dissertationes Math. (Rozprawy Mat.)* **146** (1977), 32 pp. *MR* **58** # 984 ES
LAMBEK, J.
66 *Lectures on rings and modules*, Blaisdell Publishing Co., Waltham, Mass.—Toronto, Ont.—London, 1966. *MR* **34** # 5857 EI EIH
76 Localization at epimorphisms and quasi-injectives, *J. Algebra* **38** (1976), 163—181. *MR* **56** # 12046 ES EI
LAMBEK, J. and RATTRAY, B. A.
73 Localization at injectives in complete categories, *Proc. Amer. Math. Soc.* **41** (1973), 1—9. *MR* **54** # 2750 ECS EI
LANCKAU, R.
69 Das verallgemeinerte freie Produkt in primitiven Klassen universeller Algebren, I, *Publ. Math. Debrecen* **16** (1969), 307—323. *MR* **42** # 171 SAP AP

70 Das verallgemeinerte freie Produkt in primitiven Klassen universeller Algebren, II, *Publ. Math. Debrecen* **17** (1970), 321—331. *MR* **46** # 5209 SAP AP
LAPIŃSKA, C.
77 Generalized free m-products of Boolean m-algebras with m-amalgamated m-subalgebra, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **25** (1977), 221—223. *MR* **56** # 2894 AP
LAPLAZA, M. L.
66 Note on epimorphisms in the category of rings, *Rev. Mat. Hisp.-Amer.* (4) **26** (1966), 67—72 (in Spanish). *MR* **36** # 241 ES
LAZARD, D.
68 Épimorphismes plats d'anneaux, *C. R. Acad. Sci. Paris Sér. A—B* **266** (1968), A314—A316. *MR* **39** # 197c ES
69 Autour de la platitude, *Bull. Soc. Math. France* **97** (1969), 81—128. *MR* **40** # 7310; erratum **41**, p. 1965 ES
LAZARD, D. and HUET, P.
70 Dominions des anneaux commutatifs, *Bull. Sci. Math.* (2) **94** (1970), 193—199. *MR* **42** # 5980 ES
LEE, K. B.
70 Equational classes of distributive pseudo-complemented lattices, *Canad. J. Math.* **22** (1970), 881—891. *MR* **42** # 151 RS
LEHMANN, D. and PASZTOR, A.
82 Epis need not be dense, *Theoret. Comput. Sci.* **17** (1982), 151—161. ES
LEROUX, P.
71 Sur les structures d'effacement, *Math. Z.* **121** (1971), 329—340. *MR* **46** # 1883 EI
LEVINE, J.
71 On the injective hulls of semisimple modules, *Trans. Amer. Math. Soc.* **115** (1971), 115—126. *MR* **46** # 5389 EIH
LINDERHOLM, C. E.
70 A group epimorphism is surjective, *Amer. Math. Monthly* **77** (1970), 176—177. *MR* **41** # 303 ES
LINTON, F. E. J.
66 Injective Boolean σ-algebras, *Arch. Math. (Basel)* **17** (1966), 383—387. *MR* **33** # 5539 ES
LJAPIN, E. S.
69 Independence of subsemigroups of a semigroup, *Dokl. Akad. Nauk SSSR* **185** (1969), 1229—1231 (in Russian). *MR* **40** # 259 SAP
70 Intersections of independent subsemigroups of a semigroup, *Izv. Vysš. Učebn. Zaved. Matematika* **1970**, no. 4 (95), 67—73 (in Russian). *MR* **43** # 7526 SAP
74 Independent semigroup extensions of partial groupoids, *Sovremennaya Algebra,* **2**, Leningrad. Gos. Ped. Inst., Leningrad, 1974, 55—71 (in Russian). *MR* **53** # 13450 SAP
LOEBL, R. I.
74 Injective von Neumann algebras, *Proc. Amer. Math. Soc.* **44** (1974), 46—48. *MR* **49** # 5870 EI
LORENZ, F.
69 Die Epimorphismen der Ringe von Operatoren, *Arch. Math. (Basel)* **20** (1969), 48—53. *MR* **39** # 3326 ES
LUEDEMAN, J. K., MCMORRIS, F. R. and SIM, S.-K.
78 Semigroups for which every totally irreducible S-system is injective, *Comment. Math. Univ. Carolinae* **19** (1978), 27—35. *MR* **57** # 16446 EI
L'VOV, I. V.
a Finitely approximable varieties of rings (unpublished note). RS
MACDONALD, J. L.
74 Conditions for a universal mapping of algebras to be a monomorphism, *Bull. Amer. Math. Soc.* **80** (1974), 888—892. *MR* **52** # 10551 SAP
MACLANE, S.
50 Duality for groups, *Bull. Amer. Math. Soc.* **56** (1950), 485—516. *MR* **14**—133 EI

71 *Categories for the working mathematician,* Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York—Heidelberg—Berlin, 1971. *MR* **50** # 7275 ECS

MĄCZYŃSKI, M. J.

66 Generalized free m-products of m-distributive Boolean algebras with an m-amalgamated subalgebra, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **14** (1966), 539—542. *MR* **34** # 4177 AP

69 The amalgamation of an α-distributive α-complete Boolean algebra, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **17** (1969), 415—418. *MR* **40** # 7166 AP

72 Strong $\mathfrak{m}$-extension property and $\mathfrak{m}$-amalgamation in Boolean algebras, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **20** (1972), 259—263. *MR* **46** # 3396 AP

MĄCZYŃSKY, M. J. and TRACZYK, T.

67 The m-amalgamation property for m-distributive Boolean algebras, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **15** (1967), 57—60; loose errata, *ibid.* **16** (1968), no. 2. *MR* **39** # 110 AP

MAKSIMOVA, L. L.

77 Craig's interpolation theorem and amalgamable varieties, *Dokl. Akad. Nauk SSSR* **237** (1977), 1281—1284 (in Russian). AP SAP

MAMEDOV, O. M.

78 On injectivity in semigroups, 2nd *All-Union Simp. on Semigroup Theory,* Abstracts, p. 54. Sverdlovsk (in Russian). EI

MARANDA, J.-M.

60 On pure subgroups of abelian groups, *Arch. Math. (Basel)* **11** (1960), 1—13. *MR* **22** # 5672 EI

64 Injective structures, *Trans. Amer. Math. Soc.* **110** (1964), 98—135. *MR* **29** # 1236 EI

MARKOWSKY, G.

77 Categories of chain-complete posets, *Theoret. Comput. Sci.* **4** (1977), 125—135. *MR* **56** # 11859 AP EI

MASON, G.

76 Injective and projective near-ring modules, *Compositio Math.* **33** (1976), 43—54. *MR* **54** # 7558 EI

MAZET, P.

68a Générateurs, relations et épimorphismes d'anneaux, *C. R. Acad. Sci. Paris Sér. A—B* **266** (1968), A309—A311. *MR* **39** # 197a ES

68b Charactérisations des épimorphismes par relations et générateurs, *Séminaire d'algèbre commutative P. Samuel* 1967/68, *Les épimorphismes d'anneaux,* Secrétariat math., Paris, 1968. ES

McKAY, S.

69 Surjective epimorphisms in classes of groups, *Quart. J. Math. Oxford Ser.* (2) **20** (1969), 87—90. *MR* **39** # 1558 ES

McKENZIE, R.

81 Residually small varieties of semigroups, *Algebra Universalis* **13** (1981), 171—201. *MR* **82k**: 20097 RS

82 Residually small varieties of K-algebras, *Algebra Universalis* **14** (1982), 181—196. RS

83 A note on residually small varieties of semigroups, *Algebra Universalis* **17** (1983), 143—149. RS

McKENZIE, R. and SHELAH, S.

74 The cardinals of simple models for universal theories, *Proceedings of the Tarski Symposium* (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971), Amer. Math. Soc., Providence, R. I., 1974, 53—74. *MR* **50** # 12711 RS

MEGYESI, L.

74a Extension of semigroups by means of a separating unit ideal subsemigroup, *Sovremennaya Algebra,* **2**, Leningrad. Gos. Ped. Inst., Leningrad, 1974, 72—97 (in Russian). *MR* **53** # 3163a SAP

74b Two problems on the extensions of semigroups, *Sovremennaya Algebra,* **2,** SAP

Leningrad. Gos. Ped. Inst., Leningrad, 1974, 98—108 (in Russian). *MR* **53**#3163b
77 The imbedding of semigroups in a semigroup in which the original semigroups are independent, *Acta Sci. Math. (Szeged)* **39** (1977), 329—340 (in Russian). *MR* **57**#16437 SAP
78 Construction of a semigroup amalgam that can be independently imbedded into a semigroup, *Acta Sci. Math. (Szeged)* **40** (1978), 111—120 (in Russian). *MR* **58**#22356 SAP

MELDRUM, J. P.
78 Injective near-ring modules over Z_n, *Proc. Amer. Math. Soc.* **68** (1978), 16—18. *MR* **56**#15707 EI

MESKHI, V. JU.
77 Residually small varieties of derivative algebras and pseudo-Boolean algebras (Russian), 14th *All-Union Conf. on Algebra* (Novosibirsk, 1977), Abstracts, vol. 2, p. 109. RS
79 Injective Heyting—Stone algebras with involution (Russian), 15th *All-Union Conf. on Algebra* (Krasnoyarsk, 1979), Abstracts, vol. 2, p. 107. EI

MICHLER, G. O. and VILLAMAYOR, O. E.
73 On rings whose simple modules are injective, *J. Algebra* **25** (1973), 185—201. *MR* **47**#5052 EI

MICHLER, L.
75 Über die Epimorphismen von Kategorien mehrbasiger algebraischer Systeme, *Math. Nachr.* **69** (1975), 123—132. *MR* **54**#2748 ES

MITCHELL, B.
65 *Theory of categories*, Pure and Applied Mathematics, Vol. XVII, Academic Press, New York—London, 1965. *MR* **34**#2647 EIH
72 The dominion of Isbell, *Trans. Amer. Math. Soc.* **167** (1972), 319—331. *MR* **45**#3511 ES

MONK, J. D.
67 Nontrivial *m*-injective Boolean algebras do not exist, *Bull. Amer. Math. Soc.* **73** (1967), 526—527. *MR* **35**#92 EI

MONRO, G. P.
74 The strong amalgamation property for complete Boolean algebras, *Z. Math. Logik Grundlagen Math.* **20** (1974), 499—502. *MR* **55**#2563 SAP

MORLEY, M. D. and VAUGHT, R. L.
62 Homogeneous universal models, *Math. Scand.* **11** (1962), 37—57. *MR* **27**#37 AP

MRÓWKA, S.
56 On universal spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **4** (1956), 479—481. *MR* **19**—669 ECS

MYSIOR, A.
78 The category of all zero-dimensional realcompact spaces is not simple, *General Topology Appl.* **8** (1978), 259—264. *MR* **57**#7535 ECS

NACHBIN, L.
50 A theorem of the Hahn-Banach type for linear transformations, *Trans. Amer. Math. Soc.* **68** (1950), 28—46. *MR* **11**—369 EI

NELSON, E.
75 Injectivity and equational compactness in the class of $\aleph_0$-semilattices, *Canad. Math. Bull.* **18** (1975), 387—392. *MR* **52**#13532 EI
80 An elementary proof that there are no non-trivial injective lattices, *Algebra Universalis* **10** (1981), 264—265. *MR* **81c**: 06009 EI

NÉMETI, I.
a Cylindric-relativised set algebras and Bo_α-s have strong amalgamation (preprint). SAP

NEMITZ, W. C. and WHALEY, T. P.
71 Varieties of implicative semilattices, *Pacific J. Math.* **37** (1971), 759—769. *MR* **47**#84 RS

NEUMANN, B. H.
54 An essay on free products of groups with amalgamations, *Philos. Trans. Roy. Soc. London Ser. A* **246** (1954), 503—554. *MR* **16**—10 SAP AP

60a On amalgams of periodic groups, *Proc. Roy. Soc. London Ser. A* **255** (1960), 447—489. *MR* **22** # 4758 SAP
60b Permutational products of groups, *J. Austral. Math. Soc.* **1** (1959/60), 299—310. *MR* **23** # A922 SAP

NEUMANN, B. H. and NEUMANN, H.
50 A remark on generalized free products, *J. London Math. Soc.* **25** (1950), 202—204. *MR* **12**—76 SAP
53 A contribution to the embedding theory of group amalgams, *Proc. London Math. Soc.* (3) **3** (1953), 243—256. *MR* **15**—286 SAP

NEUMANN, B. H. and WIEGOLD, J.
62 On certain embeddability criteria for group amalgams, *Publ. Math. Debrecen* **9** (1962), 57—64. *MR* **26** # 3763 SAP

NEUMANN, H.
48 Generalized free products with amalgamated subgroups, *Amer. J. Math.* **70** (1948), 590—625. *MR* **10**—233 SAP
50 Generalized free sums of cyclical groups, *Amer. J. Math.* **72** (1950), 671—685. *MR* **12**—478 SAP
51 On an amalgam of abelian groups, *J. London Math. Soc.* **26** (1951), 228—232. *MR* **12**—800 SAP

NIEDERLE, J.
74 Injektive Objekte in gewissen Kategorien der Verbände, *Arch. Math. (Brno)* **10** (1974), 63—65. *MR* **51** # 12639 EI

NORTHCOTT, D. G.
74 Injective envelopes and inverse polynomials, *J. London Math. Soc.* (2) **8** (1974), 290—296. *MR* **50** # 13003 EIH

NUMMELA, E. C.
78 On epimorphisms of topological groups, *General Topology Appl.* **9** (1978), 155—167. *MR* **58** # 11204 ES

OLIVIER, J.-P.
68 Anneaux absolument plats universels et épimorphismes d'anneaux, *C. R. Acad. Sci. Paris Sér. A—B* **266** (1968), A317—A318. *MR* **39** # 197d ES

ORDMAN, E. T.
74 Free products of topological groups with equal uniformities, I, *Colloq. Math.* **31** (1974), 37—43. *MR* **51** # 10519 SAP

OSWALD, A.
77 A note on injective modules over a d.g. near-ring, *Canad. Math. Bull.* **20** (1977), 267—269. *MR* **57** # 3200 EI

PAREIGIS, B. and SWEEDLER, M. E.
70 On generators and cogenerators, *Manuscripta Math.* **2** (1970), 49—66. *MR* **41** # 5447 ECS

PASTIJN, F.
82 Constructions of varieties that satisfy the amalgamation property or the congruence extension property, *Studia Sci. Math. Hungar.* **17** (to appear). AP CEP

PASZTOR, A.
77 A category-theoretical characterization of surjective homomorphisms of partial algebras, *Studia Sci. Math. Hungar.* **12** (1977), 251—256. *MR* **81m**: 08009 ES
82 The epis of POS(Z), *Comment. Math. Univ. Carolinae* **23** (1982), 285—299. ES
83a Epis of some categories of Z-continuous partial algebras, *Acta Cybernet.* **6** (1983), 111—123. ES
83b Surjections of complete posets and of continuous algebras, *Algebra Universalis* **17** (1983), 65—72. ES

PEDERSEN, F. D.
75 Epimorphisms in the category of abelian l-groups, *Proc. Amer. Math. Soc.* **53** (1975), 311—317. *MR* **58** # 28135 ES

PETRESCU, I.
71 Injective objects in the category of Morgan algebras, *Rev. Roumaine Math. Pures Appl.* **16** (1971), 921—926. *MR* **44** # 3940 EI

PHILLIPS, R. S.
40 On linear transformations, *Trans. Amer. Math. Soc.* **48** (1940), 516—541. *MR* **2**—318 EI

PIERCE, K. R.
72a Amalgamating abelian ordered groups, *Pacific J. Math.* **43** (1972), 711—723. *MR* **47** # 8389 AP
72b Amalgamations of lattice ordered groups, *Trans. Amer. Math. Soc.* **172** (1972), 249—260. *MR* **48** # 3835 AP
76 Amalgamated sums of abelian *l*-groups, *Pacific J. Math.* **65** (1976), 167—173. *MR* **55** # 2702 AP

PIERCE, R. S.
68 *Introduction to the theory of abstract algebras*, Holt, Rinehart and Winston, New York—Montreal, Que.—London, 1968. *MR* **37** # 2655 AP EI

PIGOZZI, D.
71 Amalgamation, congruence extension, and interpolation properties in algebras, *Algebra Universalis* **1** (1971/72), 269—349. *MR* **46** # 57 SAP AP CEP
81a On the structure of equationally complete varieties, I, *Colloq. Math.* **45** (1981), 191—201. AP RS
81b On the structure of equationally complete varieties, II, *Trans. Amer. Math. Soc.* **264** (1981), 301—319. AP RS

PITTS, A. M.
83 Amalgamation and interpolation in the category of Heyting algebras, *J. Pure Appl. Algebra* **29** (1983), 155—165. AP

PLATT, C. R.
71a One-to-one and onto in algebraic categories, *Algebra Universalis* **1** (1971/72), 117—124. *MR* **45** # 5196 ES
71b Example of a concrete category with no embedding into sets under which every epimorphism is represented by a surjection, *Algebra Universalis* **1** (1971), 128. *MR* **44** # 2685 ES

POGUNTKE, D.
70 Epimorphisms of compact groups are onto, *Proc. Amer. Math. Soc.* **26** (1970), 503—504. *MR* **41** # 8577 ES
73 Einige Eigenschaften des Darstellungsringes kompakter Gruppen, *Math. Z.* **130** (1973), 107—117. *MR* **49** # 3020 IPA

POLIN, S. V.
78 Injective envelopes in categories of functors, *Algebras and modules, Mat. Issled.* **48** (1978), 69—79 (in Russian). *MR* **58** # 5858 EIH

POPA, N.
65 Injective objects in the category of ordered Archimedean spaces with axial element, *Stud. Cerc. Mat.* **17** (1965), 1333—1336 (in Romanian). *MR* **34** # 8156 EI

POPESCU, N. and POPESCU, L.
79 *Theory of categories*, Editura Academiei, Bucharest; Sijthoff and Noordhoff International Publishers, Alphen aan den Rijn, 1979. *MR* **80i**: 18001 ECS EIH

PORST, H.-E.
81 Characterization of injective envelopes, *Cahiers Topologie Géom. Différentielle* **22** (1981), 399—406. EIH

POTHOVEN, K. L.
69 Projective and injective objects in the category of Banach spaces, *Proc. Amer. Math. Soc.* **22** (1969), 437—438. *MR* **39** # 3291 EI

PRATHER, R. E.
72 Epimorphisms in certain categories of transducers, *Information and Control* **21** (1972), 301—305. *MR* **51** # 10434 ES
76 Epimorphisms of free monoids, *Linear Algebra Appl.* **13** (1976), 201—205. *MR* **53** # 577 ES

PREHN, R.
79 Injektive Gruppen über Fastringen, *Publ. Math. Debrecen* **26** (1979), 75—90. *MR* **81c**: 16052 EI EIH

PRELLER, A.
69 Interpolation et amalgamation, *Publ. Dép. Math. (Lyon)* **6** (1969), fasc. 1, 49—65. *MR* **39**#6743 AP
PRESTON, G. B.
73 Inverse semigroups: some open questions, *Proceedings of a Symposium on Inverse Semigroups and their Generalisations* (Northern Illinois Univ., De-Kalb, Ill., 1973), Northern Illinois Univ., DeKalb, Ill., 1973, 122—139. *MR* **54**#2848 SAP
76 Free products amalgamating unitary subsemigroups, *Bull. Austral. Math. Soc.* **15** (1976), 117—124. *MR* **54**#12936 SAP
PULTR, A.
82 Subdirect irreducibility and congruences, *Category Theory. Applications to Algebra, Logic and Topology* (Proc. Conf. Gummersbach, 1981), pp. 256—262. Lecture Notes in Math., 962, Springer, Berlin—Heidelberg—New York. RS
PULTR, A. and VINÁREK, J.
77 Productive classes and subdirect irreducibility, in particular for graphs, *Discrete Math.* **20** (1977/78), 159—176. *MR* **58**#5419 RS
QUACKENBUSH, R. W.
71 Equational classes generated by finite algebras, *Algebra Universalis* **1** (1971/72), 265—266. *MR* **45**#5041 RS
72 Free products of bounded distributive lattices, *Algebra Universalis* **2** (1972), 393—394. *MR* **47**#6574 EI
74a Semi-simple equational classes with distributive congruence lattices, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **17** (1974), 15—19. *MR* **52**# 3012 AP CEP EI
74b Structure theory for equational classes generated by quasi-primal algebras, *Trans. Amer. Math. Soc.* **187** (1974), 127—145. *MR* **48**#5961 AP
76 Varieties of Steiner loops and Steiner quasigroups, *Canad. J. Math.* **28** (1976), 1187—1198. *MR* **54**#12946 CEP RS EI
RADNEV, P.
76 Injective objects in the category of *p*-rings, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **31** (1976), no. 5, 49—55 (in Russian). *MR* **55**#2999 EI
RAGHAVAN, T. G. and REILLY, I. L.
79 On projective and injective objects in some topological categories, *Categorical Topology* (Proc. Internat. Conf., Free Univ. Berlin, Berlin, 1978), Lecture Notes in Math., Vol. 719, Springer, Berlin—Heidelberg—New York, 1979, 308—315. *MR* **81i**: 18004 EI
RAMER, A.
65 Some problems on universal spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 291—294. *MR* **31**#6205 ECS
RAPHAEL, R.
75 Injective rings, *Comm. Algebra* **1** (1974), 403—414. *MR* **50**#2247 EI
76 No rings have injective effacements, *J. Pure Appl. Algebra* **8** (1976), 285—288. *MR* **53**#10843 EIH
RAVEL, J.
67 Injectivité: Généralisations et applications, extensions essentielles dans un treillis de Johnson, *Publ. Dép. Math. (Lyon)* **4** (1967), fasc. 2, 1—73. *MR* **36**#5042 EIH
REID, G. A.
70 Epimorphisms and surjectivity, *Invent. Math.* **9** (1969/70), 295—307. *MR* **41**#5450 ES
REILLY, N. R.
73 Extensions of congruences and homomorphisms, *Proceedings of a Symposium on Inverse Semigroups and their Generalisations* (Northern Illinois Univ., DeKalb, Ill., 1973), Northern Illinois Univ., DeKalb, Ill., 1973, 140—166. *MR* **52**#5836 CEP
RIBENBOIM, P.
67 On ordered modules, *J. Reine Angew. Math.* **225** (1967), 120—146. *MR* **34**#5877 EI

RIBES, L.

71 On amalgamated products of profinite groups, *Math. Z.* **123** (1971), 357—364. *MR* **45**#389 — AP

RINGEL, C. M.

71 Monofunctors as reflectors, *Trans. Amer. Math. Soc.* **161** (1971), 293—306. *MR* **45**#1989 — AP EIH

72 The intersection property of amalgamations, *J. Pure Appl. Algebra* **2** (1972), 341—342. *MR* **48**#11244 — IPA

ROBERT, E. DE

69 Projectifs et injectifs relatifs. Applications, *C. R. Acad. Sci. Paris Sér. A—B* **268** (1969), A361—A364. *MR* **39**#5664 — EI

ROBY, N.

68a Sur les épimorphismes de la catégorie des anneaux, *C. R. Acad. Sci. Paris Sér. A—B* **266** (1968), A312—A313. *MR* **39**#197b — ES

68b Diverses caractérisations des épimorphismes, *Séminaire d'algèbre commutative P. Samuel* (1967/68), *Les épimorphismes d'anneaux*, Secrétariat math., Paris, 1968. *MR* **39**#6867 — ES

ROGGENKAMP, K. W.

73 Injective modules for group rings and Gorenstein orders, *J. Algebra* **24** (1973), 465—472. *MR* **47**#3453 — EI

ROMANOWSKA, A. and FREESE, R. S.

77 Subdirectly irreducible modular double p-algebras, *Houston J. Math.* **3** (1977), 109—112. *MR* **55**#7875 — RS

ROMANOWSKA, A. and TRACZYK, T.

82 Commutative BCK-algebras. Subdirectly irreducible algebras and varieties, *Math. Japon.* **27** (1982), 35—48. — RS

ROSENTHAL, H. P.

70 On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ, *Acta Math.* **124** (1970), 205—248. *MR* **41**#2370 — EI

SABBAGH, G.

71 Embedding problems for modules and rings with application to model-companions, *J. Algebra* **18** (1971), 390—403. *MR* **43**#6259 — AP

SAIN, I.

a Strong amalgamation and epimorphisms of cylindric algebras and Boolean algebras with operators (preprint). — SAP

SANDERS, D.

75 The dominion and separable subalgebras of finitely generated algebras, *Proc. Amer. Math. Soc.* **48** (1975), 1—7. *MR* **51**#586 — ES

SANKAPPANAVAR, H. P.

79 Principal congruences of pseudocomplemented semilattices and congruence extension property, *Proc. Amer. Math. Soc.* **73** (1979), 308—312. *MR* **80f**: 06003 — CEP

SANKAR, P. B. K.

70 Injective envelopes of modules, *Progr. Math. (Allahabad)* **4** (1970), no. 2, 19—20. *MR* **46**#5306 — EIH

SAPIR, M. V.

a Semigroup varieties possessing the amalgamation property, *Semigroups. Structure and Universal Algebraic Problems* (Proc. Conf. Szeged, 1981), Colloq. Math. Soc. J. Bolyai, Vol. 39, North-Holland, Amsterdam (to appear). — SAP

ŠATALOVA, M. A.

70 The absence of o-injective partially ordered modules, *Mat. Zametki* **7** (1970), 693—696 (in Russian). *MR* **42**#165 — EI

SCHEIBLICH, H. E.

76 On epics and dominions of bands, *Semigroup Forum* **13** (1976/77), 103—114. *MR* **55**#3136 — ES

78 On coproducts of rectangular bands, *Rocky Mountain J. Math.* **8** (1978), 539—545. *MR* **57**#16433 — SAP AP

SCHEIN, B. M.
66 Homomorphisms and subdirect decompositions of semigroups, *Pacific J. Math.* **17** (1966), 529—547. *MR* **33** # 5768 RS
74 Injectives in certain classes of semigroups, *Semigroup Forum* **9** (1974/75), 159—171. *MR* **51** # 13097 EI
76 Injective commutative semigroups, *Algebra Universalis* **6** (1976), 395—397. *MR* **55** # 8219 EI
79 Injective monars over inverse semigroups, *Algebraic Theory of Semigroups* (Proc. Sixth Algebraic Conf., Szeged, 1976), Colloq. Math. Soc. J. Bolyai, Vol. 20, North-Holland, Amsterdam, 1979, 519—544. *MR* **80g**: 20082 EI
81 On two papers of B. M. Schein, *Semigroup Forum* **23** (1981), 87—88. *MR* **82j**: 20118 EI

SCHOEMAN, M. J.
76 Injective properties of torsion-complete mixed groups, *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. Ia Mat. (N. S.)* **22** (1976), 11—13. *MR* **56** # 12145 EI

SCHREIER, O.
27 Die Untergruppen der freien Gruppen, *Abh. Math. Sem. Univ. Hamburg* **5** (1927), 161—183. SAP

SCHRÖDER, J.
74 Epi und extremer Mono in T_{2a}, *Arch. Math. (Basel)* **25** (1974), 561—565. *MR* **52** # 6648 ES

SCHUBERT, H.
72 *Categories,* Springer-Verlag, Berlin—Heidelberg—New York, 1972. *MR* **50** # 2286 ECS EI EIH

SCHWAB, E.
79 Über Kongruenzen in regulären Kategorien, *Algebraische Modelle, Kategorien und Gruppoide, Studien zur Algebra und ihre Anwendungen,* **7**, Akademie-Verlag, Berlin, 1979, 45—50. *MR* **81g**: 18004 ES

SCOTT, D. S.
72 Continuous lattices, *Toposes, Algebraic Geometry and Logic* (Conf., Dalhousie Univ., Halifax, N. S., 1971), Lecture Notes in Math., Vol. 274, Springer, Berlin—Heidelberg—New York, 1972, 97—136. *MR* **53** # 7879 ECS EI

ŠEĬNBERG, M. V.
71 Relatively injective modules over Banach algebras, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **26** (1971), no. 3, 53—58 (in Russian). *MR* **43** # 6729 EI

SEMADENI, Z.
63a Free and direct objects, *Bull. Amer. Math. Soc.* **69** (1963), 63—66. *MR* **25** # 5020 ECS EI
63b Projectivity, injectivity and duality, *Rozprawy Mat.* **35** (1963), 47 pp. *MR* **27** # 4776 EI ECS

SETH, V. and TEWARI, K.
74 On injective near-ring modules, *Canad. Math. Bull.* **17** (1974), 137—141. *MR* **50** # 4670 EI

SHOCK, R. C.
71 Injectivity, annihilators and orders, *J. Algebra* **19** (1971), 96—103. *MR* **43** # 4861 EIH

SHOJI, K.
80 Right self-injective semigroups are absolutely closed, *Mem. Fac. Sci. Shimane Univ.* **14** (1980), 35—39. *MR* **82e**: 20082b IPA

SIKORSKI, R.
48 A theorem on extension of homomorphisms, *Ann. Soc. Polon. Math.* **21** (1948), 332—335. *MR* **11**—76 CEP
50 On an analogy between measures and homomorphisms, *Ann. Soc. Polon. Math.* **23** (1950), 1—20. *MR* **12**—583 CEP

SIKORSKI, R. and TRACZYK, T.
63 On free products of $\mathfrak{m}$-distributive Boolean algebras, *Colloq. Math.* **11** (1963), 13—16. *MR* **28** # 2064 CEP

SILVER, L.
67 Noncommutative localizations and applications, *J. Algebra* **7** (1967), 44—76. *MR* **36** # 205 EI

SIMMONS, H.
72 Existentially closed structures, *J. Symbolic Logic* **37** (1972), 293—310. *MR* **51** # 12518 EI AP
74 The use of injective-like structures in model theory, *Compositio Math.* **28** (1974), 113—142. *MR* **50** # 4289 AP EI
ŠIRJAEV, V. M.
77 Some properties of the category of inverse semigroups, *Izv. Vysš. Učebn. Zaved. Matematika* **1977**, no. 5, 125—136 (in Russian). *MR* **58** # 16832 ES
SKORNJAKOV, L. A.
69 The homological classification of monoids, *Sibirsk. Mat. Ž.* **10** (1969), 1139—1143 (in Russian). *MR* **40** # 5764 EI
SLAVÍK, V.
80 The amalgamation property of varieties determined by primitive lattices, *Comment. Math. Univ. Carolinae* **21** (1980), 473—478. *MR* **82b**: 06007 AP
83 A note on the amalgamation property in lattice varieties, *Contributions to Lattice Theory* (Proc. Conf. Szeged, 1980), Colloq. Math. Soc. J. Bolyai, Vol. 33, North-Holland, Amsterdam, 1983, 723—736. AP
SMITH THOMAS, B. V.
77 Categories of topological groups, *Quaestiones Math.* **2** (1977), 355—377. AP ES
STEINBERG, S.
72 Lattice-ordered injective hulls, *Trans. Amer. Math. Soc.* **169** (1972), 365—388. *MR* **47** # 1713 EIH
STENSTRÖM, B.
75 *Rings of Quotients*, Springer, Berlin, 1975. EI EIH
STEPHANI, I.
70 Injektive Operatorenideale über der Gesamtheit aller Banachräume und ihre topologische Erzeugung, *Studia Math.* **38** (1970), 105—124. *MR* **43** # 6738 EI
STORRER, H. H.
68a Sur les épimorphismes dans la catégorie des anneaux commutatifs, *C. R. Acad. Sci. Paris Sér. A—B* **266** (1968), A263—A265. *MR* **37** # 5272 ES
68b Epimorphismes d'anneaux et anneaux des fractions, *C. R. Acad. Sci. Paris Sér. A—B* **266** (1968), A322—A323. *MR* **37** # 5273 ES
68c Epimorphismen von kommutativen Ringen, *Comment Math. Helv.* **43** (1968), 378—401. *MR* **39** # 4137 ES
69 A note on quasi-Frobenius rings and ring epimorphisms, *Canad. Math. Bull.* **12** (1969), 287—292. *MR* **40** # 4306 ES
73 Epimorphic extensions of non-commutative rings, *Comment. Math. Helv.* **48** (1973), 72—86. *MR* **48** # 342 ES
STRALKA, A. R.
71 The congruence extension property for compact topological lattices, *Pacific J. Math.* **38** (1971), 795—802. *MR* **46** # 3394 CEP
72 Extending congruences on semigroups, *Trans. Amer. Math. Soc.* **166** (1972), 147—161. *MR* **45** # 3627 CEP
77 Congruence extension and amalgamation in $\mathcal{CL}$, *Semigroup Forum* **13** (1976/77), 355—375. *MR* **58** # 10640 CEP AP
SWAMINATHAN, V.
82 Injective and projective Boolean-like rings, *J. Austral. Math. Soc. Ser. A* **33** (1982), 40—49. CEP EI ES
SYDOW, W.
81 Über die Kategorie der topologischen Vektorräume, Thesis, Hagen, 1981. ECS EI
TANG, C. Y.
66 An existence theorem for generalized direct products with amalgamated subgroups, *Canad. J. Math.* **18** (1966), 75—82. *MR* **32** # 4181 AP SAP
TAYLOR, W.
71 Some constructions of compact algebras, *Ann. Math. Logic* **3** (1971), 395—435. *MR* **45** # 47 EIH
72 Residually small varieties, *Algebra Universalis* **2** (1972), 33—63. *MR* **47** # 3278 AP CEP RS ECS EI EIH

73 Products of absolute retracts, *Algebra Universalis* **3** (1973), 400—401. *MR* **50** # 2031 EI

79 Subdirectly irreducible algebras in regular permutable varieties, *Proc. Amer. Math. Soc.* **75** (1979), 196—200. *MR* **80h**: 08010 RS

THOLEN, W.

81 Injective objects and cogenerating sets, *J. Algebra* **73** (1981), 139—155. RS ECS EI EIH

82a Amalgamations in categories, *Algebra Universalis* **14** (1982), 391—397. AP IPA

82b Birkhoff's Theorem for categories, *Categorical Aspects of Topology and Analysis* (Proc. Conf. Ottawa, 1980), pp. 351—357. Lecture Notes in Math., 915, Springer, Berlin—Heidelberg—New York, 1982. RS ECS EI EIH

TIWARY, A. K.

69 Injective hulls of semi-simple modules over regular rings, *Pacific J. Math.* **31** (1969), 247—255. *MR* **41** # 6834 EIH

TONGE, A. M.

78 Translation invariance, compact semigroups and injective coalgebras, *Math. Proc. Cambridge Philos. Soc.* **83** (1978), 225—236. *MR* **56** # 15822 EI

80 La presque-périodicité et les coalgèbres injectives, *Studia Math.* **67** (1980), 103—118. *MR* **81j**: 46075 EI

TRNKOVÁ, V.

65 Sum of categories with amalgamated subcategory, *Comment. Math. Univ. Carolinae* **6** (1965), 449—474. *MR* **32** # 7622 AP

URQUHART, A.

79 Distributive lattices with a dual homomorphic operation, *Studia Logica* **38** (1979), 201—209. *MR* **80i**: 03071 RS

VIÑA ESCALAR, A.

71 *Extensiónes esenciales* [Essential extensions], Alxebra, 8, Departamento de Algebra y Fundamentos, Universidad de Santiago de Compostela, Santiago de Compostela, 1971 (in Spanish). *MR* **47** # 1906 EI EIH

VINÁREK, J.

78 Remarks on subdirect representations in categories, *Comment. Math. Univ. Carolinae* **19** (1978), 63—70. *MR* **58** # 826 ECS RS

82 On subdirect irreducibility and its variants, *Czechoslovak Math. J.* **32** (1982), 116—128. RS

VINCENT-SMITH, G. F.

67 The Hahn-Banach theorem for modules, *Proc. London Math. Soc.* (3) **17** (1967), 72—90. *MR* **35** # 766 EI

VIPERA, M. C.

81 Alcune classi di epimorfizmi d'anelli, *Ann. Univ. Ferrara Sez. VII. Sci. Mat.* **27** (1981), 125—133. ES

VIŠNJAKOVA, N. I. and KLADOV, G. K.

74 The injectivity of modules over a generalized valuation ring, *Numerical Mathematics and Computer Technology*, no. 5, Akad. Nauk Ukrain. SSR Fiz.-Tehn. Inst. Nizkih Temperatur, Kharkov, 1974, 138—140 (in Russian). *MR* **56** # 11982 EI

VISONHALER, C. and WICKLESS, W. J.

79 Injective hulls of torsion free Abelian groups as modules over their endomorphism rings, *J. Algebra* **58** (1979), 64—65. EIH

VOLKOV, M. V.

83 Varieties of associative rings in which amalgams can be embedded, *Mat. Zametki* **33** (1983), 3—13 (in Russian). SAP

WASSERMANN, S.

77 Injective W^*-algebras, *Math. Proc. Cambridge Philos. Soc.* **82** (1977), 39—47. *MR* **56** # 6418 EI

WATANABE, T. and NOGUCHI, S.

77 The amalgamation of automata, *J. Comput. System Sci.* **15** (1977), 1—16. *MR* **56** # 8239 SAP

WĘGLORZ, B.

66 Equationally compact algebras, I, *Fund. Math.* **59** (1966), 289—298. *MR* **35** # 1462 EAR EI

WEIDENFELD, G.
70 Objets injectifs et catégories de fractions, Thèse présentée à la Faculté des Sciences de l'Université de Paris pour l'obtention du doctorat 3e cycle, Spécialité: mathématiques, *Esquisses mathématiques,* **3**, Exp. No. 1, Fac. Sci. Univ. Paris, Paris, 1970. *MR* **54**#7581 EI

WEINBERG, E. C.
77 Relative injectives, *Symposia Mathematica,* Vol. XXI (Convegno sulle Misure su Gruppi e su Spazi Vettoriali, Convegno sui Gruppi e Anelli Ordinati, INDAM, Rome, 1975), Academic Press, London, 1977, 555—564. *MR* **58**#16457 EI CEP

WEINERT, H. J.
80 *S*-sets and semigroups of quotients, *Semigroup Forum* **19** (1980), 1—78. *MR* **81d**: 20062 EI EIH

WERNER, H.
78 *Discriminator-Algebras. Algebrai crepresentation and model theoretic properties,* Studien zur Algebra und Ihre Anwendungen, **6**, Akademie-Verlag, Berlin, 1978. *MR* **80f**: 08009 AP CEP RS EI

WHEELER, W. H.
79 Amalgamation and elimination of quantifiers for theories of fields, *Proc. Amer. Math. Soc.* **77** (1979), 243—250. *MR* **80j**: 03049 AP

WIEGOLD, J.
59 Nilpotent products of groups with amalgamations, *Publ. Math. Debrecen* **6** (1959), 131—168. *MR* **21**#3478 SAP
61 Some remarks on generalised products of groups with amalgamations, *Math. Z.* **75** (1960/61), 57—78. *MR* **24**#A2611 SAP
62 Embedding group amalgams in wreath products, *Math. Z.* **80** (1962), 148—153. *MR* **26**#2488 SAP
65 Soluble embeddings of group amalgams. *Publ. Math. Debrecen* **12** (1965), 227—230. *MR* **33**#195 SAP

WOLFE, J. C.
74 Injective Banach spaces of type $C(T)$, *Israel J. Math.* **18** (1974), 133—140. *MR* **50**#5445 EI
78 Injective Banach spaces of continuous functions, *Trans. Amer. Math. Soc.* **235** (1978), 115—139. *MR* **57**#1098 EI

WOOD, C.
73 The model theory of differential fields of characteristic $p\neq 0$, *Proc. Amer. Math. Soc.* **40** (1973), 577—584. *MR* **48**#8227 AP

WYLER, O.
71 On the categories of general topology and topological algebra, *Arch. Math. (Basel)* **22** (1971), 7—17. *MR* **44**#4767 ES
77 Injective spaces and essential extensions in TOP, *General Topology Appl.* **7** (1977), 247—249. *MR* **56**#16559 EI EIH

YAQUB, F. M.
68 A theorem on the existence of the generalized free α-products of Boolean algebras, *Nieuw Arch. Wisk.* (3) **16** (1968), 179—183. *MR* **39**#112 AP
72 Free products of commutative rings with amalgamation, *Rend. Ist. Mat. Univ. Trieste* **4** (1972), 46—52. *MR* **48**#10935 SAP

YASUHARA, M.
74 The amalgamation property, the universal-homogeneous models, and the generic models, *Math. Scand.* **34** (1974), 5—36. *MR* **51**#7860 AP

ZAHAROV, V. K.
73 The functional representation of the injective envelope, and tests for the injectivity of certain modules, *Izv. Vysš. Učebn. Zaved. Matematika* **1973**, no. 9 (136), 27—30 (in Russian). *MR* **48**#6174 EIH EI

ZDRAVKOVSKA, S.
79 On injective and projective objects in pro-Abelian categories, *Glas. Mat.* **14** (1979), 223—226. EI

ZEITLER, H.
76 Modeltheoretic investigations on the amalgamation property, Thesis, Ruprecht-Karl-Univ., Heidelberg, 1976. AP SAP EI CEP

81 Die Amalgamierungseigenschaft und reine Ringe, *Arch. Math. Logik Grundlag.* **21** (1981), 83—100. AP

ZELMANOWITZ, J. M.
71 Injective hulls of torsion free modules, *Canad. J. Math.* **23** (1971), 1094—1101. *MR* **44**#6755 EIH

ZIMMERMANN, W.
69 Injektive Strukturen und *M*-injektive Objekte, Thesis, Ludwig-Maximilians-Univ., München, 1969. EI EIH

(Received February 4, 1982)

E. W. Kiss and L. Márki

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

P. Pröhle

HO SI MINH TANÁRKÉPZŐ FŐISKOLA
MATEMATIKAI TANSZÉK
TEJÚT U. 10—12
H—1214 BUDAPEST
HUNGARY

W. Tholen

FACHBEREICH MATHEMATIK
FERNUNIVERSITÄT HAGEN
FEDERAL REPUBLIC OF GERMANY

Current address:

DEPARTMENT OF MATHEMATICS
YORK UNIVERSITY
DOWNSVIEW, ONTARIO
M3J 1P3
CANADA

PRINTED IN HUNGARY
Szegedi Nyomda, Szeged

RECENTLY ACCEPTED PAPERS

WEGNER, G., Über endliche Kreispackungen in der Ebene
KARSAI, J., On the global asymptotic stability of the zero solution of the equation $\ddot{x}+g(t, x, \dot{x})x+f(x)=0$
GAÁL, I., Norm form equations with several dominating variables and explicit lower bounds for inhomogeneous linear forms with algebraic coefficients
BÖRÖCZKY, K., The Tammes problem for $n=11$
CSÁKÁNY, B., Selected algebras and compatible varieties
VARMA, A. K., On optimal quadrature formulae
WEGNER, G., Extremale Groemerpackungen
ZNÁM, Š. Minimal graphs of diameter two
ELBERT, Á., Asymptotic behaviour of autonomous half-linear systems on the plane
FIALOWSKI, A., Deformations of nilpotent Kac—Moody algebras
XEKALAKI, E., Some bivariate extensions of the generalized Waring distribution
IMHOF, J. P., On Brownian bridge and excursion
MATOLCSY, K., An embedding theorem for ordered topological spaces
BIHARI, I., An asymptotic statement concerning the solutions of the equation $x''+a(t)x=0$
BIHARI, I., Note to an extension of a Sturmian comparison theorem
KELLY, D., Complete rules of inference for universal sentences
AHMAD, M., Estimation of the parameters of Burr distribution based on order statistics
KHARE, S. S., Reduction of equivariant bordism groups
SEOH, M. and PURI, M. L., Berry—Esséen theorems for signed linear rank statistics under near location alternatives
FEJES TÓTH, L., Packing of homothetic discs of n different sizes
CHIANG, C.-Y. and PURI, M. L., Tests of subhypotheses in linear regression based on rank-order statistics
FLORIAN, A. and GROEMER, H., Two remarks on the permeability of layers of convex bodies
VEIDINGER, L., On the order of convergence of a finite element method for the biharmonic equation
MARCUS F., Sur les surfaces à groupes continus G_2 de similitude projective en elles-mêmes et sur les surfaces complexes
KISS, P., Differences of the terms of linear recurrences
MILLER, H. I. and XENIKAKIS, P. J., Some results on the Cantor set
ALPÁR, L., On the change of variable of absolutely convergent power and Laurent series on the unit circle
PŁONKA, J., On the sum of a 2-semilattice ordered system of algebras
SERESS, Á., Gossips by conference calls
POYATOS, F., Archimedean decompositions of left A-semimodules and semirings
PERELLI, A. and SALERNO, S., On $2k$-dimensional density theorems
EIGEN, S. J., Putting convergent sequences into measurable sets
SIMON, L., On approximation of solutions of exterior boundary value problems
FIALOWSKI, A., On the deformations of L_1
PANNY, W. and PRODINGER, H., The expected height of paths for several notions of heights
BÖRÖCZKY, K., Closest packing and loosest covering of the space with balls
STEIN, S., Lattice-tiling by certain star bodies
BELL, H. E., On commutativity of quasi-commutative rings
GUT, A., On the law of the iterated logarithm for randomly indexed partial sums with two applications
BEAZER, R., Congruence uniform algebras with pseudocomplementation
HUIJSMANS, C. B., An inequality in complex Riesz algebras

CONTENTS

HU ISSN 0081—6906

Index: 26 748

Studia Scientiarum Mathematicarum Hungarica

VOLUME 18
NUMBERS 2—4
1983

AKADÉMIAI KIADÓ, BUDAPEST

STUDIA SCIENTIARUM MATHEMATICARUM HUNGARICA

A QUARTERLY OF THE HUNGARIAN ACADEMY OF SCIENCES

Studia Scientiarum Mathematicarum Hungarica publishes original papers on mathematics mainly in English, but also in German, French and Russian.

Studia Scientiarum Mathematicarum Hungarica is published in yearly volumes of four issues (mostly double numbers published semiannually) by

AKADÉMIAI KIADÓ
Publishing House of the Hungarian Academy of Sciences
H-1054 Budapest, Alkotmány u. 21.

Manuscripts and editorial correspondence should be addressed to

J. Merza
Managing Editor
P.O. Box 127
H—1364 Budapest

Subscription information

Orders should be addressed to

KULTURA Foreign Trading Company
P.O. Box 149
H—1389 Budapest

or to its representatives abroad.

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 143—152.

О ЧАСТИЧНОЙ УСТОЙЧИВОСТИ РЕШЕНИЙ ЛИНЕЙНЫХ АВТОНОМНЫХ ДИФФЕРЕНЦИАЛЬНО—ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ

J. TERJÉKI

1. Введение

В разных областях наук (теория автоматического регулирования, некоторые области биологии и т. п.) надо изучать такие системы и процессы, математическая модель которых является дифференциальным уравнением с отклоняющимся аргументом, или по другой терминологии, дифференциально-функциональным уравнением [10]. При исследовании практических проблем нужно знать, что данное решение устойчиво ли или частично устойчиво ли [9]. Вопрос о частичной устойчивости для обыкновенных дифференциальных уравнений поставил А. М. Ляпунов [3]. Точные определения дал В. В. Румянцев, он же получил первые результаты с помощью функцией Ляпунова [4, 5]. Для дифференциально-функциональных уравнений определения частичной устойчивости формулировал К. Кордуняну. Он доказал также и то, что, при некоторых условиях, из частичной асимптотической устойчивости тривиального решения линейной части данного уравнения следует частичная асимптотическая устойчивость тривиального решения исходного уравнения [8].

Таким образом особенно подчеркивается важность задачи о том, что при каких условиях будет частично устойчиво (асимптотически устойчиво) тривиальное решение линейного дифференциального уравнения. Эта задача решена только в специальных случаях. А. В. Луценко и. Л. В. Стадникова получили необходимые и достаточные условия частичной асимптотической устойчивости тривиального решения *обыкновенных* автономных уравнений. Ими же исследован вопрос о частичной устойчивости: получены и необходимые и достаточные условия [2]. В. И. Воротников и В. П. Прокопьев задачу о частичной асимптотической устойчивости свели к задаче об асимптотической устойчивости по всем компонентам тривиального решения некоторого другого уравнения [1].

В этом статье мы исследуем вышеупомянутую задачу о частичной устойчивости и о частичной асимптотической устойчивости и решим ее в случае автономного линейного *дифференциально-функционального* уравнения. Наш результат о частичной устойчивости является новым и для *обыкновенных* дифференциальных уравнений.

1980 *Mathematics Subject Classification*. Primary 34K20; Secondary 34D20.
Key words and phrases. Functional differential equations, characteristic equation, partial stability.

2. Обозначения, определения, вспомогательные утверждения

Рассмотрим дифференциально-функциональное уравнение

$$(2.1) \qquad \dot{x}(t)=\int_{-h}^{0} d\eta(s)\,x(t+s),$$

где h-положительная постоянная, $\eta(s)$-матричная функция типа $n\times n$ с элементами ограниченной вариации на интервале $[-h, 0]$. Введем — как обычно — банахово пространство $C=C([-h, 0]; \mathbf{C}^n)$, с нормой $\|\varphi\|=\sup\{|\varphi(s)|: -h\leqq s\leqq 0\}$. Здесь $\mathbf{C}^n$ — комплексное линейное пространство с произвольной нормой $|\cdot|$, $\mathbf{C}=\mathbf{C}^1$ множество комплексных чисел. Для любого $\varphi\in C$ обозначим через $x(t, \varphi)$ решение (2.1), проходящее через $(0, \varphi)$, т. е. непрерывную при $t\in[-h, \infty)$ функцию $x(t, \varphi)$, дифференцируемую при $t\in(0, \infty)$ и удовлетворяющую уравнению (2.1) на $(0, \infty)$. Используя теоремы существования и единственности для (2.1), легко увидеть, что совокупность решений уравнения (2.1) образует линейное пространство (принадлежащее $C([-h, \infty); \mathbf{C}^n)$). В дальнейшем используем еще обозначения

$$\mathbf{N}=\{1, 2, \dots\} \quad \text{и} \quad \mathbf{N}_0=\mathbf{N}\cup\{0\}.$$

Пусть P — матрица размеров $n\times n$.

Определение 1. Нулевое решение или тривиальное решение уравнения (2.1) будем называть *P-устойчивым*, если для любого $\varepsilon>0$ существует $\delta=\delta(\varepsilon)>0$ такое, что из $\varphi\in C$ и $\|\varphi\|<\delta$ следует неравенство $|Px(t, \varphi)|<\varepsilon$ для всех $t\geqq 0$.

Определение 2. Тривиальное решение уравнения (2.1) будем называть *асимптотически P-устойчивым*, если оно устойчиво и кроме того $|Px(t, \varphi)|\to 0$ при $t\to\infty$ для любого $\varphi\in C$.

Эти определения только формально различаются от определений К. Кордуняну [8]. Поскольку легко увидеть, что если подвергднуть (2.1) подходящему линейному преобразованию, то P-устойчивость (асимптотическая P-устойчивость) редуцируется к устойчивости (к асимптотической устойчивости) относительно первых k компонент решений $(1\leqq k\leqq n)$, или, что равносильно, можно допустить, что $P=\operatorname{diag}(\underbrace{1, \dots, 1}_{k\ \text{раз}}, 0, \dots, 0)$. В дальнейшем — если противное не оговорено — P будет обозначать произвольную матрицу.

Асимптотическое поведение решений уравнения (2.1) характеризуется решениями вида $e^{\lambda t}p(t)$, где $p(t)$ векторный полином от t. Легко можно убедится в том, что (2.1) имеет нетривиальное решение такого вида тогда и только тогда, когда $\det\left(\lambda I-\int_{h}^{0} e^{\lambda s}d\eta(s)\right)=0$, где I — единичная матрица размеров $n\times n$. Структура решений вида $e^{\lambda t}p(t)$ подробно изучена в работах [10—13]. В дальнейшем мы дадим краткий обзор тех результатов этих работ, которые необходимы для нас в этой статьи.

Пусть $\Delta(\lambda)=\lambda I-\int_h^0 e^{\lambda s}d\eta(s)$. Введем обозначения

$$\Lambda_+ = \{\lambda\in \mathbf{C}:\ \det \Delta(\lambda) = 0,\ \mathrm{Re}\,\lambda > 0\},$$

$$\Lambda_0 = \{\lambda\in \mathbf{C}:\ \det \Delta(\lambda) = 0,\ \mathrm{Re}\,\lambda = 0\}$$

и наконец пусть $\Lambda=\Lambda_+\cup\Lambda_0$. Для любых $\lambda\in\Lambda$ и $k\in\mathbf{N}$ рассмотрим матрицу размеров $kn\times kn$

$$H_k = \begin{vmatrix} \Delta(\lambda) & \frac{\Delta'(\lambda)}{1!} \dots \frac{\Delta^{(k-1)}(\lambda)}{(k-1)!} \\ 0 & \Delta(\lambda) \dots \frac{\Delta^{(k-2)}(\lambda)}{(k-2)!} \\ . & . \quad \dots \quad . \\ 0 & 0 \quad \dots \quad \Delta(\lambda) \end{vmatrix}.$$

Пусть $\sigma_k=nk-\mathrm{rank}\,H_k$ и определим величины $\alpha_0=n$, $\alpha_1=\sigma_1$ и $\alpha_j=\sigma_j-\sigma_{j-1}$ для всех $j=2, 3, \dots$. Известно [см. 11, 12], что существуют такие $r\in\mathbf{N}$, и величины $d_1, \dots, d_r\in\mathbf{N}_0$, что $0\leqq d_1<\dots<d_r$ и

$$\alpha_0=\dots=\alpha_{d_1}>\alpha_{d_1+1}=\dots=\alpha_{d_2+1}=\dots=\alpha_{d_r}>\alpha_{d_r+1}=0.$$

Пусть $m_j=\alpha_{d_j}-\alpha_{d_j+1}$ для всех $j=1, \dots, r$ и $m=d_1m_1+\dots+d_rm_r$. Заметим, что m представляет собой кратность корня λ, $m=\mathrm{rank}\,H_{d_r+N}$ для всех $N\in\mathbf{N}_0$ и $n=m_1+\dots+m_r$.

Для любого $\lambda\in\Lambda$ совокупность решений типа $e^{\lambda t}p(t)$ образует подпространство линейного пространства решений уравнения (2.1). Это подпространство—обозначим его через S_λ—представляется в виде

$$S_\lambda=\bigoplus_{\varrho=1}^{r}\bigoplus_{\mu=1}^{m_\varrho} S_{d_\varrho,\mu},$$

где $S_{d_\varrho,\mu}$ линейное пространство, $S_{d_\varrho,\mu}\subset C([-h,\infty);\mathbf{C}^n)$. В пространстве $S_{d_\varrho,\mu}$ векторные функции

$$e^{\lambda t}\sum_{j=1}^{d_\varrho-k}\frac{\gamma_{\mu,j+k}t^{j-1}}{(j-1)!}$$

при $k=0, 1, \dots, d_\varrho-1$ образуют базис, а принадлежащие $\ker H_{d_\varrho}$ векторы $\gamma_\mu=\mathrm{col}\,(\gamma_{\mu,1}, \dots, \gamma_{\mu,d_\varrho})$ при $\mu=1, \dots, m_\varrho$ линейно независимы и $\gamma_{\mu,d_\varrho}\neq 0$.

Заметим еще [см. 12, 13], что если $N\in\mathbf{N}_0$, $\lambda\in\Lambda$ и

$$\mathrm{col}\,(\gamma_1, \dots, \gamma_{d_\varrho+N})\in\ker H_{d_\varrho+N}, \quad \text{то} \quad e^{\lambda t}\sum_{j=1}^{d_\varrho+N}\frac{\gamma_j t^{j-1}}{(j-1)!}\in S_\lambda.$$

Мы знаем [см. 10, 12], что существует ограниченный линейный оператор Π: $C\to C$, и существуют такие положительные постоянные M и γ, что для любого $\varphi\in C$ выполняется следующее $x(\cdot,\Pi\varphi)\in\bigoplus_{\lambda\in\Lambda} S_\lambda$ и $|x(t,\varphi)-\Pi\varphi|\leqq$ $\leqq M\|\varphi\|e^{-\gamma t}$ при $t\geqq 0$.

3. Критерий P-устойчивости и асимптотической P-устойчивости

Определим при $k \in \mathbf{N}$ матрицы размеров $kn \times kn$:

$$P_k = \operatorname{diag} \underbrace{(P, P, \ldots, P)}_{k \text{ раз}},$$

$$Q_k = \operatorname{diag} (0, \underbrace{P, \ldots, P}_{k-1 \text{ раз}})$$

и наконец при каждом $\lambda \in \Lambda$ выберем число $N \in \mathbf{N}_0$, которое может зависеть от λ.

Теорема 3.1. а) *Тривиальное решение уравнения* (2.1) *P-устойчиво тогда и только тогда, когда для всех* $\lambda \in \Lambda_+$

$$\operatorname{rank} \operatorname{col} (P_{d_r+N}, H_{d_r+N}) = \operatorname{rank} H_{d_r+N}$$

и для $\lambda \in \Lambda_0$

$$\operatorname{rank} \operatorname{col} (Q_{d_r+N}, H_{d_r+N}) = \operatorname{rank} H_{d_r+N}.$$

б) *Тривиальное решение уравнения* (2.1) *асимптотически P-устойчиво тогда и только тогда, если для всех* $\lambda \in \Lambda$

$$\operatorname{rank} \operatorname{col} (P_{d_r+N}, H_{d_r+N}) = \operatorname{rank} H_{d_r+N}.$$

Доказательство. Мы докажем только P-устойчивость. Критерий асимптотической P-устойчивости можно получить подобным образом.

Из сказанного в части 2 следует, что если тривиальное решение уравнения (2.1) P-устойчиво, то $PS_\lambda = 0$ для $\lambda \in \Lambda_+$, а в случае $\lambda \in \Lambda_0$ функция $Px(t)$ ограничена на $[-h, \infty)$ для всех решений $x(t)$, принадлежащих S_λ. Нетрудно можно показать, что эти условия не только необходимы, но и достаточны. Действительно, в силу неравенства

$$|Px(t, \varphi)| \leqq |Px(t, \Pi\varphi)| + |Px(t, \varphi - \Pi\varphi)|$$

к семейству операторов $U_t \colon C \to \mathbf{C}^n$ при $t \geqq 0$, $U_t \varphi = Px(t, \Pi\varphi)$ применима теорема Банаха-Штейнгауза, откуда получим достаточность.

На основе структуры пространств S_λ полученный критерий можно перефразировать следующим образом:

Тривиальное решение уравнения (2.1) P-устойчиво тогда и только тогда, если

(i) из $\lambda \in \Lambda_+$, $N \in \mathbf{N}_0$ и $H_{N+d_r}\gamma = 0$, где $\gamma = \operatorname{col} (\gamma_1, \ldots, \gamma_{N+d_r})$ следует $P\gamma_i = 0$ для всех $i = 1, \ldots, N+d_r$.

(ii) из $\lambda \in \Lambda_0$, $N \in \mathbf{N}_0$ и $H_{N+d_r}\gamma = 0$, где $\gamma = \operatorname{col} (\lambda_1, \ldots, \gamma_{N+d_r})$ следует $P\gamma_j = 0$ для всех $j = 2, \ldots, N+d_r$.

Оставшаяся часть доказательства является простым следствием следующего утверждения:

Если A и B матрицы с одиноковым числом столбцов, то решения $Bx = 0$ являются решениям уравнения $Ax = 0$ в том и только в том случае, если $\operatorname{rank} \operatorname{col} (A, B) = \operatorname{rank} B$.

К доказательству этого утверждения заметим что из $Bx=0$ следует $Ax=0$ тогда и только тогда, если $\ker B = \ker \operatorname{col}(A, B)$ или, что равносильно, линейные уравнения $Bx=0$ и $\operatorname{col}(A, B)x=0$ — эквивалентны. Для выполнения последного необходимо и достаточно выполнение условия $\operatorname{rank} B = \operatorname{rank} \operatorname{col}(A, B)$, что хорошо известно в теории систем линейных уравнений.

Теорема доказана.

Замечание. В силу произвольности N для применения теоремы 3.1 не нужно точно вычислить величину d_r, а достаточно знать только ее оценку. Всегда можно выбирать N напр. так, чтобы выполнялось $N+d_r=m$. С увеличением числа N растут и размеры матрицы $\operatorname{col}(P_{d_r+N}, H_{d_r+N})$ и таким образом растет и вычислительная работа ее ранга.

4. О частичной устойчивости тривиального решения уравнений с запаздывающимся аргументом

В этой части мы применим теорему 3.1 к уравнению с запаздывающимся аргументом

$$\dot{x}(t) = Ax(t-h), \tag{4.1}$$

где A матрица порядка n, h положительная постоянная. В случае уравнения (4.1) $\Delta(\lambda)=\lambda I - Ae^{-\lambda h}$, откуда можно увидеть, что $\det \Delta(\lambda)=0$ выпольняется тогда и только тогда, когда $\mu=\lambda e^{\lambda h}$ является собственным числом матрица A. Тоже легко увидеть, что если $\lambda \neq -1/h$ и собственное число $\mu=\lambda e^{\lambda h}$ имеет кратность m, то λ является корнем кратности m уравнения $\det \Delta(\lambda)=0$.

Сейчас мы введем несколько обозначений и потом сформулируем лемму, характеризующую вещественную часть корней уравнения $\det \Delta(\lambda)=0$.

Пусть $N_\mu = \{\lambda \in \mathbf{C}\colon \lambda e^{\lambda h} = \mu\}$, где $\mu \in \mathbf{C}$,

$$\Gamma = \left\{z \in \mathbf{C}\colon \operatorname{Re} z \leqq 0, \operatorname{Im} z \geqq 0, |z| < \arg z - \frac{\pi}{2}\right\} \cup$$

$$\cup \left\{z \in \mathbf{C}\colon \operatorname{Re} z \leqq 0, \operatorname{Im} z < 0, |z| < -\arg z - \frac{\pi}{2}\right\}$$

и наконец обозначим через $\partial\Gamma$ границу множества Γ.

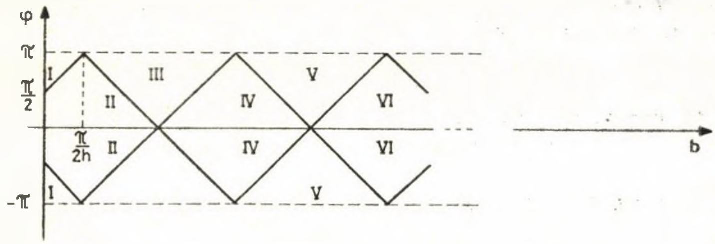
Лемма 4.1. *Каждый элемент множества N_μ имеет отрицательную вещественную часть тогда и только тогда, когда $h\mu \in \Gamma$. Если $h\mu \in \partial\Gamma$, то N_μ содержит элемент с нулевой вещественной частью, но не содержит элемент с положительной вещественной частью. Если $h\mu \notin \Gamma \cup \partial\Gamma$, то N_μ содержит элемент с положительной вещественной частью.*

Доказательство. Для случая $\mu=0$ утверждения леммы очевидны, поэтому в дальнейшем мы будем предпологать, что $\mu=be^{i\varphi}$ где $b>0$, т. е. наш квазиполином имеет вид $\Phi(z)=z-be^{-hz+i\varphi}$.

Применим так называемый метод D-разбиений см. [6], в силы которого надо разбить многообразие $b>0$, $-\pi<\varphi\leqq\pi\,(\mathrm{mod}\,2\pi)$ на области кривыми, точкам которых соответствуют квазиполиномы, имеющие хотя бы один нуль на мнимой оси. Легко вычислить, что в случае $\Phi(z)$ эти кривые:

$$(4.2)\qquad \varphi = hb+\frac{\pi}{2}\,(\mathrm{mod}\,2\pi) \quad \text{и} \quad \varphi=-hb-\frac{\pi}{2}\,(\mathrm{mod}\,2\pi),$$

которым соответствуют следующие области:



При вещественном μ квазиполином $\Phi(z)$ подробно изучен напр. в [6, 7], где доказано, что при $\varphi=\pi$ и $0<b<\pi/2h$ он имеет только нули с отрицательными вещественными частями, а при $\varphi=\pi$ и $b>\pi/2h$, а также и при $\varphi=0$ и $b>0$ $\Phi(z)$ всегда имеет нуль с положительной вещественной частью. Корни $\Phi(z)$ являются непрерывными функциями величин b и φ, поэтому в областях I, II, III, ... эти свойства сохраняются, т. е. в случае $\mu\in\mathrm{I}$ (что равносильно $h\mu\in\Gamma$) $\mathrm{Re}\,z_\mu<0$ при всех $z_\mu\in N_\mu$ а если $\mu\in\mathrm{II}\cup\mathrm{III}\cup\ldots$, то существует $z_\mu\in N_\mu$ такое, для которого $\mathrm{Re}\,z_\mu>0$. И так осталось доказать, что $\Phi(z)$ имеет нуль с положительной вещественной частью и в том случае, когда μ принадлежит кривым (4.2) при $b>\pi/2h$. Следуя методу D-разбиений, рассмотрим дифференциаль действителной части нуля квазиполинома $\Phi(z)$ когда μ переходит через (4.2). На основании работы [6]

$$dx=-\mathrm{Re}\left(\frac{\partial\Phi}{\partial b}\,db+\frac{\partial\Phi}{\partial\varphi}\,d\varphi\right)\Big/\frac{\partial\Phi}{\partial z},$$

из которого — используя $z=\mu e^{-hz}$ — получим

$$dx=\left((\mathrm{Re}\,z+h|z|^2)\,db/b-\mathrm{Im}\,z\,d\varphi\right)/|1+hz|^2,$$

которое означает, что вещественные части нули $\Phi(z)$ возрастают когда μ переходит через кривые (4.2) таким образом, что $db>0$, $d\varphi=0$.

Этим лемма польностью доказана.

К применению теоремы 3.1 надо знать ранг некоторых матриц. В случае уравнения (4.1) это можно свести к вычислению ранга некоторых легко получаемых из A матриц, имеющих порядок меньше n.

Лемма 4.2. *Пусть* $\det \Delta(\lambda)=0$ *и* $\lambda \neq -1/h$. *Тогда*

$$\operatorname{rank} H_k = (k-1)n + \operatorname{rank} \Delta^k(\lambda),$$

$$\operatorname{rank}\operatorname{col}(Q_k, H_k) = (k-1)n + \operatorname{rank}\operatorname{col}\big(P\Delta(\lambda), P\Delta^2(\lambda), \ldots, P\Delta^{k-1}(\lambda), \Delta^k(\lambda)\big),$$

$$\operatorname{rank}\operatorname{col}(P_k, H_k) = (k-1)n + \operatorname{rank}\operatorname{col}\big(P, P\Delta(\lambda), \ldots, P\Delta^{k-1}(\lambda), \Delta^k(\lambda)\big).$$

Доказательство. Достаточно доказать, что при всех матрицах $\hat{Q}$ и $\hat{P}$ размеров $n \times n$:

$$\operatorname{rank}\operatorname{col}\big(\operatorname{diag}(\hat{Q}, \underbrace{\hat{P}, \ldots, \hat{P}}_{k-1 \text{ раз}}), H_k\big) =$$

$$= \operatorname{rank}\operatorname{col}\big(\hat{Q}, \hat{P}\Delta(\lambda), \ldots, \hat{P}\Delta^{k-1}(\lambda), \Delta^k(\lambda)\big).$$

В дальнейшем для краткости будем использовать обозначение $\Delta = \Delta(\lambda)$, кроме того через одна и та же формула $q(\Delta, I)$ будет обозначать все такие различные полиномы от матриц Δ и I, структура которых нас не интересует.

Доказательство состоит из некоторых элементарных преобразований, которые совершаются в виде матриц, разложенных на блоки. В соответствии с этим, в дальнейшем напр. под m-той строкой мы будем понимать m-тую блок-строку, т. е. строки с номерами $m \cdot n$, $m \cdot n+1, \ldots, m \cdot n+n-1$.

Рассмотрим сначала матрицу H_k. Прибавив матрицы первой строки, умноженные на $-\Delta(1+\lambda h)$ и матрицы третьей строки, умноженные на соответствующую матрицу $q(\Delta, I)/(1+\lambda h)$ к второй строке H_k, далее используя коммутативность матриц $q(\Delta, I)$ и Δ, вторая строка матрицы H_k примет вид

$$-\Delta^2/(1+\lambda h) \quad 0 \quad I(1+\lambda h) \quad q(\Delta, I) \ldots q(\Delta, I).$$

Повторяя этот способ и на следующие строки матрицы H_k, можно достигнуть с помощью элементарных преобразований строк, что H_k эквивалентна матрице

$$\begin{vmatrix} \Delta & I(1+\lambda h) & q(\Delta, I) & q(\Delta, I) \ldots q(\Delta, I) \\ -\Delta^2/(1+\lambda h) & 0 & I(1+\lambda h) & q(\Delta, I) \ldots q(\Delta, I) \\ \Delta^3/(1+\lambda h)^2 & 0 & 0 & I(1+\lambda h) \ldots q(\Delta, I) \\ \cdot & \cdot & \cdot & \cdot \;\; \ldots \;\; \cdot \\ (-1)^{k-1}\Delta^k/(1+\lambda h)^{k-1} & 0 & 0 & 0 \;\; \ldots \;\; 0 \end{vmatrix}.$$

Прибавив строки матрицы H_k, умноженные на $\hat{P}$ к соответствующим строкам матрицы $\operatorname{diag}(\hat{Q}, \hat{P}, \ldots, \hat{P})$, получим, что $\operatorname{rank}\operatorname{col}\big(\operatorname{diag}(\hat{Q}, \hat{P}, \ldots, \hat{P}), H_k\big)$ равняется рангом матрицы

$$\begin{vmatrix} \hat{Q} & 0 & 0 \;\ldots\; 0 \\ \hat{P}\Delta & 0 & \hat{P}q(\Delta, I) \ldots \hat{P}q(\Delta, I) \\ \cdot & \cdot & \cdot \;\;\ldots\;\; \cdot \\ (-1)^{k-2}\hat{P}\Delta^{k-1}/(1+\lambda h)^{k-2} & 0 & 0 \;\ldots\; 0 \\ \Delta & (1+\lambda h)I & q(\Delta, I) \ldots q(\Delta, I) \\ -\Delta^2/(1+\lambda h) & 0 & (1+\lambda h)I \ldots q(\Delta, I) \\ \cdot & \cdot & \cdot \;\;\ldots\;\; \cdot \\ (-1)^{k-1}\Delta^k/(1+\lambda h)^{k-1} & 0 & 0 \;\ldots\; 0 \end{vmatrix}.$$

Используя наличие матриц $(1+\lambda h)I$ легко можно увидеть, что предыдущий ранг равно следующему

$$\operatorname{rank}\begin{pmatrix} \hat{Q} & 0 & 0 & \dots 0 \\ \hat{P}\Delta+\hat{P}q(\Delta, I)\Delta^2 & 0 & 0 & \dots 0 \\ \hat{P}\Delta^2/(1+\lambda h)+\hat{P}q(\Delta, I)\Delta^3 & 0 & 0 & \dots 0 \\ . & . & . & \dots . \\ \hat{P}\Delta^{k-1}/(1+\lambda h)^{k-2} & 0 & 0 & \dots 0 \\ 0 & (1+\lambda h)I & 0 & \dots 0 \\ 0 & 0 & (1+\lambda h)I & \dots 0 \\ . & . & . & \dots . \\ \Delta^k/(1+\lambda h)^{k-1} & 0 & 0 & \dots 0 \end{pmatrix} =$$

$$= n(k-1)+\operatorname{rank}\operatorname{col}(\hat{Q}, \hat{P}\Delta, \dots, \hat{P}\Delta^{k-1}, \Delta^k).$$

Лемма доказана.

На основании этих лемм нижеследующая теорема является следствием теоремы 3.1.

Пусть

$$\det(\mu I-A) = (\mu-\mu_1)^{m_1}\dots(\mu-\mu_s)^{m_s},$$

где $\mu_i \neq \mu_j$ при $i\neq j$. Пусть далее $(\mu-\mu_1)^{\varrho_1}\dots(\mu-\mu_s)^{\varrho_s}$ минимальный полином матрицы A, а $N_1\dots N_s \in \mathbf{N}_0$ — произвольно выбранные постоянные.

Теорема 4.1. а) *Тривиальное решение уравнения* (4.1) *P-устойчиво тогда и только тогда, когда при* $h\mu_i \notin \Gamma\cup\partial\Gamma$

$$\operatorname{rank}\operatorname{col}\big(P, P(\mu_i I-A), \dots, P(\mu_i I-A)^{\varrho_i+N_i-1}, (\mu_i I-A)^{\varrho_i+N_i}\big) = \tag{4.3}$$
$$= \operatorname{rank}(\mu_i I-A)^{\varrho_i+N_i}$$

и при $h\mu_i \in \partial\Gamma$

$$\operatorname{rank}\operatorname{col}\big(P(\mu_i I-A), P(\mu_i I-A)^2, \dots, P(\mu_i I-A)^{\varrho_i+N_i-1}, (\mu_i I-A)^{\varrho_i+N_i}\big) = \tag{4.4}$$
$$= \operatorname{rank}(\mu_i I-A)^{\varrho_i+N_i}.$$

б) *Тривиальное решение уравнения* (4.1) *асимптотически устойчиво тогда и только тогда, если выпольняется условие* (4.3) *при* $h\mu_i \notin \Gamma$.

Доказательство. $\lambda=-1/h$ является корнем уравнения $\lambda e^{-\lambda h}=\mu$ тогда и только тогда, если $\mu=-1/he$. Очевидно, что $-1/e\in\Gamma$. Пусть теперь $k\in\mathbf{N}$ и $h\mu_i\notin\Gamma$. Тогда величину rank $(\mu_i I-A)$ совпадающую с величиной rank $\Delta^k(\lambda)$ для всех $\lambda\in N_{\mu_i}$, можно выразить с помощью инвариантных величин в жордановой форме матрицы A. Специално, величина d_r в теореме 1, зависящая сейчас только от μ_i, совпадает с величиной ϱ_i. Отсюда теорема (4.1) уже непосредственно получается.

Перейдем к изучению специальново случая $P=\operatorname{diag}(\underbrace{1, \dots, 1}_{k\text{ раз}}, 0, \dots, 0)$.

В этом случае можно упростить проверку равенств (4.3) и (4.4). С этой целью представим матрицу A в виде

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

где A_1 матрица размеров $k\times k$, а A_3 размеров $k\times n-k$. Обозначим через I_k (I_{n-k}) единичную матрицу размеров $k\times k$ $(n-k\times n+k)$.

Лемма 4.3.

$$\operatorname{rank}\operatorname{col}(P, P(\mu I-A), \ldots, P(\mu I-A)^{N-1}, (\mu I-A)^N) =$$
$$= k+\operatorname{rank}\operatorname{col}(A_2, A_2A_4, \ldots, A_2A_4^{N-2}, (\mu I_{n-k}-A_4)^{N-1}),$$

$$\operatorname{rank}\begin{pmatrix} P(\mu I-A) \\ P(\mu I-A)^2 \\ \vdots \\ P(\mu I-A)^{N-1} \\ (\mu I-A)^N \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \mu I_k-A_1 & -A_2 \\ -A_2A_3 & A_2(\mu I_{n-k}-A_4) \\ -A_2A_4A_3 & A_2A_4(\mu I_{n-k}-A_4) \\ \vdots & \vdots \\ -A_2A_4^{N-3}A_3 & A_2A_4^{N-3}(\mu I_{n-k}-A_4) \\ (\mu I_{n-k}-A_4)^{N-1}A_3 & (\mu I_{n-k}-A_4)^N \end{pmatrix}$$

при всех $\mu=\mu_i$ *и* $N\geqq\varrho_i+1$ $(i=1, \ldots, s)$.

Доказательство. Ради простоты введем обозначения $Q=I-P$,

$$\tilde{A}_2=PAQ, \quad \tilde{A}_4=QAQ, \quad M_1=\ker\operatorname{col}(P, P(\mu I-A), \ldots, P(\mu I-A)^{N-1}, (\mu I-A)^N).$$
$$M_2=\ker\operatorname{col}(P, \tilde{A}_2, \tilde{A}_2\tilde{A}_4, \ldots, \tilde{A}_2\tilde{A}_4^{N-2}, (\mu Q-\tilde{A}_4)^N).$$

Покажем, что $M_1=M_2$, откуда следует справедливость первой части леммы 4.3. Действительно, пусть $x\in M_1$. Тогда $P(\mu I-A)^kx=0$ при $0\leqq k\leqq N-1$, откуда $Q(\mu I-A)^kx=(\mu I-A)^kx$ и таким образом для всех этих k

$$0=P(\mu I-A)^kx=P(\mu I-A)Q[Q(\mu I-A)Q]^{k-1}x=(-1)^k\tilde{A}_2\tilde{A}_4^{k-1}x,$$

что легко увидеть, если используем $\tilde{A}_2\tilde{A}_4^jx=0$ для $j\leqq k-1$. Нам известно еще и то, что $(\mu I-A)^Nx=0$ откуда в силу $\varrho_i<N$ получается и $(\mu I-A)^{N-1}x=$ $=(\mu I-A)^{\varrho_i}x=0$, которое равносильно равенству $Q(\mu I-A)^{N-1}x=(Q-\tilde{A}_4)^{N-1}x=0$. Таким образом получается, что $x\in M_2$ т. е. $M_1\subset M_2$. Эти преобразования можно совершить и обратно, и получить, что $M_2\subset M_1$, т. е. $M_1=M_2$.

Вторая часть леммы 4.3 можно получить аналогичным образом.

5. Применение для обыкновенных дифференциальных уравнений

В этой главе изучается P-устойчивость тривиального решения дифференциального уравнения

$$\dot{x}(t)=Ax(t), \tag{5.1}$$

где A матрица размеров $n\times n$. Здесь сохраняются прежные обозначения связанные с A. К этому уравнению можно применить рассуждения предыдущих глав, или устремить h к нулью и формально получить условия на P-устойчивость и асимптотическую P-устойчивость. Полученный таким образом результат об асимптотической P-устойчивости равносильно теореме А. В. Луценко и Л. В. Стадниковой [2]. Наша теорема о P-устойчивости является новой и в случае уравнения (5.1):

Теорема 5.1. *Пусть* $P=\operatorname{diag}(\underbrace{1, \ldots, 1}_{k \text{ раз}}, 0, \ldots, 0)$. *Тривиальное решение* $x(t, 0)=0$ *уравнения* (6) *p-устойчиво тогда и только тогда, когда выпольняются условия:*

$$\operatorname{rank} \operatorname{col}\left(A_2, A_2 A_4, \ldots, A_2 A_4^{\varrho_i+N_i-1}, (\mu_i Q-A_2)^{N_i+\varrho_i-1}\right) = n-k-m_i \quad \text{при} \quad \operatorname{Re} \mu_i>0 \quad \text{и}$$

$$n-m_i=\operatorname{rank}\begin{bmatrix} \mu_i I_k-A_1 & -A_2 \\ -A_2 A_3 & A_2(\mu_i I_{n-k}+A_4) \\ -A_2 A_4 A_3 & A_2 A_4(\mu_i I_{n-k}-A_4) \\ \vdots & \vdots \\ -A_2 A_4^{N_i+\varrho_i-3} A_3 & A_2 A_4^{N_i+\varrho_i-3}(\mu_i I_{n-k}-A_4) \\ (\mu_i I_{n-k}-A_4)^{N_i+\varrho_i-1} A_3 & (\mu_i I_{n-k}-A_4)^{N_i+\varrho_i} \end{bmatrix}$$

при $\operatorname{Re} \mu_i=0$, *где* $N_i \in \mathbf{N}$ *произвольно выбранные постоянные.*

ЛИТЕРАТУРА

[1] Воротников, В. И. и Прокопьев, В. П., Об устойчивости движения относительно части переменных для линейных систем, *Прикл. мат. мех.* **42** (1978), 268—271. *MR* **82k**: 34064.

[2] Луценко, А. В. и Стадникова, Л. В., О частичной устойчивости по первому приближению, *Дифференциальные уравнения* **9** (1973), 1530—1533, 1552. *MR* **48** #8997.

[3] Ляпунов, А. М., *Общая задача об устойчивости движения*, Москва и Ленинград, 1950.

[4] Озиранер, А. С. и Румянцев, В. В., Метод функций Ляпунова в задаче об устойчивости движения относительно части переменных, *Прикл. мат. мех.* **36** (1972), 364—384. Translated as *J. Appl. Mech.* **36** (1972), 375—381. *MR* **48** #6585.

[5] Румянцев, В. В., Об устойчивости движения по отношению к части переменных, *Вестник МГУ* **4** (1957), 9—16.

[6] Эльсгольц, Л. Э. и Норкин, С. Б., *Введение в теорию дифференциальных уравнений с отклоняющимся аргументом*, Изд. Наука, Москва, 1971. *MR* **50** #5133

[7] Bellman, R. and Cooke, K. L., *Differential-difference equations*, Academic Press, New York—London, 1963. *MR* **26** #5259; **27**—1399.

[8] Corduneanu, C., On partial stability for delay systems, *Ann. Polon. Math.* **29** (1975), 357—362. *MR* **52** #14548.

[9] Habets, P. and Peiffer, K., Classification of stability-like concepts and their study using vector Lyapunov functions, *J. Math. Anal. Appl.* **43** (1973), 537—570. *MR* **48** #11696.

[10] Hale, J., *Functional differential equations*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag New York, New York—Heidelberg, 1971. *MR* **57** #6711; *Zbl* **222**.34003.

[11] Kappel, F., A stability criterion for linear autonomous functional differential equations, *Dynamical Systems* (Proc. Internat. Sympos. Brown Univ., Providence, R. I., 1974) Ed. by J. K. Hale and J. P. LaSalle, Academic Press, New York, 1976, Vol. 2, 103—107. *MR* **58** #29020.

[12] Kappel, F. and Wimmer, H. K., An elementary divisor theory for autonomous linear functional differential equations, *J. Differential Equations* **21** (1976), 134—147. *MR* **53** #11188.

[13] Levinger, B. W., A folk theorem in functional differential equations, *J. Differential Equations* **4** (1968), 612—619.

(Received July 12, 1975)

JÓZSEF ATTILA TUDOMÁNYEGYETEM
BOLYAI INTÉZETE
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 153—157.

MHL-RINGE MIT ARTINSCHEM RADIKAL

PHAM NGOC ANH

In dieser Arbeit verstehen wir unter einem Ring immer einen assoziativen Ring, unter dem Radikal eines Ringes das Jacobsonsche Radikal. Ein Ring ist artinsch, falls er der Minimalbedingung für Linksideale genügt. Unter einem MHL-Ring verstehen wir einen Ring mit Minimalbedingung für Hauptlinksideale. Jeder artinsche Ring ist also auch ein MHL-Ring.

In [3] wurde eine vollständige Beschreibung der Struktur artinscher Ringe mit artinschem Radikal gegeben.

Es gibt zahlreiche verschiedene Verallgemeinerungen dieses Satzes. Für diese Ergebnisse weisen wir auf [1], [2], [6] hin.

In diesem Artikel wollen wir die Struktur solcher MHL-Ringe, deren Radikal ein artinscher Ring ist, untersuchen.

Zu diesem Ziel nötigen wir einige Sätze über halbeinfache MHL-Ringe von [4], [5]. Ein MHL-Ring ist genau dann halbeinfach, wenn er modultheoretische diskrete direkte Summe minimaler idempotenter Linksideale oder ringtheoretische diskrete direkte Summe von einfachen MHL-Ringen ist. Ein MHL-Ring ist genau dann einfach, wenn er ein dichter Unterring des Ringes aller linearer Transformationen von endlichem Rang eines Vektorraumes über einem Schiefkörper ist. Dieser Schiefkörper ist bis auf Isomorphie eindeutig bestimmt. Hier wird ein Ring einfach genannt, falls er keine echte Ideale hat und radikalfrei ist. Aus den obigen kann man über den Grundschiefkörper eines einfachen MHL-Ringes sprechen. Von jetzt ab bedeutet $\sum^{\boxplus}$ bzw. $\sum^{\oplus}$ die ringtheoretische bzw. modultheoretische diskrete direkte Summe. Es sei nun R ein MHL-Ring mit artinschem Radikal J.

Dann ist $\bar{R}=R/J=\sum_{\mu\in\Gamma^*}^{\boxplus}\bar{R}_\mu$ ringtheoretische diskrete direkte Summe einfacher MHL-Ringe $\bar{R}_\mu$. In diesem Artikel dient der Querstrich zur Bezeichnung von Faktorringen und ihren Elementen. Wir bezeichnen mit Γ die Menge aller μ aus Γ^*, für die Grundschiefkörper S_μ von $\bar{R}_\mu$ unendlich ist. Für jeden beliebigen Index μ aus Γ ist $\bar{R}_\mu$ diskrete direkte Summe idempotenter minimaler Linksideale $\bar{R}_{\mu\delta}(\bar{e}_{\mu\delta}^2=$ $=e_{\mu\delta},\ \delta\in\Delta_\mu)$. Deswegen ist $\bar{e}_{\mu\delta}\bar{R}_\mu\bar{e}_{\mu\delta}=\bar{R}_{\mu\delta}$ ein mit S_μ isomorpher Schiefkörper. Da das Radikal J von R ein artinscher Ring ist, so gibt es einen idempotenten Vertreter $e_{\mu\delta}$ von $\bar{e}_{\mu\delta}$ in R. Aus der Voraussetzung ist J eine abelsche Gruppe mit Minimalbedingung für Untergruppen. Da $e_{\mu\delta}Je_{\mu\delta}$ das Radikal von $e_{\mu\delta}Re_{\mu\delta}$ ist, so ist $e_{\mu\delta}Re_{\mu\delta}$ ein artinscher vollständig primärer Ring mit artinschem Radikal.

1980 *Mathematics Subject Classification* (1970). Primary 16A46; Secondary 16A21.

Key words and phrases. Minimum condition on principal left ideals, Jacobson radical, Artinian ring decompositions into direct sums.

Der Satz 2 aus [3] behauptet, daß ein artinscher vollständig primärer Ring mit artinschem Radikal notwendig endlich ist, wenn er kein Schiefkörper ist. Wegen der Unendlichkeit von $e_{\mu\delta}Re_{\mu\delta}$ ist $e_{\mu\delta}Re_{\mu\delta}$ ein mit S_μ isomorpher Schiefkörper. Wir betrachten nun für ein beliebiges, aber festes Element b aus J alle Produkte $ab\in \in J(a\in e_{\mu\delta}Re_{\mu\delta})$. Da die Ordnungen von ab durch die Ordnung von b beschränkt sind, und J höchstens endlich viele Elemente mit dieser Eigenschaft besitzen darf, gibt es Elemente $a_1, a_2\in e_{\mu\delta}Re_{\mu\delta}$, $a_1\neq a_2$ mit $a_1b=a_2b$, d. h. $(a_1-a_2)b=0$. Wegen $0\neq a_1-a_2\in e_{\mu\delta}Re_{\mu\delta}$ existiert das Inverse $a^*\in e_{\mu\delta}Re_{\mu\delta}$. Damit folgt $a^*(a_1-a_2)b= =e_{\mu\delta}b=0$, also $e_{\mu\delta}J=0$ und entsprechend $Je_{\mu\delta}=0$.

Wir bezeichnen jetzt mit R_μ die Summe aller Linksideale $Re_{\mu\delta}$ $(\delta\in\Delta_\mu)$. Zum weiteren brauchen wir

Hilfssatz 1. *R_μ ist ein Ideal in R für jeden $\mu\in\Gamma$.*

Beweis. Jedes Element a von R_μ hat die Form $a=x_1e_{\mu\delta_1}+\cdots+x_\mu e_{\mu\delta_n}$. Für ein beliebiges Element y aus R gilt $y=\sum_{\mu'\in\Gamma^*} y_{\mu'}+s$ mit $s\in J$, $y_{\mu'}\in\bar{R}_{\mu'}$ und fast alle $y_{\mu'}=0\cdot e_{\mu\delta}y_{\mu'}$ liegt in J für alle $\mu'\neq\mu$, da $\bar{R}_\mu$ direkter Summand von $\bar{R}$ ist. Wegen $e_{\mu\delta}J=0$ ist $e_{\mu\delta}y_{\mu'}=e_{\mu\delta}\cdot e_{\mu\delta}y'_\mu=0$. Dies besagt aber, daß $ay=ay_\mu$ gilt. Da das Bild von R_μ beim natürlichen Homomorphismus von R auf $\bar{R}$ $\bar{R}_\mu$ und $e_{\mu\delta}J= =Je_{\mu\delta}=0$ für alle $\delta\in\Delta_\mu$ ist, so ist $ay_\mu\in R_\mu$. Also ist R_μ ein Ideal von R. Damit ist der Beweis des Hilfssatzes 1 erbracht.

Wegen $Je_{\mu\delta}=0$ ist R_μ mit $\bar{R}_\mu$ isomorph, d. h. R_μ ist ein einfacher MHR-Ring. Es sei jetzt R^* das vollständige Urbild von $\sum_{\mu\notin\Gamma}^{\boxplus}\bar{R}_\mu$ beim natürlichen Homomorphismus von R auf $\bar{R}$. Ist $x\in R_\mu\cap R^*$ ein beliebiges Element, so ist x ein Element aus J wegen $\bar{R}_\mu\cap\bar{R}^*=0$. Aus $J\cdot e_{\mu\delta}=0$ folgt, daß $x=0$ gilt. Wir können in ähnlicher Weise $R_\mu\cap R_{\mu'}=0$ für alle μ, $\mu'\in\Gamma$, $\mu\neq\mu'$ einsehen. Dann ist evident, daß R die folgende direkte Zerlegung besitzt:

$$R=\sum_{\mu\in\Gamma}^{\boxplus}R_\mu\boxed{+}R^*.$$

Es sei neben dieser Zerlegung noch die Darstellung

$$R=\sum_{\mu'\in\Gamma'}^{\boxplus}R'_{\mu'}\boxed{+}R^{*\prime}$$

gegeben, wobei der Grundschiefkörper von $R'_{\mu'}$ für jedes $\mu'\in\Gamma'$ unendlich und der Faktorring von $R^{*\prime}$ nach dem Radikal diskrete direkte Summe einfacher MHL-Ringe mit endlichen Grundkörpern ist. Da

$$R/J\cong\sum_{\mu\in\Gamma}^{\boxplus}R_\mu\boxed{+}(R^*/J)\cong\sum_{\mu'\in\Gamma'}^{\boxplus}R'_{\mu'}\boxed{+}(R^{*\prime}/J)$$

gilt, so gibt es wegen der Eindeutigkeit von halbeinfachen MHL-Ringe eine eineindeutige Abbildung $\mu\to\mu'$ zwischen Γ und Γ', so daß $R_\mu\cong R_{\mu'}$ für jedes $\mu\in\Gamma$ ist. Also gilt

$$R^*\cong R/\sum_{\mu\in\Gamma}^{\boxplus}R_\mu\cong R/\sum_{\mu'\in\Gamma'}^{\boxplus}R'_{\mu'}\cong R^{*\prime}.$$

Damit haben wir den folgenden Satz bewiesen:

SATZ 2. *Jeder MHL-Ringe mit artinschem Radikal besitzt eine Darstellung der Form*

$$R = \sum_{\mu\in\Gamma}{}^{\boxplus} R_\mu \boxed{+} R^*$$

wobei R_μ für jedes $\mu\in\Gamma$ ein einfacher MHL-Ringe mit unendlichem Grundschiefkörper und der Faktorring von R^* nach dem Radikal diskrete direkte Summe von einfachen MHL-Ringen mit endlichen Grundkörpern ist.

Diese Darstellung ist eindeutig bis auf Isomorphie.

Im weiteren wollen wir mehrer über MHL-Ringe wissen. Für jeden direkten Summanden $\bar{R}_\mu$ mit endlichem Grundkörper S_μ von R/J gibt es zwei Fälle bezüglich seiner minimaler idempotenter Linksideale (wir bemerken hier, daß alle minimale idempotente Linksideale eines einfachen MHR-Ringes miteinander isomorph sind). Erstens können die minimale idempotente Linksideale von $\bar{R}_\mu$ endlichdimensionale Vektorräume über S_μ sein. In diesem Fall ist dann $\bar{R}_\mu$ ein endlicher voller Matiizenring über dem endlichen Körper S_μ. Zweitens können sie Vektorräume unendlicher Dimension über S_μ sein, also ist $\bar{R}_\mu$ ein unendlicher Ring. Es sei nun $\bar{R}_{\mu\delta}=\bar{R}_\mu\bar{e}_{\mu\delta}(\bar{e}_{\mu\delta}=e_{\mu\delta})$ ein beliebiges, minimales idempotentes Linksideal von $\bar{R}_\mu$ und $e_{\mu\delta}$ ein idempotenter Vertreter von $\bar{e}_{\mu\delta}$. Wir betrachten den Faktorring R/J^2. Das Ideal J/J^2 wird zu einem $\bar{R}_\mu$-Modul, indem man für ein beliebiges Element $\bar{x}=x+J$ des Faktorringes $\bar{R}_\mu$ die Verknüpfung $\bar{x}\bar{s}$ durch

$$\bar{x}\bar{s} = xs \quad (s = s+J^2\in J/J^2)$$

definiert. Es ist leicht einzusehen, daß diese Definition zulässig ist. Für ein beliebiges, aber festes Element a aus J/J^2 bilden wir den $\bar{R}_\mu$-Modul $\bar{R}_{\mu\delta}$ durch die Abbildung

$$\varphi\colon\ \bar{R}_{\mu\delta} \to J/J^2\varphi(\bar{x}) = xa,\ \bar{x}\in\bar{R}_{\mu\delta},\ x\in R$$

in J/J^2 ab, wo x irgendein Element in der Restklasse $\bar{x}$ ist. φ ist offenbar unabhängig von der Wahl des Repräsentanten aus der Restklasse $\bar{x}$. Es ist klar, daß φ ein $\bar{R}_\mu$-Homomorphismus ist. Da $\bar{R}_{\mu\delta}$ ein minimaler $\bar{R}_\mu$-Modul ist, ist φ entweder trivial oder injektiv. Das letztere ist unmöglich, da nach der Voraussetzung einerseits J/J^2 eine abelsche Gruppe mit Minimalbedingung für Untergruppen und andererseits $\bar{R}_{\mu\delta}$ diskrete direkte Summe unendlicher Exemplaren von endlicher Gruppe S_μ ist. Wir erhalten also $xa=0$, wenn $\bar{x}$ in $\bar{R}_{\mu\delta}$ liegt. Insbesondere haben wir $e_{\mu\delta}J/J^2=0$, d. h. $e_{\mu\delta}J\subseteq J^2$ gilt. Also erhalten wir $e_{\mu\delta}J\subseteq e_{\mu\delta}J^2\subseteq\ldots\subseteq e_{\mu\delta}J^n\subseteq\ldots$. Da J nilpotent ist, so gilt $e_{\mu\delta}J=0$. Da R diskrete direkte Summe minimaler idempotenter Linksideale ist, so haben wir $eJ=0$ für alle Idempotenten e in R mit $\bar{e}\in\bar{R}_\mu$. Folglich beweisen wir

HILFSSATZ 3. *Ist e ein idempotentes Element in einem MHL-Ring mit der Eigenschaft, da sein Bild nach dem Radikal in einem unendlichen, einfachen direkten Summanden mit endlichem Grundkörper liegt, so ist $eJ=0$.*

Wir bezeichnen mit $\mathscr{A}$ die Klasse aller MHL-Ringe mit artinschem Radikal J, die die folgende Eigenschaft haben:

Ist e ein idempotentes Element, dessen Bild nach dem Radikal J in einem unendlichen einfachen direkten Summanden mit endlichem Grundkörper von R/J liegt, so ist $Je=0$.

Es sei jetzt R ein Ring aus $\mathscr{A}$ und $\bar{R}_\mu$ irgendein unendlicher einfacher direkter Summand von R/J. Dann ist $\bar{R}_\mu = \sum_{\delta\in\Delta_\mu} \bar{R}_\mu \bar{e}_{\mu\delta}$ direkte Summe minimaler orthogonaler Linksideale $\bar{R}_\mu \bar{e}_{\mu\delta}(\bar{e}^2_{\mu\delta}=\bar{e}_{\mu\delta}\ \delta\in\Delta_\mu)$. Ist $\{e_{\mu\delta},\ \delta\in\Delta_\mu)$ ein System idempotenter Vertreter von $\{\bar{e}_{\mu\delta},\ \delta\in\Delta_\mu\}$ in R, so gilt $e_{\mu\delta}J=Je_{\mu\delta}=0$ für jedes $\delta\in\Delta_\mu$ nach Satz 2, Hilfssatz 3 und der Voraussetzung. Genau so wie im Beweis des Hilfssatzes 1 können wir sehen, daß die Summe R_μ aller Linksideale $Re_{\mu\delta}$, $\delta\in\Delta_\mu$ ein Ideal in R ist. Bezeichne R^* das vollständige Urbild von $\sum^{\boxplus} \bar{R}_\mu$, wobei jeder $\bar{R}_\mu$ ein voller endlicher Matrizenring über endlichem Körper ist, beim natürlichen Homomorphismus von R auf $\bar{R}$.

Mit Methoden des Beweises von Satz 2 kann man den folgenden Satz beweisen:

SATZ 4. *Es sei R ein Ring aus der Klasse $\mathscr{A}$. Dann besitzt der Ring R eine ringtheoretische direkte Zerlegung*

$$R = \sum_{\mu\in\Gamma}^{\boxplus} R_\mu \boxplus R^* \tag{1}$$

wobei R_μ für jedes $\mu\in\Gamma$ ein unendlicher einfacher MHL-Ring und der Faktorring von R^* nach dem Radikal eine direkte Summe einfacher eindlicher Ringe ist.

Wir bemerken hier, daß man die Sätze 2,4 als Gegenstücke der Ergebnisse von [1], [2] und [6] für MHL-Ringe betrachten kann.

Aus diesem Satz folgt

FOLGERUNG 5. *Die Klasse $\mathscr{A}$ ist eine erbliche Klasse, d.h. jedes Ideal eines Ringes aus $\mathscr{A}$ ein Element von $\mathscr{A}$.*

Zum Beweis braucht man nur zu erwähnen, daß jedes Ideal eines Ringes aus $\mathscr{A}$ wieder die Form (1) hat, d.h. es gehört auch zu $\mathscr{A}$.

Wir haben bei Satz 4 die Voraussetzung aufgenommen, daß R ein Ring aus der Klasse $\mathscr{A}$ ist.

Es erhebt sich das Problem, ob diese Voraussetzung notwendig ist.

Im folgenden zeigen wir die Notwendigkeit an einem Beispiel.

Es sei nun K ein endlicher Körper. Es sei ferner A der Ring aller linearer Transformationen von endlichem Rang eines Vektorraumes von Rang $\aleph_0$ über K. Dann ist A ein unendlicher einfacher MHL-Ring mit endlichem Grundkörper. Wir setzen $R=A\oplus K$ und machen R zu einem Ring durch die folgende Definition:

Für $a_1, a_2\in A$, $k_1, k_2\in K$ sei $(a_1+k_1)(a_2+k_2)=a_1a_2+k_1a_2$, wo k_1a_2 die lineare Transformation mit $k_1a_2(x)=a_2(k_1x)$ für jedes Element x des Vektorraumes ist. Es ist leicht einzusehen, daß R ein MHL-Ring mit artinschem Radikal. Das Radikal von R ist K. Es ist klar, daß R kein Element aus $\mathscr{A}$ ist. Dies erklärt, warum R keine Zerlegung der Form (1) hat.

Am End geben wir einige Ergebnisse über MHL-Ringen mit streng linear kompaktem Radikal. Dabei ist ein topologischer Ring streng linear kompakt genannt, wenn er inverses Limes von Gruppen mit Minimalbedingung für Untergruppen im algebraischen und topologischen Sinne ist. Es sei jetzt R ein MHL-Ring mit streng linear kompaktem Radikal J, so daß jedes Element a aus R die stetige Funktionen $a\to ax$, $a\to xa$, $x\in J$ induziert. Die Klasse $\mathscr{A}_1$ besteht aus diesen Ringen. Wir bezeichnen mit $\mathscr{A}_2$ die Klasse derjeniger Ringe R aus $\mathscr{A}_1$ die der folgenden Eigenschaft genügen: Ist e ein idempotentes Element, dessen Bild nach dem Radikal in einem unendlichen einfachen direkten Summanden von R/J liegt, so ist $Je=0$. Mit Methoden von dieser Arbeit und von [1], [2] kann man die folgenden zwei Sätze ohne Mühe beweisen:

SATZ 6. *Ist R ein Ring aus $\mathscr{A}_1$, dann besitzt R eine direkte Zerlegung*

$$R = \sum_{\mu\in\Gamma}{}^{\boxplus} R_\mu \boxplus R^*$$

wobei R_μ für jedes $\mu\in\Gamma$ ein einfacher MHL-Ring mit unendlichem Grundschiefkörper und der Faktorring von R^* nach dem Radikal direkte Summe einfacher MHL-Ringe mit endlichem Grundkörper ist.

Diese Zerlegung ist eindeutig bis auf Isomorphie.

SATZ 7. *Ist R ein Ring aus $\mathscr{A}_2$, so gibt es für R eine direkte Darstellung*

$$R = \Sigma^{\boxplus} R_\mu \boxplus R^*$$

wobei jeder $R_\mu(\mu\in\Gamma)$ ein unendlicher einfacher MHL-Ring und der Faktorring von R^* nach dem Radikal direkte Summe einfacher endlicher Ringe ist.

Offensichtlich sind die Sätze 6,7 Verallgemeinerungen von Sätzen 2,4.

LITERATURVERZEICHNIS

[1] ANH, P. N., Über die Struktur linear kompakter Ringe, *Acta Math. Acad. Sci. Hungar.* **31** (1978), 61—73. *MR* **57** # 16361.

[2] ANH, P. N., Über die Struktur linear kompakter Ringe, II, *Acta Math. Acad. Sci. Hungar.* **34** (1979), 245—251. *MR* **81h:** 16064.

[3] KERTÉSZ, A. und WIDIGER, A., Artinsche Ringe mit artinschem Radikal, *J. Reine Angew. Math.* **242** (1970), 8—15. *MR* **41** #6886.

[4] SZÁSZ, F., Über Ringe mit Minimalbedingung für Hauptrechtsideale I, *Publ. Math. Debrecen* **7** (1960), 54—64. *MR* **24** #A145.

[5] SZÁSZ, F., Über Ringe mit Minimalbedingung für Hauptrechtsideale II, *Acta Math. Acad. Sci. Hungar.* **12** (1961), 417—439. *MR* **26** #6207.

[6] WIDIGER, A., Die Struktur einer Klasse linear kompakter Ringe, Beiträge zur Algebra und Geometrie **3**, *Wiss. Beitr. Martin-Luther-Univ. Halle—Wittenberg Reihe M Math.* **5** (1974), 139—159. *MR* **53** #3037.

(Eingegangen am 17. Oktober 1977)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15.
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 159—164.

КРАЕВАЯ ЗАДАЧА С ПАРАМЕТРАМИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ СВЕРХНЕЙТРАЛЬНОГО ТИПА

Т. С. НИКОЛОВА и Д. Д. БАЙНОВ

Рассмотрим следующую краевую задачу

$$\ddot{x}(t) = A\lambda + B\mu + f[t, x(t), x(\tau^{x(t)}), \dot{x}(t), \dot{x}(\tau^{x(t)}), \ddot{x}(\tau^{x(t)}), \lambda, \mu], \quad 0 \leqq t \leqq T \tag{1}$$

$$x(0) = x_0, \quad \dot{x}(0) = x_0' \tag{2}$$

$$x(T) = x_T, \quad \dot{x}(T) = x_T'. \tag{3}$$

Здесь t — скалярный аргумент, $x=(x_1, \dots, x_n)$ — искомая вектор-функция, f — заданная вектор-функция, A и B — постоянные обратимые матрицы размерностями $n \times n$, λ и μ — векторные параметры; преобразованный аргумент $\tau^{x(t)}$ имеет вид

$$\tau^{x(t)} = \tau\big(t, x(t), \dot{x}(t), \ddot{x}(t)\big).$$

Отметим, что в связи с большим теоретическим и практическим значением краевых задач для дифференциальных уравнений с отклоняющимся аргументом с параметрами, эти задачи стали предметом изучения многими авторами ([1]—[8] и другие).

Будем предполагать, что краевая задача (1), (2), (3) рассматривается при выполнении следующих условий, обозначенных через (Д):

Д1. Функции $f(t, u, v, u_1, v_1, v_2, \lambda, \mu)$ и $\tau(t, u, u_1, u_2)$ определены соответственно в областях

$$G_f = [0, T] \times G_1 \times G_1 \times G_2 \times G_2 \times G_3 \times R_0 \times R_1$$

$$G_\tau = [0, T] \times G_1 \times G_2 \times G_3,$$

где

$$G_i = \{\zeta\colon\ |\zeta| \leqq g_i\}, \quad i = 1, 2, 3 \qquad (g_i = \text{const} > 0)$$

$$R_0 = \{\lambda\colon |\lambda| \leqq \varrho\}, \quad R_1 = \{\mu\colon |\mu| \leqq \varrho'\} \qquad (\varrho, \varrho' = \text{const} > 0)$$

($|\cdot|$ — некоторая норма в соответствующем конечномерном пространстве).

1980 *Mathematics Subject Classification.* Primary 34K10; Secondary 34K15.
Key words and phrases. Boundary problem, equations of neutral type.

Д2. Существует неотрицательная интегрируемая на отрезке $[0, T]$ функция $F(t)$ такая, что

$$F_0 = \sup_{t\in[0,T]} F(t) \leqq g_3; \quad |x_0'| + \int_0^T F(t)\,dt \leqq g_2;$$

$$|x_0| + |x_0'|T + \int_0^T \Big[\int_0^t F(s)\,ds\Big]\,dt \leqq g_1.$$

Д3. В области G_f функция $f(t, u, v, u_1, v_1, v_2, \lambda, \mu)$ удовлетворяет условию Липшица по всем своим аргументам кроме λ и μ с константой L и неравенству

$$|A|\varrho + |B|\varrho' + \sup\{|f(t, u, v, u_1, v_1, v_2, \lambda, \mu)|: (u, v, u_1, v_1, v_2, \lambda, \mu)\in$$

$$\in G_1\times G_1\times G_2\times G_2\times G_3\times R_0\times R_1\} \leqq F(t).$$

Д4. В области G_τ функция $\tau(t, u, u_1, u_2)$ удовлетворяет условию Липшица по всем аргументам с константой M и неравенствам

$$0 \leqq \tau^{x(t)} \leqq T.$$

Пусть $C^n[0, T]$ — пространство непрерывных n-мерных функций z: $[0, T]\to R^n$ (R-вещественная ось) с метрикой, порожденной нормой: $\|z\| = \sup\{|z(t)|: t\in[0, T]\}$ и пусть Ω — множество всех функций $z\in C^n[0, T]$, удовлетворяющих условиям

$$|z(t)| \leqq F(t), \quad t\in[0, T] \tag{4}$$

$$|z(t) - z(\bar t)| \leqq K|t - \bar t|, \quad t, \bar t\in[0, T]. \tag{5}$$

Решением задачи (1), (2) будем называть такую два раза дифференцируемую на интервале $[0, T]$ функцию $x(t)$, удовлетворяющую уравнению (1) и условию (2), вторая производная которой принадлежит Ω.

Теорема 1. *Пусть выполнены условия* (Д) *и неравенства*

$$L[1 + |x_0'| + F_0T + F_0 + M(|x_0'| + F_0T + F_0 + K)(1 + |x_0'| + F_0T + F_0 + K)] \leqq K \tag{6}$$

$$q = L\Big[T^2 + 2T + 1 + M(|x_0'| + F_0T + F_0 + K)\Big(\frac{T^2}{2} + T + 1\Big)\Big] < 1. \tag{7}$$

Тогда задача (1), (2) *имеет единственое решение.*

Доказательство. Пусть оператор Π действует в Ω по формуле

$$\Pi z(t) = A\lambda + B\mu + f[t, x(t), x(\tau^{x(t)}), y(t), y(\tau^{x(t)}), z(\tau^{x(t)}), \lambda, \mu],$$

где

$$\begin{aligned} y(t) &= x_0' + \int_0^t z(s)\,ds \\ x(t) &= x_0 + x_0't + \int_0^t \Big[\int_0^s z(u)\,du\Big]ds. \end{aligned} \tag{8}$$

Задача (1), (2) эквивалентна операторному уравнению $\Pi z = z$.

Покажем, что $\Pi\Omega \subset \Omega$. Действительно, при $z \in \Omega$ из Д3 следует, что непрерывная функция $\Pi Z(t)$ удовлетворяет условию (4).

Используя (8), Д4 и Д3 получаем последовательно

$$
(9)\qquad
\begin{aligned}
&|x(t)-x(\bar t)| \leqq (|x_0'|+F_0T)|t-\bar t|; \quad (t, \bar t \in [0, T])\\
&|y(t)-y(\bar t)| \leqq F_0|t-\bar t|;\\
&|\tau^{x(t)}-\tau^{x(\bar t)}| \leqq M(1+|x_0'|+F_0T+F_0+K)|t-\bar t|;\\
&|\Pi z(t)-\Pi z(\bar t)| \leqq\\
&\leqq L[1+|x_0'|+F_0T+F_0+M(|x_0'|+F_0T+F_0+K)(1+|x_0'|+F_0T+F_0+K)]|t_1-t_2|.
\end{aligned}
$$

Дальше из условия (6) следует, что выполнено и условие (5). Следовательно $\Pi\Omega \subset \Omega$.

Покажем ещё, что Π — сжимающий оператор на множестве Ω. Действительно, пусть $z, \bar z \in \Omega$ и пусть $\bar x$ и $\bar y$ соответствуют $\bar z$ по формулам (8). Тогда из (8), Д4 и Д3 следует

$$
\begin{aligned}
&|y(t)-\bar y(t)| \leqq T\|z-\bar z\|;\\
&|x(t)-\bar x(t)| \leqq \frac{T^2}{2}\|z-\bar z\|;\\
&|\tau^{x(t)}-\tau^{\bar x(t)}| \leqq M\left(\frac{T^2}{2}+T+1\right)\|z-\bar z\|;\\
&|\Pi z(t)-\Pi\bar z(t)| \leqq L\left[T^2+2T+1+M(|x_0'|+F_0T+F_0+K)\left(\frac{T^2}{2}+T+1\right)\right]\|z-\bar z\|,
\end{aligned}
$$

т.е.

$$\|\Pi z-\Pi\bar z\| \leqq q\|z-\bar z\|.$$

Согласно (7) $q<1$.

Все условия принципа Банаха о неподвижной точке выполнены для оператора Π. Следовательно для каждой пары $(\lambda, \mu) \in R_0 \times R_1$ существует единственная неподвижная точка оператора Π, т.е. единственное решение задачи (1), (2). Решение задачи (1), (2), соответствующее пары (λ, μ), будем обозначать через $x(t, \lambda, \mu)$ (соответсвенно первую и вторую производную решения будем обозначать через $\dot x(t, \lambda, \mu)$ и $\ddot x(t, \lambda, \mu)$).

Теорема 2. *Пусть*

1) *выполнены условия теоремы 1*;

2) *вторая производная по t существующего решения $x(t, \lambda, \mu)$ задачи* (1), (2) *удовлетворяет условию*

$$
(10)\qquad
\begin{aligned}
&|\ddot x(t, \lambda, \mu)-\ddot x(t, \bar\lambda, \bar\mu)| \leqq \delta_1|\lambda-\bar\lambda|+\delta_2|\mu-\bar\mu|\\
&(\lambda, \bar\lambda \in R_0;\ \mu, \bar\mu \in R_1);
\end{aligned}
$$

$$
3)\quad (11)\qquad
\begin{aligned}
&\max\left\{2T^{-2}|A^{-1}|\left[|x_T|+|x_0|+T|x_0'|+\frac{T^2}{2}F_0-\frac{T^2}{2}|A|\varrho\right],\right.\\
&\left.T^{-1}|B^{-1}|[|x_T'|+|x_0'|+TF_0-T|B|\varrho']\right\} \leqq r = \min\{\varrho, \varrho'\};
\end{aligned}
$$

4) *в области* G_f *функция* $f(t, u, v, u_1, v_1, v_2, \lambda, \mu)$ *удовлетворяет условию Липшица по* λ *и* μ *с константой* L;

5)

$$
\begin{aligned}
p = \max\Big\{&|A^{-1}|\big[|B|+L\big(2+(\delta_1+\delta_2)(2T^2+2T+1)+\\
&+M(|x_0'|+F_0T+F_0+K)(T^2+T+1)(\delta_1+\delta_2)\big)\big],\\
&|B^{-1}|\big[|A|+L\big(2+(\delta_1+\delta_2)(2T^2+2T+1)+\\
&+M(|x_0'|+F_0T+F_0+K)(T^2+T+1)(\delta_1+\delta_2)\big)\big]\Big\}<1.
\end{aligned}
\tag{12}
$$

Тогда существует единственное решение краевой задачи (1), (2), (3).

Доказательство. Краевая задача (1), (2), (3) эквивалентна системе

$$
\begin{aligned}
\lambda &= 2T^{-2}A^{-1}\Big\{x_T-x_0-Tx_0'-\frac{T^2}{2}B\mu-\int_0^T(T-s)f[s, x(s,\lambda,\mu), x(\tau^{x(s,\lambda,\mu)},\lambda,\mu),\\
&\qquad \dot x(s,\lambda,\mu), \dot x(\tau^{x(s,\lambda,\mu)},\lambda,\mu), \ddot x(\tau^{x(s,\lambda,\mu)},\lambda,\mu),\lambda,\mu]\,ds\Big\}\\
\mu &= T^{-1}B^{-1}\Big\{x_T'-x_0'-TA\lambda-\int_0^T f[t, x(t,\lambda,\mu), x(\tau^{x(t,\lambda,\mu)},\lambda,\mu),\\
&\qquad \dot x(t,\lambda,\mu), \dot x(\tau^{x(t,\lambda,\mu)},\lambda,\mu), \ddot x(\tau^{x(t,\lambda,\mu)},\lambda,\mu),\lambda,\mu]\,dt\Big\}.
\end{aligned}
\tag{13}
$$

Докажем однозначную разрешимость системы (13).
Вводим обозначений

$$w=\begin{pmatrix}\lambda\\ \mu\end{pmatrix};$$

$$\Gamma(w)=$$

$$
=\left|\begin{array}{c}
2T^{-2}A^{-1}\Big\{x_T-x_0-Tx_0'-\frac{T^2}{2}B\mu-\int_0^T(T-s)f[s, x(s,\lambda,\mu), x(\tau^{x(s,\lambda,\mu)},\lambda,\mu),\\
\dot x(s,\lambda,\mu), \dot x(\tau^{x(s,\lambda,\mu)},\lambda,\mu), \ddot x(\tau^{x(s,\lambda,\mu)},\lambda,\mu),\lambda,\mu]\,ds\Big\}\\
T^{-1}B^{-1}\Big\{x_T'-x_0'-TA\lambda-\int_0^T f[t, x(t,\lambda,\mu), x(\tau^{x(t,\lambda,\mu)},\lambda,\mu), \dot x(t,\lambda,\mu),\\
\dot x(\tau^{x(t,\lambda,\mu)},\lambda,\mu), \ddot x(\tau^{x(t,\lambda,\mu)},\lambda,\mu),\lambda,\mu]\,dt\Big\}
\end{array}\right|
$$

и норму $\|w\|_1=\max(|\lambda|, |\mu|)$. Обозначаем через W множество векторов $w=\begin{pmatrix}\lambda\\ \mu\end{pmatrix}(|\lambda|<\varrho, |\mu|<\varrho')$, для которых выполнено $\|w\|_1\leqq r$.

Система (13) эквивалентна операторному уравнению

$$w=\Gamma(w). \tag{14}$$

Пусть $w \in W$. Тогда из условий (Д) и (11) следует

$$\|\Gamma(w)\|_1 = \max\Big\{\Big|2T^{-2}A^{-1}\Big[x_T - x_0 - Tx_0' - \frac{T^2}{2}B\mu - \int_0^T (T-s)f(s, x(s,\lambda,\mu),$$

$$x(\tau^{x(s,\lambda,\mu)},\lambda,\mu), \dot{x}(s,\lambda,\mu), \dot{x}(\tau^{x(s,\lambda,\mu)},\lambda,\mu), \ddot{x}(\tau^{x(s,\lambda,\mu)},\lambda,\mu),\lambda,\mu)\,ds\Big]\Big|,$$

$$\Big|T^{-1}B^{-1}\Big[x_T' - x_0' - TA\lambda - \int_0^T f(t, x(t,\lambda,\mu), x(\tau^{x(t,\lambda,\mu)},\lambda,\mu), \dot{x}(t,\lambda,\mu),$$

$$\dot{x}(\tau^{x(t,\lambda,\mu)},\lambda,\mu), \ddot{x}(\tau^{x(t,\lambda,\mu)},\lambda,\mu),\lambda,\mu)\,dt\Big]\Big|\Big\} \leq$$

$$\leq \max\Big\{2T^{-2}|A^{-1}|\Big[|x_T| + |x_0| + T|x_0'| + \frac{T^2}{2}F_0 - \frac{T^2}{2}|A|\varrho\Big],$$

$$T^{-1}|B^{-1}|[|x_T'| + |x_0'| + TF_0 - T|B|\varrho']\Big\} \leq r,$$

т.е. $\Gamma W \subset W$.

Если $w = \begin{pmatrix}\lambda\\ \mu\end{pmatrix} \in W$ и $\bar{w} = \begin{pmatrix}\bar\lambda\\ \bar\mu\end{pmatrix} \in W$, согласно Д4, (10), Д3, условия 4) теоремы 2, (5) и (9), после некоторых выкладок, получаем

$$|\tau^{x(t,\lambda,\mu)} - \tau^{x(t,\bar\lambda,\bar\mu)}| \leq M[(T^2+T+1)\delta_1|\lambda-\bar\lambda| + (T^2+T+1)\delta_2|\mu-\bar\mu|];$$

$$\|\Gamma(w) - \Gamma(\bar w)\|_1 \leq \max\Big\{2T^{-2}|A^{-1}|\Big[\frac{T^2}{2}|B|\,|\mu-\bar\mu| + L\int_0^T (T-s)\big(|x(s,\lambda,\mu) -$$

$$- x(s,\bar\lambda,\bar\mu)| + |x(\tau^{x(s,\lambda,\mu)},\lambda,\mu) - x(\tau^{x(s,\bar\lambda,\bar\mu)},\bar\lambda,\bar\mu)| +$$

$$|\dot{x}(s,\lambda,\mu) - \dot{x}(s,\bar\lambda,\bar\mu)| + |\dot{x}(\tau^{x(s,\lambda,\mu)},\lambda,\mu) - \dot{x}(\tau^{x(s,\bar\lambda,\bar\mu)},\bar\lambda,\bar\mu)| +$$

$$+ |\ddot{x}(\tau^{x(s,\lambda,\mu)},\lambda,\mu) - \ddot{x}(\tau^{x(s,\bar\lambda,\bar\mu)},\bar\lambda,\bar\mu)| + |\lambda-\bar\lambda| + |\mu-\bar\mu|\big)\,ds\Big],$$

$$T^{-1}|B^{-1}|\Big[T|A|\,|\lambda-\bar\lambda| + L\int_0^T \big(|x(s,\lambda,\mu) - x(s,\bar\lambda,\bar\mu)| +$$

$$+ |x(\tau^{x(s,\lambda,\mu)},\lambda,\mu) - x(\tau^{x(s,\bar\lambda,\bar\mu)},\bar\lambda,\bar\mu)| + |\dot{x}(s,\lambda,\mu) - \dot{x}(s,\bar\lambda,\bar\mu)| +$$

$$+ |\dot{x}(\tau^{x(s,\lambda,\mu)},\lambda,\mu) - \dot{x}(\tau^{x(s,\bar\lambda,\bar\mu)},\bar\lambda,\bar\mu)| + |\ddot{x}(\tau^{x(s,\lambda,\mu)},\lambda,\mu) -$$

$$- \ddot{x}(\tau^{x(s,\bar\lambda,\bar\mu)},\bar\lambda,\bar\mu)| + |\lambda-\bar\lambda| + |\mu-\bar\mu|\big)\,ds\Big]\Big\} \leq p\|w-\bar w\|_1.$$

По условию (12) $p<1$. Следовательно Γ-оператор сжатия на множестве W. Согласно принципа сжатых отображений операторное уравнение (14) имеет единственное решение.

Теорема 2 доказана.

ЛИТЕРАТУРА

[1] Каменский, Г. А., О существовании и единственности решений краевых задач для дифференциальных уравнений с отклоняющимся аргументом, *Труды сем. по теории дифференц. уравнений с отклоняющимся аргументом* **5** (1967), 107—108.

[2] Облогина, Т. И., Краевая задача для нелинейного дифференциального уравнения второго порядка с запаздывающим аргументом, *Труды сем. по теории дифференц. уравнений с отклоняющимся аргументом* **2** (1963), 234—237. *MR* **30** #4045

[3] Исраилов, И., Об одной краевой задаче с параметром для дифференциально-экстремальных уравнений с запаздывающим аргументом, *Уч. зап. АГУ* **4** (1968), No 1, 33—37. *MR* **44** #1900

[4] Исраилов, И., Об одной краевой задаче с параметром для дифференциально-экстремальных уравнений с авторегулируемым запаздыванием, *Тр. Самаркандского университета,* вып. 202, 1972, 71—81.

[5] Сеидов, З. Б., Краевая задача с параметром для дифференциальных уравнений с запазды вающим аргументом, *Уч. зап. АГУ сер. физ.-мат. наук* **1973**, № I, 53—60. *MR* **50** #10487.

[6] Сеидов, З. Б., Краевая задача для дифференциальных уравнений с запаздывающим аргументом, *Сибирский Мат. Журнал* **14**, № 2, 1973.

[7] Бенсаад, Х. и Норкин, С. Б., Краевая задача с управлением в начальной функции для нелинейных систем дифференциальных уравнений с запаздывающим аргументом, *Украин. Мат. Ж.* **26** (1974), No. 1, 3—12, 140. *MR* **49** #5518

[8] Сеидов, З. Б., Краевая задача с параметром для систем дифференциальных уравнений второго порядка с запаздывающим аргументом, *Украин. Мат. Ж.* **26** (1974), 671—677, 717. *MR* **50** #7737

(Поступила 13-ого апреля, 1978 г.)

АСЕН ХАЛАЧЕВ 19
BG—4004 ПЛОВДИВ

ОБОРИЩЕ 23
BG—1504 СОФИЯ-4
BULGARIA

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 165—171.

THE PROBLEM OF TAMMES FOR $n=11$

K. BÖRÖCZKY

Let a_n be the maximal number with the property that one can place on the unit sphere n points so that the spherical distance of any two points is at least a_n. The problem of finding a_n together with the corresponding extremal arrangement of points was raised by the Dutch biologist Tammes [6], who was led to this problem by examining the distribution of the openings on the pollen grain of different flowers. The problem of Tammes is solved only for some special values of n. According to a result of L. Fejes Tóth [2] we have

$$a_n \leqq \arc\cos \frac{\cot^2 \dfrac{n\pi}{6n-12} - 1}{2}.$$

In this inequality equality is attained for $n=3, 4, 6$ and 12. The extremal configurations corresponding to these numbers are the vertices of a regular triangle, tetrahedron, octahedron and icosahedron, respectively. The problem of Tammes was solved for $n=5, 7, 8$ and 9 by Schütte and van der Waerden [4], for $n=10$ and 11 by Danzer [1] and for $n=24$ by Robinson [3]. Various authors contributed to the construction of good arrangements of points for further values of n. Concerning results in this direction we refer to a paper of Székely [5], who succeeded in improving many of the previous constructions by using the idea of multiple spiral systems.

The solution of the Tammes problem is particularly interesting for $n=5$ and $n=11$. For, we have $a_5=a_6$ and $a_{11}=a_{12}$, which means that 5 or 11 points cannot be arranged better than 6 and 12 points, respectively. There is a difference however: While the extremal configuration of points for $n=5$ is not unique, the position of 11 points on the sphere is uniquely determined by the assumption that the spherical distance of any two of them is at least a_{11}. This is expressed in the following

THEOREM. *If the spherical distance of any two of* 11 *points on the unit sphere is at least a_{11} then the points are distinct vertices of a regular icosahedron.*

Since the "Habilitationsschrift" of Danzer is fairly inaccessible, it seems to be worth publishing a simple alternative proof of this theorem.

Throughout this paper we shall work on the unit sphere. By the distance XY of two points X and Y we shall mean the spherical distance of X and Y. A set of points $P_1, \ldots, P_{11}$ for which $P_iP_j \geqq a_{11}$ for $i \neq j$ is said to be *extremal*. With an extre-

1980 *Mathematics Subject Classification*. Primary 52A45.
Key words and phrases. Sphere, packing and covering.

mal set of points we associate a graph with vertices $P_1, \dots, P_{11}$ and edges consisting of all spherical segments which join pairs of vertices having distance a_{11}. If all edges issuing from a vertex P_i lie on a closed hemisphere bounded by a great circle C passing through P_i then P_i can be isolated in the following way: displacing P_i perpendicularly to C into the complementary of the above hemisphere we obtain a new extremal set of points $P_1, \dots, P_i', \dots, P_{11}$ such that in the corresponding graph P_i' has valency 0. This shows that there is an extremal set of eleven points which does not contain isolable points. In what follows $P_1, \dots, P_{11}$ will denote such a set of points and $\mathcal{G}$ the graph associated with it. We shall show that $P_1, \dots, P_{11}$ are vertices of a regular icosahedron. Simultaneously we shall see that an extremal set of eleven points does not contain isolated or isolable points.

The triangle inequality implies that no two edges of $\mathcal{G}$ cross each other. This property, along with the fact that $\mathcal{G}$ does not contain isolable vertices shows that the edges of $\mathcal{G}$ divide the sphere into a finite number of convex polygons which we shall call the *faces* of $\mathcal{G}$. First we observe that $a_{11} \geqq a_{12} = \arccos \dfrac{\cot^2 \pi/5 - 1}{2} = 63.4349°\dots$ $\dots > 60°$, i.e. $6a_{11} > 360°$, showing that among the faces of $\mathcal{G}$ there is no polygon of number of sides greater than 5. Neither can $\mathcal{G}$ contain a vertex of valency greater than 5. This follows from the fact that no angle occurring in $\mathcal{G}$ can be smaller than the angle α of a regular triangle of sidelength a_{11}, and $\alpha \geqq 72°$. For later reference we record that

(∗) an angle equal to 72° can occur only in a triangle in the case when $a_{11} = a_{12}$.

We also note that by the inequality of L. Fejes Tóth mentioned above we have $a_{11} \leqq \arccos \dfrac{\cot^2 36.66°\dots - 1}{2} = 66.28°\dots$, which is equivalent with the inequality $\alpha \leqq 73.33°\dots$.

We continue to prove some simple propositions.

Proposition 1. *If in a circle of radius a_{12} there are five points with mutual distance not less than a_{12} then either all points are on the boundary of the circle or four are on the boundary and one in the centre.*

This is a simple consequence of the fact that two points of the circle other than the centre span at the centre an angle not less than 72°.

Proposition 2. *Let β_1 and β_2, $\beta_1 \leqq \beta_2$, be the angles of a rhombus of sidelength a_{11} and diagonals greater than a_{11}. Then*

$$\beta_1 \leqq \beta_2 < 2\alpha \quad \textit{and} \quad \beta_1 + \beta_2 > 3\alpha.$$

Here α denotes, in accordance with our previous notation, the angle of a regular triangle of sidelength a_{11}.

The first set of inequalities of the proposition is obvious. The inequality $\beta_1 + \beta_2 >$ $> 3\alpha$ follows from the fact that the area of the rhombus increases if the smaller angle increases. Therefore the area of the rhombus under consideration is greater than the area of a rhombus which is the union of two regular triangles.

PROPOSITION 3. *Let $\mathcal{L}$ be a simple closed polygonal line consisting of five edges of $\mathcal{G}$. Let A be the only vertex of $\mathcal{G}$ lying in the interior of one of the regions determined by $\mathcal{L}$. Then the valency of A is five.*

Let A' be the point diametrically opposite to A. There are five vertices of $\mathcal{G}$ other than A and the vertices of $\mathcal{L}$. We shall show that these vertices are contained in the closed circular disc C of radius a_{12} centred at A'.

Let B be a vertex different from A and the vertices of $\mathcal{L}$. We consider the circles C_A and C_B of radius a_{11} centred at A and B. We may assume that the boundaries of C_A and C_B intersect in two points, say, P and Q, for otherwise we have $AB \geqq 2a_{11} \geqq 2a_{12} > 180° - a_{12}$, so that B lies obviously in C. The segment AB intersects an edge, say XY of $\mathcal{L}$. It is obvious that X and Y cannot lie in the interior of the set $C_A \cup C_B$. This implies that $XY \geqq PQ$. In order to see this it is enough to observe that even the longer of the chords $XY \cap C_A$ and $XY \cap C_B$ is at least as long as PQ.

Having in mind the example of the regular icosahedron we observe that if B is a boundary-point of C then the length of the common chord of the circles of radius a_{12} centred at A and B is equal to a_{12}. It immediately follows that if $B \notin C$ or if B lies on the boundary of C and $a_{11} > a_{12}$ then $a_{11} < PQ$. Since, on the other hand $a_{11} = XY \geqq PQ$, we conclude that $B \in C$, and if B is a boundary-point of C then $a_{11} = a_{12}$. In the latter case we have $AX = AY = a_{12}$.

Applying Proposition 1 to the five vertices of $\mathcal{G}$ lying in C, we see that at least four of them lie on the boundary of C. Therefore we have $a_{11} = a_{12}$ and the end-points of those edges of $\mathcal{L}$ which are cut by a segment joining A with a vertex of $\mathcal{G}$ on the boundary of C have distance a_{11} from A. Since an edge of $\mathcal{L}$ cannot be intersected by two different segments joining A with vertices on the boundary of C, all vertices of $\mathcal{L}$ are at distance a_{11} from A. This completes the proof of Proposition 3.

PROPOSITION 4. *Let $\Pi = P_1 \ldots P_5$ be a convex equilateral pentagon with sidelength a_{11} having one of its greatest angles at P_1. Let P_1' be the point other than P_1 for which $P_1'P_2 = P_1'P_5 = a_{11}$. If $\sphericalangle P_2P_1P_5 \geqq 2\alpha$, then either $P_1'P_3 > a_{11}$ and $P_1'P_4 > a_{11}$ or $a_{11} = a_{12}$ and Π is regular.*

In the proposition the points P_3 and P_4 have symmetric roles. Therefore it is enough to show that $P_1'P_3 \geqq a_{11}$, further that the case $P_1'P_3 = a_{11}$ can occur only if $a_{11} = a_{12}$ and Π is regular. Let the equilateral triangle $P_1'P_2E$ lie on the opposite side of the great circle $P_1'P_2$ as the point P_1. Let F be the point for which EFP_5P_1' is a rhombus (Fig. 1). Write $\beta = \sphericalangle EP_1'P_5$, $\gamma = \sphericalangle P_1'EF$ and $\delta = \sphericalangle P_2P_1'P_5$. Since

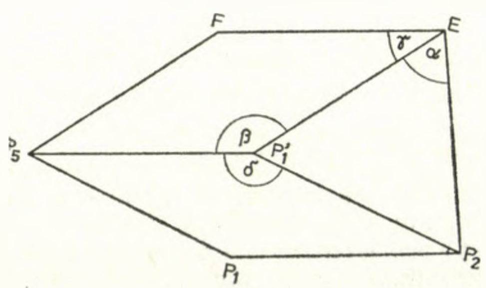


Fig. 1

$\alpha+\delta=\alpha+\sphericalangle P_2P_1P_5>3\alpha>180°$, the rhombuses $P_1P_5P_1'P_2$ and $EP_1'P_5F$ lie on different sides of the great circle $P_1'P_5$. Therefore we have $\beta=360°-\alpha-\delta$. We have on the one hand

$$\beta = 360°-\alpha-\delta > 360°-\alpha-180° = 180°-\alpha > \alpha,$$

on the other hand

$$\beta = 360°-\alpha-\delta \leqq 360°-3\alpha \leqq 360°-3\cdot 72° = 144° \leqq 2\alpha.$$

This means that the diagonals of the rhombus $EP_1'P_5F$ are not shorter than a_{11}. Applying Proposition 2 to the rhombus $EP_1'P_5F$ we obtain that $\beta+\gamma\geqq 3\alpha$. Hence we conclude that

$$\alpha+\gamma \geqq 4\alpha-\beta = 5\alpha-(\alpha+\beta) = 5\alpha-(360°-\delta) \geqq \delta.$$

The considerations above also show that the case $\alpha+\gamma=\delta$ can occur only if $\alpha=72°$, i.e. $a_{11}=a_{12}$, and $\delta=2\alpha$. In this case $P_1P_2EFP_5$ is a regular pentagon of sidelength a_{11}.

If $P_1'P_3=a_{11}$ then $P_3=E$ and $P_4=F$. Thus we have on the one hand

$$\sphericalangle P_2P_3P_4 = \alpha+\gamma \geqq \delta = \sphericalangle P_2P_1P_5,$$

on the other hand, by assumption

$$\sphericalangle P_2P_3P_4 \leqq \sphericalangle P_2P_1P_5.$$

The last two inequalities imply that $\alpha+\gamma=\delta$, so that $\Pi=P_1P_2EFP_5$ is a regular pentagon of sidelength a_{12}.

In order to finish the proof of the proposition we still have to exclude the possibility that $P_1'P_3<a_{11}$. Suppose that $P_1'P_3<a_{11}$. Then we have $\sphericalangle P_3P_2P_5<\sphericalangle EP_2P_5$. It follows that $P_3P_5<EP_5$. P_4 is a common boundary-point of the circles of radius a_{11} centred at P_3 and P_5. Therefore the inequality $P_3P_5<EP_5$ implies that P_4 lies outside the closed rhombus EFP_5P_1'. It is also obvious that P_4 cannot belong to the rhombus $P_1P_2P_1'P_5$. Since Π is convex, P_4 lies on the same side of the great circle P_1P_5 as P_2. Thus, denoting by G the point obtained from P_1 by reflecting in P_5, P_4 is a point of the open convex digon FP_5G. This means that $\sphericalangle P_2P_5P_4>\sphericalangle P_2P_5F$. The triangles P_2P_5F and $P_2P_5P_4$ have two pairs of equal sides and the angle enclosed by the respective sides is greater in the latter triangle. It follows that $P_2F<P_2P_4$. Considering now the triangles P_2EF and $P_2P_3P_4$ we conclude in a similar way that $\sphericalangle P_2EF<\sphericalangle P_2P_3P_4$. Combining this with the inequality $\sphericalangle P_2P_1P_5=\delta\leqq\alpha+\gamma=$ $=\sphericalangle P_2EF$ we obtain that $\sphericalangle P_2P_1P_5<\sphericalangle P_2P_3P_4$, which contradicts to our assumption that the angle of Π at P_1 is maximal. This contradiction shows that $P_1'P_3\geqq a_{11}$, as claimed.

The proof of Proposition 4 is finished.

We begin the investigation of the graph $\mathcal{G}$ by showing that it has no isolated vertices. For, a triangle or a quadrangle obviously cannot contain an isolated vertex. Proposition 3 shows that a pentagonal face of $\mathcal{G}$ cannot contain a single isolated vertex. We still have to show that a pentagon cannot contain two (or more) isolated points. Let a pentagonal face of $\mathcal{G}$ contain two isolated points, say P and Q. One of the two closed hemispheres determined by the great circle PQ contains at least three vertices, say A, B and C of the pentagon. The convex hull of the points A, B, C, P and Q is a convex pentagon of perimeter greater than $5a_{11}$ which lies in the original pentagon of perimeter $5a_{11}$. This is clearly a contradiction which proves our statement.

