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AKADÉMIAI KIADÓ

H-1117 Budapest, Prielle Kornélia u. 19-35

Manuscripts and editorial correspondence should be addressed to

J. Merza
Managing Editor

P.O. Box 127
H-1364 Budapest

Tel.: +36 1 118 2875 Fax: +36 1 117 7166
e-mail: merza @ math-inst.hu

Subscription information

Orders should be addressed to

AKADÉMIAI KIADÓ
P.O.Box 245
H-1519 Budapest

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IMPLICIT FUNCTION THEOREM IN UNIFORM SPACES

B. SLEZÁK

In this paper we prove an implicit function theorem for uniform spaces. As a special case an implicit function theorem is also obtained for metrizable topological groups. Moreover, an implicit function theorem is proved for Banach space case under the same assumptions as Szilágyi's generalized implicit mapping theorem in [2] with stronger consequences.

DEFINITION 1. Let H, X and Y be nonempty sets, $f\colon H\times X\to Y$ a function and $y\in \operatorname{im}(f)$. The function $g\colon H\to X$ is called the *implicit function* given by the function f, belonging to the value y and passing through the point $(t_0,x_0)\in H\times X$ if $g(t_0)=x_0$ and for every $t\in H$ the equality $f(t,g(t))=$ $=y$ holds. □

The reader can easily control that the following proposition directly follows from the definitions.

PROPOSITION 1. *Let H, X, Y and f be the same as above, $F:=(\mathrm{pr}_H, f)$, where $\mathrm{pr}_H\colon H\times X\to H$ is the projection to H.*

(i) *There exists a function $g\colon H\to X$ such that $f(t,g(t))=y$ holds for every $t\in H$ if and only if $H\times\{y\}\subseteqq F(H\times X)$.*

(ii) *Let $\mathcal{F}:=\{f(t,.)\mid t\in H\}$ and F_r be a right inverse of the function F. The equality $F_r(t,y)=(t,[f(t,.)]_r(y))$ uniquely determines the right inverse $[f(t,.)]_r$ of the function $f(t,.)$ for every fixed element t of H.*

Conversely, if for every point t of the set H a right inverse $[f(t,.)]_r$ is given, then the function $F_r(t,y)=(t,[f(t,.)]_r(y))$ is a right inverse of the function F.

(iii) *The function $g\colon H\to X$ is an implicit function of f belonging to $y\in Y$ if and only if there exists a right inverse F_r of F which satisfies the equality $F_r(.,y)=(id_H,g)$. The function g passes through the point $(t_0,x_0)\in$ $\in H\times X$ if and only if $[f(t_0,.)]_r(y)=x_0$.* □

NOTATION. Let (T,τ) be a topological space. $V\ni)t$ will note that the set $V\in\tau$ is a neighbourhood of the point $t\in T$.

DEFINITION 2. Let T be a nonempty set, X a topological space, Y a uniform space, $\mathcal{B}_Y$ a base of the uniformity of Y. Let $f\colon T\times X\to Y$ be a function and $(t_0,x_0)\in T\times X$ be a point.

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(i) The family of functions $\mathcal{F} := \{f(t,.) \mid t \in T\}$ is *equicontinuous at* x_0 with respect to T iff

$$\forall\, W \in \mathcal{B}_Y \ \ \exists V \ni)x_0 \ \ \forall t \in T\colon\ f(t,.)(V) \subseteqq W(f(t,x_0)),$$

so that every partial function $f(t,.)$ is continuous at x_0 and $f(t,.)(V) \subset \subset W(f(t,x_0))$, where V does not depend on t.

(ii) The family $\mathcal{F}$ is *equiopen at* x_0 with respect to T iff

$$\forall\, V \ni)x \ \ \exists W \in \mathcal{B}_Y \ \ \forall t \in T\colon\ f(t,.)(V) \supset W(f(t,x_0)),$$

so that every partial function $f(t,.)$ is open at x_0 and $f(t,.)(V) \supset W(f(t,x_0))$, where W does not depend on t. □

Further it is supposed that T is a topological space, X and Y are uniform spaces, $\mathcal{B}_X$ denotes a base of the uniformity of X and the elements of $\mathcal{B}_Y$ are symmetric.

PROPOSITION 2. *If the function $f(.,x_0)$ is continuous at t_0 and the family of functions $\mathcal{F} := \{f(t,.) \mid t \in T\}$ is equicontinuous at x_0 then f is continuous at (t_0,x_0).*

PROOF. We show that

$$\forall\, W \in \mathcal{B}_Y \ \ \exists H \ni)t_0 \ \ \exists V \in \mathcal{B}_X\colon\ f(H \times V(x_0)) \subseteqq W(f(t_0,x_0)).$$

Let $W_1, W_2 \in \mathcal{B}_Y$, $W_2 \circ W_1 \subseteqq W$. The neighbourhood H of t_0 can be chosen so that $f(.,x_0)(H) \subseteqq W_2(f(t_0,x_0))$, that is

$$\forall t \in H\colon\ (f(t_0,x_0), f(t,x_0)) \in W_2. \tag{1}$$

The set $V \in \mathcal{B}_X$ can be chosen so that $\forall t \in H\colon\ f(t,.)(V(x_0)) \subset W_1(f(t,x_0))$, hence

$$\forall (t,x) \in H \times V(x_0)\colon\ (f(t,x_0), f(t,x)) \in W_1. \tag{2}$$

From (1) and (2) it follows that

$$\forall (t,x) \in H \times V(x_0)\colon\ (f(t_0,x_0), f(t,x)) \in W_2 \circ W_1 \subseteqq W_1 \subseteqq W,$$

hence $f(H \times V(x_0)) \subseteqq W(f(t_0,x_0))$. □

THEOREM 1 (Implicit Function Theorem in uniform spaces). *If the function $f(.,x_0)$ is continuous at t_0 and $\mathcal{F}$ is equiopen at x_0 then*

(i) *the function $F := (\mathrm{pr}_T, f)$ is open at (t_0,x_0);*

(ii) *for every neighbourhood $V(x_0)$ of the point x_0 there exists a neighbourhood $W(f(t_0,x_0))$ of $f(t_0,x_0)$ in Y such that for every point of this neighbourhood there is an implicit function belonging to this point, having the fixed set H as a domain and having its range in $V(x_0)$, that is*

$$\forall V \in \mathcal{B}_X \quad \exists W \in B_Y \quad \exists H \ni) t_0 \quad \forall y \in W(f(t_0; x_0))$$
$$\exists g \colon H \to V(x_0) \quad \forall t \in H: \ f(t, g(t)) = y.$$

If $\mathcal{F}$ is equicontinuous at x_0 then a neighbourhood $H' \times V'(x_0) \ni (t_0, x_0)$ can be chosen so that for every point of this set there exists a $\phi \colon H \to V(x_0)$ implicit function passing through this point.

(iii) *Suppose that there exists an implicit function g which passes through the point $(h, s) \in H \times V(x_0)$ and continuous at h, furthermore $\mathcal{F}$ is equicontinuous at the point s. Then f is continuous at the point (h, s).*

(iv) *Let us suppose that every element of $\mathcal{F}$ is injective. Then the implicit functions are unique and the implicit function passing through the point (t_0, x_0) is continuous at t_0.*

PROOF. (i) Let $V \in \mathcal{B}_X$ and $U \times V(x_0)$ be a neighbourhood of (t_0, x_0). We show that $F(U \times V(x_0))$ is a neighbourhood of $F(t_0, x_0)$. It is clear that the following equalities hold:

$$\begin{aligned} F(U \times V(x_0)) &= \{(t, f(t, x)) \mid (t, x) \in U \times V(x_0)\} = \\ &= \bigcup_{t \in U} \{t\} \times f(t, .)(V(x_0)). \end{aligned} \tag{1}$$

As $\mathcal{F}$ is equiopen at x_0 the set W_1 can be chosen so that

$$\forall t \in U: \ W_1(f(t, x_0)) \subseteqq f(t, .)(V(x_0)) \tag{2}$$

holds. Let $W_2, W \in \mathcal{B}_Y$, $W_2 \circ W \subseteqq W_1$. As $f(., x_0)$ is continuous at t_0 the neighbourhood H of t_0 can be chosen so that $H \subseteqq U$ and $f(., x_0)(H) \subseteqq W_2(f(t_0, x_0))$, that is

$$\forall t \in H: \ (f(t, x_0), f(t_0, x_0)) \in W_2 \tag{3}$$

holds. It follows that for every $t \in H$ the set $W(f(t_0, x_0))$ is a subset of the set $f(t, .)(V(x_0))$. Indeed, by (3), $(f(t, x_0), f(t_0, x_0)) \in W_2$ and $(f(t_0, x_0), y) \in W$ imply that $(f(t, x_0), y) \in W_2 \circ W \subseteqq W_1$. Hence (using (2))

$$\forall t \in H: \ W(f(t_0, x_0)) \subseteqq W_1(f(t, x_0)) \subseteqq f(t, .)(V(x_0)).$$

By (1) we get that

$$H \times W(f(t_0, x_0)) \subseteqq F(H \times V(x_0)) \subseteqq F(U \times V(x_0)).$$

(ii) Let W be as above, $y \in W(f(t_0, x_0))$ be arbitrary. As $H \times W(f(t_0, x_0)) \subseteqq F(H \times V(x_0))$, we have that $\forall t \in H \ \exists g(t) = x \in V(x_0)$: $F(t, g(t)) = y$, which means that $(t, f(t, g(t))) = (t, y)$. Hence $f(t, g(t)) = y$.

If $\mathcal{F}$ is equicontinuous at (t_0, x_0) then using Proposition 2 we get that f and F are continuous at (t_0, x_0). So there exists a neighbourhood $H' \times V'(x_0)$ of the point (t_0, x_0) that the inclusion $F(H' \times V'(x_0)) \subset H \times W(f(t_0, x_0))$

holds. Let $(t', x') \in H' \times V'(x_0)$. Then $y' := f(t', x') \in W(f(t_0, x_0))$ and there exists an implicit function $g\colon H \times V(x_0)$ belonging to y'. Let

$$\phi\colon H \to V(x_0), \quad t \mapsto \begin{cases} g(t), & \text{if } t \neq t' \\ x', & \text{if } t = t'. \end{cases}$$

It is clear that ϕ is an implicit function belonging to the point y' and passing through the point (t', x').

(iii) Let $g\colon H \to V(x_0)$ be a function such that $g(h) = s$, g is continuous at h and $\forall t \in H\colon\ f(t, g(t)) = f(h, s)$. Let $W \in \mathcal{B}_Y$ be arbitrary and let $U \in \mathcal{B}_X$ be such that $\forall t \in H\colon\ f(t, .)(U(s)) \subseteqq W(f(t, s))$. Let the neighbourhood K of the point h be chosen so that the inclusions $g(K) \subseteqq U(s)$ and $K \subseteqq H$ hold. It will be shown that $f(., s)(K) \subseteqq W(f(h, s))$. Indeed, if $t \in K$ then for every $(s, x) \in U$ the set W contains the element $(f(t, s), f(t, x))$. On the other hand, if $t \in K$ then $(s, g(t)) \in U$, hence $t \in K \Rightarrow (f(t, s), f(t, g(t)) \in W$. As the equality $f(t, g(t)) = f(h, s)$ implies that $(f(h, s), f(t, s))$ is the element of W, it follows that for every t of the set K the point $f(t, s)$ is in $W(f(h, s))$. It means that $f(., s)$ is continuous at h. Using Proposition 2 we get that f is continuous at (h, s).

(iv) By Proposition 1 (ii) the function F is invertible. By Proposition 1 (iii) every implicit function is unique. As F is open at (t_0, x_0) it follows that F^{-1} is continuous at $F(t_0, x_0)$. Consequently, the function $F^{-1}(., f(t_0, x_0)) = (\mathrm{id}_T, g)$ is continuous at t_0, where g is the implicit function passing through the point (t_0, x_0). □

THEOREM 2 (Implicit Function Theorem for metrizable topological groups). *Let T be a topological space, G and Y be topological groups where the topology of G is defined by a translation invariant metric. Let us suppose that the topology of G is complete and the topology of Y is of Hausdorff-type. Let $X := B(x_0; r_0) \subseteqq G$ be a closed ball, $f\colon T \times X \to Y$ a function. Suppose that $f(., x_0)$ is continuous at t_0 and the mapping $A\colon G \to Y$ is continuous, additive and open. Suppose further that there is a number $k \in]0, 1[$ so that for every $B(x; r) \subseteqq X$ the inclusion $(A - f(t, .))(B(x; r)) \subseteqq A(B(x; kr)) - f(t, x)$ holds. Then*

(a) *For every neighbourhood W of x_0 there is a neighbourhood $H \times V$ of the point (t_0, x_0) that for every point (t, x) of $H \times V$ there is an implicit function $g\colon H \to W$ such that $f \circ (\mathrm{id}_T, g) = f(t, x)$ and $g(t) = x$.*

(b) *The following three statements are equivalent:*

(i) *Among the implicit functions passing through the point (t, x) there exists one which is continuous at t.*

(ii) *The function $f(., x)$ is continuous at t.*

(iii) *f is continuous at (t, x).*

(c) *If the mapping A is injective then the implicit functions are unique.*

PROOF. (a) By Theorem 3 of [1] the family of functions $\mathcal{F} = \{f(t, .) \mid t \in T\}$ is equiopen at every point $x \in X$. By Lemma 2 in [1] $\mathcal{F}$ is equicontinuous at every point $x \in X$ as the set W given in the proof of the Lemma depends only on the number r and on the mapping A. Using Theorem 1 (ii) we get our statement.

(b) By Theorem 1 (iv) we have that if there is an implicit function passing through the point (t,x), which is continuous at t, then $f(.,x)$ is continuous at t. Hence (i) implies (ii). By Proposition 2 it follows that (ii) implies (iii).

By (v) of Theorem 3 in [1] for every $t' \in T$ there is a right inverse $[f(t',.)]_r$ of $f(t',.)$ such that the following statement holds:

$$\forall r > 0:\quad B(x;2r) \subseteqq X \Rightarrow [f(t',.)]_r(f(t',x) + A(B(0;(1-k)r)) \subseteqq B(x;2r).$$

By Proposition 1 the function $F_r(t,y) := (t,[f(t,.)]_r(y))$ is a right inverse of the function $F = (\mathrm{id}_T, f)$ on a neighbourhood of the point $(t_0, f(t_0,x_0))$. It will be shown that F_r is continuous at $(t, f(t,x))$. Indeed, as f is continuous at (t,x) the following statements hold:

$$\forall r > 0 \quad \exists r' > 0 \quad \exists U \ni) t\colon\ r' < r \text{ and}$$
$$U \times (f(t,x) + A\Big(0; \frac{1-k}{4} r'\Big)\Big) \subseteqq F(U \times (B(x;r)).$$

If $(t',y) \in U \times (f(t,x) + A\Big(0; \dfrac{1-k}{4} r\Big)\Big)$ then

$$F_r(t',y) = (t,[f(t',.)]_r(y)) \in U \times B(x;r)$$

since $f(t',x) \in f(t,x) + A\Big(B\Big(0; \dfrac{1-k}{4} r\Big)\Big)$ and since

$$y \in f(t',x) + A\Big(B\Big(0; \frac{1-k}{4} r\Big)\Big) \Rightarrow y \in f(t,x) + A\Big(B\Big(0; \frac{1-k}{2} r\Big)\Big)$$
$$\Rightarrow [f(t',.)]_r(y) \in B(x;r).$$

Hence

$$F_r\Big(U \times \Big(f(t,x) + A\Big(0; \frac{1-k}{4} r'\Big)\Big) \subseteqq U \times B(x;r),$$

that is F_r is continuous at $(t, f(t,x))$. By Proposition 1 (iii) the implicit function g determined by the equality $F_r(., f(t,x)) = (\mathrm{id}_U, g)$ passes through the point (t,x) and it is obviously continuous at the point t. Hence (iii) implies (i).

(c) If A is injective then by Theorem 3 (vi) in [1] the set $H \times V$ can be chosen so that the functions $f(t,.)$ are homeomorphisms. By (iv) of Theorem 1 the implicit functions are unique. □

Let T be a topological space, X and Y be Banach spaces, $U \subseteqq X$ an open set, $(t_0,x_0) \in T \times X$ and $f\colon T \times U \to Y$ be a function. Suppose that f satisfies the following properties:

$$(*)\qquad \begin{array}{ll} (1) & D_2 f\colon T\times U \to \mathcal{L}(X,Y) \text{ exists and is continuous at } (t_0,x_0);\\ (2) & f(.,x_0) \text{ is continuous at } t_0;\\ (3) & D_2 f(t_0,x_0)\colon X\to Y \text{ is surjective.}\end{array}$$

THEOREM (Szilágyi [2]). *If f satisfies the properties $(*)$ then for every neighbourhood V of x_0 there is a neighbourhood H of t_0 and a function $g\colon H\to V$ such that $f\circ(\mathrm{id}_H, g) = f(t_0,x_0)$.*

It is easy to see that the theorem above is equivalent to the following statement.

There exists a neighbourhood H of t_0 and an implicit function $g\colon H\to Y$ passing through the point (t_0,x_0) which is continuous at t_0.

THEOREM 3. *If f satisfies the properties $(*)$ then for every neighbourhood W of x_0 there exists a neighbourhood $H\times V$ of (t_0,x_0) such that*

(i) $\forall (t,x)\in H\times V \quad \exists g\colon H\to W\colon f\circ(\mathrm{id}_H,g) = f(t,x)$ *and* $g(t)=x$.

(ii) *f is continuous at (t_0,x_0) and there is an implicit function passing through the point (t_0,x_0) and continuous at t_0.*

(iii) *For every $(t,x)\in H\times V$ the following three statements are equivalent:*
1. $\exists g\colon H\to W\colon f\circ(\mathrm{id}_H,g)=f(t,x)$, $g(t)=x$ *and g is continuous at t;*
2. *$f(.,x)$ is continuous at t;*
3. *f is continuous at (t,x).*

(iv) *If $D_2 f(t_0,x_0)$ is injective then the implicit functions $g\colon H\to W$ are unique.*

(v) *If T is an open subset of a Banach space and there exists an implicit function g passing through the point (t_0,x_0) which is differentiable at t_0 then f is differentiable at (t_0,x_0).*

PROOF. By Theorem 2 the statements (i), (ii), (iii) and (iv) follow from the fact that if $A := D_2 f(t_0,x_0)$ then the neighbourhood $H\times V$ of (t_0,x_0) can be chosen so that

$$\forall t\in H \quad \forall B(x;r)\subseteqq V\colon\ (A-f(t,.))(B(x;r))\subseteqq A(B(x;kr)) - f(t,x)$$

holds, where $k\in]0,1[$. Indeed, as A is open there exists a number ϱ such that $B(0;\varrho)\subseteqq Y$ and $B(0;\varrho)\subseteqq A(B(0;1))$.

As A is a strict derivative of the function $f(t_0,.)$ at the point x_0 it follows that

$$\forall \varepsilon>0\ \ \exists r_0>0\colon\ x,y\in B(x_0,r_0)\Rightarrow$$

$$\|(A-f(t_0,.))(x)-(A-f(t_0,.))(y)\| < \frac{\varepsilon}{2}\|x-y\|.$$

The continuity of the function D_2f at (t_0, x_0) implies that the neighbourhood $H \times V$ can be chosen so that if $(t,x) \in H \times V$ then $\|D_2f(t,x) - A\| < \frac{\varepsilon}{2}$ holds and

$$\forall (t,x) \in H \times V: \ \ \|(A - f(t,.))(x) - (A - f(t,.))(y)\| < \varepsilon \|x - y\|.$$

The number ε does not depend on ϱ, hence it is possible to suppose $k := \dfrac{\varepsilon}{\varrho} <$ < 1. It means that

$$(A - f(t,.))(B(x;r)) \subseteqq B(A(x); \varepsilon r) - f(t,x) \subseteqq A(B(x;kr)) - f(t,x)$$

if $B(x;r) \subseteqq B(x_0, r_0)$.

(iv) It is known that if D_2f is continuous at (t_0, x_0) and $D_1f(t_0, x_0)$ exists then $Df(t_0, x_0)$ exists, too. So it is enough to show that $D_1f(t_0, x_0)$ exists. Indeed,

$$D_1f(t_0, x_0) = -D_2f(t_0, x_0) \circ Dg(t_0)$$

as the following easy computation shows:

$$\begin{gathered}
\|f(.,x_0)(t) - f(.,x_0)(t_0) - (-D_2f(t_0,x_0) \circ Dg(t_0))(t - t_0)\| \leqq \\
\|f(t, g(t_0)) - f(t, g(t)) - D_2f(t, g(t_0))(g(t_0) - g(t))\| + \\
\|D_2f(t, x_0)(x_0 - g(t)) - D_2f(t_0, x_0) \circ Dg(t_0)(t - t_0)\| \leqq \\
\varepsilon_1 \|g(t_0) - g(t)\| + \|D_2f(t, x_0)(x_0 - g(t)) - D_2f(t_0, x_0)(x_0 - g(t))\| + \\
\|D_2f(t_0, x_0)(x_0 - g(t)) - D_2f(t_0, x_0) \circ Dg(t_0)(t - t_0)\| \leqq \\
\varepsilon_1 \|g(t_0) - g(t)\| + \varepsilon_2 \|g(t_0) - g(t)\| + \\
\|D_2f(t_0, x_0)\| \|g(t_0) - g(t) - Dg(t_0)(t - t_0)\| \leqq \\
(\varepsilon_1 + \varepsilon_2) \|g(t_0) - g(t)\| + \|D_2f(t_0, x_0)\| \varepsilon_3 \|t - t_0\| \leqq \\
(\varepsilon_1 + \varepsilon_2)(\varepsilon_4 + \|Dg(t_0)\|) \|t - t_0\| + \|D_2f(t_0, x_0)\| \varepsilon_3 \|t - t_0\| \leqq \varepsilon \|t - t_0\|,
\end{gathered}$$

if $\|t - t_0\|$ is "small enough", since

$$\|g(t) - g(t_0)\| - \|Dg(t_0)\| \|t - t_0)\| \leqq \|g(t) - g(t_0) - Dg(t_0)(t - t_0)\| \leqq$$

$$\varepsilon_4 \|t - t_0\|,$$

which implies

$$\|g(t) - g(t_0)\| \leqq (\varepsilon_4 + \|Dg(t_0)\|) \|t - t_0\|.$$

□

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VESZPRÉMI EGYETEM
MATEMATIKAI ÉS SZÁMÍTÁSTUDOMÁNYI TANSZÉK
POSTAFIÓK 158
H–8201 VESZPRÉM
HUNGARY

slezakb@almos.vein.hu

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QUASI-UNIFORM COMPLETENESS AND NEIGHBOURHOOD FILTERS

J. DEÁK

Abstract

Smyth [14] calls a quasi-uniform space *complete* if each round "Cauchy" filter is the neighbourhood filter of a unique point. We characterize the spaces that are complete, or can be completed, in this sense, and also show what the definition yields with a different meaning of Cauchy.

§ 0. Preliminaries

In the most usual terminology, a filter $\mathfrak{f}$ in a quasi-uniform space $(X,\mathcal{U})$ is *Cauchy* if for any $U\in\mathcal{U}$ there is an $x\in X$ with $Ux\in\mathfrak{f}$. Let us call $\mathfrak{f}$ *weakly hereditarily Cauchy* (*wh Cauchy*; "Cauchy" in [14]) if for any $S\in\mathfrak{f}$, $\mathfrak{f}\mid S$ is $\mathcal{U}\mid S$-Cauchy. Equivalently: for any $U\in\mathcal{U}$ and $S\in\mathfrak{f}$, there is an $x\in S$ with $Ux\in\mathfrak{f}$. (The adverb "weakly" is used here because $\mathfrak{f}$ was called *hereditarily Cauchy* in [8] if $\mathfrak{f}\mid S$ is $\mathcal{U}\mid S$-Cauchy for each $S\in\sec\mathfrak{f}$.) $\mathcal{U}$ is *Smyth complete*[1] ("complete" in [14]) if each round wh Cauchy filter is the neighbourhood filter of a unique point, where *round* means that for any $S\in\mathfrak{f}$ there are $U\in\mathcal{U}$ and $T\in\mathfrak{f}$ with $U[T]\subset S$, and the neighbourhood filters are to be understood in the topology $\mathcal{U}^{\mathrm{tp}}$ of $\mathcal{U}$. The curious part about this definition is that not the convergence or clustering of some filters (or filter pairs) is required. Since to be a neighbourhood filter is a strong assumption, few spaces are Smyth complete; in fact, we shall see that the class of spaces admitting a Smyth complete extension is small, too. Nevertheless, using neighbourhood filters when defining completeness is not a bad idea: it will turn out that a well-known notion of completeness can be characterized this way.

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[1] This is only a preliminary form of the definition. The final version will be given in the next paragraph.

Neighbourhood filters are clearly round and wh Cauchy, so any Smyth complete space is T_0. In order to encompass non-T_0 spaces, too, let us change the definition: a quasi-uniformity is *Smyth complete* if each round wh Cauchy filter is a neighbourhood filter. This condition is satisfied iff the T_0 reflexion is Smyth complete according to the original definition. (There are, of course, two other ways to deal with this detail: accept that, unlike for other notions of completeness, Smyth complete spaces have to be T_0, or work only in the class of T_0 quasi-uniformities.)

Let us recall some definitions from [5, 6]. The filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ in $(X, \mathcal{U})$ is *Cauchy* if for any $U \in \mathcal{U}$ there are $S_i \in \mathfrak{f}^i$ with $S_{-1} \times S_1 \subset U$; *round* if $\mathfrak{f}^i$ is $\mathcal{U}^i$-round ($i = \pm 1$); *linked* if $S_{-1} \cap S_1 \neq \emptyset$ whenever $S_i \in \mathfrak{f}^i$. The neighbourhood filter pair $\mathfrak{n}^0(x) = (\mathfrak{n}^{-1}(x), \mathfrak{n}^1(x))$ of $x \in X$, convergence and cluster points are to be understood in the bitopology $(\mathcal{U}^{-\mathrm{tp}}, \mathcal{U}^{\mathrm{tp}})$ of $\mathcal{U}$, where $\mathcal{U}^{-\mathrm{tp}} = (\mathcal{U}^{-1})^{\mathrm{tp}}$; e.g. $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ *converges* to x if $\mathfrak{f}^i \supset \mathfrak{n}^i(x)$ ($i = \pm 1$). Using the terminology of [7], $\mathcal{U}$ is L-*complete* (known also as bicomplete [10], pair complete [12], doubly complete [2], complete [13]) if any linked Cauchy filter pair is convergent. (Equivalently: any linked Cauchy filter pair has a cluster point; any linked round Cauchy filter pair is convergent; the uniformity $\mathcal{U}^{\mathrm{s}} = \sup\{\mathcal{U}^{-1}, \mathcal{U}\}$ is complete in the usual sense.)

§ 1. Characterizing L-completeness with neighbourhood filters

L-completeness can be characterized in a way similar to the definition of Smyth completeness. For this purpose, we call a filter L-*Cauchy* if it is the second member of a linked Cauchy filter pair. (Recall for comparison that a filter is D-*Cauchy* if it is the second member of an arbitrary Cauchy filter pair.) $\mathcal{U}(\mathfrak{f})$ will denote the $\mathcal{U}$-envelope of $\mathfrak{f}$, i.e. the finest round filter coarser than $\mathfrak{f}$; $\mathcal{U}(\mathfrak{f}) = \{U[S] \colon S \in \mathfrak{f},\ U \in \mathcal{U}\}$.

PROPOSITION. *The following conditions are equivalent for a quasi-uniformity:*

(i) *it is* L-*complete;*

(ii) *each round linked Cauchy filter pair is a neighbourhood filter pair;*

(iii) *each round* L-*Cauchy filter is a neighbourhood filter.*

PROOF. (i) $\Rightarrow$ (ii). Let $\mathfrak{f}^0 = (\mathfrak{f}^{-1}, \mathfrak{f}^1)$ be a round linked Cauchy filter pair. Such filter pairs are minimal Cauchy ([6] 7.14). If $\mathfrak{f}^0$ converges to x then it is finer than the Cauchy filter pair $\mathfrak{n}^0(x)$, thus $\mathfrak{f}^0 = \mathfrak{n}^0(x)$ by the minimality.

(ii) $\Rightarrow$ (iii). Let $\mathfrak{f}^1$ be a round L-Cauchy filter, and take $\mathfrak{f}^{-1}$ such that $\mathfrak{f}^0 = (\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is a linked Cauchy filter pair. We may assume that $\mathfrak{f}^{-1}$ is $\mathcal{U}^{-1}$-round, since $\mathfrak{f}^0$ remains linked and Cauchy if $\mathfrak{f}^{-1}$ is replaced by $\mathcal{U}^{-1}(\mathfrak{f}^{-1})$. Thus $\mathfrak{f}^0$ is round, hence a neighbourhood filter pair, and so $\mathfrak{f}^1$ is a neighbourhood filter.

(iii) $\Rightarrow$ (ii). Let $\mathfrak{f}^0 = (\mathfrak{f}^{-1}, \mathfrak{f}^1)$ be a round linked Cauchy filter pair, and pick a point x with $\mathfrak{f}^1 = \mathfrak{n}^1(x)$. Now $\mathfrak{n}^0(x) = (\mathfrak{n}^{-1}(x), \mathfrak{f}^1)$ is also Cauchy, and then so is $\mathfrak{h}^0 = (\mathfrak{f}^{-1} \cap \mathfrak{n}^{-1}(x), \mathfrak{f}^1)$. Again by [6] 7.14, $\mathfrak{f}^0$ and $\mathfrak{n}^0(x)$ are minimal Cauchy, so they coincide with the coarser Cauchy filter pair $\mathfrak{h}^0$, therefore $\mathfrak{f}^0 = \mathfrak{n}^0(x)$.

(ii) $\Rightarrow$ (i). Evident, since the round linked Cauchy filter pairs are now convergent. □

In spite of the simplicity of the proof, (iii) is somewhat surprising, since a similar one-sided modification of the original definition, namely that each L-Cauchy filter is convergent (or has a cluster point), leads to a non-equivalent notion (called complete in [2], C-complete in [1], Cs-complete in [7]).

§ 2. A characterization of Smyth completeness

We are going to characterize Smyth completeness in the following way: a space is Smyth complete iff it is L-complete and satisfies an additional condition. One more property of filters and a lemma will be needed: a filter $\mathfrak{f}$ in a quasi-uniform space $(X,\mathcal{U})$ is *stable* [3] if $\bigcap_{S\in\mathfrak{f}} U[S] \in \mathfrak{f}$ for any $U \in \mathcal{U}$.

LEMMA. *A filter is* L*-Cauchy iff it is stable and wh Cauchy.*

PROOF. According to [6] Lemma 7.17, if $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is linked and Cauchy then $\mathfrak{f}^i$ is $\mathcal{U}^i$-stable ($i = \pm 1$); hence L-Cauchy filters are stable. They are evidently wh Cauchy, too.

Conversely, let $\mathfrak{f}$ be a stable wh Cauchy filter, and define

$$\mathfrak{g} = \{\{x \in X:\ Ux \in \mathfrak{f}\} : U \in \mathcal{U}\}.$$

$(\mathfrak{g}, \mathfrak{f})$ is a Cauchy filter pair (it is enough to know for this that $\mathfrak{f}$ is stable and Cauchy, see [8] Remark 5.2). It is clear from the definition of a wh Cauchy filter that $(\mathfrak{g}, \mathfrak{f})$ is now linked. Hence $\mathfrak{f}$ is L-Cauchy. □

Two simple observations will also be needed: $\mathfrak{f}$ is stable iff $\mathcal{U}(\mathfrak{f})$ is stable ([6] 7.18); $\mathfrak{f}$ is wh Cauchy iff $\mathcal{U}(\mathfrak{f})$ is wh Cauchy (straightforward).

PROPOSITION. *A quasi-uniformity is Smyth complete iff it is* L*-complete and each wh Cauchy filter is stable.*

(Compare the additional condition with the following useful notion: a quasi-uniformity is *stable* if each D-Cauchy filter is stable, see [9, 4, 11, 8].)

PROOF. Assume that $(X,\mathcal{U})$ is Smyth complete. If $\mathfrak{f}$ is a round L-Cauchy filter then it is round and wh Cauchy, hence a neighbourhood filter. So (iii) from the Proposition in § 1 holds, i.e. $\mathcal{U}$ is L-complete. Take a wh Cauchy filter $\mathfrak{f}$. Then $\mathcal{U}(\mathfrak{f})$ is round and wh Cauchy, so it is a neighbourhood filter. Consequently, $\mathcal{U}(\mathfrak{f})$ is stable, and so is $\mathfrak{f}$ itself.

Conversely, assume that each wh Cauchy filter is stable. By the Lemma, each wh Cauchy filter is now L-Cauchy, so L-completeness implies Smyth completeness by the Proposition in § 1. □

§ 3. Which spaces have Smyth complete extensions?

The answer to this question is independent of what we exactly mean by an extension: it can be simply a superspace, or the other extreme, namely a *firm extension* [5], which means that the original space is $\mathcal{U}^{\mathrm{stp}}$-dense in the extension.

THEOREM[2]. *The following conditions are equivalent for a quasi-uniformity:*

(i) *it has a Smyth complete superspace;*
(ii) *it has a Smyth complete firm extension;*
(iii) *each wh Cauchy filter is stable.*

PROOF. (ii) $\Rightarrow$ (i). Evident.

(i) $\Rightarrow$ (iii). Let $(Y,\mathcal{V})$ be a Smyth complete superspace of $(X,\mathcal{U})$. Each of the following assumptions implies the next one: $\mathfrak{f}$ is a wh Cauchy filter in $(X,\mathcal{U})$; $\mathfrak{f}'=\mathrm{fil}_Y\,\mathfrak{f}$ is wh Cauchy; $\mathcal{V}(\mathfrak{f}')$ is wh Cauchy; $\mathcal{V}(\mathfrak{f}')$ is a neighbourhood filter; $\mathcal{V}(\mathfrak{f}')$ is stable; $\mathfrak{f}'$ is stable; $\mathfrak{f}$ is stable.

(iii) $\Rightarrow$ (ii). Any space $(X,\mathcal{U})$ has an L-complete firm extension $(Y,\mathcal{V})$ (e.g. [10] 3.33). By the Proposition in § 2 and the observations made before it, the proof will be complete if we show that each round wh Cauchy filter is stable in $(Y,\mathcal{V})$. So take a round wh Cauchy filter $\mathfrak{h}$ in $(Y,\mathcal{V})$. We are going to check that a) $\mathfrak{f}=\mathfrak{h}\,|\,X$ is wh Cauchy; b) $\mathfrak{h}=\mathcal{V}(\mathrm{fil}_Y\,\mathfrak{f})$. Then $\mathfrak{f}$ is stable by (iii), $\mathrm{fil}_Y\,\mathfrak{f}$ is $\mathcal{V}$-stable, and so is $\mathfrak{h}$.

a) Let $S\in\mathfrak{f}$. As $\mathfrak{h}$ is round, there is a $\mathcal{V}^{\mathrm{tp}}$-open $G\in\mathfrak{h}$ such that $G\cap X\subset S$. Given $V\in\mathcal{V}$, pick $a\in G$ with $Va\in\mathfrak{h}$, then a $W\in\mathcal{V}$ such that

$$Wa\subset G,\qquad W\subset V. \tag{1}$$

As X is $\mathcal{V}^{\mathrm{stp}}$-dense, there is an

$$x\in Wa\cap W^{-1}a\cap X. \tag{2}$$

Now $x\in S$ and $a\in Wx\subset Vx$, therefore $V^2x\in\mathfrak{h}$, $(V^2\,|\,X)x\in\mathfrak{f}$, implying that $\mathfrak{f}$ is wh Cauchy, since the entourages $V^2\,|\,X$ $(V\in\mathcal{V})$ form a base for $\mathcal{U}$.

b) $\mathfrak{h}\subset\mathrm{fil}_Y\,\mathfrak{f}$, so, $\mathfrak{h}$ being round, we have also $\mathfrak{h}\subset\mathcal{V}(\mathrm{fil}_Y\,\mathfrak{f})$. To prove the reverse inclusion, let $S\in\mathcal{V}(\mathrm{fil}_Y\,\mathfrak{f})$. Then there are $G\in\mathfrak{h}$ and $V\in\mathcal{V}$ such that $V[G\cap X]\subset S$; it can be assumed that G is $\mathcal{V}^{\mathrm{tp}}$-open. We claim that $G\subset V[G\cap X]$; thus clearly $S\in\mathfrak{h}$.

Indeed, let $a\in G$, and choose again $W\in\mathcal{V}$ and $x\in X$ satisfying (1) and (2). Now $x\in G\cap X$ and $a\in Wx\subset Vx\subset V[G\cap X]$. □

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[2] (iii) $\Rightarrow$ (i) is an immediate consequence of [15] Proposition 21, which states that a certain construction, completing each quasi-uniformity but leading out of the category of quasi-uniformities, remains within this category iff each round wh Cauchy filter is stable. Our proof will be different.

REMARKS. a) Using the Theorem, one can easily give quasi-uniformities that have no Smyth completion: e.g. the Euclidean neighbourhood filter of 0 is a wh Cauchy non-stable filter in the usual quasi-uniformity of the Sorgenfrey line.

b) In the realm of T_0 quasi-uniformities, the L-complete firm extension is unique up to isomorphism (e.g. [10] 3.34), so the same can be said about Smyth complete firm extensions (assuming that they exist).

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H-1364 BUDAPEST
HUNGARY

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WELL-DEFINED SOLUTION PROBLEM FOR CERTAIN GENERALIZED BOUNDARY VALUE PROBLEM

E. MIELOSZYK

Abstract

In works [5, 6, 7] the author considers certain generalized boundary value problem. This paper is partly the generalization of the results presented in [7]. We shall deal with the generalized boundary value problem

$$S^2x = f, \qquad s_q x = x_{0,q}, \qquad Ax = x_A$$

when it is a well defined solution problem.

Let the operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$ be given (the definition and properties of the operational calculus can be found in [1], [7]), and let $L^1 \subset L^0$.

Definition 1 (see [2]). L^n is the set defined by the following recurrent sequence

$$L^n := \{x \in L^{n-1} : Sx \in L^{n-1}\}, \qquad n = 2, 3, \ldots .$$

Assume that the operation $A \colon L^2 \to \operatorname{Ker} S$ is linear where $\operatorname{Ker} S := \{c \in L^1 : Sc = 0\}$. With the given assumption we shall consider the abstract differential equation

$$S^2 x = f \tag{1}$$

with the conditions

$$s_q x = x_{0,q}, \tag{2}$$

$$Ax = x_A, \tag{3}$$

where $x \in L^2$, $f \in L^0$, $x_{0,q}, x_A \in \operatorname{Ker} S$. For this purpose we shall define in Theorem 2 the operational calculus induced by the given operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$ and the operation A.

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THEOREM 1. *The abstract differential equation*

$$S^n x = f$$

with the conditions

$$s_q S^i x = x_{i,q} \quad \text{for } i = 0, 1, \ldots, n-1,$$

where $x \in L^n$, $f \in L^0$, $x_{i,q} \in \operatorname{Ker} S$ *for* $i = 0, 1, \ldots, n-1$, *has exactly one solution*

$$x = \sum_{i=0}^{n-1} T_q^i x_{i,q} + T_q^n f.$$

The proof of the theorem can be found in [2].

THEOREM 2. *If* $AT_q c = c$ *for* $c \in \operatorname{Ker} S$, *then the operations* $\widetilde{S}$, $\widetilde{T}_q$, $\widetilde{s}_q$ *defined by the formulas*

$$\widetilde{S}x := S^2 x, \qquad x \in L^2, \tag{4}$$

$$\widetilde{T}_q f := T_q^2 f - T_q A T_q^2 f, \qquad f \in L^0, \tag{5}$$

$$\widetilde{s}_q x := s_q x + T_q A x - T_q A S_q x, \qquad x \in L^2 \tag{6}$$

satisfy the definition of the operational calculus. The operation $\widetilde{S}$ *is the derivative, the operation* $\widetilde{T}_q$ *is the integral and the operation* $\widetilde{s}_q$ *is the limit condition.*

PROOF. $\widetilde{S}$, $\widetilde{T}_q$, $\widetilde{s}_q$ are linear operations. We shall prove that

$$\widetilde{S}\widetilde{T}_q f = f, \qquad f \in L^0 \qquad \text{and}$$

$$\widetilde{T}_q \widetilde{S} x = x - \widetilde{s}_q x, \qquad x \in L^2.$$

Now, directly from the definition of $\widetilde{S}$ and $\widetilde{T}_q$ and from the properties of the operations S and T we get

$$\widetilde{S}\widetilde{T}_q f = S^2 T_q^2 f - S^2 T_q A T_q^2 f = f - S A T_q^2 f = f$$

for $f \in L^0$.

We must also show that the following formula holds:

$$\widetilde{T}_q \widetilde{S} x = x - \widetilde{s}_q x, \qquad x \in L^2.$$

For this purpose we shall compute $\widetilde{T}_q \widetilde{S} x$:

$$\widetilde{T}_q \widetilde{S} x = T_q^2 S^2 x - T_q A T_q^2 S^2 x =$$

$$= x - s_q x - T_q s_q S x - T_q A x + T_q A s_q x + T_q A T_q s_q S x.$$

Using the fact that $AT_q c = c$ for $c \in \operatorname{Ker} S$ we obtain

$$\widetilde{T}_q \widetilde{S} x = x - s_q x - T_q A x + T_q A s_q x.$$

From the last relation it follows that

$$\widetilde{T}_q \widetilde{S} x = x - \widetilde{s}_q x, \qquad x \in L^2.$$

The above mentioned facts imply that the operations $\widetilde{S}$, $\widetilde{T}_q$ and $\widetilde{s}_q$ satisfy the definition of the operational calculus. □

COROLLARY 1. *The operation $\widetilde{T}_q$ is an injection. The operation $\widetilde{s}_q$ is a projection of L^2 onto* $\operatorname{Ker} S^2$.

The following formula holds

$$\widetilde{s}_{q_1} \widetilde{T}_{q_2} f = [T_{q_2}^2 - T_{q_1}^2] f - [T_{q_2} A T_{q_2}^2 - T_{q_1} A T_{q_1}^2] f.$$

THEOREM 3. *If $AT_q c = c$ for $c \in \operatorname{Ker} S$ then the abstract differential equation* (1) *with the conditions* (2), (3) *has exactly one solution defined by the formula*

$$x = x_{0,q} + T_q x_A - T_q A x_{0,q} + T_q^2 f - T_q A T_q^2 f. \tag{7}$$

PROOF. Theorem 3 follows directly from Theorems 1 and 2 because the conditions (2) and (3) are equivalent to the limit condition

$$\widetilde{s}_q x = x_{0,q} + T_q x_A - T_q A x_{0,q},$$

which corresponds to the derivative $\widetilde{S} = S^2$. □

NOTE. In [8] a formula analogous to formula (7) is given in the case when A is a limit condition. These formulas have been got using various methods.

COROLLARY 2. *If $AT_q c = c$ for $c \in \operatorname{Ker} S$ then the space L^2 is isomorphic to the direct sum* $\operatorname{Ker} S^2 \oplus L^0$, *i.e.*

$$L^2 \approx \operatorname{Ker} S^2 \oplus L^0.$$

From Theorems 1 and 2 immediately follows

THEOREM 4. *The abstract differential equation*

$$S^{2n} x = f$$

with the conditions

$$s_q S^{2i} x = x_{2i,q}$$

$$A S^{2i} x = x_{2i,A}$$

for $i = 0, 1, \ldots, n-1$, *where* $x \in L^{2n}$, $f \in L^0$ *and* $x_{2i,q}, x_{2i,A} \in \operatorname{Ker} S$ *for* $i = 0, 1, \ldots, n-1$ *has exactly one solution given in the form*

$$x = \sum_{i=0}^{n-1} (T_q^2 - T_q A T_q^2)^i (x_{2i,q} + T_q x_{2i,A} - T_q A x_{2i,q}) + (T_q^2 - T_q A T_q^2)^n f$$

as far as $A T_q c = c$ *for* $c \in \operatorname{Ker} S$.

COROLLARY 3. *If* $A T_q c = c$ *for* $c \in \operatorname{Ker} S$ *then the space* L^{2n} *is isomorphic to the direct sum*

$$L^0 \oplus \underbrace{\operatorname{Ker} S^2 \oplus \operatorname{Ker} S^2 \oplus \ldots \oplus \operatorname{Ker} S^2}_{n\text{-times}},$$

i.e.

$$L^{2n} \approx L^0 \oplus \operatorname{Ker} S^2 \oplus \operatorname{Ker} S^2 \oplus \ldots \oplus \operatorname{Ker} S^2.$$

Let L^0 be a Mikusiński space with a partial order $\leqq$ and a modulus $|\cdot|$. (The definition of the Mikusiński space can be found in [2].) Moreover, let the convergence be defined like in the work [1], [6].

THEOREM 5. *If the operations* T_q *and* A *are non-negative operations then the operation* $\tilde{T}_q$ *is regular. The operation* $\tilde{T}_q$ *is continuous, i.e.*

$$g_n \to g \quad \textit{implies} \quad \tilde{T}_q g_n \to \tilde{T}_q g.$$

PROOF. T_q is regular operation because it is the difference of non-negative operations. The second part of the theorem follows from the fact that the integral $\tilde{T}_q$, as a regular operation, is continuous. □

NOTE. The definition of the non-negative and regular operations can be found in [1].

DEFINITION 2 (see [1]). We say that the problem of solving the abstract differential equation

$$Sx = f, \qquad x \in L^1, \quad f \in L^0 \tag{8}$$

with the condition

$$s_q x = x_0, \qquad x_0 \in \operatorname{Ker} S \tag{9}$$

is a well defined solution problem if the equation (8) with the condition (9) has unique solution and if the conditions

$$Sx_n = f_n, \quad s_q x_n = x_{n,0}, \quad f_n \to f, \quad x_{n,0} \to x_0$$

imply

$$x_n \to x \quad \text{where} \quad Sx = f \quad \text{and} \quad s_q x = x_0.$$

Using the definition 2 of the well defined solution problem and Theorem 5, we may formulate the following statement.

THEOREM 6. *If T_q and A are non-negative operations and $AT_q c = c$ for $c \in \operatorname{Ker} S$ then the problem of solving the abstract differential equation* (1) *with the conditions* (2), (3) *is a well defined solution problem.*

EXAMPLE A. In [7] it has been proved that the partial differential equation

$$(10) \qquad \Big(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\Big)^2 u = f$$

with the conditions

$$(11) \qquad \{u(x_1, x_2, x_3, \dots, x_{n-1}, x_n^0)\} = \{\varphi(x_1, \dots, x_{n-1})\},$$

$$(12) \qquad \Big\{\int_{x_n^1}^{x_n^2} u\Big(x_1 - \frac{b_1}{b_n}(x_n - \tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau\Big) d\tau\Big\} = \psi,$$

where $u \in C^3(R^{n-1} \times \langle x_n^1, x_n^2\rangle, R)$, $f \in C^1(R^{n-1} \times \langle x_n^1, x_n^2\rangle, R)$, $\varphi \in C^3(R^{n-1}, R)$, $\psi \in \operatorname{Ker}\Big(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\Big)$, $b_i \in R$ for $i = 1, 2, \dots, n$, $b_n \neq 0$, $x_n^0 \in \langle x_n^1, x_n^2\rangle$ has only one solution.

Note that the operation A connected with the condition (12) is defined by formula

$$Au :=$$

$$:= \frac{2b_n}{(x_n^2)^2 - (x_n^1)^2 - 2x_n^0 x_n^2 + 2x_n^0 x_n^1} \times$$

$$\times \Big\{\int_{x_n^1}^{x_n^2} u\Big(x_1 - \frac{b_1}{b_n}(x_n - \tau), \dots, x_{n-1} - \frac{b_{n-1}}{b_n}(x_n - \tau), \tau\Big) d\tau\Big\},$$

so, for $c \in \operatorname{Ker}\Big(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}\Big)$, $AT_{x_n^0} c = c$ holds, where

$$T_{x_n^0}\{f(x_1, x_2, \dots, x_n)\} :=$$

$$:=\left\{\frac{1}{b_n}\int\limits_{x_n^0}^{x_n} f\Big(x_1-\frac{b_1}{b_n}(x_n-\tau),\dots,x_{n-1}-\frac{b_{n-1}}{b_n}(x_n-\tau),\tau\Big),d\tau\right\}$$

(cf. [3]). (The operation $T_{x_n^0}$ is the integral for the derivative $S:=\sum\limits_{i=1}^{n} b_i\frac{\partial}{\partial x}$, see [3].)

Introduce the order

$$\{f(x_1,x_2,\dots,x_n)\}=f\geqq 0 \quad\text{iff}\quad f(x_1,\dots,x_n)\geqq 0$$

for $(x_1,x_2,\dots,x_n)\in R^{n-1}\times\langle x_n^1,x_n^2\rangle$, and the modulus

$$|f|=\{|f(x_1,x_2,\dots,x_n)|\}$$

in the space $C^0(R^{n-1}\times\langle x_n^1,x_n^2\rangle,R)$. With such introduced order and modulus the space $C^0(R^{n-1}\times\langle x_n^1,x_n^2\rangle,R)$ is a Mikusiński space. Therefore, from Theorem 6, it follows that the problem of solving partial differential equation (10) with the conditions (11), (12) is well defined if $x_n^0=x_n^1$ and $b_n>0$.

Example B. The difference equation

$$\{x_{k+2}+(-p_{k+1}-p_k)x_{k+1}+p_k^2x_k\}=\{f_k\} \tag{13}$$

with conditions

$$x_{k_0}=\alpha, \tag{14}$$

$$\Big(\prod_{i=k_1}^{k_2-1}p_i\Big)\Big(\sum_{i=k_0}^{k_2-1}\frac{1}{p_i}\Big)x_{k_1}+\Big(\sum_{i=k_0}^{k_1-1}\frac{1}{p_i}\Big)x_{k_2}=\beta \tag{15}$$

where $\{x_k\}$, $\{f_k\}$, $\{p_k\}$ are real sequences, $p_k\neq 0$ for $k=0,1,\dots$, $k_0<k_1<$ $<k_2$, $\alpha,\beta\in R$ has exactly one solution.

By [4] we can rewrite the equation (13) in the form

$$\Delta_{p_k}^2\{x_k\}=\{f_k\},$$

where

$$\Delta_{p_k}\{x_k\}:=\{x_{k+1}-p_kx_k\}.$$

In this case the operation A is defined by the formula

$$A\{x_k\}:=\left\{\frac{1}{2}\Big(\prod_{i=0}^{k-1}p_i\Big)\frac{\Big(\prod\limits_{i=k_1}^{k_2-1}p_i\Big)\Big(\sum\limits_{i=k_0}^{k_2-1}\frac{1}{p_i}\Big)x_{k_1}+\Big(\sum\limits_{i=k_0}^{k_1-1}\frac{1}{p_i}\Big)x_{k_2}}{\Big(\prod\limits_{i=0}^{k_2-1}p_i\Big)\Big(\sum\limits_{i=k_0}^{k_1-1}\frac{1}{p_i}\Big)\Big(\sum\limits_{i=k_0}^{k_2-1}\frac{1}{p_i}\Big)}\right\}, \tag{16}$$

$$\prod_{i=0}^{-1} p_i := 1$$

from which it can be seen that for $\{c_k\} \in \operatorname{Ker} \Delta_{p_k}$

$$AT_{p_k,k_0}\{c_k\} = \{c_k\},$$

where the integral T_{p_k,k_0} corresponding to the derivative Δ_{p_k} is defined by the formula

$$T_{p_k,k_0}\{f_k\} := \left\{\prod_{i=0}^{k-1} p_i\right\} T_{k_0} \left\{ \frac{f_k}{\prod\limits_{i=0}^{k} p_i} \right\} \tag{17}$$

(see [4]). In the formula (17) the operation T_{k_0} is defined in the following way

$$T_{k_0}\{f_k\} := \begin{cases} 0 & \text{for } k = k_0 \\ f_{k_0} + f_{k_0+1} + \ldots + f_{k-1} & \text{for } k_0 < k \\ -f_{k_0-1} - f_{k_0-2} - \ldots - f_k & \text{for } k_0 > k \end{cases}$$

(cf. [4]). From Theorem 3 it follows that the equation (13) with the conditions (14), (15) has exactly one solution given in the form (7) in which instead of T_q and A we have to take the corresponding operations defined by the formula (16) and (17). Moreover, we need to take

$$x_{0,q} = \alpha \left\{ \frac{\prod\limits_{i=0}^{k-1} p_i}{\prod\limits_{i=0}^{k_0-1} p_i} \right\} \qquad \text{for } x_{0,q},$$

and

$$x_A = \frac{\frac{1}{2}\beta \prod\limits_{i=0}^{k-1} p_i}{\left(\prod\limits_{i=0}^{k_2-1} p_i\right)\left(\sum\limits_{i=k_0}^{k_2-1} \frac{1}{p_i}\right)\left(\sum\limits_{i=k_0}^{k_1-1} \frac{1}{p_i}\right)} \qquad \text{for } x_A.$$

It is known that the space of real sequences with the order

$$\{x_k\} = x \geqq 0 \quad \text{iff} \quad x_k \geqq 0 \quad \text{for} \quad k = 0, 1, 2, \ldots$$

and the modulus

$$|x| = \{|x_k|\}$$

is a Mikusiński space.

From Theorem 6 it follows that the problem (13), (14), (15) is well defined when $k_0 = 0$ and $p_k > 0$ for $k = 0, 1, 2, \ldots$.

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POLITECHNIKA GDAŃSKA
WYDZIAŁ FIZYKI TECHNICZNEJ I
MATEMATYKI STOSOWANEJ
UL. G. NARUTOWICZA 11/12
PL–80-952 GDAŃSK-WRZESZCZ
POLAND

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MULTIPLE SIMULATED FACTORIZATIONS

S. SZABÓ

Abstract

By an earlier result of [1] if a finite abelian group is the direct product of subsets constructed from subgroups changing at most one element, then one of the factors must be a subgroup. The paper answers the question how far this result can be extended if the product is not direct but gives the elements of the group with the same multiplicity.

1. Introduction. Let G be a finite abelian group written multiplicatively with identity element e. Let $A_1, \dots, A_n$ be subsets of G. If each element g of G is expressible in precisely k ways in the form

$$g = a_1 \cdots a_n, \quad a_1 \in A_1, \dots, a_n \in A_n,$$

then we say that the product $A_1 \cdots A_n$ is a *k-factorization* of G. When the product $A_1 \cdots A_n$ is direct it is a 1-factorization of G and will be called simply a factorization of G. If $e \in A_1 \cap \dots \cap A_n$, then the subsets and the factorization are called *normed*. In the most commonly used factorizations the factors are subgroups. If G is a direct product of cyclic subgroups of order $q_1, \dots, q_s$, then we say that G is of *type* $(q_1, \dots, q_s)$. We would like to point out that the type of a group is not uniquely determined. For example both (2,3) and (6) are types of the cyclic group of order six. However, this ambiguity will not cause any trouble.

In [1] the case where the factors are close to being subgroups was considered. The subset A_i of G is said to be *simulated* if there is a subgroup H_i of G such that $|A_i| = |H_i| \geqq 3$ and $|A_i \cap H_i| + 1 \geqq |A_i|$. It was proved that if $A_1 \cdots A_n$ is a normed factorization of G, then one of the factors is a subgroup of G, that is, there is an i, $1 \leqq i \leqq n$ such that $A_i = H_i$. This paper deals with normed k-factorizations of G by simulated subsets. We know that in the $k = 1$ case one of the factors is a subgroup. The next example shows that this is not necessarily the case when $k \geqq 2$.

Let G be of type $(2, 2, 2)$ with basis x, y, z. The subsets

$$A_1 = \{e, y, z, xyz\}, \quad A_2 = \{e, x, z, xyz\}, \quad A_3 = \{e, x, z, zy\}$$

are simulated. The corresponding subgroups are

$$H_1 = \{e, y, z, yz\}, \quad H_2 = \{e, x, z, xz\}, \quad H_3 = \{e, x, z, xz\}.$$

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The product $A_1A_2A_3$ is an 8-factorization of G and none of the factors is a subgroup of G.

We are interested in the following problems. Characterize (1) all the finite abelian groups (2) all the k, n values for which in any k-factorization $A_1 \cdots A_n$ of G by simulated factors one of the factors is a subgroup.

This paper contains the solution of these problems. The finite cyclic groups and the group of type $(2,2)$ and only these groups are solutions of the first problem. The following k, n values and only these are the solution of the second problem: $n \leqq 2$, $k \geqq 1$; $n=3$, k odd; $n \geqq 4$, $k=1$.

2. Character test. The group ring $Z(G)$ provides an adequate tool to deal with k-factorizations. We identify the subset A of G with the element

$$\overline{A} = \sum_{a \in A} a$$

of $Z(G)$. The product $A_1 \cdots A_n$ is a k-factorization of G if and only if

$$k\overline{G} = \overline{A}_1 \cdots \overline{A}_n.$$

Rédei [2] developed a method using characters of G to study factorizations of G. Characters of $Z(G)$ which are linear extensions of characters of G can be used to study multiple factorizations.

Let χ_i be the ith character of G and let g_j be the jth element of G. Let $A, B \in Z(G)$

$$A = \sum_{j=1}^{|G|} a_j g_j, \qquad B = \sum_{j=1}^{|G|} b_j g_j.$$

If $\chi_i(A) = \chi_i(B)$ for each i, $1 \leqq i \leqq |G|$, then

$$\sum_{j=1}^{|G|} (a_j - b_j)\chi_i(g_j) = 0.$$

By the standard orthogonality relations the matrix $\chi_i(g_j)$ is orthogonal and so its determinant is nonzero. Hence it follows that $a_j - b_j = 0$ for each j, $1 \leqq j \leqq |G|$. Therefore $A = B$. Thus $k\overline{G} = \overline{A}_1 \cdots \overline{A}_n$ if and only if $\chi(k\overline{G}) = \chi(\overline{A}_1 \cdots \overline{A}_n)$ for each character χ of G. For the principal character this reduces to $k|G| = |A_1| \cdots |A_n|$. For nonprincipal characters we have $0 = \chi(\overline{A}_1) \cdots \chi(\overline{A}_n)$. Therefore the product $A_1 \cdots A_n$ is a k-factorization of G if and only if $k|G| = |A_1| \cdots |A_n|$, and for each nonprincipal character χ of G there is an i, $1 \leqq i \leqq n$ such that $\chi(\overline{A}_i) = 0$.

Let A_i be a simulated subset of G and let H_i be the corresponding subgroup of G. There are elements $a_i \in A_i$ and $h_i \in H_i$ such that $a_i \notin H_i$ and $h_i \notin A_i$. Further there is an element $d_i \in G$ for which $a_i = h_i d_i$. The subgroup H_i and the element d_i do not determine A_i uniquely since there are different choices for h_i. For our purposes each of these choices plays the same role, so we will not specify A_i more closely than giving H_i and d_i.

Note that

$$\chi(\overline{H}_i) - \chi(\overline{A}_i) = \chi(h_i) - \chi(h_i d_i) = \chi(h_i)(1 - \chi(d_i)).$$

From this it follows that if $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$, then $\chi(\overline{A}_i) = 0$. We show that if $\chi(\overline{A}_i) = 0$, then $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$. To show this suppose that $\chi(\overline{A}_i) = 0$ and $\chi(\overline{H}_i) \neq 0$. Since H_i is a subgroup of G $\chi(h) = 1$ for each $h \in H_i$, this leads to the contradiction

$$3 \leqq |H_i| = \chi(\overline{H}_i) = \chi(\overline{H}_i) - \chi(\overline{A}_i) = \chi(h_i)(1 - \chi(d_i)) \leqq 2.$$

Thus $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$.

We formulate the character test we will use to study k-factorizations. If the product $A_1 \cdots A_n$ is a k-factorization of G, then $k|G| = |A_1| \cdots |A_n|$ and for each nonprincipal character χ of G there is an i, $1 \leqq i \leqq n$ for which $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$. Conversely, if there are subgroups $H_1, \dots, H_n$ and elements $d_1, \dots, d_n$ of G such that $k|G| = |H_1| \cdots |H_n|$ and for each nonprincipal character χ of G there is an i, $1 \leqq i \leqq n$ for which $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$, then there is a k-factorization $A_1 \cdots A_n$ of G.

3. Characterization in terms of groups. Let $A_1 \cdots A_n$ be a k-factorization of the finite abelian group G by simulated factors. Since H_i is a subgroup of G, $|H_i| = |A_i|$ is a divisor of $|G|$. Thus if G is of type $(2,2)$, then $|A_i| = 4$ and so $A_i = G$. If G is cyclic, then there is a character χ of G whose kernel is $\{e\}$. There is an i, $1 \leqq i \leqq n$ such that $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$. From $\chi(d_i) = 1$ it follows $d_i = e$. Therefore $A_i = H_i$.

For the remaining finite abelian groups we exhibit a k-factorization by nonsubgroup simulated factors. We start with three special cases.

Suppose G is of type $(2, 2, p^\alpha)$, where p is a prime and $\alpha \geqq 1$. Let x, y, z be a basis of G and define A_1, A_2, A_3 by

$$\begin{aligned} H_1 &= \langle y, z \rangle, & d_1 &= x, \\ H_2 &= \langle x, z \rangle, & d_2 &= y, \\ H_3 &= \langle x, z \rangle, & d_3 &= xy. \end{aligned}$$

Clearly A_i is not a subgroup of G since $d_i \notin H_i$. Let χ be a nonprincipal character of G. We show that there is an i, $1 \leqq i \leqq 3$ such that $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$. Note that one of $\chi(x)$, $\chi(y)$, $\chi(xy)$ is always 1. If $\chi(x) = 1$, then $\chi(\overline{H}_1) \neq 0$ only if $\chi(y) = 1$ and $\chi(z) = 1$. But this is impossible since χ is not the principal character. The $\chi(y) = 1$ and $\chi(xy) = 1$ cases are similar.

Suppose that G is of type $(2, 2^\alpha)$, $\alpha \geqq 2$ with basis x, y. Define the simulated subsets $A_1, \ldots, A_{\alpha+1}$ by

$$\begin{aligned}
H_1 &= \langle xy \rangle, & d_1 &= y, \\
H_2 &= \langle xy \rangle, & d_2 &= x, \\
H_3 &= \langle y \rangle, & d_3 &= xy, \\
H_4 &= \langle y \rangle, & d_4 &= xy^2, \\
&\vdots & &\vdots \\
H_{\alpha+1} &= \langle y \rangle, & d_{\alpha+1} &= xy^{2^{\alpha-1}}
\end{aligned}$$

Clearly $A_i \neq H_i$ since $d_i \notin H_i$. Let χ be a nonprincipal character of G. If $\chi(y) = 1$, then $\chi(\overline{H}_1) \neq 0$ only if $\chi(xy) = 1$. But now $\chi(x) = 1$ and χ is the principal character. The $\chi(x) = 1$ case is similar. Thus we may suppose that $\chi(x) \neq 1$ and $\chi(y) \neq 1$. One of $\chi(xy), \chi(xy^2), \ldots, \chi(xy^{2^{\alpha-1}})$ is 1. In any case there is an i, $1 \leqq i \leqq \alpha + 1$ such that $\chi(\overline{H}_i) = 0$ and $\chi(d_i) = 1$.

Suppose that G is of type (p^α, p^β), where p is an odd prime $\alpha \geqq 1$ and $\beta \geqq 1$. Let x, y be a basis of G and define $A_1, \ldots, A_{p+1}$ by

$$\begin{aligned}
H_1 &= \langle x \rangle, & d_1 &= y^{p^{\beta-1}}, \\
H_2 &= \langle y \rangle, & d_2 &= x^{p^{\alpha-1}}, \\
H_3 &= \langle y \rangle, & d_3 &= x^{p^{\alpha-1}} y^{p^{\beta-1}}, \\
H_4 &= \langle y \rangle, & d_4 &= x^{p^{\alpha-1}} y^{2p^{\beta-1}}, \\
&\vdots & &\vdots \\
H_{p+1} &= \langle y \rangle, & d_{p+1} &= x^{p^{\alpha-1}} y^{(p-1)p^{\beta-1}}.
\end{aligned}$$

Let χ be a nonprincipal character of G. If $\chi(y^{p^{\beta-1}}) = 1$, then $\chi(\overline{H}_1) \neq 0$ only if $\chi(x) = 1$. Hence $\chi(x^{p^{\alpha-1}}) = 1$. Now $\chi(\overline{H}_2) \neq 0$ only if $\chi(y) = 1$. But this is impossible since χ is not the principal character. The case when $\chi(y^{p^{\beta-1}}) = 1$ is similar. Thus we may assume that $\chi(x^{p^{\alpha-1}}) \neq 1$ and $\chi(y^{p^{\beta-1}}) \neq 1$. In this case one of $\chi(x^{p^{\alpha-1}} y^{ip^{\beta-1}})$ must be 1. Therefore $\chi(\overline{H}_{i+2}) = 0$ and $\chi(d_{i+2}) = \chi(x^{p^{\alpha-1}} y^{ip^{\beta-1}}) = 1$.

Finally, let G be of type $(q_1, \ldots, q_s)$. By the fundamental theorem of finite abelian groups we may assume that $q_1, \ldots, q_s$ are prime powers. If G is noncyclic and is not of type $(2, 2)$, then $G = K \otimes L$, where K is one of the types

$$(2, 2, p^\alpha),\ \alpha \geqq 1, \quad (2, 2^\beta),\ \beta \geqq 2, \quad (p^\alpha, p^\beta),\ p \geqq 3,\ \alpha \geqq 1, \beta \geqq 1.$$

By the previous constructions K has a k-factorization $A_1 \cdots A_n$ by nonsubgroup simulated factors. Define the simulated subset A_{n+1} by $H_{n+1} = L$ and $d_{n+1} \in K \setminus \{e\}$. Clearly $A_1 \cdots A_n A_{n+1}$ is a k-factorization of G.

4. Characterization in terms of n and k. Let $A_1 \cdots A_n$ be a k-factorization of the finite abelian group G by simulated factors. For each nonprincipal character χ of G there is an i, $1 \leqq i \leqq n$ such that $\chi(\overline{H}_i)=0$ and $\chi(d_i)=1$. From this it follows that an equation $X_1 \cdots X_n = 0$ holds in $Z(G)$, where $X_i = \overline{H}_i$ or $X_i = e - d_i$.

If $n=1$, then $e-d_1=0$ and so $d_1=e$. Thus $A_1=H_1$ no matter what is the value of k.

Suppose $n=2$. From $(e-d_1)(e-d_2)=0$ it follows that $e+d_1d_2=d_1+d_2$ and so $e=d_1$ or $e=d_2$. Thus $A_1=H_1$ or $A_2=H_2$ independently of the value of k.

Let $n=3$. Consider the equation

$$(e-d_1)(e-d_2)(e-d_3)=0. \tag{1}$$

This equation may still hold after cancelling some factors. If such a cancellation is possible, say $e-d_1=0$ or $(e-d_1)(e-d_2)=0$, then as before $d_i=e$ and $A_i=H_i$ for some i. Thus we may assume that no factor can be cancelled from (1) without destroying the equation. From (1) after multiplying out we have

$$e+d_1d_2+d_1d_3+d_2d_3=d_1+d_2+d_3+d_1d_2d_3.$$

Some term on the right side is equal to e. Clearly $e=d_1d_2d_3$ since $e-d_i \neq 0$. Now

$$d_1d_2+d_1d_3+d_2d_2=d_1+d_2+d_3.$$

Some term on the left-hand side is equal to d_1. Once again $e-d_i \neq 0$ and so $d_2d_3=d_1$. Continuing in this way we have $d_1^2=d_2^2=e$ and $d_3=d_1d_2$.

Consider $\overline{H}_1(e-d_2)(e-d_3)=0$. After multiplying out,

$$\overline{H}_1(e+d_2d_3)=\overline{H}_1(d_2+d_3), \quad \text{or} \quad \overline{H}_1(e+d_1)=\overline{H}_1(d_2+d_3).$$

G is a disjoint union of cosets modulo H_1. Hence $H_1=H_1d_2$ or $H_1=H_1d_3$. Thus $d_2 \in H_1$ or $d_3 \in H_1$. Similarly, from $(e-d_1)\overline{H}_2(e-d_3)=0$ we have $d_1 \in H_2$ or $d_3 \in H_2$, and from $(e-d_1)(e-d_2)\overline{H}_3=0$ we have $d_1 \in H_3$ or $d_2 \in H_3$.

Suppose that two different H's contain the same d, say $d_3 \in H_1$ and $d_3 \in H_2$. Now $\langle d_3 \rangle \subset H_1 \cap H_2$ and so $H_1H_2H_3$ is a k-factorization of G with even k.

Suppose that no two H's contain the same d, say $d_3 \in H_1$, $d_1 \in H_2$, $d_2 \in H_3$. Now $\langle d_1d_2 \rangle \subset H_1 \cap H_2H_3$ shows that in the k-factorization $H_1H_2H_3$ of G, k is even.

We exhibit an example to show that each even k occurs. Let G be of type $(2,2,2,2,2t)$, $t \geqq 1$ with basis x, y, u, v, w. Define the nonsubgroup simulated subsets A_1, A_2, A_3 by

$$\begin{aligned} H_1 &= \langle y, u \rangle, & d_1 &= x, \\ H_2 &= \langle x, v \rangle, & d_2 &= y, \\ H_3 &= \langle x, w \rangle, & d_3 &= xy. \end{aligned}$$

Let χ be a nonprincipal character of G.

If $\chi(x)=1$, then $\chi(\overline{H}_1)\neq 0$ only if $\chi(y)=1$ and $\chi(u)=1$. Now $\chi(\overline{H}_2)\neq$ $\neq 0$ only if $\chi(v)=1$. Since χ is not the principal character $\chi(w)\neq 1$ and so $\chi(\overline{H}_3)=0$ and $\chi(d_3)=\chi(xy)=1$.

If $\chi(y)=1$, then $\chi(\overline{H}_2)\neq 0$ only if $\chi(x)=1$, and this reduces the problem to the previous case.

If $\chi(xy)=1$, then $\chi(\overline{H}_3)\neq 0$ only if $\chi(x)=1$, which reduces the problem again to the first case.

In the resulting k-factorization $A_1A_2A_3$ of G the multiplicity is $k=2t$, $t\geqq 1$. We point out that these k-factorizations can be extended to each $n\geqq 3$. Indeed, let $G=K\otimes L$, where K is of type $(2,2,2,2,2t)$ and define A_4 by $H_4=L$ and $d_4\in K\setminus\{e\}$. Clearly, A_4 is not a subgroup of G and $A_1A_2A_3A_4$ is a k-factorization of G.

Let $n=4$ and let G be of type $(4,4,2,t)$, $t\geqq 3$ with basis x,y,z,u. Define A_1,A_2,A_3,A_4 by

$$\begin{aligned} H_1&=\langle y\rangle, & d_1&=x^2,\\ H_2&=\langle u\rangle, & d_2&=y^2,\\ H_3&=\langle z,u\rangle, & d_3&=x^2y^2,\\ H_4&=\langle x\rangle, & d_4&=u. \end{aligned}$$

Let χ be a nonprincipal character of G.

If $\chi(x^2)=1$, then $\chi(\overline{H}_1)\neq 0$ only if $\chi(y)=1$. Now $\chi(y^2)=1$ and $\chi(\overline{H}_2)\neq 0$ only if $\chi(u)=1$. Then $\chi(\overline{H}_4)\neq 0$ only if $\chi(x)=1$. Since χ is not the principal character, $\chi(z)\neq 1$ and this gives $\chi(\overline{H}_3)=0$. We see that $\chi(d_3)=\chi(x^2y^2)=1$ also holds.

If $\chi(y^2)=1$, then $\chi(\overline{H}_2)\neq 0$ only if $\chi(u)=1$. Further $\chi(\overline{H}_4)\neq 0$ only if $\chi(x)=1$, which reduces the problem to the previous case.

If $\chi(x^2y^2)=1$, then $\chi(\overline{H}_3)\neq 0$ only if $\chi(z)=1$ and $\chi(u)=1$. Now $\chi(\overline{H}_4)\neq 0$ only if $\chi(x)=1$, and this reduces the problem to the first case.

In this k-factorization $A_1A_2A_3A_4$ the multiplicity $k=t$ and $t\geqq 3$. As we have already seen, this factorization can be extended to each $n\geqq 4$ and this completes the characterization.

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BUDAPESTI MŰSZAKI EGYETEM
ÉPÍTŐMÉRNÖKI KAR
MATEMATIKA TANSZÉK
H–1521 BUDAPEST
EGRY JÓZSEF U. 1
HUNGARY

Present address:

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BAHRAIN
P.O. BOX 32038
ISA TOWN
STATE OF BAHRAIN

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ON SCHUR COMPLEMENTS IN A CONJUGATE EP MATRIX

AR. MEENAKSHI and R. INDIRA

Abstract

Necessary and sufficient conditions are determined for a Schur complement in a Conjugate EP matrix to be Conjugate EP. As an application a decomposition of a partitioned matrix into a sum of Conjugate EP matrices is obtained and the question of when sum of Conjugate EP matrices is as well a conjugate EP is studied.

1. Introduction

For an $m \times n$ complex matrix A, let $\overline{A}$, A^T, A^* denote conjugate, transpose and conjugate transpose of A, respectively. Any matrix X satisfying $AXA = A$ is called a generalized inverse of A and is denoted by A^-. A^+ is the Moore–Penrose inverse of A [6]. A square complex matrix A is said to be conjugate EP (Con-EP) if $N(A) = N(A^T)$ or $AA^+ = \overline{A^+A}$, where $N(A)$ is the null space of A. A is said to be Con-EP$_r$ if A is Con-EP and $\mathrm{rk}(A) = r$, where $\mathrm{rk}(A)$ is the rank of A [4]. In particular, if A is real, then Con-EP coincides with that of EP [7].

Throughout this paper we are concerned with $n \times n$ matrices M partitioned in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

where A and D are square matrices. With respect to this partitioning a Schur complement of A in M is a matrix of the form $M/A = D - CA^-B$. For properties of Schur complements one may refer to [1], [2], [3]. M/A is independent of the choice of A^- if and only if

$$N(A) \subseteq N(C) \text{ and } N(A^*) \subseteq N(B^*) \text{ or, equivalently,} \tag{1.2}$$

$$C = CA^-A \text{ and } B = AA^-B \text{ for every } A^- \text{ of } A. \tag{1.3}$$

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If a partitioned matrix M of the form (1.1) is Con-EP, then, in general, M/A is not Con-EP. Here we determine necessary and sufficient conditions for M/A to be Con-EP analogous to that of the results found in Meenakshi [3]. We shall deal with the cases when $\mathrm{rk}(M) \neq \mathrm{rk}(A)$ and $\mathrm{rk}(M) = \mathrm{rk}(A)$. As an application, a decomposition of a partitioned matrix into a sum of conjugate EP matrices is obtained. Further it is shown that in a Con-EP_r matrix, every principal submatrix of rank r is Con-EP_r.

Results

THEOREM 1. *Let M be a matrix of the form* (1.1) *with $N(A) \subseteqq N(C)$ and $N(M/A) \subseteqq N(B)$. Then the following are equivalent:*

(i) *M is a Con-EP matrix;*

(ii) *A and M/A are Con-EP, $N(A^*) \subseteqq N(B^*)$ and $N((M/A)^*) \subseteqq N(C^*)$;*

(iii) *Both of the matrices*

$$\begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix}$$

are Con-EP.

PROOF. (i) $\Rightarrow$ (ii) Since $N(A) \subseteqq N(C)$ and $N(M/A) \subseteqq N(B)$ by using (1.3) we can write M as $M = PQL$ where

$$P = \begin{bmatrix} I & 0 \\ CA^- & I \end{bmatrix}, \quad Q = \begin{bmatrix} I & B(M/A)^- \\ 0 & I \end{bmatrix}, \quad L = \begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix}.$$

Clearly P and Q are nonsingular. $M = PQL \Rightarrow M^- = L^- Q^{-1} p^{-1}$ is one choice of M^-. M is Con-EP $\Rightarrow N(M^T) = N(M) \Rightarrow M^T = M^T M^- M$

$$\Rightarrow \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} L^- L =$$
$$= \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} A^- & 0 \\ 0 & (M/A)^- \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix} =$$
$$= \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} A^-A & 0 \\ 0 & (M/A)^- M/A \end{bmatrix}$$

$\Rightarrow A^T = A^T A^- A$; by (1.2) and (1.3) we get $N(A) \subseteq N(A^T)$ and $\mathrm{rk}(A^T) = \mathrm{rk}(A) \Rightarrow N(A) = N(A^T) \Rightarrow A$ is Con-EP. $B^T = B^T A^- A \Rightarrow N(A^T) = N(A) \subseteqq \subseteqq N(B^T)$. Hence $N(A^*) \subseteqq N(B^*)$. Now, $C^T = C^T (M/A)^- M/A$ and $D^T = = D^T (M/A)^- M/A$. After substituting $D = M/A + (BA^- C)$ and using $C^T = C^T (M/A)^- M/A$ in $D^T = (M/A)^T + (BA^- C)^T$ we get $(M/A)^T (M/A)^- M/A$

which implies $N(M/A) \subseteqq N(M/A)^T$ and using $\mathrm{rk}(M/A) = \mathrm{rk}(M/A)^T$, we get $N(M/A) = N(M/A)^T$. Thus M/A is Con-EP. From

$$C^T = C^T (M/A)^- M/A$$

we get, $N(M/A) \subseteqq N(C^T)$. Using M/A is Con-EP, this reduces to $N((M/A)^T) \subseteqq N(C^T) \Rightarrow N((M/A)^*) \subseteqq N(C^*)$. Thus M is Con-EP $\Rightarrow A$ and M/A are Con-EP, $N(A^*) \subseteqq N(B^*)$ and $N((M/A)^*) \subseteqq N(C^*)$. Thus (ii) holds.

(ii) $\Rightarrow$ (i) Since $N(A) \subseteqq N(C)$, $N(A^*) \subseteqq N(B^*)$, $N(M/A) \Rightarrow N(B)$ and $N((M/A)^*) \subseteqq N(C^*)$ hold according to the assumption, by applying (v) of Theorem 1 of [1], M^+ is given by the formula,

$$M^+ = \begin{bmatrix} A^+ + A^+B(M/A)^+CA^+ & -A^+B(M/A)^+ \\ -(M/A)^+CA^+ & (M/A)^+ \end{bmatrix}. \tag{2.1}$$

By (1.2) M/A is invariant for every choice of A^-. Hence $M/A = D - CA^+B$. Further using $C = M/A(M/A)^+C$ and $B = AA^+B$, MM^+ is reduced to the form

$$MM^+ = \begin{bmatrix} AA^+ & 0 \\ 0 & M/A(M/A)^+ \end{bmatrix}.$$

By using $B = B(M/A)^+M/A$ and $C = CA^+A$, M^+M is reduced to the form

$$M^+M = \begin{bmatrix} A^+A & 0 \\ 0 & (M/A)^+M/A \end{bmatrix}.$$

A and M/A are Con-EP $\Rightarrow AA^+ = \overline{A^+A}$ and

$$(M/A)(M/A)^+ = \overline{(M/A)^+(M/A)} \Rightarrow MM^+ = \overline{M^+M} \Rightarrow M \text{ is Con-EP.}$$

Thus (i) holds.

(ii) $\Rightarrow$ (iii)

$$M_1 = \begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix}$$

is lower block triangular with $N(A) \subseteqq N(C)$; $N((M/A)^*) \subseteqq N(C^*)$. Hence by Corollary 7 of [5],

$$M_1^+ = \begin{bmatrix} A^+ & 0 \\ -(M/A)^+CA^+ & (M/A)^+ \end{bmatrix}$$

$$M_1M_1^+ = \begin{bmatrix} AA^+ & 0 \\ 0 & M/A(M/A)^+ \end{bmatrix}.$$

Now A and M/A are Con-EP $\Rightarrow M_1 = \begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix}$ is Con-EP.

In a similar way by applying Corollary 7 of [5] to the upper block triangular matrix

$$M_2 = \begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix}$$

with $N(A^*) \subseteq N(B^*)$; $N(M/A) \subseteq N(B)$, we have

$$M_2^+ = \begin{bmatrix} A^+ & -A^+B(M/A)^+ \\ 0 & (M/A)^+ \end{bmatrix}$$

and

$$M_2M_2^+ = \begin{bmatrix} AA^+ & 0 \\ 0 & M/A(M/A)^+ \end{bmatrix}.$$

By using A and M/A are Con-EP we get $M_2 = \begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix}$ is Con-EP. Thus (iii) holds.

(iii) $\Rightarrow$ (ii) Since $N(A) \subseteq N(C)$ by (1.3), $C = CA^-A$. Hence $M_1 = PL$ where

$$P = \begin{bmatrix} I & 0 \\ CA^- & I \end{bmatrix}, \qquad L = \begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix}, \qquad M_1 = \begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix}.$$

Suppose M_1 is Con-EP then as in the proof of (i) $\Rightarrow$ (ii) using $M_1^T = M_1^TM_1^-M_1$ we can prove that A and M/A are Con-EP, $N(M/A)^* \subseteq N(C^*)$. Similarly, $N(M/A) \subseteq N(B) \Rightarrow B = B(M/A)^-M/A \Rightarrow$

$$M_2 = \begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix} = \begin{bmatrix} I & B(M/A)^- \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix} = QL.$$

M_2 is Con-EP $\Rightarrow$ A and M/A are Con-EP and $N(A^*) \subseteq N(B^*)$. Thus $\begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix}$ and $\begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix}$ are both Con-EP $\Rightarrow$ A and M/A are Con-EP, $N(A^*) \subseteq N(B^*)$ and $N(M/A)^*) \subseteq N(C^*)$. Thus (ii) holds. The proof is complete.

THEOREM 2. *Let M be a matrix of the form* (1.1) *with* $N(A^*) \subseteq N(B^*)$ *and* $N((M/A)^*) \subseteq N(C^*)$. *Then the following are equivalent:*

(i) *M is Con-EP matrix;*

(ii) *A and M/A are Con-EP, further $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(B)$;*

(iii) *Both the matrices*

$$\begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix} \quad \textit{are Con-EP}.$$

PROOF. Theorem 2 follows immediately from Theorem 1 and from the fact that M is Con-EP $\Leftrightarrow$ M^* is Con-EP.

In the special case when $B = C^*$ we get the following.

COROLLARY 1. *Let* $M = \begin{bmatrix} A & C^* \\ C & D \end{bmatrix}$ *with* $N(A) \subseteqq N(C)$ *and* $N(M/A) \subseteqq$ $\subseteqq N(C^*)$, *then the following are equivalent:*

(i) M *is a Con-EP matrix;*

(ii) A *and* M/A *are Con-EP matrices;*

(iii) *The matrix*

$$\begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix} \quad \textit{is Con-EP.}$$

REMARK 1. The conditions taken on M in the above theorems are essential. This is illustrated in the following example. Let

$$M = \left[\begin{array}{cc:cc} i & i & i & i \\ i & i & i & i \\ \hdashline i & i & i & i \\ i & i & i & 0 \end{array}\right].$$

M is symmetric and of rank 2, hence M is Con-EP_2.

$$A = B = C = \begin{bmatrix} i & i \\ i & i \end{bmatrix} \quad \text{is Con-EP,}$$

$$M/A = D - CA^+B = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} \quad \text{is Con-EP.}$$

Clearly $N(A) \subseteqq N(C)$; $N(A^*) \subseteqq N(B^*)$. But $N(M/A) \nsubseteqq N(B)$ and $N((M/A)^*)$ $\nsubseteqq N(C^*)$. $\begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix}$ and $\begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix}$ are not Con-EP. Thus Theorems 1 and 2 as well as Corollary 1 fail.

REMARK 2. We conclude from Theorems 1 and 2 that for a Con-EP matrix M of the form (1.1) the following are equivalent:

(2.2) $$N(A) \subseteqq N(C), \quad N(M/A) \subseteqq N(B),$$

(2.3) $$N(A^*) \subseteqq N(B^*), \quad N(M/A)^* \subseteqq N(C^*).$$

However, this fails if we omit the condition that M is Con-EP. For example, let

(2.4) $$M = \left[\begin{array}{cc:cc} i & i & i & 0 \\ i & i & i & 0 \\ \hdashline i & i & i & i \\ 0 & 0 & 0 & 0 \end{array}\right]$$

be not Con-EP. Here $A=\begin{bmatrix} i & i \\ i & i \end{bmatrix}$ is Con-EP. Since $\text{rk}(A)=1$,

$$A^{+}=\frac{A^{*}}{\text{tr}\,(A^{*}A)}=-\frac{1}{4}A,$$

$$B=-C^{*}=\begin{bmatrix} i & 0 \\ i & 0 \end{bmatrix}.$$

$N(A)\subseteqq N(C)$ and $N(A^{*})\subseteqq N(B^{*})$. Hence M/A is independent of the choice of A^{-}.

$$M/A=D-CA^{+}B=\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$$

is not Con-EP. $N(M/A)^{*}\subseteqq N(C^{*})$ but $N(M/A)\nsubseteqq N(B)$. Thus (2.3) holds while (2.2) fails.

REMARK 3. It has been proved in [1] that for any matrix M its Moore–Penrose inverse M^{+} is given by the formula (2.1) if and only if both (2.2) and (2.3) hold (cf. (v) of Theorem 1 in [1]). From Remark 2, it is clear that for a Con-EP matrix M, (2.1) gives M^{+} if and only if either (2.2) or (2.3) holds.

COROLLARY 2. *Let M be a matrix of the form* (1.1) *for which M^{+} given by the formula* (2.1) *is Con-EP if and only if both A and M/A are Con-EP.*

PROOF. This follows from Theorem 1 by taking into account Remark 3.

THEOREM 3. *Let M be of the form* (1.1) *with* $\text{rk}(M)=\text{rk}(A)=r$. *Then M is a Con-EP$_r$ matrix if and only if A is Con-EP$_r$ and $CA^{+}=(A^{+}B)^{T}$.*

PROOF. Since $\text{rk}(M)=\text{rk}(A)=r$, by Corollary after Theorem 1 in [2], then $N(A)\subseteqq N(C)$; $N(A^{*})\subseteqq N(B^{*})$ and $M/A=D-CA^{+}B=0$. By (1.3) these relations are equivalent to $C=CA^{+}A$; $B=AA^{+}B$ and $D=CA^{+}B$. Let us consider the matrices

$$P=\begin{bmatrix} I & 0 \\ CA^{+} & I \end{bmatrix},\quad Q=\begin{bmatrix} I & A^{+}B \\ 0 & I \end{bmatrix},\quad L=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

P and Q are nonsingular, and, by the assumption $CA^{+}=(A^{+}B)^{T}$, it holds $P=Q^{T}$. Therefore M can be factorized as $M=PLP^{T}$. Since A is Con-EP$_r$, consequently L is as well as Con-EP$_r$. Now by Result 2.2 of [4] M is Con-EP$_r$.

Conversely, let us assume that M is Con-EP$_r$. Since $M=PLQ$, one choice of M^{-} is $Q^{-1}\begin{bmatrix} A^{+} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$. M is Con-EP$_r \Rightarrow N(M^{T})=N(M)$, by (1.3) $M^{T}=M^{T}M^{-}M$. That is

$$\begin{bmatrix} A^{T} & C^{T} \\ B^{T} & D^{T} \end{bmatrix}=\begin{bmatrix} A^{T} & C^{T} \\ B^{T} & D^{T} \end{bmatrix}\begin{bmatrix} A^{+}A & A^{+}B \\ 0 & 0 \end{bmatrix}$$

(or) equivalently, $A^T = A^T A^+ A$ and $C^T = A^T A^+ B$. Since $\mathrm{rk}(A^T) = \mathrm{rk}(A) = r$, from $A^T = A^T A^+ A$ we get $N(A^T) = N(A)$. Thus A is Con-EP$_r$, which implies $AA^+ = \overline{A^+A} = \overline{(A^+A)^*)} = (A^+A)^T$. Taking into account $C^T = A^T A^+ B$, we have $CA^+ = (A^+B)^T AA^+ = B^T(A^+)^T AA^+ = B^T(A^+)^T(A^+A)^T = B^T(A^+AA^+)^T = B^T(A^+)^T = (A^+B)^T$. The Theorem is proved.

In the special case when A is nonsingular, A is automatically EP$_r$ and Theorem 3 reduces to the following:

COROLLARY 3. *Let M be of the form* (1.1) *with A nonsingular and* $\mathrm{rk}(M) = \mathrm{rk}(A)$. *Then M is a Con-EP matrix* $\Leftrightarrow CA^+ = (A^+B)^T$.

COROLLARY 4. *Let M be an $n \times n$ matrix of rank r. Then M is Con-EP$_r$ $\Leftrightarrow$ every principal submatrix of rank r is Con-EP$_r$.*

PROOF. Suppose M is a Con-EP$_r$ matrix. Let A be any principal submatrix of M such that $\mathrm{rk}(M) = \mathrm{rk}(A) = r$. Then there exists a permutation matrix P such that

$$\widehat{M} = PMP^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $\mathrm{rk}(M) = \mathrm{rk}(A) = r$. By Result 2.2 of [4], $\widehat{M}$ is Con-EP$_r$. Now we conclude from Theorem 3 that A is Con-EP$_r$ as well. Since A was arbitrary, it follows that every principal submatrix of rank r is Con-EP$_r$. The converse is obvious.

REMARK 4. Theorem 3 fails if we relax the condition on rank of M. For example let us consider the matrix (2.4) in Remark 2, $\mathrm{rk}(M) = 2 \neq \mathrm{rk}(A)$. M is not Con-EP. However, $A = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$ is Con-EP.

$$A^+ = -\frac{1}{4}A; \quad CA^+ = (A^+B)^T.$$

Thus the theorem fails.

APPLICATION. We give conditions under which a partitioned matrix is decomposed into complementary summands of Con-EP matrices. M_1 and M_2 are called *complementary summands of M* if $M = M_1 + M_2$ and $\mathrm{rk}(M) = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$.

THEOREM 4. *Let M be of the form* (1.1) *with* $\mathrm{rk}(M) = \mathrm{rk}(A) + \mathrm{rk}(M/A)$ *where $M/A = D - CA^+B$. If A and M/A are Con-EP matrices such that $CA^+ = (A^+B)^T$ and $B(M/A)^+ = ((M/A)^+C)^T$ then M can be decomposed into complementary summands of Con-EP matrices.*

PROOF. Let us consider the matrices

$$M_1 = \begin{bmatrix} A & AA^+B \\ CA^+A & CA^+B \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} 0 & (I - AA^+)B \\ C(I - A^+A) & M/A \end{bmatrix}.$$

Taking into account that $N(A) \subseteq N(CA^+A)$, $N(A^*) \subseteq N((AA^+B)^*)$ and $M_1/A = CA^+B - (CA^+A)A^-(AA^+B) = CA^+B - CA^+B = 0$, we obtain by Corollary after Theorem 1 in [1] that $\mathrm{rk}(M_1) = \mathrm{rk}(A)$. Since A is Con-EP and $(CA^+A)A^+ = CA^+ = (A^+B)^T = (A^+AA^+B)^T$ we have from Theorem 3 that M_1 is Con-EP. Since $\mathrm{rk}(M) = \mathrm{rk}(A) + \mathrm{rk}(M/A)$, Theorem 1 of paper [1] gives $N(M/A) \subseteq N(I - AA^+)B$; $N((M/A)^*) \subseteq N((C(I - A^+A))^*)$ and $(I - AA^+)B(M/A)^-C(I - A^+A) = 0$. Thus by the Corollary after Theorem 1 in [1], we have $\mathrm{rk}(M_2) = \mathrm{rk}(M/A)$. Thus $\mathrm{rk}(M) = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$. Further using $AA^+ = \overline{A^+A} = (A^+A)^T$, we obtain

$$\begin{aligned} (I - AA^+)B(M/A)^+ &= (I - AA^+)((M/A)^+C)^T \\ &= \left(\overline{(M/A)^+C}(I - AA^+)\right)^* \\ &= \left(\overline{(M/A)^+C}\,\overline{(I - AA^+)}\right)^* \\ &= ((M/A)^+C(I - AA^+))^T . \end{aligned}$$

Thus by Theorem 3, M_2 is also Con-EP. Clearly $M = M_1 + M_2$ and $\mathrm{rk}(M) = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$. Hence M_1 and M_2 are complementary summands of Con-EP matrices.

REMARK 5. We note that any matrix that is represented as the sum of complementary summands of Con-EP matrices is Con-EP. For if $M = \sum_{i=1}^{k} M_i$ such that each M_i is Con-EP and $\mathrm{rk}(M) = \sum_{i=1}^{k} \mathrm{rk}(M_i)$, then

$$N(M) = \bigcap_{i=1}^{k} N(M_i) = \bigcap_{i=1}^{k} N(M_i^T) = N(M^T).$$

Thus M is Con-EP.

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DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
ANNAMALAI UNIVERSITY
IND–608 002 ANNAMALAINAGAR
INDIA

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SIMULTANEOUS POINTWISE APPROXIMATION OF LAGRANGE INTERPOLATION

G. MASTROIANNI[1] and P. VÉRTESI[2]

1. Introduction. Preliminary results

1.1. Let

$$X(w)=\{x_{kn}(w),k=1,\dots,n,\ n\in N\} \tag{1.1}$$

$$-1<x_{nn}(w)<x_{n-1,n}(w)<\dots<x_{1n}(w)<1,$$

be the zeros of the generalized Jacobi polynomials $p_n(w)\in\mathcal{P}_n$ orthonormal with respect to the generalized Jacobi weight function

$$w^{\alpha,\beta}(x)=w(x)=\begin{cases}\varphi(x)(1-x)^\alpha(1+x)^\beta & \text{if } |x|\leqq 1,\\ 0 & \text{if } |x|>1,\end{cases}$$

where $\alpha,\beta>-1,\varphi>0,\varphi\in C^0$ (continuous in $[-1,1]$), $\int_0^1\omega(\varphi,t)t^{-1}dt<\infty$. Here, as usual, $\omega(f,t)$ is the modulus of continuity of f.

If $L_n(f,X,x)\in\mathcal{P}_n$ denote the Lagrange interpolatory polynomials based on arbitrary interpolatory matrix $X=\{x_{kn},k=1,\dots,n,n=1,2,\dots\}$, then

$$L_n(f,X(w),x)=\sum_{k=1}^n f(x_{kn}(w))l_{kn}(X(w),x) \tag{1.2}$$

where l_{kn} are the fundamental functions of Lagrange interpolation, so with obvious short notations

$$l_{kn}(X(w),x)=\frac{p_n(x)}{p_n'(x_k)(x-x_k)},\quad k=1,2,\dots,n. \tag{1.3}$$

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For $\alpha=\beta=-1/2$, $\varphi(x)=1$, denoting $X(w)$ by T (Chebyshev nodes), O. Kis [1] proved the pointwise estimate

(1.4)
$$|L_n(f,T,x)-f(x)| \leqq c\left\{\omega\left(f,\frac{\sqrt{1-x^2}}{n}\right)\log n+\sum_{k=1}^{n}\omega\left(\frac{k}{n^2}\right)\frac{1}{k}\right\},\quad f\in C(\omega).$$

Considering the function class $C(\omega):=\{f:\omega(f,t)\leqq\omega(t)\}$, where $\omega(t)$ is a modulus of continuity, Ju. R. Vainerman [2] obtained an asymptotic expression for $\sup\limits_{f\in C(\omega)}\{|L_n(f,T,x)-f(x)|\}$, while in P. Vértesi [3] we got

$$|L_n^{(\alpha,\beta)}(f,x)-f(x)| \leqq c\sum_{k=1}^{n}\omega\left(\frac{\sqrt{1-x_k^2}}{n}\right)|l_{kn}^{(\alpha,\beta)}(x)|, \tag{1.5}$$

where $L_n^{(\alpha,\beta)}$ and $l_{kn}^{(\alpha,\beta)}$ correspond to (1.2) and (1.3) if $w(x)=(1-x)^{\alpha}\times$ $\times(1+x)^{\beta}$. Also, as it is proved in [3], the order of estimation in (1.5), in a sense, is the best possible for $C(\omega)$.

1.2. When $f^{(q)}\in C, q\geqq 0$, one may ask the simultaneous estimation of $|L_n^{(i)}(f,X,x)-f^{(i)}(x)|$, $0\leqq i\leqq q$, where the interpolatory matrix $X\subset[-1,1]$ is to be chosen. Taking the matrix T plus some additional nodes "close" to ±1, J. Szabados [4] obtained results which were best possible in order. For other $X(w)$, see P. O. Runck, P. Vértesi [5]. For a general matrix X see K. Balázs and T. Kilgore [6]. However, all three papers deal with uniform estimates. Our aim is to get good simultaneous pointwise estimations.

2. New results

2.1. Let us define $Y=\{y_{in}\}_{i=1}^{r}$ and $Z=\{z_{in}\}_{i=1}^{s}$, $(r,s\geqq 0$, given) by

$$\begin{gathered}-1\leqq z_{1n}<z_{2n}<\ldots<z_{sn}<x_{nn};\quad x_{1n}<y_{1n}<y_{2n}<\ldots<y_{rn}\leqq 1\\ z_{i+1,n}-z_{in}\sim x_{nn}-z_{sn}\sim y_{t+1,n}-y_{tn}\sim y_{1n}-x_{1n}\sim n^{-2},\end{gathered}\tag{2.1}$$

$1\leqq i<s$, $1\leqq t<r$, respectively. If

$$A_n(x)=c\prod_{i=1}^{r}(x-y_i),\qquad B_n(x)=c\prod_{i=1}^{s}(x-z_i)$$

then for the Lagrange interpolatory polynomials $L_{nrs}(f,X(w),x)$ of degree $\leqq n+r+s-1$ based on the roots of $A_n(x)B_n(x)p_n(w,x)$, by the notations

$$D_n(f,x)=D_{nrs}^{(\alpha,\beta)}(f,x)=f(x)-L_{nrs}(f,X(w^{(\alpha,\beta)}),x),$$

$$(2.2)\quad \begin{cases} \gamma=\frac{q}{2}+\frac{\alpha}{2}+\frac{1}{4}, & \delta=\frac{q}{2}+\frac{\beta}{2}+\frac{1}{4} \\ \Delta_n(x)=\frac{\sqrt{1-x^2}}{n}+\frac{2}{n^2}, & |x-x_{jn}|=\min_{1\le k\le n}|x-x_{kn}|,\ j=j(x)=j(x,n), \end{cases}$$

$$J=\min(j,n-j+1)\qquad (\text{clearly}\ \ j\sim n^2\Delta_n(x)),$$

we have

THEOREM 2.1. *Let r and s be nonnegative integers, $\alpha,\beta>-1$. If $f^{(q)}\in C$ $(q\geqq 0)$, we have*

$$(2.3)\quad \begin{aligned} |D_n(f,x)| &\leqq c\Delta_n^q(x)(1-x_{jn})^{r-\gamma}(1+x_{jn})^{s-\delta}\times \\ &\times\sum_{\substack{k=1\\k\neq j}}^{n}\omega(f^{(q)},\Delta_n(x_k))\frac{(1-x_{kn})^{\gamma-r+\frac12}(1+x_{kn})^{\delta-s+\frac12}}{n|x-x_{kn}|},\qquad |x|\leqq 1. \end{aligned}$$

2.2. From Theorem 2.1 we get an estimation of type (1.4). If

$$(2.4)\qquad S(t,n,T):=t^{2T}\sum_{k=t}^{n}\frac{\omega(f^{(q)};\frac{k}{n^2})}{k^{1+2r}},\qquad 1\leqq t\leqq n$$

then

COROLLARY 2.2. *Let us suppose that $r-\gamma=s-\delta:=\tau$. Then by the above notations and conditions*

$$(2.5)\quad |D_n(f,x)|\leqq c\Delta_n^q(x)\{\omega(f^{(q)},\Delta_n(x))\log(n^2\Delta_n(x))+S(j(|x|),n,\tau)\}$$

uniformly for $|x|\leqq 1$, whenever

$$(2.6)\qquad r\leqq 1.$$

2.3. To obtain simultaneous estimations for $|D_n^{(i)}|$, we will apply Dzjadyk's theorem.

DEFINITION. Let $\varphi(u)$ be a nonnegative increasing and continuous function defined for $c_1n^{-2}\leqq u\leqq c_2n^{-1}$. $\varphi(u)$ is *semiregular* if with a proper integer $m>0$

$$\text{(i)}\qquad \varphi(\lambda u)\leqq c_3(\lambda+1)^m\varphi(u),\qquad 1\leqq\lambda\leqq c_4n,\quad \lambda u\leqq c_2n^{-1}.$$

If, moreover

$$\text{(ii)}\qquad u\varphi(U)\leqq c_5U\varphi(u),\qquad c_1n^{-2}\leqq u\leqq U\leqq c_2n^{-1}$$

then $\varphi(u)$ is *regular* ($c_1,c_2,\dots,c_5$ are fixed and positive).

Of course a modulus of continuity $\omega(u)$ is regular. On the other hand, the function $\varphi(u)=u|\log n^2u|$ is only semiregular.