We continue to show that $\mathscr{G}$ contains either a pentagonal face or a pentahedral vertex. Suppose the contrary, i.e. that all faces of $\mathscr{G}$ are triangles or quadrangles and all vertices have valencies 3 or 4. First we observe that a triangle cannot have a trihedral vertex, neither can three triangles have a common vertex. For, if in the vertex A three or four triangles or two triangles and a quadrangle meet then the sum of the angles of these faces at A is on the one hand equal to 360°, on the other hand it is at least 4α. But this is impossible, since $4\alpha \leqq 4 \cdot 73.33°... < 360°$. Further, if in the vertex A a triangle and two quadrangles or three triangles and a quadrangle meet then A is the only vertex of $\mathscr{G}$ in the interior of the union U of the faces meeting at A. But U is now bounded by five edges of $\mathscr{G}$, so that A has valency five by Proposition 3. This is again a contradiction. Therefore if $\mathscr{G}$ has a triangular face, say ABC, then we have to scrutinize only the following two cases:

(i) On each side of ABC a quadrangle abuts and at each vertex there is a further face of $\mathscr{G}$ (Fig. 2).

(ii) On one side, say BC, a triangle $A'BC$ abuts; the faces joining to ABC and $A'BC$ along the edges AB, AC, $A'B$ and $A'C$ are quadrangles; at A, as well as at A', there is a further face of $\mathscr{G}$ (Fig. 3).

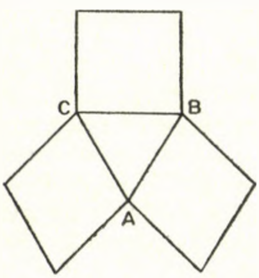


Fig. 2

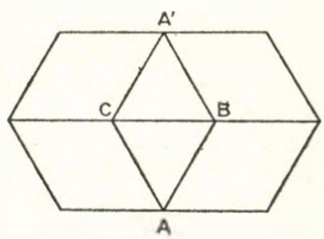


Fig. 3

In case (i) the sum of the angles of the faces meeting at A, B and C is equal to 6π. On the other hand, by Proposition 2, this sum is greater than $15\alpha \geqq 6\pi$, which is impossible.

In case (ii) we consider the angles about A, B, C and A'. The sum of these angles is, on the one hand, equal to 8π, on the other hand, by Proposition 2, greater than $20\pi \geqq 8\pi$ which, again, is a contradiction.

We still have to rule out the case when $\mathscr{G}$ has only quadrangular faces and trihedral and quadrihedral vertices. Let the number of the trihedral and quadrihedral vertices be n_3 and n_4, respectively. Since $\mathscr{G}$ has no isolated vertices, we have $n_3+n_4=$ $=11$. If f and e are the number of faces and edges of $\mathscr{G}$, then $4f=3n_3+4n_4=2e$. Since, by Euler's formula, $e=f+11-2=f+9$, we have $f=9$, $n_3=8$ and $n_4=3$.

It follows that there is a quadrangular face of $\mathcal{G}$ with at least two quadrihedral vertices. Summing up the angles of the faces meeting at the four vertices of this face we obtain a contradiction similarly as above.

This completes the proof of the statement that $\mathcal{G}$ contains either a pentagonal face or a pentahedral vertex.

First we consider the case that $\mathcal{G}$ contains a pentahedral vertex, say P_1 which we imagine to be the south pole. Referring to $(*)$ we see that $a_{11}=a_{12}$ and the faces meeting at P_1 are all triangles. Thus P_1 and the vertices $P_2, \ldots, P_6$ adjacent to P_1 are vertices of a regular icosahedron I. The five remaining vertices $P_7, \ldots, P_{11}$ of $\mathcal{G}$ lie outside the open circles of radius a_{12} centred at $P_1, \ldots, P_6$. Thus they lie in the regular pentagon whose vertices and centre are the northern vertices of I. Referring still to Proposition 1 we see that all vertices of $\mathcal{G}$ are vertices of I.

Finally, we suppose that $\mathcal{G}$ has a pentagonal face $\Pi=P_1\ldots P_5$. Let δ be the greatest angle of Π. First we show that $\delta\geqq 2\alpha$. It is easy to check that the angle ε of a regular pentagon of sidelength a_{11} is at least 2α. Thus, the inequality $\delta\geqq 2\alpha$ will be proved by showing that $\delta\geqq\varepsilon$. Suppose the contrary, i.e. that all angles of Π are smaller than ε. Let $\Pi'=P_1P_2'P_3'P_4'P_5$ be a regular pentagon lying on the same side of the great circle P_1P_5 as Π. The supposition that $\sphericalangle P_2P_1P_5<\varepsilon$ and $\sphericalangle P_1P_5P_4<\varepsilon$ obviously implies that P_2 and P_4 lie in Π'. Further, it follows from the assumptions that $\sphericalangle P_1P_2P_3<\varepsilon$ and $P_2\in\Pi'$ that P_3 lies on the same side of the great circle P_1P_3' as P_5. Similarly, we see that P_3 lies on the same side of the great circle P_5P_3' as P_2. It follows that P_3 belongs to the open triangle $P_1P_5P_3'$ and therefore of course also to the interior of Π'. Thus from the supposition that all angles of Π are smaller than ε we conclude that $\Pi\subset\Pi'$ and P_2, P_3, P_4 are interior points of Π'. But this is impossible, since Π and Π' are isoperimetric. Therefore we have, indeed, $\delta\geqq 2\alpha$.

We may assume, without loss of generality, that $\sphericalangle P_2P_1P_5=\delta$. Let P_1' be the point obtained from P_1 by reflecting in the great circle P_1P_5. Since $\delta\geqq 2\alpha$, Π satisfies the conditions of Proposition 4. Therefore we have either $P_1'P_3>a_{11}$ and $P_1'P_4>a_{11}$ or Π is a regular pentagon of sidelength a_{12}. Obviously, P_1' has from all vertices of $\mathcal{G}$ other than the vertices of Π a distance greater than a_{11}. Therefore in the first case the set of eleven points $P_1', P_2, \ldots, P_{11}$ is also extremal. But in this new set of points P_1' is isolable, which is impossible. Thus Π is a regular pentagon of sidelength a_{12}.

Consider the regular icosahedron I such that the vertices and the midpoint of Π are vertices of I. Then the points $P_6, \ldots, P_{11}$ lie in the convex hull of the remaining six vertices of I. But this is obviously possible only if they coincide with these vertices of I. Thus we obtain again, that $P_1, \ldots, P_{11}$ are distinct vertices of a regular icosahedron. This completes the proof of the theorem.

REFERENCES

[1] Danzer, L., Endliche Punktmengen auf der 2-Sphäre mit möglichst großem Minimalabstand, Habilitationsschrift, Universität Göttingen, 1965.

[2] Fejes Tóth, L., Über die Abschätzung des kürzesten Abstandes zweier Punkte eines auf einer Kugelfläche liegenden Punktsystems, *Jber. Deutsch. Math. Verein.* **53** (1943), 66—68. *MR* **8**—167.

[3] Robinson, R. M., Arrangements of 24 points on a sphere, *Math. Ann.* **144** (1961), 17—48. *MR* **24** # A3565.

[4] SCHÜTTE, K. und WAERDEN, B. L. VAN DER, Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz?, *Math. Ann.* **123** (1951), 96—124. *MR* **13**—61.
[5] SZÉKELY, E., Sur le problème de Tammes, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **17** (1974), 157—175. *MR* **52** #4141.
[6] TAMMES, P. M. L., On the origin of number and arrangement of the places of exit on the surface of pollen grains, *Rec. Trav. Bot. Neerl.* **27** (1930), 1—84.

(Received September 29, 1979)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
GEOMETRIA TANSZÉK
RÁKÓCZI ÚT 5
H—1088 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 173—182.

ON THE ERDŐS—RÉNYI GENERALIZATION OF THE BOREL—CANTELLI LEMMA

T. F. MÓRI and G. J. SZÉKELY

Dedicated to Paul Erdős on his 70th birthday

Abstract

Several lower estimates of $\mathsf{P}(\limsup A_n)$ will be given where $\{A_n\}$ is an arbitrary sequence of events. These lower estimates are better than that of Erdős and Rényi (1959).

1. Historical remarks and introduction

Let $\{A_n\}$ be an arbitrary sequence of events in a probability space and denote by A_∞ the event that infinitely many A_n occurs simultaneously, i.e. $A_\infty = \limsup_{n\to\infty} A_n$. In his fundamental paper Borel [3] proved that for independent events A_n, $n=1, 2, \ldots$

$$\Sigma \mathsf{P}(A_n) < \infty \quad \text{implies} \quad \mathsf{P}(A_\infty) = 0 \quad \text{and}$$
$$\Sigma \mathsf{P}(A_n) = \infty \quad \text{implies} \quad \mathsf{P}(A_\infty) = 1.$$

Borel also applied this result but for *not* independent events. In fact, it was only Hausdorff [11] who proved that in the first implication the condition of independence is superfluous, thus arriving to a complete proof of Borel's law of large numbers. (The first correct proof is due to Faber [11].) Somewhat later Cantelli [7], [8] also proved the generality of the first implication. (Erdős and Rényi [10] refer to Cantelli [6] but in that paper Cantelli did not deal with the Borel—Cantelli problem.) Borel applied both implications for continued fractions, where the events considered were also not independent. This is the more problematic part of Borel's paper since the second implication is not valid without any restriction on the dependence structure of $\{A_n\}$. It was only Bernstein [2] who proved the second implication for certain dependent events in order to show the correctness of Borel's continued fraction theorem (see also [4]).

In the past seven decades several conditions were published that ensure the validity of this second implication (see e.g. [1], [5], [9], [12], [13], [15], [19], [21]-[23], [25]—[27]). One of the most applicable results is due to Erdős and Rényi [10].

Theorem ER. *Let $\{A_n\}$ be an arbitrary sequence of events such that $\sum \mathsf{P}(A_n)=\infty$. Then*

$$\liminf_{n\to\infty} \frac{\sum_{i,j=1}^{n} \mathsf{P}(A_i A_j)}{\left(\sum_{i=1}^{n} \mathsf{P}(A_i)\right)^2} = 1 \tag{1}$$

implies $\mathsf{P}(A_\infty)=1$.

1980 *Mathematics Subject Classification.* Primary 60F20.
Key words and phrases. Borel—Cantelli lemma, Young functions, Luxemburg norm.

This theorem was rediscovered many times with different proofs (e.g. Lamperti [18], Kochen and Stone [16], see also Spitzer [24]) and the following generalization also appeared. Denoting the left-hand side of (1) by L

$$\mathsf{P}(A_\infty) \geqq 1/L.$$

The aim of the present paper is to give better lower estimates for $\mathsf{P}(A_\infty)$.

2. Lower estimates for $\mathsf{P}(A_\infty)$

In the following we always suppose that $\{A_n\}$ is an arbitrary sequence of events such that $\sum \mathsf{P}(A_n)=\infty$. Let I_n denote the indicator of the event A_n and put

$$\alpha_n = \sum_{i=1}^{n} I_i \Big/ \sum_{i=1}^{n} \mathsf{P}(A_i).$$

It is easy to see that

$$\mathsf{E}(\alpha_n^2) = \sum_{i,j=1}^{n} \mathsf{P}(A_i A_j) \Big/ \Big(\sum_{i=1}^{n} \mathsf{P}(A_i)\Big)^2,$$

hence we have

$$\mathsf{P}(A_\infty) \geqq \limsup_{n\to\infty} \|\alpha_n\|_2^{-2} \tag{2}$$

where $\|\cdot\|_2$ denotes the L_2-norm. A simple generalization of (2) is

PROPOSITION 1. *For $p>1$ let q be the number conjugated to p, i.e. $q=\dfrac{p}{p-1}$. Then*

$$\mathsf{P}(A_\infty) \geqq \limsup_{n\to\infty} \|\alpha_n\|_p^{-q}. \tag{3}$$

Further improvements can be obtained if instead of power functions we consider the larger class of convex Young functions Φ and replace the L_p-norm by the Luxemburg norm $\|\cdot\|_\Phi$.

Let us recall some basic facts about Young functions. A convex Young function Φ is defined as

$$\Phi(x) = \int_0^x \varphi(t)\,dt \tag{4}$$

where φ is a positive increasing function defined on the positive real numbers, $\lim_{t\to+0} \varphi(t)=0$ and $\lim_{t\to\infty} \varphi(t)=+\infty$. The number $p=\sup_{x>0} \dfrac{x\varphi(x)}{\Phi(x)}$ is called the power of Φ $(1<p\leqq+\infty)$. It is easy to see that the function $x\Phi\left(\dfrac{1}{x}\right)$ is strictly decreasing and $\lim_{x\to 0} x\Phi\left(\dfrac{1}{x}\right)=+\infty$, $\lim_{x\to\infty} x\Phi\left(\dfrac{1}{x}\right)=0$. Let us denote its inverse by Φ^*, i.e.

for $x>0$ $\Phi^*(x)$ is the only positive number satisfying

$$\Phi^*(x)\Phi\left(\frac{1}{\Phi^*(x)}\right)=x. \tag{5}$$

The Orlicz space L_Φ consists of those random variables X for which there exists a positive constant c with $\mathsf{E}\Phi(c|X|)<\infty$. The Luxemburg norm $\|X\|_\Phi=\inf\{c>0: \mathsf{E}\Phi\left(\frac{|X|}{c}\right)\le 1\}$ makes L_Φ a Banach space. For $\Phi(x)=x^p$ we obtain $L_\Phi=L_p$ and $\Phi^*(x)=x^{1-q}$. A detailed discussion of convex Young functions and the related Orlicz spaces can be found in Krasnosel'skiĭ and Rutickiĭ [17].

PROPOSITION 2. *Let Φ be an arbitrary convex Young function. Then*

$$\mathsf{P}(A_\infty)\geqq \limsup_{n\to\infty}\Phi^*(\|\alpha_n\|_\Phi)/\|\alpha_n\|_\Phi.$$

Another way of extending (2) to general Young functions is the use of $\mathsf{E}\Phi(\alpha_n)$ instead of $\|\alpha_n\|_\Phi$. In this direction we have

PROPOSITION 3. *Let Φ be an arbitrary convex Young function. Then*

$$\mathsf{P}(A_\infty)\geqq \limsup_{n\to\infty}\Phi^*(\mathsf{E}\Phi(\alpha_n)). \tag{6}$$

One can ask whether convexity has any role in Proposition 3. The answer is negative, since a similar assertion can be proved for concave Young functions. A concave Young function can be defined also by formula (4), taking a positive decreasing function φ, $\lim_{t\to+0}\varphi(t)=+\infty$, $\lim_{t\to\infty}\varphi(t)=0$. It is easy to see that concave and convex Young functions are inverses of each other.

Let us define Φ^* again by (5), then we obtain an increasing function. Though Orlicz spaces cannot be defined for concave Young functions, the right-hand side of (6) preserves its meaning.

PROPOSITION 4. *Let Φ be an arbitrary concave Young function. Then*

$$\mathsf{P}(A_\infty)\geqq \limsup_{n\to\infty}\Phi^*(\mathsf{E}\Phi(\alpha_n)).$$

A special case of Proposition 4 is when $\Phi(x)=x^{1/p}$, $p>1$. Hence we obtain

COROLLARY 1.

$$\mathsf{P}(A_\infty)\geqq \limsup_{n\to\infty}\mathsf{E}^q(\alpha_n^{1/p}),\quad p>1. \tag{7}$$

The proof of all these Propositions is given in Section 3.

3. Comparison of the estimates

In Section 2 we have seen several lower estimates for $\mathsf{P}(A_\infty)$. Now a question arises in a natural way: What is the connection between Proposition 2, 3 and 4? Which one is the strongest? In order to answer this question let us introduce some notations.

For a random variable X let $\mathcal{D}(X)$ denote the distribution of X. Further, we shall write $X_n \to_{\mathcal{D}} X$ to indicate convergence in distribution, i.e. it means that $\mathcal{D}(X_n)$ converges weakly to $\mathcal{D}(X)$. Denote by W the set of all limit points of the sequence $\{\mathcal{D}(\alpha_n)\}$. Since $\alpha_n \geqq 0$ and $\mathsf{E}(\alpha_n)=1$, it follows that $\{\mathcal{D}(\alpha_n)\}$ is a tight family of probability distributions, thus $W \neq \emptyset$. Denote further by W_1 the subset of those probability laws in W which are weak limits of uniformly integrable subsequences of $\{\alpha_n\}$. Finally, we shall write $W_{1+\varepsilon}$ for the subset of those probability laws in W_1 for which the corresponding subsequence $\{\alpha_{n(k)},\ k=1, 2, \ldots\}$ can be chosen to satisfy $\sup \mathsf{E}\alpha_{n(k)}^{1+\varepsilon} < \infty$ with some $\varepsilon > 0$. For the sake of convenience we shall write $\alpha \in W$ instead of $\mathcal{D}(\alpha) \in W$ if it does not cause any confusion.

Define

$$C_1 = \sup_{p>1} \left(\limsup_{n\to\infty} \|\alpha_n\|_p^{-q}\right),$$

$$C_2 = \sup\left\{\limsup_{n\to\infty} \|\alpha_n\|_\Phi^{-1} \Phi^*(\|\alpha_n\|_\Phi)\colon\ \Phi \text{ convex Young function}\right\},$$

$$C_3 = \sup\left\{\limsup_{n\to\infty} \Phi^*\left(\mathsf{E}\Phi(\alpha_n)\right)\colon\ \Phi \text{ convex Young function}\right\},$$

$$C_4 = \sup\left\{\limsup_{n\to\infty} \Phi^*\left(\mathsf{E}\Phi(\alpha_n)\right)\colon\ \Phi \text{ concave Young function}\right\}$$

and finally

$$C_5 = \sup_{p>1} \left(\limsup_{n\to\infty} \mathsf{E}^q(\alpha_n^{1/p})\right).$$

Using the Hölder inequality one can easily prove that $(\mathsf{E}|X|^p)^{\frac{1}{1-p}}$ is a decreasing function of p $(0<p\neq 1)$ if $\mathsf{E}|X|=1$, hence

$$C_1 = \lim_{p\to 1+0} \left(\limsup_{n\to\infty} \|\alpha_n\|_p^{-q}\right)$$

and

$$C_5 = \lim_{p\to\infty} \left(\limsup_{n\to\infty} \mathsf{E}^q(\alpha_n^{1/p})\right).$$

These quantities represent the best possible estimations achieved by Propositions 1—4 and Corollary 1, resp.

THEOREM 1.

$$C_1 = \sup_{\alpha\in W_{1+}} \exp\left(-\mathsf{E}(\alpha \log \alpha)\right) \leqq \sup_{\alpha\in W_{1+}} \mathsf{P}(\alpha > 0) \quad \textit{if} \quad W_{1+} \neq \emptyset$$

and $C_1=0$ *if* $W_{1+}=\emptyset$ $(0 \log 0 = 0$ *by definition*$)$,

$$C_2 = C_3 = \sup_{\alpha\in W_1} \mathsf{P}(\alpha > 0) \quad \textit{if} \quad W_1 \neq \emptyset \quad \textit{and} \quad C_2 = C_3 = 0 \quad \textit{if} \quad W_1 = \emptyset,$$

$$C_4 = C_5 = \sup_{\alpha\in W} \mathsf{P}(\alpha > 0).$$

Since $\alpha_n \to 0$ on the complement of A_∞, we obtain the following assertion as an immediate consequence of Theorem 1.

COROLLARY 2.

$$C_1 \leqq C_2 = C_3 \leqq C_4 = C_5 \leqq \mathsf{P}(A_\infty).$$

This proves all the results of Section 2.

PROOF OF THEOREM 1.

(C_1) For $p>1$ denote $\lambda(p)=\limsup_{n\to\infty}\|\alpha_n\|_p^{-q}$. If $W'_{1+}=\emptyset$ then $\lambda(p)=0\ \forall p>1$, i.e. $C_1=0$. Suppose that $\lambda(p')>0$ and let $\{\alpha_{n(k)}: k=1,2,\dots\}$ be a subsequence of $\{\alpha_n\}$ for which $\lim_{k\to\infty}\|\alpha_{n(k)}\|_{p'}^{-q'}=\lambda(p')$. Let $1<p<p'$, then $\{\alpha^p_{n(k)}\}$ is uniformly integrable, thus the weak limit points of $\{\alpha_{n(k)}\}$ belong to W_{1+}, therefore we may suppose $\alpha_{n(k)}\to_{\mathcal{D}}\alpha\in W_{1+}$ as $k\to\infty$. Clearly, $\|\alpha_{n(k)}\|_p\to\|\alpha\|_p$, hence $\lambda(p')=$ $=\lim_{k\to\infty}\|\alpha_{n(k)}\|_{p'}^{-q'}\leqq\lim_{k\to\infty}\|\alpha_{n(k)}\|_p^{-q}=\|\alpha\|_p^{-q}$.

On the other hand, for arbitrary $\alpha\in W_{1+}$ one can find a subsequence $\{\alpha_{n(k)}\}$ for which $\|\alpha_{n(k)}\|_p\to\|\alpha\|_p$ when $p>1$ is sufficiently small, consequently $\|\alpha\|_p^{-q}\leqq\lambda(p)$. Thus we obtain

$$C_1=\sup_{\alpha\in W_{1+}}\big(\lim_{p\to 1+0}\|\alpha\|_p^{-q}\big).$$

Here

$$\lim_{p\to1+0}\log\|\alpha\|_p^{-q}=-\lim_{p\to1+0}\frac{q}{p}\log \mathsf{E}(\alpha^p)=$$

$$=-\lim_{p\to1}\frac{1}{p-1}\big(\log\mathsf{E}(\alpha^p)-\log\mathsf{E}(\alpha)\big)=$$

$$=-\frac{d}{dp}\log\mathsf{E}(\alpha^p)|_{p=1}=-\mathsf{E}(\alpha\log\alpha).$$

Finally, using the Jensen inequality we have

$$\exp\big(-\mathsf{E}(\alpha\log\alpha)\big)\leqq\mathsf{E}\big(\alpha\exp(-\alpha)\big)\leqq\mathsf{P}(\alpha>0).$$

(C_3) Suppose $\alpha\in W_1$. Consider a uniformly integrable subsequence $\{\alpha_{n(k)}\}$ converging in distribution to α. By a theorem of de la Vallée—Poussin there exists a convex Young function $\Phi(x)=\int_0^x\varphi(t)\,dt$ for which $\sup_k\mathsf{E}\Phi(\alpha_{n(k)})<\infty$. We define another Young function Φ_1 which grows slower than Φ. Let $\Phi_1(x)=$ $=\int_0^x\sqrt{\varphi(t)}\,dt$, then $\Phi\circ\Phi_1^{-1}$ is also a convex Young function, hence $\{\Phi_1(\alpha_{n(k)})\}$ is uniformly integrable.

Now we construct a Young function Φ_2 which is nearly linear in a long interval, while for large argument values it behaves like Φ_1.

Let $c>1$, $0<\varepsilon<e^{-1}$ and M large enough. Define

$$\Phi_2(x)=\begin{cases}x^c & \text{if } x\leqq\varepsilon\\ x-(\varepsilon-\varepsilon^c) & \text{if } \varepsilon<x\leqq M\\ \dfrac{M}{\Phi_1(M)}\Phi_1(x)-(\varepsilon-\varepsilon^c) & \text{if } M<x.\end{cases}$$

Then Φ_2 is a convex Young function; in order to show it we only have to note that $c\varepsilon^{c-1}\leqq1\leqq\dfrac{M}{\Phi_1(M)}\sqrt{\varphi(M)}$.

Clearly, $\{\Phi_2(\alpha_{n(k)})\}$ is also uniformly integrable, thus $\mathsf{E}\Phi_2(\alpha_{n(k)})\to\mathsf{E}\Phi_2(\alpha)$ as $k\to\infty$. Now

$$\mathsf{E}\Phi_2(\alpha)=\mathsf{E}\big(\alpha^c I(\alpha\leqq\varepsilon)\big)+\mathsf{E}\big(\alpha I(\varepsilon<\alpha\leqq M)\big)-(\varepsilon-\varepsilon^c)\mathsf{P}(\varepsilon<\alpha\leqq M)+$$

$$+\frac{M}{\Phi_1(M)}\mathsf{E}\big(\Phi_1(\alpha)I(\alpha>M)\big)-(\varepsilon-\varepsilon^c)\mathsf{P}(\alpha>M)\leqq$$

$$\leqq\mathsf{E}(\alpha)-(\varepsilon-\varepsilon^c)\mathsf{P}(\alpha>\varepsilon)+\frac{M}{\Phi_1(M)}\mathsf{E}\big(\Phi_1(\alpha)I(\alpha>M)\big).$$

Since $\mathsf{E}(\alpha)=1$ and $\mathsf{E}\Phi_1(\alpha)<\infty$, we obtain for arbitrary fixed $\delta>0$

$$\mathsf{E}\Phi_2(\alpha)\leqq 1-(\varepsilon-\varepsilon^c)\big(\mathsf{P}(\alpha>\varepsilon)-\delta\big)$$

if M is sufficiently large.

Using that $\Phi_2^*(x)=1-(\varepsilon-\varepsilon^c)x$ if $\frac{1}{M}\leqq x\leqq\frac{1}{\varepsilon}$, we have

$$\Phi_2^*\big(\mathsf{E}\Phi_2(\alpha)\big)\geqq\mathsf{P}(\alpha>\varepsilon)-\delta$$

if the right-hand side is not less than $1/M$ (by the Markov inequality it is always less than $1/\varepsilon$).

Consequently,

$$C_3\geqq\lim_{k\to\infty}\Phi_2^*\big(\mathsf{E}\Phi_2(\alpha_{n(k)})\big)=\Phi_2^*\big(\mathsf{E}\Phi_2(\alpha)\big)\geqq\mathsf{P}(\alpha>\varepsilon)-\delta.$$

Since $\delta,\varepsilon>0$ and $\alpha\in W_1$ can be chosen arbitrarily, we obtain $C_3\geqq\sup_{\alpha\in W_1}\mathsf{P}(\alpha>0)$.

The opposite inequality is quite simple. For a given convex Young function Φ let us consider a subsequence $\{\alpha_{n(k)}\}$ for which $\lim_{k\to\infty}\mathsf{E}\Phi(\alpha_{n(k)})=\liminf_{n\to\infty}\mathsf{E}\Phi(\alpha_n)$, moreover we may suppose that $\alpha_{n(k)}$ converges in distribution to some $\alpha\in W_1$. Then $\mathsf{E}\Phi(\alpha)\leqq\lim_{k\to\infty}\mathsf{E}\Phi(\alpha_{n(k)})$, thus

$$\limsup_{n\to\infty}\Phi^*\big(\mathsf{E}\Phi(\alpha_n)\big)=\Phi^*\big(\liminf_{n\to\infty}\mathsf{E}\Phi(\alpha_n)\big)\leqq\Phi^*\big(\mathsf{E}\Phi(\alpha)\big).$$

Applying the Jensen inequality we obtain

$$\begin{aligned}\Phi^*\big(\mathsf{E}\Phi(\alpha)\big)&=\Phi^*\big(\mathsf{E}(\Phi(\alpha)\,|\,\alpha>0)\mathsf{P}(\alpha>0)\big)\leqq\\ &\leqq\Phi^*\big(\Phi(\mathsf{E}(\alpha\,|\,\alpha>0))\mathsf{P}(\alpha>0)\big)=\\ &=\Phi^*\left(\Phi\left(\frac{1}{\mathsf{P}(\alpha>0)}\right)\mathsf{P}(\alpha>0)\right)=\mathsf{P}(\alpha>0).\end{aligned}\tag{8}$$

This completes the proof for C_3.

Remark. From the construction of Φ_2 one can see that in the definition of C_3 it suffices to take supremum over such Young functions whose power is small, say less than ϱ, where $\varrho>1$ is arbitrary but fixed.

Denoting the power of a Young function Φ by $p(\Phi)$, it is easy to show that $p(\Phi_1) \leqq \sqrt{p(\Phi)}$. Further,

$$\frac{x\varphi_2(x)}{\Phi_2(x)} =$$

$$= \begin{cases} c & \text{if } 0 < x \leqq \varepsilon, \\ \dfrac{x}{x-(\varepsilon-\varepsilon^c)} \leqq \varepsilon^{1-c} & \text{if } \varepsilon < x \leqq M, \\ \dfrac{\sqrt{\varphi(x)}\,x}{\Phi_1(x)-(\varepsilon-\varepsilon^c)\Phi_1(M)/M} \leqq \dfrac{1}{1-\varepsilon/M}\,\dfrac{x\sqrt{\varphi(x)}}{\Phi_1(x)} \leqq \dfrac{p(\Phi_1)}{1-\varepsilon/M} & \text{if } M < x, \end{cases}$$

hence

$$p(\Phi_2) \leqq \max\left\{\varepsilon^{1-c}, \frac{\sqrt{p(\Phi)}}{1-\varepsilon/M}\right\}.$$

which can be pressed down below ϱ, choosing Φ, c and M suitably.

($C_2=C_3$) Let Φ be a convex Young function and let $\{\alpha_{n(k)}\}$ be a subsequence satisfying

$$\lim_{k\to\infty} \|\alpha_{n(k)}\|_\Phi = \liminf_{n\to\infty} \|\alpha_n\|_\Phi = L$$

and $\alpha_{n(k)} \to_{\mathscr{D}} \alpha \in W$. Suppose $L<\infty$, then $\|\alpha\|_\Phi \leqq L$, further $\alpha \in W_1$, since $\{\alpha_{n(k)}\}$ is uniformly integrable. Define $\Psi(x)=\Phi(\|\alpha\|_\Phi^{-1}x)$, then $\|\alpha\|_\Psi=1$, $\mathsf{E}\Psi(\alpha)\leqq 1$, thus

$$\limsup_{n\to\infty} \|\alpha_n\|_\Phi^{-1}\Phi^*(\|\alpha_n\|_\Phi) = \Phi^*(L)/L \leqq$$

$$\leqq \|\alpha\|_\Phi^{-1}\Phi^*(\|\alpha\|_\Phi) = \Psi^*(1) \leqq \Psi^*(\mathsf{E}\Psi(\alpha)).$$

Due to (8), the right-hand side is majorized by $\mathsf{P}(\alpha>0)$, hence $C_2 \leqq C_3$. On the other hand, let Φ have a finite power p. Denote $\Psi(x)=\Phi(x)/\liminf_{n\to\infty}\mathsf{E}\Phi(\alpha_n)$, then

$$\limsup_{n\to\infty} \Phi^*(\mathsf{E}\Phi(\alpha_n)) = \limsup_{n\to\infty} \Psi^*(\mathsf{E}\Psi(\alpha_n)) = \Psi^*(1).$$

Since $p(\Psi)=p$, $\|\alpha_n\|_\Psi$ lies between $\mathsf{E}\Psi(\alpha_n)$ and $(\mathsf{E}\Psi(\alpha_n))^{1/p}$, thus $\liminf_{n\to\infty}\|\alpha_n\|_\Psi=1$. Hence

$$\Psi^*(1) = \limsup_{n\to\infty} \|\alpha_n\|_\Psi^{-1}\Psi^*(\|\alpha_n\|_\Psi),$$

which implies $C_3 \leqq C_2$.

(C_4), (C_5) The inequality $C_4 \leqq \sup_{\alpha\in W}\mathsf{P}(\alpha>0)$ can be proved in a similar way as it was done for C_3. The following minor change is needed. Since $\{\Phi(\alpha_n)\}$ is uniformly integrable, there exists an $\alpha\in W$ for which $\mathsf{E}\Phi(\alpha)=\limsup_{n\to\infty}\mathsf{E}\Phi(\alpha_n)$. Inequality (8) remains valid in this case, too, and it completes the proof.

Clearly, $C_5 \leqq C_4$, thus we have to show that $C_5 \geqq \sup_{\alpha\in W}\mathsf{P}(\alpha>0)$. For arbitrary $\alpha\in W$

$$C_5 \geqq \limsup_{n\to\infty} \mathsf{E}^q(\alpha_n^{1/p}) \geqq \mathsf{E}^q(\alpha^{1/p}) \geqq \left(\varepsilon^{1/p}\mathsf{P}(\alpha>\varepsilon)\right)^q.$$

Now we can finish the proof by letting p tend to infinity then taking supremum in $\varepsilon>0$ and $\alpha\in W$.

4. Further remarks

In this section we give examples to demonstrate that in Corollary 2 the sign $\leqq$ can be nowhere replaced by equality.

In the first example $\mathsf{P}(A_\infty)=1$ and at the same time $\alpha_n\to 0$ in probability, thus every estimate in Section 2 gives 0, indicating the limits of applicability.

Let $\{B_n\}$ be a sequence of independent events such that $p_n=\mathsf{P}(B_n)$, $n=1,2,\dots$ satisfy the following conditions: $p_n>0$, $p_n\to\infty$, $\sum p_n=+\infty$. Pick an increasing sequence of positive integers $\{Q_n\}$ such that $\sum_{i=1}^{n-1} Q_i/p_nQ_n\to 0$ as $n\to\infty$. Put $A_n=B_{k+1}$ if $\sum_{i=1}^{k} Q_i<n\leqq\sum_{i=1}^{k+1} Q_i$. Then obviously $\mathsf{P}(A_\infty)=\mathsf{P}(B_\infty)=1$. On the other hand, on the complement of the event $B_k\cup B_{k+1}$

$$\alpha_n\leqq\sum_{i=1}^{k-1} Q_i\Big/\sum_{i=1}^{k} p_iQ_i$$

which tends to zero as $n\to\infty$.

The following result may be of independent interest.

THEOREM 2. *For an arbitrary nonnegative random variable α with expectation $\mathsf{E}(\alpha)\leqq 1$ there exists a sequence of events for which $\alpha_n\to\alpha$ with probability* 1.

PROOF. Let β be a nonnegative integer valued random variable with infinite expectation. Denote $b_n=\mathsf{E}\left(\frac{\beta}{n}\wedge 1\right)$ where $\wedge$ stands for minimum and put $c_n=\dfrac{nb_n}{1-\mathsf{E}\alpha+\sqrt{b_n}}$. Then c_n and n/c_n tend to infinity increasingly with n.

Define $I_{2n}=[c_n\alpha\wedge n]-[c_{n-1}\alpha\wedge(n-1)]$ where $[.]$ denotes integer part. I_{2n} is an indicator variable, since it is integer valued and $0\leqq I_{2n}\leqq 1$ as it is shown below.

On the event $\{c_{n-1}\alpha\geqq n-1\}$

$$0\leqq(c_n\alpha\wedge n)-(c_{n-1}\alpha\wedge(n-1))\leqq n-(n-1)=1,$$

and on the event $\{c_{n-1}\alpha<n-1\}$ $c_n\alpha<n$ as well, thus

$$0\leqq(c_n\alpha\wedge n)-(c_{n-1}\alpha\wedge(n-1))=(c_n-c_{n-1})\alpha\leqq\frac{c_n-c_{n-1}}{c_n}n\leqq 1.$$

Define further $I_{2n-1}=I(\beta\geqq n)$. We show that the sequence $\{\alpha_n\}$ corresponding to the indicators I_n tends to α. Since $\alpha_{2n}-\alpha_{2n-1}\to 0$ in general, it suffices to deal with the subsequence $\{\alpha_{2n}\}$. Denote $\beta_n=\sum_{i=1}^{2n} I_i$. Then $\beta_n=[c_n\alpha\wedge n]+\beta\wedge n$, thus

$$\left(\alpha\wedge\frac{n}{c_n}\right)-\frac{1}{c_n}+\frac{\beta\wedge n}{c_n}<\frac{\beta_n}{c_n}\leqq\left(\alpha\wedge\frac{n}{c_n}\right)+\frac{\beta\wedge n}{c_n}.$$

Taking expectation one can see

$$\mathsf{E}\beta_n/c_n\to\mathsf{E}\alpha+(1-\mathsf{E}\alpha)=1,$$

hence $\alpha_{2n}\sim\beta_n/c_n$, which tends to α everywhere. The proof is completed.

Using Theorem 2 one can easily construct examples showing $C_1 < C_2$ and $C_3 < C_4$. In this case the distribution of α is the only weak limit point, hence if $\alpha > 0$ and $\mathsf{E}\alpha < 1$, then $W_1 = \emptyset$, thus $C_3 = 0$, $C_4 = 1$. If $\alpha > 0$, $\mathsf{E}\alpha = 1$ and $\mathsf{E}(\alpha^{1+\varepsilon}) = +\infty$ for every positive ε, then $\alpha \in W_1$ but $\alpha \notin W_{1+}$, thus $C_1 = 0$, $C_2 = 1$.

5. Open problems

(i) All of our lower estimates for $\mathsf{P}(A_\infty)$ can be improved by computing them for arbitrary subsequences of $\{A_n\}$ and then taking their supremum for all these subsequences. One can easily see that the Erdős—Rényi lower bound becomes better (this is shown by the first example of Section 4 where the improved estimate gives 1), but in general it is still less than $\mathsf{P}(A_\infty)$. On the other hand we conjecture that this subsequence-supremum version of C_5 is equal to $\mathsf{P}(A_\infty)$.

(ii) We also conjecture that if we have an estimate of the form $\mathsf{P}(A_\infty) \geqq L_k$ where the constant L_k depends only on $\mathsf{P}(A_{i_1} A_{i_2} \dots A_{i_k})$, $1 \leqq i_1 \leqq i_2 \leqq \dots \leqq i_k$, then L_k does not exceed the improved Erdős—Rényi lower bound (i.e. taken its supremum for all subsequences of $\{A_n\}$).

REFERENCES

[1] Barndorff-Nielsen, O., On the rate of growth of the partial maxima of a sequence of independent identically distributed random variables, *Math. Scand.* **9** (1961), 383—394. *MR* **25** #2625.

[2] Bernstein, S. N., Über eine Anwendung der Mengenlehre aus der Theorie der säkularen Störungen herrührendes Problem, *Math. Ann.* **71** (1911), 417—439.

[3] Borel, É., Les probabilités dénombrables et leurs applications arithmétiques, *Rend. Circ. Mat. Palermo* **27** (1909), 247—271.

[4] Borel, É., Sur un problème de probabilités relatif aux fraction continues, *Math. Ann.* **72** (1912), 578—584.

[5] Bruss, F. T., A counterpart of the Borel—Cantelli lemma, *J. Appl. Probab.* **17** (1980), 1094—1101. *MR* **82f**: 60071.

[6] Cantelli, F. P., La tendenza ad un limite nel senso del calcolo delle probabilità, *Rend. Circ. Mat. Palermo* **41** (1916), 191—201.

[7] Cantelli, F. P., Sulla probabilità come limite della frequenza, *Rend. Accad. Naz. Lincei Roma* **26** (1917), 39—45.

[8] Cantelli, F. P., Su due applicazioni di un theorema di G. Boole alla statistica mathematica, *Rend. Accad. Naz. Lincei Roma* **26** (1917), 295—302.

[9] Dubins, L. E. and Freedman, D. A., A sharper form of the Borel—Cantelli lemma and the strong law, *Ann. Math. Statist.* **36** (1965), 800—807. *MR* **31** #6265.

[10] Erdős, P. and Rényi, A., On Cantor's series with convergent $\sum 1/q_n$, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **2** (1959), 93—109. *MR* **23** #3710.

[11] Faber, G., Über stetige Funktionen II, *Math. Ann.* **69** (1910), 372—443.

[12] Fischler, R., Borel—Cantelli type theorems for mixing sets, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 67—69. *MR* **34** #8435.

[13] Freedman, D. A., Another note on the Borel—Cantelli lemma and the strong law, with the Poisson approximation as a by-product, *Ann. Probability* **1** (1973), 910—925. *MR* **51** #6937

[14] Hausdorff, F., *Grundzüge der Mengenlehre*, Teubner, Leipzig, 1914.

[15] Iosifescu, M. and Theodorescu, R., *Random processes and learning*, Die Grundlehren der mathematischen Wissenschaften, Band 150, Springer-Verlag, New York, 1969. *MR* **45** #2781.

[16] KOCHEN, S. B. and STONE, C. J., A note on the Borel—Cantelli lemma, *Illinois J. Math.* **8** (1964), 248—251. *MR* **28** #4562.

[17] KRASNOSEL'SKIĬ, M. A. and RUTICKIĬ, JA. B., *Convex functions and Orlicz spaces*, P. Noordhoff, Ltd., Groningen, 1961. *MR* **23** #A4016 (see also *MR* **21** #5144).

[18] LAMPERTI, J., Wiener's test and Markov chains, *J. Math. Anal. Appl.* **6** (1963), 58—66. *MR* **26** #817.

[19] LÉVY, P., *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, Paris, 1937. *Zbl* **16**, 170.

[20] LOÈVE, M., On almost sure convergence, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950, 279—303. University of California Press, Berkeley and Los Angeles, 1951. *MR* **13**—853.

[21] MUCCI, A. G., Ratio limit theorems, *Canad. J. Math.* **28** (1976), 403—407. *MR* **53** #9349.

[22] NASH, S., An extension of the Borel—Cantelli lemma, *Ann. Math. Statistics* **25** (1954), 165—167. *MR* **15**—883.

[23] SERFLING, R. J., A general Poisson approximation theorem, *Ann. Probability* **3** (1975), 726—731. *MR* **52** #1843.

[24] SPITZER, F., *Principles of random walk*, The University Series in Higher Mathematics, D. Van Nostrand Co., Princeton, N. J.—Toronto—London, 1964. *MR* **30** #1521.

[25] PHO DUC TRU and TRAN VINH HIEN, The zero-one law and the strong law of large numbers for a sequence of dependent random variables, *Acta Sci. Vietnam.* **9**/**10** (1974), 100—113 (in Russian). *MR* **53** #11718.

[26] VILLENEUVE, A., Une remarque sur le lemme de Borel—Cantelli, *C. R. Acad. Sci. Paris Sér. A* **279** (1974), 569—572. *MR* **51** #6942.

[27] YOSHIHARA, K., The Borel—Cantelli lemma for strong mixing sequences of events and their applications to LIL, *Kodai Math. J.* **2** (1979), 148—157. *MR* **81m**: 60048.

(Received February 20, 1980)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
VALÓSZÍNŰSÉGSZÁMÍTÁSI TANSZÉK
MÚZEUM KRT. 6—8.
H—1088 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 183—193.

ONE-PARAMETER FAMILIES OF HARMONIC MAPS INTO SPACES OF CONSTANT CURVATURE

GÁBOR TÓTH

Introduction

A map $f\colon M\to M'$ of a compact oriented Riemannian manifold (M, g) into a complete Riemannian manifold (M', g') is harmonic [2] if its tension field $\tau(f)= =d^*f_*(=\operatorname{div} f_*)$ is zero, where f_* denotes the induced 1-form on M with values in the pull-back bundle $F=f^*(T(M'))$ and d^* is the adjoint of the operator of exterior differentiation d of the bundle $F\otimes\wedge^*(T^*(M))$. Recently, P. Baird and J. Eells [1] have defined and exploited the stress energy tensor

$$S(f)=\frac{1}{2}\|f_*\|^2 g-f^*(g')\in C(\odot^2T^*(M)),$$

in particular, if f is harmonic, then div $S(f)$ vanishes.

A vector field v along f defines a variation of f by $f_t=\exp'\circ(t\cdot v)$, $t\in R$. The purpose of this paper is to describe the behaviour of the one-parameter families of tension fields $\tau(f_t)$ and stress energy tensors $S(f_t)$, $t\in R$, when $v\in C(F)$ is fixed. Throughout this note we shall suppose that M' is locally symmetric $(\nabla'R'=0)$ and [2] will be our general reference for the notions used here. All manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞.

In Section 1 we deduce a Jacobi equation with two initial data which describe the variation of f by v. By means of the solution of this initial value problem we express the tensors $\tau(f_t)$ and div $S(f_t)$, $t\in R$, in terms of the curvature tensor R' of M'. As immediate consequences of this approach we obtain the following known results:

(a) If M' is flat then every map $f\colon M\to M'$ is homotopic to a harmonic map (Eells—Sampson homotopy theorem for flat codomain, [3]);

(b) If $f\colon M\to M'$ is a submersion almost everywhere and div $S(f_t)=0$ for all $t\in R$, then v defines a variation of f through harmonic maps, i.e. f_t is harmonic for all $t\in R$ (a recent result of Baird—Eells, [1]);

(c) If M' is negatively curved and $v\neq 0$ defines a variation of f through harmonic maps then $dv=0$ and rank $f\leqq 1$ everywhere on M. Hence, by a result of Sampson [9] either f is constant or f maps onto a closed geodesic γ of M and v is tangent to γ.

In contrast to (b), if $f\colon S^1\to S^2$ is an isometric embedding onto a great circle

1980 *Mathematics Subject Classification.* Primary 58E20; Secondary 53C21.
Key words and phrases. Variation, harmonic map, spaces of constant curvature.

and v is a parallel unit normal section of the normal bundle of f then $\operatorname{div} S(f_t)=0$ for all $t\in R$ (Corollary 2) but f_t is nonharmonic for $t\notin\frac{\pi}{2}Z$.

In Section 2 we assume that M' is a space of constant curvature σ. The initial value problem can then be solved yielding explicit expressions for $\tau(f_t)$ and $\operatorname{div} S(f_t)$. A complete characterization of variations through harmonic maps is given as follows:

THEOREM 1. *Let $f\colon M\to M'$ be harmonic. Then v defines a variation of f through harmonic maps if and only if $\|v\|=$const. and the following equations are satisfied:*
(i) trace $R'(f_*, v)f_*=\nabla^2 v$,
(ii) trace $R'(f_*, v)\mathrm{d}v=0$.

(We adopt the sign conventions of [6].)

A rigidity theorem for harmonic maps into locally symmetric nonpositively curved Riemannian manifold was recently proved by T. Sunada [12]. In the case when M' is a space of constant curvature, using a result of A. Lichnerowicz [7], we obtain an infinitesimal rigidity theorem as follows:

THEOREM 2. *Let $f\colon M\to M'$ be a harmonic Riemannian submersion almost everywhere, with M' oriented, and V be a vector field on M'. If $v=V\cdot f$ is a variation of f through harmonic maps then V is a Killing vector field and for $t\in R$ we have $f_t=h_t\cdot f$, where $h_t=\exp'\circ(tV)$ is an isometry.*

If $M=M'$ is a compact Kähler manifold and $f=$identity then the complex analogue of this result is well-known, namely, if v is a variation of f through harmonic maps then v is an infinitesimal holomorphic transformation, [10] and [11].

THEOREM 3. *Let $f\colon M\to S^n$ (or RP^n) be a map and suppose that there exists a nonzero parallel vector field v along f with $\|v\|=1$. Denoting*

$$T=\{t\in R\,|\,\tau(f_t)=0\}$$

we have $\pm\pi+T\subset T$ and the following cases can occur:
(a) *$T=R$ and either f is constant or f maps onto a closed geodesic γ with v tangent to γ and the maps f_t can be obtained by rotation;*
(b) *$T\subset\frac{\pi}{2}Z$ and the necessary and sufficient condition for $\frac{\pi}{2}\in T$ is that*

$$\langle \mathrm{d}^*f_*, v\rangle=0 \quad \text{and } \operatorname{trace}\langle f_*, v\rangle^2 v=\operatorname{trace}\langle f_*, v\rangle f_*$$

are satisfied. Moreover $\frac{\pi}{2}\in T$ implies that the Morse index of f is strictly positive, provided that f is harmonic.

We thank Prof. J. Eells for his valuable suggestions to improve the original manuscript drawing our attention to the stress energy tensor.

1. Equations determining the tension field and the stress energy tensor

Let (M, g) be a compact oriented Riemannian manifold and (M', g') be a complete locally symmetric Riemannian manifold. Given a map $f\colon M \to M'$ and a vector field v along f we define $f_t\colon M \to M'$, $t \in R$, by $f_t = \exp' \circ (t \cdot v)$. Let $F^t = (f)^* \cdot \cdot (T(M'))$ be the pull-back of the tangent bundle of M' via f_t. The canonical metric and connection on F^t will be denoted by $\langle\, ,\rangle_t$ and ∇^t, resp. If $t', t'' \in R$ then there is a canonical bundle isomorphism $\tau^{t'}_{t''}\colon F^{t'} \to F^{t''}$ defined by the parallel transport along the geodesic segments $t \mapsto f_t(x)$, $t \in [t', t'']$ (or $[t'', t']$) and $x \in M$. It extends to an isomorphism $F^{t'} \otimes \Lambda^*(T^*(M)) \to F^{t''} \otimes \Lambda^*(T^*(M))$ which is also denoted by $\tau^{t'}_{t''}$ and we omit 0's in f_0, $\tau^{t'}_0$, etc. Note that $v \in C(F)$ and $f_* \in C(F \otimes T^*(M))$.

In order to deduce a Jacobi equation for the variation of f by v, let $X_x \in T_x(M)$, $x \in M$, and choose a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, with $\gamma(0) = x$ and $\dot\gamma(0) = X_x$. Then $t \mapsto \exp'(tv_{\gamma(s)})$, $t \in R$ and $|s| < \varepsilon$, is a geodesic variation and hence [8] pp. 80, we have

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} \big((f_t)_* X_x\big) + R'\big((f_t)_* X_x, \tau_t v\big)\tau_t v = 0.$$

M' is locally symmetric and so R' commutes with the parallel transport. Applying τ^t to both sides of the equation above and omitting the tangent vector X_x we obtain

$$\text{(A)} \qquad \frac{d^2 P_v(t)}{dt^2} + R'\big(P_v(t), v\big)v = 0,$$

where $P_v(t) = \tau^t (f_t)_* \in C\big(F \otimes T^*(M)\big)$. The two initial data for the Jacobi equation are

$$\text{(B)} \qquad P_v(0) = f_* \quad \text{and} \quad \left.\frac{dP_v(t)}{dt}\right|_{t=0} = dv.$$

For fixed $v \in C(F)$, equation (A) with (B) is an initial value problem with unique solution which can be expanded into a convergent power series in t.

Our present aim is to express $\tau(f_t)$ and $S(f_t)$ in terms of the 1-forms $P_v(t)$ on M with values in F.

LEMMA. *If w is a vector field along f and X is a vector field on M then*

$$(\tau^t \circ \nabla^t_X \circ \tau_t) w - \nabla_X w = -R'\Big(\int_0^t \tau^s (f_s)_* X\, ds, v\Big) w.$$

PROOF. Straightforward, using $\nabla' R' = 0$ and the definition of the curvature tensor R' by parallel transport [4], pp. 54.

We have

$$(d^t)^* (f_t)_* = \operatorname{div}^t (f_t)_* = -\operatorname{trace}\big\{(X, Y) \to \nabla^t_X\big((f_t)_* Y\big)\big\} =$$

$$= (\tau_t \circ d^* \circ \tau^t)(f_t)_* + \tau_t \operatorname{trace}\Big\{(X, Y) \to R'\Big(\int_0^t \tau^s (f_s)_* X\, ds, v\Big)\big(\tau^t (f_t)_* Y\big)\Big\} =$$

$$= \tau_t \Big(d^* P_v(t) + \operatorname{trace}\Big\{(X, Y) \to R'\Big(\int_0^t P_v(s) X\, ds, v\Big) P_v(t) Y\Big\}\Big).$$

The map $f_t\colon M\to M'$ is harmonic if and only if

(C) $$\Psi(v,t) = d^*P_v(t) + \operatorname{trace}\Big\{R'\Big(\int_0^t P_v(s)\,ds,\ v\Big)P_v(t)\Big\} = 0.$$

Proof of (a). If M' is flat then $P_v(t)=f_*+t\,dv$ and $\Psi(v,t)=d^*P_v(t)=$ $=d^*f_*+t\cdot d^*dv$. The de Rham decomposition of f_* has the form $f_*=du+\Omega$ where $\Omega\in C(F\otimes T^*(M))$ is harmonic. Taking $v=-u$ we have $P_v(t)=(1-t)du+\Omega$ and $\Psi(v,t)=(1-t)\,d^*du$. Hence f is homotopic to a harmonic map. A vector field v along f defines a variation of f through harmonic maps if and only if f is harmonic and $dv=0$.

In order to calculate the divergence of the stress energy tensor $S(f_t)$, let Z be a vector field on M and choose a local orthonormal frame $E_1,\dots,E_m$ around a point $x\in M$ with $\nabla_{E_i}E_j(x)=0$, $i,j=1,\dots,m$. Denoting $e_i=E_i(x)$, $i=1,\dots,m$, by straightforward calculation we obtain

$$(-\operatorname{div} S(f_t))(Z_x) = \langle\nabla_Z P_v(t), P_v(t)\rangle_x + \langle d^*P_v(t), P_v(t)Z\rangle_x -$$
$$-\sum_{i=1}^m \big\langle P_v(t)e_i, (\nabla_{e_i}P_v(t))Z_x\big\rangle,$$

where we identified the metric tensor g with the inner product $\langle\,,\rangle$ of the bundle F. On the other hand, we have

$$\big\langle P_v(t), i_Z(dP_v(t))\big\rangle_x = \langle P_v(t), \nabla_Z P_v(t)\rangle_x -$$
$$-\sum_{i=1}^m \big\langle P_v(t)e_i, (\nabla_{e_i}P_v(t))(Z_x)\big\rangle$$

and thus

(D) $$(-\operatorname{div} S(f_t))(Z) = \langle d^*P_v(t), P_v(t)Z\rangle + \big\langle P_v(t), i_Z\big((dP_v(t))\big)\big\rangle.$$

Especially, $(-\operatorname{div} S(f))(Z)=\langle d^*f_*, f_*(Z)\rangle$ since f_* is closed. This latter formula can be used to calculate $\operatorname{div} S(f_t)$ in a different way as follows:

$$(-\operatorname{div} S(f_t))(Z) = \langle d^*(f_t)_*, (f_t)_*Z\rangle = \langle(\tau^t\circ d^*\circ\tau_t)P_v(t), P_v(t)Z\rangle =$$
$$= \langle d^*P_v(t), P_v(t)Z\rangle + \Big\langle \operatorname{trace}\Big\{R'\Big(\int_0^t P_v(s)\,ds,\ v\Big)P_v(t)\Big\}, P_v(t)Z\Big\rangle.$$

Thus we obtain

(D′) $$(-\operatorname{div} S(f_t))(Z) = \langle\Psi(v,t), P_v(t)Z\rangle$$

for all $t\in R$.

Proof of (b). Suppose that $f\colon M\to M'$ is a submersion almost everywhere with $\operatorname{div} S(f_t)=0$ for all $t\in R$. Then $(-\operatorname{div} S(f))(Z)=\langle d^*f_*, f_*(Z)\rangle=0$ implies that

$$\Psi(v,0) = d^*f_* = 0. \quad \text{By induction}$$

$$\left.\frac{d^r\Psi(v,t)}{dt^r}\right|_{t=0} = 0$$

and hence $\Psi(v,t)=0$ for all $t\in R$.

REMARK. Suppose that a vector field along f defines a variation of f through harmonic maps. Calculating the first three of the Taylor expansion of $\Psi(v, t)$ we obtain

(i) trace $R'(f_*, v)f_* = \nabla^2 v$,

(ii) trace $R'(f_*, v)\,\mathrm{d}v = 0$.

PROOF OF (c). If M' is negatively curved then, by (i), we have

$$0 \leqq -\int_M \operatorname{trace}\langle R'(f_*, v)v, f_*\rangle \operatorname{vol}(M) = \int_M \langle \nabla^2 v, v\rangle \operatorname{vol}(M) \leqq 0,$$

where vol (M) denotes the volume form of M. Hence $\mathrm{d}v=0$ and if $v\neq 0$ then rank $f \leqq 1$ on M.

2. Maps into spaces of constant curvature

Throughout this section M' will denote a complete manifold of constant curvature $\sigma\neq 0$. Then the solution of the initial value problem (A) with (B) has the form

$$P_v(t) = \left\langle f_* + t\cdot \mathrm{d}v, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|} + \cos\left(t\sqrt{\sigma}\,\|v\|\right)\left(f_* - \left\langle f_*, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right) +$$

$$\frac{\sin\left(t\sqrt{\sigma}\,\|v\|\right)}{\sqrt{\sigma}\,\|v\|}\left(\mathrm{d}v - \left\langle \mathrm{d}v, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}\right)$$

(if $v_x=0$ at $x\in M$ then we take the corresponding limits). Substituting it into (C) we obtain the following expression:

$$\Psi(v, t) = \cos(\alpha t)\,\mathrm{d}^* f_* - \frac{\sin(\alpha t)}{\alpha}\nabla^2 v +$$

$$+\frac{\sin(\alpha t)}{\alpha}\operatorname{trace} R'(f_*, v)f_* + \frac{2t\sin(\alpha t)}{\alpha}\operatorname{trace} R'(f_*, v)\,\mathrm{d}v +$$

$$+\frac{2\sin(\alpha t) - 2\alpha t\cos(\alpha t)}{\alpha^3}\operatorname{trace} R'(\mathrm{d}v, v)\,\mathrm{d}v -$$

$$-\sigma\frac{\cos(\alpha t)-1}{\alpha^2}\langle \mathrm{d}^* f_*, v\rangle v + \sigma\frac{\sin(\alpha t)-\alpha t}{\alpha^3}\langle \nabla^2 v, v\rangle v +$$

$$+\sigma\frac{\sin(2\alpha t) - 2\sin(\alpha t)}{2\alpha^3}\operatorname{trace}\langle R'(f_*, v)f_*, v\rangle v +$$

$$+\sigma\frac{-\cos(2\alpha t) - 2\alpha t\sin(\alpha t) + 1}{\alpha^4}\operatorname{trace}\langle R'(f_*, v)\,\mathrm{d}v, v\rangle v +$$

$$+\sigma\frac{-\sin(2\alpha t) - 4\sin(\alpha t) + 4\alpha t\cos(\alpha t) + 2\alpha t}{2\alpha^5}\operatorname{trace}\langle R'(\mathrm{d}v, v)\,\mathrm{d}v, v\rangle v,$$

where $\alpha = \sqrt{\sigma}\|v\|$.

PROOF OF THEOREM 1. Taking the Taylor expansion of $\Psi(v,t)=0$ up to the third degree equations (i)—(ii) of Theorem 1 are obtained. The fourth term of the Taylor expansion yields trace $R'(dv,v)dv=\sigma\langle\nabla^2 v,v\rangle v$. Thus $\|v\|^2\nabla^2(\|v\|^2)=2$ trace $\langle dv,v\rangle^2$ holds and hence $\nabla^2(\|v\|^2)\geqq 0$. Because M is compact we obtain that $\|v\|=$const. Substituting the equations (i)–(ii) into the expression of $\Psi(v,t)$ above we obtain that $\Psi(v,t)$ vanishes for all $t\in R$.

COROLLARY 1. *Let $f\colon M\to M'$ be a map and suppose that v is a nowhere zero vector field along f with non identically constant norm $\|v\|$. Then there are only finitely many parameter values for which f_t is harmonic.*

PROOF. Suppose that the set $T=\{t\in R\mid \tau(f_t)=0\}$ is infinite. Then T is unbounded and, by the formula above, we have

$$\int_M \langle\Psi(v,t),v\rangle\operatorname{vol}(M)=-t\int_M\langle\nabla^2 v,v\rangle\operatorname{vol}(M)-$$

$$-t\int_M\|dv\|^2\operatorname{vol}(M)+t\int_M\frac{1}{\|v\|^2}\operatorname{trace}\langle dv,v\rangle^2\operatorname{vol}(M)+O(1),$$

and hence

$$\int_M\frac{1}{\|v\|^2}\operatorname{trace}\langle dv,v\rangle^2\operatorname{vol}(M)=0.$$

Thus trace $\langle dv,v\rangle^2=0$ and by choosing an orthonormal frame $\{e_1,\dots,e_m\}\subset T_x(M)$ at some $x\in M$ we have trace $\langle dv,v\rangle^2=\frac{1}{4}\sum_{i=1}^{m}(e_i(\|v\|^2))^2=0$, i.e. it follows that $\|v\|=$const. which is contradiction.

The explicit expression of the stress energy tensor is rather complicated but in case when $\|v\|=\text{const}\neq 0$ its divergence reduces to the following:

$$(-\operatorname{div}S(f_t))(Z)=\cos^2(\alpha t)\langle d^*f_*,f_*(Z)\rangle+\frac{\sin(\alpha t)}{\alpha}\cdot$$

$$\cdot\Big\{\Big\langle\cos(\alpha t)(-\nabla^2 v+\operatorname{trace}R'(f_*,v)f_*)+\sigma\frac{\sin(\alpha t)}{\alpha}\langle d^*f_*,v\rangle v-$$

$$-\sigma\frac{2\sin(\alpha t)}{\alpha}\langle f_*,dv\rangle v,f_*(Z)\Big\rangle+\Big\langle\cos(\alpha t)\,d^*f_*+$$

$$+\frac{\sin(\alpha t)}{\alpha}(-\nabla^2 v+\operatorname{trace}R'(f_*,v)f_*),\nabla_Z v\Big\rangle\Big\},$$

where Z is a vector field on M.

PROPOSITION 1. *Let $f\colon M\to M'$ be a map and v be a vector field along f with $\|v\|=$const. Then $(\operatorname{div}S(f_t))(Z)=0$ for all $t\in R$ if and only if the following equations* *are satisfied:*

(i′) $\langle d^*f_*, f_*(Z)\rangle = 0$,

(ii′) $\langle d^*f_*, \nabla_Z v\rangle + \langle -\nabla^2 v + \operatorname{trace} R'(f_*, v)f_*, f_*(Z)\rangle = 0$,

(iii′) $\langle -\nabla^2 v + \operatorname{trace} R'(f_*, v)f_*, \nabla_Z v\rangle + \sigma(\langle d^*f_*, v\rangle -$
$-2\langle f_*, dv\rangle)\langle v, f_*(Z)\rangle = 0$.

PROOF. Straightforward, using the formula above.

COROLLARY 2. *Let $f\colon M\to M'$ be a harmonic map and v be a parallel vector field along f. Then* $\operatorname{div} S(f_t)=0$ *for all* $t\in R$.

COROLLARY 3. *Let $f\colon M\to M'$ be a map and v be a vector field along f with $\|v\| = const.\neq 0$. If*

$$\operatorname{div} S(f_{-t_0}) = \operatorname{div} S(f) = \operatorname{div} S(f_{t_0}) = 0$$

for some $t_0 \notin \dfrac{\pi}{2\sqrt{\sigma}\|v\|} Z$ *then* $\operatorname{div} S(f_t)=0$ *for all* $t\in R$.

PROOF. Let Z be a vector field on M. It is enough to show that (i′)—(iii′) are satisfied in Proposition 1. Equation (i′) is valid because $(-\operatorname{div} S(f))(Z) = \langle d^*f_*, f_*(Z)\rangle = 0$. Furthermore, using (i′), we have

$$\frac{1}{2}\{(-\operatorname{div} S(f_{t_0}))(Z) - (-\operatorname{div} S(f_{-t_0}))(Z)\} = \frac{\sin(2\alpha t_0)}{2\alpha}.$$

$$\{\langle d^*f_*, \nabla_Z v\rangle + \langle -\nabla^2 v + \operatorname{trace} R'(f_*, v)f_*, f_*(Z)\rangle\} = 0$$

and

$$\frac{1}{2}\{(-\operatorname{div} S(f_{t_0}))(Z) + (-\operatorname{div} S(f_{-t_0}))(Z)\} = \frac{\sin^2(\alpha t_0)}{\alpha^2}\times$$

$$\times\{\langle -\nabla^2 v + \operatorname{trace} R'(f_*, v)f_*, \nabla_Z v\rangle + \sigma(\langle d^*f_*, v\rangle - 2\langle f_*, dv\rangle)\langle v, f_*(Z)\rangle\} = 0$$

which implies (ii′) and (iii′), resp.

This corollary and (b) yield:

COROLLARY 4. *Let $f\colon M\to M'$ be a submersion almost everywhere and v a vector field along f with $\|v\| = const.$ If*

$$\operatorname{div} S(f_{-t_0}) = \operatorname{div} S(f) = \operatorname{div} S(f_{t_0}) = 0$$

for some $t_0 \notin \dfrac{\pi}{2\sqrt{\sigma}\|v\|} Z$ *then v is a variation of f through harmonic maps.*

Before proving Theorem 2 we have the following

PROPOSITION 2. *Let M be a totally geodesic submanifold of M' with natural inclusion $f\colon M\subset M'$ and let v be a vector field along f which defines a variation of f through harmonic maps. By the orthogonal decomposition $v = v^\perp + v^\top$, the tangential part $v^\top$ is a Killing vector field on M and $v^\perp$ satisfies the strongly elliptic equation* $\nabla^2 v^\perp + \sigma m v^\perp = 0$, *where* $m = \dim M$.

PROOF. Theorem 1 implies that $\|v\|=$const. and, since $M\subset M'$ is totally geodesic, the following equations are valid:

$$\nabla^2 v^\perp + \sigma m v^\perp = 0, \tag{1}$$

$$\nabla^2 v^\top + \sigma(m-1)v^\top = 0, \tag{2}$$

$$\langle f_*, \mathrm{d}v^\top\rangle = 0. \tag{3}$$

The first equation is strongly elliptic and has uniqueness in the Cauchy problem [10], i.e. if $v^\perp|U=0$, where $U\subset M$ is an open set, then $v^\perp=0$ on M. In order to prove that $v^\top$ is a Killing vector field on M, denote β the 1-form on M which corresponds to $v^\top$ by duality. The harmonicity of f implies that $\langle f_*, \mathrm{d}v^\top\rangle = -\mathrm{d}^*\beta=0$. Furthermore $\nabla^2\beta+\sigma(m-1)\beta=0$ is valid and so

$$\Delta\beta - 2\sigma(m-1)\beta + \mathrm{d}(\mathrm{d}^*\beta) = 0.$$

By a result of Lichnerowicz [7] it means that $v^\top$ is a Killing vector field which accomplishes the proof.

REMARK. If $M=S^m$, $M'=S^{m+k}$ and $f\colon S^m\to S^{m+k}$ is the canonical embedding then choosing globally defined linearly independent parallel sections $w_1, \dots, w_k$ of the normal bundle of f equation $\nabla^2 v^\perp + m v^\perp=0$ splits into the equations

$$\Delta\langle v^\perp, w_j\rangle + m\langle v^\perp, w_j\rangle = 0, \quad j=1, \dots, k.$$

Thus the scalar $\langle v^\perp, w_j\rangle$ on $S^m\subset R^{m+1}$, being an eigenfunction of the Laplacian, is the restriction of a homogeneous polynomial of degree 1 on R^{m+1}.

PROOF OF THEOREM 2. Let $f\colon M\to M'$ be a harmonic Riemannian submersion almost everywhere and V be a vector field on M' such that $v=V\circ f$ defines a variation of f through harmonic maps. By Theorem 1, $\|V\|=$const. and we need only to show that V is a Killing vector field on M'. Writing $f_t=h_t\circ f$, where $h_t=\exp'\circ$ $\circ(t\cdot V)$, $t\in R$, the harmonicity of f_t implies [2]

$$\mathrm{d}^*(f_t)_* = (h_t)_*\mathrm{d}^*f_* + \mathrm{trace}\,(\nabla(h_t)_*)(f_*, f_*) = 0$$

and hence trace $(\nabla(h_t)_*)(f_*, f_*)=0$. Because f is Riemannian this latter equation reduces to $\mathrm{d}^*(h_t)_*=0$, i.e. V is harmonic variation of the identity map of M'. Thus Proposition 2 implies that h_t is an isometry for every $t\in R$.

In what follows we restrict ourselves to the case when $M=S^n$ (or RP^n).

PROOF OF THEOREM 3. Let $f\colon M\to S^n$(or RP^n) be a map and suppose that v is a parallel vector field along f with $\|v\|=1$. Using the antipodal map, we have $\pm\pi+T\subset T$, where $T=\{t\in R \mid \tau(f_t)=0\}$. By the formula of $\Psi(v, t)$ at the beginning of Section 2 we obtain

$$\Psi(v, t) = \cos t\,\mathrm{d}^*f_* + (1-\cos t)\langle \mathrm{d}^*f_*, v\rangle v +$$

$$+\sin t\,\mathrm{trace}\,\langle f_*, v\rangle f_* + \frac{\sin(2t)-2\sin t}{2}\,\mathrm{trace}\,\langle f_*, v\rangle^2 v - \frac{\sin(2t)}{2}\,\|f\|^2 v$$

and hence

$$\int_M \langle \Psi(v, t), v\rangle \operatorname{vol}(M) = \frac{\sin(2t)}{2} \int_M (\operatorname{trace}\langle f_*, v\rangle^2 - \|f_*\|^2) \operatorname{vol}(M).$$

(a) Let us suppose that there exists $t_0 \in T - \frac{\pi}{2} Z$. Then

$$\int_M (\operatorname{trace}\langle f_*, v\rangle^2 - \|f_*\|^2) \operatorname{vol}(M) = 0$$

and hence trace $\langle f_*, v\rangle^2 = \|f_*\|^2$ holds, i.e. $f_* = v \otimes \omega$, where ω is the 1-form on M defined by $\omega(Z) = \langle f_*(Z), v\rangle$ for every vector field Z on M. Especially, rank $f \leqq 1$ on M. Because v is parallel we have

$$\mathrm{d}f_* = \mathrm{d}v \wedge \omega + v \otimes \mathrm{d}\omega = v \otimes \mathrm{d}\omega = 0,$$

i.e. it follows that $\mathrm{d}\omega = 0$, and

$$\mathrm{d}^* f_* = \mathrm{d}^*(v \otimes \omega) = (\mathrm{d}^*\omega) v.$$

Using these expressions we obtain $0 = \Psi(v, t_0) = (\mathrm{d}^*\omega) v$ and hence ω is a harmonic 1-form on M. It follows that f is a harmonic map and thus $\Psi(v, t) = 0$ for all $t \in R$, i.e. v is a variation of f through harmonic maps. By a result of Sampson [9] either f is constant or f maps onto a closed geodesic γ with v tangent to γ and the maps f_t can be obtained by rotation.

(b) Let us suppose that there exists a nonharmonic map f_t. By the previous reasonings it follows that $T \subset \frac{\pi}{2} Z$. Moreover, $\frac{\pi}{2} \in T$ if and only if

$$\Psi\left(v, \frac{\pi}{2}\right) = \langle \mathrm{d}^* f_*, v\rangle v + \operatorname{trace}\langle f_*, v\rangle f_* - \operatorname{trace}\langle f_*, v\rangle^2 v = 0.$$

Multiplying by v this equation splits into the two equations given in Theorem 3. If $\frac{\pi}{2} \in T$ then the value of the Hessian H_f on the pair (v, v) [2] reduces to

$$H_f(v, v) = \int_M (\operatorname{trace}\langle f_*, v\rangle^2 - \|f_*\|^2) \|v\|^2 \operatorname{vol}(M).$$

Thus, if the Morse index of f were zero then rank $f \leqq 1$ would yield a contradiction to our assumption $T \neq R$.

COROLLARY 5. *Let f: $M \to S^n$ (or RP^n) be a map and suppose that there exists a vector field v along f, $\|v\| = 1$, such that $f_* = c\,\mathrm{d}v + v \otimes \omega$ and $\mathrm{d}^*(v \otimes \omega) = 0$ are valid for some nonzero constant $c \in R$ and 1-form ω on M. Then f is homotopic to harmonic map.*

PROOF. The 1-form ω occurring in the expression of f_* is unique since $\langle f_*, v\rangle = \omega$. We have

$$0 = \mathrm{d}^*(v \otimes \omega) = (\mathrm{d}^*\omega) v - \operatorname{trace}(\mathrm{d}v \otimes \omega).$$

Multiplying by v we obtain that $\mathrm{d}^*\omega = 0$ and so trace $(\mathrm{d}v \otimes \omega) = 0$. The solution of the initial value problem (A) with (B) takes the form

$$P_v(t) = v\otimes\omega+(c\cos t+\sin t)\,dv$$

and so

$$\frac{\mathrm{d}P_v(t)}{\mathrm{d}t} = (-c\sin t+\cos t)\,\mathrm{d}v.$$

Putting $t_0=\operatorname{arc\,tg}\frac{1}{c}$ we claim that $\tau_{t_0}v$ is a parallel vector field along f_{t_0}. Our Lemma implies that

$$\tau^{t_0}\big(\mathrm{d}(\tau_{t_0}v)\big) = \mathrm{d}v - R'\Big(\int_0^{t_0} P_v(s)\,\mathrm{d}s,\ v\Big)v.$$

The right-hand side is nothing but $\left.\frac{\mathrm{d}P_v(t)}{\mathrm{d}t}\right|_{t=t_0}$ and thus $\mathrm{d}(\tau_{t_0}v)=0$. Hence Theorem 3 can be applied to the map f_{t_0} and $\tau_{t_0}v$. We state that $f_{t_0+(\pi/2)}$ is a harmonic map. This is obvious in case (a) of Theorem 3. In case (b) we have to show that $\langle \mathrm{d}^*(f_{t_0})_*, \tau_{t_0}v\rangle=0$ and trace

$$\langle (f_{t_0}), \tau_{t_0}v\rangle (f_{t_0})_* = \operatorname{trace}\langle (f_{t_0})_*, \tau_{t_0}v\rangle^2 \tau_{t_0}v$$

are satisfied. Because

$$(f_{t_0})_* = \tau_{t_0}P_v(t_0) = \tau_{t_0}v\otimes\omega+\frac{1}{\sin t_0}\tau_{t_0}(\mathrm{d}v).$$

Then

$$\sin t_0\langle \mathrm{d}^*(f_{t_0})_*, \tau_{t_0}v\rangle = \big\langle \mathrm{d}^*\big(\tau_{t_0}(\mathrm{d}v)\big), \tau_{t_0}v\big\rangle = \langle (\tau^{t_0}\circ \mathrm{d}^*\circ\tau_{t_0})(\mathrm{d}v), v\rangle =$$

$$= \langle \nabla^2 v, v\rangle + \Big\langle \operatorname{trace}\Big\{R'\Big(\int_0^{t_0} P_v(s)\,\mathrm{d}s,\ v\Big)dv\Big\}v, \Big\rangle =$$

$$= \langle \nabla^2 v, v\rangle - \operatorname{trace}\Big\langle \int_0^{t_0} P_v(s)\,\mathrm{d}s,\ \mathrm{d}v\Big\rangle = \langle \nabla^2 v, v\rangle - \|\mathrm{d}v\|^2 = 0,$$

the last but one equality is because $\int_0^{t_0} P_v(s)\,\mathrm{d}s = t_0 v\otimes\omega+\mathrm{d}v$. Hence we obtained that $\langle \mathrm{d}^*(f_{t_0})_*, \tau_{t_0}v\rangle=0$. To prove the second equality we have

$$\operatorname{trace}\langle (f_{t_0})_*, \tau_{t_0}v\rangle (f_{t_0}) = \operatorname{trace}(f_{t_0})_*\otimes\omega = \|\omega\|^2\tau_{t_0}v_0+\frac{1}{\sin t_0}\operatorname{trace}\tau_{t_0}(\mathrm{d}v)\otimes\omega =$$

$$= \|\omega\|^2\tau_{t_0}v = \operatorname{trace}\langle (f_{t_0})_*, \tau_{t_0}v\rangle^2\tau_{t_0}v.$$

Thus our Corollary is proved.

REFERENCES

[1] BAIRD, P. and EELLS, J., A conservation law for harmonic maps, Lecture Notes in Mathetics, Vol. 894.
[2] EELLS, J. and LEMAIRE, L., A report on harmonic maps, *Bull. London Math. Soc.* **10** (1978), 1—68. *Zbl* **401**. 58003; *MR* **82b**: 58033.
[3] EELLS, J. and SAMPSON, J. H., Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109—160. *MR* **29** #1603.
[4] GROMOLL, D., KLINGENBERG, W. and MEYER, W., *Riemannsche Geometrie im Grossen*, Lecture Notes in Mathematics, no. 55, Springer-Verlag, Berlin—New York, 1968. *MR* **37** #4751.
[5] HARTMAN, P., On homotopic harmonic maps, *Canad. J. Math.* **19** (1976), 673—687.
[6] KOBAYASHI, S. and NOMIZU, K., *Foundations of differential geometry*, Vol. I, Interscience Publishers, a divison of John Wiley & Sons, New York—London, 1963. *MR* **27** #2945.
[7] LICHNEROWICZ, A., Variétés kähleriennes à première classe de Chern non negative et variétés riemanniennes à courbure de Ricci généralisée non negative, *J. Differential Geometry* **6** (1971/72), 47—94. *MR* **45** #9274.
[8] MILNOR, J. W., *Morse theory*, Annals of Math. Studies, No. 51, Princeton University Press, Princeton, N. J., 1963. *MR* **29** #634.
[9] SAMPSON, J. H., Some properties and applications of harmonic mappings, *Ann. Sci. Ecole Norm. Sup.* (4) **11** (1978), 211—228. *MR* **80b**: 58031.
[10] SIMONS, J., Minimal varieties in riemannian manifolds, *Ann. of Math.* (2) **88** (1968), 62—105. *MR* **38** #1617.
[11] SMITH, R. T., The second variation formula for harmonic mappings, *Proc. Amer. Math. Soc.* **47** (1975), 229—236. *MR* **51** #11580.
[12] SUNADA, T., Rigidity of certain harmonic mappings, *Invent. Math.* **51** (1979), 297—307. *MR* **80f**: 58021.

(Received May 14, 1980)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15.
H—1053 BUDAPEST
HUNGARY

Current address:

DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
231 WEST 18TH AVENUE
COLUMBUS, OH 43210
U.S.A.

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 195—203.

ON THE KOZIOL—GREEN MODEL FOR RANDOM CENSORSHIP II

SÁNDOR CSÖRGŐ and LAJOS HORVÁTH

1. Introduction

Let $X_1^0, \ldots, X_n^0$ be independent random variables with a common continuous distribution function F^0. Another sequence, $Y_1, \ldots, Y_n$ of independent random variables, independent also of the $\{X_j^0\}$, with common continuous distribution function F^* censors the preceding one on the right, so we observe the pairs (X_j, δ_j), $1 \le j \le n$, where $X_j = \min(X_j^0, Y_j)$ and δ_j is the indicator of the event $\{X_j = X_j^0\}$. The distribution function F of X_j satisfies $1-F=(1-F^0)(1-F^*)$. This is the general random censorship model, and the Koziol—Green model is the special case when $1-F^* =(1-F^0)^\beta$ with some constant $\beta>0$. β is the so-called censoring parameter. For motivation and further references as well as for justification of some statements in this Introduction see part I of the present note. We have $\mathsf{E}\delta_j=\gamma$ with

$$0 < \gamma = (1+\beta)^{-1} < 1. \tag{1.1}$$

Consider the product-limit estimator $\hat{F}_n^0$ of F^0 defined by

$$\hat{S}_n(t) = 1-\hat{F}_n^0(t) = \begin{cases} \prod\limits_{\{1 \le j \le n:\, X_j < t\}} \{(n-R_j)/(n-R_j+1)\}^{\delta_j}, & t \le X_{n:n}, \\ 0, & t > X_{n:n}, \end{cases}$$

where $X_{n:n} = \max(X_1, \ldots, X_n)$ and R_j is the rank of $(X_j, 1-\delta_j)$ in the lexicografic ordering of the sequence $(X_1, 1-\delta_1), \ldots, (X_n, 1-\delta_n)$. The product-limit process

$$Z_n(t) = n^{1/2}\{\hat{F}_n^0(t) - F^0(t)\} \tag{1.2}$$

converges weakly to a Gaussian process $G(t)$ on $(-\infty, T^*]$, where T^* is any number satisfying

$$T^* < T^0 = \inf\{t\colon F(t) = 1\}, \tag{1.3}$$

and $\mathsf{E}G(t)=0$ and

$$\mathsf{E}G(s)G(t) = \{1-F^0(s)\}\{1-F^0(t)\} \int_{-\infty}^{\min(s,t)} (1-F^0)^{-(2+\beta)}\, dF^0.$$

This follows from the Efron—Breslow—Crowley theorem. The Efron-transformed

---

1980 *Mathematics Subject Classification.* Primary 62G10, 60F05; Secondary 62G15, 62N05, 62P10, 62E20.

Key words and phrases. Censored data, product-limit estimator, goodness of fit, Efron transformation, Brownian motion, distribution-free statistics, strong approximation, weak convergence, rates of convergence.

variant of this statement is that

$$Z_n^E(t) = [1-F^0\{a(t)\}]^{-1} Z_n\{a(t)\} \tag{1.4}$$

converges weakly on $[0, d(T^*)]$ to the Wiener process, where the functions a and d are defined in (2.1) and (2.2) below. These functions depend on the generally unknown parameter γ. The following problem was well motivated in Part I. Is it possible to estimate this γ from the sample at the beginning, with Brownian motion still being obtained as the weak limit on *each* finite interval $[0, T]$? The answer is 'yes', and this is provided by Theorem 3.5 in Section 3 here. This result was reported as Theorem 4.1 in Part I, where some corollaries and consequences of it were used. Namely, the limit theory of the corresponding Cramér—von Mises statistics is worked out there and a table is also presented for the distribution function of the limiting variable (the square-integral of the Wiener process). The present note is thus devoted to the proof of this theorem and can be viewed as a technical appendix to Part I.

2. Preliminaries

Introduce the Gaussian process

$$\begin{aligned} H(t) = & \int_{-\infty}^{t} B^0(s)\{1-F^0(s)\}^{-(2+\beta)}\, dF^0(s) + \\ & + (1+\beta) \int_{-\infty}^{t} B^1(s)\{1-F^0(s)\}^{-(2+\beta)}\, dF^0(s) - \\ & - B^1(t)\{1-F^0(t)\}^{-(1+\beta)}, \quad -\infty < t < T^0, \end{aligned}$$

where B^0 and B^1 are Brownian bridge-like stochastic processes with $\mathsf{E}B^0(t) = \mathsf{E}B^1(t) = 0$, and

$$\begin{aligned} \mathsf{E}B^0(s)B^0(t) &= \min\{F(s), F(t)\} - F(s)F(t) \\ \mathsf{E}B^1(s)B^1(t) &= \min\{F^1(s), F^1(t)\} - F^1(s)F^1(t) \\ \mathsf{E}B^0(s)B^1(t) &= \min\{F^1(s), F^1(t)\} - F(s)F^1(t), \end{aligned}$$

where F^1 is the sub-distribution function

$$F^1(t) = \mathrm{pr}\,\{X_j < t, \delta_j = 1\} = \gamma F(t)$$

with γ of (1.1). By computation $\mathsf{E}H(t) = 0$,

$$\mathsf{E}H(s)H(t) = \int_{-\infty}^{\min(s,t)} \{1-F^0\}^{-(2+\beta)} dF^0,$$

and hence $G(t) = \{1-F^0(t)\}^{-1}H(t)$ is the limit process mentioned in the Introduction. Set

$$d(t) = \int_{-\infty}^{t} \{1-F(s)\}^{-2}\, dF^1(s) = \gamma[\{1-F^0(t)\}^{-1/\gamma} - 1] = \gamma F(t)\{1-F(t)\}^{-1}. \tag{2.1}$$

The inverse function to d is

$$a(t) = G^0(1-\{(t/\gamma)+1\}^{-\gamma}), \tag{2.2}$$

where $G^0(s)=\sup\{u\colon F^0(u)\leqq s\}$ is the inverse to F^0. Clearly, $W(t)=H(a(t))$ is the standard Wiener process. For T^* satisfying (1.3) put

$$b = \{1-F(T^*)\}^{-1}. \tag{2.3}$$

Finally, for a positive ε, introduce the rate-sequence

$$r(n) = v(n)+\frac{1}{2}n^{-1/2}\left\{v(n)+3\left(\frac{\varepsilon}{2}\right)^{1/2}b^2(\log n)^{1/2}\right\}^2, \tag{2.4}$$

where

$$v(n) = \left[b^2\left\{15A_1+\left(2+\frac{15}{A_2}\right)\varepsilon+\left(\frac{2}{3}\varepsilon+\varepsilon^2\right)^{1/2}+b^3 4\varepsilon\right]n^{-1/2}\log n+$$

$$+b^2(12\varepsilon)^{1/2}n^{-1/3}(\log n)^{1/2}+b^2\left(3A_1+\frac{3\varepsilon}{A_3}\right)\varepsilon^{1/2}n^{-1/3}(\log n)^{3/2}+2bn^{-1/2}$$

with the (known) constants A_1, A_2 and A_3, D as determined in Part I. On the specially constructed probability triple $(\Omega, \mathcal{A}, \mathsf{P})$ of Theorem 3.1 of Burke et al. [1] we constructed two sequences $\{B_n^0\}$ and $\{B_n^1\}$ of replicas of B^0 and B^1, respectively, such that for Z_n^E of (1.4) and the resulting replicas $W_n(t)=H_n\{a(t)\}$ of W we have, as a very special case of Corollary 5.7 of [1] the following

LEMMA 2.1 ([1]). *If T^* satisfies* (1.3) *and* $n/\log n\geqq 2\varepsilon b^2$, *then*

$$\mathsf{P}\Big\{\sup_{0<t\leqq d(T^*)}|Z_n^E(t)-W_n(t)|>br(n)\Big\}\leqq Q_1 n^{-\varepsilon},$$

where $Q_1=30A_2+100+16D$.

Let $B(u)$, $0\leqq u\leqq 1$, be a Brownian bridge. We need the following well-known inequality.

LEMMA 2.2. *For any* $x>0$,

$$\mathrm{pr}\Big\{\sup_{0\leqq u\leqq 1}|B(u)|>x\Big\}\leqq 2\exp(-2x^2).$$

We shall also need a bound for the increments of a Brownian bridge. The following result of [2] was proved in their Lemma 1.1.1 for W in place of B, but the same proof can be applied for B. Alternatively, one may use

$$\{B(u)\colon 0\leqq u\leqq 1\}\overset{\mathcal{D}}{=}\{W(u)-uW(1)\colon 0\leqq u\leqq 1\}$$

and the next inequality follows from theirs with a somewhat worse constant. Here and later $\overset{\mathcal{D}}{=}$ denotes equality in distribution.

LEMMA 2.3 ([2]). *If* $0<h<1$ *and* $v, \delta>0$ *then*

$$\mathrm{pr}\Big\{\sup_{0\le s\le 1-h}\ \sup_{0\le t\le h} |B(s+t)-B(s)| > vh^{1/2}\Big\} \le \frac{R(\delta)}{h}\exp\Big\{-\frac{v^2}{2+\delta}\Big\},$$

where $R(\delta)=8+(32/\delta^2)$.

One more preparatory result. Let

$$\gamma_n = \frac{1}{n}\sum_{j=1}^{n}\delta_j.$$

According to the strong law of large numbers γ_n is a strongly consistent estimator of γ of (1.1). For their deviation we shall need the following bound, which is a special case of Bernstein's inequality. The most convenient form of the latter is found, e.g. in [4], p. 1.

LEMMA 2.4 (Bernstein). *For any* $x>0$,

$$\mathrm{pr}\,\{|\gamma_n-\gamma| > x\} \le 2\exp\{-nx^2/4\}.$$

3. Testing for goodness of fit: limit theory

The following inequalities are easy consequences of Lemma 2.4 and (1.1).

LEMMA 3.1. *For any integer* $n\ge 1$,

$$\mathrm{pr}\,\{|\gamma_n-\gamma| > 2\varepsilon^{1/2}n^{-1/2}(\log n)^{1/2}\} \le 2n^{-\varepsilon},$$

and if $n/\log n \ge 16(1+\beta)^2\varepsilon = 16\varepsilon/\gamma^2$, *then*

$$\mathrm{pr}\,\{(1/\gamma_n) > (2/\gamma)\} \le 2n^{-\varepsilon}.$$

As we noted in the Introduction, we want to "estimate out" γ from Z_n^E of (1.4). Since the null hypothesis completely specifies F^0 and hence its inverse G^0, this means that we have to estimate $a(t)$ in (2.2) by its empirical counterpart $a_n^0(t)$, which is

$$a_n^0(t) = G^0\big(1-\{(t/\gamma_n)+1\}^{-\gamma_n}\big).$$

LEMMA 3.2. *Assume that* F^0 *has a density* f^0 *and* $0<f^0(t)=dF^0(t)/dt<\infty$ *for all* t *in the interval* (S^0, T^0) *where* $-\infty\le S^0=\sup\{t\colon F^0(t)=0\}$, $T^0=\inf\{t\colon F^0(t)=1\}\le\infty$. *If* $M=\sup\{1/f^0(t)\colon \tilde S\le t\le \tilde T\}<\infty$, *where* $\tilde S\ge S^0$ *and* $\tilde T<T^0$, *and if* $n/\log n\ge 16\varepsilon/\gamma^2$, *then*

$$\mathrm{pr}\Big\{\sup_{d(\tilde S)\le t\le d(\tilde T)} |a_n^0(t)-a(t)| > q(n)\Big\} \le 8n^{-\varepsilon}$$

where

$$q(n) = (C_4+C_5)n^{-1/2}(\log n)^{1/2}$$

with $C_4=4M(\tilde b-1)\varepsilon^{1/2}$ *and* $C_5=2M(\log 2\tilde b)\varepsilon^{1/2}$, *where* $\tilde b=\{1-F(\tilde T)\}^{-1}$.

PROOF. By the Lagrange theorem

$$|a_n^0(t)-a(t)| \leqq \frac{1}{f^0\{a(t)\theta\}}\left|\left(\frac{t}{\gamma}+1\right)^{-\gamma}-\left(\frac{t}{\gamma_n}+1\right)^{-\gamma}\right|+$$

$$+\frac{1}{f^0\{a(t)\theta\}}\left|\left(\frac{t}{\gamma_n}+1\right)^{-\gamma}-\left(\frac{t}{\gamma_n}+1\right)^{-\gamma_n}\right|,$$

where $0<\theta=\theta(t)<1$. By applying Lagrange's theorem again to the functions $u^{-\gamma}$ and c^u with $c=(t/\gamma_n)+1$, we obtain

$$|a_n^0(t)-a(t)| \leqq M\gamma\,|(t/\gamma_n)-(t/\gamma)|+M\log\{(t/\gamma_n)+1\}\,|\gamma_n-\gamma|+$$

$$+Md(\tilde{T})(1/\gamma_n)\,|\gamma_n-\gamma|+M\log[\{d(\tilde{T})/\gamma_n\}+1]\,|\gamma_n-\gamma|.$$

Therefore, by the second statement of Lemma 3.1, we get

$$\mathrm{pr}\{\sup_{d(\tilde{S})\leqq t\leqq d(\tilde{T})}|a_n^0(t)-a(t)|>q(n)\}\leqq$$

$$\leqq 2\mathrm{pr}\,\{|\gamma_n-\gamma|>2\varepsilon^{1/2}n^{-1/2}(\log n)^{1/2}\}+4n^{-\varepsilon}\leqq 8n^{-\varepsilon},$$

using also the first statement of Lemma 3.1.

The sample size for Lemma 3.2 to hold is an increasing function of the unknown censoring parameter β. In what follows we prove the further probability inequalities only for n "large enough". Computing threshold numbers would require extra smoothness conditions on f^0 and more involved estimations good for nothing. Yet, it is still easier to display our argument leading to the result of this section if we introduce those theoretical constants which figure in our further rate-sequences.

We shall need the fact that both the functions $d(t)$ and $a(t)$ are monotone nondecreasing and that

$$\text{(3.1)}\qquad \begin{aligned}&\lim_{t\uparrow T^0} d(t)=\infty, &&\lim_{t\downarrow S^0} d(t)=0,\\ &\lim_{t\uparrow\infty} a(t)=T^0, &&\lim_{t\downarrow 0} a(t)=S^0.\end{aligned}$$

From now on S will be the following number:

$$\text{(3.2)}\qquad S=\begin{cases}0 & (\lim_{t\downarrow S^0} f^0(t)>0),\\ \eta & (\lim_{t\downarrow S^0} f^0(t)=0),\end{cases}$$

where η is an arbitrarily fixed positive number. T will be *any* fixed number such that $S<T<\infty$. Since $a(T)<T^0$ and $q(n)\downarrow 0$ as $n\to\infty$, there is a constant $R<\infty$ such that for n large enough

$$[1-F^0\{a(T)+q(n)\}]^{-(1+\beta)}\leqq[1-F^0\{a(T)+q(n)\}]^{-(2+\beta)}\leqq R,$$

and there is a constant L such that for n large enough

$$\sup\{f^0(t)\colon a(S)-q(n)\leqq t\leqq a(T)+q(n)\}\leqq L.$$

Let

$$h_j(n)=M_j q(n)(\log n)^{1/2},\qquad (j=1,2,3)$$

where $M_1=RL(\varepsilon/2)^{1/2}$, $M_2=(1+\beta)RL(\varepsilon/2)^{1/2}$, $M_3=RM_2$, and introduce also

$$h_4(n) = RL^{1/2}\left\{3\left(\frac{1}{2}+\varepsilon\right)\right\}^{1/2}\{q(n)\}^{1/2}(\log n)^{1/2}.$$

We have to estimate the increments of the Gaussian process H introduced in Section 2.

LEMMA 3.3. *Under the conditions of Lemma 3.2 for n large enough, one has*

$$\mathrm{pr}\Big\{\sup_{\substack{|x-y|<q(n)\\ a(S)-q(n)\le x,\, y\le a(T)+q(n)}} |H(x)-H(y)| > h(n)\Big\} \le Q_2 n^{-\varepsilon},$$

where $h(n)=h_1(n)+\ldots+h_4(n)$ *and* $Q_2=6+80/\{L(C_4+C_5)\}$.

PROOF. Let p_n^* denote the probability in question, and let the unspecified sup denote the supremum in this probability. For any x and y in $[a(S)-q(n), a(T)+q(n)]$ we get by Lagrange's theorem that

$$\int_x^y dF^0(u) \le L|x-y|,$$

and

$$|\{1-F^0(x)\}^{-(1+\beta)}-\{1-F^0(y)\}^{-(1+\beta)}| \le (1+\beta)R^2L|x-y|.$$

Therefore, using the representation for H given in Section 2, we obtain that

$$\begin{aligned} p_n^* &\le \mathrm{pr}\Big\{\sup R\Big|\int_x^y B^0(u)\,dF^0(u)\Big| > h_1(n)\Big\}+\\ &+\mathrm{pr}\Big\{\sup(1+\beta)R\Big|\int_x^y B^1(u)\,dF^0(u)\Big| > h_2(n)\Big\}+\\ &+\mathrm{pr}\Big\{\sup\Big|\frac{B^1(x)}{\{1-F^0(x)\}^{1+\beta}}-\frac{B^1(x)}{\{1-F^0(y)\}^{1+\beta}}\Big| > h_3(n)\Big\}+\\ &+\mathrm{pr}\{\sup R|B^1(x)-B^1(y)| > h_4(n)\} \le\\ &\le \mathrm{pr}\Big\{RLq(n)\sup_{-\infty<u<\infty}|B^0(u)| > h_1(n)\Big\}+\\ &+\mathrm{pr}\Big\{(1+\beta)RLq(n)\sup_{-\infty<u<\infty}|B^1(u)| > h_2(n)\Big\}+\\ &+\mathrm{pr}\Big\{(1+\beta)R^2Lq(n)\sup_{-\infty<u<\infty}|B^1(u)| > h_3(n)\Big\}+\\ &+\mathrm{pr}\{\sup R|B^1(x)-B^1(y)| > h_4(n)\}. \end{aligned}$$

Because of the covariance structure of B^0 and B^1 it is clear that

$$\{B^i(u)\colon -\infty<u<\infty\} \stackrel{\mathscr{D}}{=} \{B\{F^i(u)\}\colon -\infty<u<\infty\},$$

where $i=0, 1$ and B is the Brownian bridge process. Again Lagrange's theorem gives, in case of $a(S)-q(n)\leqq x, y\leqq a(T)+q(n)$, that

$$|F^1(x)-F^1(y)| = \gamma|F(x)-F(y)| = \gamma|\{1-F^0(x)\}^{1+\beta}-\{1-F^0(y)\}^{1+\beta}| \leqq L|x-y|.$$

Whence,

$$p_n^* \leqq 3\,\mathrm{pr}\Big\{\sup_{0\leqq u\leqq 1}|B(u)| > (\varepsilon/2)^{1/2}(\log n)^{1/2}\Big\}+$$

$$+\mathrm{pr}\Big\{R\sup_{-\infty<x<\infty}\ \sup_{|t|\leqq Lq(n)}|B\{F^1(x)\}-B\{F^1(y)+t\}| > h_4(n)\Big\} \leqq$$

$$\leqq 3n^{-\varepsilon}+2\,\mathrm{pr}\Big\{R\sup_{0\leqq s\leqq 1-Lq(n)}\ \sup_{0\leqq t\leqq Lq(n)}|B(s+t)-B(s)| > h_4(n)\Big\} \leqq Q_2 n^{-\varepsilon},$$

where we applied Lemma 2.2 and Lemma 2.3 with $\delta=1$.

When putting together Lemmas 2.1 and 3.2, we want to get free from the necessity of taking the supremum on an interval like $(0, d(T)]$, since $d(T)$ still contains unknown theoretical quantities. This is why we need our last lemma.

LEMMA 3.4. *If the conditions of Lemma 3.2 are satisfied, then for n large enough we have*

$$\mathrm{pr}\Big\{|d\{a_n^0(T)\}-T| > Kq(n)\Big\} \leqq 10n^{-\varepsilon},$$

where $K=[f^0\{a(T)\}+1]\{(2T/\gamma)+1\}$.

PROOF. The random variable to be estimated is

$$|d\{a_n^0(T)\}-d\{a(T)\}| \leqq d'(\theta_n)|a_n^0(T)-a(T)|,$$

where d' is the derivative of d and θ_n is a random variable between $a_n^0(T)$ and $a(T)$. A simple computation shows that

$$d'(\theta_n) \leqq \max\{(T/\gamma)+1, (T/\gamma_n)+1\}f^0(\theta_n)$$

and hence the inequality in question follows from Lemmas 3.1 and 3.2.

Now we are able to prove our main result. Our random Efron transform of the product-limit process Z_n is the process

$$Z_n^0(t) = \frac{n^{1/2}[\hat{F}_n^0\{a_n^0(t)\}-F^0\{a_n^0(t)\}]}{1-F^0\{a_n^0(t)\}} =$$

$$= n^{1/2}\left[1-\frac{\hat{S}_n\{a_n^0(t)\}}{\{\gamma_n/(\gamma_n+t)\}^{\gamma_n}}\right],$$

i.e., introducing the notation

$$V_n(s) = n^{1/2}\{1-F^0(s)\}^{-1}\{\hat{F}_n^0(s)-F^0(s)\},$$

we have $Z_n^E(t)=V_n\{a(t)\}$, $Z_n^0(t)=V_n\{a_n^0(t)\}$. On the probability space $(\Omega, \mathcal{A}, \mathsf{P})$ of Lemma 2.1 consider again the special sequence $\{H_n\}$ of copies of H and the resulting sequence $\{W_n\}$ of Wiener processes where $W_n(t)=H_n\{a(t)\}$. Let S in (3.2) be given and let T be any number such that $S<T<\infty$. Because of (3.1), there is a pair $\tilde{S}, \tilde{T}$ such that $\tilde{S}\geqq S^0$, $\tilde{T}<T^0$ and $S=d(\tilde{S})$, $T=d(\tilde{T})$. Interpret M and $\tilde{b}$ of Lemma 3.2 in terms of this pair $\tilde{S}, \tilde{T}$. Again by (3.1) we can find a number T^* such that $T^*<T^0$ and $d(T^*)=T+1$. Interpret b of (2.3) in terms of this T^*. Also $r(n)$ in (2.4), $q(n)$ in Lemmas 3.2, 3.3 and 3.4 are now defined via the just defined $M, \tilde{b}$ and b, while R, L and $h(n)$ in Lemma 3.3 by the just defined $q(n)$. Agreeing on these conventions, the result is

THEOREM 3.5. *Assume that F^0 has density function f^0 and $0<f^0(t)=dF^0(t)/dt<<\infty$ for all t in the open support (S^0, T^0) of F^0. If n is large enough, then on the probability space $(\Omega, \mathcal{A}, \mathsf{P})$ of Lemma 2.1 we have*

$$\mathsf{P}\{\sup_{S\leqq t\leqq T}|Z_n^0(t)-W_n(t)|>r_0(n)\}\leqq Q_0 n^{-\varepsilon},$$

where $r_0(n)=h(n)+br(n)$ *and* $Q_0=Q_1+Q_2+18$.

PROOF. The probability in question is not greater than

$$\mathsf{P}\{\sup_{S\leqq t\leqq T}|H_n\{a(t)\}-H_n\{a_n^0(t)\}|>h(n)\}+$$

$$+\mathsf{P}\{\sup_{S\leqq t\leqq T}|V_n\{a_n^0(t)\}-H_n\{a_n^0(t)\}|>br(n)\}=p_n^{(1)}+p_n^{(2)},\quad \text{say.}$$

Now, by Lemma 3.2,

$$p_n^{(1)}\leqq \mathsf{P}\{\sup_{S\leqq t\leqq T}\sup_{|s|\leqq q(n)}|H_n\{a(t)\}-H_n\{a(t)+s\}|>h(n)\}+8n^{-\varepsilon}\leqq (Q_2+8)n^{-\varepsilon},$$

for the last probability is majorized by p_n^*, the probability in Lemma 3.3. Let n be so large that $Kq(n)\leqq 1$. By Lemmas 3.4 and 2.1 we obtain for the second term that

$$p_n^{(2)}\leqq \mathsf{P}\{\sup_{0\leqq s\leqq a_n^0(T)}|V_n(s)-H_n(s)|>br(n)\}=$$

$$=\mathsf{P}\{\sup_{0\leqq t\leqq d\{a_n^0(T)\}}|Z_n^E(t)-W_n(t)|>br(n)\}\leqq$$

$$\leqq \mathsf{P}\{\sup_{0\leqq t\leqq T+Kq(n)}|Z_n^E(t)-W_n(t)|>br(n)\}+10n^{-\varepsilon}\leqq$$

$$\leqq \mathsf{P}\{\sup_{0\leqq t\leqq d(T^*)}|Z_n^E(t)-W_n(t)|>br(n)\}+10n^{-\varepsilon}\leqq (Q_1+10)n^{-\varepsilon},$$

and hence the theorem.

REFERENCES

[1] Burke, M. D., Csörgő, S. and Horváth, L., Strong approximations of some biometric estimates under random censorship, *Z. Wahrsch. Verw. Gebiete* **56** (1981), 87—112. *MR* **83a**: 62102.
[2] Csörgő, M. and Révész, P., *Strong approximations in probability and statistics*, Academic Press, New York, 1980.
[3] Csörgő, S. and Horváth, L., On the Koziol—Green model for random censorship, *Biometrika* **68** (1981), 391—401. *MR* **83c**: 62076.
[4] Prohorov, Yu. V., The extension of S. N. Bernšteĭn's inequalities to a multidimensional case, *Teor. Verojatnost. i Primenen.* **13** (1968), 266—274 (in Russian). *MR* **37** #5915.

(Received October 20, 1980)

JÓZSEF ATTILA TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
BOLYAI INTÉZET
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 205—210.

ON THE $\mathscr{P}_\Phi$-SPACES AND THE GENERALIZATION OF HERZ'S AND FEFFERMAN'S INEQUALITIES II

S. ISHAK and J. MOGYORÓDI

1. Introduction

In the present note we use the notions of our work [1]. Carl Herz [2] discovered the fact that the "dual" of $\mathscr{P}_1$ is the space

$$BMO_1^+ = \{X\colon \sup_{n\geq 0} \|\mathsf{E}(|X-X_n|\,|\mathscr{F}_n)\|_\infty \stackrel{\text{def}}{=} \|X\|_{BMO_1^+} < +\infty,\ X_0 = 0 \text{ a.e.}\}.$$

In establishing this fact Herz derives the following inequality:

$$|\mathsf{E}(XY)| \leqq 12\|X\|_{\mathscr{P}_1}\|Y\|_{BMO_1^+};\quad X\in\mathscr{P}_1,\ Y\in BMO_1^+.$$

Garsia [3] has proved that there is a natural way of extending the definition of $\mathsf{E}(XY)$ for all $X\in\mathscr{P}_p$ and $Y\in\mathscr{K}_q^+$, where $q=\dfrac{p}{p-1}$, $1\leqq p<+\infty$. For this purpose we have to introduce the following definition.

DEFINITION 1.1. Let $X\in L_1$ and let Φ be any Young function. Consider the class of the random variables γ defined by the formula

$$\Gamma_X^{(\Phi)} = \{\gamma\colon \gamma\in L^\Phi,\ \mathsf{E}(|X-X_n|\,|\mathscr{F}_n) \leqq \mathsf{E}(\gamma|\mathscr{F}_n) \text{ a.e., } \forall n\geqq 0\}.$$

We say that $X\in\mathscr{K}_\Phi^+$ if the set Γ_X^Φ is not empty. In this case we put

$$\|X\|_{\mathscr{K}_\Phi^+} = \inf_{\gamma\in\Gamma_X^{(\Phi)}} \|\gamma\|_\Phi.$$

It can be easily seen that $\|\cdot\|_{\mathscr{K}_\Phi^+}$ is really a norm on $\mathscr{K}_\Phi^+$.

If $\Gamma_X^{(\Phi)}$ is empty then for the sake of completeness we define

$$\|X\|_{\mathscr{K}_\Phi^+} = +\infty.$$

If $\Phi(x)=x^p$, $p>1$, then this definition leads to the classic definition of the $\mathscr{K}_p^+$-spaces (cf. Garsia [3], p. 130).

Remark that obviously we have $\mathscr{K}_\infty^+ = BMO_1^+$.

1980 *Mathematics Subject Classification.* Primary 60G42; Secondary 60G46, 42B30, 42B99.

Key words and phrases. Martingale transforms, L_p-predictable martingales and the generalization of this notion, decomposition of martingales, Herz's inequality, Fefferman's inequality, Hardy spaces, the $\mathscr{K}$-spaces of Garsia and their generalization.

Generalizing the above Herz inequality Garsia established the following one: if $X\in\mathscr{P}_p$ and $Y\in\mathscr{K}_q^+$ then

$$|\mathsf{E}(XY)| \leqq (4+4\log 2)\|X\|_{\mathscr{P}_p}\|Y\|_{\mathscr{K}_q^+}; \quad p \geqq 1, \quad q = \frac{p}{p-1}.$$

In this note we go further and we prove that for arbitrary $X\in\mathscr{P}_\Phi$ and $Y\in\mathscr{K}_\Psi^+$, where (Φ, Ψ) is any pair of conjugate Young functions, we have

$$|\mathsf{E}(XY)| \leqq (8+8\log 2)\|X\|_{\mathscr{P}_\Phi}\|Y\|_{\mathscr{K}_\Psi^+}.$$

We remark that this inequality as well as the preceding ones are "formal" since XY has not necessarily a finite Lebesgue integral when we only know that $X\in\mathscr{P}_\Phi$ and $Y\in\mathscr{K}_\Psi^+$. However, one of the by-products of Herz's inequality is that we can define $\mathsf{E}(XY)$ by setting

$$\mathsf{E}(XY) = \lim_{B\to+\infty} \mathsf{E}(X_B Y),$$

where X_B is a specially constructed bounded random variable tending to X in $\mathscr{P}_\Phi$ and a.e. when $B\to+\infty$.

To prove the above mentioned basic inequality we need some auxiliary results concerning a special martingale transform. These results and assertions are well-known, the proofs can be found e.g. in Garsia [3].

LEMMA 1.1. *Let $X\in\mathscr{P}_1$. Then the martingale transform*

$$X_0' = 0 \quad a.e., \quad X_n' = \sum_{i=1}^{n} (X_i - X_{i-1})/\lambda_{i-1}, \quad n \geqq 1,$$

is a regular martingale. Here $\{\lambda_n\}$ is an L_1-predicting sequence of X.

Without any loss of the generality we can suppose that $\lambda_0 \geqq \varepsilon$ a.e. for some $\varepsilon > 0$.

Consider the martingale $(X_n', \mathscr{F}_n)$ of the preceding lemma and for arbitrary $\alpha > 0$ put $X_0'^{(\alpha)} = 0$ a.e., further for $n \geqq 1$ let

$$X_n'^{(\alpha)} = \sum_{i=1}^{n} d_i' I(\lambda_{i-1} \leqq \alpha),$$

where $\{d_i'\}$ denotes the difference sequence of $(X_n', \mathscr{F}_n)$.

Then $(X_n'^{(\alpha)}, \mathscr{F}_n)$ is the martingale transform of the martingale $(X_n', \mathscr{F}_n)$ and converges a.e. to a limit $X'^{(\alpha)}$ (cf. Garsia [3], p. 72).

LEMMA 1.2. *For arbitrary $n \geqq 1$ we have*

$$X_n'^{(2\alpha)} - X_n'^{(\alpha)} = \sum_{i=0}^{n-1} I(\lambda_{i-1} \leqq \alpha < \lambda_i)(X_n'^{(2\alpha)} - X_i'^{(\alpha)})$$

and the terms of the sum on the right-hand side can be estimated as follows: for $i \leqq n-1$ we have

$$|X_n'^{(2\alpha)} - X_i'^{(\alpha)}| I(\lambda_{i-1} \leqq \alpha < \lambda_i) \leqq (2+2\log 2) I(\lambda_{i-1} \leqq \alpha < \lambda_i),$$

where, for convenience, we set $\lambda_{-1} = 0$.

The following assertion due to Garsia [3] will help us to establish the above announced inequality.

THEOREM 1.1. *Let* $Y\in L_1$ *be a random variable such that with some* $\gamma\in L_1$ *for* $n\geq 0$ *we have the a.e. inequality*

$$\mathsf{E}(|Y-Y_n|\,|\mathcal{F}_n)\leq \mathsf{E}(\gamma|\mathcal{F}_n).$$

If X is a bounded random variable such that $\mathsf{E}(X|\mathcal{F}_0)=0$ *a.e. then*

$$|\mathsf{E}(XY)|\leq (4+4\log 2)\,\mathsf{E}(\gamma\lambda_\infty),$$

where

$$\lambda_\infty=\lim_{n\to+\infty}\lambda_n$$

and $\{\lambda_n\}$ *is any adapted sequence which predicts X in* L_∞.

2. The generalized Herz—Garsia inequality

On the basis of the preceding section we establish now the Herz—Garsia inequality in a generalized form.

For this purpose we introduce the following notation. For arbitrary $B>0$ and for arbitrary $\lambda>0$ let

$$\Theta_B(\lambda)=1-\frac{2}{\lambda}\int_0^B I\left(\frac{\lambda}{2}\leq\alpha<\lambda\right)d\alpha.$$

It can be easily seen that

$$\Theta_B(\lambda)=1-\left(\frac{2B}{\lambda}-1\right)^+\wedge 1.$$

Note that $0\leq\theta_B(\lambda)\leq 1$ and that $\theta_B(\lambda)$ increases when λ increases.

Consider the martingale $(X_n'^{(\alpha)},\mathcal{F}_n)$ defined in the preceding section. By Lemma 1.2 it follows that

$$|X'^{(2\alpha)}-X'^{(\alpha)}|=\left|\lim_{n\to+\infty}(X_n'^{(2\alpha)}-X_n'^{(\alpha)})\right|\leq(2+2\log 2).$$

Thus by the dominated convergence theorem for arbitrary fixed $B>0$ we have

$$\int_0^B(X'^{(2\alpha)}-X'^{(\alpha)})\,d\alpha=\lim_{n\to+\infty}\int_0^B(X_n'^{(2\alpha)}-X_n'^{(\alpha)})\,d\alpha.$$

We prove now

LEMMA 2.1. *Let* $X\in\mathcal{P}_\Phi$, *where* Φ *is an arbitrary Young function. Then for any* $B>0$ *we have*

$$X-2\int_0^B(X'^{(2\alpha)}-X'^{(\alpha)})\,d\alpha\in\mathcal{P}_\Phi$$

and

$$\left\|X-2\int_0^B (X'^{(2\alpha)}-X'^{(\alpha)})\,d\alpha\right\|_{\mathscr{P}_\Phi} \leqq \|2\lambda_\infty \Theta_B(\lambda_\infty)\|_\Phi .$$

Here $\{\lambda_n\}$ *is an optimal* L^Φ*-predicting sequence of* X.

Proof. By the definition of $X_n'^{(\alpha)}$ we get

$$X_n-2\int_0^B (X_n'^{(2\alpha)}-X_n'^{(\alpha)})\,d\alpha =$$

$$= \sum_{i=1}^n d_i - 2\int_0^B \sum_{i=1}^n \frac{d_i}{\lambda_{i-1}} I\left(\frac{\lambda_{i-1}}{2} \leqq \alpha < \lambda_{i-1}\right) d\alpha =$$

$$= \sum_{i=1}^n d_i\left(1-\frac{2}{\lambda_{i-1}}\int_0^B I\left(\frac{\lambda_{i-1}}{2}\leqq\alpha<\lambda_{i-1}\right)d\alpha\right) = \sum_{i=1}^n d_i\Theta_B(\lambda_{i-1}).$$

This is a martingale transform and its a.e. limit is equal to

$$X-2\int_0^B (X'^{(2\alpha)}-X'^{(\alpha)})\,d\alpha$$

as we have remarked before the present lemma. On the other hand by rearranging and using the definition of the d_i's we get

$$\left|X_n-2\int_0^B (X_n'^{(2\alpha)}-X_n'^{(\alpha)})\,d\alpha\right| = \left|\sum_{i=1}^n d_i\Theta_B(\lambda_{i-1})\right| =$$

$$= \left|\sum_{i=1}^n (X_i-X_{i-1})\Theta_B(\lambda_{i-1})\right| = \left|X_n\Theta_B(\lambda_{n-1})+\sum_{i=1}^{n-1} X_i(\Theta_B(\lambda_{i-1})-\Theta_B(\lambda_i))\right| \leqq$$

$$\leqq 2\lambda_{n-1}\Theta_B(\lambda_{n-1}).$$

Consequently, the random variable

$$X-2\int_0^B (X'^{(2\alpha)}-X'^{(\alpha)})\,d\alpha$$

belongs to $\mathscr{P}_\Phi$ and it is predictable in L^Φ by

$$2\lambda_{n-1}\Theta_B(\lambda_{n-1}) \leqq 2\lambda_\infty \in L^\Phi.$$

This proves the lemma.

A consequence of this assertion is the following important

Theorem 2.1. *The bounded random variables are dense in the space* $\mathscr{P}_\Phi$, *where* Φ *is an arbitrary Young function.*

PROOF. Observe that

$$X_B = 2 \int_0^B (X'^{(2\alpha)} - X'^{(\alpha)})\, d\alpha$$

is bounded. By the preceding lemma we have

$$\|X - X_B\|_{\mathscr{P}_\Phi} \leqq 2\|\lambda_\infty \Theta_B(\lambda_\infty)\|_\Phi.$$

Note that $\theta_B(\lambda)\downarrow 0$ if $B \to +\infty$. Thus by the monotone convergence theorem

$$\lim_{B\to+\infty} \|\lambda_\infty \Theta_B(\lambda_\infty)\|_\Phi = 0$$

and so

$$\lim_{B\to+\infty} \|X - X_B\|_{\mathscr{P}_*} = 0.$$

This proves the theorem.

Now we are able to formulate the basic result of this paper.

THEOREM 2.2. *Let* (Φ, Ψ) *be an arbitrary pair of conjugate Young functions. Let* $X \in \mathscr{P}_\Phi$ *and* $Y \in \mathscr{K}_\Psi^+$ *and consider for any* $B>0$ *the random variables*

$$X_B = 2 \int_0^B (X'^{(2\alpha)} - X'^{(\alpha)})\, d\alpha.$$

Then

$$\lim_{B\to+\infty} \mathsf{E}(X_B Y)$$

exists and we have

$$\lim_{B\to+\infty} \mathsf{E}(X_B Y) \leqq (8+8\log 2)\|X\|_{\mathscr{P}_*}\|Y\|_{\mathscr{K}_\Psi^+}.$$

PROOF. If X is bounded then trivially $X \in \mathscr{P}_\Phi$. Consequently, by Theorem 1.1,

$$|\mathsf{E}(XY)| \leqq (4+4\log 2)\mathsf{E}(\lambda_\infty \gamma),$$

since $Y \in \mathscr{K}_\Psi^+$ implies that $Y \in L_1$. Here $\gamma \in \Gamma_X^{(\Psi)}$ and $\lambda_\infty = \lim_{n\to+\infty} \lambda_n$ where $\{\lambda_n\}$ is an L^Φ-predicting sequence of X. Using the generalized Hölder inequality to the right-hand side of the preceding inequality we get

$$|\mathsf{E}(XY)| \leqq (8+8\log 2)\|\lambda_\infty\|_\Phi \|\gamma\|_\Psi := (8+8\log 2)\|X\|_{\mathscr{P}_\Phi}\|\gamma\|_\Psi,$$

since $\{\lambda_n\}$ can be chosen to be optimal. Consequently,

$$|\mathsf{E}(XY)| \leqq (8+8\log 2)\|X\|_{\mathscr{P}_\Phi}\|Y\|_{\mathscr{K}_\Psi^+}.$$

Now let $X \in \mathscr{P}_\Phi$ and $Y \in \mathscr{K}_\Psi^+$ be arbitrary. Since the random variables

$$X_B = 2 \int_0^B (X'^{(2\alpha)} - X'^{(\alpha)})\, d\alpha$$

are bounded, namely

$$|X_B| \leqq (4+4\log 2) B,$$

we have

$$|\mathsf{E}(X_B Y)| \leqq (8+8\log 2)\|X_B\|_{\mathscr{P}_\Phi}\|Y\|_{\mathscr{K}_\Psi^+}.$$

We show that

$$\limsup_{B\to+\infty} \|X_B\|_{\mathscr{P}_\Phi} \leqq \|X\|_{\mathscr{P}_\Phi}.$$

In fact, by the triangle inequality

$$\|X_B\|_{\mathscr{P}_\Phi} \leqq \|X-X_B\|_{\mathscr{P}_\Phi}+\|X\|_{\mathscr{P}_\Phi}$$

and so by Theorem 2.1

$$\limsup_{B\to+\infty} \|X_B\|_{\mathscr{P}_\Phi} \leqq \lim_{B\to+\infty} \|X-X_B\|_{\mathscr{P}_\Phi}+\|X\|_{\mathscr{P}_\Phi} = \|X\|_{\mathscr{P}_\Phi}.$$

On the other hand we show that

$$\lim_{B\to+\infty} \mathsf{E}(X_B Y)$$

exists. To this end it is enough to prove that the expectations $\mathsf{E}(X_B Y)$ have the Cauchy property: if $B_1, B_2\to+\infty$ then

$$|\mathsf{E}(X_{B_1}Y)-\mathsf{E}(X_{B_2}Y)| \to 0.$$

Now

$$|\mathsf{E}(X_{B_1}Y)-\mathsf{E}(X_{B_2}Y)| = \big|\mathsf{E}\big((X_{B_1}-X_{B_2})Y\big)\big|.$$

Using the inequality established in the beginning of this proof to the bounded random variable $X_{B_1}-X_{B_2}$ and to $Y\in\mathscr{K}_\Psi^+$ we get

$$|\mathsf{E}(X_{B_1}Y)-\mathsf{E}(X_{B_2}Y)| \leqq (8+8\log 2)\|X_{B_1}-X_{B_2}\|_{\mathscr{P}_\Phi}\|Y\|_{\mathscr{K}_\Psi^+}.$$

Here $\|X_{B_1}-X_{B_2}\|_{\mathscr{P}_\Phi}$ tends to 0 since by the triangle inequality for the norms we have

$$\|X_{B_1}-X_{B_2}\|_{\mathscr{P}_\Phi} \leqq \|X_{B_1}-X\|_{\mathscr{P}_\Phi}+\|X_{B_2}-X\|_{\mathscr{P}_\Phi}$$

and by Theorem 2.1 the right-hand side tends to 0 as $B_1\to+\infty$ and $B_2\to+\infty$. This proves the assertion.

REFERENCES

[1] ISHAK, S. and MOGYORÓDI, J., On the $\mathscr{P}_\Phi$-spaces and the generalization of Herz's and Fefferman's inequality I, *Studia Sci. Math. Hungar.* **17** (1982), 229—234.
[2] HERZ, C., Bounded mean oscillation and regulated martingales, *Trans. Amer. Math. Soc.* **193** (1974), 199—215. *MR* **50** #5930.
[3] GARSIA, A. M., *Martingale inequalities: Seminar notes on recent progress*, Mathematics Lecture Notes Series, W. A. Benjamin, Inc., Reading, Mass. — London—Amsterdam, 1973. *MR* **56** #6844.

(Received October 27, 1980)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
VALÓSZÍNŰSÉGSZÁMÍTÁSI TANSZÉK
MÚZEUM KRT. 6—8
H—1088 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 211—219.

ON THE $\mathscr{P}_\Phi$-SPACES AND THE GENERALIZATION OF HERZ'S AND FEFFERMAN'S INEQUALITIES III

S. ISHAK and J. MOGYORÓDI

1. Introduction

Fefferman [1] discovered the remarkable fact that the space BMO_2 is the "dual" of the Hardy space $\mathscr{H}_1$ in the sense of functional analysis. We say that the random variable $X\in L_1$, with the martingale $X_n=\mathsf{E}(X|\mathscr{F}_n)$, $n\geqq 0$, $X_0=0$ a.e., belongs to BMO_2 if

$$\|X\|_{BMO_2} = \Big\|\sup_{n\geqq 1}\left(\mathsf{E}((X-X_{n-1})^2|\mathscr{F}_n)\right)^{1/2}\Big\|_\infty < +\infty.$$

Also, we say that $X\in L_1$, with the martingale $X_n=\mathsf{E}(X|\mathscr{F}_n)$, $n\geqq 0$, $X_0=0$ a.e., belongs to $\mathscr{H}_1$ if

$$\|X\|_{\mathscr{H}_1} = \mathsf{E}\left(\left(\sum_{i=1}^{\infty}(X_i-X_{i-1})^2\right)^{1/2}\right) < +\infty.$$

In establishing the above duality Fefferman has derived the following "formal" inequality: if $X\in\mathscr{H}_1$ and $Y\in BMO_2$ then

$$|\mathsf{E}(XY)| \leqq C\|X\|_{\mathscr{H}_1}\|Y\|_{BMO_2}.$$

The word "formal" is used here since XY has not necessarily a finite Lebesgue integral when only $X\in\mathscr{H}_1$ and $Y\in BMO_2$ hold.

However, we can define $\mathsf{E}(XY)$ by setting

$$\mathsf{E}(XY) = \lim_{n\to+\infty}\mathsf{E}(X_nY_n),$$

since it is proved that in this case $\lim\limits_{n\to+\infty}\mathsf{E}(X_nY_n)$ exists.

A. M. Garsia [2] (Theorem I. 3.1) went further and proved a corresponding result concerning the Hardy spaces $\mathscr{H}_p$ for $1<p\leqq 2$. The definition of these is given below.

The purpose of the present note is to extend the validity of the above inequality for every $p\geqq 1$. More generally, we prove the inequality of Garsia for any $\mathscr{H}_\Phi$- and the corresponding $\mathscr{H}_\Psi$-space, where (Φ, Ψ) is an arbitrary pair of conjugate Young-functions such that Φ has moderated growth.

1980 *Mathematics Subject Classification.* Primary 60G42; Secondary 60G46, 42B30, 42B99.

Key words and phrases. Martingale transforms, L_p-predictable martingales and the generalization of this notion, decomposition of martingales, Herz's inequality, Fefferman's inequality, Hardy spaces, the $\mathscr{K}$-spaces of Garsia and their generalization.

2. Basic notions and definitions

We continue our works [3] and [4] and we use the results and notions of them. Let (Φ, Ψ) be an arbitrary pair of conjugate Young functions.

DEFINITION 2.1. Let $X \in L_1$ and consider the set $\Delta_X^{(\Phi)}$ of the random variables γ defined by the formula

$$\Delta_X^{(\Phi)} = \{\gamma\colon \gamma \in L^\Phi, \ \mathsf{E}(|X-X_{n-1}| \,|\, \mathcal{F}_n) \leqq \mathsf{E}(\gamma \,|\, \mathcal{F}_n) \text{ a.e., } \forall n \geqq 1\}.$$

We say that $X \in \mathcal{K}_\Phi$ if $\Delta_X^{(\Phi)}$ is non-empty and in this case we put

$$\|X\|_{\mathcal{K}_\Phi} = \inf_{\gamma \in \Delta_X^{(\Phi)}} \|\gamma\|_\Phi.$$

REMARK 2.1. It is easily seen that $\|\cdot\|_{\mathcal{K}_\Phi}$ is a norm on the space $\mathcal{K}_\Phi$.

REMARK 2.2. $X \in \mathcal{K}_\Phi$ if and only if the inequality

$$\mathsf{E}(|X_n - X_{k-1}| \,|\, \mathcal{F}_k) \leqq \mathsf{E}(\gamma \,|\, \mathcal{F}_k)$$

holds a.e. for all $n \geqq k \geqq 1$ and all $\gamma \in \Delta_X^{(\Phi)}$. In this case $X_n \in \mathcal{K}_\Phi$ and $\|X_n\|_{\mathcal{K}_\Phi} \leqq \|X\|_{\mathcal{K}_\Phi}$.

REMARK 2.3. Consider the space $\mathcal{K}_\Phi^+$ defined in [4]. It is easily seen that $\mathcal{K}_\Phi \subset \mathcal{K}_\Phi^+$ and we have

$$\|X\|_{\mathcal{K}_\Phi^+} \leqq 2\|X\|_{\mathcal{K}_\Phi}.$$

In fact, let $X \in \mathcal{K}_\Phi$. Then for arbitrary $\gamma \in \Delta_X^{(\Phi)}$ the inequality

$$\mathsf{E}(|X-X_n| \,|\, \mathcal{F}_n) \leqq \mathsf{E}(|X-X_{n-1}| \,|\, \mathcal{F}_n) + \mathsf{E}(|X_n - X_{n-1}| \,|\, \mathcal{F}_n) \leqq 2\mathsf{E}(\gamma \,|\, \mathcal{F}_n)$$

holds a.e.

DEFINITION 2.2. We say that $X \in \mathcal{H}_\Phi$ if the quadratic variation $\left(\sum_{i=1}^{\infty} (X_i - X_{i-1})^2\right)^{1/2}$ belongs to L^Φ.

Let us mention that if the power p of Φ is finite then $X \in \mathcal{H}_\Phi$ is equivalent to say that $X^* = \sup_{n \geqq 1} |X_n| \in L^\Phi$ (cf. [6]).

3. A Davis-type decomposition for the random variable of the space $\mathcal{H}_\Phi$ and two interesting inequalities

THEOREM 3.1. *Let $X \in \mathcal{H}_\Phi$ where Φ is an arbitrary Young function having finite power p. Then X can be written in the form*

$$X = Y + Z$$

where $Y \in L^\Phi$ and $Y_n = \mathsf{E}(Y \,|\, \mathcal{F}_n)$, $n \geqq 0$, is a martingale such that $Y_0 = 0$ a.e. and with $X_n^* = \max_{n \geqq k \geqq 1} |X_k|$ and $X^* = \sup_{n \geqq 1} |X_n|$ we have

$$|Y| \leqq \sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \leqq 4X^* + 4 \sum_{i=1}^{\infty} \mathsf{E}(X_i^* - X_{i-1}^* | \mathcal{F}_{i-1}) \in L^\Phi.$$

Further, we have $Z\in\mathscr{P}_\Phi$ *and with* $Z_n=\mathsf{E}(Z|\mathscr{F}_n)$, $n\geqq 0$,

$$Z_0=0 \quad a.e., \quad |Z_n|\leqq 13X^*_{n-1}+4\sum_{i=1}^{n-1}\mathsf{E}(X^*_i-X^*_{i-1}|\mathscr{F}_{i-1}), \quad n\geqq 1.$$

Finally,

$$\|\sum_{i=1}^{\infty}|\mathsf{E}(Y|\mathscr{F}_i)-\mathsf{E}(Y|\mathscr{F}_{i-1})|\|_\Phi\leqq(4+4p)\|X^*\|_\Phi$$

and

$$\|Z\|_{\mathscr{P}_\Phi}\leqq(13+4p)\|X^*\|_\Phi.$$

The proof of this theorem follows in all the steps that of the case $\Phi(x)=x^p$, $p>1$ (cf. [2], p. 141, Theorem IV. 2.2).

THEOREM 3.2. *Let* $(X_n,\mathscr{F}_n)$, $n\geqq 0$, *be a martingale and suppose that* $X_0=0$ *a.e. Let us denote by* $\{d_i\}$ *the difference sequence of this martingale and suppose that for all* $i\geqq 1$ *we have a.e.*

$$|d_i|\leqq\delta_i,$$

where the random variables δ_i *are* $\mathscr{F}_i$*-measurable and such that*

$$\sum_{i=1}^{\infty}\delta_i\in L^\Phi.$$

If $Y\in\mathscr{K}_\Psi$ *then the expectation* $\mathsf{E}(X_nY_n)$ *exists and is finite. Further, we have*

$$\mathsf{E}(X_nY_n)=\sum_{i=1}^{n}\mathsf{E}(d_id_i')$$

and

$$|\mathsf{E}(X_nY_n)|\leqq 2\|\sum_{i=1}^{n}\delta_i\|_\Phi\|X_n\|_{\mathscr{K}_\Psi},$$

where $\{d_i'\}$ *denotes the martingale difference sequence corresponding to* $(Y_n,\mathscr{F}_n)$. *Moreover*

$$\lim_{n\to+\infty}\mathsf{E}(X_nY_n)$$

exists and is finite. We have

$$\left|\lim_{n\to+\infty}\mathsf{E}(X_nY_n)\right|\leqq 2\|\sum_{i=1}^{\infty}\delta_i\|_\Phi\|Y\|_{\mathscr{K}_\Psi}.$$

PROOF. Since $Y\in\mathscr{K}_\Psi$ by Remark 2.2 we deduce that for any $\gamma\in\Delta_Y^{(\Psi)}$

$$|d_i'|=\mathsf{E}(|Y_i-Y_{i-1}|\mathscr{F}_i)\leqq\mathsf{E}(\gamma|\mathscr{F}_i),\quad i\geqq 1.$$

It follows that

$$\mathsf{E}(|d_id_j'|)\leqq\mathsf{E}(\delta_i\mathsf{E}(\gamma|\mathscr{F}_j))\leqq 2\|\delta_i\|_\Phi\|\mathsf{E}(\gamma|\mathscr{F}_j)\|_\Psi\leqq$$

$$\leqq 2\|\delta_i\|_\Phi\|\gamma\|_\Psi<+\infty.$$

Here we have used the Hölder inequality. Consequently, $\mathsf{E}(X_nY_n)$ is finite and again by the Hölder inequality

$$|\mathsf{E}(X_nY_n)| = |\sum_{i=1}^{n} \mathsf{E}(d_id_i')| \leqq \sum_{i=1}^{n} \mathsf{E}(\delta_i \mathsf{E}(\gamma|\mathscr{F}_i)) =$$

$$= \mathsf{E}((\sum_{i=1}^{n} \delta_i)\gamma) \leqq 2\|\sum_{i=1}^{n} \delta_i\|_\Phi \|\gamma\|_\Psi < +\infty.$$

By the same method for arbitrary indices $m \leqq n$ we have

$$|\mathsf{E}(X_nY_n) - \mathsf{E}(X_mY_m)| = |\sum_{i=m+1}^{m} \mathsf{E}(d_id_i')| \leqq$$

$$\leqq \sum_{i=m+1}^{n} \mathsf{E}(\delta_i \mathsf{E}(\gamma|\mathscr{F}_i)) = \sum_{i=m+1}^{n} \mathsf{E}(\delta_i\gamma) =$$

$$= \mathsf{E}((\sum_{i=m+1}^{n} \delta_i)\gamma) \leqq 2\|\sum_{i=m+1}^{n} \delta_i\|_\Phi \|\gamma\|_\Psi.$$

Thus from our assumptions we deduce that $\{\mathsf{E}(X_nY_n)\}$ forms a Cauchy sequence. Consequently,

$$\lim_{n\to+\infty} \mathsf{E}(X_nY_n)$$

exists. Finally, we trivially have

$$|\lim_{n\to+\infty} \mathsf{E}(X_nY_n)| \leqq \lim_{n\to+\infty} \sum_{i=1}^{n} \mathsf{E}(|d_id_i'|) \leqq$$

$$\leqq \lim_{n\to+\infty} \sum_{i=1}^{n} \mathsf{E}(\delta_i \mathsf{E}(\gamma|\mathscr{F}_i)) = \sum_{i=1}^{\infty} \mathsf{E}(\delta_i\gamma) =$$

$$= \mathsf{E}((\sum_{i=1}^{\infty} \delta_i)\gamma) \leqq 2\|\sum_{i=1}^{\infty} \delta_i\|_\Phi \|\gamma\|_\Psi.$$

This proves the assertion.

This theorem concerns the term Y of the decomposition $X=Y+Z$ in Theorem 3.1 while the following one deals with Z.

We recall from [4] the following result. To every random variable $X \in \mathscr{P}_\Phi$ we can order a random variable X_C corresponding to arbitrary positive C, where X_C is such that $|X_C| \leqq (2+2\log 2)C$ and the a.e. and $\mathscr{P}_\Phi$ limit of it is X.

Now we are able to formulate

THEOREM 3.3. *Let (Φ, Ψ) be a pair of conjugate Young functions and suppose that $X \in \mathscr{P}_\Phi$ and $Y \in \mathscr{K}_\Psi$. Then the expectation $\mathsf{E}(X_nY_n)$ is finite for arbitrary fixed $n \geqq 1$ and we have*

$$\mathsf{E}(X_nY_n) = \sum_{i=1}^{n} \mathsf{E}(d_id_i'),$$

where $\{d_i\}$ and $\{d_i'\}$ denote the difference sequences corresponding to the martingales $(X_n, \mathscr{F}_n)$ and $(Y_n, \mathscr{F}_n)$, respectively. We also have

$$\mathsf{E}(X_n Y_n) = \lim_{C\to+\infty} \mathsf{E}(X_{n_C} Y_n)$$

and

$$|\mathsf{E}(X_n Y_n)| \leqq (16+16\log 2)\|X_n\|_{\mathscr{P}_\Phi}\|Y_n\|_{\mathscr{K}_\Psi},$$

where X_{n_C} is defined as above (see [4]*) for $X=X_n\in\mathscr{P}_\Phi$.*

Proof. $|X_n|$ is bounded by $\lambda_{n-1}\leqq\lambda_\infty\in L^\Phi$. Consequently,

$$|d_n| = |X_n - X_{n-1}| \leqq 2\lambda_{n-1}\in L^\Phi.$$

Further, if $Y\in\mathscr{K}_\Psi$ then with arbitrary $\gamma\in\Delta_Y^{(\Psi)}$ we have by Remark 2.2 that

$$|d_i'| \leqq \mathsf{E}(\gamma|\mathscr{F}_i).$$

Also, together with $\gamma\in L^\Psi$ we also have $\mathsf{E}(\gamma|\mathscr{F}_i)\in L^\Psi$ for arbitrary $i\geqq 1$. Thus $d_i'\in L^\Psi$ and by the Hölder inequality

$$\mathsf{E}(|d_i d_j'|) \leqq 2\mathsf{E}(\lambda_{i-1}\mathsf{E}(\gamma|\mathscr{F}_j)) \leqq 4\|\lambda_{i-1}\|_\Phi\|\mathsf{E}(\gamma|\mathscr{F}_j)\|_\Psi < +\infty.$$

This means that

$$|\mathsf{E}(X_n Y_n)| < +\infty \quad\text{and}\quad \mathsf{E}(X_n Y_n) = \sum_{i=1}^n \mathsf{E}(d_i d_i').$$

Now we prove that

$$\mathsf{E}(X_n Y_n) = \lim_{C\to+\infty} \mathsf{E}(X_{n_C} Y_n),$$

where X_{n_C} is the random variable defined above for arbitrary $C>0$. In [4] we have shown that for arbitrary $C>0$ we have the inequality

$$|X_n - X_{n_C}| = |\sum_{i=1}^n d_i \Theta_C(\lambda_{i-1})| \leqq 2\lambda_{n-1}\Theta_C(\lambda_{n-1}),$$

where by definition

$$0 \leqq \Theta_C(\lambda_{n-1}) = 1-\left(\frac{2C}{\lambda_{n-1}}-1\right)^+ \wedge 1 \leqq 1.$$

The limit of $\Theta_C(\lambda_{n-1})$ is 0 as $C\to+\infty$. So, $X_n - X_{n_C}\to 0$ a.e. as $C\to+\infty$ and

$$|X_n - X_{n_C}||Y_n| \leqq 2\lambda_{n-1}|Y_n|.$$

Here the right-hand side is integrable since again by the Hölder inequality

$$\mathsf{E}(\lambda_{n-1}|Y_n|) \leqq 2\|\lambda_{n-1}\|_\Phi\|Y_n\|_\Psi < +\infty.$$

Thus the Lebesgue dominated convergence theorem implies

$$\mathsf{E}(|X_n - X_{n_C}||Y_n|) \to 0 \quad\text{as}\quad C\to+\infty.$$

Finally, we prove that

$$|\mathsf{E}(X_n Y_n)| \leqq (16+16\log 2)\|X_n\|_{\mathscr{P}_\Phi}\|Y_n\|_{\mathscr{K}_\Psi}.$$

This follows from the following remarks: together with X the random variable X_n also belongs to $\mathscr{P}_\Phi$ and trivially $\|X_n\|_{\mathscr{P}_\Phi} \leqq \|X\|_{\mathscr{P}_\Phi}$. Also, if $Y \in \mathscr{K}_\Psi$, then trivially $Y_n \in \mathscr{K}_\Psi \subset \mathscr{K}_\Psi^+$ and using Remark 2.3 we can prove that

$$\|Y_n\|_{\mathscr{K}_\Psi^+} \leqq 2\|Y_n\|_{\mathscr{K}_\Psi} \leqq 2\|Y\|_{\mathscr{K}_\Psi}.$$

Consequently, by the generalized Herz inequality (cf. [4], Theorem 4.2) we get

$$|\mathsf{E}(X_n Y_n)| \leqq (8+8\log 2)\|X_n\|_{\mathscr{P}_\Phi}\|Y_n\|_{\mathscr{K}_\Psi^+} \leqq$$

$$\leqq (16+16\log 2)\|X_n\|_{\mathscr{P}_\Phi}\|Y_n\|_{\mathscr{K}_\Psi}.$$

This proves the assertion.

4. Generalization of the Fefferman—Garsia inequality

THEOREM 4.1. *Let $X \in \mathscr{H}_\Phi$ and $Y \in \mathscr{K}_\Psi$ where (Φ, Ψ) is a pair of conjugate Young functions. We suppose that Φ has finite power p. Then for arbitrary $n \geqq 1$ we have*

$$|\mathsf{E}(X_n Y_n)| \leqq C_\Phi \|X_n\|_{\mathscr{H}_\Phi}\|Y_n\|_{\mathscr{K}_\Psi},$$

where $C_\Phi > 0$ is a constant depending only on Φ. Further,

$$\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)$$

exists and is finite, moreover

$$\left|\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)\right| \leqq C_\Phi \|X\|_{\mathscr{H}_\Phi}\|Y\|_{\mathscr{K}_\Psi}.$$

PROOF. On the basis of Theorem 3.1 the random variable X can be decomposed as follows:

$$X = X' + X''.$$

Here $X' \in L^\Phi$ and $X'_n = \mathsf{E}(X' \mid \mathscr{F}_n)$, $n \geqq 0$, is such a martingale for which $X'_0 = 0$ a.e. and we can show in such a way as in Theorem 3.1 that

$$|X'_i - X'_{i-1}| \leqq \delta_i = 4(X_i^* - X_{i-1}^*) + 4\mathsf{E}(X_i^* - X_{i-1}^* \mid \mathscr{F}_{i-1})$$

with

$$\left\|\sum_{i=1}^n \delta_i\right\|_\Phi = \left\|\sum_{i=1}^n \left(4(X_i^* - X_{i-1}^*) + 4\mathsf{E}(X_i^* - X_{i-1}^* \mid \mathscr{F}_{i-1})\right)\right\|_\Phi \leqq$$

$$\leqq (4+4p)\|X_n^*\|_\Phi.$$

Also, by Theorem 3.1, $X'' \in \mathscr{P}_\Phi$ and for $X''_n = \mathsf{E}(X'' \mid \mathscr{F}_n)$, $n \geqq 0$ we have

$$X''_0 = 0 \quad \text{a.e.} \quad |X''_n| \leqq 13X_{n-1}^* + 4\sum_{i=1}^{n-1} \mathsf{E}(X_i^* - X_{i-1}^* \mid \mathscr{F}_{i-1})$$

and

$$\|X''_n\|_{\mathscr{P}_\Phi} \leqq (13+4p)\|X_n^*\|_\Phi.$$

Thus for $Y \in \mathscr{H}_\Psi$ by Theorem 3.2 we get

$$|\mathsf{E}(X_n' Y_n)| \leqq 2(4+4p)\|X_n^*\|_\Phi \|Y_n\|_{\mathscr{H}_\Psi}.$$

Also, by Theorem 3.3, we obtain

$$|\mathsf{E}(X_n'' Y_n)| \leqq (16+16\log 2)(13+4p)\|X_n^*\|_\Phi \|Y_n\|_{\mathscr{H}_\Psi}.$$

Therefore, from these,

$$|\mathsf{E}(X_n Y_n)| \leqq \big(8+8p+(16+16\log 2)(13+4p)\big)\|X_n^*\|_\Phi \|Y_n\|_{\mathscr{H}_\Psi}.$$

By the Burkholder—Davis—Gundy inequality (cf. [6], Theorem 15.1) we have that

$$\|X_n^*\|_\Phi \leqq C_\Phi' \|X_n\|_{\mathscr{H}_\Phi},$$

since p, the power of Φ, is finite. Here $C_\Phi' > 0$ is a constant depending only on Φ. So, finally,

$$|\mathsf{E}(X_n Y_n)| \leqq C_\Phi \|X_n\|_{\mathscr{H}_\Phi} \|Y_n\|_{\mathscr{H}_\Phi},$$

where

$$C_\Phi = C_\Phi'\big(8+8p+(16+16\log 2)(13+4p)\big).$$

Now we prove that $\lim\limits_{n\to+\infty} \mathsf{E}(X_n Y_n)$ exists and is bounded by $C_\Phi \|X\|_{\mathscr{H}_\Phi} \|Y\|_{\mathscr{H}_\Phi}$. To this end it suffices to show that the sequence $\{\mathsf{E}(X_n Y_n)\}$ is a Cauchy sequence. By Theorem 3.2 and by Theorem 3.3 we have that

$$\mathsf{E}(X_n Y_n) = \sum_{i=1}^{n} \mathsf{E}(d_i d_i'),$$

where $\{d_i\}$ and $\{d_i'\}$ denote the martingale difference sequences corresponding to $(X_n, \mathscr{F}_n)$ and $(Y_n, \mathscr{F}_n)$, respectively. From this it follows that if $0 \leqq m \leqq n$ are arbitrary indices then

$$|\mathsf{E}(X_n Y_n) - \mathsf{E}(X_m Y_m)| = \Big|\sum_{i=m+1}^{n} \mathsf{E}(d_i d_i')\Big| = \big|\mathsf{E}\big((X_n - X_m) Y_n\big)\big|.$$

Therefore, by the inequality we have just proved we obtain

$$|\mathsf{E}(X_n Y_n) - \mathsf{E}(X_m Y_m)| \leqq C_\Phi \|X_n - X_m\|_{\mathscr{H}_\Phi} \|Y_n\|_{\mathscr{H}_\Psi}.$$

It is trivially true that

$$\|Y_n\|_{\mathscr{H}_\Psi} \leqq \|Y\|_{\mathscr{H}_\Psi} < +\infty$$

further that $\|X_n - X_m\|_{\mathscr{H}_\Phi} \to 0$ as $n, m \to +\infty$. We get

$$|\mathsf{E}(X_n Y_n) - \mathsf{E}(X_m Y_m)| \to 0$$

as $n, m \to +\infty$. Consequently,

$$\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)$$

exists. Since $\|X_n\|_{\mathscr{H}_\Phi} \leqq \|X\|_{\mathscr{H}_\Phi}$, we also have that

$$\Big|\lim_{n\to+\infty} \mathsf{E}(X_n Y_n)\Big| \leqq C_\Phi \|X\|_{\mathscr{H}_\Phi} \|Y\|_{\mathscr{H}_\Psi}.$$

This proves the assertion.

A consequence of this is the following result which is the generalization of the well-known Fefferman—Garsia inequality.

THEOREM 4.2. *Let* $\Phi(x)=x^p/p$ *with* $1<p<+\infty$ *and consider its conjugate Young function* $\Psi(x)=x^q/q$, *where* $q=p/(p-1)$. *Suppose that* $X\in\mathscr{H}_\Phi=\mathscr{H}_p$ *and* $Y\in\mathscr{H}_\Psi=\mathscr{H}_q$. *Then*

$$|\mathsf{E}(X_nY_n)|\leqq C_p\|X_n\|_{\mathscr{H}_p}\|Y_n\|_{\mathscr{H}_q}$$

and the limit

$$\lim_{n\to+\infty}\mathsf{E}(X_nY_n)$$

exists and is finite. Moreover, we have

$$\left|\lim_{n\to+\infty}\mathsf{E}(X_nY_n)\right|\leqq C_p\|X\|_{\mathscr{H}_p}\|Y\|_{\mathscr{H}_q}.$$

Here $C_p>0$ *is a constant depending only on* p.

5. Remarks

It is needless to mention that our method of proof can also be used when proving the original inequality of Fefferman concerning the space $\mathscr{H}_1$ and the space BMO_2.

When proving his inequality for $1\leqq p\leqq 2$ Garsia has defined the $\mathscr{K}'_q$-space for the values of $q\geqq 2$. Let $X\in L_1$ and consider the set

$$\Delta_X'^{(q)}=\{\gamma:\ \gamma\in L_q,\ \mathsf{E}((X-X_{n-1})^2|\mathscr{F}_n)\leqq\mathsf{E}(\gamma^2|\mathscr{F}_n)\ \text{a.e.}\ \forall n\geqq 1\}.$$

We say that $X\in\mathscr{K}'_q$ if $\Delta_X'^{(q)}$ is not empty and in this case we put

$$\|X\|_{\mathscr{K}'_q}=\inf_{\gamma\in\Delta_X'^{(q)}}\|\gamma\|_q.$$

It is easy to see that $\|\cdot\|_{\mathscr{K}'_q}$ is a quasi-norm. Garsia's original inequality is the following: if $X\in\mathscr{H}_p$, $1\leqq p\leqq 2$, and if $Y\in\mathscr{K}'_q$, where $q=\dfrac{p}{p-1}$ $(\mathscr{K}'_\infty=BMO_2)$ then we have

$$|\mathsf{E}(X_nY_n)|\leqq\sqrt{\frac{2}{p}}\|X_n\|_{\mathscr{H}_p}\|Y_n\|_{\mathscr{K}'_q}.$$

Now we show that the space $\mathscr{K}_\Phi$, where $\Phi(x)=x^q$ $(q\geqq 2)$, and $\mathscr{K}'_q$, $(q\geqq 2)$, coincide. This will imply that our result expressed in Theorem 4.1 and Theorem 4.2 is really an extension of Garsia's result to the value of $p\geqq 2$.

In fact, suppose that $X\in\mathscr{K}'_q$, where $q\geqq 2$. Then for any $\gamma\in\Delta_X'^{(q)}$ we have

$$\mathsf{E}(|X-X_{n-1}||\mathscr{F}_n)\leqq(\mathsf{E}((X-X_{n-1})^2|\mathscr{F}_n))^{1/2}\leqq(\mathsf{E}(\gamma^2|\mathscr{F}_n))^{1/2}\leqq$$

$$\leqq\mathsf{E}\Big((\sup_{k\geqq 1}\mathsf{E}(\gamma^2|\mathscr{F}_k))^{1/2}|\mathscr{F}_n\Big).$$

If $q>2$ then by the Doob maximal inequality the random variable

$$\gamma'=(\sup_{k\geqq 1}\mathsf{E}(\gamma^2|\mathscr{F}_k))^{1/2}$$

will belong to $\Delta_X^{(\Phi)}$, where $\Phi(x)=x^q$. Namely, we have

$$\mathsf{E}(\gamma'^q) \leqq \left(\frac{q}{q-1}\right)^q \sup_{k\geqq 1} \mathsf{E}\big(\mathsf{E}(\gamma^q|\mathscr{F}_k)\big) = \left(\frac{q}{q-1}\right)^q \mathsf{E}(\gamma^q) < +\infty,$$

since $q>2$. So the set $\Delta_X^{(\Phi)}$, where $\Phi=x^q$, is not empty. It also follows that

$$\|X\|_{\mathscr{K}_\Phi} \leqq \frac{q}{q-1}\|X\|_{\mathscr{K}'_q}.$$

For $q=2$ we proceed as follows: it is known (cf. [2] Theorem II. 1.2) that the $\mathscr{H}_2$- and the $\mathscr{K}'_2$-norms are equivalent. Therefore, $X\in\mathscr{K}_2$ implies that $X^*\in L_2$ and it follows that

$$\mathsf{E}\big(|X-X_{n-1}|\,|\mathscr{F}_n\big) \leqq \mathsf{E}(2X^*|\mathscr{F}_n).$$

So $X\in\mathscr{K}'_2$ implies that $X\in\mathscr{K}_\Phi$, $\Phi=x^2$.

On the other hand it is shown (cf. [2] Theorem III. 5.2) that $X\in\mathscr{K}_\Phi$ implies $X^*\in L_q$ provided that $\Phi=x^q$, $q>1$. Consequently, for $q\geqq 2$ we have

$$\mathsf{E}\big((X-X_{n-1})^2|\mathscr{F}_n\big) \leqq \mathsf{E}(2X^{*2}|\mathscr{F}_n),$$

which means that $X\in\mathscr{K}_\Phi$ implies $X\in\mathscr{K}'_q$.

REFERENCES

[1] Fefferman, C., Characterizations of bounded mean oscillation, *Bull. Amer. Math. Soc.* **77** (1971), 587—588. *MR* **43** #6713.

[2] Garsia, A. M., *Martingale inequalities: Seminar notes on recent progress*, Mathematics Lecture Notes Series, W. A. Benjamin, Inc., Reading, Mass.—London—Amsterdam, 1973. *MR* **56** #6844.

[3] Ishak, S. and Mogyoródi, J., On the $\mathscr{P}_\Phi$-spaces and the generalization of Herz's and Fefferman's inequalities I, *Studia Sci. Math. Hungar.* **17** (1982),

[4] Ishak, S. and Mogyoródi, J., On the $\mathscr{P}_\Phi$-spaces and the generalization of Herz's and Fefferman's inequalities II, *Studia Sci. Math. Hungar.* **18** (1983), 205—210.

[5] Davis, B., On the integrability of the martingale square function, *Israel J. Math.* **8** (1970), 187—190. *MR* **42** #3863.

[6] Burkholder, D. L., Distribution function inequalities for martingales, *Ann. Probability* **1** (1973), 19—42. *MR* **51** #1944.

[7] Neveu, J., *Discrete-parameter martingales*, North-Holland Mathematical Library, Vol. 10, North-Holland Publishing Co., Amsterdam—Oxford; American Elsevier Publishing Co., Inc., New York, 1975. *MR* **53** #6729.

(Received October 27, 1980)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
VALÓSZÍNŰSÉGSZÁMÍTÁSI TANSZÉK
MÚZEUM KRT. 6—8
H—1088 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 221—228.

STANDARD AND NEUTRAL IDEALS IN PREALGEBRAIC LATTICES

GERD RICHTER

1. Introduction

This paper is thought to be a continuation of our investigations in [13] and [14].

We know that the dual of an algebraic lattice is a V_1-lattice, i.e. a lattice in which every element is the join of a family of completely join-irreducible elements (cf. Crawley—Dilworth [3], Theorem 6.1).

AC-lattices (Maeda—Maeda [11]), ZM-lattices (Fritzsche [4] and Richter [12]), ZI-lattices (Richter—Stern [16] and [17]) and Baer lattices (Stern [19] and [20]) are special V_1-lattices. Janowitz ([10], Theorem 4.6) shows that in a finite modular AC-lattice $F(L)$ is a standard ideal (cf. Grätzer [6] or [7]), where $F(L)$ is the set of those elements which are the join of a finite (possibly empty) family of atoms.

Stern ([18], Theorem 3.2 and Corollary 3.3) gives necessary and sufficient conditions in AC-lattices for $F(L)$ to be a standard ideal. Moreover, we gave in [16] (Theorem 21 and 23) necessary and sufficient conditions in ZI-lattices for the set $F(L)$ of those elements which can be represented as the join of a finite family of cycles, to be a standard ideal.

In algebraic lattices the set of precompact (inaccessible) elements (cf. Birkhoff—Frink [2], Gorbunov [5] and Grätzer—Schmidt [9]) is equal to the set of compact elements; in V_1I-lattices and therefore also in ZI-lattices and AC-lattices this set is equal to $F(L)$. Therefore, prealgebraic lattices, i.e. lattices in which every element is the join of a family of precompact elements (cf. Richter [13]—[15]), are useful generalizations of algebraic lattices, dual algebraic lattices, V_1I-, ZI-, and AC-lattices. Now the question is raised under which conditions the set of all precompact elements in a prealgebraic lattice is standard or neutral (cf. Birkhoff [1]), respectively.

We shall answer this question in some special lattices.

2. Basic notions

Let L be a complete lattice. For two elements a, b we define

$$b/a := \{x\colon\ a \leqq x \leqq b\}.$$

An element $q \in b/a$ is called *inaccessible (from below) in b/a* or in another terminology *precompact in b/a* iff $q = \bigvee T$ and $T \subseteq b/a$ imply that $q = \bigvee T'$ with $|T'| < \infty$.

1980 *Mathematics Subject Classification.* Primary 06B10.

Key words and phrases. Precompact (inaccessible) elements, prealgebraic, algebraic, V_1-, Z-lattices, isomorphism property, standard, neutral elements, standard and neutral ideals.

$Q(b/a)$ is defined to be the set of all inaccessible elements in b/a. An element $c\in b/a$ is called *compact in* b/a iff $c\leqq\bigvee S$ and $S\subseteqq b/a$ imply that there exists a finite subset $S'\subseteqq S$ with $c\leqq\bigvee S'$.

Note that q is precompact in b/a iff it is compact in q/a.

Let $K(b/a)$ be the set of all compact elements of b/a. It is easy to see that $K(b/a)\subseteqq Q(b/a)$ and that in an algebraic lattice $K(b/a)=Q(b/a)$ holds.

An element $v\in b/a$ is called *completely join-irreducible in* b/a, iff $v\leqq\bigvee T$ and $T\subseteqq b/a$ imply $v\in T$. Let $V_1(b/a)$ be the set of all completely join-irreducible elements of b/a. It is obvious that $V_1(b/a)\subseteqq Q(b/a)$.

We call $z\in V_1(b/a)$ a *cycle in* b/a iff z/a is a chain and z/x is a finite chain for each x with $a<x\leqq z$. Let $Z(b/a)$ be the set of all cycles of b/a and let $A(b/a)\subseteqq Z(b/a)$ be the set of all atoms of b/a, i.e. $p\in A(b/a)$ implies p covers $a\,(p\succ a)$.

Further we define

$$Q:=Q(L),\quad K:=K(L),\quad V_1:=V_1(L),\quad Z:=Z(L),\quad A:=A(L).$$

If every element of L is a join of precompact elements, i.e. of elements of Q, then L is called a *prealgebraic lattice* or a *Q-lattice.* L is called *compactly generated* or *algebraic,* iff every element of L is a join of compact elements. A lattice is called V_1-*lattice,* iff *every element of L is a join of completely join-irreducible elements. We call* a V_1-lattice *cyclically generated* or *Z-lattice,* iff $V_1=Z$ holds. If in a Z-lattice $A=Z$ holds, then it is called an atomistic lattice (A-lattice).

Further we say that the subset $T\subseteqq L$ satisfies the *isomorphism property* (I) iff for each $t\in T$ and for each $b\in L$ there exists an isomorphism

$$\varphi\colon\ x\mapsto\varphi(x)=x\vee b\qquad (x\in t/b\wedge t)$$

of $t/t\wedge b$ onto $t\vee b/b$ such that

$$\varphi^{-1}\colon\ y\mapsto\varphi^{-1}(y)=y\wedge t\qquad (y\in t\vee b/b)$$

holds.

If L is a prealgebraic lattice, an algebraic lattice, a V_1-lattice, a Z-lattice or an A-lattice in which Q, K, V_1, Z, A satisfy (I), then we say that L is a QI-lattice, KI-lattice, a V_1I-lattice or an AC-lattice, respectively.

The *element* $a\in L$ is called *standard* iff

$$x\wedge(a\vee y)=(x\wedge a)\vee(x\wedge y)$$

for all $x,y\in L$.

The *element* $a\in L$ is called *neutral* iff

$$(a\wedge x)\vee(x\wedge y)\vee(y\wedge a)=(a\vee x)\wedge(x\vee y)\wedge(y\vee a)$$

for all $x,y\in L$.

An *ideal I* of L is called *standard* or *neutral*, respectively, iff it is such as an element of $I(L)$, the lattice of all ideals of L.

3. Standard and neutral ideals

Every neutral ideal is standard but the converse does not hold in general. In a modular lattice every standard ideal is neutral (cf. Grätzer [7] or Grätzer—Schmidt [8]).

For our investigations we also need the following

THEOREM 1 (Grätzer—Schmidt [8], Theorem 2). *Let L be a lattice and I be an ideal of L. The following conditions are equivalent:*
(i) *I is standard.*
(ii) *For any principal ideal J of L*

$$I \vee J = \{i \vee j\colon i \in I, j \in J\},$$

where $I \vee J$ means the join of I and J in $I(L)$.

Now we are able to give a necessary and sufficient condition for the set Q of a special prealgebraic lattice to be a standard ideal. This condition is a generalization of Stern ([20], Satz 2.7.5) and Richter—Stern ([16], Theorem 23), respectively.

THEOREM 2. *If the Q-lattice L satisfies condition*
(X) *$b \in Q(1/b \wedge x)$ implies $b \vee x \in Q(1/x)$,*
then the following conditions are equivalent:
(i) *Q is a standard ideal.*
(ii) *$Q(1/x)$ is an ideal in $1/x$ for each $x \in L$ and L satisfies condition*
(*Y*) *$b \vee x \in Q(1/x)$ implies $b \in Q(1/b \wedge x)$.*

For instance, a finite modular AC-lattice is a Q-lattice satisfying (X) and (ii) (cf. Janowitz [10], Theorem 4.2 and 4.6). Before proving Theorem 2 we advance Lemmas 3 and 4.

LEMMA 3. *If the Q-lattice L satisfies condition* (X) *of Theorem 2, then the join of two precompact elements is also precompact.*

PROOF. Let $p, q \in Q$. It is obvious that $p \in Q(1/p \wedge q)$. Therefore, $p \vee q \in Q(1/q)$ by (X). Let $p \vee q = \bigvee T$ and $S := \{s\colon s = q \vee t,\ t \in T\}$. Then we get $q \vee p = q \vee \bigvee T = \bigvee S = \bigvee S'$ for some finite subset $S' = \{s\colon s = q \vee t,\ t \in T' \subseteq T,\ |T'| < \infty\} \subseteq S$, since $p \vee q \in Q(1/q)$, i.e. $p \vee q = \bigvee T' \vee q$. Since $q \in Q(1/q \wedge \bigvee T')$, we get $q \vee p = q \vee \bigvee T' \in Q(1/\bigvee T')$ by (X). Consequently, there exists a finite subset T'' of T with

$$q \vee p = q \vee \bigvee T' = \bigvee T \vee \bigvee T' = \bigvee T'' \vee \bigvee T' = \bigvee (T' \cup T'').$$

Since $T' \cup T''$ is a finite subset of T, $q \vee p$ is precompact.

LEMMA 4. *If the Q-lattice L satisfies condition* (X) *of Theorem 2, then $x \vee b \in Q(1/x)$ iff $x \vee b = x \vee q$ with $q \in Q(b/0)$.*

PROOF. If $q \in Q$, then $q \in Q(1/x \wedge q)$ and therefore $x \vee b = x \vee q \in Q(1/x)$ by (X). Let now $x \vee b \in Q(1/x)$ and

$$R := \{r\colon r = x \vee q,\ q \in Q(b/0)\}.$$

Then $x \vee b = \bigvee R$ holds. Since $x \vee b \in Q(1/x)$ and $R \subseteq 1/x$, we find a finite subset $R' \subseteq R$ such that $x \vee b = \bigvee R'$. By the definition of R there exists a finite subset Q' of $Q(b/0)$ such that $R' = \{r\colon r = x \vee q,\ q \in Q'\}$. By Lemma 3 $\bigvee Q' \in Q(b/0)$ holds.

Now we are able to give the

PROOF OF THEOREM 2. Let Q be a standard ideal. We show that $Q(1/x)$ is an ideal in $1/x$ for each $x \in L$.

Let $p, q\in Q(1/x)$. By Lemma 4 there are elements $u, v\in Q$ such that $p=x\vee u$ and $q=x\vee v$. $u\vee v$ is precompact by Lemma 3, and therefore $x\vee u\vee v=p\vee q\in Q(1/x)$ by Lemma 4.

Let now $q\in Q(1/x)$ and let $x\leqq y\leqq x\vee u=q$ with $u\in Q$. Then $y\in (Q\vee(x/0))$ holds. Since Q is standard, by Theorem 1 there exist $s\in Q$ and $z\leqq x$ such that $y=s\vee z$.

Since $z\leqq x\leqq y=z\vee s$, we get $y=s\vee x$, i.e. $y\in Q(1/x)$ by Lemma 4. Consequently, $Q(1/x)$ is an ideal in $1/x$ for each $x\in L$.

It remains to show that (Y) is satisfied.

Let $b\vee x\in Q(1/x)$. By Lemma 4 there exists an element $q\in Q(b/0)$ such that $b\vee x=q\vee x$, i.e. $b\in (Q\vee(x/0))$. By Theorem 1 we get the existence of an element $p\in Q$ and of an element $z\in x/0$ with $b=z\vee p$. Since $p\in Q(1/x\wedge p)$, we obtain $b=z\vee p\in Q(1/z)$ by (X). Consequently, since $z\leqq b\wedge x\leqq b=p\vee z$, we get $b\in Q(1/x\wedge b)$, i.e. L satisfies (Y). Thus (i) implies (ii).

Now we show that (ii) implies (i).

Let (ii) be satisfied. By Theorem 1 we have to show that $Q\vee(x/0)=\{q\vee y: q\in Q,\ y\in x/0\}$ for every principal ideal $x/0\subseteq L$.

Let x be an arbitrary element of L and let $b\in (Q\vee(x/0))$. Then there exists an element $q\in Q$ such that $b\leqq q\vee x$. By (X) $q\vee x\in Q(1/x)$ holds. Since $Q(1/x)$ is an ideal in $1/x$ and $x\leqq x\vee b\leqq x\vee q$, we get $x\vee b\in Q(1/x)$. Thus $b\in Q(1/b\wedge x)$ by (Y). By Lemma 4 there is an element $p\in Q$ with $b=(b\wedge x)\vee b=(b\wedge x)\vee p$. Consequently, Q is standard by Theorem 1.

Now we shall investigate QI-lattices (e.g., every modular algebraic lattice is a QI-lattice).

LEMMA 5. *Let L be a QI-lattice. Then every interval $b/a\subseteq L$ is a QI-lattice.*

PROOF. Lemma 4 yields that b/a is a Q-lattice. Let $q\in Q(b/a)$ and $d\in b/a$. Then q is also precompact in $b/q\wedge d$. According to Lemma 4 there exists an element $p\in Q$ with $q=(q\wedge d)\vee p$, and by (I) it follows $q/q\wedge d\cong p/p\wedge q\wedge d=p/p\wedge d$, since $p=p\wedge q$. Further we get $q\vee d=(q\wedge d)\vee p\vee d=p\vee d$. (I) yields $q\vee d/d=p\vee d/d\cong \cong p/p\wedge d$. Therefore also $q\vee d/d\cong q/q\wedge d$. Further, (I) yields that the elements of the intervals $q/q\wedge d$ and $q\vee d/d$ are exactly all elements of the form $(q\wedge d)\vee x$ and $d\vee x$, respectively, with $x\in p/p\wedge d$. We have only to show that $(z\vee d)\wedge q=z$ for all $z\in q/q\wedge d$.

Let $z\in q/q\wedge d$ and $z=(q\wedge d)\vee x$ with $x\in p/p\wedge d$. Then $z\vee d=x\vee d$. Therefore we have $q\wedge(z\vee d)=q\wedge(x\vee d)=((q\wedge d)\vee p)\wedge(x\vee d)$. Stern ([19], Proposition 3.3) showed

$$((q\wedge d)\vee p)\wedge(x\vee d)=(q\wedge d)\vee(p\wedge(x\vee d)).$$

Since $p\wedge(x\vee d)=x$ by (I), we get $q\wedge(z\vee d)=(q\wedge d)\vee x=z$.

COROLLARY 6. *Let L be a QI-lattice. Then the following conditions are equivalent:*
(i) *Q is a standard ideal.*
(ii) *Q is an ideal.*

PROOF. Condition (X) is satisfied in L by (I). It remains to show that a QI-lattice satisfies (Y) and that $Q(1/x)$ is an ideal in $1/x$ for each $x\in L$, whenever Q is an ideal.

Let $b\vee x\in Q(1/x)$ for some $x\in L$. Then, by Lemma 4, there exists an element $q\in Q(b/0)$ such that $b\vee x=q\vee x$ and $b\wedge x\leqq(b\wedge x)\vee q\leqq b$. Suppose $(b\wedge x)\vee q<b$. Then there exists an element $p\in Q(b/0)$ with $p\nleqq(b\wedge x)\vee q$. Lemma 3 yields $p\vee q\in Q$. $1/b\wedge x$ is a QI-lattice by Lemma 5. We have

$$b\wedge x\leqq(b\wedge x)\vee q<(b\wedge x)\vee(q\vee p)$$

and

$$x\vee b=x\vee q=x\vee(q\vee p),$$

i.e.

$$\varphi\big((b\wedge x)\vee q\big)=x\vee(b\wedge x)\vee q=x\vee b=x\vee(b\wedge x)\vee(q\vee p)=\varphi\big((b\wedge x)\vee(q\vee p)\big),$$

contradicting that φ is an isomorphism of $q\vee p/(q\vee p)\wedge x$ onto $x\vee q\vee p/x$, since $(q\vee p)\wedge x\leqq b\wedge x\leqq(b\wedge x)\vee q<(b\wedge x)\vee(p\vee q)$. Therefore $(b\wedge x)\vee q=b$, i.e. $b\in Q(1/b\wedge x)$ since $q\in Q(q/q\wedge b\wedge x)$. Consequently, L satisfies (Y).

Let Q be an ideal and let $x\in L$. That the join of two precompact elements of $1/x$ is precompact follows by Lemma 5 and 3. Let $q\in Q(1/x)$ and $z\in q/x$. By Lemma 4 and (I) there exists an element $p\in Q$ such that $q=x\vee p$ and

$$\varphi\colon\ r\mapsto\varphi(r)=x\vee r\qquad(r\in p/p\wedge x)$$

is an isomorphism of $p/p\wedge x$ onto $x\vee p/x$. Therefore, there exists an $r\in p/p\wedge x$ with $z=x\vee r$. Since Q is an ideal and $r\leqq p\in Q$, we have $r\in Q$. Thus $z=x\vee r\in$ $\in Q(1/x)$ by Lemma 4, i.e. $Q(1/x)$ is an ideal in $1/x$.

Corollary 7. *Let L be a modular Q-lattice. Q is a neutral ideal iff it is an ideal.*

Proof. A modular Q-lattice is a QI-lattice. In a modular lattice every standard ideal is neutral, therefore the proof is completed by Corollary 6.

In [15] (Lemma 3) we showed that every QI-lattice is semimodular. An element x of a semimodular lattice is called finite dimensional iff the interval $x/0$ is finite dimensional.

Corollary 8. *Let L be a QI-lattice. If every element of Q is finite dimensional, then Q is a standard ideal.*

Proof. Every finite dimensional element in a Q-lattice is precompact. The join of two finite dimensional elements is also a finite dimensional element. Every element which is less than a finite dimensional element is finite dimensional. Therefore Q is an ideal, and by Corollary 6 Q is standard.

Corollary 9. *Let L be a KI-lattice. If 1 is a join of finite dimensional elements, then K is a standard ideal.*

Proof. If 1 is the join of a family of finite dimensional elements, then $c\in K$ and $c\leqq 1$ imply that c is less than the join of a finite family of finite dimensional elements, i.e., c is also finite dimensional. Therefore K is an ideal and also a standard ideal by Corollary 8.

4. Standard and neutral ideals in $V_1 I$-lattices

In this section we shall deal with $V_1 I$-lattices. Examples for them include, for instance, all modular dual algebraic lattices. Applying Theorem 2 we get the following

THEOREM 10 (cf. Stern [20], Satz 2.7.5). *Let L be a $V_1 I$-lattice. Then the following conditions are equivalent:*

(i) *Q is a standard ideal.*
(ii) *Q is an ideal and L satisfies* (Y).
(iii) *$v/0 \subseteq Q$ for each $v \in V_1$ and L satisfies* (Y).

The condition

(Y) $$b \vee x \in Q(1/x) \quad \text{implies} \quad b \in Q(1/x \wedge b)$$

is the same as was used in Theorem 2.

PROOF. In [14] (Theorem 8) we showed that in a $V_1 I$-lattice $q \in Q(1/x)$ iff $q = x \vee p$ with $p \in Q$. From this we get immediately that condition (X) of Theorem 2 is satisfied. Then Theorem 2 yields that (i) implies (ii). Since $V_1 \subseteq Q$, we get $v/0 \subseteq Q$ for all $v \in V_1$ if Q is an ideal. Therefore, (ii) implies (iii).

If $v/0 \subseteq Q$, for all $v \in V_1$, then Q is an ideal by [14] (Theorem 44). Also, in [14] (Lemma 6 and 7) we showed that every interval of a $V_1 I$-lattice is a $V_1 I$-lattice. Then [14] (Corollary 2) yields that $u \in V_1(1/x)$ iff $u = x \vee v$ with $v \in V_1$. By (I) we get $u/x \cong v/v \wedge x$, i.e., $u/x \subseteq Q(1/x)$ since $v/v \wedge x \subseteq Q(1/v \wedge x)$. Therefore [14] (Theorem 44) yields that $Q(1/x)$ is an ideal in $1/x$. Thus (iii) implies (i) by Theorem 2, and the proof is complete.

For ZI-lattices (e.g. geometric lattices or subgroup lattices of primary abelian groups) we have

THEOREM 11 (cf. Richter—Stern [16], Theorem 23). *Let L be a ZI-lattice. Then the following conditions are equivalent:*

(i) *Q is a standard ideal.*
(ii) *$b \vee x \in Z(1/x)$ implies $b \in Q(1/b \wedge x)$.*

PROOF. Since in a Z-lattice $Z = V_1$ and since $z/0 \subseteq Z \subseteq Q$ for all $z \in Z$, Q is an ideal by [14] (Theorem 44). Since $Z(1/x) \subseteq Q(1/x)$, Theorem 2 yields that (i) implies (ii).

Let now (ii) be satisfied in L. We have only to show that condition (Y) of Theorem 2 is satisfied. Let $b \vee x \in Q(1/x)$. Then $b \vee x = q \vee x$ with $q \in Q(b/0)$ by [14] (Theorem 8). According to [14] (Theorem 3) we have $q = z_1 \vee z_2 \vee \ldots \vee z_n$ with $z_1, z_2, \ldots, z_n \in Z$.

For $i = 1, \ldots, n$, let $x_i := x \vee z_1 \vee \ldots \vee z_i$. Then

$$x = x_0 \leqq x_1 \leqq \ldots \leqq x_n = x \vee b$$

and $x_i \in Z(1/x_{i-1})$ hold. Further we have

$$x \wedge b = x_0 \wedge b \leqq x_1 \wedge b \leqq \ldots \leqq x_{n-1} \wedge b \leqq x_n \wedge b = b$$

and

$$x_{i-1}\wedge(x_i\wedge b) = (x_{i-1}\wedge x_i)\wedge b = x_{i-1}\wedge b.$$

Since $x_{i-1}\leqq x_{i-1}\vee(x_i\wedge b)\leqq x_i$, we get $x_{i-1}\vee(x_i\wedge b)\in Z(1/x_{i-1})$, i.e., $x_i\wedge b\in \in Q(1/x_{i-1}\wedge b)$ by (ii), and therefore, for $i=1, \ldots, n$, there are $q_i\in Q$ with $x_i\wedge b= =(x_{i-1}\wedge b)\vee q_i$. Consequently, $b=x_n\wedge b=(x_0\wedge b)\vee q_1\vee\ldots\vee q_n=(x\wedge b)\vee p$ with $p= =q_1\vee\ldots\vee q_n$. Since Q is an ideal, $p\in Q$ holds. [14] (Theorem 8) yields that b is precompact in $1/x\wedge b$, which completes the proof.

COROLLARY 12. *Let L be a modular Z-lattice, a so called ZM-lattice. Then Q is a neutral ideal.*

PROOF. A modular Z-lattice is a modular Q-lattice in which Q is an ideal. Therefore Q is a neutral ideal by Corollary 7.

Theorem 10 and Theorem 11 show that in Stern ([20], Satz 2.7.5 and Satz 2.7.7) the assumption that all elements of V_1 are finite dimensional can be dropped.

REFERENCES

[1] BIRKHOFF, G., Neutral elements in general lattices, *Bull. Amer. Math. Soc.* **46** (1940), 702—705. *MR* 2—120.

[2] BIRKHOFF, G. and FRINK, O., Representations of lattices by sets, *Trans. Amer. Math. Soc.* **64** (1948), 299—316. *MR* **10**—279.

[3] CRAWLEY, P. and DILWORTH, R. P., *Algebraic theory of lattices*, Prentice-Hall, Englewood Cliffs, N. J., 1973.

[4] FRITZSCHE, R., On Ulm's theorem in the theory of cyclically generated modular lattices, Proc. Colloq. on Universal Algebra, *Model Theory and Lattice Theory* (held in Esztergom, 1977), 305—319.

[5] GORBUNOV, V. A., Canonical decompositions in complete lattices, *Algebra i Logika* **17** (1978), 495—511, 622 (in Russian). *MR* **81c**: 06005.

[6] GRÄTZER, G., Standard ideals, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **9** (1959), 81—97 (in Hungarian). *MR* **21** #4119.

[7] GRÄTZER, G., *General lattice theory*, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Birkhäuser-Verlag, Basel—Stuttgart, 1978. *MR* **80c**: 06001a.

[8] GRÄTZER, G. and SCHMIDT, E. T., Standard ideals in lattices, *Acta Math. Acad. Sci. Hungar.* **12** (1961), 17—86. *MR* **24** #A3099.

[9] GRÄTZER, G. and SCHMIDT, E. T., On inaccessible and minimal congruence relations, I, *Acta Sci. Math. (Szeged)* **21** (1960), 337—342. *MR* **23** #3697.

[10] JANOWITZ, M. F., On the modular relation in atomistic lattices, *Fund. Math.* **66** (1969/70), 337—346. *MR* **41** #5259.

[11] MAEDA, F. and MAEDA, S., *Theory of symmetric lattices*, Die Grundlehren der mathematischen Wissenschaften, Band 173, Springer-Verlag, New York—Berlin, 1970. *MR* **44** #123.

[12] RICHTER, G., Einige Ergebnisse aus der Theorie der zyklisch erzeugten modularen Verbände, Beiträge zur Algebra und Geometrie, **8**, *Wiss. Beitr. Martin-Luther-Univ. Halle—Wittenberg M* **12** (1979), 33—53. *MR* **81e**: 06016.

[13] RICHTER, G., Strong purity in lattices, *Beiträge zur Algebra und Geometrie* **12** (1982), 7—16.

[14] RICHTER, G., On the structure of lattices in which every element is a join of join-irreducible elements, *Period. Math. Hungar.* **13** (1982), 47—69.

[15] RICHTER, G., Generalization of a theorem of Kertész, *Beiträge zur Algebra und Geometrie* **12** (1982), 117—121.

[16] RICHTER, G. and STERN, M., A unified theory of cyclically generated modular lattices and AC-lattices, *Studia Sci. Math. Hungar.* **11** (1976), 413—429. *MR* **83a**: 06009.

[17] RICHTER, G. and STERN, M., ZI-lattices, *Model Theory and Lattice Theory*, Proc. Colloq. on Universal Algebra (held in Esztergom, 1977), 635—652.

[18] STERN, M., On AC-lattices in which the ideal of the finite elements is standard, *Wiss. Beitr. Martin-Luther-Univ. Halle—Wittenberg Reihe M Math.* 7—*Beitr. Algebra Geom.* **5** (1976), 15—21. *MR* **55** #2690.
[19] STERN, M., Baer lattices, *Wiss. Z. Martin-Luther-Univ. Halle—Wittenberg Math.-Natur. Reihe* **27** (1978), 135—140. *MR* **80k**: 06007.
[20] STERN, M., Baersche Verbände, Doctoral thesis, Martin-Luther-Universität Halle—Wittenberg, 1978.

(Received December 5, 1980)

SEKTION MATHEMATIK
MARTIN-LUTHER-UNIVERSITÄT HALLE—WITTENBERG
UNIVERSITÄTSPLATZ 6
DDR—401 HALLE
GERMAN DEMOCRATIC REPUBLIC

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 229—238.

WITHIN PAIR ORDER EFFECTS IN PAIRED COMPARISONS

G. SADASIVAN

Summary

In this paper methods of estimation of order effects within each pair in a paired comparison experiment are discussed. Two methods are developed, one using the original preference data and the other using angular transformation to the original preference data. The analysis with transformation helps us to estimate the errors of the estimates as well and hence seems to be more advantageous. The methods are illustrated with the help of an example from sensory evaluation.

1. Introduction

In paired comparisons of a set of t stimuli, the final rankings of the stimuli are affected by the order in which the stimuli of a specified pair are presented to a judge who is to express his preference for one of the stimuli against the other. This effect is usually eliminated by an appropriate design for balancing the experiment for order of presentation of the stimuli in a pair. In this paper we treat two methods of isolation of the order effect in each of the $t(t-1)/2$ pairs of t stimuli in a paired comparison experiment, by devising suitable models for the purpose. In the development here, order effects are estimated without admitting ties, by giving a score '1' for the preferred stimulus and zero for the non-preferred stimulus. Scheffé [8] developed an analysis of variance for scored differences from pairs that allows for order of presentation of items within a pair. Beaver and Gokhale [2] modified the Bradley—Terry model to incorporate within pair order effects. Davidson [4] has presented a model for order-effects within the Bradley—Terry pattern by using a parameter for scaling the relative worths of the items to represent the order of presentation. Sadasivan [7] has developed a procedure using the Bradley—Terry pattern for estimating the average order effect over all the pairs in a paired comparison experiment. Davidson and Beaver [5] have considered a generalization of the Bradley—Terry model for paired comparisons in the presence of ties and which examines the effects of the order of presentation of the objects within a pair. Fienberg [3] has shown how the model of Davidson and Beaver can be represented in log-linear form and how the results for maximum likelihood estimation follow directly from known results by log-linear models fitted to categorical data. In the developments in this paper we use a variant of the Thurstone—Mosteller model [9], [10] for estimating the within pair order-effects in a paired comparison experiment.

1980 *Mathematics Subject Classification.* Primary 62F07.
Key words and phrases. Paired comparisons, Thurstone—Mosteller model, angular transformation, order effects.

2. The models

In the proposed models we use two sets (set I and set II) of observations on each pair (i, j) $(i<j;\ i, j=1, 2, \ldots, t)$ of t stimuli, one presented in the order (i, j) designated $O(i, j)$ and the other in the opposite order $O(j, i)$. Let each pair be repeated n times in each set of observations. Then for the pair of stimuli (i, j) we have under model (1)

$$
\begin{aligned}
&E(X_{i.ij}) = S_i+\theta_{ij} \quad \text{for} \quad O(i, j) \\
&E(X_{j.ij}) = S_j-\theta_{ij} \quad \text{for} \quad O(i, j) \\
&E(X_{i.ji}) = S_i+\theta_{ji} \quad \text{for} \quad O(j, i) \quad \text{and} \\
&E(X_{j.ji}) = S_j-\theta_{ji} \quad \text{for} \quad O(j, i),
\end{aligned}
\tag{2.1}
$$

where $X_{i.ij}$, $X_{j.ij}$, $X_{i.ji}$ and $X_{j.ji}$ are the observe dresponses of the ith and jth stimuli, along a linear sensation continuum for $O(i, j)$ and $O(j, i)$, respectively, S_i, S_j, the true responses for the ith and jth stimuli and θ_{ij}, θ_{ji}, the parameters of the expected effects for $O(i, j)$ and $O(j, i)$, respectively. The implication under this model is that order effect is advantageous to the object presented first and disadvantageous to the same extent to the object presented second along the postulated linear continuum. Define $\theta_{ji}=-\theta_{ij}$. In other words we assume that the expected order effect is symmetric with respect to the stimuli. From this it is evident that $\theta_{ij}>0$ and $\theta_{ji}<0$ under this model. Mosteller [9] obtains the ratings without order effects for the t stimuli using a score '1' for the preferred stimulus and a score '0' for the non-preferred stimulus under the following assumptions.

(i) There is a set of stimuli which can be located in a linear, subjective, sensation continuum.

(ii) Each stimulus when presented to an individual gives rise to a sensation in the individual.

(iii) The distribution of sensations for a particular stimulus over repetitions is normal.

(iv) The stimuli are presented in pairs to any individual thus giving rise to a sensation in the individual for each stimulus.

(v) Assume equal standard deviations for each stimulus and zero correlations between pairs of stimuli.

This latter condition in (v) has been altered to one of equal correlations with no basic change in the method in Mosteller [10]. Assumptions (i) to (iv) are valid for each set in our case. Assumption (v) regarding equal variances is also valid for each set, if we assume equal variances σ^2 for the distribution of estimates of stimuli and equal variances σ^2 for the estimates of order effects. Note that the estimates are kept identical for each data set. Thus define

$$d_{ij.ij} = X_{i.ij}-X_{j.ij}.$$

Then

$$E(d_{ij.ij}) = S_i-S_j+2\theta_{ij} \quad \text{for} \quad O(i, j).$$

Similarly, $d_{ij.ji}=X_{i.ji}-X_{j.ji}$, $E(d_{ij.ji})=S_i-S_j-2\theta_{ij}$ for $O(j,i)$. The variances of $d_{ij.ij}$ and $d_{ij.ji}$ being identical can be represented as

(2.2) $$\sigma_{d_{ij}^2} = 2(\sigma^2+2\sigma_t^2)$$

if there is no pairwise correlation among the stimuli;

(2.3) $$\sigma_{d_{ij}^2} = 2[\sigma^2(1-\varrho)+2\sigma_t^2]$$

if there is a constant pairwise correlation ϱ among the stimuli. In deductions (2.2) and (2.3) it has been assumed that estimates of stimuli and order effects are independently distributed. Note that $d_{ji.ij}$ has also the same set of variances. Let $p_{ij}^{(1)}$ be the observed proportion of preferences for stimulus 'i' against stimulus 'j' from the n repetitions of the pair (i,j) in set I. Then following Mosteller [9]—[11]

(2.4) $$p_{ij}^{(1)} = Pr(d_{ij.ij}>0) = \frac{1}{\sqrt{2\pi}\,\sigma_{d_{ij}}}\int_0^{\infty} e^{-(d_{ij.ij}-S_i+S_j-2\theta_{ij})^2/2\sigma_{d_{ij}}^2}\,dd_{ij.ij}$$

(2.5) $$= \frac{1}{\sqrt{2\pi}}\int_{-S_i'+S_j'-2\theta_{ij}'}^{\infty} e^{-y^2/2}\,dy$$

where

$$y = \frac{d_{ij.ij}-S_i+S_j-2\theta_{ij}}{\sigma_{d_{ij}}}$$

and S_i', S_j' and θ_{ij}' are estimates of S_i, S_j and θ_{ij} adjusted to make $\sigma_{d_{ij}}=1$. Thus

(2.6) $$p_{ij}^{(1)} = F(S_i'-S_j'+2\theta_{ij}')$$

or

(2.7) $$S_i'-S_j'+2\theta_{ij}' = F^{-1}(p_{ij}^{(1)}) = D_{ij}^{(1)} \quad \text{(say)}.$$

Similarly, using the data of set II, the observed proportion of preferences for 'i' against 'j' viz.

(2.8) $$p_{ij}^{(2)} = F(S_i'-S_j'-2\theta_{ij}')$$

or

(2.9) $$S_i'-S_j'-2\theta_{ij}' = F^{-1}(p_{ij}^{(2)}) = D_{ij}^{(2)} \quad \text{(say)}.$$

Similarly, it can be shown that

$$F^{-1}(p_{ji}^{(1)}) = D_{ji}^{(1)} = -D_{ij}^{(1)} \quad \text{and}$$
$$F^{-1}(p_{ji}^{(2)}) = D_{ji}^{(2)} = -D_{ij}^{(2)}.$$

(2.10) $$S_i'-S_j' = \frac{1}{2}[D_{ij}^{(1)}+D_{ij}^{(2)}] = D_{ij}' \quad \text{(say)}$$

(2.11) $$\theta_{ij}' = \frac{1}{4}[D_{ij}^{(1)}-D_{ij}^{(2)}].$$

The solutions (2.11) are unique and hence can directly be obtained from data sets I and II by using a table of normal integrals. Solution (2.10) are not unique and hence the S_i''s are estimated by least squares. Setting $S_1'=0$, the linearly independent set of normal equations can be obtained as

$$(t-1)S_i' - \sum_{j\neq i} S_j' = \sum_{j\neq i} D_{ij}' \qquad (i = 2, 3, \ldots, t). \tag{2.12}$$

(2.12) can be expressed as

$$(X'X)S = D \tag{2.13}$$

where $(X'X)$ is a $(t-1)\times(t-1)$ matrix, the transpose S' of $S=(S_2', S_3', \ldots, S_t')$ and the transpose D' of D:

$$D' = \Big[\sum_{j\neq 2} D_{2j}', \sum_{j\neq 3} D_{3j}', \ldots, \sum_{j\neq t} D_{tj}'\Big].$$

Hence the estimate of the ratings vector S^* of S is obtained as

$$S^* = (X'X)^{-1}D. \tag{2.14}$$

For the above analysis to be valid (a) $X_{i\cdot ij}$, $X_{j\cdot ij}$ should be normally distributed for set I with uniform variance; (b) $X_{i\cdot ij}$, $X_{j\cdot ij}$ should be either independent or equally correlated and (c) the scale should be additive. Similar conditions should apply to set II. If any of these conditions fail, the ratings will not reflect the true situation. It may further be noted that the assumption of normality is only a computational device for building up a rating scale. Moreover, the variability of $X_{i\cdot ij}$ will be homogeneous only if (i) the variances of S_i' are equal and (ii) variances of θ_{ij}''s are equal. The variances of the differences $(X_{i\cdot ij}-X_{j\cdot ij})$ will be homogeneous only with the further condition (iii) that the correlation between pairs $(X_{i\cdot ij}, X_{j\cdot ij})$ are either zero or equal. These conditions are not easy to attain in practical situations. For instance in sensory testing an extraneous factor as taste fatique will affect the variability of any stimulus. Further, conditions (ii) and (iii) will easily get disturbed. Thus, it may happen that different judges have entered into the preference testing giving rise to different correlations among pairs of stimuli. It is shown in the appendix how violation of conditions (i), (ii) or (iii) renders the variances heteroscedastic. The normality of responses and additivity of scale may also get disturbed under this situation. Hence to stabilise the variance and to make other conditions valid, we transform the original preference data by using angular transformation. This procedure eventually helps us to get the estimates of the errors of estimates as well. For analysis by this procedure to be valid we modify the assumptions under the model as follows. (A) The responses are additive along the transformed scale denoted by $\tilde{X}$. (B) The variability of $\tilde{X}_{i\cdot ij}$, the value of $X_{i\cdot ij}$ on the transformed scale is uniform for $(i=1, 2, \ldots, t)$. (C) Pairs of responses are equally correlated under the new scale. (D) The distribution of sensations for a particular stimulus becomes normal under the new scale.

Now for Model (2) use the inverse sine transformation

$$\theta' = \arcsin\sqrt{p'}$$

where p' is the observed proportion from a binomial sample of size n from a population with true proportion of success p. θ' is approximately normally distributed with nearly independent variance depending on whether θ' is measured in degrees or radians. It is well-known that this transformation stabilizes the variances of data expressed in the form of percentages [1]. Since a transformation intended to stabilize variances, stabilizes the covariances as well (Ref. Appendix) this transformation is used here. Now proceeding as before we get an expression of the same form as (2.5) under the modified assumptions. Now replace (2.5) by

$$F(x) = \frac{1}{2} \int_{-x}^{\pi/2} \cos y \, dy = \frac{1}{2}(1+\sin x) \quad \left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right) \tag{2.15}$$

where

$$x = \tilde{S}_i - \tilde{S}_j + 2\tilde{\theta}_{ij},$$

$\tilde{S}_i$, $\tilde{S}_j$ and $\tilde{\theta}_{ij}$ being the parametric values under the new scale. Note that x here represents an angle in radian measure. This change (2.15) is possible under the modified assumptions. Thus,

$$\begin{aligned} p_{ij}^{(1)} &= \frac{1}{2}[1+\sin(\tilde{S}_i - \tilde{S}_j + 2\tilde{\theta}_{ij})] \\ p_{ij}^{(2)} &= \frac{1}{2}[1+\sin(\tilde{S}_i - \tilde{S}_j - 2\tilde{\theta}_{ij})] \end{aligned} \tag{2.16}$$

from which

$$\tilde{S}_i' - \tilde{S}_j' + 2\tilde{\theta}_{ij}' = \sin^{-1}(2p_{ij}^{(1)} - 1) = \tilde{D}_{ij}^{(1)} \quad \text{(say)} \tag{2.17}$$

and

$$\tilde{S}_i' - \tilde{S}_j' - 2\tilde{\theta}_{ij}' = \sin^{-1}(2p_{ij}^{(2)} - 1) = \tilde{D}_{ij}^{(2)}. \tag{2.18}$$

Note that $\tilde{S}_i'$, $\tilde{S}_j'$, $\tilde{\theta}_{ij}'$ in (2.17) and (2.18) represent the estimates of $\tilde{S}_i$, $\tilde{S}_j$ and $\tilde{\theta}_{ij}$ in the new scale. Using similar notations it can be shown in the same way that

$$\begin{aligned} \sin^{-1}(2p_{ji}^{(1)} - 1) &= \tilde{D}_{ji}^{(1)} = -\tilde{D}_{ij}^{(1)} \\ \sin^{-1}(2p_{ji}^{(2)} - 1) &= \tilde{D}_{ji}^{(2)} = -\tilde{D}_{ij}^{(2)}. \end{aligned}$$

Solving (2.17) and (2.18)

$$\tilde{\theta}_{ij}' = (\tilde{D}_{ij}^{(1)} - \tilde{D}_{ij}^{(2)})/4; \quad (i < j;\ i, j = 1, 2, \ldots, t) \tag{2.19}$$

$$\tilde{S}_i' - \tilde{S}_j' = \frac{1}{2}[\tilde{D}_{ij}^{(1)} + \tilde{D}_{ij}^{(2)}] = D_{ij}' = -D_{ji}' \tag{2.20}$$

$$(i < j;\ i, j = 1, 2, \ldots, t).$$

Solutions (2.19) for the order effects are unique. Now, it can easily be shown that

$$\operatorname{Var} \tilde{D}_{ij}^{(1)} = \operatorname{Var} D_{ij}^{(2)} = \frac{1}{n}. \tag{2.21}$$

Hence

$$\operatorname{Var} \tilde{\theta}_{ij}' = (8n)^{-1}. \tag{2.22}$$

Since the system of equations (2.20) does not provide unique estimates, we use the method of least squares for solution of the system. After setting $\tilde{S}_1'=0$, the resultant independent set of normal equations can be expressed as

$$(X'X)\tilde{S} = \tilde{D} \tag{2.23}$$

where $(X'X)$ is a $(t-1)\times(t-1)$ matrix,

$$\tilde{S}' = (\tilde{S}_2', \tilde{S}_3', \ldots, \tilde{S}_t')$$

and

$$\tilde{D}' = \Big(\sum_{j\neq 2}\tilde{D}_{2j}', \sum_{j\neq 3}\tilde{D}_{3j}', \ldots, \sum_{j\neq t}\tilde{D}_{tj}'\Big),$$

$\tilde{S}'$, $\tilde{D}'$ being the transpose of the column vectors $\tilde{S}$ and $\tilde{D}$, respectively. Thus the estimates of the vector of ratings is given by

$$\tilde{S}^* = (X'X)^{-1}\tilde{D}. \tag{2.24}$$

The dispersion matrix of $\tilde{S}^*$ is given by

$$\sum_{\tilde{S}^*} = (X'X)^{-1}\sum_{\tilde{D}}(X'X)^{-1}$$

where $\sum_{\tilde{D}}$, the dispersion matrix of the vector $\tilde{D}$ can be easily evaluated.

3. An adequancy test for model

To test whether Model (1) and Model (2) are adequate we construct the chi-square statistic,

$$\chi_\nu^2 = \sum_{i<j}\frac{(np_{ij}^{(1)}-np_{ij}^{*(1)})^2}{np_{ij}^{*(1)}} + \sum_{i<j}\frac{(np_{ij}^{(2)}-np_{ij}^{*(2)})^2}{np_{ij}^{*(2)}}. \tag{3.1}$$

In (3.1) $p_{ij}^{(1)}$ and $p_{ij}^{(2)}$ are the preference proportions for i from sets I and II, respectively. Now to test for Model (1), $p_{ij}^{*(1)}$, $p_{ij}^{*(2)}$ the estimates of $p_{ij}^{(1)}$, $p_{ij}^{(2)}$ under the hypothesis of adequacy of the model are obtained by substitution of S_i', S_j', θ_{ij}' in equations (2.6) and (2.8) and reading the corresponding proportions from a table of normal integrals. In testing Model (2), $p_{ij}^{*(1)}$ and $p_{ij}^{*(2)}$ in (3.1) are obtained by substitution of $\tilde{S}_i'$, $\tilde{S}_j'$ and $\tilde{\theta}_{ij}'$ in the system (2.17) and (2.18). χ_ν^2 is here distributed as a chi-square with degrees of freedom $(t-1)(t-2)/2$. If the chi-square is not significant for any model, the corresponding model is adequate.

4. An illustration from food technology

Chappatties prepared under identical conditions from four varieties of wheat were coded 1, 2, 3, 4 and tested for their palatability using an expert judge. Each pair was offered for testing 10 times in one order and 10 times in the reverse order. Sufficient time was allowed between successive testings. The pairs were grouped into sets according to order of presentation.

Set I	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
Set II	(2,1)	(3,1)	(4,1)	(3,2)	(4,2)	(4,3).

The corresponding proportions of preferences were obtained from the data as follows:

	p_{12}	p_{13}	p_{14}	p_{23}	p_{24}	p_{34}
Set I	.4	.3	.2	.4	.3	.4
Set II	.5	.2	.1	.5	.4	.3.

Computations for the analysis without transformation were made in the following steps:

Step 1. Using the normal integral table find the values of x corresponding to

$$\int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = p_{ij},$$

the preference proportions. This gives $D_{ij}^{(1)}$ and $D_{ij}^{(2)}$.

Step 2. Obtain

$$D'_{ij} = \frac{D_{ij}^{(1)} + D_{ij}^{(2)}}{2}.$$

From these values find

$$D_i = \sum_{j \neq i} D'_{ij} \qquad (i = 2, 3, \ldots, t).$$

Step 3. Find

$$\theta'_{ij} = \frac{D_{ij}^{(1)} - D_{ij}^{(2)}}{4}.$$

Step 4. Obtain $(X'X)^{-1}$.

Step 5. Find $S^* = (X'X)^{-1}D$.

Computations gave the following result.

Set	D_{12}	D_{13}	D_{14}	D_{23}	D_{24}	D_{34}
(1)	−.253661	−.524905	−.842662	−.253661	−.524905	−.253661
(2)	0	−.842662	−1.282485	0	−.253661	−.524905
D'_{ij}	−.126830	−.683784	−1.062573	−.126830	−.389283	−.389283
Set	D_{21}	D_{31}	D_{41}	D_{32}	D_{42}	D_{43}
(1)	.253661	.524905	.842662	.253661	.524905	.253661
(2)	0	.842662	1.282485	0	.253661	.524905
D'_{ji}	.126830	.683784	1.062573	.126830	.389283	.389283

$D' = (-.389283, .421381, 1.841139)$.

Thus

$$S^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -.389283 \\ .421381 \\ 1.841139 \end{bmatrix} = \begin{bmatrix} .370988 \\ .573655 \\ .928593 \end{bmatrix}.$$

The estimates of order effects are

$$\theta'_{21} = .063\,415; \quad \theta'_{13} = .079\,439; \quad \theta'_{14} = .109\,956;$$

$$\theta'_{32} = .063\,415; \quad \theta'_{42} = .067\,811; \quad \theta'_{34} = .067\,811.$$

Computations for the method using angular transformation were made in the following steps.

Step 1. Using table XII of Hald [6] find

$$\bar{D}_{ij}^{(1)} = \sin^{-1}(2p_{ij}^{(1)} - 1) = 2\sin^{-1}\sqrt{p_{ij}^{(1)}} - \frac{\pi}{2}$$

and

$$\bar{D}_{ij}^{(2)} = \sin^{-1}(2p_{ij}^{(2)} - 1) = 2\sin^{-1}\sqrt{p_{ij}^{(2)}} - \frac{\pi}{2}.$$

Step 2. Find $\bar{D}'_{ij} = \frac{1}{2}[\bar{D}_{ij}^{(1)} + \bar{D}_{ij}^{(2)}]$; $\bar{D}'_{ji} = -\bar{D}'_{ij}$.

Step 3. Find $\bar{\theta}'_{ij} = \frac{1}{4}[\bar{D}_{ij}^{(1)} - \bar{D}_{ij}^{(2)}]$.

Step 4. Find $\bar{D}' = (\sum_{j\neq 2} \bar{D}'_{2j}, \sum_{j\neq 3} \bar{D}_{3j}, \ldots, \sum_{j\neq t} \bar{D}'_{tj})$.

Step 5. Find $(X'X)^{-1}$ and $\sum_{\bar{D}}$

Step 6. Compute $\bar{S}^* = (X'X)^{-1}\bar{D}$; $\sum_{\bar{S}^*} = (X'X)^{-1}\sum_{\bar{D}}(X'X)^{-1}$.

Computations gave the following table:

Set	D_{12}	D_{13}	D_{14}	D_{23}	D_{24}	D_{34}
(1)	$-.2020$	$-.4222$	$-.6442$	$-.2020$	$-.4222$	$-.2020$
(2)	$-.0006$	$-.6442$	$-.9278$	$-.0006$	$-.2020$	$-.4222$
$\bar{D}'_{ij}$	$-.1013$	$-.5332$	$-.7860$	$-.1013$	$-.3121$	$-.3121$
$\bar{\theta}'_{ij}$	$-.0503$	$.0555$	$.0709$	$-.0503$	$-.0551$	$.0551$

From the table $\bar{D}'=(-.3121, .3224\ 1.4102)$. Then

$$\tilde{S}^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -.3121 \\ .3224 \\ 1.4102 \end{bmatrix} = \begin{bmatrix} .2771 \\ .4332 \\ .7076 \end{bmatrix}.$$

$$\sum_{\tilde{S}^*} = \begin{bmatrix} \frac{1}{40} & \frac{1}{80} & \frac{1}{80} \\ \frac{1}{80} & \frac{1}{40} & \frac{1}{80} \\ \frac{1}{80} & \frac{1}{80} & \frac{1}{40} \end{bmatrix}.$$

The order effects were obtained as:

$$\tilde{\theta}'_{12} = -.0503; \quad \tilde{\theta}'_{13} = .0555; \quad \tilde{\theta}'_{14} = .0709;$$
$$\tilde{\theta}'_{24} = -.0551; \quad \tilde{\theta}'_{23} = -.0503; \quad \tilde{\theta}'_{34} = .0551;$$

with standard error 0.111 showing that none of the effects are significant. The χ^2's for analysis without and with transformation were respectively 0.886 and .895 for 3 degrees of freedom. This shows that the models are adequate for the present data. However, the second model has the additional advantage that it helps in the estimation of errors as well.

Appendix

Let the variance of S_i' be σ_i^2 and variance of $\theta_{ij}=\sigma_{ij}^2$. Then Var $X_{i.ij}=\sigma_i^2+\sigma_{ij}^2$. Thus these variances become unequal. If the correlations ϱ_{ij} between $X_{i.ij}$ and $X_{j.ij}$ are unequal,

$$\mathrm{Var}(X_{i.ij}-X_{j.ij}) = \sigma_i^2+\sigma_j^2+2\sigma_{ij}^2-2\varrho_{ij}\sqrt{(\sigma_i^2+\sigma_{ij}^2)(\sigma_j^2+\sigma_{ij}^2)}.$$

These variances will not get stabilized even when standardized. If angular transformation stabilizes this variance, it becomes

$$\mathrm{Var}\,(\tilde{X}_{i.ij}-\tilde{X}_{j.ij}) = 2\tilde{\sigma}^2+4\tilde{\sigma}_t^2-2\tilde{\varrho}(\tilde{\sigma}^2+\tilde{\sigma}_t^2)$$

under the transformed scale showing that the correlation gets stabilized.

REFERENCES

[1] BARTLETT, M. S., The use of transformations, *Biometrics* **3** (1947), 39—52. *MR* **8**—593.
[2] BEAVER, R. J. and GOKHALE, D. V., A model to incorporate within-pair order effects in paired comparisons, *Comm. Statist.* **4** (1975), 923—939. *MR* **53** #9533.
[3] FIENBERG, S. E., Log-linear transformation for paired comparison models with ties and within pair order effects, *Biometrics* **35** (1979), 479—481.

[4] Davidson, R. R., Beaver, R. J. and David, H. A., Discussion „Science, statistics and paired comparisons" (Biometrics **32** (1976), no. 2, 213—232) by Ralph A. Bradley, *Biometrics* **32** (1976), 233—239. *MR* **53** #11899.

[5] Davidson, R. R. and Beaver, R. J., On extending the Bradley-Terry model to incorporate within-pair order effects, *Biometrics* **33** (1977), 693—702. *MR* **58** #8051.

[6] Hald, A., *Statistical tables and formulas*, John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1952. *MR* **14**—188.

[7] Sadasivan, G., A model for order effects in paired comparisons.

[8] Scheffé, H., An analysis of variance for paired comparisons, *J. Amer. Statist. Assoc.* **47** (1952), 381—400. *MR* **14**—488.

[9] Mosteller, F., Remarks on the method of paired comparisons, I. The least squares solution assuming equal standard errors and equal correlations, *Psychometrika* **16** (1951), 3—9.

[10] Mosteller, F., Remarks on the method of paired comparisons, II. The effect of an aberrant standard deviation with equal standard error and equal correlations assumed, *Psychometrika* **16** (1951), 203—206.

[11] Mosteller, F., Remarks on the method of paired comparisons, III. A test of significance for paired comparisons when equal standard deviations and equal correlations are assumed, *Psychometrika* **16** (1951), 207—218.

(Received January 9, 1981)

INDIAN AGRICULTURAL STATISTICS
RESEARCH INSTITUTE
LIBRARY AVENUE
IND—110 012 NEW DELHI 12
INDIA

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 239—245.

DIRECT INTEGRAL OF MULTIFUNCTIONS INTO VON NEUMANN ALGEBRAS

DÉNES PETZ

Summarizing the objective of the present paper we can say that it introduces and deals with the direct integral of multifunctions defined on a Polish space and having closed values in a von Neumann algebra endowed with the strong topology in order to obtain applications in the classical reduction theory of von Neumann algebras.

Reduction theory works with notions as a measurable field of operators, von Neumann algebras etc. Roughly speaking, a field of von Neumann algebras means that for every $\gamma\in\Gamma$ a strongly closed *-subalgebra of $B(H)$, say $\mathcal{A}(\gamma)$, is given (here H is a fixed separable Hilbert space). Such a field is called measurable if there is a sequence (A_n) of measurable fields of operators such that $\mathcal{A}(\gamma)$ is generated by $\{A_n(\gamma)\colon n\in\mathbf{N}\}$ for every $\gamma\in\Gamma$. This strange notion of measurability was cleared up by E. G. Effros, who introduced a standard Borel structure on the weak* closed subspaces of $B(H)$ and showed that measurability with respect to this Borel structure is equivalent to the measurability above.

In some cases, working with reduction theory one needs more general subsets of $B(H)$ (not necessarily subspaces). Hence we follow another approach based upon the theory of multifunctions.

1. Preliminaries. In this section we collect some definitions and results, mainly from [9] and [10]. Throughout the paper $\mathcal{A}$ will be a von Neumann algebra with separable predual, Γ a complete separable metric space with a finite Borel measure μ.

Let E be a separable Banach space. An E-valued function $f\colon \Gamma\to E$ is called weakly measurable if for every $x^*\in E^*$ the complex function $\gamma\in\Gamma\mapsto\langle f(\gamma), x^*\rangle$ is μ-measurable. Denote by $L^1_B(\Gamma,\mu,E)$ the normed space of all weakly measurable E-valued functions on Γ such that

$$\|f\|_1 = \int_\Gamma \|f(\gamma)\|\, d\mu(\gamma)$$

is finite. (As usual, functions equal μ-almost everywhere are identified.) $L^1_B(\Gamma,\mu,E)$ is a Banach space and isometrically isomorphic to the projective tensor product $L^1(\Gamma,\mu)\otimes E$.

An E^*-valued function $g\colon \Gamma\to E^*$ is called weak* measurable if for every $x\in E$ the function

$$\gamma\in\Gamma\mapsto\langle x, g(\gamma)\rangle\in\mathbf{C}$$

1980 *Mathematics Subject Classification.* Primary 46L45; Secondary 28A20.
Key words and phrases. Measurable field, reduction theory, measurable selection.

is μ-measurable. $L_G^\infty(\Gamma, \mu, E^*)$ denotes the separated space of all μ-essentially bounded weak* measurable functions with the norm

$$\|g\|_\infty = \text{ess sup}\,\{\|g(\gamma)\|:\ \gamma\in\Gamma\}.$$

The spaces $L_B^1(\Gamma, \mu, E)^*$ and $L_G^\infty(\Gamma, \mu, E^*)$ can be identified in the usual way. Hence if $\mathscr{A}$ is a von Neumann algebra with separable predual then $L_G^\infty(\Gamma, \mu, \mathscr{A})$ is also a von Neumann algebra and its predual is $L_B^1(\Gamma, \mu, \mathscr{A}_*)$ (cf. [6], 3.2).

Let X be a topological space. Denote by 2^X the collection of all nonempty closed subsets of X. A function H: $\Gamma\to 2^X$ will be called multifunction with closed values. We note that H can be viewed as a subset of $\Gamma\times X$.

Let H: $\Gamma\to 2^X$ be a multifunction. H is said to be μ-measurable if the set

$$H^-(F) = \{\gamma\in\Gamma:\ H(\gamma)\cap F\neq\emptyset\}$$

is μ-measurable for every $F\in 2^X$. A sequence (φ_n) of μ-measurable functions of Γ into X is called a Castaing representation of H provided that

$$H(\gamma) = \overline{\{\varphi_n(\gamma):\ n\in\mathbf{N}\}}$$

for every $\gamma\in\Gamma$.

A Polish space means a separable topological space which is metrizable to a complete metric space. The closed unit ball $\mathscr{A}_1$ of the algebra $\mathscr{A}$ with the strong topology is a Polish space. If X is a Polish space and H: $\Gamma\to 2^X$ is a multifunction with closed values then the following conditions are equivalent.

(i) H is μ-measurable;

(ii) there is a Castaing representation for H;

(iii) there is a Borel set $\Gamma_0\subset\Gamma$ such that $\mu(\Gamma\setminus\Gamma_0)=0$ and $H|\Gamma_0$ has a Castaing representation consisting of Borel functions.

Finally, we recall that analytic space means continuous image of a Polish space. A metrizable analytic space will be called Souslin space. A Borel subset of a Souslin space is a Souslin space (i.e. a Souslin set). The measurable cross section theorem tells us the following. If X and Y are Souslin spaces, μ is a finite Borel measure on Y and F is a Borel map from X onto Y then there exists a μ-measurable map φ from Y, into X such that $F\circ\varphi(y)=y$ for every $y\in Y$.

2. Direct integrals of multifunctions. Let $\mathscr{A}$ be a von Neumann algebra with separable predual and endow $\mathscr{A}$ with the strong topology. It is quite clear that a multifunction H: $\Gamma\to 2^{\mathscr{A}}$ is μ-measurable if and only if the multifunctions $\gamma\in\Gamma\mapsto H(\gamma)\cap\{a\in\mathscr{A}:\ \|a\|\leqq n\}$ are μ-measurable for every $n\in\mathbf{N}$. Of course, if the values of H: $\Gamma\to 2^{\mathscr{A}}$ are strongly closed subspaces then for the μ-measurability of H it is necessary and sufficient that $\gamma\in\Gamma\mapsto H(\gamma)\cap\mathscr{A}_1$ be μ-measurable. This is the case when $H(\gamma)$ is a von Neumann subalgebra of $\mathscr{A}$ (possibly with different unit) for $\gamma\in\Gamma$. Hence different definitions give the same measurable fields of von Neumann algebras (cf. [9], IV. 8.3 and [4], Lemma 2.1).

THEOREM 1. *Let H: $\Gamma\to 2^{\mathscr{A}}$ be a multifunction with closed values. Then H is μ-measurable if and only if the following condition is fulfilled.*

(Λ) *There is a Borel set $\Gamma_0\subset\Gamma$ such that $\mu(\Gamma\setminus\Gamma_0)=0$ and the set $\Lambda(\Gamma_0, H)=\{(\gamma, a)\in\Gamma_0\times\mathscr{A}:\ a\in H(\gamma)\}$ is a Borel set in $\Gamma\times\mathscr{A}$.*

PROOF. It is clear that (A) is fulfilled for H if and only if it is fulfilled for all the multifunctions $\gamma\in\Gamma\mapsto H(\gamma)\cap n\cdot\mathscr{A}_1$. Hence we may assume that $H(\gamma)\subset\mathscr{A}_1$. If (A) is satisfied and $F\subset\mathscr{A}_1$ is a closed set then $A(\Gamma_0, H)\cap\Gamma\times F$ is analytic. So $H^-(F)$ is μ-measurable. Conversely, assume that H is μ-measurable. Since $\mathscr{A}_1$ is a Polish space there exists a $\Gamma_0\subset\Gamma$ such that $\mu(\Gamma\setminus\Gamma_0)=0$ and $H\mid\Gamma_0$ has a Castaing representation consisting of Borel functions $\{g_n\colon n\in\mathbf{N}\}$. Let d be a metric on $\mathscr{A}_1$. Then

$$S(n,\varepsilon)=\{(\gamma,t)\in\Gamma_0\times\mathscr{A}_1\colon d(t,g_n(\gamma))<\varepsilon\}$$

is a Borel set in $\Gamma_0\times\mathscr{A}_1$ since $(\gamma,t)\mapsto d(t,g_n(\gamma))$ is a Borel function. It is straightforward that

$$A(\Gamma_0,H)=\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}S(n,m^{-1})$$

and this shows that $A(\Gamma_0, H)$ is a Borel set in $\Gamma\times\mathscr{A}_1$.

EXAMPLE 1. If $g\in L_G^\infty(\Gamma,\mu,\mathscr{A})$ then the singleton valued "multifunction" $\gamma\mapsto\{g(\gamma)\}$ is μ-measurable.

EXAMPLE 2. If $g\in L_G^\infty(\Gamma,\mu,\mathscr{A})$ then $\gamma\mapsto\operatorname{Sp}g(\gamma)$ is a μ-measurable multifunction (scalars are identified with scalar operators). Indeed, using Lusin's lemma, for $\varepsilon>0$ we can take $F_\varepsilon\subset\Gamma$ such that F_ε is closed, $\mu(\Gamma\setminus\Gamma_\varepsilon)<\varepsilon$, $\|g(\gamma)\|\leqq\|g\|_\infty$ for $\gamma\in F_\varepsilon$ and g is continuous on F_ε. Choose a sequence $\varepsilon_n\searrow 0$ and let $\Gamma_0=\bigcup\{F_{\varepsilon_m}\colon n\in\mathbf{N}\}$.

For the sake of simplicity, we assume that $\mathscr{A}$ acts on a separable Hilbert space H, and let $\{\xi_n\}$ be dense in H. Trivially, $A\in B(H)$ has a bounded inverse if and only if there exists $q\in\mathbf{N}$ such that

(i) $$\|A\xi_i\|\geqq q^{-1}\|\xi_i\|\qquad(i\in\mathbf{N}),$$

(ii) $$\|A^*\xi_i\|\geqq q^{-1}\|\xi_i\|\qquad(i\in\mathbf{N}).$$

Now we define

$$H_\varepsilon(q,k)=\{(\gamma,\lambda)\in F_\varepsilon\times\mathbf{C}\colon \|(g(\gamma)-\lambda)\xi_k\|\geqq q^{-1}\|\xi_k\|\quad\text{and}$$
$$\|(g(\gamma)^*-\bar\lambda)\xi_k\|\geqq q^{-1}\|\xi_k\|\}.$$

This is a Borel set in $\Gamma\times\mathscr{A}$. Therefore

$$A(\Gamma_0,\operatorname{Sp}g)=\bigcup_{n=1}^{\infty}\bigcup_{q=1}^{\infty}\bigcap_{k=1}^{\infty}H_{\varepsilon_n}(q,k)$$

is analytic and Theorem 1 can be applied.

Let $H\colon\Gamma\to 2^{\mathscr{A}}$ be a μ-measurable multifunction. Its direct integral $\int_\Gamma^\oplus H(\gamma)\,d\mu(\gamma)$ is defined as the set of all $g\in L_G^\infty(\Gamma,\mu,\mathscr{A})$ such that $g(\gamma)\in H(\gamma)$ for μ-almost all $\gamma\in\Gamma$. A trivial example is the following. If $g\in L_G^\infty(\Gamma,\mu,\mathscr{A})$ then $\{g\}=$ $=\int_\Gamma^\oplus\{g(\gamma)\}\,d\mu(\gamma)$. Other examples will be shown later.

A set $A\subset L_G^\infty(\Gamma,\mu,\mathscr{A})$ is said to be decomposable if there exists a μ-measurable multifunction $H\colon \Gamma\to 2^{\mathscr{A}}$ such that $A=\int_\Gamma^\oplus H(\gamma)\,d\mu(\gamma)$.

THEOREM 2. *Let $H\colon \Gamma\to 2^{\mathscr{A}_1}$ be a μ-measurable multifunction. Then $\int_\Gamma^\oplus H(\gamma)\,d\mu(\gamma)$ is strongly closed.*

This follows easily from the following result.

LEMMA 1 ([6], 3.2.7). *Let $(g_n)\subset L_G^\infty(\Gamma,\mu,\mathscr{A})$ be a bounded sequence. If g_n converges to 0 strongly then there is a subsequence (g_{k_n}) such that $g_{k_n}(\gamma)$ converges to 0 strongly as $n\to\infty$ for μ-almost all $\gamma\in\Gamma$. On the other hand, if $g_n(\gamma)$ converges to 0 strongly for μ-almost every $\gamma\in\Gamma$ then g_n converges to 0 strongly.*

THEOREM 3. *Let $H_i\colon \Gamma\to 2^{\mathscr{A}_1}$ be μ-measurable multifunctions $(i=1,2)$. If $\int_\Gamma^\oplus H_1(\gamma)\,d\mu(\gamma)\subset\int_\Gamma^\oplus H_2(\gamma)\,d\mu(\gamma)$ then $H_1(\gamma)\subset H_2(\gamma)$ for μ-almost all $\gamma\in\Gamma$.*

PROOF. By Theorem 1 there is a Borel set $\Gamma_0\subset\Gamma$ such that $\mu(\Gamma\setminus\Gamma_0)=0$, moreover $\Lambda(\Gamma_0,H_1)$ and $\Lambda(\Gamma_0,H_2)$ are Borel sets in $\Gamma_0\times\mathscr{A}_1$. Let $S=\Lambda(\Gamma_0,H_1)\setminus\Lambda(\Gamma_0,H_2)$. Then S is a Souslin set and $\pi(S)\subset\Gamma$ is μ-measurable where $\pi\colon \Gamma\times\mathscr{A}_1\to\Gamma$ is the projection mapping. We have to prove that $\mu(\pi(S))=0$.
Take $S^1=S\cup\Lambda(\Gamma_0\setminus\pi(S),H_1)$ and apply the measurable cross section theorem. Doing so we obtain a μ-measurable function $\varphi\colon \Gamma_0\to S^1$ such that $\varphi(\gamma)\in H_1(\gamma)$. Hence $\varphi\in\int_\Gamma^\oplus H_1(\gamma)\,d\mu(\gamma)$ and $\varphi\in\int_\Gamma^\oplus H_2(\gamma)\,d\mu(\gamma)$. For $\gamma\in\pi(S)$ we have $\varphi(\gamma)\notin H_2(\gamma)$ (therefore $\mu\pi(S))=0$, indeed.
As a corollary we can establish that if $A\subset L_G^\infty(\Gamma,\mu,\mathscr{A})$ is a bounded decomposable set then its decomposition is unique up to a zero set.

THEOREM 4. *Let $A\subset L_G^\infty(\Gamma,\mu,\mathscr{A})$ be a strongly closed set in the unit ball. Then the following conditions are equivalent:*
(i) *A is decomposable;*
(ii) *if $g_1,g_2\in A$ and p is a diagonal projection in $L_G^\infty(\Gamma,\mu,\mathscr{A})$ then $pg_1+(1-p)g_2\in A$;*
(iii) *if $(g_n)\subset A$ and (p_n) is a sequence of diagonal projections in $L_G^\infty(\Gamma,\mu,\mathscr{A})$ such that $\sum_{n=1}^\infty p_n=1$ then $\sum_{n=1}^\infty p_ng_n\in A$.*

PROOF. (i)→(ii) is trivial. Let us prove (ii)→(iii). We obtain $S_n=\sum_{i=1}^n p_ig_i+(1-\sum_{i=1}^n p_i)g_1\in A$ by induction. Since $S_n\xrightarrow{s}\sum_{i=1}^\infty p_ig_i$ we have $\sum_{i=1}^\infty p_ig_i\in A$.
To show (iii)→(i) we assume (iii). Let (q_n) be dense in A. The multifunction $\gamma\mapsto H(\gamma)=\overline{\{q_n(\gamma)\colon n\in\mathbf{N}\}}$ is μ-measurable since it is given by a Castaing representation.

We intend to prove that $A=\int_\Gamma^\oplus H(\gamma)d\mu(\gamma)$. By Lemma 1 we have $A\subset\int_\Gamma^\oplus H(\gamma)d\mu(\gamma)$.

Suppose that $g\in\int_\Gamma^\oplus H(\gamma)d\mu(\gamma)$, $\varepsilon>0$,and $f_1, \dots, f_k\in L^1_B(\Gamma, \mu, \mathscr{A}_*)_+$. We are looking for $g_1\in A$ such that

$$\langle f_i, (g-g_1)^*(g-g_1)\rangle \leqq \varepsilon \quad (i\leqq k).$$

This will assure that $g\in A$.

We take a Borel set $\Gamma_0\subset\Gamma$ (to be specified later) such that $\mu(\Gamma\setminus\Gamma_0)=0$ and endow the set $X=\Gamma_0\times\{0, 1\}^{\mathbb{N}}$ with the product topology. Then X is a Souslin space. The set

$$X_1=\Big\{(\gamma, \lambda_1, \lambda_2, \dots)\in X\colon \sum_{i=1}^\infty \lambda_i=1 \quad \text{and for} \quad j\leqq k$$

$$\Big\langle f_j(\gamma), \Big[g(\gamma)-\sum_{i=1}^\infty \lambda_i q_i(\gamma)\Big]^*\Big[g(\gamma)-\sum_{i=1}^\infty \lambda_i q_i(\gamma)\Big]\Big\rangle \leqq \delta\Big\}$$

is a Borel set in X since it is defined by a Borel function on Γ_0. It is not difficult to see that Γ_0 can be chosen in this way. Then X_1 is a Souslin space. Denote by $\pi\colon X\to\Gamma_0$ the projection onto the first coordinate space. $\pi(X_1)=\Gamma_0$, and we apply the measurable cross section theorem. It gives a measurable function $\varphi\colon \Gamma_0\to X_1$ such that $\pi(\varphi(\gamma))=\gamma$ $(\gamma\in\Gamma_0)$. Then $\varphi(\gamma)=(\gamma, \lambda_1(\gamma), \lambda_2(\gamma), \dots)$, where $\lambda_i\in L^\infty_G(\Gamma, \mu, \mathscr{A})$ is a 0-1 valued function, that is a diagonal projection $(i\in\mathbb{N})$, and $\sum_{i=1}^\infty \lambda_i=1$. By condition (iii) $\sum_{i=1}^\infty \lambda_i q_i=g_1\in A$ and for $j\leqq k$

$$\langle f_j, (g-g_1)^*(g-g_1)\rangle=\int_\Gamma \langle f_i(\gamma), [g(\gamma)-g_1(\gamma)]^*[g(\gamma)-g_1(\gamma)]\rangle\, d\mu(\gamma)\leqq\delta\mu(\Gamma).$$

The proof is complete.

COROLLARY. *Let $A\subset L^\infty_G(\Gamma, \mu, \mathscr{A})$ be a decomposable strongly closed convex set. If d is a positive diagonal contraction then*

($*$) $a, b\in A$ *implies* $ad+b(1-d)\in A$.

PROOF. Let D_0 be the collection of all positive diagonal contractions such that ($*$) holds. By Theorem 4 D_0 contains every diagonal projection and evidently D_0 is strongly closed. On the other hand, simple calculation shows that D_0 is convex. Using the spectral theorem we can conclude that D_0 is the set of all positive diagonal contractions.

3. Applications. The central spectrum of an element a of a von Neumann algebra $\mathscr{A}$ is defined in the following fashion (see [8]).

$\sigma(a)=\{c\in\mathscr{C}\colon (c-a)p$ is not invertible in $\mathscr{A}p$ for any non-zero central projection $p\}$.

In case of a factor the central spectrum coincides with the spectrum if scalars are identified with scalar operators.

THEOREM 5. *Let* $\int_\Gamma^\oplus \mathcal{A}(\gamma)\,d\mu(\gamma)$ *be the central decomposition of the von Neumann algebra* $\mathcal{A}$. *Then for any* $a=\int_\Gamma^\oplus a(\gamma)\,d\mu(\gamma)\in\mathcal{A}$ *we have* $\sigma(a)=\int_\Gamma^\oplus \operatorname{Sp} a(\gamma)\,d\mu(\gamma)$.

PROOF. We have already seen that the multifunction $\gamma\mapsto\operatorname{Sp} a(\gamma)$ is μ-measurable (Example 2). If $c\notin\sigma(a)$ is a central element then $(c-a)e$ has an inverse in $\mathcal{A}e$ for a non-zero central projection e. Denote this inverse by b. Then $b(\gamma)[c(\gamma)-a(\gamma)]=$ $=[c(\gamma)-a(\gamma)]b(\gamma)=1$ and $c(\gamma)\notin\operatorname{Sp} a(\gamma)$ for μ-almost all $\gamma\in\{\gamma\in\Gamma\colon e(\gamma)=1\}$. Hence we conclude that $c\notin\int_\Gamma^\oplus \operatorname{Sp} a(\gamma)\,d\mu(\gamma)$.

Suppose that $c\notin\int_\Gamma^\oplus \operatorname{Sp} a(\gamma)\,d\mu(\gamma)$. The set $\Gamma_1=\{\gamma\in\Gamma\colon c(\gamma)\notin\operatorname{Sp} a(\gamma)\}$ is μ-measurable and $\mu(\Gamma_1)>0$. For $\gamma\in\Gamma_1$ there is $b(\gamma)\in\mathcal{A}(\gamma)$ such that $[a(\gamma)-c(\gamma)]b(\gamma)=$ $=b(\gamma)[a(\gamma)-c(\gamma)]=1$. One can find $\Gamma_2\subset\Gamma_1$ and $n\in\mathbf{N}$ with the properties $\mu(\Gamma_2)>0$ and $\|b(\gamma)\|\le n$ $(\gamma\in\Gamma_2)$. Let χ be the characteristic function of Γ_2. Then for $e=\int_\Gamma^\oplus \chi(\gamma)\,d\mu(\gamma)$ the operator $(a-c)e$ has a bounded inverse in $\mathcal{A}e$. Since $e\neq 0$ we have $c\notin\sigma(a)$.

This theorem can be used to deduce some properties of the central spectrum from those of the spectrum (cf. [8]).

THEOREM 6. *Let* $\mathcal{A}$ *be a finite von Neumann algebra with separable predual and let* τ *be a linear mapping of* $\mathcal{A}$ *into its center* $\mathcal{C}$. *Then the following statements are equivalent:*

(i) τ *is the canonical center-valued trace;*

(ii) $\tau(a)\in\overline{\operatorname{conv}\sigma(a)}^s$ *for every* $a\in\mathcal{A}$.

PROOF. A similar characterization of the center-valued trace was proved in [5]. There central-convex hull stood instead of common convex hull. So it is sufficient to see that after taking closure these two sets are the same.

Clearly, $\operatorname{conv}\sigma(a)\subset\operatorname{co}\sigma(a)$ (here $\operatorname{co}\sigma(a)=\left\{\sum_{i=1}^n p_i c_i\colon \sum_{i=1}^n p_i=1,\ c_i\in\sigma(a),\ p_i\in\mathcal{C}^+\right\}$. Take the central decomposition $\int_\Gamma^\oplus \mathcal{A}(\gamma)\,d\mu(\gamma)$ of $\mathcal{A}$. Then $\sigma(a)=\int_\Gamma^\oplus \operatorname{Sp} a(\gamma)\,d\mu(\gamma)$ and one can check without difficulty that $\overline{\operatorname{conv}\sigma(a)}=\int_\Gamma^\oplus \operatorname{conv}\operatorname{Sp} a(\gamma)\,d\mu(\gamma)$. Now the corollary of Theorem 4 gives that $\overline{\operatorname{conv}\sigma(s)}^s$ is central-convex.

REFERENCES

[1] EFFROS, E. G., Global structure in von Neumann algebras, *Trans. Amer. Math. Soc.* **121** (1966), 434—454. *MR* **33** #585.
[2] HALPERN, H., Essential central spectrum and range for elements of a von Neumann algebra, *Pacific J. Math.* **43** (1972), 349—380. *MR* **48** #2787.
[3] NEUMANN, J. VON, On rings of operators. Reduction theory, *Ann. of Math.* (2) **50** (1949), 401—485. *MR* **10**—548.
[4] PETZ, D., On central-valued states of von Neumann algebras, *Acta Sci. Math. (Szeged)* **43** (1981), 219—228.
[5] PETZ, D., A characterization of the canonical centervalued trace in finite von Neumann algebras, *J. London Math. Soc.* **23** (1981), 329—331.
[6] SAKAI, S., *C^*-algebras and W^*-algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60, Springer-Verlag, New York—Heidelberg, 1971. *MR* **56** #1082.
[7] SCHWARTZ, J. T., *W^*-algebras*, Gordon and Breach Science Publishers, New York—London—Paris, 1967. *MR* **38** #547.
[8] STRĂTILĂ, Ş. and ZSIDÓ, L., An algebraic reduction theory for W^*-algebras II, *Rev. Roumaine Math. Pures Appl.* **18** (1973), 407—460. *MR* **50** #1012.
[9] TAKESAKI, M., *Theory of operator algebras*, I, Springer-Verlag, New York—Heidelberg, 1979. *MR* **81c**: 46038.
[10] WAGNER, D. H., Survey of measurable selection theorems, *SIAM J. Control Optimization* **15** (1977), 859—903. *MR* **58** #6137.

(Received April 2, 1981)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15.
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 247—252.

ON DETACHED IMMERSIONS

ANDRÁS SZŰCS

DEFINITION 1. A fibrewise linear map $\Phi\colon E(\zeta)\to R^q$ from the total space of an n-dimensional vector bundle ζ $\big(\pi(\xi)\colon E(\xi)\to B(\xi)\big)$ into the Euclidean space R^q is called an *r-detached* bundle map (where r is a natural number or zero) if for any two different points $x, y\in B(\zeta)$ $\operatorname{rank}\{\Phi(\xi_x), \Phi(\xi_y)\}\geqq 2n-r$. (Here ξ_x stands for the fibre over x, and $\{\Phi(\xi_x), \Phi(\xi_y)\}$ denotes the linear subspace generated by $\Phi(\xi_x)$ and $\Phi(\xi_y)$.)

DEFINITION 1′. A fibrewise linear map $\Phi\colon E(\xi)\to R^q$ will be called *locally r-detached* if for any $x\in B(\xi)$ there exists a neighbourhood U_x of x in $B(\zeta)$ such that the restriction of Φ to $\pi(\xi)^{-1}(U_x)$ is r-detached.

DEFINITION 2. An immersion $f\colon M^n\to R^q$ is called *r-detached* (resp. *locally r-detached*) if df is r-detached (resp. locally r-detached).

The aim of the present paper is to show that a Smale—Hirsch—Gromov type theorem (the "h-principle") holds for r-detached maps in "metastable range" defined in the Theorem below.

THEOREM. *If $r(q-2n+r-1)>(3/2)n$ then the following statements are equivalent.*
(1) *There exists an r-detached immersion $M^n\to R^q$.*
(2) *There exists an r-detached bundle map $TM\to R^q$.*
(3) *There exists a bundle map $\Psi\colon TM\times TM\to R^q$ such that*
(a) *Ψ is equivariant in the sense that if $\Psi(u,v)=a$, $u\in T_xM$, $v\in T_yM$, $a\in R^q$, then $\Psi(v,u)=a$, too;*
(b) *the restrictions $\Psi\mid T_xM\times 0$ and $\Psi\mid 0\times T_yM$ are monomorphisms;*
(c) $$\operatorname{rank}\Psi(T_xM\times T_yM)\geqq 2n-r \quad \text{if} \quad x\neq y.$$

REMARK 1. If $r(q-2n+r-1)>(3/2)(n+1)$ then the analogue of the Theorem above for homotopies holds as well. For example, in this case two r-detached immersions are regularly homotopic through r-detached immersions if their differentials can be joined in the space of r-detached bundle maps.

PROOF of the Theorem. The implications (1)⇒(2)⇒(3) are obvious. (3)⇒(1) is analogous to Gromov's and Eliashberg's proof of Haefliger's theorem on embeddings (see [1]).

1980 *Mathematics Subject Classification.* Primary 57R42; Secondary 57R40.
Key words and phrases. Haefliger's embedding theorem, linearly independent tangent spaces.

We give some details of the proof. For simplicity we treat (mainly) the case when $r=0$. (0-detached maps are simply called detached ones.)

The map $\Psi\colon TM\times TM\to R^q$ can be considered as q-tuple of equivariant 1-forms $\Psi=(\omega_1, \dots, \omega_q)$, where $\omega_i\colon TM\times TM\to R^1$ $i=1,2,\dots,q$ is linear and $\omega_i(u,v)=\omega_i(v,u)$ for $(u,v)\in T_xM\times T_yM$ and $(v,u)\in T_yM\times T_xM$.

Our aim is to replace these 1-forms — one by one — by exact ones keeping the q-tuples of these new forms detached (compare with the introduction of [1]).

Omit one of these forms (for example the last one). Let $\tilde\Psi$ be the map $TM\times TM\to R^{q-1}$ defined by $(\omega_1, \dots, \omega_{q-1})$ and $A_{\tilde\Psi}$ be the set of pairs (x,y) such that rank $\tilde\Psi(T_xM\times T_yM)=2n-1$, $x\neq y$.

Denote $\gamma(x,y)$ the vector Ker $\tilde\Psi(x,y)$ at point $(x,y)\in A_{\tilde\Psi}$ and let γ_x, γ_y be the vectors $d\pi_1(\gamma(x,y))$, $d\pi_2(\gamma(x,y))$, respectively, where $\pi_i\colon M\times M\to M$, $\pi_i(x_1,x_2)=x_i$.

Assumptions (∗). Suppose that

1) $\overline{A_{\tilde\Psi}}$ is a manifold (the bar indicates the closure in $M\times M$).

2) $\pi_1|_{\overline{A_{\tilde\Psi}}}$ is an embedding into M^n.

3) γ_x is transversal to $\pi_1(A_{\tilde\Psi})$.

Under assumptions (∗) there exists a function $f\colon M\to R^1$ such that

$$df(\gamma_x)=\omega_q(\gamma_x)\ (\ast\ast).$$

(On the right-hand side the vector $\gamma_x\in T_xM$ is identified with the vector $(\gamma_x, 0)\in T_xM\times T_yM$, where $(x,y)\in A_{\tilde\Psi}$.) Then the map $df+df\colon TM\times TM\to R^1$, $(df+df)(u,v)=df(u)+df(v)$ can be substituted for ω_q so that

$$(\tilde\Psi, df+df)\colon TM\times TM\to R^q$$

will be a detached map. So the main problem is to achieve that assumptions (∗) hold. This can be done — off the diagonal — using transversality theorems in the same way as in the case of embeddings and we shall not repeat it in details. Instead, we shall examine what happens at the diagonal, where transversality theorems cannot be applied. (We recall that in the case of Haefliger's embedding theorem (in [1]) we used the Haefliger—Hirsch theorem on skew maps. Here we shall use the same.)

Denote $\psi\colon M\times M\to J^1(M\times M, R^q)$ and $\tilde\psi\colon M\times M\to J^1(M\times M, R^{q-1})$ the sections of the jet fibration associated with the maps Ψ and $\tilde\Psi$, respectively. The involution $M\times M\to M\times M$, $(x,y)\to(y,x)$ induces an involution τ on the jet space $J_q^1=J^1(M\times M, R^q)$. Denote Δ_q the fixed point set of τ, and Σ_q^1 the Bordman submanifold of Σ^1-type singular sets in $J^1(M\times M, R^q)$. Then Σ_q^1 and $\Delta_q\subset\Sigma_q^1$ are invariant under τ. Consider a τ-invariant tubular neighbourhood T_q of Δ_q in $J^1(M\times M, R^q)$. Let $\zeta_q\to\Delta_q$ the restriction to Δ_q of the normal bundle of Σ_q^1 in $J^1(M\times M, R^q)$ (see Figure 1).

The disc in the picture stands for the tubular neighbourhood T_q (of Δ_q in $J^1(M\times M, R^q)$); Δ_q — in the picture — is the centre of this disc; τ is the antipodal map; $T_q\cap\Sigma_q^1$ is a diameter. ζ_q can be identified with a subbundle of T_q. Denote $\mathrm{pr}_q\colon T_q\to\zeta_q$ a projection onto this subbundle. The section $\psi\colon M\times M\to J^1(M\times M, R^q)$ maps the diagonal $\Delta(M)$ of $M\times M$ into Δ_q and ψ maps a small neighbourhood U of $\Delta(M)$ in $M\times M$ into T_q equivariantly. Identifying U with the normal bundle of $\Delta(M)$ (which is TM — the tangent bundle of M) we obtain a skew map: $\mathrm{pr}_q\circ\psi$:

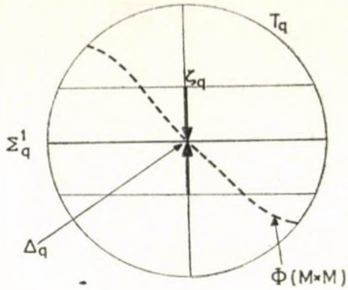


Fig. 1

$TM \to \zeta_q$ (and an equivariant map $\mathrm{pr}_{q-1} \circ \tilde{\psi}$: $TM \to \zeta_{q-1}$). By condition $\dim \zeta_q > 3/2 \dim M$. Hence by Theorem 2.3 of [3] we can change the section ψ inside T_q so that $\mathrm{pr}_q \circ \psi$ becomes a linear monomorphism. Moreover, we can suppose that $\mathrm{pr}_{q-1} \circ \tilde{\psi}$: $TM \to \zeta_{q-1}$ is generic in the sense that

1) its rank is at least $n-1$ everywhere;
2) $\operatorname{codim} \Sigma^1(\mathrm{pr}_{q-1} \circ \tilde{\psi} \mid {}_{TM}) = \dim \zeta_{q-1} - (n-1)$ and
3) the intersection of $\operatorname{Im}(\mathrm{pr}_{q-1} \circ \tilde{\psi})$ with the zero section of ζ_{q-1} is transversal outside the zero section of TM.

Denote S (in $M \times M$) the manifold $\Sigma^1(\mathrm{pr}_{q-1} \circ \tilde{\psi})$ (i.e. the set of singular points of the map $\mathrm{pr}_{q-1} \circ \tilde{\psi}$: $M \times M \xrightarrow{\tilde{\Psi}} J^1(M \times M, R^{q-1}) \xrightarrow{\mathrm{pr}_{q-1}} \zeta_{q-1}$). Then $S = \overline{A_{\tilde{\psi}}} \cap \cap \Delta(M)$. S is a submanifold of $\overline{A_{\tilde{\psi}}}$ of codimension 1. The normal bundle μ of S in $\overline{A_{\tilde{\psi}}}$ is trivial, because ω_q is nonzero on μ. Hence the function f: $M \to R^1$ can be chosen in such a way that (in addition to condition (*)) $\mathrm{d}f(\mu_x) = \omega_q(\mu_x)$ for any $x \in M$ such that $(x, x) \in S$. If f is such a function, then the map $(\tilde{\Psi}, \mathrm{d}f + \mathrm{d}f)$ will be detached in the neighbourhood of the diagonal $\Delta(M)$ of $M \times M$. This settles the case $r=0$.

For $r>0$ we need some obvious modifications:

In this case the definition of A_Ψ should read $A_\Psi =: \{(x, y) \mid \operatorname{rank} \tilde{\Psi}(T_x M + T_y M) = 2n - r - 1\}$ while the vector field γ should be the orthogonal complement (in any metric) of $\operatorname{Ker} \Psi(x, y)$ in $\operatorname{Ker} \tilde{\Psi}(x, y)$. The definition of μ is the same (replacing of course the set Σ^1_{q-1} by Σ^r_{q-1}).

Remark. To prove the h-principle, i.e. that the natural map from the space of detached immersions into the space of detached bundle maps, $f \mapsto \mathrm{d}f$, induces an isomorphism of the higher homotopy groups we have to consider a family of maps, parametrized by the points of a sphere and to perform the same constructions as before. Hence, to prove the isomorphism of the homotopy groups up to i we need the condition $\operatorname{codim} \Sigma^1_{q-1} = r(q-2n+r-1) > 3/2(n+i+1)$. So the notion of the metastability depends on i.

To give some concrete corollaries of this theorem we shall need the following two lemmas and a Remark.

LEMMA 1. *If there exists a Z_2-equivariant section*

$$\varphi\colon (M\times M\setminus U)\to (J^1(M\times M, R^q)\setminus \Sigma_q^r)$$

then there exists also an equivariant section

$$\psi\colon M\times M \to J^1(M\times M, R^q) \quad \textit{such that} \quad \psi^{-1}(\Sigma_q^r) = \Delta(M).$$

(Here U is a sufficiently small neighbourhood of $\Delta(M)$. For example we can fix a metric on M and set

$$U = \{(x, y) | \text{ distance } (x, y) < \varepsilon\}$$

where ε is less than the injectivity radius of exponential map at any point.)

PROOF. Identify U with $D_\varepsilon M$, the ε-ball bundle associated to TM. The section φ induces a map

$$\Phi\colon TM\times TM|_{M\times M\setminus U} \to R^q \quad \text{such that} \quad \Phi(u, v) = \Phi(v, u).$$

Denote $w\in\partial D_\varepsilon$ the vector which corresponds to the pair $(x, y)\in\partial U$. Since the pair (x, y) belongs to the boundary of the neighbourhood U, the map Φ is defined in this point (x, y) (or which amounts to the same, in the point $w\in TM$ corresponding to (x, y)). Extend the map Φ to the vector $\alpha w\in TM$ $0\leqq\alpha<1$ by the formula

$$\Phi(\alpha w) = \alpha\Phi(w).$$

Then the extended map Φ satisfies the requirement of the Lemma 1.

LEMMA 2. *If $q>4n-2$, then the locally detached maps form an open and dense subset in $C^\infty(M^n, R^q)$.*

Proof of Lemma 2 is at the end of the paper.

REMARK. The relative version of the Theorem also holds. For example, suppose that there exists an r-detached map $\Phi\colon TM\to R^q$, which coincides with the differential of a map over an open subset of M. Then there exists an r-detached immersion, which coincides with the given map on the given open set.

COROLLARY A.

1a. *For any n-dimensional manifold M^n there exists a detached immersion of M^n into R^{4n-1}.*

b. *Almost any locally detached immersion $M^n\to R^{4n-1}$ has even number of undetached pairs.*

2. *There are infinitely many detached maps $M^n\to R^{4n}$ which are not regularly homotopic through detached maps.*

PROOF.

1a. Denote $f\colon M^n\to R^{4n-1}$ a locally detached immersion, and U a small neighbourhood of the diagonal $\Delta(M)$ in $M\times M$, such that the restriction of $f+f$ to $U\setminus\Delta(M)$ is an immersion. The mapping $f+f\colon U\to R^{4n-1}$ maps the points (x, y) and (y, x) into the same (for $\forall x, y\in M$ such that $(x, y)\in U$), hence it induces an immersion $F\colon \left(U\setminus\frac{1}{2}U\right)\Big/Z_2\to R^{4n-1}$. Here $\frac{1}{2}U$ is an equivariant neighbourhood of

$\Delta(M)$ contained in U and $\left(U-\frac{1}{2}U\right)\Big/Z_2$ means the set $U-\frac{1}{2}U$ with identifications $(x, y)\sim(y, x)$. We can write with the obvious notation

$$F=(f+f)/Z_2 \quad \text{restricted to} \quad \left(U\setminus\frac{1}{2}U\right)\Big/Z_2.$$

Now we apply the relative version of the Whitney immersion theorem to the bounded manifold

$$M^*=\left(M\times M\setminus\frac{1}{2}U\right)\Big/Z_2.$$

We infer that there exists an immersion g of this manifold into R^{4n-1}, which coincides with F on the boundary.

Define the map Ψ: $TM\times TM\to R^{4n-1}$ as follows: Let $[x, y]$ be the point of M^* corresponding to (x, y)

$$\Psi|T_xM\times T_yM=\mathrm{d}g_{[x,y]} \quad \text{if} \quad (x, y)\notin\frac{1}{2}U$$

$$\Psi|T_xM\times T_yM=\mathrm{d}f_x+\mathrm{d}f_y \quad \text{if} \quad (x, y)\in U$$

(i.e. $\Psi(u, v)=\mathrm{d}f_x(u)+\mathrm{d}f_y(v)$ $\forall u\in T_xM$, $v\in T_xM$). This map Ψ satisfies the conditions a), b), c) of 3) in the Theorem and by (1) of the Theorem there exists a detached map $M^n\to R^{4n-1}$.

1b. By the Thom—Porteous theorem $\Sigma^1(T(M^*);\ R^{4n-1})=\overline{W}_{2n}(M^*)$ where $\Sigma^1(T(M^*), R^{4n-1})$ denotes the cohomology class (in $H^{2n}(M^*; Z_2)$) dual to the Σ^1-type singular points of a general linear map $T(M^*)\to R^{4n-1}$. It is well-known that $\overline{W}_{2n}(M^*)=0$. (For example: $\overline{W}_{2n}(M^*)$ is the only obstruction to an immersion $M^*\propto R^{4n-1}$. By Whitney's theorem there exists such an immersion, and so the obstruction must be zero.)

On the other hand for an immersion f: $M^n\to R^{4n-1}$ every undetached pair gives rise to a (Σ^1-type) singular point of the map

$$(f+f)/Z_2\colon M^*\to R^{4n-1}.$$

2. The only obstruction to regular homotopy lies in the group $H_c^{2n}(M^*, \pi_{2n}(V_{4n,2n}))$ and it is known from the obstruction theory that any class of this group can be an obstruction. Hence by the Theorem

π_0 (Detached—Immersions (M^n, R^{4n})) $\approx H_c^{2n}(M^*, \pi_{2n}(V_{4n,2n}))$. The coefficients $\pi_{2n}(V_{4n,2n})\approx Z$ are untwisted if M^* is orientable (i.e. if M is orientable and n is even) and they are twisted otherwise. In any case this cohomology group is isomorphic to $H_c^{2n}(\tilde{M}^*; Z)=Z$ (untwisted coefficients) where $\tilde{M}^*$ denotes the orientable covering space over M^*.

Remark (M. Gromov). In the statement 2 of Corollary A when $n=1$, the integer invariant of a homotopy class can be defined as the self-linking number of the spherical image (in S^3) of the curve (in R^q).

More sophisticated investigations of Becker [3, Theorem 10.5] together with our Theorem give the following

COROLLARY B. *Suppose that there exists a detached immersion* $M^n \to R^q$ *where* $q > (3{,}5)(n+1)$.

a) *If* q *is odd then the number of regular homotopy classes of detached maps is finite and a power of* 2.

b) *If* q *is even, then*

$$\pi_0(\mathrm{Det-Imm}\,(M^n, R^q)) \otimes Q \approx H^{q-2n}_{\mathrm{comp}}(M^*; Q)$$

where $\pi_0(\mathrm{Det-Imm}\,(M^n, R^q))$ *denotes the set of regular homotopy classes of detached immersions.*

PROOF of Lemma 2. The fibres of the first jet fibration $J^1(M^n, R^q) \to M^n$ can be identified with the space of $n \times q$ matrices. Denote Σ^1 the set of matrices of rank $\leqq n-1$ and let N be the complement of Σ^1. Any matrix $P \in N$ can be identified with n vectors in R^q. Let $\mathcal{K}_P$ be the submanifold of N formed by those n vectors which define a subspace in R^q such that it has common line with the space defined by the n vectors corresponding to P. In a neighbourhood of a point $P \in N$ one can define local coordinates (y and z) on $J^1(M^n, R^q)$ where $y \in R^t$ denotes the normal coordinates to $\mathcal{K}_P$ and $z \in R^s$ denotes the coordinates on $\mathcal{K}_P$ ($s = \dim \mathcal{K}_P$, $t = \operatorname{codim} \mathcal{K}_P = q - 2n + 1$). Denote p_2 the natural projection $J^2(M^n, R^q) \to J^1(M^n, R^q)$. In the fibre $p_2^{-1}(y, z)$ the coordinates are $r_1 = \frac{\partial y}{\partial x}$ and $r_2 = \frac{\partial z}{\partial x}$ (x denotes the coordinates in M^n). Denote $\sigma = \{(y, z, r_1, r_2) \in J^2(M^n, R^q) \mid \operatorname{rank} r_1 \leqq n-1\}$. Codim $\sigma = \operatorname{codim} \Sigma^1(R^n \to R^t) = t - n + 1 = q - 3n + 2$.

If codim $\sigma > n$ then any immersion can be approximated by such an immersion f that the image of the second jet section $J^2 f$ of f avoids σ. But such an immersion is locally detached.

REMARK. Codimension of $\mathcal{K}_P$ can be computed as follows. Given a matrix P of size $n \times q$ we consider those matrices A (of the same size) that the matrix (P, A) of size $2n \times q$ has rank less than $2n$. (Otherwise $A \notin \mathcal{K}_P$.) This means that the matrix A being composed with the projection $R^q \to R^q/\text{image } P$ defines a singular map $R^n \to R^{q-n}$. Codim $\Sigma^1(R^n \to R^{q-n}) = q - 2n + 1$. So $t = q - 2n + 1$.

REFERENCES

[1] SZŰCS, A., The Gromov—Eliashberg proof of Haefliger theorem on embeddings, *Studia Sci. Math. Hungar.* **17** (1982), 303—318.
[2] BECKER, J. C., Cohomology and the classification of liftings, *Trans. Amer. Math. Soc.* **133** (1968), 447—475. *MR* **38** #5217.
[3] HAEFLIGER, A. and HIRSCH, M. W., Immersions in the stable range, *Ann. of Math.* (2) **75** (1962), 231—241. *MR* **36** #784.

(Received April 16, 1981)

GYÖNGYHÁZ U. 2.
H—1132 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 253—268.

ÜBER DICHTE UND ENGE VON DOPPELGITTERFÖRMIGEN ZWEIFACHEN KREISPACKUNGEN

J. HORVÁTH und Á. H. TEMESVÁRI

Eine Menge von offenen Kreisen in der Ebene bildet *eine k-fache Packung,* wenn jeder Punkt der Ebene zu höchstens k der Kreisen gehört. Seit mehr als 25 Jahren sind mehrfache Packungen Gegenstand intensiver Forschung. Dabei wurden hauptsächlich gitterförmige Lagerungen untersucht. Eine Lagerung von kongruenten Kreisen heißt gitterförmig, wenn die Kreismittelpunkte ein ebenes Punktgitter bilden. Eine gitterförmige k-fache Packung nennen wir schlechthin *eine k-fache Gitterpackung.* Die dichteste k-fache Gitterpackung von Kreisen ist für $k \leqq 7$ bekannt [7, 1, 2]. U. Bolle [2] konstruierte für $8 \leqq k \leqq 100$ sehr günstige k-fache Gitterpackungen und er gab auch gute asymptotische Schranken für die Dichte der dichtesten k-fachen Gitterpackung von Kreisen an.

Über allgemeine k-fache Packungen ist viel weniger bekannt. Offensichtlich ist die Dichte einer k-fachen Packung höchstens k. Kürzlich fand G. Fejes Tóth [3] eine nichttriviale obere Abschätzung für die Dichte einer k-fachen Packung von kongruenten Kreisen. Er zeigte, daß die Dichte einer k-fachen Packung von kongruenten Kreisen höchstens $(\pi/6)\cot(\pi/6k)$ ist. Diese Schranke ist aber viel schwächer als die oben erwähnte Schranke von Bolle für die Dichte von k-fachen Gitterpackungen. Die folgende Konstruktion einer sehr dichten zweifachen Packung von kongruenten Kreisen stammt von Heppes [7].

Wir betrachten den Rhombus $OADB$ mit der Seitenlänge 2 (Abb. 1), wo $OD=7/2$ ist. Es sei L eine gitterförmige Lagerung von Einheitskreisen, deren Basisvektoren $\overrightarrow{OA}$ und $\overrightarrow{OB}$ sind. Wir verschieben die Lagerung L um $\overrightarrow{OC}=\frac{\overrightarrow{OD}}{7}$. Es ist leicht einzusehen, daß diese zwei gitterförmige Lagerungen gemeinsam eine 2-fache Kreispackung ergeben. Diese 2-fache Packung von Einheitskreisen wird mit $\mathcal{H}$ bezeichnet. Eine elementare Rechnung ergibt, daß die Dichte $\frac{16\pi}{7\sqrt{15}}$ beträgt. Das ist größer als $\frac{\pi}{\sqrt{3}}$, die Dichte der dichtesten zweifachen Gitterpackung von Kreisen. Daraus folgt, daß die dichteste 2-fache Packung von kongruenten Kreisen nicht gitterförmig sein kann. Es läßt sich vermuten, daß $\mathcal{H}$ die grösstmögliche Dichte unter sämtlichen 2-fachen Packungen von kongruenten Kreisen aufweist. Der Beweis dieser Vermutung dürfte sehr schwierig sein.

1980 *Mathematics Subject Classification.* Primary 52A45; Secondary 51M05.

Key words and phrases. k-fold packing, 2-fold double-lattice packing, closeness, density, closest packing, densest packing.

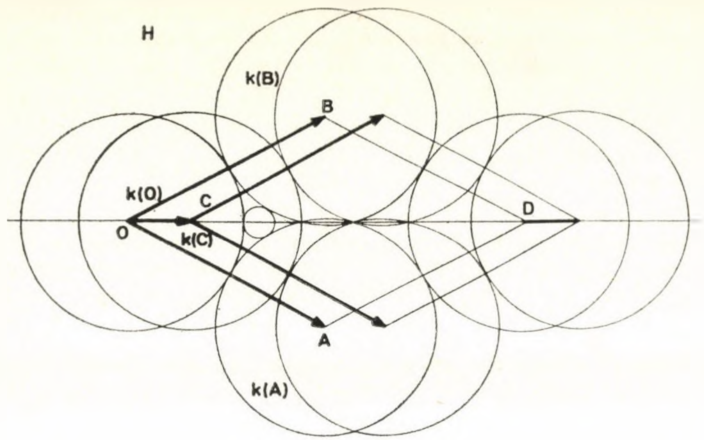


Abb. 1

Wir bemerken, daß zwar $\mathscr{H}$ nicht gitterförmig ist, doch sich als die Vereinigung von zwei kongruenten und homothetischen gitterförmigen Anordnungen darstellen läßt. Die Vereinigung von zwei kongruenten und homothetischen Punktgittern nennen wir ein *Doppelgitter*. Eine Anordnung von kongruenten Kreisen, deren Kreismittelpunkte ein Doppelgitter bilden, nennen wir *doppelgitterförmig* [8]. Im folgenden untersuchen wir die Dichte von doppelgitterförmigen 2-fachen Kreispackungen und beweisen den folgenden

SATZ 1. *Die Dichte einer doppelgitterförmigen 2-fachen Kreispackung ist* $\leqq \dfrac{16\pi}{7\sqrt{15}}$ *mit Gleichheit dann und nur dann, wenn die Kreispackung ähnlich zu* $\mathscr{H}$ *ist.*

Wir werden auch ein verwandtes Problem untersuchen. Für eine k-fache Packung L_k definieren wir nach L. Fejes Tóth die Enge $e_k(L_k)$ von L_k als das Supremum der Radien aller Kreise, die in den höchstens $(k-1)$-mal überdeckten Teilen der Ebene liegen. Mehrere Arbeiten beschäftigen sich mit der Enge von einfachen Packungen [5, 6, 10, 11]. Der Begriff der Enge ist verwandt mit dem von S. S. Ryškov eingeführten Begriff des (r, R)-Systems [9]. L. Fejes Tóth stellte die folgende Frage: Wie klein kann die Enge einer k-fachen Packung von kongruenten Kreisen sein? Er sprach die Vermutung aus, daß die Enge einer 2-fachen Packung von kongruenten Kreisen größer oder gleich 1/7 ist und Gleichheit besteht nur für eine mit $\mathscr{H}$ kongruente Kreisanordnung. Der folgende Satz bestätigt diese Vermutung in dem Spezialfall, wenn die Anordnung doppelgitterförmig ist.

SATZ 2. *Die Enge einer doppelgitterförmigen 2-fachen Einheitskreispackung ist* $\geqq 1/7$ *und Gleichheit steht nur dann, wenn die Packung mit* $\mathscr{H}$ *kongruent ist.*

Vor dem Beweis der Sätze führen wir die später gebrauchten Bezeichnungen ein und beweisen einige Hilfssätze. Wir führen in der Ebene ein Cartesisches Koordinatensystem mit dem Ursprung O ein, und bezeichnen Punkte und ihre Ortsvektoren mit demselben Symbol. Mit $|X|$ bezeichnen wir die Länge des Vektors X. Mit dem

Symbol (AB) bzw. (C, AB) bezeichnen wir das Gitter mit dem Ursprungspunkt O bzw. C, das durch die Basisvektoren A und B bestimmt ist. (AB) bzw. (C, AB) ist also die Menge der Punkte $xA+yB$ bzw. $C+xA+yB$, wo x, y ganze Zahlen sind. Die Gitter (AB) und (C, AB) gemeinsam bilden ein Doppelgitter, das mit dem Symbol (AB, C) bezeichnet wird. So ist $(AB, C)=(AB)\cup(C, AB)$.

Die Lagerung, die aus den um die Punkte von (AB), (C, AB) bzw. (AB, C) geschlagenen Einheitskreisen besteht, bezeichnen wir mit $L(AB)$, $L(C, AB)$ bzw. $L(AB, C)$. Es ist offenbar, daß $L(AB, C)=L(AB)\cup L(C, AB)$ ist. Wenn die Lagerung eine k-fache Packung ist, dann wenden wir die Bezeichnung L_k an.

$d_k(AB, C)$ und $e_k(AB, C)$ seien die Dichte und die Enge von $L_k(AB, C)$.

Wir geben das Gitter in der Ebene mit den zwei kürzesten linear unabhängigen Gittervektoren A, B an und nehmen an, daß der Winkel dieser zwei Gittervektoren nicht stumpfwinklig ist, d.h. die folgenden Bedingungen für die Vektoren A und B bestehen:

$$(1) \qquad \begin{gathered} |A| \leqq |B| \leqq |B-A| \leqq |X| \\ \sphericalangle(AOB) \leqq \frac{\pi}{2}, \end{gathered}$$

wo X ein beliebiger Gittervektor $X\neq\pm A, X\neq\pm B, X\neq 0$ ist. Ein Gitter ist von *normaler Darstellung*, wenn (1) für das Gitter gilt. Wir sprechen über *eine normale Darstellung von* (AB, C), wenn (AB) eine normale Darstellung hat und C kein äußerer Punkt des Dreiecks OAB ist. Außerdem soll $C\neq A, C\neq B$ sein.

Mit $k(P)$ bzw. $\hat{k}(P)$ bezeichnen wir den offenen Einheitskreis bzw. die Einheitskreislinie mit dem Mittelpunkt P. Wir wenden die Bezeichnung $k[PQR]$ bzw. $\hat{k}[PQR]$ für den offenen Kreis bzw. für die Kreislinie an, die durch die Punkte P, Q, R bestimmt sind. Mit $r[PQR]$ bezeichnen wir den Radius von $k[PQR]$. Es sei $PQ|R$ die Halbebene, die den Punkt R enthält und deren Grenzgerade PQ ist.

HILFSSATZ 1. *Wenn (AB, C) von normaler Darstellung ist und $|B|<2$ gilt, dann ist $L(AB, C)$ keine 2-fache Packung.*

BEWEIS. Es seien $M_1\in\hat{k}(O)\cap\hat{k}(A)$ und $M_1\in OA|B$, $M_2\in\hat{k}(O)\cap\hat{k}(B)$ und $M_2\in OB|A$, $T\in AB$ und $OT\perp AB$ (Abb. 2). Folglich gilt $T\in k(M_1)$ bzw. $T\in k(M_2)$ und deshalb überdecken $k(M_1)$ und $k(M_2)$ das Dreieck OAB mit Ausnahme der Ecken. Daraus folgt, daß $M_1\in k(C)\vee M_2\in k(C)$ ist, das heißt, $L(AB, C)$ ist keine 2-fache Packung.

HILFSSATZ 2. *Wenn $L_2(AB, C)$ von normaler Darstellung ist und $2\leqq|A|\leqq|B|$ gilt, dann sind $d_2(AB, C)\leqq\frac{\pi}{\sqrt{3}}$ und $e_2(AB, C)\geqq\frac{2}{\sqrt{3}}>\frac{1}{7}$.*

Wir gehen auf den Beweis nicht näher ein. Die Behauptung ist eine einfache Folge von der Tatsache, daß in diesem Fall $L_2(AB, C)=L_1(AB)\cup L_1(C, AB)$ für geeignete einfache Packungen $L_1(AB)$ und $L_1(C, AB)$.

HILFSSATZ 3. *Wir betrachten die Lagerung $L(AB, C)$ mit normaler Darstellung, wobei die folgenden Ungleichungen gelten:*

$$(2) \qquad 1 \leqq |A| < 2 \leqq |B|.$$

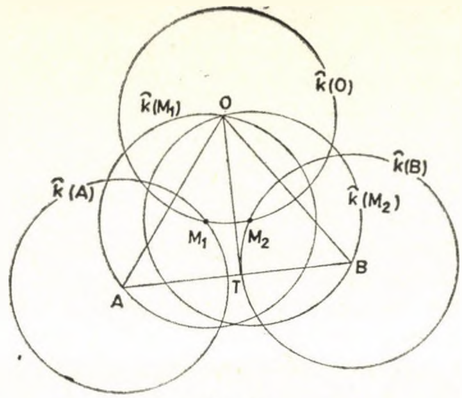


Abb. 2

Es seien $T=A/2$, $M_1\in\widehat{k}(0)\cap\widehat{k}(A)$ *und* $M_1\in OA|B$, *weiterhin sei M ein Punkt, der mit der Gleichung*

(3) $$M-M_1=2(M_1-T)$$

gegeben ist.

Wenn $L(AB, C)$ *eine zweifache Packung ist, dann gilt* $M\notin k(B)$ *und umgekehrt, wenn* $M\notin k(B)$ *ist, dann gibt es einen Punkt C, so daß* $L(AB, C)$ *eine zweifache Packung ist.*

Beweis. Es seien $M_2\in\widehat{k}(B)\cap\widehat{k}(A+B)$ und $M_2\in B(A+B)|A$, $M_3\in\widehat{k}(B)\cap\cap\widehat{k}(B-A)$ und $M_3\in B(A+B)|A$, $M_4=\widehat{k}(M_2)\cap\widehat{k}(M_3)$ und $M_4\neq B$ (Abb. 3). Aus (3) folgt die Gleichheit

(4) $$B-M_4=M-M_1$$

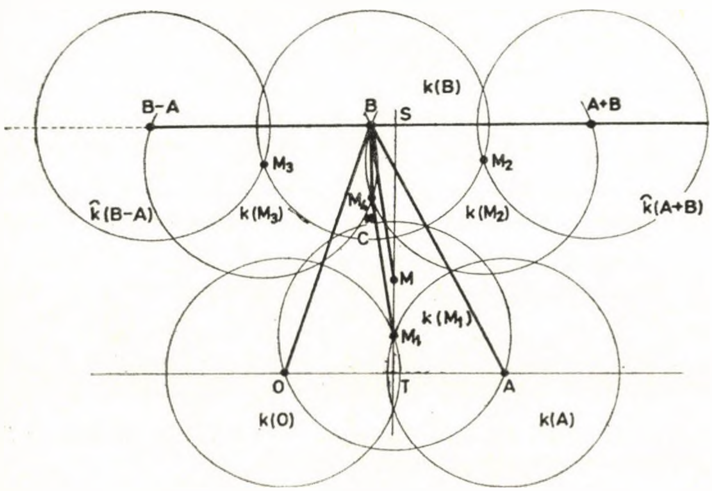


Abb. 3

und aus (4) folgt

(5) $$B-M = M_4-M_1.$$

Wenn $L(AB, C)$ eine zweifache Packung ist, dann folgt $C \notin k(M_1)$, $C \notin k(M_2)$ und $C \notin k(M_3)$. So ist

$$|C-M_1| \leqq |M_4-M_1|.$$

Auf Grund von (3), (4), (5) gilt $M \notin k(B)$.

Wenn $M \notin k(B)$ ist, dann ist $L(AB, C)$ eine zweifache Packung z.B. für den Punkt $C = M_4$.

HILFSSATZ 4. *Wenn die Ungleichungen* (1) *und* (2) *für die Seiten des Dreiecks OAB gelten und $M \notin k(B)$ ist, dann liegt das Spiegelbild des durch $\widehat{RM_1} \subset \hat{k}(O)$ begrenzten kleineren Kreisbogens bezüglich M_1 außerhalb des Kreises $k(B)$* $(R = \hat{k}(O) \cap \cap \hat{k}(A)$, $R \neq M_1$, *s. M_1 im Hilfssatz 3*$)$.

BEWEIS. Wir bezeichnen mit O' bzw. $\hat{k}(O')$ das Spiegelbild des Punktes O bzw. der Kreislinie $\hat{k}(O)$ bezüglich M_1. Es ist klar, daß die mit einem Bogen gekennzeichneten Winkel der Abb. 4 gleich sind. Wir zeigen, daß

$$\sphericalangle(BMM_1) + \sphericalangle(M_1MO') > \pi$$

gilt.

Aus den Ungleichungen

$$|B-M| \geqq |M_1| = 1$$

und

$$|B| \leqq |B-A|$$

folgt

$$\sphericalangle(BMM_1) > \sphericalangle(OM_1M).$$

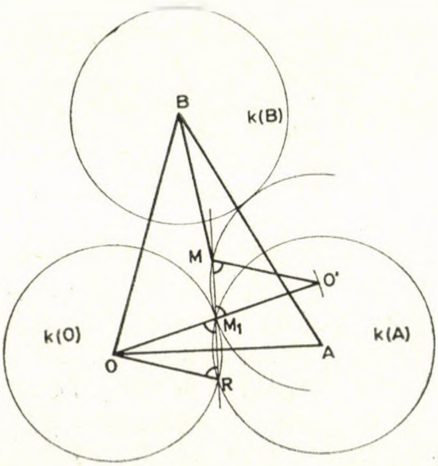


Abb. 4

Daraus ergibt sich

$$\sphericalangle(BMM_1)+\sphericalangle(M_1MO')>\sphericalangle(OM_1M)+\sphericalangle(OM_1R)=\pi.$$

Hieraus folgt, weil M kein innerer Punkt des Kreises $k(B)$ ist, daß die Punkte des Kreisbogens $\widehat{M_1M}$ mit dem Mittelpunkt O' außerhalb des Kreises $k(B)$ liegen, wobei $\widehat{M_1M}$ das Spiegelbild des oben erwähnten Kreisbogens $\widehat{RM_1}$ ist.

Hilfssatz 5. *Wenn die Ungleichungen* (1) *und* (2) *für die Seiten des Dreiecks OAB gelten und $M\notin k(B)$ ist, dann ist der Umkreisradius $r[OAB]$ des Dreiecks OAB nicht kleiner als* $8/7$. *Gleichheit tritt dann und nur dann auf, wenn*

$$OB=AB=2 \quad und \quad OA=\frac{\sqrt{15}}{2} \tag{6}$$

gelten.

Beweis. Wir verwenden die Bezeichnungen des Hilfssatzes 3.

(a) Nehmen wir an, daß $M\notin k(B)\cup\hat{k}(B)$ ist (Abb. 3). Weiterhin sei S der Schnittpunkt der Mittelsenkrechten von OA mit der Parallelen zu OA durch B. Nun verschieben wir B in Richtung von $\overrightarrow{BS}$. Der erste mögliche Fall ist, daß $M\in\hat{k}(B)$ wird. Dieser Fall wird später erörtert. Tritt dieser Fall nicht ein, verschieben wir B in den Punkt S. In diesem Fall gilt $OB=AB\geqq 2$. Bei dieser Lageänderung nimmt $r[OAB]$ ab, weil OA konstant bleibt und $\sphericalangle(OBA)$ streng wächst.

(b) Es seien $OB=AB>2$ und $M\notin k(B)\cup\hat{k}(B)$. Jetzt bewegen wir B auf der Mittelsenkrechten auf T zu. Es kann wieder der Fall eintreten, daß $M\in\hat{k}(B)$ wird, was wir später erörtern wollen. Wenn $M\notin\hat{k}(B)$ wird, tritt der Fall $OB=AB=2$ ein. Bei dieser Lageänderung nimmt $r[OAB]$ ab, weil $\sphericalangle(OBA)$ streng wächst.

(c) Es seien $OB=AB=2$ und $M\notin k(B)\cup\hat{k}(B)$. Wir drehen gleichmäßig die Kreise $k(O)$ und $k(A)$ um B auf die Mittelsenkrechte von OA zu. Durch die neuen Lagen von $k(O)$ und $k(A)$ ergibt sich die neue Lage des betrachteten Punktes M. Die Lageänderung kann dazu führen, daß $M\in\hat{k}(B)$ wird, weil $BM_4=2TM_1$ zunimmt und BM_1 abnimmt. Durch Rechnung ergibt sich wegen der Gleichungen $MB=1$ und $TM=3TM_1=3p$, daß dieser Fall auftritt, wenn $p=1/4$ ist. Man kann leicht sehen, daß $r[OAB]$ während der Drehung streng abnimmt und sein Minimum gleich $8/7$ ist. Dieser Wert tritt nur dann auf, wenn (6) gilt.

Aus dem Beweis ist ersichtlich, daß der Fall (c) nur bei $p\leqq 1/4$ auftritt und $M\in\hat{k}(B)$ nur im Falle $p=1/4$ gilt.

(d) Wir nehmen an, daß $p=TM_1>1/4$ ist und $M\in\hat{k}(B)$ (Abb. 5). Wir bewegen A (gemeinsam mit $k(A)$) auf $\hat{k}[OAB]$ derart, daß der Abstand OA zunimmt (A'). Weil der Abstand von A und O zunimmt, liegen die Schnittpunkte M_1' und R' der Kreise $\hat{k}(O)$ und $\hat{k}(A')$ auf dem von den Punkten M_1 und R begrenzten Kreisbogen von $\hat{k}(O)$. Wir spiegeln die Punkte M_1', R' über M_1. Dann liegen die Spiegelbilder $\overline{M}_1$, $\overline{R}$ auf dem Kreisbogen $\widehat{M_1M}$. Wegen des Hilfssatzes 4 gilt $\overline{M}_1$, $\overline{R}\notin k(B)\cup\hat{k}(B)$. Verschieben wir die Strecke $\overline{M}_1\overline{R}$ um $\overline{\overline{M}_1M_1'}$. Es sei M' das Bild von $\overline{R}$ bei dieser Verschiebung. Wegen der Verschiebung ist $B\overline{R}<BM'$, d.h., auch $M'\notin k(B)\cup\hat{k}(B)$.

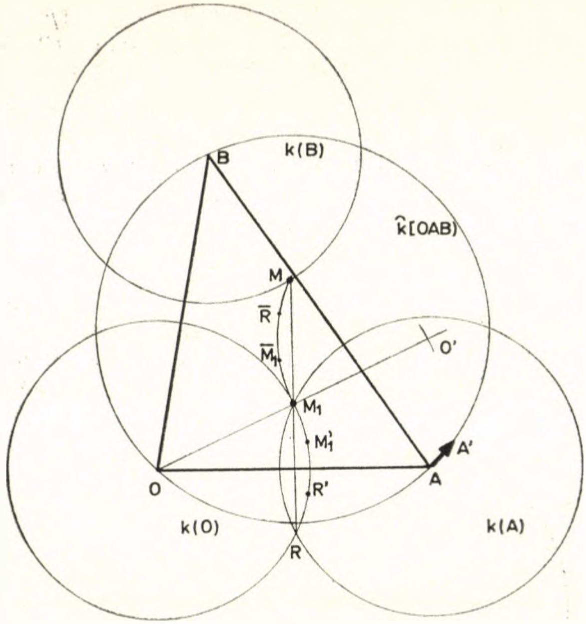


Abb. 5

M' ist das Spiegelbild des Punktes R' bezüglich M_1', d.h., M' übernimmt die in (3) definierte Rolle des Punktes M.

Wir beenden diese Lageänderung, wenn $T'M_1' \leq 1/4$ oder $AB=2$ gilt.

Im Falle $T'M_1' \leq 1/4$ wiederholen wir die in den Punkten (a), (b), (c) beschriebenen Lageänderungen. Wenn $AB=2$ und $OB>2$ sind, dann wenden wir für den Kreis $k(O)$ die im Punkt (d) beschriebene Lageänderung an. Nach der Bemerkung, die wir am Ende des Punktes (c) gemacht haben, können wir den Fall $T'M_1' \leq 1/4$ immer erreichen. So können wir $r[OAB]$ durch die in den Punkten (a), (b), (c) beschriebenen Lageänderungen vermindern. Das Minimum tritt im Fall (6) auf.