THEOREM 2.2. *If $f^{(q)} \in C$ then with any fixed $p > 0$,*

(2.7)
$$|D_n^{(i)}(f,x)| \leqq c\Delta_n^{q-i}(x)\{\omega(f^{(q)},\Delta_n(x))\log(n^2\Delta_n(x)) + S(J,n,\tau)\}, \quad 0 \leqq i \leqq q,$$

whenever $|x| \leqq 1 - pn^{-2}$. Here $c = c(p)$. Moreover, whenever

$$\Omega(u) := \omega(f^{(q)}, 2u)\log(2n^2u) + S(n^2u, n, \tau), \qquad 1 \leqq n^2u \leqq 3n/4, \tag{2.8}$$

is regular (with the same constants $c_1, \dots, c_5$, for arbitrary $n, \tau \leqq 1$ fixed) then (2.7) holds true for $|x| \leqq 1$.

Actually the argument which led to Theorem 2.2 gives the following more general

THEOREM 2.3. *Let $f^{(q)} \in C$ and let D_n be estimated as follows*

$$|D_n(f,x)| \leqq c\Delta_n^q(x)\Phi(\Delta_n(x)) \tag{2.9}$$

where $\Phi(u)$ may depend on α, β, r, s, and n, too. Then if $\Phi(u)$ is semiregular (with the same constants $c_1, c_2, \dots, c_5$ for arbitrary n; α, β, r, and s are fixed) then

$$|D_n^{(i)}(f,x)| \leqq c\Delta_n^{q-i}(x)\Phi(\Delta_n(x)), \qquad 0 \leqq i \leqq q, \tag{2.10}$$

whenever $|x| \leqq 1 - pn^{-2}$ ($p > 0$, fixed). Moreover, (2.10) *holds true for $|x| \leqq 1$, whenever $\Phi(u)$ is regular.*

2.4. To get further estimations we consider some special cases.

(1) First we estimate as follows:

$$S(j,n,\tau) \leqq cj^{2\tau}\omega\Big(\frac{j}{n^2}\Big)\sum_{k=j}^{n}\Big(\frac{k}{j}+1\Big)k^{-2\tau-1} \sim \begin{cases} \omega\Big(\dfrac{j}{n^2}\Big), & \dfrac{1}{2} < \tau \leqq 1, \\ \omega\Big(\dfrac{j}{n^2}\Big)\log\dfrac{n}{j}, & \tau = \dfrac{1}{2}, \\ \omega\Big(\dfrac{j}{n^2}\Big)\Big(\dfrac{n}{j}\Big)^{1-2\tau}, & \tau < \dfrac{1}{2}. \end{cases}$$

So we get from (2.5) by $J \sim n^2\Delta_n(x)$

$$|D_n(f,x)| \leqq \begin{cases} c\Delta_n^q(x)\omega(f^{(q)},\Delta_n(x))\log(n^2\Delta_n(x)), & \text{if } \dfrac{1}{2} < \tau \leqq 1, \\ c\Delta_n^q(x)\omega(f^{(q)},\Delta_n(x))\log n, & \text{if } \tau = \dfrac{1}{2}, \end{cases} \tag{2.11}$$

whence by Theorem 2.3

(2) If $\tau = 1/2$, for arbitrary f, with $f^{(q)} \in C$,

$$|D_n^{(i)}(f,x)| \leqq c\Delta_n^{q-i}(x)\omega(f^{(q)},\Delta_n(x))\log n, \qquad |x| \leqq 1 \tag{2.12}$$

moreover if $1/2 < \tau \leqq 1$ and $\omega(u)\log(n^2u)$ is regular (by $\omega(u) \sim u^\mu$, $0 < \mu < 1$) then, uniformly in x:
(2.13)
$$|D_n^{(i)}(f,x)| \leqq c\Delta_n^{q-i}(x)\omega(f^{(q)},\Delta_n(x))\log(n^2\Delta_n(x)),\quad 0 \leqq i \leqq q,\quad |x| \leqq 1.$$

Actually, this estimation shows that for a given $q \geqq 0$ there are infinitely many "good" r, s, α, β for which one gets (2.13). Omitting regularity, (2.13) holds only for $|x| \leqq 1 - pn^{-2}$.

(3) Let $s(t,n) := \{x;\ 0 \leqq x_{tn} \leqq |x| \leqq 1, |x - x_{j(n)}| \geqq cjn^{-2}\}$ (i.e., we exclude "small" intervals around x_k's). Obviously, for any $\varepsilon > 0$ there is $c = c(\varepsilon)$ such that $|s| \geqq 2 - 2x_{tn} - \varepsilon$. By [2, (13)], whenever $\omega(u^2) \geqq c\omega(u)$,

$$\sup_{f\in C(\omega)}\{\max_{x\in s(t,n)}|f(x) - L_n(f,T,x)|\} \leqq c\omega\Big(\frac{1}{n}\Big)\log n. \tag{2.14}$$

On the other hand, from (2.11) if $q = 0$, $\alpha = \beta = -1/2$

$$\sup_{f\in C(\omega)}\{\max_{x\in s(t,n)}|f(x) - L_{n11}(f,T,x)|\} \leqq c\omega\Big(\frac{1}{n}\Big)\log(n^2\Delta_n(x)),$$

which generally is better than (2.14). (Other values of α, β can give similar estimations: the conditions are $1/2 < \tau = r - \alpha/2 - 1/4 = s - \beta/2 - 1/4 \leqq 1$ $(q = 0)$.)

(4) If $\tau < 1/2$, we get

$$S(j,n,\tau) \leqq cj^{2\tau}\omega\Big(\frac{1}{n}\Big)\sum_{k=j}^{n}\Big(\frac{k}{n}+1\Big)k^{-2\tau-1} \sim \begin{cases} \omega\Big(\dfrac{1}{n}\Big), & \tau > 0,\\ \omega\Big(\dfrac{1}{n}\Big)\log\dfrac{n}{j}, & \tau = 0,\\ \omega\Big(\dfrac{1}{n}\Big)\Big(\dfrac{n}{j}\Big)^{-2\tau}, & \tau < 0.\end{cases}$$

Statements similar to the above ones can be obtained (if $\tau \geqq 0$), but they generally are "less pointwise" than the above ones. We omit the details.

2.5. A possible generalization is when the additional nodes are "very close" to each other, including the cases when they are equal. Then, to get estimations with the previous order sometimes we have to suppose more on $f(x)$. As an example, let $r = 2$, $y_{2n} - y_{1n} = \varepsilon_n/n^2$, $\varepsilon_n \geqq 0$, $\lim\limits_{n\to\infty}\varepsilon_n = 0$. Then $|g_n[y_1,y_2]| \leqq c\max(|g(y_1)|,|g(y_2)|)n^2$ (cf. (3.4)) does not hold anymore. Instead, to maintain the previous estimations, we have to suppose that $f' \in C$, and use $|g_n[y_1,y_2]| = |g_n'(\xi)|$, $y_1 \leqq \xi \leqq y_2$. We omit the further details.

3. Proofs

PROOF of Theorem 2.1. Let $G_n(f,x)\in\mathcal{P}_n$ be the Gopengauz polynomial (cf. [8]) so we have

$$\begin{aligned}(3.1)\qquad |f^{(i)}(x)-G_n^{(i)}(x)| &\leqq c\Big(\frac{\sqrt{1-x^2}}{n}\Big)^{q-i}\omega\Big(f^{(q)},\frac{\sqrt{1-x^2}}{n}\Big)\\ &\leqq c\Delta_n^{q-i}(x)\omega(f^{(q)},\Delta_n(x)),\quad 0\leqq i\leqq q,\quad |x|\leqq 1.\end{aligned}$$

Then by definition for $d_n(f):=f-G_n(f)$

$$\begin{aligned}(3.2)\qquad f(x)-L_{nrs}(f,X(w),x)&=d_n(f,x)+A_n(x)p_n(x)L_s\Big(\frac{d_n}{A_np_n},Z,x\Big)+\\ &+B_n(x)p_n(x)L_r\Big(\frac{d_n}{B_np_n},Y,x\Big)+\\ &+A_n(x)B_n(x)L_n\Big(\frac{d_n}{A_nB_n},X(w),x\Big).\end{aligned}$$

3.1. To estimate $B_np_nL_r$ (or $A_np_nL_s$) we use the Newton's representation whence by $g_n=d_n(B_np_n)^{-1}$

$$(3.3)\qquad L_r(g_n,Y,x)=\sum_{i=1}^{r}\Big\{g_n[y_1,y_2,\dots,y_i]\prod_{t=1}^{i-1}(x-y_t)\Big\}$$

where with

$$g_n[y_i]=g_n(y_i)$$

$$(3.4)\qquad g_n[y_1,y_2,\dots,y_i]=\frac{g_n[y_1,y_2,\dots,y_{i-1}]-g_n[y_2,y_3,\dots,y_i]}{y_1-y_i},\qquad i>1.$$

By $x_k=\cos\theta_k$, $\theta_0=0$, $\theta_{n+1}=\pi$, $x=\cos\theta$ and using (2.2) we have, by obvious short notations

$$(3.5)\qquad \theta_{k+1,n}-\theta_{k,n}\sim\frac{1}{n},\quad\text{for}\quad 0\leqq k\leqq n$$

$$(3.6)\qquad |p_n(x)|\sim\frac{n|\theta-\theta_j|}{(1-x_j)^{\frac{\alpha}{2}+\frac14}(1+x_j)^{\frac{\beta}{2}+\frac14}}$$

uniformly for $|x|\leqq 1$,

$$(3.7)\qquad |p_n'(x_k)|\sim\frac{n}{(1-x_k)^{\frac{\alpha}{2}+\frac34}(1+x_k)^{\frac{\beta}{2}+\frac34}}$$

uniformly in k and n. Hence if $K=\min(k,n-k+1)$ and $J=\min(j,n-j+1)$, $1\leqq k,j\leqq n$, we can easily obtain

$$(3.8)\qquad \begin{cases} |x-x_k|\sim \dfrac{|j-k|\min\{j+k,2n+1-j-k\}}{n^2}, & k\neq j,\\ |x-x_j|\sim \dfrac{|\theta-\theta_j|J}{n}\leqq c\dfrac{J}{n^2}, \quad 1\leqq j\leqq n, \\ 1-x_j\sim 1-x_j^2\sim \frac{j^2}{n^2}, \quad 0\leqq x_j\leqq 1 \end{cases}$$

uniformly in x,k,j and n (cf. P. Vértesi [7] for other references).

First let $0\leqq x\leqq 1$. By $g_n[y_1,\dots,y_i]\leqq c\max\limits_{1\leqq t\leqq i}|g_n(y_t)|\prod\limits_{t=1}^{i-1}|y_t-y_{t+1}|^{-1}$, (2.1), (3.6) and (3.1) we get

$$|B_n(x)p_n(x)L_r(g_n,Y,x)|\leqq c\Big(\frac{n}{j}\Big)^{\alpha+\frac12}\sum_{i=1}^{r}\max_{0\leqq t\leqq i}|g_n(y_t)|n^{2i-2}\Big(\frac{j}{n}\Big)^{2i-2}$$

$$\leqq c\sum_{i=1}^{r}\Big(\frac{n}{j}\Big)^{\alpha+\frac12}j^{2i-2}\frac{\omega(\frac{1}{n^2})}{n^{\alpha+\frac12}n^{2q}}\sim\frac{\omega(\frac{1}{n^2})}{n^{2q}}j^{2r-\alpha-2.5}:=S_1.$$

We compare S_1 with S_2 which is the "$k=1$" (or "$k=2$" if $j=1$) term of the right side of (2.3). By (3.1) and (3.8), using (2.2)

$$S_2\sim\Big(\frac{j}{n^2}\Big)^q\Big(\frac{j}{n}\Big)^{2r-2\gamma}\omega\Big(\frac{1}{n^2}\Big)\frac{1}{n^{2\gamma-2r+2}}\Big(\frac{n}{j}\Big)^2=\frac{\omega(\frac{1}{n^2})}{n^{2q}}j^{2r-\alpha-2.5}$$

whence $S_1\leqq cS_2$. Similar argument shows that $S_1\leqq cS_2$ when $-1\leqq x\leqq 0$. So $|B_np_nL_r|$ can be estimated again by the first (second) term of the right side of (2.3) (with a proper $c>0$). Similarly, $|A_np_nL_s|$ can be estimated by the "$k=n$" ($k=n-1$) term.

3.2. To estimate $A_nB_nL_n\Big(\dfrac{d_n}{A_nB_n}\Big)$ first let $0\leqq x\leqq 1$, say. If $x=x_j$, $D_n(f,x_j)=0$, so from now on we suppose $x\neq x_j$. We write

$$\Big|A_n(x)B_n(x)L_n\Big(\frac{d_n}{A_nB_n}\Big)\Big|\leqq$$

$$\leqq\Big|\sum_{k=0}^{n}A_n(x)B_n(x)d_n(x_k)A_n^{-1}(x_k)B_n^{-1}(x_k)l_k(X(w),x)\Big|\leqq$$

$$\leqq\Big|\sum_{k\neq j}\Big|+\Big|\sum_{k=j}\Big|=:S_3+S_4.$$

By $|A_n(x)| \leqq c(1-x_j)^\tau$ and $|B_n(x)| \leqq c$ (because $x \geqq 0$) further using (3.1), (3.6), (3.7) and $\Delta_n(x_k) \sim \frac{(1-x_k)^{\frac{1}{2}}}{n}$, we get

$$S_3 \leqq c\frac{(1-x_j)^{\tau-\frac{\alpha}{2}-\frac{1}{4}}}{n^q}\sum_{k\neq j}\frac{\omega(\Delta_n(x_k))}{n|x-x_k|}(1-x_k)^{\frac{q}{2}+\frac{\alpha}{2}+\frac{3}{4}-r}(1+x_k)^{\frac{q}{2}+\frac{\beta}{2}+\frac{3}{4}-s}$$

which is by $1/2 \leqq 1+x_j < 2$, the right-hand side of (2.3) if we use notations (2.2). S_4 can be estimated by the "$k=j+1$" term of (2.3) considering (3.8) and (3.6) whence $|p_n(x)|\,|x-x_j|^{-1} \sim n^2 j^{-1}(1-x_j)^{-\frac{\alpha}{2}-\frac{1}{4}}$ ($x \neq x_j$, $x \geqq$ $\geqq 0$). Similar estimation holds when $-1 \leqq x \leqq 0$. Taking into account point 3.1, we get Theorem 2.1. □

3.3. Proof of Corollary 2.2. First let $0 \leqq x \leqq 1$. We write the sum in (2.3) as

$$\sum_{k\neq j} = \sum_{1\leqq k\leqq\frac{3}{2}j}{}' + \sum_{\frac{3}{2}j<k\leqq n} =: S_1 + S_2$$

(Σ' means that $k \neq j$). For S_1 by $\Delta_n(x_k) \leqq c\Delta_n(x_j)$, $1/u \leqq 1+x_k \leqq 2$ and (3.8)

$$S_1 \sim \frac{\omega(\Delta_n(x_j))}{n^{-2r}}\sum_{k=1}^{\frac{3}{2}j}\frac{k^{1-2\tau}}{(k+j)(|j-k|+1)} \sim \omega(\Delta_n(x_j))\left(\frac{n}{j}\right)^{2\tau}\log(j+1)$$

if we desintegrate the sum according to

$$\sum_{k=1}^{\frac{3}{2}j} = \sum_{1\leqq k<\frac{j}{2}} + \sum_{\frac{j}{2}\leqq k\leqq\frac{3}{2}j}{}'$$

and consider that (by $\tau \leqq 1$) $1-2\tau \geqq -1$. Using $(1-x_j)^{\tau-\gamma}(1k+x_j)^{s-\delta} =$ $=(j/n)^{2r}$ and $n^2\Delta_n(x) \sim j$, we obtain the "log" part of (2.5).

At the estimation of S_2 we use that $|x-x_k| \sim (k/n)^2$ (by $k \geqq 3/2j$ and $x \geqq 0$ (cf. (3.8)). The remaining part is the same as above, whence $(1-x_{j,n})^{\tau-\gamma}S_2 < cj^{2\tau}\sum_{k=j}^{n}\omega(k/n^2)k^{-1-2\tau}$ which by (2.4) is the "S part" of (2.5). Finally, if $-1 \leqq x \leqq 0$, by $-x=y$, and $K=n-k+1$ we write

$$|D_n(f,x)| \leqq c\Delta_n^q(x)(1-x_{j(x)}^2)^\tau\sum_{k\neq j(x)}\omega(\Delta_n(x_k))\left(\sqrt{1-x_k^2}\right)^{1-2\tau}(n|x-x_k|)^{-1}$$

$$\sim \Delta_n^q(y)(1-x_{j(y)}^2)^\tau\sum_{K\neq j(y)}\omega(\Delta_n(x_k))\left(\sqrt{1-x_k^2}\right)^{1-2\tau}(n|y-x_k|)^{-1}$$

(cf. (2.3) and (3.8)), whence we get (2.5). □

PROOF of Theorem 2.2. We need a lemma proved essentially by V. K. Dzjadyk. For $0 < a \leqq 1$, $|x| \leqq a$ we put (cf. G. G. Lorentz [9, p. 70])

$$\Delta_n(x,a) = \frac{\sqrt{a^2 - x^2}}{n} + \frac{2}{n^2}, \qquad n = 1, 2, \dots, \quad \Delta_0(x,a) = 1.$$

LEMMA 3.1. *Let $0 < a \leqq 1$, $0 < b < a$, $t = 0, 1, \dots$, and let ϕ be a semiregular function. If an algebraic polynomial $P_n \in \mathcal{P}_N$, $N \leqq cn$, satisfies*

$$|P_n(x)| \leqq c\Delta_n^t(x,a)\varphi(\Delta_n(x,a)), \quad |x| \leqq a \tag{3.9}$$

then with $a_1 = a - bn^{-2}$ and a proper constant $M = M(a,b,m)$,

$$|P_n'(x)| \leqq M\Delta_n^{t-1}(x,a_1)\varphi(\Delta_n(x,a_1)), \quad |x| \leqq a_1. \tag{3.10}$$

If, moreover, φ is regular then

$$|P_n'(x)| \leqq M\Delta_n^{t-1}(x,a)\varphi(\Delta_n(x,a)), \quad |x| \leqq a. \tag{3.11}$$

The proof of (3.10) is a word-by-word repetition of that in [9, Lemma 4 1, p. 70], noting that it actually uses only the semiregularity of φ (cf. [9, Lemma 1, p. 67], too). From (3.9), using (ii), we get (3.11) as in [9, Theorem 3, p. 71].

Now we prove Theorem 2.2. By (3.1) and (3.2), if $Q_n := A_n p_n L_s + B_n p_n L_r + A_n B_n L_n$ then,

$$|Q_n(x)| \leqq c\Delta_n^q(x)\Omega(\Delta_n(x)), \quad |x| \leqq 1 \tag{3.12}$$

(cf. (2.4) and (2.5)). By Lemma 3.1, if we prove that $\Omega(u)$ is semiregular we can get (2.7) (using Lemma 3.1 q times) for $|x| \leqq 1 - pn^{-2}$. By $\log(\lambda 2n^2u) = \log\lambda + \log(2n^2u) < c(\lambda+1)\log(2n^2u)$ and $S(\lambda n^2u, n, \tau) < \lambda^{2\tau}(n^2u)^{2\tau} \sum_{k=[n^2u]}^{n} \dots < (\lambda+1)^T S(n^2u, n, \tau))$, $T = [|2\tau|] + 1$, we get (i) for $\Omega(u)$ with $m = \max(2, T)$.

The estimation (2.7) for $|x| \leqq 1$ comes from (3.11) and the regularity of $\Omega(u)$. □

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ISTITUTO DI MATEMATICA
UNIVERSITÀ DELLA BASILICATA
VIA N. SAURO 85
I–85100 POTENZA
ITALY

MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 51–57

AN ASYMPTOTICALLY EXACT ADDITIVE COMPLETION

I. Z. RUZSA[1]

1. Introduction

We call two sets A, B of positive integers *additive complements*, if their sum

$$A + B = \{a + b : a \in A, b \in B\}$$

contains all sufficiently large integers. A pair of additive complements obviously satisfies

$$A(x)B(x) \geqq x - K \tag{1.1}$$

with some constant K, where $A(x)$ denotes the number of elements of A up to x. Disproving a conjecture of Hanani, Danzer [1] constructed sets satisfying

$$A(x)B(x) \sim x. \tag{1.2}$$

For sets satisfying (1.2), Sárközy and Szemerédi [4] strengthened (1.1) to

$$A(x)B(x) - x \to \infty.$$

The aim of this note is to show that (1.2) can happen for very simple sets.

THEOREM. *Let $a \geqq 3$ be a positive integer and put*

$$A = \{a^k : k \geqq 0\}.$$

There is a set B of positive integers which is an additive complement of A and satisfies (1.2).

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2. Modular and global completion

Definition 2.1. Let a, m be positive integers, $a \geqq 2$, $(a,m)=1$, and let r be the order of a modulo m (the smallest number for which $a^r \equiv 1 (\operatorname{mod} m)$). We say that m is a *fissile modulus* for a if there is a set $\{d_1, \dots, d_l\}$ of integers such that the sums

$$a^i + d_j, \qquad 0 \leqq i < r, \quad 1 \leqq j \leqq l \tag{2.1}$$

represent every residue modulo m exactly once. We call r the *rank* of this modulus.

This is a finite analogue of an exact additive complement. If such a set exists, we must have $m = kl$.

We shall connect this 'modular' additive completion with the ordinary one.

Lemma 2.2. *Suppose that there exists an infinite sequence $m_1, m_2, \dots$ of fissile moduli for a of ranks $r_1, r_2, \dots$ satisfying $r_i \to \infty$ and $r_{j+1}/r_j \to 1$. Then there is a set B of positive integers which is an additive complement of A and satisfies* (2.2).

Proof. We are going to construct B in the form

$$B = \bigcup_{k=1}^{\infty} B_k,$$

where the B_k are finite sets such that

$$A + B_k \supset (a^k, a^{k+1}]. \tag{2.2}$$

Such a B is clearly an additive complement of A.

Fix an integer k, and let m be the fissile modulus with the largest rank r satisfying

$$a^{r-1} < a^k/k^2.$$

(Observe that $m=1$, $r=1$ is always a possibility.) We preserve the notations of the definition. We define

$$B_k = \{b : a^k - a^k/k^2 < b \leqq a^{k+1},\ b \equiv d_j (\operatorname{mod} m) \text{ for some } 1 \leqq j \leqq l\}.$$

First we show (2.2). Consider an integer $a^k < n \leqq a^{k+1}$. By definition we can find $0 \leqq i < r$, $1 \leqq j \leqq l$ such that

$$n \equiv a^i + d_j (\operatorname{mod} m).$$

The number $b = n - a^i$ satisfies $b \equiv d_j$ and also

$$b \leqq n \leqq a^{k+1}, \quad b \geqq n - a^{r-1} > a^k - a^k/k^2,$$

hence indeed $b \in B_k$.

Next we estimate $|B_k|$. Observe that r is the largest rank of a fissile modulus below $\log(a^k/k^2)/\log a$. Since the quotient of consecutive terms of the sequence r_k tends to 1, we have

$$r \sim \frac{\log(a^k/k^2)}{\log a} \sim k.$$

We also know by definition that $m \mid a^r - 1$, hence

$$m < a^r < a^{k+1}/k^2.$$

The set B_k is the union of $l = m/r$ residue classes modulo m, consequently

$$|B_k| = l\Big(\frac{a^{k+1} - (a^k - a^k/k^2)}{m} + O(1)\Big) = \frac{(a-1)a^k}{r} + O(a^k/k^2) \sim \frac{(a-1)a^k}{k}.$$

Now we estimate $B(x)$. Let $a^k < x \leqq a^{k+1}$. The set $B \cap [1, x]$ contains $B_1, \dots, B_{k-1}$, a part (possibly the whole) of B_k and possibly a part of B_{k+1}. From the previous estimate we deduce

$$|B_1| + \ldots + |B_{k-1}| \sim \frac{a^k}{k}.$$

B_k is the union of l residue classes modulo m, hence

$$\begin{aligned}
|B_k \cap [1, x]| &= |B_k \cap [a^k - a^k/k^2, x]| \\
&\leqq l\Big(\frac{x - (a^k - a^k/k^2)}{m} + 2\Big) \\
&= \frac{x - a^k}{r} + o(x/k) = \frac{x - a^k}{k} + o(x/k).
\end{aligned}$$

Finally, since B_{k+1} consists exclusively of numbers $\geqq a^{k+1} - a^{k+1}/(k+1)^2$, there are at most $a^{k+1}/(k+1)^2 = o(x/k)$ elements in $B_{k+1} \cap [1, x]$. Summing up we find

$$B(x) \leqq \frac{x}{k} + o(x/k) = \log a \frac{x}{\log x} + o\Big(\frac{x}{\log x}\Big).$$

This and the lower estimate given by (1.1) yield (1.2). □

3. Finding fissile moduli

In this section we construct fissile moduli for any $a > 2$. Since a fissile modulus m of rank r satisfies $r|m|a^r - 1$ and the divisibility $r|2^r - 1$ is well-known to be impossible for $r > 1$, there is no nontrivial fissile modulus for 2.

Lemma 3.1. *Let p be a prime, a a positive integer, and let s be the exponent of p in $a-1$. Assume $s \geqq 1$, and if $p=2$, then assume $s \geqq 2$. The exponent of p in a^p-1 is $s+1$.*

Proof. Write

$$a=1+tp^s, \quad p \nmid t.$$

By the binomial theorem we have

$$a^p=1+tp^{s+1}+\frac{p-1}{2}t^2p^{2s+1}+Tp^{3s}$$

with some integer T. Hence

$$\frac{a^{p-1}}{p^{s+1}}=t+p\Big(\frac{(p-1)p^{s-1}}{2}t^2+Tp^{2s-2}\Big)$$

is an integer but not a multiple of p. □

Lemma 3.2. *Assume $a \geqq 2$, $p|a-1$ and define s by $p^{s+1} \| a^p-1$. For every integer $u \geqq 1$ we have*

$$p^{s+u} \| a^{p^u}-1.$$

Proof. This follows from the previous lemma by an easy induction. □

Remark. This s is the exponent of p in $a-1$ if p is odd, and it is 1+ the highest of the exponents of 2 in $a-1$ and $a+1$ for $p=2$.

Lemma 3.3. *Assume $a \geqq 2$, $p|a-1$ and define s by $p^{s+1} \| a^p-1$. The order of a modulo p^{s+u} is p^u for every $u \geqq 1$, and also for $u=0$ if p is odd.*

Proof. Immediate consequence of the previous lemmas. □

Lemma 3.4. *Let p, a be as in the previous lemmas. There is an odd prime $q \neq p$ such that*

$$q|a^{p^2}-1, \quad q \nmid a^p-1.$$

Proof. We have

$$T \equiv p (\bmod a^p-1) \tag{3.1}$$

because each term is $\equiv 1$. Since $p^2|a^p-1$, we conclude that the exponent of p in T is 1. We know $T > a^p > p$, so T must have a prime divisor $q \neq p$. (3.1) shows that $q \nmid a^p-1$. If $p=2$, then q is odd by $a \neq p$, and if $p \neq 2$, then T is odd and hence so is q. □

From now on we fix an $a>2$, a prime $p|a-1$ and another prime q satisfying the previous lemma. We define s as in Lemma 3.2, and let t be the exponent of q in $a^{p^2}-1$.

LEMMA 3.5. *Let $u \geqq 2$, $v \geqq 0$ be integers. The order of a modulo $p^{s+u}q^{t+v}$ is $r = p^u q^v$.*

PROOF. The order of a modulo p^{s+u} is p^u by Lemma 3.3. The order modulo q is p^2 by the definition of q. By applying the same lemma after substituting q, t and a^{p^2} in the place of p, s and a we obtain that the order modulo q^{t+v} is p^2q^v. The order modulo $p^{s+u}q^{t+v}$ is the least common multiple of these orders. □

LEMMA 3.6. *If p is odd, then p^s is a fissile modulus of rank 1. If $p=2$, then 2^{s+1} is a fissile modulus of rank 2.*

PROOF. The statement for odd p is an immediate consequence of the definitions.

Consider $p=2$. The order of a modulo 2^{s+1} is 2 by the definition of s. Let w be the exponent of 2 in $a-1$; we have $1 \leqq w \leqq s-1$. Let D be the set of numbers in the form

$$j + 2^{w+1}k, \quad 0 \leqq j \leqq 2^w - 1, \quad 0 \leqq k \leqq 2^{s-w} - 1.$$

This set has 2^s elements. It is sufficient to show that the residues of $2^i + d$, $i = 0$ or 1, $d \in D$ modulo 2^s are all distinct. Assume that

$$a^i + j + 2^{w+1}k \equiv a^{i'} + j' + 2^{w+1}k' (\operatorname{mod} 2^{s+1}). \tag{3.2}$$

Since $a^i \equiv 1 (\operatorname{mod} 2^w)$ for every value of i, we conclude that $j \equiv j' (\operatorname{mod} 2^w)$, hence $j = j'$. Deleting j and j' from (3.2) we conclude

$$a^i - a^{i'} \equiv 2^{w+1}(k'-k) (\operatorname{mod} 2^{s+1}).$$

The right side is a multiple of 2^{w+1}, while the possible values of the left side are 0 and $\pm(a-1)$, consequently $i = i'$. Finally we have $2^{w+1}(k'-k) \equiv \equiv 0 (\operatorname{mod} 2^{s+1})$, that is, $k \equiv k' (\operatorname{mod} 2^{s-w})$ which yields $k = k'$ as wanted. □

LEMMA 3.7. *Assume $m|m'$, and let r and r' be the orders of a modulo m and m', resp. If $r'/r = m'/m$ and m is a fissile modulus, then so is m'.*

PROOF. Let $d_1, \dots, d_l$ be the numbers in the definition of a fissile modulus, $l = m/r$. We claim that the same numbers work for m'. We show that

$$a^i + d_j, \quad 0 \leqq i \leqq r' - 1, \ 1 \leqq j \leqq l$$

are all incongruent modulo m'. Assume the contrary, that is,

$$a^i + d_j \equiv a^{i'} + d_{j'} (\operatorname{mod} m').$$

Indeed, considering this congruence modulo m we infer $j = j'$, and then we get $a^i \equiv a^{i'} (\operatorname{mod} m')$ which yields $i = i'$.

These are $lr' = (m/r)r' = m'$ incongruent numbers, thus they form a complete residue system. □

LEMMA 3.8. *p^{s+u} is a fissile modulus of rank p^u for a for all $u \geqq 1$.*

PROOF. This is a consequence of previous three lemmas. □

LEMMA 3.9. *The number $m = p^{s+2}q^t$ is a fissile modulus of rank p^2.*

PROOF. We know that p^{s+2} is a fissile modulus of rank p^2; let $d_1, \dots, d_l$ be a complementing set for it, $l = p^s$. We claim that the following collection of $p^s q^t$ numbers works for m:

$$d_j + p^{s+2}k, \quad 1 \leqq j \leqq p^s, \quad 0 \leqq k' q^t - 1.$$

We show that the numbers

$$a^i + d_j + p^{s+2}k, \quad 0 \leqq i \leqq p^2 - 1, \ 1 \leqq j \leqq p^s, \ 0 \leqq k \leqq q^t - 1$$

are all incongruent modulo m. Suppose the contrary. This means that

$$a^i + d_j + p^{s+2}k = a^{i'} + d_{j'} + p^{s+2}k' (\operatorname{mod} p^{s+2}q^t).$$

Taking the congruence modulo p^{s+2} we see that $i = i'$ and $j = j'$. Deleting these terms we obtain $k \equiv k' (\operatorname{mod} q^t)$, hence also $k = k'$.

We proved that these sums form a complete residue system, and Lemma 3.5 implies that the order of a modulo $p^{s+2}q^t$ is p^2. □

LEMMA 3.10. *Let $u \geqq 2$, $v \geqq 0$ be integers. The number $p^{s+u}q^{t+v}$ is a fissile modulus of rank $r = p^u q^v$.*

PROOF. Follows from Lemmas 3.5, 3.7 and 3.9. □

PROOF OF THE THEOREM. By Lemma 3.10, there is a fissile modulus of rank $p^u q^v$ for every $u \geqq 2$, $v \neq 0$. The quotient of consecutive terms of this sequence tends to 1 by the irrationality of $(\log p)/\log q$. Now an application of Lemma 2.2 completes the proof. □

4. Concluding remarks

An estimate for

$$B(x) - \log a \frac{x}{\log x}$$

could be given by applying a Baker-type estimate.

I cannot decide whether a complement satisfying (1.2) exists for the powers of 2. However, with the same method, using the primes 3 and 19, one can construct a set B satisfying

$$B(x) \sim 2 \log 2 \frac{x}{\log x}$$

such that every large integer n has at least *two* representations of the form $2^k + b$, $b \in B$.

An additive complement with the weaker property

$$A(x)B(x) = O(x) \tag{4.1}$$

was constructed for the powers of 2 in Ruzsa [2], and for any linear recurrence set in Ruzsa [3].

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 59–70

HUN SEMIGROUP STRUCTURE OF POINT PROCESSES, DENSITY OF INDECOMPOSABLE DISTRIBUTIONS ON A SEMIGROUP

Z.-S. LIANG, Y.-J. HE and Z.-Q. LI

1. Introduction

Professor Kendall [5] promulgated Delphic semigroup theory, which was proved to be an effective tool in the study of the structures of renewal sequences and standard p-functions ([5]). In [6] we applied the theory to the study of the structure of the convolution semigroup $\mathbb{P}$ of the point processes on a complete separable metric space. In [3], extending the concept of the Delphic semigroups, we defined the ZH-semigroups and MD-semigroups, whose theory was proved to be a more powerful tool in the analysis of the structure of $\mathbb{P}$, especially we were able to give a new proof of the central limit theorem of point processes (that is, the limit of an infinitesimal triangular array must be infinitely divisible) by a semigroup approach.

Definition A. An abelian Hausdorff topological semigroup S with identity e is called Delphic if the following hold:

(i) For each $s \in S$, the set T_s of all factors of s is compact.

(ii) There is a continuous homomorphism D from S to the additive semigroup $(\mathbb{R}_+, +)$ of nonnegative real numbers such that $D(s) = 0$ only if $s = e$.

(iii) Each limit s of an infinitesimal triangular array (that is, $s = \lim\limits_{n\to\infty} s_{n1} \dots s_{nn}$, where $\lim\limits_{n\to\infty} \max\limits_k D(s_{nk}) = 0$) is infinitely divisible.

As it was pointed out by Ruzsa and Székely [9], assumption (ii) does not seem to be intrinsic, assumption (iii) is usually not easy to check. In [9] they freed themselves from these assumptions and put forward the elegant theory of Hun and Hungarian semigroups. They assumed the much weaker condition "normable" instead of assumption (ii), and showed that (iii) follows

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the normability and other conditions that are more easy to check. They applied their theory to the study of the structures of probability distributions on groups and to the study of other problems (for example, concerning other underlying structures for convolutions, see [9], Chapter 6). In the last section of [9] entitled "Further directions" the possibility of the application of their theories to the arithmetic properties of point processes was mentioned. In [10], Zempléni first proved a general theorem, then using it he proved that the semigroup Λ of finite point processes on a locally compact completely regular space X is a stable and normable Hun semigroup. The following very powerful theorem was given by him in [11], as a stronger version of the general theorem.

THEOREM A ([11] Theorem 4). *Let S be a stable Hun semigroup. Then the convolution semigroup $D(S)$ of compact-regular probability measures on S is a stable Hun semigroup as well.*

In this paper, we shall also realize more or less this application. We shall prove that the convolution semigroup $\mathbb{P}$ of point processes (may not be finite) on a complete separable metric space is a metrizable, stable and normable Hun semigroup and all its elements are "bald". By this we shall prove some results of [7] anew, but by a more simple way than that in [3, 7] (for example we shall prove that the infinitesimal array of point processes is equivalent to the infinitesimal array in the sense of Hun semigroup theory, and hence we shall be able to prove simply the central limit theorem of point processes by a semigroup approach), and shall obtain other new results (The Baire types of some special subsets). Finally, we shall discuss the density of indecomposable distributions on a semigroup and use the results to point processes.

In this paper, we mean by a semigroup an abelian semigroup with identity e, and by a topological semigroup a Hausdorff topological semigroup. Let S be a semigroup. As usual, for any $x, y \in S$, if there is some $z \in S$ such that $x = yz$, then we say that y is a factor of x and it is denoted by $y \mid x$. Let T_x denote the set of the factors of x. It is easy to verify that T_e is a subgroup. Sometimes we denote T_e by $U(S)$ or U. If x does not belong to U, and equation $x = yz$ always implies $y \in U$ or $z \in U$, then x is called irreducible or indecomposable. If $x \in S$ and $x^2 = x$, then x is called an idempotent. If $x \in S$ and for each natural number n, there are $y_n \in S$ and $u_n \in U$ such that $x = u_n y_n^n$, then x is called infinitely divisible or i.d. (Here the definition of i.d. follows [8] as well as [4], it is slightly different from that in [9], but if $U = \{e\}$, then they are equivalent. Since the relation $\{(x,y)\colon x \in yU\}$ is a congruence, the quotient set $S^* := S/U$ is a semigroup and the natural map $f\colon S \to S^*$ is defined by $f(s) = sU$. We usually prefer to study S^* rather than study S, but $s \in S$ is i.d. if and only if for each n there is $s_n^* \in S^*$ such that $(s_n^*)^n = f(s)$.)

DEFINITION B ([9], Definition 2.2.2). A topological semigroup S is called a Hun semigroup if the following hold:

(i) S is associate-free, that is, $s \mid t$ and $t \mid s$ imply $s = t$.
(ii) T_s is compact for each $s \in S$.

We can easily find that all Delphic semigroups are Hun semigroups, and that $U(S) = \{e\}$ for a Hun semigroup S.

DEFINITION C ([9], Definitions 2.3.1, 2.3.2 and 2.5.2). Let $(s_j)_{j \in J}$ be any collection of elements of a Hun semigroup S. We say that

$$t = \prod_{j \in J} s_j$$

is the unordered product of this system if for every neighbourhood V of t there is a finite set $B \subset J$ such that

$$\prod_{j \in C} s_j \in V$$

for every finite set C such that $B \subset C \subset J$. We say this unordered product bounded if for some $x \in S$ we have

$$\prod_{j \in B} s_j \in T_x$$

for each finite subset B of J. Let

$$\prod_{j=k}^{\infty} s_j := \lim_{n \to \infty} \prod_{j=k}^{n} s_j.$$

We say that a product $\prod\limits_{j=1}^{\infty} s_j$ is composition convergent if $\prod\limits_{j=k}^{\infty} s_j$ is convergent for each k.

Let I be a directed set. An array $(t_{ij} \colon j = 1, \dots, n(i);\ i \in I)$ is called an I-array and is denoted simply by $(t_{ij})_{i \in I}$ or (t_{ij}). If $I = \{1, 2, \dots\}$ and $n(i) = i$ for each i, then $(t_{ij})_{i \in I}$ is called a triangular array. We say that (t_{ij}) converges to t if $\lim\limits_{i} \prod\limits_{j} t_{ij} = t$.

DEFINITION D ([9], Definition 2.10.1). An I-array (t_{ij}) is called compact if for any choice $1 \leqq a(i) \leqq b(i) \leqq n(i)$ the net

$$(r_i) := \Big(\prod_{j=a(i)}^{b(i)} t_{ij},\ i \in I \Big)$$

is compact, that is, each subnet of (r_i) has a convergent subnet.

Definition E ([9], p. 47). A Hun semigroup S is called stable if for each $s \in S$ and each open set $V \supset T_s$, there is a neighbourhood W of s such that $x \in V$ whenever $y \in W$ and $x \mid y$.

Lemma A (see [9], Theorem 2.15.1). *Let S be a Hun semigroup. Then the following hold:*

(i) *S is stable if and only if each convergent I-array is compact.*

(ii) *If S is also first countable, then S is stable if and only if S has the following property:*

Let (s_n), (t_n) be sequences in S and $t_n \mid s_n$ for each n. If (s_n) is convergent, then (t_n) has a convergent subsequence.

Definition F (see [9], Definitions 2.10.5 and 2.10.6). A Hun semigroup S is called normable if for any $s \in S$ that is not idempotent there is an s-norm $N_s\colon T_s \to [0, \infty)$ with the following properties:

(i) N_s is a partial homomorphism, that is, $N_s(xy) = N_s(x) + N_s(y)$ if xy belongs to T_s.

(ii) N_s is continuous at the maximal idempotent factor $H(s)$ of s (that is, if t is an idempotent and $t \mid s$, then $t \mid H(s)$).

(iii) $N_s(s) > 0$.

Let S be a Hun semigroup without idempotent other than e. Then it is easy to see that S is normable if for each $s \neq e$ there is a map $N_s\colon \to [0, \infty)$ such that $S_n(xy) = N_s(x) + N_s(y)$ for $xy \in T_s$, N_s is continuous at e and $N_s(s) > 0$.

Definition G (see [9], Definition 2.8.2). An I-array $(t_{ij})_{i \in I}$ is called Hun-infinitesimal if for each neighbourhood V of e there is an $i_0 \in I$ such that $t_{ij} \in V$ for each $i \geqq i_0$ and $1 \leqq j \leqq n(i)$.

Theorem B (see [9], Theorem 2.10.7). *Let S be a normable Hun semigroup. Then the limit of a compact Hun-infinitesimal I-array is i.d.*

Definition H ([9], Definitions 2.8.6 and 2.8.7). Let S be a Hun semigroup and $s \in S$. Then s is called infinitesimally divisible if for each neighbourhood V of e there is a decomposition $s = s_1 \dots s_n$ such that $s_1, \dots, s_n \in$ $\in V$.

Let (X, ϱ_X) be a complete separable metric space, $\mathcal{B}$ be the ring of all the bounded Borel subsets of X. An integer-valued measure μ on $\mathcal{B}$ is called a locally finite counting measure or simply a counting measure, if $\mu(B) < \infty$ for all $B \in \mathcal{B}$. Let $\mathbb{N}$ denote all these measures on $\mathcal{B}$.

Let $\mathbb{N}$ be endowed with the coarsest topology with respect to which the mapping

$$\mathbb{N} \to \mathbb{R}\colon \ \mu \mapsto \int_X f(x)\mu(dx)$$

is continuous for each nonnegative bounded continuous function f with bounded support. Then by [7] Propositions 1.5.2 and 1.5.3, $\mathbb{N}$ is a Polish space, that is, there is some complete separable metric $\varrho_{\mathbb{N}}$ in $\mathbb{N}$ generating the topology. Thus the mapping

$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}: \quad (\mu, \nu) \mapsto \mu + \nu$$

is continuous, and so $(\mathbb{N}, +)$ is a topological semigroup.

Let $\mathcal{N}$ denote the σ-algebra generated by the class

$$\{\{\mu \in \mathbb{N}: \ \mu(B) = k\}: B \in \mathcal{B}, \ k = 0, 1, 2, \dots\}.$$

Then $\mathcal{N}$ coincides with the σ-algebra generated by the open subsets of $(\mathbb{N}, \varrho_{\mathbb{N}})$ by [7], 1.15.5.

A probability measure defined on $(\mathbb{N}, \mathcal{N})$ is called a point process. Let $\mathbb{P}$ denote all these point processes and let $\mathbb{P}$ be endowed with the coarsest topology with respect to which the mapping

$$\mathbb{P} \to \mathbb{R}: \quad P \mapsto \int_{\mathbb{N}} h(\mu) P(d\mu)$$

is continuous for each nonnegative bounded continuous function h defined on $\mathbb{N}$. Then $\mathbb{P}$ is a Polish space by [7], 3.1.2.

The convolution $P * Q$ of P and Q belonging to $\mathbb{P}$ is defined by

$$(P * Q)(Y) := (P \times Q)\{(\mu, \nu): \ \mu + \nu \in Y\}$$

for each $Y \in \mathcal{N}$. So $P * Q \in \mathbb{P}$. Sometimes we write PQ for $P * Q$. Evidently the convolution operation defined above is commutative and associative, and by [7] Proposition 3.1.10 it is continuous. Let θ be the counting measure with $\theta(X) = 0$, and let δ_θ be the point process with $\delta_\theta(\{\theta\}) = 1$. Then $P * \delta_\theta = P$ for each $P \in \mathbb{P}$. Summing up we have the following lemma:

LEMMA B. *The* $(\mathbb{P}, *)$ *is a metrizable topological semigroup with identity* $e := \delta_\theta$.

We have the following Lemma by [3], Theorem 2.2.

LEMMA C. *There are continuous homomorphisms*

$$D_k: \ \mathbb{P} \to (\mathbb{R}_+, +)$$

for $k = 1, 2, \dots$, *where* $(\mathbb{R}_+, +)$ *is the additive semigroup of nonnegative real numbers, such that* $P = \delta_\theta$ *if and only if* $D_k(P) = 0$ *for all* k.

REMARK A. In [3] Theorem 2.2 we defined

$$D_k(P) = -\log \int_{\mathbb{N}} \exp\left(- \int_X f_k(x) \mu(dx) \right) P(d\mu)$$

for all k, where $f_k(x) := \max\{0, 1-\varrho(x, B_k)\}$ with $B_k := \{x\colon \varrho(x, x_0) \leqq k\}$ for some fixed $x_0 \in X$.

We now turn to the central limit problem of $\mathbb{P}$. In [3] we proved anew the central limit theorem of point processes (Theorem B in the sequel) by the ZH-semigroup theory. We shall see that the proof may be simplified by the Hun semigroup theory.

A triangular array of $\mathbb{P}$ $(P_{ij} \in \mathbb{P}\colon j = 1, \dots, i;\ i = 1, 2, \dots)$ is called infinitesimal if for all $B \in \mathcal{B}$, $\lim\limits_{i\to\infty} \max\limits_{j} P_{ij}\{\mu\colon \mu(B) > 0\} = 0$.

THEOREM C ([7], Proposition 3.4.1). *If $P \in \mathbb{P}$ is the limit of an infinitesimal triangular array (P_{ij}), then P is i.d.*

§ 1. The Hun semigroup structure of point processes on a complete separable metric space

THEOREM 1.1. *The semigroup $\mathbb{P}$ is a Hun semigroup with identity $e = \delta_\theta$, and $\mathbb{P}$ has no idempotent other than e.*

PROOF. $\mathbb{P}$ is a topological semigroup with identity $e = \delta_\theta$ by Lemma B. By [6] or by [3] Lemma 2.1, T_s is compact for each $s \in \mathbb{P}$. Let $D_1, D_2, \dots$ be defined as in Lemma C. If $P, Q \in \mathbb{P}$, $P \mid Q$ and $Q \mid P$, then $P = RQ$ and $Q = SP$ for some $R, S \in \mathbb{P}$. So $P = RSP$ and $D_k(P) = D_k(R) + D_k(S) + D_k(P)$ for each k. Thus $D_k(R) = 0$ for each k and $R = e$ by Lemma C. So $P = Q$ and $\mathbb{P}$ is associate-free. If $P \in \mathbb{P}$ and $P^2 = P$, then $D_k(P) = D_k(P) + D_k(P)$, and $D_k(P) = 0$ for each k. So $P = e$. □

LEMMA 1.1. *Each $P \in \mathbb{P}$ always has a decomposition $P = QR$, where Q has no indecomposable factor, R is a bounded unordered product of at most countable indecomposable elements as well as a finite or composition convergent product of indecomposable elements.*

PROOF. As $\mathbb{P}$ has no idempotent other than e by Theorem 1.1, P has no idempotent factor other than e. Thus P is "bald" ([9] Definition 2.7.1). By [9] Theorem 2.7.5 P has a decomposition $P = QR$, where Q has no indecomposable factor and R is a bounded unordered product of indecomposable elements. $\mathbb{P}$ is metrizable by Lemma B, so it is first countable. By [9] Corollary 2.5.9 we know R can be represented by a bounded unordered product of at most countable indecomposable elements. By [9] Statement 2.5.3, R must be a finite or composition convergent product of indecomposable elements. □

THEOREM 1.2. *The semigroup $\mathbb{P}$ is a stable and normable Hun semigroup.*

PROOF. By Theorem 1.1 $\mathbb{P}$ is a Hun semigroup. By Lemma B $\mathbb{P}$ is first countable. By Theorem 4.1 of [3], $\mathbb{P}$ has the property listed in Lemma A (ii), therefore $\mathbb{P}$ is stable. Let $D_1, D_2, \ldots,$ be the homomorphisms as in Lemma C. For each $P \in \mathbb{P} \setminus \{e\}$ there must be k such that $D_k(P) > 0$, then we can take $N_P = D_k$ and so $\mathbb{P}$ is normable. □

REMARK 1.1. (i) In [10], Zempléni proved that the semigroup M of finite counting measures on a locally compact completely regular topological space X is a locally compact completely regular stable Hun semigroup, and using the general theorem proved in [10] he proved that the semigroup Λ of finite point processes on X is a stable and normable Hun semigroup.

(ii) Here we can easily show that $(\mathbb{N}, +)$ is a stable Hun semigroup by [7], 3.2.6. So we can deduce that $\mathbb{P}$ is a stable Hun semigroup directly from the general theorem ([10] Theorem 1) or from Theorem A.

(iii) $\mathbb{N}$ is not locally compact when X is not bounded. In fact, if $\mu \in \mathbb{N}$ and V is a neighbourhood of μ, then there are nonnegative bounded continuous functions $f_1, \ldots, f_n$ with bounded supports $F_1, \ldots, F_n$ respectively, and $c_1, \ldots, c_n > 0$ such that

$$\mu \in W := \bigcap_{1 \leqq i \leqq n} \left\{ \nu \in \mathbb{N} \colon \left| \int_X f_i d\nu - \int_X f_i d\mu \right| < c_i \right\} \subset V.$$

Let $F := \bigcup\limits_{1 \leqq i \leqq n} F_i$, $x \in X \setminus F$. For $n = 1, 2, \ldots$ and $A \in \mathcal{B}$ let $\alpha_n(A) :=$ $:= \mu(A \cap F) + n\delta_x(A)$, where $\delta_x(A) = 1$ when $x \in A$, $\delta_x(A) = 0$ when $x \notin A$. Then the sequence (α_n) is in W and (α_n) has no convergent subsequence. So $\mathbb{N}$ is not locally compact.

THEOREM 1.3. *The limit P of a Hun-infinitesimal I-array (P_{ij}) in $\mathbb{P}$ is i.d.*

PROOF. By Theorem 1.2 $\mathbb{P}$ is a stable and normable Hun semigroup. By Lemma A (i), (P_{ij}) is compact. So P is i.d. by Theorem B. □

COROLLARY 1.1. *If $P \in \mathbb{P}$ is infinitesimally divisible, then P is i.d.*

PROOF. By [9] Remark 2.8.8, P is the limit of a Hun-infinitesimal I-array, so P is i.d. by Theorem 1.3. □

COROLLARY 1.2. *If $P \in \mathbb{P}$ has no indecomposable factor, then P is i.d.*

PROOF. By Theorem 1.1, P is "bald" ([9] Definition 2.7.1). By [9] Theorem 2.8.9, P is infinitesimally divisible. By Corollary 1.1, P is i.d. □

By Lemma 1.1 and Corollary 1.2, we have the following decomposition theorem of point processes.

THEOREM 1.4. *Each $P \in \mathbb{P}$ has a decomposition $P = QR$, where Q has no indecomposable factor and is i.d., and R is a bounded unordered product of* *at most countable indecomposible elements as well as a finite or composition convergent product of indecomposable elements.*

LEMMA 1.2. *Let (P_n) be a sequence in $\mathbb{P}$. Then the following statements are equivalent.*

(i) $\lim_{n\to\infty} p_n = e$.

(ii) *For any $B_1, \dots, B_k$ of pairwise disjoint sets in $\mathcal{B}$,*

$$\lim_{n\to\infty} P_n\{\mu\colon (\mu(B_1), \dots, \mu(B_k)) \neq (0, \dots, 0)\} = 0.$$

(iii) *For each $B \in \mathcal{B}$, $\lim_{n\to\infty} P_n\{\mu\colon \mu(B) \neq 0\} = 0$.*

PROOF. It is easy to see that $\mathcal{B}_e = \mathcal{B}$, where $\mathcal{B}_e$ is defined in [7] p. 148. By [7] 3.1.9, Statement (i) and Statement (ii) are equivalent. It is obvious that

$$\{\mu\colon (\mu(B_1), \dots, \mu(B_k)) \neq (0, \dots, 0)\} = \{\mu\colon \mu(B_1 \cup \ldots \cup B_k) \neq 0\},$$

so (ii) and (iii) are equivalent. □

LEMMA 1.3. *Let $(P_{ij}\colon j = 1, \dots, i;\ i = 1, 2, \dots)$ be a triangular array in $\mathbb{P}$, $Q_1 := P_{11}$, $Q_2 := P_{21}$, $Q_3 := P_{22}$, $Q_4 := P_{31}$, $Q_5 := P_{32}, \dots$. Then the following statements are equivalent.*

(i) *(P_{ij}) is infinitesimal.*

(ii) $\lim_{n\to\infty} Q_n\{\mu\colon \mu(B) \neq 0\} = 0$ *for each $B \in \mathcal{B}$.*

(iii) *(P_{ij}) is Hun-infinitesimal.*

PROOF. By the definition we know that (i) and (ii) are equivalent. By Lemma 1.2 we know that (ii) and (iii) are equivalent. □

NEW PROOF of Theorem C. By Lemma 1.3, (P_{ij}) is Hun-infinitesimal, therefore P is i.d. by Theorem 1.3. □

THEOREM 1.5. *Let $P \in \mathbb{P}$. Then P is i.d. if and only if P is infinitesimally divisible.*

PROOF. If P is infinitesimally divisible, P must be i.d. by Corollary 1.1. Conversely, if P is i.d., then for each n we have $P_n \in \mathbb{P}$ such that $P_n^n = P$. Let $D_1, D_2, \dots$ be the continuous homomorphisms defined in [3] Theorem 2.2. Then $\lim_{n\to\infty} D_k(P_n) = 0$ for each k. By the proof of [3] Lemma 2.5, we have $\lim_{n\to\infty} P_n\{\mu\colon \mu(B) \neq 0\} = 0$ for each $B \in \mathcal{B}$. So $\lim_{n\to\infty} P_n = e$ by Lemma 1.2, it shows that P is infinitesimally divisible.

THEOREM 1.6. *Let $S_k := \{P^k\colon P \in \mathbb{P}\}$. Let I denote all the i.d. elements of $\mathbb{P}$, I_0 all the elements of $\mathbb{P}$ without indecomposable factor, $\widetilde{P}$ all the indecomposable elements of $\mathbb{P}$. Then S_k and I are closed sets, I_0 and $\widetilde{P}$ are sets of type G_δ.*

PROOF. Since $\mathbb{P}$ is a metrizable stable Hun semigroup by Lemma B and Theorem 1.2, we deduce the present theorem by [9] Statement 2.19.1 and Theorems 2.19.2 and 2.19.4. $\square$

We have the following theorem, which will be proved in the next section.

THEOREM 1.7. *Both $\widetilde{P}$ and $\mathbb{P}\setminus\widetilde{P}$ are dense subsets of $\mathbb{P}$. $\widetilde{P}$ is a second-category set and $\mathbb{P}\setminus\widetilde{P}$ is a first-category set.*

§ 2. The density of indecomposable distributions on a semigroup

DEFINITION 2.1. Let S be a semigroup. A subset A of S is called indecomposable if A is not a singleton included in $U(S)$ and for any $A_1, A_2 \subset S$ satisfying $A = A_1A_2 := \{xy\colon x \in A_1, y \in A_2\}$, either A_1 or A_2 is a singleton included in $U(S)$.

DEFINITION 2.2. A semigroup S is called semi-cancellative if for any $x, y \in S$, equation $xy = x$ implies $y = e$.

EXAMPLE 2.1. A Delphic semigroup as well as a ZH-semigroup ([3]) is a semi-cancellative semigroup. A semigroup with an idempotent other than e is not semi-cancellative.

THEOREM 2.1. *Suppose S is a semi-cancellative semigroup. If B is a finite subset of S, $B \not\subset U$ and $B\setminus U = \{s_1, \dots, s_n\}$, $s := s_1^3 \dots s_n^3$, then $A := B \cup \{e, s\}$ is an indecomposable set.*

PROOF. Let $A = A_1A_2$, where $A_1, A_2 \subset S$. Without loss of generality we can suppose $e \in A_1 \cap A_2$, so $A_1, A_2 \subset A$.

Let $s \notin A_1 \cup A_2$. Let $s = a_1a_2$ for some $a_1 \in A_1$ and some $a_2 \in A_2$. Then $a_1 \in U$ or $a_1 = s_i$ for some i, and $a_2 \in U$ or $a_2 = s_j$ for some j. From equation $a_1a_2 = s_1^3 \dots s_n^3$ we have $s_i \in U$, but this contradicts the original hypothesis. Hence $s \in A_1 \cup A_2$.

Let $s \in A_1$. Then for any $a_2 \in A_2$, $sa_2 \in A$. If $sa_2 \in U$, then $s \in U$ and $s_1 \in U$, but this contradicts the original hypothesis. Hence $sa_2 \in \{s\} \cup \{s_1, \dots, s_n\}$. If $sa_2 = s_i$ for some i, then $s_1^3 \dots s_n^3 a_2 = s_i$ and $s_i \in U$, but this contradicts the original hypothesis. Hence $sa_2 = s$, $a_2 = e$. So $A_2 = \{e\}$.

If $s \in A_2$, then we can verify that $A_1 = \{e\}$ in the same way. Thus A is an indecomposable set. $\square$

Henceforth, S is a topological semigroup, $\mathcal{F}$ is the σ-algebra generated by the open subsets of S, M_1 is the set of all probability measures on $(S, \mathcal{F})$, M_2 the set of all compact-regular probability measures on $(S, \mathcal{F})$ ($\mu \in M_2$ if and only if $\mu \in M_1$ and for each $B \in \mathcal{F}$, $\mu(B) = \sup\{\mu(K)\colon K \subset B$ and K is compact$\}$), δ_s is the degenerate probability measure on $(S, \mathcal{F})$ with $\delta_s(\{s\}) = 1$.

THEOREM 2.2. *Let S be a topological semigroup and $M=M_2$, or S be a separable metric semigroup and $M=M_1$. Then $(M,*)$ is a topological semigroup with identity δ_e, $U(M)=\{\delta_u\colon u\in U(S)\}$. If $\mu\in M$ and $\operatorname{supp}\mu$ is an indecomposable finite set, then μ is also indecomposable in M.*

PROOF. If $\mu=\lambda*\nu$, then $\operatorname{supp}\mu=\overline{(\operatorname{supp}\lambda)(\operatorname{supp}\nu)}=(\operatorname{supp}\lambda)(\operatorname{supp}\nu)$. Either $\operatorname{supp}\lambda$ or $\operatorname{supp}\nu$ is a singleton included in $U(S)$. If $\delta_e=\alpha*\beta$, then $(\operatorname{supp}\alpha)(\operatorname{supp}\beta)=\{e\}$, both $\operatorname{supp}\alpha$ and $\operatorname{supp}\beta$ are singletons included in $U(S)$. By [1] Corollary 2.3.4, $(M_2,*)$ is a topological semigroup. As for $(M_1,*)$, we can refer to [8], the proof of Theorem 3.1.1. □

THEOREM 2.3. *Let S be locally compact, $M=M_2$, or let S be separable and metric, $M=M_1$. Let $M_f:=\{\mu\in M\colon \mu$ has a finite support$\}$, $M_i:=\{\mu\in$ $\in M\colon \mu$ is indecomposable$\}$. Then the following hold:*

(i) *$M\setminus M_i$ is dense in M.*

(ii) *If $S=U$ and U is an infinite group, then M_i is dense in M.*

(iii) *If $S\neq U$ and S is semi-cancellative, then $M_i\cap M_f$ is dense in M.*

(iv) *If M_i is dense in M and is a set of type G_δ, then $M\setminus M_i$ is a first-category set. If in addition M is a second-category set, then M_i is also a second-category set.*

PROOF. (i) If $S=\{e\}$, then $M_i=\emptyset$, $M\setminus M_i=M$. If $S\neq\{e\}$, then there is $s\in S\setminus\{e\}$. Let $\mu\in M_i$ and $\mu_n:=\mu*(\delta_s/n+(n-1)\delta_e/n)$ for each n. Then $\mu_n\in M\setminus M_i$ and $\mu_n\to\mu$.

(ii) See [8], Theorem 3.4.2 or its proof.

(iii) Let $\mu\in M_f$, $\operatorname{supp}\mu=F$. Let $B:=F\cup\{x\}$ for some $x\notin U$, let s be defined as in Theorem 2.1,

$$\mu_n:=\delta_e/4n+\delta_x/4n+\delta_s/4n+(4n-3)\mu/4n$$

for each n. Then $\operatorname{supp}\mu_n=B\cup\{s,e\}$ is an indecomposable set by Theorem 2.1, $\mu_n\in M_i\cap M_f$ by Theorem 2.2, and it is obvious that $\mu_n\to\mu$. Hence $M_i\cap M_f$ is dense in M_f.

If $M=M_1$, then M_f is dense in M by [8], Theorem 2.6.3. If $M=M_2$, then for any $\mu\in M$ and for any neighbourhood of μ with the form

$$V:=\bigcap_{1\leq k\leq n}\left\{\nu\in M\colon \left|\int f_k d\nu-\int f_k d\mu\right|<c\right\},$$

where $c>0$, $f_1,\dots,f_n$ are continuous functions on S with compact supports, it is not difficult to show that $M_f\cap V$ is nonvoid. Hence M_f is dense in M.

(iv) Let $M_i=\bigcap_{1\leq k<\infty} G_k$, where each G_k is open. Then $M\setminus M_i=$ $=\bigcup_{1\leq k<\infty}(M\setminus G_k)$. Since $M\setminus G_k$ is closed and $M\setminus(M\setminus G_k)=G_k$ is dense in M, $M\setminus G_k$ is nowhere dense. Hence $M\setminus M_i$ is a first category set.

COROLLARY 2.1. *Let S be a complete separable metric semigroup as well as a semi-cancellative stable Hun semigroup and $S \neq U(S)$. $M_i :=$ $:= \{\mu \in M_1 : \mu$ is indecomposable in $M_1\}$. Then M_i is dense in M_1. $M_1 \setminus M_i$ is a first-category set and M_i is a second-category set.*

PROOF. Since S is a complete separable metric space, $M_1 = M_2$ by [8] Theorem 2.3.2, and M_1 is a complete separable metric space by [8], Theorems 2.6.2 and 2.6.4. Hence M_1 is a second-category set. By Theorem 2.3 M_i is dense in M_1. By Theorem A, M_1 is a stable Hun semigroup, so M_i is of type G_δ by [9], Theorem 2.19.2. Hence M_i is a second-category set and $M_1 \setminus M_i$ is a first-category set by Theorem 2.3. □

PROOF of Theorem 1.7. It is obvious that $(\mathbb{N}, +)$ is cancellative. Hence Theorem 1.7 holds by Corollary 2.1 and Theorem 2.3, or by Theorem 1.6 and 2.3, Remark 1.1 (ii). □

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DEPARTMENT OF MATHEMATICS
ZHONGSHAN UNIVERSITY
GUANGZHOU 510275

DEPARTMENT OF MATHEMATICS
ZHONGSHAN UNIVERSITY
GUANGZHOU 510275

STATISTIC DEPARTMENT
GUANGDONG COMMERCIAL COLLEGE
GUANGZHOU 510320
PEOPLE'S REPUBLIC OF CHINA

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ON SPERNER'S LEMMA

K. T. ATANASSOV

In memory of my grandmother Anna Bukova-Atanassova

Let $p(A_1, A_2, \dots, A_n)$ be a convex n-gon ($n \geqq 3$) with vertices marked by $A_1, A_2, \dots, A_n$, which is covered with triangles in Sperner's way [1] (see also e.g. [2]): if two triangles have a common point, then the set of their common points is either a common vertex or a common side of the two triangles. Let every vertex of these triangles be marked by the symbols $A_1, A_2, \dots, A_n$ in the following way: the vertices which are on the side A_iA_{i+1} ($1 \leqq i \leqq n$; A_{n+1} coincides with A_1) can be marked with A_i or A_{i+1}; the ones which are inside of $p(A_1, A_2, \dots, A_n)$ — with any one of the symbols $A_1, A_2, \dots, A_n$.

Let $b(p(A_1, A_2, \dots, A_n))$ be the number of the triangles in $p(A_1, A_2, \dots, A_n)$ marked with three different symbols (we shall denote these triangles by $T3$s). By $[A_i/A_j]p(A_1, A_2, \dots, A_n)$ we shall denote the replacement of the symbol A_j of all vertices of $p(A_1. A_2, \dots, A_n)$ by the symbol A_i.

LEMMA 1. *For every two natural numbers i, j:*

$$b([A_i/A_j]p(A_1, A_2, \dots, A_n)) \leqq b(p(A_1, A_2, \dots, A_n)). \tag{1}$$

PROOF. Obviously, the inequality (1) is valid for every i, j, for which $1 \leqq i = j \leqq n$; for every $i > n$; for every $j > n$.

Let $1 \leqq i$, $j \leqq n$ and $i \neq j$. The triangles of $p(A_1, A_2, \dots, A_n)$ can be divided into three groups:

triangles which have not a vertex marked with A_j;
triangles which have only one vertex marked with A_j;
triangles which have two or three vertices marked with A_j.

The number of $T3$s from the first and third groups will not change after replacing A_j by A_i while the number of $T3$s from the second group will decrease with the number of these triangles which initially contain the symbols A_i and A_j simultaneously (because after the substitution they will contain two vertices marked by A_i). Therefore the inequality (1) is valid.

LEMMA 2. *If A and B are two neighbourly vertices which take part in a marking of $p(A_1, A_2, \dots, A_n)$, then they take part jointly in at least one $T3$.*

PROOF. Let us assume (without loss of generality) that the symbols A_1 and A_2 which marked two neighbourly vertices in $p(A_1, A_2, \dots, A_n)$ do not

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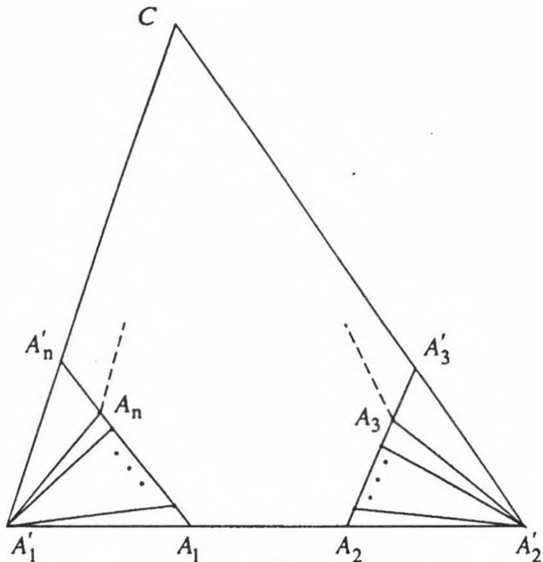

Fig. 1

mark two points of any $T3$. We construct the triangle $t(A'_1, A'_2, C)$ (see Fig. 1) which contains $p(A_1, A_2, \dots, A_n)$ and for which the arc A_1A_2 of $p(A_1, A_2, \dots, A_n)$ lies on the line on which the arc $A'_1A'_2$ of $t(A'_1, A'_2, C)$ lies. Let A'_3 and A'_n be the points of intersection of the lines on which the segments A'_2C and A_2A_3, and A'_1C and A_1A_n lie, respectively. We construct a triangle net in Sperner's way for the triangles $t'(A'_1, A_1, A'_n)$ and $t''(A_2, A'_2, A'_3)$ for which every triangle has as a vertex the points A'_1 and A'_2, respectively. The part of $t(A'_1, A'_2, C)$ which is outside of both last triangles and outside of $p(A_1, A_2, \dots, A_n)$, and which we shall mark by Q, also is covered by triangles by Sperner's way. We mark all vertices of the last figure, except those lying on the boundary of $p(A_1, \dots, A_n)$, by symbol C. Finally, we mark by new symbols the points of $t(A'_1, A'_2, C)$ constructing the triangle

$$t^*(A_1, A_2, C) = [A_1/A'_1][A_2/A'_2][C/A_3]\dots[C/A_n]t(A'_1, A'_2, C).$$

Obviously, after the change, in the triangles $t'(A'_1, A_1, A'_n)$ and $t''(A_2, A'_2, A'_3)$ there are not $T3$s, yet, by Lemma 1, in $p(A_1, A_2, \dots, A_n)$ there cannot be generated a new $T3$, because, by condition, in $p(A_1, A_2, \dots, A_n)$ initially there has not been a $T3$ with symbols A_1 and A_2; in figure Q also there are no $T3$s because all points are marked only by symbol C. Therefore there are no $T3$s in $t^*(A_1, A_2, C)$, which is a contradiction with the ordinary Sperner's lemma. Hence, the assumption that A_1 and A_2 do not mark two points in any $T3$ is false. From this it follows that Lemma 2 is valid.

LEMMA 3 (Generalization of Sperner's lemma). *For every natural number $n \geqq 3$ in every convex n-gon $p(A_1, A_2, \dots, A_n)$ there exist at least $n-2$ $T3$s.*

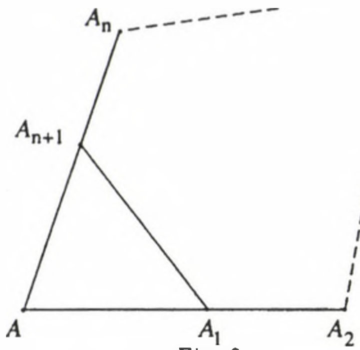

Fig. 2

PROOF. When $n = 3$ we obtain Sperner's lemma. Let us assume that the assertion is valid for some $n \geqq 3$ and let $p'(A_1, A_2, \ldots, A_n, A_{n+1})$ be a convex $(n+1)$-gon. With the exception of the case that $p'(A_1, A_2, \ldots, A_{n+1})$ is a parallelogram there exist three sides of $p'(A_1, A_2, \ldots, A_{n+1})$ (let them be A_nA_{n+1}, $A_{n+1}A_1$ and A_1A_2; see Fig. 2) for which there exists a point A common to the rays A_nA_{n+1} and A_2A_1. This follows from the convexity of $p'(A_1, A_2, \ldots, A_{n+1})$. Now we shall consider the convex n-gon $p''(A, A_2, \ldots, A_n)$ with the triangulation of Fig. 2 or Fig. 3 (we retain the edges in the triangulation of $p'(A_1, A_2, \ldots, A_{n+1})$ and there are possibly still new edges from A to interior points of A_1A_{n+1}; see e.g., Fig. 3, where the points $B_1, B_2, \ldots, B_s$ are vertices of the triangulation of $p'(A_1, A_2, \ldots, A_{n+1})$ and

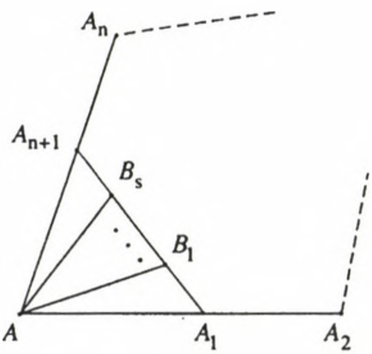

Fig. 3

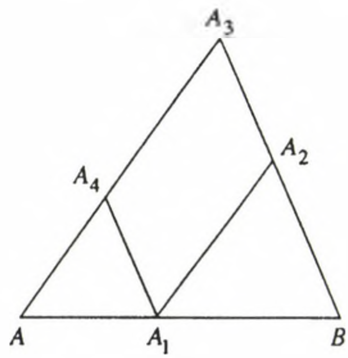

Fig. 4

$$p(A_1, A_2, \ldots, A_n) = [A_1/A][A_1/A_{n+1}]p''(A, A_2, \ldots, A_n).$$

By induction, $b(p(A_1, A_2, \ldots, A_n)) \geqq n - 2$.

There exists the following particular case when the construction of the point A is not possible: $p'(A_1, A_2, A_3, A_4)$ is a parallelogram ($n = 3$). Then we construct the points A and B (see Fig. 4) and the triangle $p''(A, B, A_3)$ and then

$$p(A_1, A_2, A_3) = [A_1/A][A_1/A_4][A_2/B]p''(A, B, A_3).$$

By Sperner's lemma it follows that $b(p(A_1, A_2, A_3)) \geqq 1$.

By Lemma 2 the symbols A_1 and A_{n+1} mark two of the vertices of at least one $T3$. Since the same triangle is no $T3$ in $p(A_1, A_2, \ldots, A_n)$, and $p(A_1, A_2, \ldots, A_n)$ does not contain a $T3$ outside $p'(A_1, A_2, \ldots, A_{n+1})$, thus the number of $T3$s in $p'(A_1, A_2, \ldots, A_{n+1})$ will be greater than the number of $T3$s in $p(A_1, A_2, \dots, A_n)$ with at least one, i.e., the inequality (1) is strict. Then

$$b(p'(A_1, A_2, \dots, A_{n+1})) > b(p(A_1, A_2, \dots, A_n)) \geqq n-2,$$

by which Lemma 3 is proved.

Observe that any simple n-gon is topologically equivalent to a convex n-gon. Further both the proof of Sperner's lemma [1] and the proofs of Lemmas 1–3 of the present paper only use the combinatorial structure of the triangulations. Thus these proofs apply to any triangulation of any domain that is topologically equivalent to a triangulation of a convex n-gon by topological arcs, in Sperner's way. Thus, in particular, we have the following

THEOREM. *Every simple n-gon, which is marked in Sperner's way, has at least $n-2$ $T3$s.*

Finally, we shall formulate the following

HYPOTHESIS. *For every n-vertex polytope P in a k-dimensional space ($n \geqq k+1; k \geqq 1$) which is covered by k-dimensional simplices in the n-dimensional analogue of Sperner's way, and every marking of the vertices of this triangulation by the vertices of P, for which the vertices of the triangulation lying on an i-face of P ($0 \leqq i \leqq k-1$) are marked by the vertices of this i-face, there exist at least $n-k$ k-dimensional simplices which are marked with $k+1$ different symbols.*

The author formulated the above described generalization of Sperner's lemma in the summer of 1970, but its first proof was published in the preprint [3] in 1989.

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MATHEMATICAL RESEARCH LABORATORY
P.O. BOX 12
BG–1113 SOFIA
BULGARIA

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 75–91

SOME DISTRIBUTION RESULTS ON TWO-SAMPLE RANK ORDER STATISTICS FOR UNEQUAL SAMPLE SIZES

J. SARAN and S. RANI

Abstract

This paper deals with the derivation of the null joint and marginal probability distributions of some rank order statistics by using the extended Dwass technique evolved by Aneja [1] and Očka [8]. The rank order statistics considered include the number of positive reflections, the index of the i^{th} positive reflection and the interval between the i^{th} and the l^{th} positive reflections.

1. Introduction

Suppose $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ $(m \geqq n)$ are two independent random samples from populations with unknown continuous distribution functions $F(x)$ and $G(x)$, respectively. Let $F_m(x)$ and $G_n(x)$ be the corresponding empirical distribution functions. Let $\{Z_k\}$, $(k=1,2,\dots,m+n)$ denote the combined set of these $m+n$ values arranged in an increasing order of magnitude and let $Z_0=-\infty$. Since the variables X's and Y's are independent and their distribution functions are continuous, the probability that any two values are equal is zero. Therefore, ties between two values are ruled out and we have $Z_1 < Z_2 < \ldots < Z_{m+n}$. If we replace X's by $(+1)$'s and Y's by (-1)'s in this ordered set, we obtain a sequence of rank order indicators whose suitable functions called rank order statistics can be studied in terms of

$$H_{m,n}(u) = mF_m(u) - nG_n(u), \qquad -\infty < u < \infty.$$

With every sequence of rank order indicators one can associate a random walk of a particle performing m upward and n downward steps. For $m=n$, Dwass [2] developed an alternate method based on the simple random walk with independent steps for obtaining the distributions of two-sample rank order statistics. Aneja [1] and Očka [8] have extended the Dwass technique when $m \neq n$ and derived the distributions of quite a few rank order statistics

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for this case. Later, using this extended Dwass technique for $m \neq n$, Mahendra Pratap [7], Kaul and Sen [5], [6] and Kaul [4] derived the distributions of various rank order statistics. In this paper we derive the null joint and marginal distributions of some rank order statistics for the case of unequal sample sizes viz, the number of positive reflections, the index of the i^{th} positive reflection and the length of the interval between the i^{th} and the l^{th} positive reflections by using the extended Dwass technique, thus generalizing the earlier results. These distributions for the case of equal sample sizes $(m = n)$ have already been obtained by the authors [9].

2. The technique

The extended Dwass technique is based on the simple random walk

$$\left\{ S_j;\ S_j = \sum_{i=1}^{j} W_i,\ S_0 = W_0 = 0 \right\}$$

generated by a sequence $\{W_i\}$ of independent random variables with common probability distribution

$$\mathbf{P}(W_i = +1) = p, \quad \mathbf{P}(W_i = -1) = q = 1 - p, \quad 1 \leqq i < \infty.$$

The result due to Aneja [1] and Očka [8] is explained here briefly.

We can suppose $m \geqq n$ and we put $d = m - n$. Then $d \geqq 0$. In the sequel, we consider d to be an arbitrary but fixed constant, while n will change. This is the main step for the generalization of Dwass technique.

The recurrent event V_d denoting a visit to the height $d = m - n$ occurs at an index j for which $S_j = H_{m,n}(Z_j) = m - n = d$.

The assumption $p < 1/2$ for the case $d \geqq 0$ implies that the event V_d is transient so that with probability one $S_j = d \geqq 0$ for only finitely many values of j. In a given realization of the random walk $\{S_j\}$, let T_d be the largest value of j for which $S_j = d$. Further, let U be a function defined on the random walk $\{S_j\}$, then U is said to satisfy assumption A_d when its value is completely determined by $W_1, W_2, \dots, W_{T_d}$. The main theorem due to Aneja [1] and Očka [8] is as follows.

THEOREM 1. *Suppose $U_{m,n}$ is a rank order statistic for every n and U is the related function satisfying assumption A_d. Define*

$$\mathbf{E}(U) = h(p), \qquad 0 \leqq p < 1/2.$$

Then the following power series in powers of pq is valid for $0 \leqq p < 1/2$:

$$h(p)/(1 - 2p)p^d = \sum_{n=0}^{\infty} \mathbf{E}(U_{m,n}) \binom{2n + d}{n} (pq)^n.$$

If ϕ is a function defined over the possible values of U then $\phi(U_{m,n})$ is also a rank order statistic. In particular if ϕ is the set indicator function of B then $\mathbf{E}(\phi(U_{m,n})) = \mathbf{P}(U_{m,n}$ in $B)$. While applying the theorem we shall let the symbols $U, U_{m,n}$ represent $\phi(U), \phi(U_{m,n})$ for the various versions of ϕ that may be convenient to the problem at hand. This implies that the coefficient of $(pq)^n$ in the power series expansion of $\mathbf{P}(U{=}K)/(1{-}2p)p^d$ equals $\binom{2n+d}{n}\mathbf{P}(U_{m,n}=k)$. Similarly, if $\mathbf{E}(t^U)$ denotes the probability generating function of the distribution of U in which case

$$\mathbf{E}(t^U) = \sum_k p_k t^k \quad \text{where } p_k = \mathbf{P}(U = k),$$

then the coefficient of $t^k(pq)^n$ in the expansion of $\mathbf{E}(t^U)/(1-2p)p^d$ equals $\binom{2n+d}{n}\mathbf{P}(U_{m,n}=k)$.

3. Definitions of rank order statistics

The following is the list of rank order statistics whose distributions will be derived. In what follows, we shall use the dual notation U, $U_{m,n}$ for these rank order statistics as suggested in Theorem 1.

I. *Return to the origin:* A 'return' to the origin occurs at an index j for which $S_j=0$.

II. *Positive and negative sojourns:* A 'sojourn' is defined as the segment between two consecutive returns to the origin. The segment between the origin and the first return point is also regarded as a sojourn. Let $0 < < i_1 < i_2 < \ldots$ be the indices for which $H_{m,n}(Z_i)=0$. If $H_{m,n}(Z_i)>0$ for $i_{j-1} < i < i_j$, we say that the j^{th} sojourn is positive and if $H_{m,n}(Z_i)<0$ for $i_{j-1} < i < i_j$, we say that the j^{th} sojourn is negative.

III. *Positive and negative reflections of height a:* A reflection at height a occurs at an index j when $S_j=a$ and $S_{j-1}=S_{j+1}=a-1$ or $S_{j-1}=S_{j+1}=a+ +1$, the reflection being positive or negative according as $S_{j-1}=S_{j+1}=a+1$ or $S_{j-1}=S_{j+1}=a-1$. Let $R_{m,n}(a)$ denote the total number of reflections of height a of which $R^+_{m,n}(a)$ are positive and $R^-_{m,n}(a)$ are negative with

$$R_{m,n}(a) = R^+_{m,n}(a) + R^-_{m,n}(a).$$

IV. *The index of the i^{th} positive reflection of height a:* Let $R^{+i}_{m,n}(a)$ denote the index of the i^{th} positive reflection at height a. Then $R^{+i}_{m,n}(a)=$ the index j where $S_j=a$ and $S_{j-1}=S_{j+1}=a+1$ for the i^{th} time, $1 \leqq i \leqq R^+_{m,n}(a)$.

V. *The length of the interval between the i^{th} and the l^{th} positive reflections of height a:* Let $M^{+(i,l)}_{m,n}(a)$ denote the length of the interval between the i^{th} and the l^{th} positive reflections of height a $(1 \leqq i < l \leqq R^+_{m,n}(a))$, then

$$M^{+(i,l)}_{m,n}(a) = R^{+l}_{m,n}(a) - R^{+i}_{m,n}(a).$$

The above mentioned statistics with respect to the origin (i.e. for $a = 0$) are denoted by the same symbols without parentheses for a, e.g., $R^+_{m,n}(0) \equiv$ $\equiv R^+_{m,n}$, $R^{+i}_{m,n}(0) \equiv R^{+i}_{m,n}$, $M^{+(i,l)}_{m,n}(0) \equiv M^{+(i,l)}_{m,n}$, etc.

4. Some basic results

Some of the results we list below concerning simple random walk appear in Feller [3] and the rest are easily derived from elementary considerations. The following list covers what is needed in the sequel.

(i) The probability generating function (PGF) for the first return time to the origin is

$$f(t) = 1 - (1 - 4pqt^2)^{1/2}$$

from which the probability of ever returning to the origin is $f(1) = 2p$.

(ii) The PGF of the length of the first passage through k is $(f(t)/2qt)^k$.

(iii) If the PGF of the length of a positive sojourn is denoted by $F^+(t)$ and that of a negative sojourn by $F^-(t)$ then

$$F^+(t) = F^-(t) = f(t)/2.$$

(iv) The PGF of the path segment between the origin and the first positive reflection is given by

$$\sum_{i=0}^{\infty}[F^-(t)]^i \sum_{j=0}^{\infty}\left[F^+(t)\sum_{i=1}^{\infty}(F^-(t))^i\right]^j F^+(t)$$
$$= F^+(t)/(1 - F^-(t) - F^+(t)F^-(t)).$$

(v) The PGF of the path segment between any two consecutive positive reflections is

$$\sum_{j=0}^{\infty}\left[F^+(t)\sum_{i=1}^{\infty}(F^-(t))^i\right]^j F^+(t)$$
$$= F^+(t)(1 - F^-(t))/(1 - F^-(t) - F^-(t)F^+(t)).$$

(vi) The probability of the path segment between any two consecutive positive reflections is

$$\sum_{j=0}^{\infty}\left(p\sum_{i=1}^{\infty}p^i\right)^j p = pq/(q - p^2).$$

(vii) The following power series expansions in powers of pq valid for positive integers i, j and k which follow immediately from Dwass [2, (14) and (16)] are frequently used in the sequel:

$$\text{(a)} \qquad \alpha^i = \{(1/2)[1-(1-4pqt^2)^{1/2}]\}^i = \sum_{r=i}^{\infty} A_{r-i}(i,2)(pqt^2)^r,$$

where $A_a(b,c) = \dfrac{b}{b+ac}\dbinom{b+ac}{a}$;

$$\text{(b)} \qquad p^j/(1-2p) = \sum_{s=j}^{\infty} \binom{2s-j}{s-j}(pq)^s;$$

$$\text{(c)} \qquad p^k = \sum_{t=k}^{\infty} A_{t-k}(k,2)(pq)^t.$$

For ease in expression, while dealing with bivariate PGF's we will abbreviate $f(s)/2$ and $f(t)/2$ by α and β, respectively, where $f(.) = 1-[1-4pq(.)^2]^{1/2}$.

5. Joint distribution of $R^+_{m,n}$, $R^{+i}_{m,n}$ and $M^{+(i,l)}_{m,n}$

THEOREM 2. *The bivariate probability generating function of the joint distribution of R^{+i}, the index of the i^{th} positive reflection on the origin and $M^{+(i,l)}$, the length of the interval between the i^{th} and the l^{th} positive reflections on the origin $(1 \leqq i < l \leqq r)$ when R^+ equals $r \geqq 0$ is given by*

$$\begin{aligned}
h(p)/(1-2p)p^d =& \mathbf{E}(s^{R^{+i}}t^{M^{+(i,l)}}; R^+ = r)/(1-2p)p^d \\
=& \sum_{j=i}^{\infty}\sum_{u=l-i}^{\infty} \mathbf{P}(R^{+i} = 2j, M^{+(i,l)} = 2u, R^+ = r)s^{2j}t^{2u} \qquad (1)\\
=& \alpha^i(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\beta^{l-i}(1-\beta)^{l-i}\times \\
& \times (1-\beta-\beta^2)^{-(l-i)}p^{r-l+d}(1-p^2/q)^{-(r-l+1)}(pq)^{-d}.
\end{aligned}$$

PROOF. Let A, B, C and D be the first, i^{th}, l^{th} and the r^{th} positive reflection points, respectively, in the random walk $\{S_j\}$ stipulated in the theorem. Let T be the last return point to the origin. Then the path comprises seven segments viz. OA, AB, BC, CD, DT, TF and a segment beyond F, where F is the last passage through d at $(2n+d, d)$. Of these, the first segment OA has the PGF $\alpha(1-\alpha-\alpha^2)^{-1}$, by (iv) of Section 4.

The PGF's of the segments AB and BC are $(\alpha(1-\alpha)/(1-\alpha-\alpha^2))^{i-1}$ and $(\beta(1-\beta)/(1-\beta-\beta^2))^{l-i}$, respectively, by (v) of Section 4. The probability associated with the segment CD is $(pq/(q-p^2))^{r-l}$ since it entails $(r-l)$ positive reflections. The segment DT occurs with probability $q/(q-p^2)$ and the segment TF can be conceived to be one of the first passage through d when viewed from F towards T with probability $(p/q)^d$. The segment beyond F entails no return to the level d with probability $(1-2p)$. The result (1) is then obtained by using the convolution theorem.

Deductions

(i) Putting $t=1$ and $s=1$ in (1), we get, respectively,

$$\begin{aligned}\mathbf{E}(s^{R^{+i}}, R^+ = r)/(1-2p)p^d = \alpha^i(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}p^{r-i+d}\times \\ \times (1-p^2/q)^{-(r-i+1)}(pq)^{-d}\end{aligned} \tag{2}$$

and

$$\begin{aligned}\mathbf{E}(t^{M^{+(i,l)}}, R^+ = r)/(1-2p)p^d = \beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}\times \\ \times p^{r-l+d+i+1}(1-p^2/q)^{-(r-l+i+1)}(pq)^{-(d+1)}.\end{aligned} \tag{3}$$

(ii) Putting $s=t=1$ in (1), we get

$$\mathbf{P}(R^+ = r)/(1-2p)p^d = p^{r+d+1}(1-p^2/q)^{-(r+1)}(pq)^{-(d+1)} \tag{4}$$

(equivalent to Aneja [1], ch. III(17)).

(iii) Summation of (1) over r from l to ∞ gives

$$\begin{aligned}\mathbf{E}(s^{R^{+i}}t^{M^{+(i,l)}})/(1-2p)p^d = \alpha^i(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\beta^{l-i}\times \\ \times(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}p^{d-1}(pq)^{-(d-1)}/(1-2p).\end{aligned} \tag{5}$$

(iv) Summing (2) over r from i to ∞ and (3) over r from l to ∞, we get, respectively,

$$\begin{aligned}\mathbf{E}(s^{R^{+i}})/(1-2p)p^d = \alpha^i(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}p^{d-1}\times \\ \times (pq)^{-(d-1)}/(1-2p)\end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathbf{E}(t^{M^{+(i,l)}})/(1-2p)p^d = \beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}\times \\ \times p^{d+i}(1-p^2/q)^{-i}(pq)^{-d}/(1-2p).\end{aligned} \tag{7}$$

Probability distributions

The following probability distributions corresponding to the PGF's (1) to (7) can be derived with the help of Theorem 1 and the power series expansions ((viii), Section 4).

$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}=2j, M^{+(i,l)}_{m,n}=2u, R^{+}_{m,n}=r)=$$
(8)
$$=\sum_{k=0}^{i-1}\sum_{a=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{a}\sum_{g=0}^{b}(-1)^{k+m+c+a+b}\binom{i-1}{k}\binom{-i}{a}\times$$
$$\times\binom{l-i}{m}\binom{-(l-i)}{b}\binom{-(r-l+1)}{c}\binom{a}{f}\binom{b}{g}A_{\psi_1}(j-\psi_1,2)\times$$
$$\times A_{\psi_2}(u-\psi_2,2)A_{\psi_3}(r-l+3c+d,2)$$

where

$$\psi_1=j-i-k-a-f,\quad \psi_2=u-l+i-m-b-g,\quad \psi_3=n-j-u-r+l-2c.$$

(9)
$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}=2j, R^{+}_{m,n}=r)=\sum_{k=0}^{i-1}\sum_{a=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{a}(-1)^{k+a+c}\times$$
$$\times\binom{i-1}{k}\binom{-i}{a}\binom{-(r-i+1)}{c}\binom{a}{f}A_{\psi_1}(j-\psi_1,2)A_{\psi_2}(r-i+3c+d,2)$$

where $\psi_1=j-i-k-a-f$, $\psi_2=n-r+i-j-2c$.

(10)
$$\binom{2n+d}{n}\mathbf{P}(M^{+(i,l)}_{m,n}=2u, R^{+}_{m,n}=r)=\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{g}(-1)^{m+b+c}\times$$
$$\times\binom{l-i}{m}\binom{-(l-i)}{b}\binom{-(r-l+i+1)}{c}\binom{b}{g}A_{\psi_2}(u-\psi_2,2)\times$$
$$\times A_{\psi_3}(r-l+3c+d+i+1,2),$$

where $\psi_2=u-l+i-m-b-g$, $\psi_3=n-r-u-i-2c+l$.

(11)
$$\binom{2n+d}{n}\mathbf{P}(R^{+}_{m,n}=r)=\sum_{c=0}^{\infty}(-1)^{c}\binom{-(r+1)}{c}A_{n-r-2c}(r+3c+d+1,2)$$

(equivalent to Aneja [1], Ch. III(27)).

$$\binom{2n+d}{n}\mathbf{P}(R^+_{m,n}=2j,\, M^{+(i,l)}_{m,n}=2u)=$$

$$(12)\quad =\sum_{k=0}^{i-1}\sum_{a=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{f=0}^{a}\sum_{g=0}^{b}(-1)^{k+m+a+b}\binom{i-1}{k}\binom{-i}{a}\binom{l-i}{m}\times$$
$$\times\binom{-(l-i)}{b}\binom{a}{f}\binom{b}{g}A_{\psi_1}(j-\psi_1,2)A_{\psi_2}(u-\psi_2,2)\binom{2n+d-2j-2u-1}{n-j-u}$$

where $\psi_1=j-i-a-k-f$, $\psi_2=u-l+i-m-b-g$.

$$(13)\quad \binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}=2j)=\sum_{k=0}^{i-1}\sum_{a=0}^{\infty}\sum_{f=0}^{a}(-1)^{a+k}\binom{i-1}{k}\times$$
$$\times\binom{-i}{a}\binom{a}{f}A_{\psi_1}(j-\psi_1,2)\binom{2n+d-2j-1}{n-j}$$

where $\psi_1=j-i-a-k-f$.

$$(14)\quad \binom{2n+d}{n}\mathbf{P}(M^{+(i,l)}_{m,n}=2u)=\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{\infty}(-1)^{m+b+c}\binom{l-i}{m}\times$$
$$\times\binom{-(l-i)}{b}\binom{b}{g}\binom{-i}{c}A_{\psi_2}(j-\psi_2,2)\binom{2n+d-2u-c-i}{n-u+c+d}$$

where $\psi_2=u-l+i-m-b-g$.

6. Joint distribution of $R^+_{m,n}(a), R^{+i}_{m,n}(a)$ and $M^{+(i,l)}_{m,n}(a)$ when $0\leqq a<d$

THEOREM 3. *The bivariate probability generating function of the joint distribution of $R^{+i}(a)$, the index of the i^{th} positive reflection of height a and $M^{+(i,l)}(a)$, the length of the interval between the i^{th} and the l^{th} positive reflections of height a $(1\leqq i<l\leqq r)$ when $R^+(a)$ equals $r\geqq 0$ is given by, for $0\leqq a<d$*

$$h(p)/(1-2p)p^d=\mathbf{E}(s^{R^{+i}(a)}t^{M^{+(i,l)}(a)};\,R^+(a)=r)/(1-2p)p^d=$$

$$(15)\quad =\sum_{j=i}^{\infty}\sum_{u=l-i}^{\infty}\mathbf{P}(R^{+i}(a)=a+2j,\, M^{+(i,l)}(a)=2u,\, R^+(a)=r)s^{a+2j}t^{2u}=$$
$$=\alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}\times$$
$$\times p^{r-l-a+d}(1-p^2/q)^{-(r-l+1)}(pq)^{-d}s^{-a}.$$

PROOF. Consider a random walk path $\{S_j\}$ as envisaged in (15). In this random walk, let A and F denote, respectively, the first and the last return points to the height a and, let B, C, D and E denote, respectively, the first, i^{th}, l^{th} and the r^{th} positive reflection points of height a. Then the path comprises eight segments, viz. OA, AB, BC, CD, DE, EF, F to $G(2n+d, d)$ and a segment beyond G. The first segment OA is a first passage through a with its length having PGF $(\alpha/qs)^a$, by (ii) of Section 4. The segment AB can be treated as the segment between the origin and the first positive reflection with PGF $\alpha(1-\alpha-\alpha^2)^{-1}$, by (iv) of Section 4. The segments BC and CD involve $(i-1)$ and $(l-i)$ positive reflections at height a with PGF's $(\alpha(1-\alpha)/(1-\alpha-\alpha^2))^{i-1}$ and $(\beta(1-\beta)/(1-\beta-\beta^2))^{l-i}$, respectively, by (v) of Section 4. The segment DE involves $r-l$ positive reflections with probability $(pq/(q-p^2))^{r-l}$, by (vi) of Section 4 and the segment EF occurs with probability $q/(q-p^2)$. The segment from F to G can be treated as a first passage through $d-a$ viewed from G as the origin with probability of occurrence $(p/q)^{d-a}$ while the segment beyond G entails no return to the level d with probability $(1-2p)$. The result (15) is then obtained by using the convolution theorem.

Deductions

(i) Putting $t=1$ and $s=1$ in (15), we get, respectively,

$$\begin{aligned}(16)\qquad & \mathbf{E}(s^{R^{+i}(a)}, R^+(a)=r)/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\times \\ & \times p^{r+d-i-a}(1-p^2/q)^{-(r-i+1)}(pq)^{-d}s^{-a}\end{aligned}$$

and

$$\begin{aligned}(17)\qquad & \mathbf{E}(t^{M^{+(i,l)}(a)}, R^+(a)=r)/(1-2p)p^d = \beta^{l-i}(1-\beta)^{l-i}\times \\ & \times (1-\beta-\beta^2)^{-(l-i)}p^{r-l+i+d+1}(1-p^2/q)^{-(r+i-l+1)}(pq)^{-(d+1)}.\end{aligned}$$

(ii) Putting $s=t=1$ in (15), we get

$$(18)\qquad \mathbf{P}(R^+(a)=r)/(1-2p)p^d = p^{r+d+1}(1-p^2/q)^{-(r+1)}(pq)^{-(d+1)}$$

(equivalent to Aneja [1], ch. III(121)).

(iii) Summation of (15) over r from l to ∞ gives

$$\begin{aligned}(19)\qquad & \mathbf{E}(s^{R^{+i}(a)}t^{M^{+(i,l)}(a)})/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}\times \\ & \times (1-\alpha-\alpha^2)^{-i}\beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}p^{d-a-1}\times \\ & \times (pq)^{-(d-1)}s^{-a}/(1-2p).\end{aligned}$$

(iv) Summing (16) over r from i to ∞ and summing (17) over r from l to ∞, we get, respectively,

$$\begin{aligned}\mathbf{E}(s^{R^{+i}(a)})/(1-2p)p^d =&\alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}p^{d-a-1}\times\\ &\times(pq)^{-(d-1)}s^{-a}/(1-2p)\end{aligned}\tag{20}$$

and

$$\begin{aligned}\mathbf{E}(t^{M^{+(i,l)}(a)})/(1-2p)p^d =&\beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}\times\\ &\times p^{i+d}(1-p^2/q)^{-i}(pq)^{-d}/(1-2p).\end{aligned}\tag{21}$$

Probability distributions

The probability distributions corresponding to the PGF's (15) to (21) are given below.

$$\begin{aligned}&\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, M^{+(i,l)}_{m,n}(a)=2u, R^{+}_{m,n}(a)=r)=\\ &=\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{h}\sum_{g=0}^{b}(-1)^{k+m+c+h+b}\binom{i-1}{k}\times\\ &\times\binom{-i}{h}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{-(r-l-1)}{c}\binom{h}{f}\binom{b}{g}A_{\psi_1}(j-\psi_1+a,2)\times\\ &\times A_{\psi_2}(u-\psi_2,2)A_{\psi_3}(r-l+3c+d-a,2)\end{aligned}\tag{22}$$

where
$\psi_1=j-i-k-h-f,\ \psi_2=u-l+i-m-b-g,\ \psi_3=n-j-u-r+l-2c.$

$$\begin{aligned}&\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, R^{+}_{m,n}(a)=r)=\\ &=\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{h}(-1)^{k+h+c}\binom{i-1}{k}\binom{-i}{a}\times\\ &\times\binom{-(r-i+1)}{c}\binom{h}{f}A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(r+d-i-a+3c,2)\end{aligned}\tag{23}$$

where $\psi_1=j-i-k-h-f$, $\psi_2=n-r-j+i-2c$.

$$\begin{aligned}&\binom{2n+d}{n}\mathbf{P}(M^{+(i,l)}_{m,n}(a)=2u, R^{+}_{m,n}(a)=r)=\\ &=\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{g}(-1)^{m+b+c}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{-(r-l+i+1)}{c}\binom{b}{g}\times\\ &\times A_{\psi_2}(u-\psi_2,2)A_{\psi_3}(r-l+i+d+3c+1,2)\end{aligned}\tag{24}$$

where $\psi_2 = u - l + i - m - b - g$, $\psi_3 = n - r - u + l - i - 2c$.

(25)
$$\binom{2n+d}{n}\mathbf{P}(R^{+}_{m,n}(a) = r) = \sum_{c=0}^{\infty}(-1)^c\binom{-(r+1)}{c}A_{n-r-2c}(r+3c+d+1,2)$$

(equivalent to Aneja [1], ch. III(129)).

(26)
$$\begin{aligned}&\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a) = a+2j, M^{+(i,l)}_{m,n}(a) = 2u) =\\ &= \sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{f=0}^{h}\sum_{g=0}^{b}(-1)^{k+m+h+b}\binom{i-1}{k}\binom{-i}{h}\times\\ &\times\binom{l-i}{m}\binom{-(l-i)}{b}\binom{h}{f}\binom{f}{b}A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(u-\psi_2,2)\times\\ &\times\binom{2n+d-2u-2j-a-1}{n-u-j},\end{aligned}$$

where $\psi_1 = j - i - k - h - f$, $\psi_2 = u - l + i - m - b - g$.

(27)
$$\begin{aligned}&\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a) = a+2j) = \sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{f=0}^{h}(-1)^{h+k}\binom{i-1}{k}\times\\ &\times\binom{-i}{h}\binom{h}{f}A_{\psi_1}(j-\psi_1+a,2)\binom{2n+d-2j-a-1}{n-j},\end{aligned}$$

where $\psi_1 = j - i - k - h - f$.

(28)
$$\begin{aligned}&\binom{2n+d}{n}\mathbf{P}(M^{+(i,l)}_{m,n}(a) = 2u) = \sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{\infty}(-1)^{m+b+c}\times\\ &\times\binom{l-i}{m}\binom{-(l-i)}{b}\binom{b}{g}\binom{-i}{c}A_{\psi_2}(u-\psi_2,2)\binom{2n+d-2u-c-i}{n+d-u+c},\end{aligned}$$

where $\psi_2 = u - l + i - m - b - g$.

Joint distributions of $R^{+}_{m,n}(a)$, $R^{+i}_{m,n}(a)$ and $M^{+(i,l)}_{m,n}(a)$ for the remaining two cases when $0 < a = d$ and when $0 \leqq d < a$ derived likewise are quoted below.

7. Joint distribution of $R^{+}_{m,n}(a)$, $R^{+i}_{m,n}(a)$ and $M^{+(i,l)}_{m,n}(a)$ when $0 < a = d$

THEOREM 4.