Damit haben wir den Beweis des Hilfssatzes 5 beendet.

Beweis des Satzes 1. Wir benützen die Bezeichnungen der Abb. 3. Weiterhin können wir auf Grund der Hilfssätze 1, 2 annehmen, daß die Ungleichungen (1), (2) für die Seiten des Dreiecks OAB bestehen.

Wir müssen den Inhalt des Dreiecks OAB so minimalisieren, daß alle Punkte der Ebene höchstens zweifach überdeckt sind. Wir haben schon gesehen, daß in einem solchen Fall $M \notin k(B)$ ist (Hilfssatz 3). Wir zeigen, daß der Inhalt des Dreiecks OAB im Fall der 2-fachen doppelgitterförmigen Packung $\mathcal{H}$ von Einheitskreisen minimal wird.

Wir können den ersten Teil des Beweises sehr knapp ausführen, da man zunächst die in Punkten (a), (b), (c) des Beweises vom Hilfssatz 5 angegebenen Lageänderungen vornehmen kann. Bei der Lageänderung (a) ändert sich die Dichte natürlich nicht, bei den Lageänderungen (b), (c) nimmt aber die Dichte zu.

Wir nehmen an, daß $M\in\widehat{k}(B)$ ist und weiterhin $TM_1>1/4$ gilt. Wenn $OB>2$ ist, dann drehen wir den Kreis $k(B)$ um M bis den Fall $OB=2$ oder der Fall $\sphericalangle(AOB)=\pi/2$ erreicht wird. Es ist offensichtlich, daß der Inhalt des Dreiecks OAB abnimmt und $M\in\widehat{k}(B)$ bleibt.

Es seien $OB=2$ und $\sphericalangle(AOB)<\pi/2$ (Abb. 6). Wegen $MM_1=RM_1$ ist R das Spiegelbild von M bezüglich M_1. Wir spiegeln den Punkt O und den Kreis $k(O)$ über M_1 und bezeichnen die Spiegelbilder mit O' bzw. $k(O')$. Es ist offenbar, daß OO' der Durchmesser des Kreises $k(M_1)$ ist, deshalb gilt

$$\sphericalangle(OAO')=\frac{\pi}{2}. \tag{7}$$

Es ist leicht einzusehen, daß auch der Winkel $\sphericalangle(AOB)$ streng zunimmt, wenn $TM_1=p$ wächst, und im Fall $p=1/2$ gerade $\pi/2$ wird. Deshalb beschäftigen wir uns nur mit dem Fall $1/4<p<1/2$.

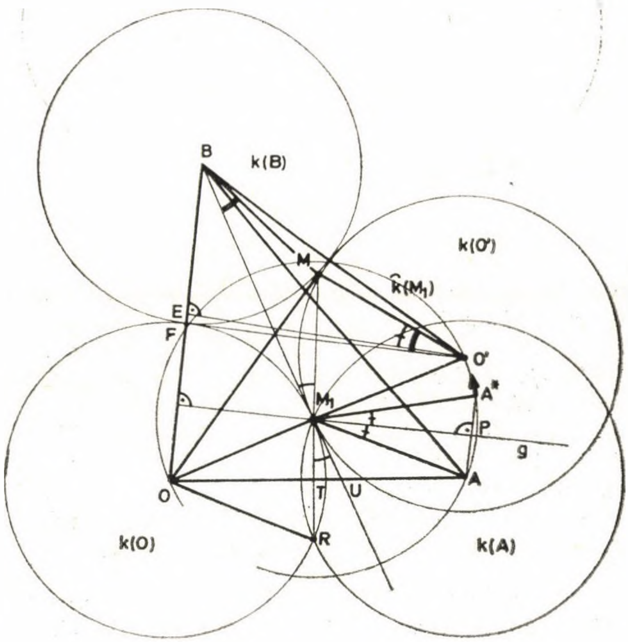


Abb. 6

Wir betrachten die Gerade g, für die $M_1\in g$, $g\perp OB$ gilt. Wenn die Punkte O und A auf verschiedenen Seiten der Geraden g liegen oder $A\in g$ ist, bewegen wir A (gemeinsam mit $k(A)$) auf der Kreislinie $\widehat{k}(M_1)$ auf den Punkt O' zu. Während der Lageänderung nimmt der Inhalt des Dreiecks OAB ab. Wenn $AB=2$ ist, beenden wir den Vorgang. Im Laufe der Erörterung des folgenden Falles werden wir sehen, daß bei dieser Lageänderung $M\notin k(B)$ wird, d.h., wegen des Hilfssatzes 3 entsteht eine 2-fache doppelgitterförmige Packung.

Wenn $O, A \notin g \mid B$ ist, spiegeln wir A und $k(A)$ über die Gerade g. Die Spiegelbilder seien A^* und $k(A^*)$. Dann bewegen wir den Punkt A^* auf $\widehat{k}(M_1)$ auf den Punkt O' zu. Wir beenden diese Bewegung, wenn $A^*B=2$ erreicht ist. Die Spiegelung ändert den Inhalt des Dreiecks OAB nicht, bei der nachfolgenden Lageänderung nimmt aber der Inhalt ab. Es ist einzusehen, daß wir diese Lageänderungen so durchführen können, daß die Packung eine 2-fache Packung bleibt.

A^* liegt auf dem durch die Punkte A, O' bestimmten kleineren Bogen von $\widehat{k}(M_1)$, weil die Ungleichung $\sphericalangle(OAA^*)>\pi/2$ wegen $g \perp OB$, $\sphericalangle(BOA)<\pi/2$ und (7) gilt.

Wir beweisen, daß der entsprechende Punkt M während dieser Lageänderungen ein äußerer Punkt von $k(B)$ bleibt. R ist das Spiegelbild von M über M_1, im Laufe der beiden früheren Lageänderungen bleibt $R \in \widehat{k}(O)$ und M_1R nimmt ab. Daher liegt M auf dem durch die Punkte M, M_1 bestimmten kürzeren Bogen von $\widehat{k}(O')$. Die Punkte dieses Bogens sind aber die äußeren Punkte von $k(B)$ auf Grund des Hilfssatzes 4.

Man muß noch zeigen, daß $BA^*>2$ ist, weil wir den Hilfssatz 3 nur in diesem Fall anwenden können. Das werden wir dadurch einsehen, daß wir die Ungleichung

$$\sphericalangle(BM_1O) < \sphericalangle(BM_1A^*) \tag{8}$$

beweisen. Aus (8) folgt $BA^*>2$, weil $OB=2$ und $OM_1=A^*M_1=1$ gilt. Statt (8) genügt es

$$\sphericalangle(OM_1U) > \sphericalangle(A^*M_1U) \tag{9}$$

zu beweisen, wo $U=OA \cap BM_1$ ist. Es ist klar, daß

$$\sphericalangle(AM_1P) = \sphericalangle(A^*M_1P) \tag{10}$$

ist, wo $P=g \cap AA^*$ ist. Weiterhin gilt

$$\sphericalangle(OM_1T) = \sphericalangle(AM_1T).$$

Aus Abb. 6 kann man ablesen, daß

$$\sphericalangle(OM_1U) = \sphericalangle(OM_1T)+ \sphericalangle(TM_1U)$$

und

$$\begin{aligned} \sphericalangle(A^*M_1U) &= \sphericalangle(AM_1T)+2\sphericalangle(AM_1P)- \sphericalangle(TM_1U) = \\ &= \sphericalangle(OM_1T)+ \sphericalangle(TM_1U)-2[\sphericalangle(TM_1U)- \sphericalangle(AM_1P)] \end{aligned} \tag{11}$$

richtig sind.

Es ist ersichtlich, daß die Ungleichung (9) gilt, wenn

$$\sphericalangle(TM_1U) > \sphericalangle(AM_1P) \tag{12}$$

ist. Aus dem Dreieck BM_1M bekommen wir

$$\sphericalangle(TM_1U) = \sphericalangle(MM_1B) > \sphericalangle(MBM_1), \tag{13}$$

weil $M_1M=2p<1=BM$ ist. Die Gerade OM ist die Symmetrieachse des Dreiecks BOO', deshalb ist

$$\sphericalangle(MBM_1) = \sphericalangle(MO'F), \tag{14}$$

wo F der Mittelpunkt von OB ist.

Wir fällen aus O' das Lot auf OB. Es sei E der Fußpunkt dieser Senkrechten. Weil $O'B<O'O=2$ gilt, ist E ein innerer Punkt der Strecke BF, d.h.,

$$\sphericalangle(MO'F) > \sphericalangle(MO'E). \tag{15}$$

Wegen $MO' \| M_1A$ gilt

$$\sphericalangle(MO'E) = \sphericalangle(PM_1A). \tag{16}$$

Auf Grund der Ungleichungen (10)—(16) ergibt sich (9), und aus (9) folgt (8). Somit gilt $BA^*>2$. Das bedeutet, daß man in der Tat den Punkt A^* bis zur Lage $BA^*=2$ bewegen kann.

Am Ende des Punktes (c) des Beweises vom Hilfssatz 5 haben wir bemerkt, daß in der Endlage $M_1T \leqq 1/4$ gilt. Wir wenden die Lageänderung an, die im Punkt (c) angegeben wurde. Dabei nimmt der Inhalt des Dreiecks OAB streng ab und schließlich bekommen wir die gewünschte doppelgitterförmige Packung $\mathscr{H}$.

Wir müssen noch den Fall $\sphericalangle(AOB)=\pi/2$, $OB \geqq 2$ untersuchen. Weil der Winkel BOA eine streng monoton wachsende Funktion von p ist, tritt dieser Fall nur im Fall $p \geqq \frac{1}{2}$ auf. Wegen $OA \geqq 1$ gilt anderseits $p \leqq \frac{\sqrt{3}}{2}$.

Der Inhalt des Dreiecks OAB ist $4p\sqrt{1-p^2}$. Der Inhalt des Dreiecks OAB von $\mathscr{H}$ in der Abbildung 1 ist $\frac{7\sqrt{15}}{16}$. Es ist leicht zu sehen, daß $4p\sqrt{1-p^2}$ nur in den Fällen $p=1/2$ und $p=\frac{\sqrt{3}}{2}$ minimal wird, das Minimum ist $\sqrt{3}$, aber es gilt $\sqrt{3}>\frac{7\sqrt{15}}{16}$.

Damit haben wir den Beweis des Satzes 1 beendet.

Beweis des Satzes 2. In der 2-fachen doppelgitterförmigen Packung $\mathscr{H}$ von Einheitskreisen kann man immer einen Kreis mit dem Radius 1/7 lagern, der auch gemeinsam mit der Packung $\mathscr{H}$ nur eine 2-fache Kreispackung bildet. Ein solcher Kreis ist z.B. der Kreis mit dem Radius 1/7, der mit dem Umkreis des Dreiecks OAB konzentrisch ist (Abb. 1). Es ist leicht einzusehen, daß man einen Kreis mit größerem Radius in der Packung $\mathscr{H}$ nicht lagern kann.

Wir werden zeigen, daß man einen Kreis, dessen Radius mindestens 1/7 ist, in den höchstens einfach überdeckten Teilen einer beliebigen 2-fachen doppelgitterförmigen Packung lagern kann, und der Fall $r=1/7$ tritt nur bei der Packung $\mathscr{H}$ auf.

Wir benützen die Bezeichnungen der Abbildung 3. Weiterhin können wir auf Grund der Hilfssätze 1, 2 annehmen, daß die Ungleichungen (1), (2) für die Seiten des Dreiecks OAB bestehen.

Wir betrachten die zu dem Punkt C gehörige Dirichletsche Zelle $\mathscr{D}$ [4]. Keine Ecke von den Ecken der Zelle $\mathscr{D}$ kann ein Punkt von $k(C)$ sein, weil anderenfalls die Überdeckung um diese Ecke 3-fach wäre. Wir können auch annehmen, daß der

Abstand zwischen den Ecken von $\mathcal{D}$ und dem Punkt C höchstens 8/7 ist, weil man im entgegengesetzten Fall um diese Ecke einen Kreis mit dem Radius $r > 1/7$ angeben kann, der keinen Kreis der Packung schneidet, so daß diese Packung auf Grund einer früheren Bemerkung nicht eng sein kann.

Aus den Bedingungen (1) und (2) folgt, daß die Seiten der Dirichletschen Zelle $\mathcal{D}$ die Mittelsenkrechten von den Strecken $C(C-A)$, $CO, CA, C(C+A), C(A+B)$ oder $C(B-A)$ und CB sind. Wegen $OA < 2$ gelten $P_1 \neq P_2$, $P_4 \neq P_5$ und im Höchstfall können nur $P_1 = P_6$ oder $P_6 = P_5$ bzw. $P_2 = P_3$ oder $P_3 = P_4$ auftreten (Abb. 7), wo P_i die Ecken von $\mathcal{D}$ sind.

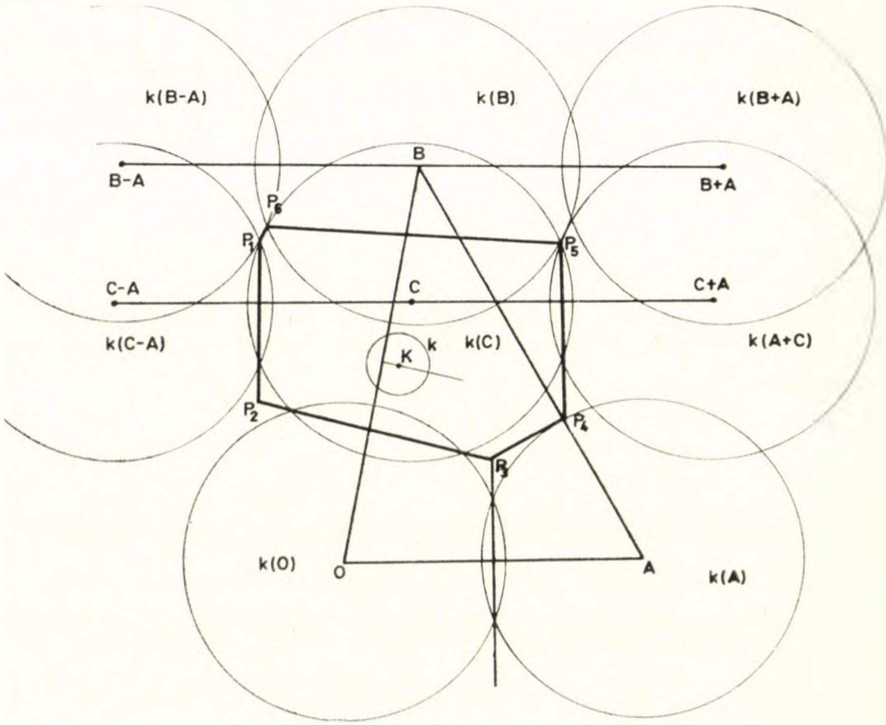


Abb. 7

Es sei K ein solcher Punkt, der im Dreieck OAB auf der Mittelsenkrechten von OB nahe am Mittelpunkt von OB liegt und ein äußerer Punkt der Kreise der Packung mit Ausnahme des Kreises $k(C)$ ist. Wählen wir um K einen Kreis k mit dem Radius $\varepsilon > 0$, der die Kreise der Packung außer $k(C)$ nicht schneidet.

Vergrößern wir den Kreis k um den Mittelpunkt K bis zur ersten Berührung. Dann, mit Beibehaltung der Berührung, vergrößern wir den Kreis weiter so, daß K auf der Mittelsenkrechte von OB bleibt und der Kreis die Kreise der Packung außer $k(C)$ nicht schneidet. Also vergrößern wir immer bis zur Berührung.

So bildet unsere 2-fache doppelgitterförmige Packung von Einheitskreisen gemeinsam mit k auch eine 2-fache Packung. Wir werden zeigen, daß der Radius von k mindestens 1/7 ist, wobei Gleichheit nur bei der Packung $\mathcal{H}$ auftritt (Abb. 1).

Wenn man den Kreis k auf die oben beschriebene Weise nicht weiter vergrößern kann, dann berührt k mindestens drei Kreise. Man kann aus den Kreisen, die den Kreis k berühren, drei Kreise so auswählen, daß die Mittelpunkte dieser Kreise ein nicht stumpfwinkliges Dreieck bestimmen und K der Mittelpunkt des Umkreises dieses Dreiecks wird. So ist es nötig zu beweisen, daß der Umkreisradius dieses Dreiecks größer oder gleich 8/7 ist, wobei Gleichheit nur bei der Packung $\mathcal{H}$ auftritt. Im Laufe der Vergrößerung von k auf die angegebene Weise treten die folgenden Berührungen auf.

1. Während der Vergrößerung berührt k zuerst die Kreise $k(O)$ und $k(B)$ (offenbar gleichzeitig), dann den Kreis $k(A)$ (Abb. 7a). Weil $L(AB, C)$ eine 2-fache Packung ist, ist $M \notin k(B)$, wo M in (3) bestimmt ist (Hilfssatz 3). Damit gelten die Voraussetzungen des Hilfssatzes 5 für OAB, also ist $r[OAB]$ größer oder gleich 8/7, d.h., der Radius von k ist nicht kleiner als 1/7. Gleichheit besteht dann und nur dann, wenn (6) gilt, d.h., $L_2(AB, C) = \mathcal{H}$ ist.

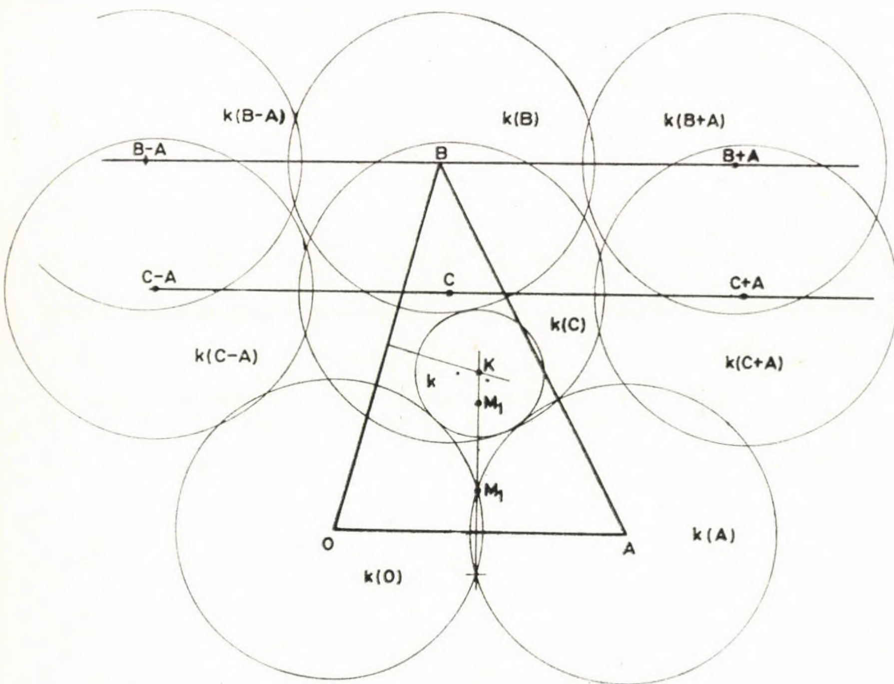


Abb. 7a

2. Bei der Vergrößerung von k berührt k zuerst den Kreis $k(C-A)$, dann $k(C+A)$ und endlich einen von den Kreisen $k(O)$, $k(A)$, $k(B)$, $k(A+B)$. Nehmen wir an, daß k z.B. den Kreis $k(O)$ berührt (Abb. 7.b). (In den übrigen Fällen verläuft der Beweis ebenso.)

Der Punkt K ist kein äußerer Punkt des Dreiecks $(C-A)O(C+A)$ und er ist der Mittelpunkt des Umkreises dieses Dreiecks, weiterhin ist die Kreispackung 2-

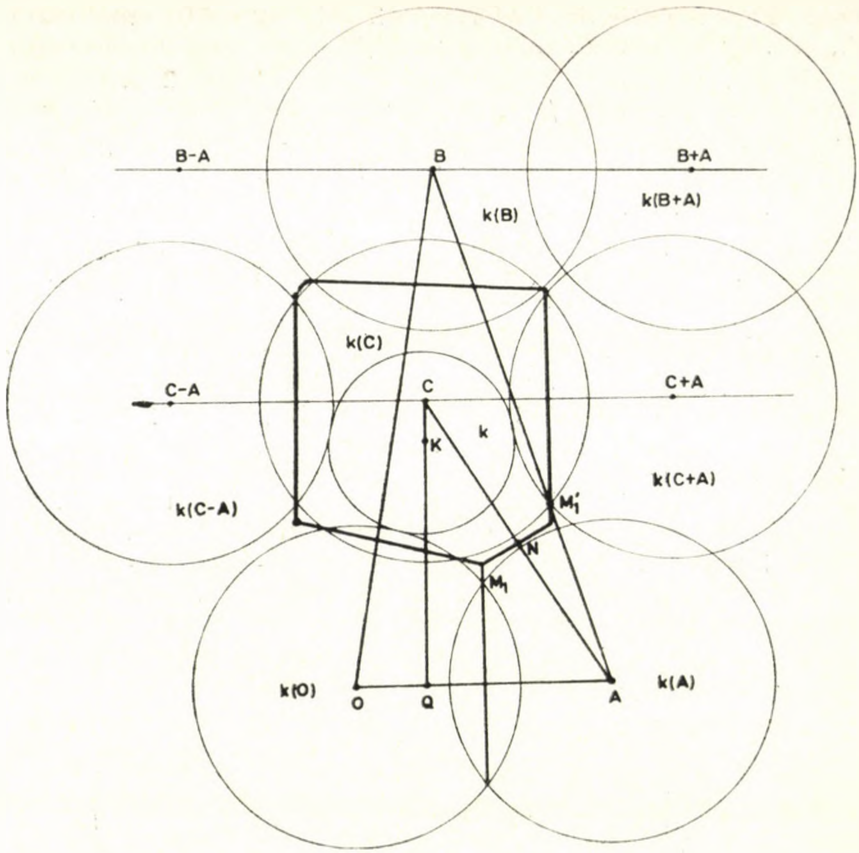


Abb. 7b

fach, deshalb sind zwei Seiten dieses Dreiecks nicht kleiner als 2 wegen des Hilfssatzes 1. Wenn jede Seite des Dreiecks mindestens 2 ist, dann ist der Umkreisradius größer als 8/7 auf Grund des Hilfssatzes 2. Daher ist der Radius von k größer als 1/7, d.h., die Kreispackung ist nicht eng.

Es ist klar, daß $|C+A-(C-A)|=2|A|\geqq 2$ ist. Nehmen wir an, daß $|C-A|<2$ gilt. Man kann leicht einsehen, daß die Gleichheit in $|C+A|\geqq 2$ nicht auftreten kann. Es sei $N=(1/2)(C+A)$. Wir spiegeln M_1 über N. Das Spiegelbild M_1' ist der Schnittpunkt von $\widehat{k}(C)$ und $\widehat{k}(C+A)$. Die Seite der Zelle $\mathscr{D}$, die durch den Punkt N geht, schneidet die Strecke M_1M_1', so liegt M_1 näher an die Gerade OA als M_1'. Es seien $Q\in OA$ und $|C-A-Q|=|C+A-Q|$. Wir verschieben O (gemeinsam mit A, $k(O)$, $k(A)$, M_1) in Q. $\overline{M}_1$ sei das Bild von M_1 bei dieser Verschiebung. Weil $\overline{M}_1M_1'\perp CA$ gilt und $\overline{M}_1$ näher an OA liegt als M_1', deshalb schneidet das Bild von $k(O)$ die Kreislinie $\widehat{k}(A+C)$ nicht, d.h., es gilt $|C+A-Q|>2$. Der Winkel $\sphericalangle(C-A, O, C+A)$ nimmt bei dieser Verschiebung zu. Wenn $\sphericalangle(C-A, Q, C+A)\leqq$ $\leqq\pi/2$ ist, dann nimmt $r[(C-A)O(C+A)]$ bei dieser Lageänderung ab. Jede Seite des Dreiecks $(C-A)Q(C+A)$ ist mindestens 2, daher ist der Umkreisradius größer als 8/7, d.h., der Radius von k ist größer als 1/7. Also ist diese Packung nicht eng. Wenn $\sphericalangle(C-A, Q, C+A)>\pi/2$ ist, dann ist $|C+A-(C-A)|>2\sqrt{2}$. Weiterhin tritt der Fall $\sphericalangle(C-A, \bar{O}, C+A)=\pi/2$ im Laufe der Lageänderung auf. Der Umkreisradius des Dreiecks $(C-A)\bar{O}(C+A)$ nimmt ab, so ist $r[(C-A)O(C+A)]$ größer als $\sqrt{2}$. Daraus folgt, daß der Radius von k größer als 1/7 ist, d.h., die Packung war nicht eng.

3. Während der Vergrößerung von k berührt k zuerst die Kreise $k(O)$, $k(B)$, dann den Kreis $k(A+C)$. Aus den Berührungen folgt, daß das Dreieck $OB(A+C)$ spitzwinklig ist. (Im entgegengesetzten Fall berührt k auch den Kreis $k(A)$.)

Wir zeigen, daß $r[OB(A+C)]$ größer als 8/7 ist. Damit ist der Radius von k größer als 1/7, d.h. die Packung ist nicht eng.

Betrachten wir das Dreieck $O(A+C)B$, wo $OB\geqq 2$ ist. Wegen (1) und (2) gilt $|A+C|\geqq|C-A|$. Wenn $|A+C|<2$, dann ist der Punkt $(1/2)(A+C)$ mindestens 3-fach überdeckt. Daher gilt $|A+C|\geqq 2$. Wenn auch $|A+C-B|\geqq 2$ gilt, dann ist $r[OB(A+C)]$ auf Grund des Hilfssatzes 2 größer als 8/7, d.h., der Radius von k ist größer als 1/7.

Es sei $|A+C-B|<2$. Weil auch $|A+C-M_2|\geqq 1$ gilt, liegt $A+C$ (s. Abb. 7c) im Inneren des Bogendreiecks WVZ oder auf dem Rand VW bzw. VZ dieses Bogendreiecks. Daher sind V, W, Z die folgenden Punkte: $W-B=2(M_2-B)$, $V\in\widehat{k}(M_2)$, $V\neq A+B$ und $V-A=\lambda B$, $|Z-B|=2$, $Z-A=\lambda_1 B$ und $\lambda>\lambda_1$. Bezüglich der

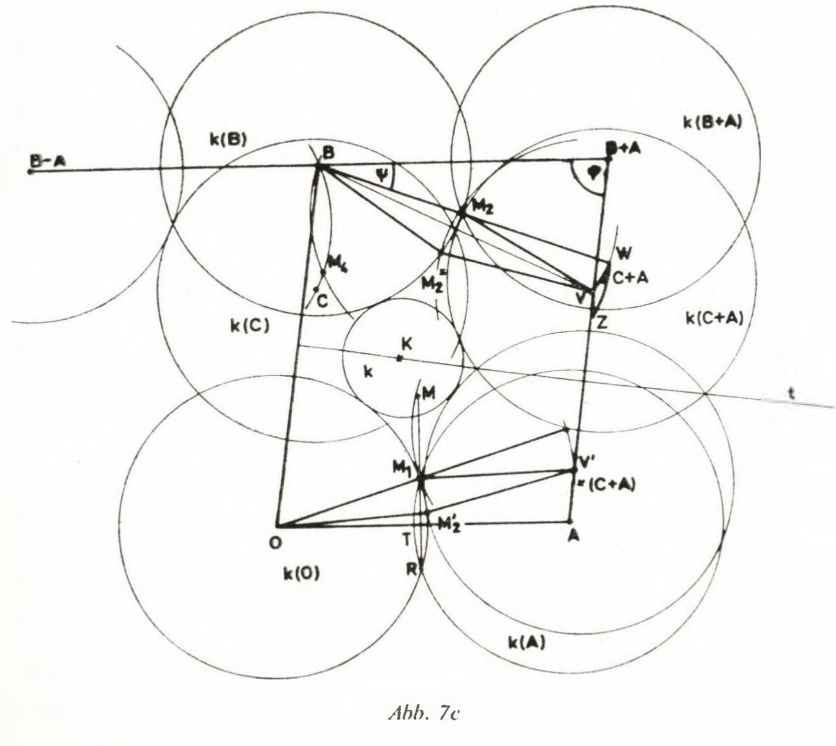


Abb. 7c

Seiten des Bogendreiecks gilt nun folgendes: VZ ist eine Strecke, VW ist der durch die Punkte V und W begrenzte kleinere Bogen von $\overset{\frown}{k}(M_2)$, ZW ist der durch die Punkte Z und W begrenzte kleinere Bogen des Kreises, dessen Mittelpunkt B und dessen Radius 2 ist.

Es sei $\varphi = \sphericalangle(AOB)$ und weiterhin $A+C=V$. Es gilt $|B-M_2|=|A+B-M_2|=|V-M_2|=1$, und M_2 ist ein innerer Punkt des Dreiecks $BV(A+B)$ $(\varphi<\pi/2)$. Deshalb ist $\sphericalangle(BM_2V)=2\varphi$. Bei der Spiegelung des Punktes M_2 über BV sei M_2^* der Bildpunkt von M_2. Es gilt $\sphericalangle(M_2BM_2^*)=\pi-2\varphi$. Es sei

$$\sphericalangle((A+B)BM_2) = \sphericalangle(TOM_1) = \psi.$$

Dann ist

$$\sphericalangle(OBM_2^*) = (\pi-\varphi)-(\pi-2\varphi)-\psi = \varphi-\psi,$$

also

$$\sphericalangle(BOM_1) = \sphericalangle(OBM_2^*). \tag{17}$$

Nun spiegeln wir über die Mittelsenkrechte t von OB. Wir kennzeichnen die Spiegelbilder mit „′". Nach (17) ist $(M_2^*)'=M_1$. Da $\sphericalangle(OA(A+B))\geqq\pi/2$ ist, liegt V' auf der Strecke $A(A+B)$. Daher liegt M_2' auf dem durch die Punkte R, M_1 begrenzten kleineren Bogen von $\overset{\frown}{k}(O)$. Wir zeigen, daß der Radius von $k[OBV']$ größer als 8/7 ist.

Der in (3) definierte Punkt M für das Dreieck OBV' ist das Spiegelbild von M_2' an M_1. Weil M_2' auf dem Bogen RM_1 liegt, liegt sein Spiegelbild auf dem Spiegelbild des Kreisbogens, d.h., auf dem Bogen $\overset{\frown}{M_1M}$. Da $M\notin k(B)$ ist (Hilfssatz 3), hat der Bogen $\overset{\frown}{M_1M}$ wegen des Hilfssatzes 4 keinen gemeinsamen inneren Punkt mit $k(B)$, d.h., der zum Dreieck $OV'B$ gehörige Punkt M ist ein äußerer Punkt von $k(B)$. Auf Grund des Hilfssatzes 5 ist daher der Radius von $k[OV'B]$ größer als 8/7.

$A+C$ sei ein Punkt des Bogendreiecks WVZ und kein Punkt des Bogens $\overset{\frown}{WZ}$. Wegen $|A+C-B|\geqq|B-V|$ und $\sphericalangle(M_2^*VZ)=\varphi+\psi>\pi/2$ liegt der Schnittpunkt von $\overset{\frown}{k}(B)$ und $\overset{\frown}{k}(A+C)$ auf dem Bogen $\overset{\frown}{M_2M_2^*}$. Nun spiegeln wir das Dreieck $OB(A+C)$ über t, sein Spiegelbild ist das Dreieck $BO(A+C)'$. Der Schnittpunkt von $\overset{\frown}{k}(O)$ und $\overset{\frown}{k}((A+C)')$ liegt auf dem Bogen $\overset{\frown}{M_1M_2'}$, also auch auf dem Bogen $\overset{\frown}{M_1R}$. Mit der im Punkt (d) des Beweises vom Hilfssatz 5 angewandten Methode kann man, da M kein innerer Punkt von $k(B)$ ist, zeigen, daß der zum Dreieck $O(A+C)'B$ gehörige Punkt M ein äußerer Punkt von $k(B)$ ist. Aus dem Hilfssatz 5 folgt, daß der Radius von $k[OB(A+C)]$ bzw. $k[OB(A+C)']$ größer als 8/7 ist, d.h., der Radius von k ist größer als 1/7 und gerade das wollten wir beweisen.

Aus (1) und (2) sowie aus der Bestimmung von k folgt, daß eine andere Art der Berührungen nicht auftreten kann. So haben wir den Satz 2 endgültig bewiesen.

Zum Abschluß bemerken wir, daß Satz 1 von Ágota H. Temesvári und Satz 2 von J. Horváth bewiesen wurde.

LITERATURVERZEICHNIS

[1] BLUNDON, W. J., Multiple packing of circles in the plane, *J. London Math. Soc.* **38** (1963), 176—182. *MR* **26** #6864.

[2] BOLLE, U., Mehrfache Kreisanordnungen in der euklidischen Ebene, Dissertation, Univ. Dortmund, 1976.

[3] FEJES TÓTH, G., Multiple packing and covering of the plane with circles, *Acta Math. Acad. Sci. Hungar.* **27** (1976), 135—140. *MR* **54** #5977.

[4] FEJES TÓTH, L., *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, Berlin—New York, 1972. *MR* **50** #5603.

[5] FEJES TÓTH, L., Close packing and loose covering with balls, *Publ. Math. Debrecen* **23** (1976), 323—326. *MR* **55** #1224.

[6] FEJES TÓTH, L., Remarks on the closest packing of convex discs, *Comment. Math. Helv.* **53** (1978), 536—541. *MR* **58** #24019.

[7] HEPPES, A., Mehrfache gitterförmige Kreislagerungen in der Ebene, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 141—148. *MR* **21** #3812.

[8] RYŠKOV, S. S., The polyhedron $\mu(m)$ and certain extremal problems of the geometry of numbers, *Dokl. Akad. Nauk SSSR* **194** (1970), 514—517 (in Russian). *MR* **43** #2613.

[9] RYŠKOV, S. S., Density of an (r, R)-system, *Mat. Zametki* **16** (1974), 447—454 (in Russian). *MR* **53** #282.

[10] HORVÁTH, J., Close lattice packing of balls in lattices of the first type, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **20** (1977), 191—194 (in Russian). *MR* **58** #5534.

[11] Хорват, Е., Об узкой решетчатой упаковке единичных шаров в пространстве E^n (Narrow latticed packing of unit balls in the space E^n), *Geometry of positive quadratic forms,* Trudy Mat. Inst. Steklov. **152** (1980), 216—231 (in Russian). *MR* **83a**: 10058.

(Eingegangen am 28. August 1981)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
ÁBRÁZOLÓ ÉS PROJEKTÍV GEOMETRIAI TANSZÉK
RÁKÓCZI ÚT 5
H—1088 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 269—275.

PERIPHERAL VERTICES IN GRAPHS

HALINA BIELAK and MACIEJ M. SYSŁO

Abstract

A vertex of maximum eccentricity (distance) in a graph is called c-peripheral (m-peripheral, resp.). Let $P_c(G)$ and $P_m(G)$ denote the set of c- and m-peripheral vertices of a graph G. In Section 1, we prove that (a) if H is regular or H is non-regular and has no vertex of degree $n(H)-1$ then there exists a graph G such that the subgraph of G induced by $P_c(G)$ is isomorphic to H, and (b) for any graph H there is a graph G such that the subgraph of G induced by $P_m(G)$ is isomorphic to H. In Section 2, we exhibit a parametric family of trees in which all possible relations between the sets of c- and m-peripheral vertices may occur.

1. Graphs induced by peripheral vertices

Let $G=(V(G), E(G))$ be a graph without loops and multiple edges, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. Let us denote $n(G)=|V(G)|$. We assume that G is connected, although some of the results can be easily extended to disconnected graphs. For each $v\in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v in G. The *distance between vertices* $v, w\in V(G)$ is denoted by $\varrho_G(v, w)$ and defined to be the least number of edges in a path joining v and w in G. For a vertex $v\in V(G)$, the *eccentricity* of v is defined by

$$e_G(v) = \max\{\varrho_G(v, w):\ w\in V(G)\},$$

and the *distance* of v to be

$$d_G(v) = \sum_{w\in V(G)} \varrho_G(v, w).$$

The subgraph of G induced by $W\subset V$ is denoted by $G\,|\,W$. Let $e_m(G)=\min\{e_G(v):\ v\in V(G)\}$ and $d_m(G)=\min\{d_G(v):\ v\in V(G)\}$. We may define now the *center* of G by

$$C(G) = \{v\in V(G):\ e_G(v) = e_m(G)\},$$

and the *median* of G to be

$$M(G) = \{v\in V(G):\ d_G(v) = d_m(G)\}.$$

1980 *Mathematics Subject Classification.* Primary 05C05; Secondary 05C99.
Key words and phrases. Distance in a graph, c- and m-peripheral vertex, tree.

This paper has been completed when the first author was with the Institute of Computer Science, University of Wrocław in the period February 1981—June 1981. Her permanent affiliation is Institute of Mathematics, Curie-Skłodowska University, Lublin, Poland.

One of the first results concerning centers and medians of graphs is due to Jordan and states that if T is a tree then $C(T)$ is K_1 or K_2 and similarly $M(T)$ is K_1 or K_2. Recently, Proskurowski [4] determined the possible centers of maximal outerplanar graphs.

Kopylov and Timofeev announced in [2] and Buckley, Miller and Slater [1] proved that for every graph H there exists a graph G such that $C(G)$ is isomorphic to H. Let $\alpha(H)=\min\{|V(G)|: C(G)=H\}$. S. T. Hedetniemi showed that $\alpha(H)\leqq |V(H)|+4$, see [2]. Similarly, Slater [6] proved that for any graph H there is a graph G such that $M(G)=H$. Let $\beta(H)=\min\{|V(G)|: M(G)=H\}$. Slater also proved that $\beta(H)$ is $O(n^3(H))$ and Miller [3] improved this bound by showing that for a graph H with no isolated vertices we have $\beta(H)\leqq 2n(H)$.

The aim of this section is to derive similar results for the sets of peripheral vertices of a graph.

Let $e_M(G)=\max\{e_G(v): v\in V(G)\}$ and $d_M(G)=\max\{d_G(v): v\in V(G)\}$. We may define $P_c(G)=\{v\in V(G): e_G(v)=e_M(G)\}$ and $P_m(G)=\{v\in V(G): d_G(v)=d_M(G)\}$. An element of $P_c(G)$ is called an *c-peripheral* vertex and of $P_m(G)$ — *m-peripheral*. Let us define also $\alpha'(H)=\min\{|V(G)|: P_c(G)=H\}$ and $\beta'(H)=\min\{|V(G)|: P_m(G)=H\}$. For simplicity, $G|P_c(G)$ and $G|P_m(G)$ will be denoted by $G|P_c$ and $G|P_m$, respectively.

First notice that the graphs $H=K_p$ $(p\geqq 1)$ and $H=C_q$ $(q\leqq 3)$, where K_p is the complete graph on p vertices and C_q is the q-vertex cycle, are isomorphic to their $C(H)$, $M(H)$, $P_c(H)$ and $P_m(H)$. However, not all graphs are isomorphic to $P_c(H)$ of some other graphs H. We have the following

Theorem 1.1. *If a graph H has a vertex of degree $n(H)-1$ and H is not complete then there exists no G such that $G|P_c=H$.*

The proof of this theorem is based on the following two very simple lemmas.

Lemma 1.1. *If $w\in P_c(G)$ and $|n(G)|\geqq 2$ then G has at least one vertex $u\neq w$ such that $\varrho_G(w,u)=e_M(G)$, therefore $u\in P_c(G)$.* □

Lemma 1.2. *If a graph H has a vertex of degree $n(H)-1$ then $e_H(u)\leqq 2$ for every $u\in V(H)$.* □

Proof of Theorem 1.1 Let $\deg_H(v)=n(H)-1$ and $\deg_H(u)<n(H)-1$. If there exists a graph G such that $G|P_c=H$ then $e_G(u)\geqq 2$. Therefore by Lemma 1.1, there exists $w\in V(G)-V(H)$ such that $e_G(w)=e_M(G)$ and w lies on the other end of a longest path starting at v. Hence, $G|P_c\neq H$, a contradiction. □

Therefore, no $K_{1,n}$ is isomorphic to $P_c(G)$ of another graph G. However, we prove now that every non-complete graph without a vertex adjacent to all other vertices is isomorphic to $P_c(G)$ of some graph G.

Theorem 1.2. *If a graph H has no vertex adjacent to all other vertices of H then there is a graph G such that H is isomorphic to $G|P_c$.*

Proof. Let us take $G=H+K_1$, where "+" means that, every vertex of H is adjacent to every vertex of K_1. Let $V(K_1)=\{z\}$. We have $e_G(z)=1$ and $e_G(v)=2$ for every $v\in V(H)$, therefore $P_c(G)=H$. □

COROLLARY 1.1. (a) $\alpha'(K_p)=p$ $(p\geqq 1)$ *and* $\alpha'(C_q)=q$ $(q\leqq 3)$.

(b) $\alpha'(H)\leqq|V(H)|+1$ *if* $\Delta(H)<n(H)-1$, *where*

$$\Delta(H)=\max\{\deg_H(v):\ v\in V(H)\}.$$

(c) *If H is not a graph of* (a) *and* (b) *then* $\alpha'(H)$ *is not defined.* □

The graph G constructed in the proof of Theorem 1.2 shows also that $|P_c(G)|-|C(G)|$ is unbounded in general.

We now turn our attention to graphs generated by $P_m(G)$. Evidently, all K_p and C_q are isomorphic to the subgraphs generated by their P_m sets. Let us consider first regular graphs. We have

LEMMA 1.3. *If a graph H on n vertices is* $(n-k)$*-regular, where* $n-k>k-2$ *then* $H|P_m=H$, *that is* $\beta'(H)=n=n(H)$.

PROOF. Let v be an arbitrary vertex of a graph H. The number of vertices not adjacent to v is $k-1$. Therefore, every vertex $u\in V(H)-N_H(v)$ must have at least one neighbour in $N_H(v)$. Hence H is connected, and $d_H(v)=(n-k)+2(k-1)=n+k-2$ for every vertex v. □

For all other regular graphs we have

THEOREM 1.3. *For every r-regular graph H, where* $r<n(H)-2$, *there exists a graph G such that* $G|P_m=H$ *and* $\beta'(H)\leqq n(H)+1$.

PROOF. It suffices to take $G=H+K_1$. Let $V(K_1)=\{z\}$. In this case we have $d_G(z)=n(H)$ and $d_G(v)=2n(H)-(r+1)>d_G(z)$ for every $v\in V(H)$. Therefore $G|P_m=H$. Notice that this theorem is also valid for disconnected graphs. □

The graph $G=H+K_1$ shows also that $|P_m(G)|-|M(G)|$ is unbounded in general.

For arbitrary graph we can prove

THEOREM 1.4. *For every graph H there exists a graph G such that* $G|P_m=H$ *and* $\beta'(H)\leqq n(H)+\Delta(H)+3$, *where* $\Delta(H)$ *is the maximum vertex degree of G.*

PROOF. First, we construct from H a graph F such that $\deg_F(v)=\Delta(H)$ for every $v\in V(H)$ and $\deg_F(u)>\Delta(H)$ for $u\in V(F)-V(H)$. Let K_r, where $r=\Delta(H)+2$, be the complete graph with the vertex set $u_1, u_2, \dots, u_r$. Assume that $V(H)=\{v_1, v_2, \dots, v_n\}$. We define $V(F)=\{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_r\}$ and $E(F)$ consists of the edges of H and K_r, and additionally every vertex v_i is connected with $\Delta(H)-\deg_H(v_i)$ consecutive vertices of K_r. Evidently, $\deg_F(v_i)=\Delta(H)$ $(i=1, \dots, n)$ and $\deg_F(u_j)>\Delta(H)$ $(j=1, \dots, r)$. Now we define G as $F+K_1$. Assume that $V(K_1)=\{z\}$. It is clear that G is of diameter 2, and $\deg_G(v_i)=\Delta(H)+1<\deg_G(u)$ for every $v_i (i=1, 2, \dots, n)$ and $u\in V(G)-V(H)$. Therefore, $d_G(v_i)=2n(H)+\Delta(H)+3$ for $i=1, \dots, n$ and $d_G(u)<d_G(v_1)$ for every $u\in V(G)-V(H)$. Thus, $G|P_m=H$. □

2. Relations between peripheral vertices of trees

The purpose of this section is to present a parametric family of trees which then is used to show that all possible relations between the sets of c- and m-peripheral vertices of a tree may occur. Thus, we solve one of the problems posed by Szamko-

9

łowicz in [7]. More precisely, we construct infinite classes of trees T with the sets $C(T)$ and $M(T)$ at arbitrary distance one from another and which satisfy one of the following relations

$$P_c(T) = P_m(T),$$

$$P_c(T) \subsetneqq P_m(T),$$

$$P_c(T) \supsetneqq P_m(T),$$

$$P_c(T) \cap P_m(T) \neq \emptyset, \quad P_m(T) \nsubseteq P_c(T) \quad \text{and} \quad P_c(T) \nsubseteq P_m(T),$$

$$P_c(T) \cap P_m(T) = \emptyset.$$

A similar problem for central vertices has been considered by Slater [5].

Let us recall first that for any tree T, graphs $T|C$ and $T|M$ are isomorphic either to K_1 or to K_2. We shall make use also of another definition of the median of a tree, see [7] and [8]. Let T be a tree, $T_u = \{T_i\}_I$ denote the family of components of $T-u$, and $f(u) = \max \{|V(T_i)|: T_i \in T_u\}$.

THEOREM 2.1 ([7] and [8]). *For every tree T*

$$M(T) = \{v: f(v) = \min \{f(u): u \in V(T)\}\}. \quad \square$$

It is easy to prove that c- and m-peripheral vertices of a tree are of degree one, see also [7] and [8]. Figure 1 shows a tree in which a wavy line is a path and

$$\varrho(u, w_i) = s \quad (i = 1, 2, \ldots, m), \quad \varrho(u, y) = s-1,$$

$$\varrho(w_i, v_i) = r_1 \quad (i = 1, 2, \ldots, m), \quad \varrho(w_i, z) = e,$$

$$\varrho(t, z) = r_2, \quad \text{and let} \quad 2s \leqq e.$$

In what follows, since we deal only with trees which can be obtained from T by fixing some of the parameters, all references to T in notations are omitted.

We shall demonstrate how to determine parameters s, r_1, r_2, m, l, k, as to obtain an infinite class of trees which satisfy exactly one of the relations between c- and

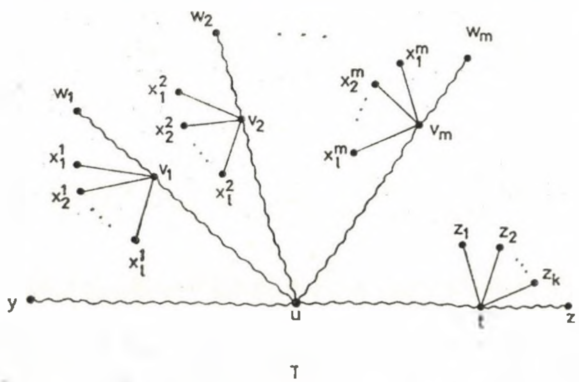


Fig. 1

m-peripheral vertices. It is evident that by the assumption $2s \leqq e$, we have $P_c \supset \supset \{z\} \cup \{w_i \colon i=1, \dots, m\}$.

Let us now calculate the values of the vertex distance function d.

$$d(w_i) = \sum_{j=1}^{e} j + m \sum_{j=s+1}^{2s-1} j + l(r_1+1) + 2s(m-1) + l(m-1)(2s-r_1+1) + k(e-r_2+1),$$
$$\text{for} \quad i = 1, 2, \dots, m,$$

$$d(y) = \sum_{j=1}^{e-1} j + m \sum_{j=s}^{2s-1} j + ml(2s-r_1) + k(e-r_2),$$

$$d(z) = \sum_{j=1}^{e-1} j + m \sum_{j=e-s+1}^{e} j + k(r_2+1) + ml(e-r_1+1),$$

$$d(z_i) = \sum_{j=1}^{e-r_2} j + m \sum_{j=e-r_2-s+2}^{e-r_2+1} j + \sum_{j=2}^{r_2+1} j + 2(k-1) + ml(e-r_1-r_2+2),$$
$$\text{for} \quad i = 1, 2, \dots, k,$$

$$d(x_i^g) = \sum_{j=1}^{e-r_1+1} j + \sum_{j=2}^{r_1+1} j + m \sum_{j=s-r_1+2}^{2s-r_1} j + 2(l-1) + k(e-r_1-r_2+2) +$$
$$(m-1)(2s-r_1+1) + l(m-1)(2s-2r_1+2),$$
$$\text{for} \quad i = 1, 2, \dots, l \quad \text{and} \quad g = 1, 2, \dots, m.$$

Proposition. 2.1. *There exists an infinite class of trees such that*

$$P_c = P_m,$$

and $\varrho(C, M)$ can be arbitrarily large.

Proof. Let $s=2$, $r_1=2$, $r_2=1$, $m=1$, $l=(e-2)h-1$ and $k=(e-4)h$, where h is a free variable. One can easily verify that in this case, $d(w_1)=d(z)=d(z_i)$ $(i= =1, 2, \dots, k)$, $d(w_1)>d(y)$, and $d(w_1)>d(x_i^1)$ for $i=1, 2, \dots, l$. Hence, $P_c=P_m= =\{w_1, z, z_i \ (i=1, 2, \dots, k)\}$.

Furthermore, if we choose $h \geqq e \geqq 4$, then vertex u is the median of the tree. To show this, note that every component of $T-u$ has at most $a=\max\{2, e+k-2\}$ vertices and every other vertex splits T into components with at least $b=\min\{e+k+ +l+1, l+4\}$ vertices. Therefore, if $h \geqq e \geqq 4$ then $b>a$. The central vertex u' of T is located on the path between u and z, and $\varrho(u, u')=(e-4)/2$. If $e=2h$, $h \geqq 2$ then $\varrho(u, u')=h-2 \geqq 0$. □

Proposition 2.2. *There exists a class of trees such that*

$$P_c \subsetneqq P_m,$$

and $\varrho(C, M)$ can be arbitrarily large.

Proof. Let $s=4$, $r_1=r_2=2$, $m=2$, $l=4h^2+3h-14$, $k=8h^2+2h-28$ $(h>1)$ and $e=4h$, for some integer h which will be determined later. Note that $k=2l-e$. One can easily verify that $d(w_1)=d(w_2)=d(z)=d(y)$ and $d(w_1)>d(x_i^g)$ $(g=1, 2;$ $i=1, 2, \dots, l)$, $d(w_1)>d(z_i)$ $(i=1, 2, \dots, k)$. Hence $P_c=\{z, w_1, w_2\}$ and $P_m=P_c \cup \{y\}$.

Using Theorem 2 we show that u is the median of T for the parameters set above, $T-u$ has four components of cardinalities 3, $l+4$, $l+4$ and $2l-4$ and it is easy to check that for every vertex x different from u, $T-x$ has a component of cardinality at least $\min(2l+12,\ 3l+4)$.

If c denotes the center of T, then we have $\varrho(u, c)=(e-2s)/2$, therefore increasing h we can make $\varrho(u, c)$ arbitrarily large. □

In the next three lemmas we assume that $s=e/2$ (e is an even number) $r_1=2$, $r_2=1$ and $m=1$. Hence, $C=\{u\}$ and $P_c=\{w_1, z, z_i\ (i=1, 2, \dots, k)\}$. We assume also that $M=\{t\}$, therefore, by Theorem 2.1, the following inequality must be satisfied

$$2+k > \frac{3}{2}e+l-2. \tag{1}$$

In this case, $\varrho(C, M)=e/2-1$, and since e will be assumed to be an arbitrary integer, the distance between the center and the median can be arbitrarily large.

Proposition 2.3. *There exists an infinite class of trees such that*

$$P_m \subsetneqq P_c,$$

and $\varrho(C, M)$ can be arbitrarily large.

Proof. We choose k, l and e such that $P_m=\{w_1\}$, that is $d(w_1)>d(z)$, $d(w_1)>d(y)$ and $d(w_1)>d(x_i^1)$ $(i=1, 2, \dots, l)$. These inequalities are equivalent to

$$k(e-2) > l(e-4),$$

$$k > l(e-5)-e/2,$$

$$k+l+3e/2 > 4$$

which are satisfied if for instance $l=1$, $k>(l+2)e$, and $e>4$. □

Proposition 2.4. *There exists an infinite class of trees such that*

$$P_c \cap P_m \neq \emptyset, \quad P_c \nsubseteq P_m, \quad P_m \nsubseteq P_c$$

and $\varrho(C, M)$ can be arbitrarily large.

Proof. We choose k, l and e such that $P_m=\{w_1, y\}$, that is $d(w_1)=d(y)$, $d(w_1)>d(z)$ and $d(w_1)>d(x_i^1)$ $(i=1, 2, \dots, l)$. These relations are satisfied if $k=l(e-5)-e/2$, $l>(2e-4)/(e-6)$ and $e\geqq 8$. □

Proposition 2.5. *There exists an infinite class of trees such that*

$$P_c \cap P_m = \emptyset$$

and $\varrho(C, M)$ can be arbitrarily large.

Proof. We choose k, l and e such that $P_m=\{y\}$, that is $d(y)>d(w_1)$, $d(y)\geqq d(z)$ and $d(y)\geqq d(x_i^1)$ $(i=1, 2, \dots, l)$. These inequalities imply that

$$l(5-e)+k+e/2 < 0,$$

$$k(3-e)+l+e/2 < 0,$$

$$(l+1)(e-4) \geqq 0.$$

The above system of inequalities together with (1) are satisfied if we take $e \geqq 8$, $l \geqq 7$ and $k = l(e-5) - e/2 - 1$. $\square$

ACKNOWLEDGEMENTS. The authors are indebted to Dr. F. Buckley and Dr. B. Zelinka for their comments which improved the presentation.

REFERENCES

[1] BUCKLEY, F., MILLER, Z. and SLATER, P. J., On graphs containing a given graph as center, *J. Graph Theory* **5** (1981), 427—434. *MR* **82k**: 05090.
[2] KOPYLOV, G. N. and TIMOFEEV, E. A., Centers and radii of graphs, *Uspehi Mat. Nauk* **32** (1977), No. 6, 226. *MR* **57** #9608.
[3] MILLER, Z., Medians and distance sequences in graphs, *Ars Combinatoria* **15** (1983), 169—177. *MR* **84h**: 05078.
[4] PROSKUROWSKI, A., Centers of maximal outerplanar graphs, *J. Graph Theory* **4** (1980), 75—79. *MR* **81a**: 05071.
[5] SLATER, P. J., Apprasing the centrality vertices in trees, *Studia Sci. Math. Hungar.* **12** (1977), 229—231 (1980).
[6] SLATER, P. J., Medians of arbitrary graphs, *J. Graph Theory* **4** (1980), 389—392. *MR* **81m**: 05125.
[7] SZAMKOŁOWICZ, L., On problems related to characteristic vertices of graphs, *Colloq. Math.* **42** (1979), 367—375.
[8] ZELINKA, B., Medians and peripherians of trees, *Arch. Math. (Brno)* **4** (1968), 87—95. *MR* **42** # 4436.

(Received September 1, 1981)

INSTYTUT MATEMATYKI
UNIWERSYTET MARII CURIE-SKŁODOWSKIEJ
PL. MARII CURIE-SKŁODOWSKIEJ 1
PL—20-931 LUBLIN

INSTYTUT INFORMATYKI
UNIWERSYTET WROCŁAWSKI
UL. PRZESMYCKIEGO 20
PL—51-151 WROCŁAW
POLAND

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 277—285.

ÜBER IONENPACKUNGEN

K. BEZDEK

Wie üblich nennen wir die positiv bzw. negativ elektrisch geladenen Teilchen ganz einfach Ionen. Aus Erfahrung wissen wir, daß die Ionen in gewisser Hinsicht durch Kugel mit einer entsprechenden Approximation modelliert werden können. Nämlich die elektrisch verschieden geladenen Teilchen wegen ihrer Abmessungen und die elektrisch gleich geladenen Teilchen wegen der Abstoßungskraft können sich nicht in beliebigem Maße nähern und so gehört jedem Ion zwei Kugel, wo eine der zwei Kugel dem Körper vom Ion die andere der Abstoßungskraft entspricht. Sowjetische Physikern (aus Novosibirsk) haben sich mit Ionen beschäftigt und einige mathematischen Probleme aufgeworfen. Die folgende Fragestellung stammt von diesem Problemenkreis.

Es sei in der euklidischen Ebene

(i) $\mathscr{K}_-$ eine Packung der Kreise vom Radius r_-, wo der Abstand der Mittelpunkte nicht kleiner als $2R_-$ ist

$$(0 < r_- \leqq R_-),$$

(ii) $\mathscr{K}_+$ eine Packung der Kreise vom Radius r_+, wo der Abstand der Mittelpunkte nicht kleiner als $2R_+$ ist

$$(0 < r_+ \leqq R_+),$$

(iii) $\mathscr{K} = \mathscr{K}_- \cup \mathscr{K}_+$ eine Packung.[1]

Wann wird die Dichte der Ionenpackung $\mathscr{K}$ mit gegebenen Radien r_-, R_-, r_+, R_+ am größten sein? Hier ist die Dichte nicht übereinandergreifender Scheiben $\{C_i\}$, die in der euklidischen Ebene irgendwie ausgestreut sind, üblicherweise durch $\overline{\lim\limits_{R\to+\infty}} \dfrac{C_i \cap C(R)}{C(R)}$ definiert, wo $C(R)$ ein Kreis vom Radius R mit dem fixen Mittelpunkt O ist.[2,3]

Zuerst hat K. Böröczky dieses Problem untersucht und den Fall $r_- = r_+ = \dfrac{1}{2}$,

1980 *Mathematics Subject Classification.* Primary 52A45; Secondary 51M05.
Key words and phrases. Ion packing, euclidean plane.

[1] Eine Anordnung nicht übereinandergreifender Scheiben wird eine Packung genannt.

[2] Im folgenden werden wir den Flächeninhalt eines Bereiches mit demselben Symbol bezeichnen wie den Bereich selbst.

[3] Dieser Wert hängt nicht von der Wahl des Ursprungspunktes O ab. Siehe dazu L. Fejes Tóth [1].

$R_- = R_+ = R$, $\frac{1}{2} \leqq R \leqq \frac{\sqrt{2}}{2}$ oder $\frac{\sqrt{3}}{2} \leqq R$ gelöst. Wir beschäftigen uns nun mit dem Fall $r_- = r$, $r_+ = 1 - r$, $\frac{1}{2} \leqq r \leqq \frac{\sqrt{2}}{2}$, $R_- = R_+ = \frac{\sqrt{2}}{2}$ und beweisen den folgenden Satz, den vorher K. Böröczky als Vermutung ausgesprochen hat.

SATZ. *Es sei* $r \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right]$ *eine gegebene reelle Zahl. Ist d die Dichte einer derartigen Packung der euklidischen Ebene aus Kreisen mit den Radien r, 1 − r, für die der Abstand der Mittelpunkte der kongruenten Kreise nicht kleiner als* $\sqrt{2}$ *ist, so gilt*

$$d \leqq d(r) = \frac{r^2 + (1-r)^2}{2} \pi.$$

BEMERKUNG. Nehmen wir das Einheitsquadratgitter mit den Kreisen von Radien r, $1-r$, die um die Gitterpunkte geschacht geschlagen sind, wie es auf der Abbildung 1 zu sehen ist, so ist hier die Dichte gleich $d(r)$.

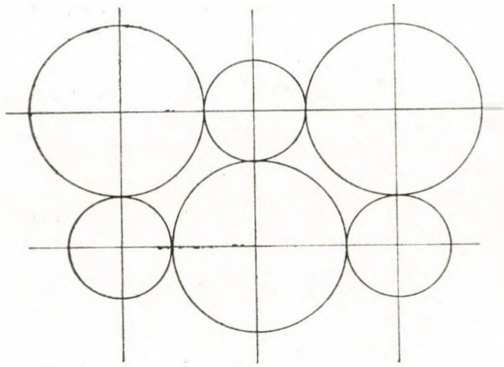


Abb. 1

BEWEIS. Wir können ohne Beschränkung der Allgemeinheit voraussetzen, daß die Packung gesättigt ist, weil wir sie sonst durch Hinzufügung weiterer Kreise sättigen könnten (d.h. die eventuell entstehenden großen Lücken durch Hinzufügung weiterer Kreise auflassen können). Dabei nimmt aber die Packungsdichte nicht ab. Unter dieser Bedingung konstruieren wir die L*-Zerlegung unserer Packung folgenderweise, die von J. Molnár [3] stammt. Ein Kreis, der in seinem Inneren keinen, auf seinem Rand aber wenigstens drei Kreismittelpunkte von Kreisen der gegebenen Packung enthält, wird Stützkreis genannt. Die auf dem Rand eines Stützkreises liegenden Kreismittelpunkte spannen ein konvexes Polygon auf, das wir ein Stützpolygon nennen. Es ist leicht einzusehen, daß die Stützpolygone ein Mosaik — die L-Zerlegung — der Ebene bilden [2]. Durch einander nicht schneidende Diagonalen zerlegen wir dann jede Zelle von L in Dreiecke, die die L′-Zerlegung der Ebene geben. Endlich betrachten wir die stumpfwinkligen Dreiecke der Zerlegung L′ und verfahren folgenderweise: in einem solchen Dreieck lassen wir die größte Seite weg, und nehmen denjenigen Streckenzug, der entsteht, wenn wir die Endpunkte der erwähnten Seite mit dem Kreismittelpunkt des zum Dreieck gehörigen Stützkreises verbinden. Damit haben wir die L*-Zerlegung als ein Mosaik der Ebene konstruiert. Der zu einer Zelle der L*-Zerlegung gehörige Gesamtinhalt der Kreise ist folgenderweise zu berechnen: nur solche Kreise (der Packung) werden betrachtet, die um eine der Ecken der gewählten Zelle geschlagen sind, dann nehmen wir diejenigen Kreissektoren der gewählten Kreise, die durch die Winkel der betrachteten Zelle bestimmt sind; schließlich müssen die Inhalte dieser Kreissektoren summiert werden (T). Dann ist die Dichte der Kreise bezüglich der betrachteten Zelle üblicherweise durch T/Δ definiert, wo Δ den Inhalt der Zelle bedeutet.

Wir werden zeigen, daß die Dichte (d) von Kreisen der Packung bezüglich beliebiger Zelle der L*-Zerlegung genügt der Ungleichung: $d \leqq d(r)$. Daraus folgt unser Satz ganz einfach.

Lemma 1. *Wir befestigen in der euklidischen Ebene zwei Kreise: mit dem Mittelpunkt A und dem Radius r_1, und mit dem Mittelpunkt B und dem Radius r_2. Es sei $T \in \overline{AB}$ ein gegebener Punkt. Ferner sei $\overline{CT} = m$; $\overline{CT} \perp \overline{AB}$ und $d(m)$ die Dichte der zwei gegebenen Kreise bezüglich des Dreiecks ABC, so ist $d(m)$ eine streng monoton fallende Funktion von m (Abb. 2).*

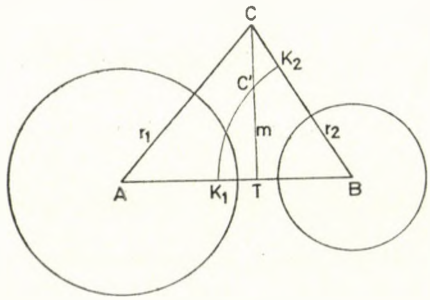


Abb. 2

Beweis. Es sei $d_1(m)$ (bzw. $d_2(m)$) die Dichte des Kreises vom Radius r_1 (bzw. r_2) bezüglich des Dreiecks ATC (bzw. BTC). Wir zeigen, daß $d_1(m)$ (bzw. $d_2(m)$) eine abnehmende Funktion von m ist, damit der Beweis des Lemmas beendet ist. Zum Beispiel für den Fall von $d_2(m)$: es sei $C' \in \overline{CT}$ $(C' \neq C)$. K_1 (bzw. K_2) ist der gemeinsame Punkt der Halbgerade $\overrightarrow{BA}$ (bzw. $\overrightarrow{BC}$) und des Randes des Kreises vom Radius BC'. Es ist leicht einzusehen, daß die Dichten des Kreises vom Radius r_2 bezüglich verschiedener Bereiche den nachfolgenden Ungleichungen genügen: $d(CC'B\Delta) < d(C'BK_2 \triangleleft) = d(K_1BC' \triangleleft) < d(TBC'\Delta)$ also $d(BTC\Delta) < d(BTC'\Delta)$.

Lemma 2. *Es seien r_1, r_2, a reelle, positive Zahlen. Ist $d(\gamma)$ die Dichte der Kreise mit dem Mittelpunkt A und dem Radius r_1, und mit dem Mittelpunkt B und dem Radius r_2 bezüglich des gleichschenkligen Dreiecks ABC, wo $AC = BC = a$ und $\sphericalangle ACB = \gamma$, so ist $d(\gamma)$ eine streng monoton fallende Funktion von γ $(0 < \gamma < \pi)$.*

BEWEIS. Es sei $2\alpha+\gamma=\pi$ (Abb. 3). Durch eine Rechnung ergibt sich:

$$d(\gamma)=\frac{r_1^2+r_2^2}{2a^2}\,\frac{2\alpha}{\sin 2\alpha}.$$

Andererseits für $0<2\alpha<\pi$ haben wir $\frac{d}{d\alpha}\left(\frac{2\alpha}{\sin 2\alpha}\right)>0$ womit der Beweis erbracht ist.

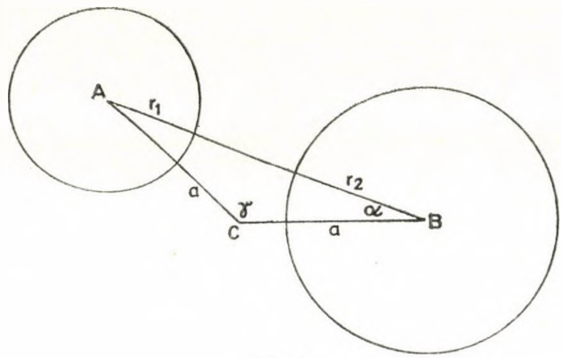


Abb. 3

Eine unmittelbare Folgerung aus Lemma 1 und 2, ist das folgende

KOROLLAR. *Es sei ABC ein gleichschenkliges Dreieck, wo $AC=BC\geqq\frac{\sqrt{2}}{2}$; $AB\geqq 1$. Ist d die Dichte der Kreise mit dem Mittelpunkt A und dem Radius r $\left(\frac{1}{2}\leqq r\leqq\frac{\sqrt{2}}{2}\right)$, und mit dem Mittelpunkt B und dem Radius $(1-r)$ bezüglich des Dreiecks ABC, so gilt*

$$d\leqq d(r)=\frac{r^2+(1-r)^2}{2}\,\pi.$$

LEMMA 3. *Es sei d die Dichte der Kreise mit dem Mittelpunkt A und dem Radius R, und mit dem Mittelpunkt B und dem Radius R bezüglich des gleichschenkligen Dreiecks ABC, wo $AC=BC$; $AB\leqq\frac{3}{2}$ und $R=r$ oder $R=1-r$ $\left(\frac{1}{2}\leqq r\leqq\frac{\sqrt{2}}{2}\right)$. Dann gilt:*

$$d<d(r)=\frac{r^2+(1-r)^2}{2}\,\pi.$$

BEWEIS. Einerseits ist offenbar $d<\frac{R^2}{(3/4)^2}\leqq\frac{16r^2}{9}$. Führen wir die Funktion $f(r)=\left(\pi-\frac{16}{9}\right)r^2-\pi r+\frac{\pi}{2}$ ein, so gilt $f\left(\frac{\sqrt{2}}{2}\right)>0$ und $f'\left(\frac{\sqrt{2}}{2}\right)<0$, woraus für

$\frac{1}{2} \leqq r \leqq \frac{\sqrt{2}}{2}$ die Ungleichung $f(r) > 0$ folgt. Also andererseits ist $\frac{16r^2}{9} < \frac{r^2+(1-r)^2}{2}\pi$. Deshalb gilt $d < d(r)$. Jetzt untersuchen wir die verschiedenen Typen der Zellen von der L^*-Zerlegung.

I. Die Zelle wird durch ein spitzwinkliges oder ein rechtwinkliges Dreieck erzeugt, die übrigens 0, oder 1, oder 2, oder 3 Einschnitte hat. Dementsprechend ist die Zelle 3-, oder 4-, oder 5-, oder 6-eck. Diese Zellen haben immer drei Kreise von der Packung, die um die Ecken desjenigen Dreiecks $K_1K_2K_3$ geschlagen sind, das die Zelle erzeugt (Abb. 4). Note: Die L^*-Zerlegung hat die folgende, wichtige Eigenschaft: ein Einschnitt = ein gleichschenkliges Dreieck.

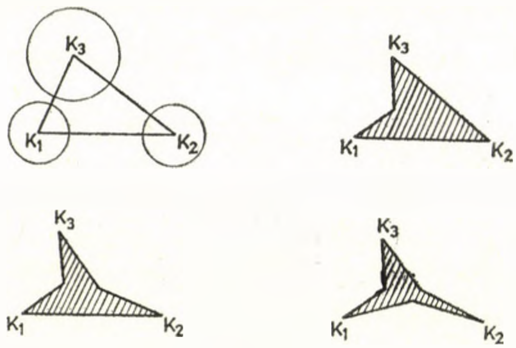


Abb. 4

I.a. Nur solche Zellen vom Typ I werden betrachtet, in denen die Kreise k_1, k_2, k_3 mit den Mittelpunkten K_1, K_2, K_3 kongruent sind.

Wir wählen eine von diesen Zellen, und bezeichnen mit d die Dichte der Packung bezüglich dieser Zelle. Es sei O der Mittelpunkt des Umkreises vom Dreieck $K_1K_2K_3$, wobei $d(OK_iK_j\Delta)$ die Dichte der Kreise k_i, k_j $(i \neq j,\ i, j=1, 2, 3)$ bezüglich des Dreiecks OK_iK_j bedeutet. Mit Rücksicht auf Lemma 1 ist die Ungleichung $d \leqq \max\{d(OK_iK_j\Delta)\ i \neq j,\ i, j=1, 2, 3\} = D$ sicher erfüllt. Ohne Beschränkung der Allgemeinheit können wir voraussetzen, daß $D = d(OK_1K_2\Delta)$ gilt. Aus dem Lemma 2 folgt die Ungleichung $\sphericalangle K_1OK_2 \leqq 2\pi/3$. Andererseits ist $K_1K_2 \geqq \sqrt{2}$. Also, es seien $K_1^*K_2^* = \sqrt{2}$, $\sphericalangle OK_1^*K_2^* = \sphericalangle OK_2^*K_1^* = \pi/6$. Und nun, schlagen wir den Kreis k_1 (bzw. k_2) um den Punkt K_1^* (bzw. K_2^*)! Betrachten wir die Dichte $d(OK_1^*K_2^*\Delta)$ der Kreise k_1, k_2 bezüglich des Dreiecks $OK_1^*K_2^*$, so ist mit Rücksicht auf Lemma 1 die Ungleichung $D \leqq d(OK_1^*K_2^*\Delta) \leqq \frac{\pi}{\sqrt{3}} r^2$ sicher erfüllt. Führen wir die Funktion $f(r) = (\sqrt{3}-1)r^2 - \sqrt{3}r + \frac{\sqrt{3}}{2}$ ein, so gilt $f\left(\frac{\sqrt{2}}{2}\right) > 0$ und $f'\left(\frac{\sqrt{2}}{2}\right) < 0$, woraus für $\frac{1}{2} \leqq r \leqq \frac{\sqrt{2}}{2}$ die Ungleichung $f(r) > 0$ folgt. Deshalb ist $\frac{\pi}{\sqrt{3}} r^2 < \frac{r^2+(1-r)^2}{2}\pi = d(r)$. Folglich gilt $d < d(r)$.

I.b. Nur solche Zellen vom Typ I werden betrachtet, in denen die Kreise k_1, k_2, k_3 mit den Mittelpunkten K_1, K_2, K_3 nicht kongruent sind.

Nehmen wir eine von diesen Zellen! d bedeutet die Dichte der Packung bezüglich der gewählten Zelle. Wir haben zu zeigen: $d \leqq d(r)$.

Zuerst beschäftigen wir uns mit dem Fall, wann die gewählte Zelle ein Dreieck ist. (Also, das Dreieck $K_1K_2K_3$ wird untersucht.) Wir wissen daß: $K_iK_j \geqq 1$ gilt, wenn k_i, k_j inkongruent sind; $K_iK_j \geqq \sqrt{2}$ gilt, wenn k_i, k_j kongruent sind ($i \neq j$, $i, j = 1, 2, 3$). Wir können ohne Beschränkung der Allgemeinheit voraussetzen, daß die Indizes i, j $i \neq j$ $i, j \in \{1, 2, 3\}$ existieren, die der Gleichung $K_iK_j = 1$ oder der Gleichung $K_iK_j = \sqrt{2}$ entsprechen. (Sonst könnten wir es durch geeignete, zentralsche Verkleinerung erreichen, die die Dichte erhöht.) Also, nur zwei Fälle sind hier möglich (Abb. 5).

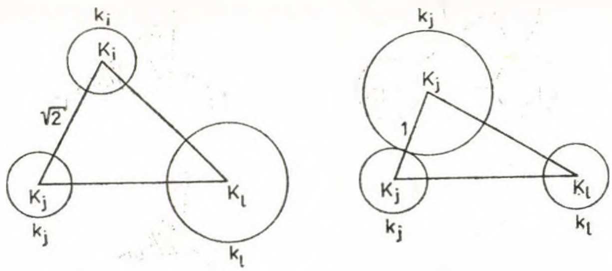


Abb. 5

Der Fall $K_iK_j = \sqrt{2}$. Wir können voraussetzen, daß der Kreis k_l wenigstens einen von den Kreisen k_i, k_j berührt. Im entgegengesetzten Fall könnten wir nämlich durch eine Bewegung vom Kreis k_l diese Situation erreichen, wenn wir den Punkt K_l auf der Höhe der Seite K_iK_j bewegten. Während die Dichte kann sicher nur erhöhen. (Siehe: Lemma 1.)

Der Fall $K_iK_j = 1$. Die Kreise k_i, k_j sind hier inkongruent. Deshalb können wir voraussetzen, daß die Strecke K_iK_l oder die Strecke K_jK_l die Länge 1 oder die Länge $\sqrt{2}$ hat. Im entgegengesetzten Fall könnten wir nämlich so denken, wie früher.

Zusammengefaßt: wir haben nur die folgenden zwei Fälle zu untersuchen

$$K_1K_2 = K_1K_3 = 1; \quad K_2K_3 \geqq \sqrt{2} \tag{1}$$

$$K_1K_2 = 1; \quad K_2K_3 = \sqrt{2}; \quad K_1K_3 \geqq 1 \tag{2}$$

und selbstverständlich ist das Dreieck $K_1K_2K_3$ in beiden Fällen ein spitzwinkliges oder ein rechtwinkliges Dreieck.

Der Fall 1 ist trivial, weil dann auch $K_2K_3 = \sqrt{2}$ gilt. Also für die Dichte d gilt: $d = \dfrac{r^2 + (1-r)^2}{2} \pi = d(r)$.

Der Fall 2. Haben die Kreise k_2, k_3 den Radius $(1-r)$, dann drehen wir um den Punkt K_2, den Kreis k_3 so weit, daß der Kreis k_3 den Kreis k_1 berührt. Bei dieser Drehung nimmt der Inhalt des Dreiecks $K_1K_2K_3$ ab, dagegen nimmt der Gesamtinhalt der Kreissektoren zu ($r \geqq 1-r$!). Also während der Drehung nimmt die Dichte zu. Deshalb ist $d \leqq d(r)$. Wir haben noch denjenigen Fall zu untersuchen, wann $K_1K_2=1$; $K_2K_3=\sqrt{2}$; $K_1K_3 \geqq 1$; die Kreise k_2, k_3 den Radius r haben und der Kreis k_1 den Radius $(1-r)$ hat. (Selbstverständlich ist das Dreieck $K_1K_2K_3$ ein spitzwinkliges oder ein rechtwinkliges Dreieck.) (Abb. 6.) Es sei $\alpha = \sphericalangle K_2K_1K_3$. Dann ist $0<\alpha^* \leqq \alpha \leqq \pi/2$ wo $\operatorname{tg}\alpha^* = \sqrt{2}$ gilt. Durch einige Rechnungen erhalten wir

$$d = \frac{\pi r^2-(2r-1)\alpha}{\sin\alpha(\sqrt{1+\cos^2\alpha}+\cos\alpha)}.$$

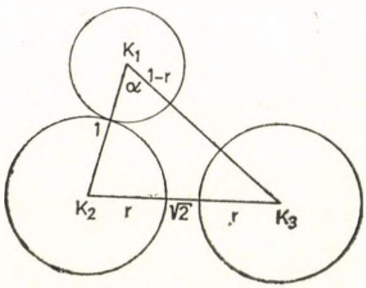


Abb. 6

Jetzt wird die folgende Funktion definiert: $f\colon [\alpha^*, \pi/2] \to \mathbf{R}$

$$\alpha \mapsto f(\alpha) = \frac{\pi r^2-(2r-1)\alpha}{\sin\alpha(\sqrt{1+\cos^2\alpha}+\cos\alpha)}.$$

Es ist leicht einzusehen:

$$f'(\alpha) = \frac{h(\alpha)}{\sin\alpha(\sqrt{1+\cos^2\alpha}+\cos\alpha)}$$

wenn

$$h(\alpha) = a+(a\alpha+b)\left(\frac{\sin\alpha}{\sqrt{1+\cos^2\alpha}}-\operatorname{ctg}\alpha\right);$$

$a=(1-2r)$; $b=\pi r^2$; $\alpha\in[\alpha^*, \pi/2]$ gelten.

Behauptung: $h'(\alpha)>0$.

Beweis: $\pi r^2-(\pi+2)r+(\pi/2+1)>0$, denn sie hat einen egative Diskriminante. Deshalb gilt $\pi r^2+(1-2r)(\alpha+1)>0$ das heißt $a\alpha+b+a>0$.

Andererseits ist

$$\frac{2\cos\alpha}{(1+\cos^2\alpha)^{3/2}}+\frac{1}{\sin^2\alpha} \geqq \frac{\sin\alpha}{\sqrt{1+\cos^2\alpha}}-\operatorname{ctg}\alpha.$$

Mit Rücksicht auf

$$h'(\alpha) = a\left(\frac{\sin\alpha}{\sqrt{1+\cos^2\alpha}} - \operatorname{ctg}\alpha\right) + (a\alpha+b)\left(\frac{2\cos\alpha}{(1+\cos^2\alpha)^{3/2}} + \frac{1}{\sin^2\alpha}\right)$$

ist also die Ungleichung der Behauptung sicher erfüllt.

Im Intervall $[\alpha^*, \pi/2]$ ist also $h(\alpha)$ eine zunehmende Funktion von α. Dann ist aber das Maximum der Funktion $f\colon [\alpha^*, \pi/2] \to \mathbf{R}$

$$f(\alpha^*) \quad \text{oder} \quad f\left(\frac{\pi}{2}\right).$$

Behauptung: $f(\alpha^*) < f(\pi/2)$.

Beweis.

$$f\left(\frac{\pi}{2}\right) - f(\alpha^*) = \left(\pi - \frac{\pi}{\sqrt{2}}\right)r^2 - (\pi - \sqrt{2}\alpha^*)r + \left(\frac{\pi}{2} - \frac{\alpha^*}{\sqrt{2}}\right) > 0$$

denn sie hat eine negative Diskriminante.

Schließlich haben wir

$$f\left(\frac{\pi}{2}\right) = \frac{r^2+(1-r)^2}{2}\pi = d(r)$$

womit der Beweis der Ungleichung $d \leqq d(r)$ erbracht ist.

Zum Schluß betrachten wir die Zellen vom Typ I, in denen die Kreise k_1, k_2, k_3 inkongruent sind und, die mindestens einen Einschnitt haben.

Wir nehmen einen von diesen Zellen. (d bedeutet die Dichte der Packung bezüglich der gewählten Zelle.) Es sei O der Mittelpunkt des Umkreises vom Dreieck $K_1K_2K_3$, wobei $d(OK_iK_j\Delta)$ die Dichte der Kreise k_i, k_j $(i \neq j,\ i, j = 1, 2, 3)$ bezüglich des Dreiecks OK_iK_j bedeutet. Mit Rücksicht auf Lemma 1 ist die Ungleichung $d \leqq \max\{d(OK_iK_j\Delta)\ i \neq j\ i, j = 1, 2, 3\} = D$ sicher erfüllt. Wir können ohne Beschränkung der Allgemeinheit voraussetzen: $D = d(OK_1K_2\Delta)$. Sind die Kreise k_1, k_2 inkongruent, so ist wegen des Korollars $d(OK_1K_2\Delta) \leqq d(r)$. Also gilt $d \leqq d(r)$.

Im weiteren sind die Kreise k_1, k_2 kongruent. Auf Grund des Lemmas 3 ist $d(OK_1K_2\Delta) < d(r)$, wenn $K_1K_2 \geqq 3/2$ gilt. Und so folglich $d < d(r)$. Für $K_1K_2 < 3/2$ haben wir die folgenden Überlegungen: Sind K_1K_3 und K_2K_3 die Seiten der gewählten Zelle, so gelten für die Dichte $d(K_1K_2K_3\Delta)$ der Kreise k_1, k_2, k_3 bezüglich des Dreiecks $K_1K_2K_3$ die untenstehenden Ungleichungen: $d \leqq d(K_1K_2K_3\Delta)$; $d(K_1K_2K_3\Delta) \leqq d(r)$. Die zweite Ungleichung wurde schon vorher bewiesen. Also ist hier $d \leqq d(r)$. Wir haben nur noch denjenigen Fall zu untersuchen, wann die gewählte Zelle mindestens zwei Einschnitte hat. Es sei z.B. ein Einschnitt bei der Strecke K_2K_3. Selbstverständlich ist dann $K_2K_3 \geqq \sqrt{3}$. (Wir wissen noch: $K_1K_2 < 3/2$; $K_1K_3 \geqq 1$.) Durch einfache Rechnung ergibt sich: $\sphericalangle K_1K_3K_2 < \pi/3$. Also ist $\sphericalangle K_1OK_2 < 2\pi/3$. Deshalb und wegen der Ungleichung $K_1K_2 \geqq \sqrt{2}$ erhalten wir $d(OK_1K_2\Delta) < d(r)$. (Siehe dazu Lemma 1.) Folglich $d < d(r)$.

II. Die Zelle wird durch das stumpfwinklige Dreieck $K_1K_2K_3$ folgenderweise erzeugt: (es sei $\sphericalangle K_2K_1K_3$ stumpfwinklig) entweder ist die Zelle das konvexe Vier-

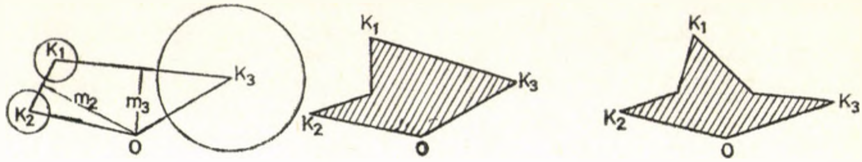


Abb. 7

eck $K_1K_2OK_3$ wo $OK_1=OK_2=OK_3$, oder ein 5-; bzw. 6-eck, das durch Hinzufügung 1 bzw. 2 Einschnitte entsteht (Abb. 7). Die Kreise k_1, k_2, k_3 mit den Mittelpunkten K_1, K_2, K_3 gehören der Packung. Wir nehmen eine von diesen Zellen wann d die Dichte der Packung bezüglich der gewählten Zelle bedeutet. Wir bezeichnen mit $d(OK_1K_2\Delta)$ [bzw. $d(OK_1K_3\Delta)$] die Dichte der Kreise k_1; k_2 [bzw. k_1; k_3] bezüglich des Dreiecks OK_1K_2 [bzw. OK_1K_3]. Mit Rücksicht auf Lemma 1 ist: $d \leqq \max \{d(OK_1K_2\Delta), d(OK_1K_3\Delta)\}$. Andererseits, hier ist es leicht einzusehen: $d(OK_1K_2\Delta) \leqq d(r)$; $d(OK_1K_3\Delta) \leqq d(r)$ $\left(m_2 \geqq \frac{1}{2};\ m_3 \geqq \frac{1}{2};\ OK_2 = OK_3 \geqq \frac{\sqrt{2}}{2}\right)$. Folglich $d \leqq d(r)$.

So ist der Beweis des Satzes erbracht.

LITERATURVERZEICHNIS

[1] Fejes Tóth, L., *Lagerungen in der Ebene, auf der Kugel und im Raum*, Zweite verbesserte und erweiterte Auflage, Die Grundlehren der mathematischen Wissenschaften, Band 65, Springer-Verlag, Berlin—New York, 1972. *MR* **50** # 5603.

[2] Fejes Tóth, L., *Reguläre Figuren*, Akadémiai Kiadó, Budapest, 1965. *MR* **30** # 3408.

[3] Molnár, J., On the ϱ-system of unit circles, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **20** (1977), 195—203. *MR* **58** # 12733.

(Eingegangen am 11. September 1981)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
GEOMETRIA TANSZÉK
RÁKÓCZI ÚT 5
H—1088 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 287—292.

BIASED RAMSEY TYPE GAMES

JÓZSEF BECK

1. Introduction

Let **N** be the set of natural numbers. If A is a set, the family of subsets of A containing exactly r elements is $[A]^r$. Let $|A|$ denote the cardinality of the set A. Given a graph G, denote by $\omega(G)$ the number of vertices of the largest complete subgraph of G. Let $\log x$ denote the natural logarithm of x.

Let us consider the following biased Ramsey type game. Two players are playing on the edges of the infinite complete graph $[\mathbf{N}]^2$. The first player selects b previously unselected edges per move and the second player selects one previously unselected edge per move. Now consider a play according to these rules. Let $e_i^{(1)}, e_i^{(2)}, \ldots, e_i^{(b)}$ and f_i denote the first and the second player's i-th move, respectively. Set

$$E_i = \{e_j^{(k)}\colon 1 \leqq j \leqq i,\ 1 \leqq k \leqq b\} \quad \text{and} \quad F_i = \{f_j\colon 1 \leqq j \leqq i\},$$

that is, E_i and F_i denote the graphs picked by the first and the second players at their first i moves. Each player wants to pick large complete subgraphs. More exactly, the first player's goal is to maximalize $\sup\limits_{1\leqq i<\infty} \omega(E_i)/\omega(F_i)$, and the second player's goal is to minimalize $\sup\limits_{1\leqq i<\infty} \omega(E_i)/\omega(F_i)$. Let this Ramsey type game be denoted by $RG(b, 1)$.

In the case $b=1$, i.e., each player selects one edge per move, the second player can force the inequality $\omega(E_i)\leqq\omega(F_i)+1$ by a simple "reflection strategy". For let $\{N_1, N_2\}$ be an infinite 2-partition of **N**, i.e., $N_1\cup N_2=\mathbf{N}$, $N_1\cap N_2=\emptyset$ and $|N_1|=|N_2|=\infty$. Let $g\colon N_1\to N_2$ be a 1—1 mapping. Given $e=\{n, m\}\in[\mathbf{N}]^2$ with $n\neq g(m)$ and $m\neq g(n)$, let e^* be the "mirror image" of e, more precisely, let

$$e^* = \begin{cases} \{g(n), g(m)\} & \text{if } n, m\in N_1 \\ \{g^{-1}(n), g^{-1}(m)\} & \text{if } n, m\in N_2 \\ \{g(n), g^{-1}(m)\} & \text{if } n\in N_1,\ m\in N_2 \\ \{g^{-1}(n), g(m)\} & \text{if } n\in N_2,\ m\in N_1. \end{cases}$$

Clearly $(e^*)^*=e$. Here is the desired strategy: If the first player has just selected $e=\{n, m\}$ with $n\neq g(m)$ and $m\neq g(n)$, then the second player selects e^*, otherwise the second player moves arbitrarily. One can easily verify that using this strategy the second player can force $\omega(E_i)\leqq\omega(F_i)+1$ for every $i\geqq 1$.

1980 *Mathematics Subject Classification.* Primary 05C55; Secondary 05C65.
Key words and phrases. Graph, strategy.

In the case $b \geqq 2$, i.e, the real biased game, this "reflection strategy" *does not* work. It is not even obvious whether the second player can force

$$\sup_{1 \leqq i < \infty} \omega(E_i)/\omega(F_i) < +\infty.$$

In this note we give a positive answer to this question.

THEOREM. *In the game $RG(b, 1)$ $(b \geqq 2)$ the second player has a strategy such that for all $i > i_0(b)$,*

$$\frac{\omega(E_i)}{\omega(F_i)} < 10b \log b.$$

Our result was motivated by a paper of Erdős and Selfridge [1]. For the proof we need two lemmas.

LEMMA 1. *In the game $RG(d, 1)$ the second player has a strategy such that for all $i \geqq 1$,*

$$\omega(F_i) > \frac{\log i}{\log (2d+2)}.$$

LEMMA 2. *In the game $RG(d, 1)$ the second player has a strategy such that for $i > i_0(d)$,*

$$\omega(E_i) < 3d \frac{\log i}{\log 2}.$$

Accept for a moment the validity of the two lemmas above. Playing on the $(2i+1)$-st moves by the strategy of Lemma 1 with $d=2b$ and on the $2i$-th moves by the strategy of Lemma 2 with $d=2b$, for all $i > i_0$ the second player can force

$$\frac{\omega(E_i)}{\omega(F_i)} < \frac{6b \log (4b+2)}{\log 2} < 10b \log b.$$

That is, the Theorem follows. Therefore, it suffices to prove the lemmas. Their proofs constitute Sections 2 and 3.

2. Proof of Lemma 1

Let $Q(n, k, d)$ denote the game that is played on the edges of a complete graph on n vertices. The first player selects d $(d \geqq 1)$ edges per move and the second player selects one edge per move. The second player wins if and only if he can choose all the $\binom{k}{2}$ edges of some complete subgraph of k $(k \geqq 2)$ vertices. We will prove by induction on k that the second player has a winning strategy $Str(k)$ for $Q\big((2d+2)^{k-1}, k, d\big)$ within $(2d+2)^{k-1}$ moves.

For $k=2$ this statement is trivial. Assume now that the statement holds for $k-1 \geqq 2$. Then the second player proceeds in two stages. In the first stage, selecting some vertex v, he will choose $(2d+1)(2d+2)^{k-2}$ edges e for which $v \in e$. By the end of the first stage, there will be a set V of vertices having the following properties:

$|V|=(2d+2)^{k-2}$, $v\notin V$, there is no edge of the first player covered by V, and for each $y\in V$ the edge $\{v, y\}$ has been selected by the second player. Here we used a corollary of the well-known Turán's theorem: Given a graph with s vertices and average degree t, one can find an independent subset of it having cardinality $\geq \frac{s}{t+1}$. In the second stage, the second player will play by $Str(k-1)$ restricting his moves to the edges of V. The induction step is complete.

Now we are in the position to define the desired strategy in Lemma 1. On the first move the second player will choose an arbitrary edge, on the next $(2d+2)^2$ moves he will choose a triangle by $Str(3)$, on the next $(2d+2)^3$ moves he will choose a complete quadrilateral by $Str(4)$, and so on. Using this complex strategy the second player can obtain for every k a complete graph of k vertices within $1+\sum_{j=2}^{k}(2d+2)^{j-1}<$ $<(2d+2)^k$ moves. This completes the proof of Lemma 1.

3. Proof of Lemma 2

Let $V_i=\bigcup_{e\in E_i} e$, that is, V_i denotes the set of vertices covered by the first player's first i moves $e_1^{(1)}, \dots, e_1^{(d)}, \dots, e_i^{(1)}, \dots, e_i^{(d)}$. Suppose now that we are after the first player's i-th move and we wish to find a good edge f_i for the second player. In order to define this strategy we introduce a weight function $\lambda(i,f)$, $f\in[\mathbf{N}]^2$ being very sensitive to the dense subgraphs of E_i: if $f\in[\mathbf{N}]^2$ and $t\geq 2$ then let

$$\psi_t(i, f) = \sum_{A\in[V_i]^t:\ [A]^2\cap F_{i-1}=\emptyset \text{ and } f\in[A]^2} 2^{|[A]^2\cap E_i|/d},$$

and

$$\lambda(i, f) = \sum_{t=2}^{|V_i|} \psi_t(i, f) i^{-t}.$$

Here is the desired strategy: for each $f\in[\mathbf{N}]^2-(E_i\cup F_{i-1})$ compute the value of $\lambda(i,f)$ and let f_i be that one for which the maximum is attained.

Set

$$\psi_t(i) = \sum_{A\in[V_i]^t:\ [A]^2\cap F_{i-1}=\emptyset} 2^{|[A]^2\cap E_i|/d},$$

and

$$\lambda(i) = \sum_{t=2}^{|V_i|} \psi_t(i) i^{-t}.$$

Since $|E_1|=d$ and $|V_1|\leq 2d$, we easily get the upper bound

$$\lambda(1) \leq \sum_{t=2}^{|V_1|} \sum_{A\in[V_1]^t} 2 = 2\sum_{t=2}^{|V_1|}\binom{|V_1|}{t} \leq 2\cdot 2^{|V_1|} \leq 2^{2d+1}.$$

The proof of Lemma 2 will be based on the following statement: using the above strategy the second player can achieve the inequality

$$\lambda(i+1) \leq \left(1+\frac{4^d}{2i}\right)\lambda(i)+\frac{4^{d+1}}{i} \quad \text{for all} \quad i\geq 1. \tag{1}$$

Assuming the validity of (1) one can easily deduce the upper estimate

(2) $$\lambda(i) < (i+1)^{4^d} \quad \text{for all} \quad i \geqq 1$$

by induction on i. Indeed, for $i=1$ (2) is true, since $\lambda(1) \leqq 2^{2d+1}$. Assume now that $\lambda(i-1) < i^{4^d}$, then by (1) we obtain

$$\lambda(i) < \left(1+\frac{4^d}{2i-2}\right) i^{4^d} + \frac{4^{d+1}}{i-1} \leqq i^{4^d} + 4^d i^{4^d-1} + 4^{d+1} < (i+1)^{4^d},$$

completing the induction step.

Using (2) we can complete the proof of Lemma 2 as follows. Assume, in the contrary that

$$\omega(E_i) \geqq 3d \log i/\log 2 \quad \text{for some} \quad i > i_0(d),$$

where $i_0(d)$ is a sufficiently large threshold index. That is, in the first i moves the first player selected all the $\binom{t}{2}$ edges of some complete graph of $t=t(i)=\lceil 3d \log i/\log 2\rceil$ (upper integral part) vertices.

Then

$$\lambda(i) \geqq \psi_t(i)\, i^{-t} \geqq 2^{\binom{t}{2}/d} i^{-t} \quad \text{with} \quad t = \lceil 3d \log i/\log 2\rceil.$$

Simple calculation shows that for $i > i_0(d)$,

$$\lambda(i) \geqq 2^{\binom{t}{2}/d} i^{-t} \geqq (i+1)^{4^d},$$

which contradicts (2).

Thus it remains to prove (1).

To verify (1) we start with the identity below that can be easily checked by the reader:

(3) $$\psi_t(i+1) = \psi_t(i) - \psi_t(i, f_i) + \beta_t(i+1) + \gamma_t(i+1),$$

where

$$\beta_t(i+1) = \sum_{A \in [V_i]^t :\, [A]^2 \cap F_i = \emptyset} \{2^{|[A]^2 \cap E_{i+1}|/d} - 2^{|[A]^2 \cap E_i|/d}\}$$

and

$$\gamma_t(i+1) = \sum_{A \in [V_{i+1}]^t :\, [A]^2 \cap F_i = \emptyset,\, A \not\subset V_i} 2^{|[A]^2 \cap E_{i+1}|/d}.$$

Now we give an upper bound to $\beta_t(i+1)$.

$$E_{i,k} = E_i \cup \{e_{i+1}^{(1)}, \ldots, e_{i+1}^{(k)}\} \qquad 0 \leqq k \leqq d,$$

that is, $E_{i,0} = E_i$ and $E_{i,d} = E_{i+1}$. Using this notation we introduce

$$\beta_t(i+1, k) = \sum_{A \in [V_i]^t :\, [A]^2 \cap F_i = \emptyset} \{2^{|[A]^2 \cap E_{i,k}|/d} - 2^{|[A]^2 \cap E_{i,k-1}|/d}\} =$$

$$= \sum_{A \in [V_i]^t :\, [A]^2 \cap F_i = \emptyset,\, e_{i+1}^{(k)} \in [A]^2} \{2^{1/d} - 1\} 2^{|[A]^2 \cap E_{i,k-1}|/d}.$$

Since $\beta_t(i+1) = \sum_{k=1}^{d} \beta_t(i+1, k)$ and $|[A]^2 \cap E_{i,k-1}| \leqq |[A]^2 \cap E_i| + k - 1$, we obtain the upper bound

$$(4)\qquad \begin{aligned}\beta_t(i+1) &\leqq \sum_{k=1}^{d} \sum_{A\in[V_i]^t:\,[A]^2\cap F_i=\emptyset,\, e_{i+1}^{(k)}\in[A]^2} \{2^{k/d}-2^{(k-1)/d}\}2^{|[A]^2\cap E_i|/d} \leqq \\ &\leqq \sum_{k=1}^{d}\{2^{k/d}-2^{(k-1)/d}\}\psi_t(i, e_{i+1}^{(k)}).\end{aligned}$$

Next we give an upper bound to $\gamma_t(i+1)$. Let $W_i=V_{i+1}\setminus V_i$. Clearly $|W_i|\leqq \leqq \left|\bigcup_{k=1}^{d} e_{i+1}^{(k)}\right|\leqq 2d$. Since $|[A]^2\cap E_{i+1}|\leqq|[A\cap V_i]^2\cap E_i|+d$, we have

$$\begin{aligned}\gamma_t(i+1) &= \sum_{k=1}^{|W_i|} \sum_{A\in[V_{i+1}]^t:\,[A]^2\cap F_i=\emptyset,\,|A\cap W_i|=k} 2^{|[A]^2\cap E_{i+1}|/d} \leqq \\ &\leqq \sum_{k=1}^{|W_i|} 2 \sum_{A\in[V_{i+1}]^t:\,[A]^2\cap F_i=\emptyset,\,|A\cap W_i|=k} 2^{|[A\cap V_i]^2\cap E_i|/d} \leqq \\ &\leqq \sum_{k=1}^{|W_i|} 2 \sum_{C\in[W_i]^k}\sum_{B\in[V_i]^{t-k}:\,[B]^2\cap F_{i-1}=\emptyset} 2^{|[B]^2\cap E_i|/d} \leqq \sum_{k=1}^{|W_i|} 2\binom{|W_i|}{k}\tilde{\psi}_{t-k}(i),\end{aligned}$$

where

$$\tilde{\psi}_j(i) = \begin{cases} \psi_j(i) & \text{for } j\geqq 2\\ i & \text{for } j=1\\ 1 & \text{for } j=0\\ 0 & \text{for } j<0.\end{cases}$$

Since $|W_i|\leqq 2d$, $\binom{|W_i|}{k}\leqq\binom{2d}{k}$. Hence

$$(5)\qquad \gamma_t(i+1)\leqq \sum_{k=1}^{2d} 2\binom{2d}{k}\tilde{\psi}_{t-k}(i).$$

By (3), (4) and (5)

$$(6)\qquad \begin{aligned}\psi_t(i+1) &\leqq \psi_t(i)-\psi_t(i,f_i)+\sum_{k=1}^{d}\{2^{k/d}-2^{(k-1)/d}\}\psi_t(i, e_{i+1}^{(k)})+ \\ &+\sum_{k=1}^{2d} 2\binom{2d}{k}\tilde{\psi}_{t-k}(i).\end{aligned}$$

Summing (6) for $t=2, \dots, |V_{i+1}|$ we conclude

$$(7)\qquad \begin{aligned}\lambda(i+1) &= \sum_{t=2}^{|V_{i+1}|}\psi_t(i+1)(i+1)^{-t}\leqq \sum_{t=2}^{|V_{i+1}|}\psi_t(i+1)i^{-t}\leqq \\ &\leqq \sum_{t=2}^{|V_{i+1}|}\Big\{\psi_t(i)-\psi_t(i,f_i)+\sum_{k=1}^{d}\{2^{k/d}-2^{(k-1)/d}\}\psi_t(i, e_{i+1}^{(k)})+ \\ &+\sum_{k=1}^{2d} 2\binom{2d}{k}\tilde{\psi}_{t-k}(i)\Big\}i^{-t} = \\ &= \lambda(i)-\lambda(i,f_i)+\sum_{k=1}^{d}\{2^{k/d}-2^{(k-1)/d}\}\lambda(i, e_{i+1}^{(k)})+ \\ &+\sum_{t=2}^{|V_{i+1}|}\Big\{\sum_{k=2}^{2d} 2\binom{2d}{k}\tilde{\psi}_{t-k}(i)\Big\}i^{-t}.\end{aligned}$$

By the maximum property of f_i, $\lambda(i, e_{i+1}^{(k)}) \leqq \lambda(i, f_i)$. From this it follows that

$$\sum_{k+1}^{d} \{2^{k/d}-2^{(k-1)/d}\}\lambda(i, e_{i+1}^{(k)}) \leqq \sum_{k=1}^{d} \{2^{k/d}-2^{(k-1)/d}\}\lambda(i, f_i) = \lambda(i, f_i).$$

Returning to (7) we have

$$\lambda(i+1) \leqq \lambda(i)+\sum_{t=2}^{|V_{i+1}|}\sum_{k=1}^{d} 2\binom{2d}{k}\tilde{\psi}_{t-k}(i)i^{-t} =$$

$$= \lambda(i)+\sum_{t=2}^{|V_{i+1}|}\sum_{k=1}^{d} 2\binom{2d}{k}\tilde{\psi}_{t-k}(i)i^{-(t-k)}i^{-k}.$$

By the definition of $\tilde{\psi}_j(i)$

$$\sum_{t=2}^{|V_{i+1}|}\sum_{k=1}^{d}\binom{2d}{k}\tilde{\psi}_{t-k}(i)i^{-(t-k)}i^{-k} \leqq \sum_{k=1}^{d}\binom{2d}{k}\{\lambda(i)+2\}i^{-k},$$

therefore

$$\lambda(i+1) \leqq \left\{1+\sum_{k=1}^{d} 2\binom{2d}{k} i^{-k}\right\}\lambda(i)+\sum_{k=1}^{d} 4\binom{2d}{k} i^{-k}.$$

Using the trivial inequality $\sum_{k=1}^{d} 4\binom{2d}{k} i^{-k} \leqq \frac{4^{d+1}}{i}$ we obtain (1). This completes the proof of Lemma 2.

ACKNOWLEDGEMENT. The author would like to thank Professor G. O. H. Katona for important remarks.

REFERENCE

ERDŐS, P. and SELFRIDGE, J., On a combinatorial game, *J. Combinatorial Theory Ser. A* **14** (1973), 298—301. *MR* **48** #5655

(Received December 1, 1981)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 293—307.

TOPOLOGIES ET PROXIMITÉS FLOUES

P. HAMBURG

Les espaces topologiques « flous » étudiés en nombreux notes — par exemple en [1], [11], [12], [14], [15] — les espaces de proximité « flous » — [9] — ou les espaces uniformes « flous » — [8], — et leur théorie unifiée en langage syntopogène — [10] — sont des cas particuliers de structures topologiques (en sens classique) dans des treillis morganiens [4]. Ce ne sont pas des ensembles flous; nous les nommerons structures *quasi-floues* (q.f.). Une topologie q.f. par exemple (et floue selon la terminologie des travaux mentionnés) est une partie $\mathscr{T}$ de l'ensemble $\mathbf{I}^X$ (où $\mathbf{I}=[0, 1]$). Une relation δ de proximité q.f. ou un ordre topogène q.f. $\sqsubset$ est un sous-ensemble (classique) de $\mathbf{I}^X \times \mathbf{I}^X$, etc. Nous nous proposons de développer les idées esquissées dans [3] en élaborant dans cette note les fondements d'une théorie des structures topologiques et proximales floues au sens propre, c'est-à-dire $\mathscr{T}$ sera une partie floue de $\mathbf{I}^X$, $\sqsubset$ un sousensemble flou de $\mathbf{I}^X \times \mathbf{I}^X$ etc. Nous employerons le langage syntopogène en considérant comme notion fondamentale celle de l'ordre topogène flou, mais nous traduirons nos définitions aussi dans les autres « langues » usuelles. Nous nous bornerons dans cette première étude aux ordres. Les structures syntopogènes feront l'objet d'un autre article. Par souci de généralité, nous remplacerons presque partout l'ensemble $\mathbf{I}^X$ par un soustreillis complet $\mathscr{L}$ de $\mathbf{I}^X$. Ainsi en choisissant $\mathscr{L}=\mathbf{I}^X$, $\mathscr{L}=\mathscr{P}(X)$, puis comme codomaine de $\sqsubset$ l'ensemble $\mathbf{I}$ où en particulier $\{0, 1\}$, nous obtiendrons a) les structures floues au sens strict, b) les structures q.f., c'est-à-dire les structures « floues » des notes citées en haut, c) les structures probabilistiques, d) les structures topologiques classiques.

Ce point de vue strictement flou apparaît pour le cas des structures proximales en [13]. On y remarque deux voies : pour la relation de proximité, les auteurs emploient *une* application $P : \mathbf{I}^X \to \mathbf{I}$; mais pour l'ordre de proximité une *famille* d'ordres q.f. Nous montrerons que toute notion topologique floue peut être caractérisée en deux manières : a) par une application dont le codomaine est $\mathbf{I}$, b) par une famille d'objets q.f. dont le domaine est $\mathbf{I}_0=]0, 1]$ mais le codomaine en est $\{0, 1\}$.

1. Ordres topogènes flous. Topologies floues

Soit X un ensemble quelconque non vide, $\mathbf{I}=[0, 1]$. Nous noterons par $\leqq$ l'ordre naturel en $\mathbf{I}$, mais aussi l'ordre produit de $\mathbf{I}^X$. Par un *treillis flou en X* nous entendons un soustreillis complet (et donc complètement distributif) $\mathscr{L}$ de $\mathbf{I}^X$. La borne inférieure (supérieure) en $\mathbf{I}$ et en $\mathscr{L}$ sera notée par $\wedge$ ($\vee$). Les mêmes symboles

1980 *Mathematics Subject Classification.* Primary 54A05; Secondary 54E05, 54E15, 54F05.
Key words and phrases. Fuzzy topological structures.

seront employés aussi pour la conjonction (disjonction) logique. 0 est le plus petit élément de $\mathscr{L}$, 1 le plus grand. $\mathscr{L}$ est *symétrique* si $1-a \in \mathscr{L}$ pour tout $a \in \mathscr{L}$.

1.1. Un *ordre semitopogène flou en* $\mathscr{L}$ est une application $\sqsubset : \mathscr{L} \times \mathscr{L} \to \mathbf{I}$ vérifiant les axiomes :

$[\sqsubset_1']$ $\sqsubset(0, 0) = \sqsubset(1, 1) = 1$,

$[\sqsubset_1'']$ $\sqsubset(a, b) > 0 \Rightarrow a \leqq b$,

$[\sqsubset_1''']$ $a_1 \leqq a, \quad b \leqq b_1 \Rightarrow \sqsubset(a, b) \leqq \sqsubset(a_1, b_1)$.

Il est *topogène* (otf) s'il vérifie aussi

$[\sqsubset_2]$ $\sqsubset(a_1, b) \wedge \sqsubset(a_2, b) \leqq \sqsubset(a_1 \vee a_2, b)$ et

$[\sqsubset_3]$ $\sqsubset(a, b_1) \wedge \sqsubset(a, b_2) \leqq \sqsubset(a, b_1 \wedge b_2)$.

Un ordre semitopogène flou est *parfait* si

$[\sqsubset_2^*]$ $\wedge\{\sqsubset(a_j, b) : j \in J\} \leqq \sqsubset(\vee\{a_j : j \in J\}, b)$.

Il est *coparfait* si

$[\sqsubset_3^*]$ $\wedge\{\sqsubset(a, b_j) : j \in J\} \leqq \sqsubset(a, \wedge\{b_j : j \in J\})$.

Un otf est *biparfait* s'il est parfait et coparfait.

De l'axiome $[\sqsubset_1''']$ on déduit que, dans les axiomes $[\sqsubset_2]$, $[\sqsubset_3]$, $[\sqsubset_2^*]$ et $[\sqsubset_3^*]$, l'inégalité peut être remplacée par l'égalité.

Soient $\sqsubset_1$, $\sqsubset_2$, $\sqsubset$ des ordres semitopogènes flous en $\mathscr{L}$. Nous définissons $\sqsubset_1 \circ \sqsubset_2$ par

$$(\sqsubset_1 \circ \sqsubset_2)(a, b) = \vee\{\sqsubset_2(a, c) \wedge \sqsubset_1(c, b) : c \in \mathscr{L}\}$$

et $\sqsubset^2 = \sqsubset \circ \sqsubset$. On a toujours $\sqsubset_1 \circ \sqsubset_2 \leqq \sqsubset_1 \wedge \sqsubset_2$, donc $\sqsubset^2 \leqq \sqsubset$. Un ordre semitopogène flou $\sqsubset$ est *dense* s'il vérifie l'inégalité

$[\sqsubset_4]$ $\sqsubset \leqq \sqsubset^2$.

Cette condition est équivalente à la suivante :

$[\sqsubset_{4'}]$ $\underset{a,b \in \mathscr{L}}{\forall} \ \underset{n \in \mathbf{N}^*}{\forall} \ \underset{c \in \mathscr{L}}{\exists} \ \sqsubset(a, b) - \frac{1}{n} \leqq \sqsubset(a, c) \wedge \sqsubset(c, b)$.

La condition $[\sqsubset_{4'}]$ est toujours vérifiée si $a \nleqq b$.

L'ordre $\sqsubset$ est *strictement dense* si

$[\sqsubset_4^*]$ $\underset{a,b \in \mathscr{L}}{\forall} \{a \leqq b \Rightarrow \underset{c \in \mathscr{L}}{\exists} [\sqsubset(a, b) \leqq \sqsubset(a, c) \wedge \sqsubset(c, b)]\}$.

La dernière inégalité peut être remplacée par l'égalité. Tout otf strictement dense est dense. La réciproque n'est pas valable.

Un otf parfait et dense s'appelle *ordre flou topologique* (oft). S'il est strictement dense, c'est un *ordre flou strictement topologique*.

Si $\mathscr{L}$ est symétrique, l'otf $\sqsubset$ s'appelle *symétrique* si

$$[\sqsubset_5] \quad \sqsubset(a, b) = \sqsubset(1-b, 1-a).$$

Un otf symétrique et dense est un *ordre flou de proximité* (ofp). S'il est strictement dense, il est un *ordre flou strictement proximal.*

Nous allons étudier d'abord les oft. Ils sont étroitement liés à la notion de générateur.

1.2. Un f-treillis en $\mathscr{L}$ est une application $\tau : \mathscr{L} \to \mathbf{I}$ jouissant des propriétés suivantes:

$[\tau_1] \quad \tau(0) = \tau(1) = 1,$

$[\tau_2] \quad \tau(c_1 \vee c_2) \geqq \tau(c_1) \wedge \tau(c_2),$

$[\tau_3] \quad \tau(c_1 \wedge c_2) \geqq \tau(c_1) \wedge \tau(c_2).$

Nous attacherons à tout f-treillis τ une autre application $\sqsubset_\tau : \mathscr{L} \times \mathscr{L} \to \mathbf{I}$ par la formule

$[\sqsubset : \tau] \quad \sqsubset_\tau(a, b) = \bigvee \{\tau(c) : a \leqq c \leqq b, c \in \mathscr{L}\}.$

Si $a \not\leqq b$, alors évidemment $\sqsubset_\tau(a, b) = 0$.

Notre construction est l'analogue de l'ordre topogène attaché à un treillis — [2]—. On montre aisément que $\sqsubset_\tau$ est un otf. Nous dirons que l'otf $\sqsubset$ est *générable* s'il y a un f-treillis τ tel que $\sqsubset = \sqsubset_\tau$. Le *générateur* $\tau = \tau_\sqsubset$ est unique et

$[\tau : \sqsubset] \quad \tau_\sqsubset(c) = \sqsubset(c, c).$

Pour que l'otf $\sqsubset$ soit générable, il faut et il suffit qu'il vérifie la condition:

$$[g] \quad \forall_{n \in \mathbf{N}^*} \forall_{a, b \in \mathscr{L}} \left\{a \leqq b \Rightarrow \exists_{c \in \mathscr{L}} \left[(a \leqq c \leqq b) \wedge \left(\sqsubset(a, b) - \frac{1}{n} \leqq \sqsubset(c, c)\right]\right\}.$$

PROPOSITION. *L'ordre topogène flou* $\sqsubset_\tau$ *est parfait si et seulement si* τ *est parfait, c'est-à-dire si*

$[\tau_2^*] \quad \bigwedge \{\tau(c_j) : j \in J\} \leqq \tau(\bigvee \{c_j : j \in J\}).$

Par une *topologie floue* en $\mathscr{L}$, nous entendons un f-treillis parfait. On voit donc qu'une telle topologie ne sépare pas comme dans le cas q.f — [1] — les ensembles flous ouverts de ceux qui ne le sont pas, mais évalue le « degré $\tau(c)$ d'ouverture » de chaque ensemble flou $c \in \mathscr{L}$.

1.3. Les conditions $[\sqsubset_4^*]$ et $[g]$ ne sont pas comparables. Mais $[g]$ est plus forte que $[\sqsubset_4]$, c'est-à-dire *tout otf générable est dense.* Mais on peut trouver des otf denses qui ne sont pas générables. Ces ordres ne sont pas parfaits, car :

PROPOSITION. *Un ordre topogène flou parfait est générable si et seulement s'il est dense.*

Démonstration. Si $\sqsubset$ est parfait et dense, $a, b \in \mathscr{L}$, $n \in \mathbf{N}^*$, $a \leqq b$, alors on choisit $c_1 \in \mathscr{L}$ tel que $a \leqq c_1 \leqq b$ et

$$\sqsubset(a, c_1) \wedge \sqsubset(c_1, b) \geqq \sqsubset(a, b) - \frac{1}{2n}.$$

Par induction, on choisit $c_k \in \mathscr{L}$ de telle manière qu'on ait

$$c_{k-1} \leqq c_k \leqq b \quad \text{et} \quad \sqsubset(c_{k-1}, c_k) \wedge \sqsubset(c_k, b) \geqq \sqsubset(c_{k-1}, b) - \frac{1}{2^k n}.$$

Alors, pour $c = \bigvee \{c_k : k \in \mathbf{N}^*\}$, on aura $a \leqq c \leqq b$ et

$$\sqsubset(c, c) \geqq \bigwedge \{\sqsubset(c_k, c) : k \in \mathbf{N}^*\} \geqq \bigwedge \{\sqsubset(c_k, c_{k+1}) : k \in \mathbf{N}^*\} \geqq \sqsubset(a, b) - \frac{1}{n}. \quad \square$$

On peut trouver des otf parfaits qui ne sont pas générables.

Corollaire. *L'application $\tau \to \sqsubset_\tau$ est une bijection entre l'ensemble des topologies floues en $\mathscr{L}$ et l'ensemble des ordres topologiques flous en $\mathscr{L}$. L'application inverse en est $\sqsubset \to \tau_{\sqsubset}$.*

1.4. L'ordre $\sqsubset_\tau$ attaché à un *f*-treillis quelconque n'est pas toujours strictement dense. Nous dirons qu'un *f*-treillis est *strict* si :

[τ_4^*] Pour chaque $a, b \in \mathscr{L}$ tels que $a \leqq b$, l'ensemble

$$B(a, b) = \{\tau(c) : c \in \mathscr{L}, a \leqq c \leqq b\}$$

a un plus grand élément. Un *f*-treillis strict et parfait s'appelle *topologie floue stricte.*

Pour que l'otf $\sqsubset$ ait un générateur strict, il faut et il suffit qu'il vérifie la condition

[g^*]. Pour chaque $a, b \in \mathscr{L}$ tels que $a \leqq b$, l'ensemble $A(a, b) = \{\sqsubset(c, c) : c \in \mathscr{L}, a \leqq c \leqq b, \sqsubset(c, c) = \sqsubset(a, b)\}$ soit nonvide.

En ce cas, nous dirons que $\sqsubset$ est strictement générable.

On démontre que, si τ est un *f*-treillis strict, alors $\sqsubset_\tau$ est strictement dense. La réciproque vaut de nouveau seulement pour les otf parfaits.

Proposition. *Un ordre topogène flou parfait est strictement dense si et seulement s'il est strictement générable. L'application $\tau \to \sqsubset_\tau$ est une bijection entre l'ensemble des topologies floues strictes et l'ensemble des ordres flous strictement topologiques en $\mathscr{L}$.*

1.5. Si l'interprétation du nombre $\tau(c)$ est simple — 1.2 — on remarque pour l'otf $\sqsubset$ que, pour un ordre topologique classique ou q.f., $a \sqsubset b$ veut dire qu'on peut trouver un ensemble ouvert entre a et b. Dans le cas flou, $\sqsubset(a, b)$ donne la borne supérieure des « degrés d'ouverture » des ensembles intermédiaires. Si le treillis flou $\mathscr{L}$ est symétrique — 1.1 — nous attachons à tout oft $\sqsubset$ une fonction σ évaluant le « degré de fermeture » et l'adhérence $\alpha(a, b)$ donnera la borne supérieure des degrés de fermeture des ensembles intermédiaires entre a et b. La fonction α sera le dual de $\sqsubset$, σ le dual de τ.

Soit donc $\mathscr{L}$ symétrique, τ une topologie floue en $\mathscr{L}$ — 1.2 —. Nous définissons

[$\sigma : \tau$] $\sigma_\tau(c) = \tau(1 - c)$.

L'application $\sigma=\sigma_\tau : \mathscr{L}\to\mathbf{I}$ jouit des propriétés suivantes:

$[\sigma_1]$ $\sigma(0)=\sigma(1)=1$,

$[\sigma_2]$ $\sigma(c_1)\wedge\sigma(c_2)\leqq\sigma(c_1\vee c_2)$,

$[\sigma_3^*]$ $\wedge\{\sigma(c_j) : j\in J\}\leqq\sigma(\wedge\{c_j : j\in J\})$.

La topologie floue τ est bien déterminée par $\sigma_\tau=\sigma$, ayant que

$[\tau : \sigma]$ $\tau(c)=\sigma(1-c)$.

La correspondance $\tau\to\sigma_\tau$ est une bijection entre l'ensemble des topologies floues en $\mathscr{L}$ et l'ensemble des *fonctions de fermeture floue* (fff) en $\mathscr{L}$, c'est-à-dire des applications $\sigma : \mathscr{L}\to\mathbf{I}$ vérifiant $[\sigma_1]$, $[\sigma_2]$, $[\sigma_3^*]$. τ est stricte si et seulement si σ vérifie

$[\sigma_4^*]$. Pour chaque $a,b\in\mathscr{L}$ tels que $a\leqq b$, l'ensemble $C(a,b)=\{\sigma(c) : c\in\mathscr{L}, a\leqq c\leqq b\}$ a un plus grand élément.

1.6. Nous introduisons, avec les conditions de 1.5, l'adhérence floue par la formule

$[\alpha : \sigma]$ $\alpha_\sigma(a,b)=\vee\{\sigma(c) : c\in\mathscr{L}, a\leqq c\leqq b\}$.

L'application $\alpha=\alpha_\sigma : \mathscr{L}\times\mathscr{L}\to\mathbf{I}$ vérifie les conditions :

$[\alpha_1']$ $\alpha(0,0)=\alpha(1,1)=1$,

$[\alpha_1'']$ $\alpha(a,b)>0\Rightarrow a\leqq b$,

$[\alpha_1''']$ $(a_1\leqq a)\wedge(b\leqq b_1)\Rightarrow\alpha(a,b)\leqq\alpha(a_1,b_1)$,

$[\alpha_2]$ $\alpha(a_1,b)\wedge\alpha(a_2,b)\leqq\alpha(a_1\vee a_2,b)$,

$[\alpha_3^*]$ $\wedge\{\alpha(a,b_j) : j\in J\}\leqq\alpha(a,\wedge\{b_j : j\in J\})$,

$[\alpha_4]$ $\alpha=\alpha^2$.

La relation α^2 est définie comme $\sqsubset^2$ — 1.1
Cette dernière condition est équivalente à $\alpha\leqq\alpha^2$, resp. à

$$[\alpha_{4'}]\qquad \underset{a,b\in\mathscr{L}}{\forall}\ \underset{n\in\mathbf{N}^*}{\forall}\ \underset{c\in\mathscr{L}}{\exists}\ \alpha(a,b)-\frac{1}{n}\leqq\alpha(a,c)\wedge\alpha(c,b).$$

Ces conditions nous montrent que α est un otf coparfait et dense.

L'application $\sigma\to\alpha_\sigma$ est une bijection de l'ensemble des fff en $\mathscr{L}$ et l'ensemble des *adhérences floues* en $\mathscr{L}$, c'est-à-dire des otf coparfaits et denses. L'application inverse est donnée par :

$[\sigma : \alpha]$ $\sigma_\alpha(c)=\alpha(c,c)$.

σ_α vérifie $[\sigma_4^*]$ si et seulement si α_σ jouit de la propriété :

$$[\alpha_4^*]\qquad \underset{a,b\in\mathscr{L}}{\forall}\left\{a\leqq b\Rightarrow\underset{c\in\mathscr{L}}{\exists}\ \alpha(a,b)\leqq\alpha(a,c)\wedge\alpha(c,b)\right\}$$

où le deuxième signe $\leqq$ peut être remplacé par $=$. Une telle adhérence sera dite *stricte.*

1.7. Les formules $[\alpha : \sigma]$, $[\sigma : \alpha]$ et les axiomes $[\alpha_i]$ relèvent une dualité des fonctions α, σ par rapport à $\sqsubset$, τ. Cette dualité devient plus nette si on cherche la liaison directe entre α et $\sqsubset$:

$$[\alpha : \sqsubset] \quad \alpha_{\sqsubset}(a, b) = \sqsubset(1-b, 1-a),$$

$$[\sqsubset : \alpha] \quad \sqsubset_{\alpha}(a, b) = \alpha(1-b, 1-a).$$

Ça veut dire que α est ce qu'on note dans le cas classique par $\sqsubset^c$ et que l'application identique de $(\mathscr{L}, \leqq)$ sur $(\mathscr{L}, \geqq)$ transforme un ordre flou topologique dans une adhérence floue, etc.

1.8. Dans la théorie classique, les voisinages sont définies à l'aide d'un ordre topologique $\sqsubset$ par une restriction : $V \in \mathscr{P}(X)$ est une voisinage de $x \in X$ si $x \sqsubset V$. Dans le cas flou nous allons restreindre la fonction $\sqsubset : \mathscr{L} \times \mathscr{L} \to \mathbf{I}$ à $\mathscr{K} \times \mathscr{L}$, où $\mathscr{K}$ est une sousbase de $\mathscr{L}$.

Une sousbase du treillis flou $\mathscr{L}$ est une partie $\mathscr{K} \subset \mathscr{L}$ pour laquelle — [4] — $0 \notin \mathscr{K}$ et

$$[B_0] \quad \underset{a \in \mathscr{L}}{\forall} \; a = \bigvee \{k \in \mathscr{K} : k \leqq a\}.$$

On démontre alors aisément la suivante

Proposition. *Si* $\sqsubset$ *est un ordre topologique flou en* $\mathscr{L}$ *et* $\mathscr{K}$ *une sousbase du treillis flou* $\mathscr{L}$, *la restriction*

$$[v : \sqsubset] \quad v_{\sqsubset} = \sqsubset | \mathscr{K} \times \mathscr{L}$$

jouit des propriétés

$$[v_1] \quad v(k, 1) = 1,$$

$$[v_2] \quad v(k, a \wedge b) = v(k, a) \wedge v(k, b),$$

$$[v_3] \quad v(k, a) > 0 \Rightarrow k \leqq a,$$

$$[v_4] \quad \underset{k \in \mathscr{K}}{\forall} \; \underset{a \in \mathscr{L}}{\forall} \; \underset{n \in \mathbf{N}^*}{\forall} \; \underset{c \in \mathscr{L}}{\exists} \; \underset{k' \in \mathscr{K}}{\forall} \left\{k' \leqq c \Rightarrow v(k, a) - \frac{1}{n} \leqq v(k, c) \wedge v(k', a)\right\};$$

$[v_5]$ *Si* $a \in \mathscr{L}$, $k, k_j \in \mathscr{K}$ *pour tout* $j \in J$ *et*

$k = \bigvee \{k_j : j \in J\}$, *alors* $\bigwedge \{v(k_j, a) : j \in J\} \leqq v(k, a)$.

1.9. Une application $v : \mathscr{K} \times \mathscr{L} \to \mathbf{I}$ vérifiant les axiomes précédents s'appelle *fonction de voisinages flous, sur la sousbase* $\mathscr{K}$. Pour une telle fonction, on peut montrer que $v(k, a) \leqq v(k', a)$ sera valable pour tout $a \in \mathscr{L}$, $k, k' \in \mathscr{K}$, si $k' \leqq k$. On en déduit que, dans l'axiome $[v_5]$, l'inégalité peut être remplacée par l'égalité.

Dans la démonstration du théorème 1.10 concernant la suffisance de la fonction de voisinages flous pour déterminer la topologie floue le résultat suivant nous sera utile :

LEMME. *Si v est une fonction de voisinages flous sur $\mathscr{K}$, $b, c, c_j \in \mathscr{L}$ pour tout $j \in J$ et si $c = \bigvee \{c_j : j \in J\}$, alors*

$$\bigwedge \{v(k, b) : k \in \mathscr{K}, k \leqq c\} = \bigwedge_{j \in J} (\bigwedge \{v(k', b) : k' \in \mathscr{K}, k' \leqq c_j\}).$$

L'inégalité $\leqq$ étant évidente, soit $k \in \mathscr{K}$, $k \leqq c$. On attache à tout $j \in J$ un ensemble A_j d'indices et les éléments $k_{j,l} \in \mathscr{K}$, où $l \in A_j$, tels que $k \wedge c_j = \bigvee \{k_{j,l} : l \in A_j\}$. Ayant $k_{j,l} \leqq c$ et $k = \bigvee_{j \in J} (k \wedge c_j)$, donc $k = \bigvee_{j \in J} \bigvee_{l \in A_j} k_{j,l}$, on déduit l'inégalité $\geqq$. □

1.10. PROPOSITION. *$\mathscr{K}$ étant une sousbase du treillis flou $\mathscr{L}$ l'application $\sqsubset \to v_{\sqsubset}$ est une bijection entre l'ensemble des ordres topologiques flous en $\mathscr{L}$ et l'ensemble des fonctions de voisinages flous sur $\mathscr{K}$. L'application inverse en est*

$[\sqsubset : v]$ $\quad \sqsubset_v(a, b) = \bigwedge \{v(k, b) : k \in \mathscr{K}, k \leqq a\}$.

DÉMONSTRATION. Si v est une fonction de voisinages, alors $\sqsubset_v$ est un otf en $\mathscr{L}$. Pour l'axiome $[\sqsubset_2^*]$, on utilise 1.9. Pour $[\sqsubset_{4'}]$, on choisit pour tout $k \leqq a$ un élément $c_k \in \mathscr{L}$ — $[v_4]$ — tel que $v(k, b) - 1/n \leqq v(k, c_k) \wedge v(k', b)$ soit valable pour tout $k' \in \mathscr{K}$, $k' \leqq c_k$, et soit $c = \bigvee \{c_k : k \in \mathscr{K}, k \leqq a\}$. Alors on a pour tout $k \leqq a$

$$\sqsubset_v(a, b) - \frac{1}{n} \leqq v(k, b) - \frac{1}{n} \leqq \bigwedge \{v(k', b) : k' \in \mathscr{K}, k' \leqq c_k\}$$

donc — 1.9 —

$$\sqsubset_v(a, b) - \frac{1}{n} \leqq \sqsubset_v(c, b);$$

d'autre part

$$\sqsubset_v(a, b) - \frac{1}{n} \leqq v(k, b) - \frac{1}{n} \leqq v(k, c_k) \leqq v(k, c)$$

donc

$$\sqsubset_v(a, b) - \frac{1}{n} \leqq \sqsubset_v(a, c).$$

L'égalité $v_{\sqsubset_v} = v$ est une conséquence de $[v_5]$, et $\sqsubset_{v_{\sqsubset}} = \sqsubset$ pour tout otf, $\sqsubset$ étant parfait. □

1.11. La proposition 1.10 assure la possibilité théorique d'exprimer toute notion topologique floue à l'aide des voisinages flous. En effet, on a la

PROPOSITION. *Dans un treillis flou $\mathscr{L}$, $\mathscr{K}$ étant une sousbase, τ une topologie floue en $\mathscr{L}$, $\sqsubset = \sqsubset_\tau$, $\alpha = \alpha_{\sqsubset}$, $\sigma = \sigma_\tau$, $v = v_{\sqsubset}$, on a les égalités suivantes*

$[v : \alpha]$ $\quad v(k, b) = \alpha(1 - b, 1 - k)$,

$[\alpha : v]$ $\quad \alpha(a, b) = \bigwedge \{v(k, 1 - a) : k \in \mathscr{K}, k \leqq 1 - b\}$,

$[v : \tau]$ $\quad v(k, b) = \bigvee \{\tau(c) : c \in \mathscr{L}, k \leqq c \leqq b\}$,

$[\tau : v]$ $\quad \tau(c) = \bigwedge \{v(k, c) : k \in \mathscr{K}, k \leqq c\}$,

$[v : \sigma]$ $\quad v(k, b) = \bigvee \{\sigma(1 - c) : c \in \mathscr{L}, k \leqq c \leqq b\}$,

$[\sigma : v]$ $\quad \sigma(c) = \bigwedge \{v(k, 1 - c) : k \in \mathscr{K}, k \leqq 1 - c\}$.

2. Opérations avec des ordres topogènes flous

2.1. Soit $\{\sqsubset_\gamma : \gamma\in\Gamma\}$ une famille d'ordres semitopogènes flous en $\mathcal{L}$ et $\sqsubset'(a, b)=\bigvee\{\sqsubset_\gamma(a, b) : \gamma\in\Gamma\}$, $\sqsubset''(a, b)=\bigwedge\{\sqsubset_\gamma(a, b) : \gamma\in\Gamma\}$. $\sqsubset'$, $\sqsubset''$ seront des ordres semitopogènes, les bornes supérieure, resp. inférieure de la famille donnée et nous noterons $\sqsubset'=\bigvee\limits_{\gamma\in\Gamma}\sqsubset_\gamma$, $\sqsubset''=\bigwedge\limits_{\gamma\in\Gamma}\sqsubset_\gamma$. Donc l'ensemble des ordres semitopogènes flous en $\mathcal{L}$ est un treillis complet distributif par rapport à l'ordre produit $\leqq$. Si $\sqsubset_1\leqq\sqsubset_2$, nous dirons que $\sqsubset_2$ est *plus fin* que $\sqsubset_1$. Nous noterons par $\mathcal{O}(\mathcal{L})$ l'ensemble des ordres topogènes flous en $\mathcal{L}$. Si $\sqsubset_\gamma\in\mathcal{O}(\mathcal{L})$ pour tout $\gamma\in\Gamma$, alors $\sqsubset''$ sera topogène, mais $\sqsubset'$ ne le sera pas toujours. Alors on peut trouver, pour tout ordre semitopogène flou $\sqsubset$, une enveloppe topogène $\sqsubset^q$, c'est-à-dire le moins fin ordre topogène flou plus fin que $\sqsubset$.

Pour construire $\sqsubset^q$, on connaît dans le cas flou la construction suivante — [5] —. Soit $(a, b)\in\mathcal{L}\times\mathcal{L}$ et $\mathcal{D}(a, b)$ l'ensemble des systèmes $d=(m, k, a_1, \dots, a_m, b_1, \dots$ $\dots, b_k)$ où $m, k\in\mathbf{N}$, $a_j, b_s\in\mathcal{L}$, $a=\bigvee\limits_{j=1}^{m} a_j$, $b=\bigwedge\limits_{s=1}^{k} b_s$. Alors

$$\sqsubset^q(a, b)=\bigvee\{n(d) : d\in\mathcal{D}(a, b)\}$$

si on note

$$n(d)=\bigwedge\{\sqsubset(a_j, b_s) : j\in\overline{1, m},\ s\in\overline{1, k}\}.$$

On remarque que l'ordre $\sqsubset^*\in\mathcal{O}(\mathcal{L})$ défini par $\sqsubset^*(a, b)=1$ si $a\leqq b$ et $\sqsubset^*(a, b)=0$ en cas contraire est le plus fin élément de $\mathcal{O}(\mathcal{L})$, le moins fin étant $\sqsubset_*$ où $\sqsubset_*(a, b)=1$ si $a=0$ ou $b=1$ et $\sqsubset_*(a, b)=0$ en cas contraire. Les ordres $\sqsubset^*$ et $\sqsubset_*$ sont denses, biparfaits, symétriques.

2.2. On constate aisément que les ordres semitopogènes flous $\sqsubset_\gamma$ étant parfaits, $\sqsubset''$ sera lui aussi parfait, donc pour tout ordre semitopogène flou $\sqsubset$ il existe une enveloppe parfait $\sqsubset^p$. On peut la construire de la manière suivante. Soit $a, b\in\mathcal{L}$; nous noterons par $\mathcal{F}(a)$ l'ensemble des familles $f=\{a_j : j\in J\}$ jouissant de la propriété $\bigvee\{a_j : j\in J\}=a$. Nous notons $l(f, b)=\bigwedge\{\sqsubset(a_j, b) : j\in J\}$.

PROPOSITION. *Pour tout ordre semitopogène flou* $\sqsubset$ *on a* $\sqsubset^p(a, b)=\bigvee\{l(f, b) : f\in\mathcal{F}(a)\}$.

Si $\sqsubset$ *est un otf, alors* $\sqsubset^p$ *l'est aussi.*

Si $\sqsubset$ est un ordre semitopogène flou et $\sqsubset_1(a, b)=\bigvee\{l(f, b) : f\in\mathcal{F}(a)\}$ alors $\sqsubset_1$ sera évidemment semitopogène. Pour vérifier $[\sqsubset_2^*]$ soit $\lambda_j=\sqsubset_1(a_j, b)(j\in J)$. Pour tout $\varepsilon\in\mathbf{R}^+$ il existe un $f_j\in\mathcal{F}(a_j)$ tel que $\lambda_j-\varepsilon<l(f_j, b)$ et soit $f_j=\{a_{jk} : k\in K_j\}$. Nous considérons $f=\{a_k : k\in K\}$, où $K=\bigcup\{K_j : j\in J\}$, $a_k=a_{j,k}$ si $k\in K_j$ (on suppose les ensembles K_j disjoints par paires). Alors on aura $f\in\mathcal{F}(\bigvee a_j)$ et

$$l(f, b)\geqq\bigwedge\{l(f_j, b) : j\in J\}>\bigwedge(\lambda_j : j\in J\}-\varepsilon$$

donc

$$\sqsubset_1(\bigvee\{a_j : j\in J\}, b)\geqq\bigwedge\{\sqsubset_1(a_j, b) : j\in J\}.$$

$\sqsubset_1$ est plus fin que $\sqsubset$ et pour tout autre ordre semitopogène flou parfait $\sqsubset_2$ plus fin que $\sqsubset$ on a $\sqsubset_2(a, b)\geqq l(f, b)$ pour tout $f\in\mathcal{F}(a)$, donc $\sqsubset_1\leqq\sqsubset_2$.

Si $\sqsubset$ est topogène, $\sqsubset_1$ le sera aussi. En effet, soit $\mu'=\sqsubset_1(a, b')$, $\mu''=\sqsubset_1(a, b'')$ et $\varepsilon\in\mathbf{R}^+$. Nous choisissons $f', f''\in\mathcal{F}(a)$ tels que $l(f', b')>\mu'-\varepsilon$, $l(f'', b'')>$

$>\mu''-\varepsilon$. Si $f'=\{a'_j : j\in J'\}$, $f''=\{a''_j : j\in J''\}$, nous noterons $J=J'\times J''$ et $a_j=a'_{j'}\wedge a''_{j''}$ si $j=(j',j'')\in J$. Alors $f=\{a_j : j\in J\}\in\mathscr{F}(a)$ et $l(f,b'\wedge b'')\geqq\mu'\wedge\mu''-\varepsilon$. Donc

$$\sqsubset_1(a,b'\wedge b'')\geqq\sqsubset_1(a,b')\wedge\sqsubset_1(a,b'').$$

2.3. Les résultats précédents montrent que $(\mathscr{O}(\mathscr{L}),\leqq)$ est un treillis complet (pourtant il n'est pas distributif).

La borne inférieure de la famille $\{\sqsubset_\gamma : \gamma\in\Gamma\}\subset\mathscr{O}(\mathscr{L})$ sera $\sqsubset''$, la borne supérieure $(\sqsubset')^q$.

L'ensemble $\mathscr{O}_p(\mathscr{L})$ des ordres topogènes parfaits en $\mathscr{L}$ est un treillis complet par rapport à la relation $\leqq$. Si $\{\sqsubset_\gamma : \gamma\in\Gamma\}\subset\mathscr{O}_p(\mathscr{L})$, alors $\sqsubset''$ sera la borne inférieure de la famille en $\mathscr{O}_p(\mathscr{L})$; $(\sqsubset')^{qp}$ en sera la borne supérieure.

2.4. Si nous considérons l'ensemble des ordres semitopogènes flous denses, nous obtenons un nouveau treillis par rapport à l'ordre produit $\leqq$. Mais maintenant c'est $\sqsubset'$ qui est dense, car

$$\sqsubset'^2(a,b)=\bigvee_{c\in\mathscr{L}}\{\sqsubset'(a,c)\wedge\sqsubset'(c,b)\}=\bigvee_{c\in\mathscr{L}}\{[\bigvee_{\gamma\in\Gamma}\sqsubset_\gamma(a,c)]\wedge[\bigvee_{\gamma\in\Gamma}\sqsubset_\gamma(c,b)]\}\geqq$$

$$\geqq\bigvee_{c\in\mathscr{L}}\bigvee_{\gamma\in\Gamma}[\sqsubset_\gamma(a,c)\wedge\sqsubset_\gamma(c,b)]=\bigvee_{\gamma\in\Gamma}\bigvee_{c\in\mathscr{L}}[\sqsubset_\gamma(a,c)\wedge\sqsubset_\gamma(c,b)]\geqq\bigvee_{\gamma\in\Gamma}\sqsubset_\gamma(a,b)=\sqsubset'(a,b).$$

Mais $\sqsubset''$ n'est pas dense en général, donc la borne inférieure de la famille sera $\sqsubset''^d$, le noyau dense de l'ordre $\sqsubset''$, dont la construction est donnée en [5].

2.5. Si $\mathscr{L}$ est symétrique et $\sqsubset$ un ordre semitopogène flou, $\sqsubset^c$ notera l'ordre définie par $\sqsubset^c(a,b)=\sqsubset(1-b,1-a)$. L'application $\sqsubset^c$ sera un ordre semitopogène, $\sqsubset$ étant symétrique si et seulement si $\sqsubset=\sqsubset^c$. Pour tout ordre $\sqsubset$, l'ordre $\sqsubset^s=(\sqsubset\vee\sqsubset^c)^q$ sera l'enveloppe symétrique topogène de $\sqsubset$, car on a $\sqsubset^{qc}=\sqsubset^{cq}$ pour tout ordre semitopogène $\sqsubset$.

Évidemment, $\sqsubset\wedge\sqsubset^c$ sera le noyau symétrique de $\sqsubset$.

2.6. Nous allons considérer des ordres topogènes flous jouissant de deux conditions, dont la première est celle de la densité. Soit $\mathscr{O}_{top}(\mathscr{L})$ l'ensemble des ordres flous topologiques en $\mathscr{L}$ (denses et parfaits).

Pour vérifier que $(\mathscr{O}_{top}(\mathscr{L}),\leqq)$ est un treillis complet, il suffit de démontrer que, pour $\{\sqsubset_\gamma : \gamma\in\Gamma\}\subset\mathscr{O}_{top}(\mathscr{L})$, l'ordre $(\sqsubset')^{qp}$ est dense, ce qui est une conséquence du

LEMME. *Pour tout ordre semitopogène flou* $\sqsubset$, *on a*

1) $\sqsubset^{2q}\leqq\sqsubset^{q2}$,

2) $\sqsubset^{2p}\leqq\sqsubset^{p2}$.

En effet, soit $a,b\in\mathscr{L}$, $d=(m,k,a_1,\ldots,a_m,b_1,\ldots,b_k)\in\mathscr{D}(a,b)$ et $n(d)=\wedge\{\sqsubset^2(a_j,b_s) : j\in\overline{1,m}, s\in\overline{1,k}\}$. Pour tout $n\in\mathbf{N}^*$, $j\in\overline{1,m}$, $s\in\overline{1,k}$ on choisit $c_{j,s}\in\mathscr{L}$ de telle manière que

$$\sqsubset^2(a_j,b_s)-\frac{1}{n}\leqq\sqsubset(a_j,c_{js})\wedge\sqsubset(c_{js},b_s).$$

Alors

$$n(d)-\frac{1}{n} \leqq \sqsubset^{q}(a, c) \wedge \sqsubset^{q}(c, b) \leqq \sqsubset^{q^{2}}(a, b),$$

où c est l'élément $\bigvee_{j=1}^{m} \bigwedge_{s=1}^{k} c_{js}$. Donc 1) est vérifiée. 2) se démontre d'une manière analogue. □

2.7. Si on préfère les topologies floues aux ordres topologiques, on peut transposer 2.6 dans ce langage ayant en vue que $\sqsubset_1 \leqq \sqsubset_2$ est équivalent pour les ordres flous topologiques à $\tau_1 \leqq \tau_2$, où $\tau_i = \tau_{\sqsubset_i}$.

Proposition. *L'ensemble des topologies floues en $\mathscr{L}$ forme un treillis complet par rapport à l'ordre produit $\leqq$.*

2.8. $\mathcal{O}_{\text{prox}}(\mathscr{L})$ désignera l'ensemble des ordres flous proximaux en $\mathscr{L}$ et nous considérons une famille $\{\sqsubset_\gamma \colon \gamma \in \Gamma\}$ de tels ordres. $\sqsubset' = \bigvee_{\gamma \in \Gamma} \sqsubset_\gamma$ sera dense — 2.4 — et alors $\sqsubset'^{q}$ est un ordre topogène dense — Lemme 2.6.1. Quant à la symétrie, elle se transmet évidemment à $\sqsubset'$ et puis, selon 2.5, aussi à $\sqsubset'^{q}$. Donc $\sqsubset'^{q}$ est l'enveloppe proximale de la famille $\{\sqsubset_\gamma \colon \gamma \in \Gamma\}$.

Proposition. $(\mathcal{O}_{\text{prox}}(\mathscr{L}), \leqq)$ *est un treillis complet.*

3. Ordres flous de proximité

3.1. Les premiers résultats concernant des ordres flous de proximité (au sens strict) sont ceux de [13]. En [6], nous avons poursuivi leur étude. Mais en continuant l'analyse de ces ordres, on se heurte à des difficultés majeures : l'ensemble des ordres de proximité flous au sens de [13] et [6] ne constitue pas un treillis par rapport à la relation $\leqq$. C'est une conséquence de l'axiome de densité qui y est adopté. En effet, ces deux travaux emploient la condition de densité stricte selon nos définitions 1.1.

On constate que ni l'ensemble des ordres flous topologiques strictes ne forment pas un treillis.

A cause de ces inconvénients, nous sommes revenus dans la présente note sur cet axiome en le remplaçant par celui défini en [3] assurant la cohérence nécessaire à la théorie des structures topologiques floues.

3.2. Soit $\sqsubset$ un ordre semitopogène flou — 1.1 —, $\mathscr{L}$ un treillis flou symétrique. Nous notons

$[\delta : \sqsubset]$ $\quad \delta(a, b) = \delta_{\sqsubset}(a, b) = 1 - \sqsubset(a, 1-b).$

Alors $\delta : \mathscr{L} \times \mathscr{L} \to \mathbf{I}$ vérifie les axiomes :

$[\delta_1']$ $\quad \delta(0, 1) = \delta(1,0) = 0,$

$[\delta_1'']$ $\quad \delta(a, b) < 1 \Rightarrow a \leqq 1-b,$

$[\delta_1''']$ $\quad (a_1 \leqq a) \wedge (b_1 \leqq b) \Rightarrow \delta(a_1, b_1) \leqq \delta(a, b).$

Nous appellerons une telle application *semirelation floue en $\mathscr{L}$*.

PROPOSITION. *L'application* $\sqsubset \to \delta_{\sqsubset}$ *est une bijection entre l'ensemble des ordres semitopogènes flous et l'ensemble des semirelations floues en* $\mathscr{L}$. *L'application inverse en est*

$$[\sqsubset : \delta] \quad \sqsubset(a, b) = 1-\delta(a, 1-b).$$

$\sqsubset$ *est topogène si et seulement si* δ *est une relation floue, c'est-à-dire vérifie aussi*

$$[\delta_2] \quad \delta(a_1 \vee a_2, b) = \delta(a_1, b) \vee \delta(a_2, b),$$

$$[\delta_3] \quad \delta(a, b_1 \vee b_2) = \delta(a_1, b_1) \vee \delta(a, b_2).$$

$\sqsubset$ *est dense si et seulement si*

$$[\delta_4] \quad \delta = \delta \triangle \delta$$

où $(\delta_1 \triangle \delta_2)(a, b) = \bigwedge \{\delta_2(a, 1-c) \vee \delta_1(c, b) : c \in \mathscr{L}, \ a \leqq c \leqq 1-b\}$.
$\sqsubset$ *est strictement dense si et seulement si*

$$[\delta_4^*] \quad \forall_{a, b \in \mathscr{L}} \ \exists_{c \in \mathscr{L}} \ \delta(a, 1-c) \vee \delta(c, b) \leqq \delta(a, b).$$

$\sqsubset$ *est symétrique si et seulement si*

$$[\delta_5] \quad \delta(a, b) = \delta(b, a).$$

Une relation floue vérifiant $[\delta_4]$ et $[\delta_5]$ s'appelle *relation floue de proximité*. Si elle vérifie aussi $[\delta_4^*]$, alors elle est *stricte*.

3.3. COROLLAIRE. *L'application* $\sqsubset \to \delta_{\sqsubset}$ *est une bijection entre l'ensemble des ordres flous (strictement) proximaux et l'ensemble des relations floues de proximité (strictes).*

Il est utile de donner une condition nécessaire et suffisante pour qu'une relation floue soit dense:

$$[\delta_{4'}] \quad \forall_{a, b \in \mathscr{L}} \ \forall_{n \in \mathbf{N}^*} \left\{ a \leqq 1-b \Rightarrow \exists_{c \in \mathscr{L}} \left(\left[\delta(a, 1-c) \vee \delta(c, b) \leqq \delta(a, b) + \frac{1}{n} \right) \right] \wedge \right.$$
$$\left. \wedge [a \leqq c \leqq 1-b] \right\}.$$

3.4. Soit $\sqsubset$ un ordre topogène flou. La borne inférieure des ordres flous topologique plus fins que $\sqsubset$ s'appelle *enveloppe topologique de l'ordre* $\sqsubset$.

PROPOSITION. *L'enveloppe topologique d'un ordre flou de proximité* $\sqsubset$ *est* $\sqsubset^p$.

Le lemme 2.6.2 montre que $\sqsubset$ étant dense, $\sqsubset^p$ le sera aussi. En particulier si $\sqsubset$ est un ofp, alors $\sqsubset^p$ sera un oft, donc l'enveloppe topologique de $\sqsubset$.

Il est utile de souligner que, pour le cas strict, nos résultats ne sont pas valables; il y a des ordres flous de proximité strictes qui n'admettent pas une enveloppe topologique stricte.

4. Ordres topogènes quasiflous

4.1. Si le codomaine d'un ordre (semi) topogène $\sqsubset$ en $\mathscr{L}$ est l'ensemble $\{0, 1\}$, nous écrivons $a \bar{\sqsubset} b$ si $\sqsubset(a, b)=1$. La relation binaire $\bar{\sqsubset} \subset \mathscr{L}\times\mathscr{L}$ est un ordre (semi) topogène dans le treillis $\mathscr{L}$ — [4] — nous l'appellerons *ordre (semi) topogène quasiflou (o(s)tqf) en* $\mathscr{L}$.

Dans le cas $\mathscr{L}=\{0, 1\}^X$ nous obtenons les ordres (semi) topogènes au sens classique — [2] — (les parties de X étant remplacées par leur fonction caractéristique).

Nous allons voir que les ordres (semi) topogènes flous peuvent être caractérisés par une famille d'o(s)tqf. À cette fin nous associons à tout ordre semitopogène flou $\sqsubset$ la famille

$$S_{\sqsubset} = (\bar{\sqsubset}_\lambda : \lambda\in \mathbf{I}_0) \quad \text{où} \quad \mathbf{I}_0 =]0,1] \quad \text{et}$$

$$a \bar{\sqsubset}_\lambda b \underset{\text{def}}{\Leftrightarrow} \sqsubset(a, b) \geqq \lambda.$$

Proposition. *La famille* $S=S_{\sqsubset}$ *de relations binaires dans* $\mathscr{L}$ *vérifie les conditions :*

$[S_0]$ *si* $\lambda_j\in \mathbf{I}_0$, $\lambda = \bigvee\{\lambda_j : j\in J\}$,

alors

$$a \bar{\sqsubset}_\lambda b \Leftrightarrow \underset{j\in J}{\forall}\, a \bar{\sqsubset}_{\lambda_j} b.$$

Pour tout $\lambda\in \mathbf{I}_0$, *on a*

$[S_1']$ $0 \bar{\sqsubset}_\lambda 0$ *et* $1 \bar{\sqsubset}_\lambda 1$,

$[S_1'']$ $a \bar{\sqsubset}_\lambda b \Rightarrow a \leqq b$,

$[S_1''']$ $a' \leqq a \bar{\sqsubset}_\lambda b \leqq b' \Rightarrow a' \bar{\sqsubset}_\lambda b'$.

Une famille $S=(\bar{\sqsubset}_\lambda : \lambda\in \mathbf{I}_0)$ de relations binaires dans $\mathscr{L}$ vérifiant ces conditions s'appelle système d'ordres semitopogènes quasiflous (système ostqf). Les axiomes $[S_1']$, $[S_1'']$, $[S_1''']$ sont ceux d'un ordre semitopogène q.f.

4.2. Théorème. *L'application*

$[S : \sqsubset]$ $\sqsubset \to S_{\sqsubset}$

est une bijection entre l'ensemble des ordres semitopogènes flous et l'ensemble des systèmes ostqf. L'application inverse en est

$[\sqsubset : S]$ $\sqsubset_S(a, b) = \bigvee\{\lambda\in \mathbf{I}_0 : a \bar{\sqsubset}_\lambda b\}$.

L'ordre $\sqsubset$ *vérifie l'axiome* $[\sqsubset_i]$ *si et seulement si* $S_{\sqsubset}$ *vérifie l'axiome* $[S_i]$, *où*

$[S_2]$ $(a_1 \bar{\sqsubset}_\lambda b)\wedge(a_2 \bar{\sqsubset}_\lambda b) \Rightarrow a_1\vee a_2 \bar{\sqsubset}_\lambda b$,

$[S_3]$ $(a \bar{\sqsubset}_\lambda b_1)\wedge(a \bar{\sqsubset}_\lambda b_2) \Rightarrow a \bar{\sqsubset}_\lambda b_1\wedge b_2$,

$[S_2^*]$ $\big(\underset{j\in J}{\forall}\, a_j \bar{\sqsubset}_\lambda b\big) \Rightarrow \bigvee\{a_j : j\in J\} \bar{\sqsubset}_\lambda b$,

$[S_3^*]$ $\big(\underset{j\in J}{\forall}\, a \bar{\sqsubset}_\lambda b_j\big) \Rightarrow a \bar{\sqsubset}_\lambda \bigwedge\{b_j : j\in J\}$,

$[S_4]$ $\underset{a,b\in\mathscr{L}}{\forall}\ \underset{\lambda,\lambda'\in \mathbf{I}_0}{\forall} [(\lambda' < \lambda)\wedge(a \bar{\sqsubset}_\lambda b) \Rightarrow \underset{c\in\mathscr{L}}{\exists} (a \bar{\sqsubset}_{\lambda'} c\wedge c \bar{\sqsubset}_{\lambda'} b)$,

$[S_5]$ $a \bar{\sqsubset}_\lambda b \Rightarrow 1-b \bar{\sqsubset}_\lambda 1-a$.

Bien sûr, on peut choisir aussi une autre définition pour $\bar{\sqsubset}_\lambda$ (p. e., $\sqsubset(a, b) > \lambda$). Alors, il faut changer les axiomes $[S_i]$.

Les axiomes $[S_i]$ sont tous naturels, imposant les mêmes conditions aux ordres q.f. $\bar{\sqsubset}_\lambda$ qu'on exige à $\sqsubset$. La seule exception est la condition $[S_4]$ laquelle ne garantit pas la densité des relations $\bar{\sqsubset}_\lambda$. En effet, la condition

$$[S_4^*] \quad \forall_{a,b\in\mathscr{L}} \ \forall_{\lambda\in\mathbf{I}_0} \left[a \bar{\sqsubset}_\lambda b \Rightarrow \exists_{c\in\mathscr{L}} (a\bar{\sqsubset}_\lambda c)\wedge(c\bar{\sqsubset}_\lambda b)\right]$$

équivaut à la *densité stricte de* $\sqsubset$.

4.3. On peut indicier aussi la topologie floue τ en notant

$$[\mathscr{G} : \tau] \quad G_\tau = (\mathscr{G}_\lambda : \lambda\in\mathbf{I}_0) \quad \text{où} \quad \mathscr{G}_\lambda = \{c\in\mathscr{L} : \tau(c) \geqq \lambda\}.$$

Proposition. *$G = G_\tau$ vérifie les propriétés suivantes :*

$[G_0]$ *Si* $\lambda_j\in\mathbf{I}_0$, $\lambda = \bigvee\{\lambda_j : j\in J\}$, *alors* $\mathscr{G}_\lambda = \bigcap\{\mathscr{G}_{\lambda_j} : j\in J\}$.

Pour tout $\lambda\in\mathbf{I}_0$, *on a*

$[G_1]$ $0, 1\in\mathscr{G}_\lambda$,

$[G_3]$ $c_1, c_2\in\mathscr{G}_\lambda \Rightarrow c_1\wedge c_2\in\mathscr{G}_\lambda$,

$[G_2^*]$ $(\forall_{j\in J} c_j\in\mathscr{G}_\lambda) \Rightarrow \bigvee\{c_j : j\in J\}\in\mathscr{G}_\lambda$.

Théorème. *L'application $\tau\to G_\tau$ est une bijection entre l'ensemble des topologies floues en $\mathscr{L}$ et l'ensemble des familles $G = (\mathscr{G}_\lambda : \lambda\in\mathbf{I}_0)$ vérifiant les axiomes $[G_0]$, $[G_1]$, $[G_2^*]$, $[G_3]$. L'application inverse en est $G\to\tau_G$ où $[\tau : G]$ $\tau_G(c) = \bigvee\{\lambda\in\mathbf{I}_0 : c\in\mathscr{G}_\lambda\}$.*

On peut indicier aussi la fonction v de voisinage, l'adhérence, la fonction de fermeture.

4.4. Soit par exemple α une adhérence floue — 1.6 — ; nous attachons à elle un *système de fermetures floues*. Soit $\alpha_\lambda : \mathscr{L}\to\mathscr{L}$ définie par

$$\alpha_\lambda(a) = \bigwedge\{b\in\mathscr{L} : \alpha(a, b) \geqq \lambda\}$$

et

$$[A : \alpha] \quad A = A_\alpha = \{\alpha_\lambda : \lambda\in\mathbf{I}_0\}.$$

Le système A vérifie les axiomes suivants :

$[A_0]$ Si $\lambda_j\in\mathbf{I}_0$ et $\lambda = \bigvee\{\lambda_j : j\in J\}$, alors $\alpha_\lambda = \bigvee_{j\in J} \alpha_{\lambda_j}$.

Pour tout $\lambda\in\mathbf{I}_0$ et $a, b\in\mathscr{L}$, on a

$[A_1]$ $\alpha_\lambda(0) = 0$,

$[A_2]$ $\alpha_\lambda(a) \geqq a$,

$[A_3]$ $\forall_{\lambda,\lambda'\in\mathbf{I}_0} \lambda' < \lambda \Rightarrow \alpha_{\lambda'}^2 \leqq \alpha_\lambda$,

$[A_4]$ $\alpha_\lambda(a\vee b) = \alpha_\lambda(a)\vee\alpha_\lambda(b)$.

Un tel système A définit l'adhérence floue, et on a

$[\alpha : A]$ $\alpha(a, b) = \bigvee \{\lambda \in \mathbf{I}_0 : \alpha_\lambda(a) \leqq b\}$.

4.5. Si on veut définir les ensembles fermés indiciés, on considère les familles

$$\mathscr{F}_\lambda = \{c \in \mathscr{L} : \sigma(c) \geqq \lambda\}$$

en notant

$[F : \sigma]$ $F_\sigma = (\mathscr{F}_\lambda : \lambda \in \mathbf{I}_0)$.

Le systèmc F_σ jouit des propriétés duales (conçernant l'ordre $\leqq$ de $\mathscr{L}$) aux propriétés de G. On trouve aisément les formules directes liant les systèmes introduites. Par exemple,

$[G : F]$ $a \in \mathscr{G}_\lambda \Leftrightarrow 1 - a \in \mathscr{F}_\lambda$,

$[S : G]$ $a \tilde{\sqsubset}_\lambda b \Leftrightarrow \underset{g \in \mathscr{G}_\lambda}{\exists} a \leqq g \leqq b$,

$[G : S]$ $a \in \mathscr{G}_\lambda \Leftrightarrow a \tilde{\sqsubset}_\lambda a$.

4.6. Outre les ordres q.f., on obtient un cas particulier si on considère des otf sur le treillis $\mathscr{L} = \{0, 1\}^X$. En ce cas, les éléments $a \in \mathscr{L}$ peuvent être identifiés aux sousensembles $A \subset X$. En employant les méthodes antérieures indiciées, on aboutit à des structures topologiques ou proximaux de type probabilistique. Si $\mathscr{L} = \{0, 1\}^X$ et les ordres sont q.f., alors on obtient les ordres topogènes classiques [2].

RÉFÉRENCES

[1] CHANG, C. L., Fuzzy topological spaces, *J. Math. Anal. Appl.* **24** (1968), 182—190. *MR* **38** # 5153.

[2] CSÁSZÁR, Á., *Grundlagen der allgemeinen Topologie*, Akadémiai Kiadó, Budapest, 1963. *MR* **26** # 6917.

[3] FLORESCU, L., Structuri topologice nuanţate, *Lucrările Sem. itin. ec. funcţ. aprox. şi convexitate*, Timişoara, 1980, 225—230.

[4] HAMBURG, P. and FLORESCU, L., Topological structures in fuzzy lattices, *J. Math. Anal. Appl.* **101** (1984), 475—490.

[5] HAMBURG, I. et BACANU, D., Operaţii cu ordini de proximitate, *Lucrările Conf. de topologie generală*, Iaşi, 1981, 13—14.

[6] HAMBURG, I. et HAMBURG, P., Proximités fuzzy, *Lucrările Conf. de topologie generală Iaşi*, 1981, 15—17.

[7] HAMBURG, P., Ordini topogene fuzzy, *Lucrările Conf. de topologie generală*, Iaşi, 1981, 18.

[8] HUTTON, B. W., Uniformities on fuzzy topological spaces, *J. Math. Anal. Appl.* **58** (1977), 559—571. *MR* **58** # 30986

[9] KATSARAS, A. K., Fuzzy proximity spaces, *J. Math. Anal. Appl.* **68** (1979), 100—110. *MR* **82j**: 54007a.

[10] KATSARAS, A. K. and PETALAS, C. G., A unified theory of fuzzy topologies, fuzzy proximities and fuzzy uniformities, *Rev. Roumaine Math. Pures Appl.* (va paraître).

[11] LOWEN, R., Topologies floues, *C. R. Acad. Sci. Paris Sér. A.* **278** (1974), 925—928. *MR* **49** # 3808.

[12] LOWEN, R., Fuzzy topological spaces and fuzzy compactness, *J. Math. Anal. Appl.* **56** (1976), 621—633. *MR* **55** # 13357.

[13] ROVENTA, E. et BINZARU, T., Spaţii de proximitate fuzzy, *Lucrările Sem. itin. ec. funcţ. aprox., ec. funcţ. şi convexitate*, Timişoara, 1980, 261—268.
[14] WARREN, R. H., Neighborhoods, bases and continuity in fuzzy topological spaces, *Rocky Mountain J. Math.* **8** (1978), 459—470. *MR* **57** #17583.
[15] WONG, C. K., Fuzzy topology: product and quotient theorems, *J. Math. Anal. Appl.* **45** (1974), 512—521. *MR* **49** #6117.

(Reçu le 2 décembre 1981)

CATEDRA DE MATEMATICA
FACULTATEA DE STIINŢE
UNIVERSITATEA DIN CRAIOVA
R—1100 CRAIOVA
ROUMANIA

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 309—317.

CATEGORY THEORETIC NOTIONS OF ULTRAPRODUCTS

ILDIKÓ SAIN and BUI HUY HIEN

In this paper we try to initiate a search for an explicit and direct definition of ultraproducts in categories which would share some of the attractive properties of products, coproducts, limits, and related category theoretic notions.

Consider products as a motivating example. Let $A \subseteq C$ be two categories, with A a full subcategory of C, and let $b \in {}^I\mathrm{Ob}\, A$ with I a set. Then if the product $\prod^C(b)$ of b exists in C and $\prod^C(b) \in \mathrm{Ob}\, A$ then $\prod^A(b)$ exists in A and $\prod^A(b) = \prod^C(b)$. Hence for any class $K \subseteq \mathrm{Mod}_t$ of models (algebraic systems in the sense of Mal'cev) and $\mathfrak{B} \in {}^I K$, if the set theoretic direct product of $\mathfrak{B}$ is a member of K then it coincides with the category theoretic product $\prod^K \mathfrak{B}$ computed in the category K.

A purely category theoretic definition of ultraproducts $\prod b/D$ was introduced and investigated in [9] Def. 4 p. 27, [5], [1], [2], [3], [8], via products and direct limits. This notion has two disadvantages:

(I) It is indirect (i.e. it is based on products and colimits);

(II) It does not have the above attractive property of products, for example if K is the category of all totally ordered (in other words linearly ordered) sets then the set theoretic ultraproducts $\mathrm{Ps}\, \mathfrak{B}/D$ do exist in K but the quoted category theoretic ones do not. We shall give an example for another axiomatizable class $L = \mathrm{Mod}\,(Ax)$ of models in which both the set theoretic and the quoted (indirect) category theoretic ultraproducts exist but they are not isomorphic, see Proposition 6.

The problem outlined above was raised by J. Rosicky [17] during the "Algebra Summer School 1980 Kromeriz".

All the proofs that are omitted from this paper are available from the authors. Requests should be sent to I. Sain.

Thanks are due to E. Makai Jr. and I. Németi for improving this paper enormously.

NOTATION. By a function we understand a class of pairs subject to the usual conditions.

Let f be a function. Then

$$\mathrm{Rng}\, f \overset{\mathrm{d}}{=} \{x \colon (\exists y) f(y) = x\} \quad \text{and}$$

$$\mathrm{Dom}\, f \overset{\mathrm{d}}{=} \{x \colon (\exists y) f(x) = y\}.$$

1980 *Mathematics Subject Classification.* Primary 18A30, 03C20; Secondary 03G30, 03C95.
Key words and phrases. Ultraproduct, universal construction, model theory, classes of models, universal algebra, reduced product.

Let $i \in \operatorname{Dom} f$. We define

$$f_i \stackrel{d}{=} f(i) \quad \text{and we let}$$

$$f \stackrel{d}{=} \langle f_i \colon i \in \operatorname{Dom} f \rangle \stackrel{d}{=} \langle f_i \rangle_{i \in \operatorname{Dom} f}.$$

Throughout this paper $C = \langle \operatorname{Ob} C, \operatorname{Mor} C \rangle$ is an arbitrary category, I is a set, D is a filter on I, and $b = \langle b_i \rangle_{i \in I} \in {}^I\operatorname{Ob} C$.

By a *cone* of C we understand a function $h \in {}^Y\operatorname{Mor} C$ such that Y is a nonempty set and $(\forall i, j \in Y)$ dom $h_i =$ dom h_j. The class of all cones of C will be denoted by Cones. Let $h \in$ Cones. We define dom h to be such that $(\forall i \in \operatorname{Dom} h)$ dom $h =$ dom h_i. Note that $h = \langle h_i : i \in \operatorname{Dom} h \rangle$, hence $\operatorname{Dom} h \neq \operatorname{dom} h \in \operatorname{Ob} C$.

Let $f \in \operatorname{Mor} C$ be such that cod $f =$ dom h. We define $h \circ f \stackrel{d}{=} \langle h_i \circ f \colon i \in \operatorname{Dom} h \rangle$. Clearly, $h \circ f \in$ Cones.

Now we define a class of cones Kon_{Db} as follows:

$$\operatorname{Kon}_{Db} \stackrel{d}{=} \{k \in \text{Cones} \colon \operatorname{Dom} k \in D, (\forall i \in \operatorname{Dom} k) \operatorname{cod} k_i = b_i\}.$$

Recall from [7], p. 46 the notion of the comma category $(C \downarrow a)$ where $a \in \operatorname{Ob} C$. □

In Definition 1 below we shall define a notion called "universal D-reduced product". Definition 1 below can be reformulated in such a way that the universal D-reduced product $\underline{P}b/D$ is defined to be the initial object of a certain functor-category. (The authors will be glad to send this functorial definition of $\underline{P}b/D$ to anyone interested.)

Definition 1 (I. Németi). We call a pair $\underline{P}b/D \stackrel{d}{=} \langle Pb/D, pb/D \rangle$ the *universal D-reduced product* of $b \in {}^I\operatorname{Ob} C$ if conditions (1)—(3) below hold.

(1) $\underline{P}b/D \in \operatorname{Ob} C$ and pb/D is a mapping, $pb/D \colon \operatorname{Kon}_{Db} \to \operatorname{Ob}(C \downarrow Pb/D)$.

(2) For all $k, h \in \operatorname{Kon}_{Db}$, statements (i) and (ii) below hold.

(i) $\{i \in I \colon h_i = k_i\} \in D$ implies $pb/D(h) = pb/D(k)$.

(ii) For all $f \in \operatorname{Mor} C$ with cod $f =$ dom h, $pb/D(h \circ f) = pb/D(h) \circ f$.

(3) For any pair $\langle E, e \rangle$ satisfying conditions (1) and (2) above, there is a unique morphism $q \colon Pb/D \to E$ of C such that $(\forall h \in \operatorname{Kon}_{Db}) e(h) = q \circ pb/D(h)$.

If D is an ultrafilter then $\underline{P}b/D$ is called the *universal D-ultraproduct* of b. If we want to indicate that $\underline{P}b/D$ is understood in the category C then we write $P^C b/D \stackrel{d}{=} \stackrel{d}{=} Pb/D$ and $\underline{P}^C b/D \stackrel{d}{=} \underline{P}b/D$.

We use the expression "algebraic system" in the sense of Mal'cev. That is, an algebraic system is nothing but a model in the sense of Chang—Keisler [4]. Hence in an algebraic system or equivalently, in a model there are both operations and relations. A typical example of such a structure is the ordered group $\mathfrak{Z} = \langle Z, \leqq, 0, + \rangle$ of the integers. Of course, either the number of relations or the number of operations may be zero, hence both algebras and relational structures are special cases of algebraic systems. E.g. the algebra $\langle Z, \circ, + \rangle$ and the structure $\langle Z, \leqq \rangle$ are both algebraic systems, i.e. models.

Let t be a similarity type containing both function symbols and relation symbols. Then Mod_t denotes the similarity class of all models of similarity type t. Let $K \subseteq \mathrm{Mod}_t$ for some type t. Then the category with class K of objects and with morphisms the usual homomorphisms between members of K will also be denoted by K. Clearly, for every $K \subseteq \mathrm{Mod}_t$, the category K is a full subcategory of the category Mod_t.

Recall that I is a set and D is a filter on I. Let t be a similarity type and let $\mathfrak{B} \in {}^I\mathrm{Mod}_t$. The usual set theoretic D-reduced product of $\mathfrak{B}$ will be denoted by $Ps\, \mathfrak{B}/D$.

Note that in Theorem 1 below the restriction that t should not contain any function symbols is not a strong one since the axioms defining our $K \subseteq \mathrm{Mod}_t$ *may* postulate that certain relation symbols denote *functions*. (E.g. the following theorem can be applied to the variety of groups.) Therefore condition (b) of Theorem 1 is satisfied for any quasivariety K of algebras!!

THEOREM 1. *Let t be an arbitrary similarity type containing no function symbols. Let $K \subseteq \mathrm{Mod}_t$ be an arbitrary first order axiomatizable class of models. Then any one of the conditions* (a), (b) *below implies that for every set I, every ultrafilter U on I, and for every $\mathfrak{B} \in {}^I K$, the universal U-ultraproduct $\underline{P}\mathfrak{B}/U$ exists in the category K and $P\mathfrak{B}/U = Ps\, \mathfrak{B}/U$.*

(a) $|t| < \omega$, *and K is axiomatizable by quantifier free formulas.*

(b) $(\forall \mathfrak{B} \in \mathrm{Mod}_t)$ (if $(*)$ *below holds for $\mathfrak{B}$ then the K-reflection of $\mathfrak{B}$ exists in the sense of* [6], *p. 275 Definition 36.1* (1)).

Now we formulate condition $(*)$:

$(*)$ There exist $\mathfrak{B} \rightarrowtail \mathfrak{C} \in K$ and $|B| < \omega$ and $\{R \in \mathrm{Dom}\, t\colon \mathfrak{B}^{(R)} \neq 0\}$ is finite.

PROOF. This is immediate by Theorem 3 below. □

COROLLARY 2. *Let K be a first order axiomatizable class of algebraic systems in which all finitely presented models exist in K. Then the universal ultraproducts exist and they coincide with the set theoretic ones in K. That is for any set I, ultrafilter U on I, and $\mathfrak{A} \in {}^I K$, $\underline{P}^K \mathfrak{A}/U$ exists and $Ps\, \mathfrak{A}/U = P^K \mathfrak{A}/U$ in the category K.*

PROOF. Immediate by (b) of Theorem 1, since if K is a class of similar algebras then there exists a similarity type t containing no function symbols such that $K \subseteq \mathrm{Mod}_t$ because every n-ary function is an $n+1$-ary relation. □

DEFINITION 2. Let $\mathfrak{A} = \langle A, R_i \rangle_{i \in I}$ and $\mathfrak{B} = \langle B, S_i \rangle_{i \in I}$ be two models. Then we define

$$\mathfrak{A} \subseteq_w \mathfrak{B} \overset{d}{\Leftrightarrow} (\forall i \in I)\, R_i \subseteq S_i$$

and $A \subseteq B$. Further

$$\mathfrak{A} \subseteq_{w\omega} \mathfrak{B} \overset{d}{\Leftrightarrow} [\mathfrak{A} \subseteq_w \mathfrak{B} \text{ and } |A| < \omega \text{ and } |\{i \in I\colon R_i \neq 0\}| < \omega].$$

If $\mathfrak{A} \subseteq_w \mathfrak{B}$ then we say that $\mathfrak{A}$ is a weak submodel of $\mathfrak{B}$. If $\mathfrak{A} \subseteq_{w\omega} \mathfrak{B}$ (for some $\mathfrak{B}$) then we say that $\mathfrak{A}$ is a doubly finite model. □

Note that if $\mathfrak{A} \subseteq_w \mathfrak{B}$ then $\mathfrak{A}$ and $\mathfrak{B}$ may belong to two different similarity types since $\mathfrak{B}$ may be an algebra (e.g. a group) and $\mathfrak{A}$ may be only a partial algebra hence only a relational structure and not an algebra. For example: $\mathfrak{A} \overset{d}{=} \langle \{1, 2\},$

$\{\langle 1, 1, 2\rangle\}\rangle \subseteq \mathfrak{B} \overset{d}{=} \langle \omega, +\rangle$ is true, but while $\mathfrak{B}$ is an algebra (actually a semigroup), $\mathfrak{A}$ is not an algebra but only a relational structure. Note that in this case $\mathfrak{A}$ is an ordinary (strong) submodel of $\mathfrak{B}$. (Of course there is a t such that $\{\mathfrak{A}, \mathfrak{B}\} \subseteq \mathrm{Mod}_t$.)

THEOREM 3. *Let K be a class of similar algebraic systems (possibly with many function symbols). Let t denote the unique similarity type containing no function symbols for which $K \subseteq \mathrm{Mod}_t$. (For the existence of t see the proof of Corollary 2.) Assume that $(**)$ below holds for K and t. Let F be a filter over some set I. Let $\mathfrak{A} \in {}^I K$. Assume that $Ps\,\mathfrak{A}/F \in K$.*

Then $\underline{P}^K \mathfrak{A}/F$ exists and $Ps\,\mathfrak{A}/F = P\mathfrak{A}/F$ in K.

$$(**) \quad \begin{array}{l} (\forall \mathfrak{A} \in K)(\forall \mathfrak{B} \in \mathrm{Mod}_t)(\exists \mathfrak{D} \in \mathrm{Mod}_t)[\mathfrak{B} \subseteq_{w\omega} \mathfrak{A} \Leftrightarrow (\mathfrak{B} \subseteq_w \mathfrak{D} \subseteq_{w\omega} \mathfrak{A} \\ \quad \text{and the } K\text{-reflection of } \mathfrak{D} \text{ exists in the category } \mathrm{Mod}_t)]. \end{array}$$

The *proof* can be found in [19]. □

DEFINITION 3 ([1], [3], [5], [8], and Definition 4 of [9], p. 27, [13], p. 35). Recall $C, I, b \in {}^I\mathrm{Ob}\,C$ and D from the Notation. The prodlim D-reduced product $\prod b/D$ is defined to be the same as in [1], [3] etc. That is $\prod b/D \overset{d}{=} \operatorname{Colim}_{X \in D} (\prod_{i \in X} b_i)$. □

About properties of $\prod b/D$ see [1], [2], [3], [8]. Note that $\prod b/D$ is only the object part of a reduced product $\underline{\prod} b/D = \langle \prod b/D, \pi b/D\rangle$ if reduced products are taken in the style of Definition 1 (and they should be!). It is easy to define $\pi b/D$, thus we omit it.

THEOREM 4. *Let C be an arbitrary category, let I be a set, let D be a filter on I, and let $b \in {}^I\mathrm{Ob}\,C$. Assume that the prodlim D-reduced product $\prod b/D$ of b exists. Then the universal D-reduced product $\underline{P}b/D$ of b exists and they coincide that is $Pb/D = \prod b/D$.* □

THEOREM 5. *There is a classical first order axiomatizable class K of models such that for every set I, ultrafilter U on I and $\mathfrak{A} \in {}^I K$, there holds $Ps\,\mathfrak{A}/U = P^K \mathfrak{A}/U$ in K but $\prod^K \mathfrak{A}/U$ does not exist for some I, some $\mathfrak{A} \in {}^I K$ and ultrafilter U on I. There is a finitely axiomatizable class of algebras with the same property and with only a finite number of function symbols (and no relation symbols of course).*

PROOF. Let K be the class of totally ordered semilattices. That is, let

$$Ax \overset{d}{=} \{(xy)z = x(yz), xy = yx, xx = x, (xy = x \vee xy = y)\}$$

and let $K \overset{d}{=} \mathrm{Mod}(Ax)$. Condition $(**)$ in the formulation of Theorem 3 holds for K. Hence, by Theorem 3, all (category theoretic) universal ultraproducts exist in K and they coincide with the set theoretic ones. However, since products do not exist in K, the prodlim ultraproducts do not exist either. □

By Theorems 4, 5 above, the universal ultraproducts do "work" in strictly more axiomatizable classes of algebras than the prodlim ultraproducts. A category theoretic notion of ultraproduct is said to work in a class K of models if it exists in K (for every system of elements of K indexed by any *set* I and for every ultrafilter on I) and coincides with the set theoretic one in K.

Propositions 6 and 7 below imply that, in the assumption of Corollary 2, the existence of finitely presented structures cannot be replaced with the existence of free structures.

PROPOSITION 6. *There is a set Ax of classical first order formulas such that in the category $K = \text{Mod}(Ax)$ the category theoretic prodlim ultraproduct $\prod \mathfrak{B}/D$ exists but is different from the set theoretic ultraproduct $Ps\, \mathfrak{B}/D$, that is $\prod \mathfrak{B}/D \not\cong Ps\, \mathfrak{B}/D$ for some ultrafilter D and some family $\mathfrak{B} \in {}^I K$ of members of K. At the same time, free models exist in K over every generator set, that is the forgetful functor Universe: $K \to$ Sets has a left adjoint. Moreover, $K \subseteq \text{Mod}_t$ with t containing only unary relation symbols.*

PROOF. Let the similarity type t consist of countably many relation symbols $Z, P_i, S_i, i \in \omega$.

$$Ax \stackrel{d}{=} \{\forall x Z(x) \to \forall x [P_i(x) \leftrightarrow \neg S_i(x)] :\ i \in \omega\}.$$

Let $K \stackrel{d}{=} \text{Mod}(Ax)$. We define $\langle \mathfrak{B}_n :\ n \in \omega \rangle \in {}^\omega K$ as follows:

$$(\forall n \in \omega)(B_n = \{a\} \text{ and } \mathfrak{B}_n \models \{\forall x Z(x)\} \cup \{P_i(x) \wedge \neg P_j(x) :\ i \leqq n < j\}).$$

By these $\mathfrak{B} \stackrel{d}{=} \langle \mathfrak{B}_n :\ n \in \omega \rangle$ is defined.

Let D be any nonprincipal ultrafilter on ω. Then $\prod \mathfrak{B}/D \not\cong Ps\, \mathfrak{B}/D$ in the category K because of the following.

Let $\mathfrak{A} \in K$ be such that $A = \{a\}$ and $\mathfrak{A} \models \neg Z(x)$ and $\mathfrak{A} \models \{P_i(x) \wedge \neg S_i(x) :\ i \in \omega\}$. We claim that $\mathfrak{A} \cong \prod^K \mathfrak{B}/D$. Clearly, $Ps\, \mathfrak{B}/D \models Z(x)$ hence $Ps\, \mathfrak{B}/D \not\cong \prod \mathfrak{B}/D$.

It is easy to check that free structures exist in K. $\square$

PROPOSITION 7. *The above Proposition 6 remains true if we replace the condition that t contains only unary relation symbols with the new condition that t consists of finitely many function symbols only.*

PROOF. Let the similarity type t consist of the function symbols f and g together with constant symbols $c_0, \ldots, c_4$. Let φ be the formula

$$(c_0 = c_1 \to [f(x) = c_3 \vee f(x) = c_4]).$$

Let $K \stackrel{d}{=} \text{Mod}(\varphi)$. Define the sequence $\mathfrak{B} \in {}^\omega K$ of algebras as follows. Let $g^0(x) \stackrel{d}{=} x$ and $g^{n+1}(x) \stackrel{d}{=} g(g^n(x))$ for all $n \in \omega$. Let $n \in \omega$. Then we require

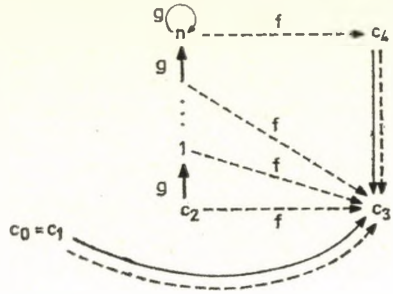
$$\mathfrak{B}_n \models \{\bigwedge_{i<n} f(g^i(c_2)) = c_3,\ f(g^n(c_2)) = c_4 \neq c_3,\ g^{n+1}(x) = g^n(x),\ c_0 = c_1\},$$

and that all the unspecified operations go to c_3 and that all the constants are different except c_0 and c_1. See the figure!

Let F be any nonprincipal ultrafilter on ω and let $Y \in F$. Then $(\prod^K \langle \mathfrak{B}_i :\ i \in Y \rangle) \models c_0 \neq c_1$. Hence $\prod^K \mathfrak{B}/F \models c_0 \neq c_1$. One can check that these exist: Indeed, let

$$\mathfrak{A} = \langle A;\ c_0, c_1, \ldots, c_4, f, g \rangle \stackrel{d}{=} Ps\, \mathfrak{B}/F.$$

$\mathfrak{B}_n$



Then

$$\mathfrak{A}^+ \stackrel{\mathrm{d}}{=} \langle \{A\} \cup A;\ A, c_1, c_2, c_3, c_4, f^+, g^+ \rangle \cong \prod^K \mathfrak{B}/F$$

where $f \subseteq f^+$ and $g \subseteq g^+$ and $f^+(A) = g^+(A) = c_3$. Now $\mathfrak{A} \models c_0 = c_1$ while $\mathfrak{A}^+ \not\models$ $\not\models c_0 = c_1$. Free algebras do exist in K since the absolutely free algebra $\mathfrak{Fr}_X$ over any set X is in K. $\square$

In the proof of Proposition 6, by omitting the premis $\forall x\, Z(x)$ from Ax and by adding a constant symbol to t, we obtain an axiomatizable K in which the category theoretic ultraproducts $\underline{\prod}\mathfrak{B}/D$ and $\underline{P}\mathfrak{B}/D$ do not exist.

FACT 8. *Let t be arbitrary and K be any reflective full subcategory of* Mod_t. *(Note that $K = SP\,K$ is sufficient but not necessary for this by* [2] *or* [8].*) Then all the prod-lim and hence all the universal reduced products exist in K. Further if K is not closed under set theoretic ultraproducts but is closed under isomorphisms then $Ps\,\mathfrak{B}/D \not\cong$ $\not\cong \underline{P}\mathfrak{B}/D$ for some ultrafilter D and $\mathfrak{B} \in {}^{UD}K$.* $\square$

COROLLARY 9. *In the categories Ds and Dts of dynamic set algebras, see Definitions 1, 4 of* [18], *p. 281 and p. 286, the category theoretic ultraproducts exist but do not coincide with the set theoretic ones.* $\square$

PROBLEM 1. Let $K \subseteq \mathrm{Mod}_t$ be axiomatizable, i.e. let $K = \mathrm{Mod}(Th)$ for some set Th of classical first order formulas. Under what conditions does (i) or (ii) below hold?

(i) $\underline{P}\mathfrak{B}/U$ exists in the category K for all ultrafilter U and all $\mathfrak{B} \in {}^I K$.
(ii) $\underline{P}\mathfrak{B}/U \cong Ps\,\mathfrak{B}/U$ for all ultrafilter U and $\mathfrak{B} \in {}^I K$. $\square$

PROBLEM 2. In which categories does Łoś lemma hold for universal ultraproducts, where validity of formulas is replaced by injectivity of small trees that is Łoś lemma is understood as in [3]. $\square$

The fundamental textbook on non category theoretic (i.e. classical) algebraic logic is [10]. It would be interesting to see how the approach to ultraproducts presented in [10] and further elaborated in [11] can be connected to the present one. E.g. Łoś lemma is true for the finitary logics of infinitary relations in the [10]—[11] approach but not in the present one, see [12].

Here we note that there is a misprint in [15], p. 44, line 8, namely there "iterated *reduced* products" should stand instead of "iterated direct products".

Remark. René Guitart [14] found a nicer definition for our universal reduced product $\underline{Pb}/D$, see Definition 1 here. Next we briefly indicate his definition.

Let $\underline{C}$ be a category, I a set, and D a filter on I. Let $b \in {}^I\mathrm{Ob}\,C$. For any $Y \in D$ we define $Y{\restriction}b \overset{\mathrm{d}}{=} \langle b_i \colon i \in Y \rangle$ to be the restriction of b to Y. We identify Y with the discrete category $\underline{Y}$ for which $Y = \mathrm{Ob}\,\underline{Y}$. We identify $Y{\restriction}b$ with the functor $Y{\restriction}b$: $\underline{Y} \to C$ which acts on $\mathrm{Ob}\,\underline{Y}$ as $Y{\restriction}b$ does. Then ${}^YC$ is a category with $(Y{\restriction}b) \in \mathrm{Ob}\,({}^YC)$. Let the diagonal embedding functor $\Delta_Y \colon C \to {}^YC$ be as defined on p. 67 of [7]. Since the diagram commutes, we have

$$\begin{tikzcd}
 & {}^YC \arrow[dr, "\mathrm{Hom}_{[{}^YC]}(-,Y{\restriction}b)"] & \\
C \arrow[ur, "\Delta_Y"] \arrow[rr, "\mathrm{Hom}_{[{}^YC]}(-,Y{\restriction}b)\circ\Delta_Y"'] & & \mathrm{Set}^{op}
\end{tikzcd}$$

$\mathrm{Hom}_{[{}^YC]}(-, Y{\restriction}b) \circ \Delta_Y \colon C^{op} \to \mathrm{Set}$.

Note that for all $a \in \mathrm{Ob}\,C$, we have

$$[\mathrm{Hom}_{[{}^YC]}(-, Y{\restriction}b)\circ\Delta_Y](a) = \mathrm{Hom}_{[{}^YC]}(\Delta_Y(a), Y{\restriction}b) =$$

$$= \{k \in \mathrm{Kon}_{Db} \colon \mathrm{Dom}\, k = Y \quad \text{and} \quad \mathrm{dom}\, k = a\}.$$

We can compute the colimit $\operatorname*{Colim}_{Y \in D}[\mathrm{Hom}_{[{}^YC]}(-, Y{\restriction}b)\circ\Delta_Y]$ in the functor category ${}^{C^{op}}\mathit{Set}$ which is indeed cocomplete. Let

$$\langle g, G\rangle \overset{\mathrm{d}}{=} \operatorname*{Colim}_{Y \in D}[\mathrm{Hom}_{[{}^YC]}(-, Y{\restriction}b)\circ\Delta_Y]$$

computed in ${}^{C^{op}}\mathit{Set}$, that is $G \in \mathrm{Ob}\,{}^{C^{op}}\mathit{Set}$ is the object part of this colimit and $g = \langle g_Y \colon Y \in D\rangle$ is the colimiting cocone of this colimit. Note that $G \colon C^{op} \to \mathit{Set}$ is a functor, and for all $Y \in D$ our

$$g_y \colon [\mathrm{Hom}_{[{}^YC]}(-, Y{\restriction}b)\circ\Delta_Y] \overset{.}{\to} G$$

is a natural transformation, hence

$$g_Y(d) \colon \mathrm{Hom}_{[{}^YC]}(\Delta_Y(d), Y{\restriction}b) \to G(d)$$

is a mapping for all $d \in \mathrm{Ob}\,C$. Let Yon: $C \to {}^{C^{op}}\mathit{Set}$ be the usual Yoneda embedding. That is, $\mathrm{Yon}\,(d) = \mathrm{Hom}_C(-, d)$ for all $b \in \mathrm{Ob}\,C$, see item (7) on p. 62 of [7].

Assume that there is a reflection $\langle \eta, \mathit{Pb}/D\rangle$ of the object G in the functor Yon, in the sense of Definition 36.1 on p. 275 of [6], that is assume that $\langle \eta, \mathit{Pb}/D\rangle$ is a

Yon-universal map for G in the sense of Definition 26.1 on p. 178 of [6]. That is $Pb/D \in \mathrm{Ob}\, C$ and the natural transformation $\eta\colon G \xrightarrow{\cdot} \mathrm{Yon}(Pb/D)$ is universal. Roughly speaking, $\eta\colon G \xrightarrow{\cdot} \mathrm{Hom}_C(-, Pb/D)$ is the reflection of G in the subcategory of representable functors. Note that we are working in ${}^{C^{op}}Set$ of which the representable functors form a subcategory.

In [14] René Guitart showed that our reduced product $\underline{P}b/D$ is nothing but a reflection $\eta\colon G \xrightarrow{\cdot} \mathrm{Hom}_C(-, Pb/D)$ of G in Yon. Our function $pb/D\colon \mathrm{Kon}_{Db} \to \mathrm{Ob}\,(C \downarrow Pb/D)$ can be recovered from the natural transformation η and the colimiting cocone g as follows. Let $k \in \mathrm{Kon}_{Db}$. Then

$$pb/D(k) = \eta_{\mathrm{dom}\,k}(g_{\mathrm{Dom}\,k}(\mathrm{dom}\,k)_k).$$

Summing up: $\underline{P}b/D$ exists in C iff the functor

$$\operatorname*{Colim}_{Y \in D}\,[\mathrm{Hom}_{[{}^YC]}(-, Y{\uparrow}b) \circ \Delta_Y]\colon C^{op} \to Set,$$

which always exists, has a reflection in the subcategory of representable functors.

We note that in the usual cases η is neither epi nor mono. Hence, usually the functor $G\colon C^{op} \to Set$ is very far from being representable.

REFERENCES

[1] Andréka, H. and Németi, I., Formulas and ultraproducts in categories, *Beiträge zur Algebra und Geometrie* 8, *Wiss. Beitr. Martin-Luther-Univ. Halle—Wittenberg. M* **12** (1979), 133—151. *MR* **81m**: 18004.

[2] Andréka, H. and Németi, I., A general axiomatizability theorem formulated in terms of cone-injective subcategories, *Universal Algebra* (Proc. Coll. Esztergom, 1977), Colloq. Math. Soc. J. Bolyai, **29**, North-Holland, 1981, 13—35.

[3] Andréka, H. and Németi, I., Łos lemma holds in every category, *Studia Sci. Math. Hungar.* **13** (1978), 361—376. *MR* **82i**: 03049.

[4] Chang, C. C. and Keisler, H. J., *Model theory*, Second edition, Studies in Logic and the Foundations of Mathematics, 73, North-Holland Publishing Co., Amsterdam—New York—Oxford, 1977. *MR* **58** #27177.

[5] Fakir, S. and Haddad, L., Objets cohérents et ultraproduits dans les catégories, *J. Algebra* **21** (1972), 410—421. *MR* **47** #304.

[6] Herrlich, H. and Strecker, G. E., *Category theory an introduction,* Allyn and Bacon Series in Advanced Mathematics, Allyn and Bacon Inc., Boston, Mass., 1973. *MR* **50** #2284.

[7] MacLane, S., *Categories for the working mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York—Berlin, 1971. *MR* **50** #7275.

[8] Németi, I. and Sain, I., Cone-implicational subcategories and some Birkhoff-type theorems, *Universal Algebra* (Proc. Coll. Esztergom, 1977), Colloq. Math. Soc. J. Bolyai, **29**, North-Holland, 1981, 535—578.

[9] Okhuma, T., Ultrapowers in categories, *Yokohama Math. J.* **14** (1966), 17—37.

[10] Henkin, L., Monk, J. D., Tarski, A., Andréka, H. and Németi, I., *Cylindric set algebras*, Lecture Notes in Mathematics, 883, Springer-Verlag, Berlin, 1981.

[11] Németi, I., Connections between cylindric algebras and initial algebra semantics of CF languages, *Mathematical Logic in Computer Science* (Proc., Salgótarján, 1978), Colloq. Math. Soc. J. Bolyai, **26**, North-Holland, Amsterdam, 1981, 561—605.

[12] Sain, I., Finitary logics of infinitary structures are compact, *Abstracts Amer. Math. Soc.*, Apr. 1982, Issue 17, Vol. 3, No. 3, p. 252.

[13] Guitart, R. and Lair, C., Calcul syntaxique des modèles et calcul des formules internes, *Diagrammes* **4**, Dec. 1980, Paris, 106 pp.

[14] Guitart, R., Mixed limits as representable functors, Personal communication.

[15] ANDRÉKA, H., MAKAI, JR. E., MÁRKI, L. and NÉMETI, I., Reduced products in categories, *Contributions to general algebra* (Proc. Klagenfurt, Conf. Klagenfurt, 1978), Verlag Johannes Heyn, Klagenfurt, 1979, 25—45. *MR* **80h**: 03049.
[16] ADÁMEK, J., Solution of a problem on reduced products in categories, Personal communication.
[17] ROSICKY, J., Personal communication.
[18] NÉMETI, I., Dynamic algebras of programs, *Fundamentals of Computation Theory '81* (Proc., Szeged, 1981), Lecture Notes in Computer Science, 117, Springer-Verlag, Berlin, 1981, 281—290.
[19] SAIN, I., Applicability of category theoretic notions of ultraproducts, *Mat. Lapok* **31** (1978—1983), 143—168 (in Hungarian).

(Received December 2, 1981)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 319—322.

CHEBYSHEV APPROXIMATION WITH WEIGHTS LARGE ON NODES

CHARLES B. DUNHAM

Discrete Chebyshev approximation with interpolation by varisolvent families is the limit of weighted Chebyshev approximation with weights large on the interpolating points and one off these, provided the best approximation is of maximum degree. This suggests use of an algorithm for weighted discrete Chebyshev approximation, for example the Remez algorithm [9], to get near best approximations.

Let X be a finite subset of the closed interval $[\alpha, \beta]$. For $g \in C(X)$ define

$$\|g\| = \max \{|g(x)|: x \in X\}.$$

Let F be an approximating family unisolvent of variable degree on $[\alpha, \beta]$ in the sense of Rice [7; 8, Chapter 7]. Let f be a fixed element of $C(X)$ which we wish to approximate. Let $I = \{x_1, \ldots, x_n\}$ be fixed distinct points of X and $Q = \{A: F(A, x_i) = f(x_i), i = 1, \ldots, n\}$. The problem of approximation with interpolation is to choose a parameter A^* to minimize $e(A) = \|f - F(A,.)\|$ over $A \in Q$. Such a parameter A^* is called best and $F(A^*,.)$ is called a best interpolating approximation to f.

We assume that the phenomenon of a constant error curve does not occur.

The best known varisolvent families are the families of Meinardus and Schwedt [6, 141 ff] with the local Haar condition, proven by Barrar and Loeb [2].

Barrar and Loeb have characterized best approximation on $[\alpha, \beta]$ in [1]. The same arguments give a characterization for best approximation on X.

DEFINITION.

$$\Pi(x) = \prod_{i=1}^{n} (x - x_i).$$

THEOREM. *Let F be of degree m at $A \in Q$, $m \geqq n$. $F(A,.)$ is a best interpolating approximation to f on X if and only if $\operatorname{sgn}(\Pi) * (f - F(A),.))$ alternates $m-n$ times on X. Best approximations are unique.*

A possible approach to determining an approximation close to the best is ot approximate with respect to a multiplicative weight function [3], which is large on I and one off I.

LEMMA 1. *Let $F(A, \cdot) \not\equiv f$ be the best interpolating approximation to f and F be of degree m at A. Let $Y = \{y_0, \ldots, y_{m-n}\}$ be an alternant of $\operatorname{sgn}(\Pi) * (f - F(A,.))$. Let $V = \{v_0, \ldots, v_m\}$ be an ordered set consisting of the union of $\{x_1, \ldots, x_n\}$ and Y.*

1980 *Mathematics Subject Classification.* Primary 41A50.
Key words and phrases. Uniform approximation, interpolation, varisolvent, weight, node.

Let w be 1 on Y and $>(\|f-F(A,\cdot)\|+\delta)/\delta$ on $\{x_1, \ldots, x_n\}$. Let $\|w(f-F(B,.))\| \leqq \|f-F(A,.)\|+\delta$. Then

$$(*)\qquad (-1)^i[F(B, v_i)-F(A, v_i)]\operatorname{sgn}[f(y_0)-F(A, y_0)] \geqq -\delta, \quad i=0, \ldots, m.$$

PROOF.

$$\operatorname{sgn}(\Pi(y_0))(-1)^i[f(y_i)-F(A, y_i)]\operatorname{sgn}(f(y_0)-F(A, y_0)) = \|f-F(A,.)\|.$$

Assume without loss of generality $\Pi(y_0)>0$ then for $v_i \in Y$,

$$(1)\qquad (-1)^i[f(v_i)-F(A, v_i)]\operatorname{sgn}(f(y_0)-F(A, y_0)) = \|f-F(A,.)\|,$$

$$(2)\qquad f(x_i)-F(A, x_i)=0,$$

$$(3)\qquad (-1)^i w(v_i)(f(v_i)-F(B, v_i))\operatorname{sgn}(f(y_0)-F(A, y_0)) \leqq \leqq \|w(f-F(B,.))\|,$$

$$(4)\qquad f(x_i)-F(B, x_i) \leqq \|w(f-F(B,.))\|/w(x_i).$$

For $v_i \in Y$, we have, subtracting (1) from (3)

$$(-1)^i(F(B, v_i)-F(A, v_i))\operatorname{sgn}(f(y_0)-F(A, y_0)) \geqq \geqq \|f-F(A,.)\|-\|w(f-F(B,.))\| \geqq -\delta.$$

For $v_i \in \{x_1, \ldots, x_n\}$, we have from (2, 4)

$$(-1)^i(F(B, v_i)-F(A, v_i))\operatorname{sgn}(f(y_0)-F(A, y_0)) \geqq -\|w(f-F(B,.))\|/w(x_i) \geqq$$

$$\geqq -\frac{[\|f-F(A,.)\|+\delta]\delta}{\|f-F(A,.)\|+\delta} = -\delta.$$

LEMMA 2. *For given δ there is $\eta(\delta)$ such that $(*)$ implies $\|F(A,.)-F(B,.)\|_{[\alpha,\beta]} < \eta$ and $\eta \to 0$ as $\delta \to 0$.*

LEMMA 3. *Let F be unisolvent of degree m at A^k, $k=0, \ldots$ and $F(A^k,.)$ converge pointwise to $F(A^0,.)$ on m distinct points, then $F(A^k,.) \to F(A^0,.)$ uniformly on $[\alpha, \beta]$.*

Generalizations of the lemmas are proven by the author in [4].

THEOREM. *Let $w_k(x_i) \to \infty$ for $i=1, \ldots, n$ and $w_k(x)=1$ for $x \neq x_i$. Let the best interpolating approximation $F(A,.)$ be of maximum degree (m). For k sufficiently large there is a best approximation $F(A^k,.)$ with respect to weight w_k and $F(A^k,.) \to F(A,.)$ uniformly on $[\alpha, \beta]$ and X.*

PROOF. Let $v_0, \ldots, v_m$ be as in Lemma 1. By definition of solvency of degree m at A there exists $\gamma>0$ such that if $|y_k-F(A, v_k)|<\gamma$, $k=1, \ldots, m$, then there exists a parameter B satisfying

$$(5)\qquad F(B, v_k) = y_k \qquad k=1, \ldots, m.$$

Using property Z and maximality of m, it is easily seen that F is unisolvent of degree m at any such B and hence B is completely determined by (5). Choose k so large that the associated δ gives $\eta(\delta)<\gamma/2$, then by Lemmas 1 and 2, we have $\|F(A,.)-$

$-F(B,.)\|_{[\alpha,\beta]}<\gamma/2$. Let w denote w_k with associated δ giving $\eta(\delta)<\gamma/2$. Now let $\|w(f-F(B_j,.))\|$ be a decreasing sequence with limit

$$\varrho(w)=\inf\{\|w(f-F(B,.))\|:\ B\in P\},$$

then for all j sufficiently large $\|F(A,.)-F(B_j,.)\|_{[\alpha,\beta]}<\gamma/2$ by Lemma 1 and 2. The m-tuples of values at the points $v_1, \ldots, v_m$ form, therefore, a bounded sequence with subsequence converging to an accumulation point $(y_1, \ldots, y_m)$ which determines a parameter B at which F is unisolvent of degree m. Using Lemma 3, we can show that for all $x\in X$, $|w(x)(f(x)-F(B,x))|\leqq\varrho(w)$ and so $F(B,.)$ is a best approximation with respect to weight w. The existence part is proven. Uniform convergence follows from Lemmas 1 and 2.

The conclusion of the theorem need not hold if the best interpolating approximation is not of maximum degree.

EXAMPLE. Let $[\alpha,\beta]=[0,1]$ and $I=\{0\}$. Let X contain at least three points. Approximate $f, f(0)=0$, $f\geqq 0$, $f\not\equiv 0$, by $R_1^0[0,1]$. As zero is the only approximation interpolating f at zero, zero is the unique best interpolating approximation. As $f-0\geqq 0$, $w(f-0)$ does not alternate for any positive weight function w, hence 0 is not best with respect to w. Thus if a best weighted approximation $F(A(w),.)$ exists, it must be nonzero and hence of degree 2. If so, $w(f-F(A(w),.))$ alternates two times on X. It follows that for $w=1$ off $\{0\}$, $f-F(A(w),.)$ is not close to $f-0$, hence $F(A(w),.)$ is not close to zero. Exactly the same thing happens if we approximate by $F(A,x)=a_1\exp(a_2x)$.

Existence of $F(A(w),.)$ is not guaranteed in general. However, if we require that wf attain its supremum on X at an interior point, then no limit of approximations which is not an approximation (namely, the functions zero except on one endpoint) can be better, and so a best approximation with respect to w must exist.

The case of approximation on the interval, that is, when $X=[\alpha,\beta]$, is also of interest. The characterization of Barrar and Loeb was for that case. The remaining lemmas and theorem of this paper also apply. It should be noted that w_k fails to be continuous at $\{x_1, \ldots, x_n\}$ but is upper semi-continuous. The alternating characterization of [5] applies, but the behaviour of the Remez algorithm with respect to non-continuous weights is completely open. The example of non-convergence when the best approximation is not of maximum degree applies.

An open question is whether a sequence of *continuous* weights exists which gives approximation with interpolation in the limit.

REFERENCES

[1] BARRAR, R. and LOEB, H. L., Best non-linear uniform approximation with interpolation, *Arch. Rational Mech. Anal.* **33** (1969), 231—237. *MR* **39** #1865.

[2] BARRAR, R. and LOEB, H. L., On the continuity of the nonlinear Tschebyscheff operator, *Pacific J. Math.* **32** (1970), 593—601. *MR* **41** #4071.

[3] DUNHAM, C., Chebyshev approximation with respect to a weight function, *J. Approximation Theory* **2** (1969), 223—232. *MR* **40** #6137.

[4] DUNHAM, C., Alternating Chebyshev approximation, *Trans. Amer. Math. Soc.* **178** (1973), 95—109. *MR* **47** # 7282.

[5] Dunham, C., Alternating Chebyshev approximation with a non-continuous weight function, *Canad. Math. Bull.* **19** (1976), 155—157. *MR* **56** #12728.
[6] Meinardus, G., *Approximation of functions. Theory and numerical methods*, Springer Tracts in Natural Philosophy, Vol. 13, Springer-Verlag, New York, Inc., New York, 1967. *MR* **36** #571.
[7] Rice, J., Tchebycheff approximations by functions unisolvent of variable degree, *Trans. Amer. Math. Soc.* **99** (1961), 298—302. *MR* **25** #374.
[8] Rice, J., *The approximation of functions*, Volume 2, Nonlinear and multivariate theory, Addison-Wesley Publishing Co., Reading, Mass.—London—Don Mills, Ont., 1969. *MR* **39** # 5989.
[9] Dunham, C., Discrete Chebyshev approximation: alternation and the Remez algorithm, *Z. Angew. Math. Mech.* **59** (1979), 326—328. *MR* **82i**: 41031.

(Received December 22, 1981)

DEPARTMENT OF COMPUTER SCIENCE
MIDDLESEX COLLEGE
THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO
N6A 5B7
CANADA

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 323—333.

О МНОЖЕСТВЕ ТОЧЕК РОСТА ФУНКЦИИ РАСПРЕДЕЛЕНИЯ АДДИТИВНОЙ ФУНКЦИИ

Н. М. ТИМОФЕЕВ

Пусть $g(n)$ аддитивная функция, принимающая вещественные значения. Будем говорить, что $g(n)$ имеет предельное распределение, если

$$\frac{1}{x}\sum_{\substack{n\le x\\ g(n)-A(x)\le uB(x)}} 1 \to F(u) \tag{1}$$

при $x\to\infty$, во всех точках непрерывности функции распределения $F(u)$. Здесь $A(x)$, $B(x)$ функции, принимающие вещественные значения и $A(x)=A([x])$, $B(x)=B([x])$. Без ограничения общности (см. [1]), можно считать $B(x)>0$. Функции распределения, которые являются предельными для аддитивных функций обладают рядом свойств. Если $B(x)=1$, то для существования предельного распределения необходимо и достаточно, чтобы существовало число «b» такое, что ряд

$$\sum_p \frac{1}{p}\|g(p)-b\log p\|^2 \tag{2}$$

сходится. Здесь $\|u\|=u$ при $|u|\le 1$, $\|u\|=1$, если $u>1$, -1, если $u<-1$. Характеристическая функция предельного закона $F(u)$ в этом случае равна

$$\frac{1}{1+i\xi b}\prod_p\left(1+\sum_{r=1}^{\infty}\frac{1}{p^r}e^{i\xi(g(p^r)-b\log p^r)}\right)\left(1-\frac{1}{p}\right)e^{-i\xi\frac{\|g(p)-b\log p\|}{p}}. \tag{3}$$

Из этого представления следует, что функция распределения чистого типа, то есть либо сингулярная, либо дискретная, либо абсолютно непрерывная. Если $B(x)\to\infty$ при $x\to\infty$, то $F(u)$ (см. [2]) так же либо сингулярная, либо абсолютно непрерывная. Здесь будет продолжено изучение предельных функций распределения.

Точку a будем называть точкой роста функции распределения $F(u)$, если для любого $\varepsilon>0$ имеем $F(a+\varepsilon)-F(a-\varepsilon)<0$.

Множество точек роста функции распределения обозначим S_F. Имеет место следующая

1980 *Mathematics Subject Classification.* Primary 10K20.

Key words and phrases. Additive function, limit law, point of growth of the limit law for additive functions.

Теорема 1. *Если аддитивная функция $g(n)$ имеет нетривиальное предельное распределение и $B(x)\to\infty$ при $x\to\infty$, то S_F — промежуток, конечный или бесконечный.*

Если $B(x)=1$, а ряд (2) сходится с $b\neq 0$ или $g(n)$ вполне аддитивная функция, то есть $g(p^\alpha)=\alpha g(p)$, $\alpha=1, 2, \ldots$, и $F(u)$ — непрерывная, то S_F также промежуток.

Заметим, что при $B(x)=1$ можно привести примеры аддитивных функций, для которых S_F не является промежутком. Действительно, положим $g(2^\alpha)=10$ и $g(p^\alpha)=\frac{1}{p^2}$ при $p\neq 2$. Аддитивная функция имеет предельное распределение, так как

$$\sum_p \frac{g^2(p)}{p}<+\infty, \quad \sum_p \frac{g(p)}{p}<+\infty.$$

Причем, в этом случае $A(x)=0$ и $B(x)=1$ и $F(u)$ непрерывная функция распределения, точки роста которой $S_F\subset[0, 1]\cup[10, 11]$, $S_F\cap[0, 1]\neq\emptyset$, $S_F\cap\cap[10, 11]\neq\emptyset$.

Если ряд (2) сходится при $b\neq 0$, то из представления (3) для характеристической функции распределения $F(u)$ следует, что $F(u)$ есть свертка функции распределения $F_1(u)$ с характеристической функцией $\frac{1}{(1+i\xi b)}$ и функции распределения $F_2(u)$ являющейся предельной для аддитивной функции $g(n)-b\log n$. Если $b>0$, то $F_1(u)=\exp\left(\frac{u}{b}\right)$ при $u\leqq 0$ и 1 при $u>0$. Следовательно,

$$F(u)=\frac{1}{b}\int_{-\infty}^{0} F_2(u-x)\exp\left(\frac{x}{b}\right)dx.$$

Из этого представления следует, что $S_F=]-\infty, a]$, где $a=\sup S_{F_2}$. Аналогично устроено множество S_F и при $b<0$.

При доказательстве остальных утверждений теоремы важную роль играет следующее неравенство Эллиота (см. [3]).

Пусть $a_1, a_2, \ldots, a_n, \ldots$ произвольные комплексные числа, тогда

$$\sum_{p\leqq x} p\left|\sum_{\substack{n\leqq x\\ p\| n}} a_n-\frac{1}{p}\sum_{n\leqq x} a_n\right|^2\leqq cx\sum_{n\leqq x}|a_n|^2.$$

Здесь $p\|n$ означает, что p делит n, а p^2 не делит n.

Пусть $a_n=1$, если $\alpha B(x)\leqq g(n)-A(x)\leqq\beta B(x)$ и $a_n=0$ в противном случае. Тогда

$$\sum_{\substack{n\leqq x\\ p\| n}}\frac{1}{p}\left|\frac{p}{x}\sum_{n\leqq x}{}'1-\frac{1}{x}\sum_{n\leqq x}{}'1\right|^2\leqq\frac{c}{x}\sum_{n\leqq x}{}'1. \tag{4}$$

Здесь $\sum'$ означает, что суммирование ведется по n удовлетворяющим условию $\alpha B(x)\leqq g(n)-A(x)\leqq\beta B(x)$.

Если вполне аддитивная функция $g(n)$ имеет предельное распределение с $B(x)=1$ и функция $F(u)$ непрерывная, то (см. [3]) сходится ряд (2) с $b=0$ (случай $b\neq 0$ рассмотрен ранее),

$$A(x)=\sum_{p\leq x}\frac{\|g(p)\|}{p}+d+o(1) \tag{5}$$

и ряд

$$\sum_{g(p)\neq 0}{}' \frac{1}{p} \tag{6}$$

расходится. Из (5) и сходимости ряда (2) следует, что

$$|A(x)-A(y)|\leq \sum_{y<p\leq x}\left|\frac{\|g(p)\|}{p}\right|+o(1)\leq \sqrt{\sum_{p>y}\frac{\|g(p)\|^2}{p}2\log\frac{\log x}{\log y}}+o(1)\to 0$$

$$\text{при}\quad \sqrt{x}\leq y\leq x \quad \text{и} \quad x\to\infty.$$

Поэтому для любого фиксированного p имеем $A(x)-A\left(\frac{x}{p}\right)\to 0$ при $x\to\infty$ и, следовательно, учитывая непрерывность $F(u)$ и полную аддитивность $g(n)$, получаем

$$\frac{p}{x}\sum_{\substack{n\leq x\\ p\| n}}{}' 1=\frac{p}{x}\sum_{\substack{n\leq x\\ \alpha\leq g(n)+g(p)-A(x)\leq\beta}}1-\frac{p}{x}\sum_{\substack{n\leq x/p^2\\ \alpha\leq g(n)+2g(p)-A(x)\leq\beta}}1=$$

$$=F(\beta-g(p))-F(\alpha-g(p))-\frac{1}{p}\left[F(\beta-2g(p))-F(\alpha-2g(p))\right]+o(1)$$

при $x\to\infty$. Отсюда и из (4) следует, что при любом фиксированном y

(7)

$$\sum_{p\leq y}\frac{1}{p}\left|F(\beta-g(p))-F(\alpha-g(p))-\frac{1}{p}\left[F(\beta-2g(p))-F(\alpha-2g(p))\right]-F(\beta)-F(\alpha)\right|^2\leq$$

$$\leq c\big(F(\beta)-F(\alpha)\big).$$

Предположим, что S_F не является промежутком. Тогда найдется интервал $]a, b[$, на котором $F(u)=\text{const}$, а точки a и b является точками роста $F(u)$. Из расходимости ряда (6) и сходимости (2) с $b=0$ вытекает существование $g(p)$ удовлетворяющего неравенству $0<|g(p)|\leq\frac{(b-a)}{4}$. Положим в (7) $\alpha=$ $=a+|g(p)|$, $\beta=b-|g(p)|$. Тогда из (7), учитывая, что $F(\beta)=F(\alpha)$ и $F(\beta-g(p))=$ $=F(\alpha-g(p))$, $F(a+|g(p)|-2g(p))-F(b-|g(p)|-2g(p))=0$, что невозможно, так как a и b точки роста функции $F(u)$. Таким образом, если $B(x)=1$ и $g(n)$ вполне аддитивная функция, то S_F промежуток.

Пусть $g(n)$ имеет предельное распределение с $B(x)\to\infty$ и таким, что $\frac{B(x^u)}{B(x)}\to 1$ для любого $0<u<1$. Тогда (см. [3], [4]) существуют число b неубывающая функция $L(u)$, $L(\pm\infty)=\lim\limits_{u\to\pm\infty} L(u)$, такие, что

$$(8)\qquad \sum_{\substack{p\leqq x\\ g(p)-b\log p\leqq uB(x)}} \left\|\frac{g(p)-b\log p}{B(x)}\right\|^2 \frac{1}{p} \to L(u) \quad \text{при} \quad x\to\infty$$

во всех точках непрерывности $L(u)$ и $A(x)=b\log x+A^*(x)+dB(x)$, где $A^*(x^u)-A^*(x)=o(B(x))$ равномерно по u на любом $[\alpha,\beta]\subset]0,+\infty[$. Заметим, что и $\dfrac{B(x^u)}{B(x)}\to 1$ равномерно по u на $[\alpha,\beta]$. Возьмем в неравенстве (4) в качестве $]\alpha,\beta[$ интервал, на котором $F(u)=\text{const}$, а α и β точки роста $F(u)$. Если S не является промежутком, то такой интервал существует. Тогда, учитывая, что

$$\frac{p}{x}\sum_{\substack{n\leqq x\\ p\| n}}{}' 1 = \frac{p}{x} \sum_{\substack{n\leqq x\\ g(n)-A\left(\frac{x}{p}\right)\leqq B(x)\left(u-\frac{-b\log p+g(p)-A^*(x)+A^*(x/p)}{B(x)}\right)}} 1+O\left(\frac{1}{p}\right)=$$

$$=F\left(\frac{B(x)}{B(x/p)}\left(u-\frac{g(p)-b\log p}{B(x)}\right)\right)+\varepsilon(x,p)+O\left(\frac{1}{p}\right),$$

где $\varepsilon(x,p)\to 0$ при $x\to\infty$ равномерно по p при $p\leqq\sqrt{x}$, $\dfrac{B(x)}{B(x/p)}\to 1$ равномерно по $p\leqq\sqrt{x}$. $F(u)$ — непрерывная функция, получим для любой функции $l(x)\to\infty$

(9)

$$\sum_{l(x)\leqq p\leqq\sqrt{x}} \left|F\left(\beta-\frac{g(p)-b\log p}{B(x)}\right)-F\left(\alpha-\frac{g(p)-b\log p}{B(x)}\right)+o(1)\right|^2 \frac{1}{p}=o(1).$$

Характеристическая функция закона $F(u)$ в этом случае (см. [4]) равна

$$\exp\left(\int\limits_{-\infty}^{+\infty} (e^{i\xi u}-1-i\xi\|u\|)\frac{1}{\|u\|^2}\,dL(u)+i\xi d\right),$$

где подинтегральная функция при $u=0$ равна $-\dfrac{\xi^2}{2}$. Следовательно, если $L(u)=A>0$ при $u>0$ и $L(u)=0$ при $u<0$, то закон $F(u)$ нормальный и $S_F=]-\infty,+\infty[$. Пусть $a\neq 0$ точка роста функции $L(u)$. Покажем, что $]0,a]\subset\subset S_L$. Так как a — точка роста, то

$$\sum_{\substack{p\leqq x\\ |g(p)-b\log p-aB(x)|\leqq\varepsilon B(x)}} \left\|\frac{g(p)-b\log p}{B(x)}\right\|^2 \frac{1}{p} \to M=L(a+\varepsilon)-L(a-\varepsilon)>0$$

при $x\to\infty$. Отсюда вытекает, что при $x\geqq x_0$

$$\sum_{p\leqq x}{}' \frac{1}{p} \geqq \frac{M}{2(|a|+\varepsilon)^2}.$$

Здесь $\sum'$ означает, что p удовлетворяют неравенству

$$(a-\varepsilon)B(x) \leqq g(p)-b\log p \leqq (a+\varepsilon)B(x).$$

Пусть $d>1$ и

$$z(x) = \sup\{y\colon\ y \geqq x,\ B(y) \leqq dB(x)\}.$$

Тогда $B(z+1)>dB(x)$ и, так как $B(z-1)\sim B(z+1)$, более того $B(x^{1+o(1)})\sim \sim B(x)$ при $x\to\infty$ (см. теорему 1 [6]), то $\frac{B(z)}{B(x)}\to d$ при $x\to\infty$. Следовательно, получаем при $x\to\infty$

$$\frac{M}{2(|a|+\varepsilon)^2} \leqq \frac{1}{k} \sum_{\substack{p\leqq z\\ (a-\varepsilon)B(x)\leqq g(p)-b\log p\leqq (a+\varepsilon)B(x)}}' \left\|\frac{g(p)-b\log p}{B(z)}\right\|^2 \frac{1}{p} \leqq$$

$$\leqq \frac{1}{k}\left[L\left(\frac{a}{d}+\frac{2\varepsilon}{d}\right)-L\left(\frac{a}{d}-\frac{2\varepsilon}{d}\right)\right],$$

где $k=\min\left(1, \frac{(|a|-\varepsilon)B(x)}{B(z)}\right)$. Так как $d>1$ любое, отсюда следует, что $]0, a]\subset S_L$. Поэтому учитывая, что $\frac{B(\sqrt{x})}{B(x)}\to 1$ находим, что при $x\geqq x_1$

$$\sum_{\substack{p\leqq\sqrt{x}\\ 0<\delta\leqq\frac{|g(p)-b\log p|}{B(x)}\leqq\frac{\beta-\alpha}{2}}} \frac{1}{p} \geqq \delta_1 > 0.$$

Последнее противоречит соотношению (9), так как α и β точки роста функции $F(u)$.

Для доказательство оставшейся части теоремы 1 нам понадобится результат аналогичный теореме 1 работы [4], в которой доказано, что, если $\frac{B(y(x))}{B(x)}\to 1$ при $x\to\infty$ для некоторой функции $x^\alpha\leqq y(x)\leqq x^\delta$, $1<\alpha<\delta$, и имеет место (1) с нетривиальной функцией распределения $F(u)$, то $\frac{B(x^u)}{B(x)}\to 1$ для любого $u>0$.

Заметим, что если $B(x)$ стремится к бесконечности и монотонная, то дальнейшие рассуждения значительно упростятся.

Теорема 2. *Пусть* $\frac{(g(n)-A(x))}{B(x)}$ *имеет нетривиальное предельное распределение. Если существует функция* $y(x)$ *такая, что* $\frac{B(y(x))}{B(x)}\to 1$ *при* $x\to\infty$ *и* $y(x)\leqq x^\gamma$, *где* $0<\gamma<1$ *фиксированное число, тогда* $\frac{B(x^u)}{B(x)}\to 1$ *при* $x\to\infty$ *для любого* $u>0$.

Доказательство. Можно считать, что $B(x)\to\infty$ при $x\to\infty$. Если $B(x)$ не стремится к бесконечности, то известны (см., например, [3]) необходимые и достаточные условия и в этом случае $B(x)$ имеет конечный предел при $x\to\infty$.

Покажем, что найдется функция $y(x)$ такая, что $\dfrac{B(y(x))}{B(x)}\to 1$ и $\dfrac{\log y(x)}{\log x}\to 0$ при $x\to\infty$. Пусть $y_1(x)=y(x)$, $y_2=y(y_1)$, …, $y_r=y(y_{r-1})$. Из условий теоремы следует, что $\log y_r\leqq\gamma^r\log x$. Следовательно, надо доказать существование $r(x)\to\infty$ такой, что $\dfrac{B(y_r(x))}{B(x)}\to 1$ при $x\to\infty$. Пусть монотонно стремящаяся к нулю функция $\varepsilon(x)$ такова, что $\left|\dfrac{B(y(x))}{B(x)}-1\right|\leqq\varepsilon(x)$ и

$$z(x)=\inf\left\{t\colon t\leqq x,\ B(t)\geqq\sqrt{B(x)}\right\}.$$

Положим $r(x)$ наибольший из r, для которого $y_r(x)\geqq z(x)$ и $r(x)\leqq\leqq\log\dfrac{1}{\varepsilon(z(x))}$. Тогда $y_r(x)\to\infty$ при $x\to\infty$ и

$$\frac{B(y_r(x))}{B(x)}=\frac{B(y_r)}{B(y_{r-1})}\frac{B(y_{r-1})}{B(y_{r-2})}\cdots\frac{B(y_1)}{B(x)}=\left(1+O(\varepsilon(z(x)))\right)^{r(x)}\to 1$$

при $x\to\infty$. Так как для любого фиксированного r $\dfrac{B(y_r(x))}{B(x)}\to 1$ и $r\leqq\log\dfrac{1}{\varepsilon(z)}$, то $r(x)\to\infty$ при $x\to\infty$.

В дальнейшем будем считать, что функция $y(x)$ удовлетворяет условию $\dfrac{\log y(x)}{\log x}\to 0$ при $x\to\infty$. Из предыдущих рассуждений вытекает существование $r(x)\to\infty$ при $x\to\infty$ такой, что $\dfrac{B(y_i(x))}{B(x)}\to 1$ и $\dfrac{\log y_i(x)}{\log y_{i-1}(x)}\to 0$ при $x\to\infty$ для $i\leqq r(x)$.

Из теоремы 1 [6] следует, что

$$\sum_{p\leqq x}\left\|\frac{g(p)-t(x)\log p}{B(x)}\right\|^2\frac{1}{p}\leqq c_3, \tag{11}$$

где $t(x)=o(B(x))$. Так как число промежутков вида $]y_i(x),\ y_{i-1}(x)]$, $i=2, 3, \ldots$, $r(x)$ стремится к ∞ при $x\to\infty$, то из (11) получаем, что существует $i(x)$, $2\leqq\leqq i(x)\leqq r(x)$ такая, что

$$\sum_{y_i<p\leqq y_{i-1}}\left\|\frac{g(p)-t(x)\log p}{B(x)}\right\|^2\frac{1}{p}\to 0\quad\text{при}\quad x\to\infty. \tag{12}$$

Рассмотрим мультипликативную по n функцию $f(n,\xi,t)$,

$$f(p^\alpha,\xi,y_{i-1}(x))=\exp\left(i\xi\left(g(p^\alpha)-t(x)\log p^\alpha\right)/B(x)\right),$$

$f(p^\alpha, \xi, t)=1$, если $t \neq y_{i-1}(x)$. Для нее выполнены условия леммы работы [1], то есть

$$\frac{1}{\log t} \sum_{p \le t} f(p, \xi, t) \frac{\log p}{p} = 1 + O(\varepsilon_1(t)),$$

где $\varepsilon_1(t) \to 0$ при $t \to \infty$. При $t \neq y_{i-1}(x)$ последнее очевидно, при $t = y_{i-1}(x)$ справедливость условия вытекает из неравенства $1 - \cos \xi u \leqq (\xi^2 + 2)\|u\|^2$ и соотношения (12).

Учитывая, что $t(x) = o(B(x))$, и применяя лемму из работы [1], получим

$$\left| \frac{1}{y_{i-1}} \sum_{n \leqq y_{i-1}} \exp\left(i\xi \frac{g(n) - A(y_{i-1})}{B(x)} \right) \right| =$$

$$= \left| \frac{1}{y_{i-1}} \sum_{n \leqq y_{i-1}} \exp\left(i\xi \frac{g(n) - t(x)\log n}{B(x)} \right) + o(1) \right| =$$

$$= \exp\left[- \sum_{p \leqq y_{i-1}} \left(1 - \cos \xi \frac{g(p) - t(x)\log p}{B(x)} \right) \frac{1}{p} \right] + o(1).$$

Левая часть последнего соотношения, учитывая, что $\dfrac{B(y_{i-1})}{B(x)} \to 1$ при $x \to \infty$, стремится к $|\tau(\xi)|$, где $\tau(\xi)$ — характеристическая функция закона $F(u)$.

Таким образом получаем

$$|\tau(\xi)| = \exp\left(- \sum_{p \leqq y_{i-1}} \left(1 - \cos \xi \frac{g(p) - t(x)\log p}{B(x)} \right) \frac{1}{p} \right) + o(1) \tag{13}$$

равномерно по ξ в любой области вида $|\xi| \leqq C$.

Покажем, что для любого $\varepsilon > 0$ при $x > x_0(\varepsilon, v)$ имеем $B(x^v) > (1-\varepsilon) B(x)$, где $0 > v > 1$.

Применим лемму 3 [5].

Лемма 3 [5]. *Пусть $f(n, \xi, x)$ мультипликативная по n функция, $|f(n, \xi, x)| \leqq 1$ и*

$$\sum_{p \leqq x} (1 - \mathrm{Re} f(p, \xi, x)) \frac{1}{p} = O(1).$$

Тогда

$$\frac{1}{x} \sum_{n \leq x} f(n, \xi, x) = \frac{1}{2\pi i} \int_{1-iK}^{1+iK} \frac{e^z}{z} \prod_{p \leq x} \left(1 - \frac{1}{p} e^{-z \frac{\log p}{\log x}} \right) \times$$

$$\times \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r, \xi, x)}{p^r} e^{-z \frac{\log p^r}{\log x}} \right) dz + O\left(\frac{1}{K^\delta} \right),$$

где $\delta > 0$, а K — любое фиксированное.

Лемму 3 применим к $f(n, \xi, x^v) = \exp\left(i\xi \dfrac{g(n) - t(x)\log n}{B(x^v)} \right)$. Справедливость условий леммы вытекает из (11) и неравенства $B(x^v) \geqq c_4 B(x)$, которое следует из неравенства $\max_{y \le x^2} B(y) \le c_1 B(x)$ (см. теорему 1 [6]). Получаем

$$\left|\frac{1}{x^v}\sum_{n\le x^v}\exp\left(i\xi\frac{g(n)-A(x^v)}{B(x^v)}\right)\right| = \left|\frac{1}{x^v}\sum_{n\le x^v}\exp\left(i\xi\frac{g(n)-t(x)\log n}{B(x)}\right)+o(1)\right| =$$

$$= \left|\frac{1}{2\pi i}\int_{1-iK}^{1+iK}\frac{e^z}{z}\exp\left[\sum_{p\le x^v}\left(\exp\left(i\xi\frac{g(p)-t(x)\log p}{B(x^v)}\right)-1\right)\frac{1}{p}e^{-z\frac{\log p}{\log x^v}}\right]dz+O\left(\frac{1}{K^\delta}\right)\right|.$$

Левая часть последнего соотношения стремится к $\tau(\xi)$. Правую часть, учитывая, что $\frac{\log y_{i-1}(x)}{\log x^v}\to 0$, можно представить в следующем виде

$$\exp\left[-\sum_{p\le y_{i-1}}\left(1-\cos\xi\frac{B(y_{i-1})}{B(x^v)}\left(\frac{g(p)-t(x)\log p}{B(y_{i-1})}\right)\right)\frac{1}{p}\right]\times$$

$$\times\left|\frac{1}{2\pi i}\int_{1-ik}^{1+ik}\frac{e^z}{z}\exp\left[\sum_{y_{i-1}<p\le x^v}\left(\exp\left(i\xi\frac{g(p)-t(x)\log p}{B(x^v)}\right)-1\right)\frac{1}{p}e^{-z\frac{\log p}{\log x^v}}\right]dz+O\left(\frac{1}{K^\delta}\right)\right|.$$

Если снова применим лемму 3 [5] к $f_1(n, \xi, x^v)$, где $f_1(p, \xi, x^v) = =\exp\left(i\xi\frac{g(p)-t(x)\log p}{B(x^v)}\right)$ при $y_{i-1}<p\le x^v$ и $f_1(p^\alpha, \xi, x^v)=1$ в противном случае, то получим, учитывая, что $B(y_{i-1}(x))\sim B(x)$, и (13),

$$|\tau(\xi)| = \left|\tau\left(\xi\frac{B(x)}{B(x^v)}\right)\frac{1}{x^v}\sum_{n\le x^v}f_1(n, \xi, x^v)+O\left(\frac{1}{K^\delta}\right)\right| \le$$

$$\le \left|\tau\left(\xi\frac{B(x)}{B(x^v)}\right)\right|+O\left(\frac{1}{K^\delta}\right).$$

Отсюда следует, что $\liminf_{x\to\infty}\frac{B(x^v)}{B(x)}\ge 1$. Напомним, что предельный закон $F(u)$ непрерывный, и поэтому $|\tau(\xi)|<1$ при $\xi\ne 0$.

Покажем, что $B(x)\le B(t)$ для всех $t\in[x^\delta, x]$ при $x\ge x_1$, где $\delta>0$ любое фиксированное. Предположим, что это не так. Тогда найдутся последовательности x_e, $z(x_e)$ такие, что $x_e^\delta\le z(x_e)\le x_e$ и $\frac{B(z(x_e))}{B(x_e)}\to d<1$ при $x_e\to\infty$. Но, как отмечалось выше, $B(x)\sim B(x^{1+o(1)})$ и поэтому можно считать $z(x_e)=x_e^v$, где $\delta\le v<1$. То есть $\liminf_{x\to\infty}\frac{B(x^v)}{B(x)}\le d<1$, что невозможно.

Докажем, наконец, что $\frac{B(x^u)}{B(x)}\to 1$ при $x\to\infty$ равномерно по u на любом $[\alpha, 1]$, $\alpha>0$. Из существования предельного распределения следует (см. лемму 1 [5])

$$(14)\quad \frac{1}{\log x}\sum_{p\le x}\tau\left(\xi\frac{B(x/p)}{B(x)}\right)\exp\left(i\xi\frac{g(p)-A(x)+A(x/p)}{B(x)}\right)\frac{\log p}{p} = \tau(\xi)+o(1)$$

равномерно по ξ при $|\xi| \leqq C$. Условия леммы выполнены, так как из существования предельного распределения следует, что $\max\limits_{y \leqq x^2} B(y) \leqq c_1 B(x)$. Функция $F(u)$ непрерывная, поэтому $|\tau(\xi)| < 1$ при $\xi \neq 0$ и $\tau(0) = 1$. Следовательно, для любого $d > 0$ найдется $\xi_0 < 1$ такое, что $|\tau(\xi_0)| > |\tau(\xi)|$ при всех $\xi \in]\xi_0, d]$. Предположим, что утверждение теоремы 2 несправедливо. Тогда найдутся последовательности $x_e, z(x_e)$ такие, что $x_e^\alpha \leqq z(x_e) \leqq x_e$ и $\dfrac{B(z(x_e))}{B(x_e)} \to d_1 > 1$. Так как $B(x) \sim B(x^{1+o(1)})$, то можно считать, что $z(x_e) = x_e^u$, где $\alpha \leqq u \leqq 1$ и (см. лемму 1 [6]) для любого $\varepsilon > 0$ существует $\delta_1 > 0$ такое, что $\left|\dfrac{B(x^t) - B(x^u)}{B(x)}\right| < \varepsilon$ при $|t - u| < \delta_1$. Возьмем $\varepsilon < d_1 - 1$, $d > c_1$, где $c_1 B(x) > \max\limits_{y \leqq x^2} B(y)$, и $0 < \xi_0 < 1$. Тогда, учитывая, что $c_1 B(x) \geqq B(t) \geqq B(x)$ при $t \in]x^\delta, x]$, где $\delta > 0$ любое, получим, что левая часть соотношения (14) не превосходит

$$\frac{1}{\log x_e} \sum_{p \leqq x_e^{1-\delta}} |\tau(\xi_0)| \frac{\log p}{p} - \frac{1}{\log x_e} \sum_{\substack{p \leqq x_e^{1-\delta} \\ x_e^{1-u-\delta_1} < p < x_e^{1-u+\delta_1}}} \frac{\log p}{p} \min_{[\gamma, \sigma]} \left(|\tau(\xi)| - |\tau(\xi_0)|\right) +$$

$$+ O(\delta) \leqq |\tau(\xi_0)| - \delta_1 \min_{[\gamma_1, \sigma]} \left(|\tau(\xi)| - |\tau(\xi_0)|\right) < |\tau(\xi_0)|.$$

Здесь $\gamma_1 = (d_1 - \varepsilon)\xi_0, \sigma = (d_1 + \varepsilon)\xi_0$. Последнее противоречит (14). Теорема 2 доказана.

Продолжение доказательства теоремы 1: Предположим, что S_F не является промежутком. Тогда найдется интервал $]a, b[$ концы которого точки роста $F(u)$, и $F(u) = \text{const}$ на $]a, b[$. Исходя из этого докажем существование функции $y(x)$ удовлетворяющей условиям теоремы 2. Но тогда $\dfrac{B(x^u)}{B(x)} \to 1$ для любого $u > 0$. В этом случае уже доказано, что S_F — промежуток.

Возьмем в неравенстве (4) $\alpha = -\infty$, $\beta = b$. Получим

$$\sum_{p \leqq x} \frac{1}{p} \left|F\left(b(x,p)(b - a(x,p))\right) - F(b) + \varepsilon_2(x, p)\right|^2 \leqq c_5$$

где

$$b(x, p) = \frac{B(x)}{B(x/p)},$$

$$a(x, p) = \frac{\left(g(p) - A(x) + A(x/p)\right)}{B(x)}$$

и $\varepsilon_1(x, p) \to 0$ при $x \to \infty$ равномерно по $p \leqq x^{1-\varepsilon}$, $\varepsilon > 0$ любое. Отсюда имеем

$$\sum_{p \leqq x}' \frac{1}{p} \leqq \frac{2c_5}{|F(b + d_2) - F(b)|^2} = A_1.$$

Здесь $\sum'$ означает, что p пробегает те p, для которых $b(x, p)\left(b - a(x, p)\right) \geqq$

$\geqq b+d_2$, $d_2=\frac{1}{4}(b-a)$. Аналогично получаем

$$\sum_p{}'' \frac{1}{p} \leqq \frac{2c_6}{|F(a-d_2)-F(a)|^2}=A_2,$$

где $\sum''$ означает, что p удовлетворяют условию $b(x,p)(a-a(x,p))\leqq a-d_2$. Выберем $0<\gamma<1$ так, чтобы

$$\sum_{x^{1-\gamma}\leqq p\leqq x} \frac{1}{p} \geqq 3\max(A_1, A_2, 1).$$

Из последних трех неравенств следует, что

$$\sum_{x^{1-\gamma}\leqq p\leqq x}{}''' \frac{1}{p} \geqq \max(A_1, A_2, 1), \tag{15}$$

здесь $\sum'''$ означает, что p удовлетворяет неравенствам

$$a-d_2 \leqq b(x,p)(a-a(x,p)),$$

$$b(x,p)(b-a(x,p)) \leqq b+d_2.$$

Возьмем в неравенстве (4) $\alpha=a$, $\beta=b$. Получим

$$\sum_{x^{1-\gamma}\leqq p\leqq x} \frac{1}{p}\left|F(b(x,p)(b-a(x,p)))-F(b(x,p)(a-a(x,p)))\right|^2=\delta(x)\to 0 \tag{16}$$

при $x\to\infty$.

Пусть $\left|\frac{B(y(x))}{B(x)}-1\right|=\min\limits_{t\leqq x^\gamma}\left|\frac{B(t)}{B(x)}-1\right|$. Покажем, что $\frac{B(y(x))}{B(x)}\to 1$ при $x\to\infty$. Предположим, что

$$\limsup_{x\to\infty}\frac{B(y(x))}{B(x)}=d_3>1$$

и пусть x_e подпоследовательность, по которой $\frac{B(y(x_e))}{B(x_e)}\to d_3$. Положим

$$z(x_e)=\inf\{t\colon t\leqq y(x_e),\ B(t)\geqq B(x_e)\}.$$

Тогда, так как $B(x)\sim B(x^{1+o(1)})$, то

$$\lim_{x_e\to\infty}\frac{B(z(x_e))}{B(x_e)}=1,$$

что противоречит определению $y(x)$. Следовательно,

$$\limsup_{x\to\infty}\frac{B(y(x))}{B(x)}=1.$$

Предположим, что $\liminf\limits_{x\to\infty}\frac{B(y(x))}{B(x)}=d_4<1$ и x_m последовательность, по которой $\frac{B(y(x_m))}{B(x_m)} \to d_4$ при $x_m \to \infty$. Тогда $\frac{B(t)}{B(x_m)} \leqq d_5$ при всех $t \leqq x_m^\gamma$, где $d_4 < d_5 < 1$. Действительно, из определения $y(x)$ следует, что либо $B(t) \leqq B(y(x_m))$, либо $B(t) \geqq B(x^m)$ при $t \leqq x_m^\gamma$. Но второй случай невозможен, так как, положив $z(x_m) = \inf \{t\colon t \leqq x_m^\gamma,\ B(t) \geqq B(x_m)\}$, опять придем к противоречию. Следовательно, $b(x_m, p) \geqq \frac{1}{d_5} > 1$ при всех $p \geqq x_m^{1-\gamma}$.

Интервал $]b(x_m, p)(a - a(x_m, p)),\ b(x_m, p)(b - a(x_m, p))[$ для $p \geqq x_m^{1-\gamma}$ будет иметь длину не меньше $\frac{1}{d_5}(b-a) > b-a$. Отсюда, учитывая (16), получаем

$$\sum_{x_m^{1-\gamma} \leqq p \leqq x_m}'' \frac{1}{p} \leqq$$

$$\leqq \frac{\delta(x_m)}{\left|F\left(b + \left(\frac{1}{d_5} - 1\right)(b-a)\right) - F(b)\right|^2} + \frac{\delta(x_m)}{\left|F\left(a - \left(\frac{1}{d_5} - 1\right)(b-a)\right) - F(a)\right|^2}$$

при $x_m \to \infty$. Последнее противоречит (15). Следовательно, $\lim_{x \to \infty} \frac{B(y(x))}{B(x)} = 1$. Теорема 1 доказана.

ЛИТЕРАТУРА

[1] Левин, Б. В. и Тимофеев, Н. М., Несколько интегральных предельных теорем для аддитивных функций, *Литовский матем. сборник* **16** (1976), № 4, 133—147, 254. *MR* **54** # 12701.

[2] Timofeev, N. M., On limit laws for additive functions, *Topics in classical number theory* (Proc. Colloq. Budapest, Hungary, 1981), ed. by G. Halász, Colloq. Math. Soc. J. Bolyai, Vol. 34, North-Holland Publ. Co., Amsterdam—New York, 1984, 1553—1570.

[3] Elliott, Peter D. T. A., *Probabilistic number theory I. Mean value theorems*, Die Grundlehren der mathematischen Wissenschaften, Bd. 239, Springer-Verlag, New York—Heidelberg, 1979. *MR* **82h**: 10002a; II. *Central limit theorems*, Die Grundlehren der mathematischen Wissenschaften, Bd. 240, Springer-Verlag, New York—Heidelberg, 1980. *MR* **82h**: 10002b.

[4] Тимофеев, Н. М., Распределение значений аддитивных функций, Исследование по теории чисел, выпуск 7, Издательство Саратовский Университет, 1978.

[5] Levin, B. V. and Timofeev, N. M., On the distribution of values of additive functions, *Acta Arith.* **26** (1974/75), 333—364. *MR* **52** # 3099.

[6] Тимофеев, Н. М., О сходимости к разрывным распределениям, *Литовский матем. сборник* **22** (1982), No. 1, 159—171.

(Поступила 3-его января 1982 г.)

ВЛАДИМИРСКИЙ ГОСУДАРСТВЕННЫЙ
ПЕДАГОГИЧЕСКИЙ ИНСТИТУТ
МАТЕМАТИЧЕСКИЙ ФАКУЛЬТЕТ
ПР. СТРОИТЕЛЕЙ, Д. 11
SU—600 024 ВЛАДИМИР 24
SOVIET UNION

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 335—341.

AN EXTREMAL PROBLEM CONCERNING KNESER'S CONJECTURE

Z. FÜREDI

Abstract

It is proved that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are k-uniform, intersecting set-systems over an n-element set ($F\cap F'\neq 0$ for all $F, F'\in\mathcal{F}_i$, $i=1,2$) and $n>6k$ then $|\mathcal{F}_1\cup\mathcal{F}_2|\leq\binom{n-1}{k-1}+\binom{n-2}{k-1}$. It is to be expected that the same holds for all $n\geqq 2k+2$.

1. Introduction

The well-known Kneser Conjecture [12], proved by Lovász [14] and Bárány [1], is equivalent to the following assertion. If $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t$ are k-uniform, intersecting set-systems over an n-element set X (i.e. $|F|=k$ for all $F\in\mathcal{F}_i$ and $F\cap F'\neq\emptyset$ for all $F, F'\in\mathcal{F}_i (1\leqq i\leqq t)$) and $n\geqq t+2k-1$ then

$$|\cup\mathcal{F}_i|<\binom{n}{k}. \tag{1}$$

In other words (1) means that Kneser's graph K^k_{t+2k-1}, obtained by connecting two k-element subsets of X whenever they have empty intersection, cannot be coloured by t colours. So the chromatic number $\chi(K^k_{t+2k-1})=t+1$.

However, Lovász's and Bárány's nice proofs of (1), even Schrijver's [16] interesting generalization, which use deep geometrical tools, do not tell the true order of magnitude of the left-hand side of (1). This geometrical method does not seem suitable for estimating the distribution of sizes of colour classes of K^k_{t+2k-1}. The determination of $\max|\cup\mathcal{F}_i|$ is in fact a problem belonging to extremal hypergraph theory. More than 10 years ago P. Erdős suggested [6] to attack the Kneser Conjecture in this way, however no "real", i.e. hypergraph theoretical proof is known. Essentially this paper is concerned with the case $t=2$.

2. Results

Let us set $f(n, k, t)=\max\left\{\left|\bigcup_{i=1}^{t}\mathcal{F}_i\right|: \mathcal{F}_i \text{ is a } k\text{-uniform, intersecting set-system over the } n\text{-element set } X\right\}$. Henceforth we shall assume that the elements of X are the integers from 1 to n. If $n\leqq t+2k-2$ then

$$f(n, k, t)=\binom{n}{k}. \tag{2}$$

1980 *Mathematics Subject Classification.* Primary 05C15; Secondary 05C65, 05A05.
Key words and phrases. Graphs, colourings, hypergraphs.

Indeed, let $\mathcal{F}_i=\{F\subset X\colon |F|=k,\ i\in F\}$ for $1\leqq i\leqq t-1$ and $\mathcal{F}_t=\{F\subset X\colon |F|=k,\ F\subset(X-\{1, \dots, t-1\})\}$. As $|X-\{1, \dots, t-1\}|\leqq 2k-1$, the set-system $\mathcal{F}_t$ is intersecting, too. For greater values of n one can replace $\mathcal{F}_t$ by the set-system whose members have the common point t, hence

$$f(n, k, t) \geqq \binom{n-1}{k-1}+\binom{n-2}{k-1}+\dots+\binom{n-t}{k-1}. \tag{3}$$

Erdős [6] conjectured that here equality holds for all $n\geqq t+2k-1$. This would be a substantial improvement of the Kneser Conjecture. The case $k=1$ is of no interest. It is easy to see that the Erdős conjecture holds for $k=2$.

PROPOSITION 1. *If* $n\geqq t+3$ *then*

$$f(n, 2, t) = \binom{n}{2}-\binom{n-t}{2}. \tag{4}$$

The Erdős—Ko—Rado [8] theorem states that the conjecture is true for $t=1$ and each k:

$$\text{If} \quad n \geqq 2k \quad \text{then} \quad f(n, k, 1) = \binom{n-1}{k-1}. \tag{5}$$

However, A. J. W. Hilton [10] observed that the Erdős conjecture is not true for $k=3$, $t=2$. Here we prove that if $k>2$ and $t>1$ then equality cannot hold in (3) for $n=t+2k-1$, i.e.

PROPOSITION 2. *If* $k\geqq 3$, $t\geqq 2$ *then*

$$f(t+2k-1, k, t) > \sum_{i=1}^{t}\binom{n-i}{k-1}. \tag{6}$$

This disproves the conjecture, but for the smallest admissible value of n, namely $n=t+2k-1$. Nevertheless, we believe that the conjecture is correct for any other value of n:

CONJECTURE. If $n\geqq t+2k$ then $f(n, k, t)=\sum_{i=1}^{t}\binom{n-i}{k-1}$. Here equality holds iff $\bigcap\mathcal{F}_i$ is nonempty for each i.

For n large enough this conjecture is supported by the following result of P. Erdős [5].

$$\textit{If} \quad n > n_0(k, t) \quad \textit{then} \quad f(n, k, t) = \sum_{i=1}^{t}\binom{n-i}{k-1}. \tag{7}$$

His proof gives an exponentially large value for $n_0(k, t)$. Later he, together with Bollobás and Daykin [3], proved that $n_0(k, t)<2k^3t$. (This bound was improved to $ckt\log t$ by P. Frankl and the author (unpublished) but our c is very large.) Below we consider the case $t=2$.

THEOREM. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are k-uniform intersecting set-systems over an n-element underlying set X ($n>6k$) and the cardinality of $\mathcal{F}_1\cup\mathcal{F}_2$ is maximal with respect* *to these assumptions, then there exist* $x_1, x_2 \in X$ *such that* $x_i \in \bigcap \mathscr{F}_i$, *i.e.*,

$$\text{(8)} \qquad \textit{if} \quad n > 6k \quad \textit{then} \quad f(n, k, 2) = \binom{n-1}{k-1} + \binom{n-2}{k-1}.$$

Thus there are several extremal families, but each of them is incident to two points.

REMARK 1. In the theorem the assumption $n > 6k$ can be replaced by $n > r_0(k)$, where r_0 is the smallest number r satisfying (13). The conjectured value of $r_0(k)$ is $3k-3$ (see [9]).

REMARK 2. However, Proposition 2 does not disprove the following nice conjecture of P. Erdős [5]:

$$g(n, k, t) = \max\left\{\sum_{i=1}^{t}\binom{n-i}{k-1};\ \binom{kt+k-1}{k}\right\},$$

where $g(n, k, t) =: \max\{|\mathscr{F}|: \mathscr{F}$ is k-uniform set-system over an n-element set X, and $\nexists$ $F_1, F_2, \dots F_{t+1}$ pairwise disjoint$\}$. This is a theorem of Erdős and Gallai [7] for $k=2$.

REMARK 3. Having learned Hilton's counterexample Erdős modified his conjecture in the following way: If $n \geqq t+2k-1$ then

$$f(n, k, t) < \binom{n-1}{k-1} + \dots + \binom{n-t}{k-1} + \binom{n-t-1}{k-1}.$$

Here the right-hand side is less than $\binom{n}{k}$, so this conjecture would imply (1), too.

3. Proofs

For the proofs of (1), (2), (3), (5) and (7) see the references. The proof of (4) is extremely easy.

LEMMA 1. *If* $\mathscr{F}$ *is a 2-uniform, intersecting set-system, then either* $\bigcap \mathscr{F} \neq \emptyset$ *(i.e.,* $\mathscr{F}$ *is a star) or* $\mathscr{F}$ *is a triangle.* □

This lemma contains the case $t=1$. We use induction on t. If in the case $t>1$ each set-system $\mathscr{F}_1, \dots, \mathscr{F}_t$ is a triangle then

$$|\bigcup \mathscr{F}_i| \leqq \Sigma|\mathscr{F}_i| = 3t < \binom{n}{2} - \binom{n-t}{2}$$

and we are ready. So we can suppose that some $\mathscr{F}_i$ is a star, say $1 \in \bigcap \mathscr{F}_1$. Then we can use the induction hypothesis for the members of $\mathscr{F}_2, \mathscr{F}_3, \dots, \mathscr{F}_t$ lying in $X - \{1\}$. □

The proof of (6) is a construction It is enough to give this construction for the case $t=2$, because for all n, k and t

$$f(n, k, t) \geqq \binom{n-1}{k-1} + f(n-1, k, t-1).$$

So $|X|=n=2k+1$, and let $X=A\cup B$, $|A|=3$, $|B|=2k-2$ and choose some $b\in B$. Let $\mathscr{F}_1=\{F\subset X\colon |F|=k,\ |F\cap A|\geqq 2\}$, and $\mathscr{F}_2=\{F\subset B\colon |F|=k\}\cup\{F\subset X\colon |F|=k,\ |F\cap B|=k-1,\ b\in F\}$. Then $\mathscr{F}_1$ and $\mathscr{F}_2$ are two disjoint set-systems and each of them is intersecting. Moreover

$$|\mathscr{F}_1|+|\mathscr{F}_2| = 3\binom{n-3}{k-2}+\binom{n-3}{k-3}+\binom{n-3}{k}+3\binom{n-4}{k-2} =$$

$$= \binom{n-1}{k-1}+\binom{n-2}{k-1}+\binom{2k-3}{k} > \binom{n-1}{k-1}+\binom{n-2}{k-1}.$$

Proof of the Theorem. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two k-uniform, intersecting set-systems over X, $X=\{1, 2, \dots, n\}$, without common member ($\mathscr{F}_1\cap\mathscr{F}_2=\emptyset$). Suppose that

$$|\mathscr{F}_1|+|\mathscr{F}_2| \geqq \binom{n-1}{k-1}+\binom{n-2}{k-1}.$$

It is enough to show that $\bigcap\mathscr{F}_1\neq\emptyset$. Use the following operation P_{ij} which was first applied by Erdős, Ko and Rado for the proof of the theorem (5). However, here we are using this P_{ij} for ordering two set-systems at the same time. For $1\leqq i<j\leqq n$ and $F\in\mathscr{F}_1\cup\mathscr{F}_2$ we have

$$P_{ij}(F) = \begin{cases} F\cup\{i\}-\{j\} & \text{if } F\in\mathscr{F}_1,\ i\notin F,\ j\in F \text{ and} \\ & \phantom{\text{if }} F\cup\{i\}-\{j\}\notin\mathscr{F}_1 \\ F\cup\{j\}-\{i\} & \text{if } F\in\mathscr{F}_2,\ j\notin F,\ i\in F \text{ and} \\ & \phantom{\text{if }} F\cup\{j\}-\{i\}\notin\mathscr{F}_2 \\ F & \phantom{\text{if }} \text{otherwise.}\end{cases}$$

Let

$$P_{ij}(\mathscr{F}_s) = \{P_{ij}(F)\colon\ F\in\mathscr{F}_s\} \qquad (s = 1, 2).$$

So we have got two new set-systems. Clearly, $|\mathscr{F}_s|=|P_{ij}(\mathscr{F}_s)|$, and with the aid of [8] it can be shown that $P_{ij}(\mathscr{F}_1)$ and $P_{ij}(\mathscr{F}_2)$ are two disjoint, intersecting set-systems, too. Applying each P_{ij} for all $1\leqq i<j\leqq n$ (perhaps several times) after finitely many steps we get two (disjoint and intersecting) set-systems $\mathscr{L}$ and $\mathscr{R}$ which are pushed to the left and to the right, respectively, i.e.,

(9) $\qquad$ If $L\in\mathscr{L}$, $i<j$, $i\notin L$, $j\in L$ then $L\cup\{i\}-\{j\}\in\mathscr{L}$.

Similarly,

If $R\in\mathscr{R}$, $i<j$, $i\in R$, $j\notin R$ then $R-\{i\}\cup\{j\}\in\mathscr{R}$. Denote by

$$\mathscr{L}(1, n) =: \{L\in\mathscr{L}\colon\ 1\in L,\ n\in L\},$$

$$\mathscr{L}(1, \bar{n}) =: \{L\in\mathscr{L}\colon\ 1\in L,\ n\notin L\}, \quad \text{etc.}$$

Now we give an upper estimation for the cardinalities of these parts of $\mathscr{L}$ and $\mathscr{R}$. Clearly,

$$|\mathscr{L}(1, n)|+|\mathscr{R}(1, n)| \leqq \binom{n-2}{k-2}. \tag{10}$$

LEMMA 2. $|\mathcal{L}-\mathcal{L}(1,n)|\leqq\binom{n-2}{k-1}$, *furthermore if here equality holds, then* $1\in\bigcap\mathcal{L}$ (*so* $\mathcal{L}(\bar{1},n)=\emptyset$ *and* $\mathcal{L}(\bar{1},\bar{n})=\emptyset$).

PROOF of Lemma 2. Let us write $|\mathcal{L}(\bar{1},\bar{n})|=m$. The case $m=0$ is trivial, so suppose $m\geqq 1$. Then

$$|\mathcal{L}(\bar{1},n)| \leqq m\frac{k}{n-k+1}. \tag{11}$$

In fact, using (9) to each $L\in\mathcal{L}(\bar{1},n)$ there corresponds $n-k+1$ members of $\mathcal{L}(\bar{1},\bar{n})$, namely, the sets $L-\{n\}\cup\{i\}$ $(2\leqq i\leqq n-1,\ i\notin L)$. Moreover, in this way we can get each member of $\mathcal{L}(\bar{1},\bar{n})$ at most k times.

$$\text{If}\quad n>6k\quad\text{then}\quad m\leqq\binom{n-4}{k-2}=\binom{n-4}{n-k-2}. \tag{12}$$

In fact, the set-system $\mathcal{L}(\bar{1},\bar{n})$ is 2-intersecting, i.e., if $A,B\in\mathcal{L}(\bar{1},\bar{n})$ then $|A\cap B|\geqq 2$. (This follows from the fact that $A\cap B=\{x\}$ implies $(A-\{x\}\cup\{1\})\cap B=\emptyset$). So we can apply the sharper form of the Erdős—Ko—Rado theorem which is due to P. Frankl [9]:

If $\mathcal{H}$ is a k-uniform, 2-intersecting set-system over an r-element set and $r\geqq 6k-1$ then

$$|\mathcal{H}|\leqq\binom{r-2}{k-2}. \tag{13}$$

Now the cardinality of $\mathcal{L}(1,\bar{n})$ will be estimated by an upper bound depending on m. The main idea of the following argument is due to Daykin [4], who gave an original proof for theorem (5) in this way. We need the Kruskal—Katona theorem [11], [13]. We use a weaker but much more simple form of this theorem which is due to Lovász [15]:

If $\mathcal{A}$ is an arbitrary a-uniform set-system and

$$|\mathcal{A}|=\binom{x}{a},\quad\text{then}\quad|\Delta_b(\mathcal{A})|\geqq\binom{x}{b}. \tag{14}$$

Here $a\geqq b>0$ are integers, $x\geqq a$ is a real number,

$$\binom{x}{a}=:\frac{1}{a!}x(x-1)\dots(x-a+1)$$

and

$$\Delta_b(\mathcal{A})=:\{B\colon |B|=b,\ \exists A\in\mathcal{A}\ B\subset A\}.$$

Denote by $\mathcal{A}$ the set of the complements of the members of $\mathcal{L}(\bar{1},\bar{n})$ with respect to $X-\{1,n\}$. $|\mathcal{A}|=m=\binom{x}{n-k-2}$, where $x\geqq n-k-2$. If $|B|=k-1$, $B\subset A\in\mathcal{A}$

then $B\cup\{1\}\notin\mathscr{L}(1,\bar{n})$, because $\mathscr{L}$ is intersecting. Hence

$$\mathscr{L}(1,\bar{n})\leqq\binom{n-2}{k-1}-|\Delta_{k-1}(\mathscr{A})|\leqq\binom{n-2}{k-1}-\binom{x}{k-1}. \tag{15}$$

Now (12) gives that $x\leqq n-4$, so applying (11) we get

$$\begin{gathered}|\mathscr{L}(\bar{1},n)|+|\mathscr{L}(\bar{1},\bar{n})|\leqq m\frac{n+1}{n-k+1}=\\ \binom{x}{n-k-2}\frac{n+1}{n-k+1}\leqq\binom{x}{k-2}\frac{n+1}{n-k+1}<\binom{x}{k-1}.\end{gathered} \tag{16}$$

Finally, summing (15) and (16) we obtain Lemma 2. Moreover here equality can hold only if $m=0$, so $\mathscr{L}(\bar{1},\bar{n})=\emptyset$, $\mathscr{L}(\bar{1},n)=\emptyset$ and $\mathscr{L}(1,\bar{n})=\{L\subset X\colon |L|=k, 1\in L,\ n\notin L\}$. □

Returning to the proof of the Theorem we can get an estimation for $\mathscr{R}$, similarly to Lemma 2. Hence together with (10) we have:

$$|\mathscr{L}|+|\mathscr{R}|\leqq 2\binom{n-2}{k-1}+\binom{n-2}{k-2}=\binom{n-1}{k-1}+\binom{n-2}{k-1}.$$

Finally, it is easy to check that if $\bigcap\mathscr{L}=\{1\}$ and $\bigcap\mathscr{R}=\{n\}$ then before carrying out the operations P_{ij} $\bigcap\mathscr{F}_s\neq\emptyset$, too. □

Added in proof. Further new results are in P. Frankl and Z. Füredi „Extremal problems concerning Kneser-graphs" (submitted to *J. Combinatorial Theory Ser. B*). Especially, it is proved that (8) is true for $n>\frac{1}{2}(3+\sqrt{5})k\sim 2.62k$ and it does not hold for $n<2k+o(\sqrt{k})$ disproving my conjecture.

REFERENCES

[1] Bárány, I., A short proof of Kneser's conjecture, *J. Combin. Theory Ser. A* **25** (1978), 325—326. *MR* **81g**: 05056.

[2] Berge, C., *Graphs and hypergraphs*, North-Holland Mathematical Library, Vol. 6, North-Holland Publ. Co., Amsterdam—London; American Elsevier Publ. Co., Inc., New York, 1973. *MR* **50** #9640.

[3] Bollobás, B., Daykin, D. E. and Erdős, P., Sets of independent edges of a hypergraph, *Quart. J. Math. Oxford Ser.* (2) **27** (1976), 25—32. *MR* **54** #159.

[4] Daykin, D. E., A simple proof of the Kruskal—Katona theorem, *J. Combinatorial Theory Ser. A* **17** (1974), 252—253. *MR* **54** #4994.

[5] Erdős, P., A problem on independent r-tuples, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **8** (1965), 93—95. *MR* **41** #5223. *Zbl.* **136**, 213.

[6] Erdős, P., Problems and results on finite and infinite combinatorial analysis, *Infinite and finite sets* (Colloq. Keszthely, 1973), Vol. I, Colloq. Math. Soc. J. Bolyai, **10**, North-Holland, Amsterdam, 1975, 403—424. *MR* **52** #10438.

[7] Erdős, P. and Gallai, T., On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337—356. *MR* **22** #5591.

[8] Erdős, P., Ko, C. and Rado, R., Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser.* (2) **12** (1961), 313—320. *MR* **25** #3839.

[9] Frankl, P., The Erdős—Ko—Rado theorem is true for $n=ckt$, *Combinatorics* (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, Colloq. Math. Soc. János Bolyai, Vol. 18, Ed. by A. Hajnal and V.T. Sós, North-Holland, Amsterdam, 1978, 365—375. *MR* **80c**: 05014.
[10] Hilton, A. J. W., unpublished, see [6].
[11] Katona, G. O. H., A theorem of finite sets, *Theory of graphs* (Proc. Colloq., Tihany, 1966), Ed. by P. Erdős and G. Katona, Academic Press, New York, 1968, 187—207. *MR* **45** #76.
[12] Kneser, M., Aufgabe 300, *Jber. Deutsch. Math. Ver.* **58** (1955).

(Received January 15, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 343—369.

IRREGULARITIES IN THE DISTRIBUTION OF PRIME IDEALS II

SZILÁRD GYÖRGY RÉVÉSZ

1. Denote by K an algebraic number field, by n and Δ the degree and the discriminant of it. Let $F(m)$ be the number of ideals in K with norm m, and

$$G(m) = \sum_{\substack{P,k \\ NP^k=m}} \frac{\log m}{k}, \tag{1.1}$$

where P runs over the prime ideals of K, NI is the norm of the ideal I, and k runs over the positive integers. Denote by $\zeta_K(s)$ the Dedekind zeta function of K. We have the absolutely convergent expansions

$$\zeta_K(s) = \sum_{m=1}^{\infty} \frac{F(m)}{m^s} \quad (\sigma > 1), \tag{1.2}$$

and

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{m=1}^{\infty} \frac{G(m)}{m^s} \quad (\sigma > 1). \tag{1.3}$$

Let us denote by $\Delta_K(x)$ the remainder term in the prime ideal theorem:

$$\Delta_K(x) := \Psi_K(x) - x = \sum_{m \leqq x} G(m) - x. \tag{1.4}$$

The aim of this paper is to investigate the connection between the domain in which $\zeta_K(s)$ does not vanish and the oscillation of $\Delta_K(x)$. In this connection W. Staś and K. Wiertelak [18] proved the following theorems:

THEOREM (W. Staś—K. Wiertelak). *Suppose that* $\zeta_K(s) \neq 0$ *in the domain*

$$\sigma > 1 - c_K \eta(|t|) \quad c_K \leqq 1 \tag{1.5}$$

where c_K is a constant[1] *depending on K, and $\eta(t)$ is for $t \geqq 0$ a decreasing function,*

1980 *Mathematics Subject Classification.* Primary 12A70; Secondary 10H22.
Key words and phrases. Zeros of the Dedekind zeta-functions, oscillation of the remainder term in the prime ideal formula.

[1] c always denote an explicitly calculable effective constant, which is absolute if it has only an integer index, and depends on some object only when this object is denoted in the index of c.

having a continuous derivative $\eta'(t)$ and satisfying the following conditions:

$$0 < \eta(t) \leqq \frac{1}{2}, \tag{1.6}$$

$$\eta'(t) \to 0 \quad as \quad t \to \infty, \tag{1.7}$$

$$\frac{1}{\eta(t)} = O(\log t) \quad as \quad t \to \infty. \tag{1.8}$$

Let α be a fixed number in $0<\alpha<1$, and

$$\omega(x) = \min_{t \geqq 1} (\eta(t) \log x + \log t). \tag{1.9}$$

Then for $x \to \infty$

$$|\Delta_k(x)| < c_{\alpha,\eta} \left\{ \frac{n^2 \log^2(|\Delta|+1)}{c_K^2} x \exp\left(-\frac{c_K \alpha}{2} \omega(x)\right) \right\}. \tag{1.10}$$

Theorem (W. Staś—K. Wiertelak). *Let $\eta(t)$ be a function which satisfies the conditions of the preceding theorem and also the condition*

$$\eta(t) \leqq c_1 \quad for \quad |t| > c_2 \tag{1.11}$$

where c_1 is a sufficiently small constant and $\omega(x)$ is defined as in (1.9). *Suppose further that* (1.10) *holds. Then $\zeta_K(s) \neq 0$ in the domain*

$$\sigma > 1 - \frac{\log t}{400 \log \left[\omega^{-1}\left(\frac{4}{\alpha c_0} \log t\right)\right]} \tag{1.12}$$

$$t > \max\left\{c_3, \left(\frac{n}{c_K} \log(|\Delta|+1)\right)^{10}, |\Delta|+1, \eta^{-1}(e^{-n^2})\right\},$$

where ω^{-1} and η^{-1} denotes the inverse functions for ω and η.

2. We have to remark, that in the special case $K=\mathbf{Q}$, where $\Delta(x)$ is the remainder of the prime number theorem and $\zeta(s)$ is the Riemann zeta function, a long development was done from the time of the first proof of the prime number theorem in 1896. A general theorem of Ingham ([4], Theorem 22), which uses first the functions η and ω with the definitions and properties listed above, handle the problem of obtaining an estimate for $\Delta(x)$ from a general zero-free domain of $\zeta(s)$. In the other direction E. Schmidt [14] and J. E. Littlewood [9] began investigations, and a decisive step was done by P. Turán (see [19], p. 150), whose power-sum method played an essential role in the further improvement reached by Staś [17]. Finally, J. Pintz [12] could strengthen Ingham's result to such an extent, that it was optimal (apart from constants and ε in the exponent), and he also could prove its optimality by giving a converse of it, which is naturally also optimal and so essentially settles the question in this important special case. His theorems sounds as follows:

THEOREM (J. Pintz). *Suppose that* $\zeta(s)\neq 0$ *in the domain*

$$\sigma > 1-\eta(|t|) \quad \left(0 < \eta(t) \leqq \frac{1}{2}\right), \tag{2.1}$$

where $\eta(t)$ *is a continuous decreasing function for* $t\geqq 0$. *Let* $0<\varepsilon<1$ *be fixed and* $\omega(x)$ *as in* (1.9). *Then we have*

$$\Delta(x) = O\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right). \tag{2.2}$$

THEOREM (J. Pintz). *Suppose that* $\zeta(s)$ *has an infinity of zeros in the domain* (2.1) *where* $\eta(t)$ *is a continuous decreasing function, but we make the further assumption that for* $g(u):=\eta(e^u)$

$$g'(u) \nearrow 0 \quad \text{for} \quad u\to\infty, \tag{2.3}$$

by which we now mean that $g'(u)$ *tends to* 0 *monotonically increasingly for* $u>c_4$ *and if* $\lim_{u\to\infty} g(u)=0$ *then* $g'(u)$ *tends to* 0 *strictly monotonically increasingly for* $u>c_4$. *Let* ε *be a fixed real number with* $0<\varepsilon<1$ *and let* ω *be the function defined in* (1.9) *which has now the form*

$$\omega(x) = \min_{u\geqq 0}\,(g(u)\log x+u). \tag{2.4}$$

Then we have

$$\Delta(x) = \Omega_\pm\left(\frac{x}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{2.5}$$

3. In Pintz's theorems well-known deep results were combined with a new method, which was built also on the powersum method. Following Pintz's method, we shall prove the undermentioned results, which corresponds to those of Pintz, and so are also optimal (apart from the constants implied by the 0-symbol and ε in the exponent).

THEOREM 1. *Suppose that* $\zeta_K(s)\neq 0$ *in the domain*

$$\sigma > 1-\eta(|t|), \tag{3.1}$$

where $\eta(t)$ *is for* $t\geqq 0$ *a continuous nonincreasing function and* $0<\eta(t)\leqq 1/2$.

Let $0<\varepsilon<1$ *be fixed, further let*

$$\omega(x) = \min_{t\geqq 1}\,(\eta(t)\log x+\log t). \tag{3.2}$$

Then we have

$$\Delta_K(x) = O_{K,\varepsilon,\eta}\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right) \tag{3.3}$$

where the O-constant can be explicitely determined in dependence of K, ε *and the function* $\eta(t)$.

Roughly speaking, this theorem improves the corresponding result of Staś—Wiertelak by a factor 2 in the exponent and at the same time we suppose less about the function $\eta(t)$. For the proof of the result in the other direction, described in Theorem 3, we also need Theorem 2. So we state

THEOREM 2. *Let* $0<\varepsilon<0.1$ *and let us assume the existence of a zero* $\varrho_0=\beta_0+i\gamma_0$ *of* $\zeta_K(s)$ *with the condition, that with the constant*[2]

$$G_K := \max\left\{(n+\log|\Delta|)^2, \frac{1}{1-E_K}\right\}, \tag{3.4}$$

where

$$E_K := \max\{\beta_0 : \zeta_K(\beta_0)=0\}, \tag{3.5}$$

and for a sufficiently large absolute constant c_5

$$\log\gamma_0 > \left(\frac{c_5 n}{\varepsilon}\right)^{3/2} \max\left\{\frac{G_K}{n}, \exp\left(\left(\frac{c_5 n}{\varepsilon}\right)^2\right)\right\}. \tag{3.6}$$

Then for every Y *for which*

$$\log Y > \left(\frac{c_5 n}{\varepsilon}\right)^5 \log\gamma_0 \tag{3.7}$$

we have in the interval

$$I = [Y, Y^{1+\varepsilon}] \tag{3.8}$$

an x *for which*

$$|\Delta_K(x)| > \frac{x^{\beta_0}}{\gamma_0^{1+\varepsilon}}. \tag{3.9}$$

THEOREM 3. *Suppose that* $\zeta_K(s)$ *has an infinity of zeros in the domain*

$$\sigma \geqq 1-\eta(|t|) \qquad \left(0<\eta(|t|)\leqq\frac{1}{2}\right), \tag{3.10}$$

where $\eta(t)$ *is for* $t\geqq 0$ *a continuous nonincreasing function, for which with the notation*

$$g(u) := \eta(e^u) \qquad u \geqq 0 \tag{3.11}$$

we have

$$g'(u) \nearrow 0 \quad \text{for} \quad u\to\infty \tag{3.12}$$

with the meaning described after (2.3).

Let $0<\varepsilon<0.1$ *be fixed and* $\omega(x)$ *be the function*

$$\omega(x) = \min_{t\geqq 1}\left(\eta(t)\log x+\log t\right) = \min_{u\geqq 0}\left(g(u)\log x+u\right). \tag{3.13}$$

[2] We remark that by Lemma 13 $G_K \ll \max(n, |\Delta|^{c_6})$.

Then we have for an infinite sequence of x_k values tending to infinity

$$|\Delta_K(x_k)| > \frac{x_k}{e^{(1+\varepsilon)\omega(x_k)}}. \tag{3.14}$$

I am deeply indebted to J. Pintz for drawing my attention to the problem and giving significant help by his comments during my work.

4. Before we begin the proofs, let us fix some notations. We take an element ϑ from K, which generates K, and denote its canonical polynomial by f, so n is the degree of f, and f has r_1 real and $2r_2$ complex roots. Denote $r=r_1+r_2-1<r_1+2r_2=n$, then r is the number of elements of any system of fundamental unit elements of K (see [7], Satz 137). M denotes the absolute value of the determinant of any such system which depends only on K (see [7], Satz 138, 139), and w denote the (by [7], Satz 135 finite) number of unity roots in K, and $d_{r+1}=1$ if $r_2=0$ and 2 if $r_2>0$. h will stand for the (by [7], Satz 125 finite) class number of K. Following Landau, we introduce the constants

$$\lambda = \frac{2^n \pi^{r_2} M}{w d_{r+1} \sqrt{|\Delta|}}, \tag{4.1}$$

and

$$A = 2^{-r_2}\pi^{-n/2}\sqrt{|\Delta|}. \tag{4.2}$$

By $\varrho=\beta+i\gamma$ we denote a non-trivial zero of $\zeta_K(s)$, $\sum_\varrho$ means a sum extended over all of them, and for restricted sums we shall subscribe the special restrictions. We define

$$\Theta_K = \sup\{\mathrm{Re}\,\varrho:\ \zeta_K(\varrho)=0\} = \sup_\varrho \beta, \tag{4.3}$$

so $1/2\leqq\Theta_K\leqq 1$ and the Riemann hypothesis for K means $\Theta_K=1/2$. We will use the facts written in [7] Satz 155 about the pole and the trivial zeros of $\zeta_K(s)$ without repeated citations, and write

$$\frac{\zeta_K'}{\zeta_K}(s) = \frac{r}{s} + \sum_{j=0}^{\infty} a_j s^j, \tag{4.4}$$

$$\zeta_K(s) = \frac{h\lambda}{s-1} + \sum_{j=0}^{\infty} b_j (s-1)^j, \tag{4.5}$$

$$\frac{\zeta_K'}{\zeta_K}(s) = \frac{-1}{s-1} + \sum_{j=0}^{\infty} e_j (s-1)^j. \tag{4.6}$$

We denote the distance of a real σ and the nearest integer by $\|\sigma\|$:

$$\|\sigma\| := \min_{n\in\mathbf{Z}} |n-\sigma|. \tag{4.7}$$

The constants implied by the O and $\ll$ symbols are effective constants, and absolute ones except when the dependence of some parameters is explicitly stated.