(29)
$$\begin{aligned}&\mathbf{E}(s^{R^{+i}(a)}t^{M^{+(i,l)}(a)}; R^{+}(a) = r)/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}\times\\ &\times(1-\alpha-\alpha^2)^{-i}\beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}p^{r-l+2}\times\\ &\times(1-p^2/q)^{-(r-l+1)}pq^{-(a+1)}s^{-a}.\end{aligned}$$

Deductions

(i) Putting $t=1$ and $s=1$ in (29), we get, respectively,

$$\begin{aligned}(30)\qquad &\mathbf{E}(s^{R^{+i}(a)}, R^+(a)=r)/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\times\\ &\times p^{r+2-i}(1-p^2/q)^{-(r-i+1)}(pq)^{-(a+1)}s^{-a}\end{aligned}$$

and

$$\begin{aligned}(31)\qquad &\mathbf{E}(t^{M^{+(i,l)}(a)}, R^+(a)=r)/(1-2p)p^d = \beta^{l-i}(1-\beta)^{l-i}\times\\ &\times(1-\beta-\beta^2)^{-(l-i)}p^{r-l+a+i+3}(1-p^2/q)^{-(r-l+i+1)}(pq)^{-(a+2)}.\end{aligned}$$

(ii) Putting $s=t=1$ in (29), we get

$$(32)\qquad \mathbf{P}(R^+(a)=r)/(1-2p)p^d = p^{r+a+3}(1-p^2/q)^{-(r+1)}(pq)^{-(a+2)}.$$

(iii) Summation of (29) over r from l to ∞ gives

$$\begin{aligned}(33)\qquad &\mathbf{E}(s^{R^{+i}(a)}, t^{M^{+(i,l)}(a)})/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\times\\ &\times\beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}p(pq)^{-a}s^{-a}/(1-2p).\end{aligned}$$

(iv) Summing (30) over r from i to ∞ and (31) over r from l to ∞, we get, respectively,

$$\begin{aligned}(34)\qquad &\mathbf{E}(s^{R^{+i}(a)})/(1-2p)p^d =\\ &=\alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}p(pq)^{-a}s^{-a}/(1-2p)\end{aligned}$$

and

$$\begin{aligned}(35)\qquad &\mathbf{E}(t^{M^{+(i,l)}(a)})/(1-2p)p^d = \beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}\times\\ &\times p^{a+i+2}(1-p^2/q)^{-i}(pq)^{-(a+1)}/(1-2p).\end{aligned}$$

Probability distributions

$$\begin{aligned}(36)\qquad &\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, M^{+(i,l)}_{m,n}(a)=2u, R^+_{m,n}(a)=r)=\\ &=\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{h}\sum_{g=0}^{b}(-1)^{k+m+c+h+b}\times\\ &\times\binom{i-1}{k}\binom{-i}{h}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{-(r-l-1)}{c}\binom{h}{f}\binom{b}{g}\times\\ &\times A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(u-\psi_2,2)A_{\psi_3}(r-l+2+3c,2),\end{aligned}$$

where

$\psi_1 = j-i-k-h-f,\quad \psi_2 = u-l+i-m-b-g,\quad \psi_3 = n-j-u-r+l-2c-1.$

$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, R^{+}_{m,n}(a)=r)=$$

$$(37)\qquad =\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{h}(-1)^{k+h+c}\binom{i-1}{k}\binom{-i}{a}\binom{h}{f}\times$$
$$\times\binom{-(r-i+1)}{c}A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(r+2-i+3c,2),$$

where $\psi_1 = j-i-k-h-f$, $\psi_2 = n-j-r-2c+i-1$.

$$\binom{2n+d}{n}\mathbf{P}(M^{+(i,l)}_{m,n}(a)=2u, R^{+}_{m,n}(a)=r)=$$

$$(38)\qquad =\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{\infty}(-1)^{m+b+c}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{b}{g}\times$$
$$\times\binom{-(r-l+i+1)}{c}A_{\psi_2}(u-\psi_2,2)A_{\psi_3}(r-l+a+i+3c+3,2),$$

where $\psi_2 = u-l+i-m-b-g$, $\psi_3 = n-r-u+l-i-2c-1$.

$$(39)\qquad \binom{2n+d}{n}\mathbf{P}(R^{+}_{m,n}(a)=r)=\sum_{c=0}^{\infty}(-1)^{c}\binom{-(r+1)}{c}\times$$
$$\times A_{n-r-2c-1}(r+a+3c+3,2).$$

$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, M^{+(i,l)}_{m,n}(a)=2u)=$$

$$(40)\qquad =\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{f=0}^{h}\sum_{g=0}^{b}(-1)^{k+m+h+b}\times$$
$$\times\binom{i-1}{k}\binom{-i}{h}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{h}{f}\binom{f}{b}\times$$
$$\times A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(u-\psi_2,2)\binom{2n-2j-2u-1}{n-j-u-1},$$

where $\psi_1 = j-i-k-h-f$, $\psi_2 = u-l+i-m-b-g$.

$$(41)\qquad \binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j)=\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{f=0}^{h}(-1)^{k+h}\times$$
$$\times\binom{i-1}{k}\binom{-i}{a}\binom{h}{f}A_{\psi_1}(j-\psi_1+a,2)\binom{2n-2j-1}{n-j-1},$$

where $\psi_1 = j - i - k - h - f$.

$$\binom{2n+d}{n}\mathbf{P}(M_{m,n}^{+(i,l)}(a) = 2u) = \tag{42}$$
$$= \sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{\infty}(-1)^{m+b+c}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{b}{g}\times$$
$$\times\binom{-i}{c}A_{\psi_2}(u-\psi_2, 2)\binom{2n+d-2u-i-c}{n+d-u+c+1},$$

where $\psi_2 = u - l + i - m - b - g$.

8. Joint distribution of $R_{m,n}^{+}(a)$, $R_{m,n}^{+i}(a)$ and $M_{m,n}^{+(i,l)}$ when $0 \leqq d < a$

THEOREM 5.

$$\mathbf{E}(s^{R^{+i}(a)}t^{M^{+(i,l)}(a)}; R^{+}(a) = r)/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}\times \tag{43}$$
$$\times(1-\alpha-\alpha^2)^{-i}\beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}(1-p^2/q)^{-(r-l+1)}\times$$
$$\times p^{r-l-d+a+2}(pq)^{-(a+1)}s^{-a}.$$

Deductions

(i) Putting $t = 1$ and $s = 1$ in (43), we get, respectively,

$$\mathbf{E}(s^{R^{+i}(a)}, R^{+}(a) = r)/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\times \tag{44}$$
$$p^{r-d-i+a+2}(1-p^2/q)^{-(r-i+1)}(pq)^{-(a+1)}s^{-a}$$

and

$$\mathbf{E}(t^{M^{+(i,l)}(a)}, R^{+}(a) = r)/(1-2p)p^d = \beta^{l-i}(1-\beta)^{l-i}\times \tag{45}$$
$$\times(1-\beta-\beta^2)^{-(l-i)}p^{r-l-d+i+2a+3}(1-p^2/q)^{-(r-l+i+1)}(pq)^{-(a+2)}.$$

(ii) Putting $s = t = 1$ in (43), we get

$$\mathbf{P}(R^{+}(a) = r)/(1-2p)p^d = p^{r-d+2a+3}(1-p^2/q)^{-(r+1)}(pq)^{-(a+2)}. \tag{46}$$

(iii) Summation of (43) over r from l to ∞ gives

$$\mathbf{E}(s^{R^{+i}(a)}t^{M^{+(i,l)}(a)})/(1-2p)p^d = \alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}\times \tag{47}$$
$$\times\beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}p^{a-d+1}(pq)^{-a}s^{-a}/(1-2p).$$

(iv) Summing (44) over r from i to ∞ and (45) over r from l to ∞, we get, respectively,

(48)
$$\mathbf{E}(s^{R^{+i}(a)})/(1-2p)p^d = \\ = \alpha^{a+i}(1-\alpha)^{i-1}(1-\alpha-\alpha^2)^{-i}p^{a-d+1}(pq)^{-a}s^{-a}/(1-2p)$$

and

(49)
$$\mathbf{E}(t^{M^{+(i,l)}(a)})/(1-2p)p^d = \beta^{l-i}(1-\beta)^{l-i}(1-\beta-\beta^2)^{-(l-i)}\times \\ \times p^{2a+i-d+2}(1-p^2/q)^{-i}(pq)^{-(a+1)}/(1-2p).$$

Probability distributions

(50)
$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, M^{+(i,l)}_{m,n}(a)=2u, R^{+}_{m,n}(a)=r) = \\ = \sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{h}\sum_{g=0}^{b}(-1)^{k+m+c+h+b}\times \\ \times\binom{i-1}{k}\binom{-i}{h}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{-(r-l-1)}{c}\binom{h}{f}\binom{b}{g}\times \\ \times A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(u-\psi_2,2)A_{\psi_3}(r-l+a+3c-d+2,2)$$

where $\psi_1 = j-i-k-h-f$, $\psi_2 = u-l+i-m-b-g$, $\psi_3 = n-j-u-r+ +l+d-a-2c-1$.

(51)
$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, R^{+}_{m,n}(a)=r) = \\ = \sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{c=0}^{\infty}\sum_{f=0}^{h}(-1)^{k+h+c}\binom{i-1}{k}\binom{-i}{a}\binom{h}{f}\times \\ \times\binom{-(r-i+1)}{c}A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(r-d-i+a+3c+2,2)$$

where $\psi_1 = j-i-k-h-f$, $\psi_2 = n-r-j+d+i-a-2c-1$.

(52)
$$\binom{2n+d}{n}\mathbf{P}(M^{+(i,l)}_{m,n}(a)=2u, R^{+}_{m,n}(a)=r) = \\ = \sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{\infty}(-1)^{m+b+c}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{b}{g}\times \\ \times\binom{-(r-l+i+1)}{c}A_{\psi_2}(u-\psi_2,2)A_{\psi_3}(r-l-d+i+2a+3c+3,2)$$

where $\psi_2 = u-l+i-m-b-g$, $\psi_3 = n-r-u+l+d-i-a-2c-1$.

$$\binom{2n+d}{n}\mathbf{P}(R^{+}_{m,n}(a)=r)= \tag{53}$$
$$=\sum_i \binom{r+i}{i}\frac{r-d+3+2a+3i}{2n+d-r-i+1}\binom{2n+d-r-i+1}{n+d-r-2i-a-1}$$

(equivalent to Aneja [1], ch. III(83)).

$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j, M^{+(i,l)}_{m,n}(a)=2u)= \tag{54}$$
$$=\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{f=0}^{h}\sum_{g=0}^{b}(-1)^{k+m+h+b}\times$$
$$\times\binom{i-1}{k}\binom{-i}{h}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{h}{f}\binom{f}{b}\times$$
$$\times A_{\psi_1}(j-\psi_1+a,2)A_{\psi_2}(u-\psi_2,2)\binom{2n+d-2j-2u-a-1}{n+d-j-u-a-1},$$

where $\psi_1 = j-i-k-h-f$, $\psi_2 = u-l+i-m-b-g$.

$$\binom{2n+d}{n}\mathbf{P}(R^{+i}_{m,n}(a)=a+2j)=\sum_{k=0}^{i-1}\sum_{h=0}^{\infty}\sum_{f=0}^{h}(-1)^{k+h}\times \tag{55}$$
$$\times\binom{i-1}{k}\binom{-i}{a}\binom{h}{f}A_{\psi_1}(j-\psi_1+a,2)\binom{2n+d-2j-a-1}{n+d-j-a-1},$$

where $\psi_1 = j-i-k-h-f$.

$$\binom{2n+d}{n}\mathbf{P}(M^{+(i,l)}_{m,n}(a)=2u)= \tag{56}$$
$$=\sum_{m=0}^{l-i}\sum_{b=0}^{\infty}\sum_{g=0}^{b}\sum_{c=0}^{\infty}(-1)^{m+b+c}\binom{l-i}{m}\binom{-(l-i)}{b}\binom{b}{g}\times$$
$$\times\binom{-i}{c}A_{\psi_2}(u-\psi_2,2)\binom{2n+d-c-2u-i}{n+d-a-i-u-2c-1},$$

where $\psi_2 = u-l+i-m-b-g$.

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DEPARTMENT OF STATISTICS
FACULTY OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELHI
IND–110 007 DELHI
INDIA

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TRANSLATIONAL AND HOMOTHETIC CLOUDS FOR A CONVEX BODY

K. BÖRÖCZKY and V. SOLTAN

Abstract

For a convex body $A \subset E^d$, distinct types of clouds formed by translates or by homothetic copies of A are investigated, and the relations between the minimum cardinalities of these clouds are studied.

1. Introduction

The following cloud problem suggested by H. Hornich (see [4]) is well-known in discrete geometry. Let $S, S_1, \dots, S_N$ be a family of pairwise non-overlapping congruent spheres in the Euclidean space E^3. One says that $S_1, \dots, S_N$ form a *cloud* for S if every ray emanating from the center of S intersects at least one of $S_1, \dots, S_N$. The *cloud problem* is to determine the minimum number N of spheres $S_1, \dots, S_N$ forming a cloud for S.

L. Fejes Tóth [4] obtained the first result in this direction proving the inequality $N \geqq 19$. A. Heppes [7] improved this estimate showing that $N \geqq 24$. The known best lower bound $N \geqq 30$ belongs to G. Csóka [2]. Earlier, L. Danzer [3] constructed a cloud consisting of 42 spheres, and the upper bound $N \geqq 42$ still remains the best known.

The cloud problem can be obviously generalized for the case of centrally symmetric convex bodies in E^d.

There exists another variant of the cloud problem. As above, let $S, S_1, \dots, S_M$ denote a family of pairwise non-overlapping congruent spheres in E^3. Following L. Fejes Tóth, we say that the spheres $S_1, \dots, S_M$ form a *dark cloud* for S if every ray with the apex in S intersects at least one of $S_1, \dots, S_M$. The *dark cloud* problem is to find the minimum number M of spheres forming a dark cloud for S. There is known only one result, due to J. Schopp (unpublished), about this number: $M \leqq 326$. Note that this result is based on the following assertion proved by K. Böröczky [1]: for any plane H in E^3, there exists a dark cloud consisting of unit balls which are placed in a strip of finite width bounded by two planes parallel to H, w.r.t. rays starting but not lying in H. A. Heppes and L. Danzer observed that the last assertion can be generalized as follows: for any $\varepsilon > 0$ and for any hyperplane

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H in E^d, there exists a dark cloud consisting of balls of radii $\varepsilon > 0$, such that they are placed in a strip of finite width bounded by two hyperplanes parallel to H, and the distance between any two of these balls is at least 1, w.r.t. rays like above. A review of the above mentioned results can be found in A. Florian's paper [5].

In this paper, we consider some variations of the dark cloud problem for the case of arbitrary convex bodies.

Definitions and main results

Let $A, A_1, \dots, A_n$ be a family of convex bodies in E^d, $d \geqq 1$. We say that $A_1, \dots, A_n$ form a *dark cloud* for A if every ray with the apex in A intersects at least one of $A_1, \dots, A_n$. Similarly, $A_1, \dots, A_n$ form a *deep cloud* for A if every ray with the apex in A intersects the interior of at least one of $A_1, \dots, A_n$.

Below we will distinguish two cases: 1) all $A_1, \dots, A_n$ are translates of A, 2) all $A_1, \dots, A_n$ are positive homothetic copies of A.

Denote by $r(A)$ (respectively, by $s(A)$) the minimum number n of translates $A_1, \dots, A_n$ of A forming a dark cloud for A such that $A, A_1, \dots, A_n$ are pairwise non-overlapping (respectively, pairwise disjoint). Similarly, denote by $p(A)$ (respectively, by $q(A)$) the minimum number n of positive homothetic copies $A_1, \dots, A_n$ of A forming a dark cloud for A such that $A, A_1, \dots, A_n$ are pairwise non-overlapping (respectively, pairwise disjoint). If there is no finite family of translates (or homothetic copies) forming one of the above defined clouds, we let the respective number to be infinite.

Note that A. B. Harazishvili [6] studied a similar problem on the minimum number of translates of A (non-overlapping with A but, possibly, pairwise overlapping) forming a dark cloud for A.

Write $r'(A)$, $s'(A)$, $p'(A)$, and $q'(A)$ for the respective numbers in the above definitions if *deep clouds* are considered instead of dark clouds. Trivially,

$$
\begin{aligned}
&p(A) \leqq q(A) \leqq q'(A), \quad p(A) \leqq p'(A) \leqq q'(A), \\
&r(A) \leqq s(A) \leqq s'(A), \quad r(A) \leqq r'(A) \leqq s'(A).
\end{aligned} \tag{1}
$$

THEOREM 1. *For a convex body $A \subset E^d$,*

$$
\begin{gathered}
d+1 \leqq p(A) = p'(A) = q(A) = q'(A) \leqq 2d, \\
2d \leqq r(A) \leqq r'(A) = s(A) = s'(A).
\end{gathered}
$$

Any of the equalities $p(A) = 2d$, $r(A) = 2d$ holds if and only if A is a parallelotope.

Note that the question on the feasibility of the equality $r'(A) = s(A)$ has been posed by G. Csóka in private conversation.

The next theorem implies the finiteness of the number $s'(A)$.

THEOREM 2. *For any given real number $\gamma > 0$, there exists a constant $k = k(\gamma, d)$ such that for any convex body $A \subset E^d$ there is a deep cloud for A consisting of at most k pairwise disjoint and disjoint to A homothetic copies of A, with coefficient of homothety γ.*

PROBLEM. Determine the least natural numbers $n = n(d)$, $k = k(d)$ and the greatest natural number $m = m(d)$ such that $r(A) \leqq n$ and $m \leqq s'(A) \leqq k$ for any convex body $A \subset E^d$.

The proof of Theorem 2 is based on the following assertion.

THEOREM 3. *Let R, r be real positive numbers, and let B_R be a ball of radius R. There exists a finite deep cloud consisting of pairwise disjoint and disjoint to B_R balls of radii r, such that the distance between the centers of any two of them is at least one.*

Proof of Theorem 1

1. First we prove the inequality $d + 1 \leqq p(A)$. Let $A, A_1, \ldots, A_d$ be a family of pairwise non-overlapping convex bodies in E^d. Denote by Q_i a closed half-space containing A and disjoint to $\operatorname{int} A_i$, $i = 1, \ldots, d$. It is easily seen that $Q = Q_1 \cap \ldots \cap Q_d$ is an unbounded polyhedral convex body containing A. Therefore there exists a ray $l \subset \operatorname{int} Q$ with the apex in A. Hence l intersects none of $A_1, \ldots, A_d$. The last means that $A_1, \ldots, A_d$ is not a dark cloud for A; i.e., $d + 1 \leqq p(A)$.

2. Next we prove the inequality $q'(A) \leqq 2d$. Without loss of generality we may consider the origin O of E^d to be interior for A. For any regular boundary point x of A, denote by e_x the unit vector in E^d such that $x + e_x$ is the outer unit normal to A at x. Since A is compact, the set $\mathcal{G} = \{e_x : x \in \operatorname{bd} A\}$ positively generates E^d. Choose in $\mathcal{G}$ a positive basis $L = \{e_1, \ldots, e_n\}$ of minimum cardinality. It is well-known that $d + 1 \leqq n \leqq 2d$, with $n = 2d$ if and only if L is of the form $\{z_1, -z_1, \ldots, z_d, -z_d\}$ with linearly independent $z_1, \ldots, z_d$ (see, for instance, [9]).

Denote by $x_1, \ldots, x_n$ regular points in $\operatorname{bd} A$ corresponding to $e_1, \ldots, e_n$; i.e., $x_i + e_i$ is the outer unit normal to A at x_i for all $i = 1, \ldots, n$. Let S be a sphere with center O and radius r containing A in its interior. Denote by P_i the closed half-space in E^d containing S and supporting S at the point $-re_i$, $i = 1, \ldots, n$. Since the vectors $-re_1, \ldots, -re_n$ positively generate E^d, the polyhedral body $P = P_1 \cap \ldots \cap P_n$ is bounded; i.e., P is a convex polytope. Trivially, P has n facets, say $F_1, \ldots, F_n$, such that $-re_i \in \operatorname{rint} F_i$, $i = 1, \ldots, n$, where $\operatorname{rint} F_i$ means the relative interior of F_i. Denote by C_i the cone with apex O generated by F_i:

$$C_i = \{\lambda x : \lambda \geqq 0, x \in F_i\}, \qquad i = 1, \ldots, n.$$

Every C_i is an acute convex closed cone, and $C_1 \cup \ldots \cup C_n = E^d$. It is easily seen that $\operatorname{aff} F_i$ is the unique common supporting hyperplane for the bodies

P and $A_i' = A - (re_i + x_i)$ passing through the point $-re_i \in P \cap A_i'$. Since C_i is acute and closed and since $-re \cdot e_i$ is a regular boundary point for A_i', there exists a real number $\mu_i > 1$ such that for any $\mu \geqq \mu_i$ and $z \in F_i$, the ray $[0, z\rangle = \{\lambda z : \lambda \geqq 0\}$ intersects the interior of the positive homothetic copy

$$A_i'(\mu) = \mu(A_i' + re_i) - re_i.$$

Moreover, since A is compact, the number μ_i can be chosen so large that the ray $l_{yx} = y + [0, z\rangle$ intersects the interior of $A_i'(\mu)$ for any $y \in A$ and $z \in F_i$.

Now we construct a deep cloud for A consisting of pairwise disjoint positive homothetic copies $A_1'', \dots , A_n''$ of A. Put $A_1'' = A_1'(\mu_1)$. If the sets $A_1'', \dots , A_i''$, $1 \leqq i < n$ are determined, choose a number $\mu'' \geqq \mu_{i+1}$ such that for a suitable number $\nu > 0$ the set $A_{i+1}'' = \mu'' A_{i+1}'(\mu_{i+1}) - \nu e_i$ is disjoint to $A_1'' \cup \dots \cup A_i'' \cup S$ and the ray $l_{yz} = y + [0, z\rangle$ intersects the interior of A_{i+1}'' for any $y \in A$ and $z \in F_i$. Since every ray with the apex in A can be represented in the form $l_{yz} = y + [0, z\rangle$ mentioned above, the family $A_1'', \dots , A_n''$ is a deep cloud for A. Hence $q'(A) \leqq n \leqq 2d$.

3. We are going to prove the equality $p(A) = p'(A) = q(A) = q'(A)$. Due to (1), it remains to show that $q'(A) \leqq p(A)$. Note that $p(A)$ is finite, because of $p(A) \leqq q'(A) \leqq 2d$. Put $k = p(A)$ and let $A_1, \dots , A_k$ be a family of pairwise non-overlapping positive homothetic copies of A forming a dark cloud for A. Choose any point $a \in \operatorname{int} A$ and denote by B a homothetic copy of A with center of homothety a and a sufficiently small coefficient $\lambda > 0$ such that $B \subset \operatorname{int} A$ and every set

$$X_i = \cap \{\operatorname{conv}(b \cup A_i) \setminus A_i : b \in B\}, \qquad i = 1, \dots , k$$

has non-empty interior. Fix any points $c_i \in \operatorname{int} X_i$, $i = 1, \dots , k$.

Since $B \subset A$, the sets $A_1, \dots , A_k$ form a dark cloud for B. We are going to construct a deep cloud for B consisting of k pairwise disjoint positive homothetic copies of A. For this purpose, consider the sets

$$D_i = \cup\{(1 - \nu)x + \nu A_i : \nu \geqq 1, x \in B\}, \qquad i = 1, \dots , k,$$

called by V. Klee penumbras (see [10]). D_i is the union of all rays of the form

$$m_{xy} = \{(1 - \nu)x + \nu y : \nu \geqq 1\},$$

where $x \in B$, $y \in A_i$. It is easily verified that every D_i is an unbounded convex set. Since $B \cap A_i = \emptyset$, the set D_i is closed and the characteristic cone of D_i is acute. Hence, due to the choice of c_i, there is a number $\mu_i > 1$ such that for any real $\mu \geqq \mu_i$ and points $x \in B$, $y \in A_i$, the ray m_{xy} intersects the interior of the homothetic copy $A_i(\mu) = (1 - \mu)c_i + \mu A_i$.

Now we are ready to construct a deep cloud for B consisting of k pairwise disjoint homothetic copies of A. Put $A_1' = A_1(\mu_1)$. If the sets $A_1', \dots , A_i'$, $1 \leqq$ $\leqq i < k$ are determined, choose a number $\mu' \geqq \mu_{i+1}$ such that $A_{i+1}' = A_{i+1}(\mu')$

is disjoint to $A'_1 \cup \ldots \cup A'_i$ and every ray of the form m_{xy} intersects the interior of A'_{i+1} for all $x \in B$ and $y \in A_i$. It remains to verify that $A'_1, \ldots, A'_k$ form a deep cloud for B. Let l_x be a ray with apex x in B. Since the sets $A_1, \ldots, A_k$ form a dark cloud for B, the ray l_x intersects some set A_i. Therefore l_x can be represented as

$$l_x = [x, y\rangle = \{(1-\nu)x + \nu y : \nu \geqq 0\}$$

for some point $y \in A_i$. By the above said, $[x, y\rangle$ intersects the interior of A'_i. Hence $A'_1, \ldots, A'_k$ form a deep cloud for B.

If Φ is the homothety of E^d with center a and coefficient λ, then the sets $\Phi^{-1}(A'_1), \ldots, \Phi^{-1}(A'_k)$ are pairwise disjoint homothetic copies of A forming a deep cloud for A. Thus $q'(A) \leqq k = p(A)$.

4. Assume A to be a parallelotope in E^d, and let $A_1, \ldots, A_m$ be a dark cloud for A consisting of pairwise non-overlapping positive homothetic copies of A. Denote by a the center of A, and let $b_1, \ldots, b_{2d}$ be the centers of all facets of A. Consider the rays $l_1, \ldots, l_{2d}$ having a common apex a and passing through $b_1, \ldots, b_{2d}$, respectively. It is easily seen that any body A_i intersects at most one of the rays $l_1, \ldots, l_{2d}$. Therefore $m \geqq 2d$. By the above demonstrated inequality $p(A) \leqq p'(A) \leqq 2d$, therefore $p(A) = 2d$.

Now assume that A is not parallelotope. Then the minimal set $\mathcal{G} = \{e_x : x \in \operatorname{bd} A\}$ (x are regular for A) of unit vectors positively generating E^d is different from any set of the form $\{z_1, -z_1, \ldots, z_d, -z_d\}$ with linearly independent $z_1, \ldots, z_d$. In this situation, there is a positive basis $L \subset \mathcal{G}$ of cardinality n less than $2d$ (see [9]). By the above demonstrated, $p(A) \leqq n < 2d$. Hence the relation $p(A) = 2d$ is fulfilled for parallelotopes only.

5. The inequality $2d \leqq r(A)$ and the respective characterization of the parallelotope by means of the equality $2d = r(A)$ follow from [6].

6. Now we are going to prove the relation $r'(A) = s(A) = s'(A)$. Due to (1), it is sufficient to prove the inequalities $s'(A) \leqq r'(A)$ and $s'(A) \leqq s(A)$.

First we show that $s'(A) \leqq r'(A)$. If $r'(A) = \infty$, the inequality $s'(A) \leqq r'(A)$ is trivial. Assume that $r'(A)$ is finite. Put $n = r'(A)$, and let $A_1, \ldots, A_n$ be pairwise non-overlapping translates of A forming a deep cloud for A. Fix a point $x \in A$, and choose a ray l_x with apex x. The ray l_x intersects the interior of some of the bodies $A_1, \ldots, A_n$, say $A_{i_1}, \ldots, A_{i_k}$. For any point $y \in l_x \cap A_{i_j}$, denote by $\delta_j(y)$ the maximum radius of a ball with center y contained in A_{i_j}. Put

$$\delta(y) = \max\{\delta_j(y) : 1 \leqq j \leqq k\}$$

and

$$\delta(l_x) = \max\{\delta(y) : y \in l_x \cap [\, A_{i_1} \cup \ldots \cup A_{i_k} \,]\}.$$

The function $\delta(l_x)$ is positive and continuous on the family $\mathcal{R}_x$ of all rays in E^d with apex x. Since $\mathcal{R}$ is compact (in the standard metric), the minimum

$$\varrho(x) = \min\{\delta(l_x) : l_x \in \mathcal{R}_x\}$$

exists and is positive. It is easily seen that $\varrho(x)$ is a positive-valued continuous function on the compact A. Hence the minimum ϱ of $\varrho(x)$ on A is positive. In other words, for any point $x \in A$ and any ray l_z with apex z, there exist a body A_i and a point $v \in l_z \cap \operatorname{int} A_i$ such that the ball of radius ϱ and center v is contained in A_i.

Denote by $A_i(\varrho/2)$ the inner parallel body of A_i; i.e., $A_i(\varrho/2)$ is the set of all points in A_i whose distance from the boundary of A_i is at least $\varrho/2$. By the above said, the sets $A_1(\varrho/2), \dots, A_n(\varrho/2)$ form a deep cloud for A. Choose some homothetic copies $A', A'_1, \dots, A'_n$ of A corresponding to the same coefficient $\mu \in [0,1[$ of homothety such that $A' \subset \operatorname{int} A$ and $A_i(\varrho/2) \subset \subset A'_i \subset \operatorname{int} A_i$ for all $i = 1, \dots, n$. The bodies $A', A'_1, \dots, A'_n$ are pairwise disjoint and $A'_1, \dots, A'_n$ form a deep cloud for A'. If Ω is the homothety of E^d mapping A onto A', then the sets $\Omega^{-1}(A'_1), \dots, \Omega^{-1}(A'_n)$ are pairwise disjoint translates of A forming a deep cloud for A. Hence $s'(A) \leqq n = r'(A)$.

7. Now we prove the inequality $s'(A) \leqq s(A)$. As above, this inequality is trivial in case $s(A) = \infty$. Hence we can assume that $s(A)$ is finite. Put $n = s(A)$ and let $A_1, \dots, A_n$ be pairwise disjoint translates of A, disjoint to A and forming a dark cloud for A. Every A_i is of the form $A_i = a_i + A$ for some vector $a_i \in E^d$, $i = 1, \dots, n$. Since $A, A_1, \dots, A_n$ are pairwise disjoint, there is a real number $\beta \in]0,1[$ such that the sets A, and $A'_i = \beta a_i + A$, $i = 1, \dots, n$ are pairwise disjoint. We want to show that $A'_1, \dots, A'_n$ form a deep cloud for A.

Let v be any point in A. Since $A_1, \dots, A_n$ form a dark cloud for A, any ray l_v with apex v intersects at least one of $A_1, \dots, A_n$.

In other words, the union of the cones

$$C_i(v) = \{(1-\lambda)v + \lambda z : z \in A_i, \lambda \geqq 0\}, \qquad i = 1, \dots n$$

is E^d. We claim that the union of the cones

$$C'_i(v) = \{(1-\lambda)v + \lambda z : z \in A'_i, \lambda \geqq 0\}, \qquad i = 1, \dots, n$$

also is E^d. For this purpose, it is sufficient to prove the inclusion $C_i(v) \subset \subset C'_i(v)$ for all $i = 1, \dots, n$. Indeed, for any point $z \in A_i$, there exists a point

$$w \in]\beta a_i + v, (\beta - 1)a_i + z[\subset \beta a_i + A = A'_i$$

such that $[v, z\rangle = [v, w\rangle$. Hence $C_i(v) \subset C'_i(v)$, and the union of cones $C'_i(v)$, $i = 1, \dots, n$ is E^d. The last means that the bodies $A'_1, \dots, A'_n$ form a dark cloud for A.

It remains to show that $A'_1, \dots, A'_n$ form a deep cloud for A. In other words, we will prove that every ray l_v with apex $v \in A$ intersects the interior of a body A'_i, $i = 1, \dots, n$.

Denote by $A_{i_1}, \dots, A_{i_k}$ all the bodies in the family $\{A_1, \dots, A_n\}$ intersected by l_v. For any $j = 1, \dots, k$, let F_j be the Minkowski sum of A and the ray $[0, a_{i_j}\rangle$.

We claim that if l_v intersects the interior of some F_j, then l_v intersects the interior of A'_{i_j}. Indeed, assume that the ray $l_v = [v, z\rangle$, $z \in A_{i_j}$ intersects the interior of a set F_j. Then one can choose a point $x \in]\,v, z\,[\cap \operatorname{int} F_j$. Let

$$e \in]\,v, z - a_{ij}\,[\subset A, \quad f \in]\,a_{i_j} + v, z\,[\subset A_{i_j}$$

be the points such that $[e, f]$ is parallel to $[0, a_{ij}]$ and contains x. Since $x \in \operatorname{int} F_j$, one has $f \in \operatorname{int} A_{i_j}$. In this case, the point

$$w := l_v \cap]\beta a_{i_j} + v, (\beta - 1)a_{i_j} + z[$$

belongs to int A'_{i_j}.

Hence it remains to verify that l_v intersects the interior of at least one of the F_j, $j = 1, \dots, k$. Assume, in order to obtain a contradiction, that l_v intersects the interior of none of F_j. Then the line l containing l_v also intersects the interior of none of the F_j. By the separation property, there exists a hyperplane H_j through l and supporting F_j. In this case $v \in H_j$. Let P_j denote the closed half-space bounded by H_j and containing F_j, and let Q_j be the open half-space that is the complement of P_j. Since every half-space P_j contains the open cone $C = \cup \{\,]\,v, z\rangle : z \in \operatorname{int} A\}$ with vertex $v \in H_j$, the opposite open cone $C' = 2v - C$ with vertex v belongs to $Q_1 \cap \cap \dots \cap Q_k$. By the assumption, l_v intersects the bodies $A_{i_1}, \dots, A_{i_k}$ only. Hence it is possible to choose in C' a point w sufficiently close to v such that the ray l_w parallel to and similarly oriented as l_v and having apex w intersects none of $\{A_1, \dots, A_n\} \setminus \{A_{i_1}, \dots, A_{i_k}\}$. At the same time $l_w \subset Q_1 \cap \cap \dots \cap Q_j$. Hence l_w intersects none of $A_{i_1}, \dots, A_{i_k}$, i.e., l_w intersects none of $A_1, \dots, A_n$. We may even suppose that this last statement holds for any point $(1 - \lambda)v + \lambda w$, where $0 < \lambda \leqq 1$, rather than w. Now we can choose a point $u \in l_w$ sufficiently far from v such that the ray $[v, u\rangle$ intersects none of $A_1, \dots, A_n$. The last is impossible, since $A_1, \dots, A_n$ form a dark cloud for A. The obtained contradiction shows that $A'_1, \dots, A'_n$ form a deep cloud for A. □

Proof of Theorem 3

Without loss of generality, we can assume that $r < \frac{1}{2}$ and the origin O of E^d is the center of B_R. Consider an orthogonal basis $e_1, \dots, e_d$ in E^d, and denote by $\mathcal{L}$ the regular orthogonal lattice consisting of all points $x = (\eta_1, \dots, \eta_d)$ in E^d with integer coordinates $\eta_1, \dots, \eta_d$ (relative to $e_1, \dots, e_d$). Choose in E^d a vector $v = (\xi_1, \dots, \xi_d)$ with positive rationally independent coordinates $\xi_1, \dots, \xi_d$, and let l be the ray generated by v: $l = \{\lambda v : \lambda \geqq 0\}$. It is well-known that for any given $\varepsilon > 0$, and any given translate of l, there are infinitely many points in $\mathcal{L}$ whose distance from the given translate of l is less than ε, and whose first coordinates are arbitrarily large. Now fix natural numbers $m < n$, and put

$$\mathcal{L}_{mn} = \{x = (\eta_1, \dots, \eta_d) \in \mathcal{L} : m \leqq \eta_1 \leqq n\}.$$

Denote by H the $(d-1)$-dimensional subspace orthogonal to e_1.

LEMMA. *There exists a real number $\mu > 0$ (depending on l, r and $n-m$) such that every ray with the apex in H forming with l an angle of at most μ intersects some open ball of radius r and center in $\mathcal{L}_{mn}$.*

PROOF. Denote by Q the unit cube of H, i.e.,

$$Q = \Big\{x = \sum_{i=2}^{d} \zeta_i e_i : 0 \leqq \zeta_i \leqq 1,\ i = 2, \dots, d\Big\}.$$

For any point $z \in Q$, denote by l_z the ray with apex z, parallel to l. Let $\delta_{mn}(z)$ be the distance between l_z and a point in $\mathcal{L}_{mn}$ nearest to l_z. Obviously, $\delta_{mn}(z)$ is a continuous function on Q, and $\delta_{mn}(z)$ tends to zero if m is constant and n tends to infinity. Since Q is compact, we can choose number n so big that $\delta_{mn}(z) < r/2$ for every point $z \in Q$. In other words, l_z intersects some open ball of radius $r/2$ with the center in $\mathcal{L}_{mn}$. Trivial continuity arguments show the existence of a positive number μ depending on r, m, n such that for any point $z \in Q$, every ray l'_z with apex z forming with l an angle at most μ intersects some open ball of radius r with the center in $\mathcal{L}_{mn}$.

Now we observe that the above considerations can be extended from Q to the whole plane H. Indeed, if $z \in H$, then $z - v \in Q$ for some vector $v \in H$ with integer coordinates. If l'_z is a ray with the apex z forming with l an angle at most μ, then, by the above, the ray $l'_z - v$ intersects some open ball B of radius r and center in $\mathcal{L}_{mn}$. It remains to mention that $B + v$ is an open ball of radius r with the center in $\mathcal{L}_{mn}$ and $l'_z \cap (B + v) \neq \emptyset$.

Finally, we will show that in fact μ depends on r and $n - m$ only. Indeed, let $\mathcal{L}_{kl}$ be a lattice with $k > 0$, $l - k = n - m$, and let l'_z be a ray with apex $z \in H$ forming with l an angle at most μ. Denote by l' the line containing l'_z, and let w be the point of intersection of l' and the hyperplane $G = (k - m)e_1 + H$. The distance between G and $\mathcal{L}_{kl}$ is m. Then, by the above, the ray l'_w with apex w forming with l an angle at most μ intersects some open ball of radius r with center in $\mathcal{L}_{kl}$. This easily implies that l'_z intersects the same ball in $\mathcal{L}_{kl}$. □

We continue the proof of Theorem 3. Choose a lattice $\mathcal{L}_{mn}$ such that $m > R + 1$ and every ray with the apex in B_R and parallel to l intersects some open ball of radius $r/2$ with center in $\mathcal{L}_{mn}$. As it is proved in the Lemma, there exists a real number $\mu > 0$ (depending on R, r, and $n - m$) with the following property: every ray l'_w with apex w in B_R, forming with l an angle at most μ intersects some open ball of radius r with center in $\mathcal{L}_{mn}$. Since B_R is compact, there is a natural number p (depending on R, r, m, n, and μ) such that every ray l'_w with apex w in B_R, forming with l an angle at most μ intersects some open ball of radius r with center in

$$\mathcal{L}_{mn}(p) = \{x = (\eta_1, \dots, \eta_d) \in \mathcal{L}_{mn} : |\eta_i| \leqq p, \quad i = 2, \dots, d\},$$

which is a bounded part of the lattice $\mathcal{L}_{mn}$.

Now cover the whole space E^d by a finite number of infinite cones over $(d-1)$-balls as bases, say $L_1, \dots, L_t$, with common apex O, each of them having the half aperture at most μ. Let l_i be the interior ray of symmetry for L_i. Without loss of generality, put $l_1 = l$ and $m_1 = m$, $n_1 = n$, $p_1 = p$. For any $i = 2, \dots, t$, choose consecutively a set

$$\mathcal{L}_{m_i n_i}(p_i), \qquad n_i - m_i = n - m, \quad m_i > R + 1$$

(in a suitable basis $e_1^i, \dots, e_d^i$) satisfying the following properties:

(i) every ray with apex in B_R, forming with l_i an angle at most μ intersects some open ball of radius r with center in $\mathcal{L}_{m_i n_i}(p_i)$ (this is possible due to the Lemma);

(ii) the distance between the sets

$$\operatorname{conv} \mathcal{L}_{m_i n_i}(p_i), \quad \operatorname{conv}(\mathcal{L}_{m_1 n_1}(p_1) \cup \dots \cup \mathcal{L}_{m_{i-1} n_{i-1}}(p_{i-1}))$$

is at least 1.

These two conditions mean that we have constructed a finite deep cloud for B_R consisting of balls of radii r such that the distance between the centers of any two of them is at least one. These balls are pairwise disjoint by $r < \frac{1}{2}$, and are disjoint to B_R by $m_i > R + 1$. □

Proof of Theorem 2

Since the value $k = k(\gamma, d)$ is constant under affine transformations of E^d, we can apply a suitable affine transformation such that the ratio of the radii of two concentric spheres one of them containing A and the other contained in A is at most d. The last is possible due to the following well-known result, proved in [8]: for any convex body $A \subset E^d$, there are concentric and homothetic ellipsoids S and T such that $S \subset A \subset T$ and T is d times greater than S.

By the above said, we may consider A to be in the ball $\frac{1}{3\gamma}B$ and to contain the ball $\frac{1}{3\gamma d}B$, where B is the unit ball about the origin in E^d. By Theorem 3, there is a deep cloud for $\frac{\gamma+1}{3\gamma}B$ consisting of a finite number, say k, of pairwise disjoint balls of radii $\frac{1}{3d}$ such that the distance between the centers of any two of them is at least one, and all these balls are disjoint to $\frac{\gamma+1}{3\gamma}B$. From Theorem 3 it follows that the number k of hiding balls depends on γ and d only. Denote by $x_1, \dots, x_k$ the centers of these balls, and consider the family $\mathcal{N} = \{x_1 + \gamma A, \dots, x_k + \gamma A\}$ of homothetic copies of A. Since $x_i + \gamma A$ contains the ball $x_i + \frac{1}{3d}B$, $\mathcal{N}$ is a deep cloud for $\frac{\gamma+1}{3\gamma}B$. At the same time $A \subset \frac{\gamma+1}{3\gamma}B$ and $x_1 + \gamma A, \dots, x_k + \gamma A$ belong, respectively, to the pairwise disjoint balls $x_1 + \frac{1}{3}B, \dots, x_k + \frac{1}{3}B$. Hence $\mathcal{N}$ is a deep cloud for A, consisting of pairwise disjoint homothetic copies of A. Lastly we show A is disjoint to the bodies $x_i + \gamma A$. In fact, the balls $x_i + \frac{1}{3d}B$ were chosen to be disjoint to $\frac{\gamma+1}{3\gamma}B$, hence $\|x_i\| > \frac{\gamma+1}{3\gamma}$, therefore $A \cap (x_i + \gamma A) \subset (\frac{1}{3\gamma}B) \cap$ $\cap (x_i + \frac{1}{3}B) = \emptyset$. □

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EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
GEOMETRIA TANSZÉK
RÁKÓCZI ÚT 5
H–1088 BUDAPEST
HUNGARY

MATHEMATICAL INSTITUTE OF THE
ACADEMY OF SCIENCES
STR. ACADEMIEI 5
MD–2028 CHIŞINĂU
REPUBLICA MOLDOVA

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A REMARK ON AFFINE COMPLETE RINGS

H. WORACEK

The notion of affine complete universal algebras has been studied by a number of authors. One of the problems investigated is the following: When is a direct product of affine complete universal algebras affine complete?

Grätzer [3] for example shows the affine completeness of Boolean algebras, Iskander [4] determines the affine complete subdirect products of finite prime-fields. Nöbauer [7] shows that the finite direct product of rings with unity is affine complete if and only if all factors are affine complete. All affine complete abelian groups have also been determined (see [5], [6], [7]). Dorninger and Nöbauer [2] and Dorninger and Eigenthaler [1] have obtained a number of results on affine complete lattices. Further examples of affine complete algebras can be found e.g. in the papers of Werner [8], [9].

In this note we will characterize the affine completeness of an infinite direct product of commutative rings with unity.

By I we denote an arbitrary index set. Further let $\mathcal{R}_i$ for each $i \in I$ be a commutative ring with unity, and denote by $\mathcal{R}$ the direct product $\mathcal{R} = \prod_{i\in I} \mathcal{R}_i$. The theorem we are going to prove is stated as follows:

THEOREM 1. *Let $k \in \mathbb{N}$. The direct product $\mathcal{R}$ of commutative rings with unity $(\mathcal{R}_i)_{i\in I}$ is k-affine complete if and only if the following two conditions are satisfied:*

(1) *All rings $\mathcal{R}_i$ are k-affine complete,*

(2) *There exists a number $M_1 \in \mathbb{N}$, such that every unary polynomial function on the ring $\mathcal{R}_i$ for all but finitely many indices $i \in I$, can be realized by a polynomial of degree at most M_1.*

The first of the two following lemmata generalizes Condition (2) to functions of more than one variable. We will use this generalization in the proof of the above theorem.

LEMMA 1. *Assume Condition* (2) *holds for the family $(\mathcal{R}_i)_{i\in I}$ of commutative rings with unity. Then a stronger condition is also true as follows:*

For each $k \in \mathbb{N}$ and family $(p_i)_{i\in I}$ where p_i denotes a k-place polynomial of $\mathcal{R}_i$, we can find some $M_k \in \mathbb{N}$ and another family of polynomials $(q_i)_{i\in I}$,

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such that for each $i \in I$ the degree of q_i is at most M_k and the function induced by q_i coincides with that induced by p_i.

We note that this number M_k also depends on the family $(p_i)_{i\in I}$ we start with.

PROOF. To prove this statement we use induction on k. For $k=1$ our assertion is an immediate consequence of (2). So let $k\in\mathbb{N}$, $k>1$ and suppose our assertion is true for $k-1$. We start with a family of k-place polynomials $(p_i)_{i\in I}$. Let

$$p_i(x_1,\dots,x_k)=a_{i,n}(x_1,\dots,x_{k-1})x_k^n+\dots+a_{i,0}(x_1,\dots,x_{k-1}).$$

In this representation n depends on i. From Condition (2) we get a cofinite set $I'\subseteq I$ and a number M_1 such that we find polynomials $h_{i,n}$ for each $i\in I'$ and $n\in\mathbb{N}$ of degree at most M_1 that represent as functions the monomials x^n, that is $h_{i,n}(x)=x^n$ for each $x\in\mathcal{R}_i$. So we may write (as functions)

$$\begin{aligned}p_i(x_1,\dots,x_k)=&a_{i,n}(x_1,\dots,x_{k-1})h_{i,n}(x_k)+\dots+a_{i,0}(x_1,\dots,x_{k-1})=\\&=b_{i,M_1}(x_1,\dots,x_{k-1})x_k^{M_i}+\dots+b_{i,0}(x_1,\dots,x_{k-1})\end{aligned}$$

for each $i\in I'$. As a consequence of the inductive hypothesis we find polynomials $c_{i,l}(x_1,\dots,x_{k-1})$ for each $l\in\{0,\dots,M_1\}$ and $i\in I$ which induce the same functions as $b_{i,l}(x_1,\dots,x_{k-1})$ and have bounded degree, i.e. there are numbers $m_l\in\mathbb{N}$ such that

$$\deg(c_{i,l}(x_1,\dots,x_{k-1}))\leqq m_l \quad \text{for } i\in I',\ l=0,\dots,M_1$$

holds. So we get

$$(*)\qquad p_i(x_1,\dots,x_k)=c_{i,n}(x_1,\dots,x_{k-1})x_k^{M_1}+\dots+c_{i,0}(x_1,\dots,x_{k-1})$$

as functions for $i\in I'$. The degree of the polynomial q_i on the right-hand side of equation $(*)$ is at most $\max(m_1,\dots,m_{M_1})+M_1$. For $i\notin I'$ we take $q_i=p_i$. Obviously the family $(q_i)_{i\in I}$ satisfies the desired properties. □

LEMMA 2. *For each $i\in I$ let f_i be a compatible k-place function of $\mathcal{R}_i$. Then the product function f defined on $\mathcal{R}$ by*

$$f((a_i^1)_{i\in I},\dots,(a_i^k)_{i\in I})=(f_i(a_i^1,\dots,a_i^k))_{i\in I}$$

is again compatible.

PROOF. It is sufficient to show that f is compatible with every principle congruence. This follows from the fact that in a direct product of rings with unity each principle ideal is a product ideal. □

Now we are in the position to prove Theorem 1.

PROOF. To prove the 'only if' part of the theorem, suppose $\mathcal{R}$ is k-affine complete. Let f be any compatible k-place function on $\mathcal{R}_{i_0}$ for some $i_0\in I$

and consider the function $\hat{f}$ on $\mathcal{R}$ defined as $\hat{f}((a_i^1)_{i\in I}, \dots, (a_i^k)_{i\in I}) = (b_i)_{i\in I}$ with

$$b_i = \begin{cases} f(a_{i_0}^1, \dots, a_{i_0}^k) & \text{for } i = i_0, \\ 0 & \text{for } i \neq i_0. \end{cases}$$

Due to Lemma 2 $\hat{f}$ is compatible with every ideal of $\mathcal{R}$. We find then a k-place polynomial $\hat{p}$ of $\mathcal{R}$, which realizes $\hat{f}$. Taking the i_0-th projections of each coefficient of $\hat{p}$ we obtain a polynomial p_{i_0} of $\mathcal{R}_{i_0}$ which clearly realizes f. That shows that $\mathcal{R}_{i_0}$ is k-affine complete.

Now suppose, on the contrary, that Condition (2) fails to be true. We can then find a sequence $i_l \in I$ $(l \in \mathbb{N})$ of distinct indices, and polynomial functions f_{i_l} of $\mathcal{R}_{i_l}$ which cannot be written as polynomials with degree at most l. We consider the function $\hat{f}$ of $\mathcal{R}$ which is defined as $\hat{f}((a_i)_{i\in I}) = (b_i)_{i\in I}$ where

$$b_i = \begin{cases} f_{i_l}(a_{i_l}) & \text{if } i = i_l \text{ and } l \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

This function again is compatible on $\mathcal{R}$, and therefore must be represented by some polynomial $\hat{p}$ of $\mathcal{R}$. If the degree of $\hat{p}$ equals n, every function f_{i_l} is realized as a polynomial of $\mathcal{R}_{i_l}$ of degree at most n, again by taking the i_l-the projection of each coefficient. This is a contradiction to our choice of f_{i_l} for $l > n$.

It remains to show the sufficiency of our Conditions (1) and (2). Let f be a compatible k-place function on $\mathcal{R}$, then f decomposes into a direct product of functions f_i which are compatible in $\mathcal{R}_i$. This means

$$f((a_i^1)_{i\in I}, \dots, (a_i^k)_{i\in I}) = (b_i)_{i\in I}, \quad \text{with}$$

$$b_i = f_i(a_i^1, \dots, a_i^k).$$

From Assumption (1) we find a polynomial p_i for each $i \in I$ representing the function f_i. By Lemma 1 we obtain other polynomials q_i, which induce the same functions as p_i, but have bounded degree. We can therefore define a polynomial q as 'product polynomial', i.e. we just take the families of corresponding coefficients of the polynomials q_i as coefficients of q. The function f is clearly realized by q. □

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INSTITUT FÜR ANALYSIS, TECHNISCHE MATHEMATIK
UND VERSICHERUNGSMATHEMATIK
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRASSE 8–10
A–1040 WIEN
AUSTRIA

hworacek@pop.tuwien.ac.at

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ON THE RADICAL THEORY OF GRADED RINGS WHICH ARE INVERSIVE HEMIRINGS

C. ROOS and R. WIEGANDT[1]

Abstract

Talat Shaheen and Yusuf [5] studied the radical theory of inversive hemirings, Krempa and Terlikowska-Osłowska [2] investigated that of semigroup graded rings, two seemingly totally different structures. Using Piochi's [3] description of congruences of inversive hemirings we exhibit that inversive hemirings are in fact semigroup graded rings whose additive semigroup is a strong semilattice of groups. Also we show that the powerful method of Puczyłowski [4] applies to a category of inversive hemirings, or equivalently, to a category of semigroup graded rings, and in this way we settle three major issues of their radical theory: the ADS-property of radicals, Sands' characterization of semisimple classes and the termination of the Kurosh lower radical construction at the first limit ordinal. Thus beside recovering results of [5], we contribute to the radical theory of inversive hemirings as well as to that of semigroup graded rings.

1. Preliminaries

A *hemiring* is a nonempty set A with two operations $+$ and $\cdot$ where $(A,+)$ is a commutative semigroup with identity 0, $(A,\cdot)$ is a semigroup, $\cdot$ distributes over $+$ from both sides and $0\cdot a=a\cdot 0=0$ holds for all $a\in\in A$. A hemiring is called an *additively inversive hemiring* (briefly: *inversive hemiring*) if $(A,+)$ is an inverse semigroup, that is, for every element $a\in A$ there exists exactly one element $(-a)$ such that $a+(-a)+a=a$ and $(-a)++a+(-a)=(-a)$. In view of [3] Lemma 1.1 it holds $a(-b)=-(ab)=(-a)b$ for all $a,b\in A$. As usual, we shall omit parentheses and write simply $a-b$ for $a+(-b)$.

A subset I of an inversive hemiring A is said to be an *ideal* of A, denoted by $I\triangleleft A$, if I is an inversive subhemiring such that $A\cdot I\cup I\cdot A\subseteq I$ and I contains all additive idempotents of A. The set E of all additive idempotents of A is an ideal in A (cf. [5]), and is called the *trivial ideal* of A. Clearly, the additive semigroup E^+ of E is a semilattice.

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PROPOSITION 1.1. *For any elements a,b of an inversive hemiring A with $a-a=e$ and $b-b=f$ it holds $af=ef=eb$.*

PROOF. $ef=(a-a)f=af+(-a)f=af-af=a(f-f)=af$.

Let A be any inversive hemiring with trivial ideal E, and let κ be a congruence on A. The restriction $\operatorname{tr}\kappa$ of κ to E is called the *trace* of κ and the subset $\ker\kappa$ of all elements of A which are congruent to some additive idempotent modulo κ, is called the *kernel* of κ. It is readily seen that $\ker\kappa$ is an ideal of A. If I is an ideal of A and ξ a congruence on the trivial ideal E of A such that

(1.1) $\quad\forall a\in A$ and $\forall e\in E$, $a+e\in I$ and $(a-a)\xi e$ imply $a\in I$,

then (ξ,I) is said to be a *congruence pair* on A.

PROPOSITION 1.2 ([3] Theorem 1.5). *For every congruence κ on an inversive hemiring A, $(\operatorname{tr}\kappa,\ \ker\kappa)$ is a congruence pair, and conversely, for every congruence pair (ξ,I) on A there exists exactly one congruence κ on A such that $\operatorname{tr}\kappa=\xi$ and $\ker\kappa=I$. In particular, if ε_E denotes the identity relation on E, then (ε_E,I) is a congruence pair for every ideal I of A.*

Let us notice that condition (1.1) effects rather the congruence ξ on E and not the ideal I of A: (1.1) means that the congruence ξ on E extends to a congruence on $A/(\varepsilon_E,I)$, in particular, on $A\cong A/(\varepsilon_E,E)$.

A mapping $\varphi\colon A\longrightarrow B$ between two inversive hemirings is said to be a *hemiring homomorphism* if it preserves addition and multiplication. By the homomorphism theorem and Proposition 1.2, if φ is a surjective hemiring homomorphism, then

$$B\cong A/\kappa=A/(\operatorname{tr}\kappa,\ker\kappa)$$

so we may say that $\operatorname{tr}\kappa$ is the *trace* of φ and $\ker\kappa$ is the kernel of φ, and we may write $\operatorname{tr}\varphi=\operatorname{tr}\kappa$ and $\ker\varphi=\ker\kappa$. A hemiring homomorphism $\varphi\colon A\longrightarrow B$ is called a P-homomorphism (for principal homomorphism), if $\operatorname{tr}\varphi=\varepsilon_E$ is the identity relation on the trivial ideal E of A. For the sake of brevity we shall sometimes write A/I for $A/(\varepsilon_E,I)$ and A/ξ for $A/(\xi,E)$.

From the previous considerations it follows that every hemiring homomorphism φ factors as

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[d, "\mu"'] & & B \\
A/\ker\varphi \arrow[r, "\nu"] & (A/\ker\varphi)/(\operatorname{tr}\varphi,E) \cong A/(\operatorname{tr}\varphi,\ker\varphi) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where μ is a P-homomorphism, ν maps E to φE (effecting A as demanded in (1.1)) and ψ is an embedding (and as such a P-homomorphism), moreover,

φ factors also as

$$\begin{array}{ccc} A & \xrightarrow{\quad\quad\quad\varphi\quad\quad\quad} & B \\ \vartheta \Big\downarrow & & \Big\uparrow \psi \\ A/\operatorname{tr}\varphi & \xrightarrow{\quad\tau\quad} & (A/\operatorname{tr}\varphi)/(\operatorname{tr}\varphi, E) \cong A/(\operatorname{tr}\varphi, \ker\varphi) \end{array}$$

where ϑ maps E onto φE and τ is a P-homomorphism with kernel $\ker\varphi/(\operatorname{tr}\varphi, E)$.

2. Inversive hemirings and graded rings

An additive semigroup A is called a *strong semilattice of groups* over an additive meet-semilattice Λ, if $A = \bigcup\limits_{\lambda\in\Lambda} A_\lambda$ where each A_λ is a group such that

(2.1) for each $\lambda, \mu \in \Lambda$ with $\lambda \geqq \mu$ there exists a homomorphism $\varphi^\lambda_\mu \colon A_\lambda \to$ $\to A_\mu$ such that φ^λ_λ is the identical mapping and $\varphi^\mu_\nu \varphi^\lambda_\mu = \varphi^\lambda_\nu$ for every $\lambda \geqq \mu \geqq \nu$,

(2.2) for each $\lambda, \mu \in \Lambda$ and $a \in A_\lambda, b \in A_\mu$ one has

$$a + b = \varphi^\lambda_{\lambda+\mu}(a) + \varphi^\mu_{\lambda+\mu}(b) \in A_{\lambda+\mu}.$$

As is well-known, every commutative inverse semigroup is a strong semilattice of groups (cf.[1], IV.2.1). Hence the additive semigroup $(A, +)$ of an inversive hemiring A is a strong semilattice of groups $A = \bigcup\limits_{\lambda\in\Lambda} A_\lambda$. We recall Piochi's [3] Theorem 2.4.

PROPOSITION 2.1. *Let* $A = \bigcup\limits_{\lambda\in\Lambda} A_\lambda$ *be an inversive hemiring.*

(i) *For every* $\lambda, \mu \in \Lambda$ *there exists a* $\nu \in \Lambda$ *such that* $A_\lambda A_\mu \subseteqq A_\nu$*; we put* $\lambda * \mu = \nu$.

(ii) *For every* $\lambda, \mu, \nu \in \Lambda$, $(\lambda + \mu) * \nu = \lambda * \nu + \mu * \nu$.

(iii) *For every* $\lambda, \mu, \nu \in \Lambda$, *if* $\lambda \leqq \mu$, *then* $\lambda * \nu \leqq \mu * \nu$ *and* $\nu * \lambda \leqq \nu * \mu$; $\lambda * \lambda \leqq \lambda * \mu \leqq \mu * \mu$ *and* $\lambda * \lambda \leqq \mu * \lambda \leqq \mu * \mu$.

Let R be a ring and Λ a multiplicative semigroup. Assume that the additive group R^+ is a direct sum $R^+ = \bigoplus\limits_{\lambda\in\Lambda} A^+_\lambda$ of its subgroups A^+_λ such that $A_\lambda \cdot A_\mu \subseteqq A_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$. The subset $A = \bigcup\limits_{\lambda\in\Lambda} A_\lambda$ is called a Λ-*graded ring*. Let us mention that some authors call R a Λ-graded ring;

for our purpose the previous definition will be more convenient. Anyway, the difference between the two definitions is only of a technical nature. A homomorphism φ: $A \to B$ between two Λ-graded rings A and B is said to be a Λ-*homomorphism* if $\varphi(A_\lambda) \subseteqq B_\lambda, \lambda \in \Lambda$. A Λ-*ideal* I of a Λ-graded ring A is defined as a Λ-graded ring I satisfying $I \subseteqq A$ and $A \cdot I \cup I \cdot A \subseteqq I$.

After these preparations we establish a natural connection between inversive hemirings and graded rings.

THEOREM 2.2. *If A is an inversive hemiring with trivial ideal E, then A is an $E^\times$-graded ring $A = \bigcup_{e \in E} A_e$ and A^+ is a strong semilattice of (commutative) groups over the additive semilattice E^+. Conversely, let E be an inversive hemiring consisting of additive idempotents. If A is an $E^\times$-graded ring such that $A^+ = \bigcup_{e \in E} A_e^+$ is a strong semilattice of (commutative) groups over the additive semilattice E^+, then A is an inversive hemiring with trivial ideal*

$$E_A = \{0_e \in A | 0_e + 0_e = 0_e, e \in E\}$$

consisting of additive idempotents, and $E_A \cong E$.

PROOF. The point is in both directions of the proof that in view of Proposition 2.1 there is a one-to-one correspondence between the elements of the trivial ideal E_A of the inversive hemiring and of the grading semilattice E of the graded ring.

Let A be any inversive hemiring. Now A^+ is a strong semilattice of groups where the semilattice is isomorphic to E^+. Moreover, by Proposition 2.1 and the observation thereafter one has $A_e \cdot A_f \subseteqq A_{ef}$ showing that A is an $E^\times$-graded ring.

Conversely, let A be any $E^\times$-graded ring such that $A^+ = \bigcup_{e \in E} A_e$ is a strong semilattice of groups. Then A^+ is an inverse semigroup. Since E is an inversive hemiring, the semilattice E^+ contains a greatest element o. By the distributivity of A we see that E_A is an ideal of A. The element 0_o is clearly an additive neutral element of A; since 0_o is the zero element of E_A which is an ideal of A, 0_o must be a multiplicative zero element of A. Hence A is an inversive hemiring.

Notice that the element 0 of the inversive hemiring $A = \bigcup_{e \in E} A_e$ is the greatest element with respect to the relation $\leqq$ defined on E, as $e + f$ is the greatest lower bound of e and f.

PROPOSITION 2.3. *A P-homomorphism between inversive hemirings is an $E^\times$-homomorphism between the corresponding $E^\times$-graded rings, and vice versa. An ideal of an inversive hemiring is an $E^\times$-graded ideal of the corresponding graded ring and vice versa.*

PROOF. This follows easily from the previous considerations.

Let $\varphi\colon A \to B$ be any hemiring homomorphism. Then φ maps the trivial ideal E of A into the trivial ideal F of B. In the language of graded rings this means that φ maps the $E^{\times}$-graded ring A into the $F^{\times}$-graded ring B such that $\varphi(A_e) \subseteqq B_{\varphi(e)}, e \in E$. Thus each inversive hemiring E consisting of additive idempotents determines a category $\mathcal{C}(E)$ consisting of all inversive hemirings with trivial ideal E and all P-homomorphisms. Further, $\mathcal{C}(E)$ can be viewed as the category of all $E^{\times}$-graded rings A, with A^{+} a strong semilattice of groups over the additive semilattice E^{+}, and of all $E^{\times}$-homomorphisms. Furthermore, we can define the category $\mathcal{C}(\mathbf{E})$ of all inversive hemirings with trivial ideals in the class $\mathbf{E}$ of inversive hemirings consisting of additive idempotents, its objects are all hemirings with trivial ideal $E \in \mathbf{E}$, and its morphisms are all hemiring morphisms between the objects. Expressed equivalently, $\mathcal{C}(\mathbf{E})$ is the category of all $E^{\times}$-graded rings A such that A^{+} is a strong semilattice of groups over the additive semilattice E^{+}, E runs through the elements of $\mathbf{E}$, and the morphisms are hemiring homomorphisms, that is, homomorphisms preserving addition and multiplication.

3. A category catering a decent radical theory

We start with fixing the variety $\mathbf{E}$ of all inversive hemirings subject to the following identities:

$$e + e = e, \tag{3.1}$$
$$ef = fe, \tag{3.2}$$
$$efg = e^2 fg \tag{3.3}$$

for all $e, f, g \in E \in \mathbf{E}$.

Let us observe that in the context of (3.1) and (3.2) the identity (3.3) means that every element $e \in E \in \mathbf{E}$ induces a hemiring homomorphism

$$\varphi_e\colon E \to eE, \qquad \varphi_e(g) = eg, \qquad g \in E. \tag{3.4}$$

Clearly, every distributive lattice E satisfies (3.1),(3.2) and (3.3). Another example is an inversive hemiring E such that $E^2 = \{0\}$.

Let us define $A^{(e)} = \bigcup\limits_{f \in eE} A_f$ for any inversive hemiring $A = \bigcup\limits_{e \in E} A_e$. For any element $a \in A^{(e)}$ and $b \in A$ there exist $f, g \in E$ such that $a \in A_{ef}$ and $b \in A_g$. Hence

$$ab \in A_{ef} A_g \subseteqq A_{efg} \subseteqq A^{(e)}.$$

Since by (3.2) the multiplication in E is commutative, we have also

$$ba \in A_g A_{ef} \subseteqq A_{gef} = A_{egf} \subseteqq A^{(e)}.$$

Hence $A^{(e)} \cdot A \cup A \cdot A^{(e)} \subseteqq A^{(e)}$ holds, although $A^{(e)}$ need not be an ideal of A because $eE \neq E$ may happen.

In the sequel we shall work in the category $\mathcal{C}(\mathbf{E})$ of inversive hemirings with all hemiring homomorphisms as introduced in Section 2. In order to apply the results of Puczyłowski [4], we must ensure that axioms A1 to A6 of that paper are satisfied in our category $\mathcal{C}(\mathbf{E})$. The validity of the first two axioms is obvious.

A1. The set $l(A)$ of all ideals of an inversive hemiring A is a complete lattice.

A2. If $K \triangleleft A$ and $K \subseteqq I \triangleleft A$, then $K \triangleleft I$.

A3. The lattice $l(A/I)$ is isomorphic to the lattice of all ideals K of A with $I \subseteqq K \triangleleft A$ (cf.[9] Theorem 13).

Before stating axiom $A4$, we define a relation $\sim$ between inversive hemirings from $\mathcal{C}(\mathbf{E})$ as follows.

$B \sim C \iff$ there exists an $A \in \mathcal{C}(\mathbf{E})$ such that $B \cong A/(\xi, E)$ and $C \cong A/(\eta, E)$ where (ξ, E) and (η, E) are congruence pairs on A.

One readily sees that $\sim$ is an equivalence. Moreover, in view of Proposition 1.2 there is a one-to-one correspondence between the ideals of A and those of $A/(\xi, E)$. Hence it holds

A4. If $B \sim C$, then the lattices $l(B)$ and $l(C)$ are isomorphic.

A5. The isomorphism theorems are valid for every $I \triangleleft A$ and $K \triangleleft A$:

$$K/(K \cap I) \cong (K+I)/I$$

and if $K \subseteqq I$, then

$$(A/K)/(I/K) \cong A/I$$

(cf. [9] Theorems 13 and 14).

Remark 1. Our considerations remain true also for the following weaker version of the equivalence relation $\sim$: $B \sim C \Leftrightarrow$ there exist an $A \in \mathcal{C}(\mathbf{E})$ with trivial ideal E and elements $e, f \in E$ such that $B \cong A/(\xi, E)$ and $C \cong A/(\eta, E)$ where ξ and η stand for the traces of the congruences determined by the hemiring homomorphisms φ_e and φ_f as in (3.4), respectively.

Remark 2. Puczyłowski [4] assumed the existence of an up to isomorphism unique zero algebra O which is the smallest element in each lattice $l(A)$. This is not the case in $\mathcal{C}(\mathbf{E})$: the trivial ideals are, in general, not isomorphic, though the trivial ideals of $\sim$-equivalent inversive hemirings are obviously $\sim$-equivalent, and this suffices.

Puczyłowski's axiom A6 is a version of Terlikowska-Osłowska's condition in [6], and it is decisive in obtaining the desired results in radical theory. Prior to proving the validity of axiom A6, we need some preparations.

Proposition 3.1. *Let $A = \bigcup_{e \in E} A_e$ be an inversive hemiring and $K \triangleleft I \triangleleft A$.*

(i) *For every* $a \in A$ *with* $a \in A_e$ *the mapping*

$$\varphi\colon K \to (aK + K^{(e)})/K^{(e)}$$

defined by $\varphi(k) = ak + K^{(e)}, k \in K$, *is a surjective hemiring homomorphism.*

(ii) φ *factors as*

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\mu"'] & (aK + K^{(e)})/K^{(e)} \\ K/(\operatorname{tr}\varphi, E) \arrow[r, "\nu"] & K/(\operatorname{tr}\varphi, \ker\varphi) \arrow[u, "\psi"'] \end{tikzcd}$$

where ν *is a P-homomorphism and* ψ *is an isomorphism.*

(iii) $((aK + K^{(e)})/K^{(e)})^2 \cong eE$.

(iv) $aK + K^{(e)} \triangleleft I^{(e)}$.

(v) $\ker\varphi \triangleleft I$.

Proof. Clearly, φ is surjective and preserves addition and additive inverses.

Let $k_1, k_2 \in K$ be arbitrary elements with $k_1 \in K_f = K \cap A_f$ and $k_2 \in \in K_g = K \cap A_g$. Since

$$ak_1k_2 \in A_eK_fK_g \subseteqq (A_{ef} \cap I)K_g = I_{ef}K_g \subseteqq I_{efg} \cap K = K_{efg} \subseteqq K^{(e)},$$

it follows

$$\varphi(k_1k_2) = ak_1k_2 + K^{(e)} = K^{(e)},$$

and since

$$ak_1ak_2 \in A_eK_fA_eK_g \subseteqq (A_{efe} \cap I)K_g = I_{efe}K_g \subseteqq I_{efeg} \cap K = K_{efg} \subseteqq K^{(e)},$$

we get

$$\varphi(k_1)\varphi(k_2) = (ak_1 + K^{(e)})(ak_2 + K^{(e)}) = ak_1ak_2 + K^{(e)} = K^{(e)}.$$

Thus φ is a homomorphism, and the proof yields also (iii).

Assertion (ii) is obvious in view of the observations made after Proposition 1.2.

A straightforward computation verifies statement (iv).

For proving (v), let us consider elements $k \in K_f = A_f \cap K$ and $i \in I$ with $i \in I_g = A_g \cap I$. Now by $k \in \ker\varphi$ we have $ak \in A_eK_f \cap K \subseteqq K_{ef}$, and so

$$\varphi(ki) = aki + K^{(e)} \subseteqq K_{ef}I_g + K^{(e)} \subseteqq (I_{efg} \cap K) + K^{(e)} = K_{efg} + K^{(e)},$$

and
$$\varphi(ik) = aik + K^{(e)} \subseteq A_e I_g K_f + K^{(e)} \subseteq (A_{eg} \cap I)K_f =$$
$$= I_{eg}K_f \subseteq I_{egf} \cap K = K_{egf} \subseteq K^{(e)}.$$
Hence $(\ker\varphi)I \cup I(\ker\varphi) \subseteq \ker\varphi$ is valid. Furthermore, by $\ker\varphi \triangleleft K \triangleleft I$, the trivial ideal of $\ker\varphi$ is that of I (and A). Thus also $\ker\varphi \triangleleft I$ has been established.

The next statement is, in fact, axiom A6 of Puczyłowski [4] translated into the terms of inversive hemirings.

PROPOSITION 3.2. *Let $A \in \mathcal{C}(\mathbf{E})$ be an inversive hemiring, $K \triangleleft I \triangleleft A$. Let us suppose that whenever J and M are ideals of I such that*

(i) $M \subseteq K \subseteq J$,

(ii) $K/M \cong J/K$,

(iii) $\overline{Y} \triangleleft \overline{X} \triangleleft \overline{K} := K/M$ *implies* $\overline{Y} \triangleleft \overline{K}$, *then* $M = K = J$.

Under these hypotheses $K \triangleleft A$.

PROOF. Suppose that the assertion is not true, that is, K is not an ideal of A. Then there exists an element $a \in A$ such that $aK \nsubseteq A$ or $Ka \nsubseteq A$. Obviously it suffices to deal only with the first case.

Let us consider the mapping
$$\varphi\colon K \to (aK + K^{(e)})/K^{(e)} := J^{(e)}/K^{(e)},$$
as given in Proposition 3.1. By Proposition 3.1 we have
$$K/(\operatorname{tr}\varphi, \ker\varphi) \cong J^{(e)}/K^{(e)}.$$
Since $K/\operatorname{tr}\varphi \cong K^{(e)}$ and
$$K/(\operatorname{tr}\varphi, \ker\varphi) \cong K^{(e)}/(\ker\varphi)^{(e)},$$
we conclude
$$K^{(e)}/(\ker\varphi)^{(e)} \cong J^{(e)}/K^{(e)}.$$
We have also $(\ker\varphi)^{(e)} \triangleleft K^{(e)} \triangleleft I^{(e)}$, and by Proposition 3.1 (iv) and (v)
$$(\ker\varphi)^{(e)} \triangleleft I^{(e)} \qquad \text{and} \qquad J^{(e)} \triangleleft I^{(e)}.$$
In view of Proposition 3.1 (iii) we have $(J^{(e)}/K^{(e)})^2 \cong eE$, and hence condition (iii) of this Proposition is fulfilled. Thus the hypothesis yields
$$(\ker\varphi)^{(e)} = K^{(e)} = J^{(e)},$$
and consequently
$$(aK + K^{(e)})/K^{(e)} = J^{(e)}/K^{(e)} \cong eE,$$
which implies $aK + K^{(e)} = K^{(e)}$. Since the inversive hemiring $K^{(e)}$ possesses a 0-element, it follows $aK \subseteq K^{(e)} \subseteq K$, a contradiction. Thus Proposition 3.2 has been proved.

4. Radical theory for inversive hemirings

As exhibited in [2], a category $\mathcal{C}(E)$, $E \in \mathbf{E}$, with $E^{\times}$- homomorphisms only, does not have enough morphisms for proving the ADS-property of radicals. This explains why the ADS-property was proved only for homomorphically closed radical classes in [5]. And this is the reason why we consider the category $\mathcal{C}(\mathbf{E})$ with all homomorphisms.

In what follows we apply Puczyłowski's [4] method and in this way we settle clue issues of the radical theory of inversive hemirings and, of course, of graded rings which are inversive hemirings.

All classes of inversive hemirings considered in the sequel, are assumed to be subclasses of objects of the category $\mathcal{C}(\mathbf{E})$ of Section 3 and to be *closed under the equivalence relation* $\sim$.

A class $\mathbf{R}$ of inversive hemirings is called a *(Kurosh-Amitsur) radical class*, if

(4.1) $\mathbf{R}$ is closed under P-homomorphisms,

(4.2) every inversive hemiring $A \in \mathcal{C}(\mathbf{E})$ contains a largest $\mathbf{R}$-ideal $\mathbf{R}(A)$,

(4.3) $\mathbf{R}(A/\mathbf{R}(A))$ is the trivial ideal for all $A \in \mathcal{C}(\mathbf{E})$.

We shall use the standard notions and notations of radical theory (cf. for instance [8]).

PROPOSITION 4.1. *Every radical class $\mathbf{R}$ is homomorphically closed and also closed under trace extensions: if $A/\xi \in \mathbf{R}$, then also $A \in \mathbf{R}$ for every congruence ξ on the trivial ideal of A satisfying* (1.1).

PROOF. The assertion is a straightforward consequence of the assumption that all classes considered are closed under the equivalence relation $\sim$.

Every radical class $\mathbf{R}$ determines its semisimple class

$$\mathcal{S}(\mathbf{R}) = \{A \in \mathcal{C}(\mathbf{E}) | \mathbf{R}(A) \text{ is the trivial ideal}\}.$$

In fact, the semisimple class $\mathcal{S}\mathbf{R}$ is closed under the equivalence relation $\sim$, or otherwise expressed, we have

PROPOSITION 4.2. *Every semisimple class is closed under trace extensions and under trace homomorphisms: if $A \in \mathcal{S}\mathbf{R}$ and ξ is any congruence on the trivial ideal of A satisfying* (1.1), *then $A/\xi \in \mathcal{S}\mathbf{R}$.*

PROOF. The proof is a straightforward application of Proposition 4.1.

Having verified the validity of axioms A1 to A6 in $\mathcal{C}(\mathbf{E})$, we can apply the results of [4] and obtain the following Theorem. For short, we shall write $\sim$-radical and $\sim$-semisimple class for a radical and a semisimple class being closed under the equivalence relation $\sim$.

THEOREM 4.3. (i) *Every $\sim$-radical $\mathbf{R}$ in $\mathcal{C}(\mathbf{E})$ is an ADS-radical: $\mathbf{R}(I) \triangleleft \triangleleft A$ for all $I \triangleleft A \in \mathcal{C}(\mathbf{E})$. Hence every $\sim$-semisimple class is hereditary: $I \triangleleft \triangleleft A \in \mathcal{S}\mathbf{R}$ implies $I \in \mathcal{S}\mathbf{R}$.*

(ii) *A subclass* $\mathbf{S}$ *of* $C(\mathbf{E})$ *is a* $\sim$*-semisimple class if and only if* $\mathbf{S}$ *is regular (if* $E \neq I \triangleleft A \in \mathbf{S}$*, then there exists a* $K \triangleleft I$ *with* $E \neq I/K \in \mathbf{S}$*), coinductive (for any descending chain* $\{I_\alpha\}$ *of ideals of* $A \in C(\mathbf{E})$ *such that* $A/I_\alpha \in \mathbf{S}$ *for each* α*, also* $A/\cap I_\alpha \in \mathbf{S}$ *holds), and closed under ideal extensions (*$I \in \mathbf{S}, A/I \in \mathbf{S}$ *imply* $A \in \mathbf{S}$*), trace extensions and trace homomorphisms.*

(iii) *Starting from any subclass* $\mathbf{M}$ *of* $C(\mathbf{E})$ *which is closed under homomorphisms and trace extensions, the Kurosh lower radical construction stops at the first limit ordinal* (for the lower radical construction we refer to [4]).

REMARK 3. As we may see from the proofs of Propositions 3.1 and 3.2, trace extensions do not occur there at all, and also in proving Theorem 4.3 (i) we do not need them. Hence it suffices to consider only homomorphic images with respect to hemiring homomorphisms φ_e in (3.4). Thus every homomorphically closed radical class $\mathbf{R} \subseteq C(\mathbf{E})$ has the ADS-property without demanding that $\mathbf{R}$ be a $\sim$-radical.

Finally we show how the results of Talat Shaheen and Yusuf [5] fit into the scheme of our theory. In [5] a radical class $\mathbf{R}$ has been defined by (4.1), (4.2) and (4.3), and it has been demanded that $\mathbf{R}$ be homomorphically closed. Moreover, in [5] the ADS-property of $\mathbf{R}$ has been proved under the assumptions

$$ef = efe = fe \qquad \text{for all} \qquad e, f \in E \in \mathbf{E}, \tag{4.4}$$

$$\mathbf{R}(A^{(e)}) = \mathbf{R}(A) \cap A^{(e)} \qquad \text{for all} \qquad e \in E \in \mathbf{E} \text{ and } A \in C(\mathbf{E}). \tag{4.5}$$

Obviously, (4.4) implies (3.2) and (3.3). Hence in view of Remark 3 our results infer the ADS-property for homomorphically closed radical classes of [5]. Furthermore, we have also

PROPOSITION 4.4. *Any* $\sim$*-radical class* $\mathbf{R}$ *satisfies* (4.5).

PROOF. A straightforward set theoretic computation yields

$$\mathbf{R}(A) \cap A^{(e)} = \bigcup_{f \in eE} (\mathbf{R}(A) \cap A_f) = (\mathbf{R}(A))^{(e)}.$$

Since $\mathbf{R}$ is homomorphically closed, it follows

$$(\mathbf{R}(A))^{(e)} \cong \mathbf{R}(A)/(\xi, E) \in \mathbf{R}$$

with an appropriate congruence ξ on E. Hence $(\mathbf{R}(A))^{(e)} \triangleleft A^{(e)}$ implies

$$\mathbf{R}(A) \cap A^{(e)} = (\mathbf{R}(A))^{(e)} \subseteq \mathbf{R}(A^{(e)}).$$

On the other hand, from $\mathbf{R}(A^{(e)}) \triangleleft A^{(e)}$ we conclude

$$\mathbf{R}(A^{(e)}) \cong I/(\xi, E)$$

with a suitable ideal I of A. Since $\mathbf{R}$ is closed under trace extensions, it follows $I \in \mathbf{R}$, whence $I \subseteqq \mathbf{R}(A)$. Thus we get

$$\mathbf{R}(A^{(e)}) = I \cap A^{(e)} \subseteqq \mathbf{R}(A) \cap A^{(e)}.$$

and (4.5) has been established.

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DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
MEKELWEG 4
NL–2628 CD DELFT
THE NETHERLANDS

c.roos@math.tudelft.nl

MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

wiegandt@math-inst.hu

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 119–126

A MINIMAL CONDITION FOR STOCHASTIC APPROXIMATION

D.-Q. TUYEN

Abstract

A sharp condition is found for noises in Robbins–Monro procedures, which assures the convergence of the procedures if the noises obey the Strong Law of Large Numbers, without further assumptions on dependences between the noises.

1. Introduction

For Robbins–Monro procedure (R.M.P.) (see [1], [2])

$$X_{n+1} = X_n - a_n(f(X_n) + \varepsilon_n), \tag{1}$$

where $X_n, a_n \geqq 0$ and ε_n are real numbers, $n = 1, 2, \ldots,$ (X_1 is arbitrary) a question is raised what is the best condition for noises ε_n under which the procedure still converges. For the identity function $f(x) = x$ and a_n such that $\lim a_n = 0$, $\sum a_n = \infty$, $X_n \to 0$ if and only if

$$\sum_{i=1}^{n} \prod_{k=i+1}^{n} (1 - a_k) a_i \varepsilon_i \to 0, \tag{2}$$

since X_{n+1} equals just the term in (2) plus $X_1 \prod_1^n (1 - a_k)$. We will show that under quite natural and well-known conditions on the (non-continuous) function $f(x)$ and a_n, condition (2) is also sufficient for the convergence of X_n in (1).

In probabilistic language this means that if ε_n satisfy the Strong Law of Large Numbers (S.L.L.N.) (in the sense below) then X_n converges a.s.. So the problem of the convergence of R.M.P.s with dependent noises is reduced to the one of S.L.L.N. for dependent random variables (r.v.s). Many known S.L.L.N.s for mixing sequences (in any sense) or martingale-like ones therefore can apply to R.M.P.s.

Also there are given two examples to show the results below are not easy to improve.

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2. Main results

The following lemma is needed.

LEMMA. *Suppose a real function $f\colon R \to R$ and real numbers a_n, ε_n satisfy:*

(i) $|f(x)| \leqq K(1+|x|)$ *for some constant* K,

(ii) $a_n \geqq 0$ *and* $\lim\limits_n a_n = 0$,

(iii) $\lim\limits_n a_n \varepsilon_n = 0$,

and for X_n in (1) *suppose* $\lim X_n$ *does not exist. Then, for any* $[a,b] \subset \subset (\liminf X_n, \limsup X_n)$, $\varepsilon > 0$ *and* N, *there exist* $n \geqq m > N$ *and* $n' \geqq m' > > N$ *such that*

$$X_i \in [a,b] \quad for \quad m \leqq i \leqq n, \quad X_{n+1} > b \quad and \quad X_{m-1} < a \leqq X_m < a + \varepsilon,$$

$$X_i \in [a,b] \quad for \quad m' \leqq i \leqq n', \quad X_{n'+1} < a \quad and \quad X_{m'-1} > b \geqq X_{m'} > b - \varepsilon.$$

PROOF. For $[a,b]$ and any N there are $p > q > N$ such that $X_q < a$ and $X_p > b$. Then there exists $q < m \leqq p$ such that $X_{m-1} < a$ and $X_i \geqq a$ for $m \leqq i \leqq p$. Define n between m and p such that $X_{n+1} > b$ and $X_i \leqq b$ for $m \leqq i \leqq n$. n will exist if we show $X_m < a + \varepsilon < b$. Since $X_{m-1} < a$ by (i) we have

$$\begin{aligned} X_m \leqq & X_{m-1} + a_{m-1}K(1+|X_{m-1}|) - a_{m-1}\varepsilon_{m-1} \leqq \\ & \leqq \max\{a(1+\varepsilon'), a(1-\varepsilon')\} + 2\varepsilon' \end{aligned}$$

if N is so large that $|a_{m-1}K| < \varepsilon'$ and $|a_{m-1}\varepsilon_{m-1}| < \varepsilon'$ for $\varepsilon' < 1$, using (ii) and (iii). Obviously, we can choose ε' so small that the last term $< a + \varepsilon < b$ for any given ε. Similarly, we can prove the second conclusion of the lemma.

For any function f define $f_+ := \{x; f(x) > 0\}$ and $f_- := \{x; f(x) < 0\}$. A finite point a is called a weak root of f if the set $\lim f(a) := \{\lim\limits_n f(x_n);$ for all sequences (x_n) such that $x_n \to a\}$ contains either zero or two non-zero values with different signs.

THEOREM 1. *Suppose a real function f and real numbers a_n, ε_n satisfy:*

(i) $|f(x)| \leqq K(1+|x|)$ *for some constant* K,

(ii) *there are* $c < d$ *such that* $\sup f(-\infty, c] \leqq 0$ *and* $\inf f[d, \infty) \geqq 0$,

(iii) $a_n \geqq 0$, $\lim\limits_n a_n = 0$ *and* $\sum a_n = \infty$,

(iv) $\sum\limits_1 a_i \varepsilon_i$ *converges.*

Then:

(I) $0 \in [\liminf f(X_n), \limsup f(X_n)]$,

(II) *if* $\lim X_n$ *exists then it is a weak root of f, otherwise*

$$(\liminf X_n, \limsup X_n) \subset \bar{f}_+ \cap \bar{f}_- \subset [c,d].$$

Consequently, if ε_n are r.v.s on some probability space $(\Omega, \mathcal{F}, P)$ for which (iv) *holds with probability* 1 *then* (I) *and* (II) *hold with probability* 1, *too.*

Note that

$$\bar{f}_+ \cap \bar{f}_- \subset \{\text{weak roots of } f\}.$$

PROOF. By induction, for $n \geqq m > 0$

$$(X_{n+1} - X_m) + \sum_m^n a_i f(X_i) = -\sum_m^n a_i \varepsilon_i. \tag{3}$$

(a) Suppose $\lim X_n$ does not exist. For any $[a,b] \subset (\liminf X_n, \limsup X_n)$ and $0 < \varepsilon < (b-a)/2$ and for large enough N, by (iv) the last term of (3) is less than ε in absolute values for $n \geqq m > N$. Applying the lemma to such $[a,b]$, ε and N there are $n \geqq m > N$ such that $X_{n+1} - X_m > b - a - \varepsilon > \varepsilon$ and $X_i \in [a,b]$ for $m \leqq i \leqq n$. Consequently, by (3) for such m, n $\sum_m^n a_i f(X_i) < 0$. So there exists at least one $i > N$ such that $X_i \in [a,b]$ and $f(X_i) < 0$. Similarly, using the second conclusion of the lemma there is an $i > N$ such that $X_i \in [a,b]$ and $f(X_i) > 0$. So $f_- \cap [a,b] \neq 0$ and $f_+ \cap [a,b] \neq 0$, that is (II) is verified, since $[a,b]$ can be arbitrarily chosen. Also (I) holds, too, since N can be taken arbitrarily large.

(b) Suppose $\lim X_n$ exists. If $\lim X_n = \infty$, then there is a N such that $X_n > d$ for $n > N$. Then $X_{n+1} - X_N \to \infty$ $(n \to \infty)$ and $\sum_N^n a_i f(X_i) \geqq 0$ by (ii), which contradict (3) and (iv), so $\lim X_n < \infty$. By similar way we can see $\lim X_n > -\infty$. If $a = \lim X_n$ is finite then for any $n \geqq m > N$, N large enough, the middle term of (3) is less than any ε in absolute values since this holds for the other terms of (3). Hence by choosing m and n such that $\sum_m^n a_i \geqq 1$ which is possible by (iii) we see that there are at least one $m \leqq i \leqq n$ such that $f(X_i) \leqq \varepsilon$ and at least one $m \leqq j \leqq n$ such that $f(X_j) \geqq -\varepsilon$. Since ε can be arbitrarily small these facts exclude the case when $\liminf f(X_n)$ and $\limsup f(X_n)$ are not zero and have the same signs (or equal one of $\pm\infty$ both). So (I) holds. Since $\liminf f(X_n)$ and $\limsup f(X_n)$ belong to $\lim f(a)$, by (I) a is a weak root of f.

THEOREM 2. *Suppose a real function f and real numbers a_n, ε_n satisfy:*

(i) $|f(x)| \leqq K(1+|x|)$ *for some constant* K,

(ii) *there exist $c < d$ such that* $\inf f[d,\infty) > 0$ *and* $\sup f(-\infty, c] < 0$,

(iii) $a_n \geqq 0$, $\lim a_n = 0$ *and* $\sum a_n = \infty$,

(iv) *condition* (2) *holds, i.e.* $\lim_n \sum_1^n \prod_i^n a_i \varepsilon_i = 0$, *where*

$$\prod_i^n = \begin{cases} \prod_{k=i+1}^{n} (1-a_k), & \textit{for } i<n \\ 1, & \textit{for } i=n. \end{cases}$$

Then:

(I) $0 \in [\liminf f(X_n), \limsup f(X_n)]$,

(II) $[\liminf X_n, \limsup X_n] \subset \{\textit{weak roots of } f\} \subset [c,d]$.

So, if ε_n *are r.v.s for which* (iv) *holds with probability* 1 *then* (I) *and* (II) *are true with probability* 1, *too.*

We note that condition (iv) is implied by (iv) of Theorem 1 by Kronecker's Lemma, and both of them imply (iii) of the lemma since

$$a_n \varepsilon_n = \sum_1^n \prod_i^n a_i \varepsilon_i - (1-a_n) \sum_1^{n-1} \prod_i^{n-1} a_i \varepsilon_i.$$

PROOF. For the procedure in (1) without loss of generality we can suppose $0 < a_n < 1$ for all n, so $0 < \prod_i^n < 1$ for all $1 \leqq i < n$. Writing (1) in the form

$$X_{n+1} = X_n(1-a_n) - a_n(f(X_n) - X_n + \varepsilon_n),$$

by induction we have

$$X_{n+1} - \prod_{m-1}^{n} X_m = -\sum_{i=m}^{n} \prod_i^n a_i(f(X_i) - X_i + \varepsilon_i)$$

for any $n \geqq m \geqq 1$. So, using the equality

$$\prod_{m-1}^{n} + \sum_{i=m}^{n} \prod_i^n a_i = 1 \tag{4}$$

$(n \geqq m \geqq 1)$ we have

$$\begin{aligned} \prod_{m-1}^{n} (X_{n+1} - X_m) + \sum_m^n \prod_i^n a_i(X_{n+1} - X_i) + \sum_m^n \prod_i^n a_i f(X_i) = \\ = -\sum_m^n \prod_i^n a_i \varepsilon_i. \end{aligned} \tag{5}$$

Let A_{mn}, B_{mn}, C_{mn} and D_{mn} denote in turns the four sums in the last equality. Like the role of (3) in the previous proof we shall use (5) to derive the conclusions. Note that since $D_{mn} = D_{1n} - \prod_{m-1}^{n} D_{1,m-1}$ by (iv) $D_{mn} \to 0$ as $m, n \to \infty$.

(a) Suppose $\liminf X_n < \limsup X_n$. For any $[a,b] \subset \subset (\liminf X_n, \limsup X_n)$, $0 < \varepsilon < (b-a)/2$ and for any large enough N $|D_{mn}| < \varepsilon$ for $n \geqq m > N$. Applying the lemma to such $[a,b]$, ε and N there exist $n \geqq m > N$ such that $B_{mn} \geqq 0$ and $A_{mn} \geqq \prod_{m-1}^{n} (b - a - \varepsilon) > \prod_{m-1}^{n} \varepsilon$. Consequently, by (5), $C_{mn} < \left(1 - \prod_{m-1}^{n}\right)\varepsilon$. Also by (4) we have

$$C_{mn} \geqq \sum_{i=m}^{n} \prod_{i}^{n} a_i \inf_i f(X_i) = \left(1 - \prod_{m-1}^{n}\right) \inf_i f(X_i), \tag{6}$$

consequently $\inf_{m \leqq i \leqq n} f(X_i) < \varepsilon$. Since ε and N can be arbitrary and $X_i \in \in [a,b]$ we obtain that $\inf f[a,b] \leqq 0$ and $\liminf f(X_n) \leqq 0$. For a point $a \in \in (\liminf X_n, \limsup X_n)$ take a sequence $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset a$ such that $\cap [a_i, b_i] = a$ then $\inf f[a_i, b_i] \uparrow 0$ or $\uparrow p < 0$, hence 0 or p belongs to $\lim f(a)$, respectively. Similarly using the other conclusion of the lemma we have $\sup f[a,b] \geqq 0$ and $\limsup f(X_n) \geqq 0$ and hence 0 or any $q > 0$ belongs to $\lim f(a)$. So we obtain (I) and (II) noting that the set $\{$weak roots of $f\}$ is closed for any functions f satisfying (i).

(b) Suppose $\lim X_n$ exists. To show it is finite, suppose in the contrary, $\lim X_n = \infty$. Then there are infinitely many n_i such that $X_{n_i} = \max\{X_1, X_2, \dots, X_{n_i}\}$ and a N such that $X_n > d$ and $|D_{mn}| < \inf f[d, \infty)/2$ for any $n \geqq m \geqq N$. For any $n_i > N$ we have $A_{N, n_i - 1} \geqq 0$ and $B_{N, n_i - 1} \geqq 0$. Since by (iii) $\prod_{N-1}^{n} \leqq \exp\left(-\sum_{N+1}^{n} a_i\right) \to 0$ as $n \to \infty$ we can find $n_i > N$ such that $\prod_{N-1}^{n_i - 1} < 1/2$, hence by (6) $C_{N, n_i - 1} > \inf f[d, \infty)/2$ which contradicts (5). Similarly we can see $\lim X_n > -\infty$. Since $\lim X_n$ is finite, for any $\varepsilon > 0$ there is an N such that for any $n \geqq m > N$ $|D_{mn}| < \varepsilon$ and by (4) $|A_{mn} + B_{mn}| \leqq \leqq \prod_{m-1}^{n} \varepsilon + \left(1 - \prod_{m-1}^{n}\right)\varepsilon = \varepsilon$, hence by (5) we obtain $|C_{mn}| < 2\varepsilon$. So, for any $n > m > N$ such that $\prod_{m-1}^{n} < 1/2$ by (6) $\inf_{m \leqq i \leqq n} f(X_i) < 4\varepsilon$. Similarly we have $C_{mn} \leqq \left(1 - \prod_{m-1}^{n}\right) \sup_{m \leqq i \leqq n} f(X_i)$, consequently $\sup_{m \leqq i \leqq n} f(X_i) > -4\varepsilon$. As ε can be arbitrary we obtain (I) and (II) similarly to the last part of the previous proof.

Condition (iv) of Theorem 2 can be presented in a known form of probability theory.

DEFINITION. The r.v.s ε_n having common expectations $E\varepsilon_n = E\varepsilon_1$ are said to obey the S.L.L.N. with the double sequence of real numbers (b_{in}) $i \leqq n$ if

(i) $\sum_1^n b_{in} \to 1$ and $\max_i(b_{in}) \to 0$ as $n \to \infty$,

(ii) $\sum_1^n b_{in}\varepsilon_i \to E\varepsilon_1$ a.s. as $n \to \infty$.

Then (iv) of Theorem 2 can be replaced by

(iv′) the r.v.s ε_n having zero expectations obey the S.L.L.N. with the double sequence $b_{in} = \prod_i^n a_i$.

This replacement is possible because by (iii) of Theorem 2 and (4) $\sum_1^n b_{in} \to 1$ and by supposing $a_n < 1$

$$\max_i(b_{in}) \leqq \max_{1 \leqq i < g(n)} \prod_i^n + \max_{g(n) \leqq i \leqq n} a_i \leqq \prod_{g(n)-1}^n + \max_{g(n) \leqq i \leqq n} a_i \to 0$$

where $g(n)$ is the first k such that $\sum_1^k a_i > \left(\sum_1^n a_i\right)/2$ for which $g(n) \to \infty$ and

$$\sum_{g(n)}^n a_i = \sum_1^n a_i - \sum_1^{g(n)-1} a_i \geqq \left(\sum_1^n a_i\right)/2 \to \infty \quad \text{as} \quad n \to \infty,$$

using (iii) of Theorem 2 again.

For any continuous function f, $\bar{f}_+ \cap \bar{f}_-$ contains only separate points (no interval) or is empty, and a weak root of f is the root of f.

COROLLARY 1.1. *If in addition to the conditions of Theorem 1 the function f is supposed to be continuous then X_n always converges to one root of f.*

THEOREM 3. *If f is a continuous function satisfying* (i) *and* (ii) *of Theorem 2, $a_n = 1/n$ and the r.v.s ε_n, having zero expectations, satisfy the usual S.L.L.N. then X_n in* (1) *converges a.s. to one root or to one interval of roots of f.*

3. Two examples

Two conclusions: $\lim X_n$ is a weak root of f in Theorem 1 and the statement (II) of Theorem 2 cannot be replaced by the better ones: $\lim X_n \in \in \bar{f}_+ \cap \bar{f}_-$ and $[\liminf X_n, \limsup X_n] \subset \bar{f}_+ \cap \bar{f}_-$, respectively, as is shown by the following examples.

Let f be such that: $f(x)=x$ for $x<0$, $f(x)=0$ for $0\leqq x\leqq 1$, and $f(x)=$ $=x-1$ for $x>1$. Let a_n,ε_n be such that: $a_n\downarrow 0$, $a_n<1/2$, $a_{n+1}/a_n>1/2$, $\sum a_n=\infty$, $\sum a_n^2<\infty$, and $\varepsilon_n=1$ if $X_n>1$ and $\varepsilon_n=-1$, otherwise. Let $X_1=1$. Then

$$X_{n+1}=\begin{cases} X_n+a_n & \text{if } 0\leqq X_n\leqq 1,\\ X_n(1-a_n) & \text{if } X_n>1.\end{cases}$$

Since X_{n+1} steps forward (grows) if $0\leqq X_n\leqq 1$ and backward if $X_n>1$, by induction we can see that $0\leqq X_n\leqq 2$ and $|X_{n+1}-X_n|<1$. Hence using the above equations we have $|X_{n+1}-X_n|\leqq 2a_n\downarrow 0$. Also by induction we have $|X_n-1|\leqq 2a_n$ since if this holds for n then $|X_{n+1}-1|\leqq a_n<2a_{n+1}$, using the equations and that $-2a_n\leqq X_n-1\leqq 0$ for $0\leqq X_n\leqq 1$ and $0\leqq X_n-1\leqq 2a_n$ for $X_n>1$. So $X_n\to 1$. To show $\sum_1^n a_i\varepsilon_i$ converges, hence that all conditions of Theorem 1 hold, by (3) we need to show $\sum_1^n a_if(X_i)$ converges. We have for any $m\leqq n$

$$\Big|\sum_m^n a_if(X_i)\Big|=\sum_{\{m\leqq i\leqq n;X_i>1\}} a_i(X_i-1)\leqq\sum_m^n 2a_i^2.$$

Since $\sum a_i^2<\infty$, $\sum_1^n a_if(X_i)$ converges. So all conditions of Theorem 1 are satisfied, $\lim X_n=1$, but 1 does not belong to $\bar f_+\cap\bar f_-$ which is empty in this case.

Now for the same function f let $a_n=1/n$, ε_n be such that $1>|\varepsilon_n|\downarrow 0$, $\sum a_n|\varepsilon_n|=\infty$, and the sign of ε_n equals the sign of $f(X_n)$ for $f(X_n)\neq 0$ and equals the sign of ε_{n-1} otherwise. Let $X_1=3/2$. So

$$X_{n+1}=\begin{cases} X_n-a_n(X_n-1)-a_n|\varepsilon_n| & \text{if } X_n>1,\\ X_n+a_n|X_n|+a_n|\varepsilon_n| & \text{if } X_n<0,\\ X_n\pm a_n|\varepsilon_n| & \text{otherwise.}\end{cases}$$

We can see that: (a) using these equations and using induction for $n\geqq$ $\geqq 2$ we have $-1<X_n<2$ and $|X_{n+1}-X_n|<1$ for all n, (b) using also the equations and (a) we obtain $|X_{n+1}-X_n|\leqq 2a_n\downarrow 0$, (c) any step X_n out of $[0,1]$ will make following steps $X_{n+1},X_{n+2},\dots$ be in the direction toward $[0,1]$ and keep that direction until to another step taking out of $[0,1]$ at the other end, (d) since for any m there is an n such that $\sum_m^n a_i|\varepsilon_i|\geqq 2$, the steps in $[0,1]$ without changing the direction always lead to the one which is out of $[0,1]$, consequently there are infinitely many X_n which are out of $[0,1]$ at both ends, so by (b) $[\liminf X_n,\limsup X_n]=[0,1]$, (e) the term in (iv) of

Theorem 2 equals $\left(\sum_1^n \varepsilon_i\right)/n$, so it tends to zero since $|\varepsilon_n| \downarrow 0$, that is (iv) of Theorem 2 is satisfied. So all conditions of Theorem 2 hold for this example but $[0,1] = [\liminf X_n, \limsup X_n] \nsubseteq \bar{f}_+ \cap \bar{f}_-$, because the last one is empty.

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INSTITUTE OF MATHEMATICS
P.O. BOX 631
BO HO
HANOI
VIETNAM

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 127–140

ORDER OF BEST APPROXIMATION BY POLYNOMIALS IN $H_q^p (p \geqq 1, q > 1)$ SPACES

F. XING

Abstract

In this paper we improve a Hardy–Littlewood type theorem of [1] and extend the estimation for order of best approximation by polynomials in $H_q^p (p \geqq 1, q \geqq 2)$ spaces obtained in [1], to $H_q^p (p \geqq 1, q > 1)$ spaces.

If a function $f(z)$ is analytic in the unit disc $|z| < 1$ and with the parameters p and q satisfies the condition

$$\|f(z)\|^p \overset{\text{def}}{=} \iint\limits_{|z|<1} (1 - |z|^2)^{q-2} |f(z)|^p dx\, dy < +\infty, \qquad z = x + iy,$$

then we say that the function $f(z)$ belongs to H_q^p spaces.

As the integral

$$\iint\limits_{|z|<1} (1 - |z|^2)^{q-2} dx\, dy$$

exists only for $q > 1$, we must assume $q > 1$ in the definition of H_q^p spaces. However, the parameter p can be an arbitrary positive real number. But $p \geqq 1$ and $0 < p < 1$ make H_q^p spaces having different properties. When $p \geqq 1$, according to the norm $\|f(z)\|$ mentioned above, H_q^p spaces are Banach spaces. When $0 < p < 1$, the $\|f(z)\|$ defined above does not satisfy the triangle inequality. So $H_q^p (0 < p < 1, q > 1)$ are not normed spaces, but Fréchet spaces.

Hardy–Littlewood type theorems are very important for estimating the order of best approximation by polynomials in H_q^p spaces. In [2] there is a Hardy–Littlewood type theorem for functions of $\operatorname{Lip} \alpha$ $(0 < \alpha \leqq 1)$ classes of H_q^p spaces when $q = 2$, $p \in [1, +\infty)$. But there are no proofs in [2]. In [1] we obtained a Hardy–Littlewood type theorem for functions in $\operatorname{Lip} \alpha$ $(0 < \alpha \leqq 1)$ classes and an estimation for order of best approximation by polynomials in H_q^p $(p \geqq 1, q \geqq 2)$ spaces. In this paper we investigate the Hardy–Littlewood type theorem for general H_q^p $(0 < p < +\infty, 1 < q < +\infty)$ spaces and extend the estimation for order of best approximation by polynomials in H_q^p $(p \geqq 1, q \geqq 2)$ spaces obtained in [1], to H_q^p $(p \geqq 1, q > 1)$ spaces.

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Besides the notation $\|f(z)\|$ we have mentioned, the following notations are needed in this paper:

$$\|f(z)\|^p_{(D)} \stackrel{\text{def}}{=} \int_D (1-|z|^2)^{q-2}|f(z)|^p dx\,dy, \qquad z=x+iy,$$

where D is a certain subset of the unit disc $|z|<1$;

$$\omega(\tau,f) \stackrel{\text{def}}{=} \sup_{-\tau\leqq h\leqq\tau} \{\|f(ze^{ih})-f(z)\|\};$$

$$\rho^{(n)}(f) \stackrel{\text{def}}{=} \inf_{Q_n(z)\in\Pi_n} \{\|f(z)-Q_n(z)\|\},$$

where Π_n is the set of all algebraic polynomials of degree at most n.

The main results in this paper are Theorem 1 and Theorem 2.

THEOREM 1 (Hardy–Littlewood type theorem). *Suppose that* $f(z)\in H^p_q$ $(0<p<+\infty, 1<q<+\infty)$, $\mu=\min\{p,1\}$, $\Omega(t)$ *is a non-negative function, and*

$$\int_0^1 \frac{\Omega^\mu(t)}{t}dt<+\infty.$$

Suppose for any $0<t<h$ *we have*

$$\frac{\Omega^\mu(h)}{h}\leqq\frac{2\Omega^\mu(t)}{t},$$

and for any $\varrho\in(0,1)$

$$\|f'(z)\|_{(|z|<\varrho)}\leqq\frac{\Omega(1-\varrho)}{1-\varrho}.$$

Then for any $\tau\in[0,1]$ *we have*

$$\omega(\tau,f)\leqq c_{p,q}\left(\int_0^\tau \frac{\Omega^\mu(t)}{t}dt\right)^{1/\mu},$$

where $c_{p,q}$ *is a constant depending only on* p *and* q.[1]

To prove Theorem 1, we divide it into three parts. At first we prove

[1] In this paper we denote constants depending only on p and q by $c_{p,q}$. The same $c_{p,q}$ at different places may take different values. For example, we can write $c_{p,q}+1=c_{p,q}$ etc. In this paper we will denote by c_p constants depending only on p. The same c_p at different places may take different values, too.

LEMMA 1. *Let $r \in (0,1)$, $0 < h < r$, $0 < p < 1$. Then for any function $f(z)$ analytic in the unit disc $|z| < 1$ we have*

$$\int_0^{2\pi} |f(re^{i\theta}) - f[(r-h)e^{i\theta}]|^p d\theta \leqq c_p \int_{h/2}^{h} (h-t)^{p-1} \left(\int_0^{2\pi} |f'[(r-h+t)e^{i\theta}]|^p d\theta \right) dt.$$

PROOF. Let $\{h_k\}_{k=0}^{\infty}$ be a series such that $h_0 = 0$, $h_k \to h$ monotone increasingly. By using the triangle inequality in Fréchet spaces we obtain

$$\begin{aligned} |f(re^{i\theta}) - f[(r-h)e^{i\theta}]|^p &\leqq \sum_{k=0}^{\infty} |f[(r-h+h_k)e^{i\theta}] - f[(r-h+h_{k+1})e^{i\theta}]|^p \leqq \\ &\leqq \sum_{k=0}^{\infty} \left\{ \int_{h_k}^{h_{k+1}} |f'[(r-h+t)e^{i\theta}]| dt \right\}^p \leqq \\ &\leqq \sum_{k=0}^{\infty} (h_{k+1} - h_k)^p \sup_{h_k < t < h_{k+1}} |f'[(r-h+t)e^{i\theta}]|^p. \end{aligned}$$

Using the Hardy–Littlewood maximal theorem in H^p spaces (see [3], § 1.6) gives

$$\int_0^{2\pi} \sup_{h_k < t < h_{k+1}} |f'[(r-h+t)e^{i\theta}]|^p d\theta \leqq c_p \int_0^{2\pi} |f'[(r-h+h_{k+1})e^{i\theta}]|^p d\theta.$$

It follows that

$$\begin{aligned} &\int_0^{2\pi} |f(re^{i\theta}) - f[(r-h)e^{i\theta}]|^p d\theta \leqq \\ (1) \qquad &\leqq c_p \sum_{k=0}^{\infty} (h_{k+1} - h_k)^p \int_0^{2\pi} |f'[(r-h+h_{k+1})e^{i\theta}]|^p d\theta. \end{aligned}$$

If, in particular, $h_k = (1-2^{-k})h$, $k = 0, 1, 2, \dots$, then

$$(h_{k+1} - h_k)^p = (h_{k+1} - h_k)(h_{k+1} - h_k)^{p-1} = 2(h_{k+2} - h_{k+1})(h - h_{k+1})^{p-1}.$$

Both $(h-t)^{p-1}$ and $\int_0^{2\pi} |f'[(r-h+t)e^{i\theta}]|^p d\theta$ are monotone increasing with respect to $t \in [0, h]$. Therefore from (1) we obtain

$$\int_0^{2\pi} |f(re^{i\theta}) - f[(r-h)e^{i\theta}]|^p d\theta \leqq$$

$$\leqq 2c_p \sum_{k=0}^{\infty} (h_{k+2}-h_{k+1})(h-h_{k+1})^{p-1} \int_0^{2\pi} |f'[(r-h+h_{k+1})e^{i\theta}]|^p d\theta \leqq$$

$$\leqq c_p \sum_{k=0}^{\infty} \int_{h_{k+1}}^{h_{k+2}} (h-t)^{p-1} \left(\int_0^{2\pi} |f'[(r-h+t)e^{i\theta}]|^p d\theta \right) dt =$$

$$= c_p \int_{h/2}^{h} (h-t)^{p-1} \left(\int_0^{2\pi} |f'[(r-h+t)e^{i\theta}]|^p d\theta \right) dt.$$

The proof of Lemma 1 is complete.

LEMMA 2. *If a function $F(z)$ is analytic on the closed unit disc $|z| \leqq 1$, then*

$$\int_0^{2\pi} |F(e^{i(\theta+h)}) - F(e^{i\theta})|^p d\theta \leqq c_p |h|^p \int_0^{2\pi} |F'(e^{i\theta})|^p d\theta.$$

PROOF. See [4].

PROOF OF THEOREM 1 (case $0<p<1$, $2 \leqq q < +\infty$). According to the definition of $\omega(\tau, f)$, we start from estimating $|f(ze^{ih}) - f(z)|$ and $\|f(ze^{ih}) - f(z)\|$.

$\|f(ze^{ih}) - f(z)\|$ is an even function of h. So what we need to consider is only the case $h > 0$.

Setting $r_0 = \min\{1/2, (1/2)|2q-3|^{-1/2}\}$, we consider the case $\tau \in [0, \frac{1}{2}r_0]$ at first. At this moment we have $h \in [0, \frac{1}{2}r_0]$, and for $r_0 < |z| < 1$, $z = re^{i\theta}$,

$$\begin{aligned} &|f(ze^{ih}) - f(z)| \leqq \\ &\leqq |f(re^{i\theta}) - f[(r-h)e^{i\theta}]| + |f[(r-h)e^{i\theta}] - f[(r-h)e^{i(\theta+h)}]| + \\ &\quad + |f[(r-h)e^{i(\theta+h)}] - f(re^{i(\theta+h)})| \\ &\overset{\text{def}}{=} \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

By using Lemma 1 of [1] and the triangle inequality in Fréchet spaces we obtain

$$\|f(ze^{ih}) - f(z)\|^p \leqq 2 \|f(ze^{ih}) - f(z)\|^p_{(r_0<|z|<1)} \leqq 2 \sum_{k=1}^{3} \|\Delta_k\|^p_{(r_0<|z|<1)}.$$

We need estimates of $\|\Delta_k\|^p_{(r_0<|z|<1)}$ $(k=1,2,3,)$, respectively.

$$\text{(i)}\qquad \|\Delta_1\|^p_{(r_0<|z|<1)} = \int_{r_0}^{1} r(1-r^2)^{q-2}\Big(\int_0^{2\pi} |f(re^{i\theta}) - f[(r-h)e^{i\theta}]|^p d\theta\Big) dr.$$

Using Lemma 1 of this paper and Fubini theorem gives

$$\|\Delta_1\|^p_{(r_0<|z|<1)} \leqq c_p \int_{r_0}^{1} r(1-r^2)^{q-2}\Big[\int_{h/2}^{h}(h-t)^{p-1}\Big(\int_0^{2\pi}|f'[(r-h+t)e^{i\theta}]|^p d\theta\Big) dt\Big] dr =$$

$$= c_p \int_{h/2}^{h}(h-t)^{p-1}\Big[\int_{r_0}^{1} r(1-r^2)^{q-2}\Big(\int_0^{2\pi}|f'[(r-h+t)e^{i\theta}]|^p d\theta\Big) dr\Big] dt =$$

$$= c_p \int_{h/2}^{h}(h-t)^{p-1}\Big[\int_{r_0-h+t}^{1-h+t}(\rho+h-t)[1-(\rho+h-t)^2]^{q-2}\Big(\int_0^{2\pi}|f'(\rho e^{i\theta})|^p d\theta\Big) d\rho\Big] dt \leqq$$

$$\leqq c_p \int_{h/2}^{h}(h-t)^{p-1}\Big[\int_{r_0-h+t}^{1-h+t} 2\rho(1-\rho^2)^{q-2}\Big(\int_0^{2\pi}|f'(\rho e^{i\theta})|^p d\theta\Big) d\rho\Big] dt \leqq$$

$$\leqq c_p \int_{h/2}^{h}(h-t)^{p-1} \|f'(z)\|_{(|z|<1-h+t)}\, dt.$$

$$\text{(ii)}\qquad \|\Delta_2\|^p_{(r_0<|z|<1)} = \int_{r_0}^{1} r(1-r^2)^{q-2}\Big(\int_0^{2\pi} |f[(r-h)e^{i\theta}] - f[(r-h)e^{i(\theta+h)}]|^p d\theta\Big) dr.$$

For any fixed r and h, $0<r-h<1$, let

$$F(z) \stackrel{\text{def}}{=} f[(r-h)z].$$

Then $F(z)$ is analytic on $|z| \leqq 1$. By using Lemma 2 we obtain

$$\|\Delta_2\|^p_{(r_0<|z|<1)} \leqq \int_{r_0}^{1} r(1-r^2)^{q-2}\Big(\int_0^{2\pi} c_p(r-h)^p h^p |f'[(r-h)e^{i\theta}]|^p d\theta\Big) dr \leqq$$

$$\leqq c_p h^p \int_{r_0}^{1} r(1-r^2)^{q-2}\Big(\int_0^{2\pi} |f'[(r-h)e^{i\theta}]|^p d\theta\Big) dr =$$

$$= c_p h^p \int_{r_0-h}^{1-h} (r+h)[1-(r+h)^2]^{q-2} \left(\int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \right) dr \leqq$$

$$\leqq c_p h^p \int_{r_0-h}^{1-h} 2r(1-r^2)^{q-2} \left(\int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \right) dr \leqq$$

$$\leqq c_p h^p \|f'(z)\|^p_{(|z|<1-h)}.$$

(iii) Obviously we have

$$\|\Delta_3\|^p_{(r_0<|z|<1)} = \|\Delta_1\|^p_{(r_0<|z|<1)} .$$

Thus we obtain

$$\sum_{k=1}^{3} \|\Delta_k\|^p_{(r_0<|z|<1)} \leqq c_p \left(\int_{h/2}^{h} (h-t)^{p-1} \|f'(z)\|^p_{(|z|<1-h+t)} dt + h^p \|f'(z)\|^p_{(|z|<1-h)} \right) \leqq$$

$$\leqq c_p \left(\int_{h/2}^{h} (h-t)^{p-1} \frac{\Omega^p(h-t)}{(h-t)^p} dt + h^p \frac{\Omega^p(h)}{h^p} \right) =$$

$$= c_p \left(\int_{h/2}^{h} \frac{\Omega^p(h-t)}{h-t} dt + \Omega^p(h) \right) \leqq$$

$$\leqq c_p \left(\int_{0}^{h} \frac{\Omega^p(h-t)}{h-t} dt + \Omega^p(h) \right) =$$

$$= c_p \left(\int_{0}^{h} \frac{\Omega^p(t)}{t} dt + \Omega^p(h) \right).$$

From the hypotheses of the theorem we know that

$$\Omega^p(h) = \int_0^h \frac{\Omega^p(h)}{h} dt \leqq 2 \int_0^h \frac{\Omega^p(t)}{t} dt.$$

Therefore we have

$$\sum_{k=1}^{3} \|\Delta_k\|^p_{(r_0<|z|<1)} \leqq c_p \int_0^h \frac{\Omega^p(t)}{t} dt,$$

$$\|f(ze^{ih}) - f(z)\|^p \leqq 2\sum_{k=1}^{3} \|\Delta_k\|^p_{(r_0<|z|<1)} \leqq c_p \int_0^h \frac{\Omega^p(t)}{t} dt,$$

and

$$\omega(\tau, f) = \sup_{-\tau \leqq h \leqq \tau} \{\|f(ze^{ih}) - f(z)\|\} \leqq$$

$$\leqq \sup_{0 \leqq h \leqq \tau} \left\{ \left(c_p \int_0^h \frac{\Omega^p(t)}{t} dt \right)^{1/p} \right\} =$$

$$= c_p \left(\int_0^\tau \frac{\Omega^p(t)}{t} dt \right)^{1/p}.$$

This completes the proof for $\tau \in [0, \frac{1}{2}r_0]$.

If $\tau \in (\frac{1}{2}r_0, 1]$, by using properties of $\omega(\tau, f)$ we obtain

$$\omega^p(\tau, f) \leqq \omega^p(1, f) \leqq \left[\frac{2}{r_0} + 1\right] \omega^p\left(\frac{r_0}{z}, f\right) \leqq$$

$$\leqq \left[\frac{2}{r_0} + 1\right] c_p \int_0^{r_0/2} \frac{\Omega^p(t)}{t} dt \leqq$$

$$\leqq c_{p,q} \int_0^\tau \frac{\Omega^p(t)}{t} dt.$$

The proof of Theorem 1 (case $0 < p < 1$, $2 \leqq q < +\infty$) is complete.

Proof of Theorem 1 (case $1 \leqq p < +\infty$, $2 \leqq q < +\infty$). For $p \geqq 1$, H^p_q spaces are Banach spaces, so the proof is even simpler; we omit the details.

In order to finish the proof of Theorem 1, we need some more preparation.

Definition (see [5]). Let a function $f(z)$ be analytic in the unit disc $|z| < 1$, $f(z) = \sum_{n=0}^{\infty} c_n z^n$, and β an arbitrary real number. Then the operators D^β and I^β are defined as

$$D^\beta f(z) = \sum_{n=0}^{\infty} (n+1)^\beta c_n z^n,$$

$$I^{\beta} f(z)=\sum_{n=0}^{\infty}(n+1)^{-\beta} c_{n} z^{n}.$$

It is not difficult to prove that the operators D^{β} and I^{β} have the following properties:

(i) $D^{\beta} I^{\beta} f(z) = I^{\beta} D^{\beta} f(z) = f(z)$;

(ii) $D^{\beta} D^{\alpha} f(z)=D^{\alpha} D^{\beta} f(z)=D^{\alpha+\beta} f(z)$;

(iii) $I^{\beta} I^{\alpha} f(z) = I^{\alpha} I^{\beta} f(z) = I^{\alpha+\beta} f(z)$;

(iv) Setting $D_{*}^{\alpha} f(ze^{i\theta})=D^{\alpha} f(\zeta)|_{\zeta=ze^{i\theta}}$, we have

$$D_{*}^{\alpha} f(ze^{i\theta})=D^{\alpha} f(ze^{i\theta}).$$

In fact, according to the definitions of $D_{*}^{\alpha} f(ze^{i\theta})$ and $D^{\alpha} f(ze^{i\theta})$, we have

$$D_{*}^{\alpha} f(ze^{i\theta})=\sum_{n=0}^{\infty}(n+1)^{\alpha} c_{n}(ze^{i\theta})^{n}=\sum_{n=0}^{\infty}(n+1)^{\alpha} c_{n} z^{n} e^{in\theta},$$

$$D^{\alpha} f(ze^{i\theta})=D^{\alpha} \sum_{n=0}^{\infty} c_{n} e^{in\theta} z^{n}=\sum_{n=0}^{\infty}(n+1)^{\alpha} c_{n} e^{in\theta} z^{n}.$$

These are equal. So the property (iv) holds.

LEMMA 3 (see [6], [7], [8]). *Let $p>0$, $q>1$, β a real number, $q+\beta p>1$. Then D^{β} is a bounded linear operator from H_q^p onto $H_{q+\beta p}^p$. The inverse operator I^{β} is bounded, too. This means that there exist suitable constants $c_{p,q}>0$ such that the inequalities*

$$c_{p,q}\|D^{\beta} f(z)\|_{H_{q+\beta p}^{p}} \leqq \|f(z)\|_{H_{q}^{p}} \leqq c_{p,q}\|D^{\beta} f(z)\|_{H_{q+\beta p}^{p}}$$

hold for any $f(z) \in H_q^p$, where the subscripts indicate the spaces where the $\|\cdot\|$ are taken. (If a $\|\cdot\|$ does not involve the whole unit disc but a certain subset of the unit disc, we will mark it by an appropriate subscript.)

We can formulate the result of Lemma 3 as

$$\|D^{\beta} f(z)\|_{H_{q+\beta p}^{p}} \sim \|f(z)\|_{H_{q}^{p}}.$$

LEMMA 4. *Let $p>0$, $q>1$, β a real number, $q+\beta p>1$, $\rho \in(0,1)$. Then we have*

$$\|D^{\beta} f'(z)\|_{H_{q+\beta p}^{p}(|z|<\rho)} \sim \|(D^{\beta} f(z))'\|_{H_{q+\beta p}^{p}(|z|<\rho)}.$$

PROOF. From [5] we know that there are appropriate constants $c_p>0$ such that the inequalities

$$c_p \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \leqq \int_0^{2\pi} |D^1 f(re^{i\theta})|^p d\theta \leqq c_p \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta$$

hold for any positive number p. Thus, in both spaces H_q^p and $H_{q+\beta p}^p$ we have appropriate constants $c_p > 0$ such that

$$c_p \|f'(z)\|^p_{(|z|<\rho)} \leqq \|D^1 f(z)\|^p_{(|z|<\rho)} \leqq c_p \|f'(z)\|^p_{(|z|<\rho)}.$$

Furthermore, by using Lemma 3 we obtain

$$\begin{aligned}\|D^\beta f'(z)\|_{H^p_{q+\beta p}(|z|<\rho)} \sim \|f'(z)\|_{H^p_q(|z|<\rho)} \sim \|D^1 f(z)\|_{H^p_q(|z|<\rho)} \sim \|D^\beta D^1 f(z)\|_{H^p_{q+\beta p}(|z|<\rho)} \sim \\ \sim \|D^1 D^\beta f(z)\|_{H^p_{q+\beta p}(|z|<\rho)} \sim \|(D^\beta f(z))'\|_{H^p_{q+\beta p}(|z|<\rho)}.\end{aligned}$$

The proof of Lemma 4 is complete.

LEMMA 5. *Let $p>0$, $q>1$, β a real number, $q+\beta p>1$. Then we have*

$$\omega_{H^p_{q+\beta p}}(\tau, D^\beta f(z)) \sim \omega_{H^p_q}(\tau, f(z)),$$

where the $H^p_{q+\beta p}$ and H^p_q in the subscripts indicate the spaces where the moduli of continuity are taken.

PROOF. From the properties of D^β we obtain

$$D^\beta_* f(ze^{ih}) - D^\beta f(z) = D^\beta f(ze^{ih}) - D^\beta f(z) = D^\beta (f(ze^{ih}) - f(z)).$$

So from Lemma 3 we have

$$\begin{aligned}&\|D^\beta_* f(ze^{ih}) - D^\beta f(z)\|_{H^p_{q+\beta p}} = \\ &= \|D^\beta (f(ze^{ih}) - f(z))\|_{H^p_{q+\beta p}} \sim \|f(ze^{ih}) - f(z)\|_{H^p_q}.\end{aligned}$$

Taking supremum for $h \in [-\tau, \tau]$ gives

$$\omega_{H^p_{q+\beta p}}(\tau, D^\beta f(z)) \sim \omega_{H^p_q}(\tau, f(z)).$$

The proof of Lemma 5 is complete.

PROOF OF THEOREM 1 (case $0 < p < +\infty$, $1 < q < 2$). Let $q \in (1,2)$ be fixed. We choose an arbitrary natural number k such that $kq = Q \geqq 2$. Denoting $\beta = (Q-q)/p$, by Lemma 3 we know that D^β is a bounded linear operator from H^p_q onto H^p_Q.

By assumption

$$\|f'(z)\|_{H^p_q(|z|<\rho)} \leqq \frac{\Omega(1-\rho)}{1-\rho}.$$

So by using Lemma 3 we obtain

$$\|D^\beta f'(z)\|_{H^p_Q(|z|<\rho)} \leqq c_{p,q}\frac{\Omega(1-\rho)}{1-\rho}.$$

Furthermore, from Lemma 4, we have

$$\|(D^\beta f(z))'\|_{H^p_Q(|z|<\rho)} \leqq c_{p,q}\frac{\Omega(1-\rho)}{1-\rho}.$$

Using the already proved parts of Theorem 1 gives

$$\omega_{H^p_Q}(\tau, D^\beta f) \leqq c_{p,q}\left(\int_0^\tau \frac{\Omega^\mu(t)}{t}dt\right)^{1/\mu},$$

where $\mu = \min\{p,1\}$.

Finally, from Lemma 5, we obtain

$$\omega_{H^p_q}(\tau, f) \leqq c_{p,q}\left(\int_0^\tau \frac{\Omega^\mu(t)}{t}dt\right)^{1/\mu}.$$

The proof of Theorem 1 is complete.

COROLLARY. *Suppose that* $f(z) \in H^p_q$ $(0<p<+\infty,\ 1<q<+\infty)$, $\mu = \min\{p,1\}$, *and there exists a constant* M *such that for any* $\rho \in (0,1)$ *we have*

$$\|f'(z)\|_{(|z|<\rho)} \leqq M(1-\rho)^{\alpha-1}, \qquad 0<\alpha\leqq 1.$$

Then for any $\tau \in [0,1]$ *we have*

$$\omega(\tau, f) \leqq c_{p,q}\alpha^{-1/\mu}M\tau^\alpha.$$

This is a consequence of Theorem 1 if we put $\Omega(t) = Mt^\alpha$.

LEMMA 6. *If a function* $f(z)$ *is analytic in the unit disc* $|z|<1$ *and* $f'(z) \in H^p_q$ $(1 \leqq p < +\infty, 1<q<+\infty)$, *then we have also* $f(z) \in H^p_q$ $(1\leqq p < +\infty,\ 1<q<+\infty)$.

PROOF. We have

$$f(z) - f(0) = \int_0^r \frac{\partial}{\partial t} f(te^{i\theta})dt = \int_0^r e^{i\theta} f'(te^{i\theta})dt,$$

$$|f(z)-f(0)| \leqq \int_0^r |f'(te^{i\theta})|dt = \int_0^1 |f'(tre^{i\theta})|r\,dt,$$

$$\|f(z)-f(0)\| \leqq \left[\iint\limits_{|z|<1} (1-|z|^2)^{q-2}\left(\int_0^1 |f'(tre^{i\theta})|r\,dt\right)^p dx\,dy\right]^{1/p} \leqq$$

$$\leqq \int_0^1 \left(\iint\limits_{|z|<1} (1-|z|^2)^{q-2} r^p |f'(tre^{i\theta})|^p dx\,dy\right)^{1/p} dt \leqq$$

$$\leqq \int_0^1 \left(\iint\limits_{|z|<1} (1-|z|^2)^{q-2} |f'(tre^{i\theta})|^p dx\,dy\right)^{1/p} dt =$$

$$= \int_0^1 \left[\int_0^1 r(1-r^2)^{q-2}\left(\int_0^{2\pi} |f'(tre^{i\theta})|^p d\theta\right) dr\right]^{1/p} dt \leqq$$

$$\leqq \int_0^1 \left[\int_0^1 r(1-r^2)^{q-2}\left(\int_0^{2\pi} |f'(re^{i\theta})|^p d\theta\right) dr\right]^{1/p} dt =$$

$$= \int_0^1 \|f'(z)\| dt =$$

$$= \|f'(z)\| <$$

$$< +\infty.$$

By using the triangle inequality in Banach spaces we obtain

$$\|f(z)\| \leqq \|f'(z)\| + \|f(0)\| < +\infty.$$

THEOREM 2. *Let $f^{(m)}(z) \in H_q^p$ $(1 \leqq p < +\infty,\ 1 < q < +\infty)$, where m is a non-negative integer. Then for any natural number n we have*

$$\rho^{(n)}(f) \leqq \frac{A}{n^m}\omega\left(\frac{1}{n}, f^{(m)}\right),$$

where A is a constant independent of n and f.

PROOF. Let

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad |z| < 1.$$

For any natural number n, set

$$n' = 2\left[\frac{n+1}{2}\right] - 2 = 2\left[\frac{n-1}{2}\right] < n$$

and

$$P_n(f,m,z) = c_0 + \sum_{k=1}^{n'} (1-\lambda_k^{m+1}) c_k z^k,$$

where $\lambda_k + 1 - j_k$, j_k $(k = 1, 2, \dots, n')$ are the coefficients of Jackson kernel of order $[\frac{n+1}{2}]$:

$$J_{[\frac{n+1}{2}]}(t) = \frac{1}{2} + \sum_{k=1}^{n'} j_k \cos kt.$$

We want to prove that for any natural number n we have

$$\|f(z) - P_n(f,m,z)\| \leqq \frac{12^{m+1} c_{p,q}}{n^m} \omega\left(\frac{1}{n}, f^{(m)}\right).$$

We use mathematical induction with respect to $m \geqq 0$.

(i) In case $m = 0$ we have

$$P_n(f,0,z) = c_0 + \sum_{k=1}^{n'} (1-\lambda_k) c_k z^k = P_{J[\frac{n+1}{2}]}(f,z), \tag{2}$$

where

$$P_{J[\frac{n+1}{2}]}(f,z)$$

is the Jackson polynomial of degree $2[\frac{n+1}{2}] - 2$ generated by $f(z)$.

From the proof of Theorem 2 of [1] we know that

$$\|f(z) - P_n(f,0,z)\| \leqq 6\omega\left(\frac{1}{[\frac{n+1}{2}]}, f\right) \leqq 6\omega\left(\frac{2}{n}, f\right) \leqq 12\omega\left(\frac{1}{n}, f\right). \tag{3}$$

(In fact, Theorem 2 of [1] is the case $m = 0$ of Theorem 2 of this paper. Here we use a new proof similar to the proof of Jackson theorem so as to prove the case $m > 0$.)

(ii) Suppose that the proposition for $m = r$ is true (r is a non-negative integer). Let us investigate the case $m = r + 1$. Now the condition is

$$f^{(r+1)}(z) = (f'(z))^{(r)} \in H_q^p \ (1 \leqq p < +\infty,\ 1 < q < +\infty).$$

We know that

$$P_n'(f,r,z) = P_n(f',r,z).$$

So according to the induction hypothesis we have

$$\|[f(z)-P_n(f,r,z)]'\| =$$
$$= \|f'(z)-P_n(f',r,z)\| \leqq \frac{12^{r+1}c_{p,q}}{n^r}\omega\Big(\frac{1}{n},(f')^{(r)}\Big) \overset{\text{def}}{=} M(n,r,p,q).$$

From Lemma 6 we know that $f(z)-P_n(f,r,z)\in H_q^p$ $(1\leqq p<+\infty,\ 1<q<+ +\infty)$. And by using the corollary of Theorem 1 (the case $\alpha=1$) we obtain

$$\omega\Big(\frac{1}{n},f(z)-P_n(f,r,z)\Big) \leqq \frac{c_{p,q}}{n}M(n,r,p,q).$$

Therefore from (3) and (2) we have

$$\|f(z)-P_n(f,r,z)-P_{J[\frac{n+1}{2}]}[f(t)-P_n(f,r,t),z]\| \leqq 12\omega\Big(\frac{1}{n},f(z)-P_n(f,r,z)\Big) \leqq$$
$$\leqq \frac{12c_{p,q}}{n}M(n,r,p,q) = \frac{12^{r+2}c_{p,q}}{n^{r+1}}\omega\Big(\frac{1}{n},f^{(r+1)}\Big).$$

From $1-j_k=\lambda_k$ and (2) we have

$$P_n(f,r,z)+P_{J[\frac{n+1}{2}]}[f(t)-P_n(f,r,t),z] =$$
$$= c_0+\sum_{k=1}^{n'}(1-\lambda_k^{r+1})c_kz^k + c_0+\sum_{k=1}^{n'}j_kc_kz^k - \Big(c_0+\sum_{k=1}^{n'}j_k(1-\lambda_k^{r+1})c_kz^k\Big) =$$
$$= c_0+\sum_{k=1}^{n'}(1-\lambda_k^{r+2})c_kz^k =$$
$$= P_n(f,r+1,z).$$

So we obtain

$$\|f(z)-P_n(f,r+1,z)\| \leqq \frac{12^{r+2}c_{p,q}}{n^{r+1}}\omega\Big(\frac{1}{n},f^{(r+1)}\Big).$$

This means that the proposition is true for the case $m=r+1$. The proof of Theorem 2 is complete.

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DEPARTMENT OF MATHEMATICS
CENTRAL INSTITUTE FOR NATIONALITIES
27, BAI SHI QIAO ROAD
100081 BEIJING
PEOPLE'S REPUBLIC OF CHINA

and

BUDAPESTI MŰSZAKI EGYETEM
VILLAMOSMÉRNÖKI ÉS INFORMATIKAI KAR
MATEMATIKA TANSZÉK
H–1521 BUDAPEST
EGRY JÓZSEF U. 1
HUNGARY

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EXTENDING A FAMILY OF CAUCHY STRUCTURES IN A LIMIT SPACE. I

J. DEÁK

Abstract

Given compatible Cauchy structures on some subspaces of a limit space or pseudotopological space, we look for necessary and sufficient conditions for the existence of a simultaneous extension of these structures. Some related problems will also be considered.

The following simultaneous extension problem was considered in [8]–[11], [5], [6] for several kinds of topological structures: Let X be a set, σ a structure (e.g. a closure) on X, $\{X_i : i \in I\}$ a system of subsets of X, and assume that a richer structure (e.g. a proximity) Σ_i is given on each X_i; under what condition is there a common extension of the structures Σ_i compatible with σ? (The analogous problem with no prescribed structure on X was dealt with in [12]–[15].) In particular, Császár [6] considered simultaneous extensions of Cauchy structures[1] in a closure space (in the sense of Čech [3]). He only obtained results for extensions satisfying an additional condition (Riesz or Lodato axiom); the reason is that it is not even clear which closures can be induced by a Cauchy structure (cf. [7]; following [6] and [7], a Cauchy structure will be called from now on a Cauchy screen), while it is quite easy to describe those closures that can be induced by a Riesz or Lodato Cauchy screen (shortly: CR screen, respectively CL screen).

We aim at solving the problem of extending Cauchy screens, although in a convergence or pseudotopological space instead of a closure space. (A convergence induced by a Cauchy screen is always a limitation. A pseudotopology is usually defined as, or identified with, a convergence or limitation satisfying an additional axiom; such an identification would, however, result in losing some interesting information on extensions.) Being able to solve the problem in convergence (bot not in closure) spaces seems to back up the assumption that convergences provide a more natural framework than

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[1] All the notions mentioned in this introduction will be defined in § 1.

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closures for investigating Cauchy screens (cf. [2], [19], [21], [22], [24], [25] versus [6], [7]).

We shall begin with solving the much simpler problem of extending a family of screens in a convergence or pseudotopological space. It will also be observed that extending a family of Riesz, Lodato, CR or CL screens in a convergence or pseudotopological space is equivalent to the same problem in a closure space, which was solved in [5], [6]. The proofs of the main results will make use of a theorem from [14], which gives a necessary and sufficient condition for the existence of a Cauchy extension of screens in the case when no structure is prescribed on X.

CONTENTS

Part I

Part II

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(Not containing the notations and terminology introduced in § 0.)

§ 0. Notations and terminology

0.1. Notations. $\operatorname{Fil} X$ and $\operatorname{Ult} X$ denote the collection of the proper filters, respectively of the ultrafilters, on X. $\operatorname{Fil}^+ X = \operatorname{Fil} X \cup \{\exp X\}$, $\operatorname{Ult}^+ X = \operatorname{Ult} X \cup \{\exp X\}$. A filter $\mathfrak{s}$ is *fixed at* x if $x \in \bigcap \mathfrak{s}$; $\mathfrak{s}$ is *fixed* if it is fixed at some point; $\mathfrak{s}$ is *free* if it is not fixed. $\operatorname{Fil}^{\mathrm{f}} X$ and $\operatorname{Ult}^{\mathrm{f}} X$ denote the collection of the free elements of $\operatorname{Fil} X$, respectively of $\operatorname{Ult} X$. $\dot{x}$ denotes the filter $\{A \subset X : x \in A\}$, or the same in a subset of X (it will always be clear from the context which subset is meant). For $\mathfrak{s} \in \operatorname{Fil}^+ X$, $\mathfrak{s} \,|\, X_0$ denotes the trace of $\mathfrak{s}$ on $X_0 \subset X$; the same notation $|X_0$ will be used for restrictions of structures to subsets. For $\mathfrak{s}_0 \in \operatorname{Fil}^+ X_0$ (where $X_0 \subset X$), $\mathfrak{s}_0^1 = \operatorname{fil} \mathfrak{s}_0$ where, for a filter base $\mathfrak{b}$, $\operatorname{fil} b = \operatorname{fil}_X \mathfrak{b}$ is the filter in X generated by $\mathfrak{b}$. (A *filter base* is allowed to contain the empty set.)

For $\mathfrak{a}, \mathfrak{b} \subset \exp X$, $\mathfrak{a} \triangle \mathfrak{b}$ iff $A \cap B \neq \emptyset$ ($A \in \mathfrak{a}$, $B \in \mathfrak{b}$); $\mathfrak{a} \triangle_P \mathfrak{b}$ iff $\bigcap \mathfrak{a} \cap \bigcap \mathfrak{b} \neq$ $\neq \emptyset$; $\mathfrak{a} \overline{\triangle} \mathfrak{b}$ iff $\mathfrak{a} \triangle \mathfrak{b}$ does not hold. For $\mathfrak{a} \subset \exp X$, $\sec \mathfrak{a} = \sec_X \mathfrak{a}$ consists of the subsets of X meeting each element of $\mathfrak{a}$. For structures Σ, Σ' on X, $\Sigma < \Sigma'$ means that Σ' is finer than Σ; suprema and infima are to be understood with respect to this partial order.

0.2. Families of structures. With the notations from the first paragraph of the introduction, the structures Σ_i $(i \in I)$ form a *family of structures in the space* (X, σ) if

$$\sigma(\Sigma_i) = \sigma_i \qquad (i \in I), \tag{1}$$

where $\sigma(\ )$ denotes the induced structure and $\sigma_i = \sigma | X_i$ (*compatibility*), and

$$\Sigma_i | X_{ij} = \Sigma_j | X_{ij} \qquad (i, j \in I), \tag{2}$$

where $X_{ij} = X_i \cap X_j$ (*accordance*). We speak about a *family of structures in the set* X if (2) holds. Given a family of structures in a set (or in a space), it will be understood implicitly that the set is denoted by X, the subsets by X_i $(i \in I)$, the structures are Σ_i (and the structure on X is σ); the letters Σ and σ will always be replaced by those typically used for the structures in question (e.g. $\mathfrak{S}_i$ and λ for a family of screens in a convergence space). The notation $\Sigma_{(i)}$ $(i \in I)$ will be preferred when structures on the same set are given.

For a family of structures in a set, Σ is an *extension* if

$$(3) \qquad \Sigma \mid X_i = \Sigma_i \qquad (i \in I).$$

Σ is an *extension* of a family of structures given in a space if, in addition,

$$(4) \qquad \sigma(\Sigma) = \sigma.$$

When wishing to emphasize that (4) is assumed (i.e. that Σ is not just an extension in the set), we shall speak about an *extension in the space* (or a *compatible extension*). (2) (and (1)) figure in the definition of a family of structures, because they are clearly necessary for the existence of an extension (at least when the structures satisfy some natural conditions, cf. the introduction in [8]).

0.3. Conventions. We shall assume (except in definitions) that $X \neq \emptyset$. The special case $I = \emptyset$ of the extension problem in a space means looking for structures Σ compatible with σ; this case will always be dealt with separately in a lemma, and then we shall assume $I \neq \emptyset$ in the statements on extensions.

Without these conventions, some statements would have to be worded more carefully: e.g. 2.1 (1) is false for $X = \emptyset$, and Corollary 2.2 for $I = \emptyset$.

§ 1. Definitions

In addition to the definitions, we shall formulate some simple statements on the connexions between the notions. Nothing in this section is new; see [2] for more historical data. As the continuity of non-injective maps is irrelevant from the point of view of extensions (cf. [12] 1.1), we shall only define restrictions to subsets (instead of continuity of maps) and the relation finer/coarser between the structures on the same set. This means defining concrete categories over the category of sets with the injective maps as morphisms (cf. [12] 1.2). We shall, however, avoid categorical terminology in the present paper [adding some categorical comments in brackets; such remarks can always be skipped].

1.1. A *convergence* ([18], see also [23], [1]) on the set X is a function $\lambda\colon X \longrightarrow \exp \mathrm{Fil}^+ X$ such that

C1. $\dot{x} \in \lambda(x) \qquad (x \in X)$;

C2. if $\mathfrak{s} \in \lambda(x)$ and $\mathfrak{s} \subset \mathfrak{t} \in \mathrm{Fil}^+ X$ then $\mathfrak{t} \in \lambda(x)$;

C3. if $\mathfrak{s} \in \lambda(x)$ then $\mathfrak{s} \cap \dot{x} \in \lambda(x)$.

$\mathfrak{s}$ is said to *converge* to x if $\mathfrak{s} \in \lambda(x)$; we shall also write $\mathfrak{s} \to x$ or, more precisely, $\mathfrak{s} \xrightarrow{\lambda} x$. $\lambda < \lambda'$ iff $\lambda(x) \supset \lambda'(x)$ $(x \in X)$. For $I \neq \emptyset$, $(\sup_{i \in I} \lambda_{(i)})(x) = \bigcap_{i \in I} \lambda_{(i)}(x)$, and the infimum is similar, with union. The restriction to a subset is defined by $\mathfrak{s}_0 \xrightarrow{\lambda|X_0} x$ iff $\mathfrak{s}_0^1 \xrightarrow{\lambda} x$ (equivalently: iff there is an $\mathfrak{s} \in \lambda(x)$ with $\mathfrak{s}_0 = \mathfrak{s}|X_0$). Suprema commute with restriction [the convergences form a topological category Conv], and so do infima [Conv is a simple extension category in the sense of [12]]. Put $\tilde{\lambda}(x) = \lambda(x) \setminus \{\exp X\}$.

C3 is sometimes omitted from the definition of a convergence, cf. [17], [24], [27], [28]; we shall not need this more general notion.

A convergence is *symmetric* if $\mathfrak{s} \to x$, $y \in \bigcap \mathfrak{s}$ imply $\mathfrak{s} \to y$. This property was introduced in [18] as weakly uniformizable, but the word symmetric is now widely used, see e.g. [23]. (In [12], we called an arbitrary structure *symmetric* provided that the map interchanging the points is an isomorphism in any two-point subspace; for convergences, this means that $\dot{x} \to y$ implies $\dot{y} \to x$, a condition strictly weaker than the one introduced above.) A convergence λ is *reciprocal* if $\tilde{\lambda}(x) \cap \tilde{\lambda}(y) \neq \emptyset$ implies $\lambda(x) = \lambda(y)$. (Introduced in [24] as S_1, but note that reciprocal convergences are analogous to the S_2, not the S_1, closures from [7]; "reciprocal" is the word used in the works of Lowen-Colebunders.) A reciprocal convergence is symmetric. A restriction of a symmetric or reciprocal convergence has the same property; both properties are also preserved by suprema. [The symmetric, respectively reciprocal, convergences form a concretely reflective subcategory of Conv. These subcategories are also strongly reflective in the sense of [12], hence they are extension categories.] The infimum of symmetric convergences is also symmetric. [The symmetric convergences form a strongly coreflective (see [12]) subcategory of Conv, hence this extension category is simple. But it is not concretely coreflective in the usual sense: quotients do not preserve symmetry if non-injective maps are allowed.]

1.2. A convergence is a *limitation* provided that it satisfies the following stronger version of C3:

C3$'$. if $\mathfrak{s}, \mathfrak{t} \to x$ then $\mathfrak{s} \cap \mathfrak{t} \to x$.

Following [20], [16], the German term "Limitierung" is used by several authors. A limitation is called a convergence in some papers where the more general notion is not needed, see e.g. [19], [2]. A *limit space* is a set endowed with a limitation.

For limitations, symmetry can also be described in the following equivalent ways: (i) if $\mathfrak{s} \to x$ and $\dot{y} \to x$ then $\mathfrak{s} \to y$ (cf. Axiom S_0 in [24]); (ii) if $\dot{x} \to y$ then $\lambda(x) = \lambda(y)$ (e.g. [21] 1.2.9). Suprema and restrictions of limitations are again limitations. [The limitations form a concretely reflective subcategory Lim of Conv. It is also strongly reflective, and the extra condition in [12] 2.4

is satisfied, thus Lim is a simple extension category. Concerning the subcategories made up by the symmetric, respectively reciprocal, limitations, the same can be said as in Conv.]

1.3. A *pseudotopology* on X is a function $\pi\colon X \to \exp \operatorname{Ult} X$ such that

Ps. $\quad \dot{x} \in \pi(x) \qquad (x \in X)$.

$\mathfrak{u}$ *converges* to x if $\mathfrak{u} \in \pi(x)$; we shall also write $\mathfrak{u} \to x$ or $\mathfrak{u} \xrightarrow{\pi} x$. $\pi < \pi'$ iff $\pi(x) \supset \pi'(x)$ $(x \in X)$. Suprema, infima and restrictions can be obtained in the same way as for convergences. Suprema as well as infima commute with restrictions. [The pseudotopologies form a simple extension category PsTop, cf. [13] § 7.]

A pseudotopology is *symmetric* if $\dot{x} \to y$ implies $\dot{y} \to x$ (differently from the symmetry of convergences, this is a special case of the general definition from [12]); it is *reciprocal* if $\pi(x) \cap \pi(y) \neq \emptyset$ implies $\pi(x) = \pi(y)$. A reciprocal pseudotopology is symmetric. The statements on symmetric or reciprocal convergences hold for pseudotopologies, too [including the categorical results; [12] Proposition 3.2 also applies].

A convergence λ induces a pseudotopology $\pi(\lambda)$ defined by

$$\pi(\lambda)(x) = \lambda(x) \cap \operatorname{Ult} X \qquad (x \in X).$$

λ is also said to be compatible with $\pi(\lambda)$. If $\lambda < \lambda'$ then $\pi(\lambda) < \pi(\lambda')$. Suprema and infima commute with taking the induced pseudotopology. For each pseudotopology π, there are a coarsest and a finest compatible convergence $\lambda^0(\pi)$, respectively $\lambda^1(\pi)$:

$$\mathfrak{s} \xrightarrow{\lambda^0(\pi)} x \ \text{ iff } \ \mathfrak{u} \xrightarrow{\pi} x \ \text{ whenever } \ \mathfrak{s} \subset \mathfrak{u} \in \operatorname{Ult} X;$$

$$\lambda^1(\pi)(x) = \pi(x) \cup \{\mathfrak{u} \cap \dot{x}\colon \mathfrak{u} \in \pi(x)\} \cup \{\exp X\}.$$

Not requiring the convergences to satisfy C3 would have the advantage that the simpler formula $\lambda^1(\pi)(x) = \pi(x) \cup \{\exp X\}$ holds [and our $\lambda^1(\pi)$ is its reflexion in Conv]. $\lambda^0(\pi)$ is always a limitation. There is also a finest limitation compatible with π, namely $\lambda^1_{\mathrm{lim}}(\pi) = (\lambda^1(\pi))_{\mathrm{lim}}$, where, for a convergence λ, λ_{lim} denotes the finest one of the limitations coarser than λ [the Lim-reflexion of λ], i.e. $\lambda_{\mathrm{lim}}(x)$ consists of the finite intersections of elements of $\lambda(x)$, and $\lambda^1_{\mathrm{lim}}(\pi)(x)$ of the finite intersections of elements of $\pi(x)$ (including $\exp X = \bigcap \emptyset$).

A convergence of the form $\lambda^0(\pi)$ is called *pseudotopological*. λ is pseudotopological iff it satisfies the following axiom stronger than C3′:

C3″. $\quad$ if $\mathfrak{s} \in \operatorname{Fil} X$ and $(\mathfrak{s} \subset \mathfrak{u} \in \operatorname{Ult} X \Rightarrow \mathfrak{u} \to x)$ then $\mathfrak{s} \to x$.

Most authors call a pseudotopological convergence a pseudotopology; our definition of a pseudotopology has been taken from [1]. [The pseudotopological convergences form a concretely reflective subcategory of Conv, and of Lim; this subcategory is isomorphic to PsTop.]

1.4. A closure is usually defined as a map $\exp X \to \exp X$, but the following version (a "neighbourhood structure") is more suitable for our purpose: a *closure* on X is a map $\mathfrak{n}\colon X \to \operatorname{Fil} X$ such that

Cl. $\quad x \in \bigcap \mathfrak{n}(x) \qquad (x \in X)$

(cf. [3] 14 B. 11). The elements of $\mathfrak{n}(x)$ are the *neighbourhoods* of x. $\mathfrak{n} < \mathfrak{n}'$ iff $\mathfrak{n}(x) \subset \mathfrak{n}'(x)$ $(x \in X)$; $(\mathfrak{n} \mid X_0)(x) = \mathfrak{n}(x) \mid X_0$. [The closures form a simple extension category Clos, see [13] § 4.] A closure $\mathfrak{n}$ is *symmetric* if $x \in \bigcap \mathfrak{n}(y)$ implies $y \in \bigcap \mathfrak{n}(x)$; it is *reciprocal* ($S_2$ in [7], [13]) if $\mathfrak{n}(x) \,\Delta\, \mathfrak{n}(y)$ implies $\mathfrak{n}(x) = \mathfrak{n}(y)$.

A pseudotopology π induces a closure $\mathfrak{n}(\pi)$ defined by

$$\mathfrak{n}(\pi)(x) = \bigcap \pi(x) \qquad (x \in X).$$

For each closure $\mathfrak{n}$, there is a coarsest compatible pseudotopology $\pi^0(\mathfrak{n})$:

$$\mathfrak{u} \xrightarrow{\pi^0(\mathfrak{n})} x \quad \text{iff} \quad \mathfrak{n}(x) \subset \mathfrak{u} \in \operatorname{Ult} X.$$

There is in general no finest compatible pseudotopology. A convergence also induces a closure in two steps: $\mathfrak{n}(\lambda) = \mathfrak{n}(\pi(\lambda))$, i.e.

$$\mathfrak{n}(\lambda)(x) = \bigcap \lambda(x) \qquad (x \in X).$$

For each closure $\mathfrak{n}$, there is a coarsest compatible convergence (which is a limitation), namely $\lambda^0(\mathfrak{n}) = \lambda^0(\pi^0(\mathfrak{n}))$;

$$\mathfrak{s} \xrightarrow{\lambda^0(\mathfrak{n})} x \quad \text{iff} \quad \mathfrak{n}(x) \subset \mathfrak{s} \in \operatorname{Fil}^+ X;$$

there is no finest one. A convergence or pseudotopology of the form $\lambda^0(\mathfrak{n})$, respectively $\pi^0(\mathfrak{n})$, is *pretopological*. A pretopological convergence is often called a pretopology, and it is identified with the corresponding closure; we shall, however, distinguish between the two notions [although the pretopological convergences form a concretely reflective subcategory of Conv isomorphic with Clos; the same holds for the pretopological pseudotopologies in PsTop]. A convergence is pretopological iff it satisfies the following condition stronger than C3″:

C3‴. $\quad \bigcap \lambda(x) \in \lambda(x) \qquad (x \in X)$.

1.5. A *screen* (or filter merotopy) on X is a system $\emptyset \neq \mathfrak{S} \subset \operatorname{Fil}^+ X$ such that

S1. $\quad \dot{x} \in \mathfrak{S} \qquad (x \in X)$;

S2. $\quad$ if $\mathfrak{s} \in \mathfrak{S}$ and $\mathfrak{s} \subset \mathfrak{s}' \in \operatorname{Fil}^+ X$ then $\mathfrak{s}' \in \mathfrak{S}$.

$\mathfrak{S} < \mathfrak{S}'$ iff $\mathfrak{S} \supset \mathfrak{S}'$; supremum means intersection, infimum means union. $\mathfrak{S} \mid X_0 = \{\mathfrak{s} \mid X_0 \colon \mathfrak{s} \in \mathfrak{S}\}$. Both suprema and infima commute with restrictions. [The screens form a simple extension category Scr, see [14] § 13.]

$\mathfrak{B} \subset \mathfrak{S}$ is a *base* for $\mathfrak{S}$ if for any $\mathfrak{s} \in \mathfrak{S}$ there is an $\mathfrak{s}' \in \mathfrak{B}$ with $\mathfrak{s}' \subset \mathfrak{s}$. If $\mathfrak{B}$ is a base for $\mathfrak{S}$ then $\{\mathfrak{s} \mid X_0 : \mathfrak{s} \in \mathfrak{B}\}$ is a base for $\mathfrak{s} \mid X_0$.

The screen $\mathfrak{S}$ is *Cauchy* (a "Cauchy structure") if

SC. $\quad \mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$, $\mathfrak{s} \,\Delta\, \mathfrak{s}'$ imply $\mathfrak{s} \cap \mathfrak{s}' \in \mathfrak{S}$.

(The expression Cauchy structure was first used in [24], but with a more general meaning.) The restriction of a Cauchy screen is Cauchy, and so is the supremum of Cauchy screens. [The Cauchy screens form a concretely reflective subcategory CScr of Scr.] Let us call a sequence $\mathfrak{r}_1, \dots, \mathfrak{r}_n$ of filter bases a *chain* if $\mathfrak{r}_m \,\Delta\, \mathfrak{r}_{m+1}$ $(1 \leqq m < n)$; it is an $\mathfrak{S}$-*chain* if, in addition, $\mathfrak{r}_m \in \mathfrak{S}$ $(1 \leqq m \leqq n)$. Given a screen $\mathfrak{S}$, the intersections of the $\mathfrak{S}$-chains form a base for the finest Cauchy screen $\mathfrak{S}_{\mathrm{C}}$ coarser than $\mathfrak{S}$ [the Cauchy reflexion of $\mathfrak{S}$].

A screen $\mathfrak{S}$ induces a convergence $\lambda(\mathfrak{S})$, a pseudotopology $\pi(\mathfrak{S}) = \pi(\lambda(\mathfrak{S}))$ and a closure $\mathfrak{n}(\mathfrak{S}) = \mathfrak{n}(\lambda(\mathfrak{S})) = \mathfrak{n}(\pi(\mathfrak{S}))$:

$$\mathfrak{s} \xrightarrow{\lambda(\mathfrak{S})} x \quad \text{iff} \quad \mathfrak{s} \in \mathrm{Fil}^+ X,\ \mathfrak{s} \cap \dot{x} \in \mathfrak{S};$$

$$\mathfrak{u} \xrightarrow{\pi(\mathfrak{S})} x \quad \text{iff} \quad \mathfrak{u} \in \mathrm{Ult}^+ X,\ \mathfrak{u} \cap \dot{x} \in \mathfrak{S};$$

$$\mathfrak{n}(\mathfrak{S})(x) = \bigcap \{\mathfrak{s} \in \mathfrak{S} : x \in \bigcap \mathfrak{s}\} \qquad (x \in X).$$

If $\mathfrak{S}$ is Cauchy then $\lambda(\mathfrak{S})$ is a limitation. In each case, taking the induced structure commutes with restrictions.

A screen $\mathfrak{S}$ is *Riesz* if $\mathfrak{n}(\mathfrak{S})(x) \in \mathfrak{S}$ $(x \in X)$, *Lodato* if $\mathfrak{n}(\mathfrak{S})$ is a topology and there is a base for $\mathfrak{S}$ consisting of $\mathfrak{n}(\mathfrak{S})$-open filters, *pointwise Cauchy* (C in [2]) if $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$, $\mathfrak{s} \,\Delta_{\mathrm{P}}\, \mathfrak{t}$ imply $\mathfrak{s} \cap \mathfrak{t} \in \mathfrak{S}$, *fully Cauchy* (separated in [2]) if for each $\emptyset \notin \mathfrak{s} \in \mathfrak{S}$ there is a coarsest one among the elements of $\mathfrak{S}$ coarser than $\mathfrak{s}$ ("coarsest" can be replaced by "minimal" if we already know that $\mathfrak{S}$ is Cauchy), $CR =$ Riesz Cauchy, $CL =$ Lodato Cauchy, $FL =$ Lodato fully Cauchy. All these classes of screens are closed for restrictions and suprema. [They form concretely reflective subcategories of Scr. These subcategories are also strongly reflective, hence they are extension categories.] $\mathfrak{S}_{\mathrm{P}}$ and $\mathfrak{S}_{\mathrm{F}}$ will denote the finest one of the pointwise Cauchy, respectively fully Cauchy, screens coarser than $\mathfrak{S}$ [the pointwise/fully Cauchy reflexion of $\mathfrak{S}$]. $\mathfrak{S}_{\mathrm{P}}$ can be obtained similarly to $\mathfrak{S}_{\mathrm{C}}$, substituting Δ_{P} for Δ in the definition of a chain (such chains will be called *strong*). $\mathfrak{S} \cup \{\mathfrak{n}(\mathfrak{S})(x) : x \in X\}$ is a base for $\mathfrak{S}_{\mathrm{R}}$, the finest one of the Riesz screens coarser than $\mathfrak{S}$ [the Riesz reflexion of $\mathfrak{S}$]. The following implications hold:

$$\begin{tikzcd}
\text{Lodato} \arrow[dr, Rightarrow] & & \\
& \text{Riesz} \arrow[dr, Rightarrow] & \\
\text{fully Cauchy} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{pointwise Cauchy} \\
& \text{Cauchy} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

§ 2. Screens in a convergence space

2.1. LEMMA (partly in [23] 4.2.3.4)[2]. *A convergence λ can be induced by screens iff it is symmetric. If so then there exist a finest and a coarsest compatible screen, namely*

$$\mathfrak{S}^1(\lambda) = \bigcup_{x \in X} \lambda(x), \qquad \mathfrak{S}^0(\lambda) = \mathfrak{S}^1(\lambda) \cup \mathrm{Fil}^{\mathrm{f}} X. \tag{1}$$

PROOF. 1° Let $\mathfrak{S}$ be a screen, $\lambda = \lambda(\mathfrak{S})$. Assume that $\mathfrak{s} \to x$, $y \in \bigcap \mathfrak{s}$. Then $\mathfrak{s}' = \mathfrak{s} \cap \dot{x} \in \mathfrak{S}$ and $\mathfrak{s}' \cap \dot{y} = \mathfrak{s}'$, thus $\mathfrak{s}' \to y$, implying $\mathfrak{s} \to y$, i.e. λ is symmetric.

2° Let λ be symmetric, and define $\mathfrak{S}^1 = \mathfrak{S}^1(\lambda)$ and $\mathfrak{S}^0 = \mathfrak{S}^0(\lambda)$ by (1); both are clearly screens. Put $\kappa = \lambda(\mathfrak{S}^1)$. If $\mathfrak{s} \xrightarrow{\lambda} x$ then $\mathfrak{s} \cap \dot{x} \xrightarrow{\lambda} x$, thus $\mathfrak{s} \cap \dot{x} \in \mathfrak{S}^1$ and $\mathfrak{s} \xrightarrow{\kappa} x$. Conversely, assume that $\mathfrak{s} \xrightarrow{\kappa} x$. Then $\mathfrak{s}' = \mathfrak{s} \cap \dot{x} \in \mathfrak{S}^1$, thus $\mathfrak{s}'$ is λ-convergent and $x \in \bigcap \mathfrak{s}'$; now $\mathfrak{s}' \xrightarrow{\lambda} x$ by the symmetry, so $\mathfrak{s} \xrightarrow{\lambda} x$. Hence $\mathfrak{S}^1$ is compatible with λ, and so is $\mathfrak{S}^0$, since adding some free filters does not change the induced convergence.

3° Let $\mathfrak{S}$ be compatible with λ. If $\mathfrak{s} \in \mathfrak{S}^1$ then $\mathfrak{s}$ converges to some x, thus $\mathfrak{s} \cap \dot{x} \in \mathfrak{S}$, $\mathfrak{s} \in \mathfrak{S}$, i.e. $\mathfrak{S}^1 \subset \mathfrak{S}$. To prove $\mathfrak{S} \subset \mathfrak{S}^0$, let $\mathfrak{s} \in \mathfrak{S}$. If $\mathfrak{s}$ is free then evidently $\mathfrak{s} \in \mathfrak{S}^0$; if $x \in \bigcap \mathfrak{s}$ then $\mathfrak{s} = \mathfrak{s} \cap \dot{x} \in \mathfrak{S}$, thus $\mathfrak{s} \to x$, $\mathfrak{s} \in \mathfrak{S}^1 \subset \mathfrak{S}^0$. □

Two screens induce the same convergence iff they contain the same fixed filters. The analogous statement for pseudotopologies or closures is false: let $\mathfrak{S}^*$ consist of those elements of $\mathfrak{S}$ that are the intersections of at most two ultrafilters; then $\pi(\mathfrak{S}^*) = \pi(\mathfrak{S})$ and $\mathfrak{n}(\mathfrak{S}^*) = \mathfrak{n}(\mathfrak{S})$, but, in general, $\mathfrak{S}^*$ does not contain all the fixed elements of $\mathfrak{S}$.

2.2. According to [14] 13.1, a family of screens in a set always has extensions; the finest and the coarsest ones can be described as follows:

$$\mathfrak{S}^1(X, \mathfrak{S}_i) = \{\mathfrak{s}_i^1 : i \in I, \mathfrak{s}_i \in \mathfrak{S}_i\} \cup \{\dot{x} : x \in X\} \cup \{\exp X\}; \tag{1}$$

$$\mathfrak{S}^0(X, \mathfrak{S}_i) = \{\mathfrak{s} \in \mathrm{Fil}^+ X : \mathfrak{s} \mid X_i \in \mathfrak{S}_i \ (i \in I)\}. \tag{2}$$

In (1), the filters $\dot{x}$ and $\exp X$ had to be added, because it is now not assumed that $X = \bigcup_{i \in I} X_i$.

PROPOSITION. *A family of screens in a symmetric convergence space has compatible extensions iff*

$$\mathfrak{s}_i \in \mathrm{Fil}\, X_i, \quad \mathfrak{s}_i^1 \to x \quad \textit{imply} \quad \mathfrak{s}_i \in \mathfrak{S}_i. \tag{3}$$

---

[2] For better readability, proofs of some well-known simple statements will be included. We give references whenever possible, but have not tried to trace the original sources.

If so then there are a finest and a coarsest extension, namely

(4) $\mathfrak{S}^1(\lambda, \mathfrak{S}_i) = \mathfrak{S}^1(\lambda) \cup \mathfrak{S}^1(X, \mathfrak{S}_i) = \mathfrak{S}^1(\lambda) \cup \{\mathfrak{s}_i^1 : i \in I, \mathfrak{s}_i \in \mathfrak{S}_i\}$;

(5) $\mathfrak{S}^0(\lambda, \mathfrak{S}_i) = \mathfrak{S}^0(\lambda) \cap \mathfrak{S}^0(X, \mathfrak{S}_i) = \mathfrak{S}^1(\lambda) \cup \{\mathfrak{s} \in \mathrm{Fil}^{\mathrm{f}} X : \mathfrak{s} \mid X_i \in \mathfrak{S}_i \ (i \in I)\}$.

PROOF. 1° If $\mathfrak{S}$ is a compatible extension and $\mathfrak{s}_i$ is as in (3) then $\mathfrak{s}_i^1 \cap \cap \dot{x} \in \mathfrak{S}$, $\mathfrak{s}_i^1 \in \mathfrak{S}$, $\mathfrak{s}_i = \mathfrak{s}_i^1 \mid X_i \in \mathfrak{S} \mid X_i = \mathfrak{S}_i$, thus (3) holds.

2° Assume conversely that (3) is satisfied. $\mathfrak{S}' = \mathfrak{S}^1(\lambda) \cup \mathfrak{S}^1(X, \mathfrak{S}_i)$ is a screen (being the infimum of two screens). $\mathfrak{S}' < \mathfrak{S}^1(\lambda)$ implies $\kappa = \lambda(\mathfrak{S}') < \lambda$. To prove $\lambda < \kappa$, assume $\mathfrak{s} \xrightarrow{\kappa} x$. Then $\mathfrak{s}' = \mathfrak{s} \cap \dot{x} \in \mathfrak{S}'$. If $\mathfrak{s}' \in \mathfrak{S}^1(\lambda)$ then $\mathfrak{s} \xrightarrow{\lambda} x$ is clear. Otherwise, $\mathfrak{s}' = \mathfrak{s}_i^1$ with some $i \in I$ and $\mathfrak{s}_i \in \mathfrak{S}_i$; now $x \in X_i$ and $\mathfrak{s}_i = \mathfrak{s}_i \cap \dot{x} \xrightarrow{\lambda_i} x$, implying $\mathfrak{s}' \xrightarrow{\lambda} x$, $\mathfrak{s} \xrightarrow{\lambda} x$. Thus $\mathfrak{S}'$ is compatible.

From $\mathfrak{S}' < \mathfrak{S}^1(X, \mathfrak{S}_i)$ we have $\mathfrak{S}' \mid X_i < \mathfrak{S}_i$. To prove $\mathfrak{S}' \mid X_i > \mathfrak{S}_i$, it is enough to check that $\mathfrak{S}^1(\lambda) \mid X_i \subset \mathfrak{S}_i$, since $\mathfrak{S}^1(X, \mathfrak{S}_i)$ is an extension. Take $\mathfrak{s} \in \mathfrak{S}^1(\lambda)$. Then $\mathfrak{s}$ is convergent, and so is $(\mathfrak{s} \mid X_i)^1 \supset \mathfrak{s}$; hence $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$ follows from (3). Thus $\mathfrak{S}'$ is an extension.

3° If $\mathfrak{S}$ is a compatible extension then $\mathfrak{S} < \mathfrak{S}^1(\lambda)$ and $\mathfrak{S} < \mathfrak{S}^1(X, \mathfrak{S}_i)$, so $\mathfrak{S} < \mathfrak{S}'$. Therefore $\mathfrak{S}'$ is the finest compatible extension. The second equality in (4) is clear from (1) and 2.1 (1).

Put $\mathfrak{S}'' = \mathfrak{S}^0(\lambda) \cap \mathfrak{S}^0(X, \mathfrak{S}_i)$ (the supremum of the two screens). Analogously to the case of $\mathfrak{S}'$, we have $\mathfrak{S}'' < \mathfrak{S}$. Now $\mathfrak{S}^0(\lambda) < \mathfrak{S}'' < \mathfrak{S}$ implies that $\mathfrak{S}''$ is compatible; $\mathfrak{S}''$ is an extension, as $\mathfrak{S}^0(X, \mathfrak{S}_i) < \mathfrak{S}'' < \mathfrak{S}$. Hence $\mathfrak{S}''$ is the coarsest compatible extension. According to (2) and 2.1 (1), $\mathfrak{s} \in \mathfrak{S}''$ iff $\mathfrak{s} \mid X_i \in S_i$ $(i \in I)$ and $\mathfrak{s}$ is either convergent or free; but all the convergent filters $\mathfrak{s}$ satisfy $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$ if there are compatible extensions (see in 2°), thus the second equality in (5) holds. □

COROLLARY. *If each member of a family of screens in a convergence space has extensions then so has the whole family.* □

(3) does not always hold:

EXAMPLE. On $X = \mathbb{N} \cup \{0\}$, let $\lambda(x) = \dot{x} \cup \{\exp X\}$ for $x \neq 0$ and $\mathfrak{s} \to \to 0$ iff $\bigcap \mathfrak{s} \subset \{0\}$. The discrete screen $\mathfrak{S}_0$ on $X_0 = \mathbb{N}$ is compatible, λ is symmetric, but (3) does not hold for the proper free filters in X_0, thus $\mathfrak{S}_0$ has no compatible extension. □

2.3. A necessary and sufficient condition for the existence of an extension in a closure space was given in [5] 2.6; reformulating it with neighbourhood filters, we have:

PROPOSITION. *A family of screens in a symmetric closure space has extensions iff for each $x \in X$ and $i \in I$, the trace filter $\mathfrak{n}(x) \mid X_i$ is the intersection of some elements of $\mathfrak{S}_i$.* □

This condition and 2.2 (3) remain valid if each $\mathfrak{S}_i$ is replaced by a coarser screen; hence there is an extension compatible with λ iff $\mathfrak{S}_i < \mathfrak{S}^1(\lambda) \mid X_i$, while nothing similar holds in a closure space, since there is in general no finest screen compatible with a closure ([4] 3.15).

2.4. 2.2 (3) remains also valid if λ is replaced by a finer extension of the induced convergences. The analogous statement for closures is false: Take a screen $\mathfrak{S}_0$ on $X_0 \subset X$ in a symmetric closure space $(X, \mathfrak{n})$ such that there is no extension (e.g. Example 2.2, with λ replaced by $\mathfrak{n}(\lambda)$, or [5] 2.2). By the proposition below, there are extensions $\mathfrak{n}'$ and $\mathfrak{n}''$ of $\mathfrak{n}(\mathfrak{S}_0) = \mathfrak{n} \mid X_0$ such that $\mathfrak{n}'' < \mathfrak{n} < \mathfrak{n}'$, and $\mathfrak{S}_0$ has extensions compatible with $\mathfrak{n}''$, and also with $\mathfrak{n}'$.

PROPOSITION. *Let a family of screens be given in a set,* $\mathfrak{n}_i = \mathfrak{n}(\mathfrak{S}_i)$ $(i \in I)$. *Denote by* $\mathfrak{n}^1$ *and* $\mathfrak{n}^0$ *the finest, respectively coarsest, extension of the closures* $\mathfrak{n}_i$.

a) *The screens* $\mathfrak{S}_i$ *have an extension compatible with* $\mathfrak{n}^1$.

b) *If* I *is finite then there is also an extension compatible with* $\mathfrak{n}^0$.

PROOF. $\mathfrak{n}^1$ and $\mathfrak{n}^0$ are symmetric ([13] 4.4 or a simple direct proof using the formulas given below).

a) According to [13] 4.5 (1),

$$\mathfrak{n}^1(x) = \bigcap \{\mathfrak{n}_i(x)^1 : x \in X_i, i \in I\} \qquad (x \in \bigcup_{i \in I} X_i),$$

and $\mathfrak{n}^1(x) = \dot{x}$ otherwise. In the first case,

$$\mathfrak{n}^1(x) \mid X_j = \bigcap \{\mathfrak{t}_{(i)} : x \in X_i, i \in I\},$$

where $\mathfrak{t}_{(i)}$ is the filter in X_j generated by $\mathfrak{n}_i(x) \mid X_{ij}$. By the compatibility, $\mathfrak{n}_i(x)$ is the intersection of the elements of $\mathfrak{S}_i$ fixed at x, so the accordance implies that $\mathfrak{t}_{(i)}$ is the intersection of some elements of $\mathfrak{S}_j$. Hence the conditions of Proposition 2.3 are satisfied (the second case is trivial).

b) $\mathfrak{n}^0(x)$ is the coarsest filter $\mathfrak{s}$ on X for which $\mathfrak{s} \mid X_i = \mathfrak{n}_i(x)$ whenever $x \in X_i$. If I is finite then

$$\mathfrak{n}^0(x) = \mathfrak{n}^1(x) \cap \bigcap \{\dot{y} : y \notin \bigcup \{X_i : x \in X_i, i \in I\}\}.$$

Thus Proposition 2.3 can be applied again, since $\mathfrak{S}_j$ contains the fixed ultrafilters in X_j. □

§ 3. Screens in a pseudotopological space

3.1. The results are similar to those valid in a convergence space (rather than in a closure space).

LEMMA. a) *A symmetric convergence induces a symmetric pseudotopology.*

b) *A pseudotopology can be induced by screens iff it is symmetric. If so then there exist a finest and a coarsest compatible screen:*

(1) $$\{\dot{x} \cap \mathfrak{u} \colon \operatorname{Ult} X \ni \mathfrak{u} \to x \in X\} \text{ is a base for } \mathfrak{S}^1(\pi);$$

(2) $$\mathfrak{S}^0(\pi) = \{\mathfrak{s} \in \operatorname{Fil}^+ X \colon (x \in \bigcap \mathfrak{s},\ \mathfrak{u} \in \operatorname{Ult} X,\ \mathfrak{u} \supset \mathfrak{s}) \Rightarrow \mathfrak{u} \to x\}.$$

PROOF. a) Evident.

b) The necessity follows from a) and Lemma 2.1. Assume that π is symmetric and put $\pi(k) = \pi(\mathfrak{S}^k(\pi))$ $(k = 0, 1)$. It is clear (even without the symmetry) that $\pi(1) < \pi < \pi(0)$. Let $\mathfrak{s} = \mathfrak{u} \cap \dot{x}$ with $\mathfrak{u} \xrightarrow{\pi} x$. If $y \in \bigcap \mathfrak{s}$, $\mathfrak{v} \in \operatorname{Ult} X$, $\mathfrak{v} \supset \mathfrak{s}$ then either $y = x$, and then $\mathfrak{v} \xrightarrow{\pi} x$ $(\mathfrak{v} = \mathfrak{u}$ or $\mathfrak{v} = \dot{y})$, or $\mathfrak{u} = \dot{y}$, thus $\dot{x} \xrightarrow{\pi} y$, and $\mathfrak{v} \xrightarrow{\pi} y$ again $(\mathfrak{v} = \dot{x}$ or $\mathfrak{v} = \dot{y})$. Hence $\mathfrak{s} \in \mathfrak{S}^0(\pi)$, implying $\mathfrak{S}^1(\pi) \subset \mathfrak{S}^0(\pi)$, $\pi(0) < \pi(1)$, i.e. both screens are compatible. It is straightforward that if $\mathfrak{S}$ is compatible then $\mathfrak{S}^0(\pi) < \mathfrak{S} < \mathfrak{S}^1(\pi)$. □

REMARK. π is symmetric iff $\mathfrak{n}(\pi)$ is so. On the other hand, the symmetry of π does not imply that each compatible convergence is symmetric, see the details in 9.2. In particular, it can happen that a pseudotopology (or closure) can be induced by a screen, but not the corresponding pseudotopological (or pretopological) convergence, cf. Lemma 9.2.

3.2. PROPOSITION. *A family of screens in a symmetric pseudotopological space has extensions iff*

(1) $$\mathfrak{u}_i \in \operatorname{Ult} X_i,\ \mathfrak{u}_i^1 \to x \ \text{ imply } \ \mathfrak{u}_i \in \mathfrak{S}_i.$$

If so then there are a finest and a coarsest extension:

(2) $$\mathfrak{S}^1(\pi, \mathfrak{S}_i) = \mathfrak{S}^1(\pi) \cup \mathfrak{S}^1(X, \mathfrak{S}_i) = \mathfrak{S}^1(\pi) \cup \{\mathfrak{s}_i^1 : i \in I,\ \mathfrak{s}_i \in \mathfrak{S}_i\};$$

(3) $$\mathfrak{S}^0(\pi, \mathfrak{S}_i) = \mathfrak{S}^0(\pi) \cap \mathfrak{S}^0(X, \mathfrak{S}_i).$$

PROOF. Analogous to the proof of Proposition 2.2. We only mention one detail, where the reasoning is somewhat different (the second paragraph of 2°): To show $\mathfrak{S}^1(\pi) \mid X_i \subset \mathfrak{S}_i$, take $\mathfrak{s} \in \mathfrak{S}^1(\pi)$. Then $\mathfrak{s} \supset \mathfrak{u} \cap \dot{x}$ with $\mathfrak{u} \xrightarrow{\pi} x$. By (1), $\mathfrak{u} \mid X_i \in \mathfrak{S}_i$. Now if $x \notin X_i$ then $\mathfrak{s} \mid X_i \supset \mathfrak{u} \mid X_i \in \mathfrak{S}_i$; if $x \in X_i$ then $\mathfrak{u} \mid X_i \xrightarrow{\pi_i} x$, thus $\mathfrak{s} \mid X_i \supset (\mathfrak{u} \mid X_i) \cap \dot{x} \in \mathfrak{S}_i$. □

COROLLARY. *If each member of a family of screens in a pseudotopological space has extensions then so has the whole family.* □

In spite of Remark 3.1, it is important to differentiate between closures and pretopological pseudotopologies:

EXAMPLE. Let X, X_0 and λ be as in Example 2.2. With $\pi=\pi(\lambda)$ and $\mathfrak{n}=\mathfrak{n}(\lambda)$, we have $\pi=\pi^0(\mathfrak{n})$. Take $\mathfrak{v}_0\in \mathrm{Ult}^{\mathrm{f}} X_0$, and let $\mathfrak{S}_0=\mathrm{Ult}^+X_0\setminus\{\mathfrak{v}_0\}$. Then $\pi(\mathfrak{S}_0)=\pi_0$, $\mathfrak{n}(\mathfrak{S}_0)=\mathfrak{n}_0$, and $\mathfrak{S}_0$ has an extension in $(X,\mathfrak{n})$, but not in (X,π). □

Similarly to 2.3 and 2.4, (1) remains valid if each $\mathfrak{S}_\iota$ is replaced by a coarser screen, or π by a finer extension of the induced pseudotopologies.

§ 4. Cauchy screens in a convergence space

4.1. It is not clear which closures can be induced by Cauchy screens; moreover, the existence of a compatible Cauchy screen does not imply that there is either a finest or a coarsest one, cf. [7]. The situation is much simpler in a convergence space. It was already mentioned in § 1 that a convergence induced by a Cauchy screen is a limitation.

LEMMA (partly [2] p. 35). *A limitation can be induced by Cauchy screens iff it is reciprocal; if so then $\mathfrak{S}^1(\lambda)$ is Cauchy, and*

(1) $$\mathfrak{S}^0_{(\mathrm{C})}(\lambda)=\mathfrak{S}^1(\lambda)\cup\{\mathfrak{s}\in \mathrm{Fil}^{\mathrm{f}} X:\ \mathfrak{s}\,\bar{\Delta}\,\mathfrak{t}\ (\mathfrak{t}\in\mathfrak{S}^1(\lambda))\}$$

is the coarsest compatible Cauchy screen.[3]

($\mathrm{Fil}^{\mathrm{f}}$ can be replaced by Fil in (1), since if $x\in\bigcap\mathfrak{s}$ then $\mathfrak{s}\,\Delta\,\dot{x}\in\mathfrak{S}^1(\lambda)$.)

PROOF. 1° Assume that $\lambda=\lambda(\mathfrak{S})$, $\mathfrak{S}$ is Cauchy, $\mathfrak{s}\in\widetilde{\lambda}(x)\cap\widetilde{\lambda}(y)$, $\mathfrak{s}'\to x$. Then $\mathfrak{s}'\cap\dot{x}$, $\mathfrak{s}\cap\dot{x}$, $\mathfrak{s}\cap\dot{y}$ is an $\mathfrak{S}$-chain, thus $\mathfrak{s}'\cap\dot{y}\in\mathfrak{S}$, $\mathfrak{s}'\to y$. Hence $\lambda(x)=\lambda(y)$.

2° Let λ be reciprocal, $\mathfrak{s},\mathfrak{t}\in\mathfrak{S}^1(\lambda)$, $\mathfrak{s}\,\Delta\,\mathfrak{t}$. Take x,y such that $\mathfrak{s}\to x$, $\mathfrak{t}\to y$. From $\mathfrak{s}\,\Delta\,\mathfrak{t}$ we have $\widetilde{\lambda}(x)\cap\widetilde{\lambda}(y)\neq\emptyset$, thus $\lambda(x)=\lambda(y)$, and so $\mathfrak{t}\to x$. λ being a limitation, $\mathfrak{s}\cap\mathfrak{t}\to x$, therefore $\mathfrak{s}\cap\mathfrak{t}\in\mathfrak{S}^1(\lambda)$.

3° $\mathfrak{S}^0(\lambda)\supset\mathfrak{S}^0_{(\mathrm{C})}(\lambda)\supset\mathfrak{S}^1(\lambda)$ (see 2.1 (1)), thus $\mathfrak{S}'=\mathfrak{S}^0_{(\mathrm{C})}(\lambda)$ is compatible. It is also Cauchy, since if $\mathfrak{s},\mathfrak{s}'\in\mathfrak{S}'$, $\mathfrak{s}\,\Delta\,\mathfrak{s}'$ then either $\mathfrak{s},\mathfrak{s}'\in\mathfrak{S}^1(\lambda)$ or $\mathfrak{s}\,\bar{\Delta}\,\mathfrak{t}\,\bar{\Delta}\,\mathfrak{s}'$ for each $\mathfrak{t}\in\mathfrak{S}^1(\lambda)$; in the second case, $\mathfrak{s}\cap\mathfrak{s}'\,\bar{\Delta}\,\mathfrak{t}$ $(\mathfrak{t}\in\mathfrak{S}^1(\lambda))$, thus $\mathfrak{s}\cap\mathfrak{s}'\in\mathfrak{S}'$.

Let $\mathfrak{S}$ be a compatible Cauchy screen; we show that $\mathfrak{S}\subset\mathfrak{S}'$. Take $\mathfrak{s}\in\in\mathfrak{S}\setminus\mathfrak{S}'$. There is a $\mathfrak{t}\in\mathfrak{S}^1(\lambda)$ with $\mathfrak{s}\,\Delta\,\mathfrak{t}$. As $\mathfrak{S}^1(\lambda)\subset\mathfrak{S}$, we have $\mathfrak{t}\in\mathfrak{S}$, thus $\mathfrak{s}\cap\mathfrak{t}\in\mathfrak{S}$. Now there is an x with $\mathfrak{t}\to x$, so $\mathfrak{t}\cap\dot{x}\in\mathfrak{S}$, $\mathfrak{t}\cap\dot{x}\,\Delta\,\mathfrak{s}\cap\mathfrak{t}$ (since $\mathfrak{t}\neq\exp X$ follows from $\mathfrak{s}\,\Delta\,\mathfrak{t}$), hence $\mathfrak{s}\cap\dot{x}\in\mathfrak{S}$, $\mathfrak{s}\to x$, $\mathfrak{s}\in\mathfrak{S}^1(\lambda)\subset\mathfrak{S}'$, a contradiction. □

[3] We have put the letter C into parentheses to avoid confusion with $(\mathfrak{S}^0(\lambda))_{\mathrm{C}}$.

4.2. The following result will be needed:

PROPOSITION ([14] 14.1 and 14.3). *A family of Cauchy screens in a set has Cauchy extensions iff for each*[4] $n \in \mathbb{N}$ *and different indices* $i, j_0, j_1, \ldots, j_n \in I$,

$$\text{(1)} \qquad \textit{if } \mathfrak{s}_i, \mathfrak{s}_i' \in \mathfrak{S}_i,\ \mathfrak{s}_{j_m} \in \mathfrak{S}_{j_m}\ (0 \leqq m \leqq n) \textit{ and } \mathfrak{s}_i, \mathfrak{s}_{j_0}, \ldots, \mathfrak{s}_{j_n}, \mathfrak{s}_i' \textit{ is a chain then } \mathfrak{s}_i \cap \mathfrak{s}_i' \in \mathfrak{S}_i.$$

If so then $\mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i)$ *is the finest Cauchy extension.* □

$\mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i)$ stands here for $(\mathfrak{S}^1(X, \mathfrak{S}_i))_{\mathrm{C}}$. The same convention applies in similar situations; cf. the footnote to 4.1.

4.3. If a family of screens in a convergence space has Cauchy extensions then the screens are Cauchy, the convergence is a reciprocal limitation, 2.2 (3) and 4.2 (1) hold. Adding one more assumption, we obtain a set of necessary and sufficient conditions:

THEOREM. *A family of Cauchy screens in a reciprocal limit space has Cauchy extensions iff* 2.2 (3) *and* 4.2 (1) *hold, and*

$$\text{(1)} \qquad \mathfrak{s}_i \in \mathfrak{S}_i,\ \mathfrak{s}_i \subset \mathfrak{t}_i \in \operatorname{Fil} X_i,\ \mathfrak{t}_i^1 \to x \ \textit{ imply } \ \mathfrak{s}_i^1 \to x.$$

If so then $\mathfrak{S}^1_{\mathrm{C}}(\lambda, \mathfrak{S}_i)$ *is the finest Cauchy extension, and*

$$\text{(2)} \qquad \mathfrak{S}^1_{\mathrm{C}}(\lambda, \mathfrak{S}_i) = \mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i) \cup \mathfrak{S}^1(\lambda).$$

If $|I| \leqq 2$ *then* 4.2 (1) *is superfluous.*

PROOF. 1° Assume that $\mathfrak{S}$ is a compatible Cauchy extension, and let $\mathfrak{s}_i$, $\mathfrak{t}_i$ and x be as in (1). Then $\mathfrak{s}_i^1 \in \mathfrak{S}$, $\mathfrak{t}_i^1 \cap \dot{x} \in \mathfrak{S}$ and $\mathfrak{s}_i^1 \,\Delta\, \mathfrak{t}_i^1 \cap \dot{x}$, thus $\mathfrak{s}_i^1 \cap \dot{x} \in \mathfrak{S}$, $\mathfrak{s}_i^1 \to x$, i.e. (1) holds.

2° Assume that all the conditions hold, and put $\mathfrak{S}' = \mathfrak{S}^1_{\mathrm{C}}(\lambda, \mathfrak{S}_i)$, which is a Cauchy screen. By Proposition 2.2, $\mathfrak{S}^1(\lambda, \mathfrak{S}_i)$ is a compatible extension, thus $\lambda(\mathfrak{S}') < \lambda$ and $\mathfrak{S}' \mid X_i < \mathfrak{S}_i$. We are going to show that equality holds in both cases; but let us first prove (2):

3° $\mathfrak{S}^1(\lambda, \mathfrak{S}_i) < \mathfrak{S}^1(X, \mathfrak{S}_i)$ and $\mathfrak{S}^1(\lambda, \mathfrak{S}_i) < \mathfrak{S}^1(\lambda)$, thus $\mathfrak{S}' < \mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i)$ and $\mathfrak{S}' < \mathfrak{S}^1_{\mathrm{C}}(\lambda) = \mathfrak{S}^1(\lambda)$ (Lemma 4.1), showing that $\mathfrak{S}' < \mathfrak{S}''$ where $\mathfrak{S}''$ denotes the right-hand side of (2). To prove the converse, take $\mathfrak{s} \in \mathfrak{S}'$. There is an $\mathfrak{S}^1(\lambda, \mathfrak{S}_i)$-chain $\mathfrak{s}_{(1)}, \ldots, \mathfrak{s}_{(n)}$ with $\mathfrak{s} \supset \bigcap_{m=1}^{n} \mathfrak{s}_{(m)}$; choose this chain with the smallest possible n. Consider $\mathfrak{S}^1(\lambda, \mathfrak{S}_i)$ in the form given by the right-hand side of 2.2 (4). If $\mathfrak{s}_{(m)} \in \mathfrak{S}^1(\lambda)$ for each m then $\mathfrak{s} \in \mathfrak{S}^1(\lambda)$ (as $\mathfrak{S}^1(\lambda)$ is Cauchy), thus $\mathfrak{s} \in \mathfrak{S}''$. If each $\mathfrak{s}_{(m)}$ is of the form $\mathfrak{t}^1_{(m)}$ with suitable $i_m \in I$

[4] [14] 14.1 in itself gives this with $n \in \mathbb{N} \cup \{0\}$. (The indexing is different there.)

and $\mathfrak{t}_{(m)} \in \mathfrak{S}_{i_m}$ then $\mathfrak{s}_{(m)} \in \mathfrak{S}^1(X, \mathfrak{S}_i)$, $\mathfrak{s} \in \mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i) \subset \mathfrak{S}''$. So there remains only one possibility:

Assume there is an m such that $\mathfrak{s}_{(m)} \in \mathfrak{S}^1(\lambda)$, $\mathfrak{s}_{(m+1)} = \mathfrak{s}_i^1$, $i \in I$, $\mathfrak{s}_i \in \mathfrak{S}_i$ (or $\mathfrak{s}_{(m-1)} = \mathfrak{s}_i^1$, which makes no difference). Define $\mathfrak{t}_i = \mathfrak{s}_i(\cap)(\mathfrak{s}_{(m)} \mid X_i)$. Now $\mathfrak{t}_i \neq \exp X_i$, since $\mathfrak{s}_{(m)} \Delta \mathfrak{s}_{(m+1)}$. From $\mathfrak{s}_{(m)} \in \mathfrak{S}^1(\lambda)$ we have $\mathfrak{s}_{(m)} \to x$, thus $\mathfrak{t}_i^1 \to x$, and (1) yields that $\mathfrak{s}_i^1 = \mathfrak{s}_{(m+1)} \to x$. Hence, λ being a limitation, $\mathfrak{s}_{(m)} \cap \mathfrak{s}_{(m+1)} \to x$, and so $\mathfrak{s}_{(m)}$ and $\mathfrak{s}_{(m+1)}$ can be replaced in the chain by $\mathfrak{s}_{(m)} \cap \mathfrak{s}_{(m+1)} \in \mathfrak{S}^1(\lambda)$, contradicting the minimality of n. Thus (2) has been proved.

4° Note also that $\mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i)$ can be replaced in (2) by its free elements: Let $\mathfrak{s} \in \mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i)$, $x \in \bigcap \mathfrak{s}$, and take a chain $\mathfrak{t}_{(1)}, \ldots, \mathfrak{t}_{(n)}$ such that $\mathfrak{t}_{(m)} \in \mathfrak{S}_{i_m}$, $\mathfrak{s} \supset \bigcap\limits_{m=1}^{n} \mathfrak{t}^1_{(m)}$. Then $x \in \bigcap \mathfrak{t}_{(m)}$ for some m. Now $\mathfrak{t}_{(m)} \xrightarrow{\lambda_{i_m}} x$, $\mathfrak{t}^1_{(m)} \xrightarrow{\lambda} x$, implying $\mathfrak{t}^1_{(m)} \in \mathfrak{S}^1(\lambda)$. The reasoning in the second paragraph of 3° gives that $\mathfrak{t}^1_{(m)} \cap \mathfrak{t}^1_{(m+1)} \in \mathfrak{S}^1(\lambda)$, and similarly $\mathfrak{t}^1_{(m-1)} \cap \mathfrak{t}^1_{(m)} \in \mathfrak{S}^1(\lambda)$. By repeating the reasoning, we finally get $\mathfrak{s} \in \mathfrak{S}^1(\lambda)$.

5° $\mathfrak{S}' \mid X_i \subset \mathfrak{S}_i$ is clear from (2), since $\mathfrak{S}^1(\lambda) \subset \mathfrak{S}^1(\lambda, \mathfrak{S}_i)$ gives

$$\mathfrak{S}' \subset \mathfrak{S}^1_{\mathrm{C}}(X, \mathfrak{S}_i) \cup \mathfrak{S}^1(\lambda, \mathfrak{S}_i),$$

and both screens on the right-hand side are extensions.

6° $\mathfrak{S}'$ is compatible, since, according to (2) and 4°, it contains the same fixed filters as the compatible screen $\mathfrak{S}^1(\lambda)$.

7° If $\mathfrak{S}$ is a Cauchy extension compatible with λ then $\mathfrak{S} < \mathfrak{S}^1(\lambda, \mathfrak{S}_i)$, so $\mathfrak{S} < \mathfrak{S}'$, i.e. $\mathfrak{S}'$ is indeed the finest Cauchy extension.

8° If $|I| \leqq 2$ then we cannot take different indices i, j_0, j_1 in 4.2 (1). □

COROLLARY. *A family of screens has Cauchy extensions in a convergence space iff any finite subfamily has one. A family of two screens has Cauchy extensions in a convergence space iff the screens taken separately have Cauchy extensions.* □

Differently from Lemma 4.1, there is no coarsest compatible Cauchy extension:

EXAMPLE. Let X_0 and $X \setminus X_0$ be infinite, $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathrm{Ult}^{\mathrm{f}} X$, $X_0 \in \mathfrak{u}, \mathfrak{v}$, $X_0 \notin \mathfrak{w}$. Take the discrete convergence λ on X, and the screen $\mathfrak{S}_0$ on X_0 obtained by adding $\mathfrak{u} \mid X_0$ and $\mathfrak{v} \mid X_0$ to the discrete screen. Then there are Cauchy extensions containing either of the filters $\mathfrak{u} \cap \mathfrak{w}$ and $\mathfrak{v} \cap \mathfrak{w}$, but not both. □

4.4. 2.2 (3) and 4.3 (1) are both needed in Theorem 4.3, even when $|I| = 1$; 4.2 (1) cannot be dropped either if $|I| \geqq 3$:

EXAMPLES. a) Let X_0 be infinite, $X \setminus X_0 = \{z\}$, $\mathfrak{u}, \mathfrak{v} \in \mathrm{Ult}^{\mathrm{f}} X$, $\mathfrak{u} \neq \mathfrak{v}$, $\lambda(x) = \dot{x} \cup \{\exp X\}$ $(x \neq z)$, and $\mathfrak{s} \to z$ iff $\mathfrak{s} \supset \mathfrak{u} \cap \dot{z}$. Consider the following

Cauchy screens compatible with $\lambda \mid X_0$: $\mathfrak{S}_0$ is the discrete one, and $\mathfrak{S}_0 \cup \{(\mathfrak{u} \cap \cap \mathfrak{v}) \mid X_0\}$ is a base for $\mathfrak{S}'_0$. Now $\mathfrak{S}_0$ satisfies 4.3 (1), but not 2.2 (3), and $\mathfrak{S}'_0$ conversely.

b) Let A_1, A_2, A_3 be disjoint infinite sets, $X = \bigcup_{i=1}^{3} A_i$, $X_i = X \setminus A_i$, $A_i \in \in \mathfrak{u}_{(i)} \in \mathrm{Ult}^{\mathrm{f}} X$. Take the discrete convergence λ on X, and define the screens $\mathfrak{S}_i$ by the following bases:

$$\{\dot{x} \colon x \in X_i\} \cup \{\bigcap_{j=1}^{3} \mathfrak{u}_{(j)} \mid X_i\} \qquad (i = 1, 2),$$

$$\{\dot{x} \colon x \in X_3\} \cup \{\mathfrak{u}_{(1)} \mid X_3, \mathfrak{u}_{(2)} \mid X_3\}.$$

All the conditions of Theorem 4.3 are satisfied, excepting 4.2 (1). □

4.5. COROLLARY. *A family of Cauchy screens in a reciprocal limit space has Cauchy extensions iff*

(1) $$\mathfrak{S}^0_{(\mathrm{C})}(\lambda)|X_i < \mathfrak{S}_i < \mathfrak{S}^1(\lambda)|X_i \qquad (i \in I)$$

and 4.2 (1) *holds.*

PROOF. The necessity being obvious, let us assume (1) and 4.2 (1). By Theorem 4.3, 2.2 (3) holds for the family $\mathfrak{S}^1(\lambda) \mid X_i$, and 4.3 (1) for the family $\mathfrak{S}^0_{(\mathrm{C})}(\lambda) \mid X_i$. Now 2.2 (3) (respectively 4.3 (1)) remains valid if the screens are replaced by coarser (respectively finer) ones, thus all the assumptions of Theorem 4.3 are fulfilled for the screens $\mathfrak{S}_i$. □

Example 4.4 b) shows that 4.2 (1) is needed in this corollary.

Let us say that x is a *cluster point* of $\mathfrak{s} \in \mathrm{Fil}^+ X$ in (X, λ) if there is a proper filter (equivalently: an ultrafilter) $\mathfrak{s}' \supset \mathfrak{s}$ with $\mathfrak{s}' \to x$. x is a cluster point of $\mathfrak{s}$ iff there is a $\mathfrak{t} \in \lambda(x)$ with $\mathfrak{t} \Delta \mathfrak{s}$. Thus 4.1 (1) can be expressed as follows: $\mathfrak{S}^0_{(\mathrm{C})}(\lambda)$ consists of the convergent filters and of those having no cluster point. Extending the notion of convergence and cluster points to filter bases, we can also say: A family of Cauchy screens in a reciprocal limit space has Cauchy extensions iff there is a Cauchy extension in the set, and for each i, $\mathfrak{S}_i$ consists of all the filters in X_i that are λ-convergent, and of some having no λ-cluster point.

REMARK. If $\mathfrak{s} \to x$ in a reciprocal limit space and y is a cluster point of $\mathfrak{s}$ then $\mathfrak{s} \to y$. Hence $\mathfrak{S}^0_{(\mathrm{C})}(\lambda)$ consists of the filters that converge to their cluster points; cf. the notion of a strongly compressed filter in a closure space, [7] § 2.

4.6. Assume that $\lambda' < \lambda < \lambda''$ and $\{\mathfrak{S}_i \colon i \in I\}$ is a family of screens in all three spaces. It can occur that there are Cauchy extensions in (X, λ') and (X, λ'') but not in (X, λ), although λ is a reciprocal limitation:

EXAMPLE. With the notations of Example 4.4 a), let λ'' be the discrete convergence, and $\lambda'(x) = \lambda(x)$ for $x \neq z$, $\mathfrak{s} \xrightarrow{\lambda'} z$ iff $\mathfrak{s} \supset \mathfrak{u} \cap \mathfrak{v} \cap \dot{z}$. Now $\mathfrak{S}_0'$ has extensions in (X, λ') and (X, λ''), but not in (X, λ). □

4.7. $\lambda' < \lambda$ does not imply $\mathfrak{S}^0_{(\mathrm{C})}(\lambda') < \mathfrak{S}^0_{(\mathrm{C})}(\lambda)$ [$\mathfrak{S}^0_{(\mathrm{C})}$ is not a functor]:

EXAMPLE. Let λ' and λ be as in Example 4.6, and take $\mathfrak{w} \in \mathrm{Ult}^{\mathrm{f}} X$, $\mathfrak{u} \neq \mathfrak{w} \neq \mathfrak{v}$. Then $\mathfrak{v} \cap \mathfrak{w} \in \mathfrak{S}^0_{(\mathrm{C})}(\lambda) \setminus \mathfrak{S}^0_{(\mathrm{C})}(\lambda')$, thus $\mathfrak{S}^0_{(\mathrm{C})}(\lambda') \not< \mathfrak{S}^0_{(\mathrm{C})}(\lambda)$. □

The following positive result will be needed in the next section:

LEMMA. *If λ' and λ are reciprocal limitations, $\lambda' < \lambda$ and $\pi(\lambda') = \pi(\lambda)$ then $\mathfrak{S}^0_{(\mathrm{C})}(\lambda') < \mathfrak{S}^0_{(\mathrm{C})}(\lambda)$.*

PROOF. The cluster points are determined by the induced pseudotopology. □

§ 5. Cauchy screens in a pseudotopological space

5.1. LEMMA. *Let λ be a convergence, π a pseudotopology.*

a) *If λ is reciprocal then so are λ_{lim} and $\pi(\lambda)$.*

b) *If $\pi(\lambda)$ is reciprocal and $\pi(\lambda)(x) = \pi(\lambda)(y)$ implies $\lambda(x) = \lambda(y)$ then λ is reciprocal, too.*

c) *If π is reciprocal then so are $\lambda^0(\pi)$ and $\lambda^1_{\mathrm{lim}}(\pi)$.*

PROOF. Assume that λ is reciprocal, $\emptyset \notin \mathfrak{s} \xrightarrow{\lambda_{\mathrm{lim}}} x, y$. With $\mathfrak{s} \subset \mathfrak{u} \in \mathrm{Ult}\, X$, $\mathfrak{u} \xrightarrow{\lambda_{\mathrm{lim}}} x, y$, implying $\mathfrak{u} \xrightarrow{\lambda} x, y$, hence $\lambda(x) = \lambda(y)$, $\lambda_{\mathrm{lim}}(x) = \lambda_{\mathrm{lim}}(y)$, and so λ_{lim} is reciprocal. b) and the other part of a) are evident. c) follows from b). □

5.2. LEMMA. *A pseudotopology can be induced by Cauchy screens iff it is reciprocal; if so then $\mathfrak{S}^1_{\mathrm{C}}(\pi)$ is the finest compatible Cauchy screen, and*

(1)
$$\mathfrak{S}^0_{(\mathrm{C})}(\pi) = \mathfrak{S}^0_{(\mathrm{C})}(\lambda^0(\pi)) = \{\mathfrak{s} \in \mathrm{Fil}^+ X \colon (\mathfrak{u}, \mathfrak{v} \supset \mathfrak{s}, \mathfrak{u} \xrightarrow{\pi} x, \mathfrak{v} \in \mathrm{Ult}\, X) \Rightarrow \mathfrak{v} \xrightarrow{\pi} x\}$$

is the coarsest one.

PROOF. The necessity is clear from Lemmas 4.1 and 5.1 a). Assume that π is reciprocal. Then $\lambda^0(\pi)$ is a reciprocal limitation (Lemma 5.1 c)), so $\mathfrak{S}' = \mathfrak{S}^0_{(\mathrm{C})}(\lambda^0(\pi))$ is a Cauchy screen compatible with π (Lemma 4.1). Let $\mathfrak{S}$ be another compatible Cauchy screen, $\lambda = \lambda(\mathfrak{S})$. Then $\pi(\lambda^0(\pi)) = \pi = \pi(\lambda)$ and $\lambda^0(\pi) < \lambda$, thus $\mathfrak{S}' < \mathfrak{S}^0_{(\mathrm{C})}(\lambda)$ (Lemma 4.7), therefore $\mathfrak{S}' < \mathfrak{S}$ (Lemma 4.1). Moreover, $\mathfrak{S} < \mathfrak{S}^1(\pi)$, and the latter is the finest compatible screen (Lemma 3.1 b)), thus $\mathfrak{S} < \mathfrak{S}^1_{\mathrm{C}}(\pi)$, and $\mathfrak{S}^1_{\mathrm{C}}(\pi)$ is a compatible Cauchy screen, hence the finest one. The second equality in (1) follows from 4.1 (1), cf. the observation after Corollary 4.5. □

5.3. If a family of screens in a pseudotopological space has Cauchy extensions then the screens are Cauchy, π is reciprocal, 4.2 (1) and 3.2 (1) hold. One might expect that these conditions together with an analogue of 4.3 (1) are sufficient; but this is not the case: 3.2 (1) has to be replaced by a stronger assumption. The reason is that we need $\mathfrak{S}_i < \mathfrak{S}^1_{\mathrm{C}}(\pi) \mid X_i$, while 3.2 (1) only guarantees $\mathfrak{S}_i < \mathfrak{S}^1(\pi) \mid X_i$ (and $\mathfrak{S}^1_{\mathrm{C}}(\pi)$ can be different from $\mathfrak{S}^1(\pi)$, while $\mathfrak{S}^1_{\mathrm{C}}(\lambda) = \mathfrak{S}^1(\lambda)$, see Lemma 4.1).