5. First we state some lemmas, most of them being well known. For the sake of completeness and to obtain explicit dependence on the parameters of the field (where it is needed) we give their proofs (or exact references for them).

LEMMA 1.

$$G(m) \leqq \frac{n}{\log 2} \log^2 m. \tag{5.1}$$

LEMMA 2.

$$|\Delta_K(x)| < \frac{n}{\log 2} x \log^2 x. \tag{5.2}$$

PROOF. Lemma 1 is Lemma 2 of [13], and Lemma 2 is a trivial consequence.

LEMMA 3. *For any* $\sigma > 1$ *and* $-\infty < t < \infty$ *we have*

$$|\zeta_K(\sigma + it)| \leqq \zeta(\sigma)^n, \tag{5.3}$$

$$\left|\frac{1}{\zeta_K(\sigma+it)}\right| \leqq \zeta_K(\sigma) \leqq \zeta(\sigma)^n. \tag{5.4}$$

PROOF. The second inequality of (5.4) is a special case of (5.3) which is Corollary 3 on p. 295 of [10], while the first inequality of (5.4) is Corollary 2 on p. 295 of [10].

LEMMA 4. *Let* $0 < B \leqq 1/2$ *be a nonnegative parameter. Then in the domain*

$$\mathscr{D}_B = \{s = \sigma + it \colon \sigma \leqq -B \text{ and if } |t| \leqq B \text{ then } \|\sigma\| > B\} \tag{5.5}$$

we have

$$\left|\frac{\zeta_K'}{\zeta_K}(s)\right| < C_B n \log(|s|+1) + \log|\Delta|. \tag{5.6}$$

PROOF. By [7], p. 112, formula (184) we have from the functional equation

$$\frac{\zeta_K'}{\zeta_K}(s) = -\frac{\zeta_K'}{\zeta_K}(1-s) + n \log 2\pi - \log|\Delta| + (r_1 + r_2)\frac{\pi}{2}\operatorname{ctg}\frac{\pi s}{2} -$$

$$-r_2 \frac{\pi}{2} \operatorname{tg}\frac{\pi s}{2} - n\frac{\Gamma'}{\Gamma}(1-s). \tag{5.7}$$

As $\sigma < -B$, $1-\sigma > 1+B > 1$ and by Lemma 1

$$\left|\frac{\zeta_K'}{\zeta_K}(1-s)\right| \leqq \sum_{m=1}^{\infty} \frac{G(m)}{m^{1+B}} < \frac{n}{\log 2} \sum_{m=1}^{\infty} \frac{\log^2 m}{m^{1+B}} = nc_1(B). \tag{5.8}$$

By the periodicity of $\operatorname{ctg}(\pi s/2)$ and $\operatorname{tg}(\pi s/2)$ by 2 it suffices to consider them only in $\{s = \sigma + it \colon -2 - B \leqq \sigma < -B\} \cap \mathscr{D}_B$. But there are no poles of these meromorf functions in this domain and for $-2-B \leqq \sigma \leqq -B$, $|t| > 1$, we have the trivial estimates

$$\left|\operatorname{tg}\frac{\pi s}{2}\right| < \frac{e^{\pi/2}+e^{-\pi/2}}{e^{\pi/2}-e^{-\pi/2}}, \qquad \left|\operatorname{ctg}\frac{\pi s}{2}\right| < \frac{e^{\pi/2}+e^{-\pi/2}}{e^{\pi/2}-e^{-\pi/2}}, \tag{5.9}$$

so these functions have an absolute bound depending only on B in the whole domain.

As for $\Gamma'/\Gamma(z)$, $z=x+iy$, $x\geqq 1+B$, we can use the identity

$$\frac{\Gamma'}{\Gamma}(z)=\frac{\Gamma'}{\Gamma}(z-1)+\frac{1}{z-1} \tag{5.10}$$

$[x]-1$ times, and obtain $\left(\text{as in } 1\leqq\sigma<2 \quad \left|\frac{\Gamma'}{\Gamma}(s)\right|=O(\log(|t|+2))\right)$

$$\left|\frac{\Gamma'}{\Gamma}(z)\right|\leqq\left|\sum_{k=1}^{[x]-1}\frac{1}{x-k}\right|+\left|\frac{\Gamma'}{\Gamma}(z-[x]+1)\right|<\log x+O(\log(y+2))\ll\log(|z|+2).$$

If $1-z\in\mathcal{D}_B$, $\|x-[x]+1\|\geqq B$ or $|y|\geqq B$, and so for $s\in\mathcal{D}_B$

$$\left|\frac{\Gamma'}{\Gamma}(1-s)\right|\ll\log(|1-s|+2)<c_2(B)\log(|s|+1). \tag{5.11}$$

Writing (5.11), (5.9) and (5.8) in (5.7), by $r_1+r_2\leqq n$ we get (5.6).

LEMMA 5. *If* $T\in\mathbf{R}$ *and* $L\in\mathbf{N}$, *we have*

$$\sum_{\substack{\varrho\\|\gamma-T|\leqq 1}}1\ll n\log(|T|+2)+\log|\Delta|, \tag{5.12}$$

$$\sum_{\substack{\varrho\\|\gamma-T|\leqq L}}1\ll nL\log(|T|+L+2)+L\log|\Delta|, \tag{5.13}$$

$$\sum_{\substack{\varrho\\0\leqq\gamma\leqq T}}1\ll n|T|\log(|T|+2)+(|T|+2)\log|\Delta|. \tag{5.14}$$

Further, if $|T|\geqq 2$, *then all these estimates hold with* $|T|$ *instead of* $|T|+2$.

PROOF. The assertion of (5.12) is Lemma 4 of [13], while the others are trivial implications of it.

LEMMA 6. *For* $-1/2\leqq\sigma\leqq 4$ *we have*

$$\frac{\zeta_K'}{\zeta_K}(s)=\sum_{\substack{\varrho\\|\gamma-t|\leqq 1}}\frac{1}{s-\varrho}-\frac{1}{s-1}+\frac{r}{s}+O((n+\log|\Delta|)\log^2(|t|+2)). \tag{5.15}$$

Consequently, if $|t|\geqq 2$ *we have*

$$\frac{\zeta_K'}{\zeta_K}(s)=\sum_{\substack{\varrho\\|\gamma-t|\leqq 1}}\frac{1}{s-\varrho}+O((n+\log|\Delta|)^2\log^2|t|). \tag{5.16}$$

PROOF. Since the number b appearing in Satz 179 in [7] is

$$b=-\sum_{\varrho}{}^{*}\frac{1}{\varrho}-\log A$$

(where the $*$ means, that the $\varrho'\neq 1/2$ numbers of the sum are taken together with their pairs ϱ'' for which $\varrho'+\varrho''=1$ for the sake of convergence). We get from it

(5.17) $$\frac{\zeta_K'}{\zeta_K}(s)=b-\frac{1}{s-1}-\frac{1}{s}-\frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)-r_2\frac{\Gamma'}{\Gamma}(s)+\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right).$$

Since the sum is absolute convergent, we can take together members as in $\sum^*$, and after it the $\sum^*$ for b and the $\sum_{\varrho}\frac{1}{\varrho}$ part of the together taken sum cancel each other. Further in $-1/2\leqq\sigma\leqq 4$ we can use

(5.18) $$\frac{\Gamma'}{\Gamma}(s)=-\frac{1}{s}+O\left(\log(|t|+2)\right).$$

Finally, if we write separately those finite many members of the transformed sum for which $|\gamma|\leqq 1$, we get

(5.19) $$\frac{\zeta_K'}{\zeta_K}(s)=\frac{r}{s}-\frac{1}{s-1}+\sum_{\substack{\varrho\\ |\gamma|\leqq 1}}\frac{1}{s-\varrho}+$$
$$+\sum_{\substack{\varrho\\ \gamma>1}}{}'\left(\frac{1}{s-\varrho}+\frac{1}{s-(1-\varrho)}\right)+O\big((n+\log|\Delta|)\log(|t|+2)\big).$$

We can suppose $t\geqq 0$. Making use of (5.12) we obtain:

(5.20)
$$\left|\sum_{\substack{\varrho\\ \gamma>t+1}}\left(\frac{1}{s-\varrho}+\frac{1}{s-(1-\varrho)}\right)\right|=\left|\sum_{\substack{\varrho\\ \gamma>t+1}}\frac{2s-1}{(s-\varrho)(s-1+\varrho)}\right|\ll\sum_{\substack{\varrho\\ \gamma>t+1}}\frac{t+1}{(\gamma-t)(\gamma+t)}\ll$$
$$\ll(t+1)\sum_{j=1}^{\infty}\frac{n\log(t+j+2)+\log|\Delta|}{j(2t+j)}\ll(t+1)(n+\log|\Delta|)\int_{t+1}^{\infty}\frac{1+\log x}{(x-t)(x+t)}\,dx<$$
$$<(n+\log|\Delta|)(t+1)\left|\frac{1+\log(2t+2)}{t+1}\int_{t+1}^{2t+2}\frac{1}{x-t}dx+\int_{2t+2}^{\infty}\frac{1+\log x}{\left(\frac{x^2}{2}\right)}\,dx\right|\ll$$
$$\ll(n+\log|\Delta|)\log^2(t+2),$$

(5.21)
$$\left|\sum_{\substack{\varrho\\ 1<\gamma<t-1}}\left(\frac{1}{s-\varrho}+\frac{1}{s-(1-\varrho)}\right)\right|\ll(t+1)\sum_{\substack{\varrho\\ 1<\gamma<t-1}}\frac{1}{(t-\gamma)(t+\gamma)}<\sum_{\substack{\varrho\\ 1<\gamma<t-1}}\frac{1}{t-\gamma}\ll$$
$$\ll(n+\log|\Delta|)\sum_{j=1}^{[t]}\frac{\log(t+2)}{j}<(n+\log|\Delta|)\log^2(t+2),$$

(5.22) $$\left|\sum_{\substack{\varrho\\ |\gamma|\leqq 1\\ |t-\gamma|>1}}\frac{1}{s-\varrho}\right|+\left|\sum_{\substack{\varrho\\ \gamma>1\\ |t-\gamma|\leqq 1}}\frac{1}{s-(1-\varrho)}\right|<\sum_{\substack{\varrho\\ |\gamma|\leqq 1}}1+\sum_{\substack{\varrho\\ |t-\gamma|\leqq 1}}1\ll$$
$$\ll(n+\log|\Delta|\log(t+2).$$

Now (5.19)—(5.22) imply the Lemma.

Lemma 7. *For any t there exist some T, $|T-t|\leqq 1$ for which*

$$\left|\frac{\zeta_K'}{\zeta_K}(\sigma+iT)\right| < c_6(n+\log|\Delta|)^2\log^2(|t|+2) \quad \textit{for all } \sigma \textit{ in } -\frac{1}{2}\leqq\sigma\leqq 4. \tag{5.23}$$

Proof. Let $t\geqq 0$ and consider the poles of ζ_K'/ζ_K in $[0,1]\times[t-1,t+1]$. By Lemma 5 there are not more than $c_7(n+\log|\Delta|)\log(|t|+2)$, and so we have a horizontal line $y=T$ which avoid all of them at least $R=\dfrac{1}{c_7(n+\log|\Delta|)\log(|t|+2)+1}$ far, and $|t-T|\leqq 1-R$. By Lemma 6 and Lemma 5

$$\left|\frac{\zeta_K'}{\zeta_K}(\sigma+iT)\right| \leqq \sum_{\substack{\varrho \\ |\gamma-T|\leqq 1}}\left|\frac{1}{\sigma+iT-\varrho}\right| + O\big((n+\log|\Delta|)\log^2(|t|+2)\big) <$$

$$< \sum_{\substack{\varrho \\ |\gamma-T|\leqq 1}}\big(c_7(n+\log|\Delta|)\log(|t|+2)+1\big) + O\big((n+\log|\Delta|)\log^2(|t|+2)\big) < \tag{5.24}$$

$$< c_6(n+\log|\Delta|)^2\log^2(|t|+2).$$

Lemma 8. *Let*

$$\delta(y)=\begin{cases} 0 & \textit{if}\quad 0<y<1\\ \dfrac{1}{2} & \textit{if}\quad y=1\\ 1 & \textit{if}\quad y>1.\end{cases}$$

For arbitrary $y>0$, $H>0$, $T>0$ we have

$$\left|\frac{1}{2\pi i}\int_{H-iT}^{H+iT}\frac{y^s}{s}\,ds-\delta(y)\right| < \begin{cases}\dfrac{H}{T} & \textit{if}\quad y=1\\ y^H\min\left(1,\dfrac{1}{T\,|\log y|}\right) & \textit{if}\quad y\neq 1.\end{cases} \tag{5.25}$$

Proof. This is the Lemma of Chapter 17 in [2], p. 105.

Lemma 9. *For any $Q>1$, $T>0$ and $x=[x]+1/2>1$ we have*

$$\left|\Psi_K(x)-\frac{1}{2\pi i}\int_{Q-iT}^{Q+iT}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds\right| \ll n\,\frac{X^Q}{T}\left(\log^3 x+\frac{1}{(Q-1)^3}\right). \tag{5.26}$$

Proof. Let $k=[x]$, so $x=k+1/2$. If $(3/4)x\leqq m\leqq k$ then

$$\left|\log\frac{x}{m}\right| = -\log\left(1-\frac{x-m}{x}\right) \geqq \frac{x-m}{x} \geqq \frac{3}{4}\,\frac{x-m}{m},$$

and if $m\geqq k+1$, then

$$\left|\log\frac{x}{m}\right| = -\log\left(1-\frac{m-x}{m}\right) \geqq \frac{m-x}{m}.$$

Applying the above inequalities and Lemma 8 we get by Lemma 1

$$\left|\Psi_K(x)-\frac{1}{2\pi i}\int_{Q-iT}^{Q+iT}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds\right| < \sum_{m=1}^{\infty} G(m)\left(\frac{x}{m}\right)^{Q}\frac{1}{T\left|\log\frac{x}{m}\right|} \ll$$

$$\ll \frac{x^Q}{T}n\left\{\sum_{m\leq(3/4)x}\frac{\log^2 m}{m}+\sum_{(3/4)x<m<(5/4)x}\frac{\log^2 m}{m\left|\frac{m-k-1/2}{m}\right|}+\sum_{m\geq(5/4)x}\frac{\log^2 m}{m^Q}\right\} \ll$$

$$\ll \frac{x^Q n}{T}\left\{\log^3 x+\log^2 x\sum_{l=1}^{x}\frac{1}{1/2l}+\int_x^\infty\frac{\log^2 t}{t^Q}\right\} \ll$$

$$\ll \frac{nx^Q}{T}\left(\log^3 x+\frac{1}{(Q-1)^3}\right).$$

LEMMA 10. *If* $x=[x]+1/2>1$, $T>3$ *and* a_0 *is the constant defined by* (4.4) *then we have*

$$\Delta_K(x)=-(r\log x+a_0)-\frac{r_1}{2}\log\left(1-\frac{1}{x^2}\right)-r_2\log\left(1-\frac{1}{x}\right)-\sum_{\varrho}\frac{x^\varrho}{\varrho} \tag{5.27}$$

and

$$\begin{aligned}\Delta_K(x)&=-(r\log x+a_0)-\frac{r_1}{2}\log\left(1-\frac{1}{x^2}\right)-r_2\log\left(1-\frac{1}{x}\right)-\\&-\sum_{\substack{\varrho\\|\gamma|<T}}\frac{x^\varrho}{\varrho}+O\left(\frac{(n+\log|\Delta|)^2x}{T}\left(\log^3 x+\frac{\log^2 T}{\log x}\right)\right).\end{aligned} \tag{5.28}$$

PROOF. It suffices to show (5.28) for those values of T, which satisfy (5.23) (with T in place of $|t|+2$) and $T>2$, since for other $T'>3$ Lemma 7 guarantees the existence of a T in the interval $[T'-1, T'+1]$ with this property, and by Lemma 5

$$\left|\sum_{\min(T,T')\leq\gamma<\max(T,T')}\frac{x^\varrho}{\varrho}\right| < \sum_{\substack{\varrho\\|\gamma-T'|\leq1}}\frac{x}{T'-1} \ll$$

$$\ll (n+\log|\Delta|)\log T\frac{x}{T} < \frac{(n+\log|\Delta|)x}{T}\max\left(\log^3 x,\ \frac{\log^2 T}{\log x}\right).$$

Let $Q=1+1/\log x$ and $R=[R]+1/2>0$. We define the broken line

$$L=L_1\cup L_2\cup L_3, \tag{5.29}$$

where

$$\begin{aligned}L_1&=[Q-iT,-R-iT],\quad L_2=[-R-iT,-R+iT],\\L_3&=[-R+iT,Q+iT].\end{aligned} \tag{5.30}$$

By the residuum principle

$$\frac{1}{2\pi i}\int_{Q-iT}^{Q+iT}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds = \tag{5.31}$$

$$= \frac{1}{2\pi i}\int_L \left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds + \sum_{\substack{z_j \\ -R<\mathrm{Re}\, z_j<Q \\ |\mathrm{Im}\, z_j|<T}} \mathrm{Res}\left[\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s};\ z_j\right].$$

As ζ_K has one simple pole at $s=1$, there ζ_K'/ζ_K has residuum -1, while at the nontrivial zeros from the region $|\mathrm{Im}\, z_j|<T$ we get the sum over the ϱ-s in (5.28). At $s=0$ with the notation (4.4)

(5.32)

$$\mathrm{Res}\left[\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s};\ 0\right] = \mathrm{Res}\left[-\left(\frac{r}{s}+a_0+\dots\right)(1+s\log x+\dots)\frac{1}{s};\ 0\right] = -r\log x - a_0,$$

while from the multiplicity of the negative zeros of ζ_K follows that at the negative integers we obtain, as the sum over the corresponding residues

$$r_2\sum_{l=-1}^{-[R]}\frac{x^l}{l} + r_1\sum_{l=-1}^{-[R/2]}\frac{x^{2l}}{2l}. \tag{5.33}$$

Collecting the residues and applying Lemma 9, since $x^Q=ex$, we get from (5.31)

$$\left|\Psi_K(x) - \left\{\frac{1}{2\pi i}\int_L\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds + x - \right.\right. \tag{5.34}$$

$$\left.\left. - \sum_{\substack{\varrho \\ |\gamma|<T}}' \frac{x^\varrho}{\varrho} - r\log x - a_0 - r_2\sum_{l=1}^{[R]}\frac{x^{-l}}{l} - \frac{r_1}{2}\sum_{l=1}^{[R/2]}\frac{x^{-2l}}{l}\right\}\right| \ll n\frac{x}{T}\log^3 x.$$

We can estimate the integrals over L_1 and L_3, using in $-1/2\leqq\sigma\leqq Q$ that T satisfies (5.23) and in $\sigma\leqq -1/2$ Lemma 4 (e.g. with $B=1/3$), and get independently of R for $j=1$ and $j=3$

$$\left|\frac{1}{2\pi i}\int_{L_j}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds\right| \ll$$

$$\ll (n+\log|\Delta|)^2\left(\max_{\sigma\leqq -1/2}\frac{\log^2(|\sigma+iT|+1)}{|\sigma+iT|} + \frac{\log^2 T}{T}\right)\int_{-\infty}^{Q} x^\sigma\, d\sigma \ll \tag{5.35}$$

$$\ll \frac{(n+\log|\Delta|)^2 x\log^2 T}{T\log x},$$

and

$$\left|\frac{1}{2\pi i}\int_{L_2}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}ds\right| \ll \frac{(n+\log|\Delta|)\log(T+R+1)x^{-R}}{R}T, \tag{5.36}$$

so letting $R\to\infty$ and using in (5.34) that by $x>1$ the series over l are convergent Taylor expansions, we get from (5.34) by (5.35) and (5.36)

$$\Psi_K(x)-x-\left\{-r\log x-a_0-r_2\log\left(1-\frac{1}{x}\right)-\frac{r_1}{2}\log\left(1-\frac{1}{x^2}\right)-\sum_{\substack{\varrho\\ |\gamma|<T}}\frac{x^\varrho}{\varrho}\right\}=$$
(5.37)
$$=O\left((n+\log|\Delta|)^2\frac{x}{T}\left(\log^3 x+\frac{\log^2 T}{\log x}\right)\right)$$

which gives (5.28). (5.27) follows by letting $T\to\infty$.

LEMMA 11. *We have for any* $0<\varepsilon<(1-F_K)/2$

(5.38) $$\left|\sum_{\substack{\varrho\\ |\gamma|\le 1}}\frac{x^\varrho}{\varrho}\right|\ll\frac{(n+\log|\Delta|)^2}{\varepsilon^2}x^{F_K+\varepsilon}$$

where

(5.39) $$F_K:=\max\left\{\frac{1}{2},\ \max\{\beta:\ \zeta_K(\beta+i\gamma)=0,\ |\gamma|\le 4\}\right\}.$$

PROOF. By Lemma 7 we can find to $t=2$ a T in $1\le T\le 3$ for which

(5.40) $$\left|\frac{\zeta_K'}{\zeta_K}(\sigma+iT)\right|\ll(n+\log|\Delta|)^2\quad\text{for all}\quad\sigma\quad\text{in}\quad -\frac{1}{2}\le\sigma\le 4.$$

By symmetry it holds also for $\sigma-iT$ values if $-1/2\le\sigma\le 4$. On the segments

$$L_1=[F_K+\varepsilon-iT,\ F_K+\varepsilon+iT],\quad L_3=[1-F_K-\varepsilon-iT,\ 1-F_K-\varepsilon+iT]$$

by the definition of F_K and $\varepsilon<(1-F_K)/2$ we avoid the poles of ζ_K'/ζ_K at least with ε, and so by Lemma 6 and Lemma 5

(5.41)
$$\left|\frac{\zeta_K'}{\zeta_K}(s)\right|\ll\frac{n}{\varepsilon}+\sum_{\substack{\varrho\\ |\gamma|<T+1}}\frac{1}{\varepsilon}+O(n+\log|\Delta|)\ll\frac{n+\log|\Delta|}{\varepsilon}\quad\text{for}\quad s\in L_1\quad\text{and}\quad s\in L_3.$$

Now we apply the residuum theorem on the rectangle R with vertical sides L_1 and L_3, and get (as by the symmetry of the zeros of ζ_K to $\sigma=1/2$ we have all the nontrivial zeros of ζ_K with $|\gamma|\le T$ (<4) in R)

(5.42) $$-\sum_{\substack{\varrho\\ |\gamma|<T}}\frac{x^\varrho}{\varrho}=\frac{1}{2\pi i}\int_{\partial R}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds.$$

By (5.39) and (5.41), since $\varepsilon<1-F_K-\varepsilon$,

(5.43) $$\left|\frac{1}{2\pi i}\int_{\partial R}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)\frac{x^s}{s}\,ds\right|\ll$$

$$\ll(n+\log|\Delta|)^2\frac{x^{F_K+\varepsilon}}{T}+\frac{n+\log|\Delta|}{\varepsilon}\left(\frac{x^{F_K+\varepsilon}}{F_K+\varepsilon}+\frac{x^{1-F_K-\varepsilon}}{1-F_K-\varepsilon}\right)\ll\frac{(n+\log|\Delta|)^2x^{F_K+\varepsilon}}{\varepsilon^2}.$$

Further by Lemma 5, and $1<T<4$

$$\left|\sum_{\substack{\varrho \\ 1<|\gamma|<T}} \frac{x^\varrho}{\varrho}\right| < \sum_{\substack{\varrho \\ 1<|\gamma|<T}} \frac{x^{F_K}}{1} \ll (n+\log|\Delta|)x^{F_K}. \tag{5.44}$$

Now (5.42), (5.43) and (5.44) give the Lemma.

LEMMA 12. *Let us denote by $N_K(\sigma, T)$ the number of zeros of ζ_K in the parallelogram $\beta>\sigma$, $|\gamma|<T$. Then we have*

$$N_K(\sigma, T) \ll_K T^{(n+2)(1-\sigma)} (\log^2 T)^{2n^2+4+1-\sigma}. \tag{5.45}$$

PROOF. This is the Corollary of Sokolovskii's paper [15].

LEMMA 13. $\zeta_K(s)\neq 0$ *in the domain*

$$\sigma \geqq 1-\frac{c_8}{n\log(|t|+2)+\log|\Delta|}, \tag{5.46}$$

except for at most one real and simple zero β_0 for which

$$\beta_0 \leqq 1-|\Delta|^{-c_9}. \tag{5.47}$$

PROOF. This is Lemma 2.3 of [6].

LEMMA 14. *For the coefficients a_0 and e_0 in* (4.4) *and* (4.6) *we have*

$$|a_0| \ll \frac{n+\log|\Delta|}{1-F_K}, \tag{5.48}$$

and

$$|e_0| \ll \frac{n+\log|\Delta|}{1-F_K}. \tag{5.49}$$

Consequently, by Lemma 13 and the definition of F_K in (5.39) *we have*

$$|a_0| < \max\left(c_{10}(n+\log|\Delta|)^2, c_{11}(n+\log|\Delta|)|\Delta|^{c_9}\right) \ll n^2|\Delta|^{c_9+2}, \tag{5.50}$$

and

$$|e_0| < \max\left(c_{12}(n+\log|\Delta|)^2, c_{13}(n+\log|\Delta|)|\Delta|^{c_9}\right) \ll n^2|\Delta|^{c_9+2}. \tag{5.51}$$

PROOF. By the functional equation (for this form see, e.g. [7] formula (184) on p. 112)

(5.52)

$$\frac{\zeta_K'}{\zeta_K}(s) = -\frac{\zeta_K'}{\zeta_K}(1-s)+n\log 2\pi-\log|\Delta|+(r_1+r_2)\frac{\pi}{2}\operatorname{ctg}\frac{\pi s}{2}-r_2\frac{\pi}{2}\operatorname{tg}\frac{\pi s}{2}-n\frac{\Gamma'}{\Gamma}(1-s).$$

Using the Laurent expansions of $\frac{\pi}{2}\operatorname{ctg}\frac{\pi s}{2}$, $\frac{\pi}{2}\operatorname{tg}\frac{\pi s}{2}$, $\frac{\Gamma'}{\Gamma}(1-s)$ and of $\frac{\zeta_K'}{\zeta_K}(s)$ (see

(4.4), (4.6)) in (5.2) we get

$$\frac{\zeta_K'}{\zeta_K}(s) = \frac{r}{s}+a_0+\sum_{j=1}^{\infty} a_j s^j = \frac{-1}{s}-e_0-\sum_{j=1}^{\infty} e_j(-s)^j+n\log 2\pi-\log|\Delta|+$$

$$+(r_1+r_2)\left\{\frac{1}{s}+\sum_{j=0}^{\infty} d_{2j+1}s^{2j+1}\right\}+r_2\sum_{j=0}^{\infty} f_{2j+1}s^{2j+1}-n\frac{\Gamma'}{\Gamma}(1)+n\sum_{j=1}^{\infty} g_j s^j.$$

Equating coefficients in (5.53) we find

$$a_0 = n\left(\log 2\pi-\frac{\Gamma'}{\Gamma}(1)\right)-\log|\Delta|-e_0, \tag{5.54}$$

so it suffices to prove (5.49). By the maximum principle it follows, that for any $0<R<1-F_K$

$$|e_0| \leqq \max_{|s-1|=R}\left|\frac{\zeta_K'}{\zeta_K}(s)+\frac{1}{s-1}\right|. \tag{5.55}$$

From Lemmas 5 and 6 and from $r<n$ we get

$$\max_{|s-1|=\frac{1-F_K}{2}}\left|\frac{\zeta_K'}{\zeta_K}(s)+\frac{1}{s-1}\right| \ll \frac{n+\log|\Delta|}{\frac{1-F_K}{2}} \ll \frac{n+\log|\Delta|}{1-F_K}. \tag{5.56}$$

For the proof of Theorem 2 Turán's powersum method is essential. We shall apply it in the following continuous form:

Lemma 15. *Let $\alpha_j\in\mathbf{C}$ for $j=1, \dots, N$ and $d>0$ be arbitrary. Then we have*

$$\max_{a\leqq t\leqq a+d}\frac{\left|\sum_{j=1}^{N} e^{\alpha_j t}\right|}{\max_{1\leqq j\leqq N}|e^{\alpha_j t}|} \geqq \left(\frac{1}{4e\left(\frac{a}{d}+1\right)}\right)^N. \tag{5.57}$$

Proof. For $z_j\in\mathbf{C}$ $(j=1, \dots, N)$ and $m>0$ we have by the second main theorem of the powersum method (see [5])

$$\max_{m\leqq\nu\leqq m+N}\frac{\left|\sum_{j=1}^{N} z_j^\nu\right|}{\max_{1\leqq j\leqq N}|z_j^\nu|} \geqq \left(\frac{N}{4e(m+N)}\right)^N.$$

Choosing $m=Na/d$, $z_j=e^{\alpha_j(a/m)}=e^{\alpha_j(d/N)}$ $(j=1, \dots, N)$, we get the continuous form.

Our last lemma is a somewhat modified form of a result proved in the Appendix of [20].

Lemma 16. *Let $T>30$, $\delta>q=\dfrac{\log\log\log T}{\log\log T}$ and denote the number of roots of ζ_K in the rectangle $|t-T|\leqq\delta/2$, $a-\delta\leqq\sigma<a$ by $M_K(a, T, \delta)$. Then if ζ_K has no*

zeros in the domain $\sigma \geqq a$, $|t-T| \leqq 2$, *then we have*

$$M_K(a, T, \delta) \ll \delta(G_K + n \log T) \tag{5.58}$$

where G_K *is defined by* (3.4).

PROOF. For $\delta > 1/20$ Lemma 5 can be used, so we can suppose $q < \delta < 1/20$. For any $\varrho = \beta + i\gamma$ and $s = \sigma + iT$

$$\operatorname{Re}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) = \frac{\sigma - \beta}{(\sigma-\beta)^2 + (T-\gamma)^2} + \frac{\beta}{\beta^2+\gamma^2},$$

hence for any $s = \sigma + iT$ with $\sigma > a$ by the condition

$$\left|\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)\right| \geqq \operatorname{Re} \sum_{\substack{\varrho \\ |\gamma - T| \leqq 2}} \left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right) - \left| \sum_{\substack{\varrho \\ |\gamma - T| > 2}} \left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)\right| \geqq$$

$$\geqq \sum_{\substack{\varrho \\ \beta \geqq a - \delta \\ |\gamma - T| \leqq \delta/2}} \frac{\sigma - a}{(\sigma + \delta - a)^2 + \dfrac{\delta^2}{4}} - \left| \sum_{\substack{\varrho \\ |\gamma - T| > 2}} \left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)\right| \geqq \tag{5.59}$$

$$\geqq \frac{\sigma - a}{2(\sigma - a + \delta)^2} M_K(a, T, \delta) - \left| \sum_{\substack{\varrho \\ |\gamma - T| > 2}} \left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)\right|.$$

Let $z = 2 + iT$, then by Lemma 5 and $T > 30$ (>2)

$$\left| \sum_{\substack{\varrho \\ |\gamma - T| > 2}} \left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right) - \sum_{\substack{\varrho \\ |\gamma - T| > 2}} \left(\frac{1}{z-\varrho}+\frac{1}{\varrho}\right)\right| = \left| \sum_{\substack{\varrho \\ |\gamma - T| > 2}} \left(\frac{1}{s-\varrho}-\frac{1}{z-\varrho}\right)\right| = \tag{5.60}$$

$$= \left| \sum_{\substack{\varrho \\ |\gamma - T| > 2}} \frac{s-z}{(s-\varrho)(z-\varrho)}\right| \ll |s-z| \sum_{m=2}^{\infty} \frac{n \log(T+m) + \log|\Delta|}{m^2} \ll n \log T + \log|\Delta|.$$

Also by Lemma 5, since $|z-\varrho| \geqq 2 - \beta > 1$ and $T > 30$

$$\left| \sum_{\substack{\varrho \\ |\gamma - T| \leqq 2}} \left(\frac{1}{z-\varrho}+\frac{1}{\varrho}\right)\right| \ll n \log T + \log|\Delta|. \tag{5.61}$$

Now collecting (5.59), (5.60) and (5.61) we see

(5.62)

$$M_K(a, T, \delta) \ll \frac{(\sigma - a + \delta)^2}{\sigma - a} \left\{ \left|\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)\right| + \left|\sum_{\varrho}\left(\frac{1}{z-\varrho}+\frac{1}{\varrho}\right)\right| + n \log T + \log|\Delta| \right\}.$$

We shall estimate $\sum_{\varrho}\left(\frac{1}{w-\varrho}+\frac{1}{\varrho}\right)$ where $w = u + iv$ $u \geqq 1/2$, $v \geqq 2$. We use $\Gamma'/\Gamma(w) = O(\log v)$ for $1/4 \leqq u \leqq 2$ and (with the notation described there) formula (5.17), and get by Lemmas 5 and 13

$$(5.63)\quad \left|\sum_{\varrho}\left(\frac{1}{w-\varrho}+\frac{1}{\varrho}\right)\right| \leqq \left|\frac{\zeta_K'}{\zeta_K}(w)\right| + |\log A| + \left|{\sum_{\varrho}}^{*}\frac{1}{\varrho}\right| + O(1) + O(n\log v) \ll$$

$$\ll \left|\frac{\zeta_K'}{\zeta_K}(w)\right| + n + \log|\Delta| + n\log v + \sum_{j=1}^{\infty}\frac{(n+\log|\Delta|)\log(j+2)}{j^2} + (n+\log|\Delta|)^2 \min_{\varrho}\left|\frac{1}{\varrho}\right|.$$

By Lemma 11 (1.3), and the definition of the constants G_K, E_K in (3.4)—(3.5), we get for z from (5.63)

$$(5.64)\qquad \left|\sum_{\varrho}\left(\frac{1}{z-\varrho}+\frac{1}{\varrho}\right)\right| \ll G_K + n\log T.$$

Similarly for $s=\sigma+iT$, since $T>4$

$$(5.65)\qquad \left|\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)\right| \ll G_K + n\log T + \left|\frac{\zeta_K'}{\zeta_K}(s)\right|.$$

In $u>a$, $|v-T|\leqq 2$ $\zeta_K(w)\neq 0$, and so the $\sigma\geqq 1$ halfplane with this rectangle is a domain, where $\log\zeta_K(w)$ is an analytic function. Let us define the following disks around $z=2+iT$

$$(5.66)\qquad \begin{aligned} \mathscr{D}_1 &= \{w\colon\ |w-z| \leqq R = 2-a\} \\ \mathscr{D}_2 &= \left\{w\colon\ |w-z| \leqq r = 2-a-\frac{1}{2}q\right\} \\ \mathscr{D}_3 &= \{w\colon\ |w-z| \leqq r_3 = 2-a-6q\} \\ \mathscr{D}_4 &= \{w\colon\ |w-z| \leqq r_4 = 2-a-7q\} \\ \mathscr{D}_5 &= \left\{w\colon\ |w-z| \leqq r_5 = \frac{1}{2}\right\}. \end{aligned}$$

Since in $u>-1$ and $v>2$ (see, e.g. [16], Lemma 7) $|\zeta_K(w)|<|\Delta|^{c_{14}}v^{c_{15}n}$, Lemma 1 leads at once to

$$(5.67)\qquad \begin{aligned} \operatorname{Re}\log\frac{\zeta_K(w)}{\zeta_K(z)} &= \log\left|\frac{\zeta_K(w)}{\zeta_K(z)}\right| \ll \\ \ll n\log v + \log|\Delta| + \sum_{m=2}^{\infty}\frac{n\log m}{m^2} &\ll n\log v + \log|\Delta|;\quad w\in\mathscr{D}_1. \end{aligned}$$

Using the Borel—Carathéodory theorem (see, e.g. [3], p. 53)

$$\max_{|w-z|\leqq r}|f(w)-f(z)| \leqq \frac{2r}{R-r}\left\{\max_{|w-z|\leqq R}\operatorname{Re}f(w)-\operatorname{Re}f(z)\right\}$$

for the analytic function $f(w)=\log\dfrac{\zeta_K(w)}{\zeta_K(z)}$, $R=2-a$, $r=2-a-1/2\,q$, we get from (5.67)

$$|\log \zeta_K(w)| \ll \frac{1}{q}\{n\log T+\log|\Delta|\} \quad \text{for} \quad w\in\mathcal{D}_2. \tag{5.68}$$

Now we apply the three-circle theorem to $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_5$ (where $|\log \zeta_K(w)|=O(n)$ for $w\in\mathcal{D}_5$ by Lemma 1 and (1.1)—(1.3)), and get

$$\begin{aligned}|\log \zeta_K(w)| &\ll \left(\frac{1}{q}\left(n+\frac{\log|\Delta|}{\log T}\right)\log T\right)^{\alpha} n^{1-\alpha} \ll \\ &\ll \left(n+\frac{\log|\Delta|}{\log T}\right)\left(\frac{\log T}{q}\right)^{\alpha} \quad \text{for} \quad w\in\mathcal{D}_3,\end{aligned} \tag{5.69}$$

where

$$\begin{aligned}\alpha &= \frac{\log r_3-\log r_5}{\log r-\log r_5} = \frac{\log(4-2a-12q)}{\log(4-2a-q)} = 1+\frac{\log\left(1-\frac{11q}{4-2a-q}\right)}{\log(4-2a-q)} < \\ &< 1-\frac{11q}{(4-2a-q)\log(4-2a-q)} < 1-\frac{11q}{3\log 3} < 1-3q\end{aligned} \tag{5.70}$$

since $q<\delta<1/20$ and $a\geqq 1/2$ can be supposed because if $a<1/2$ then by the symmetry of the ζ_K-roots to $\sigma=1/2$ the condition gives $M_K(a, T, \delta)=0$. Now from (5.69) and (5.70)

$$\begin{aligned}&|\log \zeta_K(w)| \ll \\ \ll \left(n+\frac{\log|\Delta|}{\log T}\right)\frac{\log T}{q}&(\log T)^{-2\frac{\log\log\log T}{\log\log T}} \ll \left(n+\frac{\log|\Delta|}{\log T}\right) q\log T \quad \text{for} \quad w\in\mathcal{D}_3.\end{aligned} \tag{5.71}$$

We apply Cauchy's coefficient estimate in $\mathcal{D}_4$, and get

$$\left|\frac{\zeta_K'}{\zeta_K}(w)\right| \ll \left(n+\frac{\log|\Delta|}{\log T}\right)\log T = n\log T+\log|\Delta| \quad \text{for} \quad w\in\mathcal{D}_4. \tag{5.72}$$

Let $\sigma=a+6q+\delta$, so $s=\sigma+iT\in\mathcal{D}_4$, and (5.72) applies to s. From (5.72), (5.62), (5.64) and (5.65) we obtain

$$M_K(a, T, \delta) \ll \frac{(6q+2\delta)^2}{6q+\delta}\{n\log T+\log|\Delta|+G_K\} \ll \delta(n\log T+G_K). \tag{5.73}$$

6. Proof of Theorem 1. We shall handle the case $\lim\limits_{t\to\infty}\eta(t)=H>0$ and the case $\lim\limits_{t\to\infty}\eta(t)=0$ separately. First let

$$\lim_{t\to\infty}\eta(t) = H > 0, \tag{6.1}$$

i.e. we suppose the quasi-Riemann hypothesis (if $H=1/2$, the Riemann hypothesis). It follows, that

$$F_K \leqq \Theta_K \leqq 1-H. \tag{6.2}$$

From Lemma 11 for all $\delta < H/2$ we have

$$\left|\sum_{\substack{\varrho \\ |\gamma|\le 1}} \frac{x^\varrho}{\varrho}\right| \ll (n+\log|\Delta|)^2 \frac{1}{\delta^2} x^{F_K+\delta}. \tag{6.3}$$

Now we apply the explicit formula described in Lemma 10, and get (choosing $T=x$ in (5.28)) for any $x=[x]+1/2>3$

$$\Delta_K(x) = -a_0 - \sum_{\substack{\varrho \\ |\gamma|<x}} \frac{x^\varrho}{\varrho} + O\big((n+\log|\Delta|)^2 \log^3 x\big). \tag{6.4}$$

The contribution of the zeros with $|\gamma|\le 1$ is estimated in (6.3), while Lemma 5 can be used to the remaining part. If we choose $\delta = 1/\log x$, we get

$$\begin{aligned}\left|\sum_{\substack{\varrho \\ |\gamma|<x}} \frac{x^\varrho}{\varrho}\right| &\ll (n+\log|\Delta|)^2 \frac{1}{\delta^2} x^{F_K+\delta} + \sum_{m\le x} \frac{x^{1-H}(n+\log|\Delta|)\log(m+2)}{m} \ll \\ &\ll (n+\log|\Delta|)^2 x^{1-H}\left(\log^2 x + \sum_{m\le x}\frac{\log(m+2)}{m}\right) \ll \\ &\ll_{H,\varepsilon'} (n+\log|\Delta|)^2 x^{1-H+H\varepsilon'}.\end{aligned} \tag{6.5}$$

A short reflection to Lemma 14 and (6.2) turns (6.4) by (6.5) to

$$|\Delta_K(x)| \ll_{H,\varepsilon'} (n+\log|\Delta|)^2 x^{1-H+H\varepsilon'} \tag{6.6}$$

which is now valid for any $x>3$, since $\Delta_K(x)$ changes in the interval $[[x], [x]+1)$ not more than $2n\log^2 x$ by Lemma 1. Now if $t>t_0(\varepsilon,\eta)$ then $\eta(t)<H(1+\varepsilon/3)$, and so for

$$x > x_0 := t_0^{\frac{3}{H\varepsilon}}(\varepsilon,\eta) \tag{6.7}$$

we have

$$\begin{aligned}\omega(x) &\le \eta(x^{\frac{H\varepsilon}{3}})\log x + \log x^{\frac{H\varepsilon}{3}} < H\left(1+\frac{\varepsilon}{3}\right)\log x + \frac{H\varepsilon}{3}\log x = \\ &= H\left(1+\frac{2}{3}\varepsilon\right)\log x < \frac{1}{1-\varepsilon} H\left(1-\frac{\varepsilon}{3}\right)\log x,\end{aligned} \tag{6.8}$$

so if $x>x_0(\varepsilon,\eta)$ and $\varepsilon'=\varepsilon/3$,

$$(1-\varepsilon)\omega(x) < H(1-\varepsilon')\log x,$$

and thus by (6.6)

$$|\Delta_K(x)| \ll_{\varepsilon,\eta} (n+\log|\Delta|)^2 \frac{x}{e^{(1-\varepsilon)\omega(x)}}, \tag{6.9}$$

proving our theorem in this case.

The case $\lim\limits_{t\to\infty}\eta(t)=0$ is much deeper. Here we shall use the analogue of Carlson's density theorem, described in Lemma 12. For any $0<\varepsilon<1/2$ this lemma gives (as $n\geqq 2$)

$$N_K(1-\varepsilon, T)\ll_K T^{(n+2)\varepsilon}(\log^2 T)^{7n^2}\ll_{K,\varepsilon} T^{9n\varepsilon}. \tag{6.10}$$

We apply the explicit formula (5.28) of Lemma 10 to obtain (6.4), and for $\delta\leqq\eta(4)/2$ we have (6.3), too, by Lemma 11. By Lemma 14 we can estimate a_0, and as $F_K\leqq \leqq 1-\eta(4)$, we get for $\delta=\eta(4)/2$

$$\Delta_K(x)=-\sum_{\substack{\varrho\\ 1<|\gamma|<x}}\frac{x^\varrho}{\varrho}+O\left((n+\log|\Delta|)^2\log^3 x+\frac{(n+\log|\Delta|)^2}{\eta(4)^2}x^{1-\frac{\eta(4)}{2}}\right). \tag{6.11}$$

Now we estimate the weights of the x-powers by Lemma 5 and get

$$\sum_{\substack{\varrho\\ 1<|\gamma|<x}}\frac{1}{|\varrho|}\ll(n+\log|\Delta|)\sum_{m\leqq x}\frac{\log(m+2)}{m}\ll(n+\log|\Delta|)\log^2 x. \tag{6.12}$$

Let φ be choosen later, then the contribution of zeros with $\beta\leqq 1-\varphi$ in (6.11) is by (6.12)

$$\left|\sum_{\substack{\varrho\\ 1<|\gamma|<x\\ \beta\leqq 1-\varphi}}\frac{x^\varrho}{\varrho}\right|\ll(n+\log|\Delta|)x^{1-\varphi}\log^2 x. \tag{6.13}$$

Finally, we have to consider the contribution of zeros near $\sigma=1$. By (6.10) for any $\varepsilon>9n\varphi$ we obtain

$$\begin{aligned}\left|\sum_{\substack{\varrho\\ 1<|\gamma|<x\\ \beta>1-\varphi}}\frac{x^\varrho}{\varrho}\right| &\ll_{K,\varphi}\sum_{k=1}^{[\log x]+1}\frac{x^{1-\eta(e^k)}}{e^{k-1}}(e^k)^{9n\varphi}\ll x\sum_{k=1}^{[\log x]+1}\frac{e^{-(\varepsilon-9n\varphi)k}}{e^{\eta(e^k)\log x+k(1-\varepsilon)}}< \\ &<\frac{x}{e^{(1-\varepsilon)\omega(x)}}\sum_{k=1}^{\infty}e^{-(\varepsilon-9n\varphi)k}\ll\frac{x}{e^{(1-\varepsilon)\omega(x)}}\frac{1}{\varepsilon-9n\varphi}.\end{aligned} \tag{6.14}$$

Setting $\varphi=\varepsilon/10n$ and collecting our estimates (6.11), (6.13) and (6.14) we arrive to

$$|\Delta_K(x)|\ll_{K,\varepsilon}\frac{x}{e^{(1-\varepsilon)\omega(x)}}+\frac{1}{\eta(4)^2}x^{1-\frac{\eta(4)}{2}}+x^{1-\frac{\varepsilon}{10n}}\log^2 x. \tag{6.15}$$

Since $\log x=O_\delta(x^\delta)$ and $e^{\omega(x)}=O_{\eta,\varepsilon'}(x^{\varepsilon'})$, from (6.15) follows

$$|\Delta_K(x)|\ll_{K,\varepsilon,\eta}\frac{x}{e^{(1-\varepsilon)\omega(x)}}.$$

7. Proof of Theorem 2. In the following we will use always that c_5 is chosen sufficiently large. We introduce the notations

$$\varepsilon'=\frac{\varepsilon}{c_5 n}=\frac{1}{\lambda},\quad \mu=k\lambda^2 \tag{7.1}$$

where k is a real number to be chosen later in the range

$$\varepsilon'^2(1+4\varepsilon')\log Y \leqq k \leqq \varepsilon'^2(1+5\varepsilon')\log Y, \tag{7.2}$$

or, equivalently

$$(1+4\varepsilon')\log Y \leqq \mu \leqq (1+5\varepsilon')\log Y. \tag{7.3}$$

First we find a convenient zero of ζ_K. Let $\varrho_1=\beta_1+i\gamma_1$ be a ζ_K-zero with the maximal real part β_1 among the zeros for which

$$0 \leqq \gamma \leqq \gamma_0$$

and $\varrho_{j+1}=\beta_{j+1}+i\gamma_{j+1}$ a zero with maximal real part among those satisfying

$$\gamma_j \leqq \gamma \leqq \gamma_j+2\lambda, \quad \beta > \beta_j+\frac{1}{\log Y}. \tag{7.4}$$

After not more than [1/2 log Y] steps this process cannot be continued, and we get a zero $\varrho_M=\beta_M+i\gamma_M$ for which

$$\gamma_M \leqq \gamma_1+(2M-2)\lambda \leqq \gamma_0+\log^2 Y \tag{7.5}$$

and for which the regions

$$0 \leqq t \leqq \gamma_M, \quad \sigma > \beta_M \tag{7.6}$$

and

$$\gamma_M \leqq t \leqq \gamma_M+2\lambda, \quad \sigma > \beta_M+\frac{1}{\log Y} \tag{7.7}$$

are zerofree.

If $\gamma_j \leqq \gamma_0$, by $\beta_j \geqq \beta_0$ trivially

$$\frac{x^{\beta_j}}{\gamma_j^{1+\varepsilon}} \geqq \frac{x^{\beta_0}}{\gamma_0^{1+\varepsilon}}, \tag{7.8}$$

and if $\gamma_j \geqq \gamma_0$, it means that $\beta_j > \beta_{j-1}+1/\log Y$ and further $\gamma_j < 2\gamma_{j-1}$, which results for every $x \geqq Y$ the inequality

$$\frac{x^{\beta_j}}{\gamma_j^{1+\varepsilon}} \geqq \frac{x^{\beta_{j-1}} e^{\frac{\log x}{\log Y}}}{(2\gamma_{j-1})^{1+\varepsilon}} > \frac{x^{\beta_{j-1}}}{\gamma_{j-1}^{1+\varepsilon}}. \tag{7.9}$$

Hence for any $x \geqq Y$ we can prove from (7.8) and (7.9) by induction (7.8) for every j without the condition $\gamma_j > \gamma_0$. This, using (7.8) for $j=M$, gives that it is enough to prove (3.9) with ϱ_M instead of ϱ_0. Now we distinguish two cases, according to

$$\begin{aligned} &\gamma_M > \gamma_0^{\varepsilon'^{3/2}} \quad \text{(Case I), or} \\ &\gamma_M \leqq \gamma_0^{\varepsilon'^{3/2}} \quad \text{(Case II).} \end{aligned} \tag{7.10}$$

Let us define

$$\gamma' = \begin{cases} \gamma_M & \text{in Case I} \\ \gamma_0 & \text{in Case II.} \end{cases} \tag{7.11}$$

If we prove (3.9) for $\beta_M+i\gamma'$ (which, of course, in Case II need not to be a zero of ζ_K) in place of ϱ_0, it implies (3.9), since in Case I we work with ϱ_M instead of ϱ_0, which was justified just before, and in Case II we can simply apply $\beta_M \geqq \beta_0$. So our aim will be to prove, that the indirect assumption

$$|\Delta_K(x)| \leqq \frac{x^{\beta_M}}{\gamma'^{1+\varepsilon}} \quad \text{for} \quad x\in I \tag{7.12}$$

leads to a contradiction: this contradiction will prove the theorem. As the trivial consequences of our notations and conditions (3.6), (3.7), (7.1), (7.2), (7.3), (7.5), (7.10), (7.11), we will use the estimates

$$\gamma' \geqq \gamma_0^{\varepsilon'^{3/2}} > e^{\lambda^2} > \lambda^2, \tag{7.13}$$

$$\gamma_M \leqq \gamma' \leqq \gamma_0 + \log^2 Y \leqq Y^{\varepsilon'^3} + \log^2 Y < e^{\varepsilon'^3\mu} + \mu^2. \tag{7.14}$$

For $\sigma > 1$ we define the analytic function

$$H(s) = \int_1^\infty \Delta_K(x) \frac{d}{dx}(x^{-s})\, dx \tag{7.15}$$

for which in $\sigma > 1$ by partial integration

$$H(s) = \frac{\zeta_K'}{\zeta_K}(s) + \frac{s}{s-1} \tag{7.16}$$

so $H(s)$ can be defined as a meromorf function in the whole complex plane. We shall use the well-known integral formula

$$\frac{1}{2\pi i} \int_{(2)} e^{Vs^2+Zs}\, ds = \frac{1}{2\sqrt{\pi V}} \exp\left(-\frac{Z^2}{4V}\right) \tag{7.17}$$

valid for any $V > 0$ and arbitrary complex Z. Let further

$$\begin{aligned} U := \frac{1}{2\pi i} \int_{(2)} H(s+i\gamma_M) e^{ks^2+\mu s}\, ds &= \frac{1}{2\pi i} \int_{(2)} \int_1^\infty \Delta_K(x) \frac{d}{dx}(x^{-s-i\gamma_M}) e^{ks^2+\mu s}\, dx ds = \\ &= \int_1^\infty \Delta_K(x) \frac{d}{dx}\left\{x^{-i\gamma_M} \frac{1}{2\pi i} \int_{(2)} e^{ks^2+(\mu-\log x)s}\, ds\right\} dx = \\ &= \int_1^\infty \Delta_K(x) \frac{d}{dx}\left\{x^{-i\gamma_M} \frac{1}{2\sqrt{\pi k}} \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\right\} dx = \\ &= \frac{1}{2\sqrt{\pi k}} \int_1^\infty \frac{\Delta_K(x)}{x} x^{-i\gamma_M}\left(-i\gamma_M + \frac{\mu-\log x}{2k}\right) \exp\left(-\frac{(\mu-\log x)^2}{4k}\right) dx. \end{aligned} \tag{7.18}$$

The idea is to estimate U from above using (7.12) in the latter expression, and find a contradiction by giving a greater lower estimate using the defining formula of U

and choosing a suitable k satisfying (7.2). We split up U into three parts, namely

$$U_1 = \int_1^{e^{\mu-3\lambda k}}, \quad U_2 = \int_{e^{\mu-3\lambda k}}^{e^{\mu+3\lambda k}}, \quad U_3 = \int_{e^{\mu+3\lambda k}}^{\infty}, \tag{7.19}$$

so by (7.1) and (7.3) we have

$$[e^{\mu-3\lambda k}, e^{\mu+3\lambda k}] \subset I. \tag{7.20}$$

Taking account (7.13), (7.14), (7.12) and (7.20), we obtain

$$|U_2| \leqq \frac{1}{2\sqrt{\pi k}} \int_{e^{\mu-3\lambda k}}^{e^{\mu+3\lambda k}} \left|\frac{\Delta_K(x)}{x}\right| \left(\gamma_M + \frac{|\mu - \log x|}{2k}\right) \exp\left(-\frac{(\mu-\log x)^2}{4k}\right) dx \leqq$$

$$\leqq \frac{1}{2\sqrt{\pi k}} \int_{e^{\mu-3\lambda k}}^{e^{\mu+3\lambda k}} \left(\gamma_M + \frac{3}{2}\lambda\right) \frac{x^{\beta_M - 1}}{\gamma'^{1+\varepsilon}} \exp\left(-\frac{(\log x - \mu)^2}{4k}\right) dx \leqq$$

$$\leqq \frac{\left(1+\frac{3}{2}\varepsilon'\right)\gamma'}{2\sqrt{\pi k}\,\gamma'^{1+\varepsilon}} \int_{-3\lambda k}^{3\lambda k} e^{\beta_M(\mu+y) - \frac{y^2}{4k}}\, dy < \tag{7.21}$$

$$< \frac{e^{\mu\beta_M + k\beta_M^2}\left(1+\frac{3}{2}\varepsilon'\right)}{\gamma'^{\varepsilon}\, 2\sqrt{\pi k}} \int_{-\infty}^{\infty} e^{-\left(\frac{y}{2\sqrt{k}} - \beta_M\sqrt{k}\right)^2} dy = \left(1+\frac{3}{2}\varepsilon'\right) \frac{e^{\mu\beta_M + k\beta_M^2}}{\gamma'^{\varepsilon}}.$$

For U_1 and U_3 Lemma 2 and (7.14) gives

$$|U_1| \leqq \frac{1}{2\sqrt{\pi k}} \int_1^{e^{\mu-3k\lambda}} \frac{|\Delta_K(x)|}{x} \left(\gamma_M + \frac{\lambda^2}{2}\right) \exp\left(-\frac{(\log x-\mu)^2}{4k}\right) dx \leqq n\mu^2 2\gamma' e^{\mu - \frac{9}{4}\mu} < 1, \tag{7.22}$$

$$|U_3| \leqq \frac{1}{2\sqrt{\pi k}} \int_{e^{\mu+3k\lambda}}^{\infty} \frac{n}{\log 2} \log^2 x \left(\gamma' + \frac{\log x - \mu}{2k}\right) \exp\left(-\frac{(\log x-\mu)^2}{4k}\right) dx \leqq$$

$$\leqq \frac{2n\gamma'}{\log 2\sqrt{\pi}} \int_{e^{\mu+3\lambda k}}^{\infty} \left(\frac{\log x}{2\sqrt{k}}\right)^2 \frac{(\log x - \mu)}{2\sqrt{k}} \exp\left(-\frac{(\log x-\mu)^2}{4k}\right) dx =$$

$$= \frac{4n\gamma'\sqrt{k}}{\log 2\sqrt{\pi}} \int_{\frac{3}{2}\lambda\sqrt{k}}^{\infty} \left(y + \frac{\mu}{2\sqrt{k}}\right)^2 y e^{-y^2} e^{2\sqrt{k}y+\mu}\, dy < \tag{7.23}$$

$$< 4n\gamma'\sqrt{k}\, e^{\mu+k} \int_{\frac{3}{2}\lambda\sqrt{k}}^{\infty} \frac{\left(\frac{3}{2}\lambda\sqrt{k} + \frac{\mu}{2\sqrt{k}}\right)^2 \frac{3}{2}\lambda\sqrt{k}}{\left(\frac{3}{2}\lambda\sqrt{k} - \sqrt{k}\right)^3} (y-\sqrt{k})^3 e^{-(y-\sqrt{k})^2}\, dy <$$

$$< n\gamma' e^{\mu+k}\mu \int_{\left(\frac{3}{2}\lambda - 1\right)\sqrt{k}}^{\infty} u^3 e^{-u^2}\, du < e^{(3/2)\mu} \int_{2\mu}^{\infty} v e^{-v}\, dv < 1.$$

So we conclude from (7.21), (7.22) and (7.23) that (7.12) implies

$$|U| < 2\frac{e^{\mu\beta_M + k\beta_M^2}}{\gamma'^{\varepsilon}}. \tag{7.24}$$

Now in (7.18) we shall use the first form, and shift the line of the integration to $\sigma=-1/2$, where by $\log(a+b)<\log(a+1)+\log(b+1)$, (7.14), (7.16) and Lemma 4

$$\begin{aligned} &\left|\frac{1}{2\pi i}\int_{(-1/2)} H(s+i\gamma_M)e^{ks^2+\mu s}\,ds\right| \ll \\ &\ll \int_{-\infty}^{\infty} (n+\log|\Delta|)^2 \log(|t|+\gamma_M+2)e^{-\frac{\mu}{2}+\frac{1}{4}-t^2}\,dt < 1, \end{aligned} \tag{7.25}$$

and so by the residuum principle and $r \leqq n$

$$\begin{aligned} U &= \sum_{\varrho} e^{k(\varrho-i\gamma_M)^2+\mu(\varrho-i\gamma_M)} + re^{-k\gamma_M^2-i\mu\gamma_M} + O(1) = \\ &= \sum_{\varrho} \{e^{(\varrho-i\gamma_M)^2+\lambda^2(\varrho-i\gamma_M)}\}^k + O(n). \end{aligned} \tag{7.26}$$

The contribution of the zeros with $|\gamma-\gamma_M|>2\lambda$ is by Lemma 5

$$\ll (n+\log|\Delta|)\sum_{j=2[\lambda]}^{\infty} e^{\mu+k(1-j^2)}\log(j+\gamma_M+2) < 1. \tag{7.27}$$

Now we divide the remaining zeros into two classes, namely

$$\begin{aligned} \mathscr{C}_1 &:= \left\{\varrho\colon\ \zeta_K(\varrho)=0,\ |\gamma-\gamma_M|<\varepsilon'^{3/2},\ \left|\beta-\beta_M-\frac{1}{\log Y}\right|<\varepsilon'^{3/2}\right\}, \\ \mathscr{C}_2 &:= \{\varrho\colon\ \zeta_K(\varrho)=0,\ |\gamma-\gamma_M|\leqq 2\lambda\}\setminus\mathscr{C}_1. \end{aligned} \tag{7.28}$$

The distinction of the two cases in (7.10) was needful for the proof of

$$M_K\left(\beta_M+\frac{1}{\log Y}, \gamma_M, \varepsilon'^{3/2}\right) = \sum_{\varrho\in\mathscr{C}_1} 1 < c_{14}\varepsilon'^{3/2} n\log\gamma'. \tag{7.29}$$

In Case I by (3.6) and (7.10)

$$\varepsilon'^{3/2} > \frac{\log\log\log\gamma_0^{\varepsilon'^{3/2}}}{\log\log\gamma_0^{\varepsilon'^{3/2}}} > \frac{\log\log\log\gamma_M}{\log\log\gamma_M}, \tag{7.30}$$

and so (7.29) follows from the application of Lemma 16 while in Case II we use (7.10) and Lemma 5 to get

$$\begin{aligned} M_K\left(\beta_M+\frac{1}{\log Y}, \gamma_M, \varepsilon'^{3/2}\right) &< \sum_{\substack{\varrho \\ |\gamma-\gamma_M|\leqq 1}} 1 \ll n\log\gamma_M+\log|\Delta| \ll \\ &\ll n\varepsilon'^{3/2}\log\gamma_0 = n\varepsilon'^{3/2}\log\gamma'. \end{aligned}$$

So (7.29) is valid, and by Lemma 15 we can find a k satisfying (7.2) for which by (7.1)

$$\left|\sum_{\varrho\in\mathscr{C}_1}\{e^{(\varrho-i\gamma_M)^2+\lambda^2(\varrho-i\gamma_M)}\}^k\right| \geqq \left(\frac{1}{4e\dfrac{1+5\varepsilon'}{\varepsilon'}}\right)^{c_{14}\varepsilon'^{3/2}n\log\gamma'} e^{k\beta_M^2+\mu\beta_M} \geqq$$

(7.31)

$$\geqq \gamma'^{-c_{14}n\varepsilon'^{3/2}\log\frac{20}{\varepsilon'}} e^{k\beta_M^2+\mu\beta_M} > 4\frac{e^{k\beta_M^2+\mu\beta_M}}{\gamma'^{\varepsilon}}.$$

As for the remaining zeros of $\mathscr{C}_2$, we have $|\gamma-\gamma_M|\leqq 2\lambda$ and so by (7.6) and (7.7)

$$\beta < \beta_M+\frac{1}{\log Y} < \beta_M+\frac{2}{\mu},$$

which gives by Lemma 5 and the definition of $\mathscr{C}_2$ in (7.28), using (3.6)

$$\left|\sum_{\varrho\in\mathscr{C}_2} e^{k(\varrho-i\gamma_M)^2+\mu(\varrho-i\gamma_M)}\right| \ll$$

$$\ll (n+\log|\Delta|)\lambda\log(\gamma_M+2\lambda+2)e^{k\left\{\left(\beta_M+\frac{2}{\mu}\right)^2-\varepsilon'^8\right\}+\mu\left(\beta_M+\frac{2}{\mu}\right)} <$$

(7.32)

$$< c_{15}(n+\log|\Delta|)\lambda\log\gamma' e^{\mu\beta_M+k\beta_M^2}e^{-\frac{\mu}{\lambda^5}} <$$

$$< \frac{\gamma'^{\varepsilon}\gamma'^{\varepsilon}e^{\mu\beta_M+k\beta_M^2}}{Y^{\varepsilon'^5}} < \frac{e^{\mu\beta_M+k\beta_M^2}}{\gamma'^{\varepsilon}}.$$

Finally, collecting (7.26), (7.27), (7.31) and (7.32) we get

$$|U| > 3\frac{e^{\mu\beta_M+k\beta_M^2}}{\gamma'^{\varepsilon}}+O(n) > 2\frac{e^{\mu\beta_M+k\beta_M^2}}{\gamma'^{\varepsilon}},$$

which gives the contradiction to (7.24), whence the theorem.

8. Proof of Theorem 3. First we shall investigate the case $\lim\limits_{u\to\infty} g(u)=0$. We define

(8.1) $$u := \log t,\quad r := \log x,\quad \tilde\omega(r) := \omega(x).$$

If $r>c_g$, then for $u<c_{16}$

(8.2) $$g(u)r+u \geqq g(c_{16})r > g(\sqrt{r})r+\sqrt{r} = o(r),$$

and so $g(u)r+u$ takes its minimal value for $u\geqq c_{16}$ — and this minimal value is taken only in one place, since by (3.12) at most for one $u\geqq c_{16}$

(8.3) $$0 = \frac{d}{du}(g(u)r+u) = g'(u)r+1 = 0 \Leftrightarrow g'(u) = -\frac{1}{r}.$$

Thus we can define the unique solution of (8.3) as the function of r, and denote it by $u_0(r)$, and for $u>c_g'$ we can define $r_0(u)$ as the unique solution of (8.3) in r, since

g' is strictly monotone by (3.12). Further for $\delta>0$ and $u>\delta r$ we have

$$g(u)r+u > \delta r > g(\sqrt{r})r+\sqrt{r} = o(r), \tag{8.4}$$

and thus we get

$$u_0(r) = o(r), \quad \text{i.e.} \quad \lim_{u\to\infty} \frac{r_0(u)}{u} = \lim_{r\to\infty} \frac{r}{u_0(r)} = \infty. \tag{8.5}$$

Now let us consider a number u_0, and order the given zeros with $\gamma>e^{u_0}$ in (3.10) according to the increasing imaginary parts, i.e. let

$$\varrho_k = \beta_k+i\gamma_k = \beta_k+ie^{u_k}, \quad \beta_k \geqq 1-g(u_k), \quad u_k \geqq u_{k-1}. \tag{8.6}$$

We want to apply Theorem 2 with an $\varepsilon<1/2-g(u_0)$, for which we suppose by $g(u)\to 0$ that we fixed a u_0 for which $g(u_0)<1/2$. Since $\gamma_k\to\infty$, for $k>k_1(\varepsilon, K)$ γ_k satisfies (3.6), and if we define

$$Y_k = e^{r_0(u_k)} \tag{8.7}$$

we see by (8.5) that for $k>k_2(\varepsilon, n, g)$ (3.7) is satisfied, too. Hence for $k>k_0(\varepsilon, K, g)$ $=\max(k_1, k_2)$ we can apply Theorem 2, and get an x_k with

$$r_0(u_k) \leqq \log x_k \leqq (1+\varepsilon)r_0(u_k), \tag{8.8}$$

for which

$$\begin{aligned} |\Delta_K(x)| &> \frac{x_k^{\beta_k}}{\gamma_k^{1+\varepsilon}} = \frac{x_k}{e^{(1-\beta_k)\log x_k+(1+\varepsilon)u_k}} \geqq \\ &\geqq \frac{x_k}{e^{g(u_k)(1+\varepsilon)r_0(u_k)+(1+\varepsilon)u_k}} \geqq \frac{x_k}{e^{(1+\varepsilon)\omega(Y_k)}} \geqq \frac{x_k}{e^{(1+\varepsilon)\omega(x_k)}} \end{aligned} \tag{8.9}$$

since ω is trivially an increasing function of x. In the case $\lim\limits_{u\to\infty} g(u)=H>0$ we have $H>1-\Theta_K$ and we have a zero on $\sigma=\Theta_K$ or a sequence of zeros with $\beta_k\to\Theta_K-0$ by (4.3). Hence with fixed $\varepsilon>0$ we have at least one $\varrho_0=\beta_0+i\gamma_0$ with $1-\beta_0<$ $<(1+\varepsilon/2)H$. Applying the Theorem of [13] we have for a sequence of sufficiently large x-values tending to infinity the estimate

$$|\Delta_K(x)| > \frac{x^{\beta_0}}{\sqrt{\gamma_0^2+50}} > \frac{x}{e^{(1-\beta_0)\log x+c(\gamma_0)}} > \frac{x}{e^{(1+\varepsilon/2)H\log x+\varepsilon/2H\log x}} \tag{8.10}$$

and so for any $u\geqq 0$

$$|\Delta_K(x)| > \frac{x}{e^{(1+\varepsilon)H\log x}} > \frac{x}{e^{(1+\varepsilon)(g(u)\log x+u)}} \geqq \frac{x}{e^{(1+\varepsilon)\omega(x)}}. \tag{8.11}$$

9. Finally, we can collect our knowledge about the oscillation of $\Delta_K(x)$ as follows. Giving a domain $\sigma\geqq 1-\eta(t)$ by a function η (or g) satisfying the conditions, the following cases are possible:

1) If the domain is zerofree, then

$$\Delta_K(x) = O_{K,\varepsilon,\eta}\left(\frac{x}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{9.1}$$

2) If the domain contains finitely many zeros $\varrho_1, \dots, \varrho_N$ but all of them have real parts not exceeding $\lim\limits_{u\to\infty} 1-g(u)=1-H$, then (taking into account also $\zeta_K(1+it)\neq 0$) we have

$$\Delta_K(x) = O_{K,\eta,\varepsilon,\varrho_1,\dots,\varrho_N}\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right). \tag{9.2}$$

3) If the domain has finitely many zeros, $\varrho_1, \dots, \varrho_N$, but some of them have real parts $>1-H$, we can see similarly to Theorem 1

$$\Delta_K(x) = -\sum_{j=1}^{N}\frac{x^{\varrho_j}}{\varrho_j} + O_{K,\varepsilon,\eta}\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right) = \Omega(x^{1-H}) = \Omega\left(\frac{x}{e^{\omega(x)}}\right). \tag{9.3}$$

4) If the domain contains an infinity of zeros, then

$$\Delta_K(x) = \Omega\left(\frac{x}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{9.4}$$

Till this time the only information known about the real situation is that $\sigma=1$ is zerofree, moreover 1) is true with

$$\eta(t) = \frac{1}{cn^{11}|\Delta|^3 \log^{2/3} t\,(\log\log t)^{1/3}} \quad \text{for} \quad t \geqq 4, \tag{9.5}$$

as can be seen in [1], and, on the other hand if $H=1/2$ then 4) must hold since there is an infinity of zeros with Re $\varrho \geqq 1/2$, and so on the boundary or in the interior of the given domain. In this latter case we know a little bit more, since we have by a theorem of Landau [8]

$$\Delta_K(x) = \Omega(\sqrt{x}\log\log\log x), \tag{9.6}$$

which is a generalization of Littlewood's result [9].

REFERENCES

[1] Bartz, K. M., On a theorem of A. V. Sokolovskiĭ, *Acta Arith.* **34** (1977/78), 113—126. *MR* **57** # 5923.

[2] Davenport, H., *Multiplicative number theory*, Second edition, Graduate Texts in Mathematics, 74, Springer-Verlag, New York—Berlin, 1980. *MR* **82m**: 10001.

[3] Holland, A. S. B., *Introduction to the theory of entire functions*, Pure and Applied Mathematics, Vol. 56, Academic Press, New York—London, 1973. *MR* **56** # 5882. *Zbl* **278.** 30001.

[4] Ingham, A. E., *The distribution of prime numbers*, Cambridge Tracts in Math. and Math. Phys. Nr. 30, Cambridge University Press, Cambridge, 1932. *Zbl* **6**, 397.

[5] Kolesnik, G. and Straus, E. G., On the sum of powers of complex numbers, *Studies in Pure Mathematics, To the memory of Paul Turán*, ed. by P. Erdős et al., Akadémiai Kiadó, Budapest, 1983, 427—442.

[6] LAGARIAS, J. C., MONTGOMERY, H. L., and ODLYZKO, A. M., A bound for the least prime ideal in the Chebotarev density theorem, *Invent. Math.* **54** (1979), 271—296.

[7] LANDAU, E., *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Teubner, Leipzig—Berlin, 1918.

[8] LANDAU, E., Über Ideale und Primideale in Idealklassen, *Math. Z.* **2** (1918), 52—154.

[9] LITTLEWOOD, J. E., Sur la distribution des nombres premiers, *C. R. Acad. Sci. Paris* **158** (1914), 1869—1872.

[10] NARKIEWICZ, W., *Elementary and analytic theory of algebraic numbers*, Monografie Matematyczne, Tom 57, PWN-Polish Scientific Publishers, Warszawa, 1974. *MR* **50** #268.

[11] PINTZ, J., On the remainder of the prime number formula I. On a problem of Littlewood, *Acta Arith.* **36** (1980), 341—365.

[12] PINTZ, J., On the remainder of the prime number formula II. On a theorem of Ingham, *Acta Arith.* **37** (1980), 209—220.

[13] RÉVÉSZ, SZ. GY., Irregularities in the distribution of prime ideals I, *Studia Sci. Math. Hungar.* **18** (1983), 57—67.

[14] SCHMIDT, E., Über die Anzahl der Primzahlen unter gegebener Grenze, *Math. Ann.* **57** (1903), 195—204.

[15] SOKOLOVSKII, A. V., Density theorems for a class of zeta functions, *Izv. Akad. Nauk. UzSSR Ser. Fiz. Mat. Nauk* **10** (1966), no 3, 33—40 (in Russian). *MR* **33** #7311.

[16] STAŚ, W., Über eine Anwendung der Methode von Turán auf die Theorie des Restgliedes im Primidealsatz, *Acta Arith.* **5** (1959), 179—195. *MR* **21** #6355.

[17] STAŚ, W., Über die Umkehrung eines Satzes von Ingham, *Acta Arith.* 6 (1960/61), 435—446. *MR* **26** #3679.

[18] STAŚ, W. and WIERTELAK, K., Some estimates in the theory of Dedekind zeta-functions, *Acta Arith.* **23** (1973), 127—135. *MR* **49** #2668.

[19] TURÁN, P., *Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953. *MR* **15**—688.

[20] TURÁN, P., On the so-called density-hypothesis in the theory of the zeta-function of Riemann, *Acta Arith.* **4** (1958), 31—56. *MR* **20** #2304.

(Received January 25, 1982)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
ANALÍZIS TANSZÉK
MÚZEUM KRT. 6—8
H—1088 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 371—375.

LOCALLY SEPARABLE CIRCLE PACKINGS

A. BEZDEK

A set of open domains arranged in the Euclidean plane is said to be totally separable if any two of them can be separated by a straight line avoiding all of the domains. This notion was introduced in a joint paper by G. Fejes Tóth and L. Fejes Tóth [1]. Let us recall the following theorem [1]. The (upper) density d of the densest totally separable packing of the plane with congruent convex discs s satisfies the inequality $d \leq \frac{A(s)}{A(q)}$ where $A(s)$ denotes the area of the disc s and $A(q)$ the area of a quadrangle of least area containing s. As an immediate consequence we phrase the following corollary: The density of a totally separable packing of equal circles is at most $\pi/4$. Equality holds for the face-incircles of the regular tiling $\{4, 4\}$.

We introduce the notion of locally separability. A triplet of disjoint open domains is said to be separable if there is a straight line not intersecting any of them, but containing on both sides one. We say a packing locally separable, if any triplet of the domains is separable. Obviously, any totally separable packing of domains is locally separable.

We shall prove that the above corollary remains true by replacing the term "totally separable" by "locally separable". On the other hand we shall give a construction to show that the statement arising from the above theorem by replacing the term "totally separable" by "locally separable" is false. Our counterexample will contain domains being in non-parallel position. It may be conjectured that the theorem under consideration is true under the restriction that the packing consists only of translates of a convex domain.

THEOREM. *The density of a locally separable packing of equal circles is at most* $\pi/4$.

First we remark that equality holds for the face incircles of the regular tiling $\{4, 4\}$. But this is not the only extremal configuration. For example, we get an other extremal configuration, if we translate a row of circles of the above packing in the direction of the row through a certain distance, or if we delete a set of circles of density 0.

PROOF. Let us consider a locally separable packing of unit circles $\{C_i\}$ with centres $\{O_i\}$. Without loss of generality we may suppose that any circle C of radius 4 contains a centre O_i. Since otherwise the unit circle concentric with C can be added to the packing preserving the property of local separability. Under this condition we

1980 *Mathematics Subject Classification.* Primary 52A45; Secondary 52A40.
Key words and phrases. Convex domain, locally separable packing, lattice, density.

can construct the tiling L^*, which was introduced by Molnár [2]. Associating with the point O_i the set D_i of all points lying nearer to O_i than to any other point O_j, we obtain the Dirichlet cell D_i. Considering all the Dirichlet cells $\{D_i\}$ we get a partition of the plane in cells $\{D_i\}$ forming a D-tiling (Dirichlet-tiling, [3]). Connecting the centres O_i, O_j of neighbouring Dirichlet cells D_i, D_j we obtain the dual L-tiling (Delaunay—Voronoi-tiling, [4]) of the plane. Dividing the faces of the L-tiling of more than three sides by non-intersecting diagonals we obtain the L'-tiling. The L^* tiling arises by replacing each edge O_iO_j of L' separating the face $O_iO_jO_k$ of L' from the centre V of its circumcircle by the broken line O_iVO_j [2]. Let the triangle $\triangle$ be a face of L'. Denote by $\triangle^*$ the face of L^* derived from $\triangle$.

Let P be a polygon. We consider those vertices of P which belong to the set $\{O_i\}$. Let α be the sum of angles of P at these vertices. We define the density of the circles relative to P by $d(P)=\frac{\pi\alpha}{A(P)}$. It is sufficient to prove that $d(\triangle^*)\leqq\frac{\pi}{4}$. We distinguish three types of faces $\triangle^*$. (1) $\triangle$ is an acute or right-angled triangle and the radius of its circumcircles is less than or equal to 2. (This means that $\triangle^*=\triangle$.) (2) $\triangle$ is an acute or right-angled triangle not of type (1). (3) $\triangle$ is an obtuse-angled triangle.

Type (1). Since the packing is locally separable, $\triangle$ has an altitude greater than or equal to 2. Thus $d(\triangle^*)\leqq\pi/4$. We shall need the following

LEMMA. *Let ABC and $A'B'C'$ be triangles such that $AB=BC\geqq A'B'=B'C'$ and $AC\geqq A'C'$. Then*

$$\frac{\sphericalangle BAC}{A(ABC)}\leqq\frac{\sphericalangle B'A'C'}{A(ABC)}.$$

PROOF. Consider the triangle $A''B''C''$ such that $A''B''=B''C''=AB$ and $A''C''=A'C'$. If A'', B'' are centres of unit circles then — $\frac{x}{\sin x}$ and $\frac{-x}{\tan x}$ being increasing functions —

$$\frac{\sphericalangle BAC}{A(ABC)}=\frac{\sphericalangle BAC}{\frac{AC}{2}\frac{AB}{2}\sin(\sphericalangle BAC)}\leqq\frac{\sphericalangle B''A''C''}{\frac{A''C''}{2}\frac{AB}{2}\sin(\sphericalangle B''A''C'')}=$$

$$=\frac{\sphericalangle B''A''C''}{\frac{A''C''}{2}\frac{A''C''}{2}\tan(\sphericalangle B''A''C'')}\leqq\frac{\sphericalangle B'A'C'}{\frac{A'C'}{2}\frac{A'C'}{2}\tan(\sphericalangle B'A'C')}=$$

$$=\frac{\sphericalangle B'A'C'}{A(A'B'C')}.$$

Type (2). Let V be the centre of the circumcircle of $\triangle=O_1O_2O_3$ (Fig. 1/a). Obviously, $d(\triangle^*)\leqq\max_{i=1,2,3} d(\triangle^*\cap O_iVO_{i+1})$ where $O_4\equiv O_1$. It follows from the Lemma that

$$d(\triangle^*\cap O_iVO_{i+1})\leqq d(O_iVO_{i+1}).$$

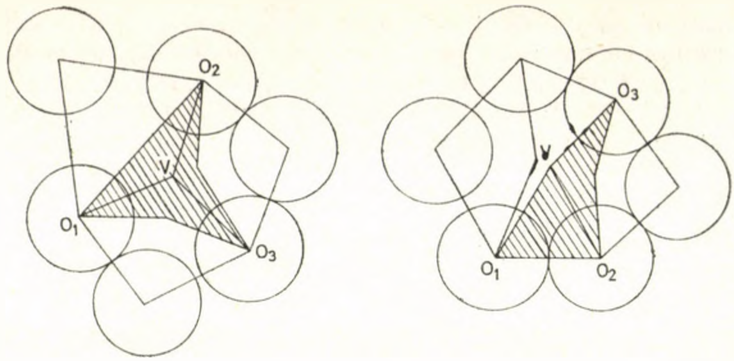


Fig. 1

Since $O_iV \geqq \sqrt{2}$ and $O_iO_{i+1} \geqq 2$ we have, by the Lemma, $d(O_iVO_{i+1}) \leqq \pi/4$.

Type (3). Without loss of generality we may suppose that the side O_1O_3 separates V from $\triangle$ (Fig. 1/b). Thus

$$d(\triangle^*) \leqq \max_{i=1,2} d(\triangle^* \cap O_iVO_{i+1}).$$

The proof can be finished similarly as in the case of Type (2). □

Now we describe the counter-example mentioned in the introduction (Fig. 4). The packing consists of centro-symmetric hexagons $H = H_1H_2 \dots H_6$ inscribed in rectangles $R = R_1R_2R_3R_4$ of sides $R_1R_2 = 1$ and $R_2R_3 = 10 - \sqrt{2} = l$ in the following way (Fig. 2). The vertex H_1 is the midpoint of R_1R_2, and the vertices H_2 and H_2

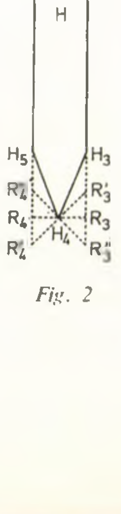


Fig. 2

are on R_2R_3 so that $R_2H_2=R_3H_3=1/2+\sqrt{2}$. It is easy to see that R is a parallelogram of least area containing H. Let $R'=R'_1...R'_4$ and $R''=R''_1...R''_4$ be the parallelogram derived from R by rotating the straight lines R_1R_2 and R_1R_3 about H_1 and H_4 through $+45°$ and $-45°$, respectively. Obviously, R, R' and R'' are of the same area. Since $\sqrt{2}$ and 1 are incommensurable, there exist for any $\varepsilon>0$ integers n, m such that $0<\frac{l}{\sqrt{2}}n-m=2\varepsilon<\varepsilon_0$. Let the rectangle $A=A_1...A_4$ be the union of 10 translates of R such that $A_1A_2=l$ and $A_2A_3=10$ (Fig. 3).

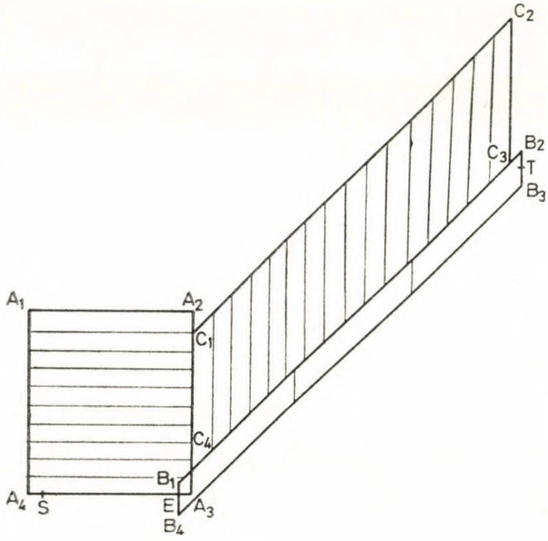


Fig. 3

Further let the parallelogram $B=B_1...B_4$ be the union of n translates of R'' such that $B_1B_2=nl$ and $B_2B_3=\sqrt{2}$. At last let the parallelogram $C=C_1...C_4$ be the union of m translates of R' such that $C_1C_2=m\sqrt{2}$ and $C_2C_3=l$. Let A, B, C be placed in such a way that 1) the midpoint E of B_1B_4 should be on A_4A_3; 2) $EA_3=$ $=\varepsilon$; 3) C_1C_4 should be on A_2A_3; 4) C_3C_4 should be on B_1B_2. Denote with D_ε the union of A, B and C.

Let S be a point on A_4A_3 such that $A_4S=\varepsilon$ and let T be the midpoint of B_2B_3.

Translate the parallelograms contained in D_ε by the lattice vectors of the lattice generated by the vectors $\overrightarrow{A_1A_4}$ and $\overrightarrow{ST}$. Inscribe in each parallelogram a hexagon. The set of hexagons has density greater than $\frac{A(H)}{A(R)}$, since the parallelograms cover the plane and some of them overlap (Fig. 4). Choose ε_0 so small that the triplet of hexagons denoted by star in Fig. 4 should be separable. It is easy to verify that the set of hexagons forms a packing which is locally separable.

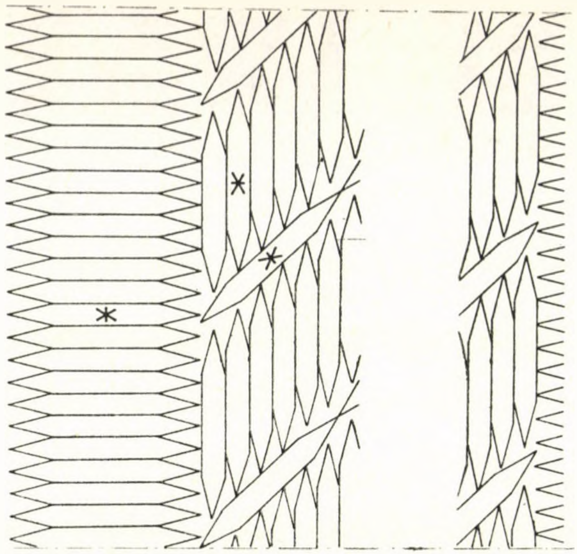


Fig. 4

REFERENCES

[1] FEJES TÓTH, G. and FEJES TÓTH, L., On totally separable domains, *Acta Math. Acad. Sci. Hungar.* **24** (1973), 229—232. *MR* **48** # 1052.

[2] MOLNÁR, J., On the ϱ-system of unit circles, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **20** (1977), 195—203. *MR* **58** # 12733.

[3] DIRICHLET, G. L., Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen, *J. Reine Angew. Math.* **40** (1850), 209—227.

[4] DELAUNAY, B., Sur la sphère vide, *Bull. Acad. Sci. URSS*, 7. S. Nr. 6 (1934), 793—800. *MR* **10**, 411.

(Received February 16, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15.
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 377—387.

EMBEDDING ORIENTED n-TREES IN TOURNAMENTS

K. B. REID and N. C. WORMALD

Abstract

Results are obtained in support of the conjecture of Sumner, that every tournament with $2n-2$ vertices contains every oriented tree on n vertices. The main results verify the conjecture for all trees in case the tournament is near-regular, and for all tournaments in case the oriented tree is an oriented path or an oriented caterpillar whose spine is a directed path.

Introduction

In 1980 Sumner [14] conjectured that every $(2n-2)$-tournament contains every oriented n-tree. Of course, the n-tree $K_{1,n-1}$ can be given an orientation so that the out-degree (or in-degree) of one vertex is $(n-1)$, and such an oriented n-tree is not contained in the regular $(2n-3)$-tournament. So, in a sense Sumner's conjecture is best possible. However, it can be refined to reveal some connection with other work as follows: given a specific n-tree W, what is the smallest integer $m=m(W)$ so that every m-tournament contains every orientation of W? For example, if W is an undirected path and $n\geqq 8$, then Rosenfeld [12] conjectured that $m(W)=n$. That conjecture generalizes the well-known result that every tournament contains a Hamiltonian path. Rosenfeld's conjecture has been established for directed paths with exactly one arc reversed by Grünbaum (Harary [7], p. 211, ex. 16.26), for antidirected paths by Grünbaum [6] and Rosenfeld [11], for all orientations in case n is a power of 2 by Forcade [4], for orientations yielding a directed path followed by the reverse of a directed path by Alspach and Rosenfeld [1] and by Straight [13], and for orientations yielding alternating "blocks" of directed paths and reversed directed paths provided the i-th block has at least $i+1$ vertices by Alspach and Rosenfeld [1]. Also, C. Thomassen [15] obtained some results on antidirected Hamiltonian paths in denumerably infinite tournaments. Sumner's conjecture restated is that $m(W)\leqq 2n-2$, for every n-tree W.

In this paper Sumner's conjecture will be proved for caterpillars of diameter at most 4, for paths, and for various special orientations of caterpillars. In some cases the present work will yield a better upper bound than in Sumner's conjecture.

Most of the terminology is as in [10]. In this paper a subset of vertices of a tournament is often identified with the subtournament induced by the set.

It is easy to see that the transitive n-tournament contains every oriented n-tree. Gyárfás' generalization [8] is that every acyclic orientation of any n-chromatic graph

1980 *Mathematics Subject Classification.* Primary 05C20; Secondary 05C05, 05C35.
Key words and phrases. Oriented trees, tournaments, near-regular tournaments, caterpillar.

contains all oriented n-trees. That result is reminiscent of Gallai's theorem [5] that every orientation of an n-chromatic graph contains a path of length $n-1$ and Chvátal's generalization [3] that if the arcs of any n-chromatic digraph without loops are k-colored and $n_1, n_2, \ldots, n_k$ are positive integers such that $n > n_1 n_2 \ldots n_k$, then for some j, $1 \leqq j \leqq k$, there exists a monochromatic directed path of length n_j.

A generalization of Sumner's conjecture which is attributed to Burr [2] is as follows: every orientation of any $(2n-2)$-chromatic graph contains all oriented n-trees. In [2] this conjecture is verified for all orientations of the star, and weaker assertions, obtained by replacing $2n-2$ by a larger number, are discussed. In particular, it is shown that every orientation of an $(n-1)^2$-chromatic graph contains all oriented n-trees $(n \geqq 3)$.

Results

The most nontransitive $(2n-2)$-tournaments are the near-regular $(2n-2)$-tournaments. They also contain every oriented n-tree as is seen next.

THEOREM 1. *Every near-regular* $(2n-2)$*-tournament contains every oriented* n*-tree.*

PROOF. Let X be an oriented n-tree, and let T be a near-regular $(2n-2)$-tournament. Let u be an endvertex in X, and let v be the vertex of X adjacent to u. A mapping α from the vertices of X into the vertices of T is described which will induce an isomorphism from X to an oriented n-tree contained in T. If v dominates u in X, let $\alpha(v)$ be any vertex of T with score $n-1$; otherwise, let $\alpha(v)$ be any vertex of T with score $n-2$. In the following S denotes the subset of X on which α has been defined. Initially $S=\{v\}$. After sufficient repetitions of the following operation S will contain all vertices of X other than u. The operation consists of three steps:

1. Choose a vertex $x \neq u$ of X such that x is not in S but is adjacent to some vertex s in S.

2. If s dominates x, let $\alpha(x)$ be any vertex of T in the out-set of $\alpha(s)$ which is not in $\alpha(S)$; otherwise, let $\alpha(x)$ be any vertex of T in the in-set of $\alpha(s)$ which is not in $\alpha(S)$.

3. Update S to be $S \cup \{x\}$. If S consists of all vertices of X other than u, then stop; otherwise return to step 1.

Note that if $|S| \leqq n-2$, then step 1 can be carried out. Moreover, if $|S| \leqq n-2$, then step 2 can be carried out because $d_T^+(\alpha(s)) \geqq n-2$, $d_T^-(\alpha(s)) \geqq n-2$ and $|\alpha(S)-\{\alpha(s)\}| \leqq n-3$. But when $|S|=n-1$, the operation ceases and α has been defined for all vertices of X other than u. The image $\alpha(u)$ of u then is defined by letting $x=u$ and $s=v$ and performing step 2 above. Step 2 can be carried out in this case for the following reasons. If v dominates u in X, then $d_T^+(\alpha(v))=n-1$, whereas $|\alpha(S)-\{\alpha(v)\}|=n-2$. Hence, the outset of $\alpha(v)$ contains some vertex not in $\alpha(S)$. And, if u dominates v in X, then $d_T^+(\alpha(v))=n-2$ and $d_T^-(\alpha(v))=n-1$, whereas $|\alpha(S)-\{\alpha(v)\}|=n-2$. Hence, the in-set of $\alpha(v)$ contains some vertex not in $\alpha(S)$.

Thus, α is defined on all vertices of X and by construction induces an isomorphism from X to an oriented n-tree in T. Theorem 1 follows.

In the proofs of the remaining results considerable use is made of the scores (i.e. outdegrees) of the vertices of the tournaments under consideration. The following Lemma and Corollaries are useful tools for these proofs.

LEMMA 2 (Landau [9]). *If* $(s_1, s_2, \ldots, s_N)$ *is the score sequence for an N-tournament, then for each* j, $1 \leqq j \leqq N$, $\frac{j-1}{2} \leqq s_j \leqq \frac{N+j-2}{2}$.

COROLLARY 3. *Suppose that T is a N-tournament and that r is a positive integer. If T contains both a vertex of score less than a and a vertex of score greater than $a+(N/2)-2+r$, then T contains at least $2r$ vertices with scores in the interval $[a, a+(N/2)-2+r]$.*

PROOF. Let $(s_1, s_2, \ldots, s_N)$ be the score sequence of T. Let s_i be the largest score less than a, and let s_{i+k} be the smallest score greater than $a+(N/2)-2+r$. Then

$$s_{i+k}-s_i \geqq \frac{N}{2}+r.$$

By Landau's result (Lemma 2), for each j, $1 \leqq j \leqq 2n-2$,

$$(j-1)/2 \leqq s_j \leqq \frac{N+j-2}{2},$$

so that

$$s_{i+k}-s_i \leqq \frac{N+k-1}{2}.$$

Thus,

$$\frac{N+k-1}{2} \geqq \frac{N+2r}{2}, \quad \text{or}$$

$$k-1 \geqq 2r.$$

That is, at least $2r$ scores are in the interval $[a, a+(N/2)-2+r]$.

COROLLARY 4. *Suppose that T is a $(2n-2)$-tournament and that a is an integer, $0 \leqq a \leqq n-1$. Then T contains at least two vertices with scores in the interval $[a, a+n-2]$ unless either $a=0$ and T contains a vertex with score 0 or $a=n-1$ and T contains a vertex with score $2n-3$.*

PROOF. Suppose that T does not contain two vertices with scores in $[a, a+n-2]$. Either T contains no vertices with scores in $[a, a+n-2]$ or T contains exactly one vertex with score in $[a, a+n-2]$. In the former case, by Corollary 3 either all scores are less than a or all scores are greater than $a+n-2$. But some score is at least $(2n-2)/2$ and some score is no greater than $n-2$ by Landau's result (Lemma 2). So, either $a>n-1$ or $a+n-2<n-2$, contrary to the choice of a.

If T contains exactly one vertex with score in $[a, a+n-2]$, then, by Lemma 3, either all scores save one are less than a or all scores save one are greater than $a+n-2$. As a is in $[0, n-1]$, either all scores save one are no greater than $n-2$ or all scores save one are no less than $n-1$. So, the sum of all scores save one is either no greater than $(2n-3)(n-2)=\binom{2n-2}{2}-(2n-3)$ or no less than $\binom{2n-2}{2}$. That is, the excluded score is either $2n-3$ (and $a=n-1$) or 0 (and $a=0$).

Theorem 1 suggests one approach to Sumner's conjecture, namely to prove that a specific class of $(2n-2)$-tournaments contains every oriented n-tree. Another approach, adopted primarily in this paper, is to prove that every $(2n-2)$-tournament contains every orientation of a specific n-tree (or at least many orientations of a specific n-tree). For example, Burr [2] proved the following result:

THEOREM 5. *Let A denote the orientation of $K_{1,n-1}$ with exactly one source, and let B denote the orientation of $K_{1,n-1}$ with exactly one sink. The smallest N such that every N-tournament contains every orientation of $K_{1,n-1}$ save A and B is given by $N=2n-3$. Also, every $(2n-2)$-tournament contains A and B, but no regular $(2n-3)$-tournament contains either A or B.*

Theorem 5 will serve to launch an induction proof to Theorem 7 below. Of course, combining the second remark in the Introduction with Theorem 5 yields

$$m(K_{1,n-1}) = 2n-2.$$

The next result concerning a specific tree, a path, is connected with the conjecture of Rosenfeld discussed in the Introduction. In order to state the result some definitions are required.

DEFINITION. Let Q be an orientation of an undirected path P with n vertices given by $v_1, v_2, \dots, v_n$ (where v_i is adjacent in P to v_{i-1} and v_{i+1}, $2\leqq i\leqq n-1$). Let Q_i denote the sub-digraph of Q induced by $\{v_{i+1}, v_{i+2}, \dots, v_n\}$, $0\leqq i\leqq n-1$. If $Q_0=Q$ is a directed path or its converse, then set $b(Q)=0$. If $Q_0=Q$ is not a directed path or its converse, define $b(Q)$ inductively as follows: let i denote the smallest index greater than 1 so that v_i is a source or sink in Q_0, and let $b(Q)=b(Q_i)+1$.

EXAMPLES. If Q is a directed path or its converse, then $b(Q)=0$. If Q is an anti-directed path, then $b(Q)=\left\lfloor\frac{n-1}{2}\right\rfloor$. If Q results from a directed path of length at least 2 by reversing exactly one arc, then $b(Q)=1$.

THEOREM 6. *Let Q denote an orientation of a path with n vertices, $n\geqq 1$. Then any $(n+b(Q))$-tournament contains a copy of Q.*

PROOF. The notation of the previous definition is used in this proof which proceeds by induction on $n\geqq 1$. If $n\leqq 2$, then $b(Q)=0$ as Q is the n-tournament, so the result follows. Let $n\geqq 3$. Suppose that every orientation Q' of a path with m vertices, $m<n$, is contained in every $(m+b(Q'))$-tournament. Let Q be an orientation of a path with n vertices given by $v_1, v_2, \dots, v_n$. Let T be any $(n+b(Q))$-tournament. Since $b(Q)\geqq b(Q-v_n)$, the induction hypothesis implies that T contains a copy of $Q-v_n$. Denote that copy by Q' and identify $v_1, v_2, \dots, v_{n-1}$ with the corresponding vertices in Q'. Let W denote the vertices of T distinct from $v_1, v_2, \dots,$ $\dots, v_{n-1}$. If v_{n-1} dominates v_n in Q and if any vertex of W is dominated by v_{n-1} in T, then T contains a copy of Q. Similarly, if v_n dominates v_{n-1} in Q and if any vertex of W dominates v_{n-1} in T, then T contains a copy of Q. This idea is extended in statement (1) of the following claim:

Either T contains a copy of Q or for $j=0, 1, \ldots, n-2$

(1) if v_{n-j-1} is dominated by v_{n-j} in Q, then in T, v_{n-j-1} dominates every vertex in W, and if v_{n-j-1} dominates v_{n-j} in Q, then in T, v_{n-j-1} is dominated by every vertex in W, and

(2) for every subset W_j of W of size $1+b(Q_{n-j-2})$, the subtournament of T induced by $W_j \cup \{v_{n-j-1}, v_{n-j}, \ldots, v_{n-1}\}$ contains a copy of Q_{n-j-2} which starts in W_j, and no arc of the copy joins two vertices in W_j.

The claim is applied when $j=n-2$ to establish that T contains a copy of Q. It is proved by induction on j. So, suppose that T contains no copy of Q. For $j=0$, statement (1) is the remark above preceeding the claim. For $j=0$ in statement (2), Q_{n-2} is a single arc, $b(Q_{n-2})=0$, W_0 is a singleton, and the subtournament of T induced by $W_0 \cup \{v_{n-1}\}$ is a single arc. So statement (2) follows for $j=0$. Inductively, assume that (1) and (2) hold for $0, 1, 2, \ldots, j$, where $0 \leqq j < n-2$. Suppose that the first half of statement (1) fails for $j+1$, i.e. suppose that v_{n-j-2} is dominated by v_{n-j-1}, but for some w_0 in W, w_0 also dominates v_{n-j-2}. Since $1+b(Q) \geqq \geqq 1+b(Q_{n-j-2})$, the induction hypothesis for (2) implies that the subtournament of T induced by $W \cup \{v_{n-j-1}, v_{n-j}, \ldots, v_{n-1}\}$ contains a copy Q'' of Q_{n-j-2} which starts with w_0. Follow Q' from v_1 to v_{n-j-2}, adjoin arc (w_0, v_{n-j-2}), then adjoin Q'' to obtain a copy of Q in T. Thus, if T contains no copy of Q, then the first part of (1) holds for $j+1$. The second part of (1) is established for $j+1$ in a similar manner.

In order to establish (2) for $j+1$, the four possibilities for the subdigraph of Q induced by $\{v_{n-j-2}, v_{n-j-1}, v_{n-j}\}$ are considered. Let W_{j+1} be any subset of W of size $1+b(Q_{n-j-3})$, and let T_{j+1} be the subtournament of T induced by $W_{j+1} \cup \cup \{v_{n-j-2}, v_{n-j-1}, \ldots, v_{n-1}\}$. First, suppose that v_{n-j-1} is dominated by both v_{n-j-2} and v_{n-j}. Let w_1 be any vertex in W_{j+1}. By the induction hypothesis (2) for $j-1$, the subtournament $T_{j+1}-\{w_1, v_{n-j-2}, v_{n-j-1}\}$ contains a copy Q'' of Q_{n-j-1} which starts in $W_{j+1}-\{w_1\}$ (say at w_2), and no arc of Q'' joins two vertices in $W_{j+1}-\{w_1\}$. That hypothesis is applicable since $|W_{j+1}-\{w_1\}|=b(Q_{n-j-3}) \geqq b(Q_{n-j-1})+1$ as Q_{n-j-3} contains the sink v_{n-j-1}. Thus, the arcs (w_1, v_{n-j-2}) and (w_2, v_{n-j-2}) followed by Q'' yield a copy of Q_{n-j-3} as desired. Second, the case in which v_{n-j-1} dominates both v_{n-j-2} and v_{n-j} is treated similarly. Third, suppose that v_{n-j-2} dominates v_{n-j-1} and that v_{n-j-1} dominates v_{n-j}. Let w_1 be any vertex of W_{j+1}. If Q_{n-j-3} is a directed path, then arc (w_1, v_{n-j-2}) (in T by statement (1)) followed by arcs of T_{j+1} (v_{n-j-2}, v_{n-j-1}), (v_{n-j-1}, v_{n-j}), $\ldots$, (v_{n-2}, v_{n-1}) yield a copy of Q_{n-j-3} in T_{j+1} as desired. So, suppose that Q_{n-j-3} is not a directed path, and let l be the largest integer, $1 \leqq l \leqq j$, so that v_{n-l} is a sink. Recall from statement (1) that every vertex of W dominates every vertex in $\{v_{n-j-2}, v_{n-j-1}, v_{n-j}, \ldots, v_{n-l-1}\}$, but v_{n-l} dominates every vertex of W. By the induction hypothesis, the subtournament $T_{j+1}-\{w_1, v_{n-j-2}\}$ contains a copy Q'' of Q_{n-l} which starts in $W_{j+1}-\{w_1\}$ say at w_2, and no arc of Q'' joins two vertices in $W_{j+1}-\{w_1\}$. That hypothesis is applicable since $|W_{j+1}-\{w_1\}| = =b(Q_{n-j-3}) \geqq 1+b(Q_{n-l})$ as Q_{n-j-3} contains the sink v_{n-l}. Thus, the arcs (w_1, v_{n-j-2}), (v_{n-j-2}, v_{n-j-1}), $\ldots$, (v_{n-l-2}, v_{n-l-1}), (w_2, v_{n-l-1}) followed by Q'' yield a copy of Q_{n-j-3} as required. Fourth, the case in which v_{n-j-2} is dominated by v_{n-j-1} and v_{n-j-1} is dominated by v_{n-j} is treated similarly. This completes the inductive step for (2).

By induction (1) and (2) follow. But, when $j=n-2$, (2) implies the existence of a copy of Q in T. So, in any case the theorem follows for n-paths. By induction the theorem follows for all $n \geqq 1$.

COROLLARY 7. *Every* $\left(n+\left[\frac{n-1}{2}\right]\right)$*-tournament contains every orientation of a path with* n *vertices.*

Orientations of caterpillars are a next step of complexity in the problem at hand. Recall that a caterpillar is a tree such that when its endvertices are removed the graph remaining is a path, called the *spine*. Caterpillars of diameter 2 were treated in Theorem 5. Orientations of caterpillars in which the spine is a directed path are treated in the next result.

THEOREM 8. *Every* $(2n-2)$*-tournament contains every oriented caterpillar with* n *vertices whose spine is a directed path.*

PROOF. Let Q denote an oriented caterpillar with n vertices whose spine is a directed path, say given by $x_1, x_2, \ldots, x_k$, where x_i dominates x_{i+1} in Q. Let $i_j = d_Q^-(x_j)-1$ for $2 \leqq j \leqq k$, $i_1 = d_Q^-(x_1)$, $e_j = d_Q^+(x_j)-1$ for $1 \leqq j \leqq k-1$, $e_k = d_Q^+(x_j)$. Note that $n = \sum_{j=1}^{k}(1+i_j+e_j)$. Let T denote an $(2n-2)$-tournament. The proof proceeds by induction on k. The case $k=1$ is implied by Theorem 5 above. Inductively assume that an oriented caterpillar with m vertices whose spine is a directed path with fewer than k vertices, $k>1$, is contained in every $(2m-2)$-tournament. Assume also that $k>1$. Let

$$N_0 = 2n-i_1-4 \quad \text{and for} \quad 1 \leqq j \leqq k \quad \text{set}$$

$$N_j = 2\left(n-\left(j+\sum_{p=1}^{j} i_p+\sum_{p=1}^{j} e_p\right)\right)-2,$$

$$\hat{N}_j = n-\left(j+\sum_{p=1}^{j} i_p+\sum_{p=1}^{j-1} e_p\right),$$

$$M_j = N_{j-1}+1, \quad \text{and}$$

$$\hat{M}_j = \hat{N}_j+(n-2),$$

where $\sum_{p=1}^{j-1} e_p$ is understood to mean 0 when $j=1$.

LEMMA A. *If T contains a vertex with score in the interval $[N_j, \hat{M}_j]$ for some j, $1 \leqq j \leqq k-1$, then T contains Q.*

PROOF. Pick j to be the smallest such index. Suppose $j=1$. Let z_1 denote a vertex of T with score in the interval $[N_1, \hat{M}_1]$. Let Q_1 denote the subcaterpillar of Q of order $n-(1+i_1+e_1)$ obtained by removing x_1 and the i_1+e_1 endvertices adjacent to and from x_1. The spine of Q_1 contains $k-1$ vertices and $d_T^+(z_1) \geqq N_1 = 2(n-(1+i_1+ +e_1))-2$, so by the induction hypothesis the out-set of z_1 contains a copy of Q_1. As $d_T^+(z_1) \leqq \hat{M}_1$, $d_T^-(z_1) \geqq i_1$, so let I_1 be any set of i_1 vertices in the in-set of z_1. If $N_1 \geqq \hat{N}_1$, then $d_T^+(z_1)-|V(Q_1)| \geqq e_1$, so let E_1 be any e_1 vertices in the out-set of

z_1 in T which are not in Q_1. Then the copy of Q_1, I_1, E_1, and z_1 yield a copy of Q in T. If $\hat{N}_1 \geqq N_1$, then as $\hat{M}_1 - \hat{N}_1 = n-2$ Corollary 4 implies that there exists a vertex of T with score in $[\hat{N}_1, \hat{M}_1]$. That vertex may serve as z_1 above. In any case the result holds in case $j=1$.

Suppose $j>1$. If the vertex x_j in Q and all the i_j+e_j endvertices of Q adjacent to and from x_j are removed from Q, two subcaterpillars are formed. Let P_j denote the one containing x_1 and let Q_j denote the one containing x_k. The proof of the case $j>1$ follows from the existence of a vertex z_j in T so that

$$\text{(a)}\quad d_T^+(z_j) \geqq N_j, \qquad \text{(c)}\quad d_T^-(z_j) \geqq 2\left[(j-1)+\sum_{p=1}^{j-1} i_p + \sum_{p=1}^{j-1} e_p\right]-2,$$

$$\text{(b)}\quad d_T^+(z_j) \geqq \hat{N}_j, \qquad \text{(d)}\quad d_T^-(z_j) \geqq (j-1)+\sum_{p=1}^{j} i_p + \sum_{p=1}^{j-1} e_p.$$

By induction, (a) and (c) insure the existence of copies of Q_j and P_j in the out-set of z_j and the in-set of z_j, respectively, and (b) and (d) insure sets of cardinality e_j and i_j in the out-set of z_j and the in-set of z_j, respectively, which are disjoint from those copies of Q_j and P_j. So those copies of Q_j and P_j, those sets of cardinality e_j and i_j, and the vertex z_j form a copy of Q in T.

Notice that (c) and (d) are equivalent to the following:

$$\text{(c}'\text{)}\quad d_T^+(z_j) = 2n-3-d_T^-(z_j) \leqq N_{j-1}+1 = M_j,$$

and

$$\text{(d}'\text{)}\quad d_T^+(z_j) = 2n-3-d_T^-(z_j) \leqq n-2+\hat{N}_j = \hat{M}_j.$$

Thus, a vertex z_j of T is required so that

$$(*)\qquad \max(N_j, \hat{N}_j) \leqq d_T^+(z_j) \leqq \min(M_j, \hat{M}_j).$$

Four cases are treated.

(i) $N_j \leqq \hat{N}_j$ and $\hat{M}_j \leqq M_j$. As $\hat{M}_j - \hat{N}_j = n-2$, a vertex z_j satisfying $(*)$ exists by Corollary 4.

(ii) $\hat{N}_j \leqq N_j$ and $\hat{M}_j \leqq M_j$. A vertex z_j satisfying $(*)$ exists by the choice of the index j.

(iii) $N_j \leqq \hat{N}_j$ and $M_j \leqq \hat{M}_j$. As $\hat{M}_{j-1} - \hat{N}_j = (n-2)+i_j+e_{j-1}+1 > n-2$, there is a vertex z_j of T with score in $[\hat{N}_j, \hat{M}_{j-1}]$. Now, by the choice of the index j, $d_T^+(z_j)$ is not in $[N_{j-1}, \hat{M}_{j-1}]$. But $N_{j-1}+1 = M_j$, so z_j satisfies $(*)$.

(iv) $\hat{N}_j \leqq N_j$ and $M_j \leqq \hat{M}_j$. As $\hat{M}_j < \hat{M}_{j-1}$, there is a vertex z_j in T with score in $[N_j, \hat{M}_{j-1}]$ by the choice of the index j. Now $d_T^+(z_j)$ is not in $[N_{j-1}, \hat{M}_{j-1}]$. But $N_{j-1}+1 = M_j$, so z_j satisfies $(*)$.

This completes the proof of Lemma A.

Lemma B. *If T contains a vertex with score in the interval* $[e_k, \min(M_k, \hat{M}_k)]$, *then T contains Q.*

Proof. Let z_k be such a vertex. Let Q_k denote the subcaterpillar of Q of order $n-(1+i_k+e_k)$ obtained by removing x_k and the i_k+e_k endvertices adjacent to and from x_k. The spine of Q_k contains $k-1$ vertices and $d_T^-(z_k)\geqq 2n-3-M_k=2n-4-N_{k-1}=2(n-(1+i_k+e_k))-2$, so by the induction hypothesis the in-set of z_k contains a copy of Q_k. Moreover, as $d_T^-(z_k)\geqq 2n-3-\hat{M}_k=n-1-\hat{N}_k=n-1-e_k$, $d_T^-(z_k)-|V(Q_k)|\geqq i_k$, so there exists a set I_k of i_k vertices in the in-set of z_k which is disjoint from the above copy of Q_k. Let E_k be any set of e_k vertices in the out-set of z_k in T (recall that $d_T^+(z_k)\geqq e_k$). Then the copy of Q_k, sets I_k and E_k, and z_k yield a copy of Q in T. This completes the proof of Lemma B.

To complete the proof of the theorem, by Lemmas A and B it suffices to show that if T contains no vertex with score in $\bigcup_{j=1}^{k-1} [N_j, \hat{M}_j]$, then there exists a vertex z_k with score in $[e_k, \min(M_k, \hat{M}_k)]$. Two cases arise.

(i) $\hat{M}_k\leqq M_k$. As $\hat{M}_k-e_k=n-2$, there exists a vertex z_k in T with score in $[e_k, \hat{M}_k]$ by Corollary 4.

(ii) $M_k\leqq \hat{M}_k$. As in case (i), there exists a vertex z_k in T with score in $[e_k, \hat{M}_k]$. Now $d_T^+(z_k)$ is not in $[N_{k-1}, \hat{M}_{k-1}]$ by the assumption above. And $\hat{M}_k<\hat{M}_{k-1}$, so $d_T^+(z_k)$ is not in $[N_{k-1}, \hat{M}_k]$. Thus $d_T^+(z_k)$ is in $[e_k, M_k]$ as $M_k=N_{k-1}+1$.

In any case, T contains a copy of Q if $k>1$. By induction, the proof of Theorem 8 is complete.

Actually, Theorem 8 yields a bit more. Let i_j and e_j, $1\leqq j\leqq k$, be $2k$ non-negative integers; let A_j (B_j) be a digraph of order i_j (respectively, e_j) which is known to exist in every i_j-tournament (respectively, e_j-tournament), $1\leqq j\leqq k$. For example, these digraphs could be any oriented path for which Rosenfeld's conjecture (discussed in the Introduction) is true. Let Q denote a digraph obtained from a directed path $x_1x_2\dots x_k$ by appending A_j and B_j to x_j, $1\leqq j\leqq k$, as follows: add arcs from (to) some vertices of A_j (B_j) to (from) x_j. Let $n=k+\sum_{j=1}^{k}(i_j+e_j)$. The caterpillar described in the proof of Theorem 8 is contained in every $(2n-2)$-tournament, so Q is contained in every $(2n-2)$-tournament.

It should be noted that if Q' denotes the directional dual of the oriented n-tree Q, and if every $(2n-2)$-tournament contains Q, then every $(2n-2)$-tournament contains Q'. For, if $\mathscr{I}$ denotes the set of m-tournaments and if T' denotes the dual of a tournament T, then

$$\{T'\colon T \text{ is in } \mathscr{I}\}=\mathscr{I}.$$

The oriented caterpillars of order n and diameter not exceeding 4 not treated by Theorem 8 are those whose spine is an antidirected path with 3 vertices. Thus, the following result completes the treatment of oriented caterpillars of order n and diameter 4.

Theorem 9. *Let Q be an oriented caterpillar of order n whose spine is an antidirected path with 3 vertices one of which dominates the other two. Then Q is contained in every $(2n-2)$-tournament.*

PROOF. Only a sketch of the proof will be given as the technique is similar to the technique used in the proof of Theorem 8, save at a certain point two scores are required in a certain interval. Let the vertices on the spine of Q be denoted by x_1, x_2, x_3, where x_2 dominates x_1 and x_3. Let $i_1 = d_Q^-(x_j) - 1$, $e_1 = d_Q^+(x_1)$, $i_2 = d_Q^-(x_2)$, $e_2 = d_Q^+(x_2) - 2$, $i_3 = d_Q^-(x_e) - 1$, $e_3 = d_Q^+(x_3)$. There is no loss of generality in assuming that $i_1 + e_1 \geqq i_3 + e_3$. Let T be any $(2n-2)$-tournament.

If there exists a vertex z_1 in T such that

$$d_T^-(z_1) \geqq 2(n - i_1 - e_1 - 1) - 2,$$

$$d_T^-(z_1) \geqq (n - i_1 - e_1 - 1) + i_1,$$

and

$$d_T^+(z_1) \geqq e_1,$$

then by Theorem 8 Q is in T (where z_1 plays the role of x_1 in the proof of Theorem 8). These conditions imply that the existence of a score in $[e_1, 2i_1 + 2e_1 + 1]$ is sufficient for finding a copy of Q in T.

Next, if there exists a vertex z_2 in T such that

$$d_T^+(z_2) \geqq 2(i_1 + e_1 + 1) - 2,$$

$$d_T^+(z_2) \geqq (i_1 + e_1 + 1) + (2(i_3 + e_3 + 1) - 2),$$

$$d_T^+(z_2) \geqq (i_1 + e_1 + 1) + (i_3 + e_3 + 1) + e_3, \quad \text{and}$$

$$d_T^-(z_2) \geqq i_2,$$

then, by Theorem 5, Q is in T (where z_2 plays the role of x_2). It should be noted here that if $2i_3 + 2e_3 \leqq i_1 + e_1 + 1$, then $i_1 + e_1 + 2i_3 + 2e_3 + 1 \leqq 2i_1 + 2e_1 + 2$. So, by Corollary 4, T contains a vertex with score in $[e_1, 2n - 3 - i_2] = [e_1, 2i_1 + 2e_1 + 1] \cup [i_1 + e_1 + 2i_3 + 2e_3 + 1, 2n - 3 - i_2]$, a vertex that serves as z_1 or z_2 above. So henceforth assume that $2i_3 + 2e_3 > i_1 + e_1 + 1$. Corollary 4, together with the conclusions of the last paragraph and these conditions, imply that the existence of a score in $[i_1 + e_1 + 2i_3 + 2e_3 + 1, 2n - 3 - i_2]$ is sufficient for finding a copy of Q in T.

Also, if there exists a vertex z_3 in T such that

$$d_T^-(z_3) \geqq 2(n - i_3 - e_3 - 1) - 2,$$

$$d_T^-(z_3) \geqq (n - i_3 - e_3 - 1) + i_1,$$

and

$$d_T^+(z_3) \geqq e_3,$$

then by Theorem 8, Q is in T (where z_3 plays the role of x_3). These conditions imply that the existence of a score in $[e_3, 2i_3 + 2e_3 + 1]$ is sufficient for finding a copy of Q in T. By the assumption above that $2i_3 + 2e_3 > i_1 + e_1 + 1$ it follows that $2i_3 + 2e_3 + 1 \geqq e_1$. So a score in $[e_3, e_1]$ (provided $e_3 \leqq e_1$) will yield z_3.

Combining the above remarks it is seen that the existence of a score in $[\min\{e_1, e_3\}, 2i_1 + 2e_1 + 1]$ or in $[i_1 + e_1 + 2i_3 + 2e_3 + 1, 2n - 3 - i_2]$ implies the existence of a copy of Q in T. So suppose that no such score exists.

Now if T contains two vertices z_2, z_3 such that z_2 dominates z_3 and

$$d_T^+(z_2) \geqq (2(i_1 + e_1 + 1) - 2) + 1,$$

and

$$d_T^+(z_2) \geq (i_1+e_1+1)+e_2+1,$$

and

$$d_T^-(z_3) \geq (i_1+e_1+1)+(e_2+1)+i_3,$$

and

$$d_T^+(z_3) \geq (i_1+e_1+1)+e_2+e_3,$$

and

$$d_T^-(z_2) \geq i_3+e_3+i_2,$$

then T contains a copy of Q (by Theorem 5). In the event that $e_2 \leq i_1+e_1-e_3$, there exist vertices z_2 and z_3 with scores in $[\min\{e_1, e_3\}, (n-2)+\min\{e_1, e_3\}]$ by Corollary 4. But by the first part of the proof, these scores may be assumed to lie in $[2e_1+2i_1+2, n-2+\min\{e_1, e_3\}]$, and so can be seen to satisfy the five inequalities above. In the event that $e_2 \geq i_1+e_1-e_3-1$, there exists vertices z_2 and z_3 with scores in $[n-1-i_2, 2n-3-i_2]$ by Corollary 4. Again by the first part of the proof, these scores may be assumed to lie in $[n-1-i_2, i_1+e_1+2i_3+2e_3]$, and so can be seen to satisfy the five inequalities above.

Theorem 9 follows.

Corollary 10. *Every orientation of a caterpillar of order n and diameter at most 4 is contained in every $(2n-2)$-tournament.*

Of course, as seen in Theorems 5 and 6, for specific orientations of an n-tree the bound $2n-2$ can be improved. In conclusion, it is demonstrated that for some trees the best possible bound depends heavily on the particular orientation. For example, the following argument shows that one orientation of the broom-like tree consisting of a $(k-1)$-path (handle) with $n-k$ endvertices attached to one end of the path is contained in all $(2n-2k+2)$-tournaments, but another orientation of the same tree is not in some $(2n-k-2)$-tournament. To see this let P denote a directed path with k vertices, say given by $x_1x_2\ldots x_k$. Let Q_1 denote the oriented n-tree obtained from P by adjoining $n-k$ vertices $y_1, y_2, \ldots, y_{n-k}$ and $n-k$ arcs (x_k, y_i), $1 \leq i \leq n-k$. Let Q_2 denote the oriented n-tree obtained from P by adjoining $n-k$ vertices $y_1, y_2, \ldots, \ldots, y_{n-k}$ and $n-k$ arcs (y_i, x_k), $1 \leq i \leq n-k$. Then Q_1 is not in any $(2n-k-2)$-tournament consisting of any $(k-1)$-tournament each vertex of which dominates every vertex of any regular $\big(2(n-k)-1\big)$-tournament. However, by inducting on k, it can be seen that Q_1 is in every $(2n-k-1)$-tournament. On the other hand, Q_2 is in the $(2n-k-2)$-tournaments above provided $n \geq 2k-1$. Moreover, Q_2 is contained in every $(2n-2k+2)$-tournament if $n \geq 3k-5$. To see this, let x denote a vertex of largest in-degree in a $(2n-2k+2)$-tournament T. Then $d_T^-(x)=n-k+\alpha$, for some $1 \leq \alpha \leq n-k+1$. First, treat the case $k-1 \leq \alpha \leq n-k+1$. Let R denote any path with $k-1$ vertices in the subtournament induced by the in-set of x (e.g. the first $k-1$ vertices on a Hamiltonian path in that subtournament), and let S denote any $n-k$ vertices in the in-set of x, none of which is on R. As $k-1 \leq \alpha$, such a set S exists. The $n-k$ arcs of the form (s, x), $s \in S$, together with the path R followed by x yield a copy of Q_2 in T. On the other hand, consider the case $1 \leq \alpha \leq k-2$. Let y denote the last vertex in some Hamiltonian path U in the subtournament induced by the out-set of x. By the maximality of $d_T^-(x)$, y dominates some vertex z in the in-set of x. Let S be any set of $n-k$ vertices of the in-set of x, none of which is z (use $\alpha \geq 1$).

Note that the number of vertices in the path U followed by the path yzx is equal to $d_T^+(x)+2=n-k-\alpha+3\geqq k$ (since $\alpha\leqq k-2$ and $n\geqq 3k-5$). The $n-k$ arcs of the form (s, x), $s\in S$, together with the path U followed by the path yzx yield a copy of Q_2 in T. In any case, Q_2 is in T.

REFERENCES

[1] Alspach, B. and Rosenfeld, M., Realization of certain paths in tournaments, *Discrete Math.*

[2] Burr, S. A., Subtrees of directed graphs and hypergraphs, *Proceedings of the Eleventh Southeastern Conference of Combinatorics, Graph Theory and Computing* (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol. I, *Congr. Numer.* **28** (1980), 227—239, Utilitas Math. Winnipeg, Man. *MR* **82f**: 05030.

[3] Chvátal, V., Monochromatic paths in edge-colored graphs, *J. Combinatorial Theory Ser. B* **13** (1972), 69—70. *MR* **46** # 8890.

[4] Forcade, R., Parity of paths and circuits in tournaments, *Discrete Math.* **6** (1973), 115—118.

[5] Gallai, T., On directed paths and circuits, *Theory of Graphs* (Proc. Colloq., Tihany, 1966), ed. by P. Erdős and G. Katona, Academic Press, New York, 1968, 115—118. *MR* **38** # 2054.

[6] Grünbaum, B., Antidirected Hamiltonian paths in tournaments, *J. Combinatorial Theory Ser. B* **11** (1971), 249—257. *MR* **45** # 116.

[7] Harary, F., *Graph theory*, Addison—Wesley, Publishing Co., Reading, Mass.—Menlo Park, Calif.—London, 1969. *MR* **41** # 1566.

[8] Gyárfás, A., Private communication, 1981.

[9] Landau, H. G., On dominance relations and the structure of animal societies II. Some effects of possible social factors, *Bull. Math. Biophys.* **13** (1951), 245—262. *MR* **14**—1000.

[10] Reid, K. B. and Beineke, L. W., Tournaments, *Selected Topics in Graph Theory*, ed. by L. W. Beineke and R. J. Wilson, Academic Press, London, 1978, 169—204. *MR* **81e**: 05059.

[11] Rosenfeld, M., Antidirected Hamiltonian paths in tournaments, *J. Combinatorial Theory Ser. B* **12** (1972), 93—99.

[12] Rosenfeld, M., Antidirected Hamiltonian circuits in tournaments, *J. Combinatorial Theory Ser. B* **16** (1974), 234—242. *MR* **49** # 4857.

[13] Straight, J., The existence of certain types of semiwalks in tournaments, *Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing* (Florida Atlantic Univer., Boca Raton, Fla., 1980), Vol. II, *Congr. Numer.* **29** (1980), 901—908, Utilitas Math., Winnipeg, Man. *MR* **82g**: 05047.

[14] Sumner, D., Private communication, 1980.

[15] Thomassen, C., Antidirected Hamiltonian circuits and paths in tournaments, *Math. Ann.* **201** (1973), 231—238. *MR* **50** # 1960.

(Received February 18, 1982)

DEPARTMENT OF MATHEMATICS
LOUISIANA STATE UNIVERSITY
BATON ROUGE, LA 70803
U. S. A.

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 389—400.

MIXED SYSTEMS WITH ISOMORPHIC LIMITS

C. G. CHEHATA, M. EL-GENDY and I. A. ASSEM

Introduction

The concept of mixed systems in categories was introduced in (2). In that paper, we have proved that such a system has two limits and that there exists a canonical morphism from one of these limits into the other. It was also noticed that this morphism is not in general an isomorphism.

In this work, we obtain sufficient conditions under which the canonical morphism is an isomorphism in the category Ens of sets and maps, that is, it is bijective. The same results and proofs hold in any category in which the underlying objects are sets (together with an additional structure) and in which a bijective morphism is an isomorphism such as the categories of groups, rings, modules but not for instance the category of topological spaces.

1. Preliminary notions

DEFINITION. A mixed system in a category $\mathscr{C}$ over $I\times L$ where I and L are two pre-ordered sets in a system $(E_\alpha^\lambda, f_{\alpha\beta}^{\mu\lambda})I\times L$ such that to each pair $(\alpha, \lambda)\in I\times L$ is associated an object E_α^λ of $\mathscr{C}$ and to each couple of pairs (α, μ) and (β, λ) of $I\times L$ such that $\alpha\leqq\beta$ and $\lambda\leqq\mu$ is associated a morphism $f_{\alpha\beta}^{\mu\lambda}$: $E_\beta^\lambda\to E_\alpha^\mu$ such that:

(MS1) For any $(\alpha, \lambda)\in I\times L$, we have:

$$f_{\alpha\alpha}^{\lambda\lambda}=1_{E_\alpha^\lambda}.$$

(MS2) If $\alpha\leqq\beta\leqq\gamma$ in I and $\lambda\leqq\mu\leqq\nu$ in L, then

$$f_{\alpha\gamma}^{\nu\lambda}=f_{\alpha\beta}^{\nu\mu}f_{\beta\gamma}^{\mu\lambda}.$$

To simplify notations, we shall denote from now on the morphisms $f_{\alpha\alpha}^{\mu\lambda}$ by $h_\alpha^{\mu\lambda}$ and the morphisms $f_{\alpha\beta}^{\lambda\lambda}$ by $g_{\alpha\beta}^\lambda$.

Given a mixed system $(E_\alpha^\lambda, f_{\alpha\beta}^{\mu\lambda})_{I\times L}$, the system $(E_\alpha^\lambda, h_\alpha^{\mu\lambda})_L$ (when α is kept constant) is an inductive system. We shall denote its limit by:

$$(E_\alpha, h_\alpha^\lambda)=\varinjlim_{\lambda\in L}(E_\alpha^\lambda, h_\alpha^{\mu\lambda}).$$

1980 *Mathematics Subject Classification.* Primary 18A30; Secondary 18A23.

Key words and phrases. Category, morphism, mixed systems, pre-ordered sets, maximal elements, canonical map.

Similarly, $(E_\alpha^\lambda, g_{\alpha\beta}^\lambda)$ is a projective system with limit

$$(E^\lambda, g_\alpha^\lambda) = \varprojlim_{\alpha\in I} (E_\alpha^\lambda, g_{\alpha\beta}^\lambda).$$

The family of morphisms $(g_{\alpha\beta}^\lambda)_{\lambda\in L}$ forms an inductive system of morphisms from $(E_\beta^\lambda, h_\beta^{\mu\lambda})_L$ into $(E_\alpha^\lambda, h_\alpha^{\mu\lambda})_L$. We shall denote its limit by

$$g_{\alpha\beta} = \varinjlim_{\lambda\in L} g_{\alpha\beta}^\lambda .$$

Similarly, $(h_\alpha^{\mu\lambda})_{\alpha\in I}$ forms a projective system of morphisms from $(E_\alpha^\lambda, g_{\alpha\beta}^\lambda)_I$ into $(E_\alpha^\mu, g_{\alpha\beta}^\mu)_I$, with limit

$$h^{\mu\lambda} = \varprojlim_{\alpha\in I} h_\alpha^{\mu\lambda} .$$

It is easily seen that $(E_\alpha, g_{\alpha\beta})_I$ is a projective system, with limit

$$(E, g_\alpha) = \varprojlim_{\alpha\in I} (E_\alpha, g_{\alpha\beta})$$

and that $(E^\lambda, h^{\mu\lambda})_L$ is an inductive system, with limit

$$(F, h^\lambda) = \varinjlim_{\lambda\in L} (E^\lambda, h^{\mu\lambda}).$$

It follows that Figure (1.1) is a commutative diagram (for $\alpha \leqq \beta$ and $\lambda \leqq \mu$).

$$\begin{tikzcd}
E_\alpha^\lambda \arrow[d, "h_\alpha^{\mu\lambda}"'] & E_\beta^\lambda \arrow[l, "g_{\alpha\beta}^\lambda"'] \arrow[d, "h_\beta^{\mu\lambda}"'] & & E^\lambda \arrow[ll, "g_\beta^\lambda"'] \arrow[d, "h^{\mu\lambda}"'] \\
E_\alpha^\mu \arrow[d, "h_\alpha^\mu"'] & E_\beta^\mu \arrow[l, "g_{\alpha\beta}^\mu"'] \arrow[d, "h_\beta^\mu"'] & & E^\mu \arrow[ll, "g_\beta^\mu"'] \arrow[d, "h^\mu"'] \\
E_\alpha & E_\beta \arrow[l, "g_{\alpha\beta}"'] & E \arrow[l, "g_\beta"'] & F
\end{tikzcd}$$

Fig. 1.1

E and F will be called the limits of the mixed system. By definition:

$$E = \varprojlim_{\alpha\in I} \varinjlim_{\lambda\in L} E_\alpha^\lambda$$

and

$$F = \varinjlim_{\lambda\in L} \varprojlim_{\alpha\in I} E_\alpha^\lambda .$$

It was shown (2) that there exists a unique morphism f: $F \to E$ such that

$$(\forall\, (\alpha, \lambda)\in I\times L)\colon\ g_\alpha f h^\lambda = h_\alpha^\lambda g_\alpha^\lambda .$$

f is called the canonical morphism of the system. It is thus the only morphism from F to E that makes Figure (1.2) a commutative

$$\begin{tikzcd}
& & E^\lambda_\alpha \arrow[dll, "h^\lambda_\alpha"'] & & \\
E_\alpha & & & & E^\lambda \arrow[ull, "g^\lambda_\alpha"'] \arrow[dl, "h^\lambda"] \\
& E \arrow[ul, "g_\alpha"] & & F \arrow[ll, "f"] &
\end{tikzcd}$$

Fig. 1.2

2. Mixed systems of sets

We now consider the case where the category $\mathscr{C}$ is the category Ens of sets and maps.

(A) *The limit set F.* For each index $\lambda \in L$, $E^\lambda = \varprojlim_{\alpha\in I} E^\lambda_\alpha$ is the subset of $\prod_{\alpha\in I} E^\lambda_\alpha$ consisting of all elements $x^\lambda = (x^\lambda_\alpha)_{\alpha\in I}$ for which

$$x^\lambda_\alpha = g^\lambda_{\alpha\beta}(x^\lambda_\beta) \quad \text{if} \quad \alpha \leqq \beta \tag{2.1}$$

(cf. (1) or (3)). The maps $(g^\lambda_\alpha\colon E^\lambda \to E^\lambda_\alpha)_{\alpha\in I}$ are the projections, that is, for each $\alpha\in I$, we have:

$$g^\lambda_\alpha(x^\lambda) = x^\lambda_\alpha. \tag{2.2}$$

To construct $F = \varinjlim_{\lambda\in L} E^\lambda$, we define an equivalence relation R on the set-theoretic sum $\coprod_{\lambda\in L} E^\lambda$ of the family $(E^\lambda)_{\lambda\in L}$ (cf. (3)). We shall say that $x^\lambda\in E^\lambda$ is equivalent to $x^\mu\in E^\mu$ (and write $x^\lambda R x^\mu$) if there exist two finite sequences:

$$\lambda_0 = \lambda, \quad \lambda_1, \lambda_2, \ldots, \lambda_{2n} = \mu \quad \text{in} \quad L$$

and

$$x^{\lambda_0} = x^\lambda, \quad x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_{2n}} = x^\mu \quad \text{in} \quad \coprod_{\lambda\in L} E^\lambda$$

such that

$$\lambda_{2k-1} \geqq \lambda_{2k-2}, \lambda_{2k}$$

and

$$h^{\lambda_{2k-1}\lambda_{2k-2}}(x^{\lambda_{2k-2}}) = h^{\lambda_{2k-1}\lambda_{2k}}(x^{\lambda_{2k}})$$

for all $k = 1, 2, \ldots, n$.

In case the index set L is directed, the relation R can be expressed in a much simpler way:

$x^\lambda\in E^\lambda$ is equivalent to $x^\mu\in E^\mu$ if there exists a $\nu\in L$ such that $\nu \geqq \lambda, \mu$ and

$$h^{\nu\lambda}(x^\lambda) = h^{\nu\mu}(x^\mu).$$

The quotient set $\left(\coprod_{\lambda\in L} E^\lambda\right)/R$ is equal to F. It has for elements the equivalence classes $h^\lambda(x^\lambda)$. Clearly

(2.3) $$x^\lambda R x^\mu \quad \text{iff} \quad h^\lambda(x^\lambda) = h^\mu(x^\mu).$$

(B) *The limit set E.* For each $\alpha \in I$, we have $E_\alpha = \varinjlim_{\lambda \in L} E_\alpha^\lambda$. We then define an equivalence R_α on $\coprod_{\lambda \in L} E_\alpha^\lambda$ by saying that $x_\alpha^\lambda \in E_\alpha^\lambda$ is equivalent to $x_\alpha^\mu \in E_\alpha^\mu$ (in symbols $x_\alpha^\lambda R_\alpha x_\alpha^\mu$) if there exist two finite sequences:

$$\lambda_0 = \lambda, \lambda_1, \lambda_2, \ldots, \lambda_{2n} = \mu \quad \text{in} \quad L$$

and

$$x_\alpha^{\lambda_0} = x_\alpha^\lambda, x_\alpha^{\lambda_1}, x_\alpha^{\lambda_2}, \ldots, x_\alpha^{\lambda_{2n}} = x_\alpha^\mu \quad \text{in} \quad \coprod_{\lambda \in L} E_\alpha^\lambda$$

such that

$$\lambda_{2k-1} \geqq \lambda_{2k-2}, \lambda_{2k},$$

and

$$h_\alpha^{\lambda_{2k-1}\lambda_{2k-2}}(x_\alpha^{\lambda_{2k-2}}) = h_\alpha^{\lambda_{2k-1}\lambda_{2k}}(x_\alpha^{\lambda_{2k}})$$

for all $k = 1, 2, \ldots, n$.

In case L is directed, x_α^λ is equivalent to x_α^μ if there exists a $\nu \in L$ such that $\nu \geqq \lambda, \mu$ and

$$h_\alpha^{\nu\lambda}(x_\alpha^\lambda) = h_\alpha^{\nu\mu}(x_\alpha^\mu).$$

Then $E_\alpha = (\coprod_{\lambda \in L} E_\alpha^\lambda)/R_\alpha$. In other words, an element of E_α is an equivalence class $\bar{x}_\alpha = h_\alpha^\lambda(x_\alpha^\lambda)$ (where x_α^λ is a representative of $\bar{x}_\alpha$). Again:

(2.4) $$x_\alpha^\lambda R_\alpha x_\alpha^\mu \quad \text{iff} \quad h_\alpha^\lambda(x_\alpha^\lambda) = h_\alpha^\mu(x_\alpha^\mu).$$

Finally, $E = \varprojlim_{\alpha \in I} E_\alpha$ is the subset of $\prod_{\alpha \in I} E_\alpha$ consisting of all $\bar{x} = (\bar{x}_\alpha)_{\alpha \in I}$ such that:

(2.5) $$\bar{x}_\alpha = g_{\alpha\beta}(\bar{x}_\beta) \quad \text{whenever} \quad \alpha \leqq \beta.$$

The maps $(g_\alpha\colon E \to E_\alpha)_{\alpha \in I}$ are the projections, hence

(2.6) $$g_\alpha(\bar{x}) = \bar{x}_\alpha \quad \text{for any} \quad \alpha \in I.$$

Our first result concerns the relation between the equivalences R and $(R_\alpha)_{\alpha \in I}$.

PROPOSITION 2.1. *If $x^\lambda = (x_\alpha^\lambda)_{\alpha \in I}$ and $x^\mu = (x_\alpha^\mu)_{\alpha \in I}$ are equivalent, then each of their respective coordinates are equivalent, that is*

$$x^\lambda R x^\mu \quad \textit{implies} \quad x_\alpha^\lambda R_\alpha x_\alpha^\mu \quad \textit{for any} \quad \alpha \in I.$$

PROOF. If $\nu \geqq \lambda$, the definition of $h^{\nu\lambda}$ implies

$$g_\alpha^\nu h^{\nu\lambda}(x^\lambda) = h_\alpha^{\nu\lambda} g_\alpha^\lambda(x^\lambda) = h_\alpha^{\nu\lambda}(x_\alpha^\lambda).$$

There exist two finite sequences

$$\lambda_0 = \lambda, \lambda_1, \ldots, \lambda_{2n} = \mu \quad \text{in} \quad L$$

and

$$x^{\lambda_0} = x^\lambda, x^{\lambda_1}, \ldots, x^{\lambda_{2n}} = x^\mu \quad \text{in} \quad \coprod_{\lambda \in L} E^\lambda$$

such that, for all $k=1, 2, \ldots, n$

$$\lambda_{2k-1} \geqq \lambda_{2k-2},\ \lambda_{2k}$$

and

$$h^{\lambda_{2k-1}\lambda_{2k-2}}(x^{\lambda_{2k-2}}) = h^{\lambda_{2k-1}\lambda_{2k}}(x^{\lambda_{2k}}),$$

hence, for any $\alpha \in I$

$$g_\alpha^{\lambda_{2k-1}} h^{\lambda_{2k-1}\lambda_{2k-2}}(x^{\lambda_{2k-2}}) = g_\alpha^{\lambda_{2k-1}} h^{\lambda_{2k-1}\lambda_{2k}}(x^{\lambda_{2k}})$$

and so

$$h_\alpha^{\lambda_{2k-1}\lambda_{2k-2}}(x_\alpha^{\lambda_{2k-2}}) = h_\alpha^{\lambda_{2k-1}\lambda_{2k}}(x_\alpha^{\lambda_{2k}})$$

for all k. Hence $x_\alpha^\lambda R_\alpha x_\alpha^\mu$ for any $\alpha \in I$.

The converse of this proposition is not true in general. However, we have

Proposition 2.2. *If* (1) *L has a maximal element or* (2) *L is directed and I finite then, for two elements* $x^\lambda=(x_\alpha^\lambda)_{\alpha\in I}$ *of* E^λ *and* $x^\mu=(x_\alpha^\mu)_{\alpha\in I}$ *of* E^μ,

$$x^\lambda R x^\mu \quad \textit{iff} \quad x_\alpha^\lambda R_\alpha x_\alpha^\mu \quad \textit{for any} \quad \alpha\in I.$$

Proof. We first notice that (1) and (2) mean, in particular, that L is directed. Assume that $x_\alpha^\lambda R_\alpha x_\alpha^\mu$ for each $\alpha \in I$, then, by definition of R_α

$$(\forall \alpha\in I)(\exists \nu_\alpha \in L)\colon\ \nu_\alpha \geqq \lambda, \mu \quad \text{and} \quad h_\alpha^{\nu_\alpha\lambda}(x_\alpha^\lambda) = h_\alpha^{\nu_\alpha\mu}(x_\alpha^\mu).$$

If there exists a unique ν corresponding to all α's, or equivalently, if there exists $\nu \geqq \nu_\alpha$ for all α, then, for such a ν we could write

$$h_\alpha^{\nu\lambda}(x_\alpha^\lambda) = h_\alpha^{\nu\lambda} g_\alpha^\lambda(x^\lambda) = g_\alpha^\nu h^{\nu\lambda}(x^\lambda)$$

and similarly,

$$h_\alpha^{\nu\mu}(x_\alpha^\mu) = g_\alpha^\nu h^{\nu\mu}(x^\mu),$$

and since, for all $\alpha \in I$, we have

$$h_\alpha^{\nu\lambda}(x_\alpha^\lambda) = h_\alpha^{\nu\mu}(x_\alpha^\mu).$$

Then

$$g_\alpha^\nu h^{\nu\lambda}(x^\lambda) = g_\alpha^\nu h^{\nu\mu}(x^\mu).$$

Since the family $(g_\alpha^\nu)_{\alpha\in I}$ is monomorphic,

$$h^{\nu\lambda}(x^\lambda) = h^{\nu\mu}(x^\mu),$$

that is,

$$x^\lambda R x^\mu.$$

But the existence of ν is not guaranteed by the mere fact that L is directed (except when I is finite: if this is the case, the family $\{\nu_\alpha\colon \alpha\in I\}$ is finite, and the directedness of L implies that it has an upper bound $\nu \geqq \nu_\alpha$ for all $\alpha \in I$). We must in general assume that L is such that any subset has an upper bound, that is, that L has a maximal element.

(C) *The canonical map.* Let $x \in F$, then, by definition,

$$(\exists \lambda\in L)(\exists x^\lambda \in E^\lambda)\colon\ x = h^\lambda(x^\lambda),$$

and

$$f(x) = fh^\lambda(x^\lambda).$$

Therefore

$$g_\alpha f(x) = g_\alpha f h^\lambda(x^\lambda) = h_\alpha^\lambda g_\alpha^\lambda(x^\lambda) = h_\alpha^\lambda(x_\alpha^\lambda),$$

where $x^\lambda = (x_\alpha^\lambda)_{\alpha\in I}$. It follows that

(2.7) $$f(x) = \big(h_\alpha^\lambda(x_\alpha^\lambda)\big)_{\alpha\in I}.$$

This definition is not ambiguous, for if $\lambda, \mu\in L$ and $x^\lambda\in E^\lambda$, $x^\mu\in E^\mu$ are such that

$$x = h^\lambda(x^\lambda) = h^\mu(x^\mu)$$

then, by Proposition (2.1), for any $\alpha\in I$,

$$h_\alpha^\lambda(x_\alpha^\lambda) = h_\alpha^\mu(x_\alpha^\mu).$$

3. General properties of the map f

THEOREM 3.1. *The canonical map f is injective iff for any $x^\lambda = (x_\alpha^\lambda)_{\alpha\in I}$ in E^λ and $x^\mu = (x_\alpha^\mu)_{\alpha\in I}$ in E^μ, the following implication holds true:*

$$x_\alpha^\lambda R_\alpha x_\alpha^\mu \quad \textit{for all} \quad \alpha\in I \quad \textit{implies} \quad x^\lambda R x^\mu.$$

PROOF. We first assume that f is injective and that $x^\lambda\in E^\lambda$, $x^\mu\in E^\mu$ are such that, for all $\alpha\in I$,

$$x_\alpha^\lambda R_\alpha x_\alpha^\mu$$

that is,

$$h_\alpha^\lambda(x_\alpha^\lambda) = h_\alpha^\mu(x_\alpha^\mu).$$

Then by equation (2.7),

$$fh^\lambda(x^\lambda) = fh^\mu(x^\mu).$$

Since f is injective,

$$h^\lambda(x^\lambda) = h^\mu(x^\mu)$$

or $x^\lambda R x^\mu$.

Conversely, suppose that the stated implication holds true, and let $x, y\in F$ be such that

$$f(x) = f(y)$$

$$(\exists\lambda\in L)(\exists x^\lambda\in E^\lambda)\colon\ x = h^\lambda(x^\lambda)$$

$$(\exists\mu\in L)(\exists x^\mu\in E^\mu)\colon\ y = h^\mu(x^\mu).$$

Then

$$fh^\lambda(x^\lambda) = fh^\mu(x^\mu).$$

By equation (2.7), this gives, for all $\alpha\in I$

$$h_\alpha^\lambda(x_\alpha^\lambda) = h_\alpha^\mu(x_\alpha^\mu)$$

or

$$x_\alpha^\lambda R_\alpha x_\alpha^\mu.$$

Since this holds true for all $\alpha\in I$, it implies by hypothesis $x^\lambda R x^\mu$, that is, $h^\lambda(x^\lambda) = h^\mu(x^\mu)$, or $x = y$. This shows that f is injective.

COROLLARY 3.1. *If* (1) *L has a maximal element or* (2) *L is directed and I finite then* $f\colon F \to E$ *is injective.*

PROOF. This follows from Theorem (3.1) and Proposition (2.2)

LEMMA 3.1. *The canonical map* $f\colon F \to E$ *is surjective if both of the following conditions are satisfied:*

(i) *L has a maximal element or L is directed and I finite.*

(ii) *All maps* $(h_\alpha^\lambda)_{I \times L}$ *are injective.*

PROOF. To prove that f is surjective, we take $\bar{x} \in E$ and try to find $\lambda \in L$ and $x^\lambda \in E^\lambda$ such that

$$\bar{x} = fh^\lambda(x^\lambda).$$

But $\bar{x} = (\bar{x}_\alpha)_{\alpha \in I}$ is such that, if $\alpha \leqq \beta$,

$$\bar{x}_\alpha = g_{\alpha\beta}(\bar{x}_\beta).$$

Each $\bar{x}_\alpha$ is an equivalence class, that is

$$(\forall \alpha \in I)(\exists \lambda_\alpha \in L)(\exists x_\alpha^{\lambda_\alpha} \in E_\alpha^{\lambda_\alpha})\colon \bar{x}_\alpha = h_\alpha^{\lambda_\alpha}(x_\alpha^{\lambda_\alpha}).$$

By (i), there exists a $\lambda \in L$ such that $\lambda \geqq \lambda_\alpha$ for all α, hence setting $x_\alpha^\lambda = h_\alpha^{\lambda\lambda_\alpha}(x_\alpha^{\lambda_\alpha})$, we get

$$\bar{x}_\alpha = h_\alpha^\lambda h_\alpha^{\lambda\lambda_\alpha}(x_\alpha^{\lambda_\alpha}) = h_\alpha^\lambda(x_\alpha^\lambda).$$

Let us put $x^\lambda = (x_\alpha^\lambda)_{\alpha \in I}$, then if $\alpha \leqq \beta$

$$g_{\alpha\beta}(\bar{x}_\beta) = g_{\alpha\beta} h_\beta^\lambda(x_\beta^\lambda) = h_\alpha^\lambda g_{\alpha\beta}^\lambda(x_\beta^\lambda).$$

On the other hand,

$$h_\alpha^\lambda(x_\alpha^\lambda) = \bar{x}_\alpha = g_{\alpha\beta}(\bar{x}_\beta) = h_\alpha^\lambda g_{\alpha\beta}^\lambda(x_\beta^\lambda).$$

Hence, by (ii),

$$x_\alpha^\lambda = g_{\alpha\beta}^\lambda(x_\beta^\lambda),$$

thus proving that $x^\lambda \in E^\lambda$. Finally, for any $\alpha \in I$

$$g_\alpha(\bar{x}) = \bar{x}_\alpha = h_\alpha^\lambda(x_\alpha^\lambda) = g_\alpha fh^\lambda(x^\lambda),$$

and since, the family $(g_\alpha)_{\alpha \in I}$ is monomorphic,

$$\bar{x} = fh^\lambda(x^\lambda).$$

Obviously:

COROLLARY 3.2. *The canonical map* $f\colon F \to E$ *is bijective if both of the following conditions are satisfied:*

(i) *L has a maximal element or L is directed and I finite;*

(ii) *All maps* $(h_\alpha^\lambda)_{I \times L}$ *are injective.*

4. Sufficient conditions for f to be bijective

THEOREM 4.1. *If L has a maximal element, the canonical map f: $F \to E$ is bijective.*

PROOF. We first remark that, since L has a maximal element λ (say) then, for any $\alpha \in I$, h^λ_α: $E^\lambda_\alpha \to E_\alpha$ is an isomorphism (cf. (1)).

Also the maximality of λ implies, by Corollary (3.1), that f is injective. It thus remains to prove it is also surjective. Let us take $\bar{x}=(\bar{x}_\alpha)_{\alpha\in I}$ in E. Since h^λ_α is bijective for any $\alpha \in I$ then

$$(\forall \alpha \in I)(\exists x^\lambda_\alpha \in E^\lambda_\alpha)\colon \bar{x}_\alpha = h^\lambda_\alpha(x^\lambda_\alpha).$$

We can then define $x^\lambda=(x^\lambda_\alpha)_{\alpha\in I}$. If $\alpha \leqq \beta$

$$h^\lambda_\alpha g^\lambda_{\alpha\beta}(x^\lambda_\beta) = g_{\alpha\beta} h^\lambda_\beta(x^\lambda_\beta) = g_{\alpha\beta}(\bar{x}_\beta) = \bar{x}_\alpha = h^\lambda_\alpha(x^\lambda_\alpha).$$

But h^λ_α is injective, hence the result. Moreover

$$\bar{x} = f h^\lambda(x^\lambda)$$

since, for any $\alpha \in I$, we have

$$g_\alpha f h^\lambda(x^\lambda) = h^\lambda_\alpha(x^\lambda_\alpha) = \bar{x}_\alpha.$$

THEOREM 4.2. *If I has a maximal element, the canonical map f: $F \to E$ is bijective.*

PROOF. Let α_0 denote the maximal element of I.

(1) We first prove that f is surjective. Let $\bar{x}=(\bar{x}_\alpha)_{\alpha\in I}$ belong to E. Corresponding to $\alpha_0 \in I$, there is a $\lambda \in L$ and an $x^\lambda_{\alpha_0} \in E^\lambda_{\alpha_0}$ such that,

$$\bar{x}_{\alpha_0} = h^\lambda_{\alpha_0}(x^\lambda_{\alpha_0}).$$

We can thus define, for any $\alpha \in I$, $\alpha \neq \alpha_0$,

$$x^\lambda_\alpha = g^\lambda_{\alpha\alpha_0}(x^\lambda_{\alpha_0}). \tag{4.1}$$

We now prove that $x^\lambda=(x^\lambda_\alpha)_{\alpha\in I}$ belongs to E^λ. If $\alpha \leqq \beta$, then

$$g^\lambda_{\alpha\beta}(x^\lambda_\beta) = g^\lambda_{\alpha\beta} g^\lambda_{\beta\alpha_0}(x^\lambda_{\alpha_0}) = g^\lambda_{\alpha\alpha_0}(x^\lambda_{\alpha_0}) = x^\lambda_\alpha.$$

The set $\{x^\lambda_\alpha\colon \alpha \in I\}$ is a set of representatives of the classes $\{\bar{x}_\alpha\colon \alpha \in I\}$, indeed,

$$h^\lambda_\alpha(x^\lambda_\alpha) = h^\lambda_\alpha g^\lambda_{\alpha\alpha_0}(x^\lambda_{\alpha_0}) = g_{\alpha\alpha_0} h^\lambda_{\alpha_0}(x^\lambda_{\alpha_0}) = g_{\alpha\alpha_0}(\bar{x}_{\alpha_0}) = \bar{x}_\alpha.$$

Finally, we prove as usual that

$$\bar{x} = f h^\lambda(x^\lambda).$$

(2) We now try to prove that f is injective; assume $x^\lambda \in E^\lambda$ and $x^\mu \in E^\mu$ are such that

$$f h^\lambda(x^\lambda) = f h^\mu(x^\mu).$$

For any $\alpha \in I$, this implies as usual

$$g_\alpha f h^\lambda(x^\lambda) = g_\alpha f h^\mu(x^\mu)$$
$$h^\lambda_\alpha(x^\lambda_\alpha) = h^\mu_\alpha(x^\mu_\alpha)$$
$$x^\lambda_\alpha R_\alpha x^\mu_\alpha .$$

This holds for all $\alpha \in I$, in particular for the maximal element α_0,

$$x^\lambda_{\alpha_0} R_{\alpha_0} x^\mu_{\alpha_0} .$$

Thus there exist two finite sequences

$$\lambda_0 = \lambda, \lambda_1, \ldots, \lambda_{2n} = \mu \quad \text{in} \quad L$$

$$x^{\lambda_0}_{\alpha_0} = x^\lambda_{\alpha_0}, x^{\lambda_1}_{\alpha_0}, \ldots, x^{\lambda_{2n}}_{\alpha_0} = x^\mu_{\alpha_0} \quad \text{in} \quad \coprod_{\lambda \in L} E^\lambda_{\alpha_0}$$

such that, for all $k = 1, 2, \ldots, n$

$$\lambda_{2k-1} \geqq \lambda_{2k-2}, \lambda_{2k}$$

and

$$h^{\lambda_{2k-1}\lambda_{2k-2}}_{\alpha_0}(x^{\lambda_{2k-2}}_{\alpha_0}) = h^{\lambda_{2k-1}\lambda_{2k}}_{\alpha_0}(x^{\lambda_{2k}}_{\alpha_0}). \tag{4.2}$$

We now define elements $x^{\lambda_i} \in E^{\lambda_i}$ $(i = 1, 2, \ldots, 2n-1)$ as follows: we let the α_0th coordinate of x^{λ_i} be $x^{\lambda_i}_{\alpha_0}$ and for $\alpha \in I$, $\alpha \neq \alpha_0$,

$$x^{\lambda_i}_\alpha = g^{\lambda_i}_{\alpha\alpha_0}(x^{\lambda_i}_{\alpha_0}).$$

By the first part of the proof $x^{\lambda_i} \in E^{\lambda_i}$. Equation (4.2) implies

$$g^{\lambda_{2k-1}}_{\alpha_0} h^{\lambda_{2k-1}\lambda_{2k-2}}(x^{\lambda_{2k-2}}) = g^{\lambda_{2k-1}}_{\alpha_0} h^{\lambda_{2k-1}\lambda_{2k}}(x^{\lambda_{2k}}).$$

Now the maximality of α_0 in I implies that $g^{\lambda_{2k-1}}_{\alpha_0}$: $E^{\lambda_{2k-1}} \to E^{\lambda_{2k-1}}_{\alpha_0}$ is bijective for all k (cf. (1)). Hence

$$h^{\lambda_{2k-1}\lambda_{2k-2}}(x^{\lambda_{2k-2}}) = h^{\lambda_{2k-1}\lambda_{2k}}(x^{\lambda_{2k}})$$

and

$$x^\lambda R x^\mu$$

proving that

$$h^\lambda(x^\lambda) = h^\mu(x^\mu).$$

THEOREM 4.3. *If I is the finite union of disjoint classes, each of which has a maximal element, and L is directed, the canonical map f: $F \to E$ is bijective.*

PROOF. Let

$$I = \bigcup_{i=1}^{n} K_i$$

with

$$K_i \cap K_j = \emptyset \quad \text{for} \quad i \neq j.$$

We shall assume that this partition of I is compatible with the given pre-order in I in the sense that two elements of I are comparable only if they belong to the same K_i, and we shall let α_i denote the maximal element of the class K_i.

(1) To prove that f is surjective, let $\bar{x}=(\bar{x}_\alpha)_{\alpha\in I}$ belong to E, then

$$(\forall i)(1 \leqq i \leqq n)(\exists \lambda_i\in L)(\exists x_{\alpha_i}^{\lambda_i}\in E_{\alpha_i}^{\lambda_i}) = \bar{x}_{\alpha_i} = h_{\alpha_i}^{\lambda_i}(x_{\alpha_i}^{\lambda_i}).$$

Since L is directed, there is a $\lambda\in L$ such that $\lambda\geqq\lambda_i$ for all $1\leqq i\leqq n$. We thus obtain

$$\bar{x}_{\alpha_i} = h_{\alpha_i}^{\lambda_i}(x_{\alpha_i}^{\lambda_i}) = h_{\alpha_i}^{\lambda} h_{\alpha_i}^{\lambda\lambda_i}(x_{\alpha_i}^{\lambda_i}) = h_{\alpha_i}^{\lambda}(x_{\alpha_i}^{\lambda})$$

where $x_{\alpha_i}^{\lambda}\in E_{\alpha_i}^{\lambda}$ is defined by

$$x_{\alpha_i}^{\lambda} = h_{\alpha_i}^{\lambda\lambda_i}(x_{\alpha_i}^{\lambda_i}).$$

We have thus found a unique λ and representatives $x_{\alpha_i}^{\lambda}$ of each class $\bar{x}_{\alpha_i}$ $(1\leqq i\leqq n)$. There remains to find representatives corresponding to those indices $\beta\in I$ which do not belong to the subset $\{\alpha_1, \ldots, \alpha_n\}$ of maximal elements. Since $\beta\in I$, there exists exactly one i such that $\beta\in K_i$. Put

$$x_\beta^{\lambda} = g_{\beta\alpha_i}^{\lambda}(x_{\alpha_i}^{\lambda}).$$

This is indeed a representative of the class $\bar{x}_\beta$,

$$h_\beta^{\lambda}(x_\beta^{\lambda}) = h_\beta^{\lambda} g_{\beta\alpha_i}^{\lambda}(x_{\alpha_i}^{\lambda}) = g_{\beta\alpha_i} h_{\alpha_i}^{\lambda}(x_{\alpha_i}^{\lambda}) = g_{\beta\alpha_i}(\bar{x}_{\alpha_i}) = \bar{x}_\beta.$$

Now $x^\lambda=(x_\gamma^\lambda)_{\gamma\in I}$ belongs to E^λ. Indeed, $\alpha\leqq\beta$ implies that both α and β belong to the same K_i, hence

$$g_{\alpha\beta}^{\lambda}(x_\beta^{\lambda}) = g_{\alpha\beta}^{\lambda} g_{\beta\alpha_i}^{\lambda}(x_{\alpha_i}^{\lambda}) = g_{\alpha\alpha_i}^{\lambda}(x_{\alpha_i}^{\lambda}) = x_\alpha^{\lambda}.$$

And we show as usual that $\bar{x}=fh^\lambda(x^\lambda)$.

(2) To prove that f is injective, assume that $\lambda\in L$ and two elements $x^\lambda, y^\lambda\in E^\lambda$ are given such that

$$fh^\lambda(x^\lambda) = fh^\lambda(y^\lambda)$$

(cf. (1)), then, for each $\alpha\in I$,

$$h_\alpha^{\lambda}(x_\alpha^{\lambda}) = h_\alpha^{\lambda}(y_\alpha^{\lambda}).$$

In particular, for any $1\leqq i\leqq n$,

$$h_{\alpha_i}^{\lambda}(x_{\alpha_i}^{\lambda}) = h_{\alpha_i}^{\lambda}(y_{\alpha_i}^{\lambda}).$$

Hence, by definition of the equivalence R_{α_i},

$$(\forall i)(1 \leqq i \leqq n)(\exists \mu_i\in L)(\mu_i \geqq \lambda):\ h_{\alpha_i}^{\mu_i\lambda}(x_{\alpha_i}^{\lambda}) = h_{\alpha_i}^{\mu_i\lambda}(y_{\alpha_i}^{\lambda}).$$

Since L is directed, there exists a $\mu\in L$ such that $\mu\geqq\mu_i$ for all $1\leqq i\leqq n$, hence

$$h_{\alpha_i}^{\mu\lambda}(x_{\alpha_i}^{\lambda}) = h_{\alpha_i}^{\mu\lambda}(y_{\alpha_i}^{\lambda}),$$

$$g_{\alpha_i}^{\mu} h^{\mu\lambda}(x^\lambda) = g_{\alpha_i}^{\mu} h^{\mu\lambda}(y^\lambda).$$

Let now $\alpha \in I$ be arbitrary and K_i be the (only) class to which it belongs, then

$$g^{\mu}_{\alpha} h^{\mu\lambda}(x^{\lambda}) = g^{\mu}_{\alpha\alpha_i} g^{\mu}_{\alpha_i} h^{\mu\lambda}(x^{\lambda}) = g^{\mu}_{\alpha\alpha_i} g^{\mu}_{\alpha_i} h^{\mu\lambda}(y^{\lambda}) = g^{\mu}_{\alpha} h^{\mu\lambda}(y^{\lambda}).$$

Thus

$$h^{\mu\lambda}(x^{\lambda}) = h^{\mu\lambda}(y^{\lambda}),$$

and finally

$$h^{\lambda}(x^{\lambda}) = h^{\lambda}(y^{\lambda}).$$

COROLLARY 4.1. *If I is finite and trivially ordered, and L is directed, the canonical map $f\colon F \to E$ is bijective.*

PROPOSITION 4.1. *If $(E^{\lambda}_{\alpha}, f^{\mu\lambda}_{\alpha\beta})_{I\times L}$ is a mixed system such that:* (i) *I is finite and L directed, and* (ii) *All maps $f^{\mu\lambda}_{\alpha\alpha}$ are injective, then $f\colon F \to E$ is bijective.*

PROOF. This follows from Corollary 3.2 since (ii) implies that all the maps h^{λ}_{α} are injective.

COROLLARY 4.2. *If $(E^{\lambda}_{\alpha}, f^{\mu\lambda}_{\alpha\beta})_{I\times L}$ is a mixed system such that:* (i) *I is finite and L directed and* (ii) *All maps $f^{\mu\lambda}_{\alpha\beta}$ are bijective, then $f\colon F \to E$ is bijective.*

LEMMA 4.1. *Let J be a cofinal subset of I, then the canonical map f of the mixed system $(E^{\lambda}_{\alpha}, f^{\mu\lambda}_{\alpha\beta})_{I\times L}$ is injective (bijective) iff the canonical map f^{*} of the mixed system $(E^{\lambda}_{\alpha}, f^{\mu\lambda}_{\alpha\beta})_{J\times L}$ is injective (bijective).*

PROOF. If we let $E^{*\lambda} = \varprojlim_{\alpha\in J} E^{\lambda}_{\alpha}$ and $E^{*} = \varprojlim_{\alpha\in J} E_{\alpha}$ then $E^{*\lambda}$ is isomorphic to E^{λ} for all λ, and E^{*} to E (cf. (3)). Let $j^{\lambda}\colon E^{\lambda} \to E^{*\lambda}$ denote the first bijection and $i\colon E \to E^{*}$ the second. Moreover, the set $F^{*} = \varinjlim_{\lambda\in L} E^{*\lambda}$ is isomorphic to $F = \varinjlim_{\lambda\in L} E^{\lambda}$. Let $j\colon F \to F^{*}$ be the latter isomorphism. Then Figure 4.1 is a commutative diagram and the result follows from the relation $f = i^{-1} f^{*} j$.

$$\begin{tikzcd}
E \arrow[d, "i"'] & F \arrow[l, "f"'] \arrow[d, "j"] & E^{\lambda} \arrow[l, "h^{\lambda}"'] \arrow[d, "j^{\lambda}"] \\
E^{*} & F^{*} \arrow[l, "f^{*}"'] & E^{*\lambda} \arrow[l, "h^{*\lambda}"']
\end{tikzcd}$$

Fig. 4.1

This lemma gives at once

PROPOSITION 4.2. (1) *If I has a finite cofinal subset and L is directed, then f is injective;*

(2) *If I has a finite cofinal subset J, L is directed and all the maps $\{f^{\mu\lambda}_{\alpha\alpha}\colon \lambda \leqq \mu, \alpha\in J\}$ are injective then f is bijective;*

(3) *Consequently, if I has a finite cofinal subset J, L is directed, and all the maps $\{f^{\mu\lambda}_{\alpha\beta}\colon \lambda \leqq \mu,\ \alpha \leqq \beta;\ \alpha, \beta \in J\}$ are injective then f is bijective.*

(4) *If I has a finite trivially ordered cofinal subset J and L is directed, then $f\colon F \to E$ is bijective.*

REFERENCES

[1] BOURBAKI, N., *Éléments de mathématique. Théorie des ensembles*, Hermann, Paris, 1970. *MR* **43** # 1849.
[2] CHEHATA, C. G., EL-GENDY, M. and ASSEM, I. A., Limits of bifunctors into a category, *Acta Math. Acad. Sci. Hungar.* **34** (1979), 3—15. *MR* **80i**: 18005.
[3] LAFON, J. P., *Les formalismes fondamentaux de l'algèbre commutative*, Collection Enseignement des Sciences, No. 20, Hermann, Paris, 1974. *MR* **51** # 465.

(Received March 8, 1982)

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF ALEXANDRIA
ALEXANDRIA
EGYPT

MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE
KING ABDUL-AZIZ UNIVERSITY
JEDDAH
SAUDI ARABIA

MATHEMATICS DEPARTMENT
CARLETON UNIVERSITY
OTTAWA, ONTARIO
K1S 5B6
CANADA

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 401—406.

AN UPPER BOUND FOR DIAGONAL RAMSEY NUMBERS

JÓZSEF BECK

1. Introduction

This paper deals with finite, simple and undirected graphs only. If F, G, H are graphs, write $F \to (G, H)$ to mean that if the edges of F are colored red and blue (say) in any fashion, then either the red subgraph of F contains a copy of G, or the blue subgraph of F contains a copy of H.

Let K_r denote the complete graph on r vertices. The classical Ramsey theorem yields that for every pair (G, H), there is a *Ramsey number* $r(G, H)$, which is the smallest integer r such that $K_r \to (G, H)$. If $G = H$ we write simple $r(G)$.

The first nontrivial exact result is due to L. Gerencsér and A. Gyárfás [5]. They determined $r(P_n, P_m)$, where P_n denotes the path of length n. Following the work of A. Bondy and P. Erdős [1], Vera Rosta [7] and independently R. Faudree and R. Schelp [4] determined $r(C_n, C_m)$, where C_n denotes the circuit of length n. Concerning further results, see the survey [2] and the excellent book [6].

For more complex graphs, however, there is a tremendous gap in our knowledge.

Our first result is an upper bound on Ramsey numbers for bipartite graphs.

THEOREM 1.1. *Let $K_{n,n}$ denote the complete $n \times n$ bipartite graph and let G be an arbitrary bipartite graph of n vertices and maximal degree Δ. Then*

$$r(K_{n,n}, G) < n^{2\Delta}.$$

Note that here the upper bound $n^{2\Delta}$ cannot be replaced by $n^{c\Delta}$, where c is a constant small enough. Indeed, using Lovász Local Lemma one can easily prove that if Δ is fixed and $n \to \infty$ then

$$r(K_{n,n}, K_{\Delta,\Delta}) > n^{c_0 \Delta}$$

with an absolute constant $c_0 > 0$ (see the argument in Spencer [8]).

Theorem 1.1 directly follows from Lemma 2.2 (see Section 2). The next result is an upper bound on diagonal Ramsey numbers.

THEOREM 1.2. *Assume that G has n vertices, maximal degree Δ and chromatic number k, then*

$$r(G) < (2n)^{2\Delta^{2k-3}}.$$

We remark that very recently E. Szemerédi proved (oral communication) the following basic theorem: $r(G) < c(\Delta) \cdot n$, *where $c(\Delta)$ is an absolute constant depending*

1980 *Mathematics Subject Classification.* Primary 05C55; Secondary 05A17.
Key words and phrases. Ramsey number of a graph, Turán numbers for uniform hypergraphs, "ramification" procedure.

only on the maximal degree Δ. This means that if Δ is fixed and $n\to\infty$ then the Ramsey number is *linear* as a function of the number of vertices (note that in this case Theorem 1.2 implies only that the Ramsey number is polynomial in n). However, if $\Delta\to\infty$ with $n\to\infty$, Szemerédi's proof gives a much weaker upper bound than Theorem 2.1. Szemerédi applied his famous "regular partition lemma", so in his proof the factor $c(\Delta)$ seems to be much more greater than e.g. the 3-fold iterated exponential function of Δ.

Finally, we mention the following corollary of Theorem 1.2: the Ramsey number of the *log n-dimensional cube* is less than $n^{c\log n}$. Erdős suspects that this Ramsey number is *polynomial* in n.

2. Proof of Theorem 1.2

Let $|X|$ denote the cardinality of the set X. Given a graph G and two disjoint subsets A, B of the set of vertices of G, denote by $G[A]$ and $G[A,B]$ the subgraph of G induced by A and the bipartite subgraph of G induced by $\{A, B\}$, i. e. the set of edges starting from A and terminating at B.

We say that a bipartite graph $G=G(A,B)$ has *type* (t,d) if $|A|=t$ and for each i-subset A_0 of A with $0<i\le d$ one can find at least t vertices in B being adjacent to the elements of A_0. For instance, the complete $t\times t$ bipartite graph $K_{t,t}$ is of type (t,d) for each $1\le d\le t$.

LEMMA 2.1. *Let $G=G(A,B)$ be a bipartite graph of type (t,d). Then given any bipartite graph H with bipartition $\{C,D\}$, $|C|+|D|\le t$ and with maximal degree $\le d$, and given any $|C|$-element subset A_0 of A with a one-to-one mapping $\varphi\colon C\to A_0$, there exist a $|D|$-element subset B_0 of B, a subgraph G_0 of the induced bipartite subgraph $G[A_0,B_0]$ with the same bipartition $\{A_0,B_0\}$ and a one-to-one mapping $\psi\colon C\cup D\to A_0\cup B_0$ such that ψ induces an isomorphism between H and G_0, and ψ is an extension of φ, i.e., $\psi|_C\equiv\varphi$.*

The proof is trivial. □

The proof of Theorem 1.2 will be based on the following Ramsey type lemma on bipartite graphs.

LEMMA 2.2. *Assume that $l=t^{2d}$. Then two-coloring the edges of $K_{l,l}$ red and blue in any fashion, there must exist either a red copy of $K_{t,t}$ or a blue copy of a bipartite graph of type (t,d).*

PROOF. Let f be a two-coloring of the edges of $K_{l,l}$. Let $\{X,Y\}$ denote the bipartition of the vertices of $K_{l,l}$. Set

$$A_i=\{x\in X\colon \{x,y_i\} \text{ has color } blue \text{ by } f\}\qquad i=1,\dots,l$$

where $Y=\{y_1,y_2,\dots,y_l)$.

We can express Lemma 2.2 in terms of A_i's as follows.

(1) Given any l not necessarily distinct subsets $A_1,\dots,A_l$ of an l-element set X, either there is a t-element subsequence $A_{i_1},\dots,A_{i_t}$ such that $|X\setminus\bigcup_{j=1}^{t}A_{i_j}|\ge t$, or

there is a t-subset T of X such that for each d-subset D of T one can find a t-element subsequence $A_{j_1}, \dots, A_{j_t}$ with $\bigcap_{i=1}^{t} A_{j_i} \supseteq D$.

In order to prove (1) we require two lemmas. First we recall Turán's numbers $\tau(l, t, d)$. Let $\tau(l, t, d)$ denote the smallest natural number q such that there exists a d-uniform hypergraph with l vertices, q d-edges, and with no *independent set* of size t. In other words, $\tau(l, t, d)$ is the smallest number of d-subsets of an l-set X such that every t-subset of X contains at least one of these subsets. For $d=2$ the well-known Turán theorem determines the exact values of τ. Although for $d>2$ there is no even asymptotically exact result, it will be suffices to use the following lower bound due to G. Katona, T. Nemetz and M. Simonovits (cf. [3], §13, a better estimate is due to J. Spencer [3], §13).

LEMMA 2.3.

$$\binom{l}{d} \Big/ \binom{t}{d} \leqq \tau(l, t, d).$$

For the sake of completeness we include here the simple *proof*. Let F be a minimal d-uniform hypergraph on an l-element set X with no t independent vertices. Each d-edge of F is contained in exactly $\binom{l-d}{t-d}$ t-sets. Since each t-set contains some d-edge, $|F| \binom{l-d}{t-d} \geqq \binom{l}{t}$. Therefore,

$$\tau(l, t, d) = |F| \geqq \binom{l}{t} \Big/ \binom{l-d}{t-d} = \binom{l}{d} \Big/ \binom{t}{d}. \quad \square$$

(1) and Lemma 2.3 motivates the following definition. We will denote by $S(l, m, t, d)$ the smallest integer s such that given any m not necessarily distinct subsets $A_1, \dots, A_m$ of an l-element set X, either one can find a t-element subsequence $A_{i_1}, \dots, A_{i_t}$ with $|X \setminus \bigcup_{j=1}^{t} A_{i_j}| \geqq t$, or the set

$\{D \subset X$: $|D|=d$ and *does not* exist $A_{j_1}, \dots, A_{j_t}$ with $\bigcap_{i=1}^{t} A_{j_i} \supseteq D\}$ has cardinality $\leqq s$.

LEMMA 2.4. *If* $m \geqq (2t)^d$ *then*

$$S(l, m, t, d) < \sum_{i=0}^{d-2} l^i \left\{ \binom{l}{d-i} - \binom{l-2t}{d-i} \right\} + l^{d-1}(2t).$$

PROOF. First we prove the following recursion formula:

$$S(l, m, t, d) < \binom{l}{d} - \binom{l-2t}{d} + lS\left(l, \frac{m}{2t}, t, d-1\right). \tag{2}$$

Let us be given m subsets $A_1, \dots, A_m$ of an l-element set X. Set

$$F_i = \{A_{(i-1)t+1}, \dots, A_{it}\} \quad \text{and} \quad B_i = \bigcup_{A \in F_i} A,$$

$1 \leqq i \leqq [m/t]$ (integral part). In what follows we assume that *does not* exist a t-element subsequence $A_{i_1}, \dots, A_{i_t}$ with $|X \setminus \bigcup_{j=1}^{t} A_{i_j}| \geqq t$. Then $|X \setminus B_i| < t$ for each i, $1 \leqq i \leqq$ $\leqq [m/t]$. Let Ω denote the set of the elements x of X for which $x \notin B_i$ for at least $1/2[m/t]$ indices i. We have

$$t\left[\frac{m}{t}\right] > \sum_{i=1}^{[m/t]} |X \setminus B_i| = \sum_{x \in X} \sum_{i: x \notin B_i} 1 \geqq |\Omega| \frac{1}{2}\left[\frac{m}{t}\right].$$

Therefore, $|\Omega| < 2t$. For later purpose we note that (3) if $d=1$ and $m/2t \geqq t$, then $S(l, m, t, d) \leqq |\Omega| < 2t$. Set $X_0 = X \setminus \Omega$. Denote by $m_0(x)$ the number of A_i's containing $x \in X$. From the definition of Ω it follows that for each $x \in X_0$ one can find at least $m/2t$ indices i with $x \in B_i$. Consequently, $m_0(x) \geqq m/2t$ for each $x \in X_0$. Set $|X_0| = l_0$. Observe that

$$\left|\left\{D \subset X\colon |D| = d \text{ and does not exist } A_{j_1}, \dots, A_{j_t} \text{ with } \bigcap_{i=1}^{t} A_{j_i} \supseteq D\right\}\right| \leqq$$

$$\leqq \left|\left\{D \subset X\colon |D| = d \quad\text{and}\quad D \cap \Omega \neq \emptyset\right\}\right| + \sum_{x \in X_0} S(l_0, m_0(x), t, d-1).$$

Since $l_0 > l - 2t$ and $m_0(x) \geqq m/2t$ for each $x \in X_0$, we conclude

$$\left|\left\{D \subset X\colon |D| = d \text{ and does not exist } A_{j_1}, \dots, A_{j_t} \text{ with } \bigcap_{i=1}^{t} A_{j_i} \supseteq D\right\}\right| <$$

$$< \left\{\binom{l}{d} - \binom{l-2t}{d}\right\} + lS(l, m/2t, t, d-1),$$

which completes the proof of (2).

By repeated application of (2) we obtain

$$S(l, m, t, d) < \left\{\binom{l}{d} - \binom{l-2t}{d}\right\} + lS\left(l, \frac{m}{2t}, t, d-1\right) \leqq$$

$$\leqq \left\{\binom{l}{d} - \binom{l-2t}{d}\right\} + l\left\{\binom{l}{d-1} - \binom{l-2t}{d-1}\right\} + l^2 S\left(l, \frac{m}{(2t)^2}, t, d-2\right) \leqq \tag{4}$$

$$\leqq \sum_{i=1}^{d-2} l^i \left\{\binom{l}{d-i} - \binom{l-2t}{d-i}\right\} + l^{d-1} S\left(l, \frac{m}{(2t)^{d-1}}, t, 1\right).$$

Lemma 2.4 follows from (4) and (3). □

Choosing $l=m$, $t=l^{1/(2d)}$, Lemma 2.4 yields after some easy calculation that

$$S(l, l, l^{1/2d}, d) < l^{-1/2}\binom{l}{d}. \tag{5}$$

Now let us return to (1). Estimate (5) gives that either there is a t-element subsequence $A_{i_1}, \dots, A_{i_t}$ with $|X \setminus \bigcup_{j=1}^{t} A_{i_j}| \geqq t$, and we are done, or the set

$$\left\{D \subset X\colon |D| = d \text{ and does not exist } A_{j_1}, \dots, A_{j_t} \text{ with } \bigcap_{i=1}^{t} A_{j_i} \supseteq D\right\}$$

has cardinality less than $l^{-1/2}\binom{l}{d}$. If the second alternative holds, we shall apply Lemma 2.3. Since for $t=l^{1/(2d)}$, $\binom{t}{d}\leqq l^{1/2}$, we conclude that there exists a t-subset T of X such that for each d-subset D of T one can find a t-element subsequence $A_{j_1}, \ldots, \ldots, A_{j_t}$ with $\bigcap_{i=1}^{t} A_{j_i} \supseteq D$. This completes the proof of Lemma 2.2. □

Now we are in the position to finish the proof of Theorem 1.2. It will be a repeated application of Lemma 2.1 and Lemma 2.2. Let us be given a two-coloring of the edges of the complete graph K_N. By Lemma 2.2 there exists a monochromatic subgraph $F_1=F_1(A_1, B_1)$ of type (N_1, d), where $N_1=(N/2)^{1/(2d)}$ (K_{N_1,N_1} is of type (N_1, d)). Again by Lemma 2.2 the induced complete subgraph $K_N[A_1]$ contains a monochromatic subgraph $F_2=F_2(A_2, B_2)$ of type (N_2, d), where $N_2=(N_1/2)^{1/(2d)}$. Repeating this argument we obtain a sequence of monochromatic subgraphs

$$F_i = F_i(A_i, B_i) \subseteqq K_N[A_{i-1}]$$

of type (N_i, d) where

$$N_i = (N_{i-1/2})^{1/(2d)}, \quad 1 \leqq i \leqq 2k-3$$

(we recall that k denotes the chromatic number of G). Since there are two colors, there must exist a subsequence $F_{i_1}, F_{i_2}, \ldots, F_{i_{k-1}}$ of length $k-1$ of (say) blue subgraphs.

Now let $\{V_1, V_2, \ldots, V_k\}$ denote the k-partition of the vertices of G. Choosing $N=(2n)^{(2d)^{2k-3}}$ we get $N_{2k-3}\geqq n$. Thus by Lemma 2.1 we conclude that $F_{i_{k-1}}$ contains a copy of the induced bipartite subgraph $G[V_{k-1}, V_k]$ of G. Again by Lemma 2.1 we obtain that $F_{i_{k-1}}\cup F_{i_{k-2}}$ contains a copy of $G[V_{k-1}, V_k]\cup G[V_{k-2}, V_{k-1}\cup \cup V_k]$.

By repeated application of Lemma 2.1 finally we obtain that $F_{i_{k-1}}\cup F_{i_{k-2}}\cup \ldots \ldots \cup F_{i_1}$ contains a copy of

$$G[V_{k-1}, V_k]\cup G[V_{k-2}, V_{k-1}\cup V_k]\cup \ldots \cup G[V_1, V_2\cup \ldots \cup V_k] = G.$$

Theorem 1.2 follows with $\Delta=d$. □

REFERENCES

[1] Bondy, J. A. and Erdős, P., Ramsey numbers for cycles in graphs, *J. Combinatorial Theory Ser. B* **14** (1973), 46—54. *MR* **47** #6540.

[2] Burr, S. A., Generalized Ramsey theory for graphs — a survey, *Graphs and Combinatorics* (Proc. Capital Conf. George Washington Univ., Washington, D. C., 1973), Lecture Notes in Math., Vol. 406, Springer, Berlin, 1974, 52—75. *MR* **52** #116.

[3] Erdős, P. and Spencer, J., *Probabilistic methods in combinatorics*, Probability and Mathematical Statistics, Vol. 17, Academic Press, New York—London, 1974. *MR* **52** #2895.

[4] Faudree, R. J. and Schelp, R. H., All Ramsey numbers for cycles in graphs, *Discrete Math.* **8** (1974), 313—329. *MR* **49** #10596.

[5] Gerencsér, L. and Gyárfás, A., On Ramsey type problems, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **10** (1967), 167—170. *MR* **39** #1351.

[6] GRAHAM, R. L., ROTHSCHILD, B. L. and SPENCER, J., *Ramsey theory*, Wiley-Interscience Series in Discrete Mathematics, A, Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1980. *MR* **82b**: 05001.
[7] ROSTA, V., On a Ramsey-type problem of J. A. Bondy and P. Erdős, I—II, *J. Combinatorial Theory Ser. B* **15** (1973), 94—104, 105—120. *MR* **48** # 10894.
[8] SPENCER, J., Asymptotic lower bounds for Ramsey functions, *Discrete Math.* **20** (1977/78), 69—76. *MR* **58** # 10600.
[9] SZEMERÉDI, E., oral communication.

(Received March 16, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 407—412.

DECOMPOSING REGULAR r-UNIFORM HYPERGRAPHS INTO REGULAR FACTORS

A. J. W. HILTON

1. Introduction

In this note we consider the problem of whether there is for hypergraphs an analogue of Petersen's theorem [3] that a regular (multi-) graph of even degree is the union of 2-factors. We shall consider r-uniform hypergraphs $H=(V, \mathcal{E})$, where V is a set of vertices and $\mathcal{E}$ is a collection (or multiset) of edges, an edge being here a subset of V of cardinality r; thus $\mathcal{E}$ may contain an edge e several times, the number of times being the *multiplicity* $m(e)$ of e. The *order* of H is the number of vertices; the *degree* of a vertex is the number of edges (counting multiplicities) which contain a vertex. A hypergraph is *regular* of degree d if the degree of each vertex is d. A regular hypergraph $H=(V, \mathcal{E})$ of degree d is said to have a d_1-factor and a d_2-factor, where $d=d_1+d_2$, if there are regular hypergraphs $H_1=(V, \mathcal{E}_1)$ and $H_2=(V, \mathcal{E}_2)$ of degrees d_1 and d_2, respectively, such that for $e\in\mathcal{E}$, $m_1(e)+m_2(e)=m(e)$, where $m_1(e)$ and $m_2(e)$ denote the multiplicities of e in H_1 and H_2, respectively. We say that H is the union of H_1 and H_2.

Neumann-Lara [2] asked if it is true that for $m\geqq 1$, $r\geqq 2$ every regular r-uniform hypergraph of degree mr is the union of m factors, each of degree r. If this were true it would be a nice generalization of Petersen's theorem — the loopless version of Petersen's theorem is the special case of Neumann-Lara's conjecture when $r=2$. However, the conjecture is not true as we show below in Section 2. R. M. Wilson informed me that he also had found some counterexamples. However I do not know of any counterexample to the weaker conjecture that for $m\geqq 1$, $r\geqq 2$, every regular r-uniform hypergraph of degree $m(r!)$ is the union of m r-uniform hypergraphs of degree $r!$. If true, this would also be a nice generalization of Petersen's theorem. Of course, the question of whether the basic regular factors should be of degree r or $r!$ is not really the main point at this stage — the important question is whether there are such basic regular factors at all.

Petersen's theorem and this conjecture are independent of the order of the graph or hypergraph. We obtain in Section 3 some decomposition theorems for some special kinds of regular r-uniform hypergraphs which are weaker than the conjecture mentioned above because the degree of the basic regular factors depends not only on the size of the edges but on the order of the hypergraph as well.

1980 *Mathematics Subject Classification.* Primary 05C65.
Key words and phrases. Regular r-uniform hypergraphs.

2. Counter-example to Neumann-Lara's conjecture

Let $r=3$. Let G be a regular loopless graph of degree 6 which is not the union of two regular graphs of degree three and let $|V(G)|$ be even. [An example of such a graph G is the disjoint union of two copies of K_7. For K_7 is not the union of two 3-factors since a regular graph of degree 3 has an even number of vertices.]

Now let H be the 3-uniform hypergraph formed from G as follows. First observe that since $2|E(G)|=6|V(G)|$ and $|V(G)|$ is even, then $6 \mid |E(G)|$. Collect the edges of G into sets of 6 edges each, take $|E(G)|/6$ new vertices and, for each such new vertex, select one of the sets of 6 edges and adjoin the new vertex to each of the 6 edges. Then each of the edges of G is extended to a 3-set, i.e. an edge of a 3-uniform hypergraph. The 3-uniform hypergraph so formed is H. It is easy to see that it is regular of degree 6; also it is not the union of two 3-uniform hypergraphs of degree 3 since, if it were, the restriction of $V(H)$ to $V(G)$ would give two regular factors of G of degree 3, a contradiction.

3. Decompositions of regular r-uniform hypergraphs

Let $r\geqq 2$. The regular r-uniform hypergraphs $H=(V, \mathscr{E})$ we consider are of a special kind known as star-regular. A *star-regular* r-uniform hypergraph is either

(i) A regular r-uniform hypergraph of degree divisible by r and of order $r+2$;

or

(ii) A regular graph of even degree if $r=2$;

or

(iii) For $r\geqq 3$ and $n>r+2$, a regular r-uniform hypergraph satisfying the following inductive relationship: there is a vertex $v\in V$ such that

(a) the $(r-1)$-uniform hypergraph $H'=(V', \mathscr{E}')$ is star-regular, where $V'=V\setminus\{v\}$ and $\mathscr{E}'$ is a multiset containing the edge $e\setminus\{v\}$ $m(e)$ times for each $e\in\mathscr{E}$ such that $v\in e$, and

(b) the r-uniform hypergraph $H''=(V'', \mathscr{E}'')$ is star-regular, where $V''=V'$ and $\mathscr{E}''$ is a multiset containing the edge e $m(e)$ times for each $e\in\mathscr{E}$ such that $v\notin e$.

For $n-2\geqq r\geqq 2$, let $K(r; n)$ be the set of positive integers s which have the property that there is a star-regular r-uniform hypergraph of degree s and order n, and, for each positive integer k, every star-regular r-uniform hypergraph H of degree ks and order n contains an s-factor. Let $k(r; n)$ be the smallest integer $\geqq 1$ contained by $K(r; n)$; it is a consequence of the results below that $k(r; n)$ exists.

LEMMA 1. $k(2, 4)=1$; $k(2, n)=2$ *for* $n\geqq 5$.

PROOF. By Petersen's theorem, any regular graph of even degree is the union of edge-disjoint 2-factors. Therefore $2\in K(2, n)$ for $n\geqq 5$. However, any loopless regular graph on 4 vertices of degree 2 is the union of two 1-factors. Consequently, $k(2, 4)=1$. On the other hand, if $n\geqq 5$ there is a loopless regular graph of degree 2 on n vertices which contains an odd circuit and which, therefore, is not the union of two 1-factors. Therefore $1\notin K(2, n)$ for $n\geqq 5$. Therefore $k(2, n)=2$ for $n\geqq 5$.

LEMMA 2. *Let* $n-2\geqq r\geqq 2$. *If* $g(r, n)\in K(r, n)$ *and* t *is a positive integer then* $tg(r, n)\in K(r, n)$ *also.*

PROOF. Obvious from the definitions.

LEMMA 3. *Let* $n-2 \geqq r \geqq 2$. *If* $g(r, n) \in K(r, n)$ *then* $r \mid ng(r, n)$.

PROOF. Consider any regular r-uniform hypergraph $H=(V, \mathscr{E})$ of degree $g(r, n)$ and order n. Then $\sum_{v \in V} d(v) = ng(r, n)$, where $d(v)$ denotes the degree of the vertex v. But $\sum_{v \in V} d(v) = \sum_{e \in \mathscr{E}} r$. The result now follows.

LEMMA 4. *Let* $n-2 > r > 2$. *If* $g(r, n) \in K(r, n)$ *then* $(n-1) \mid (r-1)g(r, n)$.

PROOF. Consider any star-regular r-uniform hypergraph $H=(V, \mathscr{E})$ of degree $g(r, n)$ and order n. Let $v \in V$ be the vertex to which part (iii) of the definition of star-regular applies. Let $H'=(V', \mathscr{E}')$ be the $(r-1)$-uniform hypergraph of part (iii)(a) of the definition. Then $|\mathscr{E}'| = g(r, n)$. Therefore by the argument of Lemma 3, $(r-1)g(r, n) = \sum_{w \in V'} d'(w) = (n-1)d'(H')$, where $d'(H')$ is the degree in H' of each vertex of H'. Lemma 4 now follows.

LEMMA 5. *If* $r \geqq 3$ *then* $k(r, r+2) = r$.

PROOF. Let $H=(V, \mathscr{E})$ be a star-regular r-uniform hypergraph of degree sr and order $r+2$. Let G be the regular graph of degree $2s$ with the vertex set V and edge multi-set $\mathscr{E}^*$ in which $V \setminus e$ is contained by $\mathscr{E}^*$ $m(e)$ times for each $e \in \mathscr{E}$. By Petersen's theorem $\mathscr{E}^*$ is the edge-disjoint union of s 2-factors. Therefore $\mathscr{E}$ is the union of s r-factors. Therefore $r \in K(r, r+2)$.

Conversely, suppose $p \in K(r, r+2)$. Let $H=(V, \mathscr{E})$ be a star-regular r-uniform hypergraph of degree sp. Then H is the union of s p-factors. Let G be the graph constructed as above. Then G is the union of s regular factors, each of degree $2p/r$. Therefore $r \mid 2p$. Therefore $r \mid p$, except possibly when r is even and $r/2 \mid p$. Now if r is even and $\geqq 4$ there is a graph G of degree 2 and order $r+2$ which contains an odd circuit and therefore contains no 1-factor. Let H be the r-uniform star-regular hypergraph of degree r corresponding to G (as above). Then H is not the union of $(r/2)$-factors (if it were, then G would be the union of 1-factors). Therefore $r/2 \notin K(r, r+2)$. Therefore $k(r, r+2) = r$. This proves Lemma 5.

THEOREM 1. *For* $n-2 \geqq r \geqq 3$ *let* $g(r, n) \in K(r, n)$, $g(r-1, n) \in K(r-1, n)$ *and*

$$h(r, n+1) = \frac{n}{n-r+1} \,\text{l.c.m}\left(g(r, n), \frac{n-r+1}{r-1} g(r-1, n)\right).$$

Then $h(r, n+1) \in K(r, n+1)$.

PROOF. First we show that a star-regular r-uniform hypergraph H of degree $h(r, n+1)$ and order $n+1$ exists. Let $H'=(V, \mathscr{E}')$ be a star-regular $(r-1)$-uniform hypergraph of order n and degree $g(r-1, n)$, and let $H''=(V, \mathscr{E}'')$ be a star-regular r-uniform hypergraph of order n and degree $g(r, n)$ [note that their vertex sets are the same].

By Lemma 3, $\frac{n}{r-1} g(r-1, n)$ is an integer, so it follows that

$$\left(\frac{n}{r-1} - 1\right) g(r-1, n) = \frac{n-r+1}{r-1} g(r-1, n)$$

is also an integer. Since l.c.m $\left(g(r,n), \frac{n-r+1}{r-1}\, g(r-1,n)\right)$ is a multiple of $\frac{n-r+1}{r-1}$ $g(r-1,n)$, it follows that

$$\frac{r-1}{n-r+1}\,\text{l.c.m}\left(g(r,n), \frac{n-r+1}{r-1}\, g(r-1,n)\right)$$

is a multiple of $g(r-1,n)$.

We form H as follows. Let $v^*(\notin V)$ be a vertex; take $\frac{1}{g(r-1,n)}\,\frac{r-1}{n-r+1}$ l.c.m $\left(g(r,n), \frac{n-r+1}{r-1}\, g(r-1,n)\right)$ copies of H' and from each edge of each copy of H' form an r-edge by adjoining v^*. Let H'^* denote the hypergraph of order $n+1$ formed as the multiset union of the enlarged edges of all these copies of H'. Now to H'^* adjoin on V

$$\frac{1}{g(r,n)}\,\text{l.c.m}\left(g(r,n), \frac{n-r+1}{r-1}\, g(r-1,n)\right)$$

copies of H''. The hypergraph now formed is H.

We show now that H has the required properties. Clearly H is an r-uniform hypergraph of order $n+1$. If $v\in V$ then the degree of v is

$$\left\{\frac{1}{g(r-1,n)}\,\frac{r-1}{n-r+1}\,\text{l.c.m}\left(g(r,n), \frac{n-r+1}{r-1}\, g(r-1,n)\right)\right\} g(r-1,n)+$$

$$+\left\{\frac{1}{g(r,n)}\,\text{l.c.m}\left(g(r,n), \frac{n-r+1}{r-1}\, g(r-1,n)\right)\right\} g(r,n) = h(r,n+1).$$

The degree of v^* is

$$\left\{\frac{1}{g(r-1,n)}\,\frac{r-1}{n-r+1}\,\text{l.c.m}\left(g(r,n), \frac{n-r+1}{r-1}\, g(r,n)\right)\right\}\left\{g(r-1,n)\frac{n}{r-1}\right\} =$$

$$= h(r,n+1)$$

also. Thus H is regular of degree $h(r,n+1)$.

Now let $H_t=(V_t, \mathscr{E}_t)$ be a star-regular r-uniform hypergraph of degree $th(r,n+1)$, for some positive integer t. Let $v\in V_t$ have the property (iii) of the definition of star-regular, so that $H_t'=(V_t', \mathscr{E}_t')$ is a star-regular $(r-1)$-uniform hypergraph of order n, $V_t'=V_t\setminus\{v\}$, and $H_t''=(V_t, \mathscr{E}_t'')$ is a star-regular r-uniform hypergraph of order n. Then the degree of H_t' is

$$t\frac{(r-1)}{n}h(r,n+1) = t\frac{(r-1)}{n-r+1}\,\text{l.c.m}\left(g(r,n), \frac{n-r+1}{r-1}\, g(r-1,n)\right)$$

and the degree of H_t'' is

$$th(r, n+1) - \frac{t(r-1)}{n} h(r, n+1) = t\,\frac{(n-r+1)}{n} h(r, n+1) =$$

$$= t\,\text{l.c.m}\left(g(r, n),\ \frac{n-r+1}{r-1} g(r-1, n)\right).$$

Since H_t' is a star-regular $(r-1)$-uniform hypergraph of order n of degree (as explained earlier) a multiple of $g(r-1, n)$, it follows that H_t' is the union of t regular factors; their degrees are all

$$\frac{r-1}{n-r+1}\,\text{l.c.m}\left(g(r, n),\ \frac{n-r+1}{r-1} g(r-1, n)\right).$$

Likewise, since H_t'' is a star-regular r-uniform hypergraph of order n, it follows that H_t'' is the union of t regular factors of degree

$$\text{l.c.m}\left(g(r, n),\ \frac{n-r+1}{r-1} g(r-1, n)\right).$$

Now adjoin the vertex v to each of the $(r-1)$-edges of H_t' to obtain an r-uniform hypergraph H_t^*. Then in H_t^* the degree of v is $th(r, n+1)$, so H_t^* is the union of t factors, in each of which v has degree $h(r, n+1)$ and the other vertices have degree

$$\frac{r-1}{n-r+1}\,\text{l.c.m}\left(g(r, n),\ \frac{n-r+1}{r-1} g(r-1, n)\right).$$

Now H_t is the union of H_t^* and H_t'', and so is the union of t factors in which v has degree $h(r, n+1)$ and the other vertices have degree

$$\left(1+\frac{r-1}{n-r+1}\right)\text{l.c.m}\left(g(r, n),\ \frac{n-r+1}{r-1} g(r-1, n)\right) = h(r, n+1)$$

also. Therefore $h(r, n+1) \in K(r, n+1)$.

THEOREM 2. *For* $n-2 \geqq r \geqq 3$,

$$\frac{(n-1)(n-2)\dots(n-r+1)}{3.4\dots(r-1)} \in K(r, n),$$

where the denominator is taken to be 1 *when* $r=3$.

PROOF. First consider the case when $n=r+2$. In this case the formula reduces to

$$r(r+1) \in K(r, r+2).$$

This is correct by Lemmas 2 and 5. From now we may suppose that $n>r+2$. We shall use induction on n.

Suppose now that $r=3$. By Lemma 1, we may put $g(r-1, n)=2$ and by induction on n we may put $g(r, n)=(n-1)(n-2)$ in Theorem 1. Then

$$\text{l.c.m}\left(g(r, n), \frac{n-2}{2} g(r-1, n)\right) = \text{l.c.m}\left((n-1)(n-2), (n-2)\right) = (n-1)(n-2).$$

Therefore $h(3, n+1)=\frac{n}{(n-2)}(n-1)(n-2)=n(n-1)$. Therefore by induction on n and Theorem 1, our formula is true in this case.

Now suppose that $r>3$. Then by induction on n and r we may take $g(r, n)$ and $g(r-1, n)$ to be given by

$$g(r, n) = \frac{(n-1)(n-2)\dots(n-r+1)}{3.4\dots(r-1)}$$

and

$$g(r-1, n) = \frac{(n-1)(n-2)\dots(n-r+2)}{3.4\dots(r-2)}$$

when we apply Theorem 1. Since

$$\text{l.c.m}\left(g(r, n), \frac{n-r+1}{r-1} g(r-1, n)\right) = \frac{(n-1)\dots(n-r+1)}{3.4\dots(r-1)}$$

it follows that

$$h(r, n+1) = \frac{n(n-1)\dots(n-r+2)}{3.4\dots(r-1)}.$$

Therefore, by induction, our formula holds in all cases.

4. Concluding remarks

The results of Section 3 are not very strong because they only deal with a special kind of regular r-uniform hypergraph, because the numbers in Theorem 2 are so large, and because the results all depend on the order of the hypergraph. However, they may be of use as a yardstick against which one may test the value of other results and, possibly, the consistency of other conjectures. It is hoped that this note will stimulate interest in Neumann-Lara's very nice problem. Thanks are due to the referee for his useful comments.

REFERENCES

[1] BERGE, C., *Graphs and hypergraphs*, North-Holland Mathematical Library, Vol. 6, North-Holland Publishing Co., Amsterdam—London; American Elsevier Publishing Co., Inc., New York, 1973. *MR* **50** #9640.

[2] NEUMANN-LARA, V., oral communication.

[3] PETERSEN, J., Die Theorie der regularen Graphen, *Acta Math.* **15** (1891), 193—220.

(Received March 16, 1982)

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF READING
WHITEKNIGHTS
READING
RG6 2AX
ENGLAND

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 413—426.

AN EMBEDDING THEOREM FOR SEPARABLE METRIZABLE SPACES

J. DEÁK

A characterization of the class of subspaces of a Euclidean space will be proved.

DEFINITION (E. Deák [2, 3]). A *pseudo-direction* of a space X is a linearly ordered family $\mathscr{R}=(\mathscr{R}, <^{\mathscr{R}})$ of ordered pairs of subsets of X such that

(i) if $(G, F)\in\mathscr{R}$ then G is open, F is closed in X and[1] $G\subset F$;

(ii) if $(G_1, F_1)<^{\mathscr{R}}(G_2, F_2)$ then $F_1\subset G_2$.

A *pseudo-directional structure* of X is a set of pseudo-directions of X. A pseudo-directional structure $\mathfrak{R}$ of X is *compatible* if

$$\bigcup_{\mathscr{R}\in\mathfrak{R}} (\mathscr{G}_{\mathscr{R}}\cup\mathscr{H}_{\mathscr{R}})$$

is an open subbase for the topology of X where

$$\mathscr{G}_{\mathscr{R}} = \{G\colon\ \exists F, (G, F)\in\mathscr{R}\}\quad (\mathscr{R}\in\mathfrak{R})$$

and

$$\mathscr{H}_{\mathscr{R}} = \{X-F\colon\ \exists G, (G, F)\in\mathscr{R}\}\quad (\mathscr{R}\in\mathfrak{R}).$$

THEOREM. *If a separable metrizable space admits a finite compatible pseudo-directional structure $\mathfrak{R}$ then it can be topologically embedded into the Euclidean space of dimension*[2] $|\mathfrak{R}|$.

This theorem generalizes a result due to E. Deák [1] (for details, see later); it yields indeed a characterization of the subspaces of $\mathbf{R}^{|\mathfrak{R}|}$, for the converse to the theorem is a triviality.

Preliminaries

In this section, X will always be a topological space, $\mathscr{R}$ a pseudo-direction and $\mathfrak{R}$ a pseudo-directional structure of X. In addition to the notations $\mathscr{G}_{\mathscr{R}}$ and $\mathscr{H}_{\mathscr{R}}$ already introduced, let

$$\mathscr{F}_{\mathscr{R}} = \{F\colon\ \exists G, (G, F)\in\mathscr{R}\}$$

1980 *Mathematics Subject Classification.* Primary 54F45; Secondary 54C25, 54F65, 54E35, 54E99.

Key words and phrases. Separable metrizable space, Euclidean space, embedding, (pseudo-)direction, (compatible) (pseudo-)directional structure.

[1] The symbol $\subset$ means not necessarily proper containing. (On the other hand, $<^{\mathscr{R}}$ is to be understood in the strict sense.)

[2] The symbol $|\,|$ will denote cardinality as well as absolute value — there will be no danger of misunderstanding.

(i.e. $F\in\mathcal{F}_{\mathcal{R}}$ iff $X-F\in\mathcal{H}_{\mathcal{R}}$) and

$$\mathcal{O}_{\mathcal{R}}=\mathcal{G}_{\mathcal{R}}\cup\mathcal{H}_{\mathcal{R}},\quad \mathcal{O}_{\mathfrak{R}}=\bigcup_{\mathcal{R}\in\mathfrak{R}}\mathcal{O}_{\mathcal{R}}.$$

With this notation, $\mathfrak{R}$ is compatible iff $\mathcal{O}_{\mathfrak{R}}$ is a subbase of X.

Definition (E. Deák [1]). A pseudo-direction $\mathcal{R}$ is a *direction* if $\mathcal{G}_{\mathcal{R}}$ contains the union of an arbitrary subfamily of it and a similar condition holds for $\mathcal{H}_{\mathcal{R}}$. A pseudo-direction $\mathcal{R}$ of X is *orderly* if

$$\bigcup\{F-G\colon (G,F)\in\mathcal{R}\}=X.$$

A(n *orderly*) *(pseudo-)directional structure* of X is a system of (orderly) (pseudo-) directions of X.

Remark. As the empty subfamily is not excluded in the above definition, we have $\emptyset\in\mathcal{G}_{\mathcal{R}}$ and $\emptyset\in\mathcal{H}_{\mathcal{R}}$ for any direction $\mathcal{R}$. Thus there are sets F_0 and G_0 with

$$(\emptyset, F_0)\in\mathcal{R},\quad (G_0, X)\in\mathcal{R}.$$

Consequently, each direction has a first and a last element because for any $G\in\mathcal{G}_{\mathcal{R}}$ $(F\in\mathcal{F}_{\mathcal{R}})$, the condition $(G,F)\in\mathcal{R}$ can be satisfied by at most two sets $F\in\mathcal{F}_{\mathcal{R}}$ $(G\in\mathcal{G}_{\mathcal{R}})$. In E. Deák's original definition, the additional conditions

$$(\emptyset,\emptyset)\in\mathcal{R},\quad (X,X)\in\mathcal{R}$$

are imposed on the directions. These conditions, however, are not needed anywhere in the theory of directional structures; what counts in some cases (but not in the present paper) is the existence of a first and a last element.

E. Deák has proved a characterization of the class of subspaces of $\mathbf{R}^n$:

Theorem A. *A separable metrizable space is homeomorphic to a subspace of* $\mathbf{R}^n$ *iff it has a compatible directional structure of cardinality* $\leqq n$ $(n=1,2,\dots)$.[3]

The necessity of the condition can be easily seen by taking the restriction to the given subspace of $\mathbf{R}^n$ of the natural directions

$$\mathcal{R}_n^i=\{(\emptyset,\emptyset)\}\cup\{(\Gamma_n^i[\alpha],\Phi_n^i[\alpha])\colon \alpha\in\mathbf{R}\}\cup\{(\mathbf{R}^n,\mathbf{R}^n)\}\qquad (1\leqq i\leqq n)$$

where[4]

$$\Gamma_n^i[\alpha]=\{p\in\mathbf{R}^n\colon p^i<\alpha\},\quad \Phi_n^i[\alpha]=\{p\in\mathbf{R}^n\colon p^i\leqq\alpha\}$$

and the ordering of $\mathcal{R}_n^i$ is the obvious one.

Our theorem is clearly a generalization of the non-trivial part of Theorem A: the directions have been replaced by pseudo-directions. The following example will show that our theorem is not only seemingly more general.

---

[3] This theorem is evidently true for $n=0$, supposing that $\emptyset$ is regarded as a subbase for any indiscrete topology.

[4] For a point $p\in\mathbf{R}^n$, p^i denotes the ith coordinate of p. Otherwise, upper indices will range over the set $\{1,\dots,n\}$ or over a subset of it; and they will have nothing to do with exponentation, except when applied to concrete numbers, e.g. $2^{\aleph}$.

Take the set

$$X = \mathbf{R}^2 - \{(0, \beta) : 0 < |\beta| < 1\}$$

and let the local bases at the points[5] $x \neq 0$ consist of the Euclidean neighbourhoods of x, while

$$\{\{0\} \cup \{(\alpha, \beta) : -\varepsilon < \alpha < 0, |\beta| < \varepsilon\} : \varepsilon > 0\}$$

is taken as a local base at 0 (Fig. 1). X is a separable metrizable space. A slight modification of the directional structure

$$\mathfrak{R}^* = \{\mathscr{R}_2^1 | X, \mathscr{R}_2^2 | X\}$$

provides a compatible pseudo-directional structure of X: the point 0 has to be added to $\Gamma_2^1[0]$; let this pseudo-directional structure be denoted by

$$\mathfrak{R} = \{\mathscr{R}^1, \mathscr{R}^2 = \mathscr{R}_2^2 | X\}.$$

$\mathscr{R}^2$ is a direction and there is just one "place" where $\mathscr{R}^1$ fails to be a direction:

$$\bigcup_{\alpha<0} \Gamma_2^1[\alpha] = \Gamma_2^1[0] \notin \mathscr{G}_{\mathscr{R}^1}.$$

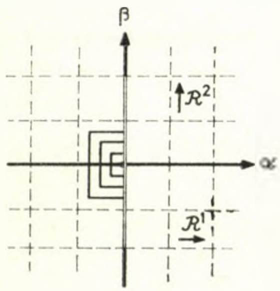


Fig. 1

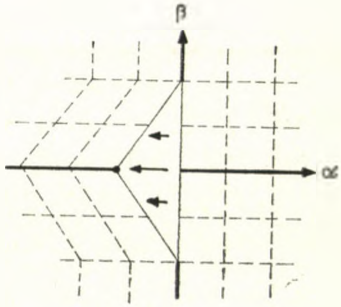


Fig. 2

X can easily be embedded into $\mathbf{R}^2$: we have only to pull away the point 0 from the points which are near to it in $\mathbf{R}^2$ but not in the topology of X (see Fig. 2). Theorem A, however, does not yield an embedding: if we correct the fault of $\mathfrak{R}$ in order to get a directional structure, we are back at $\mathfrak{R}^*$ which is a directional structure of X, but not a compatible one.

The above example gives an idea how to prove the theorem: take a compatible pseudo-directional structure $\mathfrak{R}$ of X, $|\mathfrak{R}| = n$; construct a directional structure $\mathfrak{R}^*$ with changing $\mathfrak{R}$ as little as possible; copying the proof of Theorem A, define a continuous mapping $f: X \to \mathbf{R}^n$ (which will not be an embedding if $\mathfrak{R}^*$ is not compatible; f may even fail to be one-to-one); finally, change f by using a procedure suitably generalizing the "pulling away" shown in Fig. 2.

The directional structure $\mathfrak{R}^*$ will be obtained through

[5] The symbol 0 will stand not only for the number zero, but also for the origin in any Euclidean space; the meaning will always be clear from the context.

THEOREM B (E. Deák [3], 53.3 and 53.4 1°). *Let $\mathscr{R}$ be a pseudo-direction of a space X. Then there are a direction $\mathscr{R}^*$ of X and an order-preserving mapping $*:\mathscr{R}\to\mathscr{R}^*$ such that $\Gamma\subset G$, $F\subset\Phi$ whenever $(G, F)^*=(\Gamma, \Phi)$.*

One of the proofs of Theorem A is based on

THEOREM C (E. Deák [3], 63.7). *If $\mathscr{R}$ is an orderly direction of a second countable space and $\mathscr{R}$ is endowed with the order topology then there is an order-preserving topological embedding of $\mathscr{R}$ into $\mathbf{R}$.*

In addition to Theorems B and C, we shall need some elementary properties of the pseudo-directions; they can be readily deduced from the definitions. (Or see in [3].)

PROPOSITION D. a) $\mathscr{G}_{\mathscr{R}}\cap\mathscr{H}_{\mathscr{R}}\subset\{\emptyset, X\}$.

b) *If* (G_1, F_1), $(G_2, F_2)\in\mathscr{R}$, $(G_1, F_1)\neq(G_2, F_2)$ *then* $(F_1-G_1)\cap(F_2-G_2)=\emptyset$.

c) *If $\mathscr{R}$ is an orderly pseudo-direction then the canonical mapping χ: $X\to\mathscr{R}$ defined by*

$$\chi(x) = (G, F) \quad \textit{iff} \quad x\in F-G, (G, F)\in\mathscr{R}$$

is continuous.

d) *If $\mathscr{R}$ is orderly, χ is the canonical mapping and $(G, F)\in\mathscr{R}$ then*

$$\chi(x) <^{\mathscr{R}} (G, F) \quad \textit{iff} \quad x\in G$$

and

$$(G, F) <^{\mathscr{R}} \chi(x) \quad \textit{iff} \quad x\in X-F.$$

Finally, wee need

THEOREM E (E. Deák [1]). *If $\mathscr{R}$ is a direction of a perfectly normal space X, then there is an orderly direction $\bar{\mathscr{R}}$ of X such that $\bar{\mathscr{R}}\supset\mathscr{R}$.*

Proof of the theorem

1° Let $\mathfrak{R}=\{\mathscr{R}^1, ..., \mathscr{R}^n\}$ be a compatible pseudo-directional structure of the separable metrizable space X, $n\geqq 1$. Applying Theorem B to $\mathscr{R}^i$, we get a direction $\mathscr{R}^{i*}$ and a mapping $*(i)$: $\mathscr{R}^i\to\mathscr{R}^{i*}$. By Theorem E, there is an orderly direction containing $\mathscr{R}^{i*}$. As the properties of $\mathscr{R}^{i*}$ and $*(i)$ guaranteed by Theorem B remain evidently valid if $\mathscr{R}^{i*}$ is replaced by a larger direction, we may suppose that $\mathscr{R}^{i*}$ itself is an orderly direction. So we have an orderly (but not necessarily compatible) directional structure $\mathfrak{R}^*=\{\mathscr{R}^{1*}, ..., \mathscr{R}^{n*}\}$ of X.

In order to simplify the notations, let

$$\mathscr{G}^i = \mathscr{G}_{\mathscr{R}^i}, \quad \mathscr{F}^i = \mathscr{F}_{\mathscr{R}^i}, \quad \mathscr{H}^i = \mathscr{H}_{\mathscr{R}^i}, \quad \mathscr{O}^i = \mathscr{O}_{\mathscr{R}^i} \quad (1\leqq i\leqq n);$$

$$\mathscr{O} = \mathscr{O}_{\mathfrak{R}}\left(=\bigcup_{i=1}^{n}\mathscr{O}^i\right), \quad \mathscr{O}^* = \mathscr{O}_{\mathfrak{R}^*}.$$

As X is second countable, the subbase $\mathscr{O}$ contains a countable subbase $\mathscr{V}$ of X. If X is finite then it can be clearly embedded into $\mathbf{R}^n$, so we may suppose that X is infinite; consequently, $|\mathscr{V}|=\omega$. We may assume $\emptyset$, $X\notin\mathscr{V}$. Let us take disjoint sets

$\mathscr{V}^i \subset \mathcal{O}^i$ $(1 \leqq i \leqq n)$ such that $\mathscr{V} = \bigcup_{i=1}^{n} \mathscr{V}^i$. Thus for each $V \in \mathscr{V}$, there is a unique $i[V] \in \{1, \ldots, n\}$ with $V \in \mathscr{V}^{i[V]}$. Now exactly one of the statements $V \in \mathscr{G}^{i[V]}$ and $V \in \mathscr{H}^{i[V]}$ holds: if both were true then, by Proposition D a), we would have $V = \emptyset$ or $V = X$.

If $V \in \mathscr{G}^{i[V]}$, let $G[V] = V$ and pick an $F[V] \in \mathscr{F}^{i[V]}$ with

$$(G[V], F[V]) \in \mathscr{R}^{i[V]}. \tag{1.1}$$

Put

$$H[V] = X - F[V]. \tag{1.2}$$

On the other hand, if $V \in \mathscr{H}^{i[V]}$ then let $H[V] = V$, $F[V] = X - H[V]$ and pick now a $G[V] \in \mathscr{G}^{i[V]}$ satisfying (1.1). Thus for each $V \in \mathscr{V}$, we have three sets

$$G[V] \in \mathscr{G}^{i[V]}, \quad F[V] \in \mathscr{F}^{i[V]}, \quad H[V] \in \mathscr{H}^{i[V]}$$

satisfying (1.1) and (1.2) such that

$$V = G[V] \quad \text{or} \quad V = H[V]$$

and only one of these equalities holds (the first iff $V \in \mathscr{G}^{i[V]}$, the second iff $V \in \mathscr{H}^{i[V]}$).

For each $V \in \mathscr{V}$, let the sets $G^*[V]$, $F^*[V]$, $H^*[V]$ and V^* be defined by

$$(G^*[V], F^*[V]) = (G[V], F[V])^{*(i[V])}, \quad H^*[V] = X - F^*[V],$$

$$V^* = \begin{cases} G^*[V] & \text{if } V = G[V], \\ H^*[V] & \text{if } V = H[V]. \end{cases}$$

It follows from Theorem B that

$$G^*[V] \subset G[V] \subset F[V] \subset F^*[V],$$

$$H^*[V] \subset H[V], \quad V^* \subset V.$$

For $x \in X$, let

$$\mathscr{V}^*(x) = \{V^* : x \in V^*, V \in \mathscr{V}\}, \quad \mathscr{V}(x) = \{V \in \mathscr{V} : x \in V - V^*\}.$$

Clearly,

$$\mathscr{S}(x) = \mathscr{V}^*(x) \cup \mathscr{V}(x)$$

is a local subbase at x. For each $i \in \{1, \ldots, n\}$, we have

$$|\mathscr{V}(x) \cap \mathscr{V}^i| \leqq 1. \tag{1.3}$$

Indeed, if $V \in \mathscr{V}(x) \cap \mathscr{V}^i$ then either

$$x \in G[V] - G^*[V] \subset F^*[V] - G^*[V] \tag{1.4}$$

or

$$x \in H[V] - H^*[V] = F^*[V] - F[V] \subset F^*[V] - G^*[V] \tag{1.5}$$

and this determines $(G^*[V], F^*[V])$, by Proposition D b) applied to $\mathscr{R}^{i*}$, thus $(G[V], F[V])$ is determined, too, since $*(i)$ is one-to-one; x can belong only to one of the disjoint sets $G(V)$ and $H(V)$; now if $V, V' \in \mathscr{V}(x) \cap \mathscr{V}^i$ then $G[V]=G[V']$, $H[V]=H[V']$ and if V is equal to one of these sets then V' has to be equal to this same set, i.e. $V=V'$.

2° Let $f\colon X \to \mathbf{R}^n$ be defined by

$$f^i(x) = \varphi^i(\chi^i(x)) \quad (x \in X,\ 1 \leqq i \leqq n)$$

where $\chi^i\colon X \to \mathscr{R}^{i*}$ is the canonical mapping [Proposition D c)] and $\varphi^i\colon \mathscr{R}^{i*} \to \mathbf{R}$ is an order-preserving embedding (Theorem C). Each coordinate of f is the composite of two continuous functions, so f is continuous.

Let $i \in \{1, \dots, n\}$ and $(\Gamma, \Phi) \in \mathscr{R}^{i*}$ be fixed, and let $\alpha = \varphi^i((\Gamma, \Phi))$. For an $x \in X$, $x \in \Gamma$ iff $\chi^i(x) <^{\mathscr{R}^{i*}} (\Gamma, \Phi)$, by Proposition D d). As φ^i is order-preserving,

$$x \in \Gamma \quad \text{iff} \quad f^i(x) < \alpha. \tag{2.1}$$

Similarly, using the second part of Proposition D d),

$$x \in X - \Phi \quad \text{iff} \quad f^i(x) > \alpha. \tag{2.2}$$

Since $\mathscr{V}^*(x) \subset O^*$, we have

3° Proposition. *If $x \in X$, $\mathscr{W} \subset \mathscr{V}^*(x)$ and $|\mathscr{W}| < \omega$ then there is an open set $S \subset \mathbf{R}^n$ such that $f(y) \in S$ iff $y \in \bigcap \mathscr{W}$.*

4° Let

$$\Sigma = \left\{(V^1, \dots, V^n)\colon V^i \in \mathscr{V}^i \cup \{X\}\ (1 \leqq i \leqq n),\ \bigcap_{i=1}^{n} V^i \neq X\right\}.$$

$|\Sigma| = \omega$, so there is a series

$$\{(V_k^1, \dots, V_k^n)\colon k \in \omega\} = \Sigma, \quad (V_k^1, \dots, V_k^n) \neq (V_l^1, \dots, V_l^n) \quad (k \neq l).$$

If $V_k^i \neq X$, let

$$G_k^i = G[V_k^i], \quad F_k^i = F[V_k^i], \quad H_k^i = H[V_k^i],$$
$$G_k^{i*} = G^*[V_k^i], \quad F_k^{i*} = F^*[V_k^i], \quad H_k^{i*} = H^*[V_k^i].$$

(If $V_k^i = X$, the sets G_k^i, F_k^i etc. are not defined.) By 1° and $V_k^i \in \mathscr{V}^i$, we have

$$V_k^i = G_k^i \quad (\text{iff } V_k^i \in \mathscr{G}^i - \{X\}) \quad \text{or} \quad V_k^i = H_k^i \quad (\text{iff } V_k^i \in \mathscr{H}^i - \{X\}),$$
$$(G_k^i, F_k^i) \in \mathscr{R}^i, \quad (G_k^{i*}, F_k^{i*}) \in \mathscr{R}^{i*},$$
$$G_k^{i*} \subset G_k^i \subset F_k^i \subset F_k^{i*},$$
$$H_k^i = X - F_k^i, \quad H_k^{i*} = X - F_k^{i*}, \quad H_k^{i*} \subset H_k^i.$$

Let

$$\alpha_k^i = \varphi^i((G_k^{i*}, F_k^{i*})), \quad P_k^i = \{p \in \mathbf{R}^n\colon p^i = \alpha_k^i\}$$

(if $V_k^i = X$ then α_k^i and P_k^i are not defined). Further, put

$$C_k = \bigcup \{P_l^i\colon 1 \leqq i \leqq n,\ 0 \leqq l < k,\ V_l^i \neq X \neq V_k^i,\ \alpha_l^i \neq \alpha_k^i\},$$

$$Q_k = \bigcap_{i=1}^{n} V_k^i, \quad D_k = f(\operatorname{Fr} Q_k) \quad (k \in \omega).$$

Let the mappings $\psi_k\colon X \to \mathbf{R}$ be defined by

$$\psi_k(x) = \varrho\big(f(x), C_k \cup D_k\big) \qquad (x \in X,\ k \in \omega)$$

(if $C_k \cup D_k = \emptyset$, we take $\psi_k \equiv 1$). Instead of the usual Euclidean metric, let

$$|a| = \max_{1 \le i \le n} |a^i| \quad (a \in \mathbf{R}^n), \qquad \varrho(a, b) = \min\{|a-b|, 1\} \quad (a, b \in \mathbf{R}^n).$$

Both f and ϱ are continuous, so each ψ_k is continuous, too. Let us define $\hat{k} \in \mathbf{R}^n$ by

$$\hat{k}^i = \begin{cases} -1 & \text{if } V_k^i \in \mathscr{G}^i - \{X\} \\ \ \ 1 & \text{if } V_k^i \in \mathscr{H}^i - \{X\} \\ \ \ 0 & \text{if } V_k^i = X \end{cases} \qquad (k \in \omega,\ 1 \le i \le n)$$

and let the mappings $g_k\colon X \to \mathbf{R}^n$ be defined by

$$g_k(x) = \begin{cases} \psi_k(x)\hat{k} & \text{if } x \in Q_k \\ 0 & \text{if } x \notin Q_k \end{cases} \qquad (k \in \omega).$$

g_k is continuous, for $\psi_k = 0$ in $\operatorname{Fr} Q_k$. From $0 \le \psi_k \le 1$ and $|\hat{k}| = 1$, we have $|g_k| \le 1$. Take positive numbers ε_k such that

$$\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k < \infty.$$

We claim that

$$h = f + \sum_{k=0}^{\infty} \varepsilon_k g_k$$

is an embedding if ε is small enough. h is evidently continuous, since the series in the definition of h is uniformly convergent.

5° With the notations introduced in 4°, (2.1) and (2.2) mean

(5.1) $$f^i(x) < \alpha_k^i \quad \text{iff} \quad x \in G_k^{i*},$$

(5.2) $$f^i(x) > \alpha_k^i \quad \text{iff} \quad x \in H_k^{i*}.$$

Consequently,

(5.3)
$$[x \ne y,\ |\{x, y\} \cap V_k^i| = 1] \Rightarrow [f^i(x) \le \alpha_k^i \le f^i(y) \quad \text{or} \quad f^i(y) \le \alpha_k^i \le f^i(x)].$$

Indeed, the role of x and y being symmetrical, we may assume that $x \in V_k^i$, $y \notin V_k^i$. If $V_k^i = G_k^i$ then $x \notin H_k^{i*}$, $y \notin G_k^{i*}$, so (5.1) and (5.2) give $f^i(x) \le \alpha_k^i \le f^i(y)$. Similarly, if $V_k^i = H_k^i$ then $x \notin G_k^{i*}$, $y \notin H_k^{i*}$, so now $f^i(y) \le \alpha_k^i \le f^i(x)$.

Furthermore,

(5.4) $$[f^i(x) < f^i(y),\ y \in G_k^i] \Rightarrow x \in G_k^i,$$

(5.5) $$[f^i(x) < f^i(y),\ x \in H_k^i] \Rightarrow y \in H_k^i.$$

Indeed, if $f^i(x)<f^i(y)$ and $y\in G_k^i$ then $y\notin H_k^{i*}$, so, by (5.2), $f^i(y)\leqq\alpha_k^i$, $f^i(x)<\alpha_k^i$, thus $x\in G_k^{i*}\subset G_k^i$, by (5.1). The proof of (5.5) is the same, only the order of applying (5.2) and (5.1) has to be reversed.

6° For the time being, let $x, y\in X$ be fixed. We need estimations for $|h(y)-h(x)|$. Let

$$\sigma_k = \varepsilon_k[g_k(y)-g_k(x)] \quad (k\in\omega), \quad \sigma_A = \sum_{k\in A}\sigma_k \ (A\subset\omega),$$

$$\Delta = f(y)-f(x).$$

We shall define several subsets of ω; they all depend on x and y, but this dependance will not be shown in the notations. First, let

$$B = \{k\in\omega\colon x, y\in Q_k\},$$

$$T = \{k\in\omega\colon [g_k(y)\neq 0, x\notin Q_k] \text{ or } [g_k(x)\neq 0, y\notin Q_k]\}.$$

As $g_k=0$ outside Q_k, we have

(6.1) $$|Q_k\cap\{x, y\}| = 1 \qquad (k\in T).$$

Conversely, if $|Q_k\cap\{x,y\}|=1$ for some $k\in\omega$ then either $k\in T$ or $g_k(x)=0=$ $=g_k(y)$. Thus

(6.2) $$h(y)-h(x) = \Delta+\sigma_B+\sigma_T,$$

since if $x, y\notin Q_k$ then $\sigma_k=0$. Further,

(6.3) $$\sigma_k = \varepsilon_k g_k(y)\neq 0 \quad\text{or}\quad \sigma_k = -\varepsilon_k g_k(x)\neq 0 \quad (k\in T).$$

Now we shall define by recursion disjoint non-empty sets T_λ such that

(6.4) $$T = \bigcup_{\lambda=1}^{\nu} T_\lambda$$

and $0\leqq\nu\leqq n$ ($\nu=0$ iff $T=\emptyset$). Suppose that $\lambda\geqq 1$ and the sets $T_\varkappa$ have been defined for $k<\lambda$. Now if $T=\bigcup\limits_{\varkappa=1}^{\lambda-1} T_\varkappa$, we take $\nu=\lambda-1$. Otherwise, let

$$t_\lambda = \min\left(T-\bigcup_{\varkappa=1}^{\lambda-1} T_\varkappa\right).$$

By (6.1) and the definition of Q_k, there is an index i_λ such that

(6.5) $$|V_{t_\lambda}^{i_\lambda}\cap\{x, y\}| = 1$$

(if there are more than one such indices, choose one of them). According to (5.3),

(6.6) $$f^{i_\lambda}(x)\leqq\alpha_{t_\lambda}^{i_\lambda}\leqq f^{i_\lambda}(y) \quad\text{or}\quad f^{i_\lambda}(y)\leqq\alpha_{t_\lambda}^{i_\lambda}\leqq f^{i_\lambda}(x).$$

Let

$$T_\lambda = \left\{k\in T-\bigcup_{\varkappa=1}^{\lambda-1} T_\varkappa\colon V_k^{i_\lambda}\neq X\right\}.$$

Clearly, $V_{t_\lambda}^{i_\lambda} \neq X$ follows from (6.5), thus $t_\lambda \in T_\lambda$, $t_\lambda = \min T_\lambda$ (so $T_\lambda \neq 0$). From the definition of T_λ we have

(6.7) $$[1 \leqq \lambda < \mu \leqq \nu,\ k \in T_\mu] \Rightarrow V_k^{i_\lambda} = X.$$

Consequently, the indices $i_1, i_2, \ldots$ are different. But they all belong to the set $\{1, \ldots, \ldots, n\}$, so the construction comes to an end in at most n steps, i.e. (6.4) holds with $\nu \leqq n$.

Let

$$\left.\begin{aligned} L_\lambda &= \{k \in T_\lambda \colon \alpha_k^{i_\lambda} = \alpha_{t_\lambda}^{i_\lambda}\} \\ M_\lambda &= \{k \in T_\lambda \colon \alpha_k^{i_\lambda} \neq \alpha_{t_\lambda}^{i_\lambda}\} \end{aligned}\right. \quad (1 \leqq \lambda \leqq \nu).$$

Clearly,

$$L_\lambda \cap M_\lambda = \emptyset, \quad T_\lambda = L_\lambda \cup M_\lambda \quad (1 \leqq \lambda \leqq \nu).$$

Furthermore, $t_\lambda \in L_\lambda$, i.e.

(6.8) $$t_\lambda = \min L_\lambda < k \quad (1 \leqq \lambda \leqq \nu,\ k \in M_\lambda).$$

Put

$$L = \bigcup_{\lambda=1}^{\nu} L_\lambda, \quad M = \bigcup_{\lambda=1}^{\nu} M_\lambda.$$

Observe that

(6.9) $$\sigma_L^{i_\lambda} = \sum_{\varkappa=1}^{\lambda} \sigma_{L_\varkappa}^{i_\lambda} \quad (1 \leqq \lambda \leqq \nu)$$

because for $k \in L_\mu \subset T_\mu$ $(\lambda < \mu \leqq \nu)$, we have $V_k^{i_\lambda} = X$ [cf. (6.7)], so $\hat{k}^{i_\lambda} = 0$, $g^{i_\lambda} \equiv 0$ and $\sigma_k^{i_\lambda} = 0$. Further, by (6.2),

(6.10) $$h(y) - h(x) = \Delta + \sigma_B + \sigma_L + \sigma_M.$$

7° Proposition. *If* $1 \leqq \lambda \leqq \nu$ *and* $k \in L_\lambda$ *then*

$$|\{x, y\} \cap V_k^{i_\lambda}| = 1, \quad G_k^{i_\lambda} = G_{t_\lambda}^{i_\lambda}, \quad H_k^{i_\lambda} = H_{t_\lambda}^{i_\lambda}.$$

Proof. $\alpha_k^{i_\lambda} = \alpha_{t_\lambda}^{i_\lambda}$, so $(G_k^{i_\lambda *}, F_k^{i_\lambda *}) = (G_{t_\lambda}^{i_\lambda *}, F_{t_\lambda}^{i_\lambda *})$, i.e. $(G_k^{i_\lambda}, F_k^{i_\lambda}) = (G_{t_\lambda}^{i_\lambda}, F_{t_\lambda}^{i_\lambda})$. By the definition of i_λ, one of the disjoint sets

(7.1) $$G_{t_\lambda}^{i_\lambda} = G_k^{i_\lambda}, \quad H_{t_\lambda}^{i_\lambda} = H_k^{i_\lambda}$$

contains exactly one of the points x, y, thus the other set in (7.1) contains at most one of these points. So $|V_k^{i_\lambda} \cap \{x, y\}| \leqq 1$. But $x, y \notin V_k^{i_\lambda}$ would mean $x, y \notin Q_k$, in contradiction to the definition of $T \supset L_\lambda$.

8° Proposition. *Let* λ *be fixed,* $1 \leqq \lambda \leqq \nu$.

a) *If* $f^{i_\lambda}(x) < f^{i_\lambda}(y)$ *then*

(8.1) $$\sigma_k^{i_\lambda} > 0 \quad (k \in L_\lambda).$$

b) *If* $f^{i_\lambda}(x) > f^{i_\lambda}(y)$ *then*

$$\sigma_k^{i_\lambda} < 0 \quad (k \in L_\lambda). \tag{8.2}$$

c) *If* $f^{i_\lambda}(x) = f^{i_\lambda}(y)$ *then either* (8.1) *or* (8.2) *holds.*

Proof of a). By Proposition 7°, either

$$x \in V_k^{i_\lambda}, \quad y \notin V_k^{i_\lambda} \tag{8.3}$$

or

$$x \notin V_k^{i_\lambda}, \quad y \in V_k^{i_\lambda}. \tag{8.4}$$

First assume (8.4). Then $V_k^{i_\lambda} = G_k^{i_\lambda}$ is impossible by (5.4), thus $V_k^{i_\lambda} = H_k^{i_\lambda}$, hence $\hat{k}^{i_\lambda} = 1$ and we have

$$\sigma_k^{i_\lambda} = \varepsilon_k [g_k^{i_\lambda}(y) - g_k^{i_\lambda}(x)] = \varepsilon_k g_k^{i_\lambda}(y) = \varepsilon_k \psi_k(y) > 0$$

(since (i) $x \notin V_k^{i_\lambda}$ implies $x \notin Q_k$; (ii) $\psi_k(y) \geqq 0$; (iii) $\psi_k(y) = 0$ would imply $g_k(y) = 0$, contradicting the definition of T).

Similarly, if (8.3) holds then $V_k^{i_\lambda} = G_k^{i_\lambda}$ [cf. (5.5)], $\hat{k}^{i_\lambda} = -1$ and

$$\sigma_k^{i_\lambda} = -\varepsilon_k g_k^{i_\lambda}(x) = -\varepsilon_k(-\psi_k(x)) > 0.$$

Proof of b). Apply a) to x and y interchanged.

Proof of c). By Proposition 7°, each $V_k^{i_\lambda}$ contains exactly one of the points x and y $(k \in L_\lambda)$.

Case I: each $V_k^{i_\lambda}$ contains the same of the points x and y. We may suppose without loss of generality that $x \in V_k^{i_\lambda}$ $(k \in L_\lambda)$, since interchanging x and y is equivalent to multiplying each σ_k by -1. By Proposition 7°, either $V_k^{i_\lambda} = G_k^{i_\lambda}$ for each $k \in L_\lambda$ or $V_k^{i_\lambda} = H_k^{i_\lambda}$ for each $k \in L_\lambda$ (because only one of the disjoint sets $G_{t_\lambda}^{i_\lambda} = G_k^{i_\lambda}$ and $H_{t_\lambda}^{i_\lambda} = H_k^{i_\lambda}$ can contain x). Consequently, $\hat{k}^{i_\lambda}$ is independent of k $(k \in L_\lambda)$ and

$$\sigma_k^{i_\lambda} = -\varepsilon_k g_k^{i_\lambda}(x) = -\varepsilon_k \hat{k}^{i_\lambda} \psi_k(x) = \begin{cases} -\varepsilon_k \psi_k(x) < 0 & (k \in L_\lambda) \\ \text{or} & \\ \varepsilon_k \psi_k(x) > 0 & (k \in L_\lambda). \end{cases}$$

Case II: there are $k_1, k_2 \in L_\lambda$ such that $x \in V_{k_1}^{i_\lambda}$, $y \in V_{k_2}^{i_\lambda}$. One of the disjoint sets

$$G_{t_\lambda}^{i_\lambda} = G_{k_1}^{i_\lambda} = G_{k_2}^{i_\lambda}, \quad H_{t_\lambda}^{i_\lambda} = H_{k_1}^{i_\lambda} = H_{k_2}^{i_\lambda}$$

contains x and the other contains y. Now we may suppose without loss of generality that $x \in G_{t_\lambda}^{i_\lambda}$, $y \in H_{t_\lambda}^{i_\lambda}$. Take a $k \in L_\lambda$. If $V_k^{i_\lambda} = G_k^{i_\lambda}$ then $\hat{k}^{i_\lambda} = -1$, $x \in G_{t_\lambda}^{i_\lambda} = V_k^{i_\lambda}$ and

$$\sigma_k = -\varepsilon_k g_k(x) = -\varepsilon_k(-\psi_k(x)) > 0;$$

if $V_k^{i_\lambda} = H_k^{i_\lambda}$ then $\hat{k}^{i_\lambda} = 1$, $y \in H_{t_\lambda}^{i_\lambda} = V_k^{i_\lambda}$ and

$$\sigma_k = \varepsilon_k g_k(y) = \varepsilon_k \psi_k(y) > 0$$

again.

9° PROPOSITION. $|\sigma_{L_\lambda}|=|\sigma^{i_\lambda}_{L_\lambda}|$ $(1\leqq\lambda\leqq\nu)$.

PROOF. By (6.3), the non-zero coordinates of a σ_k $(k\in T)$ are of the same absolute value and by Proposition 8°, $\sigma^{i_\lambda}_k$ is positive (negative) for each $k\in L_\lambda(\subset T)$, thus $|\sigma^{i}_{L_\lambda}|$ has the largest possible value at $i=i_\lambda$.

10° PROPOSITION. *If* $\eta>0$ *and* $|\Delta+\sigma_L|<2\eta$ *or* $|\sigma_L|<2\eta$ *then*

$$|\sigma_{L_\lambda}|<2^\lambda\eta \qquad (1\leqq\lambda\leqq\nu). \tag{10.1}$$

PROOF. Suppose that $1\leqq\mu\leqq\nu$ and (10.1) holds for $\lambda<\mu$. By (6.9)

$$\left|c+\sum_{\lambda=1}^{\mu-1}\sigma^{i_\mu}_{L_\lambda}+\sigma^{i_\mu}_{L_\mu}\right|<2\eta$$

where $c=\Delta^{i_\mu}$ or $c=0$. By the induction hypothesis,

$$\left|\sum_{\lambda=1}^{\mu-1}\sigma^{i_\mu}_{L_\lambda}\right|\leqq\left|\sum_{\lambda=1}^{\mu-1}\sigma_{L_\lambda}\right|<(2^\mu-2)\eta,$$

so

$$|c+\sigma^{i_\mu}_{L_\mu}|<2^\mu\eta. \tag{10.2}$$

In case $c=0$, this means

$$|\sigma^{i_\mu}_{L_\mu}|<2^\mu\eta, \tag{10.3}$$

but (10.3) follows from (10.2) even if $c=\Delta^{i_\mu}$, since, by Proposition 8°, either each $\sigma^{i_\mu}_k$ is positive $(k\in L_\mu)$ and Δ^{i_μ} is non-negative, or each $\sigma^{i_\mu}_k$ is negative and Δ^{i_μ} is non-positive. (10.3) and Proposition 9° yield (10.1) for $\lambda=\mu$.

11° PROPOSITION. *If* $k\in B\cup M$ *then* $|\sigma_k|\leqq\varepsilon_k|\Delta|$. *Consequently,*

$$|\sigma_B+\sigma_M|\leqq\varepsilon|\Delta|. \tag{11.1}$$

PROOF. a) If $k\in B$ then

$$|\sigma_k|=\varepsilon_k|g_k(y)-g_k(x)|=\varepsilon_k|\hat{k}||\psi_k(y)-\psi_k(y)|=$$
$$=\varepsilon_k|\psi_k(y)-\psi_k(x)|\leqq\varepsilon_k|f(y)-f(x)|,$$

by the triangle inequality applied to the definition of ψ_k.

b) Assume now $k\in M$. There is a λ such that $k\in M_\lambda$. By (6.8), $t_\lambda<k$. From the definition of M_λ we have $\alpha^{i_\lambda}_k\neq\alpha^{i_\lambda}_{t_\lambda}$, thus $P^{i_\lambda}_{t_\lambda}\subset C_k$. According to (6.6),

$$\varrho(f(z),P^{i_\lambda}_{t_\lambda})\leqq|\Delta^{i_\lambda}|\leqq|\Delta|\qquad(z=x,y),$$

so

$$\psi_k(z)=\varrho(f(z),C_k\cup D_k)\leqq|\Delta|\qquad(z=x,y)$$

and, by (6.5),

$$|\sigma_k|=\varepsilon_k|g_k(z)|=\varepsilon_k\psi_k(z)\leqq\varepsilon_k|\Delta|$$

where $z=x$ or $z=y$.

c) (11.1) follows now from $B \cap M = \emptyset$.

12° PROPOSITION. *If ε is small enough then*

(12.1) $$|h(y)-h(x)| \geqq \varepsilon |f(y)-f(x)| \quad (x, y \in X).$$

PROOF. Let $\varepsilon > 0$ be fixed and assume that (12.1) does not hold for this ε. Choose points $x, y \in X$ such that

(12.2) $$|h(y)-h(x)| < \varepsilon |f(y)-f(x)|.$$

In all the notations introduced in 6°, let the suppressed parameters x and y be now equal to the points we have just chosen. Thus (12.2) means

(12.3) $$|h(y)-h(x)| < \varepsilon |\Delta|.$$

By (6.10) and Proposition 11°,

$$|\Delta + \sigma_L| < 2\varepsilon |\Delta|.$$

Proposition 10° gives now

$$|\sigma_{L_\lambda}| < 2^\lambda \varepsilon |\Delta| \quad (1 \leqq \lambda \leqq \nu),$$

so

(12.4) $$|\sigma_L| < (2^{\nu+1}-2)\varepsilon |\Delta|.$$

By (6.10), (12.3), (12.4) and Proposition 11°,

$$|\Delta| < 2^{\nu+1}\varepsilon |\Delta| \leqq 2^{n+1}\varepsilon |\Delta|,$$

so $\varepsilon > 1/2^{n+1}$, i.e. (12.1) holds whenever $\varepsilon \leqq 1/2^{n+1}$.

13° Fix now the numbers ε_k such that

(13.1) $$\varepsilon = \frac{1}{2^{n+1}}.$$

Then (12.1) holds, as we have just shown. We are going to prove that h is an embedding in this case.

If $x \in X$ and $\mathscr{V}(x) \neq \emptyset$, let $q(x) \in \omega$ be defined by

$$V^i_{q(x)} \in \mathscr{V}(x) \cap \mathscr{V}^i \quad \text{iff} \quad \mathscr{V}(x) \cap \mathscr{V}^i \neq \emptyset,$$

$$V^i_{q(x)} = X \quad \text{iff} \quad \mathscr{V}(x) \cap \mathscr{V}^i = \emptyset.$$

According to (1.3), there is exactly one $q(x)$ satisfying these conditions. Observe that

(13.2) $$V \in \mathscr{V}(x) \Rightarrow Q_{q(x)} \subset V \quad (x \in X).$$

14° PROPOSITION. *If $x \in X$ and $\mathscr{V}(x) \neq \emptyset$ then $\psi_{q(x)}(x) \neq 0$.*

PROOF. Let x be fixed and $q = q(x)$. What we have to show is

(14.1) $$\varrho(f(x), C_q) > 0 \quad \text{if} \quad C_q \neq \emptyset$$

and

$$\varrho(f(x), D_q) > 0 \quad \text{if} \quad D_q \neq \emptyset. \tag{14.2}$$

Put

$$I = \{i \in \{1, \dots, n\} \colon V_q^i \neq X\}.$$

For any $i \in I$, $V_q^i \in \mathscr{V}(x) \cap \mathscr{V}^i$, thus (1.4) and (1.5) imply

$$x \in F^*[V_q^i] - G^*[V_q^i] = F_q^{i*} - G_q^{i*},$$

so

$$f^i(x) = \varphi^i((G_q^{i*}, F_q^{i*})) = \alpha_q^i.$$

If $l < q$, $i \in \{1, \dots, n\}$, $V_l^i \neq X \neq V_q^i$ and $\alpha_l^i \neq \alpha_q^i$ then $i \in I$ and

$$\varrho(f(x), P_l^i) = |f^i(x) - \alpha_l^i| = |\alpha_q^i - \alpha_l^i| > 0.$$

C_q is the union of a finite collection of such planes P_l^i, thus we have (14.1).

Take a neighbourhood U of x such that $\overline{U} \subset Q_q$. As $\mathscr{S}(x)$ is a local subbase at x, U can be chosen in the form

$$U = Q_q \cap \bigcap \mathscr{W}, \quad \mathscr{W} \subset \mathscr{V}^*(x), \quad |\mathscr{W}| < \omega \tag{14.3}$$

[cf. (13.2)]. By Proposition 3°,

$$\varrho(f(x), f(X - \bigcap \mathscr{W})) > 0$$

if $X - \bigcap \mathscr{W} \neq \emptyset$. To prove (14.2), it is enough to show that

$$\operatorname{Fr} Q_q \cap \bigcap \mathscr{W} = \emptyset. \tag{14.4}$$

Indirectly, assume that z belongs to the left side of (14.4). $X - \overline{U}$ and $\bigcap \mathscr{W}$ are neighbourhoods of z, but by (14.3),

$$[(X - \overline{U}) \cap \bigcap \mathscr{W}] \cap Q_q = (X - \overline{U}) \cap [Q_q \cap \bigcap \mathscr{W}] = (X - \overline{U}) \cap U = \emptyset,$$

thus $z \notin \operatorname{Fr} Q_q$, a contradiction.

15° Proposition. *For each $x \in X$ with $\mathscr{V}(x) \neq \emptyset$, there is a $\delta = \delta(x) > 0$ such that if $y \notin Q_{q(x)}$ and $|f(y) - f(x)| < \delta$ then $|h(y) - h(x)| \geqq \varepsilon\delta$.*

Proof. Let x be fixed and assume that the conclusion of the proposition does not hold for some $\delta > 0$. Take a point $y \notin Q_{q(x)}$ such that

$$|f(y) - f(x)| < \delta, \quad |h(y) - h(x)| < \varepsilon\delta. \tag{15.1}$$

Again, we write q for $q(x)$ and use the notations introduced in 6°. By Proposition 14°, $\psi_q(x) \neq 0$, so $g_q(x) \neq 0$ (since $x \in Q_q$) and $q \in T$. Choose λ such that $q \in T_\lambda$. As $y \notin Q_q$, we have $\sigma_q = -\varepsilon_q g_q(x)$. By the definition of T_λ, $V_q^{i_\lambda} \neq X$, so $|\hat{q}^{i_\lambda}| = 1$ and

$$|\sigma_q^{i_\lambda}| = \varepsilon_q |g_q(x)| = \varepsilon_q \psi_q(x). \tag{15.2}$$

a) Suppose that $q \in L_\lambda$. From (6.10), (15.1) and Proposition 11° we have

$$|\sigma_L| < |\Delta| + \varepsilon\delta + \varepsilon|\Delta| < (1 + 2\varepsilon)\delta < 2\delta$$

[cf. (13.1)]. By Proposition 10°,

$$|\sigma_L^{i_\lambda}| \leqq |\sigma_L| < 2^\lambda \delta \leqq 2^\nu \delta \leqq 2^n \delta.$$

For each $k \in L_\lambda$, $\sigma_k^{i_\lambda}$ is positive (negative) by Proposition 8°. So $|\sigma_q^{i_\lambda}| < 2^n \delta$ and from (15.2) we have

(15.3) $$\delta > 2^{-n} \varepsilon_q \psi_q(x).$$

b) Suppose now that $q \in M_\lambda$. Then $|\sigma_q^{i_\lambda}| \leqq |\sigma_q| \leqq \varepsilon_k |\Delta| < \varepsilon_k \delta < \delta$, by Proposition 11°. (15.2) gives now $\delta > \varepsilon_q \psi_q(x)$, so (15.3) holds again.

Thus the proposition holds with δ equal to the right-hand side of (15.3).

16° In order to prove that h is one-to-one and h^{-1} is continuous, it is enough to show that for each $x \in X$ and $W \in \mathscr{S}(x)$, there is an $\eta = \eta(x, W) > 0$ with

(16.1) $$|h(y) - h(x)| > \eta \qquad (y \in X - W).$$

a) If $W \in \mathscr{V}^*(x)$ then for each $y \in X - W$,

$$|h(y) - h(x)| \geqq \varepsilon |f(y) - f(x)| \geqq \varepsilon \varrho\big(f(x), f(X-W)\big) > 0$$

by Propositions 12° and 3°, so (16.1) holds now.

b) If $W \in \mathscr{V}(x)$ then $Q_{q(x)} \subset W$ by (13.2). Take a $y \in X - W$. If $|f(y) - f(x)| \geqq \geqq \delta(x)$ then

(16.2) $$|h(y) - h(x)| \geqq \varepsilon \delta(x)$$

by Proposition 12°. On the other hand, if $|f(y) - f(x)| < \delta(x)$ then (16.2) holds again, now by Proposition 15°. So we have (16.1) in this case, too.

REFERENCES

[1] Deák, E., Eine vollständige Charakterisierung der Teilräume eines euklidischen Raumes mittels der Richtungsdimension, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1964), 437—465. *MR* **32** #4663*b*.

[2] Deák, E., Untersuchungen über Richtungsstrukturen, III. Über die Reichweite des $\mathfrak{R}$-Kompaktifizierungsverfahrens, *Studia Sci. Math. Hungar.* **11** (1976), 437—449. *MR* **81k**: 54052.

[3] Deák, E., *Dimension und Konvexität* (in preparation).

(Received March 17, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 427—428.

ABELIAN GROUPS DEFINED BY LA-SEMIGROUPS

QAISER MUSHTAQ

An LA-semigroup (left almost semigroup) [1] is a groupoid $(G, \cdot)$, satisfying the condition

(1) $$(ab)c = (cb)a \quad \text{for all} \quad a, b, c \quad \text{in } G.$$

It is known [1] that $(G, \cdot)$ is medial, that is

(2) $$(ab)(cd) = (ac)(bd) \quad \text{for all} \quad a, b, c, d \quad \text{in } G.$$

In [3] it is proved that every Abelian group $(G, *)$ induces an LA-semigroup $(G, \cdot)$, where the operation $(\cdot)$ is defined by

(3) $$a \cdot b = b * a^{-1} \quad \text{for all } a, b \text{ in } G.$$

In this note, following the method used by Margado [2], we have shown that conversely, provided that a necessary and sufficient condition is satisfied by an LA-semigroup, it can induce an Abelian group satisfying the condition (3). Some additional characteristics of such LA-semigroups are also noted.

THEOREM. *In an LA-semigroup $(G, \cdot)$, the following conditions are equivalent:*

(i) $a = (cc \cdot ab)b$ *for all* a, b, c *in* G.

(ii) *There exists an Abelian group* $(G, *)$ *such that* $a \cdot b = b * a^{-1}$ *for all* a, b *in* G.

(iii) $(G, \cdot)$ *is cancellative with left identity e and with the property that* $a^2 = e$ *for all* a *in* G.

(iv) $(G, \cdot)$ *has a left identity e and* $a^2 = e$ *for all a in* G.

PROOF. Suppose (i) holds in G. Then $aa = ((cc \cdot ab)b)((cc \cdot ab)b)$, and so by successive application of (2), we get

$$\begin{aligned} aa &= ((cc \cdot ab)b)((cc \cdot ab)b) = (cc \cdot ab)(cc \cdot ab))bb \\ &= ((ca \cdot cb)(ca \cdot cb))bb = ((ca \cdot ca)(cb \cdot cb))bb \\ &= ((ca \cdot ca)(cc \cdot bb))bb = cc. \end{aligned}$$

This implies that

(4) $$aa = cc \quad \text{for all } a, c \text{ in } G.$$

This means that the element aa does not depend on a and so we can fix

(5) $$aa = e.$$

1980 *Mathematics Subject Classification.* Primary 20N99.
Key words and phrases. Left almost semigroups.

It is immediate that e is the left identity of $(G, \cdot)$. By (5), (4) and (1), we get $ex= =aa\cdot x=(xx\cdot xx)x=x$ and so $ex=x$ for all x in G. Also, by virtue of the existence of the left identity, $(cc\cdot ab)b=a$ becomes $(ab)b=a$ for all a, b in G. Let us now define an operation $(*)$ in G, as follows

$$x*y=(ye)x \quad \text{for all } x, y \text{ in } G. \tag{6}$$

Let x be an arbitrary element of G, then $x*e=ee\cdot x=ex=x$ and so $x*e=x$ for all x in G. Hence e is the right identity of $(G, *)$. Because of (6) and (1) we have

$$(a*b)*c=(ce)(be\cdot a)=\big((be\cdot a)e\big)c=\big((ea)(be)\big)c=(c\cdot be)a=(a\cdot be)c.$$

Also,

$$(c*b)*a=(be\cdot c)*a=(ae)(be\cdot c)=(a\cdot be)(ec)=(a\cdot be)c.$$

Thus the identity (1) holds in $(G, *)$ and so $(G, *)$ becomes an LA-semigroup with right identity e and consequently (see [3]) it becomes an Abelian monoid. Now, $(ae)*a=(ae)(ae)=(aa)(ee)=(aa)e=aa=e$ and so $(ae)*a=e$ for all a in G. Hence ae is the left inverse of a in $(G, *)$. Hence, the existence of left inverses in Abelian monoid $(G, *)$, proves that it is an Abelian group. Finally $y*x^{-1}=y*(xe)= =(xe\cdot e)y=(ex)y=xy$. That is $y*x^{-1}=xy$ for all x, y in G.

The condition (iii) is an easy consequence of (ii). For if a, b, c in G are such that $ab=ac$ then by condition (ii), we have $b*a^{-1}=c*a^{-1}$ and hence $b=c$. Thus $(G, \cdot)$ is left cancellative and consequently, (see [3]), a cancellative LA-semigroup.

By definition (3) it is trivial that the identity of $(G, *)$ is the left identity in $(G, \cdot)$ and that $a^2=e$ holds for all a in G.

Condition (iii) implies (iv) is obvious.

Now we show that (iv) implies (i). If a, b, c are in G then, $a=e(ea)=e(bb\cdot a)= =(ee)(bb\cdot a)=(e\cdot bb)(ea)=(bb)(ea)=(be)(ba)=(be)(ab\cdot e)=(b\cdot ab)(ee)=(b\cdot ab)e= =\big(b(cc\cdot ab)\big)e=\big(e(cc\cdot ab)\big)b=(cc\cdot ab)b$. Thus $a=(cc\cdot ab)b$ for all a, b, c in G. Hence the proof.

I acknowledge my thanks to the referee for making many valuable suggestions.

REFERENCES

[1] Kazim, M. and Naseeruddin, M., On almost semigroups, *Aligarh Bull. Math.* **2** (1972), 1—7. *MR* **54** # 7662.

[2] Morgado, J., A theorem on entropic groupoids, *Portugal. Math.* **26** (1967), 449—452. *MR* **41** # 375.

[3] Mushtaq, Q. and Yusuf, S. M., On left almost semigroups, *Aligarh Bull. Math.* **8** (1978).

(Received March 17, 1982)

MATHEMATICAL INSTITUTE
24—29 St. GILES
OXFORD
OX1 3LB
ENGLAND

Permanent address:

DEPARTMENT OF MATHEMATICS
QUAID-I-AZAM UNIVERSITY
ISLAMABAD
PAKISTAN

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 429—433.

ON DISCRETE LINEAR OPERATORS

P. VÉRTESI

1. Introduction

Let $I_n(f, z)$ be the Lagrange interpolatory polynomials of degree $\leqq n-1$ based on the nodes

$$z_{jn} = \exp(2j\pi i/n), \quad j = 1, 2, \ldots, n; \tag{1.1}$$

for a function $f \in AC$ $(=f$ is analytic if $|z|<1$ and continuous for $|z| \leqq 1)$.

D. Gaier [1] asked whether the operators

$$A_n(f, z) = \frac{1}{n} \sum_{k=1}^{n} I_k(f, z) \tag{1.2}$$

uniformly tend to $f(z)$ on $|z| \leqq 1$ for arbitrary $f \in AC$ whenever $n \to \infty$.

Actually the negative answer can be immediately obtained using a recent result of G. Somorjai [2] (see 2.5, Theorem 1). Moreover, from my paper [4] it comes that $\|A_n\| \sim \ln n$ where, as usual,

$$\|A_n\| = \sup_{\|f\| \leqq 1} \|A_n f\|, \quad f \in AC, \quad \|g\| = \max_{|z|=1} |g(z)|$$

(see 2.5, Theorem 2).

The proofs strongly depend on the next statement proved by Rudin (see, e.g. T. Gamelin [3], p. 58):

Let $H_n \subset \Gamma = \{z;\ |z|=1\}$ be a closed set with $|H_n|=0$ ($|\cdot|$ stands for the angular Lebesgue measure). If $f \in C$ $(=f$ is continuous for $f \in \Gamma)$ and $\|f\| \leqq 1$, then there exists a $g \in AC$ for which $\|g\| \leqq 1$ and $g(z)=f(z)$ whenever $z \in H_n$.

Very recently, in a letter, D. Gaier raised the problem to give a direct proof for the theorem $\|A_n\| \sim \ln n$.

Now I intend to obtain this result directly not using the mentioned Rudin-theorem (Actually, we shall apply the explicit form of the mentioned f and g.) Some similar problems will be treated as well.

1980 *Mathematics Subject Classification.* Primary 41A35; Secondary 41A20.
Key words and phrases. Discrete linear operators, Lagrange interpolation.

2. Results

2.1. Consider the linear bounded operator $A_n(f, z)$ which maps the Banach space X into the Banach space Y (shortly $A_n \in B(X, Y)$). We shall suppose that X and Y are subspaces of C and AC, respectively, endowed with the maximum norm. (We defined $A_n(f, z)$ by (1.2).)

A. We wish to give a direct elementary proof for the fact

(2.1) $$\|A_n\| \sim \ln n \quad \text{if} \quad A_n \in B(AC, AC).$$

Indeed let us consider the "test functions"

(2.2) $$h_{nm\varrho}(z) = z^{n!} \sum_{\substack{s=-m \\ s\neq 0}}^{m} \frac{(\varrho z)^s}{s},$$

where $m \leqq n$ and $|\varrho| = 1$ is arbitrary fixed. Obviously, $h_{nm\varrho} \in AC$, moreover

$$|h_{nm\varrho}(z)| = |z^{n!}| \left| \sum_{\dots} \frac{(\varrho z)^s}{s} \right| = \left| 2i \sum_{s=1}^{m} \frac{\sin s \arg(\varrho z)}{s} \right| \leqq d_2$$

if $z \in \Gamma$ ($d, d_1, d_2, \dots$ are absolute constants), i.e.

$$\|h_{nm\varrho}\| \leqq d_2 \quad \text{for arbitrary} \quad |\varrho| = 1 \quad \text{and} \quad m \leqq n.$$

Further, if we use the fact that

(2.3) $$z_{jk}^{n!} = 1 \quad \text{if} \quad 1 \leqq j \leqq k, \quad 1 \leqq k \leqq n,$$

we can write by (2.2)

$$I_k(h_{nm\varrho}, z) = \sum_{\substack{s=-m \\ s\neq 0}}^{m} \frac{\varrho^s}{s} I_k(t^s, z), \qquad 1 \leqq k \leqq n,$$

which means

(2.4) $$A_n(h_{nm\varrho}, z) = \sum_{\substack{s=-m \\ s\neq 0}}^{m} \frac{\varrho^s}{s} A_n(t^s, z) \quad \text{if} \quad |z| \leqq 1, \quad |\varrho| = 1, \quad m \leqq n.$$

(Remark, that the expressions $I_k(f, z)$ and $A_n(f, z)$ have meaning for any $f \in C$.) I.e. we can write with *any fixed* $\varrho = 1/z_0$, $|z_0| = 1$,

(2.5) $$\|A_n\| \geqq \frac{\|A_n(h_{nm\varrho}, z)\|}{\|h_{nm\varrho}\|} \geqq \frac{\max\limits_{|z|=1} \left| \sum\limits_{\substack{s=-m \\ m\neq 0}}^{m} \frac{\varrho^s}{s} A_n(t^s, z) \right|}{d_2} \geqq \frac{\left| \sum\limits_{\substack{s=-m \\ m\neq 0}}^{m} \frac{A_n(t^s, z_0)}{s z_0^s} \right|}{d_2}.$$

So, to obtain a lower bound for $\|A_n\|$ we have to estimate (from below) the expression

(2.6) $$g_{nm}(z) \stackrel{\text{def}}{=} \sum_{\substack{s=-m \\ s\neq 0}}^{m} \frac{A_n(t^s, z)}{s z^s}, \qquad |z| = 1.$$

If we use the fact that with $\varepsilon_{ns}(z)\in C$, $|\varepsilon_{ns}(z)|\leqq\varepsilon<1$

$$A_n(t^s, z) = z^s+\varepsilon_{ns}(z) \quad \text{if} \tag{2.7}$$

$$|z| = 1, \quad 1 \leqq s \leqq [\sqrt{n}] \overset{\text{def}}{=} N, \quad n \geqq n_0$$

(which can be proved by (1.2), $I_k(t^s, z)=z^s$ $(k\geqq s+1)$ and $\|I_k\|\leqq d_3 \ln(k+1)$ as follows:

$$|A_n(t^s, z)-z^s| \leqq \left|\frac{1}{n}\sum_{k=1}^{s}[I_k(t^s, z)-z^s]\right| \leqq \frac{d_3 \ln n}{\sqrt{n}},$$

which is more than (2.7)), we can write if $z=\exp i\vartheta$ (see [4], 2.2)

$$2\pi\|g_{nN}(z)\| \geqq \left|\int_{|z|=1} g_{nN}(z)\,d\vartheta\right| =$$

$$= \left|\sum_{s=1}^{N}\left[\int_{|z|=1}\frac{A_n(t^s, z)}{sz^s}\,d\vartheta - \int_{|z|=1}\frac{z^sA_n(t^{-s}, z)}{s}\,d\vartheta\right]\right| = \tag{2.8}$$

$$= \left|\sum_{s=1}^{N}\int_{|z|=1}\left(\frac{1}{s}+\frac{\varepsilon_{ns}(z)}{sz^s}\right)d\vartheta - 0\right| \geqq 2\pi\sum_{s=1}^{N}s^{-1}(1-\varepsilon) > 2\pi(1-\varepsilon)\ln N.$$

(Here we used that by the Cauchy integral theorem

$$\int_{|z|=1} z^sA_n(t^{-s}, z)\,d\vartheta = \frac{1}{i}\int_{|z|=1} z^{s-1}A_n(t^{-s}, z)\,dz = 0$$

if $s=1, 2, \ldots,$ because $A_n(t^{-s}, z)\in AC$.)

By (2.5), (2.6) and (2.8), $\|A_n\|\geqq d_4 \ln n$ $(n\geqq n_0)$. On the other hand

$$\|A_n(f, z)\| \leqq d_5 n^{-1}\|f\|\sum_{k=1}^{n}\ln n = d\|f\|\ln n,$$

from where we obtain (2.1).

2.2. *Now let us consider an arbitrary interpolatory matrix*

$$Z = \{z_{jn} = \exp(2\pi i\vartheta_{jn})\}, \quad j = 1, 2, \ldots, n; \quad n = 1, 2, \ldots \tag{2.9}$$

and the corresponding processes $I_n(f, Z, z)$ and $A_n(f, Z, z)$ (see Part 1).

B. Using again an "elementary" argument we state

$$\lim_{n\to\infty}\|A_n(Z)\| = \infty \quad \text{if} \quad A_n(Z)\in B(AC, AC). \tag{2.10}$$

To obtain (2.10) first let ϑ_{jn} be rational for any j and n. If $\vartheta_{jn}=p_{jn}/q_{jn}$, let

$$P = P(n) = \prod_{k=1}^{n}\prod_{j=1}^{k} q_{jk}$$

moreover let N be an arbitrary fixed positive integer. If for a fixed $0<\varepsilon<1$ we have the inequalities

$$E(A_n, N) \overset{\text{def}}{=} \max_{1\leqq s\leqq N}\|A_n(t^s, Z, z)-z^s\| \leqq \varepsilon < 1 \quad \text{if} \quad n \geqq n_0(N), \tag{2.11}$$

then using the "test functions"

$$h_{nN\varrho}(z) = z^P \sum_{\substack{s=-N \\ s\neq 0}}^{N} \frac{(\varrho z)^s}{s} \in AC, \qquad |\varrho| = 1, \tag{2.12}$$

we obtain that $\|h\| \leqq d_2$, moreover $z_{jk}^P = 1$ if $1 \leqq j \leqq k$, $1 \leqq k \leqq n$. Using the argument of 2.1, and (2.11) instead of (2.7), we obtain, sometimes omitting the superfluous notations,

$$\|A_n\| \geqq d_6 \ln N \quad n \geqq n_0(N). \tag{2.13}$$

If we prove the relation (2.11) we are ready. But

$$E(A_n, N) = \max_{1 \leqq s \leqq N} \left\| \frac{1}{n} \sum_{k=1}^{s} [I_k(t^s, z) - z^s] \right\| \leqq$$

$$\leqq d_7 \frac{N}{n} \max_{1 \leqq k \leqq N} \|I_k\| \leqq \varepsilon \quad \text{if} \quad n \geqq n_0(N)$$

which was to be proved.

Now let at least one ϑ_{jn} be irrational. Using that not so difficult to construct the functions $\varphi_n(z) \in AC$ for which

$$\varphi_n(z_{jk}) = \frac{1}{z_{jk}^N}, \quad 1 \leqq j \leqq k, \quad k = 1, 2, \ldots, n, \quad \text{and} \quad \|\varphi_n\| \leqq 2$$

(see e.g. S. J. Alper [5]; N is as in (2.11)), we can take

$$h_{nN\varrho}(z) = z^N \varphi_n(z) \sum_{\substack{s=-N \\ s\neq 0}}^{N} \frac{(\varrho z)^s}{s} \in AC, \qquad |\varrho| = 1$$

as test functions. The remaining parts are left to the reader.

2.3. The above "direct" argument can be applied for the operators

$$L_n(f, Z, z) = \sum_{k=1}^{n} \sum_{j=1}^{k} f(z_{jk}) t_{jk}(z) \tag{2.14}$$

where $t_{jk} \in AC$ for arbitrary j and k. As above, we can prove as follows.

C. *If $L_n \in B(AC, AC)$, moreover if for certain fixed N and m, $E(L_m, N) \leqq \varepsilon < 1$, then*

$$\|L_m\| \geqq d_8 \ln N.$$

(The definition of $E(L_n, N)$ corresponds to (2.11).) This statement — which is a certain analogy to the Lozinskii—Harsiladze theorem — can be used, among others, for the Hermite—Birkhoff interpolation, the (C, α)-means of $\{I_n\}$ a. s. o. (see e.g. [4]). We omit the further details.

2.4. As an application of the above statement we answer another question of D. Gaier.

D. *If $\{a_{nk}\}$ is an arbitrary permanent row-finite summability matrix then for any Z*

$$\lim_{n\to\infty} \|G_n(Z)\| = \infty,$$

where $G_n(Z)\in B(AC, AC)$ is defined by

$$G_n(f, Z, z) = \sum_{k=1}^{n} a_{nk} I_k(f, Z, z)$$

(see 2.2).

Indeed, by definition $\sum_{k=1}^{n} a_{nk} = 1+\varepsilon_n$ where $\lim_{n\to\infty} \varepsilon_n = 0$, from where for any s, $1 \leqq s \leqq n$, N fixed

$$\|G_n(t^s, z) - z^s\| = \Big\|\sum_{k=1}^{n} a_{nk}[I_k(t^s, z) - z^s] + \varepsilon_n z^s\Big\| =$$

$$= \Big\|\sum_{k=1}^{s} a_{nk}[I_k(t^s, z) - z^s] + \varepsilon_n z^s\Big\| \leqq (\lambda_N^* + 1) \sum_{k=1}^{N} |a_{nk}| + \varepsilon_n \leqq \varepsilon < 1$$

if n is big enough. Here $\lambda_N^* = \max_{1\leqq k\leqq N} \|I_k(Z)\|$, moreover we used the fact that for any fixed k, $\lim_{n\to\infty} a_{nk} = 0$. I.e., by statement C we obtain that for any fixed N, $\|G_n\| \geqq \geqq d_8 \ln N$ whenever $n \geqq n_0(N)$, which was to be proved.

2.5. Remark 1. The arguments applied here are similar to those in [2] and [4].

Remark 2. For sake of completeness we quote the two theorems mentioned in **1**.

Theorem 1 ([2]). *Let $H_n \subset \Gamma$ $(n=1, 2, \ldots)$ be closed sets of angular Lebesgue measure zero. Suppose that L_n are arbitrary operators with $L_n \in B(C, AC)$ for which $L_n(f, z) \equiv L_n(g, z)$ whenever $f=g$ on the set H_n. Then there is an $f\in AC$ for which*

$$\overline{\lim_{n\to\infty}} \|L_n(f, z) - f(z)\| > 0.$$

Theorem 2 ([4]). *Let L_n are as in Theorem 1. Moreover let us suppose that $E(L_n, N) \leqq \varepsilon < 1$ if $n \geqq n_1$. Then $\|L_n\| \geqq d_9 \ln N$ whenever $n \geqq n_1$.*

REFERENCES

[1] Gaier, D., *Period. Math. Hungar.* **12** (1981), 161.
[2] Somorjai, G., On discrete linear operators in the function space A, *Constructive Function Theory '77*, Sofia, 1980, 489—496.
[3] Gamelin, T. W., *Uniform algebras*, Prentice—Hall, Inc., Englewood Cliffs, N. J., 1969. *MR* **53** # 14137.
[4] Vértesi, P., Linear operators on the roots of unity, *Studia Sci. Math. Hungar.* **15** (1980), 241—245.
[5] Al'per, S. Ya., On the convergence of Lagrange's interpolational polynomials in the complex domain, *Uspehi Mat. Nauk. N. S.* **11** (1956), no. 5 (71), 44—50 (in Russian). *MR* **18**—728.

(Received March 25, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 435—459.

CONTROLLING FUNCTION CLASSES AND COVERING EUCLIDEAN SPACE

ENDRE MAKAI, JR. and JÁNOS PACH

Abstract

Given any class $\mathcal{F}$ of functions $\mathbf{R}^m \to \mathbf{R}^n$, we call a sequence $\langle \mathbf{x}_i \rangle \subseteq \mathbf{R}^m$ *$\mathcal{F}$-controlling*, if there exists another sequence $\langle \mathbf{y}_i \rangle \subseteq \mathbf{R}^n$ with the property that for every $f \in \mathcal{F}$

$$\inf_i \|f(\mathbf{x}_i) - \mathbf{y}_i\| \leqq 1.$$

How sparsely can an $\mathcal{F}$-controlling sequence be distributed in $\mathbf{R}^m$? The aim of this paper is to answer this question for several function classes, in particular, for the class of linear functions, polynomials, functions satisfying Lipschitz condition, etc. It turns out that these problems are strongly connected with coverings of the euclidean space by convex sets. (This connection is demonstrated by a sample problem, in Section 2.) The last section contains some related, purely geometric results. We give there, among others, a necessary and sufficient condition for a countable class of convex sets to permit a covering of $\mathbf{R}^n$. This solves a problem of Groemer. The main results of the paper and some open problems are listed in Section 3.

1. Introduction — the basic problem

In this paper we shall investigate a question which is, at first glance, of purely analytic character, but which, however, turns out to be closely related to problems of covering the euclidean space by convex sets. To formulate our question we need some simple definitions.

Let n be a fixed natural number and let $\mathcal{F}$ be any class of functions f: $\mathbf{R} \to \mathbf{R}^n$.

DEFINITION 1.1. A set of pairs

$$\{(\mathbf{x}_i, \mathbf{y}_i) \mid \mathbf{x}_i \in \mathbf{R}, \mathbf{y}_i \in \mathbf{R}^n;\ i = 1, 2, \dots\}$$

is said to form an *$\mathcal{F}$-controlling system* if for each function $f \in \mathcal{F}$ there exists a natural number i satisfying

$$\|f(\mathbf{x}_i) - \mathbf{y}_i\| \leqq 1.$$

Roughly speaking, a controlling system is a subset S of $\mathbf{R} \times \mathbf{R}^n$ with the property that for each function $f \in \mathcal{F}$ one can find *at least* one point in S, "sufficiently close" to the graph of f.

DEFINITION 1.2. A sequence $\langle x_i \rangle$ of real numbers is called an *$\mathcal{F}$-controlling sequence* if one can choose points $\mathbf{y}_i \in \mathbf{R}^n$ $(i=1, 2, \dots)$ so that the pairs $(x_i, \mathbf{y}_i)$ form an $\mathcal{F}$-controlling system.

1980 *Mathematics Subject Classification.* Primary 05—XX; Secondary 52A45.

Key words and phrases. Controlling system, controlling sequence, linear functions, Lipschitz functions, space coverings, translative coverings, packings, convex sets.

The aim of this paper is to characterize controlling sequences for various classes $\mathcal{F}$ of functions, in particular, for the class of linear functions, polynomials, functions satisfying Lipschitz condition, etc.

In Section 2 we go into a sample problem in order to point out how the solution of questions of the above type may lead to nice and natural *geometric* problems about coverings of the space by convex sets. In Section 3 we summarize the results of this paper and we formulate some open problems and conjectures. The next two sections contain the proofs. In Section 6 we are going to deal with some related, purely geometric questions.

2. A sample problem — connection with geometry

Throughout this section, let $\mathcal{F}$ be the class of all linear functions from $\mathbf{R}$ into $\mathbf{R}$, i.e.

$$\mathcal{F} = \mathcal{L} := \{ax+b \mid a, b \in \mathbf{R}\}.$$

In what follows, we give a necessary and sufficient condition for a sequence $\langle x_i | i = 1, 2, \ldots \rangle$ of real numbers to be *$\mathcal{L}$-controlling.*

THEOREM 2.1. *A sequence $\langle x_i \rangle$ of real numbers is $\mathcal{L}$-controlling if and only if $\Sigma(1+|x_i|)^{-1} = +\infty$.*

PROOF. Suppose first that $\langle x_i \rangle$ is a controlling sequence. This means that there exists another sequence $\langle y_i \rangle$ of real numbers, with the property that, for every linear function $ax+b$, one can find at least one i satisfying

$$|ax_i + b - y_i| \leqq 1$$

or, equivalently,

$$y_i - 1 \leqq ax_i + b \leqq y_i + 1. \tag{1}$$

Let us fix an index i, and consider those points $(a, b) \in \mathbf{R}^2$ for which (1) holds. They clearly form a strip S_i of the plane, whose width is equal to $2(1+x_i^2)^{-1/2}$. It is now clear, by our assumption, that the strips S_i $(i=1, 2, \ldots)$ together will cover the whole plane. Thus, we can use the following result known as "Tarski's plank problem" (see Tarski [23] and, for a solution, Bang [1, 2, 3]).