THEOREM. *A family of Cauchy screens in a reciprocal pseudotopological space has Cauchy extensions iff* 4.2 (1) *and the following conditions hold:*

(1) $$\mathfrak{u}_i, \mathfrak{v}_i \in \operatorname{Ult} X_i,\ \mathfrak{u}_i^1, \mathfrak{v}_i^1 \to x \quad \textit{imply} \quad \mathfrak{u}_i \cap \mathfrak{v}_i \in \mathfrak{S}_i;$$

(2) $$\mathfrak{u}_i, \mathfrak{v}_i \in \operatorname{Ult} X_i,\ \mathfrak{u}_1^1 \to x,\ \mathfrak{u}_i \cap \mathfrak{v}_i \in \mathfrak{S}_i \quad \textit{imply} \quad \mathfrak{v}_i^1 \to x.$$

If so then $\mathfrak{S}^1_{\mathrm{C}}(\pi, \mathfrak{S}_i)$ is the finest Cauchy extension. If $|I| \leqq 2$ then 4.2 (1) *is superfluous.*

PROOF. 1° Let $\mathfrak{S}$ be a compatible Cauchy extension, $\mathfrak{u}_i, \mathfrak{v}_i \in \operatorname{Ult} X_i$, $\mathfrak{u}_i^1 \to$ $\to x$; then $\mathfrak{u}_i^1 \cap \dot{x} \in \mathfrak{S}$. If $\mathfrak{v}_i^1 \to x$ then also $\mathfrak{v}_i^1 \cap \dot{x} \in \mathfrak{S}$, thus, $\mathfrak{S}$ being Cauchy, $\mathfrak{u}_i^1 \cap \mathfrak{v}_i^1 \in \mathfrak{S}$, $\mathfrak{u}_i \cap \mathfrak{v}_i \in \mathfrak{S}_i$. Conversely, if $\mathfrak{u}_i \cap \mathfrak{v}_i \in \mathfrak{S}_i$ then $\mathfrak{u}_i^1 \cap \mathfrak{v}_i^1 = (\mathfrak{u}_i \cap \mathfrak{v}_i)^1 \in \mathfrak{S}$, $\mathfrak{v}_i^1 \cap \dot{x} \in \mathfrak{S}$, $\mathfrak{v}_i^1 \to x$. Hence the conditions are necessary.

2° Assume that all the conditions are satisfied. (1) with $\mathfrak{u}_i = \mathfrak{v}_i$ yields 3.2 (1), so $\mathfrak{S}' = \mathfrak{S}^1(\pi, \mathfrak{S}_i)$ is a compatible extension (the finest one) by Proposition 3.2. We claim that $\mathfrak{S}'_{\mathrm{C}}$ is also a compatible extension (consequently, it is the finest compatible Cauchy extension). The reverse inequalities being evident, it is enough to show that $\pi < \pi(S'_{\mathrm{C}})$ and $\mathfrak{S}_i < \mathfrak{S}'_{\mathrm{C}} \mid X_i$.

3° For $\mathfrak{s} \in \mathfrak{S}'_{\mathrm{C}}$, take an $\mathfrak{S}'$-chain $\mathfrak{s}_{(1)}, \ldots, \mathfrak{s}_{(n)}$ with $\mathfrak{s} \supset \bigcap_{m=1}^{n} \mathfrak{s}_{(m)}$. According to 3.2 (2), each $\mathfrak{s}_{(m)}$ has one of the following forms:

(3) $$\mathfrak{s}_{(m)} = \dot{x}_m \cap \mathfrak{u}_{(m)}, \qquad \mathfrak{u}_{(m)} \to x_m;$$

(4) $$\mathfrak{s}_{(m)} = \mathfrak{u}_{(m)}, \qquad \mathfrak{u}_{(m)} \to x_m;$$

(5) $$\mathfrak{s}_{(m)} = \mathfrak{t}^1_{(m)}, \qquad \mathfrak{t}_{(m)} \in \mathfrak{S}_{i_m}.$$

Each $\mathfrak{s}_{(m)}$ of the form (4) can be replaced by $\dot{x}_m \cap \mathfrak{u}_{(m)}$, so we may assume that (3) holds for $m \in H \subset \{1, \ldots, n\}$, and (5) for $m \notin H$. Moreover, it can be assumed that if $H = \emptyset$ then each $\mathfrak{s}_{(m)}$ is free: if not then $x \in \bigcap \mathfrak{s}_{(m)}$ with suitable x and m, and $\mathfrak{s}_{(m)}$ can be replaced in the chain by $\mathfrak{s}_{(m)}, \dot{x}, \mathfrak{s}_{(m)}$; here $\dot{x}$ is of the type (3), thus H has been made non-empty.

Let $|H| \geqq 2$, and denote its elements, in increasing order, by $m_1, \dots, m_p$. If $m_2 = m_1 + 1$ then $\dot{x}_{m_1} \cap \mathfrak{u}_{(m_1)} \,\Delta\, \dot{x}_{m_2} \cap \mathfrak{u}_{(m_2)}$ implies $\pi(x_{m_1}) = \pi(x_{m_2})$ (as π is reciprocal). If $m_2 > m_1 + 1$ then pick $\mathfrak{v}_{(k)} \in \operatorname{Ult} X$ $(m_1 \leqq k < m_2)$ such that

$$(6) \qquad \mathfrak{v}_{(m_1)} = \mathfrak{u}_{(m_1)} \quad \text{or} \quad \mathfrak{v}_{(m_1)} = \dot{x}_{m_1},$$

$$(7) \qquad X_{i_k} \in \mathfrak{v}_{(k)}, \quad \mathfrak{v}_{(k)} \,|\, X_{i_k} \supset \mathfrak{t}_{(k)} \qquad (k \neq m_1),$$

$$(8) \qquad X_{i_{k+1}} \in \mathfrak{v}_{(k)}, \quad \mathfrak{v}_{(k)} \,|\, X_{i_{k+1}} \supset \mathfrak{t}_{(k+1)} \qquad (k \neq m_2 - 1),$$

$$(9) \qquad \mathfrak{v}_{(m_2-1)} = \mathfrak{u}_{(m_2)} \quad \text{or} \quad \mathfrak{v}_{(m_2-1)} = \dot{x}_{m_2};$$

this is possible, because $\mathfrak{s}_{(k)} \,\Delta\, \mathfrak{s}_{(k+1)}$. Now by (7) and (8),

$$(\mathfrak{v}_{(m_1)} \cap \mathfrak{v}_{(m_1+1)}) \,|\, X_{m_1+1} \in \mathfrak{S}_{m_1+1}$$

and $\mathfrak{v}_{(m_1)} \to x_{m_1}$ (see (3) and (6)), so $\mathfrak{v}_{(m_1+1)} \to x_{m_1}$ follows from (2). By induction,

$$(10) \qquad \mathfrak{v}_{(k)} \to x_{m_1} \qquad (m_1 \leqq k < m_2),$$

too, thus $\pi(x_{m_1}) = \pi(x_{m_2})$, since $\mathfrak{v}_{(m_2-1)} \to x_{m_2}$ (see (3) and (9)), and π is reciprocal. Hence an induction gives that, assuming $H \neq \emptyset$, there is a point x_0 such that

$$(11) \qquad \pi(x_m) = \pi(x_0) \qquad (m \in H).$$

4° The following holds for each m: if $x \in \bigcap \mathfrak{s}_{(m)}$ then $\pi(x) = \pi(x_0)$. Indeed, if $m \in H$ then either $x = x_m$ or $\mathfrak{u}_{(m)} = \dot{x}$, $\pi(x) = \pi(x_m) = \pi(x_0)$ in both cases. If $m \notin H$ then (as earlier) we can insert $\dot{x}$ into the chain, and apply (11) to this longer chain.

5° Assume $H \neq \emptyset$. Starting from an arbitrary $m \in H$ instead of m_1 (or also backwards from m_1), we can pick $\mathfrak{v}_{(k)} \in \operatorname{Ult} X$ for each $k \notin H$ such that conditions similar to (6), (7), (8) are satisfied; then an analogue of (10) can be obtained, which gives (taking (11) into account) that

$$(12) \qquad X_{i_k} \in \mathfrak{v}_{(k)} \in \operatorname{Ult} X, \quad \mathfrak{v}_{(k)} \,|\, X_{i_k} \supset \mathfrak{t}_{(k)}, \quad \mathfrak{v}_{(k)} \to x_0 \quad (k \notin H).$$

6° To prove $\pi < \pi(S'_{\mathrm{C}})$, let $\mathfrak{s} = \mathfrak{u} \cap \dot{x} \in \mathfrak{S}'_{\mathrm{C}}$, $\mathfrak{u} \in \operatorname{Ult} X$, and take a chain $\mathfrak{s}_{(1)}, \dots, \mathfrak{s}_{(n)}$ as in 3°. $H \neq \emptyset$, since $\mathfrak{s}$ is fixed. Take m and m' such that $\mathfrak{u} \supset \mathfrak{s}_{(m)}$, $\dot{x} \supset \mathfrak{s}_{(m')}$ (if each $\mathfrak{s}_{(m)}$ contained a set not in $\mathfrak{u}$ then the union of these sets were in $\mathfrak{s} \setminus \mathfrak{u}$). According to 4° (applied to m'), $\pi(x) = \pi(x_0)$, so it is enough to show that $\mathfrak{u} \to x_0$. This is clear if $m \in H$ (see (3) and (11)). If $m \notin H$ then take $\mathfrak{v}_{(m)}$ according to (12). Now $\mathfrak{v}_{(m)} \to x_0$, $(\mathfrak{v}_{(m)} \cap \mathfrak{u}) \mid X_{i_m} \in \in \mathfrak{S}_{i_m}$, $X_{i_m} \in \mathfrak{v}_{(m)}, \mathfrak{u}$, so $\mathfrak{u} \to x_0$ by (2).

7° Let us prove now that $\mathfrak{S}_i < \mathfrak{S}'_C \mid X_i$. Take $\mathfrak{s} \in \mathfrak{S}'_C$ such that $X_i \in \sec \mathfrak{s}$, and pick again a chain as in 3°. If $H = \emptyset$ then $\mathfrak{s} \in \mathfrak{S}^1_C(X, \mathfrak{S}_i)$, which is an extension by Proposition 4.2, thus $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$.

Assume $H \neq \emptyset$. With $K = \{m\colon X_i \in \sec \mathfrak{s}_{(m)}\}$,

$$\mathfrak{s}|X_i \supset \bigcap_{m \in K} \mathfrak{s}_{(m)}|X_i. \tag{13}$$

For $m \in K$, choose $\mathfrak{w}_{(m)} \in \operatorname{Ult} X$ such that $X_i \in \mathfrak{w}_{(m)} \supset \mathfrak{s}_{(m)}$. If $m \notin H$ then $\mathfrak{w}_{(m)} \to x_0$ (the last sentence of 6°, with $\mathfrak{w}_{(m)}$ instead of $\mathfrak{u}$); if $m \in H$ then $\mathfrak{w}_{(m)} \to x_0$ again, since $\mathfrak{w}_{(m)} \to x_m$ by (3), and so (11) can be applied. Now

$$\mathfrak{z}_i = \bigcap_{m \in K} \mathfrak{w}_{(m)} \mid X_i \in \mathfrak{S}_i$$

follows from (1), using induction. $\mathfrak{z}_i \,\Delta\, \mathfrak{s}_{(m)} \mid X_i$ for each $m \in K$ (as $\mathfrak{w}_{(m)} \mid X_i$ is finer than both filters), thus the Cauchy property of $\mathfrak{S}_i$ implies that $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$.

8° The last statement holds for the same reason as in Theorem 4.3. □

Statements analogous to Corollaries 4.3 and 4.5 are valid in pseudotopological spaces, too. (In 4.5, replace $\mathfrak{S}^1(\lambda)$ by $\mathfrak{S}^1_C(\pi)$, not by $\mathfrak{S}^1(\pi)$.) The proofs are the same.

5.4 All the conditions are needed in Theorem 5.3: replace λ by $\pi(\lambda)$ in Examples 4.4 a) and b). Moreover, 5.3 (1) cannot be replaced by the weaker assumption 3.2 (1):

Example. With the notations of Example 4.4 a), let $\pi(x) = \dot{x}$ $(x \neq \neq z)$, $\pi(z) = \{\mathfrak{u}, \mathfrak{v}, \dot{z}\}$, and consider the screen $\operatorname{Ult}^+ X_0$. The conditions of Theorem 5.3 are satisfied, except that only 3.2 (1) holds instead of 5.3 (1).□

There is in general no coarsest Cauchy extension in a pseudotopological space: substitute the discrete pseudotopology for the discrete convergence in Example 4.3. An analogue of Example 4.6 can also be obtained, replacing each convergence by the induced pseudotopology.

§ 6. Riesz screens

6.1. It is clear from the definition that $\mathfrak{S}$ is Riesz iff $\lambda(\mathfrak{S})$ is pretopological. The same is false for $\pi(\mathfrak{S})$: on an infinite set X, let $\mathfrak{S}$ consist of the filters that can be written as the intersection of a finite collection of ultrafilters; then $\pi(\mathfrak{S})$ is pretopological, although $\mathfrak{S}$ is not Riesz. Nevertheless, if $\mathfrak{S}$ is Riesz then $\pi(\mathfrak{S})$ is pretopological; hence a pseudotopology π can be induced by Riesz screens iff π is pretopological and $\mathfrak{n}(\pi)$ can be induced by a Riesz screen.

Let us be given a family of Riesz screens in a pseudotopological space (X, π). If there is a Riesz extension in this space then π is pretopological and $\mathfrak{S}$ is an extension in $(X, \mathfrak{n}(\pi))$. Conversely, if $\mathfrak{S}$ is a Riesz extension of a family of screens in a closure space $(X, \mathfrak{n})$ then it is also an extension in $(X, \pi^0(\mathfrak{n}))$. This means that the problem of Riesz extensions in pseudotopological spaces is equivalent to the same problem in closure spaces. We can say the same about Riesz extensions in convergence spaces, and also about special classes of Riesz screens (Lodato, CR, CL). Such extensions in closure spaces were investigated in [5], [6], [7]; we cannot add anything essentially new.

Note also that if there are Riesz extensions in a convergence space then each compatible extension is Riesz (cf. the first sentence of this section).

6.2. Let us call a closure $\mathfrak{n}$ *pointwise reciprocal* if

(1) $$\mathfrak{n}(x) \,\Delta_{\mathrm{P}}\, \mathfrak{n}(y) \ \text{ implies } \ \mathfrak{n}(x) = \mathfrak{n}(y);$$

equivalently:

(2) $$y \in \bigcap \mathfrak{n}(x) \ \text{ implies } \ \mathfrak{n}(x) = \mathfrak{n}(y)$$

(weakly separated in [5], S_1 in [7]).

LEMMA. *A closure $\mathfrak{n}$ is pointwise reciprocal iff $\lambda^0(\mathfrak{n})$ is symmetric.*

PROOF. Let $\mathfrak{n}$ be pointwise reciprocal, $\mathfrak{s} \to x$ (in $\lambda^0(\mathfrak{n})$), $y \in \bigcap \mathfrak{s}$. Then $\mathfrak{s} \supset \mathfrak{n}(x)$, $y \in \bigcap \mathfrak{n}(x)$, so $\mathfrak{n}(x) = \mathfrak{n}(y)$ by (2), $\mathfrak{s} \supset \mathfrak{n}(y)$, $\mathfrak{s} \to y$.

Conversely, if $\lambda^0(\mathfrak{n})$ is symmetric and $y \in \bigcap \mathfrak{n}(x)$ then $\mathfrak{n}(x) \to x$ implies $\mathfrak{n}(x) \to y$, so $\mathfrak{n}(x) \supset \mathfrak{n}(y)$. Moreover, if $\lambda^0(\mathfrak{n})$ is symmetric then so is $\mathfrak{n}$, thus $x \in \bigcap \mathfrak{n}(y)$, and $\mathfrak{n}(y) \supset \mathfrak{n}(x)$, too. $\square$

According to [5] 2.7, a family of Riesz screens in a closure space has Riesz extensions iff the closure is pointwise reciprocal, and

(3) $$\mathfrak{n}(x) \mid X_i \in \mathfrak{S}_i \qquad (x \in X,\ i \in I).$$

This result can also be obtained from Proposition 2.2, the above lemma and 6.1, Proposition 2.2 also gives the finest and the coarsest Riesz extensions (cf. [5] 2.7 and 2.8).

6.3. The following version of [6] 2.8 and 2.14 can be deduced from Theorem 4.3:

THEOREM. *A family of CR screens in a closure space has CR extensions iff there is a Cauchy extension in the set, the closure is reciprocal, and each proper trace filter is a minimal element of the corresponding screen. If $|I| \leqq 2$*

then the existence of a Cauchy extension in the set does not have to be assumed. □

See [6] 2.8 and 2.15 for two descriptions of the finest CR extension; both can be obtained from 4.3 (2), too.

[6] 3.1 gives a necessary and sufficient condition for the existence of a CL extension in a closure space; it seems to be impossible to deduce that result from Theorem 4.3. Lodato extensions in a closure space are dealt with in [5] 2.9–2.17. For Riesz, Lodato, CR and CL extensions in a set, see [15].

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 165–173

STRONG TOPOLOGY AND STRONG PERTURBATIONS OF NORMAL ELEMENTS IN GW*-ALGEBRAS

J. KRISTÓF

The simplest perturbation theorem for bounded normal operators in a Hilbert space $\mathcal{H}$ can be formulated as follows. Let L be a bounded normal operator on $\mathcal{H}$ and consider a sequence $(L_n)_{n\in\mathbb{N}}$ of bounded normal operators approximating L in the strong operator topology (i.e. $\lim\limits_{n\to\infty} L_n\zeta = L\zeta$ for all $\zeta\in\mathcal{H}$). If φ is a locally bounded complex Borel function defined on the Borel subset $B\subseteqq\mathbb{C}$ and $\mathrm{Sp}(L)\subseteqq B$ and $\mathrm{Sp}(L_n)\subseteqq B$ $(n\in\mathbb{N})$ then the problem is to clarify whether $\lim\limits_{n\to\infty}\varphi(L_n)=\varphi(L)$ in the strong operator topology or not?

It is known that the answer is positive if φ is continuous, however, it may be negative if φ has a large enough number of discontinuities. A thorough examination of the problem revealed that we obtain an affirmative answer if the set of discontinuities of φ is negligible with respect to the spectral resolution of the operator L. This is the classical form of the perturbation theorem due to F. Rellich [see 5]. Related topics are discussed, e.g., in [1, Ch. X, § 7] and [6, § 134–136].

In this paper we will expose a generalization of this perturbation theorem for a special type of C^*-algebras. It will be proved that for GW^*-algebras introduced and studied by the author in a series of papers we can formulate and verify an analogous theorem. It is worth mentioning that our proof will not make use of the classical theorem, thus it will provide an independent proof of the classical result. Moreover, our generalization requires and contains the introduction of the "strong topology" in GW^*-algebras which cannot be done for arbitrary C^*-algebras due to the representation dependent nature of "pointwise convergence" in an operator algebra.

1. Preliminaries

The notion of GW^*-algebras was introduced in [3], however, for the sake of completeness and to fix the terminology, here we repeat the basic notions and notations. We also need a brief summary of the spectral theorem for normal elements in GW^*-algebras detailed in [4].

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If A is a unital $*$-algebra (whose unit will always be denoted by 1) and S is a non-void and pointwise bounded set of positive linear forms on A then $\| \|_S$ denotes the function

$$A \to \mathbb{R}_+; \quad a \mapsto \|a\|_S := \sup_{p \in S} \sqrt{p(a^*a)}.$$

It is obvious that $\| \|_S$ is a seminorm on A, and it is a norm if S separates the points of A. If p is a positive linear form on A then write $\| \|_p$ instead of $\| \|_{\{p\}}$.

If A is a $*$-algebra and f is a linear form on A then for all $a \in A$ we denote by $a.f$ and $f.a$ the linear forms on A defined by $(a.f)(b) := f(ab)$ and $(f.a)(b) := f(ba)$, respectively, for $b \in A$. Then for $a, b \in A$, $a.f.b$ stands for $(a.f).b$.

The pair (A, P) is called a *GW^*-algebra* if A is a unital $*$-algebra and P is a separating set of positive linear forms on A satisfying:

(I) The set $P(1) := \{p \in P \mid p(1) \leq 1\}$ is pointwise bounded.

(II) $\lambda p \in P$, $a^*.p.a \in \overline{\mathrm{co}}(P)$ and $a.p \in \overline{\mathrm{sp}}(P)$ for all $\lambda \in \mathbb{R}_+$, $a \in A$ and $p \in P$, where $\overline{\mathrm{co}}(P)$ and $\overline{\mathrm{sp}}(P)$ denotes the functional $\| \|_{P(1)}$-closed convex and linear hull of P, respectively.

(III) If $(a_n)_{n \in \mathbb{N}}$ is a $\| \|_{P(1)}$-bounded sequence in A and for all $p \in P$ the sequence $(p(a_n))_{n \in \mathbb{N}}$ is convergent then there is an element $a \in A$ such that $\lim_{n \to \infty} p(a_n) = p(a)$ for every $p \in P$ (see [3]).

If (A, P) is a GW^*-algebra then the $\sigma(A, \mathrm{sp}(P))$ and $\sigma(A, \overline{\mathrm{sp}}(P))$ topologies will be called the *P-weak* and *P-ultraweak* topologies on A, respectively.

If (A, P) is a GW^*-algebra then

The $*$-algebra A is a C^*-algebra whose C^*-norm equals $\| \|_{P(1)}$ (see [2]).

The P-weak and P-ultraweak topologies coincide in every C^*-norm bounded subset of A (see [2]).

The set $\mathbb{P}(A)$ of projections (i.e. self-adjoint idempotents) of A, equipped with the partial ordering $e \leq f$ iff $e = ef$ and the orthocomplementation $e^\perp := 1 - e$ ($e \in \mathbb{P}(A)$) is a σ-complete orthomodular lattice admitting a separating set of σ-additive states (see [3]).

If (A, P) is a GW^*-algebra then a *σ-additive projection-valued measure in A* defined on a measurable space $(T, \mathcal{B})$ is a map $m : \mathcal{B} \to \mathbb{P}(A)$ which is a σ-orthohomomorphism between the σ-complete orthomodular lattices $\mathcal{B}$ and $\mathbb{P}(A)$. More precisely, for m we have the following properties:

(a) $m(T) = 1$,

(b) $m(\bigcup_{n \in \mathbb{N}} E_n) = \bigvee_{n \in \mathbb{N}} m(E_n)$ for every sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise disjoint subsets in $\mathcal{B}$.

Given a σ-additive projection-valued measure m in A defined on a measurable space $(T, \mathcal{B})$, there is a unique unit preserving $*$-homomorphism $\overline{m}$

from the C^*-algebra of bounded complex $\mathcal{B}-\mathcal{B}(\mathbb{C})$ measurable functions on T into A such that $\overline{m}(\chi_E)=m(E)$ for $E\in\mathcal{B}$ and $p\circ m:\mathcal{B}\to\mathbb{R}_+$ is a σ-additive measure for all $p\in P$ (see [4]). This $*$-homomorphism is called the *integral* defined by m. For every bounded $\mathcal{B}-\mathcal{B}(\mathbb{C})$ measurable function $\varphi:T\to\mathbb{C}$ we write $\int\varphi dm$ instead of $\overline{m}(\varphi)$.

As for the spectral properties of normal elements in a GW^*-algebra we have the following basic result.

To every normal element $a\in A$ there is a unique σ-additive projection valued measure m_a in A defined on the Borel σ-algebra of the spectrum $\mathrm{Sp}(a)$ of a such that

$$a=\int\limits_{\mathrm{Sp}(a)}\mathrm{id}_{\mathrm{Sp}(a)}dm_a$$

(see [4]). Then m_a is referred to as the *spectral resolution* of a. The spectral resolution of a normal element in A lives on the spectrum of the given element (see [4]).

Given a normal element a in the GW^*-algebra (A,P), for every locally bounded complex Borel function φ, defined on a Borel subset of $\mathbb{C}$ containing $\mathrm{Sp}(a)$, we may give $\varphi(a)$ as the integral of $\varphi\,|\,_{\mathrm{Sp}(a)}$ with respect to the spectral resolution of a. This is meaningful since $\varphi\,|\,_{\mathrm{Sp}(a)}$ is a bounded Borel function on $\mathrm{Sp}(a)$.

There are two important examples for GW^*-algebras.

Example 1. Let A be a von Neumann algebra over the Hilbert space $\mathcal{H}$ and for all $\zeta\in\mathcal{H}$ define $p_\zeta:A\to\mathbb{C};\ a\to(a\zeta\mid\zeta)$. If $P_A:=\{p_\zeta\mid\zeta\in\mathcal{H}\}$ then (A,P_A) is a GW^*-algebra. Here the $\sigma(A,\mathrm{sp}(P_A))$ and $\sigma(A,\overline{\mathrm{sp}}(P_A))$ topologies coincide with the weak and ultraweak operator topologies in A, respectively [2].

Example 2. Let $(T,\mathcal{B})$ a measurable space and let A denote the $*$-algebra of complex bounded $\mathcal{B}$-measurable functions defined on T. Let P be the set of integrals on A arising from positive σ-additive finite measures defined on $\mathcal{B}$. Then (A,P) is a GW^*-algebra for which the $\sigma(A,\mathrm{sp}(P))$ and $\sigma(A,\overline{\mathrm{sp}}(P))$ topologies are equal. A sequence $(a_n)_{n\in\mathbb{N}}$ in A tends to $a\in A$ with respect to the $\sigma(A,\mathrm{sp}(P))$ topology if and only if $(a_n)_{n\in\mathbb{N}}$ is uniformly bounded and $a_n\to a$ $(n\to\infty)$ pointwise on T.

2. Strong topology in GW^*-algebras

If A is a von Neumann algebra then one can define the strong operator topology in A besides the weak, ultraweak and C^*-norm topologies. It is an easy task to generalize this notion for the case of GW^*-algebras as follows.

DEFINITION. If (A, P) is a GW^*-algebra, then the *P-strong topology* of A is the locally convex topology defined by the family $(\| \|_p)_{p\in P(1)}$ of seminorms on A.

REMARK 1. Let (A, P_A) be the GW^*-algebra associated with the von Neumann algebra A on the Hilbert space $\mathcal{H}$. If $(a_\iota)_{\iota\in I}$ is a net in A, then the statement "$\lim_\iota a_\iota = 0$ in the strong operator topology" is equivalent to "$\lim_\iota a_\iota\zeta = 0$ in the Hilbert space $\mathcal{H}$ for all $\zeta \in \mathcal{H}$", i.e. "$\lim_\iota (a_\iota^* a_\iota\zeta \mid \zeta) = = \lim_\iota \|a_\iota\zeta\|^2 = 0$ for all $\zeta \in \mathcal{H}$". By the definition of P_A, the latter statement is equivalent to "$\lim_\iota p(a_\iota^* a_\iota) = \lim_\iota \|a_\iota\|_p^2 = 0$ for all $p \in P_A$", i.e. $\lim_\iota a_\iota = 0$ in the P_A-strong topology. This means that the P_A-strong topology introduced in this section coincides with the strong operator topology in the case of GW^*-algebras associated with von Neumann algebras.

REMARK 2. If (A, P) is a GW^*-algebra then the P-weak topology on A (i.e. $\sigma(A, \mathrm{sp}(P))$) equals the locally convex topology on A defined by the family $(|p|)_{p\in P}$ of seminorms on A. If $p \in P$, then

$$|p(a)|^2 \leqq p(1)\, p(a^*a) = p(1)\|a\|_p^2$$

showing that the P-strong topology on A is finer than the P-weak topology. Further, we have for $p \in P$

$$\|a\|_p^2 := p(a^*a) \leqq \|p\|\,\|a\|^2$$

showing that the C^*-norm topology of A is finer than the P-strong topology.

REMARK 3. In a GW^*-algebra (A, P), the P-strong and P-ultraweak (i.e. $\sigma(A, \overline{\mathrm{sp}}(P))$) topologies are incomparable, in general. However, the restriction of the P-strong topology to a C^*-norm bounded subset is finer than the restriction to the same subset of the P-ultraweak topology.

It is well known that in a B^*-algebra A we have

$$\|ab\|_p^2 = p(b^*a^*ab) \leqq \|a^*a\| p(b^*b) = \|a\|^2 \|b\|_p^2, \tag{1}$$

for every continuous positive linear form p on A and for all $a, b \in A$.

PROPOSITION 1. *Let (A, P) be a GW^*-algebra, $a, b \in A$ and $(a_\iota)_{\iota\in I}$, $(b_\iota)_{\iota\in I}$ nets in A. Then:*

(i) *If $\lim_\iota b_\iota = 0$ in the P-strong topology and $\sup_{\iota\in I} \|a_\iota\| < \infty$ then $\lim_\iota a_\iota b_\iota = = 0$ in the P-strong topology.*

(ii) *If $\lim_\iota a_\iota = a$ and $\lim_\iota b_\iota = b$ in the P-strong topology and $\sup_{\iota\in I} \|a_\iota\| < \infty$, then $\lim_\iota a_\iota b_\iota = ab$ in the P-strong topology.*

PROOF. (i) is an immediate consequence of the inequality

$$\|a_\iota b_\iota\|_p \leqq \|a_\iota\|\,\|b_\iota\|_p \quad (p \in P,\ i \in I).$$

(ii) First assume that $b_\iota = b$ for all $\iota \in I$. If $\iota \in I$ and $p \in P$ then

$$\|a_\iota b - ab\|_p^2 = p(b^*(a_\iota - a)^*(a_\iota - a)b) = (b^*pb)((a_\iota - a)^*(a_\iota - a)).$$

Clearly, we have $\lim_\iota (a_\iota - a)^*(a_\iota - a) = 0$ in the P-weak topology and this sequence is C^*-norm bounded. Consequently, $\lim_\iota (a_\iota - a)^*(a_\iota - a) = 0$ in the P-ultraweak topology of A. Since $b^*pb \in \overline{\mathrm{co}}(P) \subseteqq \overline{\mathrm{sp}}(P)$ this implies that $\lim_\iota \|a_\iota b - ab\|_p^2 = 0$, i.e. $\lim_\iota a_\iota b = ab$ in the P-strong topology. Now suppose that $a_\iota = a$ for all $\iota \in I$. Then (i) implies that $\lim_\iota ab_\iota = ab$ in the P-strong topology.

In the general case, for every $\iota \in I$ we have

$$a_\iota b_\iota - ab = (a_\iota - a)(b_\iota - b) + (a_\iota - a)b + a(b_\iota - b). \tag{2}$$

Since $\lim_\iota b_\iota = b$ in the P-strong topology and $\sup_{\iota \in I} \|a_\iota - a\| < \infty$, by (i) we infer that the first summand in the right side of (2) tends to 0 in the P-strong topology. Since $\lim_\iota a_\iota = a$ in the P-strong topology, by our former result we deduce that the second summand in the right side of (2) also tends to 0 with respect to the same topology. At last, as we have shown before, the third term also tends to 0 in the P-strong topology. □

Our next statement has a vital importance in the proof of the strong perturbation theorem for normal elements in GW^*-algebras.

PROPOSITION 2. *Let (A, P) be a GW^*-algebra, $a \in A$ a normal element and $(a_\iota)_{\iota \in I}$ a net of normal elements in A converging to a in the P-strong topology. If $\sup_\iota \|a_\iota\| < \infty$ then $\lim_\iota a_\iota^* = a^*$ with respect to the P-strong topology.*

PROOF. Let $\iota \in I$ be a fixed index. Since a and a_ι are both normal elements in A, we have $\|a\|_p = \|a^*\|_p$ and $\|a_\iota\|_p = \|a_\iota^*\|_p$ for all $p \in P$. Every seminorm $\| \ \|_p$ $(p \in P)$ is continuous with respect to the P-strong topology evidently, thus $\lim_\iota \|a_\iota^*\|_p = \|a^*\|_p$ $(p \in P)$. For every $p \in P$ we have the equalities

$$\begin{aligned}\|a_\iota^* - a^*\|_p^2 &= p((a_\iota - a)(a_\iota - a)^*) = p(a_\iota a_\iota^*) + p(aa^*) - 2\mathrm{Re}\,(p(a_\iota a^*)) = \\ &= p(a_\iota a_\iota^*) - p(aa^*) + 2\mathrm{Re}\,(p((a - a_\iota)a^*)) \\ &= \|a_\iota^*\|_p^2 - \|a^*\|_p^2 + 2\mathrm{Re}\,((pa^*)(a - a_\iota)).\end{aligned}$$

Since the net $(a - a_\iota)_{\iota \in I}$ is C^*-norm bounded in A and tends to 0 in the P-strong topology, we conclude that it converges to 0 in the P-ultraweak topology. But $p.a^* \in \overline{\mathrm{sp}}(P)$ for every $p \in P$, thus the third term in the right side of the equalities above converges to 0 for all $p \in P$. □

If (A, P_A) is the GW^*-algebra associated with a von Neumann algebra A then every sequence in A converging in the P_A-weak or P_A-strong topology is necessarily C^*-norm bounded. This is not the case for an arbitrary GW^*-algebra as the following example shows.

EXAMPLE. Let T be a set and A be the C^*-algebra of *all* the complex bounded functions defined on T. For every $t \in T$ we write ε_t for the positive linear form on A defined by the formula $\varepsilon_t(a) := a(t)$ $(a \in A)$. If

$$P := \{\lambda\varepsilon_t \mid \lambda \in \mathbb{R}_+,\ t \in T\}$$

then it is not a difficult task to check that (A, P) is a GW^*-algebra such that the P-weak topology on A coincides with the topology of pointwise convergence on T. If $\lambda \in \mathbb{R}_+$ and $t \in T$ then for all $a \in A$ we have

$$\|a\|_{\lambda\varepsilon_t} = \sqrt{\lambda}\,|a(t)| = |\sqrt{\lambda}\varepsilon_t|(a),$$

i.e. the P-strong and P-weak topologies on A coincide. Assume that T is infinite and let $u : \mathbb{N} \to T$ be an arbitrary injection. Define the sequence $(a_n)_{n \in \mathbb{N}}$ in A as follows

$$a_n(t) := \begin{cases} 0 & \text{if } t \in T \setminus u(\mathbb{N}), \\ n & \text{if } t = u(k) \text{ and } k \geqq n, \\ 0 & \text{if } t = u(k) \text{ and } k < n. \end{cases}$$

Then $(a_n)_{n \in \mathbb{N}}$ converges to 0 pointwise on T, i.e. $\lim\limits_{n \to \infty} a_n = 0$ in both the P-weak and P-strong topologies, but $\sup\limits_{t \in T} |a_n(t)| = n$ $(n \in \mathbb{N})$, i.e. this sequence is not C^*-bounded in A.

The above example shows that it is not a superflous condition on a sequence in a GW^*-algebra to be C^*-norm bounded even if it is convergent in the strong topology.

3. Strong perturbation of normal elements in GW^*-algebras

This section contains the main perturbation theorem for normal elements in GW^*-algebras. We denote the sup-norm of a bounded complex function φ defined on a set T by the symbol $|||\varphi|||_T$.

THEOREM. *Let (A, P) be a GW^*-algebra and $(a_n)_{n \in \mathbb{N}}$ a sequence of normal elements in A. Suppose that $a \in A$ is a normal element with $\lim\limits_{n \to \infty} a_n = a$ in the P-strong topology and $\sup\limits_{n \in \mathbb{N}} \|a_n\| < \infty$. Let B be a Borel subset of $\mathbb{C}$ and $\varphi : B \to \mathbb{C}$ a locally bounded Borel function such that $\bigcup\limits_{n \in \mathbb{N}} \mathrm{Sp}(a_n) \cup \mathrm{Sp}(a) \subseteqq B$ and there is a closed subset E of $\mathbb{C}$ such that $E \subseteqq B$ and E contains all the* *points of discontinuity of φ and satisfies $m_a(E\cap\mathrm{Sp}(a))=0$, where m_a is the spectral resolution of a. Then $\lim\limits_{n\to\infty}\varphi(a_n)=\varphi(a)$ in the P-strong topology.*

PROOF. Since $\sup\limits_{n\in\mathbb{N}}\|a_n\|<\infty$, the set $T:=\overline{\bigcup\limits_{n\in\mathbb{N}}\mathrm{Sp}(a_n)}\cup\mathrm{Sp}(a)$ is compact in $\mathbb{C}$ and, by the hypothesis, it is contained in B. Let $\mathbb{C}(T)$ and $\mathbb{B}(T)$ denote the C^*-algebra of complex continuous functions and complex bounded Borel functions defined on T, respectively. Then we set

$$\mathcal{A}:=\{\psi\in\mathbb{B}(T)\mid \lim_{n\to\infty}\psi(a_n)=\psi(a) \text{ in the } P\text{-strong topology}\}.$$

First we will show that $\mathcal{A}$ is a C^*-subalgebra of $\mathbb{B}(T)$ such that $\mathbb{C}(T)\subseteq\mathcal{A}$.

Since the P-strong topology of A is a linear topology, $\mathcal{A}$ is a linear subspace of $\mathbb{B}(T)$. If $\psi\in\mathcal{A}$ then $(\psi(a_n))_{n\in\mathbb{N}}$ is a C^*-norm bounded sequence of normal elements in A converging to $\psi(a)$ in the P-strong topology, so, by Proposition 2, we obtain that $\lim\limits_{n\to\infty}\psi(a_n)^*=\psi(a)^*$ in the same topology, i.e. $\lim\limits_{n\to\infty}\overline{\psi}(a_n)=\overline{\psi}(a)$ in the P-strong topology showing that $\overline{\psi}\in\mathcal{A}$.

If $\psi,\psi'\in\mathcal{A}$ then $\lim\limits_{n\to\infty}\psi(a_n)=\psi(a)$ and $\lim\limits_{n\to\infty}\psi'(a_n)=\psi'(a)$ in the P-strong topology of A and the sequence $(\psi(a_n))_{n\in\mathbb{N}}$ is C^*-norm bounded, so, by Proposition 1, we infer that $\lim\limits_{n\to\infty}(\psi\psi')(a_n)=\lim\limits_{n\to\infty}\psi(a_n)\psi'(a_n)=$ $=\psi(a)\psi'(a)=(\psi\psi')(a)$, i.e. $\psi\psi'\in\mathcal{A}$, thus $\mathcal{A}$ is a $*$-subalgebra of $\mathbb{B}(T)$.

In order to prove that $\mathcal{A}$ is sup-norm closed in $\mathbb{B}(T)$ take a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ converging uniformly to a function $\psi\in\mathbb{B}(T)$. Then we obtain the following inequalities for all $n,m\in\mathbb{N}$ and $p\in P(1)$:

$$\begin{aligned}\|\psi(a_n)-\psi(a)\|_p&=\|(\psi(a_n)-\psi_m(a_n))-(\psi(a)-\psi_m(a))+(\psi_m(a_n)-\psi_m(a))\|_p\\&\leqq\|(\psi-\psi_m)(a_n)\|_p+\|(\psi-\psi_m)(a)\|_p+\|\psi_m(a_n)-\psi_m(a)\|_p\leqq\\&\leqq|||\psi-\psi_m|||_{\mathrm{Sp}(a_n)}+|||\psi-\psi_m|||_{\mathrm{Sp}(a)}+\|\psi_m(a_n)-\psi_m(a)\|_p\leqq\\&\leqq 2|||\psi-\psi_m|||_T+\|\psi_m(a_n)-\psi_m(a)\|_p.\end{aligned}$$

Given a number $\varepsilon>0$ and $p\in P(1)$, there is a number $m\in\mathbb{N}$ such that $|||\psi_m-\psi|||_T\leqq\varepsilon/4$. Since $\psi_m\in\mathcal{A}$, we have $\lim\limits_{n\to\infty}\psi_m(a_n)=\psi_m(a)$ in the P-strong topology, thus there is a number $n\in\mathbb{N}$ with the property that for every $n\in\mathbb{N}$, $n\geqq N$ we have $\|\psi_m(a_n)-\psi_m(a)\|_p\leqq\varepsilon/2$. Then, taking into account the former inequalities, we arrive at $\|\psi(a_n)-\psi(a)\|_p\leqq 2(\varepsilon/4)+(\varepsilon/2)=\varepsilon$ for all $n\in\mathbb{N}$, $n\geqq N$. This means that $\lim\limits_{n\to\infty}\psi(a_n)=\psi(a)$ in the P-strong topology, i.e. $\psi\in\mathcal{A}$, showing that $\mathcal{A}$ is a C^*-subalgebra of $\mathbb{B}(T)$.

Since $\lim\limits_{n\to\infty}a_n=a$ in the P-strong topology, we have $\mathrm{id}_T\in\mathcal{A}$ as well as $\overline{\mathrm{id}_T}\in\mathcal{A}$. Clearly, every constant function on T belongs to $\mathcal{A}$, so we have $\{\mathrm{id}_T,\overline{\mathrm{id}_T},1_T\}\subseteq\mathcal{A}$. Then the theorem of Stone–Weierstrass implies that $\mathbb{C}(T) \subseteq \mathcal{A}$. In other words, for every continuous function $\psi : T \to \mathbb{C}$ the sequence $(\psi_n(a))_{n\in\mathbb{N}}$ converges to $\psi(a)$ in the P-strong topology.

Now take a locally bounded Borel function $\varphi : B \to \mathbb{C}$ and assume that E is a closed subset of B containing the points of discontinuity of φ and satisfying $m_a(E \cap \mathrm{Sp}(a)) = 0$, where m_a is the spectral resolution of a.

Then $E \cap T$ is a closed subset of $\mathbb{C}$ containing the points of discontinuity of $\varphi\,|\,_T$ satisfying $m_a((E\cap T)\cap \mathrm{Sp}(a)) = 0$. By the theorem of Urysohn there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathbb{C}(T)$ with the property that for all $m \in \mathbb{N}$ and $z \in T$, if $\mathrm{dist}(z, E\cap T) \geqq 1/m$ then $\varphi_m(z) = 1$ else if $z \in E \cap T$ then $\varphi_m(z) = 0$. We may assume that $0 \leqq \varphi_m \leqq 1$ for all $m \in \mathbb{N}$. We claim that $\lim\limits_{n\to\infty} \varphi_n(a) = 1$ in A with respect to the P-strong topology.

Indeed, we have $\lim\limits_{n\to\infty} \varphi_m = \chi_{T\setminus E}$ pointwise on T and $\sup\limits_{m\in\mathbb{N}} |||\varphi_m|||_T \leqq \leqq 1$, thus the sequence $(|\varphi_m - \chi_{T\setminus E}|^2)_{m\in\mathbb{N}}$ tends to 0 on T pointwise and is uniformly bounded. Then we have $\lim\limits_{m\to\infty} |\varphi_m - \chi_{T\setminus E}|^2(a) = 0$ with respect to the $\sigma(A, \mathrm{sp}(P))$ topology. Therefore

$$\|\varphi_m(a) - \chi_{T\setminus E}(a)\|_p^2 = p\left(|\varphi_m - \chi_{T\setminus E}|^2(a)\right) \to 0 \qquad (m \to \infty),$$

i.e. $\lim\limits_{m\to\infty} \varphi_m(a) = \chi_{T\setminus E}(a)$ in the P-strong topology. On the other hand:

$$\begin{aligned}\chi_{T\setminus E}(a) &= \chi_{\mathrm{Sp}(a)\cap(T\setminus E)}(a) = m_a(\mathrm{Sp}(a)\setminus E) = \\ &= m_a(\mathrm{Sp}(a)) - m_a(E\cap \mathrm{Sp}(a)) = 1\end{aligned}$$

i.e. $\lim\limits_{m\to\infty} \varphi_m(a) = 1$ in A with respect to the P-strong topology.

By the definition of the sequence $(\varphi_m)_{m\in\mathbb{N}}$, $(\varphi\,|\,_T)\varphi_m \in \mathbb{C}(T)$ for $m \in \mathbb{N}$, so we obtain that $\lim\limits_{n\to\infty} \varphi(a_n)\varphi_m(a) = \varphi(a)\varphi_m(a)$ in the P-strong topology for all $m \in \mathbb{N}$. If $m, n \in \mathbb{N}$ then

$$\begin{aligned}\varphi(a) - \varphi(a_n) = {}&(\varphi(a)1 - \varphi(a)\varphi_m(a)) + (\varphi(a)\varphi_m(a) - \varphi(a_n)\varphi_m(a_n)) + \\ &+ (\varphi(a_n)\varphi_m(a_n) - \varphi(a_n)\varphi_m(a)) + (\varphi(a_n)\varphi_m(a) - \varphi(a_n)1).\end{aligned}$$

From this and (1) we conclude that for every $p \in P(1)$ and $m, n \in \mathbb{N}$

$$\begin{aligned}\|\varphi(a) - \varphi(a_n)\|_p \leqq {}&\|\varphi(a)\|\|1 - \varphi_m(a)\|_p + \|\varphi(a)\varphi_m(a) - \varphi(a_n)\varphi_m(a_n)\|_p + \\ &+ \|\varphi(a_n)\|\|\varphi_m(a_n) - \varphi_m(a)\|_p + \|\varphi(a_n)\|\|\varphi_m(a) - 1\|_p.\end{aligned}$$

Let M be a positive number such that $\max(|||\varphi|||_T, \sup\limits_{n\in\mathbb{N}} |||\varphi_n|||_T) \leqq M$. Then, for $\varepsilon > 0$ and $p \in P(1)$ there is a number $m \in \mathbb{N}$ such that $\|1 - \varphi_m(a)\|_p \leqq \leqq \frac{\varepsilon/3}{2M+1}$. On the other hand, to m we can choose a number $N_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geqq N_1$ the inequality $\|\varphi(a_n)\varphi_m(a_n) - \varphi(a)\varphi_m(a)\|_p \leqq \varepsilon/3$ holds, since $\lim\limits_{n\to\infty} \varphi(a_n)\varphi_m(a_n) = \varphi(a)\varphi_m(a)$ in the P-strong topology. But

$\varphi_m \in \mathbb{C}(T) \subseteqq \mathcal{A}$, i.e. $\lim_{n\to\infty} \varphi_m(a_n) = \varphi_m(a)$ in the P-strong topology, so there is a number $N_2 \in \mathbb{N}$ such that

$$\|\varphi_m(a_n) - \varphi_m(a)\|_p \leqq \frac{\varepsilon/3}{M+1} \qquad \text{for } n \in \mathbb{N},\ n \geqq N_2.$$

From this it follows that for $n \in \mathbb{N}$, $n \geqq \max(N_1, N_2)$ we have

$$\|\varphi(a) - \varphi(a_n)\|_p \leqq M\frac{\varepsilon/3}{2M+1} + \frac{\varepsilon}{3} + M\frac{\varepsilon/3}{M+1} + M\frac{\varepsilon/3}{2M+1} \leqq \varepsilon.$$

This is true for every $p \in P(1)$ and $\varepsilon > 0$, so $\lim_{n\to\infty} \varphi(a_n) = \varphi(a)$ in the P-strong topology. □

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EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
ALKALMAZOTT ANALIZIS TANSZÉK
MUZEUM KRT. 6–8
H–1088 BUDAPEST
HUNGARY

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AN ALTERNATIVE CONDITION OF FIXED POINT OF NON-CONTINUOUS MAPPINGS IN METRIC SPACES

M. ANGRISANI and M. CLAVELLI

Introduction

A theorem concerning the fixed points of non-continuous self-mappings in metric spaces is given in this paper, with conditions different from the usual fixed point theorems, by using the notion (see [1] or [2]) of r.g.i. function (property weaker than l.s.c.). Some corollaries and two examples of applications are given.

We recall the following definition.

DEFINITION. Let G be a real function defined on a metric space (X,d), and put $L_c=\{x\in X|G(x)\leqq c\}$. G is r.g.i. (regular-global-inf) in $x\in X$ if and only if $G(x)>\inf_x(G)$ implies $d(x,L_c)>0$ for some real number c such that $c>\inf_x(G)$. G is r.g.i. in (X,d) if and only if it is r.g.i. in every $x\in X$.

§ 1. The main theorem

LEMMA 1.1. *Let (X,d) be a metric space. $G:X\to\mathbb{R}$ is r.g.i. in (X,d) if and only if for each minimizing G sequence $\{x_n\}_n$, $\{x_n\}_n\to x^*$ implies that x^* is an absolute minimum point of G.*

PROOF. Let G be r.g.i. So, $\{x_n\}_n\to x^*$ implies that the neighbourhoods of x^* intersect all the level sets $L_c=\{x|G(x)\leqq c\}$ where $c>\inf_x(G)$ (i.e. $d(x^*,L_c)=0$ when $c>\inf_x(G)$) and therefore, as G is r.g.i. in x^*, x^* is an absolute minimum point of G.

On the other hand, we suppose that, for each minimizing G sequence $\{x_n\}_n$, $\{x_n\}_n\to x^*$ implies that x^* is an absolute minimum point of G; i.e. we suppose the minimizing sequences may not tend to points which are not absolute minimum points of G.

Let, ad absurdo, G not be r.g.i. in (X,d): this would imply that there is, at least, a point x^* which is not an absolute minimum point of G, whose

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neighbourhoods intersect all the level sets L_c with $c > \inf_x(G)$. On the other hand, we may construct a sequence minimizing G which tends to x^*, and therefore, x^* is an absolute minimum point of G, which is absurd.

MAIN THEOREM 1.2. *Let (X,d) be a metric space, and $T: X \to X$ be given, and $F: X^4 \to \mathbb{R}$ be u.s.c. jointly on the first two variables in all points of the kind (x,x,x,Tx) with $x \neq Tx$, let $F(x,x,x,Tx) > 0$ for $x \neq Tx$, let $F(x,Tx,y,Ty) \leqq 0$ for $x \neq y$, and defining $G(x) = d(x,Tx)$, let $\inf_x(G) = 0$. Then G is r.g.i.*

PROOF. Let, ad absurdo, G not be r.g.i. Then there is, by Lemma 1.1, a sequence $\{x_n\}_n$ minimizing G and tending to a point x^* which is not an absolute minimum point of G (so $x^* \neq Tx^*$). We have, by hypothesis, because $x_n \neq x^*$:

(1) $$F(x_n, Tx_n, x^*, Tx^*) \leqq 0 \text{ by definition.}$$

As, by hypothesis, $d(x_n, Tx_n) \to 0$ ($\inf_x(G) = 0$) and $x_n \to x^*$, we also have $Tx_n \to x^*$. Going to limit in (1) for $n \to +\infty$ we have $F(x^*, x^*, x^*, Tx^*) \leqq 0$, which is impossible.

COROLLARY 1.3. *Let (X,d) be a metric space, $T: X \to X$ be given, and $F^*: X^4 \to \mathbb{R}$ be u.s.c. jointly on the first two variables in all points of the kind (x,x,x,Tx) with $x \neq Tx$, let $F^*(x,x,x,Tx) < d(x,Tx)$ for $x \neq Tx$, and $d(Tx,Ty) \leqq F^*(x,Tx,y,Ty)$ for $x \neq y$. Assuming that $G(x) = d(x,Tx)$, let $\inf_x(G) = 0$. Then G is r.g.i.*

PROOF. Let $F(x,y,z,w) = d(y,w) - F^*(x,y,z,w)$. So, F and G verify the hypothesis of the main theorem.

PROPOSITION 1.4. *Let (X,d) be a complete metric space, and $T: X \to \to X$, and $G(x) = d(x,Tx)$ be an i.g.r. function on X such that $\inf_x G = 0$, and $\{x_n\}_n$ be a Cauchy sequence and a minimizing sequence of G. Thus T has a fixed point.*

PROOF. Let x^* be the limit point of $\{x_n\}_n$, by Lemma 1.1, x^* is an absolute minimum point of G, consequently, by $\inf_x G = 0$ we have $d(x^*, Tx^*) = G(x^*) = 0$, therefore x^* is a fixed point of T.

REMARK 1.5. Let $S: X \to X$ and $V: X \to X$, let $SVx = x \Rightarrow Vx = x$ (this is a condition which sometimes appears in the literature: see, as example, [3]). Thus, we can prove that there are common fixed points of S and V, proving that there are fixed points of $T = SV$, for example verifying the hypothesis of some of the above theorems for T.

§ 2. First example

We are presenting a strengthening of the main theorem of [3], when $T = S$ (our hypothesis $\min(b,c) < 1$ is better than the hypothesis $bc < 1$).

THEOREM 2.1. *Let (X,d) be a complete metric space, and $T : X \to X$ be such that if $x = T^2x$, then $x = Tx$. We suppose that there are $b, c \geqq 0$ with $\min(b,c) < 1$ such that*

$$(*) \qquad d(Tx,Ty) \leqq \frac{bd(x,Tx)d(x,Ty) + cd(y,Tx)d(y,Ty)}{d(x,Ty) + d(y,Tx)}$$

(when the denominator does not vanish). Thus, T has a fixed point which is unique.

PROOF. We must apply Corollary 1.3. Let

$$F^*(x,y,z,w) = (bd(x,y)d(x,w) + cd(z,y)d(z,w))/(d(x,w) + d(z,y)).$$

We define $F^* = 0$, when $x = w$ and $z = y$ (i.e. when it is not defined). We intend to verify the hypothesis of Corollary 1.3: F^* is continuous (and therefore in particular u.s.c.) in the points (x,x,x,Tx) with $x \neq Tx$, by the continuity of the distance, and because the denominator does not vanish in the points (x,x,x,y) with $x \neq y$.

For each $x \neq Tx$,

$$\begin{aligned} &F^*(x,x,x,Tx) = \\ &\quad = (bd(x,x)d(x,Tx) + cd(x,x)d(x,Tx))/(d(x,Tx) + d(x,x)) = 0 < \\ &\quad < d(x,Tx). \end{aligned}$$

$d(Tx,Ty) \leqq F^*(x,Tx,y,Ty)$ is valid by hypothesis, and $\inf_x(G) = 0$ because $\{T^n x\}_n$ is a Cauchy sequence. It can be seen that $\{T^n x\}_n$ is a Cauchy sequence in the following way: If for each n, $T^n x \neq T^{n+2}x$, we have, using $(*)$, as the denominator does not vanish,

$$\begin{aligned} &d(T^{n+1}x, T^{n+2}x) \leqq \\ &\leqq \frac{bd(T^n x, T^{n+1}x)d(T^n x, T^{n+2}x) + cd(T^{n+1}x, T^{n+1}x)d(T^{n+1}x, T^{n+2}x)}{d(T^n x, T^{n+2}x) + d(T^{n+1}x, T^{n+1}x)} \\ &= bd(T^n x, T^{n+1}x), \end{aligned}$$

but also we have

$$\begin{aligned} &d(T^{n+1}x, T^{n+2}x) = d(T^{n+2}x, T^{n+1}x) \\ &\leqq \frac{bd(T^{n+1}x, T^{n+2}x)d(T^{n+1}x, T^{n+1}x) + cd(T^n x, T^{n+2}x)d(T^n x, T^{n+1}x)}{d(T^{n+1}x, T^{n+1}x) + d(T^n x, T^{n+2}x)} \\ &= cd(T^n x, T^{n+1}x). \end{aligned}$$

Then
$$d(T^{n+1}x, T^{n+2}x) \leqq \min(b,c) d(T^n x, T^{n+1}x)$$
and we conclude by standard arguments that $\{T^n x\}_n$ is a Cauchy sequence.

If for some n we have $T^n x = T^{n+2}x$, we also have, by hypothesis, $T^n x = T^{n+1}x$ and hence the sequence is still a Cauchy sequence, since it is definitively constant.

So, by Corollary 1.3, G is r.g.i.

Since X is complete and $\{T^n x\}_n$ is a Cauchy sequence, $\{T^n x\}_n$ is a minimizing sequence of G which is r.g.i., and $\inf_x(G) = 0$, by Proposition 1.4 there is a fixed point of T.

To prove unicity we observe that if x, y are fixed points of T and $x \neq y$ (i.e. $d(x,y) > 0$), we have by (*) $d(x,y) \leqq 0$, which is absurd.

§ 3. Second example

We are giving a new proof of the existence of fixed points of Kannan-contractive self-mappings which can be generalized to other conditions of contractivity.

THEOREM 3.1. *Let (X,d) be a complete metric space, and let $T: X \to X$ be so that $d(Tx,Ty) \leqq a(d(x,Tx) + d(y,Ty))$ with $0 < a < 1/2$. Thus T has a (unique) fixed point.*

PROOF. We will use Corollary 1.3 and Proposition 1.4. Let
$$F^*(x,y,z,w) = a(d(x,y) + d(z,w)).$$
F^* is clearly continuous, and
$$F^*(x,x,x,Tx) = a(d(x,x) + d(x,Tx)) < d(x,Tx).$$
By hypothesis
$$d(Tx,Ty) \leqq a(d(x,Tx) + d(y,Ty)) = F^*(x,Tx,y,Ty).$$
Now we prove that $\{T^n x\}_n$ is a Cauchy sequence. Effectively, by
$$d(Tx,T^2x) < a(d(x,Tx) + d(Tx,T^2x)),$$
we have
$$d(T^{n+1}x, T^{n+2}x) < (a/(1-a)) d(T^n x, T^{n+1}x)$$
where $(a/(1-a)) < 1$ and we conclude by the triangular inequality and some calculations that $\{T^n x\}_n$ is a Cauchy sequence. The fact that $\{T^n x\}_n$ is a Cauchy sequence implies also that the infimum of $G(x) = d(x,Tx)$ vanishes.

So we can use Corollary 1.3 and conclude that G is r.g.i. Using Proposition 1.4 we conclude that T has a fixed point. The unicity is obvious.

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UNIVERSITÀ DI ROMA ,,LA SAPIENZA"
VIA DEL CASTRO LAURENZIANO, 9
I–00161 ROMA
ITALY

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THE LAPLACE TRANSFORM OF THE SQUARE IN THE CIRCLE AND DIVISOR PROBLEMS

A. IVIĆ

§ 1. Introduction and statement of results

The circle problem and the divisor problem consist of the estimation of the functions

$$(1.1)\qquad \begin{aligned} P(x) &= {\sum_{n\le x}}' r(n) - \pi x + 1, \\ \Delta(x) &= {\sum_{n\le x}}' d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4}, \end{aligned}$$

respectively. Here, as usual, ${\sum_{n\le x}}'$ denotes that the last term in the sum is to be halved if x is an integer, γ is Euler's constant, $r(n) = \sum_{a^2+b^2=n} 1$ and $d(n) = \sum_{ab=n} 1$ denote the number of ways n $(\geqq 1)$ may be written as a sum of two integer squares and as a product of two natural numbers, respectively. These two problems have a long and rich history (see, for example, Chapter 3 and Chapter 13 of [8]). Pointwise estimates of $P(x)$ and $\Delta(x)$ depend on intricate techniques for the estimation of certain exponential sums, and H. Iwaniec and C. J. Mozzochi [10] proved

$$(1.2)\qquad P(x) \ll x^{7/22+\varepsilon}, \quad \Delta(x) \ll x^{7/22+\varepsilon}$$

for any given $\varepsilon > 0$. Better results than (1.2) may be obtained in the mean square sense, which is often the case in analytic number theory. Thus

$$(1.3)\qquad \int_0^X P^2(x)\,dx = \Big(\frac{1}{3\pi^2}\sum_{n=1}^{\infty} r^2(n)n^{-3/2}\Big)X^{3/2} + Q(X)$$

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and

$$\int_0^X \Delta^2(x)dx = \left(\frac{1}{6\pi^2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}\right)x^{3/2} + F(x), \tag{1.4}$$

where

$$Q(X) \ll X\log^2 X \tag{1.5}$$

and

$$F(X) \ll X\log^4 X \tag{1.6}$$

is known. The bound in (1.5) is due to I. Kátai [12]. It was reproved by E. Preissmann [16], who also proved (1.6). Namely $F(X) \ll X\log^5 X$ was proved long ago by K. -C. Tong [17]. More recently T. Meurman [14] reproved Tong's bound by a new, simpler method. The saving of the log-power given by (1.6) comes from the observation, made by E. Preissmann [16], that at a certain place in the proof a variant of Hilbert's inequality may be successfully used.

In many instances problems involving the functions $P(x)$ and $\Delta(x)$ can be successfully dealt with by means of the classical explicit formulas involving the Bessel functions. These are

$$P(x) = x^{1/2}\sum_{n=1}^{\infty} r(n)n^{-1/2}J_1(2\pi\sqrt{xn}) \qquad (x>0) \tag{1.7}$$

and

$$\Delta(x) = -\frac{2}{\pi}x^{1/2}\sum_{n=1}^{\infty} d(n)n^{-1/2}(K_1(4\pi\sqrt{xn}) + \frac{\pi}{2}Y_1(4\pi\sqrt{xn})) \quad (x>0), \tag{1.8}$$

where both series are boundedly convergent, and J_1, K_1 and Y_1 are the familiar Bessel functions. The above formulas are due to G. H. Hardy [3] and G. F. Voronoï [17], respectively. They are special cases of formulas for general number-theoretic error terms, which involve coefficients of Dirichlet series satisfying certain types of functional equations. The general theory was worked out by K. Chandrasekharan and R. Narasimhan [1], [2], and later by other researchers. The expressions (1.7) and (1.8) are certainly the ones that are best known. Especially striking is (1.7), since the J-Bessel function is less difficult to handle than the Y-function.

To work with the infinite series in (1.7) and (1.8) is not easy in practice, so that one often uses the so-called truncated formulas

$$P(x) = -\frac{x^{1/4}}{\pi}\sum_{n\le N} r(n)n^{-3/4}\cos(2\pi\sqrt{xn} + \frac{\pi}{4}) + O(x^{\varepsilon} + x^{1/2+\varepsilon}N^{-1/2}) \tag{1.9}$$

and

$$(1.10)\quad \Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le N} d(n) n^{-3/4} \cos(4\pi\sqrt{xn} - \frac{\pi}{4}) + O(x^{\varepsilon} + x^{1/2+\varepsilon} N^{-1/2}).$$

Both formulas are valid for $x \geqq 1$, $1 \leqq N \leqq x^A$, where $A > 0$ is any fixed constant and $\varepsilon > 0$ is a constant which may be arbitrarily small. The formula (1.10) is well-known (see Chapter 3 of [8] for a proof), while (1.9) can be established along similar lines by using the functional equation

$$L(s) = \pi^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} L(1-s),$$

$$L(s) = \sum_{n=1}^{\infty} r(n) n^{-s} \qquad (\operatorname{Re} s > 1).$$

Nevertheless (1.9) and (1.10) alone are not sufficient for the proof of (1.5) and (1.6), which require additional arguments and techniques.

The aim of this paper is to consider the Laplace transforms

$$(1.11)\qquad a(s) = \int_0^{\infty} P^2(x) e^{-sx} dx, \qquad b(s) = \int_0^{\infty} \Delta^2(x) e^{-sx} dx$$

and seek their asymptotic behaviour as $s \to 0_+$. The evaluation of Laplace transforms is fairly common in analytic number theory. For example, H. Kober [13] found a precise asymptotic formula for the Laplace transform of $|\zeta(1/2+ix)|^2$ as $s \to 0_+$. It might appear that it is perhaps more natural to ask for the Laplace transform of $P(x)$ than of $P^2(x)$. This is not so, since in view of (1.7) and the well-known formula (it follows from (2.8))

$$\int_0^{\infty} e^{-sx} x^{\nu/2} J_{\nu}(2\sqrt{ax}) dx = e^{-a/s} a^{\nu/2} s^{\nu-1} \quad (\operatorname{Re} s > 0,\ \operatorname{Re} \nu > -1)$$

the Laplace transform of $P(x)$ may be evaluated exactly. We obtain

$$\int_0^{\infty} P(x) e^{-sx} dx = \pi s^{-2} \sum_{n=1}^{\infty} r(n) e^{-\pi^2 n/s} \qquad (\operatorname{Re} s > 0),$$

and the right-hand side decays exponentially as $s \to 0_+$. Putting $s = 1/T$ in (1.11) one may suppose that $T \to \infty$, and then the problem of the evaluation of $a(1/T)$ and $b(1/T)$ becomes similar to the problem of the evaluation of the integrals in (1.3) and (1.4). In fact, we can use (1.3)–(1.6) and integration by parts to obtain, in the circle problem,

$$\begin{aligned}\int_0^\infty P^2(x)e^{-x/T}dx &= \frac{1}{T}\int_0^\infty\Big(\int_0^x P^2(t)dt\Big)e^{-x/T}dx = \\ &= \frac{1}{T}\int_0^\infty\Big\{\Big(\frac{1}{3\pi^2}\sum_{n=1}^\infty r^2(n)n^{-3/2}\Big)x^{3/2}+Q(x)\Big\}e^{-x/T}dx = \\ &= \Big(\frac{1}{3\pi^2}\sum_{n=1}^\infty r^2(n)n^{-3/2}\Gamma\Big(\frac{5}{2}\Big)\Big)T^{3/2}+O(T\log^2 T) = \\ &= \frac{1}{4}\Big(\frac{T}{\pi}\Big)^{3/2}\sum_{n=1}^\infty r^2(n)n^{-3/2}+O(T\log^2 T),\end{aligned} \tag{1.12}$$

and likewise

$$\int_0^\infty \Delta^2(x)e^{-x/T}dx = \frac{1}{8}\Big(\frac{T}{\pi}\Big)^{3/2}\sum_{n=1}^\infty d^2(n)n^{-3/2}+O(T\log^4 T). \tag{1.13}$$

It will turn out, however, that the asymptotic formulas (1.12) and (1.13) can be considerably sharpened if the integrals in question are evaluated directly. In the case of the circle problem the simplicity of the series expansion (1.7) comes into play by means of the identity

$$\begin{aligned}&\int_0^\infty e^{-st}tJ_1(a\sqrt{t})J_1(b\sqrt{t})dt = \\ &= e^{-\frac{a^2+b^2}{4s}}(4s^3)^{-1}\Big\{2abI_0\Big(\frac{ab}{2s}\Big)-(a^2+b^2)I_1\Big(\frac{ab}{2s}\Big)\Big\},\end{aligned} \tag{1.14}$$

valid for $\mathrm{Re}\, s>0$ and a,b real. The asymptotic expansion of the Bessel function $I_\nu(x)$, for fixed ν and $|x|\geqq 1$, is

$$I_\nu(x)=\frac{e^x}{\sqrt{2\pi x}}\Big\{1-\frac{4\nu^2-1}{8x}+\frac{(4\nu^2-1)(4\nu^2-9)}{128x^2}+O\Big(\frac{1}{|x|^3}\Big)\Big\}, \tag{1.15}$$

and in fact an asymptotic formula exists with any assigned degree of accuracy. For this, and other properties of the Bessel functions, the reader is referred to G. N. Watson's monograph [19]. As (1.15) contains neither sines nor cosines, the problem of the evaluation of $a(1/T)$ (and similarly of $b(1/T)$) is eventually reduced to the evaluation of certain ordinary arithmetic sums, and not of exponential sums. The quality of the final results depends on the values of two constants α and β which satisfy $1/2\leqq\alpha<1$, $1/2\leqq\beta<1$ and

$$\sum_{n \leqq x} r(n)r(n+h) = \frac{(-1)^h 8x}{h} \sum_{d|h} (-1)^d d + O(x^{\alpha+\varepsilon}), \tag{1.16}$$

$$\sum_{n \leqq x} d(n)d(n+h) = x \sum_{i=0}^{2} (\log x)^i \sum_{j=0}^{2} c_{ij} \sum_{d|h} \frac{(\log d)^j}{d} + O(x^{\beta+\varepsilon}), \tag{1.17}$$

uniformly for $1 \leqq h \leqq x^{1/2}$, where ε is an arbitrarily small, positive constant. In (1.17) the c_{ij}'s are certain absolute constants, and in particular $c_{22} = c_{21} = 0$, $c_{20} = 6\pi^{-2}$. Our results are

THEOREM 1. *For any given* $\varepsilon > 0$

$$\int_0^\infty P^2(x) e^{-x/T} dx = \frac{1}{4}\left(\frac{T}{\pi}\right)^{3/2} \sum_{n=1}^\infty r^2(n) n^{-3/2} - T + O(T^{\alpha+\varepsilon}). \tag{1.18}$$

THEOREM 2. *There exist constants* A_1, A_2, A_3 *such that, for any* $\varepsilon > 0$,

$$\begin{aligned} &\int_0^\infty \Delta^2(x) e^{-x/T} dx = \\ &= \frac{1}{8}\left(\frac{T}{\pi}\right)^{3/2} \sum_{n=1}^\infty d^2(n) n^{-3/2} + T(A_1 \log^2 T + A_2 \log T + A_3) + O(T^{\beta+\varepsilon}). \end{aligned} \tag{1.19}$$

The estimations of the error terms in (1.16) and (1.17) represent classical problems of analytic number theory, with a long and rich history. The first significant results were obtained by T. Estermann in the 1930's. D. R. Heath-Brown [4] and D. Ismoilov [5], [6], working independently, obtained $\beta = 5/6$ in (1.17) uniformly for $1 \leqq h \leqq x^{5/6}$. Recently, Y. Motohashi [15] employed powerful methods from the spectral theory of automorphic forms and obtained very precise results on $\sum_{n \leqq x} d(n)d(n+h)$, which improve all previous results. In particular, he showed that $\beta = 2/3$ holds uniformly for $1 \leqq h \leqq x^{20/27}$ and that (already for $h = 1$) $\beta < 1/2$ cannot hold. Thus the assumption that $\beta \geqq 1/2$ (and likewise $\alpha \geqq 1/2$) is a reasonable one to make. The method of D. Ismoilov [5], [6] is fully explained in his monograph [7], where it is indicated that its application to the circle problem yields $\alpha = 5/6$ uniformly for $1 \leqq h \leqq x^{5/6-\varepsilon_1}$. Therefore (1.18) and (1.19) certainly hold with $\alpha = 5/6$ and $\beta = 2/3$, respectively. It remains to be seen whether the methods of spectral theory can be employed to decrease the value of α to $\alpha = 2/3$, which is reasonable to expect to be possible.

The values of the constants A_j in (1.19) may be written down in closed form, although the expressions in question would not be simple, because they depend on the constants c_{ij} in (1.17).

There is an aspect of Theorem 1 and Theorem 2 which deserves to be mentioned. Namely, in view of the best known bounds (1.5) and (1.6) for $Q(X)$ and $F(X)$, respectively, it is immaterial whether $P(x)$ and $\Delta(x)$ are defined as in (1.1), or as

$$P(x)=\sum_{n\leqq x} r(n)-\pi x, \quad \Delta(x)=\sum_{n\leqq x} d(n)-x(\log x+2\gamma-1), \tag{1.20}$$

which is also customary. In the case of the Laplace transforms $a(1/T)$ and $b(1/T)$ this distinction is vital, since the constants in (1.1) contribute linear terms T and $T/16$ in (1.18) and (1.19), respectively.

It may be also asked what are the correct orders of magnitude of the functions $Q(X), F(X)$, and likewise of the error terms in (1.18) and (1.19), which we may denote by $Q_1(T)$ and $F_1(T)$, respectively. Since one may reasonably conjecture that $\alpha=\beta=1/2$ holds uniformly for $1\leqq h\leqq x^{1/2}$, it is plausible to conjecture, for any given $\delta,\varepsilon>0$,

$$\begin{aligned} Q_1(T)&=O(T^{1/2+\varepsilon}), \quad Q_1(T)=\Omega(T^{1/2-\delta}),\\ F_1(T)&=O(T^{1/2+\varepsilon}), \quad F_1(T)=\Omega(T^{1/2-\delta}). \end{aligned} \tag{1.21}$$

The omega-results in (1.21) are not analogous to the omega-results

$$Q(T)=\Omega(T^{3/4-\delta}), \quad F(T)=\Omega(T^{3/4-\delta}), \tag{1.22}$$

which hold unconditionally. For $F(T)$ this is Theorem 13.6 of [8], and even a sharper results is proved by M. Ouellet and the author [9]. For $Q(T)$ the proof is analogous to the one that works for $F(T)$. It is unclear whether in (1.3) and (1.4) there will be another main term of the form $TR(\log T)$ present, where $R(y)$ is a polynomial with non-zero leading coefficient. As I already noted, the shape (or even the existence) of such a new main term depends on the definitions (1.1) or (1.20) for $P(x)$ and $\Delta(x)$. But the proof of (1.22) shows that, regardless of the definition of $\Delta(x)$, if

$$\int_0^x \Delta^2(x)dx=\Big(\frac{1}{6\pi^2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}\Big)X^{3/2}+XR(\log X)+F_2(X)$$

holds with

$$F_2(X)=o(X) \qquad (X\to\infty),$$

then certainly

$$F_2(X) = \Omega(X^{3/4-\delta})$$

is true with any given $\delta > 0$. A similar discussion can be made for the circle problem. Probably it is true that

$$F_2(X) = O(X^{3/4+\varepsilon})$$

holds for any given $\varepsilon > 0$. This is a very strong conjecture, since it easily implies the classical conjecture $\Delta(x) = O(x^{1/4+\varepsilon})$, and this is known to be quite deep. An argument similar to (1.12) shows that $F_2(X) = O(X^{\theta+\varepsilon})$ with $3/4 \leqq \theta < 1$ implies (1.19) with $F_1(T) = O(T^{\theta+\varepsilon})$. However, the converse implication cannot hold, since we know that $F_1(T) = O(T^{\theta+\varepsilon})$ holds with $\theta = 2/3$, but (1.22) shows that $F(T) = O(T^{2/3+\varepsilon})$ is impossible for $0 < \varepsilon < 1/12$.

Generalizations of (1.18) and (1.19) to number-theoretic error terms of the type investigated by K. Chandrasekharan and R. Narasimhan [1], [2], are, of course, possible. This was not done here for several reasons: to keep the exposition as clear as possible, because in the general case no simple analogous of (1.14) seems to exist, and because the size of the error term is determined by the intrinsic properties of the arithmetic function involved in the problem. The last fact is clearly reflected in different forms of the main terms in (1.16) and (1.17), which account for different forms of the second main term in (1.18) and (1.19), respectively. In particular, the analogue of (1.16) and (1.17) holds for the "normalized" function $\tilde{a}(n) = a(n)n^{1/2(1-\kappa)}$, where $a(n)$ is the n-th Fourier coefficient of a cusp form of weight $\kappa = 2k$ for the full modular group. This was kindly indicated to me by Prof. M. Jutila, who pointed out that it was proved in [11] that one can obtain

$$\sum_{n \leqq x} \tilde{a}(n)\tilde{a}(n+h) = O(x^{\gamma+\varepsilon}) \tag{1.23}$$

with $\gamma = 2/3$, uniformly for $1 \leqq h \leqq x^{2/3}$. Thus an appropriate analogue of (1.18) holds for $A(x) = \sum'_{n \leqq x} \tilde{a}(n)$, without the linear term and with error term $O(x^{\gamma+\varepsilon})$.

The plan of the paper is as follows. In the next section the lemmas necessary for the proof of (1.18), including a proof of (1.14), are given. The proof of (1.18) will be given in § 3, while the modification necessary for the proof of (1.19) will be given in § 4. Special care is given to keep the exposition as self-contained as possible, but to avoid excessive length.

§ 2. Lemmas needed for the circle problem

The purpose of the following two lemmas is to truncate the series for $P(x)$ given by (1.7). In this way questions involving convergence are avoided when we evaluate the integral in (1.18).

LEMMA 1. *Let $1 \leqq x \leqq M \leqq x^A$, where A (>1) is any fixed constant. If $\|x\|$ denotes the distance of x to the nearest integer and $\varepsilon > 0$ is any given constant, then we have uniformly*

$$x^{1/2} \sum_{n \geqq M} r(n) n^{-1/2} J_1(2\pi\sqrt{xn}) =$$

$$(2.1) \qquad = \begin{cases} O(x^{\varepsilon}) & \textit{always}, \\ O(x^{5/4} M^{-1/2} + x^{1/2+\varepsilon} M^{-1/2} \|x\|^{-1} + x^{1/4} M^{-1/4}) & \\ & \textit{if } x \textit{ is not an integer.} \end{cases}$$

PROOF. Results similar to Lemma 1 are given in Chapter 13 of [8] for $P(x)$ and by T. Meurman [14] for $\Delta(x)$. The basic idea is to feed back the Voronoï-type formula (1.7), in integrated form, to itself. The first bound in (2.1) follows easily from (1.7), (1.9) and the asymptotic formula

(2.2)
$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left\{ \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) - \frac{(4\nu^2 - 1)}{8x} \sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \right\} + O(x^{-5/2}),$$

valid for ν fixed and $x \geqq 1$. To obtain the second bound write

$$\sum_{n \geqq M} r(n) n^{-1/2} J_1(2\pi\sqrt{xn} = \int_{M-0}^{\infty} t^{-1/2} J_1(2\pi\sqrt{xt}) d\Big(\sum_{n \leqq t}{}' r(n)\Big) + O((xM)^{-1/4}) =$$

$$= \pi \int_M^\infty t^{-1/2} J_1(2\pi\sqrt{xt})dt + \int_{M-0}^\infty t^{-1/2} J_1(2\pi\sqrt{xt}) dP(t) + O((xM)^{-1/4}) =$$

$$= O((xM)^{-1/4}) - M^{-1/2} J_1(2\pi\sqrt{xM}) P(M) - \int_M^\infty P(t)(t^{-1/2} J_1(2\pi\sqrt{xt}))' dt =$$

$$= O((xM)^{-1/4}) - \pi x^{1/2} \int_M^\infty P(t) t^{-1} J_0(2\pi\sqrt{xt}) dt.$$

Here we used the well-known bound $P(x) = O(x^{1/3})$, (2.2),

$$(2.3) \qquad J'_\nu(x) = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x),$$

and the familiar first derivative test (Lemma 2.1 of [5]) for exponential integrals. From (1.7) and (2.2) we obtain

(2.4)

$$P_1(x) := \int_0^x P(t)dt = -\frac{x^{3/4}}{\pi^2}\sum_{n=1}^{\infty} r(n)n^{-5/4}\cos(2\pi\sqrt{xn}-\frac{\pi}{4}) + O(x^{1/4}) \quad (x \geqq 1),$$

which is the weakened form of the full Voronoï-type expansion analogous to (1.7), but for our purposes (2.4) is sufficient. Integration by parts, and (2.2)–(2.4) give then

(2.5)

$$x^{1/2}\int_M^{\infty} P(t)t^{-1}J_0(2\pi\sqrt{xt})dt = O(x^{1/4}M^{-1/2}) - \pi x\int_M^{\infty} P_1(t)t^{-3/2}J_{-1}(2\pi\sqrt{xt})dt =$$

$$= O(x^{3/4}M^{-1/2}) + \frac{x}{\pi}\int_M^{\infty} t^{-3/4}J_{-1}(2\pi\sqrt{xt})\sum_{n=1}^{\infty} r(n)n^{-5/4}\cos\left(2\pi\sqrt{tn}-\frac{\pi}{4}\right)dt =$$

$$= O(x^{3/4}M^{-1/2}) + \frac{x^{3/4}}{\pi^2}\sum_{n=1}^{\infty} r(n)n^{-5/4}\int_M^{\infty} t^{-1}\cos\left(2\pi\sqrt{tn}-\frac{\pi}{4}\right)\cos\left(2\pi\sqrt{xt}+\frac{\pi}{4}\right)dt,$$

where the inversion of summation and integration is justified by the absolute convergence of the series. Now if x is not an integer we use

$$\cos\left(2\pi\sqrt{tn}-\frac{\pi}{4}\right)\cos\left(2\pi\sqrt{xt}+\frac{\pi}{4}\right) =$$

$$= \frac{1}{2}\Big\{\cos(2\pi\sqrt{t}(\sqrt{n}+\sqrt{x})) + \sin(2\pi\sqrt{t}(\sqrt{n}-\sqrt{x}))\Big\}$$

and the first derivative test to obtain from (2.5), since $r(n) \ll n^{\varepsilon}$,

$$x^{1/2}\int_M^{\infty} P(t)t^{-1}J_0(2\pi\sqrt{xt})dt \ll x^{3/4}M^{-1/2}\Big(1+\sum_{n=1}^{\infty} r(n)n^{-5/4}|\sqrt{n}-\sqrt{x}|^{-1}\Big) \ll$$

$$\ll x^{3/4}M^{-1/2}\Big(1 + \sum_{x/2<n\leqq 2x} r(n)n^{-5/4}|\sqrt{n}-\sqrt{x}|^{-1}\Big) \ll$$

$$\ll x^{3/4}M^{-1/2}\Big(1 + x^{\varepsilon/2-3/4}\sum_{x/2<n\leqq 2x} |n-x|^{-1}\Big) \ll$$

$$\ll x^{3/4}M^{-1/2}\Big(1 + x^{\varepsilon-3/4}\|x\|^{-1}\Big),$$

which gives (2.1).

LEMMA 2. *For $x>0$ and $T\geqq 2$ define $F(x,T)$ by*

$$P(x)=x^{1/2}\sum_{n\leqq T^{10}} r(n)n^{-1/2}J_1(2\pi\sqrt{xn})+F(x,T).$$

Then for $1\leqq x\leqq T\log^2 T$ and any given $\varepsilon>0$ we have uniformly

$$F(x,T)\ll\begin{cases} T^{\varepsilon} & \textit{always},\\ T^{-1} & \textit{if } \|x\|>T^{-3},\end{cases}$$

while for $T^{-10}\leqq x<1$ we have uniformly

$$F(x,T)\ll T^{-5/2}+T^{-5}(1-x)^{-1}.$$

PROOF. Suppose first $1\leqq x\leqq T\log^2 T$, and write (1.7) as

$$P(x)=x^{1/2}\sum_{n\leqq T^{10}} r(n)n^{-1/2}J_1(2\pi\sqrt{xn})+x^{1/2}\sum_{n>T^{10}} r(n)n^{-1/2}J_1(2\pi\sqrt{xn}).$$

The sum in which $n>T^{10}=M$ is estimated by Lemma 1. For $\|x\|>T^{-3}$ we have

$$x^{1/2+\varepsilon}M^{-1/2}\|x\|^{-1}\ll T^{\frac12+2\varepsilon}T^{-5}T^3\ll T^{-1}$$

for $0<\varepsilon<\frac12$. Since trivially

$$x^{1/4}M^{-1/4}+x^{5/4}M^{-1/2}\ll T^{-1},$$

the first part of the lemma follows.

If $T^{-10}\ll x<1$, then for $n>T^{10}=M$ we have $2\pi\sqrt{xn}\geqq 2\pi$, so that we may estimate the sum with $n>M$ as in the proof of Lemma 1 by using the asymptotic formula (2.2). The only difference is that now (2.5) will give

$$x^{1/2}\int_M^\infty P(t)t^{-1}J_0(2\pi\sqrt{xt})dt\ll x^{3/4}M^{-1/2}\Big(1+\sum_{n=1}^\infty r(n)n^{-5/4}(\sqrt{n}-\sqrt{x})^{-1}\Big)\ll$$
$$\ll M^{-1/2}(1-x)^{-1},$$

since $x<1$. Thus

$$F(x,T)\ll M^{-1/4}+M^{-1/2}(1-x)^{-1}\ll T^{-5/2}+T^{-5}(1-x)^{-1}.$$

LEMMA 3. *For $\operatorname{Re}s>0$, a and b real we have*

$$\int_0^\infty e^{-st}tJ_1(a\sqrt{t})J_1(b\sqrt{t})dt=e^{-\frac{a^2+b^2}{4s}}(4s^3)^{-1}\Big\{2abI_0\Big(\frac{ab}{2s}\Big)-(a^2+b^2)I_1\Big(\frac{ab}{2s}\Big)\Big\}. \tag{2.6}$$

PROOF. For $\operatorname{Re} s>0$, $\nu>-1$ and a,b real we have

$$\int_0^\infty e^{-st}J_\nu(a\sqrt{t})J_\nu(b\sqrt{t})dt=\frac{1}{s}e^{-\frac{a^2+b^2}{4s}}I_\nu\Big(\frac{ab}{2s}\Big). \tag{2.7}$$

If we set $\nu=1$, differentiate both sides of (2.7) with respect to s and use the fact that

$$I_1'(z)=I_0(z)-z^{-1}I_1(z),$$

we obtain (2.6). We could invoke (2.7) from known results on Laplace transforms, but we sketch a proof for the completeness of the exposition. First we insert the series expansions

$$\begin{aligned} J_\nu(z)&=\Big(\frac{z}{2}\Big)^\nu\sum_{k=0}^\infty\frac{(-\frac14 z^2)^k}{k!\Gamma(\nu+k+1)},\\ I_\nu(z)&=\Big(\frac{z}{2}\Big)^\nu\sum_{k=0}^\infty\frac{(\frac14 z^2)^2}{k!\Gamma(\nu+k+1)}\end{aligned} \tag{2.8}$$

in both sides of (2.7). By integrating termwise and equating the coefficients of s^{-k} in the resulting power series it follows that (2.7) holds if

$$\sum_{m+n=k}\frac{a^{2m}b^{2m}\Gamma(k+\nu+1)}{m!n!\Gamma(\nu+m+1)\Gamma(\nu+n+1)}=\sum_{m+2n=k}\frac{(a^2+b^2)^m(ab)^{2n}}{m!n!\Gamma(\nu+n+1)}$$

for $k=0,1,2,\dots$, where m and n are also nonnegative integers. If $b=0$ the above identity is trivial. If $b\neq 0$, we put $x=(a/b)^2$ and develop the binomial on the right-hand side. We obtain

(2.9)
$$\sum_{m+n=k;m,n\geq 0}\frac{x^m\Gamma(k+\nu+1)}{m!n!\Gamma(\nu+m+1)\Gamma(\nu+n+1)}=\sum_{q+r+2n=k;q,r,n\geq 0}\frac{x^{r+n}}{q!r!n!\Gamma(\nu+n+1)}.$$

Since (2.9) is a polynomial identity, because of symmetry it will hold if the coefficients of x^m for $0\leqq m\leqq k/2$ on both sides of (2.9) are equal, which reduces to

$$\binom{\nu+k}{k-m}=\sum_{n=0}^m\binom{m}{m-n}\binom{\nu+k-m}{k-m-n} \tag{2.10}$$

in view of

$$\binom{y}{l}=\frac{y(y-1)\dots(y-l+1)}{l!}=\frac{\Gamma(y+1)}{l!\Gamma(y-l+1)}\qquad(l=0,1,2,\dots).$$

As both sides of (2.10) represent the coefficient of z^{k-m} in the series expansion, for $|z|<1$, of

$$(1+z)^{\nu+k}=(1+z)^m(1+z)^{\nu+k-m},$$

the proof is finished.

This section is concluded with a lemma which provides an arithmetic ingredient necessary for the proof of Theorem 1. This is

LEMMA 4. *Let*

$$g(h)=\frac{8(-1)^h}{h}\sum_{d|h}(-1)^d d.$$

There exists a constant D such that

$$\sum_{h\le x}\frac{g(h)}{h}=\pi^2\log x+D+O\Big(\frac{\log x}{x}\Big). \tag{2.11}$$

PROOF. If all the variables under the summation signs denote natural numbers, then

(2.12)

$$\begin{aligned}\sum_{h\le x}\frac{g(h)}{h}&=8\sum_{km\le x}\frac{(-1)^{km}}{k^2m^2}(-1)^k k=8\sum_{m\le x}\frac{1}{m^2}\sum_{k\le x/m}\frac{((-1)^{m+1})^k}{k}=\\&=8\sum_{m_1\le x/2}\frac{1}{4m_1^2}\sum_{k\le x/(2m_1)}\frac{(-1)^k}{k}+8\sum_{m_2\le\frac12(x+1)}\frac{1}{(2m_2-1)^2}\sum_{k\le x/(2m_2-1)}\frac{1}{k}.\end{aligned}$$

Since

$$\sum_{n\le y}\frac{(-1)^n}{n}=-\log 2+\sum_{n>y}\frac{(-1)^n}{n}=-\log 2+O\Big(\frac1y\Big),$$

$$\sum_{n\le y}\frac1n=\log y+\gamma+O\Big(\frac1y\Big),$$

one obtains (2.11) from (2.12) after some rearrangement. The value of the constant D may be easily written down explicitly, but for our purposes it is unimportant.

§ 3. The circle problem

Theorem 1 follows from two asymptotic formulas. They are

$$\sum_{m,n\leqq T^{10}} r(m)r(n)(mn)^{-1/2}\int_0^\infty e^{-x/T}xJ_1(2\pi\sqrt{xm})J_1(2\pi\sqrt{xn})dx = \tag{3.1}$$
$$= \frac{1}{4}\left(\frac{T}{\pi}\right)^{3/2}\sum_{n=1}^\infty r^2(n)n^{-3/2}\sum_{n=1}^\infty r^2(n)n^{-3/2} - T + O(T^{\alpha+\varepsilon})$$

and

$$\int_0^\infty P^2(x)e^{-x/T}dx = \tag{3.2}$$
$$= \sum_{m,n\leqq T^{10}} r(m)r(n)(mn)^{-1/2}\int_0^\infty e^{-\frac{x}{T}}xJ_1(2\pi\sqrt{xm})J_1(2\pi\sqrt{xn})dx + O(T^{1/4}).$$

To obtain (3.1) we start from (2.6) with $s=1/T$, $a=2\pi\sqrt{m}$, $b=2\pi\sqrt{n}$, which gives

$$\sum_{m,n\leqq T^{10}} r(m)r(n)(mn)^{-1/2}\int_0^\infty e^{-\frac{x}{T}}xJ_1(2\pi\sqrt{xm})J_1(2\pi\sqrt{xn})dx =$$
$$= \pi^2T^3\sum_{m,n\leqq T^{10}} r(m)r(n)(mn)^{-1/2}e^{-\pi^2(m+n)t}\times$$
$$\times\Big\{2\sqrt{mn}I_0(2\pi^2\sqrt{mn}T) - (m+n)I_1(2\pi^2\sqrt{mn}T)\Big\} = \tag{3.3}$$
$$= \frac{\sqrt{\pi}}{2}T^{5/2}\sum_{m,n\leqq T^{10}}\frac{r(m)r(n)}{(mn)^{3/4}}e^{-\pi^2T(\sqrt{m}-\sqrt{n})^2}\times$$
$$\times\Big\{-(\sqrt{m}-\sqrt{n})^2 + \frac{3(m+n)+2\sqrt{mn}}{16\pi^2\sqrt{mn}T}\Big\} + O(T^{1/2}) =$$
$$= \frac{1}{4}\left(\frac{T}{\pi}\right)^{3/2}\sum_{n=1}^\infty r^2(n)n^{-3/2} + \sum(T) + O(T^{1/2}),$$

where we used the asymptotic formula (1.15) and we set

$$\sum(T) := \frac{\sqrt{\pi}}{2}T^{5/2}\sum_{m\neq n\leqq T^{10}}\frac{r(m)r(n)}{(mn)^{3/4}}e^{-\pi^2T(\sqrt{m}-\sqrt{n})^2}\times \tag{3.4}$$
$$\times\Big\{-(\sqrt{m}-\sqrt{n})^2 + \frac{3(m+n)+2\sqrt{mn}}{16\pi^2\sqrt{mn}T}\Big\}.$$

Since $\frac{1}{2} \leqq \alpha < 1$, the proof of (3.1) is then reduced to the proof of

$$\sum(T) = -T + O(T^{\alpha+\varepsilon}). \tag{3.5}$$

To evaluate the double sum in (3.4) we may suppose, because of symmetry, that $n > m$ and set $h = n - m (\geqq 1)$. The contribution of the terms for which $h > m$ is trivially $\ll 1$. By using $r(n) \ll n^{\varepsilon}$ we obtain that the contribution of the terms for which $m^{1/2} < h \leqq m$ is

$$\begin{aligned}
&\ll T^{5/2} \sum_{m \leqq T^{10}} \sum_{m^{1/2} < h \leqq m} \frac{r(m)r(m+h)}{m^{3/2}} e^{-\frac{\pi^2 T h^2}{8m}} \left(\frac{h^2}{m} + \frac{1}{T}\right) \ll \\
&\ll T^{1/2} \sum_{m \leqq T^{10}} m^{\varepsilon - 3/2} \sum_{m^{1/2} < h \leqq m} \frac{m}{h^2} \max_{h \geqq 1} \left\{ \left(\frac{Th^2}{m} + \left(\frac{Th^2}{m}\right)^2\right) e^{-\frac{Th^2}{m}} \right\} \ll \\
&\ll T^{1/2} \sum_{m \leqq T^{10}} m^{\varepsilon - 1/2} \sum_{h > m^{1/2}} h^{-2} \ll \\
&\ll T^{1/2} \sum_{m \leqq T^{10}} m^{\varepsilon - 1} \ll T^{\frac{1}{2} + 10\varepsilon}.
\end{aligned}$$

Thus we have, changing ε to $\varepsilon/10$,

$$\begin{aligned}
\sum(T) = O(T^{1/2+\varepsilon}) + \sqrt{\pi} T^{5/2} \sum_{m \leqq T^{10}} \sum_{h \leqq m^{1/2}} \frac{r(m)r(m+h)}{(m(m+h))^{3/4}} e^{-\pi^2 T(\sqrt{m+h} - \sqrt{m})^2} \times \\
\times \left\{ -(\sqrt{m+h} - \sqrt{m})^2 + \frac{3(2m+h) + 2\sqrt{m(m+h)}}{16\pi^2 \sqrt{m(m+h)} T} \right\}.
\end{aligned}$$

To effect further simplification set

$$f(t,h) := \left\{ -(\sqrt{t+h} - \sqrt{t})^2 + \frac{3(2t+h) + 2\sqrt{t(t+h)}}{16\pi^2 \sqrt{t(t+h)} T} \right\} t^{-3/4} (t+h)^{-3/4},$$

and as in Lemma 4

$$g(h) := \frac{8(-1)^h}{h} \sum_{d \mid h} (-1)^d d.$$

Then we have, using (1.16),

$$\sum(T) = \sqrt{\pi} T^{5/2} \sum_{h \leqq T^5} \int\limits_{h^2-0}^{T^{10}+0} e^{-\pi^2 T(\sqrt{t+h}-\sqrt{t})^2} f(t,h) d\Big(\sum_{n \leqq t} r(n) r(n+h)\Big) +$$

$$+ O(T^{1/2+\varepsilon}) =$$

$$= \sqrt{\pi} T^{5/2} \sum_{h \leqq T^5} g(h) \int\limits_{h^2}^{T^{10}} e^{-\pi^2 T(\sqrt{t+h}-\sqrt{t})^2} f(t,h) dt + O(T^{1/2+\varepsilon}) + \tag{3.6}$$

$$+ O\Big(T^{5/2} \sum_{h \leqq T^5} \int\limits_{h^2-0}^{T^{10}+0} t^{\alpha+\varepsilon} \Big| d\Big\{ f(t,h) e^{-\pi^2 T(\sqrt{t+h}-\sqrt{t})^2} \Big\} \Big| \Big).$$

For $h^2 \ll t \ll T^{10}$ we have

$$\frac{df(t,h)}{dt} \ll h^2 t^{-7/2} + T^{-1} t^{-5/2},$$

so that the contribution of the second O-term in (3.6) is

$$\ll T^{5/2} \sum_{h \leqq T^5} \int\limits_{h^2}^{T^{10}} t^{\alpha+\varepsilon} (h^2 t^{-7/2} + T^{-1} t^{-5/2} + T h^4 t^{-9/2}) e^{-\frac{Th^2}{2t}} dt \ll$$

$$\ll T^{\alpha+\varepsilon} \sum_{h \leqq T^5} h^{2\alpha+2\varepsilon-3} \int\limits_0^\infty (x^{3/2-\alpha-\varepsilon} + x^{1/2-\alpha-\varepsilon} + x^{5/2-\alpha-\varepsilon}) e^{-x} dx \ll$$

$$\ll T^{\alpha+\varepsilon},$$

since $\frac{1}{2} \leqq \alpha < 1$ and ε may be arbitrarily small. This gives

$$\sum(T) = \sqrt{\pi} T^{5/2} \sum_{h \leqq T^5} g(h) \int\limits_{h^2}^{T^{10}} e^{-\pi^2 T(\sqrt{t+h}-\sqrt{t})^2} f(t,h) dt + O(T^{\alpha+\varepsilon}). \tag{3.7}$$

Next note that

$$(t+h)^{-3/4} = t^{-3/4} + O(ht^{-7/4}),$$

$$(\sqrt{t+h} - \sqrt{t})^2 = \frac{h^2}{4t} + O\Big(\frac{h^3}{t^2}\Big),$$

$$\frac{3(2t+h) + 2\sqrt{t(t+h)}}{\sqrt{t(t+h)}} = 3\Big(\sqrt{\frac{t}{t+h}} + \sqrt{\frac{t+h}{t}}\Big) + 2 = 8 + O\Big(\frac{h}{t}\Big).$$

This means that if in (3.7) we replace $f(t,h)$ by

$$t^{-3/2}\Big(-\frac{h^2}{4t}+\frac{1}{2\pi^2T}\Big),$$

we shall obtain an error term in which the integral is by a factor of ht^{-1} of a lower order of magnitude than the main term, which is easily seen to be $\ll T^{1+\varepsilon}$. Change of variable $t=Th^2x^{-1}$ in the error term produces an integral which is then by a factor of $xT^{-1}h^{-1}$ smaller, hence we obtain from (3.7)

$$(3.8)\quad \sum(T)=\sqrt{\pi}T^{5/2}\sum_{h\leqq T^5}g(h)\int\limits_{h^2}^{T^{10}}\Big(\frac{1}{2\pi^2T}-\frac{h^2}{4t}\Big)t^{-3/2}e^{-\pi^2T(\sqrt{t+h}-\sqrt{t})^2}dt+O(T^{\alpha+\varepsilon}).$$

Now we make the change of variable

$$\pi^2T(\sqrt{t+h}-\sqrt{t})^2=x,$$

so that

$$t=\frac{\pi^2Th^2}{4x}+\frac{x}{4\pi^2T}-\frac{h}{2},$$

$$dt=\Big(-\frac{\pi^2Th^2}{4x^2}+\frac{1}{2\pi^2T}\Big)dx.$$

Therefore (3.8) gives

$$(3.9)\quad \sum(T)=\frac{1}{4}\pi^{5/2}T^{5/2}\sum_{h\leqq T^5}g(h)\int\limits_{U}^{V}\frac{Th^2}{x^2}\Big(\frac{1}{2\pi^2T}-\frac{h^2}{4t}\Big)t^{-3/2}e^{-x}dx+O(T^{\alpha+\varepsilon}),$$

since the contribution of $(4\pi^2T)^{-1}$ in dt is negligible, and where

$$U=\pi^2T((T^{10}+h)^{1/2}-T^5)^2,$$

$$V=\pi^2Th^2((1+h^{-1)})^{1/2}-1)^2.$$

We note that

$$U-\frac{\pi^2h^2}{4T^9}\ll\frac{h^3}{T^{19}},$$

so that in (3.9) we may replace U by $\pi^2h^2/(4T^9)$ with the total error which is $\ll 1$. Furthermore

$$t^{-3/2}=\Big(\frac{\pi^2Th^2}{4x}\Big)^{-3/2}\Big\{1+O\Big(\frac{x}{Th^2}\Big(\frac{x}{T}+h\Big)\Big)\Big\},$$

$$\frac{h^2}{4t}=\frac{x}{\pi^2T}\Big(1+O\Big(\frac{x^2}{T^2h^2}\Big)\Big),$$

and the contribution of both error terms above to (3.9) is also $\ll 1$. It follows that (3.9) reduces to

$$\sum(T) = 2\pi^{-5/2}T \sum_{h \leqq T^5} \frac{g(h)}{h} \int\limits_{\frac{\pi^2 h^2}{4T^9}}^{\infty} \Big(\frac{1}{2}x^{-1/2} - x^{1/2}\Big)e^{-x}dx + O(T^{\alpha+\varepsilon}). \tag{3.10}$$

The sum in (3.10) is evaluated by partial summation, setting

$$H(t) := \int\limits_{\frac{\pi^2 t^2}{4T^9}}^{\infty} \Big(\frac{1}{2}x^{-1/2} - x^{1/2}\Big)e^{-x}dx$$

and writing Lemma 4 as

$$G(t) := \sum_{h \leqq t} \frac{g(h)}{h} = \pi^2 \log t + D + K(t),$$

$$K(t) = O\Big(\frac{\log t}{t}\Big).$$

Then we have

$$\sum_{h \leqq T^5} \frac{g(h)}{h} H(h) = \int\limits_{1-0}^{T^5+0} H(t)dG(t) = \int\limits_{1}^{T^5} \pi^2 H(t)\frac{dt}{t} + \int\limits_{1-0}^{T^5+0} H(t)dK(t) = I_1 + I_2,$$

say. Integration by parts gives

$$I_2 = H(T^5)K(T^5+0) - H(1)K(1-0) - \int\limits_{1}^{T^5} K(t)dH(t) =$$

$$= O(T^{-1}) - H(0)K(1-0) + O\Big(\int\limits_{1}^{T^5} \frac{\log t}{t}\Big(\Big(\frac{t^2}{T^9}\Big)^{-1/2} + \Big(\frac{t^2}{T^9}\Big)^{1/2}\Big)\frac{t}{T^9}e^{-t^2/T^9}dt\Big) =$$

$$= O(T^{-1}),$$

since in view of

$$\Gamma(x+1) = x\Gamma(x) \tag{3.11}$$

it follows that

$$H(0) = \int\limits_{0}^{\infty} \Big(\frac{1}{2}x^{-1/2} - x^{1/2}\Big)e^{-x}dx = \frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) - \Gamma\Big(\frac{3}{2}\Big) = 0. \tag{3.12}$$

Similarly change of variable $\pi^2 t^2/(4T^9) = x$ and the use of (3.12) give

$$I_1 = \pi^2 \int_1^{T^5} H(t)\frac{dt}{t} = \pi^2 H(t)\log t\Big|_1^{T^5} - \pi^2 \int_1^{T^5} H'(t)\log t\, dt =$$

$$= \pi^2 \int_{\frac{\pi^2}{4T^9}}^{\infty} \Big(\log\frac{2}{\pi} + \frac{1}{2}\log x + \frac{9}{2}\log T\Big)\Big(\frac{1}{2}x^{-1/2} - x^{1/2}\Big)e^{-x}\,dx + O(T^{-1}) =$$

$$= \frac{\pi^2}{2}\int_0^{\infty}\Big(\frac{1}{2}x^{-1/2} - x^{1/2}\Big)e^{-x}\log x\, dx + O(T^{-1}).$$

Finally, differentiation of (3.11) gives

$$\Gamma'(x+1) = x\Gamma'(x) + \Gamma(x),$$

hence

$$\frac{\pi^2}{2}\int_0^{\infty}\Big(\frac{1}{2}x^{-1/2} - x^{1/2}\Big)e^{-x}\log x\, dx = \frac{\pi^2}{2}\Big(\frac{1}{2}\Gamma'\Big(\frac{1}{2}\Big) - \Gamma'\Big(\frac{3}{2}\Big)\Big) =$$

$$= \frac{\pi^2}{2}\Big(-\Gamma\Big(\frac{1}{2}\Big)\Big) = -\frac{1}{2}\pi^{5/2}.$$

Therefore (3.10) gives

$$\sum(T) = 2\pi^{-5/2}\Big(-\frac{1}{2}\pi^{5/2}\Big)T + O(T^{\alpha+\varepsilon}) = -T + O(T^{\alpha+\varepsilon}),$$

proving (3.5) and hence also (3.1).