LEMMA 2.2. *If a convex set C (in $\mathbf{R}^d$) is covered by a system of strips, then the sum of their widths is not less than the width of C (that is, the minimal width of a strip containing C).*

Consequently, the sum of widths of S_i

$$\sum w(S_i) = \sum 2(1+x_i^2)^{-1/2} = +\infty, \tag{2}$$

which is equivalent to the condition in the theorem.

Assume now $\Sigma 2(1+x_i^2)^{-1/2} = +\infty$. We are going to show that $\langle x_i \rangle$ is a controlling sequence. Let us define the strip S_i^* as the set of those points $(a, b) \in \mathbf{R}^2$ which satisfy

$$-1 \leqq ax_i + b \leqq 1. \tag{1*}$$

We clearly have $w(S_i^*)=2(1+x_i^2)^{-1/2}$ and hence $\Sigma w(S_i^*)=+\infty$. Observe that, for each translated copy S_i of S_i^*, one can find a real number y_i such that S_i is defined by the inequalities (1). Thus, if we proved that the strips S_i^* have suitable translates S_i such that $\bigcup S_i$ covers the whole plane, then it would imply that the corresponding sequence $\langle y_i\rangle$ meets the requirements of Definition 1.2, in other words, $\langle x_i\rangle$ is a controlling sequence. But this is an immediate consequence of the following lemma.

LEMMA 2.3. *Let C be a bounded convex set in the plane and let $\langle S_i^*|i=1, 2, \dots, m\rangle$ be a finite sequence of strips with total width*

$$\sum_{i=1}^{m} w(S_i^*) \geqq p(C). \tag{3}$$

Then C can be covered by translates S_i of S_i^*. ($p(C)$ denotes the perimeter of C.)

PROOF of Lemma 2.3. Fix, as usual, any rectangular system (x, y) of coordinates on the plane. Let ϑ_i denote the angle between the normal vector of S_i^* and the positive x axis. We may clearly assume

$$0<\vartheta_1\leqq\vartheta_2\leqq\dots\leqq\vartheta_m<\pi,$$

by changing the direction of some normal vectors to the opposite.

We are going to define the translates S_i of S_i^* by induction. Let S_1 be that translated copy of S_1^* whose upper boundary line supports C from above. (Here the words "upper" and "above" are used according to the direction of the y axis.) Assume that $S_1, \dots, S_{i-1}$ have already been determined. Put $C^i := C\setminus\bigcup_{k=1}^{i-1} S_k$. Now, let S_i be defined as the translate of S_i^* whose upper boundary supports C^i from above.

It is easy to see that the uncovered part of the boundary Bd C of C decreases by at least $w(S_i^*)$ at the i-th stage, which completes the proof of the lemma, and hence the theorem. ∎

We note that Groemer [16, 17] proved the assertion of Lemma 2.3 under the essentially stronger condition

$$\sum_{i=1}^{m} \min\,(w^{3/2}(S_i^*), 1) > p(C)+a(C), \tag{3'}$$

where $a(C)$ denotes the area of C.

Let us close this section by mentioning a straightforward strengthening of Theorem 2.1. To formulate this, we need a further definition which is, in fact, a slight modification of 1.1.

DEFINITION 2.4. Let $\mathscr{F}$ be a class of functions from $\mathbf{R}$ into $\mathbf{R}^n$. A set of pairs $\{(x_i, \mathbf{y}_i)|x_i\in\mathbf{R},\ \mathbf{y}_i\in\mathbf{R}^n;\ i=1, 2, \dots\}$ is called a *strongly $\mathscr{F}$-controlling system* if, for every $f\in\mathscr{F}$ and every $\varepsilon>0$, one can find an i satisfying

$$\|f(x_i)-\mathbf{y}_i\|\leqq\varepsilon.$$

Accordingly, a *strongly $\mathscr{F}$-controlling sequence* is defined as a sequence $\langle x_i\rangle$ such that $\{(x_i, \mathbf{y}_i)|i=1, 2, \dots\}$ forms a strongly $\mathscr{F}$-controlling system, for some suitably chosen vectors $\mathbf{y}_i\in\mathbf{R}^n$.

Intuitively speaking, a strongly controlling system is a set of points in $\mathbf{R}\times\mathbf{R}^n$, which comes arbitrarily near to the graph of every function $f\in\mathcal{F}$.

Now, a stronger version of Theorem 2.1 can be formulated as follows.

THEOREM 2.5. *Let* $\langle x_i\rangle$ *be a sequence of real numbers.*

A. *If* $\Sigma(1+|x_i|)^{-1}<+\infty$, *then* $\langle x_i\rangle$ *is not an* $\mathcal{L}$*-controlling sequence. Moreover, for every sequence* $\langle y_i\rangle$ *of real numbers and for every (large)* $K>0$, *there is a linear function* $l\in\mathcal{L}$ *satisfying*

$$|l(x_i)-y_i|>K, \quad \textit{for all } i.$$

B. *If* $\Sigma(1+|x_i|)^{-1}=+\infty$, *then* $\langle x_i\rangle$ *is a strongly* $\mathcal{L}$*-controlling sequence.*

The proof is essentially the same as that of Theorem 2.1, and we leave the minor changes to the reader. Anyway, it is worth mentioning that all of our theorems establishing necessary and sufficient conditions for a sequence $\langle x_i\rangle$ to be $\mathcal{F}$-controlling (regarding different classes of functions), can be strengthened in the above way. In particular, the sets of "controlling" and "strongly controlling" sequences are almost always identical.

3. Survey of results and conjectures

It is obvious that the Definitions 1.1, 1.2 and 2.4 of (strongly) controlling systems and sequences can be directly extended to classes $\mathcal{F}$ of functions $X\to\mathbf{R}^n$, where X is any abstract set.

Let $\mathcal{L}(\mathbf{R}^m,\mathbf{R}^n)$ denote the class of all non-homogeneous linear functions $\mathbf{R}^m\to\mathbf{R}^n$. In the next two theorems we deal with the problem, how *sparsely* an $\mathcal{L}(\mathbf{R}^m,\mathbf{R}^n)$-controlling sequence $\langle\mathbf{x}_i\rangle$ can be distributed in $\mathbf{R}^m$. This question is clearly more general than that investigated in the previous section. However, owing to the lack of an adequate generalization of Lemma 2.3, our general result is somewhat weaker, and we have to use a different geometric technique for the proof.

THEOREM 3.1.A. *Let* m, n *be natural numbers, and let* $\langle\mathbf{x}_i\mid i=1,2,\dots\rangle$ *be any sequence of points in* $\mathbf{R}^m$.

If $\Sigma(1+\|\mathbf{x}_i\|)^{-n}<+\infty$, *then* $\langle\mathbf{x}_i\rangle$ *is not an* $\mathcal{L}(\mathbf{R}^m,\mathbf{R}^n)$*-controlling sequence. Moreover, for every sequence* $\langle\mathbf{y}_i\rangle\subseteq\mathbf{R}^n$ *and for every (large)* $K>0$, *there exists a function* $l\in\mathcal{L}(\mathbf{R}^m,\mathbf{R}^n)$ *satisfying*

$$\|l(\mathbf{x}_i)-\mathbf{y}_i\|>K, \quad \textit{for all } i.$$

THEOREM 3.1.B. *Let* m, n *be natural numbers, and let* $\langle\xi_i\mid i=1,2,\dots\rangle$ *be any sequence of nonnegative real numbers. If* $\Sigma(1+\xi_i)^{-n}=+\infty$, *then there exists a strongly* $\mathcal{L}(\mathbf{R}^m,\mathbf{R}^n)$*-controlling sequence* $\langle\mathbf{x}_i\rangle\subseteq\mathbf{R}^m$, *with* $\|\mathbf{x}_i\|=\xi_i$ *for all* i.

In the following we make an attempt to generalize Theorem 2.1 in another direction. Let $\mathcal{P}_n$ denote the class of all polynomials $p\colon\mathbf{R}\to\mathbf{R}$, whose degree is at most n.

THEOREM 3.2.A. *Let* n *be a natural number, and let* $\langle x_i|i=1,2,\dots\rangle$ *be any sequence of real numbers. If* $\Sigma(1+|x_i|)^{-n}<+\infty$, *then* $\langle x_i\rangle$ *is not a* $\mathcal{P}_n$*-controlling* *sequence. Moreover, for every sequence* $\langle y_i \rangle \subseteq \mathbf{R}$ *and for every (large)* $K>0$, *there exists a polynomial* $p \in \mathcal{P}_n$ *such that*

$$|p(x_i)-y_i| > K, \quad \textit{for all } i.$$

On the other hand, we are unable to prove that the condition $\Sigma(1+|x_i|)^{-n}=+\infty$ is also *sufficient* for $\langle x_i \rangle$ to be a $\mathcal{P}_n$-controlling sequence. Nevertheless, we think that this is the case.

CONJECTURE 3.2.B. Let n be a natural number and let $\langle x_i \rangle$ be any sequence of real numbers satisfying $\Sigma(1+|x_i|)^{-n}=+\infty$. Then $\langle x_i \rangle$ is a strongly $\mathcal{P}_n$-controlling sequence.

A closed convex subset of $\mathbf{R}^n$ is called a *slab* (or strip) *of thickness* w if its boundary consists of two parallel hyperplanes having mutual distance w. As we shall see in the next section, the above conjecture would be an easy corollary to the following

CONJECTURE 3.3. Let $\langle S_i \mid i=1, 2, \dots \rangle$ be a sequence of slabs in $\mathbf{R}^n$, with total thickness $\Sigma w(S_i)=+\infty$. Then the n-dimensional unit ball B^n can be covered by translates of the slabs S_i.

3.2.B (and 3.3) are verified only for $n \leqq 1$, (2). All other cases are unsettled. This is, in our opinion, the most challenging open problem of the topic. To support Conjecture 3.3 we note that, if we choose randomly translated copies of S_i intersecting B^n then, with probability 1 they will cover all points of B^n with the exception of a set of Lebesgue measure 0. (Apply Fubini's theorem on the product measure space $B^n \times \{$systems of translated slabs$\}$; analogous holds for cylinders like in Groemer [14].)

For the class

$$\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n \tag{4}$$

of all polynomials, we have the following result.

THEOREM 3.4. *A sequence* $\langle x_i \rangle$ *of real numbers is* $\mathcal{P}$*-controlling if and only if* $\Sigma(1+|x_i|)^{-n}=+\infty$ *holds for each natural number* n.

As a matter of fact, in Section 4 we prove the above results in a more general form, for finitely generated function classes.

In Section 5 we deal with controlling of Lipschitz functions.

DEFINITION 3.5. Let m, n be natural numbers. A function f: $\mathbf{R}^m \to \mathbf{R}^n$ is called a Lipschitz function (or, in notation, $f \in \mathrm{Lip}(\mathbf{R}^m, \mathbf{R}^n)$) if there exists a constant $C_f>0$ satisfying

$$\|f(\mathbf{x})-f(\mathbf{y})\| < C_f \|\mathbf{x}-\mathbf{y}\|,$$

for every pair $\mathbf{x}, \mathbf{y} \in \mathbf{R}^m$.

We prove the following two complementary theorems.

THEOREM 3.6.A. *Let* m, n *be natural numbers, and let* $\langle \mathbf{x}_i \mid i=1, 2, \dots \rangle$ *be any sequence of points in* $\mathbf{R}^m$ *with* $\|\mathbf{x}_1\| \leqq \|\mathbf{x}_2\| \leqq \dots$. *If*

$$\liminf_{i \to \infty} (\|\mathbf{x}_{(i+1)^n}\| - \|\mathbf{x}_{i^n}\|) > 0, \tag{5}$$

then $\langle \mathbf{x}_i \rangle$ *is not a* Lip $(\mathbf{R}^m, \mathbf{R}^n)$*-controlling sequence. Moreover, for every sequence* $\langle \mathbf{y}_i \rangle \subseteq \mathbf{R}^n$ *and for every* $K>0$, *there exists a function* $f \in \mathrm{Lip}\,(\mathbf{R}^m, \mathbf{R}^n)$ *satisfying*

$$\|f(\mathbf{x}_i)-\mathbf{y}_i\| > K, \quad \textit{for all } i.$$

THEOREM 3.6.B. *Let* m, n *be natural numbers, and* $\langle \xi_i \mid i=1, 2, \ldots \rangle$ *be any increasing sequence of nonnegative real numbers. If*

$$\lim_{i\to\infty} (\xi_{(i+1)^n} - \xi_{i^n}) = 0,$$

then there exists a strongly Lip $(\mathbf{R}^m, \mathbf{R}^n)$*-controlling sequence* $\langle \mathbf{x}_i \rangle \subseteq \mathbf{R}^m$ *with* $\|\mathbf{x}_i\| = \xi_i$ *(for all* i*).*

As we have already mentioned, all the proofs of our theorems of type B are based on corresponding results assuring the existence of a covering of the space by congruent copies (or translates) of certain convex sets. So it is worth considering here the following general problem:

Find necessary and sufficient conditions that a given sequence $\langle C_i \rangle$ of convex sets in $\mathbf{R}^n$ *permits a covering* (resp. *translative covering*) of the space, i.e. there are rigid motions (resp. translations) σ_i such that

$$\mathbf{R}^n \subseteq \bigcup_{i=1}^{\infty} \sigma_i C_i. \tag{6}$$

In Section 6 we deal with this problem. To formulate our results we need some notations (cf. [7]).

Let $C \subseteq \mathbf{R}^n$ be any closed convex set. We are going to define a sequence of parameters $d_i(C) \in [0, +\infty]$, $i=1, 2, \ldots, n$. If C is *bounded,* then let $d_1(C)$ be defined as the diameter of C. Let $[\mathbf{x}, \mathbf{y}]$ be a segment of maximal length in C, and let $H_{\mathbf{x},\mathbf{y}}$ be any hyperplane perpendicular to $[\mathbf{x}, \mathbf{y}]$. The orthogonal projection of C onto $H_{\mathbf{x},\mathbf{y}}$ is denoted by $N(C)$ and is called a *normal projection* of C. (However, $N(C)$ is not uniquely determined, but we always assume that some normal projection has been selected and we keep it fixed.) Now, let $d_2(C)$ be the diameter of $N(C)$, that is

$$d_2(C) = d_1(N(C))$$

and, further,

$$\begin{aligned} d_3(C) &= d_1(N(N(C))) = d_1(N^2(C)) \\ &\vdots \qquad\quad \vdots \\ d_n(C) &= d_1(N^{n-1}(C)). \end{aligned}$$

If C is *unbounded,* then let us consider the set $\mathscr{V}$ of all vectors $\mathbf{v} \in \mathbf{R}^n$ (starting from the origin) for which C contains arbitrarily long segments parallel to $\mathbf{v}$. It is easy to see that $\mathscr{V}$ is a subspace of $\mathbf{R}^n$. In case $\dim \mathscr{V} = k$, set

$$d_1(C) = d_2(C) = \ldots = d_k(C) = +\infty,$$

and denote by $N^k(C)$ the orthogonal projection of C onto $\mathscr{V}^\perp$, where $\mathscr{V}^\perp$ is the maximal subspace perpendicular to $\mathscr{V}$.

PROPOSITION 3.7. $N^k(C)$ *is bounded.*

Using this, we can define $d_{k+1}(C), \dots, d_n(C)$ as above, that is

$$\begin{aligned} d_{k+1}(C) &= d_1(N^k(C)) \\ &\vdots \\ d_n(C) &= d_1(N^{n-1}(C)). \end{aligned}$$

Let $\mathbf{p}$ be any point of C. Consider the union of all halflines from $\mathbf{p}$, contained entirely in C. This is clearly a cone, the so-called *characteristic cone* of C. It is obvious that the characteristic cone does not depend on the choice of $\mathbf{p}$ (up to translation), so, for the sake of simplicity, we may suppose that the apex of the characteristic cone is at the origin.

In Section 6, solving a problem of Groemer, we prove among others the following

THEOREM 3.8. *Let n be a natural number, and let K be a non-degenerate convex cone with apex at the origin $\mathbf{0}\in\mathbf{R}^n$. A sequence $\langle C_i \mid i=1,2,\dots\rangle$ of closed convex sets of $\mathbf{R}^n$ permits a covering of K if and only if one of the following relations holds:*

(1) $$\sum_{d_1(C_i)\le 1} d_1(C_i)\, d_2(C_i)\dots d_n(C_i) = +\infty,$$

(2) $$\sum_{\substack{d_1(C_i)>1\\ d_2(C_i)\le 1}} d_2(C_i)\dots d_n(C_i) = +\infty,$$

$\vdots$

(n) $$\sum_{\substack{d_{n-1}(C_i)>1\\ d_n(C_i)\le 1}} d_n(C_i) = +\infty,$$

($n+1$) $$\sum_{d_n(C_i)>1} 1 = +\infty,$$

($n+2$) *the characteristic cones of C_i with apices at the origin permit a covering of K.*

In particular, for $K=\mathbf{R}^n$, we obtain the necessary and sufficient condition that $\langle C_i\rangle$ permits a covering of $\mathbf{R}^n$. The special cases, when $\langle C_i\rangle$ is a finite sequence or all C_i are bounded, were proved by Groemer [15], and Chakerian—Groemer [6, 7], resp. The problem of finding a general result was stated by Groemer in [13].

Similar necessary and sufficient conditions are (or can be) proved for $\langle C_i\rangle$ to permit a packing (or a translative packing) in the plane, resp. a translative almost covering of the space. (An *almost covering* of $\mathbf{R}^n$ is defined as a covering of $\mathbf{R}^n\setminus S$ where S is an arbitrary set of measure 0.) The main unsolved problem of this kind is to find a non-trivial characterization of those systems of convex sets that permit translative coverings of the space. The proof (or disproof) of Conjecture 3.3 might be an important step in this direction.

4. Proofs — controlling finitely generated function classes

The aim of this section is to prove Theorems 3.1.A and B, 3.2.A and 3.4. As a matter of fact, we prove them in a more general setting.

Let $f_1, \dots, f_k$ be a fixed system of functions $X\to\mathbf{R}$, where X is any abstract set. Consider the class $\mathscr{L}$ of all functions $X\to\mathbf{R}^n$ whose coordinates are linear combinations of f_i. That is,

(7) $$\mathcal{L} := \{(\lambda_{11}f_1+\ldots+\lambda_{1k}f_k, \ldots, \lambda_{n1}f_1+\ldots+\lambda_{nk}f_k)\,|\,\lambda_{ij}\in\mathbf{R} \text{ for all } i, j\}.$$

THEOREM 4.1.A. *Let X be an abstract set, and let the class $\mathcal{L}$ of functions $X\to\mathbf{R}^n$ be defined as in* (7). *Further, let $\langle x_t|t=1, 2, \ldots\rangle$ be a sequence of points in X. If $\sum_{t=1}^{\infty}(\sum_{j=1}^{k}f_j^2(x_t))^{-n/2}<+\infty$, then $\langle x_t\rangle$ is not an $\mathcal{L}$-controlling sequence. Moreover, for every sequence $\langle \mathbf{y}_t\rangle\subseteq\mathbf{R}^n$ and for every $K>0$, there exists a function $l\in\mathcal{L}$ satisfying*

$$\inf_t \|l(x_t)-\mathbf{y}_t\| > K.$$

For $X=\mathbf{R}^m$, $k=m+1$, $f_j(\mathbf{x})=$the j-th coordinate of $\mathbf{x}$ $(j\leqq m)$ and $f_{m+1}\equiv1$, the above assertion is clearly equivalent to Theorem 3.1.A. If $X=\mathbf{R}$, $n=1$, $k=m+1$, $f_j(x)=x^{j-1}$ $(j\leqq m+1)$, then we obtain Theorem 3.2.A.

PROOF. Suppose, in contradiction with 4.1.A, that there are a sequence $\langle\mathbf{y}_t\rangle\subseteq\mathbf{R}^n$ and a positive number K, such that for each $l\in\mathcal{L}$ there exists a t satisfying

$$\|l(x_t)-\mathbf{y}_t\| \leqq K.$$

This implies that, for every $n\times k$-matrix $\Lambda=(\lambda_{ij})_{1\leqq i\leqq n}^{1\leqq j\leqq k}$ with real entries, one has

(8) $$\max_{1\leqq i\leqq n}|\lambda_{i1}f_1(x_t)+\ldots+\lambda_{ik}f_k(x_t)-y_t^{(i)}| \leqq K,$$

for some t. (Here $y_t^{(i)}$ denotes the i-th coordinate of $\mathbf{y}_t\in\mathbf{R}^n$.)

Let us regard the set of all $n\times k$-matrices as an nk-dimensional euclidean space $\mathbf{R}^{nk}$, and fix a natural number t. Consider now all those matrices $\Lambda=(\lambda_{ij})$ which satisfy (8). They clearly form a cylinder C_t in $\mathbf{R}^{nk}$, whose base is an n-dimensional cube of side length $2K(\sum_{j=1}^{k}f_j^2(x_t))^{-1/2}$. The union of these cylinders must cover, by our assumption, the whole space $\mathbf{R}^{nk}$. Let $B(r)\subseteq\mathbf{R}^{nk}$ denote a ball of radius r, and let ω_s be the volume of the s-dimensional unit ball. Taking into account that

$$\operatorname{vol}(C_t\cap B(r)) < 2^nK^n(\sum_{j=1}^{k}f_j^2(x_t))^{-n/2}\omega_{nk-n}r^{nk-n},$$

$$\operatorname{vol} B(r) = \omega_{nk}r^{nk},$$

and $\bigcup C_t\supseteq B(r)$, we get

$$\sum_{t=1}^{\infty}2^nK^n(\sum_{j=1}^{k}f_j^2(x_t))^{-n/2} > \frac{\omega_{nk}}{\omega_{nk-n}}r^n$$

for an arbitrarily large r. This contradicts the convergence criterion in the theorem. ∎

Let us note that, instead of the last step of the proof, we could have used a general form of 2.2 (see e.g. Bang [3]), but this is much stronger than that we actually need.

We have the feeling that the condition $\sum_{i=1}^{\infty}\left(\sum_{j=1}^{k} f_j^2(x_i)\right)^{-n/2}=+\infty$ is not only necessary but also sufficient for a sequence $\langle x_i\rangle$ to be $\mathscr{L}$-controlling. However, we can prove this only for $n=1$ and $k\leqq 2$, and in some other special cases.

PROOF of Theorem 3.1.B. It is clearly enough to prove the assertion for $m=1$. We may suppose $\xi_i>0$ for every i.

Let $\langle \mathbf{q}_k\rangle$ be the sequence of all rational points of $\mathbf{R}^n$, in an arbitrary order. (A point is called *rational* if all of its coordinates are rational.) Define, for each natural number k, a subclass $\mathscr{L}_k\subseteqq\mathscr{L}(\mathbf{R},\mathbf{R}^n)$, as follows:

$$\mathscr{L}_k := \{l\in\mathscr{L}(\mathbf{R},\mathbf{R}^n)\,|\,l(0)=\mathbf{q}_k \text{ and } \|l(1)-l(0)\|\leqq k\}.$$

Let us split now $\langle\xi_i\rangle$ into infinitely many subsequences $\langle\xi_i^{(k)}\rangle$ satisfying

$$\sum_{i=1}^{\infty}(1+\xi_i^{(k)})^{-n}=+\infty \qquad (k=1,2,\dots). \tag{9}$$

We are going to prove that, for each k, there is a sequence $\langle \mathbf{y}_i^{(k)}\,|\,i=1,2,\dots\rangle\subseteqq\mathbf{R}^n$ with the property that if l is any function belonging to $\mathscr{L}_k$, then one can find an index i, for which

$$\|l(\xi_i^{(k)})-\mathbf{y}_i^{(k)}\|\leqq\frac{1}{k} \tag{10}$$

holds, and from this the theorem easily follows.

Let k be fixed. We set up a one-to-one correspondence φ between the functions of $\mathscr{L}_k$ and the ball $B(k)=\{\mathbf{y}\in\mathbf{R}^n\,|\,\|\mathbf{y}-\mathbf{q}_k\|\leqq k\}$, in the following natural way:

$$\varphi(l):=l(1).$$

If i is any natural number and $\mathbf{y}_i^{(k)}\in\mathbf{R}^n$ is arbitrarily fixed, then we consider the φ-image of those functions $l\in\mathscr{L}_k$ which satisfy (10). Obviously, this is the intersection of $B(k)$ and a ball of radius $(\xi_i^{(k)}k)^{-1}$ and centre

$$\mathbf{q}_k+\frac{1}{\xi_i^{(k)}}(\mathbf{y}_i^{(k)}-\mathbf{q}_k).$$

We have to prove only that the points $\mathbf{y}_i^{(k)}$ $(i=1,2,\dots)$ can be chosen so that the above balls form a covering of $B(k)$. But this can be done, since, by (9), the sum of volumes of the balls is

$$\sum_{i=1}^{\infty}\omega_n(\xi_i^{(k)}k)^{-n}=+\infty,$$

which completes the proof. ∎

We remark that in the concluding step of the proof we used the following simple result which is a special case of the Auerbach—Banach—Mazur—Ulam problem (see [7, 18, 19, 20, 21]): If $\mathscr{B}$ is any system of balls in $\mathbf{R}^n$ with total volume $+\infty$, then $\mathscr{B}$ permits a covering of $\mathbf{R}^n$.

We close this section with the proof of Theorem 3.4. We shall prove it not only for the class $\mathcal{P}$ of polynomials but for any *countably generated* function class $\mathcal{L}^*$ satisfying some weak condition on the generating functions.

Let $f_1, f_2, \dots$ be a fixed sequence of functions $X \to \mathbf{R}$, where X is an arbitrary set. Let $\mathcal{L}^*$ denote the class of all functions $l = (l^1, l^2, \dots, l^n)\colon X \to \mathbf{R}^n$ whose coordinates l^i are finite linear combinations of some f_j's. That is,

$$(7^*)\qquad \mathcal{L}^* := \Big\{\Big(\sum_{j=1}^{\infty} \lambda_{1j} f_j, \dots, \sum_{j=1}^{\infty} \lambda_{nj} f_j\Big) \Big| \text{ there are finitely many } \lambda_{ij} \neq 0\Big\}.$$

The following is an immediate consequence of 4.1.A.

COROLLARY 4.2. *Let X be an abstract set, and let the class $\mathcal{L}^*$ of functions from X into $\mathbf{R}^n$ be defined as in* (7*). *If $\langle x_t \rangle \subseteq X$ is an $\mathcal{L}^*$-controlling sequence, then*

$$\sum_{t=1}^{\infty} \Big(\sum_{j=1}^{N} f_j^2(x_t)\Big)^{-n/2} = +\infty$$

is valid for every natural number N.

Conversely, we prove

THEOREM 4.3. *Let $f_1, f_2, \dots$ be a sequence of functions $X \to \mathbf{R}$ having the property that, for every pair of natural numbers k and l, there is a constant $C = C(k, l)$ satisfying*

$$|f_k(x)|^l \leqq C \sum_{j=1}^{C} |f_j(x)| \qquad (\forall x \in X).$$

Further, let the class $\mathcal{L}^*$ of functions $(X \to \mathbf{R}^n)$ be defined as in (7*). *If $\langle x_t \rangle \subseteq X$ is such a sequence that*

$$(11)\qquad \sum_{t=1}^{\infty} \Big(\sum_{j=1}^{N} f_j^2(x_t)\Big)^{-n/2} = +\infty$$

holds for all natural numbers N, then $\langle x_t \rangle$ is a strongly $\mathcal{L}^*$-controlling sequence.

PROOF. Let $\mathcal{L}^{(k)}$ denote the family of those functions of $\mathcal{L}^*$ whose coefficients λ_{ij} $(1 \leqq i \leqq n,\ 1 \leqq j < +\infty)$ satisfy

$$\lambda_{ij} = 0 \quad \text{for all } j > k \qquad (1 \leqq i \leqq n).$$

We note that $\mathcal{L}^{(k)}$ coincides with the class $\mathcal{L}$ defined in (7), $\mathcal{L}^{(1)} \subseteq \mathcal{L}^{(2)} \subseteq \dots$, and $\bigcup_{k=1}^{\infty} \mathcal{L}^{(k)} = \mathcal{L}^*$.

It is easy to see that the sequence $\langle x_t \rangle$ in the theorem can be divided into countably many subsequences $\langle x_t^{(k)} | t = 1, 2, \dots \rangle$, $(1 \leqq k < +\infty)$ such that for each of them (11) remains true. That is,

$$(11')\qquad \sum_{t=1}^{\infty} \Big(\sum_{j=1}^{N} f_j^2(x_t^{(k)})\Big)^{-n/2} = +\infty$$

for all N $(1 \leqq k < +\infty)$.

It suffices to prove that, for every k, one can choose a sequence $\langle \mathbf{y}_t^{(k)} | t = 1, 2, \ldots \rangle \subseteq \mathbf{R}^n$ such that, if l is any function belonging to $\mathscr{L}^{(k)}$, then there is an index t satisfying

$$\|l(x_t^{(k)}) - \mathbf{y}_t^{(k)}\| \leqq \frac{1}{k}.$$

Let k be fixed. Consider now the nk-dimensional space $\mathbf{R}^{nk}$ of all matrices $\Lambda = (\lambda_{ij})_{1 \leqq i \leqq n}^{1 \leqq j \leqq k}$ with real entries. It is obvious that every Λ corresponds to a function $l \in \mathscr{L}^{(k)}$ and, conversely, for each $l \in \mathscr{L}^{(k)}$ there is (at least) one matrix determining l.

We take, for any point $\mathbf{y} \in \mathbf{R}^n$, the set of those matrices $\Lambda = (\lambda_{ij})$ which correspond to some $l \in \mathscr{L}^{(k)}$ satisfying

$$\|l(x_t^{(k)}) - \mathbf{y}\| \leqq \frac{1}{k}.$$

It is clear, that this set contains a cylinder $C_t(\mathbf{y}) \subseteq \mathbf{R}^{nk}$ defined by the inequalities

$$|\lambda_{i1} f_1(x_t^{(k)}) + \ldots + \lambda_{ik} f_k(x_t^{(k)}) - y_i| \leqq \frac{1}{k\sqrt{n}} \qquad (i = 1, 2, \ldots, n),$$

where y_i denotes the i-th coordinate of $\mathbf{y}$. It is easy to compute that the base of $C_t(\mathbf{y})$ is an n-dimensional cube of side length $2k^{-1}n^{-1/2}(\sum_{j=1}^{k} f_j^2(x_t^{(k)}))^{-1/2}$. Observe that any translate of $C_t(\mathbf{y})$ (in $\mathbf{R}^{nk}$) is equal to $C_t(\mathbf{y}')$, for some other $\mathbf{y}' \in \mathbf{R}^n$.

Thus, if we can prove that the cylinders $C_t(\mathbf{0}) = C_t$ $(t = 1, 2, \ldots)$ permit a translative covering of $\mathbf{R}^{nk}$, then we are done. However, each C_t contains an nk-dimensional cube of volume

$$(2k^{-1}n^{-1/2})^{nk}(\sum_{j=1}^{k} f_j^2(x_t^{(k)}))^{-nk/2}.$$

We show that these cubes already permit a translative covering of the space. To see this, it is enough to prove that the sum of their volumes is infinite. But this is true, since, by Hölder's inequality and the condition in the theorem

$$(\sum_{j=1}^{k} f_j^2(x_t^{(k)}))^{k/2} \leqq k^{\frac{k}{2}-1} \sum_{j=1}^{k} |f_j(x_t^{(k)})|^k \leqq k^{\frac{k}{2}-1} C \sum_{j=1}^{C} |f_j(x_t^{(k)})| \leqq$$

$$\leqq Ck^{k/2}(\sum_{j=1}^{C} f_j^2(x_t^{(k)}))^{1/2},$$

where C is a suitable constant and $k \geqq 2$. This, in turn, implies

$$\sum_{t=1}^{\infty} (\sum_{j=1}^{k} f_j^2(x_t^{(k)}))^{-nk/2} \geqq C' \sum_{t=1}^{\infty} (\sum_{j=1}^{C} f_j^2(x_t^{(k)}))^{-n/2},$$

which is divergent, by (11′). ∎

If $X = \mathbf{R}$, $n = 1$ and $f_j(x) = x^{j-1}$ $(j = 1, 2, \ldots)$, then 4.2 and 4.3 provide the "only if" and "if" parts of Theorem 3.4, respectively.

5. Proofs — controlling Lipschitz functions

This section contains the proofs of Theorems 3.6.A and B. Let $\mathbf{Z}$ and $\mathbf{R}^+$ denote the set of all integers and non-negative reals, respectively. First, we prove Theorem 3.6.A in the following slightly stronger form (where $\mathrm{Lip}(X, \mathbf{R}^n)$ is defined in analogy to Definition 3.5).

THEOREM 5.1. *Let X be any metric space. Let a sequence $\langle x_i \rangle \subseteq X$ be given, such that*

$$\left|\{i \mid r \leqq d(x^*, x_i) \leqq r+1\}\right| \leqq Cr^{n-1} \qquad (r > r_0),$$

for some $x^* \in X$, $C>0$. Then $\langle x_i \rangle$ is not a $\mathrm{Lip}(X, \mathbf{R}^n)$*-controlling sequence. Moreover, for any (large) $K>0$ and any sequence $\langle \mathbf{y}_i \rangle$ in $\mathbf{R}^n$, there is a function $f \in \mathrm{Lip}(X, \mathbf{R}^n)$ with*

$$\inf_i \|f(x_i) - \mathbf{y}_i\| \geqq K.$$

3.6.A is an easy consequence of 5.1.

For the proof we need the following discrete version of the Brunn—Minkowski inequality.

LEMMA 5.2. *Let B be any finite subset of $\mathbf{Z}^n$ $(n \geqq 1)$, and let*

$$B_\beta = \{\mathbf{z} \in \mathbf{Z}^n \mid \exists \mathbf{b} \in B, \|\mathbf{b} - \mathbf{z}\| \leqq \beta\},$$

where β is a positive integer and $\|\cdot\|$ denotes the maximum norm (i.e. $\|(x_1, \dots, x_n)\| = \max\{|x_1|, \dots, |x_n|\}$). Then, for the cardinality of B_β

$$|B_\beta| \geqq (|B|^{1/n} + 2\beta)^n$$

holds, with equality if and only if $|B|^{1/n}$ is an integer and B consists of all lattice points in a cube, with edges parallel to the coordinate axes and all vertices having integral coordinates.

PROOF. Let the "bodies" $\bar{B}$ and $\bar{B}_\beta \subseteq \mathbf{R}^n$ be defined by

$$\bar{B} = B + \left[-\frac{1}{2}, \frac{1}{2}\right]^n,$$

$$\bar{B}_\beta = B_\beta + \left[-\frac{1}{2}, \frac{1}{2}\right]^n.$$

Then we obviously have $\bar{B}_\beta = \bar{B} + [-\beta, \beta]^n$, $\mathrm{Vol}\, \bar{B} = |B|$ and $\mathrm{Vol}\, \bar{B}_\beta = |B_\beta|$. Using a variant of the Brunn—Minkowski inequality, due to Busemann [5], we obtain

$$(\mathrm{Vol}\, \bar{B}_\beta)^{1/n} \geqq (\mathrm{Vol}\, \bar{B})^{1/n} + 2\beta,$$

with equality if and only if $\bar{B}$ is homothetic to $[-\beta, \beta]^n$. ∎

The following consequence of Lemma 5.2 is basic to our purposes.

LEMMA 5.3. *For every* $\alpha>0$, *there is a natural number* $\beta(\alpha)$ *such that if* $\langle A_i\rangle$ *is any sequence of subsets of* $\mathbf{Z}^n$, *satisfying*

$$|A_i| \leqq \alpha i^{n-1} \qquad (i \geqq i_0),$$

then one can find points $\mathbf{b}_i \in \mathbf{Z}^n$ *with the property that*

$$\mathbf{b}_i \notin A_i \quad \text{and} \quad \|\mathbf{b}_{i+1}-\mathbf{b}_i\| \leqq \beta(\alpha) \qquad (0 \leqq i < +\infty).$$

(If $\|\cdot\|$ *denotes the maximum norm, then any* $\beta(\alpha) > \frac{1}{2}\alpha^{1/n}\left(1-\frac{1}{n}\right)^{(1/n)-1}$ *meets the requirements.)*

PROOF. Let β be any integer such that

$$0 < \frac{\alpha}{\beta^n} < 2^n\left(1-\frac{1}{n}\right)^{n-1}. \tag{12}$$

(Let $0^0=1$.) We are going to prove that 5.3 is valid with $\beta(\alpha)=\beta$.

For any $\mathbf{b}\in\mathbf{Z}^n$, let $B_k(\mathbf{b})$ $(0\leqq k<+\infty)$ be defined as the set of those points $\mathbf{z}\in\mathbf{Z}^n$ for which one can find a sequence $\langle \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_k\rangle$ such that $\mathbf{b}_0=\mathbf{b}$, $\mathbf{b}_k=\mathbf{z}$ and

$$\mathbf{b}_i \notin A_i \qquad (i = 0, 1, \dots, k),$$
$$\|\mathbf{b}_{i+1}-\mathbf{b}_i\| \leqq \beta \quad (i = 0, 1, \dots, k-1).$$

By (12), we can choose a real number γ $(0<\gamma<1)$ such that

$$0 < \frac{\alpha}{\beta^n} < 2^n n(\gamma^{\frac{n-1}{n}} - \gamma). \tag{12'}$$

We shall prove that there exists a $\mathbf{b}\in\mathbf{Z}^n$ satisfying

$$|B_k(\mathbf{b})| \geqq \gamma(2\beta k)^n \quad \text{for all} \quad k\in\mathbf{N}. \tag{13}$$

The proof is by induction on k. Let k_0 be sufficiently large and let $\mathbf{b}\in\mathbf{Z}^n$ be a point with

$$|B_k(\mathbf{b})| = (2\beta k+1)^n \quad \text{for all} \quad k = 0, 1, \dots, k_0.$$

(Such a point always exists.) Suppose now that (13) has already been proved for $k\leqq l$ $(l\geqq k_0)$. Then, using Lemma 5.2 and (12′), we have

$$\begin{aligned} |B_{l+1}(\mathbf{b})| &\geqq \left|(B_l(\mathbf{b}))_\beta\right| - |A_{l+1}| \geqq \\ &\geqq |B_l(\mathbf{b})| + 2n\beta|B_l(\mathbf{b})|^{(n-1)/n} - \alpha(l+1)^{n-1} \geqq \\ &\geqq \gamma[2\beta(l+1)]^n, \end{aligned}$$

as desired. From here the lemma easily follows, using a simple diagonal process. ∎

As to the sharpness of 5.3, we note that, for a fixed large β, the supremum of those α for which the lemma holds is somewhere around $2^n\beta^n$. (12) shows that this value is at least $\sim 2^n\beta^n/e$.

We are now in a position to prove 5.1.

PROOF of Theorem 5.1. We may suppose, by homogeneity, that $K=1$, and $\mathbf{R}^n$ is supplied with the maximum norm. Further, it suffices to prove the assertion for $X=\mathbf{R}^+$ and $x^*=0$. Indeed, assume that the theorem has been proved for this case, and let X be any metric space. If $\langle x_i\rangle \subseteq X$ satisfies the condition in the theorem, then, for every sequence $\langle \mathbf{y}_i\rangle \subseteq \mathbf{R}^n$, there is a function $g\in \mathrm{Lip}(\mathbf{R}^+, \mathbf{R}^n)$ such that

$$\inf_i \|g(d(x^*, x_i))-\mathbf{y}_i\| \geqq 1.$$

Put $f(x):=g(d(x^*, x))\in \mathrm{Lip}(X, \mathbf{R}^n)$. This clearly has the property required in 5.1.

Let now a sequence $\langle x_i\rangle \subseteq \mathbf{R}^+$ be given, satisfying

(14) $$\left|\{i\,|\,r \leqq x_i \leqq r+1\}\right| \leqq Cr^{n-1} \qquad (r > r_0),$$

and let $\langle \mathbf{y}_i\rangle \subseteq \mathbf{R}^n$. We fix a constant $\varrho>0$, such that

$$\left(\frac{2}{3}\right)^{n-1}\left(1-\frac{1}{n}\right)^{n-1} > C\varrho^{n-1}.$$

Let us consider the set L_j of those points $(k_1, k_2, \dots, k_n)\in \mathbf{Z}^n$, for which there exists an index i such that

$$\varrho j \leqq x_i \leqq \varrho(j+1) \quad \text{and} \quad \|(k_1, \dots, k_n)-\mathbf{y}_i\| \leqq 1.$$

($\|\cdot\|$ denotes the maximum norm.) By (14), we have

$$|L_j| \leqq C3^n(\varrho j)^{n-1} \qquad (j > r_0/\varrho).$$

Hence, applying Lemma 5.3 to the sets L_j, we obtain a sequence of points $\langle \mathbf{b}_j\rangle \subseteq \mathbf{Z}^n$, with

$$\mathbf{b}_j \notin L_j \quad \text{and} \quad \|\mathbf{b}_{j+1}-\mathbf{b}_j\| \leqq 1.$$

Define a function $f\in \mathrm{Lip}(\mathbf{R}^+, \mathbf{R}^n)$, as follows.

$$f(j) = \mathbf{b}_j \qquad (j = 0, 1, 2, \dots),$$

and f is linear on each interval $[j, j+1]$. It is now clear that

$$\inf_i \|f(x_i)-\mathbf{y}_i\| \geqq 1,$$

which completes the proof. ∎

It seems to be very likely that the following strengthening of 5.1 (for $X=\mathbf{R}^m$) is also true.

CONJECTURE 5.4. Let $\langle \mathbf{x}_i\rangle \subset \mathbf{R}^m$ be any sequence of points such that, for every $\mathbf{x}\in \mathbf{R}^m$, the number of $\mathbf{x}_i$-s contained in the unit ball around $\mathbf{x}$ is at most $K(\|\mathbf{x}\|^{n-1}+1)$. Then $\langle \mathbf{x}_i\rangle$ is not a $\mathrm{Lip}(\mathbf{R}^m, \mathbf{R}^n)$-controlling sequence.

Our conjecture, if true, is sharp in the sense that $K(\|\mathbf{x}\|^{n-1}+1)$ cannot be replaced by a function of larger order of magnitude.

We now turn to the proof of Theorem 3.6.B, a converse of 5.1.

PROOF of Theorem 3.6.B. As a matter of convenience, we suppose $\|\cdot\|$ is the usual maximum norm. Since every $\mathrm{Lip}(\mathbf{R}, \mathbf{R}^n)$-controlling sequence is $\mathrm{Lip}(\mathbf{R}^m, \mathbf{R}^n)$-controlling, too, it suffices to prove our assertion for $m=1$.

For each natural number k, we define the class $\mathrm{Lip}\,(k)\subseteq \mathrm{Lip}\,(\mathbf{R},\mathbf{R}^n)$ as the set of all functions $f\in \mathrm{Lip}\,(\mathbf{R},\mathbf{R}^n)$ satisfying

$$\|f(0)\| \leqq k$$

and

$$\|f(x)-f(y)\| \leqq k\,|x-y|,$$

for all $x, y\in \mathbf{R}$.

We clearly have $\mathrm{Lip}\,(k)\subseteq \mathrm{Lip}\,(k+1)$ and $\bigcup_{k=1}^{\infty} \mathrm{Lip}\,(k)=\mathrm{Lip}\,(\mathbf{R},\mathbf{R}^n)$. Further, observe that the graph of any function belonging to $\mathrm{Lip}\,(k)$ is entirely contained in the double cone

$$C(k):=\{\langle x,\mathbf{y}\rangle\in \mathbf{R}\times\mathbf{R}^n\,|\,\|\mathbf{y}\|\leqq k(1+|x|)\}.$$

Let $x_i=\xi_i$, for all i. We are going to define a finite sequence of points $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{i_1}\in\mathbf{R}^n$ with the property that for any $f\in\mathrm{Lip}\,(1)$ there exists some $i\leqq i_1$ such that

(15) $$\|f(x_i)-\mathbf{y}_i\|\leqq 1.$$

We note first that the condition in the theorem easily implies

$$\lim_{i\to+\infty}(x_{i+[ci^{(n-1)/n}]}-x_i)=0,$$

for any fixed $c>0$. Put $c=5n$, and suppose that j_0 is a sufficiently large natural number for which

(16) $$x_{i+[5ni^{(n-1)/n}]}-x_i\leqq\frac{1}{2}\quad\text{if}\quad i\geqq j_0.$$

Let $\mathbf{y}_1=\mathbf{y}_2=\ldots=\mathbf{y}_{j_0}=\mathbf{0}$, and consider the following sets of lattice points

$$Z_t:=\{\mathbf{z}\in\mathbf{Z}^n\,|\,t-1<\|\mathbf{z}\|\leqq t\},\qquad (t=0,1,2,\dots).$$

Put $j_t=j_0+(2t+1)^n-1$, and define the points $\mathbf{y}_i$ $(i>j_0)$ in the following way:

$$\{\mathbf{y}_{j_t+1},\mathbf{y}_{j_t+2},\dots,\mathbf{y}_{j_{t+1}}\}=Z_{t+1},\qquad (t=0,1,2,\dots).$$

Consider now the boxes

$$B_i:=\left\{\langle x,\mathbf{y}\rangle\in\mathbf{R}\times\mathbf{R}^n\,|\,|x-x_i|\leqq\frac{1}{2}\ \text{and}\ \|\mathbf{y}-\mathbf{y}_i\|\leqq\frac{1}{2}\right\},$$

for all $i=1,2,\dots$. It is clear, by our assumption, that one can find a natural number T such that all the boxes B_i, $i>j_T$, are outside the double cone $C(1)$. On the other hand, using (16), it is easy to check that $x_{j_{t+2}}-x_{j_t}\leqq 1/2$ for all t. This yields that

$$\bigcup_{i=j_0}^{\infty}B_i\supseteq\bigcup_{t=0}^{\infty}\left([x_{j_t},x_{j_{t+1}}]\times\{\mathbf{y}\in\mathbf{R}^n\,|\,\|\mathbf{y}-\mathbf{z}\|\leqq\frac{1}{2}\ \text{for some}\ \mathbf{z}\in Z_t\}\right).$$

Putting these two things together, we obtain that $\bigcup_{i=j_0}^{j_T}B_i$ cuts the double cone $C(1)$ into two parts, and the graph of any function $f\in\mathrm{Lip}\,(1)$ will intersect at least one of the boxes B_i $(j_0\leqq i\leqq j_T)$. This means that (15) holds with $i_1=j_T$.

Similarly, one can define indices $i_2, i_3, \dots$, and points $\mathbf{y}_i\in\mathbf{R}^n$ $(i>i_1)$ such that, for each $k \geqq 2$, the system

$$\{\langle x_i, \mathbf{y}_i\rangle \mid i_{k-1} < i \leqq i_k\}$$

controls the Lip (k)-functions in the sense that, for every $f \in \mathrm{Lip}\,(k)$, there is an i $(i_{k-1} < i \leqq i_k)$ satisfying

$$\|f(x_i) - \mathbf{y}_i\| \leqq \frac{1}{k}.$$

This completes the proof. ▮

The substance of Theorems 3.6.A and B is that the "maximal order of magnitude" of $|\{i \mid \|\mathbf{x}_i\| \leqq r\}|$, where $\langle \mathbf{x}_i \rangle$ is not a Lip $(\mathbf{R}^m, \mathbf{R}^n)$-controlling sequence, is somewhere around cr^n. Note that the proof of both theorems is evident, for $n=1$. However, we are unable to characterize exactly the Lip $(\mathbf{R}^m, \mathbf{R}^n)$-controlling sequences.

Instead of Lipschitz functions, one can investigate functions satisfying $\|f(\mathbf{x}) - f(\mathbf{y})\| \leqq \mathrm{const} \cdot \varphi(\|\mathbf{x}-\mathbf{y}\|)$, where φ is any fixed non-negative real function. The case of all uniformly continuous functions seems to be of interest, too. Further, one can replace $\mathbf{R}^n$ by other metric spaces.

6. Packings and coverings of euclidean space by convex sets

In this section we are going to prove Theorem 3.8 and some related results on packings and coverings.

Let $K \subseteqq \mathbf{R}^n$ be an m-dimensional convex cone $(m \leqq n)$, and let $C \subseteqq \mathbf{R}^n$ be any Lebesgue measurable set. Define the *upper density* of C with respect to K, as

$$\limsup_{r \to \infty} V_m(C \cap K \cap B(r))/V_m(K \cap B(r)),$$

where $B(r)$ denotes the ball of radius r (with centre at the apex of K), and V_m is the m-dimensional Lebesgue measure (volume).

The following two theorems are generalizations of the results in [7], [14] and [15]. Their proof is based on a fairly natural extension of the ideas used there.

THEOREM 6.1. *Let $\langle C_i \rangle$ be a (possibly finite) sequence of closed convex sets in $\mathbf{R}^n$, and let $K \subseteqq \mathbf{R}^n$ be an m-dimensional convex cone. Then the following three assertions are equivalent:*

(A) *$\langle C_i \rangle$ permits a covering of K;*

(B) *$\langle C_i \rangle$ permits an arrangement with union of upper density* 1 *with respect to K;*

(C) *$\langle C_i \rangle$ satisfies at least one of the following conditions*

$$(1) \qquad \sum_{d_1(C_i) \leqq 1} d_1(C_i)\, d_2(C_i) \dots d_m(C_i) = +\infty$$

$$(2) \qquad \sum_{d_2(C_i) \leqq 1 < d_1(C_i)} d_2(C_i) \dots d_m(C_i) = +\infty$$

$$\vdots \qquad \qquad \vdots$$

$$(m) \qquad \sum_{d_m(C_i) \leqq 1 < d_{m-1}(C_i)} d_m(C_i) = +\infty$$

$$(m+1) \qquad \sum_{1 < d_m(C_i)} 1 = +\infty$$

$(m+2)$ *The characteristic cones of C_i (with apices at the apex of K) permit a covering of K.*

(The notion of the characteristic cone of a convex set, and the numbers $d_j(C)$ were introduced at the end of Section 3.) We remark that 3.8 is a special case of 6.1.

THEOREM 6.2. *Let $\langle C_i\rangle$ be a sequence of closed convex sets in $\mathbf{R}^m$ and let $K\subseteq \mathbf{R}^m$ be an m-dimensional (non-degenerate) convex cone. Then the following three assertions are equivalent:*

(A′) *$\langle C_i\rangle$ permits a translative almost covering of K;*

(B′) *$\langle C_i\rangle$ permits a translative arrangement with union of upper density 1 with respect to K;*

(C′) *$\langle C_i\rangle$ satisfies at least one of the first $m+1$ conditions of 6.1 (C), or*

$(m+2)'$ *the characteristic cones of C_i (with apices at the apex of K) form a covering of K.*

Before we sketch the proof, we have to show that our definition of the numbers $d_j(C)$, $(1\leqq j\leqq n)$ was reasonable. For this, we need only the

PROOF of Proposition 3.7. Let the set $\mathscr{V}$ consist of all those vectors $\mathbf{v}\in\mathbf{R}^n$ for which C contains arbitrarily long segments parallel with $\mathbf{v}$. Evidently, $\mathbf{v}\in\mathscr{V}$ if and only if $C-C$ contains the line $\{\alpha\mathbf{v}\mid\alpha\in\mathbf{R}\}$. $C-C$ is a convex set, hence $\mathscr{V}$ must form a subspace of $\mathbf{R}^n$. Denote its dimension by k. To prove that $N^k(C)$ is bounded, it is enough to show that the projection of $C-C$ with kernel $\mathscr{V}$ is bounded. Suppose this is not true. Then the projection of $C-C$ should contain a line l through the origin. Using the convexity of $C-C$, we obtain

$$l\subseteq \mathscr{V}+l\subseteq C-C,$$

which contradicts the definition of $\mathscr{V}$. ∎

PROOF of Theorems 6.1 and 6.2. (A)→(B), (A′)→(B′) are obviously valid. (C)→(A) and (C′)→(A′) can easily be deduced from the results of [7] and [14], resp., where the same implications were proved for the case when all sets C_i are bounded. We have to note only that every convex set C has a convex compact subset C' satisfying

$$d_i(C')>d_i(C)/2\qquad (d_i(C)\leqq 1)\quad\text{and}$$
$$d_i(C)>1\Rightarrow d_i(C')>1,$$

for all $i=1,2,\dots,n$.

We are going to prove (B)→(C). The proof is indirect. Suppose that none of the $m+2$ conditions is satisfied. This yields that there are real numbers $a_0,\dots,a_m$ such that

$$\sum_{d_1(C_i)\leqq 1} d_1(C_i)\,d_2(C_i)\dots d_m(C_i)=a_0$$
$$\vdots$$
$$\sum_{1<d_m(C_i)} 1=a_m.$$

Assume further that the sets C_i are already so arranged that the upper density of $\bigcup C_i$, with respect to K, is equal to 1. Then it is easy to see that

$$V_m\Big(\bigcup_{d_m(C_i)\leqq 1} C_i\cap K\cap B(r)\Big)\leqq a_0+\sum_{j=1}^{m-1} a_j\omega_j r^j,$$

where ω_j denotes the volume of the j-dimensional unit ball. From here

$$\limsup_{r\to\infty} V_m\Big(\bigcup_{1<d_m(C_i)} C_i\cap K\cap B(r)\Big)/V_m(K\cap B(r)) = 1,$$

or, equivalently,

$$\liminf_{r\to\infty} V_m\Big[(K\cap B(r))\setminus \bigcup_{1<d_m(C_i)} C_i\Big]/V_m[K\cap B(r)] = 0. \tag{17}$$

But there are only a_m (i.e. finitely many) C_i-s with $d_m(C_i)>1$. Let us consider their characteristic cones K_i, and blow them up slightly. That is, take a sufficiently small $\varepsilon>0$, and define the cone K_i^ε, as the set of all points $\mathbf{x}\in\mathbf{R}^n$ for which there exists a $\mathbf{y}\in K_i$ satisfying $\sphericalangle(\mathbf{x0y})\leqq\varepsilon$. (We suppose that the apex of both K_i and K is at the origin $\mathbf{0}$.) Using the fact that the characteristic cones are closed sets and, by our assumption,

$$\bigcup_{1<d_m(C_i)} K_i \not\supseteq K,$$

we obtain that, if ε is small enough, then

$$K\setminus \bigcup_{1<d_m(C_i)} K_i^\varepsilon$$

contains an m-dimensional convex cone K^*. Observe now that there is a real number r_0 such that

$$(K\cap B(r))\setminus \bigcup_{1<d_m(C_i)} C_i \supseteq K^*\cap B(r)\setminus B(r_0).$$

This contradicts (17). The proof of (B′)→(C′) is essentially the same. ∎

We strongly suspect that if a sequence $\langle C_i\rangle$ permits a translative almost covering of $\mathbf{R}^m$, then it will also permit a covering. More generally, the three equivalent assertions in Theorem 6.2 could be supplemented by a fourth one

(D′) *$\langle C_i\rangle$ permits a translative covering of K.*

Note that a similar question was posed under [17], for the special case $K=\mathbf{R}^m$ and all C_i are slabs (cf. [14], Corollary 2, too).

So far we have dealt with the characterization of those sequences $\langle C_i\rangle$ of convex sets which permit a covering or a translative covering of a convex *cone*. But what happens if we would like to cover a *bounded* region?

THEOREM 6.3. *Let $\langle C_i\rangle$ be a sequence of (possibly unbounded) closed convex sets in $\mathbf{R}^n$, and let $C\subseteq\mathbf{R}^n$ be an m-dimensional bounded convex region ($m\leqq n$). Then there exist two absolute constants k_1 and k_2 (depending only on m) such that*

(i) *if $\langle C_i\rangle$ permits a covering of C, then*

$$\sum_i \prod_{j=1}^m \min(d_j(C_i), d_j(C)) \geqq k_1 \prod_{j=1}^m d_j(C),$$

(ii) *if*

$$\sum_i \prod_{j=1}^m \min(d_j(C_i), d_j(C)) \geqq k_2 \prod_{j=1}^m d_j(C),$$

then $\langle C_i\rangle$ permits a covering of C.

PROOF. Suppose that all C_i are bounded. (The problem can easily be reduced to this case.)

We shall make use of the following simple fact (see, e.g. [7]). Let D be any bounded convex set in $\mathbf{R}^n$. Then one can inscribe in D an ellipsoid (or a rectangular box) with axes (resp. with sides) $l(n)d_1(D)$, $l(n)d_2(D)$, ..., $l(n)d_n(D)$, where $l(n)$ is a suitable constant which does not depend on D. Note that $d_1(D) \geqq d_2(D) \geqq \ldots \geqq d_n(D) \geqq 0$, and if D is degenerate (i.e. $\dim D < n$), then so is the ellipsoid (box) in question.

Let $D' \subseteq D$, $\dim D' = m$. Inscribe an ellipsoid E' in D' (guaranteed by the above assertion). Then $d_j(D)$ is the diameter of a projection of D to an $(n-j+1)$-flat, which is at least the diameter of a projection of E' to an $(n-j+1)$-flat. The minimum of this latter is just the length of the j-th axis (in decreasing order of magnitude) of E', which in turn equals $d_j(E')$ $\left(=l(m)d_j(D')\right)$. Thus we obtain

$$l(m)\,d_j(D') \leqq d_j(D) \qquad (j = 1, 2, \ldots, n).$$

Using this relation for $D' = C_i \cap C$, $D = C_i$ (or $D = C$), we get

$$d_j(C_i \cap C) \leqq \frac{1}{l(m)} \min\left(d_j(C_i), d_j(C)\right) \qquad (j = 1, 2, \ldots, n).$$

From here the m-dimensional volume of $C_i \cap C$

$$V_m(C_i \cap C) \leqq \prod_{j=1}^{m} d_j(C_i \cap C) \leqq \frac{1}{l^m(m)} \prod_{j=1}^{m} \min\left(d_j(C_i), d_j(C)\right).$$

Further, we have

$$V_m(C) \geqq l^m(m) \prod_{j=1}^{m} d_j(C),$$

because C contains a box of sides $l(m)d_1(C)$, $l(m)d_2(C)$, ..., $l(m)d_m(C)$. Assume now $\bigcup_i C_i \supseteq C$. Then $\sum_i V_m(C_i \cap C) \geqq V_m(C)$, and combining this with the above two inequalities, we obtain

$$\frac{1}{l^m(m)} \sum_i \prod_{j=1}^{m} \min\left(d_j(C_i), d_j(C)\right) \geqq l^m(m) \prod_{j=1}^{m} d_j(C),$$

which proves (i).

For the proof of (ii) we have to note only that C is contained in an m-dimensional box B of sides $d_1(C)$, ..., $d_m(C)$. On the other hand, there are boxes $B_i \subseteq C_i$ with sides $l(m) \min\left(d_1(C_i), d_1(C)\right)$, ..., $l(m) \min\left(d_m(C_i), d_m(C)\right)$, where $l(m) < 1$. Suppose that each B_i is already in a position parallel to B. If the condition of (ii) is satisfied then

$$\sum_i V_m(B_i) \geqq \frac{k_2}{l^m(m)} V_m(B).$$

Now, if k_2 is large enough, we can apply a well-known theorem of Moon and Moser [21] (see also Groemer [12]), which states that the boxes B_i can be translated so as to cover B. This completes the proof. ∎

For *packings* we can state analogously the following

THEOREM 6.4. *Let $\langle C_i\rangle$ be a sequence of bounded convex sets in $\mathbf{R}^n$, and let $C\subseteq\mathbf{R}^n$ be a non-degenerate bounded convex region. Then there exist four constants k_1', k_2', k_3' and k_4' depending only on n, such that*

(i) *if $\langle C_i\rangle$ permits a packing in C, then*

$$d_j(C_i)\leqq k_1' d_j(C) \qquad (j=1,2,\dots,n)$$

$$\sum_i \prod_{j=1}^n d_j(C_i)\leqq k_2' \prod_{j=1}^n d_j(C),$$

(ii) *if the above $n+1$ conditions hold with k_3' and k_4' (instead of k_1' and k_2', resp.), then $\langle C_i\rangle$ permits a packing in C.* ∎

In 6.1—4 we have always supposed that the set C is either bounded, or it is a convex cone. We remark that this assumption is not only technical. Let $\langle C_i\rangle$ be a sequence of (possibly unbounded) convex sets in $\mathbf{R}^n$, and let $C\subseteq\mathbf{R}^2$ be an *arbitrary* unbounded convex region. Then, in terms of the numbers d_j and the characteristic cones, no necessary and sufficient condition can be formulated for $\langle C_i\rangle$ to permit a (translative) packing in, or covering of C. To see this, consider the following simple example. Put $n=2$, $C=\{(x,y)\in\mathbf{R}^2\,|\,y\geqq x^2\}$, and let

$$C_1=\{(x,y)\in\mathbf{R}^2\,|\,y\geqq x^2/2\},\quad C_2=C_3=\dots=\emptyset,$$

$$C_1'=\{(x,y)\in\mathbf{R}^2\,|\,y\geqq 2x^2\},\quad C_2=C_3=\dots=\emptyset.$$

Then $\langle C_i\rangle$ permits a covering of C, but $\langle C_i'\rangle$ does not. And similarly, $\langle C_i'\rangle$ permits a packing in C, but $\langle C_i\rangle$ does not.

In what follows, we concern ourselves with packings. Our next theorem gives a characterization of those sequences of plane convex sets which permit a packing in $\mathbf{R}^2$. This solves a problem of Groemer [13]. To formulate our result, we need a few notations.

Let $\mathscr{C}=\langle C_i\rangle$ be a sequence of convex sets in $\mathbf{R}^2$. The width and the characteristic angle of C_i will be denoted by $w(C_i)$ and $\alpha(C_i)$, respectively. (The *characteristic angle* is the angle of the characteristic cone. If C_i is a bounded region, or a strip, we put $\alpha(C_i)=0$.) Let us partition $\mathscr{C}$ into six disjoint subsequences, as follows.

$$\mathscr{C}_b=\langle C_i\in\mathscr{C}\,|\,C_i \text{ is bounded}\rangle$$

$$\mathscr{C}_s=\langle C_i\in\mathscr{C}\,|\,C_i \text{ is a strip}\rangle$$

$$\mathscr{C}_{s1}=\langle C_i\in\mathscr{C}\,|\,\alpha(C_i)=0,\ w(C_i)<\infty,\ \text{but } C_i\notin\mathscr{C}_b\cup\mathscr{C}_s\rangle$$

$$\mathscr{C}_{s2}=\langle C_i\in\mathscr{C}\,|\,\alpha(C_i)=0,\ w(C_i)=\infty\rangle$$

$$\mathscr{C}_{c1}=\langle C_i\in\mathscr{C}\,|\,\alpha(C_i)>0,\ C_i \text{ can be covered by a cone of angle } \alpha(C_i)\rangle$$

$$\mathscr{C}_{c2}=\langle C_i\in\mathscr{C}\,|\,\alpha(C_i)>0 \text{ and } C_i\notin\mathscr{C}_{c1}\rangle.$$

Note that an unbounded set $C_i\in\mathscr{C}$ belongs to $\mathscr{C}_{s1}$ if and only if C_i can be covered by a strip, but is not a strip itself. (This formulation is similar to the definition of $\mathscr{C}_{c1}$.)

However, a set belonging to $\mathscr{C}_{s1}$ can be covered not only by a strip, but also by a suitable half-strip. Note further, that a straight line is considered a (degenerate) strip.

THEOREM 6.5. *A sequence $\mathscr{C}=\langle C_i\rangle$ of convex proper subsets of the plane permits a packing in $\mathbf{R}^2$, if and only if $\sum \alpha(C_i)\leqq 2\pi$ and at least one of the following conditions holds.*

I. $\mathscr{C}_s=\emptyset$ *and either*

(i) $\sum \alpha(C_i) < 2\pi$, *or*

(ii) $|\mathscr{C}_{c1}\cup\mathscr{C}_{c2}| = \infty$, *or*

(iii) $\mathscr{C}_{s2} = \mathscr{C}_{c2} = \emptyset$, $\quad \sup\limits_{\mathscr{C}_b} w(C_i) < \infty$, $\quad \sum\limits_{\mathscr{C}_{s1}} w(C_i) < \infty$.

II. $\sum\limits_{\mathscr{C}_s} w(C_i)<\infty$, *and there is a partition $\mathscr{C}=\mathscr{C}^{(1)}\cup\mathscr{C}^{(2)}$ such that each $\mathscr{C}^{(j)}$ $(j=1,2)$ satisfies $\sum\limits_{\mathscr{C}^{(j)}} \alpha(C_i)\leqq\pi$ and either*

(i) $\sum\limits_{\mathscr{C}^{(j)}} \alpha(C_i) < \pi$, *or*

(ii) $|\mathscr{C}_{c1}^{(j)}\cup\mathscr{C}_{c2}^{(j)}| = \infty$, *or*

(iii) $\mathscr{C}_{s2}^{(j)} = \mathscr{C}_{c2}^{(j)} = \emptyset$, $\quad \sup\limits_{\mathscr{C}_b^{(j)}} w(C_i) < \infty$, $\quad \sum\limits_{\mathscr{C}_{s1}^{(j)}} w(C_i) < \infty$.

III. $\sum \alpha(C_i)\leqq\pi$ *and either*

(i) $\sum \alpha(C_i) < \pi$, *or*

(ii) $|\mathscr{C}_{c1}\cup\mathscr{C}_{c2}| = \infty$, *or*

(iii) $\mathscr{C}_{s2} = \mathscr{C}_{c2} = \emptyset$.

PROOF. It is easy to see that there exists an absolute constant k such that the inradius of any convex region $C\subseteq\mathbf{R}^2$ is at least $w(C)/k$. (See, e.g. [18].) Using this fact, the proof of the »only if« part is rather pedestrian. Thus, we deal only with the sufficiency of the above conditions.

If I (i) holds, then $\mathscr{C}$ permits a packing. To see this, choose an arbitrary sequence $\langle\varepsilon_i\rangle$ of positive numbers satisfying

$$\sum \left(\alpha(C_i)+\varepsilon_i\right) \leqq 2\pi.$$

Observe now that every C_i can be covered by an (infinite) cone of angle $\alpha(C_i)+\varepsilon_i$, and these cones can be arranged so as to form a packing.

The case I (ii) can be settled, as follows. We establish a one-to-one correspondence between $\mathscr{C}=\langle C_i|i=1,2,\ldots\rangle$ and the set $\mathbf{Q}$ of all rational numbers. Let q_i denote the number corresponding to C_i. Suppose further that

$$\alpha(C_i) = 0 \Rightarrow q_i\in\mathbf{N} \quad (i = 1, 2, \ldots).$$

For every natural number j, define an angle φ_j $(0<\varphi_j<2\pi)$, by

$$\varphi_j := \sum_{\{i \mid q_i < q_j\}} \alpha(C_i). \tag{18}$$

Assume now that each C_i is already turned to such a position that its characteristic cone coincides with the angular domain $[\varphi_i, \varphi_i+\alpha(C_i)]$. (The position of any bounded C_i may be arbitrary.) We are going to show that in this case $\langle C_i \rangle$ permits also a *translative* packing.

Let us suppose that $C_1, \dots, C_{i-1}$ have already been placed, and denote their (pairwise disjoint) translates by $C'_1, \dots, C'_{i-1}$. It is easy to see, by (18), that there exists a sufficiently small $\varepsilon>0$, satisfying

$$\Big(\bigcup_{j=1}^{i-1} [\varphi_j-\varepsilon, \varphi_j+\alpha(C_j)+\varepsilon]\Big) \cap [\varphi_i-\varepsilon, \varphi_i+\alpha(C_i)+\varepsilon] = \emptyset.$$

For every j, take a cone K_j^ε which contains C'_j and whose angular domain is $[\varphi_j-\varepsilon, \varphi_j+\alpha(C_j)+\varepsilon]$. The set

$$\mathbf{R}^2 \setminus \bigcup_{j=1}^{i-1} K_j^\varepsilon$$

obviously contains an (infinite) cone K^* with angular domain $[\varphi_i-\varepsilon, \varphi_i+\alpha(C_i)+\varepsilon]$. Now we are done, since C_i can be translated so as to be completely covered by K^*.

Condition I (iii) is clearly sufficient.

The cases II and III can be treated similarly. ∎

It is not surprising that, in terms of characteristic cones and widths, one can give a similar necessary and sufficient condition for a given sequence of convex sets to permit a translative plane packing. The formulation of this result is rather tedious, so the reader will probably forgive us for omitting that. We also note that, by some minor changes, Theorem 6.5 becomes valid for packings in any fixed angular domain.

To obtain stronger results for translative coverings and packings, one requires more information about the relative position of the sets C_i and C. We illustrate this by an example in the plane. Let C be a rectangle with sides 1 and n, where n is a large number. Further, let C_1 be a strip of width 10, parallel with the long side of C. It is now clear that C can be covered by a translate of C_1. However, the existence of this covering cannot be guaranteed by 2.3, since $w(C_1) \ll p(C)$. We can overcome this difficulty in the following way.

Let C be any bounded convex set in the plane, and let $\langle S_i \rangle$ be a sequence of strips. Let $w_i(C)$ denote the minimal width of a strip which is parallel to S_i and contains C. Then the ratio $w(S_i)/w_i(C)$ is called the *relative width* of S_i (with respect to C). Following the terminology of Groemer [16], [17], we shall say that $\langle S_i \rangle$ permits a *non-dissecting translative covering* of C, if there are suitable translations σ_i such that $\bigcup \sigma_i(S_i) \supseteq C$ and

$$C \setminus \bigcup_{i=1}^{k} \sigma_i(S_i)$$

is connected (and consequently convex), for every k.

The following assertion already applies to the example mentioned above.

PROPOSITION 6.6. *Let C be a bounded convex domain, and let $\langle S_i \rangle$ be a sequence of strips. If the sum of the relative widths of S_i (with respect to C) is at least 4, then some rearrangement of $\langle S_i \rangle$ permits a non-dissecting translative covering of C.*

PROOF. It is not difficult to show (see e.g. Behrend [4], p. 716) that there exists an affine image C' of C, such that the perimeter of C' is at most $4w(C')$. Observe that it suffices to prove 6.6 for C' (instead of C), since the statement is affine invariant. We have

$$\sum_i w(S_i) \geqq w(C') \sum_i \frac{w(S_i)}{w_i(C')} \geqq w(C') \cdot 4 \geqq p(C'),$$

thus 2.3 can be applied. The covering constructed there is obviously non-dissecting. ∎

Note that 6.6 is a converse of the following theorem of Ohmann (announced in [22], proved correctly in [23]). If a sequence of strips $\langle S_i \rangle$ forms a covering of a convex domain C, then the sum of the relative widths of S_i is at least 1.

7. Remarks

7.1. Our original question on controlling of function classes was partly motivated by L. Fejes Tóth [11]. He, P. Erdős and E. G. Straus [10] raised the following related problem. Let $\mathcal{S}$ be an arbitrary system of surfaces in $\mathbf{R}^n$. How sparsely can a sequence $\langle \mathbf{x}_i \rangle \subseteqq \mathbf{R}^n$ be distributed in $\mathbf{R}^n$ which is controlling in the sense that for every $S \in \mathcal{S}$ there is an $\mathbf{x}_i$ whose distance from S is at most 1. This has been solved for the class $\mathcal{S}_m$ of all m-flats in $\mathbf{R}^n$, by P. Erdős—J. Pach [8]. There are no other non-trivial results in this direction.

7.2. We have learned from P. Erdős that he and E. G. Straus some years ago also found a proof of Lemma 2.3 (unpublished).

7.3. Let $\mathcal{P}_n$ denote the class of all polynomials $(\mathbf{R} \to \mathbf{R})$ having degree at most n. Our Theorem 3.2.A states that a necessary condition for a sequence $\langle x_i \rangle$ to be $\mathcal{P}_n$-controlling is $\sum (1+|x_i|)^{-n} = +\infty$. On the other hand, it easily follows from the results of H. Groemer [17] that the stronger condition $\sum (1+|x_i|)^{-n(n+1)/2} = +\infty$ is already sufficient. This rather large gap between the two conditions can be reduced in the following way. Let $\langle x_i \rangle \subseteqq \mathbf{R}^+$ and put $N(r) = |\{i \mid x_i \leqq r\}|$.

If $\lim\limits_{r\to\infty} \dfrac{N(r)}{r^n} = +\infty$, *then $\langle x_i \rangle$ is a $\mathcal{P}_n$-controlling sequence.* (We leave the simple proof to the reader.) This assertion shows that the necessary condition above cannot be far from the truth, which supports Conjecture 3.3.

7.4. It is easy to see that, in the case of the plane, the implications (i)$\leftrightarrow$(ii)$\to$$\to$(iii)$\leftrightarrow$(iv) hold, where

(i) There exists a constant C such that any sequence of strips with total width at least C permits a translative covering of the unit circle.

(ii) There exists a constant C such that any sequence $\langle C_i \rangle$ of convex sets of diameter at most 1 and total area at least C permits a translative covering of the unit circle.

(iii) Any sequence of strips whose sum of widths is infinite permits a translative covering of $\mathbf{R}^2$.

(iv) Any sequence $\langle C_i \rangle$ of convex sets of diameter at most 1 and infinite total area permits a translative covering of $\mathbf{R}^2$.

(ii)→(i), (iv)→(iii), (i)→(iii) are immediate. (i)→(ii), resp. (iii)→(iv) follows by classifying the convex sets C_i according to the values $[\log_2 d_1(C_i)]$ and $[\log_2 d_2(C_i)]$, and then applying the simple observation that, for the covering of a circle of diameter $\leq d_1(C_i)/2$, C_i can be replaced by a strip of width $d_2(C_i)/4$.

Since (i) is true (see 2.3), we obtain, in particular, that (iv) is also valid. This implies that if $n=2$, then the conditions in Theorem 6.2 which are necessary for $\langle C_i \rangle$ to permit a translative almost covering of the plane, are already sufficient to permit a translative covering, too. This answers a question of Groemer [13].

An analogue of Theorem 6.3 for translative covering of a bounded convex set C in $\mathbf{R}^n$ $(\dim C=m)$ with a sequence of convex sets $\langle C_i \rangle$ would be the following:

$$\sum_i \sup_{\mathbf{x}_i} V_m[C \cap (C_i+\mathbf{x}_i)]$$

sufficiently large implies $\langle C_i \rangle$ permits a translative covering of C. This follows from (ii) for $m \leq 2$ (and for each m for which (ii) holds). In fact, this problem is affine invariant, and C has an affine image containing the unit m-sphere and contained in an m-sphere of radius const.

Similar equivalences can be stated in case of higher dimensions, showing that e.g. Conjecture 3.3 would have some interesting consequences. However, we do not know whether the analogue of (iii)→(i) holds.

7.5. Concerning 5.2, it might be interesting to find the exact minimum of $|B_\beta|$, for arbitrary $|B|$. An analogous result for the case when $\mathbf{Z}^n$ is supplied with the l_1-norm and $\beta=1$, was proved by D.-L. Wang—P. Wang [25].

7.6. When finishing our paper H. Groemer was kind to send us two of his preprints, "Covering and packing properties of bounded sequences of convex sets", *Mathematika* **29** (1982), 18—31 (*MR* **84b**: 52021), "On space coverings by unbounded convex sets", *J. Combinatorial Theory Ser. A* **34** (1983), 71—79 (*MR* **84b**: 52022), which contain results partly overlapping with some results of our Sections 2, 6 and 7. The most complete recent survey of packing and covering problems of such type is H. Groemer, Covering and packing by sequences of convex sets, preprint (1983).

REFERENCES

[1] BANG, TH., On covering by parallel-strips, *Mat. Tidsskr. B* **13** (1950), 49—53. *MR* **12**—352.

[2] BANG, TH., A solution of the "plank problem", *Proc. Amer. Math. Soc.* **2** (1951), 990—993. *MR* **13**—769.

[3] BANG, TH., Some remarks on the union of convex bodies, *Tolfte Skandinaviska Matematikerkongressen*, Lund, 1953, Lunds Universitets Matematiska Institution, Lund, 1954, 5—11. *MR* **16**—395.

[4] BEHREND, F., Über einige Affininvarianten konvexer Bereiche, *Math. Ann.* **113** (1937), 713—747. *Zbl* **15**, 367.

[5] BUSEMANN, H., The isoperimetric problem for Minkowski area, *Amer. J. Math.* **71** (1949), 743—762. *MR* **11**—200.

[6] CHAKERIAN, G. D. and GROEMER, H., On classes of convex sets that permit plane coverings, *Israel J. Math.* **19** (1974), 305—312. *MR* **51** #4060.

[7] CHAKERIAN, G. D. and GROEMER, H., On coverings of Euclidean space by convex sets, *Pacific J. Math.* **75** (1978), 77—86. *MR* **58** #18162.
[8] DROBOT, S., Sur un problème de la théorie des corps convexes, *Colloq. Math.* **1** (1948), 341.
[9] ERDŐS, P. and PACH, J., On a problem of L. Fejes Tóth, *Discrete Math.* **30** (1980), 103—109. *MR* **81e**: 51008.
[10] ERDŐS, P. and STRAUS, E. G., unpublished.
[11] FEJES TÓTH, L., Remarks on the dual to Tarski's plank problem, *Mat. Lapok* **25** (1974), 13—20 (in Hungarian). *MR* **56** #1204.
[12] GROEMER, H., On a covering property of convex sets, *Proc. Amer. Math. Soc.* **59** (1976), 346—352. *MR* **54** #1091.
[13] GROEMER, H., Some covering and packing problems, *Amer. Math. Monthly* **83** (1976), 726—727. *Zbl* **339**. 52003.
[14] GROEMER, H., Space coverings by translates of convex sets, *Pacific J. Math.* **82** (1979), 379—386. *MR* **80j**: 52012.
[15] GROEMER, H., On finite classes of convex sets that permit space coverings, *Amer. Math. Monthly* **87** (1980), 286—288. *MR* **81b**: 52012.
[16] GROEMER, H., On coverings of plane convex sets by translates of strips, *Aequationes Math.* **22** (1981), 215—222. *MR* **83j**: 52012.
[17] GROEMER, H., On coverings of convex sets by translates of slabs, *Proc. Amer. Math. Soc.* **82** (1981), 261—266. *MR* **82i**: 52027.
[18] JAGLOM, I. M. und BOLTJANSKI, W. G., *Konvexe Figuren*, Hochschulbücher für Mathematik, Bd. 24, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. *MR* **18**—146.
[19] KOSIŃSKI, A., A proof of an Auerbach—Banach—Mazur—Ulam theorem on convex bodies, *Colloq. Math.* **4** (1957), 216—218. *MR* **19**—165.
[20] MEIR, A. and MOSER, L., On packing of squares and cubes, *J. Combinatorial Theory* **5** (1968), 126—134. *MR* **37** #4716.
[21] MOON, J. W. and MOSER, L., Some packing and covering theorems, *Colloq. Math.* **17** (1967), 103—110. *MR* **35** #6040.
[22] OHMANN, D., Eine Abschätzung für die Dicke bei Überdeckung durch konvexe Körper, *J. Reine Angew. Math.* **190** (1952), 125—128. *MR* **14**—788.
[23] OHMANN, D., Kurzer Beweis einer Abschätzung für die Breite bei Überdeckung durch konvexe Körper, *Arch. Math.* **8** (1957), 150—152. *MR* **20**—1948.
[24] TARSKI, A., Further remarks about the degree of equivalence of polygons, *Parametr* **2** (1932), 310—314 (in Polish).
[25] WANG, D. L. and WANG, P., Discrete isoperimetric problems, *SIAM J. Appl. Math.* **32** (1977), 860—870. *MR* **55** #7844.

(Received March 29, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE
GOWER STREET
LONDON
WC1E 6BT
ENGLAND

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 461—464.

PACKING THE PLANE WITH MINKOWSKIAN SUMS OF CONVEX SETS

G. FEJES TÓTH and L. FEJES TÓTH

Let c be a convex domain. We consider a convex domain C swept over by translates of c, i.e. the Minkowskian sum of c and some convex set. We shall consider a packing of such domains and give an upper bound for its density which involves as special cases several known bounds. We shall denote a domain and its area with the same symbol.

THEOREM. *Let c_0 be a convex domain of unit area. Let $e(k)$ be the excess over 1 of the area of the k-gon of minimal area circumscribed about c_0. Let $c_1, \ldots, c_n$ be affine images of c_0 such that for any two domains c_i and c_j with $1 \leqq i, j \leqq n$ we have*

$$\frac{c_i}{c_j} \leqq \frac{e(5)-e(6)}{e(6)-e(7)}.$$

Let C_i be a convex domain swept over by translates of c_i. Write $C=(C_1+\ldots+C_n)/n$, $c=(c_1+\ldots+c_n)/n$ and $e=e(6)$. If the domains $C_1, \ldots, C_n$ are packed in a convex polygon h with at most six sides then the density $d=nC/h$ of the packing satisfies the inequality

$$d \leqq \frac{C}{C+ec}. \tag{1}$$

In the special case when the domains c_i have the same area and $C_i=c_i$ this inequality has been proved in [5]. Another special case when $c_1, \ldots, c_n$ are equal circles and the domains $C_1, \ldots, C_n$ are congruent is equivalent with an inequality of Groemer [8]. Our theorem implies also that similar centro-symmetric convex domains of "not very different area" cannot be packed denser than congruent ones. This was first observed by K. Böröczky. It can be deduced from a more general theorem proved in [4]. A proof for circles is given in [1].

PROOF. Using a known method (see [2, 5, 6]) we "inflate" the domains C_i so as to obtain non-overlapping convex polygons P_i such that $C_i \subset P_i \subset h$, and, denoting the number of sides of P_i with k_i,

$$k_1+\ldots+k_n \leqq 6n.$$

Suppose that c_i is in a position such that $c_i \subset C_i$. Translate the sides of P_i which do not touch C_i inwards until they touch C_i obtaining a polygon P_i'. Again,

1980 *Mathematics Subject Classification.* Primary 52A45.
Key words and phrases. Packing of convex sets.

translate the sides of P_i' which do not touch c_i inwards until they touch c_i obtaining a polygon p_i. Let B_i be the biggest domain contained in P_i' swept over by translates of c_i. Since $B_i \supset C_i$ and, by definition, $p_i - c_i \geqq c_i e(k_i)$, we have

$$P_i \geqq P_i' = B_i + p_i - c_i \geqq C_i + c_i e(k_i).$$

Thus

$$h \geqq \sum_{i=1}^{n} P_i \geqq nC + \sum_{i=1}^{n} c_i e(k_i).$$

Now we recall Dowker's theorem [3] according to which the sequence $e(3)$, $e(4)$, ... is convex: $e(3)-e(4) \geqq e(4)-e(5) \geqq \ldots$. Suppose that for some i and j we have $k_i < 6 < k_j$. Then, because of the above sequence of inequalities and the condition made on the c_i's, we have

$$c_i e(k_i) + c_j e(k_j) \geqq c_i e(k_i+1) + c_j e(k_j-1).$$

Thus the sum $\sum_{i=1}^{n} c_i e(k_i)$ decreases or remains constant by replacing k_i by k_i+1 and k_j by k_j-1. Since the average value of the k_i's is at most 6, we can continue these replacements until no k_i will exceed 6. Since furthermore $e(3) \geqq e(4) \geqq \ldots$, we have

$$h \geqq nC + e(6) \sum_{i=1}^{n} c_i = n(C+ec),$$

so that, in accordance with the theorem,

$$d = \frac{nC}{h} \leqq \frac{C}{C+ec}.$$

Let $c_1, c_2, \ldots$ and $C_1, C_2, \ldots$ be infinite sets of uniformly bounded convex domains defined similarly as in the theorem. Distribute the domains C_i in the plane so as to form a packing. We consider a summation $\sum_R$ which extends to those domains of the packing which are contained in a circle $K(R)$ of radius R centred at a fixed point. Let $N(R) = \sum_R 1$ be the number of domains contained in $K(R)$. Suppose that the average area C of the domains C_i, defined by

$$C = \lim_{R\to\infty} \sum_R C_i / N(R),$$

exists. We also assume that the average area

$$c = \lim_{R\to\infty} \sum_R c_i / N(R)$$

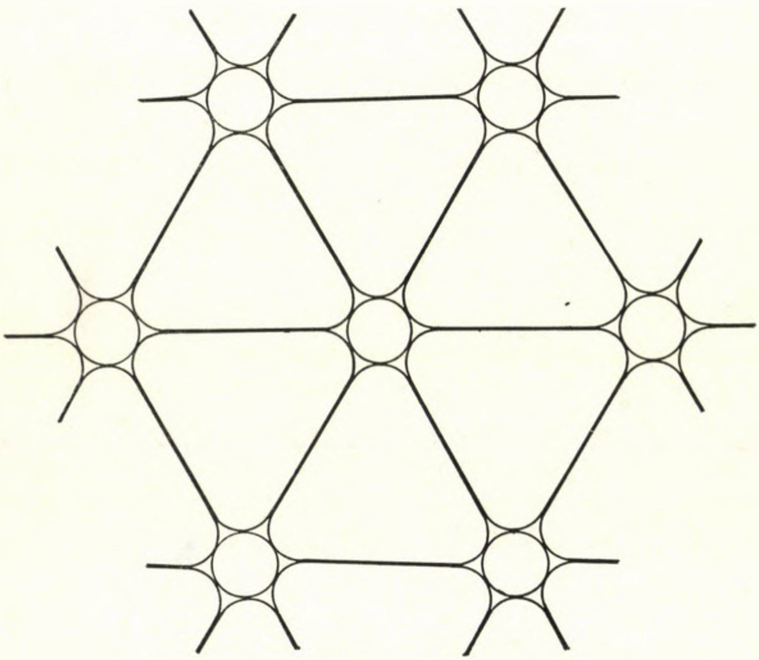
of the domains c_i associated with the domains C_i exists. Applying the inequality (1), it is not difficult to show that, under the above conditions, the upper density D of the packing (see [7]) satisfies the inequality

$$D \leqq \frac{C}{C+ec}. \tag{2}$$

Equality can be attained in various cases. Let us consider the case when the packing consists of congruent copies of $C_1, \dots, C_n$ each set of copies generated by different c_i's having the same number-density δ (see [7]). Then the whole set of copies has a density which is equal to $D=nC\delta$.

Following the steps of the above proof one sees that simultaneous fulfilment of the following conditions implies equality in (2). 1. The plane can be decomposed into convex hexagons each circumscribed about one of the copies. 2. For each $i=1, \dots, n$ any copy of C_i is the biggest domain contained in the respective hexagon swept over by translates of c_i. 3. For each $i=1, \dots, n$ the hexagon circumscribed about c_i which has the same directions of the outer normals of sides as a hexagon containing a copy of C_i is a hexagon of smallest area circumscribed about c_i.

These conditions are fulfilled in the packing exhibited in the figure. Here the set $c_1, \dots, c_n$ consists of three unit circles; the set $C_1, \dots, C_n$ consists of one unit circle and two congruent regular triangles with corners rounded off by arcs of a unit circle.



REFERENCES

[1] Blind, G., Über Unterdeckungen der Ebene durch Kreise, *J. Reine Angew. Math.* **236** (1969), 145—173. *MR* **43** # 1048; *Zbl* **174**, 255.

[2] Cassels, J. W. S., *An introduction to the geometry of numbers*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 99, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1959. *MR* **28** # 1175.

[3] Dowker, C. H., On minimum circumscribed polygons, *Bull. Amer. Math. Soc.* **50** (1944), 120—122. *MR* **5**—153.

[4] Fejes Tóth, G., Covering the plane by convex discs, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 263—270. *MR* **47** #9431.
[5] Fejes Tóth, L., Some packing and covering theorems, *Acta Sci. Math. Szeged* **12** Pars A (1950), 62—67. *MR* **12**—352.
[6] Fejes Tóth, L., *Regular figures*, The Macmillan Co., New York, 1964. *MR* **29** #2705.
[7] Fejes Tóth, L., *Lagerungen in der Ebene, auf der Kugel und im Raum*, Zweite verbesserte und erweiterte Auflage, Die Grundlehren der mathematischen Wissenschaften, Band 65, Springer-Verlag, Berlin—New York, 1972. *MR* **50** #5603.
[8] Groemer, H., Lagerungs- und Überdeckungseigenschaften konvexer Bereiche mit gegebener Krümmung, *Math. Z.* **76** (1961), 217—225. *MR* **24** #A1258.

(Received March 31, 1982)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15
H—1053 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 465—469.

EINE ABSCHÄTZUNG FÜR DIE DICHTE DER DICHTESTEN PACKUNG MIT REULEAUX—DREIECKEN

G. BLIND und R. BLIND

1. Die Dichte der dichtesten Unterdeckung der euklidischen Ebene durch kongruente Exemplare einer konvexen Scheibe läßt sich nach einem allgemeinen Verfahren abschätzen, nämlich durch den Quotienten von Flächeninhalt der Scheibe durch Flächeninhalt des flächenkleinsten umbeschriebenen Sechsecks [1, S. 86]. Für zentralsymmetrische Scheiben ist diese Schranke scharf, und ihr entspricht eine Gitterpackung; dadurch ist für zentralsymmetrische Scheiben das Problem, eine dichteste Packung mit kongruenten Exemplaren anzugeben, grundsätzlich gelöst.

Bei Packungsproblemen mit nicht-zentralsymmetrischen konvexen Scheiben spielt das Reuleaux-Dreieck eine wichtige Rolle als "sehr" nicht-zentralsymmetrische und zugleich „einfache" Scheibe. Zwar gibt es eine Vermutung über die dichteste Packung mit Reuleaux-Dreiecken, aber bisher keinerlei Abschätzungen für ihre Dichte. Deshalb soll diese Dichte abgeschätzt werden nach dem obengenannten Verfahren, wodurch zugleich die Güte dieses Verfahrens im nicht-zentralsymmetrischen Fall getestet werden soll.

Folgende Bezeichnungen werden verwendet: $\mathcal{R}$ sei ein Reuleaux-Dreieck, R sein Schwerpunkt, $\mathcal{S}$ sei ein beliebiges, durch die Symmetrieachsen von $\mathcal{R}$ gebildetes offenes Sechstel von $\mathcal{R}$. $\mathcal{K}$ sei der Kreis, der $\mathcal{S}$ berandet, K sei sein Mittelpunkt. s_1 und s_2 seien die $\mathcal{S}$ begrenzenden Symmetrieachsen von $\mathcal{R}$, wobei s_1 orthogonal zu $\mathcal{K}$ sei. $F(...)$ bezeichne den Flächeninhalt.

2. Um ein flächenkleinstes Sechseck $\mathcal{E}$ zu bestimmen, das $\mathcal{R}$ umbeschrieben ist, beachte man, daß nach [4]

(1) die Seitenmittelpunkte von $\mathcal{E}$ auf $\mathcal{R}$ liegen,

(2) die Seitenmittelpunkte von $\mathcal{E}$ keine Ecken von $\mathcal{R}$ sind,

und daß nach [2] o.B.d.A.

(3) $\mathcal{E}$ wie $\mathcal{R}$ 3-fach rotationssymmetrisch ist.

Weiter gilt

(4) $\mathcal{E}$ berührt jedes durch die Symmetrieachsen von $\mathcal{R}$ gebildete offene Sechstel von $\mathcal{R}$ genau einmal.

Beweis: Aus (2) und (3) folgt, daß $\mathcal{K}\cap\mathcal{R}$ von $\mathcal{E}$ in genau 2 Punkten berührt wird, die verschieden von den Ecken von $\mathcal{R}$ sind. Lägen diese Berührpunkte beide in $\mathcal{S}\cup s_1$, so folgte aus (1), daß der näher bei s_1 gelegene Berührpunkt in der Mitte von 2 Berührpunkten von $\mathcal{E}$ mit $\mathcal{K}\cap\mathcal{R}$ liegen müßte.

1980 *Mathematics Subject Classification.* Primary 52A45.
Key words and phrases. Packing, non-symmetrical convex body, Reuleaux-triangle, density.

3. Es gilt

HILFSSATZ. *Ein $\mathscr{R}$ umbeschriebenes flächenkleinstes Sechseck $\mathscr{E}$ hat o.B.d.A. die Symmetrieachsen von $\mathscr{R}$ als Symmetrieachsen.*

BEWEIS (vgl. Abb. 1). $\mathscr{E}$ ist nach **2** (3) o.B.d.A. 3-fach rotationssymmetrisch; $\mathscr{E}'$ sei das Spiegelbild von $\mathscr{E}$ bzgl. einer der Symmetrieachsen von $\mathscr{R}$. B und B' seien die Berührpunkte von $\mathscr{E}$ bzw. $\mathscr{E}'$ mit $\mathscr{S}$, M sei der Mittelpunkt von B und B' auf dem Rand von $\mathscr{S}$. Wird nun die Stützgerade von $\mathscr{R}$ in M wiederholt an den Symmetrieachsen von $\mathscr{R}$ gespiegelt, so begrenzen ihre Bilder ein Sechseck $\mathscr{T}$, das $\mathscr{R}$ umbeschrieben ist und die Symmetrieachsen von $\mathscr{R}$ als Symmetrieachsen hat. Der Hilfssatz ist bewiesen, wenn $F(\mathscr{T}) \leqq F(\mathscr{E})$ gilt.

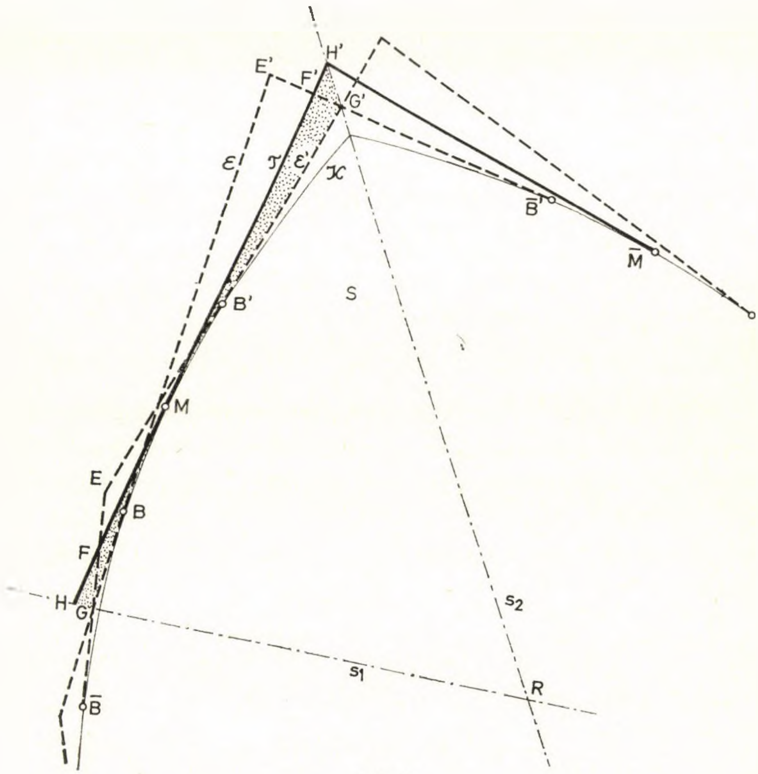


Abb. 1

Hierzu wird gezeigt: $2F(\mathscr{T}) \leqq F(\mathscr{E}') + F(\mathscr{E})$; aus Symmetriegründen genügt es, dies in $\mathscr{S}$ zu zeigen.

Folgende Bezeichnungen werden eingeführt: O.B.d.A. sei $B \neq B'$ und B liege näher bei s_1 als B'. $\bar{B}$ sei das Spiegelbild von B bzgl. s_1, und $\bar{M}$ bzw. $\bar{B}'$ das von M bzw. B' bzgl. s_2. $n\ldots$ und $t\ldots$ seien die Orthogonalen auf $\mathscr{K}$ bzw. die Stützgeraden von $\mathscr{R}$ durch den indizierenden Punkt. s_1 werde von t_B in G und von t_M in H geschnitten, und s_2 werde von $t_{B'}$ in G' und von t_M in H' geschnitten. t_B schneide $t_{B'}$ in E und

t_M in F, und $t_{B'}$ schneide t_B in E' und t_M in F'. Im folgenden werden Vielecke betrachtet, die durch $\mathscr{K}$-Kreisbogen $\widehat{BM}$, $\widehat{B'M}$ oder $\widehat{BB'}$ und ansonsten durch Strecken begrenzt werden; sie werden durch Angabe ihrer Ecken gekennzeichnet, ebenso ihr Flächeninhalt.

So ist zu zeigen: $2MBGH+2MB'G'H' \leqq B'BGE+BB'G'E'$, wozu $2\,MBGH = B'BGE$ und $2MB'G'H' \leqq BB'G'E'$ gezeigt wird.

Zum Beweis der Gleichung sei n_E die Normale auf $\mathscr{K}$ durch E, ihr Schnittpunkt mit t_M sei J. Sei $\alpha := \sphericalangle n_B n_M = \sphericalangle n_{B'} n_M$ und $\beta := \sphericalangle s_1 n_B = \sphericalangle s_1 n_B$. Dann ist $\alpha+\beta = \sphericalangle s_1 n_M = \sphericalangle n_E n_{B'}$. Daher ist $MBGH = B'MJE$ und $GFH = JFE$. Also ist $2MBGH = B'MJE + MBGH = B'MJE + MBGF + GFH = B'MJE + MBGF + JFE = B'BGE$, was gezeigt werden sollte.

Zum Beweis der Ungleichungen beachte man, daß $\sphericalangle t_B t_{B'} = \sphericalangle t_M t_{\overline{M}}$ ist, weil entsprechende Geraden jeweils den Winkel α einschließen. Es sei J' der Schnittpunkt der Winkelhalbierenden von $\sphericalangle t_B t_{B'}$ mit t_M. Weil nun $\sphericalangle E'\overline{M}H' < \alpha$, liegt K außerhalb des α-Faßkreises durch E' und H'. Deshalb ist $\sphericalangle E'KH' < \alpha$ und also $\overline{MH'} < \overline{BE'}$. Daher ist $MB'G'H' \leqq BMJ'E'$ und $G'F'H' \leqq J'F'E'$. Also ist $2MB'G'H' \leqq BMJ'E' + MB'G'H' = MBJ'E' + MB'G'F' + G'F'H' \leqq MBJ'E' + MB'G'F' + J'F'E' = BB'G'E'$, was gezeigt werden sollte.

Damit ist der folgende Satz 1 bewiesen, der ein $\mathscr{R}$ umbeschriebenes kleinstes Sechseck vollständig beschreibt.

SATZ 1. *Ein $\mathscr{R}$ umbeschriebenes flächenkleinstes Sechseck $\mathscr{E}$ hat o.B.d.A. die Symmetrieachsen von $\mathscr{R}$ als Symmetrieachsen, seine Ecken liegen auf den Symmetrieachsen von $\mathscr{R}$, und die Seiten werden durch ihre Berührpunkte mit $\mathscr{R}$ halbiert.*

4. Um schließlich gemäß **1** eine obere Schranke für die Dichte D der dichtesten Unterdeckung der Ebene durch kongruente Exemplare von $\mathscr{R}$ anzugeben, sei $\mathscr{E}$ ein $\mathscr{R}$ umbeschriebenes kleinstes Sechseck nach Satz 1, und es bleibt, $\dfrac{F(\mathscr{R})}{F(\mathscr{E})}$ numerisch auszuwerten.

Dazu sei M der Berührpunkt von $\mathscr{E}$ mit $\mathscr{S}$, und S_1, S_2 seien die Ecken von $\mathscr{E}$ auf s_1 bzw. s_2. Mit

$$\tau := \sphericalangle S_1 RM \quad \text{und} \quad \varphi := \sphericalangle RMK = \arcsin \frac{\sin\tau}{\sqrt{3}}$$

gilt dann für das Dreieck $S_1 S_2 R$

$$F(S_1 S_2 R) = \frac{1}{\sqrt{3}} \frac{\sin^2(\tau-\varphi)\cos\varphi}{\cos(\tau-\varphi)\sin\varphi}.$$

Aus $\overline{S_1 M} = \overline{S_2 M}$ nach **2** (1) und wegen

$$\overline{S_1 M} = \frac{\sin(\tau-\varphi)}{\cos(\tau-\varphi)}, \qquad \overline{S_2 M} = \frac{\sin\left(\dfrac{\pi}{3}-\tau\right)\sin(\tau-\varphi)}{\sin\left(\dfrac{\pi}{6}+\tau-\varphi\right)\sin\tau}$$

ergibt sich für τ die Bestimmungsgleichung $2\,752\sin^8\tau-5\,448\sin^6\tau+3\,537\sin^4\tau-$ $-918\sin^2\tau+81=0$. Ihre Nullstelle $\sin^2\tau=0{,}327\,180\ldots$, d. h. $\tau=0{,}608\,938\ldots$ erfüllt die Bedingung $\overline{S_1M}=\overline{S_2M}$, wodurch τ eindeutig bestimmt ist. Damit ist $F(S_1S_2R)=0{,}123\,999\ldots$, und zusammen mit $F(\mathfrak{R})=\frac{\pi}{2}-\frac{\sqrt{3}}{2}$ erhält man das gesuchte Ergebnis

SATZ 2. *Für die Dichte D der dichtesten Unterdeckung der Ebene durch kongruenten Exemplare von $\mathfrak{R}$ gilt $D\leqq 0{,}947\,275\ldots$.*

5. Die Güte dieser Abschätzung ergibt sich durch Vergleich mit der Dichte D vermutlich dichtesten Packung mit Reuleaux-Dreiecken (siehe Abb. 2)

$$D=\frac{2(\pi-\sqrt{3})}{\sqrt{15}+\sqrt{7}-2\sqrt{3}}=0{,}922\,887\ldots.$$

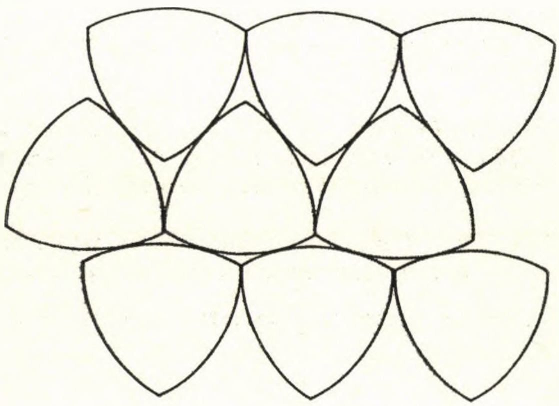


Abb. 2

Angesichts des Aufwands zur Bestimmung des kleinsten umbeschriebenen Sechsecks befriedigt das Ergebnis der oberen Abschätzung nicht ganz. Für nicht-zentralsymmetrische konvexe Scheiben wäre zur Abschätzung der Dichte ihrer dichtesten Packung ein Verfahren wünschenswert, das entweder bessere Ergebnisse mit noch ertäglichem Aufwand liefert, oder aber noch brauchbare Ergebnisse mit weniger Aufwand. Ein solches Verfahren wurde von L. Fejes Tóth [3] während der Fertigstellung dieser Arbeit gefunden. Man erhält damit

$$D\leqq 0{,}950\,248\ldots.$$

LITERATURVERZEICHNIS

[1] Fejes Tóth, L., *Lagerungen in der Ebene, auf der Kugel und im Raum*, Zweite verbesserte und erweiterte Auflage, Die Grundlehren der mathematischen Wissenschaften, Band 65, Springer-Verlag, Berlin—New York, 1972. *MR* **50** # 5603.

[2] Fejes Tóth, G. and Fejes Tóth, L., Remark on a paper of C. H. Dowker, *Period. Math. Hungar.* **3** (1973), 271—274. *MR* **48** # 12309.

[3] Fejes Tóth, L., Packing and covering with r-convex discs, *Studia Sci. Math. Hungar.* **18** (1983), 69—73.

[4] Reinhardt, K., Über die dichteste gitterförmige Lagerung kongruenter Bereiche in der Ebene und eine besondere Art konvexer Kurven, *Abh. Math. Semin. Hamburg. Univ.* **10** (1934), 216—230. *MR* **9**, 320.

(Eingegangen am 31. März 1982)

MATHEMATISCHES INSTITUT
DER UNIVERSITÄT
PFAFFENWALDRING 57
D—7000 STUTTGART 80

WALDBURGSTRASSE 88
D—7000 STUTTGART 80
FEDERAL REPUBLIC OF GERMANY

Studia Scientiarum Mathematicarum Hungarica 18 (1983), 471—480.

NORM CONDITIONS FOR THE PATH REGULARITY OF RANDOM PROCESSES

OLAV KALLENBERG

Abstract

Norm continuous L_p processes X of bounded L_p-variation $(p>1)$ are shown to be a.s. absolutely continuous with respect to some fixed measure. Moreover, the rate of convergence of the norm sums for finite interval partitions turns out to provide rather precise information on the sample path behaviour of X and its density Y, such as Lipschitz conditions on X or Y, and absolute continuity of Y. A basic role is played by a general norm inequality for L_p random variables of independent interest.

1. Introduction

In [2] it was shown that a random measure with finite total variation in L_p-norm $(p>1)$ is a.s. absolutely continuous with respect to some fixed measure. This result generalizes easily to random processes X on the real line which are not necessarily monotone. Less obvious is perhaps the fact that the rate of convergence of the norm sums related to finite partitions, as the mesh size tends to zero, gives rather precise information on the sample path regularity of X. The properties considered here include Lipschitz conditions on X and its density Y, as well as the a.s. absolute continuity of Y with respect to some fixed measure. It is interesting to compare these results with the classical work of Loève and others (cf. [3], [1]).

For the sake of clarity, the exposition is divided into two parts (§§ 2 and 3) which are rather loosely connected. In § 2 we prove a general norm inequality for random variables in L_p (Lemma 2.1), which is then applied to yield norm estimates of the second order differences of X in terms of the rate of convergence and smoothness of the L_p variation. The basic inequality will be given in a more general form than required in this paper, because of its independent interest, and because the greater generality is actually needed in the multi-dimensional case (which, for the sake of brevity, will not be treated here).

The sample path behavior of X is not considered until § 3. There we prove the existence of a density when the L_p-variation is finite, and we show how norm estimates of the second order differences may provide further information on the sample path behavior of X and its density. The only known result in the latter direction seems to be the Corollary in § 4.3 of [1], where conditions are provided for X to possess a continuous density Y. This result arises here as a boundary case when we examine the possible Lipschitz conditions on Y. (Indeed, a slightly greater generality is achieved by stating our conditions globally.) The remaining results are entirely new.

1980 *Mathematics Subject Classification.* Primary 60G17.

Key words and phrases. L_p-variation, norm inequality, absolute continuity, Lipschitz conditions.

We conclude this section with some remarks on terminology and notation. All random variables and processes under consideration may be taken to be defined on some fixed measurable space with probability P and expectation E. A random variable X lies by definition in L_p, if $\|X\|_p^p = \mathsf{E}|X|^p < \infty$. We shall always assume that $p > 1$, and we put $p' = p \vee 2$. When there is no risk for confusion, we write $\|X\|_p = \|X\|$.

The L_p-variation of an L_p process X over an interval $[s, t]$ and with respect to a partition $s = t_0 < t_1 < \ldots < t_n = t$ is defined as $\sum_i \|X(t_i) - X(t_{i-1})\|_p$, and the total variation is the supremum of this sum over all finite partitions. Since the norm is subadditive, the fundamental theorem on subadditive interval functions in [4], p. 25, applies to the variation above and shows that, if X is L_p-continuous and has locally bounded L_p-variation, the total variation over $[s, t]$ equals $\mu[s, t]$ for some diffuse measure μ. It is often convenient to denote the total L_p-variation by $\int \|X(ds)\|_p$.

We finally point out that the first and second order differences of a function f are given by

$$\Delta_h f_t = f_{t+h} - f_t, \quad \Delta_h^2 f_t = f_{t+2h} - 2f_{t+h} + f_t,$$

and that $f \lesssim g$ means the same as $f = O(g)$.

2. A norm inequality

We begin by stating and proving our basic norm inequality for L_p random variables. Later in this section, it will be seen how this result may be applied to yield norm estimates of the second order differences of random processes.

Lemma 2.1. *Let $p > 1$ and $n \in N$ be fixed, and put $p' = p \vee 2$. Then we have, uniformly for all $\xi_1, \ldots, \xi_n \in L_p$,*

$$\sum_i \sum_j \|\xi_i - \xi_j\|_p^{p'} \lesssim \Big(\sum_i \|\xi_i\|_p\Big)^{p'-2} \sum_i \sum_j (\|\xi_i\|_p - \|\xi_j\|_p)^2 +$$
$$+ \Big(\sum_i \|\xi_i\|_p\Big)^{p'-1} \Big(\sum_i \|\xi_i\|_p - \Big\|\sum_i \xi_i\Big\|_p\Big).$$

(Here the term "uniformly" means that the majorizing constant is independent of the joint distribution of $\xi_1, \ldots, \xi_n$. It may of course depend on n and p.)

Proof. For arbitrary $\alpha_1, \ldots, \alpha_n \geqq -1$ with $\sum \alpha_i = 0$, we get by Jensen's inequality $\sum_i (1 + \alpha_i)^p \geqq n$, with equality iff $|\alpha|^2 = \sum \alpha_i^2 = 0$. At the origin, we further have the Taylor expansion

$$\sum_i (1 + \alpha_i)^p = n + \frac{p(p-1)}{2} |\alpha|^2 + O(|\alpha|^3).$$

Moreover,

$$(\alpha_i - \alpha_j)^2 \leqq 2\alpha_i^2 + 2\alpha_j^2 \leqq 2|\alpha|^2.$$

Solving for $|\alpha|^2$ in these relations yields

$$(1) \qquad (\alpha_i - \alpha_j)^2 \lesssim |\alpha|^2 \lesssim \sum_i (1 + \alpha_i)^p - n,$$

at least near the origin. But then (1) must be generally true, since the domain is compact and since all three members are continuous and strictly positive outside the origin. Note also that

$$|\alpha_i-\alpha_j|^p \lesssim (\alpha_i-\alpha_j)^2, \qquad p\geqq 2.$$

For arbitrary $x_1, \ldots, x_n \geqq 0$ with $|x|>0$, we may substitute

$$1+\alpha_i = nx_i/\sum_j x_j, \qquad i = 1, \ldots, n,$$

into the last two relations to obtain

$$|x_i-x_j|^p \lesssim n^{p-1}\sum_i |x_i|^p - \Big|\sum_i x_i\Big|^p, \qquad p \geqq 2. \tag{2}$$

Note that this remains true for $|x|=0$, and also when all the x_i are $\leqq 0$. Directly from (1), we further obtain

$$(x_i-x_j)^2/\Big(\sum_k |x_k|\Big)^{2-p} \lesssim n^{p-1}\sum_i |x_i|^p - \Big|\sum_i x_i\Big|^p, \qquad p \leqq 2, \tag{3}$$

provided that the left side is interpreted as zero when $|x|=0$. Even (3) remains valid when the x_i are instead $\leqq 0$.

Now suppose that there are numbers x_i of different signs, and assume without loss that $\sum x_i \geqq 0$. Put $m = \# \{i\colon x_i \geqq 0\}$, and let $\sum'$ denote summation over the corresponding index set. Then clearly

$$n^{p-1}\sum_i |x_i|^p - \Big|\sum_i x_i\Big|^p \geqq n^{p-1}\sum_i{}' x_i^p - \Big(\sum_i{}' x_i\Big)^p \geqq$$

$$\geqq (n^{p-1}-m^{p-1})\sum_i{}' x_i^p \gtrsim \sum_i{}' x_i^p \gtrsim \Big(\sum_i{}' x_i\Big)^p \gtrsim \Big(\sum_i |x_i|\Big)^p \gtrsim |x_i-x_j|^p,$$

so (2) remains true, and so does (3) since for $p\leqq 2$

$$(x_i-x_j)^2 \leqq |x_i-x_j|^p(|x_i|+|x_j|)^{2-p} \leqq |x_i-x_j|^p\Big(\sum_k |x_k|\Big)^{2-p}.$$

Substituting ξ_i for x_i and taking expected values, we get from (2)

$$\|\xi_i-\xi_j\|_p^p \lesssim n^{p-1}\sum_i \|\xi_i\|_p^p - \Big\|\sum_i \xi_i\Big\|_p^p, \qquad p \geqq 2. \tag{4}$$

For $p<2$, we get instead from (3) and Hölder's inequality

$$\|\xi_i-\xi_j\|_p^p \leqq \mathsf{E}\Big(\big((\xi_i-\xi_j)^2/\big(\sum_k |\xi_k|\big)^{2-p}\big)^{p/2}\big(\sum_k |\xi_k|\big)^{(2-p)p/2}\Big) \leqq$$

$$\leqq \Big(\mathsf{E}(\xi_i-\xi_j)^2/\big(\sum_k |\xi_k|\big)^{2-p}\Big)^{p/2}\Big(\mathsf{E}\big(\sum_k |\xi_k|\big)^p\Big)^{1-p/2} \lesssim \tag{5}$$

$$\lesssim \Big(n^{p-1}\sum_i \|\xi_i\|_p^p - \Big\|\sum_i \xi_i\Big\|_p^p\Big)^{p/2}\Big(\sum_k \|\xi_k\|_p^p\Big)^{1-p/2}.$$

(Here the set $\{|\xi|=0\}$ requires special attention.)

For arbitrary $x_1, \ldots, x_n \geqq 0$ with $|x|>0$, it is further seen from (1) that

$$0 \leqq n^{p-1} \sum_i x_i^p - (\sum_i x_i)^p \lesssim (\sum x_i)^{p-2} \sum_i (nx_i - \sum_j x_j)^2.$$

$$\lesssim (\sum_i x_i)^{p-2} \sum_i \sum_j (x_i - x_j)^2.$$

Substituting $\|\xi_i\|_p$ for x_i, we get

(6) $$0 = n^{p-1} \sum_i \|\xi_i\|_p^p - (\sum_i \|\xi_i\|_p)^p \lesssim (\sum_i \|\xi_i\|_p)^{p-2} \sum_i \sum_j (\|x_i\|_p - \|\xi_j\|_p)^2.$$

Finally note that, for $x \geqq y \geqq 0$,

$$x^p - y^p \asymp \int_y^x u^{p-1}\, du \leqq x^{p-1}(x-y),$$

and substitute $x = \sum_i \|\xi_i\|_p$, $y = \left\|\sum_i \xi_i\right\|_p$, to obtain

(7) $$0 \leqq (\sum_i \|\xi_i\|_p)^p - \left\|\sum_i \xi_i\right\|_p^p \lesssim (\sum_i \|\xi_i\|_p)^{p-1} (\sum_i \|\xi_i\|_p - \left\|\sum_i \xi_i\right\|_p).$$

The asserted relation now follows by combination of (6) and (7) with (4) or (5), respectively, as $p \geqq 2$ or $p<2$, and then summing over i and j. □

In the special case of two random variables $\xi, \eta \in L_p$, we obtain

$$\|\xi - \eta\|^{p'} \lesssim (\|\xi\| + \|\eta\|)^{p'-2} (\|\xi\| - \|\eta\|)^2 + (\|\xi\| + \|\eta\|)^{p'-1} (\|\xi\| + \|\eta\| - \|\xi + \eta\|).$$

Applying this to the adjacent differences $\Delta_h X_t = X_{t+h} - X_t$ of a process X, and assuming that $\|\Delta_h X_t\| = O(h)$ (which is a reasonable assumption when the L_p-variation exists), we get

(8) $$\|\Delta_h^2 X_t\|^{p'} \lesssim h^{p'-2} (\|\Delta_h X_{t+h}\| - \|\Delta_h X_t\|)^2 + h^{p'-1} (\|\Delta_h X_t\| + \|\Delta_h X_{t+h}\| - \|\Delta_{2h} X_t\|).$$

Here the first term measures the inhomogeneity of the L_p-variation, while the second term expresses the rate of convergence of the norm sum as the partition gets finer. Note that the second term is dominated by

$$h^{p'-1} \Big(\int_t^{t+2h} \|dX(s)\| - \|\Delta_{2h} X_t\| \Big).$$

For the sake of simplicity, we take X to be defined on the unit interval [0, 1] and introduce the notations

$$Q_h^2 = \sum_i (\|\Delta_h X_{ih}\| - \|\Delta_h X_{(i-1)h}\|)^2,$$

$$R_h = 2 \int_0^1 \|dX(s)\| - \sum_i \|\Delta_{2h} X_{ih}\| - \|\Delta_h X_0\| - \|\Delta_h X_{1-h}\|.$$

Summation in (8) then yields

$$\sum_i \|\Delta_h^2 X_{ih}\|^{p'} \lesssim h^{p'-2} Q_h^2 + h^{p'-1} R_h.$$

By Hölder's inequality, we further obtain

$$\sum_i \|\Delta_h^2 X_{ih}\| \leqq h^{1/p'-1}\Big(\sum_i \|\Delta_h^2 X_{ih}\|^{p'}\Big)^{1/p},$$

and moreover, if $p<2=p'$,

$$\sum_i \|\Delta_h^2 X_{ih}\|^p \leqq h^{p/p'-1}\Big(\sum_i \|\Delta_h^2 X_{ih}\|^{p'}\Big)^{p/p'}.$$

This leads easily to the following result.

COROLLARY 2.2. *If* $\|\Delta_h X_t\|_p = O(h)$ *uniformly in* t, *then*

$$\sum_i \|h^{-1}\Delta_h^2 X_{ih}\|_p^p \lesssim h^{-1}(Q_h^2/h+R_h)^{p/p'},$$

$$\sum_i \|h^{-1}\Delta_h^2 X_{ih}\|_p \lesssim h^{-1}(Q_h^2/h+R_h)^{1/p'}.$$

For the purposes of the next section, we are hence left with the problem of estimating the orders of magnitude of Q_h^2 and R_h as $h\to 0$. For an illustration, note that Q_h^2/h will typically be of the order $O(h^2)$, whereas for R_h we will normally get an estimate of the form $O(h^c)$ with $c\in[0,2]$. In this case, the second term in our estimates will dominate, and we will get the bounds

$$\sum_i \|h^{-1}\Delta_h^2 X_{ih}\|_p^p \lesssim h^{cp/p'-1}, \quad \sum_i \|h^{-1}\Delta_h^2 X_{ih}\|_p \lesssim h^{c/p'-1}. \tag{9}$$

3. Path regularity

Throughout this section, let X denote a separable random process in L_p. Our first concern is the absolute continuity of X. The following result extends the main theorem in [2].

PROPOSITION 3.1. *For a* $p>1$, *let* X *be an* L_p*-continuous process with locally bounded* L_p*-variation* μ. *Then* X *is a.s. absolutely continuous with respect to* μ, *and the density has a version* Y *satisfying* $\|Y\|_p \equiv 1$.

PROOF. The argument is similar to the one in [2]. Consider a sequence of ultimately dense nested partitions $\{I_{nj}\}$ of a fixed interval I, and let ηI_{nj} denote the increment of X over I_{nj}. Writing $I_n(s)$ for the interval I_{nj} containing s, and putting $Y_n(s) = \eta I_n(s)/\mu I_n(s)$ (which may e.g. be interpreted as 1 when the denominator vanishes), we get by Hölder's inequality (with $p^{-1}+q^{-1}=1$)

$$\mathsf{E}\int_I |Y_n(s)|^{1+p/q}\mu(ds) = \sum_j \mathsf{E}|\eta I_{nj}|\left|\frac{\eta I_{nj}}{\mu I_{nj}}\right|^{p/q} \leqq$$

$$\leqq \sum_j (\mathsf{E}|\eta I_{nj}|^p)^{1/p}\left(\mathsf{E}\left|\frac{\eta I_{nj}}{\mu I_{nj}}\right|^p\right)^{1/q} \leqq \sum_j \|\eta I_{nj}\| \leqq \mu I < \infty,$$

where we have used the inequality $\|\eta I_{nj}\| \leqq \mu I_{nj}$ twice. Since the μ-integral on the left is non-decreasing, it follows that Y_n is a.s. uniformly μ-integrable, and so we may conclude as in [2] that $Y_n \to$ some Y a.e. $\mu\times P$. and that Y satisfies $\eta I = \int_I Y\,d\mu$ a.s. By our hypothesis on separability, it follows that X is a.s. absolutely continuous with respect to μ and has Y as a density.

To calculate $\|Y\|$, we first conclude from Theorem 1 in [2] that, for any interval I,

$$\|\eta I\| = \Big\|\int_I Y\,d\mu\Big\| \leqq \Big\|\int_I |Y|\,d\mu\Big\| \leqq \int_I \|Y\|\,d\mu.$$

Summing over disjoint intervals I_j partitioning I, we get $\mu I \leqq \int_I \|Y\|\,d\mu$. The converse inequality follows as in [2] from Fatou's lemma and the fact that $Y_n \to Y$ a.e. $\mu\times P$. Hence $\mu I = \int_I \|Y\|\,d\mu$, and this extends easily to arbitrary Borel sets. It follows that $\|Y\| = 1$ a.e. μ, and by changing the definition of Y on a set of μ-measure zero, we may arrange that $\|Y\| = 1$ holds identically. □

In the remainder of this section, we shall assume the hypotheses of Proposition 3.1 to be fulfilled.

Our next aim is to examine when X fulfils Lipschitz conditions of different orders. Since the norms of the first order differences cannot be uniformly of smaller order than $O(h)$ unless X is a.s. constant, the classical conditions (cf. [1], p. 74) can only ensure X to be Lipschitz of orders $< 1 - p^{-1}$. By considering instead the second order differences, we can get closer to 1 for fixed p.

PROPOSITION 3.2. *Let $p > 1$ and $a \in (0, 1]$ be fixed, and suppose that*

$$\sum \|\Delta_h^2 X_{ih}\|_p^p = O(h^r) \tag{1}$$

for some $r > ap$. Then X is a.s. locally Lipschitz of order a.

Note in particular that X is a.s. Lipschitz of order 1 and hence has a locally bounded Lebesgue density, if (1) holds for some $r > p$. The next proposition shows than the density has even a continuous version in this case. (This is essentially the corollary in [1], p. 69.)

PROOF. We may clearly take X to be defined on [0, 1]. Fix $b \in (a, r/p)$. By (1) we get

$$\mathsf{E} \sum_i |h^{-b} \Delta_h^2 X_{ih}|^p = O(h^{r-bp}),$$

so

$$\mathsf{E} \sum_{h=2^{-n}} \sum_i |h^{-b} \Delta_h^2 X_{ih}| < \infty.$$

But then the integrand must be a.s. finite, so

$$\sum_i |h^{-b} \Delta_h^2 X_{ih}|^p \to 0 \quad \text{a.s.,} \quad h = 2^{-n} \to 0,$$

which implies that

$$\max_i |\Delta_h^2 X_{ih}| = O(h^b) \quad \text{a.s.,} \quad h = 2^{-n} \to 0. \tag{2}$$

Let us now consider an arbitrary dyadic interval I_n of length 2^{-n}, and let I_{n-1}, ..., I_0 be the dyadic intervals of lengths 2^{-n+1}, ..., 1 containing I_n. Write ξ_k for the increment of X over I_k, and note that

$$2\xi_k = \xi_{k-1} \pm \delta_k, \quad k = 1, \dots, n, \tag{3}$$

where δ_k is the second order difference for the two 2^{-k}-length intervals in I_{k-1}. Multiplying (3) by 2^{k-1} and summing over $k=1, \dots, n$ yields

$$2^n \xi_n = \xi_0 + \sum_{k=1}^{n} (\pm 2^{k-1}\delta_k),$$

so by (2)

$$|\xi_n| \lesssim 2^{-n}|\xi_0| + 2^{-n} \sum_{k=1}^{n} 2^{k-1-bk}.$$

Here the series on the right is of the order $2^{n(1-b)}$ if $b<1$, n if $b=1$, and 1 if $b>1$, so in all three cases we get $|\xi_n| = O(2^{-na})$. Note also that this estimate is a.s. uniform over all dyadic intervals. This proves that

$$\max_i |\Delta_h X_{ih}| = O(h^a) \quad \text{a.s.}, \quad h = 2^{-n} \to 0. \tag{4}$$

We next consider arbitrary $s, t \in [0, 1]$, and let m be the largest integer satisfying $h = |s-t| \leqq 2^{-m}$. Writing s_n and t_n for the closest n-th order lattice points to the left of s and t, respectively, we get

$$|X_s - X_t| \leqq |X_s - X_{s_m}| + |X_{s_m} - X_{t_m}| + |X_{t_m} - X_t|.$$

Since X is a.s. continuous, it is further seen that

$$|X_s - X_{s_m}| \leqq \sum_{n=m}^{\infty} |X_{s_{n+1}} - X_{s_n}|,$$

and similarly for $|X_t - X_{t_m}|$. Noting that $|s_m - t_m| = 0$ or 2^{-m} and that $|s_{n+1} - s_n|$ and $|t_{n+1} - t_n|$ are equal to 0 or 2^{-n-1}, it follows from (4) that

$$|X_s - X_t| \lesssim 2^{-ma} + 2 \sum_{n=m}^{\infty} 2^{-(n+1)a} \lesssim 2^{-ma} \lesssim h^a.$$

Since this bound is a.s. uniform, we obtain the desired estimate

$$\sup_t |\Delta_h X_t| = O(h^a) \quad \text{a.s.}, \quad h \to 0. \quad \square$$

Estimates of the same type may be used to draw conclusions on the Lebesgue density of X:

PROPOSITION 3.3. *Let $p>1$ and $a \in (0, 1]$ be fixed, and suppose that*

$$\sum_i \|h^{-1}\Delta_h^2 X_{ih}\|_p^p = O(h^r) \tag{5}$$

for some $r > ap$. Then X has a.s. a derivative which is locally Lipschitz of order a.

PROOF. For simplicity, we take X to be defined on $[0, 1)$. Proceeding as before, we may conclude from (5) that

(6) $$\max_i |h^{-1}\Delta_h^2 X_{ih}| = O(h^a) \quad \text{a.s.}, \quad h = 2^{-n} \to 0.$$

Fix an arbitrary $t \in [0, 1)$, and let I_n be the half-open dyadic interval of length 2^{-n} containing t. Write $Y_n = Y_n(t)$ for the difference quotient of X over I_n, and note that, by (3),

(7) $$Y_n = Y_{n-1} \pm 2^{n-1}\delta_n.$$

Since the series $\sum 2^{n-1}|\delta_n|$ converges by (6), it follows that Y_n tends towards some $Y = Y(t)$. Note that this convergence is a.s. uniform in t. Now it is clear that, for binary rational t and for large enough n,

$$X_t - X_0 = \int_0^t Y_n(s)\,ds,$$

and by dominated convergence, this remains true with Y_n replaced by Y. Since X is continuous, we may finally turn to arbitrary t. This proves that X is a.s. absolutely continuous with derivative Y.

We next consider arbitrary $s, t \in [0, 1)$, and let m be the largest integer satisfying $h = |s-t| \leqq 2^{-m}$. Then s and t lie in the same or in adjacent dyadic intervals of length 2^{-m}. Abbreviating $Y'_n = Y_n(s)$ and $Y''_n = Y_n(t)$ etc., we get

$$|Y' - Y''| \leqq |Y' - Y'_m| + |Y'_m - Y''_m| + |Y''_m - Y''|,$$

and moreover,

$$|Y' - Y'_m| \leqq \sum_{n=m}^{\infty} |Y'_{n+1} - Y'_n|.$$

Combining these relations and using (6) and (7), it is seen that

(8) $$|Y' - Y''| \lesssim 2^{-ma} + 2\sum_{n=m}^{\infty} 2^{-(n+1)a-1} \lesssim 2^{-ma} \lesssim h^a,$$

and since this estimate is a.s. uniform, it follows that indeed

$$\sup_t |\Delta_h Y_t| = O(h^a) \quad \text{a.s.}$$

An obvious refinement of the above argument yields a new proof of the corollary in [1], p.69.

Our final aim is to examine the possible a.s. absolute continuity of the derivative Y.

PROPOSITION 3.4. *Fixed $p > 1$, suppose that (5) holds for some $r > 0$ and that moreover*

(9) $$\sum_i \|h^{-1}\Delta_h^2 X_{ih}\|_p = O(1), \quad h \to 0.$$

Then X has a.s. a derivative which is absolutely continuous with respect to some fixed diffuse measure.

(The measure occurring here may of course differ from the Lebesgue measure.) Note in particular that (9) holds if $p \geqq 2$ and if the exponent c in (2.9) takes the boundary value 2.

PROOF. The preceding proposition shows that X has a.s. a continuous derivative Y. Since by (5)

$$\max_i \|h^{-1}\Delta_h^2 X_{ih}\|_p = O(h^a),$$

where $a = r/p$, it follows from (7) that $Y_n \to Y$ holds in L_p also. Moreover, it may be seen as in (8) that $\|Y_s - Y_t\|_p = O(|s-t|^a)$, which shows that Y is L_p-continuous. By Proposition 3.1, it is hence enough to prove that Y has a bounded total L_p-variation.

As before, we may take X to be defined on [0, 1). Once more, we shall use the fact that

$$\|Y_n - Y_{n-1}\|_p = \frac{1}{2}\|2^n \delta_n\|_p.$$

Consider both sides as functions of t, and replace t by a uniformly distributed random variable θ. Then the norm on the right will be uniformly distributed among the even terms in (9) with $h = 2^{-n}$, and since there are 2^{n-1} of them and their sum is bounded, it follows that

$$\mathsf{E}\|Y_n(\theta) - Y_{n-1}(\theta)\|_p = O(2^{-n}).$$

(Here we assume the norm to be evaluated with θ fixed.) Summing over n yields

$$\mathsf{E}\|Y(\theta) - Y_n(\theta)\|_p = O(2^{-n}). \tag{10}$$

Now define $\theta_i = \theta + i2^{-n} \pmod 1$, $i = 1, \ldots, 2^n$. Then (10) remains valid for each θ_i, and from (9) it is further seen that

$$\sum_i \|Y_n(\theta_i) - Y_n(\theta_{i-1})\|_p = O(1).$$

Combining these facts, we get

$$\mathsf{E}\sum_i \|Y(\theta_i) - Y(\theta_{i-1})\| \leqq \mathsf{E}\sum_i \|Y_n(\theta_i) - Y_n(\theta_{i-1})\| + 2\sum_i \mathsf{E}\|Y(\theta_i) - Y_n(\theta_i)\| = O(1).$$

For each n, it is therefore possible to choose 2^n points with equal span 2^{-n}, such that the corresponding L_p-variation of Y remains bounded as $n \to \infty$. Let Π_n denote the n-th of these partitioning sets.

If the total L_p-variation of Y is unbounded, there exists for every constant $b > 0$ a finite partition Π such that the variation over Π is $> b$. Now there exists for every $t \in \Pi$ some $t_n \in \Pi_n$ with $t_n \to t$, and we get $Y_{t_n} \to Y_t$ in L_p since Y is L_p-continuous. If Π_n' is the set of these t_n for fixed n, it follows that the variation over Π_n' is also $> b$ for sufficiently large n. But since the variation is non-decreasing under refinements of the partitioning set, the variation over Π_n must be at least as large. The bound b being arbitrary, it follows that the latter variation tends to infinity. This contradiction shows that Y must indeed have bounded total L_p-variation. □

REFERENCES

[1] Cramér, H. and Leadbetter, M. R., *Stationary and related stochastic processes. Sample function properties and their applications*, John Wiley & Sons, Inc., New York—London—Sydney, 1967. *MR* **36** #949.
[2] Kallenberg, O., L_p-intensities of random measures, *Stochastic Process. Appl.* **9** (1979), 155—161. *MR* **80j**: 60081.
[3] Loève, M., Fonctions aléatoires du second ordre, in: P. Lévy, *Processus stochastiques et mouvement brownien*, Suivi d'une note de M. Loève, Gauthier-Villars, Paris, 1948, 299—352. *MR* **10**—551. See also: Fonctions aléatoires de second ordre, *Revue Sci.* **84** (1946), 195—206. *MR* **8**—281.
[4] Riesz, F. and Sz.-Nagy, B., *Functional analysis*, Frederick Ungar Publishing Co., New York, 1955. *MR* **17**—175.

(Received August 16, 1982)

DEPARTMENT OF MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY
AND THE UNIVERSITY OF GÖTEBORG
S—412 96 GÖTEBORG
SWEDEN

PRINTED IN HUNGARY
Szegedi Nyomda, Szeged

RECENTLY ACCEPTED PAPERS

POPENDA, J., On the boundedness of the solutions of second order differential equations
Дарбинян, С. Х., Панцикличность орграфов при условии Мейнила
FRIED, E. and GRÄTZER, G., Congruence classes of filtral varieties
FRANKL, P., Asymptotic solution of a locally-Turán problem
LÉNÁRD, M., Spline interpolation in two variables
ISAC, G., Branches continues de vecteurs propres généralisés. Applications aux équations de coincidences
KOLIBIAR, M., Weak homomorphisms in some classes of algebras
SIMON, L., On approximation of the solutions of strongly nonlinear equations in unbounded domains
IVIĆ, A., The distribution of primitive abundant numbers
IVIĆ, A., On squarefree numbers with restricted prime factors
DUGGAL, B. P., On the spectrum of a class of integral transforms II
DUGGAL, B. P., On the spectrum of a class of integral transforms III
BERG, G., Steinness and the vanishing of cohomology
SRIVASTAVA, K. B., A remark on Mathur's paper "On a proof of Jackson's theorem through an interpolation process"
BECK, J., On the number of circles determined by n points in the plane
BLIND, G. and BLIND, R., Ein Kreisüberdeckungsprobleme
JAYARAM, C., 0-modular semilattices
HARMAN, G., PINTZ, J. and WOLKE, D., A note on the Möbius and Liouville functions
GRIMMETT, G. R., The largest components in a random lattice
GAÁL, I., Effective bounds for the solutions of norm form equations
LÖFSTRÖM, J., Best approximation in $L_p(w)$ by algebraic polynomials
MOÓR, A. und NADJ, Dj. F., Über die autoparallele Abweichung von Finsler—Otsukischen Räumen mit speziellen P-Tensoren
KHAN, L. A., Separability in the uniform topology
DAMASCHKE, P. and STERN, M., A characterization of generalized matroid lattices
WILKER, J. B., Inversive geometry and the Hopf fibration
MÜLLER, H. R., Eine Kennzeichnung der Schraubenbewegungen des euklidischen Raumes
REIMNITZ, P., An arcsine-law for the oscillating random walk
CHAKERIAN, G. D. and FILLIMAN, P., The measures of the projections of a cube
DANZER, L., Zur Lösung des Gallaischen Problems über Kreisscheiben in der euklidischen Ebene
HANÁK, G., Stopping rules for stochastic approximation methods
GÓŹDŹ, Cz. and POLAK, M., An approximation theorem for the Poisson processes defined on an abstract space
KLEE, V., Facet centroids and volume minimization
RUZSA, I. Z., Iterated difference sets
GRUBER, P. M., Typical convex bodies have surprisingly few neighbours in densest lattice packings
SALERNO, S., On the distribution of $x_1^k+\ldots+x_s^k$ in the arithmetic progressions
WERNICKE, B., Möbiusebenen in spiegelungsgeometrischer Darstellung
DEBRUNNER, H. E., Tiling three-space with handlebodies
SACHS, H., No more than nine unit balls can touch a closed unit hemisphere
FLORIAN, A., Packing and covering with convex discs
BALOG, A., FRIEDLANDER, J. and PINTZ, J., Large prime factors of integers in an arithmetic progression

CONTENTS

HU ISSN 0081—6906 **Index: 26 748**