It remains to prove (3.2). If $F(x,T)$ is as in Lemma 2, then

$$\int_0^{\infty} P^2(x)e^{-x/T}dx = \int_1^{T\log^2 T} P^2(x)e^{-x/T}dx + O(1) =$$

$$= \int_1^{T\log^2 T}\Big(x^{1/2}\sum_{n\leq T^{10}} r(n)n^{-1/2}J_1(2\pi\sqrt{xn})\Big)^2 e^{-x/T}dx +$$

$$+ 2\int_1^{T\log^2 T} F(x,T)x^{1/2}\sum_{n\leq T^{10}} r(n)n^{-1/2}J_1(2\pi\sqrt{xn})e^{-x/T}dx +$$

$$+\int_1^{T\log^2 T} F^2(x,T)e^{-x/T}dx + O(1) =$$
$$= I_1 + 2I_2 + I_3 + O(1),$$

say. From Lemma 2 we obtain

$$I_3 = \int_1^{T\log^2 T} F^2(x,T)e^{-x/T}dx \ll T^{-1} + T^{-2}\int_1^{T\log^2 T} e^{-x/T}dx \ll T^{-1}, \tag{3.13}$$

so that by the Cauchy–Schwarz inequality for integrals, (3.1) and (3.13) it follows that

$$I_2 = \int_1^{T\log^2 T} F(x,T)x^{1/2}\sum_{n\leqq T^{10}} r(n)n^{-1/2}J_1(2\pi\sqrt{xn})dx \ll T^{1/4}.$$

Hence

(3.14)
$$\int_0^\infty P^2(x)e^{-x/T}dx =$$
$$= \sum_{m,n\leqq T^{10}} r(m)r(n)(mn)^{-1/2}\int_1^{T\log^2 T} e^{-x/T}xJ_1(2\pi\sqrt{xm})J_1(2\pi\sqrt{xn})dx + O(T^{1/4}) =$$
$$= \sum_{m,n\leqq T^{10}} r(m)r(n)(mn)^{-1/2}\int_1^{\infty} e^{-x/T}xJ_1(2\pi\sqrt{xm})J_1(2\pi\sqrt{xn})dx + O(T^{1/4}),$$

and (3.2) will follow from (3.14) and
(3.15)
$$\sum_{m,n\leqq T^{10}} r(m)r(n)(mn)^{-1/2}\int_0^1 e^{-x/T}xJ_1(2\pi\sqrt{xm})J_1(2\pi\sqrt{xn})dx = O(1).$$

Write the integral in (3.15) as

$$\int_0^1 e^{-x/T}xJ_1(2\pi\sqrt{xm})J_1(2\pi\sqrt{xn})dx = \int_0^{T^{-10}} + \int_{T^{-10}}^{1-T^{-10}} + \int_{1-T^{-10}}^{1} =$$

$$=I^{(1)}+I^{(2)}+I^{(3)},$$

say. For x real we have

$$|J_1(x)|=\left|\frac{1}{\pi}\int\limits_0^\pi \cos(x\sin\theta-\theta)d\theta\right|\leqq 1,$$

so that the contribution of $I^{(1)}$ to the sum in (3.15) is trivially $\ll 1$, and likewise the contribution of $I^{(3)}$ is also $\ll 1$. The contribution of $I^{(2)}$ is, by the second part of Lemma 2,

$$\leqq \int\limits_{T^{-10}}^{1-T^{-10}} (P(x)-F(x,T))^2dx \leqq 2\int\limits_0^1 P^2(x)dx+2\int\limits_{T^{-10}}^{1-T^{-10}} F^2(x,T)dx \ll$$

$$\ll 1+T^{-10}\int\limits_{T^{-10}}^{1-T^{-10}} \frac{dx}{(1-x)^2}\ll 1.$$

This proves (3.2) and completes the proof of Theorem 1.

§ 4. The divisor problem

In this section we shall give the modifications in the preceding proof needed for the proof of Theorem 2. Since for ν fixed and $x\to\infty$

$$K_\nu(x)\sim\left(\frac{\pi}{2x}\right)^{1/2}e^{-x},$$

it follows that the contribution of the terms with $K_1(4\pi\sqrt{xn})$ in (1.8) is $\ll$ $\ll e^{-x}$, and therefore can be altogether discarded. However, one of the main differences between the proof of Theorem 1 and Theorem 2 is that no direct analogue of (1.14) for Y_1 seems available. The reason for this is that the series expansion

$$Y_1(z)=\frac{1}{\pi}\sum_{k=0}^\infty\frac{(-1)^k(\frac12 z)^{2k+1}}{k!(k+1)!}\left\{2\log\left(\frac12 z\right)-\psi(k+1)-\psi(k+2)\right\},\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$$

is more complicated than the series expansion for $J_1(z)$, furnished by (2.8). On the other hand, the asymptotic formula $(x\geqq 1)$

$$Y_\nu(x)=\left(\frac{2}{\pi x}\right)^{\frac12}\left\{\sin\left(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)+\frac{4\nu^2-1}{8x}\cos\left(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\right\}+O(x^{-5/2}) \tag{4.1}$$

is completely analogous to (2.2). Coupled with (1.8) it gives, for $x \geqq 1$,

$$\begin{aligned}\Delta(x) = &\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n=1}^{\infty} d(n)n^{-3/4}\cos\left(4\pi\sqrt{xn}-\frac{\pi}{4}\right)- \\ &-\frac{3x^{-1/4}}{32\pi^2\sqrt{2}}\sum_{n=1}^{\infty} d(n)n^{-5/4}\sin\left(4\pi\sqrt{xn}-\frac{\pi}{4}\right)+O(x^{-3/4}),\end{aligned} \tag{4.2}$$

$$\int_0^x \Delta(t)dt = \frac{x^{3/4}}{2\sqrt{2}\pi^2}\sum_{n=1}^{\infty} d(n)n^{-5/4}\sin\left(4\pi\sqrt{xn}-\frac{\pi}{4}\right)+O(x^{1/4}). \tag{4.3}$$

Equally important is that, since $d(n) \ll_\varepsilon n^\varepsilon$, the analogues of Lemma 1 and Lemma 2 will hold for the divisor problem, in the sense that the series in (4.2) may be truncated at $M = T^{10}$ with error terms similar to those in Lemma 1 and Lemma 2. We obtain

$$\int_0^\infty \Delta^2(x)e^{-x/T}dx = \int_1^\infty e^{-x/T}q^2(x,T)dx + \int_0^1 \Delta^2(x)e^{-x/T}dx + O(T^{1/4}), \tag{4.4}$$

where we set

$$\begin{aligned}q(x,T) := &\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n\leqq T^{10}} d(n)n^{-3/4}\cos\left(4\pi\sqrt{xn}-\frac{\pi}{4}\right)- \\ &-\frac{3x^{-1/4}}{32\pi^2\sqrt{2}}\sum_{n\leqq T^{10}} d(n)n^{-5/4}\sin\left(4\pi\sqrt{xn}-\frac{\pi}{4}\right).\end{aligned}$$

In the integral over $[0,1]$ in (4.4) we replace $\Delta(x)$ by

$$\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n\leqq T^{10}} d(n)n^{-3/4}\cos\left(4\pi\sqrt{xn}-\frac{\pi}{4}\right)$$

plus an error term, which will make a small contribution after integration. In this way we obtain from (4.4)

$$\int_0^\infty \Delta^2(x)e^{-x/T}dx = S_1 + S_2 + O(T^{1/2}), \tag{4.5}$$

where

$$
\begin{aligned}
(4.6) \qquad S_1 :=& \frac{1}{2\pi^2} \sum_{m,n \le T^{10}} d(m)d(n)(mn)^{-3/4} \times \\
&\times \int_0^\infty e^{-x/T} x^{1/2} \cos\left(4\pi\sqrt{xm} - \frac{\pi}{4}\right) \cos\left(4\pi\sqrt{xn} - \frac{\pi}{4}\right) dx,
\end{aligned}
$$

$$
\begin{aligned}
(4.7) \qquad S_2 :=& -\frac{3}{32\pi^3} \sum_{m,n \le T^{10}} d(m)d(n)m^{-3/4}n^{-5/4} \times \\
&\times \int_1^\infty e^{-x/T} \cos\left(4\pi\sqrt{xm} - \frac{\pi}{4}\right) \sin\left(4\pi\sqrt{xn} - \frac{\pi}{4}\right) dx \\
\ll& \sum_{m,n \le T^{10}} d(m)d(n)m^{-3/4}n^{-5/4} \times \\
&\times \left| \int_1^{T\log^2 T} e^{-x/T} \cos\left(4\pi\sqrt{xm} - \frac{\pi}{4}\right) \sin\left(4\pi\sqrt{xn} - \frac{\pi}{4}\right) dx \right| + 1.
\end{aligned}
$$

We treat the integrals for which $m = n$ and $m \ne n$ separately, using for each the first derivative test. We obtain

$$
S_2 \ll T^{1/2} \log T \Big(1 + \sum_{m \ne n \le T^{10}} d(m)d(n)m^{-3/4}n^{-5/4} |\sqrt{m} - \sqrt{n}|^{-1}\Big) \ll T^{1/2} \log T.
$$

To evaluate the integral in (4.6) write

$$
\begin{aligned}
(4.8) \qquad & \cos\left(4\pi\sqrt{xm} - \frac{\pi}{4}\right) \cos\left(4\pi\sqrt{xn} - \frac{\pi}{4}\right) = \\
&= \frac{1}{2}\{\cos(4\pi\sqrt{x}(\sqrt{m} - \sqrt{n})) + \sin(4\pi\sqrt{x}(\sqrt{m} + \sqrt{n}))\}.
\end{aligned}
$$

The sine integrals become, after change of variable $x = t^2 T$,

$$
2T^{3/2} \int_0^\infty e^{-t^2} t^2 \sin(4\pi t\sqrt{T}(\sqrt{m} + \sqrt{n}))dt \ll T^{1/2}(\sqrt{m} + \sqrt{n})^{-2}
$$

if we integrate twice by parts. Hence these integrals contribute a total of $O(T^{1/2})$ to S_1. To evaluate the ensuing cosine integrals note that

$$
f(A) := \int_{-\infty}^\infty e^{-Bx^2 + Ax} dx = \left(\frac{\pi}{B}\right)^{1/2} e^{\frac{A^2}{4B}}
$$

is a regular function of A for a given B such that $\operatorname{Re} B > 0$. The derivatives of $f(A)$ may be found by differentiating under the integral sign, hence

$$f''(A) = \int_{-\infty}^{\infty} x^2 e^{-Bx^2+Ax} dx = \frac{\sqrt{\pi}}{2} B^{-3/2}\Big(1 + \frac{A^2}{2B}\Big) e^{\frac{A^2}{4B}}.$$

Changing A to Ai we obtain

$$\int_0^{\infty} x^2 e^{-Bx^2} \cos(Ax) dx = \frac{\sqrt{\pi}}{4} B^{-3/2}\Big(1 - \frac{A^2}{2B}\Big) e^{-\frac{A^2}{4B}} \qquad (\operatorname{Re} B > 0). \tag{4.9}$$

We replace x by x^2 in (4.6), use (4.8) and apply (4.9) with $B = 1/T$, $A = 4\pi(\sqrt{m} - \sqrt{n})$ to obtain

$$S_1 = \frac{1}{8}\Big(\frac{T}{\pi}\Big)^{3/2} \sum_{n=1}^{\infty} d^2(n) n^{-3/2} + \sum{}'(T) + O(T^{1/2}), \tag{4.10}$$

where

(4.11)
$$\sum{}'(T) := \frac{1}{8}\pi^{-3/2} T^{5/2} \sum_{m \ne n \le T^{10}} \frac{d(m)d(n)}{(mn)^{3/4}} e^{-4\pi^2 T(\sqrt{m}-\sqrt{n})^2} \Big(\frac{1}{T} - 8\pi^2(\sqrt{m} - \sqrt{n})^2\Big).$$

The sum $\sum'(T)$ is completely analogous to the sum $\sum(T)$ in (3.4), and it is evaluated in a similar way, so there is no need to repeat the details. The only essential difference is that instead of $r(n)$ the function $d(n)$ appears. Thus instead of (1.16) we shall use (1.17), where the main term is more complicated and accounts for the appearance of $A_1 \log^2 T + A_2 \log T + A_3$ in (1.19). Note that there will be no term of the form $A_0 T \log^3 T$ in (1.19) since, similarly as in the preceding proof, in view of (3.12) it will turn out that $A_0 = 0$. Thus we shall obtain

$$\sum{}'(T) = T(A_1 \log^2 T + A_2 \log T + A_3) + O(T^{\beta+\varepsilon}),$$

and (1.19) follows from (4.10).

In concluding it may be mentioned that the procedure used for $\Delta(x)$ could have been also used for $P(x)$. The analogue of (4.2) would be

$$\begin{aligned} P(x) = &-\frac{x^{1/4}}{\pi} \sum_{n=1}^{\infty} r(n) n^{-3/4} \cos\Big(2\pi\sqrt{xn} + \frac{\pi}{4}\Big) + \\ &+ \frac{3x^{-1/4}}{16\pi^2} \sum_{n=1}^{\infty} r(n) n^{-5/4} \sin\Big(2\pi\sqrt{xn} + \frac{\pi}{4}\Big) + O(x^{-3/4}). \end{aligned} \tag{4.12}$$

However, the direct use of (2.6) in the proof of Theorem 1 seemed appropriate for several reasons. One is that the last formula in (3.3) may be easily sharpened to

$$\begin{gathered}\frac{1}{4}\Big(\frac{T}{\pi}\Big)^{3/2}\sum_{n=1}^{\infty} r^2(n)n^{-3/2} + \sum(T) + CT^{1/2} + O(T^{-1/2}),\\ C = \frac{3}{64\pi^{7/2}}\sum_{n=1}^{\infty} r^2(n)n^{-5/2},\end{gathered}\tag{4.13}$$

while by using (4.12) (i.e. the method used for $\Delta(x)$) it seems quite difficult to obtain (4.13). To see why (4.13) holds, note that by using (1.15) it is seen that the terms in (3.3) for which $m=n$ contribute

$$2\pi^2T^3\sum_{n\le T^{10}} r^2(n)e^{-2\pi^2nT}\{I_0(2\pi^2nT) - I_1(2\pi^2nT)\} =$$

$$= \frac{1}{4}\Big(\frac{T}{\pi}\Big)^{3/2}\sum_{n=1}^{\infty} r^2(n)n^{-3/2} + CT^{1/2} + O(T^{-1/2}).$$

The terms in (3.3) for which $m \ne n$ contribute $\sum(T)$ plus an error which is

$$\ll T^{5/2}\sum_{m\ne n\le T^{10}} \frac{r(m)r(n)}{(mn)^{3/4}}\,\frac{m+n}{mnT^2}e^{-\pi^2T(\sqrt{m}-\sqrt{n})^2} \ll$$

$$\ll T^{1/2}\sum_{m\le T^{10}}\sum_{1\le h\le m} \frac{r(m)r(m+h)}{m^{3/2}}\,\frac{1}{m}e^{-Th^2/m} + T^{-1/2} \ll$$

$$\ll T^{1/2}\sum_{m\le T^{10}} m^{\varepsilon-3/2}\sum_{h\le m}\frac{1}{Th^2}\Big(\max_{h\ge1}\frac{Th^2}{m}e^{-Th^2/m}\Big) + T^{-1/2} \ll T^{-1/2}.$$

This discussion clearly shows that the most delicate part in the proof of Theorem 1 is the evaluation of the sum $\sum(T)$.

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KATEDRA MATEMATIKE RGF-A
UNIVERSITETA U BEOGRADU
DJUŠINA 7
YU–11000 BEOGRAD
SERBIA

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METRIC THEOREMS ON MINIMAL BASES AND MAXIMAL NONBASES

M. B. NATHANSON and A. SÁRKÖZY

1. We denote the set of the sequences of non-negative integers by Σ. A sequence $\mathcal{A}\in\Sigma$ is said to be an asymptotic basis of order k if for $n>n_0$, n can be represented in the form

$$a_1+a_2+\ldots+a_k=n \quad \text{where} \qquad a_1\in\mathcal{A},\ a_2\in\mathcal{A},\ldots,a_k\in\mathcal{A}.$$

If $\mathcal{A}$ is an asymptotic basis of order k, but no proper subset of $\mathcal{A}$ is an asymptotic basis of order k, then $\mathcal{A}$ is said to be a minimal asymptotic basis of order k. If $\mathcal{A}$ is not an asymptotic basis of order k, but $\mathcal{A}\cup\{b\}$ is an asymptotic basis of order k for all $b\notin\mathcal{A}$, then $\mathcal{A}$ is said to be a maximal (asymptotic) nonbasis of order k. We denote the set of the sequences which are *not* asymptotic bases of order k by $\Delta^{(k)}$, and we write $\Delta=\Delta^{(2)}$. The set of the minimal asymptotic bases and maximal asymptotic non-bases of order k is denoted by $\phi^{(k)}$ and $\psi^{(k)}$, respectively, and we write $\phi=\phi^{(2)}$, $\psi=\psi^{(2)}$.

If $\mathcal{A}\in\Sigma$, $\Gamma\subset\Sigma$ and $n=0,1,2,\ldots$, then we write

$$\mathcal{A}_n=\mathcal{A}\cap\{0,1,\ldots,n\}$$

and

$$\Gamma_n=\{\mathcal{B}_n:\mathcal{B}\in\Gamma\}.$$

If $\mathcal{A}\in\Sigma$, $\mathcal{B}\in\Sigma$ and

$$\mathcal{A}\cap[n,+\infty)=\mathcal{B}\cap[n,+\infty)$$

for some n, then we write $\mathcal{A}\sim\mathcal{B}$.

The lower asymptotic density of a sequence $\mathcal{A}\in\Sigma$ is denoted by $\underline{d}(\mathcal{A})$.

If $\mathcal{A}\in\Sigma$, then we write

$$\varrho(\mathcal{A})=\sum_{a\in\mathcal{A}}\frac{1}{2^{a+1}}.$$

(In this way, we map Σ into the interval $[0,1]$.) If $\Gamma\subset\Sigma$, then we put

$$\varrho(\Gamma)=\{\varrho(\mathcal{A}):\ \mathcal{A}\in\Gamma\}.$$

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We need the concepts of Hausdorff measure and Hausdorff dimension.

The length of the interval I is denoted by $|I|$.

Let $M \subset [0,1]$. Let us assume that the set S satisfies the following conditions:

(i) $M \subset S \subset [0,1]$.

(ii) S is of the form

$$S = \bigcup_{k=1}^{+\infty} I_k$$

where $I_1, I_2, \dots$ are intervals such that

$$|I_k| \leqq \varrho.$$

Then S is said to be a ϱ-covering of the set M.

Let

$$f_\varrho(M,\alpha) = \inf_S \sum_{k=1}^{+\infty} |I_k|^\alpha \qquad (0 < \alpha \leqq 1)$$

where the infimum is taken for all ϱ-coverings S of M. It can be shown easily that $f_\varrho(M,\alpha)$ is a decreasing function of ϱ, thus the limit

$$\lim_{\varrho \to 0+0} f_\varrho(M,\alpha) = L(M,\alpha)$$

exists. $L(M,\alpha)$ is said to be the α-dimensional Hausdorff measure of the set M.

It can be shown that $L(M,\alpha) < +\infty$ and $\alpha_1 > \alpha$ imply that $L(M,\alpha_1) = 0$ while $L(M,\alpha) > 0$ and $\alpha_1 < \alpha$ imply that $L(M,\alpha_1) = +\infty$. In view of these facts, we define the Hausdorff dimension of M (denoted by $\dim M$) in the following way:

$$\dim M = \begin{cases} \inf\{\alpha\colon\ L(M,\alpha) = 0\} & \text{if}\ \ L(M,1) = 0 \\ 1 & \text{if}\ \ L(M,1) > 0. \end{cases}$$

Finally, for $0 \leqq x \leqq 1$ we define the function $\xi(x)$ by

$$\xi(x) = \begin{cases} -\dfrac{1}{\log 2}(x \log x + (1-x)\log(1-x)) & \text{for}\ \ 0 < x < 1 \\ 0 & \text{for}\ \ x = 0\ \ \text{and}\ \ x = 1. \end{cases}$$

(This function is continuous on $[0,1]$, increasing on $[0,1/2]$, $\xi(1/2) = 1$ and $\xi(1-x) = \xi(x)$ for $0 \leqq x \leqq 1$.)

2. Sárközy [7] proved that

$$\dim \varrho(\Delta) = \frac{\log 3}{\log 4} \tag{2.1}$$

and

$$\lim_{k\to+\infty} \dim \varrho(\Delta^{(k)}) = \frac{1}{2}.$$

In this paper, our goal is to estimate $\dim \varrho(\phi)$, $\dim \varrho(\phi^{(k)})$, $\dim \varrho(\psi)$ and $\dim \varrho(\psi^{(k)})$. In fact, we will prove the following theorems:

THEOREM 1.

$$\dim \varrho(\phi) = \frac{\log 3}{\log 4}. \tag{2.2}$$

THEOREM 2. *For $k = 3, 4, \dots$, we have*

$$\dim \varrho(\phi^{(k)}) \leqq \xi\left(\frac{1}{k}\right).$$

THEOREM 3.

$$\dim \varrho(\psi) = \frac{\log 3}{\log 4}.$$

THEOREM 4. *For $k = 3, 4, \dots$, we have*

$$\dim \varrho(\psi^{(k)}) \leqq \xi\left(\frac{1}{k}\right). \tag{2.3}$$

Note that we provide only upper bounds for the Hausdorff dimensions of the sets $\varrho(\phi^{(k)})$, $\varrho(\psi^{(k)})$ assigned to the set $\phi^{(k)}$ of minimal asymptotic bases of order $k \geqq 3$ and the set $\psi^{(k)}$ of maximal asymptotic nonbases of order $k \geqq 3$. In general, it is extremely difficult to construct examples of these kinds of extremal sets in additive number theory. Most of the known examples are constructed by one of the following two methods: They consist of numbers whose expansions to the base $g \geqq 2$ satisfy certain arithmetical restrictions, or they come from an application of the Erdős–Rényi probability method. Both constructions provide only a small set of examples, and it is difficult even to conjecture whether the Hausdorff dimensions of the sets $\varrho(\phi^{(k)})$ and $\varrho(\psi^{(k)})$ are positive or not for $k \geqq 3$.

3. To prove Theorem 1, first we will show that

$$\dim \varrho(\phi) \leqq \frac{\log 3}{\log 4}. \tag{3.1}$$

Let $\phi(a)$ denote the set of those sequences $\mathcal{A}$ for which $a \in \mathcal{A}$ holds, and there exist infinitely many positive integers n with the property that

$$a' + a'' = n, \quad a' \leqq a'', \quad a' \in \mathcal{A}, \quad a'' \in \mathcal{A}$$

holds if and only if $a' = a$, $a'' = n - a$.

Clearly,

$$\phi \subset \bigcap_{n=0}^{+\infty} \bigcup_{a=n}^{+\infty} \phi(a),$$

hence

$$\dim \varrho(\phi) \leqq \max_{a=0,1,\dots} \dim \varrho(\phi(a)). \tag{3.2}$$

In order to estimate $\dim \varrho(\phi(a))$, we need the following

LEMMA 1. *Let $\theta^{(j)} \in \Sigma$ for $j = 0, 1, 2, \dots$ and let*

$$\theta \subset \bigcap_{i=0}^{+\infty} \Big(\bigcup_{j=i}^{+\infty} \theta^{(j)} \Big). \tag{3.3}$$

For fixed j, let $\varphi(\theta^{(j)})$ denote the cardinality of $\theta_j^{(j)}$ (i.e., $\varphi(\theta^{(j)})$ is the number of those sequences $\mathcal{B} \subset \{0, 1, \dots, j\}$ for which there exists a sequence $\mathcal{A}$ such that $\mathcal{A}_j = \mathcal{B}$ and $\mathcal{A} \in \theta^{(j)}$ hold). Let $0 \leqq \mu < 1$ and let us assume that for all $\varepsilon > 0$, $j > j_0(\varepsilon)$ implies

$$\varphi(\theta^{(j)}) < 2^{(\mu+\varepsilon)j} \qquad (j > j_0(\varepsilon)). \tag{3.4}$$

Then

$$\dim \varrho(\theta) \leqq \mu.$$

This is Lemma 3 in [7].

Let $\phi^{(j)}(a)$ denote the set of the sequences $\mathcal{A}$ such that $a \in \mathcal{A}$, $j - a \in \mathcal{A}$, and the only representation of j in the form $a' + a'' = j$, $a' \leqq a''$, $a' \in \mathcal{A}$, $a'' \in \mathcal{A}$ is the representation with $a' = a$, $a'' = j - a$. Clearly,

$$\phi(a) = \bigcap_{i=0}^{+\infty} \Big(\bigcup_{j=i}^{+\infty} \phi^{(j)}(a) \Big)$$

so that (3.3) in Lemma 1 holds with $\phi(a)$ and $\phi^{(j)}(a)$ in place of θ and $\theta^{(j)}$, respectively. Now we will estimate $\varphi(\phi^{(j)}(a))$ (where $\varphi(\theta^{(j)})$ is the function defined in Lemma 1).

Assume that $j \geqq 2a$, $\mathcal{B} \subset \{0, 1, \dots, j\}$ and there exists a sequence $\mathcal{A}$ with $\mathcal{A} \in \phi^{(j)}(a)$ and $\mathcal{A}_j = \mathcal{B}$. Then by the definition of $\phi^{(j)}(a)$,

$$a \in \mathcal{B} \quad \text{and} \quad j - a \in \mathcal{B}, \tag{3.5}$$

(3.6)

if $0 \leqq i \leqq \left[\frac{j}{2}\right]$, $i \neq a$ then $i \in \mathcal{B}$ and $j - i \in \mathcal{B}$ cannot hold simultaneously.

By (3.6), to construct such a sequence $\mathcal{B}$, we have at most three possibilities to select numbers from a pair i, $j-i$ with $0 \leqq i \leqq \left[\frac{j}{2}\right]$, $i \neq a$. (In fact, for $i \neq j/2$ these three possibilities are: $i \notin \mathcal{B}$, $j-i \notin \mathcal{B}$; $i \in \mathcal{B}$, $j-i \notin \mathcal{B}$; and $i \notin \mathcal{B}$, $j-i \in \mathcal{B}$.) Furthermore, i with $0 \leqq i \leqq \left[\frac{j}{2}\right]$, $i \neq a$ can be chosen in $\left[\frac{j}{2}\right] \leqq \frac{j}{2}$ ways. (While by (3.5), both a and $j-a$ must belong to $\mathcal{B}$.) Thus the sequence $\mathcal{B}$ can be chosen in

$$\varphi(\phi^{(j)}(a)) \leqq 3^{j/2} = 2^{\frac{\log 3}{\log 4} j}$$

ways, so that for $j \geqq 2a$, (3.4) in Lemma 1 holds with $\phi^{(j)}(a)$ and $\frac{\log 3}{\log 4}$ in place of $\theta^{(j)}$ and μ, respectively. Thus Lemma 1 can be applied, and we obtain that

$$\dim \varrho(\phi^{(a)}) \leqq \frac{\log 3}{\log 4} \qquad (\text{for } a = 0, 1, \dots). \tag{3.7}$$

(3.2) and (3.7) imply (3.1).

4. Now we will show that

$$\dim \varrho(\phi) \geqq \frac{\log 3}{\log 4}. \tag{4.1}$$

This will complete the proof of Theorem 1 since (2.2) follows from (3.1) and (4.1).

We need two lemmas.

Lemma 2. *Let $0 < a < b$ and $\varepsilon > 0$. There exist a positive number $\delta = \delta(a, b, \varepsilon)$ and a positive integer $m_0(a, b, \varepsilon)$ such that*

$$m \geqq m_0(a, b, \varepsilon),$$

$$|u - bm| < \delta m$$

and

$$|v - am| < \delta m$$

imply that

$$2^{(b\xi(a/b) - \varepsilon)m} < \binom{u}{v} < 2^{(b\xi(a/b) + \varepsilon)m}.$$

This is Lemma 2 in [7], and it can be proved easily by using Stirling's formula.

LEMMA 3. *Suppose I_k $(k=1,2,\dots)$ is a linear set consisting of N_k closed intervals each of length δ_k. Let each interval of I_k contain*

$$n_{k+1} \geqq 2 \tag{4.2}$$

closed intervals of I_{k+1} so distributed that their minimum distance apart is

$$\varrho_{k+1} > \delta_{k+1}. \tag{4.3}$$

Let

$$P = \bigcap_{k=1}^{+\infty} I_k.$$

Then

$$\liminf_{k\to+\infty} N_{k+1}\varrho_{k+1}\delta_k^{s-1} > 0 \tag{4.4}$$

implies that

$$\dim P \geqq s. \tag{4.5}$$

This is Lemma 4 in [7] and it follows from a result of Eggleston [1]; see also [3].

To prove (4.1), first we will define a subset Γ of Σ.

Let us define the sequence $\mathcal{B}=\{b_0,b_1,\dots\}$ by the recursion

$$b_0=90, \quad b_{i+1}=b_i^2 \quad \text{for} \quad i=0,1,2,\dots,$$

and let us write

$$\mathcal{C}^{(i)}=\{b_i, b_i+b_{i-1}, b_i+2b_{i-1},\dots,b_{i+1}-b_i\} \quad \text{for} \quad i=1,2,\dots$$

(note that $b_{i-1}|b_i|b_{i+1}$, so that $b_{i-1}|b_{i+1}-b_i$) and

$$\mathcal{C}=\bigcup_{i=1}^{+\infty}\mathcal{C}^{(i)}.$$

Let Γ denote the set of those sequences $\mathcal{A}\in\Sigma$ for which the following conditions hold:

$$\text{If } 0\leqq n\leqq b_1, \text{ then } n\in\mathcal{A}. \tag{4.6}$$

$$\text{If } i=1,2,\dots,\quad b_i<n\leqq b_i+2b_{i-1}, \text{ then } n\in\mathcal{A}. \tag{4.7}$$

Moreover, write $\mathcal{A}=\{a_1,a_2,\dots\}$ (where $a_1<a_2<\dots$) and denote the i^{th} term of the sequence $a_1, a_1, a_2, a_1, a_2, a_3, a_1, \dots, a_1, a_2, \dots, a_k, a_1, \dots$ by a_i^*. Then clearly,

$$0\leqq a_i^* < b_{i-1} \qquad (\text{for } i=1,2,\dots).$$

(4.8)
$$\text{If } i=1,2,\ldots, b_{i+1}-b_i-2b_{i-1} \leqq n \leqq b_{i+1} \text{ and } n \neq b_{i+1}-a_i^*, \text{ then } n \notin \mathcal{A}.$$

$$\text{If } i=1,2,\ldots, \text{ then } b_{i+1}-a_i^* \in \mathcal{A}. \tag{4.9}$$

$$\text{If } i=1,2,\ldots, a\in\mathcal{A}, a \leqq \frac{b_{i+1}}{2}, a \neq a_i^*, \text{ then } b_{i+1}-a \notin \mathcal{A}. \tag{4.10}$$

(4.11)
$$\text{If } i=1,2,\ldots, c_j\in\mathcal{C}, b_i+2b_{i-1} < c_j \leqq b_{i+1}-b_i-2b_{i-1}, \text{ then } c_j-1\in\mathcal{A}.$$

$$\text{If } i=1,2,\ldots, c_j\in\mathcal{C}, -4\leqq k\leqq 4, k\neq -1, \tag{4.12}$$

$$b_i+2b_{i-1} < c_j+k \leqq b_{i+1}-b_i-2b_{i-1}, \text{ then } c_j+k\notin\mathcal{A}.$$

$$\text{If } i=1,2,\ldots, \; c_j\in\mathcal{C}, \; b_i+2b_{i-1}\leqq c_j < b_{i+1}-b_i-2b_{i-1}, \text{ then} \tag{4.13}$$

$$A(c_{j+1})-A(c_j)=\frac{c_{j+1}-c_j}{3}.$$

We will prove that the set Γ has the following three properties:

P1. If $\mathcal{A}\in\Gamma$, then $\mathcal{A}$ is an asymptotic basis of order 2.

P2. If $\mathcal{A}\in\Gamma$, then $\mathcal{A}$ is a *minimal* asymptotic basis of order 2.

P3.

$$\dim \varrho(\Gamma) \geqq \frac{\log 3}{\log 4}. \tag{4.14}$$

This will complete the proof of Theorem 1 since it follows from P1 and P2 that $\Gamma\subset\phi$, so that by P3,

$$\dim\varrho(\phi) \geqq \dim\varrho(\Gamma) \geqq \frac{\log 3}{\log 4}$$

which proves (4.1).

Proof of P1. We will show that if $\mathcal{A}\in\Gamma$ and $n>b_1$, then n can be represented in the form

$$a+a'=n, \quad a\in\mathcal{A}, \quad a'\in\mathcal{A}. \tag{4.15}$$

To prove this, define i by

$$b_i < n \leqq b_{i+1}. \tag{4.16}$$

We have to distinguish two cases.

Case 1.

$$b_i < n < b_{i+1}. \tag{4.17}$$

It follows from (4.6), (4.7) and (4.11) that for

$$-2b_{i-1} < j < b_{i+1} - b_i - 2b_{i-1} \tag{4.18}$$

we have

$$\mathcal{A} \cap [j, j + 2b_{i-1}) \neq \emptyset. \tag{4.19}$$

Writing $j = n - b_i - 2b_{i-1}$, (4.18) holds by (4.17), so that by (4.19) there exists an $a \in \mathcal{A}$ with

$$j = n - b_i - 2b_{i-1} \leqq a < j + 2b_{i-1} = n - b_i,$$

or, in equivalent form,

$$b_i < n - a \leqq b_i + 2b_{i-1}.$$

Thus writing $a' = n - a$, we have $a' \in \mathcal{A}$ by (4.7), so that a and a' provide a representation of n in form (4.15).

Case 2. $n = b_{i+1}$. Then in view of (4.9), (4.15) holds with $a = a_i^*$, $a' = b_{i+1} - a_i^*$.

PROOF of P2. By the definition of the sequence $a_1^*, a_2^*, \dots$, for all $a \in \mathcal{A}$ there exist infinitely many subscripts i with $a_i^* = a$. By (4.9) and (4.10), for all these i's the unique representation of b_{i+1} in form (4.15) (with b_{i+1} in place of n and with $a \leqq a'$) is the one with $a_i^* = a$ and $b_{i+1} - a_i^* = b_{i+1} - a$ in place of a and a', respectively, which proves the minimality of the basis $\mathcal{A}$.

PROOF of P3. The proof of (4.14) will be based on Lemma 3. Let us write

$$\mathcal{D} = \bigcup_{i=1}^{+\infty} \{b_i + 3b_{i-1} + 1, b_i + 4b_{i-1} + 1, \dots, b_{i+1} - b_i - 2b_{i-1} + 1\}$$

(note that $b_{i-1} | b_i | b_{i+1}$), and let us denote the j^{th} element of $\mathcal{D}$ by d_j so that $\mathcal{D} = \{d_1, d_2, \dots\}$, $d_1 < d_2 < \dots$. Then all the d_j's are of the form $c_k + 1$ (where $c_k \in \mathcal{C}$), and $c_k + 1 \in \mathcal{D}$ except for k's satisfying $b_{i+1} - b_i - 2b_{i-1} < c_k \leqq b_{i+1} + 2b_i$ for some i.

In order to use Lemma 3, first we have to define the sets I_k. For $k = 1, 2, \dots$, let

$$I_k = \bigcup_{\mathcal{A} \in \Gamma} \left[\varrho(\mathcal{A}_{d_k}), \varrho(\mathcal{A}_{d_k}) + \frac{1}{2^{d_k+1}}\right]$$

and

$$J_k = \bigcup_{\mathcal{A} \in \Gamma} \left[\varrho(\mathcal{A}_{d_k}), \varrho(\mathcal{A}_{d_k}) + \frac{1}{2^{d_k+1}}\right).$$

Clearly,

$$\varrho(\Gamma)=\bigcap_{k=1}^{+\infty} J_k. \tag{4.20}$$

Furthermore, if $\mathcal{A}\in\Gamma$, then by (4.12), $d_k\notin\mathcal{A}$ and $d_k+1\notin\mathcal{A}$ (for $k=1,2,\dots$) and hence

$$\varrho(\mathcal{A}_{d_k})+\frac{1}{2^{d_k+1}}\notin I_{k+1}.$$

Thus

$$\bigcap_{k=1}^{+\infty} I_k=\bigcap_{k=1}^{+\infty} J_k. \tag{4.21}$$

By (4.20) and (4.21),

$$\varrho(\Gamma)=\bigcap_{k=1}^{+\infty} I_k.$$

We have to show that the sets I_k satisfy (4.2), (4.3) and (4.4) $\left(\text{with } \dfrac{\log 3}{\log 4}-\varepsilon\right.$ $\left.\text{in place of } s\right)$.

To prove that (4.3) holds, we have to estimate ϱ_{k+1} $\left(\text{obviously, } \delta_{k+1}=\right.$ $\left.=\dfrac{1}{2^{d_{k+1}+1}}\right)$. If $\mathcal{A}\in\Gamma$, $\mathcal{A}'\in\Gamma$ and $\varrho(\mathcal{A}_{d_{k+1}})>\varrho(\mathcal{A}'_{d_{k+1}})$, then by (4.12),

$$\begin{aligned}\varrho(\mathcal{A}_{d_{k+1}})-(\varrho(\mathcal{A}'_{d_{k+1}})+\delta_{k+1})=&(\varrho(\mathcal{A}_{d_{k+1}})-\varrho(\mathcal{A}'_{d_{k+1}}))-\delta_{k+1}\geqq\\ &\geqq\frac{1}{2^{(d_{k+1}-2)+1}}-\delta_{k+1}=3\delta_{k+1},\end{aligned}$$

hence

$$\varrho_{k+1}\geqq 3\delta_{k+1} \tag{4.22}$$

which proves (4.3).

To show that also (4.2) holds, let us introduce the following notation: for some positive integer k and for a sequence $\mathcal{G}\in\Gamma_{d_k}$, let $f(\mathcal{G},k)$ denote the number of those sequences $\mathcal{H}$ for which $\mathcal{H}_{d_k}=\mathcal{G}$ and $\mathcal{H}\in\Gamma_{d_{k+1}}$ hold. In other words, $f(\mathcal{G},k)$ denotes the number of those sequences $\mathcal{H}\subset\{0,1,\dots,d_{k+1}\}$ for which there exists a sequence

$$\mathcal{A}\in\Gamma \tag{4.23}$$

with

$$\mathcal{A}_{d_k}=\mathcal{G} \tag{4.24}$$

and

$$\mathcal{A}_{d_{k+1}} = \mathcal{H}. \tag{4.25}$$

By the definition of the sequences $\mathcal{C}$ and $\mathcal{D}$, for $d_k > b_2$ there exists a uniquely determined positive integer j such that one of the following inequalities holds:

$$\begin{aligned} d_k = & b_j - b_{j-1} - 2b_{j-2} + 1 < b_j < d_{k+1} = b_j + 3b_{j-1} + 1 = \\ & = d_k + 4b_{j-1} + 2b_{j-2}, \end{aligned} \tag{4.26}$$

$$b_j + 3b_{j-1} + 1 \leqq d_k < d_{k+1} = d_k + b_{j-1} = \frac{b_{j+1}}{2} + 1 \tag{4.27}$$

$\left(\text{note that } \dfrac{b_{j+1}}{2} + 1 \in \mathcal{D}\right)$ or

$$\frac{b_{j+1}}{2} + 1 \leqq d_k < d_{k+1} = d_k + b_{j-1} \leqq b_{j+1} - b_j - 2b_{j-1} + 1. \tag{4.28}$$

Assume first that (4.27) holds. If (4.23) and (4.24) hold, then (4.23) and (4.25) imply by (4.12) that the sequence

$$\mathcal{H} \cap \{d_k + 1, d_k + 2, \dots, d_{k+1}\} \tag{4.29}$$

must not contain the elements $d_k + 1$, $d_k + 2$, $d_k + 3$, $d_{k+1} - 5$, $d_{k+1} - 4$, $d_{k+1} - 3$, $d_{k+1} - 1$, d_{k+1}, by (4.11) it must contain the element $d_{k+1} - 2$, and by (4.13) the number of its elements is

$$H(d_{k+1}) - H(d_k) = \frac{d_{k+1} - d_k}{3}$$

(and these are the only restrictions on the sequence (4.29)). Thus to obtain the sequence (4.29), we have to select $\frac{1}{3}(d_{k+1} - d_k) - 1$ $\left(= \frac{1}{3} b_{j-1} - 1\right)$ of the $d_{k+1} - d_k - 9$ integers

$$d_k + 4,\ d_k + 5, \dots,\ d_{k+1} - 6 \tag{4.30}$$

(note that $3 \mid d_{k+1} - d_k = b_{j-1}$). Thus

$$f(\mathcal{G}, k) = \binom{d_{k+1} - d_k - 9}{\frac{1}{3}(d_{k+1} - d_k) - 1} \quad \left(= \binom{b_{j-1} - 9}{\frac{1}{3} b_{j-1} - 1} \right) \qquad \text{(in case (4.27))}. \tag{4.31}$$

The case (4.26) can be studied similarly. In this case, the sequence (4.29) is uniquely determined except for the $\dfrac{b_{j-1}}{3} - 1$ integers chosen from the $b_{j-1} - 9$ integers

$$\{b_j + 2b_{j-1} + 5,\ b_j + 2b_{j-1} + 6, \dots, b_j + 3b_{j-1} - 5\}.$$

Thus in this case

$$f(\mathcal{G},k)=\binom{b_{j-1}-9}{\frac{1}{3}b_{j-1}-1} \qquad \text{(in case (4.26))}. \tag{4.32}$$

Assume now that (4.28) holds. Then the sequence (4.29) must satisfy one more condition. Namely, by (4.10), b_{j+1} cannot be represented in the form $h+h'=b_{j+1}$, where $b_j<h\leqq h'$, $h\in\mathcal{H}$, $h'\in\mathcal{H}$. Thus if

$$n\in\{d_k+4,\ d_k+5,\dots,d_{k+1}-6\} \tag{4.33}$$

and

$$b_{j+1}-n\in\mathcal{G} \tag{4.34}$$

hold, then n must not belong to the sequence (4.29). Let us put $m=b_{j+1}-n$. Then the conditions (4.33) and (4.34) can be rewritten in the form

$$m\in\{b_{j+1}-d_{k+1}+6,\ b_{j+1}-d_{k+1}+7,\dots,b_{j+1}-d_k-4\}, \tag{4.35}$$

$$m\in\mathcal{G}. \tag{4.36}$$

The construction of the sequence $\mathcal{C}$ implies that $b_{j+1}-d_{k+1}+1$ and $b_{j+1}-d_k+1$ are consecutive elements of $\left[b_j+2b_{j-1},\frac{b_{j+1}}{2}\right]\cap\mathcal{C}$. Let us write $c_r=b_{j+1}-d_{k+1}+1$ and $c_{r+1}=b_{j+1}-d_k+1$. Then (4.11), (4.12), (4.13), (4.23) and (4.24) imply that (4.35) and (4.36) hold for

$$\begin{aligned}G(b_{j+1}-d_k-4)-G(b_{j+1}-d_{k+1}+5)&=G(c_{r+1}-5)-G(c_r+4)=\\&=G(c_{r+1})-G(c_r)-1=\frac{c_{r+1}-c_r}{3}-1=\frac{d_{k+1}-d_k}{3}-1\end{aligned}$$

integers m. In other words, when we construct the sequence (4.29), $\frac{1}{3}(d_{k+1}-d_k)-1$ of the $d_{k+1}-d_k-9$ integers in (4.33) must be excluded. Thus we choose $\frac{1}{3}(d_{k+1}-d_k)-1$ of

$$(d_{k+1}-d_k-9)-\left(\frac{1}{3}(d_{k+1}-d_k)-1\right)=\frac{2}{3}(d_{k+1}-d_k)-8$$

integers:

$$f(\mathcal{G},k)=\binom{\frac{2}{3}(d_{k+1}-d_k)-8}{\frac{1}{3}(d_{k+1}-d_k)-1} \qquad \text{(in case (4.28))}. \tag{4.37}$$

(4.31), (4.32) and (4.37) show that in fact, $f(\mathcal{G},k)$ is independent of $\mathcal{G}$ thus we may put $n_{k+1}=f(\mathcal{G},k)$ and, in view of $d_{k+1}-d_k \geqq 9$ (for all k),

(4.38)
$$n_{k+1}=\binom{d_{k+1}-d_k-9}{\frac{1}{3}(d_{k+1}-d_k)-1} \quad \text{for} \quad b_j+3b_{j-1}+1 \leqq d_k < d_{k+1} \leqq \frac{b_{j+1}}{2}+1,$$

(4.39)
$$n_{k+1}=\binom{\frac{2}{3}(d_{k+1}-d_k)-8}{\frac{1}{3}(d_{k+1}-d_k)-1} \quad \text{for} \quad \frac{b_{j+1}}{2}+1 \leqq d_k < d_{k+1} \leqq b_{j+1}-b_j-2b_{j-1}+1$$

and

(4.40)
$$n_{k+1} \geqq 2 \quad \text{for all } k.$$

Finally, to prove that also (4.4) holds $\left(\text{with } s=\frac{\log 3}{\log 4}-\varepsilon\right)$, we have to give a lower estimate for N_{k+1} (for large k). Since

(4.41)
$$N_{k+1}=N_1 \prod_{i=1}^{k} \frac{N_{i+1}}{N_i}=N_1 \prod_{i=1}^{k} n_{i+1},$$

it suffices to estimate n_i for large i.

Let us fix some $\varepsilon>0$. Applying Lemma 2, we obtain from (4.38) and (4.39) that

(4.42)
$$n_{i+1}>2^{(\xi(1/3)-\varepsilon)(d_{i+1}-d_i)} \quad \text{for} \quad j>j_0(\varepsilon),$$
$$b_j+3b_{j-1}+1 \leqq d_i < d_{i+1} \leqq \frac{b_{j+1}}{2}+1$$

and

(4.43)
$$n_{i+1}>2^{(\frac{2}{3}\xi(1/2)-\varepsilon)(d_{i+1}-d_i)}=2^{(\frac{2}{3}-\varepsilon)(d_{i+1}-d_i)}$$
$$\text{for} \quad j \geqq j_1(\varepsilon), \quad \frac{b_{j+1}}{2}+1 \leqq d_i < d_{i+1} \leqq b_{j+1}-b_j-2b_{j-1}+1,$$

respectively.

By the definition of the sequence $\mathcal{D}$, $b_j+3b_{j-1}+1 \in \mathcal{D}$ and $\frac{b_{j+1}}{2}+1 \in \mathcal{D}$ for $j=1,2,\dots$; let us write $b_j+3b_{j-1}+1=d_{u_j}$ and $\frac{b_{j+1}}{2}+1=d_{v_j}$. We will show that if

(4.44)
$$j \geqq j_0(\varepsilon), \qquad d_{u_j}<d_t \leqq d_{u_{j+1}},$$

then

(4.45)
$$n_{u_j+1}n_{u_j+2}\dots n_t>2^{(\frac{\log 3}{\log 4}-3\varepsilon)(d_t-d_{u_j})}.$$

Assume first that $d_t \leqq \frac{b_{j+1}}{2} + 1 = d_{v_j}$. Then by (4.42),

$$n_{u_j+1} n_{u_j+2} \dots n_t > \prod_{i=u_j}^{t-1} 2^{(\xi(1/3)-\varepsilon)(d_{i+1}-d_i)} = \tag{4.46}$$
$$= 2^{(\xi(1/3)-\varepsilon)(d_t - d_{u_j})} \qquad \left(\text{for } \ d_t \leqq \frac{b_{j+1}}{2} + 1\right).$$

Since

$$\xi(1/3) = -\frac{1}{\log 2}\left(\frac{1}{3}\log\frac{1}{3} + \frac{2}{3}\log\frac{2}{3}\right) = \tag{4.47}$$
$$= -\frac{2}{3} + \frac{\log 3}{\log 2} > -\frac{\log 3}{2\log 2} + \frac{\log 3}{\log 2} = \frac{\log 3}{\log 4},$$

(4.46) implies (4.45) in this case.

Assume now that $\frac{b_{j+1}}{2} + 1 = d_{v_j} < d_t \leqq d_{u_{j+1}-1} = b_{j+1} - b_j - 2b_{j-1} + 1$. Then by (4.43) and (4.46),

$$n_{u_j+1} n_{u_j+2} \dots n_t = (n_{u_j+1} n_{u_j+2} \dots n_{v_j})(n_{v_j+1} n_{v_j+2} \dots n_t) >$$
$$> 2^{(\xi(1/3)-\varepsilon)(d_{v_j}-d_{u_j})} \prod_{i=v_j}^{t-1} 2^{(\frac{2}{3}-\varepsilon)(d_{i+1}-d_i)} = \tag{4.48}$$
$$= 2^{(\xi(1/3)-\varepsilon)(d_{v_j}-d_{u_j})} 2^{(\frac{2}{3}-\varepsilon)(d_t-d_{v_j})} =$$
$$= 2^{\xi(1/3)(d_{v_j}-d_{u_j}) + \frac{2}{3}(d_t - d_{v_j})} 2^{-\varepsilon(d_t - d_{u_j})}.$$

In view of (4.47), the exponent of the first factor can be estimated in the following way:

$$\xi\left(\frac{1}{3}\right)(d_{v_j} - d_{u_j}) + \frac{2}{3}(d_t - d_{v_j}) =$$
$$= \frac{\log 3}{\log 4}(d_t - d_{u_j}) + \left(\frac{\log 3}{\log 4} - \frac{2}{3}\right)(2d_{v_j} - d_t - d_{u_j}) = \tag{4.49}$$
$$= \frac{\log 3}{\log 4}(d_t - d_{u_j}) + \left(\frac{\log 3}{\log 4} - \frac{2}{3}\right)(b_{j+1} - b_j - 3b_{j-1} + 1 - d_t).$$

We have

$$b_{j+1} - b_j - 3b_{j-1} + 1 - d_t = d_{u_{j+1}-1} - b_{j-1} - d_t \geqq -b_{j-1}, \tag{4.50}$$

$$0 < \frac{\log 3}{\log 4} - \frac{2}{3} \tag{4.51}$$

and

(4.52) $$b_{j-1}=(b_j)^{1/2}=o(d_{v_j}-d_{u_j})=o(d_t-d_{u_j}).$$

It follows from (4.48), (4.49), (4.50), (4.51) and (4.52) that

(4.53) $$\begin{aligned} n_{u_j+1}n_{u_j+2}\dots n_t &> 2^{\frac{\log 3}{\log 4}(d_t-d_{u_j})-\varepsilon(d_t-d_{u_j})}2^{-\varepsilon(d_t-d_{u_j})}= \\ &=2^{(\frac{\log 3}{\log 4}-2\varepsilon)(d_t-d_{u_j})} \qquad (\text{for } d_{v_j}<d_t\leqq d_{u_{j+1}-1}) \end{aligned}$$

so that (4.45) holds also in this case.

Finally, by $d_{u_{j+1}}-d_{u_{j+1}-1}=o(d_{u_{j+1}}-d_{u_j})$, it follows from (4.40) and (4.53) that

$$\begin{aligned} n_{u_j+1}n_{u_j+2}\dots n_{u_{j+1}} &= (n_{u_j+1}n_{u_j+2}\dots n_{u_{j+1}-1})n_{u_{j+1}} > \\ &> n_{u_j+1}n_{u_j+2}\dots n_{u_{j+1}-1} > 2^{(\frac{\log 3}{\log 4}-2\varepsilon)(d_{u_{j+1}-1}-d_{u_j})} = \\ &= 2^{(\frac{\log 3}{\log 4}-2\varepsilon)(d_{u_{j+1}}-d_{u_j})-(\frac{\log 3}{\log 4}-2\varepsilon)(d_{u_{j+1}}-d_{u_{j+1}-1})} > \\ &> 2^{(\frac{\log 3}{\log 4}-3\varepsilon)(d_{u_{j+1}}-d_{u_j})} \end{aligned}$$

which completes the proof of (4.45) (assuming that (4.44) holds).

Starting out from (4.41) and using (4.45), N_{k+1} can be estimated easily. Define the non-negative integer q by

$$u_q<k+1\leqq u_{q+1}$$

(where u_j is defined by $d_{u_j}=b_j+3b_{j-1}+1$). Then we get from (4.41) and (4.45) that for $k\geqq k_0(\varepsilon)$,

(4.54) $$\begin{aligned} N_{k+1} &= N_1\prod_{i=1}^{k} n_{i+1} = \\ &= N_1\prod_{i=1}^{u_{j_0}-1} n_{i+1}\prod_{j=j_0}^{q-1}\Big(\prod_{i=u_j+1}^{u_{j+1}} n_i\Big)\prod_{i=u_q+1}^{k+1} n_i > \\ &> \prod_{j=j_0}^{q-1} 2^{(\frac{\log 3}{\log 4}-3\varepsilon)(d_{u_{j+1}}-d_{u_j})}2^{(\frac{\log 3}{\log 4}-3\varepsilon)(d_{k+1}-d_{u_q})} = \\ &= 2^{(\frac{\log 3}{\log 4}-3\varepsilon)(d_{k+1}-d_{u_{j_0}})} > 2^{(\frac{\log 3}{\log 4}-4\varepsilon)}d_{k+1}. \end{aligned}$$

Let us put $s=\dfrac{\log 3}{\log 4}-5\varepsilon$. Then it follows from (4.22) and (4.54) that for $k\geqq k_0(\varepsilon)$,

$$\begin{aligned} N_{k+1}\varrho_{k+1}\delta_k^{s-1} &> 2^{(\frac{\log 3}{\log 4}-4\varepsilon)d_{k+1}}3\delta_{k+1}\delta_k^{s-1} = \\ &= 3\cdot 2^{(s+\varepsilon)d_{k+1}}\frac{1}{2^{d_{k+1}+1}}\cdot\frac{1}{2^{(d_k+1)(s-1)}} = \\ &= 3\cdot 2^{\varepsilon d_{k+1}-(1-s)(d_{k+1}-d_k)-s}. \end{aligned}$$

By the construction of the sequence $\mathcal{D}$, $d_{k+1} - d_k = o(d_{k+1})$. Thus for $k \geqq \geqq k_1(\varepsilon)$,

$$N_{k+1}\varrho_{k+1}\delta_k^{s-1} > 3 \cdot 2^{\varepsilon d_{k+1} - (\varepsilon/2)d_{k+1}} = 3 \cdot 2^{(\varepsilon/2)d_{k+1}}.$$

Hence

$$\lim_{k \to +\infty} \inf N_{k+1}\varrho_{k+1}\delta_k^{s-1} = +\infty$$

which proves (4.4).

Summarizing: we have proved that all the conditions in Lemma 3 hold with $P = \varrho(\Gamma)$, $s = \frac{\log 3}{\log 4} - 5\varepsilon$. Thus applying Lemma 3, we get that

$$\dim \varrho(\Gamma) \geqq \frac{\log 3}{\log 4} - 5\varepsilon.$$

This inequality holds for all $\varepsilon > 0$ which implies (4.14) and thus also (4.1), and this completes the proof of Theorem 1.

5. Proof of Theorem 2. We need the following lemma:

Lemma 4. *Let $0 \leqq \alpha \leqq 1/2$ and let $\Gamma(\alpha)$ denote the set of the sequences $\mathcal{A}$ for which $\underline{d}(\mathcal{A}) \leqq \alpha$ holds. Then*

$$\dim \varrho(\Gamma(\alpha)) = \xi(\alpha).$$

This is a trivial consequence of a result of Volkmann [8].

Furthermore, we need the following result of the authors [6]:

Lemma 5. *If $\mathcal{A}$ is a minimal asymptotic basis of order k, then we have $\underline{d}(\mathcal{A}) \leqq \frac{1}{k}$.*

Combining Lemmas 4 and 5, we obtain Theorem 2.

6. Proof of Theorem 3. The upper bound

$$\varrho(\psi) \leqq \frac{\log 3}{\log 4}$$

follows from (2.1) (since $\psi \subset \Delta$). Thus it suffices to show that

$$\varrho(\psi) \geqq \frac{\log 3}{\log 4}. \tag{6.1}$$

This will be proved in a similar way as (4.1) in the proof of Theorem 1, i.e., the proof will be based on Lemma 3. By the similarity of the two proofs, we will give only a sketch of the proof.

Again, we define the sequence $\mathcal{B} = \{b_0, b_1, \dots\}$ by the recursion

$$b_0 = 90, \quad b_{i+1} = b_i^2 \quad \text{for} \quad i = 0, 1, 2, \dots,$$

and we write

$$\mathcal{C}^{(i)} = \{b_i, b_i + b_{i-1}, b_i + 2b_{i-1}, \dots, b_{i+1} - b_{i-1}\} \quad \text{for} \quad i = 1, 2, \dots$$

(note that $b_{i-1} | b_i | b_{i+1}$ so that $b_{i-1} | b_{i+1} - b_i$) and

$$\mathcal{C} = \bigcup_{i=1}^{+\infty} \mathcal{C}^{(i)}.$$

Let Γ^* denote the set of those sequences $\mathcal{A} \in \Sigma$ for which the following conditions hold:

(6.2) If $0 \leqq n \leqq b_1$, then $n \in \mathcal{A}$.

(6.3) If $i = 1, 2, \dots, b_i < n \leqq b_i + 2b_{i-1}$, then $n \in \mathcal{A}$.

(6.4) If $i = 1, 2, \dots, c_j \in \mathcal{C}$, $b_i + 2b_{i-1} < c_j \leqq b_{i+1} - b_i - 2b_{i-1}$, then

$$c_j - 1 \in \mathcal{A}.$$

(6.5) If $i = 1, 2, \dots, c_j \in \mathcal{C}$, $-4 \leqq k \leqq 4$, $k \neq -1$,

$$b_i + 2b_{i-1} < c_j + k \leqq b_{i+1} - b_i - 2b_{i-1}, \text{ then } c_j + k \notin \mathcal{A}.$$

(6.6) If $i = 1, 2, \dots$, $b_{i+1} + 2b_{i-1} \leqq c_j < b_{i+1} - b_i - 2b_{i-1}$, then

$$A(c_{j+1}) - A(c_j) = \frac{c_{j+1} - c_j}{3}.$$

(6.7) If $i = 1, 2, \dots$, $b_i + 2b_{i-1} < a \leqq \frac{b_{i+1}}{2}$, $a \in \mathcal{A}$, then $b_{i+1} - a \notin \mathcal{A}$.

(6.8) If $i = 1, 2, \dots$, $b_{i+1} - b_i - 2b_{i-1} < n \leqq b_{i+1}$, then $n \in \mathcal{A}$ if and only if

$$b_{i+1} - n \notin \mathcal{A}.$$

Now we will sketch the proof of that that Γ^* has the following three properties:

P1. If $\mathcal{A} \in \Gamma^*$, then $\mathcal{A}$ is *not* an asymptotic basis of order 2.

P2. If $\mathcal{A} \in \Gamma^*$, then $\mathcal{A}$ is a *maximal* asymptotic nonbasis of order 2.

P3.

$$\dim \varrho(\Gamma^*) \geqq \frac{\log 3}{\log 4}.$$

This will complete the proof of Theorem 3, since it follows from P1 and P2 that $\Gamma^* \subset \psi$, so that by P3,

$$\dim \varrho(\psi) \geqq \dim \varrho(\Gamma^*) \geqq \frac{\log 3}{\log 4}$$

which proves (6.1).

PROOF of P1. It follows from (6.7) and (6.8) that for $\mathcal{A}\in\Gamma^*$ and $i=$ $=1,2,\dots,$ b_{i+1} cannot be represented in the form

$$a+a'=b_{i+1},\quad a\in\mathcal{A},\quad a'\in\mathcal{A}.$$

PROOF of P2. First we will show that if $n\notin\mathcal{B}$, then n can be represented in the form

$$a+a'=n,\quad a\in\mathcal{A},\quad a'\in\mathcal{A}. \tag{6.9}$$

If $n\leqq b_1$, then by (6.2), (6.9) holds with $a=0$, $a'=n$. Assume now that $n>b_1$ and define i by

$$b_i<n<b_{i+1}. \tag{6.10}$$

It follows from (6.2), (6.3) and (6.4) that for

$$-2b_{i-1}<j<b_{i+1}-b_i-2b_{i-1} \tag{6.11}$$

we have

$$\mathcal{A}\cap[j,j+2b_{i-1})\neq\emptyset. \tag{6.12}$$

Writing $j=n-b_i-2b_{i-1}$, (6.11) holds by (6.10), so that by (6.12) there exists an $a\in\mathcal{A}$ with

$$j=n-b_i-2b_{i-1}\leqq a<j+2b_{i-1}=n-b_i,$$

or, in equivalent form,

$$b_i<n-a\leqq b_i+2b_{i-1}.$$

Thus writing $a'=n-a$, we have $a'\in\mathcal{A}$ by (6.3), so that these a and a' provide a representation of n in form (6.9).

To complete the proof of property P2, we have to show that if u is a non-negative integer with $u\notin\mathcal{A}$, then $\mathcal{A}^*=\mathcal{A}\cup\{u\}$ is an asymptotic basis of order 2, in other words, any large non-negative integer n can be represented in the form

$$a+a'=n,\quad a\in\mathcal{A}^*,\quad a'\in\mathcal{A}^*. \tag{6.13}$$

If $n\notin\mathcal{B}$, $n=b_0$ or $n=b_1$, then as we showed above, n has a representation of form (6.9). By $\mathcal{A}\subset\mathcal{A}^*$, this is also a representation of form (6.13). Finally, if $n\in\mathcal{B}$ and n is large enough in terms of u, then $u\notin\mathcal{A}$ implies by (6.8) that $n-u\in\mathcal{A}$. Thus writing $a=u$, $a'=n-u$, (6.13) holds which completes the proof of P2.

PROOF of P3. By using Lemma 3, this can be proved in the same way as P3 in the proof of Theorem 1; we leave the details to the reader.

7. PROOF OF THEOREM 4. For $\mathcal{A}\in\Sigma$ and $g=1,2,\dots$, define the sequence $\mathcal{A}^{[g]}$ by $\mathcal{A}^{[g]}=\bigcup_{a\in\mathcal{A}}\{a,a+g,a+2g,\dots\}$ (in other words, $\mathcal{A}^{[g]}$ is the set of the integers n for which $n\equiv a(\operatorname{mod} g)$, $n\geqq a$ for some $a\in\mathcal{A}$). We need the following lemma:

LEMMA 6. *Let $\mathcal{A}$ be a maximal asymptotic nonbasis of order k such that $\underline{d}(\mathcal{A})=\frac{1}{k}+\delta$ for some $\delta>0$. Then there exists a positive integer $g\leqq\frac{k-2}{k\delta}$ such that $\mathcal{A}=\mathcal{A}^{[g]}$.*

PROOF OF LEMMA 6. Since $\underline{d}(k\mathcal{A})\leqq 1<k\underline{d}(\mathcal{A})$, it follows from Kneser's theorem [5] (see also [4]) that there is an integer $g\geqq 1$ such that $k\mathcal{A}\sim k\mathcal{A}^{[g]}$ and

$$\underline{d}(k\mathcal{A})\geqq k\underline{d}(\mathcal{A})-\frac{k-1}{g}=1+k\delta-\frac{k-1}{g}.$$

Since $k\mathcal{A}^{[g]}$ is a union of congruence classes mod g and $k\mathcal{A}^{[g]}\sim k\mathcal{A}\not\sim\mathbb{N}$ ($\mathbb{N}$ denotes the set of positive integers), it follows that

$$1+k\delta-\frac{k-1}{g}\leqq\underline{d}(k\mathcal{A})=d(k\mathcal{A})\leqq 1-\frac{1}{g}$$

and so

$$g\leqq\frac{k-2}{k\delta}.$$

Let $u\in\mathbb{N}\setminus\mathcal{A}$. Then $\mathcal{A}$ *maximal* implies that $\mathcal{B}=\mathcal{A}\cup\{u\}$ is an asymptotic basis of order k, and so $k\mathcal{B}\sim\mathbb{N}$. Since $k\mathcal{A}^{[g]}\not\sim\mathbb{N}$, it follows that $u\notin\mathcal{A}^{[g]}$, and so $\mathcal{A}^{[g]}\setminus\mathcal{A}=\emptyset$; that is, $\mathcal{A}=\mathcal{A}^{[g]}$, which completes the proof of the lemma.

Note that it is difficult to construct maximal asymptotic nonbases that are *not* unions of congruence classes. The first examples were found by Erdős and Nathanson [2]. The argument in Lemma 6, which depends on Kneser's theorem, was also used by Nathanson and Sárközy [6] to prove that if $\mathcal{A}$ is a minimal asymptotic basis of order k, then $\underline{d}(\mathcal{A})\leqq\frac{1}{k}$.

In order to prove (2.3), for some $\varepsilon>0$ write $\psi^{(k)}$ in the form

$$\psi^{(k)}=\psi_1^{(k)}\cup\psi_2^{(k)} \tag{7.1}$$

where $\psi_1^{(k)}$ is the set of the sequences $\mathcal{A}$ with $\mathcal{A}\in\psi^{(k)}$ and $\underline{d}(\mathcal{A})\leqq\frac{1}{k}+\varepsilon$, while $\psi_2^{(k)}$ is the set of the sequences $\mathcal{A}$ with $\mathcal{A}\in\psi^{(k)}$ and

$$\underline{d}(\mathcal{A})>\frac{1}{k}+\varepsilon. \tag{7.2}$$

(7.1) implies that

$$\dim \varrho(\psi^{(k)}) = \max(\dim \varrho(\psi_1^{(k)}), \dim \varrho(\psi_2^{(k)})). \tag{7.3}$$

By Lemma 4, we have

$$\dim \varrho(\psi_1^{(k)}) \leqq \xi\Big(\frac{1}{k} + \varepsilon\Big). \tag{7.4}$$

On the other hand, by Lemma 6, if $\mathcal{A}$ satisfies (7.2), then it is of the form

$$\mathcal{A} = \bigcup_{i=1}^{k} \{a_i, a_i + g, a_i + 2g, \dots\}$$

so that the $k+1$-tuple $(g, a_1, a_2, \dots, a_k)$ uniquely determines $\mathcal{A}$. But the set of all finite n-tuples of integers is countable, and hence the set $\psi_2^{(k)}$ is countable. This implies that

$$\dim \varrho(\psi_2^{(k)}) = 0. \tag{7.5}$$

It follows from (7.3), (7.4) and (7.5) that

$$\dim \varrho(\psi^{(k)}) \leqq \xi\Big(\frac{1}{k} + \varepsilon\Big).$$

This holds for all $\varepsilon > 0$ which implies (2.3).

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DEPARTMENT OF MATHEMATICS
LEHMAN COLLEGE
CITY UNIVERSITY OF NEW YORK
BRONX, NY 10468
U.S.A

MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

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CHAIN CONDITIONS ON DIRECT SUMMANDS

N. V. SANH

Abstract

For an infinite cardinal $\aleph$ and a unitary right CS-module M over a ring R with identity, we show that if every ascending chain of direct summands of $M/\mathrm{Soc}\,(M)$ has cardinality less than $\aleph$ then M is a direct sum of a semisimple module and a module in which every independent family of submodules has cardinality less than $\aleph$.

1. Introduction

Chain conditions on a module appear in many contexts. Let $\aleph$ be an infinite cardinal number. A partially ordered set $\mathcal{S}$ has the ascending $\aleph$-chain condition (briefly, $\aleph$-acc) if and only if for every ordinal κ such that there is a chain $\{N_\alpha, \alpha<\kappa\}$ of subsets of $\mathcal{S}$ with $N_\beta<N_\alpha$ for all $\beta<\alpha\in\kappa$, we have $|\kappa|<\aleph$. Thus the usual acc (ascending chain condition) is $\aleph_0$-acc. Modules satisfying ascending $\aleph$-chain condition have been studied by many authors (see [10] and authors cited therein). In this note we follow this investigation and show that for a CS-module M, if $M/\mathrm{Soc}(M)$ has $\aleph$-acc on direct summands then M is a direct sum of a semisimple module and a module in which every independent family of submodules has cardinality less than $\aleph$. For $\aleph=\aleph_0$ we get a result of N. V. Dung recently obtained in [9].

2. Preliminaries

Throughout this paper R is an associative ring with identity and Mod-R is the category of unitary right R-modules. M will denote a right R-module and $\aleph$ an infinite cardinal. For any set $\mathcal{S}$, $|\mathcal{S}|$ will denote the cardinality of $\mathcal{S}$. A complement submodule of a module M is a submodule N of M for which there is a submodule L of M such that N is maximal with respect to $L\cap N=0$. A module is called a CS-module if every complement submodule is a direct summand. A family of submodules $\{N_\iota; \iota\in\mathcal{I}\}$ is called independent if the sum $\sum\limits_{\iota\in\mathcal{I}} N_\iota$ is direct and all N_ι are non-zero. From the definition, we

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see that M satisfies $\aleph$-acc on sets of independent submodules if and only if any independent family of submodules has cardinality less than $\aleph$, and M satisfies $\aleph$-acc on direct summands if M does not contain a direct sum $\bigoplus_{i\in\Lambda} A_i$ of submodules A_i with $|\Lambda| \geqq \aleph$, in which $\bigoplus_{i\in\Lambda'} A_i$ is a direct summand of M for every subset Λ' of Λ.

LEMMA 1 ([6, Folgerung 9.1.5]). *If S is the socle of a direct sum $\bigoplus_{\alpha\in\kappa} N_\alpha$, then $S = \bigoplus_{\alpha\in\kappa} \operatorname{soc}(N_\alpha)$. Hence*

$$(\bigoplus_{\alpha\in\kappa} N_\alpha)/S \cong \bigoplus_{\alpha\in\kappa} (N_\alpha/\operatorname{Soc}(N_\alpha)). \qquad \square$$

LEMMA 2. *Let M be a module and $S = \operatorname{Soc}(M)$, then:*
(1) *If A and B are submodules of M with $A \cap B = 0$, then*

$$((A+S)/S) \cap ((B+S)/S) = 0;$$

(2) *If A is a direct summand of M, then $(A+S)/S$ is a direct summand of M/S;*
(3) *If $\bigoplus_{i\in I} A_i$ is a direct sum of submodules of M, then $\bigoplus_{i\in I}((A_i+S)/S)$ is also a direct sum of submodules in M/S.*

PROOF. This proof is given by N. V. Dung in [9] and we present it here for completeness.
Let $f\colon M \to M/S$ be the canonical map.
(1) Suppose that A and B are submodules of M with $A \cap B = 0$. Set $\overline{V} = f(A) \cap f(B)$. Then there exists a submodule V of A such that $f(V) = \overline{V}$. Clearly $V \subseteqq B + S = B \oplus T$ for some submodule T of S. Since $V \cap B = 0$, it follows that V is isomorphic to a submodule of T. Hence $V \subseteqq S$ which implies that $\overline{V} = 0$. Therefore we have $f(A) \cap f(B) = 0$.
(2) Let A be a direct summand of M. Then $M = A \oplus B$ for some submodule B of M. Clearly $M/S = f(A) + f(B)$. By (1) we have $f(A) \cap f(B) = 0$. Thus $f(A)$ is a direct summand of M/S.
(3) This is an immediate consequence of (1). $\square$

3. Results

Before stating the main theorem, we need some lemmas, the first one is straightforward.

LEMMA 3. *If M satisfies $\aleph$-acc on direct summands, then every direct summand of M also has $\aleph$-acc on direct summands.* $\square$

For an ordinal κ, we consider the following property:

A right R-module M is said to satisfy property (P) if for any ordinal κ and for any independent family $\{N_\alpha, \alpha \in \kappa\}$ of submodules of M, there exists an ascending chain $\{M_\alpha, \alpha \in \kappa\}$ of direct summands of M such that $\bigoplus_{\alpha \leqq \beta} N_\alpha$ is essential in M_β for every $\beta \in \kappa$.

LEMMA 4. *Let M be a right R-module. Then M is a CS-module if and only if M has property (P).*

PROOF. One direction is clear.

Suppose now that M is CS. Let $\{N_\alpha, \alpha \in \kappa\}$ be an independent family of submodules of M, where κ is an ordinal. We may assume that all N_α are direct summands of M. We define the system $\{M_\alpha, \alpha \in \kappa\}$ by transfinite induction. For the least element α_0 of κ, let $M_{\alpha_0} = N_{\alpha_0}$. If $\gamma \in \kappa$ is not a limit ordinal, then $\gamma = \beta + 1$ for some $\beta \in \kappa$, so we take M_γ to be a maximal essential extension of $M_\beta \oplus N_{\beta+1}$ in M. If $\lambda \in \kappa$ is a limit ordinal, then M_λ is defined to be a maximal essential extension of $(\bigcup_{\gamma<\lambda} M_\gamma) \oplus N_\lambda$ in M. From this we obtain an ascending chain of direct summands of M:

$$\{M_\alpha, \alpha \in \kappa\}$$

such that $\bigoplus_{\gamma \leqq \beta} N_\gamma$ is essential in M_β for every $\beta \in \kappa$. This shows that M has (P). □

LEMMA 5. *Let M be a CS-module. Then M satisfies $\aleph$-acc on direct summands if and only if any independent family of submodules of M has cardinality less than $\aleph$.*

PROOF. Let κ be an ordinal and $\{N_\alpha, \alpha \in \kappa\}$ be an independent family of submodules of M. For each $\alpha \in \kappa$ let X_α be a maximal essential extension of N_α in M. Then the set $\{X_\alpha, \alpha \in \kappa\}$ is an independent family of direct summands of M, since M is CS. By Lemma 4, there exists an ascending chain $\{M_\alpha, \alpha \in \kappa\}$ of direct summands of M such that $\bigoplus_{\alpha \leqq \beta} X_\alpha$ is essential in M_β for every $\beta \in \kappa$. Since M has $\aleph$-acc on direct summands, we must have

$$|\kappa| < \aleph.$$

The converse is from Lemma 4. □

The main result of this note is the following theorem:

THEOREM 6. *Let M be a CS-module and $\aleph$ be an infinite cardinal number. If $M/\mathrm{Soc}(M)$ has $\aleph$-acc on direct summands then M is a direct sum $K \oplus L$ where L is semisimple and K is a module in which any independent family of submodules has cardinality less than $\aleph$.*

PROOF. Let M be a CS-module such that $M/\mathrm{Soc}(M)$ has $\aleph$-acc on direct summands. Put $S = \mathrm{Soc}(M)$. Then $M = M_1 \oplus M_2$ where S is essential in M_1. Clearly, M_2 is isomorphic to a direct summand of M/S. Thus, by Lemma 3, M_2 has $\aleph$-acc on direct summands. Since M_2 is CS, then any independent family of submodules of M_2 has cardinality less than $\aleph$ by Lemma 5. Therefore, without loss of generality, we may assume that M has essential socle. If $M = \mathrm{Soc}(M)$ so the statement is trivially true. Hence we suppose that $M \neq \mathrm{Soc}(M)$, i.e. M is not semisimple.

Let $\mathcal{F} = \{N_\alpha, \alpha \in \kappa\}$ be an independent family of all direct summands of M such that for every $\alpha \in \kappa$ (where κ is an ordinal) $\mathrm{Soc}(N_\alpha) \neq N_\alpha$ and $\mathrm{Soc}(N_\alpha)$ is generated by fewer than $\aleph$ elements. If we denote by K the maximal essential extension of $\sum_{\alpha\in\kappa} N_\alpha$ in M, then $M = K \oplus L$.

First, we show that every independent family of submodules of K has cardinality less than $\aleph$. Since M is CS, then by Lemma 4 there exists an ascending chain $\{M_\alpha, \alpha \in \kappa\}$ of direct summands of M such that $\bigoplus_{\alpha\leqq\beta} N_\alpha$ is essential in M_β for every $\beta \in \kappa$. By Lemma 2, $\{\overline{M}_\alpha, \alpha \in \kappa\}$ is an ascending chain of direct summands of M/S, where $\overline{M}_\alpha = (M_\alpha + S)/S$. By hypothesis, we have $|\kappa| < \aleph$.

By Lemma 1, $\mathrm{Soc}(K) = \mathrm{Soc}(\sum_{\alpha\in\kappa} N_\alpha) = \sum_{\alpha\in\kappa} \mathrm{Soc}(N_\alpha)$ and therefore $\mathrm{Soc}(K)$ is generated by fewer than $\aleph$ elements. Since $\mathrm{Soc}(K)$ is semisimple and essential in K, then every submodule of K has non-zero socle, therefore every independent family of submodules of K has cardinality less than $\aleph$.

Now, it remains to show that L is semisimple. Note that $\mathrm{Soc}(L)$ is essential in L. Assume on the contrary that L is not semisimple. Then there exists a finitely generated submodule E of L which is not semisimple. Let H be a maximal essential extension of E in L, we have $\mathrm{Soc}(H) = \mathrm{Soc}(E) \neq H$. If $\mathrm{Soc}(E)$ is generated by fewer than $\aleph$ elements then H is in $\mathcal{F}$. This contradicts the choice of $\mathcal{F}$. Hence $\mathrm{Soc}(E)$ is not generated by fewer than $\aleph$ elements. (If a module is not generated by fewer than $\aleph$ elements we will roughly say below that it is $\aleph$-generated.)

We will show that $\mathrm{Soc}(E)$ must contain an $\aleph$-generated submodule which is a complement in H. Since $\aleph.\aleph = \aleph$, there exists an independent family $\{T_\beta, \beta \in \Lambda\}$ of cardinality $\aleph$ and for each $\beta \in \Lambda$, $T_\beta \subset \mathrm{Soc}(E)$ and T_β is $\aleph$-generated. Since all T_β are independent, we may assume that Λ is an ordinal and since H is a CS-module, there exists an ascending chain $\{C_\gamma, \gamma \in \Lambda\}$ of direct summands of H such that $\bigoplus_{\beta\leqq\gamma} T_\beta$ is essential in C_γ for every $\gamma \in \Lambda$ (see Lemma 4).

Hence, $\overline{C}_\gamma = (C_\gamma + \mathrm{Soc}(H))/\mathrm{Soc}(H)$ is a direct summand of $H/\mathrm{Soc}(H)$ for every $\gamma \in \Lambda$ (Lemma 2) and therefore $\{\overline{C}_\gamma, \gamma \in \Lambda\}$ is an ascending chain of direct summands of $H/\mathrm{Soc}(H)$. Since $H/\mathrm{Soc}(H)$, being isomorphic to a direct summand of $L/\mathrm{Soc}(L)$, also has $\aleph$-acc on direct summands (Lemma 3) and $|\Lambda| = \aleph$ there exists a $\theta \in \Lambda$ such that

$$\overline{C}_{\theta+1} = \overline{C}_\theta. \tag{1}$$

Since $\mathrm{Soc}(H) = (\mathrm{Soc}(H) \cap C_{\theta+1}) \oplus U$ for some $U \subseteq \mathrm{Soc}(H)$ we have $C_{\theta+1} + \mathrm{Soc}(H) = C_{\theta+1} \oplus U$. Since $\mathrm{Soc}(H) = \mathrm{Soc}(C_{\theta+1}) \oplus U$ and $C_\theta \oplus T_{\theta+1}$ is essential in $C_{\theta+1}$ by our construction, we have

$$C_\theta + \mathrm{Soc}(H) = C_\theta \oplus T_{\theta+1} \oplus U. \tag{2}$$

From this and (1) we have

$$(C_\theta \oplus T_{\theta+1} \oplus U)/\mathrm{Soc}(H) = (C_{\theta+1} \oplus U)/\mathrm{Soc}(H). \tag{3}$$

Since $C_\theta \oplus T_{\theta+1} \subseteq C_{\theta+1}$ it follows $C_\theta \oplus T_{\theta+1} = C_{\theta+1}$. This shows that $T_{\theta+1}$ is a direct summand of H and hence of E. But this is impossible because E is finitely generated and $T_{\theta+1}$ is $\aleph$-generated by our assumption. This contradiction shows that every finitely generated submodule of L must be semisimple. Therefore L is semisimple and the proof is complete. □

REMARK. The following example of B. L. Osofsky shows that the converse is not true. Let $\Lambda = \mathrm{End}(V_F)$ for V an $\aleph_0$-dimensional vector space over the field F, and $S = \mathrm{Soc}(\Lambda)$. Then Λ_Λ is injective and every independent family of submodules of Λ_Λ has cardinality less than $\aleph_1$. Therefore every independent family of submodules of S has cardinality less than $\aleph_1$, and we have S is essential in Λ. Since Λ is a von Neumann regular ring, so is $\Lambda/\mathrm{Soc}(\Lambda)$. Observe that the direct summands of a von Neumann regular ring are the cyclic right ideals, an intersection of cyclic right ideals is again cyclic, and a countably generated right ideal is a direct sum of cyclics. Now let I be any countably but not finitely generated right ideal of $\Lambda/\mathrm{Soc}(\Lambda)$. Then I is not equal to the cyclic module of $\Lambda/\mathrm{Soc}(\Lambda)$. Express I as a countable direct sum of non-zero modules

$$I = \bigoplus_{\alpha=0}^{\infty} x_\alpha(\Lambda/\mathrm{Soc}(\Lambda)),$$

where the x_α are projections onto infinite dimensional subspaces of V. By independence modulo linear transformations of finite rank and the fact that

$$x_\alpha A \cap \sum_{\beta=0}^{\alpha-1} x_\beta \Lambda$$

is cyclic and hence a direct summand, we may select the x_α so that the sum $\sum_{\alpha\in\omega} x_\alpha \Lambda$ is direct in Λ_Λ. Now replace each x_α with a projection e_α onto a subspace of its image with codimension 1 and whose kernel contains $\ker(x_\alpha)$. Then $\mathrm{rank}(x_\alpha - e_\alpha) = 1$. We thus have

$$\sum_{\alpha\in\omega} x_\alpha(\Lambda/\mathrm{soc}(\Lambda)) = \sum_{\alpha\in\omega} e_\alpha(\Lambda/\mathrm{Soc}(\Lambda))$$

and $W = \sum_{\alpha} e_\alpha V$ is of infinite codimension in the vector space V. If $\varepsilon \in \Lambda$ projects V onto W, then $\varepsilon(\Lambda/\mathrm{Soc}(\Lambda))$ is a direct summand of $\Lambda/\mathrm{Soc}(\Lambda)$ with

$$(1-\varepsilon)(\Lambda/\mathrm{Soc}(\Lambda)) \neq 0 \quad \text{and} \quad \varepsilon(\Lambda/\mathrm{Soc}(\Lambda)) \supset \sum_{\alpha} x_\alpha(\Lambda/\mathrm{Soc}(\Lambda)).$$

Now let $\mathcal{I}$ denote the poset of chains

$$\mathcal{I} = \{x_\alpha(\Lambda/\mathrm{Soc}(\Lambda)) \mid \alpha \in \Omega,\ \Omega \text{ an ordinal, } x_\alpha V \text{ has codimension } \infty, \\ \mid x_\alpha(\Lambda/\mathrm{Soc}(\Lambda)) \subset x_\beta(\Lambda/\mathrm{Soc}(\Lambda)) \text{ for } \alpha < \beta \in \Omega\}$$

ordered by "is an initial segment of". Observe that the empty chain is in $\mathcal{I}$ and a union of a chain in $\mathcal{I}$ is again in $\mathcal{I}$, so Zorn's Lemma shows that $\mathcal{I}$ has maximal elements. The previous discussion shows that no maximal element in $\mathcal{I}$ can be countable. Thus $\mathcal{I}$ contains a chain of order type ω_1. This shows that $\Lambda/\mathrm{Soc}(\Lambda)$ does not satisfy $\aleph_1$-acc on direct summands but Λ does.

Theorem 6 is a generalization of the following result obtained by N. V. Dung in [9].

COROLLARY 7 ([9]). *Let M be a CS-module. If $M/\mathrm{Soc}(M)$ has acc (or dcc) on direct summands then M is a direct sum of a semisimple module and a module with finite Goldie dimension.*

PROOF. Applying Theorem 6 with $\aleph = \aleph_0$ and the fact that a module has acc on direct summands if and only if it has dcc on direct summands. □

COROLLARY 8. *Let M be a CS-module such that $M/\mathrm{Soc}M$ is a module in which any independent family of submodules has cardinality less than $\aleph$. Then M is a direct sum of a semisimple module and a module in which any independent family of submodules has cardinality less than $\aleph$.*

PROOF. This follows from Theorem 6, since $M/\mathrm{Soc}(M)$ has $\aleph$-acc on direct summands. □

COROLLARY 9. *Let M be a CS-module such that $M/\mathrm{Soc}(M)$ has finite Goldie dimension. Then M is a direct sum of a semisimple module and a module with finite Goldie dimension.* □

COROLLARY 10 ([10, Theorem 4]). *Let M be a CS-module. If M has $\aleph$-acc on essential submodules, then $M = S \oplus N$, where S is semisimple and N has $\aleph$-acc on all submodules.*

PROOF. Since M has $\aleph$-acc on essential submodules, $M/\mathrm{Soc}(M)$ has $\aleph$-acc on all submodules by [10, Theorem 3], and therefore by Theorem 6, $M = S \oplus N$ where S is semisimple and N is a module in which any independent family of submodules has fewer than $\aleph$ elements. Since $N/\mathrm{Soc}N$ has $\aleph$-acc on all submodules, so N has $\aleph$-acc on all submodules by [10, Theorem 1]. □

COROLLARY 11 ([3, Proposition 5]). *Let M be a CS-module. If M has acc on essential submodules, then $M = S \oplus N$ where S is semisimple and N is noetherian.*

PROOF. Applying Corollary 10 with $\aleph = \aleph_0$. $\square$

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DEPARTMENT OF MATHEMATICS
HUE UNIVERSITY OF EDUCATION
32 LE LOI STREET
HUE
VIETNAM

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CURVATURE IN THE GEOMETRY OF CANONICAL CORRELATION

F. HIAI, D. PETZ and G. TOTH

The state space of a finite quantum system, i.e., the set of all positive definite matrices of trace 1, is a convex set and becomes naturally a differentiable manifold. The canonical correlation (or Mori scalar product of selfadjoint matrices) defines a Riemannian structure on it ([8], [11]) whose study is the objective of this paper. In particular, the scalar curvature and the sectional curvatures will be in the center of interest.

Parametric families of measures occur in mathematical statistics. The monograph [1] treats geometric methods of statistics in detail. Our investigations may be regarded as first steps towards a noncommutative (or quantum) generalization. From the side of physics, [2] stresses the importance of the geometry of the canonical correlation and more generally, that of the geometric viewpoint.

Roughly speaking, we replace the Fisher information metric (of probability distributions) by the Mori product of linear response theory ([6]). The induced geometry will be rather different: Instead of constant positive curvature we observe both negative and positive sectional curvatures. Actually, we compute several sectional curvatures and find a simple formula for the scalar curvature at the tracial state. We make the conjecture that the scalar curvature takes its maximum here.

1. Introduction. In the Hilbert space formalism of quantum mechanics a pure state of a system is described by a state vector $|\Phi\rangle$ belonging to a complex Hilbert space $\mathcal{H}$. The mean (or expectation) value of an observable $A=A^*\in B(\mathcal{H})$ is the scalar product

$$\langle A\rangle = \langle\Phi|A|\Phi\rangle. \tag{1.1}$$

The statistical operator (or density matrix) of a mixed state is a positive compact operator with trace 1. We denote by $\mathfrak{S}$ the set of all such operators acting on the basic Hilbert space $\mathcal{H}$. Sometimes $\mathfrak{S}$ is called state space. In a mixed state $D\in\mathfrak{S}$ the mean value of the observable A is

$$\langle A\rangle = \operatorname{Tr} DA \tag{1.2}$$

which extends (1.1).

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In the present paper, we assume that the underlying complex Hilbert space $\mathcal{H}$ is finite dimensional and we write n for its dimension. So operators on $\mathcal{H}$ may be represented by matrices and a statistical operator corresponds to a positive semidefinite matrix of trace 1. Physically, a finite dimensional Hilbert space appears, for example, when one deals with a spin. Our aim is to endow the state space with a differentiable structure. Actually, we are going to study a Riemannian geometry on the space $\mathcal{S}$ of all invertible density matrices. Motivated by quantum statistical mechanics, we write elements of $\mathcal{S}$ in the form

$$\frac{e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}}, \tag{1.3}$$

where $\beta > 0$ is a real constant and H is a selfadjoint operator. If the difference $H' - H$ is a multiple of the identity then H and H' give the same density matrix in (1.3). Therefore, to get a suitable parametrization we have to identify selfadjoint operators differing in a constant multiple of the identity I. For example, we may consider only traceless selfadjoint matrices, their space is denoted by $\mathcal{K}_0^0$. The real linear space $\mathcal{K}_0^0$ has dimension $n^2 - 1$ and will be identified with the Euclidean space $\mathbb{R}^{n^2-1}$ by means of the following linearly independent matrices.

$$\begin{aligned}
\sigma_1^{k,l} &= E_{k,l} + E_{l,k} && (1 \leqq k < l \leqq n), \\
\sigma_2^{k,l} &= -\,\mathrm{i}E_{k,l} + \mathrm{i}E_{l,k} && (1 \leqq k < l \leqq n), \\
\sigma_3^{m} &= \sum_{i=1}^{m} E_{i,i} - mE_{m+1,m+1} && (1 \leqq m \leqq n-1),
\end{aligned} \tag{1.4}$$

where (E_{ij}) is a system of matrix units. (Observe that for $n=2$ exactly the Pauli matrices show up here.) The homeomorphism

$$h:\, D \longmapsto -\frac{1}{\beta} \log D + \frac{1}{\beta} \operatorname{Tr} \log D \tag{1.5}$$

maps onto $\mathcal{K}_0^0 \equiv \mathbb{R}^{n^2-1}$ and endows $\mathcal{S}$ with a differentiable structure. The mapping h is called the logarithmic coordinate system at the inverse temperature β. $\mathcal{S}$ becomes a differentiable manifold with an atlas containing a single chart. Differentiation of a function $f\colon \mathcal{S} \to \mathbb{R}$ along the curve

$$t \longmapsto D_t = \frac{e^{-\beta(H+tA)}}{\operatorname{Tr} e^{-\beta(H+tA)}} \qquad (A, H \in \mathcal{K}_0^0)$$

is the same as differentiation of $f \circ h^{-1}$ in the direction A. Hence we could identify the tangent space $\mathbf{T}_D(\mathcal{S})$ with $\mathcal{K}_0^0$ but below we prefer another representation.

It is an idea from mathematical statistics that an informational distance between probability measures gives rise to a Riemannian metric. As it was proposed in [8] we shall use Umegaki's relative entropy in the role of informational distance between density matrices. Recall that the relative entropy

$$S(D_1, D_2) \equiv \operatorname{Tr} D_1(\log D_1 - \log D_2) \tag{1.6}$$

of the densities D_1 and D_2 measures the information between the corresponding states (see [12] and [9]). We note that the relative entropy is not a metric in the common sense, for example, it is not a symmetric function of its two variables. If

$$D_t^1 = \frac{e^{-\beta(H+tA)}}{\operatorname{Tr} e^{-\beta(H+tA)}} \quad \text{and} \quad D_s^2 = \frac{e^{-\beta(H+sB)}}{\operatorname{Tr} e^{-\beta(H+sB)}}$$

are some curves in $\mathcal{S}$ then

$$\begin{aligned} &\frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} S(D_t^1, D_s^2) = \\ &= -\beta \int_0^\beta \operatorname{Tr}\Big(\frac{e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} e^{xH} A e^{-xH} B\Big)dx + \beta^2 \frac{\operatorname{Tr}(e^{-\beta H} A)}{\operatorname{Tr}(e^{-\beta H})} \frac{\operatorname{Tr}(e^{-\beta H} B)}{\operatorname{Tr}(e^{-\beta H})}. \end{aligned} \tag{1.7}$$

Using the notation

$$(A, B) = \frac{1}{\beta} \int_0^\beta \operatorname{Tr}\Big(\frac{e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} e^{xH} A e^{-xH} B\Big)dx$$

we have

$$\begin{aligned} -\frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} S(D_t^1, D_s^2) &= \beta^2\Big((A, B) - \frac{\operatorname{Tr} e^{-\beta H} A}{\operatorname{Tr} e^{-\beta H}} \frac{\operatorname{Tr} e^{-\beta H} B}{\operatorname{Tr} e^{-\beta H}}\Big) = \\ &= \beta^2((A, B) - (A, I)(I, B)), \end{aligned} \tag{1.8}$$

which is often called the canonical correlation of A and B. The scalar product $(.,.)$ is an important ingredient of linear response theory and bears the names Mori scalar product, Bogoliubov inner product or Duhamel two-point function ([2], [5], [6], [11]).

Our aim is to study the Riemannian geometry induced by the canonical correlation (1.7). For the sake of simplicity, the inverse temperature β is chosen to be 1. Let us anticipate a few remarks on the geometry of the canonical correlation. The analogous geometry in the probabilistic case (which is the starting point of these ideas) is rather simple, one obtains a manifold with a constant positive curvature ([1]). The case of 2×2 density matrices is very different, this manifold has nonpositive sectional curvatures and the scalar curvature is not bounded ([11]). For a higher spin the geometry is rather complicated and some of its features will be discussed here.

What we need from differential geometry is standard, and we follow the books [3] and [7]. Concerning the Mori product, we refer to the monograph [6]. Finally, we note that from the mathematical point of view, [2] is devoted to the same geometry as discussed here. (The slight difference is in the fact that in the present paper the state space is considered while [2] treats the geometry of all positive definite matrices without the normalization constraint.)

2. Riemannian metric. Let $\mathcal{H}$ be a complex Hilbert space (of dimension n) and let $\mathcal{K}$ be the real linear space of all selfadjoint operators acting on $\mathcal{H}$. We parametrize the set S of all invertible density operators by the factor space $\mathcal{K}/\mathbb{R}I$. For $H \in \mathcal{K}/\mathbb{R}I$ the corresponding density matrix is

$$R(H) = \frac{e^{-H}}{\operatorname{Tr} e^{-H}}. \tag{2.1}$$

The factor space $\mathcal{K}/\mathbb{R}I$ has several representations and we will always use a convenient one.

The space $\mathcal{K}$ is a real linear subspace of the space $\mathfrak{L}$ of all operators on the Hilbert space $\mathcal{H}$. $\mathfrak{L}$ is sometimes called Liouville space and operators on it are termed superoperators. The Mori scalar product

$$\langle A, B\rangle_H = \int_0^1 \operatorname{Tr} R(H) e^{xH} A^* e^{-xH} B \, dx \tag{2.2}$$

makes a complex Hilbert space $\mathfrak{L}_H$ from $\mathfrak{L}$. The real subspace $\mathcal{K}_H$ is a real inner product space. The Liouville (super)operator is defined as

$$\mathbf{L}_H(A) = HA - AH. \tag{2.3}$$

Since

$$e^{x\mathbf{L}_H}(A) = e^{xH} A e^{-xH} \qquad (x \in \mathbb{R}) \tag{2.4}$$

we may write

$$\langle A, B\rangle_H = \operatorname{Tr}\left(BR(H) \int_0^1 e^{x\mathbf{L}_H}(A^*) dx\right),$$

where the integral is called, by some authors, the Kubo transform of A. Decompose $\mathcal{K}_H$ as

$$\mathbb{R}I \oplus \mathcal{K}_H^0, \tag{2.5}$$

where $\mathcal{K}_H^0 = \{A \in \mathcal{K}_H : \mathrm{Tr}\, R(H)A = 0\}$ and $A \mapsto A - (\mathrm{Tr}\, R(H)A)I$ is the orthogonal projection P_H onto $\mathcal{K}_H^0$. We write $\tilde{A} = P_H(A)$ and this is the fluctuation of the observable A around its mean value $\langle A \rangle = \mathrm{Tr}\, R(H)A$.

We identify the tangent space $\mathbf{T}_H(\mathcal{S})$ of the manifold $\mathcal{S}$ at the density matrix $R(H)$ by the space $\mathcal{K}_H^0$ of fluctuations. The Riemannian metric is given by the Mori product:

$$g(\tilde{A},\tilde{B})(H) = \langle \tilde{A},\tilde{B} \rangle_H \qquad (\tilde{A},\tilde{B} \in \mathbf{T}_H(\mathcal{S})). \tag{2.6}$$

Observe that this Riemannian metric is unitarily covariant. Let U be a unitary. If $\tilde{A},\tilde{B} \in \mathbf{T}_H(\mathcal{S})$ then $U\tilde{A}U^*, U\tilde{B}U^* \in \mathbf{T}_{UHU^*}(\mathcal{S})$ and

$$g(U\tilde{A}U^*, U\tilde{B}U^*)(UHU^*) = g(\tilde{A},\tilde{B})(H) \tag{2.7}$$

is obvious from (2.2). Hence, computing, for example curvature, at H, one may assume that the matrix H is diagonal.

3. The Levi–Civita connection. The basic geometric quantities are derived from the Levi–Civita (or Riemannian) connection ∇ induced by the metric. The Kostant formula (cf. (2.52) in [7]) says that

$$\begin{aligned} 2g(Y, \nabla_Z X) = {} & Xg(Y,Z) + Zg(X,Y) - Yg(X,Z) - \\ & - g([X,Y],Z) - g([Z,Y],X) - g([X,Z],Y). \end{aligned} \tag{3.1}$$

When the Lie brackets $[X,Y]$, $[Y,Z]$, $[Z,X]$ vanish, as it happens when X, Y, Z are coordinate fields, (3.1) becomes simpler. Since we want to have the convenience of vanishing Lie brackets, we restrict our discussion to linear combinations of coordinate fields (with constant coefficients). A tangent vector $\tilde{A} \in \mathbf{T}_H(\mathcal{S})$ can uniquely extend to a vector field X which is the linear combination of coordinate fields with constant coefficients. From now on, every vector field, usually denoted by X, Y, Z, is supposed to be a linear combination of coordinate fields.

If $X(H) = \tilde{A}$, $Y(H) = \tilde{B}$, and $Z(H) = \tilde{C}$, then we understand $\tilde{A}\langle \tilde{B},\tilde{C} \rangle$ as $X\langle Y, Z \rangle(H)$. The starting point of our computations is the quantity $\tilde{A}\langle \tilde{B},\tilde{C} \rangle$. It is easy to see that if $\mathrm{Tr}\, e^{-H} = 1$ then

$$\begin{aligned} \tilde{A}\langle \tilde{B},\tilde{C} \rangle &= \frac{\partial^3}{\partial s \partial t \partial u}\bigg|_{s=t=u=0} \mathrm{Tr}\, e^{-H+s\tilde{A}+t\tilde{B}+u\tilde{C}} = \\ &= \frac{\partial}{\partial u}\bigg|_{u=0} \int_0^1 \mathrm{Tr}\, e^{(x-1)(H-u\tilde{C})}\tilde{A}e^{-x(H-u\tilde{C})}\tilde{B}\, dx, \end{aligned} \tag{3.2}$$

which is a symmetric function of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. Standard perturbation theory

([4]) gives

$$e^{(x-1)(H-u\tilde{C})} = e^{-(1-x)H} + u \int_0^{1-x} e^{-(1-x-v)H} \tilde{C} e^{-vH}\, dv + O(u^2),$$

$$e^{-x(H-u\tilde{C})} = e^{-xH} + u \int_0^{x} e^{-(x-v)H} \tilde{C} e^{-vH}\, dv + O(u^2),$$

and we obtain the formula

$$\tilde{A}\langle\tilde{B},\tilde{C}\rangle = \int_{x=0}^{1}\int_{v=0}^{x} \operatorname{Tr} e^{-(1-x)H}\tilde{A}e^{-(x-v)H}\tilde{B}\, dv\, dx + \int_{x=0}^{1}\int_{v=0}^{x} \operatorname{Tr} e^{-(x-v)H}\tilde{C}e^{-vH}\tilde{A}e^{-(1-x)H}\tilde{B}\, dv\, dx. \tag{3.3}$$

Observe that the two terms on the right-hand side are similar, exchanging $\tilde{A}$ with $\tilde{B}$ one gets from each term the other one. Moreover, both terms are invariant under cyclic permutations of $\tilde{A}, \tilde{B}$ and $\tilde{C}$. Hence the full symmetry of their sum is clear. Due to the symmetry, we have

$$\langle X, \nabla_Y Z\rangle(H) = \frac{1}{2}\tilde{A}\langle\tilde{B},\tilde{C}\rangle. \tag{3.4}$$

We are interested in the sectional curvatures of $\mathcal{S}$. Let $\tilde{A},\tilde{B} \in \mathbf{T}_H(\mathcal{S})$ and let X, Y be vector fields such that $X(H) = \tilde{A}$ and $Y(H) = \tilde{B}$. The sectional curvature for the plane spanned by $\tilde{A}$ and $\tilde{B}$ is

$$K(\tilde{A},\tilde{B}) = \frac{\langle\nabla_Y\nabla_X X, Y\rangle - \langle\nabla_X\nabla_Y X, Y\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} \tag{3.5}$$

by definition ([3], [7]). Since

$$\langle\nabla_Y\nabla_X X, Y\rangle = Y\langle\nabla_X X, Y\rangle - \langle\nabla_X X, \nabla_Y Y\rangle$$
$$\langle\nabla_X\nabla_Y X, Y\rangle = X\langle\nabla_Y X, Y\rangle - \langle\nabla_Y X, \nabla_X Y\rangle$$

and

$$Y\langle\nabla_X X, Y\rangle = \frac{1}{2}YX\langle X, Y\rangle = \frac{1}{2}XY\langle X, Y\rangle = X\langle\nabla_Y X, Y\rangle$$

due to $[X, Y] = 0$ and (3.4), we have

$$K(\tilde{A},\tilde{B}) = \frac{\langle\nabla_Y X, \nabla_X Y\rangle - \langle\nabla_X X, \nabla_Y Y\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}. \tag{3.6}$$

The scalar curvature is the sum of the sectional curvatures for all pairs of basis vectors and it is independent of the basis. For an orthogonal basis $\tilde{A}_1, \dots, \tilde{A}_{n^2-1}$ of the tangent space $\mathbf{T}_H(\mathcal{S}) \equiv \mathcal{K}_H^0$, the scalar curvature is defined as

$$\text{Scal}\,(H) = 2 \sum_{1 \le i < j \le n^2-1} K(\tilde{A}_i, \tilde{A}_j) \tag{3.7}$$

(see 3.19 in [7]).

4. Curvature. To compute sectional curvatures, we shall use the coordinate fields given by the fluctuations of the basis matrices (1.4). So we have the advantage that Lie brackets vanish. At the origin, that is, at $H = 0$, the coordinate fields are pairwise orthogonal and (3.4) takes a rather simple form. Therefore, we first compute the sectional and scalar curvature for the tracial state.

If $\tilde{A}H = H\tilde{A}$ then from (3.3)

$$\tilde{A}\langle \tilde{B}, \tilde{C}\rangle = \int_0^1 x \text{Tr}\, R(H) \left(e^{xH}\tilde{B}e^{-xH}\tilde{A}\tilde{C} + \tilde{C}\tilde{A}e^{-xH}\tilde{B}e^{xH} \right) dx \tag{4.1}$$

at the point H. When in addition $\tilde{B}H = H\tilde{B}$ holds, one gets

$$\tilde{A}\langle \tilde{B}, \tilde{C}\rangle = \frac{1}{2}\langle \tilde{A}\tilde{B} + \tilde{B}\tilde{A}, \tilde{C}\rangle, \tag{4.2}$$

where ordinary matrix multiplications stand on the right-hand side. By a bit more computation we may arrive at (4.2) if the commutation $\tilde{B}\tilde{A} = \tilde{A}\tilde{B}$ is assumed instead of $\tilde{B}H = H\tilde{B}$.

Let the vector fields $X_1, X_2, \dots, X_{n^2-1}$ be obtained by linear combination from the basis fields such that $X_1(H), X_2(H), \dots, X_{n^2-1}(H)$ form an orthonormal basis in $\mathbf{T}_H(\mathcal{S})$. Choose X and Y from $X_1, X_2, \dots, X_{n^2-1}$ and write $\tilde{A}$ and $\tilde{B}$ for $X(H)$ and $Y(H)$, respectively. If (4.2) holds, we have

$$\begin{aligned} \langle \nabla_X Y, X_i\rangle(H) &= \frac{1}{2} X\langle Y, X_i\rangle(H) = \\ &= \frac{1}{4}\langle (\tilde{A}\tilde{B} + \tilde{B}\tilde{A}), X_i\rangle = \langle \nabla_Y X, X_i\rangle(H) \end{aligned}$$

for every i. Consequently,

$$\begin{aligned} \nabla_X Y(H) = \nabla_Y X(H) &= \frac{1}{4} P_H((\tilde{A}\tilde{B} + \tilde{B}\tilde{A})) = \\ &= \frac{1}{4}(\tilde{A}\tilde{B} + \tilde{B}\tilde{A}) - \frac{1}{4}\langle \tilde{A}\tilde{B} + \tilde{B}\tilde{A}, I\rangle I, \end{aligned}$$

and similarly

$$\nabla_X X(H) = \frac{1}{2}(\tilde{A}^2 - I), \qquad \nabla_Y Y(H) = \frac{1}{2}(\tilde{B}^2 - I).$$

We obtain

$$\langle \nabla_Y X, \nabla_X Y\rangle(H) = \frac{\langle \tilde{A}\tilde{B} + \tilde{B}\tilde{A}, \tilde{A}\tilde{B} + \tilde{B}\tilde{A}\rangle_H}{16},$$
$$\langle \nabla_X X, \nabla_Y Y\rangle(H) = \frac{\langle \tilde{A}^2, \tilde{B}^2\rangle_H - 1}{4},$$

and we arrive at the following

THEOREM 4.1. *Let* $\tilde{A},\tilde{B} \in \mathbf{T}_H(\mathcal{S})$ *be orthonormal. If* $\tilde{A}H = H\tilde{A}$ *and* $\tilde{B}H = H\tilde{B}$, *then*

$$K(\tilde{A},\tilde{B})(H) = \frac{4 + \mathrm{Tr}\,(R(H)(\tilde{A}\tilde{B} - \tilde{B}\tilde{A})^2)}{16}.$$

If $\tilde{A}H = H\tilde{A}$ *and* $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$, *then*

$$K(\tilde{A},\tilde{B})(H) = \frac{1 + \langle \tilde{A}\tilde{B}, \tilde{A}\tilde{B}\rangle_H - \mathrm{Tr}\,(R(H)\tilde{A}^2\tilde{B}^2)}{4}.$$

Since $(\tilde{A}\tilde{B} - \tilde{B}\tilde{A})^2 \leqq 0$, the sectional curvatures do not exceed $1/4$ at the tracial state $H = 0$. (It is shown below that $K(\tilde{A},\tilde{B})(H) \leqq 1/4$ holds also in the second case of the previous theorem.) Positive sectional curvature will appear if $n \geqq 3$ because then $\tilde{A}\tilde{B} - \tilde{B}\tilde{A} = 0$ may happen. It seems that the 2×2 case is exceptional from the point of view of curvature. Recall that for $n = 2$ the sectional curvatures are strictly negative except for the tracial state where they vanish ([11]).

Next we compute the scalar curvature at the tracial state. Let $X_1(0)$, $X_2(0), \dots, X_{n^2-1}(0)$ be the same orthonormal basis in $\mathbf{T}_0(\mathcal{S})$ as above and write $\tilde{A}_i$ for $X_i(0)$. Then we infer

$$\mathrm{Scal}\,(0) = \frac{(n^2-1)(n^2-2)}{4} - \frac{1}{16}\sum_{i=1}^{n^2-1}\sum_{j=1}^{n^2-1} 2\tau(\tilde{A}_i^2\tilde{A}_j^2 - \tilde{A}_i\tilde{A}_j\tilde{A}_i\tilde{A}_j)$$

from Theorem 4.1, where $\tau = \mathrm{Tr}\,/n$ is the tracial state. To compute the sum we use the Liouville operators $\mathbf{L}_i \equiv \mathbf{L}_{\tilde{A}_i}$ and the trace functional $\mathbf{Tr}$ of superoperators. We have

$$\langle \mathbf{L}_i^2(\tilde{A}_j), \tilde{A}_j\rangle = 2\tau(\tilde{A}_i^2\tilde{A}_j^2 - \tilde{A}_i\tilde{A}_j\tilde{A}_i\tilde{A}_j), \tag{4.3}$$

where the Liouville operators $\mathbf{L}_A$ are defined according to (2.3). We have

$$\sum_{j=1}^{n^2-1} \langle \mathbf{L}_i^2(\tilde{A}_j), \tilde{A}_j \rangle = \mathbf{Tr}\, \mathbf{L}_i^2.$$

For any selfadjoint operator B with spectral decomposition $\sum_k \mu_k p_k$, the operator $\mathbf{L}_B^2$ has eigenvalues $(\mu_k - \mu_l)^2$ (cf. (3.1.19) in [6]) and we have

$$\mathbf{Tr}\, \mathbf{L}_B^2 = \sum_{k,l} (\mu_k - \mu_l)^2 = 2n \mathrm{Tr}\, B^2 - 2(\mathrm{Tr}\,(B))^2. \tag{4.4}$$

Since $\tilde{A}_i$ is a fluctuation, $\mathrm{Tr}\, \tilde{A}_i = 0$ and from the normalization we have $\mathrm{Tr}\, \tilde{A}_i^2 = n$. So (4.4) yields $\mathbf{Tr}\, \mathbf{L}_i^2 = 2n^2$ in this particular case. Therefore

$$\sum_{i=1}^{n^2-1} \sum_{j=1}^{n^2-1} 2\tau(\tilde{A}_i^2 \tilde{A}_j^2 - \tilde{A}_i \tilde{A}_j \tilde{A}_i \tilde{A}_j) = \sum_{i=1}^{n^2-1} \mathbf{Tr}\, \mathbf{L}_i^2 = (n^2-1)2n^2,$$

and we conclude the following

THEOREM 4.2. *The scalar curvature of $\mathcal{S}_n$ is given by the formula*

$$\mathrm{Scal}\,(0) = \frac{n^2-1}{8}(n^2-4)$$

at the tracial state corresponding to $H = 0$.

Hence the scalar curvature at the tracial state is positive for $n \geqq 3$.

Now we fix a diagonal density $e^{-H} = \mathrm{Diag}(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n})$ and for $B = B^*$ define a (super)operator $\mathbf{D}_B$ on the real Liouville space $\mathcal{K}_H$ by its bilinear form as follows:

$$\langle A, \mathbf{D}_B C \rangle = \frac{\partial^3}{\partial s \partial t \partial u}\Big|_{s=t=u=0} \mathrm{Tr}\, e^{-H+sA+tB+uC}. \tag{4.5}$$

The operator $\mathbf{D}_B$ is strongly related to the covariant derivation ∇_Y when $Y(H) = B$ as it is clear from (3.2) and (3.4). Having fixed the coordinate fields, we regard ∇_Y as a linear operator on the tangent space $\mathcal{K}_H^0 \subset \mathcal{K}_H$ and this operator is the compression of $\frac{1}{2}\mathbf{D}_B$ to the space of fluctuations:

$$\langle X, \nabla_Y Z \rangle = \frac{1}{2} \langle \tilde{A}, \mathbf{D}_{\tilde{B}} \tilde{C} - (\mathrm{Tr}\, R(H) \mathbf{D}_{\tilde{B}} \tilde{C}) I \rangle = \frac{1}{2} \langle \tilde{A}, \mathbf{B}_{\tilde{B}} \tilde{C} \rangle \tag{4.6}$$

where $X(H) = \tilde{A}$, $Y(H) = \tilde{B}$, and $Z(H) = \tilde{C}$. By computing the double integrals of (3.3) we obtain

$$(\mathbf{D}_B C)_{ij} = \sum_{k=1}^{n} (b_{ik} c_{kj} + c_{ik} b_{kj}) d_{ikj}, \tag{4.7}$$

where

$$d_{ikj} = \frac{1}{\lambda_i - \lambda_k}\left(\frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j} - \frac{e^{\lambda_k} - e^{\lambda_j}}{\lambda_k - \lambda_j}\right)\frac{\lambda_i - \lambda_j}{e^{\lambda_i} - e^{\lambda_j}}. \tag{4.8}$$

This expression is formally well-defined when all the values of λ_i, λ_j, and λ_k are different but it has a limit when some of them coincide. (For example, $d_{ikj} = 1/2$ when $\lambda_i = \lambda_j = \lambda_k$.) Observe that d_{ikj} is built from logarithmic means. The logarithmic mean of $\alpha, \beta > 0$ is defined as $\mathrm{Lm}(\alpha, \beta) = (\alpha - \beta)/(\log\alpha - \log\beta)$. Using the notation m_{ij} for $\mathrm{Lm}(e^{\lambda_i}, e^{\lambda_j})$ we may write

$$d_{ikj} = \frac{m_{ij} - m_{kj}}{\lambda_i - \lambda_k}\frac{1}{m_{ij}}. \tag{4.9}$$

We introduce the notation c_{ijk} for the following symmetric function of the variables λ_i, λ_j, and λ_k.

$$\begin{aligned} c_{ikj} &= \frac{e^{\lambda_i}}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)} + \frac{e^{\lambda_k}}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} + \frac{e^{\lambda_j}}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)} = \\ &= \frac{m_{ij} - m_{kj}}{\lambda_i - \lambda_k}. \end{aligned} \tag{4.10}$$

Among c, d and m we have the relation

$$d_{ikj} = \frac{c_{ikj}}{m_{ij}} \tag{4.11}$$

which shows that $d_{ikj} = d_{jki}$ holds. (Below d's will be identified as Christoffel symbols, see (5.3).)

Definition (4.7) of $\mathbf{D}_B$ extends to the complex Liouville space $\mathcal{L}_H$ when it is wished. Formula (3.5) for the sectional curvature may be rewritten in terms of the operators $\mathbf{D}$. Assume that $\tilde{A}$ and $\tilde{B}$ are orthonormal in $\mathcal{K}_H^0$. Then

$$K(\tilde{A}, \tilde{B})(H) = \frac{1}{4} + \frac{1}{4}\langle \mathbf{D}_{\tilde{B}}\tilde{A}, \mathbf{D}_{\tilde{B}}\tilde{A}\rangle - \frac{1}{4}\langle \mathbf{D}_{\tilde{A}}\tilde{A}, \mathbf{D}_{\tilde{B}}\tilde{B}\rangle. \tag{4.12}$$

(Note that dependence of $\mathbf{D}$ on H is suppressed from the notation.)

Our next aim is to express (4.12) explicitly in terms of the entries of the matrices $\tilde{A}$ and $\tilde{B}$. It is easy to verify that

$$\langle A, B\rangle_H = \sum_{i,j} a_{ij} b_{ji} m_{ij} \quad \text{for} \quad e^{-H} = \mathrm{Diag}(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}). \tag{4.13}$$

By means of (4.7) and (4.12) we arrive at

$$
\begin{aligned}
\langle \mathbf{D}_B A, \mathbf{D}_B A\rangle &= \sum_{i,j}\Big(\sum_k (b_{ik}a_{kj}+a_{ik}b_{kj})d_{ikj}\sum_l (b_{jl}a_{li}+a_{jl}b_{li})d_{jli}\Big)m_{ij} = \\
&= \sum_{i,j,k,l} a_{ik}a_{kj}b_{jl}b_{li}\frac{2c_{lik}c_{kjl}}{m_{lk}} + \sum_{i,j,k,l} a_{ik}a_{jl}b_{kj}b_{li}\Big(\frac{c_{lik}c_{kjl}}{m_{lk}}+\frac{c_{ijk}c_{ijl}}{m_{ij}}\Big)
\end{aligned} \tag{4.14}
$$

and similarly

$$
\langle \mathbf{D}_A A, \mathbf{D}_B B\rangle = \sum_{i,j,k,l} a_{ik}a_{kj}b_{jl}b_{li}\frac{4c_{ijk}c_{ijl}}{m_{ij}}, \tag{4.15}
$$

which hold for arbitrary selfadjoint A and B.

LEMMA 4.3. *If H and A are diagonal then*

$$\langle \mathbf{D}_B A, \mathbf{D}_B A\rangle \leqq \langle \mathbf{D}_A A, \mathbf{D}_B B\rangle.$$

Furthermore, equality holds if and only if $AB = BA$ provided that the spectrum of H is free of multiplicities.

PROOF. Let $e^{-H} = \mathrm{Diag}(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n})$ and $A = \mathrm{Diag}(a_1, a_2, \dots, a_n)$. Without loss of generality, we may assume that $\lambda_i > \lambda_j$ when $i > j$. Specialization of formulas (4.14) and (4.15) to this situation gives

$$
\begin{aligned}
&\langle \mathbf{D}_A A, \mathbf{D}_B B\rangle - \langle \mathbf{D}_B A, \mathbf{D}_B A\rangle = \\
&= 2\sum_{i>j}|b_{ij}|^2\left\{\frac{a_i^2e^{\lambda_i}-a_j^2e^{\lambda_j}}{\lambda_i-\lambda_j} - \frac{(a_i^2-a_j^2)(e^{\lambda_i}-e^{\lambda_j})}{(\lambda_i-\lambda_j)^2} - \right.\\
&\qquad\left. - \frac{e^{\lambda_i}-e^{\lambda_j}}{\lambda_i-\lambda_j}\Big(\frac{a_ie^{\lambda_i}-a_je^{\lambda_j}}{e^{\lambda_i}-e^{\lambda_j}} - \frac{a_i-a_j}{\lambda_i-\lambda_j}\Big)^2\right\} = \\
&= 2\sum_{i>j}|b_{ij}|^2\frac{(a_i-a_j)^2e^{2\lambda_j}}{(\lambda_i-\lambda_j)^3(e^{\lambda_i}-e^{\lambda_j})}\left(-x_{ij}^2e^{x_{ij}} + x_{ij}(e^{2x_{ij}}-1) - (e^{x_{ij}}-1)^2\right),
\end{aligned}
$$

where x_{ij} is written for $\lambda_i - \lambda_j$. Hence we may prove that for any $x \geqq 0$

$$-x^2e^x + x(e^{2x}-1) - (e^x-1)^2 \geqq 0.$$

The power series expansion of the above left-hand side is

$$\sum_{k=4}^{\infty}\frac{k2^{k-1} - 2^k - k(k-1) + 2}{k!}x^k.$$

It is easy to check that

$$k2^{k-1} - 2^k - k(k-1) + 2 > 0 \quad \text{for} \quad k \geqq 4.$$

This completes the proof of the inequality in the lemma.

If the spectrum of H is free of multiplicities, that is, $\lambda_i \neq \lambda_j$ for $i \neq j$, then the condition for equality is

$$2\sum_{i>j} |b_{ij}|^2 (a_i - a_j)^2 = 0$$

which is the same as $AB = BA$. □

The previous lemma may be interesting on its own right. (In the 2×2 case, the inequality in the lemma is verified for arbitary selfadjoint A and B.) An upper bound for some sectional curvatures follows directly from the lemma.

THEOREM 4.4. *Assume that $\tilde{A}, \tilde{B} \in \mathbf{T}_h(\mathcal{S})$, $\tilde{A}H = H\tilde{A}$ and $\tilde{A}, \tilde{B}$ are linearly independent. Then*

$$K(\tilde{A}, \tilde{B})(H) \leqq \frac{1}{4}$$

holds for the sectional curvature. Furthermore, if the spectrum of H is free of multiplicities and $\tilde{A}\tilde{B} \neq \tilde{B}\tilde{A}$, then the inequality is strict.

One can get all sectional curvatures from (4.12) by means of formulas (4.14) and (4.15). They are mostly rather complicated expressions of $\lambda_1, \ldots, \lambda_n$. Here are some examples.

$$\begin{aligned} K(\tilde{A}, \sigma_k^{ij}) &= \frac{1}{4} + \frac{(a_i c_{iij} + a_j c_{jji})^2}{4m_{ij}^2} - \frac{a_i^2 c_{iij} + a_j^2 c_{jji}}{4m_{ij}} = \\ &= \frac{1}{4} + \frac{1}{4}(a_i d_{iij} + a_j d_{jji})^2 - \frac{1}{4}(a_i^2 d_{iij} + a_j^2 d_{jji}), \end{aligned} \tag{4.16}$$

when $\tilde{A} = \mathrm{Diag}\,(a_1, a_2, \ldots, a_n)$ is assumed to be normalized, $1 \leqq i < j \leqq n$, and $k \in \{1, 2\}$. (Note that $\tilde{A}$ and $\tilde{\sigma}_k^{ij}$ are orthogonal but the latter is not normalized: $\langle \sigma_k^{ij}, \sigma_k^{ij} \rangle = 2m_{ij}$.) We know from Theorem 4.4 that the curvatures (4.16) do not exceed $1/4$.

Because of the relations $\mathbf{D}_{\sigma_1^{ij}} \sigma_2^{ij} = 0 = \mathbf{D}_{\sigma_2^{ij}} \sigma_1^{ij}$ and $\nabla_{\sigma_1^{ij}} \sigma_1^{ij} = \nabla_{\sigma_2^{ij}} \sigma_2^{ij}$, it is clear from (3.6) that $K(\tilde{\sigma}_1^{ij}, \tilde{\sigma}_2^{ij})$ must be negative. In fact, we have

$$K(\tilde{\sigma}_1^{ij}, \tilde{\sigma}_2^{ij}) = \frac{1}{4} - \frac{d_{iji}^2 e^{\lambda_i} + d_{jij}^2 e^{\lambda_j}}{4m_{ij}^2}. \tag{4.17}$$

If i, j, i', j' are all different then σ_k^{ij} commutes with $\sigma_{k'}^{i'j'}$ but the corresponding sectional curvature is

$$K(\tilde{\sigma}_k^{ij}, \tilde{\sigma}_{k'}^{i'j'}) = \frac{1}{4} \tag{4.18}$$

for $k, k' \in \{1,2\}$. The case where the upper indices are not all different is more interesting and it shows something new compared with spin 1/2. Let $1 \leqq i < j < k \leqq n$. Then for $A = \sigma_t^{ij}$ and $B = \sigma_u^{jk}$ we have

$$\langle \mathbf{D}_B A, \mathbf{D}_B A\rangle = 2d_{ijk}^2 m_{ik}, \qquad \langle \mathbf{D}_A A, \mathbf{D}_B B\rangle = 4d_{jij}d_{jkj}e^{\lambda_j}.$$

Therefore, we obtain

$$K(\tilde{\sigma}_t^{ij}\tilde{\sigma}_u^{jk}) = \frac{1}{4} + \frac{d_{ijk}^2 m_{ik}}{8m_{ij}m_{jk}} - \frac{d_{jij}d_{jkj}e^{\lambda_j}}{4m_{ij}m_{jk}}. \tag{4.19}$$

Performing the limit $\lambda_i = \lambda_k \to -\infty$ we find that the term with negative sign converges to $e^{-\lambda_j}$ while the term with positive sign goes to $+\infty$. This shows that approaching to the boundary of $\mathcal{S}_n$ with $n > 2$ one can detect some sectional curvatures which can be arbitrarily large.

5. Discussion. First we consider the 8-dimensional space $\mathcal{S}_3$ in more detail. The basis fields $\{\sigma_i^{kl} : 1 \leqq k < l \leqq 3,\ 1 \leqq j \leqq 2\}$ are pairwise orthogonal and they are orthogonal to σ_3^1, σ_3^2 at a diagonal density $\mathrm{Diag}\,(e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3})$. So the only nonvanishing offdiagonal entry of the matrix of the metric is

$$\langle \tilde{\sigma}_3^1, \tilde{\sigma}_3^2\rangle = e^{\lambda_1} - e^{\lambda_2} - (e^{\lambda_1} - e^{\lambda_2})(e^{\lambda_1} + e^{\lambda_2} - 2e^{\lambda_3}). \tag{5.1}$$

We have

$$\begin{aligned} c_{123} &= \langle \sigma_1^{12}, \nabla_{\sigma_1^{13}}\sigma_1^{23}\rangle = \langle \sigma_1^{12}, \nabla_{\sigma_2^{13}}\sigma_2^{23}\rangle = \\ &= \langle \sigma_2^{12}, \nabla_{\sigma_2^{13}}\sigma_1^{23}\rangle = -\langle \sigma_2^{12}, \nabla_{\sigma_1^{13}}\sigma_2^{23}\rangle \end{aligned} \tag{5.2}$$

and symmetrically. (When for $i, i', i'' \in \{1,2\}$ the scalar product $\langle \sigma_i^{kl}, \nabla_{\sigma_{i'}^{k'l'}}\sigma_{i''}^{k''l''}\rangle$ is not determined by (5.2), then it vanishes.) The related connection coefficients are obtained from $\pm c$ by division by m. For example,

$$\begin{aligned} &\Gamma^{\sigma_1^{23}}_{\sigma_1^{12},\sigma_1^{13}} = d_{132} \\ &\Gamma^{\sigma_2^{23}}_{\sigma_2^{12},\sigma_1^{13}} = -d_{213} \\ &\Gamma^{\sigma_2^{23}}_{\sigma_2^{12},\sigma_1^{11}} = 0 \end{aligned} \tag{5.3}$$

etc. In the case

$$R(H) = \mathrm{Diag}\,(e^{\lambda}, e^{\lambda}, e^{\mu}), \tag{5.4}$$

(5.1) vanishes and all Christoffel symbols are conveniently obtained. We do not list them.

In principle, the scalar curvature of $\mathcal{S}_n$ can be expressed at a diagonal point $R(H) = \mathrm{Diag}\,(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n})$ and it must be a complicated expression of λ_i's. The range of the scalar curvature function is not determined by the results of Section 4. However, we make the following

CONJECTURE. *The scalar curvature of S_n is maximal at the tracial trace (corresponding to $H = 0$).*

The conjecture is fully verified for $n = 2$ in [11]. For $n = 3$, the following are known to us in the case (5.4).

(1) Scal (H) is bounded from above.

(2) Scal $(H) \to -\infty$ as $(e^\lambda, e^\lambda, e^\mu) \to (0,0,1)$.

(3) Scal $(H) \to -\infty$ as $(e^\lambda, e^\lambda, e^\mu) \to (1/2,1/2,0)$.

Moreover, there are some numerical results supporting the conjecture for $n = 3$ and 4. For example:

$R(H) = $ Diag		Scal (H)
$R(H) = $ Diag	(0.335, 0.333, 0.332)	=4.9999
	(0.339, 0.336, 0.325)	4.99769
	(0.36, 0.34, 0.3)	4.95935
	(0.35, 0.4, 0.25)	4.72556
	(0.35, 0.5, 0.15)	3.13543
	(0.5, 0.4, 0.1)	1.16438
	(0.7, 0.2, 0.1)	−0.668265
	(0.99, 0.001, 0.009)	−568.425
	(0.999, 0.0001, 0.0009)	−5212.51
	(0.99999, 0.000001, 0.000009)	−459501

This suggests that the scalar curvature attains the maximum at the tracial state. (A stronger conjecture about the scalar curvature is formulated in [10].)

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BOOK REVIEW

Cohen, A. M., Gastel, L. van and Lunel, S. V., Computer Algebra in Industry, Vol. 2. Problem Solving in Practice, John Wiley and Sons, Ltd., Chichester–New York–Brisbane–Toronto–Singapore, 1995. ISBN 0 471 95529 9.

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ON THE ORDER OF CONVERGENCE OF INTERPOLATORY PROCESSES

H. H. GONSKA, J. PRASAD and $\boxed{\text{A. K. VARMA}}$

Abstract

Using a certain $(0,1,2,3)$-Hermite–Fejér-type interpolation process, it is shown that there exist interpolation polynomials satisfying the DeVore–Gopengauz-type estimate (i.e., the pointwise Jackson-type estimate involving interpolation at the endpoints of $[-1,1]$ and in terms of the second order modulus of smoothness) for algebraic polynomial approximation.

1. Introduction

We denote by

$$-1 = x_n < x_{n-1} < \ldots < x_2 < x_1 = 1 \tag{1.1}$$

the n distinct zeros of

$$\Pi_n(x) = (1-x^2)P'_{n-1}(x). \tag{1.2}$$

Here $P_n(x)$ is the Legendre polynomial of degree n with $P_n(1)=1$. In the earlier paper [16] the second two authors considered the Hermite–Fejér interpolation operators based on the zeros of (1.2). These operators, denoted by R_n and mapping $C[-1,1]$ into the space of algebraic polynomials of degree not exceeding $2n-1$, are given by

$$R_n(f,x) = \sum_{k=1}^{n} f(x_k)h_k(x), \qquad f \in C[-1,1], \quad x \in [-1,1]. \tag{1.3}$$

Here

$$h_k(x) = l_k^2(x), \qquad k = 2,3,\ldots,n-1; \tag{1.4}$$

$$l_k(x) = \frac{\Pi_n(x)}{(x-x_k)\Pi'_n(x_k)}, \qquad k = 1,\ldots,n,$$

$$h_1(x) = \left[1 + \frac{n(n-1)}{2}(1-x)\right] l_1^2(x), \quad h_n(x) = h_1(-x). \tag{1.5}$$

Concerning $R_n(f,x)$, the following was proved in [16]:

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THEOREM A. *There exists an absolute constant* $c_1 > 0$ *such that, for all* $f \in C[-1,1]$ *and* $-1 \leqq x \leqq 1$, *one has*

$$|R_n(f,x) - f(x)| \leqq \frac{c_1}{n} \sum_{k=1}^{n} w_1\left(\frac{\sqrt{1-x^2}}{k}\right). \tag{1.6}$$

Here, $w_1(\delta)$ *is the (first order) modulus of continuity of* f.

However, better inequalities are available for polynomial approximation with interpolation at the endpoints of $[-1,1]$. In 1976 DeVore [6, Theorem 7.2] proved the following

THEOREM B. *For each* $f \in C[-1,1]$ *there exists an algebraic polynomial* $\mu_n(f)$ *of degree* $\leqq n$ *for which*

$$|f(x) - \mu_n(f;x)| \leqq c_2 w_2\left(\frac{\sqrt{1-x^2}}{n}\right), \qquad -1 \leqq x \leqq 1, \qquad n \geqq 2. \tag{1.7}$$

Here, c_2 *is a constant independent of* f, n *and* x, *and* $w_2(\delta)$ *denotes the second order modulus of smoothness of* f.

Following up DeVore's contribution, a number of further papers were published, in which (1.7) was proved for other operators, and also for such satisfying certain side conditions as, for example:
shape preservation ([5], [13], [18]),
discrete definition ([3], [17]),
ease of computation ([3], [4]).
One further approach to be mentioned is the one bridging the gap between pointwise estimates as discussed in this note and similar inequalities in terms of the so-called Ditzian–Totik modulus. In [7], Ditzian and his collaborators succeeded in developing a new technique to prove "interpolatory results" which relate the two types of estimates mentioned to each other, while also guaranteeing the preservation of monotonicity and convexity.

However, so far it has been an open question if there exist (true) interpolation operators of the DeVore–Gopengauz type. It is the aim of this note to answer this question in the affirmative by constructing interpolation polynomials $S_n(f,\cdot)$ satisfying (1.7) for arbitrary $f \in C[-1,1]$. To be more specific, below we will consider a $(0,1,2,3)$-Hermite–Fejér-type interpolation process $S_n(f,x)$ yielding polynomials in Π_{4n-1}, and satisfying the conditions

$$S_n(f,x_k) = f(x_k); \quad S_n^{(j)}(f,x_k) = \mu_n^{(j)}(x_k), \quad j = 1,2,3; \quad k = 1,2,\dots,n. \tag{1.8}$$

Here $\mu_n = \mu_n(f;\cdot)$ can by *any* sequence of algebraic polynomials satisfying (1.7) (and not necessarily that from the proof of Theorem B).

This approach of "smoothing" the given function f by polynomials with "good" derivatives is a standard technique in interpolation theory which, however, always requires complicated estimates for the fundamental functions of the interpolation process and for related quantities. It is probably due to Fejér [9] who used the derivatives of the polynomials of best approximation in the context of (0, 1)-Hermite–Fejér interpolation. Note that only recently some progress was made in regard to cheap computation of polynomials $\mu_n(f)$ satisfying (1.7); see, e.g., [3] and [4]. It is also in this sense that the construction in (1.8) is quite constructive.

It is our firm conviction that the approach taken below confirms and stresses the power of interpolatory processes; for an excellent partial survey of the work done in that field the reader should refer to [15].

For the (0, 1, 2, 3)-Hermite–Fejér type interpolation process S_n described above we shall prove the following

THEOREM 1. *There exists an absolute positive constant c_3 such that, for all $f \in C[-1,1]$ and $-1 \leqq x \leqq 1$,*

$$|S_n(f,x) - f(x)| \leqq c_3 w_2\left(\frac{\sqrt{1-x^2}}{n}\right). \tag{1.9}$$

2. Explicit representation of the fundamental polynomials

The explicit representation of $S_n(f,x)$ is given by

$$\begin{aligned} S_n(f,x) = {} & \sum_{k=1}^{n} f(x_k)A_k(x) + \sum_{k=1}^{n} \mu_n'(x_k)B_k(x) + \\ & + \sum_{k=1}^{n} \mu_n''(x_k)C_k(x) + \sum_{k=1}^{n} \mu_n'''(x_k)D_k(x). \end{aligned} \tag{2.1}$$

Here,

$$D_k(x) = \frac{1}{6}(x-x_k)^3 l_k^4(x), \qquad k = 1, \dots, n, \tag{2.2}$$

$$C_k(x) = \frac{1}{2}(x-x_k)^2 l_k^4(x), \qquad k = 2, 3, \dots, n-1, \tag{2.3}$$

$$C_1(x) = \frac{(1-x^2)^2(1+x)^2[P_{n-1}'(x)]^4}{2n^4(n-1)^4} - 3n(n-1)D_1(x), \tag{2.4}$$

$$C_n(x) = \frac{(1-x^2)^2(1-x)^2[P_{n-1}'(x)]^4}{2n^4(n-1)^4} + 3n(n-1)D_n(x), \tag{2.5}$$

$$B_k(x) = (x-x_k)l_k^4(x) + \frac{4n(n-1)}{1-x_k^2}D_k(x), \quad k = 2, 3, \dots, n-1, \tag{2.6}$$

$$B_1(x) = \frac{(x-1)(1+x)^4[P'_{n-1}(x)]^4}{n^4(n-1)^4} - 2n(n-1)C_1(x) - \left[\frac{11n^2(n-1)^2 - 4n(n-1)}{12}\right] D_1(x), \tag{2.7}$$

$$B_n(x) = \frac{(1+x)(1-x)^4[P'_{n-1}(x)]^4}{n^4(n-1)^4} - 2n(n-1)C_n(x) + \left[\frac{11n^2(n-1)^2 - 4n(n-1)}{12}\right] D_n(x), \tag{2.8}$$

$$A_k(x) = l_k^4(x) + \frac{2\Pi_n^2(x)l_k^2(x)[1-x_k^2 - x_k(x_k - x)]}{3n(n-1)(1-x_k^2)^2P_{n-1}^2(x_k)}, \quad k = 2, 3, \dots, n-1, \tag{2.9}$$

$$A_1(x) = l_1^4(x) - 4l_1'(1)B_1(x) - [4l_1''(1) + 12(l_1'(1))^2]C_1(x) - [4l_1'''(1) + 24(l_1'(1))^3 + 36l_1'(1)l_1''(1)]D_1(x), \tag{2.10}$$

and

$$A_n(x) = A_1(-x). \tag{2.11}$$

3. Preliminaries

Here we shall state a few known results that will be needed later. For the polynomials $l_k(x)$ we have the following (see [1], [8]):

$$\sum_{k=1}^{n} l_k^2(x) \leqq 1, \qquad -1 \leqq x \leqq 1, \quad |l_j(x)| \leqq 1, \quad j = 1, 2, \dots, n. \tag{3.1}$$

From the inequality due to S. N. Bernstein [2] we have

$$|P'_{n-1}(x)| \leqq \frac{n(n-1)}{2}, \qquad (1-x^2)^{1/2}|P'_{n-1}(x)| \leqq n-1, \qquad (1-x^2)^{3/4}|P'_{n-1}(x)| \leqq \sqrt{2n}. \tag{3.2}$$

It is also known that for $c_4 > 0$,

$$(1-x_k^2)^{1/2}|P_{n-1}^2(x_k)| > \frac{c_4}{n}, \qquad k = 2, 3, \dots, n-1, \tag{3.3}$$

$$(3.4) \qquad (1-x_k^2) > \begin{cases} \dfrac{k^2}{4(n-1)^2}, & k=2,3,\dots,\left[\dfrac{n}{2}\right], \\ \dfrac{(n+1-k)^2}{4(n-1)^2}, & k=\left[\frac{n}{2}\right]+1,\dots,n-1, \end{cases}$$

and

$$(3.5) \qquad \frac{(k-3/2)\pi}{n-1} < \theta_k < \frac{k\pi}{n-1}, \qquad k=2,3,\dots,n-1, \quad x_k=\cos\theta_k.$$

A well-known property of the modulus of smoothness of second order (see [14, p. 48]), namely

$$(3.6) \qquad w_2(\lambda\delta) \leqq (1+\lambda)^2 w_2(\delta), \qquad \lambda,\delta>0,$$

implies

$$(3.7) \qquad w_2(\lambda\delta) \leqq 3(1+\lambda^2) w_2(\delta), \qquad \lambda,\delta>0.$$

4. Estimates

Here we provide certain estimates which will be used later in the proof of the main result.

LEMMA 4.1. *Let $-1 \leqq x \leqq 1$ and let x_j be that zero of P'_{n-1} which is closest to x. Then for a suitable numerical constant $c_5>0$ one has*

$$(4.1) \qquad \frac{\sqrt{1-x^2}}{\sqrt{1-x_j^2}} < c_5, \qquad \frac{\sqrt{1-x^2}}{\sqrt{1-x_{j-1}^2}} < c_5, \qquad \frac{\sqrt{1-x^2}}{\sqrt{1-x_{j+1}^2}} < c_5.$$

LEMMA 4.2 (see [10, Lemma 10]). *Let $A_k(x)$ be defined as in* (2.9). *Then there exists an absolute constant $c_6>0$ such that, for $-1 \leqq x \leqq 1$,*

$$(4.2) \qquad \sum_{k=2}^{n-1} |A_k(x)| \leqq c_6.$$

Using Hermann's notation for the moment, the validity of Lemma 4.2 can be verified by putting $m=p=q=4$ in [10], so that $\alpha(p)=\beta(q)=-1$, and thus $a=b=-2$. Hence Lemma 10 from [10] applies indeed.

REMARK 4.3. Note that (4.2) also follows from an unpublished general result of G. Kook [12, Satz 6.8] dating from 1984. Kook derived the estimate in the more general framework of Hermite–Fejér interpolation of order 3 subject to certain additional boundary conditions at ± 1. In his thesis Kook used the reduction method of Knoop [11].

Next we formulate an auxiliary result which will be needed below several times.

LEMMA 4.4. *Let $x=\cos\theta\in[-1,1]$ be fixed and let x_j be a zero of P'_{n-1} which is closest to x. Then for $2\leqq k\leqq n-1$ we have*

$$\frac{1}{\sin\dfrac{|\theta-\theta_k|}{2}}\leqq\frac{4(n-1)}{2i-3}\leqq\frac{8n}{i},\qquad k\neq j,\quad k\neq j\pm 1. \tag{4.3}$$

Here $x_k=\cos\theta_k$, and $k=j\pm i$ for an appropriate $i\geqq 2$.

The following estimate concerning another quantity involving the fundamental functions A_k, $2\leqq k\leqq n-2$, will also be needed below. This is the subject of

LEMMA 4.5. *Let $A_k(x)$ be given as in (2.9). Then there is an absolute constant $c_7>0$ so that for all $-1\leqq x\leqq 1$ one has*

$$\sum_{k=2}^{n-1}\frac{1-x_k^2}{1-x^2}|A_k(x)|\leqq c_7. \tag{4.4}$$

PROOF. In order to prove (4.4) first observe that from (2.9) it is evident that

$$\begin{aligned}
&\sum_{k=2}^{n-1}\frac{(1-x_k^2)}{(1-x^2)}|A_k(x)|\leqq\\
&\leqq\sum_{k=2}^{n-1}\frac{(1-x_k^2)}{(1-x^2)}l_k^4(x)+\sum_{k=2}^{n-1}\frac{\Pi_n^2(x)l_k^2(x)}{n(n-1)(1-x^2)P_{n-1}^2(x_k)}+\\
&\qquad+\sum_{k=2}^{n-1}\frac{\Pi_n^2(x)l_k^2(x)|x-x_k|}{n(n-1)(1-x^2)(1-x_k^2)P_{n-1}^2(x_k)}=:\\
&=:J_1+J_2+J_3.
\end{aligned} \tag{4.5}$$

We first note that

$$\begin{aligned}
J_1=&\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{(1-x_k^2)}{(1-x^2)}l_k^4(x)+\frac{(1-x_j^2)}{(1-x^2)}l_j^4(x)+\\
&+\frac{(1-x_{j+1}^2)}{(1-x^2)}l_{j+1}^4(x)+\frac{(1-x_{j-1}^2)}{1-x^2}l_{j-1}^4(x).
\end{aligned} \tag{4.6}$$

But

$$1-x_j^2=\sin^2\theta_j=(\sin\theta_j-\sin\theta+\sin\theta)^2\leqq 2(\sin\theta_j-\sin\theta)^2+2\sin^2\theta.$$

So, on using (3.1)–(3.4), we obtain

$$
\begin{aligned}
\frac{(1-x_j^2)}{(1-x^2)} l_j^2(x) &\leqq 2\frac{(\sin\theta_j-\sin\theta)^2}{(1-x^2)} l_j^2(x) + 2l_j^2(x) \leqq \\
&\leqq \frac{\sin^2\left(\frac{\theta_i-\theta}{2}\right)(1-x^2)[P'_{n-1}(x)]^2}{n^2(n-1)^2 P_{n-1}^2(x_j)\sin^2\left(\frac{\theta+\theta_j}{2}\right)\sin^2\left(\frac{\theta_j-\theta}{2}\right)} + 2 \leqq \\
&\leqq \frac{2n(n-1)^2(1-x_j^2)^{1/2}}{c_5 n^2(n-1)^2\sin^2\left(\frac{\theta+\theta_j}{2}\right)} + 2 \leqq \\
&\leqq \frac{4}{c_5\, n\, \sin\left(\frac{\theta+\theta_j}{2}\right)} + 2 \leqq c_8.
\end{aligned}
\tag{4.7}
$$

Hence, on using (3.1) again we get

$$
\frac{1-x_j^2}{1-x^2} l_j^4(x) \leqq c_8. \tag{4.8}
$$

Similarly we obtain

$$
\frac{1-x_{j+1}^2}{1-x^2} l_{j+1}^4(x) \leqq c_9, \tag{4.9}
$$

and

$$
\frac{1-x_{j-1}^2}{1-x^2} l_{j-1}^4(x) \leqq c_{10}. \tag{4.10}
$$

In the earlier work [16] it was shown that if x_j is one of the zeros of $P'_{n-1}(x)$ closest to x, then

$$
|l_k(x)| \leqq \frac{c_{11}}{i}, \qquad k=j+i,\ k=j-i,\ k\neq j,\ j\pm 1. \tag{4.11}
$$

So we have, due to (3.3), (3.2), (4.11) and (4.3)

$$
\begin{aligned}
\frac{(1-x_k^2)l_k^2(x)}{1-x^2} &= \left[\frac{(1-x^2)+(x^2-x_k^2)}{1-x^2}\right] l_k^2(x) \leqq \\
&\leqq \frac{c_{11}^2}{i^2} + \frac{2|x-x_k|}{1-x^2} l_k^2(x) \leqq \\
&\leqq \frac{c_{12}}{i^2} + \frac{(1-x^2)[P'_{n-1}(x)]^2 n(1-x_k^2)^{1/2}}{c_4 n^2(n-1)^2\sin\left(\frac{\theta+\theta_k}{2}\right)\sin\left(\frac{|\theta-\theta_k|}{2}\right)} \leqq \\
&\leqq \frac{c_{12}}{i^2} + \frac{c_{13}}{n\,\sin\left(\frac{|\theta-\theta_k|}{2}\right)} \leqq \\
&\leqq \frac{c_{14}}{i}, \qquad k\neq j,\ k\neq j\pm 1.
\end{aligned}
\tag{4.12}
$$

Consequently, from (4.11) and (4.12) it follows that

$$\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{(1-x_k^2)l_k^4(x)}{1-x^2}\leqq c_{11}^2c_{14}\sum_{i=2}^{n-1}\frac{1}{i^3}\leqq c_{15}. \tag{4.13}$$

Hence from (4.6), (4.8), (4.9), (4.10) and (4.13) we obtain

$$J_1\leqq c_{16}. \tag{4.14}$$

Next, from the definition of J_2 we have

$$\begin{aligned} J_2=\sum_{k=2}^{n-1}\frac{\Pi_n^2(x)l_k^2(x)}{n(n-1)(1-x^2)P_{n-1}^2(x_k)}&=\\ =\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{\Pi_n^2(x)l_k^2(x)}{n(n-1)(1-x^2)P_{n-1}^2(x_k)}&+\frac{(1-x^2)[P'_{n-1}(x)]^2l_j^2(x)}{n(n-1)P_{n-1}^2(x_j)}+\\ +\frac{(1-x^2)[P'_{n-1}(x)]^2l_{j+1}^2(x)}{n(n-1)P_{n-1}^2(x_{j+1})}&+\frac{(1-x^2)[P'_{n-1}(x)]^2l_{j-1}^2(x)}{n(n-1)P_{n-1}^2(x_{j-1})}. \end{aligned}\tag{4.15}$$

On using (3.1)–(3.3) we observe that

$$\begin{aligned}\frac{(1-x^2)[P'_{n-1}(x)]^2l_j^2(x)}{n(n-1)P_{n-1}^2(x_j)}&\leqq\frac{(1-x^2)[P'_{n-1}(x)]^2(1-x_j^2)^{1/2}}{c_4(n-1)}\leqq\\ &\leqq\frac{(1-x^2)[P'_{n-1}(x)]^2}{c_4(n-1)}\{|\sin\theta_j-\sin\theta|+\sin\theta\}\leqq\\ &\leqq\frac{c_{17}(1-x^2)[P'_{n-1}(x)]^2}{(n-1)}+\frac{(1-x^2)^{3/2}[P'_{n-1}(x)]^2}{c_4(n-1)}\leqq\\ &\leqq c_{18}.\end{aligned}$$

Similarly, we obtain

$$\frac{(1-x^2)[P'_{n-1}(x)]^2l_{j+1}^2(x)}{n(n-1)P_{n-1}^2(x_{j+1})}\leqq c_{19}, \tag{4.17}$$

and

$$\frac{(1-x^2)[P'_{n-1}(x)]^2l_{j-1}^2(x)}{n(n-1)P_{n-1}^2(x_{j-1})}\leqq c_{20}. \tag{4.18}$$

Further, we have

$$\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{\Pi_n^2(x)l_k^2(x)}{n(n-1)(1-x^2)P_{n-1}^2(x_k)}=$$

(4.19)

$$=\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{(1-x^2)^3[P'_{n-1}(x)]^4}{n^3(n-1)^3|x-x_k|^2P_{n-1}^4(x_k)}\leqq\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{(2n)^2(1-x_k^2)n^2}{c_4^2n^3(n-1)^3|x-x_k|^2}\leqq$$

$$\leqq\frac{c_{21}}{n^2}\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{1-x_k^2}{|x-x_k|^2}\leqq\frac{c_{22}}{n^2}\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{1}{\sin^2((\theta-\theta_k)/2)}\leqq c_{23},$$

where in the last step (4.3) was used again. Consequently, from (4.16)–(4.19) it follows that

(4.20) $$J_2\leqq c_{24}$$

For J_3 we have

(4.21)

$$J_3=\sum_{k=2}^{n-1}\frac{\Pi_n^2(x)l_k^2(x)|x-x_k|}{n(n-1)(1-x^2)(1-x_k^2)P_{n-1}^2(x_k)}=$$

$$=\sum_{\substack{k=2\\k\neq j,j\pm1}}^{n-1}\frac{(1-x^2)^3[P'_{n-1}(x)]^4}{n^3(n-1)^3(1-x_k^2)P_{n-1}^4(x_k)|x-x_k|}+$$

$$+\frac{(1-x^2)^2|P'_{n-1}(x)|^3|l_j(x)|}{n^2(n-1)^2(1-x_j^2)|P_{n-1}(x_j)|^3}+\frac{(1-x^2)^2|P'_{n-1}(x)|^3|l_{j+1}(x)|}{n^2(n-1)^2|P_{n-1}(x_j)|^3(1-x_{j+1}^2)}+$$

$$+\frac{(1-x^2)^2|P'_{n-1}(x)|^3|l_{j-1}(x)|}{n^2(n-1)^2(1-x_{j-1}^2)|P_{n-1}(x_j)|^3}.$$

First, due to (3.1)–(3.4) we note that

(4.22)

$$\frac{(1-x^2)^2|P'_{n-1}(x)|^3|l_j(x)|}{n^2(n-1)^2(1-x_j^2)|P_{n-1}(x_j)|^3}\leqq$$

$$\leqq\frac{c_{25}(1-x^2)^{3/2}[P'_{n-1}(x)]^2(1-x^2)^{1/2}|P'_{n-1}(x)|n^{3/2}}{n^2(n-1)^2(1-x_j^2)^{1/4}}\leqq$$

$$\leqq\frac{c_{26}(2n)(n-1)n^{3/2}}{n^2(n-1)^2(1-x_j^2)^{1/4}}\leqq\frac{c_{27}}{(n-1)^{1/2}(1-x_j^2)^{1/4}}\leqq c_{28}.$$

Similarly, we obtain

$$\frac{(1-x^2)^2|P'_{n-1}(x)|^3|l_{j+1}(x)|}{n^2(n-1)^2(1-x_{j+1}^2)|P_{n-1}(x_{j+1})|^3} \leqq c_{29}, \tag{4.23}$$

and

$$\frac{(1-x^2)^2|P'_{n-1}(x)|^3|l_{j-1}(x)|}{n^2(n-1)^2(1-x_{j-1}^2)|P_{n-1}(x_{j-1})|} \leqq c_{30}. \tag{4.24}$$

Also from (3.2) and (3.3) we get

$$\sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{\Pi_n^2(x)l_k^2(x)|x-x_k|}{n(n-1)(1-x^2)(1-x_k^2)P_{n-1}^2(x_k)} =$$

$$= \sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{(1-x^2)^3[P'_{n-1}(x)]^4}{n^3(n-1)^3|x-x_k|(1-x_k^2)[P_{n-1}(x_k)]^4} \leqq$$

$$\leqq \sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{(2n)^2n^2}{c_4^2n^3(n-1)^3|x-x_k|} \leqq \frac{c_{31}}{n^2} \sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{1}{|x-x_k|}.$$

The latter sum, namely

$$\sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{1}{|x-x_k|},$$

can be written as

$$\sum_{k\neq j,j\pm 1}^{n-1} \frac{1}{|\cos\theta-\cos\theta_k|} \leqq \sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{1}{\sin^2\frac{|\theta-\theta_k|}{2}} \leqq \sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{(8n)^2}{i^2},$$

where in the last summation the i's were chosen as in Lemma 4.4. This shows that

$$\sum_{\substack{k=2\\ k\neq j,j\pm 1}}^{n-1} \frac{\Pi_n^2(x)l_k^2(x)|x-x_k|}{n(n-1)(1-x^2)(1-x_k^2)P_{n-1}^2(x_k)} \leqq c_{32}. \tag{4.25}$$

Finally, from (4.21) through (4.25) we obtain

$$J_3 \leqq c_{33}. \tag{4.26}$$

Consequently, from (4.5), (4.14), (4.20) and (4.26) inequality (4.4) follows.

5. Proof of Theorem 1

On using (1.7), (2.1) and the uniqueness of Hermite interpolation, we have

$$S_n(f,x)-\mu_n(x)=\sum_{k=1}^{n}(f(x_k)-\mu_n(x_k))A_k(x)=$$
$$=\sum_{k=2}^{n-1}(f(x_k)-\mu_n(x_k))A_k(x).$$

Now from (1.7) and (3.7) it follows that

$$|S_n(f,x)-\mu_n(x)|\leqq c_2\sum_{k=2}^{n-1}w_2\left(\frac{\sin\theta_k}{n}\right)|A_k(x)|\leqq$$
$$\leqq 3c_2\sum_{k=2}^{n-1}\left(1+\frac{1-x_k^2}{1-x^2}\right)w_2\left(\frac{\sin\theta}{n}\right)|A_k(x)|\leqq \tag{5.1}$$
$$\leqq 3c_2w_2\left(\frac{\sin\theta}{n}\right)\left\{\sum_{k=2}^{n-1}|A_k(x)|+\sum_{k=2}^{n-1}\frac{(1-x_k^2)}{(1-x^2)}|A_k(x)|\right\}.$$

The estimates from Lemmas 4.2 and 4.5 then imply

$$|S_n(f,x)-\mu_n(x)|\leqq c_{34}w_2\left(\frac{\sin\theta}{n}\right). \tag{5.2}$$

On combining (5.2) with (1.7) we obtain (1.9). This completes the proof of Theorem 1.

6. Appendix

In the early paper [16] the second two authors also considered the Hermite interpolation process Q_n given by

$$Q_n(f,x)=\sum_{k=1}^{n}f(x_k)h_k(x)+\sum_{k=1}^{n}\mu_n'(x_k)\sigma_k(x). \tag{6.1}$$

Here the h_k are given as in (1.4) and (1.5),

$$\sigma_k(x)=(x-x_k)l_k^2(x),\qquad k=1,2,\dots,n, \tag{6.2}$$

and μ_n is given as in (1.7). Regarding $Q_n(f,x)$ as defined by (6.1), the following was claimed in [16]:

THEOREM C. *There exists an absolute constant $c_{35} > 0$ such that, for all $f \in C[-1,1]$ and all $-1 \leqq x \leqq 1$, there holds*

$$|Q_n(f,x) - f(x)| \leqq c_{35} w_2 \left(\frac{\sqrt{1-x^2}}{n} \right). \tag{6.3}$$

There is, however, an error in the proof of Theorem C due to an error in the proof of inequality (3.22) given in [16]. We note that Theorem C should be replaced by the following

THEOREM D. *There exists an absolute positive constant c_{36} such that, for all $f \in C[-1,1]$ and all $-1 \leqq x \leqq 1$, the following inequality holds.*

$$|Q_n(f,x) - f(x)| \leqq c_{36} \sum_{k=1}^{n} \frac{1}{i^2} w_2 \left(\frac{i\sqrt{1-x^2}}{n} \right). \tag{6.4}$$

Note that this inequality is weaker than (6.3) (for example, consider the case $f' \in \mathrm{Lip}\, 1$).

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GERHARD-MERCATOR-UNIVERSITÄT
GESAMTHOCHSCHULE DUISBURG
FACHBEREICH 11 MATHEMATIK
FACHGEBIET INFORMATIK
LOTHARSTRASSE 63
D–47048 DUISBURG
GERMANY

gonska@informatik.uni-duisburg.de

DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY
LOS ANGELES, CA 90032
U.S.A.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF FLORIDA
GAINESVILLE, FL 32611
U.S.A.

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RANDOM WALK WITH ALTERNATING EXCURSIONS

E. CSÁKI,[1] A. FÖLDES[2] and P. RÉVÉSZ

Abstract

We investigate the properties of a modification of simple symmetric random walk which has positive and negative excursions alternately. It turns out that many properties of this new random walk remains the same as those of the simple symmetric random walk, but there are essential differences, too.

1. Introduction

Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mathbf{P}(X_j=1)=\mathbf{P}(X_j=-1)=$ $=1/2$ and put $S_0=0$, $S_n=\sum_{j=1}^{n} X_j$, $n=1,2,\dots$. $\{S_n\}_{n=1}^{\infty}$ is the so called simple symmetric random walk, or briefly SSRW.

Let $\varrho_0=0$, $\varrho_k=\min\{i\colon i>\varrho_{k-1}, S_i=0\}$, $k=1,2,\dots$. The section between two consecutive ϱ's is called excursion. Introduce

$$S_n^*=S_n \qquad \text{if} \ \ 0 \leqq n \leqq \varrho_1$$

and

$$S_n^*=(-1)^k X_1 |S_n| \quad \text{if} \ \ \varrho_k \leqq n \leqq \varrho_{k+1}, \ \ k=1,2,\dots .$$

Observe that in $\{S_n^*\}_{n=1}^{\infty}$ the first excursion keeps its original sign and the sign of the others alternate. Thus it is a modified random walk in the sense that instead of having excursions with random signs, we have the same sequence of excursions but with alternating signs. We will call this walk Random Walk with Alternating Excursions, or briefly RWAE. Thus in an RWAE part of the random nature of the SSRW is eliminated. Our concern in this paper is to investigate how much resemblance is shown by $\{S_n^*\}$ to $\{S_n\}$. Is there any essential difference between them? We will see that though most of the properties of RWAE remains the same as those of the SSRW, they have important differences, too.

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2. Number of paths and their probabilities

LEMMA 2.1. *The number of different paths in the first $2n$ steps in RWAE is* $2\binom{2n}{n}$.

PROOF. For SSRW it is well-known that

$$\mathbf{P}(S_1 \geqq 0, \dots, S_{2n} \geqq 0) = \frac{\binom{2n}{n}}{2^{2n}}. \tag{2.1}$$

Thus in $\{|S_i|\}_{i=1}^{\infty}$ the number of paths in the first $2n$ steps is $\binom{2n}{n}$. However, from each path of the nonnegative walk $\{|S_i|\}_{i=1}^{\infty}$ arises 2 paths in $\{S_i^*\}_{i=1}^{\infty}$.

We will say that in a path of length n there are k excursions if $\varrho_{k-1} < n \leqq \varrho_k$.

REMARK. We use asterisks to denote RWAE for distinction from SSRW. The only exception is ϱ_k, since they are the same for both $\{S_i\}$ and $\{S_i^*\}$.

LEMMA 2.2. *If a path of length n of the RWAE has k excursions, then the probability of that path is 2^{k-n-1}.*

PROOF. Every path $(S_1, ..., S_n)$ has probability 2^{-n}. If a path $(S_1^*, ..., S_n^*)$ has k excursions, then attaching random signs to each of the excursions, except to the first one, we get 2^{k-1} paths of $(S_1, \dots, S_n)$. All of these paths correspond to one single path in the RWAE. Hence our lemma follows.

LEMMA 2.3.

$$\mathbf{P}(S_n^* = k) = \mathbf{P}(S_n = k) \qquad n = 0, 1, 2, \dots; \quad k = 0, \pm 1, \pm 2, \dots. \tag{2.2}$$

PROOF. Introduce

$$L_n = \max\{k\colon 0 \leqq k \leqq n, S_k = 0\}$$

and

$$L_n^* = \max\{k\colon 0 \leqq k \leqq n, S_k^* = 0\}$$

the last return time to zero. Clearly $L_n^* = L_n$. Observe that

$$\mathbf{P}(S_n^* = k) = \sum_{s=0}^{n-|k|} \mathbf{P}(S_n^* = k \mid L_n^* = s)\mathbf{P}(L_n^* = s). \tag{2.3}$$

Moreover, for $k \neq 0$

$$\mathbf{P}(S_n^* = k \mid L_n^* = s) = \frac{1}{2}\mathbf{P}(|S_n| = |k| \mid L_n = s) = \mathbf{P}(S_n = k \mid L_n = s). \tag{2.4}$$

(2.3)–(2.4) imply our statement.

REMARK. On the other hand, it is easy to see, that the two-dimensional joint distributions do not necessarily match. As an example

$$\mathbf{P}(S_1 = 1, S_3 = 1) = \frac{1}{4} \quad \text{and} \quad \mathbf{P}(S_1^* = 1, S_3^* = 1) = \frac{1}{8}. \tag{2.5}$$

3. Distribution of the maximum

In this section we compare the distributions of

$$\nu_{2n} = \max_{0 \leqq i \leqq \varrho_{2n}} S_i \quad \text{and} \quad \nu^*_{2n} = \max_{0 \leqq i \leqq \varrho_{2n}} S^*_i .$$

The following well-known result will be applied (see e.g. Révész [7], p. 23).

LEMMA A. *Let $0 \leqq i \leqq k$. Then for any $m \geqq i$*

(3.1)

$$p(0,i,k) = \mathbf{P}(\min\{j\colon j \geqq m, S_j = 0\} < \min\{j\colon j \geqq m, S_j = k\} \mid S_m = i) = \frac{k-i}{k}.$$

Introduce

$$\pi_{2n} = \#\{l\colon 0 < l \leqq 2n, X_{\varrho_{l-1}+1} = 1\} \tag{3.2}$$

the number of positive excursions among the first $2n$ ones of $\{S_i\}$. Then π_{2n} is clearly binomial $(1/2, 2n)$ and hence

$$\mathbf{P}(\nu_{2n} < k) = \sum_{j=0}^{2n} \binom{2n}{j} \frac{1}{2^{2n}} \left(1 - \frac{1}{k}\right)^j = \left(1 - \frac{1}{2k}\right)^{2n}. \tag{3.3}$$

On the other hand, among the first $2n$ excursions of $\{S^*_i\}$ exactly n are positive, so we obtain

$$\mathbf{P}(\nu^*_{2n} < k) = \left(1 - \frac{1}{k}\right)^n. \tag{3.4}$$

Thus the exact distribution of ν_{2n} and ν^*_{2n} are different. It is easy to see, however, that they have the same limit distribution, namely

THEOREM 3.1.

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\nu_{2n}}{n} < x\right) = e^{-1/x}, \qquad x > 0, \tag{3.5}$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\nu^*_{2n}}{n} < x\right) = e^{-1/x}, \qquad x > 0. \tag{3.6}$$

4. Local time

The local times of $\{S_i\}$ and $\{S^*_i\}$ are defined as follows:

$$\xi(k,n) = \#\{i\colon 0 < i \leqq n, S_i = k\} \tag{4.1}$$

and

(4.2) $$\xi^*(k,n) = \#\{i\colon 0 < i \leqq n, S_i^* = k\}.$$

From the definition it is obvious that

$$\mathbf{P}(\xi^*(0,n) = j) = \mathbf{P}(\xi(0,n) = j).$$

On the other hand, $\xi^*(k,n)$ and $\xi(k,n)$ for $k \neq 0$ have different distributions. From (2.5) we have

$$\mathbf{P}(\xi(1,3) = 2) = \frac{1}{4} \qquad \text{and} \qquad \mathbf{P}(\xi^*(1,3) = 2) = \frac{1}{8}.$$

We recall from Csörgő and Révész [3] that

LEMMA B. *For any $k \neq 0$*

$$\mathbf{P}(\xi(k,\varrho_1) = 0) = 1 - \frac{1}{2|k|},$$

$$\mathbf{P}(\xi(k,\varrho_1) = m) = \left(\frac{1}{2|k|}\right)^2 \left(1 - \frac{1}{2|k|}\right)^{m-1}, \qquad m = 1, 2, \ldots$$

$$\mathbf{E}(\xi(k,\varrho_1)) = 1, \qquad \operatorname{Var} \xi(k,\varrho_1) = 4|k| - 2.$$

For the RWAE we determine the distribution of $\xi^*(k,\varrho_2)$, since in the first two excursions exactly one is positive.

LEMMA 4.1. *For any $k \neq 0$*

(4.3) $$\mathbf{P}(\xi^*(k,\varrho_2) = 0) = 1 - \frac{1}{|k|},$$

(4.4) $$\mathbf{P}(\xi^*(k,\varrho_2) = m) = \frac{1}{2k^2}\left(1 - \frac{1}{2|k|}\right)^{m-1}, \qquad m = 1, 2, \ldots$$

(4.5) $$\mathbf{E}(\xi^*(k,\varrho_2)) = 2, \qquad \operatorname{Var} \xi^*(k,\varrho_2) = 8|k| - 6.$$

PROOF. The idea of the proof is the same as that of Lemma B. From the two excursions considered exactly one is positive. Lemma A gives immediately (4.3). To get (4.4) observe that after hitting k first there are exactly $m-1$ returns to k before hitting 0. Thus

(4.6) $$\mathbf{P}(\xi^*(k,\varrho_2) = m) = \frac{1}{|k|} \sum_{j=0}^{m-1} \binom{m-1}{j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\frac{|k|-1}{|k|}\right)^{m-1-j} \frac{1}{2|k|} = \\ = \frac{1}{2k^2}\left(1 - \frac{1}{2|k|}\right)^{m-1}.$$

(4.5) follows easily from (4.3) and (4.4).

As a simple consequence of Lemma 4.1, the strong law of large numbers implies that

$$\lim_{l\to\infty}\frac{\xi^*(k,\varrho_{2l})}{2l}=1 \quad \text{a.s..} \tag{4.7}$$

Moreover, applying (4.7) for the subsequence $\{\xi^*(0,n)\}_{n=1}^{\infty}$ one can easily deduce

THEOREM 4.1. *For any fixed k*

$$\mathbf{P}\Big(\lim_{n\to\infty}\frac{\xi^*(k,n)}{\xi^*(0,n)}=1\Big)=1. \tag{4.8}$$

Obviously for SSRW

$$\mathbf{P}\Big(\lim_{n\to\infty}\frac{\xi(k,n)}{\xi(0,n)}=1\Big)=1. \tag{4.9}$$

Since $\xi(0,n)=\xi^*(0,n)$, (4.8) and (4.9) imply that

$$\mathbf{P}\Big(\lim_{n\to\infty}\frac{\xi^*(k,n)}{\xi(k,n)}=1\Big)=1. \tag{4.10}$$

Theorem 4.1 and (4.10) also imply that $\xi^*(k,n)/\sqrt{n}$ and $\xi(k,n)/\sqrt{n}$ have the same limit distribution, namely we have for any fixed k and $x>0$

$$\lim_{n\to\infty}\mathbf{P}\Big(\frac{\xi^*(k,n)}{\sqrt{n}}<x\Big)=2\Phi(x)-1. \tag{4.11}$$

5. The arcsine law

One of the most appealing result about the SSRW is the celebrated arcsine law. It is quite natural to investigate whether this law remains valid for the RWAE. First we give a recursion for the time spent by RWAE on the positive side. Define

$$\begin{aligned}\mu_n&=\#\{i\colon 1\leqq i\leqq n, S_i>0\},\\ \mu_n^*&=\#\{i\colon 1\leqq i\leqq n, S_i^*>0\}.\end{aligned} \tag{5.1}$$

$$\begin{aligned}q_{k,n}&=\mathbf{P}(\mu_n=k\mid X_1=1),\\ q_{k,n}^*&=\mathbf{P}(\mu_n^*=k\mid X_1^*=1).\end{aligned} \tag{5.2}$$

Then we have

$$q^*_{2k,2n} = \sum_{l=1}^{k} \mathbf{P}(\varrho_1 = 2l) q^*_{2n-2k,2n-2l} \quad \text{for } 0 < k < n \tag{5.3}$$

and

$$q^*_{0,2n} = 0, \quad q^*_{2n,2n} = \mathbf{P}(\varrho_1 \geqq 2n). \tag{5.4}$$

It is known that

$$\mathbf{P}(\varrho_1 = 2l) = \frac{1}{l 2^{2l-1}} \binom{2l-2}{l-1}, \qquad l = 1, 2, \dots .$$

To see (5.3) it is enough to observe that the RWAE starting with a positive step spends $2k$ (out of $2n$) steps on the positive side if the length of the first excursion is $2l$ $(0 < l \leqq k)$ and from the remaining $2n - 2l$ steps (the first of which is now negative) $2n - 2k$ steps are spent on the negative side. But this probability is $q^*_{2n-2k,2n-2l}$.

By symmetry we have

$$\mathbf{P}(\mu^*_{2n} = 2k) = \frac{1}{2}(q^*_{2k,2n} + q^*_{2n-2k,2n}). \tag{5.5}$$

From this recursion we get

$$\begin{aligned} \mathbf{P}(\mu^*_8 = 8) = \mathbf{P}(\mu^*_8 = 8) &= \frac{5}{32} \\ \mathbf{P}(\mu^*_8 = 2) = \mathbf{P}(\mu^*_8 = 6) &= \frac{7}{32} \\ \mathbf{P}(\mu^*_8 = 4) &= \frac{1}{4}. \end{aligned}$$

For comparison, the corresponding probabilities for SSRW are:

$$\begin{aligned} \mathbf{P}(\mu_8 = 0) = \mathbf{P}(\mu_8 = 8) &= \frac{35}{2^7} \\ \mathbf{P}(\mu_8 = 2) = \mathbf{P}(\mu_8 = 6) &= \frac{5}{2^5} \\ \mathbf{P}(\mu_8 = 4) &= \frac{9}{2^6}. \end{aligned}$$

Thus for fixed n the two distributions are different. To get the limit distributions from (5.3)–(5.5) seems to be difficult. Later we shall see, however, from Donsker's theorem that the limit distribution of μ^*_n is also the arcsine distribution. In what follows we deal with μ^* at excursion endpoints and show that in this context we can get arcsine law, too, in the limit.

THEOREM 5.1. *For* $0 \leq a < b \leq 1$ *we have*

$$\lim_{k\to\infty} \mathbf{P}\left(a < \frac{\mu^*_{\varrho_k}}{\varrho_k} < b\right) = \int_a^b \frac{1}{\pi} \frac{1}{(x(1-x))^{1/2}} dx. \tag{5.6}$$

PROOF. Introduce

$$Z_i = \varrho_i - \varrho_{i-1}, \qquad i = 1, 2, \ldots$$

the length of the ith excursion. $\{Z_i\}_{i=1}^{\infty}$ is an i.i.d. sequence. Since exactly k out of $2k$ excursions are positive, we get that

$$\frac{\mu^*_{\varrho_{2k}}}{\varrho_{2k}} \quad \text{and} \quad \frac{\sum_{i=1}^{k} Z_i}{\sum_{i=1}^{2k} Z_i}$$

have the same distribution. Hence we get for $0 < x < 1$

$$\mathbf{P}\left(\frac{\mu^*_{\varrho_{2k}}}{\varrho_{2k}} < x\right) = \mathbf{P}\left(\frac{\sum_{i=1}^{k} Z_i}{\sum_{i=1}^{2k} Z_i} < x\right) = \mathbf{P}\left((1-x)\sum_{i=1}^{k} Z_i < \sum_{i=k+1}^{2k} Z_i\right). \tag{5.7}$$

Observing the independence of

$$T_k = \frac{\sum_{i=1}^{k} Z_i}{k^2} \quad \text{and} \quad T_k^{(1)} = \frac{\sum_{i=k+1}^{2k} Z_i}{k^2}$$

and using their well-known limit distribution

$$\lim_{k\to\infty} \mathbf{P}(T_k < u) = \lim_{k\to\infty} \mathbf{P}(T_k^{(1)} < u) = 2\left(1 - \Phi\left(\frac{1}{\sqrt{u}}\right)\right), \tag{5.8}$$

we get

$$\lim_{k\to\infty} \mathbf{P}\left(\frac{\mu^*_{\varrho_{2k}}}{\varrho_{2k}} < x\right) = \frac{4}{\sqrt{2\pi}} \int_0^\infty \left(1 - \Phi\left(s\sqrt{\frac{1-x}{x}}\right)\right) e^{-s^2/2}\, ds. \tag{5.9}$$

On applying the formula 6.2851 in Gradshteĭn and Ryzhik [5] we arrive at (5.6).

Similar arcsine law holds also for SSRW, i.e.

THEOREM 5.2. *For* $0 \leqq a < b \leqq 1$ *we have*

$$\lim_{k\to\infty} \mathbf{P}\Big(a < \frac{\mu_{\varrho_k}}{\varrho_k} < b\Big) = \int_a^b \frac{1}{\pi} \frac{1}{(x(1-x))^{1/2}}\,dx. \tag{5.10}$$

PROOF. Though it would be possible to prove this theorem directly, we want to demonstrate that this kind of arcsine law (and perhaps many other results) is equivalent for SSRW and RWAE. Therefore we think that the following proof is interesting on its own right and has also other consequences.

Assume that we have $2n$ excursions with lengths $Z_1, \dots, Z_{2n}$. Denote by $Y_1 \geqq Y_2 \geqq \dots \geqq Y_{2n}$ these lengths in nonincreasing order. Define

$$\varepsilon_i = \begin{cases} 1, & \text{if the excursion in SSRW with length } Y_i \text{ is positive} \\ 0, & \text{if the excursion in SSRW with length } Y_i \text{ is negative,} \end{cases}$$

and

$$\varepsilon_i^* = \begin{cases} 1, & \text{if the excursion in RWAE with length } Y_i \text{ is positive} \\ 0, & \text{if the excursion in RWAE with length } Y_i \text{ is negative.} \end{cases}$$

Then

$$\mu_{\varrho_{2n}} = \sum_{i=1}^{2n} \varepsilon_i Y_i$$

and

$$\mu^*_{\varrho_{2n}} = \sum_{i=1}^{2n} \varepsilon_i^* Y_i.$$

Now let $k \leqq 2n$ and consider

$$A = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$$

and

$$A^* = (\varepsilon_1^*, \varepsilon_2^*, \dots, \varepsilon_k^*).$$

Then since $\varepsilon_1, \dots, \varepsilon_k$ are independent, any possible value of A has probability $1/2^k$. The variables $\varepsilon_1^*, \dots, \varepsilon_k^*$, however, are not independent. The possible values of A^* have the probabilities

$$\frac{\binom{n}{j}\binom{n}{k-j}}{\binom{2n}{k}\binom{k}{j}}, \qquad j = \sum_{i=1}^{k} \varepsilon_i^*.$$

We use the following lemma.

Lemma 5.1. *Let A and A^*, resp. two random quantities with possible values $a_1, \dots, a_r$ and distributions $p_1, \dots, p_r$ and $q_1, \dots, q_r$, resp. Then one can define a joint distribution of (A, A^*) such that*

$$\mathbf{P}(A \neq A^*) \leqq \frac{1}{2} \sum_{i=1}^{r} |p_i - q_i|. \tag{5.11}$$

This result is well-known but can be proved very easily by taking $\mathbf{P}(A = a_i, A^* = a_i) = \min(p_i, q_i)$ and arbitrary otherwise with attention to the given marginal distributions.

By this lemma, we can redefine $(\varepsilon_1, \dots, \varepsilon_k)$ and $(\varepsilon_1^*, \dots, \varepsilon_k^*)$ without changing their distributions such that

$$\mathbf{P}((\varepsilon_1, \dots, \varepsilon_k) \neq (\varepsilon_1^*, \dots, \varepsilon_k^*)) \leqq \frac{1}{2} \sum \left| \frac{1}{2^k} - \frac{\binom{n}{j}\binom{n}{k-j}}{\binom{2n}{k}\binom{k}{j}} \right| =$$

$$= \frac{1}{2} \sum_{j=0}^{k} \binom{k}{j} \left| \frac{1}{2^k} - \frac{\binom{n}{j}\binom{n}{k-j}}{\binom{2n}{k}\binom{k}{j}} \right| = \frac{1}{2} \sum_{j=0}^{k} \frac{1}{2^k} \binom{k}{j} \left| 1 - \frac{2^k \binom{2n-k}{n-j}}{\binom{2n}{n}} \right| \leqq \frac{ck^2}{n},$$

where in the last step we used Stirling's formula. Hence

$$\mathbf{P}\left(\sum_{i=1}^{k} \varepsilon_i Y_i \neq \sum_{i=1}^{k} \varepsilon_i^* Y_i \right) \leqq \frac{ck^2}{n}. \tag{5.12}$$

By repeating this procedure for every n and choosing $k = k_n = [\log n]$, it follows that

$$\frac{\sum_{i=1}^{k_n} \varepsilon_i Y_i}{\sum_{i=1}^{k_n} \varepsilon_i^* Y_i} \xrightarrow{p} 1 \qquad \text{as } n \to \infty. \tag{5.13}$$

On the other hand, it follows from Csáki et al. [2] that

$$\frac{\sum_{i=k_n+1}^{2n} Y_i}{\sum_{i=1}^{2n} Y_i} \xrightarrow{p} 0 \qquad \text{as } n \to \infty. \tag{5.14}$$

This combined with (5.13) yields

$$\frac{\sum_{i=1}^{2n} \varepsilon_i Y_i}{\sum_{i=1}^{2n} \varepsilon_i^* Y_i} \xrightarrow{p} 1 \qquad \text{as } n \to \infty. \tag{5.15}$$

6. A new representation and invariance principles for RWAE

It is obvious from the definition of $\{S_i^*\}$ that its excursions are either identical with or the reflections of the excursions of $\{S_i\}$. Hence either $S_i^* = S_i$ or $S_i^* = -S_i$. In the latter case S_i^* and S_i may be "far" from each other and we cannot claim that they have the same limit properties. Therefore we can ask what is the limit process associated with RWAE? In this section we give another representation of RWAE in terms of SSRW and show that they are "close" to each other so that they have the same limit process and as a consequence, Donsker's theorem holds for RWAE.

Let $\{X_n\}_{n=1}^\infty$ and $\{S_n\}_{n=1}^\infty$ be as in Section 1. Define S_n^{**} as follows.

$$S_0^{**}=0, \qquad S_n^{**}=\sum_{i=1}^n X_i^{**}=\sum_{i=1}^n \delta_i X_i, \tag{6.1}$$

where

$$\delta_i=\begin{cases} 1, & \text{if } i=1 \text{ or } S_{i-1}^{**}\neq 0, \quad i=2,3,\dots \\ X_{i-1}X_i, & \text{if } S_{i-1}^{**}=0, \quad i=2,3,\dots . \end{cases} \tag{6.2}$$

It can be easily seen that $\{S_n^*\}_{n=1}^\infty$ and $\{S_n^{**}\}_{n=1}^\infty$ have the same distribution, hence the latter is also an RWAE. Thus from now on when talking about RWAE, we can use either of the two definitions according to our convenience. We show that $\{S_n\}$ and $\{S_n^{**}\}$ are close to each other.

LEMMA 6.1. *For any $\varepsilon>0$ we have as $n\to\infty$*

$$|S_n^{**}-S_n|=O(n^{\frac{1}{4}+\varepsilon}) \quad a.s.. \tag{3.6}$$

PROOF. From the definition

$$S_n^{**}-S_n=\sum_{i=1}^n (X_i^{**}-X_i)=\sum_{i=1}^n X_i^{**}I\{S_{i-1}^{**}=0\}-\sum_{i=1}^n X_i I\{S_{i-1}^{**}=0\}. \tag{6.4}$$

Since $X_i^{**}I\{S_{i-1}^{**}=0\}$ are alternately $+1$ and -1, we have

$$\Big|\sum_{i=1}^n X_i^{**}I\{S_{i-1}^{**}=0\}\Big|\leq 1. \tag{6.5}$$

On the other hand,

$$\sum_{i=1}^n X_i I\{S_{i-1}^{**}=0\}=\sum_{j=1}^{\xi^{**}(0,n)} Z_j, \tag{6.6}$$

where Z_j are i.i.d. with $\mathbf{P}(Z_j=1)=\mathbf{P}(Z_j=-1)=1/2$. Since $\{\xi^{**}(0,n)\}_{n=1}^\infty$ has the same distribution as $\{\xi(0,n)\}_{n=1}^\infty$, (6.3) follows from (6.5), (6.6) and the laws of the iterated logarithm for $\sum Z_j$ and $\xi(0,n)$.

Lemma 6.1 has many consequences. It follows e.g. that Donsker's theorem holds for $\{S_n^{**}\}$, i.e. we have

THEOREM 6.1. *Let*

$$S_n^{**}(t) = n^{-1/2}(S_{[nt]}^{**} + X_{[nt]+1}^{**}(nt - [nt])). \tag{6.7}$$

Then

$$S_n^{**}(t) \Rightarrow W(t) \qquad as \quad n \to \infty, \tag{6.8}$$

where $W(t)$ is a standard Wiener process.

This theorem implies, e.g., that the arcsine law holds also for the RWAE.
It follows moreover from Lemma 6.1 that we have strong approximations, too, for RWAE, i.e.

THEOREM 6.2. *On an appropriate probability space one can define an RWAE $\{S_n^{**}\}_{n=1}^{\infty}$ and a standard Wiener process $\{W(t), t \geqq 0\}$ such that as $n \to \infty$*

$$S_n^{**} - W(n) = o(n^{\frac{1}{4}+\varepsilon}) \quad a.s. \tag{6.9}$$

for any $\varepsilon > 0$.

This theorem has many consequences. It follows e.g. that Strassen's LIL and Chung's LIL hold for RWAE. Moreover, we have also a.s. central limit theorem (see Theorem 1 in Lacey and Philipp [6])

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\delta(S_k(t))}{k} \Rightarrow W(t) \quad \text{a.s.}, \tag{6.10}$$

where $\delta(x)$ is the point mass at $x \in C[0,1]$. In particular, we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\{S_n^{**} < 0\}}{k} = \frac{1}{2} \quad \text{a.s.}. \tag{6.11}$$

7. Dobrushin's Theorem for RWAE

A theorem of Dobrushin [4] reads as follows.

THEOREM A. *Let $\{S_n\}_{n=1}^{\infty}$ be a SSRW and let $f(x)$ be a real valued function on integers with finite support such that $\sum_{x=-\infty}^{\infty} f(x) = 0$. Then*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\sum_{i=1}^{n} f(S_i)}{dn^{1/4}} < z \right) = \mathbf{P}(U\sqrt{|V|} < z), \tag{7.1}$$

where U and V are two independent standard normally distributed random variables and

$$d^2 = 4 \sum_{x=-\infty}^{\infty} x f^2(x) + 8 \sum_{-\infty<x<y<\infty} x f(x) f(y) - \sum_{x=-\infty}^{\infty} f^2(x). \tag{7.2}$$

This theorem has been extended in the literature under more general conditions. In particular, the condition of $f(x)$ being of finite support can be replaced by the weaker condition

$$\sum_{x=-\infty}^{\infty} |x f(x)| < \infty \tag{7.3}$$

(see Csáki et al. [1]).

An analogue result is true also for RWAE.

THEOREM 7.1. *Let $\{S_n^*\}_{n=1}^{\infty}$ be an RWAE and let $f(x)$ be a real valued function on integers satisfying* (7.3) *and* $\sum\limits_{x=-\infty}^{\infty} f(x) = 0$. *Then*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\sum\limits_{i=1}^{n} f(S_i^*)}{d^* n^{1/4}} < z \right) = \mathbf{P}(U\sqrt{|V|} < z), \tag{7.4}$$

where U and V are two independent standard normally distributed random variables and

$$d^{*2} = d^2 - \left(f(0) + 2\sum_{x=1}^{\infty} f(x) \right)^2. \tag{7.5}$$

PROOF. This theorem can be proved along the lines of proof in Csáki et al. [1]. The idea there was to split the sum $\sum f(S_i)$ into i.i.d. terms, one term representing the summation for an excursion. First considering fixed number of excursions, i.e.

$$\sum_{i=1}^{\varrho_{2n}} f(S_i^*) = \sum_{j=1}^{n} \sum_{i=\varrho_{2j-2}+1}^{\varrho_{2j}} f(S_i^*) =$$

$$= \sum_{j=1}^{n} \sum_{x=-\infty}^{\infty} f(x)(\xi^*(x, \varrho_{2j}) - \xi^*(x, \varrho_{2j-2})) = \sum_{i=1}^{n} Z_i^*$$

we observe that this sum can be written as a sum of i.i.d. random variables, hence by the central limit theorem we may conclude that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\sum\limits_{i=1}^{\varrho_{2n}} f(S_i^*)}{d^*(2n)^{1/2}} < z \right) = \Phi(z),$$

where $2d^{*2} = \operatorname{Var} Z_j^*$. Here we used that by (4.5) $\mathbf{E}Z_j^* = 0$. The asymptotic independence of

$$\frac{\sum_{i=1}^{\varrho_{2n}} f(S_i^*)}{d^*(2n)^{1/2}} \quad \text{and} \quad \frac{\varrho_{2n}}{n^2}$$

and the statement of Theorem 7.1 can be proved exactly as in Csáki et al. [1].

The constant d^* can be calculated in the following manner.

Let $Z = \sum_{i=1}^{\varrho_1} f(S_i)$. Then we know that $d^2 = \mathbf{E}Z^2$. On the other hand,

$$Z = \begin{cases} \sum_{x=0}^{\infty} f(x)\xi(x,\varrho_1) = f(0) + \sum_{x=1}^{\infty} f(x)\xi^*(x,\varrho_2), & \text{if } X_1 = 1 \\ \sum_{x=-\infty}^{0} f(x)\xi(x,\varrho_1) = f(0) + \sum_{x=-\infty}^{-1} f(x)\xi^*(x,\varrho_2), & \text{if } X_1 = -1. \end{cases}$$

We get

$$d^2 = \mathbf{E}Z^2 = \frac{1}{2}\left(\mathbf{E}\Big(f(0) + \sum_{x=1}^{\infty} f(x)\xi^*(x,\varrho_2)\Big)^2 + \mathbf{E}\Big(f(0) + \sum_{x=-\infty}^{-1} f(x)\xi^*(x,\varrho_2)\Big)^2\right) =$$

$$= \frac{1}{2}\mathbf{E}(Z_1^*)^2 - \mathbf{E}\Big(f(0) + \sum_{x=1}^{\infty} f(x)\xi^*(x,\varrho_2)\Big)\Big(f(0) + \sum_{x=-\infty}^{-1} f(x)\xi^*(x,\varrho_2)\Big) =$$

$$= d^{*2} - \Big(f(0) + 2\sum_{x=1}^{\infty} f(x)\Big)\Big(f(0) + 2\sum_{x=-\infty}^{-1} f(x)\Big).$$

Taking into account that $\sum_{x=-\infty}^{\infty} f(x) = 0$, we arrive at (7.5).

Following Csáki et al. [1] we can also see that the next strong approximation result holds:

THEOREM 7.2. *Let $f(x)$ be a real valued function on integers such that*

$$\sum_{x=-\infty}^{\infty} |x|^{1+\delta}|f(x)| < \infty$$

for some $\delta > 0$ and $\sum_{x=-\infty}^{\infty} f(x) = 0$. Then on a suitable probability space one can define an RWAE $\{S_n^*\}_{n=1}^{\infty}$ and independent standard Wiener processes $\{W^{(1)}(t), W^{(2)}(t), t \geqq 0\}$ such that

$$\sum_{i=1}^{n} f(S_i^*) - d^* W^{(2)}(L^{(1)}(n)) = O(n^{1/4-\varepsilon}) \quad a.s., \tag{7.6}$$

and

$$\xi^*(0,n) - L^{(1)}(n) = O(n^{1/2-\varepsilon}) \quad a.s. \tag{7.7}$$

for all ε *small enough, where* $L^{(1)}(n)$ *is the local time at zero of* $W^{(1)}(t)$.

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

CITY UNIVERSITY OF NEW YORK
130 STUYVESANT PLACE
STATEN ISLAND, NY 10301
U.S.A.

INSTITUT FÜR WAHRSCHEINLICHKEITSRECHNUNG UND
MATHEMATISCHE STATISTIK
TECHNISCHE UNIVERSITÄT WIEN
WIEDNER HAUPTSTRASSE 8–10/107
A–1040 WIEN
AUSTRIA

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EXTENDING A FAMILY OF CAUCHY STRUCTURES IN A LIMIT SPACE. II

J. DEÁK

Abstract

The problem mentioned in the title was in fact dealt with in Part I*. Now we consider the following related problems: (i) extending less/more general structures (fully/pointwise Cauchy screens) in a limit or pseudotopological space; (ii) extending convergences, limitations and pseudotopologies.

§ 7. Fully Cauchy screens

7.1 For the elements of the screen $\mathfrak{S}$, consider the following equivalence relation: $\mathfrak{s} \sim \mathfrak{t}$ iff there is an $\mathfrak{S}$-chain joining $\mathfrak{s}$ and $\mathfrak{t}$. Denote by $\mathfrak{C}(\mathfrak{s})$ the equivalence class containing $\mathfrak{s}$, $\operatorname{St}\mathfrak{s} = \bigcap \mathfrak{C}(\mathfrak{s})$, and let $\mathfrak{S}_{\mathrm{E}}$ be the screen for which $\{\operatorname{St}\mathfrak{s} \colon \mathfrak{s} \in \mathfrak{S}\}$ is a base. If $\mathfrak{S}$ is Cauchy then $\mathfrak{s} \sim \mathfrak{t}$ iff $\mathfrak{s} \cap \mathfrak{t} \in \mathfrak{S}$. We have $\mathfrak{S}_{\mathrm{F}} < \mathfrak{S}_{\mathrm{E}} < \mathfrak{S}_{\mathrm{C}}$, $\mathfrak{S}_{\mathrm{E}} < \mathfrak{S}_{\mathrm{R}}$ and $\mathfrak{S}_{\mathrm{E}} = (\mathfrak{S}_{\mathrm{C}})_{\mathrm{E}}$.

The proof of the following result is similar to that of [15] Proposition 17.1, but simpler:

PROPOSITION. *If a countable family of fully Cauchy screens in a set has Cauchy extensions then it has fully Cauchy extensions, too;*

$$\mathfrak{S}^1_{\mathrm{F}}(X, \mathfrak{S}_i) = \mathfrak{S}^1_{\mathrm{E}}(X, \mathfrak{S}_i) \tag{1}$$

is the finest one.

PROOF. Put $\mathfrak{S}^1 = \mathfrak{S}^1(X, \mathfrak{S}_i)$. It is enough to show that $\mathfrak{S}^1_{\mathrm{E}}$ is a fully Cauchy extension.

1° $\mathfrak{S}^1$ being an extension, we have $\mathfrak{S}^1_{\mathrm{E}} \mid X_i < \mathfrak{S}_i$. To prove the converse, let $\emptyset \notin \mathfrak{s} \in \mathfrak{S}^1_{\mathrm{E}}$, $i \in I$; it has to be proved that $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$. Pick $\mathfrak{t} \in \mathfrak{S}^1$ such that

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$\mathfrak{s} \supset \mathrm{St}\,\mathfrak{t}$; then either $\mathfrak{t} = \mathfrak{s}_j^1$ with some $j \in I$, $\emptyset \notin \mathfrak{s}_j \in \mathfrak{S}_j$ or $\mathfrak{t} = \dot{x}$, $x \in X \setminus \bigcup_{j \in I} X_j$. In the second case, $\mathrm{St}\,\mathfrak{t} = \dot{x}$, too, thus $\mathfrak{s} \mid X_i = \exp X_i \in \mathfrak{S}_i$.

Assume $\mathfrak{t} = \mathfrak{s}_j^1$, $X_i \in \sec \mathrm{St}\,\mathfrak{t}$. Then

$$(\mathrm{St}\,\mathfrak{t}) \mid X_i = \bigcap \{\mathfrak{p} \mid X_i : \mathfrak{p} \sim \mathfrak{t},\ X_i \in \sec \mathfrak{p}\}. \tag{2}$$

Pick one of these filters $\mathfrak{p} \mid X_i = \mathfrak{p}_i$. As $\mathfrak{S}_i$ is fully Cauchy, there is a coarsest $\mathfrak{q}_i \in \mathfrak{S}_i$ coarser than $\mathfrak{p}_i$. Now $\mathfrak{q}_i^1 \sim \mathfrak{t}$ follows from $\mathfrak{q}_i^1 \mathbin{\triangle} \mathfrak{p} \sim \mathfrak{t}$, thus $\mathfrak{q}_i^1$ is among the filters $\mathfrak{p}$ in (2), and so $(\mathrm{St}\,\mathfrak{t}) \mid X_i \subset \mathfrak{q}_i$. If $\mathfrak{p}_i' = \mathfrak{p}' \mid X_i$ is another filter from the right-hand side of (2) then $\mathfrak{q}_i^1 \sim \mathfrak{t} \sim \mathfrak{p}'$, $\mathfrak{q}_i^1 \cap \mathfrak{p}' \in \mathfrak{S}_C^1$, which is an extension if there are Cauchy extensions, $\mathfrak{q}_i \cap \mathfrak{p}_i' \in \mathfrak{S}_i$, implying $\mathfrak{p}_i' \supset \mathfrak{q}_i$. The intersection of all these filters $\mathfrak{p}_i'$ is also finer than $\mathfrak{q}_i$, thus

$$(\mathrm{St}\,\mathfrak{t}) \mid X_i = \mathfrak{q}_i. \tag{3}$$

Therefore $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$. Moreover, $\mathfrak{t} \sim ((\mathrm{St}\,\mathfrak{t}) \mid X_i)^1$.

2°. We are going to show that if $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}^1$ are not equivalent then $\mathrm{St}\,\mathfrak{s} \mathbin{\overline{\triangle}} \mathrm{St}\,\mathfrak{t}$ (forget the notations used in 1°); then the filters $\mathrm{St}\,\mathfrak{s}$ ($\emptyset \notin \mathfrak{s} \in \mathfrak{S}^1$) are the coarsest filters required in the definition of a fully Cauchy screen. The other cases being trivial, let us assume that $\mathfrak{s} = \mathfrak{s}_i^1$, $\mathfrak{t} = \mathfrak{t}_j^1$, $i, j \in I$, $\emptyset \notin \mathfrak{s}_i \in \mathfrak{S}_i$, $\emptyset \notin \mathfrak{t}_j \in \mathfrak{S}_j$. Put $\mathfrak{p} = \mathrm{St}\,\mathfrak{s}$, $\mathfrak{q} = \mathrm{St}\,\mathfrak{t}$, $\mathfrak{p}_k = \mathfrak{p} \mid X_k$, $\mathfrak{q}_k = \mathfrak{q} \mid X_k$ $(k \in I)$. According to 1°, $\mathfrak{p}_k, \mathfrak{q}_k \in \mathfrak{S}_k$. We claim that

$$\mathfrak{p} = \bigcap \{\mathfrak{p}_k^1 : k \in I,\ \emptyset \notin \mathfrak{p}_k\}, \tag{4}$$

and similarly for $\mathfrak{q}$. It is enough to show that $\mathfrak{p}$ is finer than the intersection, since the converse is evident.

Assume $\mathfrak{s} \sim \mathfrak{s}'$. Then $\mathfrak{s}' = \mathfrak{f}_h^1$ with some $h \in I$, $\emptyset \notin \mathfrak{f}_h \in \mathfrak{S}_h$ (the elements of $\mathfrak{S}^1$ not of this type cannot occur in a chain starting from $\mathfrak{s} = \mathfrak{s}_i^1$). $\mathfrak{s}' \supset \mathfrak{p}$ implies $\mathfrak{f}_h \supset \mathfrak{p}_h$, thus $\mathfrak{s}'$ is finer than the right-hand side of (4). This proves (4), since $\mathfrak{p}$ is the intersection of all these filters $\mathfrak{s}'$.

If $\emptyset \notin \mathfrak{p}_k$, $\emptyset \notin \mathfrak{q}_k$ then $\mathfrak{s} \sim \mathfrak{p}_k^1$, $\mathfrak{t} \sim \mathfrak{q}_k^1$ (see at the end of 1°). Now $\mathfrak{p}_k \mathbin{\triangle} \mathfrak{q}_k$ would imply $\mathfrak{s} \sim \mathfrak{t}$. Hence

$$\mathfrak{p}_k \mathbin{\overline{\triangle}} \mathfrak{q}_k \qquad (k \in I). \tag{5}$$

Put $I = \{i_1, i_2, \dots\}$ (possibly only a finite sequence), and pick $A_n \in \mathfrak{p}_{i_n}$, $B_n \in \mathfrak{q}_{i_n}$ such that $A_n = \emptyset$ if $\emptyset \in \mathfrak{p}_{i_n}$, $B_n = \emptyset$ if $\emptyset \in \mathfrak{q}_{i_n}$, and $A_n \cap B_n = \emptyset$ (see (5)). It can also be assumed (cf. (4)) that

$$A_n \cap X_{i_m} \subset A_m, \quad B_n \cap X_{i_m} \subset B_m \qquad (m < n). \tag{6}$$

Now $A = \bigcup_n A_n \in \mathfrak{p}$, $B = \bigcup_n B_n \in \mathfrak{q}$. $A \cap B = \emptyset$ follows from (6), thus $\mathfrak{p} \mathbin{\overline{\triangle}} \mathfrak{q}$, indeed. □

COROLLARY. *A countable family of screens in a set has a fully Cauchy extension iff each finite subfamily has one. Two fully Cauchy screens in a set always have a fully Cauchy extension.*

PROOF. Proposition 4.2. □

REMARKS. a) Let I be finite. With $\mathfrak{s},\mathfrak{p}$ and $\mathfrak{p}_k$ as in 2°, $\mathfrak{s}\sim\mathfrak{p}_k^1$ whenever $\emptyset\notin\mathfrak{p}_k$. Thus there is an $\mathfrak{S}^1$-chain containing $\mathfrak{s}$ and all these filters $\mathfrak{p}_k^1$. The intersection of this chain is finer than $\mathfrak{p}$, thus, by (4), it is in fact equal to $\mathfrak{p}$. Therefore $\mathfrak{S}_{\mathrm{F}}^1(X,\mathfrak{S}_i)=\mathfrak{S}_{\mathrm{C}}^1(X,\mathfrak{S}_i)$.

b) Let I be countably infinite. Similarly to a), the intersections of the infinite $\mathfrak{S}^1$-chains form a base for $\mathfrak{S}_{\mathrm{F}}^1(X,\mathfrak{S}_i)$.

c) Let I be uncountable. The examples below show that the statement of the proposition is now false (even the existence of a CL extension does not imply that there is a fully Cauchy extension), and also that (1) does not necessarily hold even when there do exist fully Cauchy extensions. The corollary does not hold either for uncountable families.

d) In Example 4.3 (considered without the convergence), there is no coarsest one among the fully Cauchy extensions.

EXAMPLES. a) (A modification of [15] Example 17.3.) Let $X=(\omega_1\times\mathbb{N})\cup P\cup Q$ with P and Q infinite, and the members of the union disjoint,

$$X_i=\begin{cases}(\{i\}\times\mathbb{N})\cup P & \text{if } i\in\omega_1,\\ (\omega_1\times\{-i\})\cup Q & \text{if } -i\in\mathbb{N},\\ P\cup Q & \text{if } i=0.\end{cases}$$

Take $\mathfrak{u},\mathfrak{v}\in\mathrm{Ult^f}\,X$ such that $P\in\mathfrak{u}$, $Q\in\mathfrak{v}$. Let a base for $\mathfrak{S}_i$ consist of $\dot{x}$ ($x\in X_i$) and the following filters: $\mathfrak{t}_i$ for $i\neq 0$ where $\mathfrak{t}_i\mid P=\mathfrak{u}\mid P$ and $\mathfrak{t}_i\mid(X_i\setminus P)$ consists of the cofinite sets ($i\in\omega_1$), $\mathfrak{t}_i\mid Q=\mathfrak{v}\mid Q$ and $\mathfrak{t}_i\mid(X_i\setminus Q)$ consists of the cocountable sets ($-i\in\mathbb{N}$); for $i=0$, take $\mathfrak{u}\mid X_0$ and $\mathfrak{v}\mid X_0$. This is a family of FL screens (fully Cauchy, because $\mathfrak{s}_i\,\overline{\triangle}\,\mathfrak{s}'_i$ for different elements from the base; Lodato, because $\mathfrak{n}(\mathfrak{S}_i)$ is discrete). $\mathfrak{S}_{\mathrm{C}}^1$ is a CL extension. $\mathfrak{u},\mathfrak{v}\in\mathfrak{S}^1$ and $\mathrm{St}\,\mathfrak{u}\triangle\mathrm{St}\,\mathfrak{v}$, thus a fully Cauchy extension would contain $\mathfrak{u}\cap\mathfrak{v}$, contradicting $(\mathfrak{u}\cap\mathfrak{v})X_0\notin\mathfrak{S}_0$.

b) (Cf. [15] Example 17.4.) Drop $(X_0,\mathfrak{S}_0)$ from the above example. Then $\dot{x}$ ($x\in X$) and the filter $\mathfrak{s}=\mathrm{St}\,\mathfrak{u}\cap\mathrm{St}\,\mathfrak{v}$ form a base for $\mathfrak{S}_{\mathrm{F}}^1$, which is now an extension, but $\mathfrak{s}\notin\mathfrak{S}_{\mathrm{E}}^1$. □

7.2 Let us consider now fully Cauchy screens in a closure space. Fully Cauchy screens are Riesz, so, according to 6.1, we do not have to deal with the same problem in convergence or pseudotopological spaces.

LEMMA (cf. [2] p. 38). *A closure $\mathfrak{n}$ can be induced by fully Cauchy screens iff it is reciprocal; if so then $\mathfrak{S}^1(\lambda^0(\mathfrak{n}))$ is the finest compatible fully Cauchy screen.*

PROOF. The necessity follows from the fact that even CR screens induce reciprocal closures ([6] 2.1). Conversely, if $\mathfrak{n}$ is reciprocal then $\mathfrak{S}^1 = \mathfrak{S}^1(\lambda^0(\mathfrak{n}))$ is fully Cauchy, since the filters $\mathfrak{n}(x)$ $(x \in X)$ form a base for $\mathfrak{S}^1$ (cf. 2.1 (1)), and $\mathfrak{n}(x) \overline{\triangle} \mathfrak{n}(y)$ whenever the two filters are different. If $\mathfrak{S}$ is a compatible fully Cauchy screen then, $\mathfrak{S}$ being Riesz, $\lambda(S) = \lambda^0(\mathfrak{n})$, thus $\mathfrak{S} < \mathfrak{S}^1$. □

Differently from the case of CR screens (see [7] 2.4), there is no coarsest compatible fully Cauchy screen:

EXAMPLE. Let X be infinite, $\mathfrak{u}, \mathfrak{v} \in \mathrm{Ult}^{\mathrm{f}} X$, $\mathfrak{u} \neq \mathfrak{v}$, $z \in X$, $\mathfrak{n}(x) = \dot{x}$ $(x \neq z)$, $\mathfrak{n}(z) = \dot{z} \cap \mathfrak{u}$. For $\mathfrak{w} \in \mathrm{Ult}^{\mathrm{f}} X \setminus \{u\}$, $\mathfrak{S}^1(\lambda^0(\mathfrak{n})) \cup \{\mathfrak{v} \cap \mathfrak{w}\}$ is a base for a compatible fully Cauchy screen. If $\mathfrak{S}$ is a fully Cauchy screen coarser than all these screens (with each $\mathfrak{w}$) then $\mathfrak{s} = \bigcap(\mathrm{Ult}^{\mathrm{f}} X \setminus \{u\}) \in \mathfrak{S}$. Now $\mathfrak{s}$ consists of the cofinite sets, thus $\mathfrak{u} \supset \mathfrak{s}$, $\mathfrak{n}(z) \triangle \mathfrak{s}$, and so $\mathfrak{n}(z) \cap \mathfrak{s} \in \mathfrak{S}$, i.e. $\mathfrak{S}$ cannot be compatible. □

7.3 LEMMA. *A countable family of fully Cauchy screens in a reciprocal closure space has fully Cauchy extensions iff there is a Cauchy extension in the set, a Riesz extension in the space, and for any $x \in X$ and any infinite $\mathfrak{S}^1(X, \mathfrak{S}_i)$-chain $\mathfrak{s}_{(1)}, \mathfrak{s}_{(2)}, \ldots,$*

$$\text{(1)} \qquad \mathfrak{n}(x) \triangle \bigcap_{n \in \mathbb{N}} \mathfrak{s}_{(n)} \quad \textit{implies} \quad \mathfrak{s}_{(n)} \supset \mathfrak{n}(x) \quad (n \in \mathbb{N}).$$

If so then

$$\text{(2)} \qquad \mathfrak{S}^1_{\mathrm{E}}(\lambda^0(\mathfrak{n}), \mathfrak{S}_i) = \mathfrak{S}^1_{\mathrm{E}}(X, \mathfrak{S}_i) \cup \mathfrak{S}^1(\lambda^0(\mathfrak{n}))$$

is the finest one.

PROOF. The necessity is obvious. Assume that all the conditions are satisfied, and denote by $\mathfrak{S}'$ the right-hand side of (2), which is the infimum of an extension in the set (Proposition 7.1) and a compatible screen. Therefore, to prove that $\mathfrak{S}'$ is a compatible extension, it is enough to show that $\mathfrak{n}(\mathfrak{S}') > \mathfrak{n}$ and $\mathfrak{S}' \mid X_i > \mathfrak{S}_i$.

Let $\mathfrak{s} \in \mathfrak{S}^1_{\mathrm{E}}(X, \mathfrak{S}_i)$ be fixed at x. Then $\mathfrak{s}$ is finer than the intersection of an infinite $\mathfrak{S}^1(X, \mathfrak{S}_i)$-chain (Remark 7.1 b)), thus $\mathfrak{s} \supset \mathfrak{n}(x)$ follows from (1), hence $\mathfrak{s} \in \mathfrak{S}^1(\lambda^0(\mathfrak{n}))$. This means that $\mathfrak{S}'$ contains the same fixed filters as the compatible screen $\mathfrak{S}^1(\lambda^0(\mathfrak{n}))$, thus $\mathfrak{S}'$ is compatible, too.

If $\mathfrak{s} \in \mathfrak{S}^1_{\mathrm{E}}(X, \mathfrak{S}_i)$ then $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$, since this screen is an extension in the set. If $\mathfrak{s} \in \mathfrak{S}^1(\lambda^0(\mathfrak{n}))$ then $\mathfrak{s} \supset \mathfrak{n}(x)$ for some $x \in X$, thus $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$ by the result cited after Lemma 6.2. So we have proved that $\mathfrak{S}'$ is a compatible extension.

Take the filters $\mathfrak{n}(x)$ $(x \in X)$ and those minimal elements of $\mathfrak{S}^1_{\mathrm{E}}(X, \mathfrak{S}_i)$ for which the premissa of (1) does not hold for any $x \in X$. By (1), these filters form a base $\mathfrak{B}$ for $\mathfrak{S}'$. If $\mathfrak{s}'$, $\mathfrak{s}''$ are different elements of $\mathfrak{B}$ then $\mathfrak{s}' \overline{\triangle} \mathfrak{s}''$

follows from Proposition 7.1, Lemma 7.2 and (1) (since, by Remark 7.1 b), if $\mathfrak{s}'$ is not a neighbourhood filter then it is the intersection of an infinite chain). Thus $\mathfrak{S}'$ is fully Cauchy.

By Proposition 2.2,

$$\mathfrak{S}^1(\lambda^0(\mathfrak{n}),\mathfrak{S}_i)=\mathfrak{S}^1(X,\mathfrak{S}_i)\cup\mathfrak{S}^1(\lambda^0(\mathfrak{n})). \tag{3}$$

Applying the operation E to the right-hand side of (3) yields the right-hand side of (2), since $\mathfrak{s}\triangle\mathfrak{n}(x)$, $\mathfrak{s}\in\mathfrak{S}^1(X,\mathfrak{S}_i)$ imply $\mathfrak{s}\supset\mathfrak{n}(x)$. Thus the equality in (2) holds. Any fully Cauchy extension is coarser than $\mathfrak{S}''=\mathfrak{S}^1(\lambda^0(\mathfrak{n}),\mathfrak{S}_i)$, hence it is coarser than $\mathfrak{S}''_{\mathrm{F}}<\mathfrak{S}''_{\mathrm{E}}$. □

THEOREM. *If a finite family of fully Cauchy screens has CR extensions in a closure space then the finest compatible CR extension is fully Cauchy.*

(Thus Theorem 6.3 gives a necessary and sufficient condition for the existence of a fully Cauchy extension of a finite family.)

PROOF. $\mathfrak{n}$ is reciprocal (as there is a compatible CR screen), thus the above lemma can be applied if we show that (1) holds. It is enough to check (1) for infinite chains in which $\mathfrak{s}_{(n)}=\mathfrak{t}^1_{(n)}$ and $\mathfrak{t}_{(n)}$ is a minimal element of $\mathfrak{S}_{i_n}$ $(n\in\mathbb{N})$. If $i_m=i_n$, $m<n$ then $\mathfrak{s}_{(m)},\mathfrak{s}_{(m+1)},\dots,\mathfrak{s}_{(n)}$ is an $\mathfrak{S}^1(X,\mathfrak{S}_i)$-chain, thus $\mathfrak{s}_{(m)}\cap\mathfrak{s}_{(n)}\in\mathfrak{S}^1_{\mathrm{C}}(X,\mathfrak{S}_i)$, which is now an extension, and so $\mathfrak{t}_{(m)}=\mathfrak{t}_{(n)}$ (as both are minimal). Hence there are only a finite number of different members of the infinite chain, i.e. its intersection is the same as that of a finite chain. Thus if $\mathfrak{S}$ is a CR extension then $\mathfrak{s}=\bigcap_m\mathfrak{s}_{(m)}\in\mathfrak{S}$, $\mathfrak{n}(x)\triangle\mathfrak{s}$, so $\mathfrak{s}\supset\mathfrak{n}(x)$.

The finest CR extension is given in [6] 2.8 in the following form:

$$\mathfrak{S}^1_{\mathrm{C}}(X,\mathfrak{S}_i)\cup\mathfrak{S}^1(\lambda^0(\mathfrak{n})).$$

By Remark 7.1 a), this is the same as the right-hand side of (2). □

REMARK. If we are given a finite family of fully Cauchy screens in a set (or space) and $X=\bigcup_{i\in I}X_i$ then each Cauchy extension is fully Cauchy.

Indeed, let $\mathfrak{S}$ be a Cauchy extension, $\emptyset\notin\mathfrak{s}\in\mathfrak{S}$. Define

$$I_0=\{i\in I\colon \exists\mathfrak{s}_{(i)}\in\mathfrak{S},\ \mathfrak{s}_{(i)}\subset\mathfrak{s},\ X_i\in\sec\mathfrak{s}_{(i)}\}. \tag{4}$$

For each $i\in I_0$, pick an $\mathfrak{s}_{(i)}$ as in (4), and let $\mathfrak{t}_i$ be the minimal element of $\mathfrak{S}_i$ coarser than $\mathfrak{s}_{(i)}\mid X_i$. Now $\mathfrak{t}=\bigcap_{i\in I_0}\mathfrak{t}^1_i$ is a minimal element of $\mathfrak{S}$ coarser than $\mathfrak{s}$. (If $\mathfrak{s}'\in\mathfrak{S}$ is strictly coarser than $\mathfrak{t}$ then, as I is finite, $\mathfrak{s}'\mid X_i\in\mathfrak{S}_i$ is strictly coarser than $\mathfrak{t}\mid X_i$ for some i. By (4), $i\in I_0$, thus $\mathfrak{s}'\mid X_i\subsetneq\mathfrak{t}_i$, a contradiction.)

7.4 The statement of Theorem 7.3 is false for countably infinite families:

EXAMPLE. Let $X=[0,\infty]$, $I=\mathbb{N}$, $A_i=]i-1,i[$, $X_i=A_i\cup A_{i+1}$ $(i\in I)$. Pick $\mathfrak{u}_{(i)}\in \mathrm{Ult}^{\mathrm{f}}X$ with $A_i\in\mathfrak{u}_{(i)}$ $(i\in I)$. Let $\mathfrak{n}(x)=\dot{x}$ $(x\neq\infty)$, $\mathfrak{n}(\infty)=\mathrm{fil}\{[x,\infty]\colon x\neq\infty\}$. Consider on X_i the screen $\mathfrak{S}_i$ for which the filters $\dot{x}$ $(x\in X_i)$ and $(\mathfrak{u}_{(i)}\cap\mathfrak{u}_{(i+1)})\mid X_i$ form a base. The conditions of Theorem 6.3 are satisfied, so there are CR extensions. But 7.3 (1) fails for $x=\infty$, $\mathfrak{s}_{(n)}=\mathfrak{u}_{(n)}\cap\mathfrak{u}_{(n+1)}$. □

§ 8. FL screens

8.1 If $\mathfrak{S}$ is FL then it is CL, thus $\mathfrak{n}(\mathfrak{S})$ is a reciprocal topology. Conversely, if $\mathfrak{n}$ is a reciprocal topology then $\mathfrak{S}^1(\lambda^0(\mathfrak{n}))$ is a compatible FL screen, namely the finest one (Lemma 7.2, using that the neighbourhood filters are open in a topological space). There is no coarsest compatible FL screen: in Example 7.2, $\mathfrak{n}$ is a topology, and the screens considered are Lodato.

THEOREM. *If a finite family of FL screens has CL extensions in a closure space then the finest compatible CL extension is FL.*

(A necessary and sufficient condition for the existence of a CL extension, and also a description of the finest one, can be found in [6] 3.1. The index set is arbitrary in that result, and it seems to be unlikely that assuming finiteness would simplify the conditions.)

PROOF. 1° Assume first that $I=\{0\}$. Let $\mathfrak{S}$ denote the finest Lodato extension. According to [5] 2.17, the filters $\mathfrak{n}(\mathfrak{s}_0^1)$ $(\mathfrak{s}_0\in\mathfrak{S}_0)$ and the neighbourhood filters form a base for $\mathfrak{S}$, where, for $\mathfrak{s}\in\mathrm{Fil}\,X$, $\mathfrak{n}(\mathfrak{s})$ is the filter generated by the open elements of $\mathfrak{s}$. As $\mathfrak{S}_0$ is fully Cauchy, it is enough to take the minimal elements of $\mathfrak{S}_0$;

$$(1)\qquad \{\mathfrak{n}(\mathfrak{s}_0^1)\colon \mathfrak{s}_0 \text{ is minimal in } \mathfrak{S}_0,\ \mathfrak{s}_0\neq\mathfrak{n}(x)\mid X_0\ (x\in X)\}\cup\mathfrak{S}^1(\lambda^0(\mathfrak{n}))$$

is also a base for $\mathfrak{S}$ (the filters $\mathfrak{s}_0=\mathfrak{n}(x)\mid X_0$ are not needed, since in this case $\mathfrak{n}(\mathfrak{s}_0^1)\supset\mathfrak{n}(x)$). The theorem will be proved for $|I|=1$ if we show that $\mathfrak{s}\,\overline{\triangle}\,\mathfrak{t}$ for different $\mathfrak{s}$ and $\mathfrak{t}$ from the base (1), since then $\mathfrak{S}$ is fully Cauchy, hence an FL extension.

Let $\mathfrak{S}'$ denote the finest CL extension. By [6] 3.6, $\mathfrak{S}'=\mathfrak{S}_C$; but we shall only need the evident fact that $\mathfrak{S}'<\mathfrak{S}$. Let $\mathfrak{s}$ and $\mathfrak{t}$ be elements of (1) such that $\mathfrak{s}\triangle\mathfrak{t}$. If $\mathfrak{s}=\mathfrak{n}(\mathfrak{s}_0^1)$, $\mathfrak{t}=\mathfrak{n}(\mathfrak{t}_0^1)$ then $\mathfrak{s},\mathfrak{t}\in\mathfrak{S}'$, $\mathfrak{s}\cap\mathfrak{t}\in\mathfrak{S}'$, thus, $\mathfrak{S}'$ being an extension, $\mathfrak{s}_0\cap\mathfrak{t}_0\in\mathfrak{S}_0$, $\mathfrak{s}_0=\mathfrak{t}_0$ (since both are minimal), $\mathfrak{s}=\mathfrak{t}$. If $\mathfrak{s}=\mathfrak{n}(x)$, $\mathfrak{t}=\mathfrak{n}(y)$ then $\mathfrak{s}=\mathfrak{t}$ again, as $\mathfrak{n}$ is reciprocal. Finally, the case $\mathfrak{s}=\mathfrak{n}(\mathfrak{s}_0^1)$, $\mathfrak{t}=\mathfrak{n}(x)$ is impossible, since then $\mathfrak{s}\supset\mathfrak{n}(x)$ (as $\mathfrak{s}$ is in the compatible CR screen $\mathfrak{S}'$), so $\mathfrak{s}_0^1\supset\mathfrak{n}(x)$, $\mathfrak{s}_0\supset\mathfrak{n}(x)\mid X_0$, implying $\mathfrak{s}_0=\mathfrak{n}(x)\mid X_0$ (as the latter is in $\mathfrak{S}_0$, cf. 6.1 (3), and the former is minimal); but the filters $\mathfrak{s}_0$ with this property were excluded in (1).

2° Assume now that $|I|>1$. Denote by $\mathfrak{S}'$ the finest compatible CL extension, and let $Y=\bigcup_{i\in I} X_i$, $\mathfrak{S}''=\mathfrak{S}'\mid Y$. Then $\mathfrak{S}''$ is fully Cauchy by Remark 7.3. $\mathfrak{S}'$ is the finest CL extension of $\mathfrak{S}''$, thus $\mathfrak{S}'$ is fully Cauchy by 1°. □

REMARK. Theorem 7.3 could also be proved analogously, in two steps; this proof does not use Lemma 7.3 or Proposition 7.1.

COROLLARY. *If a finite family of FL screens has CL extensions in a set then there are FL extensions;* $\mathfrak{S}^1_{\mathrm{C}}(X,\mathfrak{S}_i)$ *is the finest one.*

PROOF. If $\mathfrak{S}$ is the finest CL extension in the set then it is also the finest CL extension in $(X,\mathfrak{n}(\mathfrak{S}))$, thus the theorem can be applied. According to [15] Proposition 20.2, $\mathfrak{S}=\mathfrak{S}^1_{\mathrm{C}}(X,\mathfrak{S}_i)$. □

[15] Proposition 20.2 gives a necessary and sufficient condition for the existence of a CL extension in a set (only for finite families).

8.2 In Example 7.4, the finest CR extension is CL, but there are no FL extensions. Thus Theorem 8.1 does not hold for countably infinite families. It is even possible that a countable family of screens in a closure space has fully Cauchy extensions as well as CL extensions, but no FL extensions:

EXAMPLE. Let $(X,\mathfrak{n})$ be the Tikhonov corkscrew*, $I=\mathbb{N}$. Denote by X_i $(i\in\mathbb{N})$ the copy of ω or ω_1 (alternately) in which the neighbouring Tikhonov planks are joined. Let z be the point that makes the space non-completely regular. Take in X_i the filter $\mathfrak{t}_i$ that consists of the cofinite sets if X_i is countable and of the cocountable sets otherwise. Let $\mathfrak{S}_i$ be the screen for which $\{\mathfrak{t}_i\}\cup\{\dot{x}\colon x\in X_i\}$ is a base. This is a family of fully Cauchy screens. $\mathfrak{S}^1(\lambda^0(\mathfrak{n}),\mathfrak{S}_i)$ is a fully Cauchy extension, and

$$\mathfrak{S}^1(\lambda^0(\mathfrak{n}))\cup\Big\{\bigcap_{i=1}^{n}\mathfrak{n}(\mathfrak{t}_i^1)\colon n\in\mathbb{N}\Big\}$$

is a base for a CL extension. If $\mathfrak{S}$ is an FL extension then $\mathfrak{n}(\mathfrak{t}_i^1)\bigtriangleup\mathfrak{n}(\mathfrak{t}_{i+1}^1)$ implies $\mathfrak{s}=\bigcap_{i\in I}\mathfrak{n}(\mathfrak{t}_i^1)\in\mathfrak{S}$; now $\mathfrak{s}\bigtriangleup\mathfrak{n}(z)$ and $\mathfrak{n}(z)\in\mathfrak{S}$, thus $\mathfrak{s}\cap\mathfrak{n}(z)\in\mathfrak{S}$, and this filter is strictly coarser than $\mathfrak{n}(z)$, a contradiction. Hence there is no FL extension. □

We do not know whether there exists a similar example in a set instead of a space.

§ 9. Pointwise Cauchy screens

9.1 According to [15] Proposition 18.1, a family of pointwise Cauchy screens in a set has pointwise Cauchy extensions iff the following holds for

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* We build up the corkscrew from the version of the Tikhonov plank in which the points of ω_1 are isolated in ω_1+1 (called the "Dieudonné plank" in [26] Example 89). Differently from [26] Example 90, the corkscrew is spiralling in one direction only.

any $n \in \mathbb{N}$ and $i, j_0, j_1, \ldots, j_n \in I$:

$$\text{if } \mathfrak{s}_{(m)} \in \mathfrak{S}_{i_m} \ (0 \leqq m \leqq n) \text{ and } \mathfrak{s}_{(0)}, \mathfrak{s}_{(1)}, \ldots, \mathfrak{s}_{(n)} \text{ is a strong chain then}$$
$$\bigcap_{m=0}^{n} \mathfrak{s}^1_{(m)} \mid X_i \in \mathfrak{S}_i. \tag{1}$$

Although not as clear from (1) as for Cauchy extensions from Proposition 4.2, two pointwise Cauchy screens in a set always have pointwise Cauchy extensions ([15] Proposition 18.4).

A screen $\mathfrak{S}$ is pointwise Cauchy iff $\lambda(\mathfrak{S})$ is a limitation (evident), thus a family of screens in a convergence space has pointwise Cauchy extensions iff the convergence is a limitation and the conditions of Proposition 2.2 are satisfied, and then each extension is pointwise Cauchy. The same problem in pseudotopological spaces will be more interesting.

9.2 Let us call a pseudotopology π *pointwise reciprocal* if $\dot{x} \to y$ implies $\pi(x) = \pi(y)$; equivalently: $\dot{z} \to x$, $\dot{z} \to y$ imply $\pi(x) = \pi(y)$ (cf. 6.2 (1), (2)). Reciprocal $\Rightarrow$ pointwise reciprocal $\Rightarrow$ symmetric.

LEMMA. *The following conditions are equivalent for a pseudotopology π:*
(i) *π is pointwise reciprocal,*
(ii) *$\lambda^0(\pi)$ is symmetric,*
(iii) *$\lambda^1_{\lim}(\pi)$ is symmetric,*
(iv) *π can be induced by a symmetric limitation.*

PROOF. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii): Let $\lambda = \lambda^0(\pi)$ or $\lambda = \lambda^1_{\lim}(\pi)$, $\mathfrak{s} \to x$, $y \in \bigcap \mathfrak{s}$. Then $\dot{y} \xrightarrow{\lambda} x$, so $\dot{y} \xrightarrow{\pi} x$, $\pi(x) = \pi(y)$, $\lambda(x) = \lambda(y)$, $\mathfrak{s} \xrightarrow{\lambda} y$.

(ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv): Evident.

(iv) $\Rightarrow$ (i): Let λ be a symmetric limitation, $\pi(\lambda) = \pi$, $\dot{x} \xrightarrow{\pi} y$, $\mathfrak{u} \xrightarrow{\pi} y$. Then $\mathfrak{s} = \dot{x} \cap \mathfrak{u} \xrightarrow{\lambda} y$, $x \in \bigcap \mathfrak{s}$, thus $\mathfrak{s} \xrightarrow{\lambda} x$, $\mathfrak{u} \xrightarrow{\pi} x$. Hence $\pi(y) \subset \pi(x)$. The symmetry of λ also implies that π is symmetric, thus $\dot{y} \xrightarrow{\pi} x$, and then the above reasoning gives $\pi(x) \subset \pi(y)$, too. □

9.3 LEMMA. *A pseudotopology π can be induced by pointwise Cauchy screens iff it is pointwise reciprocal; if so then $\mathfrak{S}^1(\lambda^1_{\lim}(\pi))$ is the finest and $\mathfrak{S}^0(\pi)$ the coarsest compatible pointwise Cauchy screen.*

PROOF. If $\mathfrak{S}$ is pointwise Cauchy then $\lambda(\mathfrak{S})$ is a symmetric limitation (cf. 9.1), thus $\pi(\mathfrak{S})$ is pointwise reciprocal by Lemma 9.2. Conversely, assume that π is pointwise reciprocal. Then, again by Lemma 9.2, $\lambda^0(\pi)$ is a symmetric limitation, thus $\mathfrak{S} = \mathfrak{S}^0(\lambda^0(\pi))$ is a compatible pointwise Cauchy screen (cf. 9.1). $\mathfrak{S}^0(\pi) < \mathfrak{S}$, thus $\lambda(\mathfrak{S}^0(\pi)) = \lambda^0(\pi)$, implying $\mathfrak{S} < \mathfrak{S}^0(\pi)$, $\mathfrak{S} = \mathfrak{S}^0(\pi)$. Hence $\mathfrak{S}^0(\pi)$ is the coarsest compatible pointwise Cauchy screen. $\mathfrak{S}' = \mathfrak{S}^1(\lambda^1_{\lim}(\pi))$ is also a compatible pointwise Cauchy screen (by the same reasoning as $\mathfrak{S}$). Assume that $\mathfrak{S}''$ is another compatible pointwise Cauchy screen. Then $\lambda(\mathfrak{S}'')$ is a limitation, so $\lambda'' = \lambda(\mathfrak{S}'') < \lambda^1_{\lim}(\pi)$, $\mathfrak{S}'' < \mathfrak{S}^1(\lambda'')$. Now $\mathfrak{S}'' < \mathfrak{S}'$ follows from the fact that $\lambda'' < \lambda$ implies $\mathfrak{S}^1(\lambda'') < \mathfrak{S}^1(\lambda)$ (clear from 2.1 (1)). $\square$

THEOREM. *A family of pointwise Cauchy screens in a pointwise reciprocal pseudotopological space has pointwise Cauchy extensions iff the following condition holds for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and different indices $i_1, \dots, i_n \in I$:*

$$\text{(1)}\quad \begin{gathered} \textit{if}\ \ x_1,\dots,x_n \notin X_{i_1},\quad \mathfrak{s}_{i_m} \in \mathfrak{S}_{i_m},\quad x_m \in \bigcap \mathfrak{s}_{i_m},\quad \pi(x_m) = \pi(x_1)\quad (2 \leqq m \leqq n) \\ \textit{and}\ \ \emptyset \notin \mathfrak{s}_{i_1} \in \lambda^1_{\lim}(\pi)(x_1) \mid X_{i_1}\ \ \textit{then}\ \ \bigcap_{m=1}^{n} \mathfrak{s}^1_{i_m} \mid X_{i_1} \in \mathfrak{S}_{i_1}. \end{gathered}$$

If so then $\mathfrak{S}^1_{\mathrm{P}}(\pi, \mathfrak{S}_i)$ is the finest pointwise Cauchy extension.

PROOF. *Necessity.* Let $\mathfrak{S}$ be a pointwise Cauchy extension. Then $\mathfrak{s}^1_{i_m} \in \mathfrak{S}$ is clear for $m \geqq 2$, but it holds for $m = 1$, too, since $\mathfrak{s}^1_{i_1} \in \lambda^1_{\lim}(\pi)(x_1)$, $\mathfrak{s}^1_{i_1} = \bigcap \mathfrak{U}$ where $\mathfrak{U}$ is a finite collection of ultrafilters with $\mathfrak{u} \to x_1$ $(\mathfrak{u} \in \mathfrak{U})$; now $\mathfrak{u} \cap \dot{x}_1 \in \mathfrak{S}$ for each $\mathfrak{u}$, thus $\mathfrak{s}^1_{i_1} \cap \dot{x}_1 \in \mathfrak{S}$. $\pi(x_{m-1}) = \pi(x_m)$ implies that $\dot{x}_{m-1} \to x_m$, $\dot{x}_{m-1} \cap \dot{x}_m \in \mathfrak{S}$.

$$\text{(2)}\qquad \mathfrak{s}^1_{i_1} \cap \dot{x}_1,\ \dot{x}_1 \cap \dot{x}_2,\ \mathfrak{s}^1_{i_2}, \dots, \dot{x}_{n-1} \cap \dot{x}_n,\ \mathfrak{s}^1_{i_n}$$

is a strong $\mathfrak{S}$-chain (take only $\mathfrak{s}^1_{i_1} \cap \dot{x}_1$ if $n = 1$), so its intersection is in $\mathfrak{S}$, implying that the conclusion of (1) holds.

Sufficiency. 1° (1) remains valid if $x_m \in X_{i_1}$ is allowed: Consider first the case when $x_m \in X_{i_1}$ for each m. Then the traces of the filters in (2) form a strong $\mathfrak{S}_{i_1}$-chain ($\dot{x}_{m-1} \cap \dot{x}_m \in \mathfrak{S}_{i_1}$, because $\pi(x_{m-1}) = \pi(x_m)$ holds in the subspace, too, and $\mathfrak{S}_{i_1}$ is compatible); thus, $\mathfrak{S}_{i_1}$ being pointwise Cauchy, the conclusion of (1) holds

If there are $x_p \notin X_{i_1}$, $x_q \in X_{i_1}$ then from $\pi(x_p) = \pi(x_1) = \pi(x_q)$ we have $\mathfrak{s}_{i_1} \in \lambda^1_{\lim}(\pi)(x_p) \mid X_{i_1}$, and similarly with x_q, thus x_1 can be replaced by x_p or x_q, and also $\mathfrak{s}_{i_1}$ by $\mathfrak{s}_{i_1} \cap \dot{x}_q \in \lambda^1_{\lim}(\pi)(x_1) \mid X_{i_1}$. Consider now the collections

$$\begin{aligned} \mathfrak{A} &= \{\mathfrak{s}^1_{i_1} \cap \dot{x}_q\} \cup \{\mathfrak{s}^1_{i_m} : m \geqq 2, x_m \notin X_{i_1}\}, \\ \mathfrak{B} &= \{\mathfrak{s}^1_{i_1} \cap \dot{x}_q\} \cup \{\mathfrak{s}^1_{i_m} : m \geqq 2, x_m \in X_{i_1}\}, \end{aligned}$$

with x_p, respectively x_q, playing the role of x_1. We have $\bigcap \mathfrak{A} \mid X_{i_1} \in \mathfrak{S}_{i_1}$ from (1), while $\bigcap \mathfrak{B} \mid X_{i_1} \in \mathfrak{S}_{i_1}$ was proved above. Both intersections are fixed at x_q, thus their intersection is in $\mathfrak{S}_{i_1}$, too.

2° (1) (with the generalization from 1°) remains valid for not necessarily different indices: To avoid using the same symbol for different filters, let us write $\mathfrak{q}_{(m)}$ instead of $\mathfrak{s}_{i_m}$. If $m, p > 1$, $i_m = i_p$ then $\mathfrak{q}_{(m)}, \dot{x}_m \cap \dot{x}_p, \mathfrak{q}_{(p)}$ is a strong $\mathfrak{S}_{i_m}$-chain, thus the filters $\mathfrak{q}_{(m)}$ and $\mathfrak{q}_{(p)}$ can be replaced by their intersection. Hence it is enough to consider the case when, say, $i_1 = i_2$, and the indices $i_2, \dots, i_n$ are different. Now $x_2 \in X_{i_1}$ and $\pi(x_2) = \pi(x_1)$, thus $\mathfrak{q}_{(1)} \in \lambda^1_{\lim}(\pi)(x_2) \mid X_{i_1}$, and the same holds for $\mathfrak{q}_{(1)} \cap \dot{x}_2$. (1) can be applied to the filters $\mathfrak{q}_{(1)} \cap \dot{x}_2, \mathfrak{q}_{(3)}, \dots, \mathfrak{q}_{(n)}$, since the indices are now different. The intersection of the traces of these filters as well as $\mathfrak{q}_{(2)}$ are fixed at x_2, thus their intersection is in $\mathfrak{S}_{i_1}$.

3° 3.2 (1) follows from (1) with $n = 1$, so we can take the finest extension $\mathfrak{S}^1 = \mathfrak{S}^1(\pi, \mathfrak{S}_i)$. We are going to prove that $\mathfrak{S}^1_{\mathrm{P}}$ is a compatible extension (hence clearly the finest compatible pointwise Cauchy extension). Consider $\mathfrak{S}^1$ in the form given by the right-hand side of 3.2 (2). Let $\mathfrak{s}_{(1)}, \dots, \mathfrak{s}_{(N)}$ be a strong $\mathfrak{S}^1$-chain, $\mathfrak{s} = \bigcap_{m=1}^{N} \mathfrak{s}_{(m)}$. $\mathfrak{S}^1$ being a compatible extension, it is enough to show that $\mathfrak{S}^1_{\mathrm{P}}$ is still fine enough, i.e. that

$$x \in \bigcap \mathfrak{s},\ \operatorname{Ult} X \ni \mathfrak{u} \supset \mathfrak{s} \ \text{ imply } \ \mathfrak{u} \to x; \tag{3}$$

$$\mathfrak{s} \mid X_i \in \mathfrak{S}_i \qquad (i \in I). \tag{4}$$

Both statements are evident if $N = 1$, so assume $N > 1$, and take points y_m such that $y_m \in \bigcap \mathfrak{s}_{(m)} \cap \bigcap \mathfrak{s}_{(m+1)}$; put $y_N = y_{N-1}$. Now $\dot{y}_{m-1} \cap \dot{y}_m \supset \mathfrak{s}_{(m)}$, thus $\dot{y}_{m-1} \cap \dot{y}_m \in \mathfrak{S}^1$, $\dot{y}_{m-1} \to y_m$, and so $\pi(y_m) = \pi(y_1)$ $(1 < m \leqq N)$.

4° *Proof of (3).* Pick m and p such that $\mathfrak{u} \supset \mathfrak{s}_{(m)}$, $x \in \bigcap \mathfrak{s}_{(p)}$. Then $\dot{x} \to y_p$, $\mathfrak{u} \to y_m$, $\pi(y_m) = \pi(y_p) = \pi(x)$, thus $\mathfrak{u} \to x$.

5° *Proof of (4).* Define

$$J = \{m\colon 1 \leqq m \leqq N,\ \mathfrak{s}_{(m)} \in \mathfrak{S}^1(\pi)\}, \qquad H = \{1, \dots, N\} \setminus J.$$

For $m \in H$, $\mathfrak{s}_{(m)}$ can be written as $\mathfrak{t}^1_{(m)}$ with some $\mathfrak{t}_{(m)} \in \mathfrak{S}_{j_m}$, $j_m \in I$. For $m \in J$, $\mathfrak{s}_{(m)} = \mathfrak{u}_{(m)} \cap \dot{z}$, $\mathfrak{u}_{(m)} \to z$. By $y_m \in \bigcap \mathfrak{s}_{(m)}$, either $z = y_m$ or $\mathfrak{u}_{(m)} = \dot{y}_m$ and $\dot{z} \to y_m$; in both cases $\mathfrak{s}_{(m)} = \mathfrak{v}_{(m)} \cap \dot{y}_m$, $\mathfrak{v}_{(m)} \to y_m$.

Put $K = \{m\colon X_i \in \sec \mathfrak{s}_{(m)}\}$. It is enough to consider the case $K \neq \emptyset$. If $m \in K \cap H$ then take $\mathfrak{v} \in \operatorname{Ult} X$ such that $X_i \in \mathfrak{v} \supset \mathfrak{s}_{(m)}$. Now $\mathfrak{v} \to y_m$, thus $\mathfrak{s}' = \mathfrak{v} \cap \dot{y}_m \in \mathfrak{S}^1(\pi)$ can be inserted into the chain without changing its intersection: $\dots, \mathfrak{s}_{(m)}, \mathfrak{s}', \mathfrak{s}_{(m)}, \dots$. Hence $K \cap J \neq \emptyset$ can be assumed. (1), with the generalization from 1° and 2°, can now be applied to $i_1 = i$,

$$\mathfrak{q}_{(1)} = \bigcap \{\mathfrak{s}_{(m)} \mid X_i\colon m \in K \cap J\},$$

$\mathfrak{q}_{(2)}, \mathfrak{q}_{(3)}, \dots$ the filters $\mathfrak{t}_{(m)}$ $(m \in K \cap H)$, $x_2, x_3, \dots$ the corresponding points y_m. Thus $\mathfrak{s} \mid X_i \in \mathfrak{S}_i$, indeed. □

COROLLARY. *If any finite subfamily of a family of screens in a pseudotopological space has pointwise Cauchy extensions then so has the whole family.* □

9.4 Condition 9.3 (1) gets much simpler if $|I|=1$:

Corollary. *A single pointwise Cauchy screen $\mathfrak{S}_0$ in a pointwise reciprocal pseudotopological space has pointwise Cauchy extensions iff the following condition holds for each $x \in X \setminus X_0$ and $n \in \mathbb{N}$:*

$$(1) \qquad \textit{if} \ \ \mathfrak{u}_{(m)} \in \operatorname{Ult} X_0, \ \ \mathfrak{u}^1_{(m)} \to x \quad (1 \leqq m \leqq n) \ \ \textit{then} \ \ \bigcap_{m=1}^{n} \mathfrak{u}_{(m)} \in \mathfrak{S}_0.$$

Proof. There is no index different from i_0, thus $n=1$ in 9.3 (1). □

In Theorem 5.3, the weaker condition 5.3 (1) was enough, since then (1) above follows from the Cauchy property.

There is no coarsest pointwise Cauchy extension:

Example. With X, X_0 and λ from Example 2.2, let $\pi=\pi(\lambda)$, A and B disjoint infinite sets, $A \cup B = X_0$. Let $\mathfrak{p}_0, \mathfrak{q}_0 \in \operatorname{Fil} X_0$ be generated by the cofinite subsets of A, respectively of B. Consider the screen $\mathfrak{S}_0$ with the following base:

$$\{\dot{x}\colon x \in X_0\} \cup \{\mathfrak{p}_0 \cap \mathfrak{s}_0, \mathfrak{q}_0 \cap \mathfrak{s}_0\colon \mathfrak{s}_0 \in \lambda'(0) \mid X_0\},$$

where $\lambda' = \lambda^1_{\lim}(\pi)$. Then

$$\{\mathfrak{s}_0^1\colon \mathfrak{s}_0 \in \mathfrak{S}_0\} \cup \{\mathfrak{p}_0^1 \cap \mathfrak{s}\colon \mathfrak{s} \in \lambda'(0)\}$$

is a base for a compatible pointwise Cauchy extension $\mathfrak{S}'$. Now $\dot{0} \cap \mathfrak{p}_0^1 \in \mathfrak{S}'$, and there is a similar extension containing $\dot{0} \cap \mathfrak{q}_0^1$. A pointwise Cauchy screen $\mathfrak{S}$ coarser than both has to contain $\mathfrak{p}_0^1 \cap \mathfrak{q}_0^1$, thus $\mathfrak{p}_0 \cap \mathfrak{q}_0 \in \mathfrak{S} \mid X_0$, i.e. $\mathfrak{S}$ cannot be an extension. □

9.5 In contrast to Corollary 4.3, it is possible that two screens have compatible pointwise Cauchy extensions separately, but not simultaneously:

Example. Let X, X_0, π, A, B, $\mathfrak{p}_0$ and λ' be as in Example 9.4, $X_1 = A \cup \{0\}$. Generate $\mathfrak{S}_0$ and $\mathfrak{S}_1$ by the following bases:

$$\{\dot{x}\colon x \in X_0\} \cup \{\mathfrak{p}_0\} \cup \lambda'(0) \mid X_0, \qquad \{\dot{x}\colon x \in X_1\} \cup \{\dot{0} \cap \mathfrak{p}_0^1 \mid X_1\}.$$

9.4 (1) holds for both screens, but 9.3 (1) fails for $n=2$, $i_1=0$, $i_2=1$, $x_1=x_2=0$, any $\mathfrak{s}_{(0)} \in \operatorname{Ult}^{\mathrm{f}} X_0$ containing B and $\mathfrak{s}_{(1)} = \dot{0} \cap \mathfrak{p}_0^1 \mid X_1$. □

9.6 Let us consider now pointwise Cauchy screens in a closure space. A closure can be induced by pointwise Cauchy screens iff it is pointwise reciprocal ([15] 18.1), so, in a closure space, pointwise Cauchy screens behave better than Cauchy screens. $\mathfrak{S}^0(\lambda^0(\mathfrak{n}))$ is the coarsest compatible pointwise Cauchy screen, while there is no finest one, e.g. in [4] 3.15 (replacing the free ultrafilters by their finite intersections). But we cannot give a complete solution of the problem of pointwise Cauchy extensions in closure spaces.

A single pointwise Cauchy screen $\mathfrak{S}_0$ in a pointwise reciprocal closure space has pointwise Cauchy extensions iff for each $x \in X$ (or: $x \in X \setminus X_0$) there is $\mathfrak{F}_0(x) \subset \mathfrak{S}_0$ such that $\mathfrak{n}(x) \mid X_0 = \bigcap \mathfrak{F}_0(x)$ and $\mathfrak{F}_0(x)$ is closed for finite intersections. To prove the sufficiency, modify first the collections $\mathfrak{F}_0(x)$ such that $\mathfrak{F}_0(x) = \mathfrak{F}_0(y)$ whenever $\mathfrak{n}(x) = \mathfrak{n}(y)$, and also $\mathfrak{F}_0(x) = \{\mathfrak{s}_0 \in \mathfrak{S}_0 : x \in \bigcap \mathfrak{s}_0\}$ if $x \in X_0$. The following filters will make up a base for a pointwise Cauchy extension:

$$\mathfrak{s}_0^1 \cap (\mathfrak{n}(x) \mid (X \setminus X_0))^1 \qquad (x \in X, \mathfrak{s}_0 \in \mathfrak{F}_0(x))$$

and $\mathfrak{s}_0^1$ $(\mathfrak{s}_0 \in \mathfrak{S}_0)$. This construction does not yield a specific extension: it depends on the choice of the collections $\mathfrak{F}_0(x)$. There is in fact neither a finest nor a coarsest extension: in Example 9.4, take $\mathfrak{n} = \mathfrak{n}(\pi)$ ($\mathfrak{S}'$ and $\mathfrak{S}''$ are compatible with $\mathfrak{n}$, and we saw in 9.4 that no pointwise Cauchy screen coarser than both can be an extension even in the set).

A similar condition for $|I| > 1$ would have to contain an assumption connecting the collections $\mathfrak{F}_i(x)$ and $\mathfrak{F}_j(x)$ $(i \neq j)$: taking $\mathfrak{n} = \mathfrak{n}(\pi)$ in Example 9.5, we obtain two screens in a closure space that have pointwise Cauchy extensions separately, but not simultaneously. (A compatible extension would contain $\dot{0} \cap \mathfrak{p}_0^1$ and also $\dot{0} \cap \mathfrak{u}$ with some $\mathfrak{u} \in \mathrm{Ult}^{\mathrm{f}} X$ containing B; the trace of the intersection of these filters is not in $\mathfrak{S}_0$.)

§ 10. Convergences in a pseudotopological space

10.1 There are several separation axioms for convergences, but we confine our attention to properties arising from screens, namely symmetry and (pointwise) reciprocity. It was mentioned in the introduction that a limitation is symmetric iff $\dot{x} \to y$ implies $\lambda(x) = \lambda(y)$; a convergence having this property will be called *pointwise reciprocal* (compare with the similar definitions for pseudotopologies and closures). Equivalently: $\dot{z} \to x$, $\dot{z} \to y$ imply $\lambda(x) = \lambda(y)$. A pointwise reciprocal convergence is symmetric, but not conversely, see the example below.

Let us call a function $\beta \colon X \to \exp \mathrm{Fil}^+ X$ a *base* for the convergence λ if

$$\lambda(x) = \{\mathfrak{s} \in \mathrm{Fil}^+ X : \ \exists \mathfrak{t} \in \beta(x),\ \mathfrak{t} \subset \mathfrak{s}\} \qquad (x \in X).$$

E.g. $\beta(x) = \{\mathfrak{n}(x)\}$ defines a base for $\lambda^0(\mathfrak{n})$.

EXAMPLE. Let $X = \{x, y, z\}$, $\beta(x) = \{\dot{x} \cap \dot{z}\}$, $\beta(y) = \{\dot{y} \cap \dot{z}\}$, $\beta(z) = \beta(x) \cup \beta(y)$. This β is a base for a symmetric convergence, which is not pointwise reciprocal. □

10.2 Let us first consider convergences without separation properties. The proof of the following statement is straightforward: a family of convergences in a set always has extensions;

$$\lambda^1(X, \lambda_i)(x) = \{\mathfrak{s}_i^1 : i \in I,\ x \in X_i,\ \mathfrak{s}_i \in \lambda_i(x)\} \cup \{\dot{x}, \exp X\}, \qquad (x \in X)$$

is the finest extension $\left(\dot{x}\right.$ and $\exp X$ do not have to be added if $\left.x \in \bigcup_{i\in I} X_i\right)$, while the coarsest extension $\lambda^0 = \lambda^0(X, \lambda_i)$ is defined by

$$\mathfrak{s} \xrightarrow{\lambda^0} x \text{ iff } \mathfrak{s} \,|\, X_i \xrightarrow{\lambda_i} x \text{ for each } i \in I \text{ with } x \in X_i.$$

[[12] Proposition 2.1 can be applied.]

PROPOSITION. *Any family of convergences in a pseudotopological space has extensions;*

$$\lambda^1(\pi, \lambda_i) = \inf\{\lambda^1(X, \lambda_i), \lambda^1(\pi)\} \tag{1}$$

is the finest, and

$$\lambda^0(\pi, \lambda_i) = \sup\{\lambda^0(X, \lambda_i), \lambda^0(\pi)\} \tag{2}$$

the coarsest extension. □

10.3 Let us be given now a family of symmetric convergences in a set. Then the finest extension $\lambda^1(X, \lambda_i)$ is also symmetric (straightforward). [Or [12] Proposition 1.12 b).] $\lambda^0(X, \lambda_i)$ is not necessarily symmetric. (Example: Let $X = \{x, y, z\}$, $X_1 = \{x\}$, $X_2 = \{y, z\}$, λ_1 and λ_2 discrete. $\mathfrak{s} = \{X\} \in \lambda^0(x)$, $y \in \cap\mathfrak{s}$, but $\mathfrak{s} \notin \lambda^0(y)$.) Nevertheless, there exists a coarsest symmetric extension, too, namely $\lambda^0_{(\mathrm{s})}(X, \lambda_i)$, where, for a convergence λ, $\lambda_{(\mathrm{s})}$ denotes the coarsest one of the symmetric convergences finer than λ [the symmetric coreflexion of λ in Conv over mSet], i.e.

$$\mathfrak{s} \xrightarrow{\lambda_{(\mathrm{s})}} x \text{ iff } \dot{x} \cap \mathfrak{s} \xrightarrow{\lambda} y \text{ for each } y \in \bigcap(\dot{x} \cap \mathfrak{s}).$$

[[12] Proposition 2.3 can be applied.]

A pseudotopology can be induced by symmetric convergences iff it is symmetric (Lemma 3.1); $\lambda^1(\pi) = \lambda(\mathfrak{S}^1(\pi))$ is the finest and $\lambda^0_{(\mathrm{s})}(\pi) = \lambda(\mathfrak{S}^0(\pi))$ is the coarsest compatible symmetric convergence. ($\lambda^0(\pi)$ is not always symmetric, see Lemma 9.2.)

PROPOSITION. *A family of symmetric convergences in a symmetric pseudotopological space always has symmetric extensions; $\lambda^1(\pi, \lambda_i)$ is the finest, and $\lambda^0_{(\mathrm{s})}(\pi, \lambda_i)$ the coarsest symmetric extension.*

PROOF. $\lambda^1(\pi, \lambda_i)$ is symmetric, because it is the infimum of two symmetric convergences, see 10.2 (1). □

The first part of the proposition can also be deduced from Proposition 3.2: Replacing each λ_i by $\mathfrak{S}^0(\lambda_i)$, we obtain a family of screens in (X, π), since the operation $\mathfrak{S}^0$ commutes with restrictions (but $\mathfrak{S}^1$ does not). Now 3.2 (1) holds, because $\mathfrak{S}^0(\lambda_i)$ contains all the ultrafilters. Hence

$\lambda(\mathfrak{S}^0(\pi, \mathfrak{S}^0(\lambda_i)))$ is a compatible symmetric extension (the coarsest one in fact). If λ is a symmetric extension then $\mathfrak{S}^0(\lambda)$ is an extension of the screens $\mathfrak{S}^0(\lambda_i)$ in (X, π), thus $\mathfrak{S}^0(\lambda) < \mathfrak{S}^1(\pi, \mathfrak{S}^0(\lambda_i))$, implying that $\lambda(\mathfrak{S}^1(\pi, \mathfrak{S}^0(\lambda_i)))$ is the finest compatible extension.

10.4 Suprema of pointwise reciprocal convergences have the same property, so for each convergence there is a finest one among the pointwise reciprocal convergences coarser than λ; it will be denoted by λ_{P} [this is a strong reflexion]. It will be sufficient for our purposes to describe λ_{P} for symmetric convergences λ (although the following construction works in the more general case when $\pi(\lambda)$ is symmetric).

LEMMA. *Given a symmetric convergence* λ,

$$(1)\quad \beta(x) = \left\{\mathfrak{s} \cap \dot{x}_1 \cap \ldots \cap \dot{x}_n : n \in \mathbb{N},\ \dot{x}_m \xrightarrow{\lambda} x_{m+1}\ (1 \leq m < n),\ x_1 = x,\ \mathfrak{s} \xrightarrow{\lambda} x_n\right\}$$

is a base for λ_{P}.

PROOF. Let $x_1, \ldots, x_n$ and $\mathfrak{s}$ be as in (1). $\lambda_{\mathrm{P}} < \lambda$, so $\dot{x}_m \xrightarrow{\lambda_{\mathrm{P}}} x_{m+1}$, and therefore $\lambda_{\mathrm{P}}(x_1) = \lambda_{\mathrm{P}}(x_2) = \ldots = \lambda_{\mathrm{P}}(x_n)$. From $\mathfrak{s} \xrightarrow{\lambda} x_n$ we have $\mathfrak{s} \xrightarrow{\lambda_{\mathrm{P}}} x_n$, $\mathfrak{s} \cap \dot{x}_n \xrightarrow{\lambda_{\mathrm{P}}} x_n$, hence $\mathfrak{s} \cap \dot{x}_n \xrightarrow{\lambda_{\mathrm{P}}} x_{n-1}$ and $\mathfrak{s} \cap \dot{x}_n \cap \dot{x}_{n-1} \xrightarrow{\lambda_{\mathrm{P}}} x_{n-1}$; by induction, $\mathfrak{s} \cap \dot{x}_n \cap \ldots \cap \dot{x}_1 \xrightarrow{\lambda_{\mathrm{P}}} x_1$, i.e. λ_{P} is coarser than the convergence λ' for which β is a base (λ' is a convergence, since the elements of $\beta(x)$ are fixed at x). Thus it is enough to check that λ' is pointwise reciprocal. (It is clearly coarser than λ.)

Assume $\dot{y} \xrightarrow{\lambda'} x$. Then there are $\mathfrak{s}$ and $x_1, \ldots, x_n$ such that $\dot{\mathfrak{y}} \supset \mathfrak{s} \cap \dot{x}_1 \cap \ldots \cap \dot{x}_n$, i.e. either $y \in \bigcap \mathfrak{s}$ or $y = x_k$ with some k. If $y \in \bigcap \mathfrak{s}$ then $\dot{y} \xrightarrow{\lambda} x_n$, so $\dot{x}_n \xrightarrow{\lambda} y$, and $x_{n+1} = y$, $x_{n+2} = x_n$ can be added to the sequence of points, i.e. we can assume without loss of generality that $y = x_k$.

If $\mathfrak{s}' \xrightarrow{\lambda'} y$ then $\mathfrak{s}' \supset \dot{y}_1 \cap \ldots \cap \dot{y}_q \cap \mathfrak{t}$ where $\dot{y}_p \xrightarrow{\lambda} y_{p+1}$ $(1 \leq p < q)$, $y_1 = y$ and $\mathfrak{t} \xrightarrow{\lambda} y_q$. Now $\mathfrak{s}' \supset \dot{x}_1 \cap \ldots \cap \dot{x}_k \cap \dot{y}_1 \cap \ldots \cap \dot{y}_q \cap \mathfrak{t}$ and $x_k = y_1$, so $\mathfrak{s}' \xrightarrow{\lambda'} x$, $\lambda'(y) \subset \lambda'(x)$. The converse follows in the same way, since, by the symmetry of λ, $\dot{x}_m \xrightarrow{\lambda} x_{m+1}$ implies $\dot{x}_{m+1} \xrightarrow{\lambda} x_m$. $\square$

PROPOSITION. *A family of pointwise reciprocal convergences in a set has pointwise reciprocal extensions iff the following condition holds for any* $n > 1$, *different points* $x_1, \ldots, x_n$ *and different indices* $i, j_1, \ldots, j_n$:

(2)

$$\text{if } x_1 \in X_i,\ \dot{x}_m \xrightarrow{\lambda_{j_m}} x_{m+1}\ (1 \leq m < n) \text{ and } \mathfrak{s}_{j_n} \xrightarrow{\lambda_{j_n}} x_n \text{ then } \mathfrak{s}^1_{j_n} \mid X_i \xrightarrow{\lambda_i} x_1;$$

if so then $\lambda^1_{\mathrm{P}}(X, \lambda_i)$ *is the finest pointwise reciprocal extension.*

PROOF. *Necessity.* If λ is a pointwise reciprocal extension then $\dot{x}_m \xrightarrow{\lambda} x_{m+1}$, thus $\lambda(x_1) = \lambda(x_2) = \ldots = \lambda(x_n)$, $\mathfrak{s}^1_{j_n} \xrightarrow{\lambda} x_n$, $\mathfrak{s}^1_{j_n} \xrightarrow{\lambda} x_1$, and $\mathfrak{s}^1_{j_n} \mid X_i \xrightarrow{\lambda_i} x_1$.

Sufficiency. 1° By the accordance, (2) holds for $n=1$, too (even when $i=j_1$). We are going to show by induction on n that (2) remains valid for not necessarily different indices.

For $1 \leqq u < v \leqq n$ and $h \in I$, we have:

$$\text{(3)} \qquad \text{if } x_u, x_v \in X_h \text{ then } \lambda_h(x_u) = \lambda_h(x_v).$$

Indeed, $\dot{x}_{v-1} \xrightarrow{\lambda_{j_{v-1}}} x_v$ implies $\dot{x}_v \xrightarrow{\lambda_{j_{v-1}}} x_{v-1}$, thus the induction hypothesis can be applied to the points $x_u, \ldots, x_{v-1}$, the filter $\dot{x}_v$ (in X_{v-1}), and h in lieu of i; thus $\dot{x}_v \xrightarrow{\lambda_h} x_u$, and so $\lambda_h(x_u) = \lambda_h(x_v)$ (as λ_h is pointwise reciprocal).

Assume first that $j_p = j_q$, $p < q$. If $q < n$ then (3) with $u = p$, $v = q+1$, $h = j_p = j_q$ yields $\dot{x}_p \xrightarrow{\lambda_{j_p}} x_{q+1}$, thus the induction hypothesis can be applied to $x_1, \ldots, x_p, x_{q+1}, \ldots, x_n$ and $\mathfrak{s}_{j_n}$. For $q = n$, take $u = p$, $v = n$, $h = j_p = j_n$; then $\lambda_h(x_p) = \lambda_h(x_n)$, thus $\mathfrak{s}_{j_n} \xrightarrow{\lambda_{j_p}} x_p$, and the induction hypothesis applies now to $x_1, \ldots, x_p$ and $\mathfrak{s}_{j_n}$.

Assume $i = j_n$. With $u = 1$, $v = n$, $h = i$ we obtain $\lambda_i(x_1) = \lambda_i(x_n)$, thus the conclusion of (2) holds. Finally, if $i = j_q$, $q < n$ then $\lambda_i(x_1) = \lambda_i(x_{q+1})$ follows from (3) with $u = 1$, $v = q+1$, $h = i$; the induction hypothesis gives $\mathfrak{s}^1_{j_n} \mid X_i \xrightarrow{\lambda_i} x_{q+1}$.

(2) remains also valid for not necessarily different points: if $x_p = x_q$, $p < q$ then drop the points $x_{p+1}, \ldots, x_q$. (More precisely, this has to be done by induction, too.)

2° $\lambda^1 = \lambda^1(X, \lambda_i)$ is an extension (10.2), and $\lambda^1_{\mathrm{P}} < \lambda^1$, so it is enough to prove that $\lambda^1_{\mathrm{P}} \mid X_i > \lambda_i$. Assume $\mathfrak{t} \xrightarrow{\lambda^1_{\mathrm{P}}} x \in X_i$; we have to check that $\mathfrak{t} \mid X_i \xrightarrow{\lambda_i} x$.

λ^1 is symmetric (10.3), so, by the lemma, there are $x_1, \ldots, x_n$ and $\mathfrak{s}$ such that $\mathfrak{t} \supset \dot{x}_1 \cap \ldots \cap \dot{x}_n \cap \mathfrak{s}$, $x_1 = x$, $\dot{x}_m \xrightarrow{\lambda_1} x_{m+1}$, $\mathfrak{s} \xrightarrow{\lambda^1} x_n$. $x_1 \in X_i$ implies that each x_m is in $\bigcup\limits_{j \in I} X_j$, thus, by 10.2, there are $j_1, \ldots, j_n$ and $\mathfrak{s}_{j_n}$ with $\dot{x}_m \xrightarrow{\lambda_{j_m}} x_{m+1}$, $\mathfrak{s} = \mathfrak{s}^1_{j_n}$, $\mathfrak{s}_{j_n} \xrightarrow{\lambda_{j_n}} x_n$. Denote by $m_1, \ldots, m_s$ (in increasing order) those m for which $x_m \in X_i$; then $m_1 = 1$ and

$$\text{(4)} \qquad \mathfrak{t} \mid X_i \supset \Big(\bigcap_{p=1}^{s} \dot{x}_{m_p} \Big) \cap (\mathfrak{s} \mid X_i).$$

(2) applied to different segments of the sequence $x_1, \ldots, x_n$ gives $\dot{x}_{m_{p+1}} \xrightarrow{\lambda_i} x_{m_p}$, hence $\dot{x}_{m_p} \xrightarrow{\lambda_i} x_{m_{p+1}}$, and also $\mathfrak{s} \mid X_i \xrightarrow{\lambda_i} x_{m_s}$. Now $\mathfrak{t} \mid X_i \xrightarrow{\lambda_i} x$ follows from the lemma, since $(\lambda_i)_{\mathrm{P}} = \lambda_i$. □

The condition in the proposition is vacuous if $|I| = 2$. There is no coarsest pointwise reciprocal extension: take the discrete convergence in a two-point subset of a three-point set.

10.5 A pointwise reciprocal convergence induces a pseudotopology with the same property. Conversely, if π is pointwise reciprocal then so are $\lambda^0(\pi)$ and $\lambda^1(\pi)$ (evident), but not all the convergences compatible with π.

PROPOSITION. *A family of pointwise reciprocal convergences in a pointwise reciprocal pseudotopological space has pointwise reciprocal extensions iff the following condition holds for each $i, j \in I$:*

$$(1) \qquad \textit{if } x \in X_i \setminus X_j,\ y \in X_j \setminus X_i,\ \pi(x) = \pi(y), \mathfrak{s}_j \xrightarrow{\lambda_j} y \textit{ then } \mathfrak{s}_j^1 \mid X_i \xrightarrow{\lambda_i} x;$$

if so then $\lambda_{\mathrm{P}}^1(\pi, \lambda_i)$ is the finest one, and there exists a coarsest one as well.

PROOF. 1° The necessity is obvious. Before proving the sufficiency, let us note that if (1) holds then the same is valid more generally for each $x \in X_i$, $y \in X_j$: If $y \in X_i$ then, by accordance, $\mathfrak{s}_j^1 \mid X_i \xrightarrow{\lambda_i} y$, thus $\mathfrak{s}_j^1 \mid X_i \xrightarrow{\lambda_i} x$, since $\pi(x) = \pi(y)$ implies $\dot{x} \xrightarrow{\pi} y$, thus $\dot{x} \xrightarrow{\lambda_i} y$, $\lambda_i(x) = \lambda_i(y)$. The case $x \in X_j$ can be dealt with similarly, reversing the order of the two steps in the reasoning.

2° $\lambda^1 = \lambda^1(\pi, \lambda_i)$ is a symmetric extension (Proposition 10.3). It has to be proved that λ_{P}^1 is a compatible extension, i.e. that $\pi(\lambda_{\mathrm{P}}^1) > \pi$ and $\lambda_{\mathrm{P}}^1 \mid X_i > \lambda_i$. The existence of a coarsest pointwise reciprocal extension is then a consequence of the following observation: the infimum of pointwise reciprocal convergences inducing the same pseudotopology (or just the same closure) is pointwise reciprocal, too.

3° To prove $\pi(\lambda_{\mathrm{P}}^1) > \pi$, assume $\mathfrak{u} \xrightarrow{\pi(\lambda_{\mathrm{P}}^1)} x$. Then $\mathfrak{u} \xrightarrow{\lambda_{\mathrm{P}}^1} x$, and, by Lemma 10.4, there are $x_1, \ldots, x_n$ and $\mathfrak{s}$ as in 10.4 (1) (with $\lambda = \lambda^1$) such that $\mathfrak{u} \supset \mathfrak{s} \cap \dot{x}_1 \cap \ldots \cap \dot{x}_n$; hence $\mathfrak{u} \supset \mathfrak{s}$ or $\mathfrak{u}$ is of the form $\dot{x}_m$. In both cases, $\mathfrak{u} \xrightarrow{\pi} x_m$ with some m. From $\dot{x}_m \xrightarrow{\lambda^1} x_{m+1}$ we have $\dot{x}_m \xrightarrow{\pi} x_{m+1}$, thus $\pi(x_1) = \pi(x_2) = \pi(x_n)$, and $\mathfrak{u} \xrightarrow{\pi} x_1 = x$.

4° To prove $\lambda_{\mathrm{P}}^1 \mid X_i > \lambda_i$, assume $\mathfrak{t} \xrightarrow{\lambda_{\mathrm{P}}^1} x \in X_i$; we have to show that $\mathfrak{t} \mid X_i \xrightarrow{\lambda_i} x$. Pick $x_1, \ldots, x_n$ and $\mathfrak{s}$ as in 10.4 (1) (again with $\lambda = \lambda^1$) such that $\mathfrak{t} \supset \mathfrak{s} \cap \dot{x}_1 \cap \ldots \cap \dot{x}_n$. Just like in 3°, $\pi(x_1) = \pi(x_2) = \pi(x_n)$. Let $m_1, \ldots, m_s$ denote the same as at the end of the proof of Proposition 10.4; then 10.4 (4) holds. Now $\dot{x}_{m_p} \xrightarrow{\lambda_i} x_{m_{p+1}}$ follows from $\dot{x}_{m_p} \xrightarrow{\pi} x_{m_{p+1}}$. Moreover,

$$\mathfrak{s} \mid X_i \xrightarrow{\lambda_i} x_{m_s}, \tag{2}$$

since, according to Proposition 10.2, $\mathfrak{s}$ is either an ultrafilter, or it is of the form $\mathfrak{s}_j^1$ with $\mathfrak{s}_j \xrightarrow{\lambda_j} x_j$, $j \in I$; in the first case, $\mathfrak{s} \xrightarrow{\pi} x_n$, so $\mathfrak{s} \xrightarrow{\pi} x_{m_s}$, and (2) evidently holds; in the second case, (1), or rather its form proved in 1°, can be applied with $x = x_{m_s}$, $y = x_j$. The last sentence of the proof of Proposition 10.4 completes this proof, too. □

(1) is clearly satisfied for $|I| = 1$, but not for $|I| = 2$:

EXAMPLE. Let X be infinite $y_1, y_2 \in X$, $y_1 \neq y_2$, $X_i = X \setminus \{y_i\}$, $\pi(y_i) = \{\dot{y}_1, \dot{y}_2\} \cup \mathrm{Ult}^{\mathrm{f}} X$ $(i = 1, 2)$, $\pi(x) = \{\dot{x}\}$ otherwise, $\lambda_1 = \lambda^1_{\mathrm{lim}}(\pi \mid X_1)$, $\lambda_2 = \lambda^0(\pi \mid X_2)$. We have defined a family of reciprocal limitations in a reciprocal pseudotopological space, but there is no pointwise reciprocal extension, since (1) fails for $i = 1$, $j = 2$, $x = y_2$, $y = y_1$, $\mathfrak{s}_2 = \mathfrak{n}(\lambda_2)(y_1)$. □

10.6 We are going to consider now reciprocal extensions in a pseudotopological space. (The same problem in a set seems to be much more difficult.) The case $I = \emptyset$ was dealt with in Lemma 5.1.

LEMMA. *If λ is a pointwise reciprocal convergence and $\pi(\lambda)$ is reciprocal then so is λ.*

PROOF. Assume $\emptyset \notin \mathfrak{s} \xrightarrow{\lambda} x, y$, and let $\mathfrak{s} \subset \mathfrak{u} \in \mathrm{Ult}\, X$. Then $\mathfrak{u} \xrightarrow{\pi(\lambda)} x, y$, thus $\pi(x) = \pi(y)$, $\dot{x} \xrightarrow{\pi(\lambda)} y$, $\dot{x} \xrightarrow{\lambda} y$, and so $\lambda(x) = \lambda(y)$. □

Consequently, if a family of convergences in a reciprocal pseudotopological space has pointwise reciprocal extensions (see Proposition 10.5) then each pointwise reciprocal extension is reciprocal; in particular, $\lambda^1_{\mathrm{p}}(\pi, \lambda_i)$ is the finest reciprocal extension. Condition 10.5 (1) is not superfluous in this case either, see Example 10.5.

§ 11. Limitations in a pseudotopological space

11.1 We are going to consider extensions of limitations in sets and in pseudotopological spaces. An *extension of limitations* will mean an extension that is a limitation, too. A family of limitations in a set always has extensions; $\lambda^1_{\mathrm{lim}}(X, \lambda_i)$ is the finest and $\lambda^0(X, \lambda_i)$ the coarsest one. (Straightforward.) [Or [12] Propositions 2.4 and 1.12 a) can be applied.]

PROPOSITION. *Any family of limitations in a pseudotopological space has extensions; $\lambda^1_{\mathrm{lim}}(\pi, \lambda_i)$ is the finest and $\lambda^0(\pi, \lambda_i)$ the coarsest one.*

PROOF. $\lambda^0(\pi, \lambda_i)$ is a limitation, since, by 10.2 (2), it is the supremum of two limitations. □

11.2 LEMMA. *If λ is a pointwise reciprocal convergence then so is λ_{lim}.*

PROOF. $\dot{x} \to y$ means the same with respect to λ and λ_{lim}. □

Consequently, $\lambda_{\mathrm{Plim}} = (\lambda_{\mathrm{P}})_{\mathrm{lim}}$ is the finest one among the symmetric (= pointwise reciprocal) limitations coarser than the convergence λ. Given a symmetric convergence λ, a base for λ_{Plim} is defined by

$$\beta(x) = \Big\{ \bigcap_{m=1}^{n} \mathfrak{s}_{(m)} : n \in \mathbb{N}, \mathfrak{s}_{(m)} \xrightarrow{\lambda} x_m \ (1 \leqq m \leqq n), \\ x_1 = x, \quad \dot{x}_m \xrightarrow{\lambda} x_{m+1} \ (1 \leqq m < n) \Big\}.$$

The proof is easier than that of Lemma 10.4. (It is again enough to know that $\pi(\lambda)$ is symmetric.)

PROPOSITION. *A family of symmetric limitations in a set has symmetric extensions iff the condition in Proposition 10.4 holds; if so then $\lambda^1_{\mathrm{Plim}}(X, \lambda_i)$ is the finest symmetric extension.*

PROOF. Proposition 10.4 and the lemma above, using that the operation lim commutes with restrictions. □

There is no coarsest symmetric extension, see after the proof of Proposition 10.4.

11.3 By Lemma 9.2, a pseudotopology π can be induced by symmetric limitations iff it is pointwise reciprocal; if so then $\lambda^1_{\mathrm{lim}}(\pi)$ and $\lambda^0(\pi)$ are symmetric.

PROPOSITION. *A family of symmetric limitations in a pointwise reciprocal pseudotopological space has symmetric extensions iff 10.5 (1) holds; if so then $\lambda^1_{\mathrm{Plim}}(\pi, \lambda_i)$ is the finest one, while the coarsest one is the same as in Proposition 10.5.*

PROOF. The necessity is clear from Proposition 10.5. The converse as well as the additional statements follow from the observation that if λ is a compatible extension (now not required to be a limitation) then so is $\lambda_{\mathrm{lim}} < \lambda$ (which is symmetric if λ was pointwise reciprocal, see Lemma 11.2).

10.5 (1) is not superfluous in this proposition, see Example 10.5.

11.4 Concerning reciprocal extensions of limitations, the same can be said as in 10.6.

§ 12. Pseudotopologies in a closure space

12.1 According to [13] 7.4, a family of pseudotopologies in a set always has extensions;

$$\pi^1(X, \pi_i)(x) = \{\mathfrak{u}_i^1 : i \in I, x \in X_i, \mathfrak{u}_i \in \pi_i(x)\} \cup \{\dot{x}\}$$

is the finest one, while the coarsest extension $\pi^0 = \pi^0(X, \pi_i)$ is defined by

$$\mathfrak{u} \xrightarrow{\pi_0} x \text{ iff } \mathfrak{u} \mid X_i \xrightarrow{\pi_i} x \text{ for each } i \in I \text{ with } x \in X_i \in \mathfrak{u}.$$

In particular, $\pi^0(x) = \operatorname{Ult} X$ if $x \notin \bigcup_{i \in I} X_i$.

PROPOSITION. *Any family of pseudotopologies in a closure space has extensions;*

$$(1) \qquad \pi^0(\mathfrak{n}, \pi_i) = \sup\{\pi^0(\mathfrak{n}), \pi^0(X, \pi_i)\}$$

is the coarsest one.

PROOF. 1° Denoting the right-hand side of (1) by π,

$$(2) \qquad \mathfrak{u} \xrightarrow{\pi} x \text{ iff } \mathfrak{u} \supset \mathfrak{n}(x) \text{ and } \mathfrak{u} \mid X_i \xrightarrow{\pi_i} x \text{ for each } i \text{ with } x \in X_i \in \mathfrak{u}.$$

We have to show that $\mathfrak{n}(\pi) < \mathfrak{n}$ and $\pi \mid X_i < \pi_i$.

2° The proof of $\pi \mid X_i < \pi_i$ is straightforward. To prove $\mathfrak{n}(\pi) < \mathfrak{n}$, it has to be checked that $\mathfrak{n}(\pi)(x) \subset \mathfrak{n}(x)$, i.e. that $\mathfrak{n}(\pi)(x) \subset \mathfrak{u}$ for each ultrafilter $\mathfrak{u} \supset \mathfrak{n}(x)$. This is evident if $\mathfrak{u} \xrightarrow{\pi} x$. Otherwise, there is an $i \in I$ with $x \in X_i \in \mathfrak{u}$. Now $\mathfrak{u} \mid X_i \supset \mathfrak{n}(x) \mid X_i$, thus $\mathfrak{u} \mid X_i \supset \bigcap \pi_i(x)$. For each $\mathfrak{u}_i \xrightarrow{\pi_i} x$, we have $\mathfrak{u}_i^1 \xrightarrow{\pi} x$. Therefore $\mathfrak{u} \supset \bigcap \pi(x) = \mathfrak{n}(\pi)(x)$. □

REMARK. (2) could also be formulated in the following equivalent way: $\mathfrak{u} \xrightarrow{\pi} x$ iff either $\mathfrak{u} \supset \mathfrak{n}(x)$ and there is no i with $x \in X_i \in \mathfrak{u}$, or there is an i with $\mathfrak{u} \mid X_i \xrightarrow{\pi_i} x$.

12.2 If each π_i is symmetric then so are $\pi^0(X, \pi_i)$ and $\pi^1(X, \pi_i)$. [[12] Corollary 3.2.] The problem of symmetric extensions in a closure space is of no interest, since π is symmetric iff $\mathfrak{n}(\pi)$ is so.

12.3 The analogue of Proposition 10.4 holds for pointwise reciprocal extensions of pseudotopologies in a set (replace the filters by ultrafilters). Proof of the sufficiency: the convergences $\lambda^0(\pi_i)$ form a family satisfying 10.4 (2), and they are pointwise reciprocal (Lemma 9.2, remembering that $\lambda^0(\pi_i)$ is a limitation), thus $\pi(\lambda^1_{\mathrm{P}}(X, \lambda^0(\pi_i)))$ is an extension, and it is pointwise reciprocal, see 10.5. The finest pointwise reciprocal pseudotopology π_{P} coarser than the symmetric pseudotopology π can be described as follows: $\mathfrak{u} \xrightarrow{\pi_{\mathrm{P}}} x$ iff there are $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ such that $x = x_1$, $\dot{x}_m \xrightarrow{\pi} x_{m+1}$ $(1 \leqq m < n)$, $\mathfrak{u} \xrightarrow{\pi} x_n$. With this notation, $\pi^1_{\mathrm{P}}(X, \pi_i)$ is the finest pointwise reciprocal extension. The extension given in the proof above is in fact also equal to the finest pointwise reciprocal extension: Let π be a pointwise reciprocal extension. Then $\lambda^0(\pi)$ is a pointwise reciprocal extension of the convergences $\lambda^0(\pi_i)$, thus $\lambda^0(\pi) < \lambda^1_{\mathrm{P}}(X, \lambda^0(\pi_i))$; take now the induced pseudotopologies on both sides.

If π is pointwise reciprocal then so is $\mathfrak{n}(\pi)$; conversely, if $\mathfrak{n}$ is pointwise reciprocal then so is $\pi^0(\mathfrak{n})$.

PROPOSITION. *A family of pointwise reciprocal pseudotopologies in a pointwise reciprocal closure space has pointwise reciprocal extensions iff the following condition holds for each* $i, j \in I$:
(1)
$$\text{if } x \in X_i \setminus X_j,\ y \in X_j \setminus X_i,\ \mathfrak{n}(x) = \mathfrak{n}(y),\ \mathfrak{u}_i \xrightarrow{\pi_i} x,\ X_j \in \mathfrak{u}_i^1 \text{ then } \mathfrak{u}_i^1 \mid X_j \xrightarrow{\pi_j} y;$$

if so then there exists a coarsest pointwise reciprocal extension.

PROOF. 1° The necessity is obvious. If (1) holds then it remains valid for each $x \in X_i$, $y \in X_j$: Assume first $y \in X_i$. Then $\mathfrak{n}(x) = \mathfrak{n}(y)$ implies $\mathfrak{n}_i(x) = \mathfrak{n}_i(y)$, $\dot{y} \in \bigcap \pi_i(x)$, $\dot{y} \in \pi_i(x)$, $\pi_i(x) = \pi_i(y)$; hence $\mathfrak{u}_i \xrightarrow{\pi_i} y$, and $\mathfrak{u}_i^1 \mid X_j \xrightarrow{\pi_j} y$ by the accordance. In the case $x \in X_j$, use the same reasoning, but begin with applying the accordance. Put
(2)
$$\mathfrak{u} \xrightarrow{\pi} x \text{ iff } \mathfrak{u} \supset \mathfrak{n}(x), \text{ and } \mathfrak{u} \mid X_i \xrightarrow{\pi_i} y \text{ whenever } i \in I, y \in X_i \in \mathfrak{u},\ \mathfrak{n}(x) = \mathfrak{n}(y).$$

π is a pointwise reciprocal pseudotopology: if $\dot{z} \xrightarrow{\pi} x$ then $z \in \bigcap \mathfrak{n}(x)$, thus $\mathfrak{n}(z) = \mathfrak{n}(x)$, hence $\pi(z) = \pi(x)$, since $\pi(x)$ is determined by $\mathfrak{n}(x)$. If we show that π is an extension in $(X, \mathfrak{n})$ then it is clearly the coarsest pointwise reciprocal extension.

2° *π is an extension.* $\pi > \pi^0(\mathfrak{n}, \pi_i)$ (cf. 12.1 (2)), which is an extension, so we have only to prove that $\pi \mid X_i < \pi_i$. Assume $\mathfrak{u}_i \xrightarrow{\pi_i} x$, $\mathfrak{u} = \mathfrak{u}_i^1$. Then $\mathfrak{u} \supset \mathfrak{n}(x)$ (as π_i is compatible). If $j \in I$, $y \in X_j \in \mathfrak{u}$, $\mathfrak{n}(x) = \mathfrak{n}(y)$ then, by (1), $\mathfrak{u} \mid X_j \xrightarrow{\pi_j} y$; thus $\mathfrak{u} \xrightarrow{\pi} x$.

3° *π is compatible.* $\mathfrak{u} \xrightarrow{\pi} x$ implies $\mathfrak{u} \supset \mathfrak{n}(x)$, thus $\mathfrak{n}(\pi)(x) \supset \mathfrak{n}(x)$. To prove the converse, let $\mathfrak{n}(x) \subset \mathfrak{u} \in \operatorname{Ult} X$. If $\mathfrak{u} \notin \pi(x)$ then there are $i \in I$, $y \in X_i \in \mathfrak{u}$ such that $\mathfrak{n}(x) = \mathfrak{n}(y)$. Now $\mathfrak{u} \supset \mathfrak{n}(y)$, and we can proceed as in 2° of the proof of Proposition 12.1. □

12.4 It is not clear which closures can be induced by a reciprocal pseudotopology; by Lemma 5.2, these closures are the same as the ones that can be induced by Cauchy structures.

§ 13. Convergences and limitations in a closure space

13.1 A family of convergences in a closure space always has extensions; this can be proved similarly to Proposition 12.1, but it can also be obtained in two steps: the induced pseudotopologies have an extension by Proposition 12.1, and then the convergences can be extended in this pseudotopological space (Proposition 10.2).

$$\lambda^0(\mathfrak{n}, \lambda_i) = \sup\{\lambda^0(\mathfrak{n}), \lambda^0(X, \lambda_i)\} = \lambda^0(\pi^0(\mathfrak{n}, \pi(\lambda_i)), \lambda_i)$$

is the coarsest extension. If each λ_i is a limitation then so is $\lambda^0(\mathfrak{n}, \lambda_i)$, since it is the supremum of two limitations (cf. 11.1).

13.2 A symmetric convergence induces a symmetric closure; conversely, a symmetric closure can be induced by symmetric convergences; $\lambda^0_{(s)}(\mathfrak{n})$ is the coarsest one (12.2 and 10.3). A family of symmetric convergences in a symmetric closure space always has symmetric extensions; $\lambda^0_{(s)}(\mathfrak{n}, \lambda_i)$ is the coarsest one (12.2 and Proposition 10.3).

13.3 A pointwise reciprocal convergence induces a closure with the same property; conversely, if $\mathfrak{n}$ is a pointwise reciprocal closure then the limitation $\lambda^0(\mathfrak{n})$ is pointwise reciprocal = symmetric (12.3 and 10.5).

PROPOSITION. *A family of pointwise reciprocal convergences in a pointwise reciprocal closure space has pointwise reciprocal extensions iff* 10.5 (1) *holds with* π *replaced by* $\mathfrak{n}$*; if so then there exists a coarsest pointwise reciprocal extension.*

PROOF. *Sufficiency.* The conditions of Proposition 12.3 are satisfied for the pseudotopologies $\pi(\lambda_i)$, so they have a compatible pointwise reciprocal extension π. Now $\pi(x) = \pi(y)$ implies $\mathfrak{n}(x) = \mathfrak{n}(y)$, thus 10.5 (1) holds, and so Proposition 10.5 yields a pointwise reciprocal extension compatible with π, hence with $\mathfrak{n}$. The infimum of all these extensions is also a pointwise reciprocal extension, see in 2° of the proof of Proposition 10.5. □

If each λ_i is a limitation then so is the coarsest pointwise reciprocal extension. (See the proof of Proposition 11.3.)

13.4 A closure can be induced by a reciprocal convergence (or reciprocal limitation) iff it can be induced by a Cauchy screen. (Apply Lemma 4.1: If $\mathfrak{S}$ is a compatible Cauchy screen then $\lambda(\mathfrak{S})$ is a compatible reciprocal limitation; if λ is a compatible reciprocal convergence then there are Cauchy screens compatible with λ_{lim}, hence with the closure, cf. Lemma 5.1 a).)

§ 14. Extensions in two steps

In § 13, we obtained some results for extensions of convergences in a closure space in two steps: first extending the induced pseudotopologies, and then the convergences in the pseudotopological space. This method can clearly be applied if there are extensions in both steps without any additional assumption (as in 13.1, see also [10] 5.7, 6.3, [13] 8.6), but sometimes also in more interesting cases (e.g. Proposition 13.3, see also [10] 5.12, and the Riesz and Lodato extensions in a set in [14], [15]), although not always (see e.g. [10] 5.26).

Some results on extensions of screens in a pseudotopological space can also be obtained through an extension of the induced convergences. E.g. the second half of the proof of Theorem 9.3 (from 3° on) can be replaced by the following:

With $\lambda_i = \lambda(\mathfrak{S}_i)$, the conditions of Proposition 10.5 are satisfied: apply 9.3 (1) to $n = 2$, $x_1 = x$, $x_2 = y$, $i_1 = i$, $i_2 = j$ and the filters $\dot{x}$, $\mathfrak{s}_j \cap \dot{y}$. Thus, by Proposition 11.3, the symmetric limitation $\lambda = \lambda^1_{\mathrm{Plim}}(\pi, \lambda_i)$ is an extension.

According to 9.1, it is enough to show that the screens $\mathfrak{S}_i$ have an extension compatible with λ, since all these extensions are pointwise Cauchy. This means that only 2.2 (3) has to be checked; but it is equivalent to the generalized form of 9.3 (1) obtained in 1° and 2° (use 10.2 (1) and Lemma 10.4; in the latter $\dot{x}_m \to x_{m+1}$ with $\lambda^1(\pi, \lambda_i)$ is equivalent to $\pi(x_m) = \pi(x_{m+1})$).

Theorem 5.3 cannot be obtained in two steps:

EXAMPLE. Let $X = \mathbb{N} \cup \{0\}$, $X_0 = \mathbb{N}$, $\pi(x) = \{\dot{x}\}$ if $x \neq 0$, $\pi(0) = \mathrm{Ult}^{\mathrm{f}}\, X \cup \{\dot{0}\}$, $\mathfrak{S}_0 = \mathrm{Ult}^+\, X_0 \cup \mathrm{Fil}^{\mathrm{f}}\, X_0$. The conditions of Theorem 5.3 are satisfied, but there is no Cauchy extension compatible with the finest reciprocal extension of $\lambda(\mathfrak{S}_0)$ = the discrete limitation, which is $\lambda^1_{\mathrm{Plim}}(\pi, \lambda(\mathfrak{S}_0))$ by 11.4, and is now equal to $\lambda^1_{\lim}(\pi)$: 4.3 (1) does not hold for $\mathfrak{t}_0 \in \mathrm{Ult}^{\mathrm{f}}\, X_0$ and the filter $\mathfrak{s}_0$ consisting of the cofinite sets.

Taking the coarsest reciprocal extension of $\lambda(\mathfrak{S}_0)$ (see 11.4 and Proposition 11.3) is no good either: Let $\mathfrak{S}'_0$ consist of the finite intersections of free ultrafilters (and of the obligatory elements). The conditions of Theorem 5.3 are again satisfied. $\lambda^0(\pi)$ is the coarsest reciprocal extension of $\lambda(\mathfrak{S}'_0)$. Now $\mathfrak{S}'_0$ has no extension compatible with $\lambda^0(\pi)$, since 2.2 (3) fails for $\mathfrak{s}_0$ consisting of the cofinite sets. □

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

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MEAN CONVERGENCE OF EXTENDED HERMITE INTERPOLATION OF HIGHER ORDER[1]

B. DELLA VECCHIA, G. MASTROIANNI and P. VÉRTESI

Abstract

The authors establish necessary and sufficient conditions for the weighted L^p-convergence at given rates of Hermite interpolation of higher order based on extended Jacobi matrices plus the endpoints ± 1. Theorems on simultaneous approximation are also proved.

1. Introduction

Let w be a Jacobi weight, i.e.

$$w(x)=v^{\alpha,\beta}(x):=(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta>-1, \quad |x|\leqq 1,$$

and put $w^*(x)=(1-x^2)w(x)$. Let us denote by $p_n^{\alpha,\beta}(x):=p_n(w;x)$ the nth orthonormal Jacobi polynomial corresponding to the weight w, by $x_{i,m+1}=x_i$, $i=1,\dots,m+1$, the zeros of $p_{m+1}(w;x)$, by $p_m(w^*;x)$ the mth orthonormal Jacobi polynomial corresponding to the weight w^* and by x_i^*, $i=1,\dots,m$, the zeros of $p_m(w^*)$. It was proved in [7] that the zeros of the polynomial $q_{2m+1}:=p_{m+1}(w)p_m(w^*)$ are simple and in $(-1,1)$ they have an arcsin distribution. This property allows us to introduce the so called extended interpolation matrix based on the zeros of q_{2m+1} plus the endpoints ± 1 and to consider the corresponding extended Hermite interpolation polynomial of higher order. Indeed, for a given function $f\in C^{q-1}([-1,1])$, $q\geqq 1$, we denote by $H_{mq,r,s}(w;w^*;f)$ the unique polynomial of degree at most $(2m+1)q+r+s-1$ defined by

$$H_{mq,r,s}^{(i)}(w;w^*;f;x_k)=f^{(i)}(x_k), \quad i=0,\dots,q-1, \quad k=1,\dots,m+1,$$
$$H_{mq,r,s}^{(i)}(w;w^*;f;x_k^*)=f^{(i)}(x_k^*), \quad i=0,\dots,q-1, \quad k=1,\dots,m,$$
$$H_{mq,r,s}^{(j)}(w;f;-1)=f^{(j)}(-1), \quad j=0,\dots,s-1,$$

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$$H^{(i)}_{mq,r,s}(w;f;1)=f^{(i)}(1), \qquad i=0,\dots,r-1,$$

where r,s are nonnegative integers.

If $r=s=0$, then we put $H_{mq,0,0}(w;w^*;f)=H_{mq}(w;w^*;f)$, with $H_{mq}(w;w^*;f)$ the interpolating polynomial defined by

$$H^{(i)}_{mq,r,s}(w;w^*;f;x_k)=f^{(i)}(x_k), \quad i=0,\dots,q-1, \quad k=1,\dots,m+1,$$
$$H^{(i)}_{mq,r,s}(w;w^*;f;x^*_k)=f^{(i)}(x^*_k), \quad i=0,\dots,q-1, \quad k=1,\dots,m.$$

In particular, if $r=s=0$ and $q=1$, we get extended Lagrange interpolatory polynomials, while $q=2$ gives extended Hermite interpolatory polynomials.

If $r=s=0$ and $q>2$, then $H_{mq}(w;w^*;f)$ is the extended Hermite interpolating polynomial of higher order.

In [2, 5, 8, 13] the weighted L_p-convergence of $H_{mq,r,s}(w;w^*;f)$ for $q=1,2$ was studied and sufficient conditions for the simultaneous approximation were established.

In the present paper we establish both necessary and sufficient conditions for the weighted L^p-convergence at given rates of polynomial $H_{mq,r,s}(w;w^*;f)$, $\forall q\geqq 1$. Necessary and sufficient conditions for the simultaneous approximation are also given. The two main new tools are: the asymptotic formula for q_{2m+1}, whence we get a fairly precise asymptotic for its roots (cf. Lemma 3.4, (3.10) and (3.9)). By these, we can prove our fundamental Lemma 3.4 (cf. [25, Lemma 4.3]).

2. Main results

We say that the function u is a *generalized Jacobi weight* ($u\in \mathrm{GJ}$) if

$$u(x)=\psi(x)\prod_{j=0}^{\sigma+1}|\bar t_j-x|^{\gamma_j}, \qquad |x|\leqq 1, \tag{2.1}$$

where ψ is a positive continuous function in $[-1,1]$ and its modulus of continuity $\omega(\psi)$ satisfies $\int_0^1\omega(\psi;t)t^{-1}<\infty$, $-1=\bar t_0<\bar t_1<\ldots<\bar t_{\sigma+1}=1$, further $\gamma_j>-1$, $j=0,\dots,\sigma+1$. If $\gamma_j>0$, $j=1,\dots,\sigma$, then we say that u is a generalized positive Jacobi weight ($u\in \mathrm{GPJ}$). Now, putting $\|\cdot\|$ the supremum norm on $[-1,1]$, we denote by $E_n(f)=\min_{P\in\mathcal{P}_n}\|f-P\|$ the best uniform approximation error, where $\mathcal{P}_n$ is the set of algebraic polynomials of degree at most n. Then we denote by $\|\cdot\|_p$, $1\leqq p<\infty$, the usual L_p-norm and, if $0<p<1$, keep this notation for convenience.

Throughout this paper we let $N=(2m+1)q+r+s-1$ and $M=\max\{q-1,r-1,s-1\}$. Then we state the main results (cf. [25, Theorem 3.1]).

THEOREM 2.1. *Let* $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1$, $w^*(x) = (1-x^2)w(x)$, $u \in \mathrm{GJ}$ *and* $0 < p < \infty$. *Let* $f \in C^{\rho}([-1,1])$, $\rho \geqq M$. *If*

$$v^{\frac{\rho}{2}-r+q, \frac{\rho}{2}-s+q} w^q \in L^1, \qquad u v^{-\frac{l}{2}, -\frac{l}{2}} \in L^p, \tag{2.2}$$

with l *integer, such that* $0 \leqq l \leqq q-1$, *and* τ *positive real, then*

$$\|[f - H_{mq,r,s}(w; w^*; f)]^{(l)} u\|_p \leqq \text{const} \, \frac{E_{N-\rho}(f^{(\rho)})}{m^{\rho - l - \tau}}, \tag{2.3}$$

for some constant independent of f *and* m, *whenever*

$$F(x) := \frac{u^p(x) v^{p\frac{\tau}{2}+1+rp, p\frac{\tau}{2}+1+sp}(x)}{[v^{\frac{l}{2}+q, \frac{l}{2}+q}(x) w^q(x)]^p} \leqq K, \qquad in \ [-1,1]. \tag{2.4}$$

If, additionally, $u \in \mathrm{GPJ}$, $1 \leqq p < \infty$ *and* $\rho = q-1$, *then*

$$\|[f - H_{mq,r,s}(w; w^*; f)]^{(l)} u\|_p \leqq \text{const} \, \frac{\|f^{(\rho)}\|}{m^{\rho - l - \tau}}, \qquad \forall f \in C^{\rho}([-1,1]), \tag{2.5}$$

implies (2.4).

If we want a better rate of convergence ($\tau = 0$), then we have to replace (2.4) by a stronger condition on the weight u. Indeed (cf. [25, Theorem 3.2]),

THEOREM 2.2. *Under the assumptions of Theorem 2.1, we have*

$$\|[f - H_{mq,r,s}(w; w^*; f)]^{(l)} u\|_p \leqq \frac{\text{const}}{m^{\rho - l}} E_{N-\rho}(f^{(\rho)}), \tag{2.6}$$

for some constant independent of f *and* m, *if*

$$u v^{r - \frac{l}{2} - q, s - \frac{l}{2} - q} w^{-q} \in L^p. \tag{2.7}$$

If, additionally, $u \in \mathrm{GPJ}$ *and* $\rho = q-1$, *then*

$$\|[f - H_{mq,r,s}(w; w^*; f)]^{(l)} u\|_p \leqq \text{const} \, \frac{\|f^{(\rho)}\|}{m^{\rho - l}}, \qquad \forall f \in C^{\rho}([-1,1]), \tag{2.8}$$

implies (2.7).

From Theorems 2.1 and 2.2, for $u \in \mathrm{GPJ}$, $\rho = q-1$ and $1 \leqq p < \infty$, we get the following corollaries.

COROLLARY 2.3. *Under the assumptions of Theorem 2.1, for* $0 \leqq l \leqq q-2$,

$$\lim_{m \to \infty} \|[f - H_{mq,r,s}(w; w^*; f)]^{(l)} u\|_p = 0, \qquad \forall f \in C^{q-1}([-1,1]), \tag{2.9}$$

iff

$$\frac{u^p(x)v^{p\frac{q-1-l}{2}+1+rp,p\frac{q-1-l}{2}+1+sp}(x)}{[v^{\frac{l}{2}+q,\frac{l}{2}+q}(x)w^q(x)]^p} \leqq K, \qquad in \ \ [-1,1]. \tag{2.10}$$

PROOF. First we prove that (2.10) implies (2.9). Indeed, if in Theorem 2.1 we put $\tau=q-1-l$, then assumption (2.4) is equal to (2.10) and therefore, from (2.3), for $\rho-l-\tau=0$, we get (2.9).

On the other hand, if (2.9) holds, then we get (2.5) with $\rho-l-\tau=0$, and by Theorem 2.1 we get (2.4), that is (2.10). □

Working similarly, from Theorem 2.2, for $l=q-1$, we get

COROLLARY 2.4. *Under the assumptions of Theorem 2.2*

$$\lim_{m\to\infty} \|[f-H_{mq,r,s}(w;w^*;f)]^{(q-1)}u\|_p=0, \qquad \forall f\in C^{q-1}([-1,1]), \tag{2.11}$$

iff

$$uv^{r-\frac{3}{2}q+\frac{1}{2},s-\frac{3}{2}q+\frac{1}{2}}w^{-q}\in L^q. \tag{2.12}$$

REMARKS. First note that, if $q=1$ or $q=2$, Theorems 2.1 and 2.2 generalize previous results given in [8, 13] and [5], respectively, in the sense that they give necessary and sufficient (not only sufficient) conditions for the weighted L^p-convergence at given rates, even for higher order derivatives.

From Theorem 2.1 and 2.2, for the particular case $\rho=q-1$, $u\in$ GPJ and $1\leqq p<\infty$ it follows that (2.3) is equivalent to (2.4) and (2.6) to (2.7), respectively.

In Theorem 2.1 we need the additional assumption $1\leqq p<\infty$ to prove that (2.5) implies (2.4). This follows from the proof of Theorem 2.1, where we used Hölder inequality (cf. also [25]).

We also remark that the number $\tau>0$ in Theorem 2.1 is not necessarily integer, hence we have infinite possibilities. (However, if $\tau=0$, we have to suppose (2.7) which is stronger than (2.4).) Moreover, comparing (2.10) in Corollary 2.3 and (2.12) in Corollary 2.4 shows the different behaviour of the ith derivative of $H_{mq,r,s}(w;w^*;f)$, $0\leqq i\leqq q-2$, and $H^{(q-1)}_{mq,r,s}(w;w^*;f)$ (cf. [25]).

Finally we remark that we can get results analogous to Statements 2.1–2.4, if we consider the extended Hermite interpolation of higher order on the zeros of the product polynomial [7, 9] $Q_{2m}(x)=p_m^{\alpha+1,\beta}(x)p_m^{\alpha,\beta+1}(x)$ plus the endpoints ±1. We omit the proofs because they are very similar to the above case. Here also the main tools are the asymptotic formula for $Q_{2m}(x)$ (cf. Lemma 3.4)

$$Q_{2m}(x)=\frac{1}{2\sqrt{\pi}\sqrt{m}}\left(p_m^{2\alpha+\frac{3}{2},2\beta+\frac{3}{2}}(x)+\frac{O(1)}{m^{\frac{3}{2}}(\sin\frac{\theta}{2})^{2\alpha+3}(\cos\frac{\theta}{2})^{2\beta+3}}\right),$$

with $x=\cos\theta$ and relations analogous to (3.9), (3.11) and (3.12) for its roots.

3. Proofs

If a and b are two quantities depending on some parameters, we write $a \sim b$, iff $a/b \leqq \text{const}$ and $b/a \leqq \text{const}$, uniformly for the parameters in question. Throughout this paper C and const stand for positive constants, which can assume different values in different formulas.

Let

$$w(x) = v^{\gamma,\delta}(x)$$

be a Jacobi weight and let $x_{i,m}(w) := x_{i,m} = x_i$, $i = 1, 2, \ldots, m$, be the zeros of the mth Jacobi polynomial $p_m(w) = p_m^{\gamma,\delta}$ corresponding to the weight w. We denote by $\lambda_{i,m}(w) = \lambda_m(w; x_{i,m}(w))$, $i = 1, 2, \ldots, m$, the Cotes numbers, where

$$\lambda_m(w;x) = \Big[\sum_{i=0}^{m-1} p_i^2(w;x)\Big]^{-1},$$

is the mth Christoffel function.

We collect some useful estimates in the following lemma (see [12, 24, 25] for references).

Lemma 3.1. *Set $x_{i,m}(w) = \cos\theta_{i,m}$, for $0 \leqq i \leqq m+1$, where $x_{0,m}(w) = 1$, $x_{m+1,m}(w) = -1$ and $0 \leqq \theta_{i,m} \leqq \pi$. Then*

$$\theta_{i,m} - \theta_{i+1,m} \sim m^{-1}, \tag{3.1}$$

uniformly for $0 \leqq i \leqq m$, $m \in N$;

$$\lambda_{i,m}(w) \sim m^{-1}(1 - x_{i,m}(w))^{\gamma+1/2}(1 + x_{i,m}(w))^{\delta+1/2}, \tag{3.2}$$

uniformly for $1 \leqq i \leqq m$, $m \in N$;

$$|p_m(w;x)| \leqq \text{const}(\sqrt{1-x} + m^{-1})^{-\gamma-\frac{1}{2}}(\sqrt{1+x} + m^{-1})^{-\delta-\frac{1}{2}}, \tag{3.3}$$

uniformly for $-1 \leqq x \leqq 1$ and $m \in N$. Furthermore,

$$|p_{m-1}(w; x_{i,m}(w))| \sim w(x_{i,m})^{-1/2}(1 - x_{i,m}^2(w))^{1/4} \tag{3.4}$$

uniformly for $-1 \leqq x \leqq 1$ and $m \in N$ and

$$l_{m,k}(x) = \frac{\gamma_{m-1}(w)}{\gamma_m(w)} \frac{\lambda_{m,k}(w) p_{m-1}(w; x_k) p_m(w;x)}{x - x_k}, \tag{3.5}$$

with $l_{m,k}$ the kth fundamental Lagrange polynomial and $\gamma_m(w)$ the leading coefficient of $p_m(w;x)$.

Let $w^*(x) = (1 - x^2)w(x)$. Denoting the zeros of $p_m(w^*;x)$ by $x_{i,m}^*$, $i = 1, \ldots, m$, it results [3, 5, 7, 8, 9]

$$(1 - x_{i,m+1}^2) p_m(w^*; x_{i,m+1}) = A_m p_m(w; x_{i,m+1}), \qquad i = 1, \ldots, m+1, \tag{3.6}$$

with

$$A_m = \frac{\gamma_m(w)}{\gamma_m(w^*)} + \frac{\gamma_m(w^*)\gamma_m(w)}{\gamma_{m+1}^2(w)} > 0,$$

and

$$p_{m+1}(w; x_{i,m}^*) = -B_m p_{m-1}(w^*; x_{i,m}^*), \qquad i = 1, \ldots, m, \tag{3.7}$$

with

$$B_m = \frac{\gamma_{m-1}(w^*)}{\gamma_{m+1}(w)} + \frac{\gamma_{m+1}(w)\gamma_{m-1}(w^*)}{\gamma^2(w^*)} > 0.$$

Moreover, if u is a GJ weight (see (2.1)), then there exists a constant $\overline{c} > 0$ such that for every polynomial Q of degree at most m [15]

$$\| |Q|^p u \|_1 \leqq C \left\| |Q|^p u 1_m^{\overline{c}} \right\|_1, \tag{3.8}$$

where $1_m^c = \chi_{\Delta_m(c)}$ denotes the characteristic function of

$$\Delta_m(c) = [-1 + cm^{-2}, 1 - cm^{-2}] \setminus \bigcup_{i=1}^{\sigma} [\overline{t}_i - cm^{-1}, \overline{t}_i + cm^{-1}].$$

To prove Theorems 2.1–2.2 the following lemmas will be useful.

LEMMA 3.2 (see, e.g. [1, 4, 5, 10–12]). *For every $f \in C^\rho([-1,1])$, with $\rho \geqq 0$, there exists a sequence of polynomials $\{G_m\}$ of degree $m \geqq 4\rho + 5$, such that*

$$|f^{(i)}(x) - G_m^{(i)}(x)| \leqq \text{const} \left[\frac{\sqrt{1-x^2}}{m} \right]^{\rho - i} E_{m-\rho}(f^{(\rho)}),$$

with $|x| \leqq 1$, $i = 0, \ldots, \rho$ and for some constant independent of f and m.

LEMMA 3.3 (see, e.g. [1, 4, 5, 8, 10, 12]). *Let $1 < p < \infty$, $0 < c \leqq 1$, $\mu \in \mathrm{GJ}$ and $\phi \in \mathrm{GJ}$. Let A be a polynomial of degree $l(m-1)$, with l positive integer, such that $|A(x) p_m(\mu; x)| \leqq \phi(x)$ for $x \in (-1,1)$ and $m = 1, 2, \ldots$. Given nonnegative integers r and s, and a function $u \in L^1$, if $v^{(r,s)}\mu \in L^1$, $\phi u \in L^p$ and $\phi u v^{(r,s)} \mu \in (L \log^+ L)^p$ then*

$$\sum_{i=1}^{m} \lambda_{i,m}(\mu) v^{(r,s)}(x_{i,m}(\mu)) \left| \int_{-1}^{1} 1_m^c(x) F^{p-1}(x) u(x) \frac{A(x) p_m(\mu; x)}{x - x_{i,m}(\mu)} dx \right| \leqq$$
$$\leqq \text{const} \, \| 1_m^c F \|_p^{p-1}, \qquad m = 1, 2, \ldots,$$

for every function $F \geqq 0$ such that $F \in (L \log^+ L)^p$ with some constant independent of m and F.

Letting $\varepsilon_m(a) = [-1 + am^{-2}, 1 - am^{-2}]$, $a \geqq 0$, we have our fundamental lemma (cf. [25, Lemma 4.3]).

LEMMA 3.4. *Let w be a Jacobi weight, $w^*(x)=(1-x^2)w(x)$ and $q_{2m+1}=p_{m+1}(w)p_m(w^*)$. For arbitrary $d_1,d_2\geqq 0$, fixed, $u\in \mathrm{GJ}$ and $0\leqq i\leqq q$, if $u\phi^{-i}\in L^p$, we have*

$$\|(q_{2m+1}^q)^{(i)}1_m^{d_1}u\|_p\sim\|(q_{2m+1}^q)^{(i)}\chi_{\varepsilon_m(d_2)}u\|_p\sim\|m^i v^{-\frac12,-\frac12}q_{2m+1}^q 1_m^{d_1}u\|_p\sim$$
$$\sim\|m^i v^{-\frac12-q,-\frac12-q}w^{-q}\chi_{\varepsilon_m(d_1)}u\|_p,$$

where the equivalence $\sim$ depends on d_1,d_2 and p, but is independent of m.

PROOF. We give here only a sketch.

Step 1. First we prove that, if $t_{k,2m+1}=t_k=\cos\tau_{k,2m+1}$, $k=1,\dots,2m+1$, are the roots of $q_{2m+1}(x)=p_{m+1}(w;x)p_m(w^*;x)=p_{m+1}^{\alpha,\beta}(x)p_m^{\alpha+1,\beta+1}(x)$, then

$$t_{k,2m+1}=\frac{2k+2\alpha+1}{4m+2\alpha+2\beta+6}\pi+\xi_{k,2m+1}, \tag{3.9}$$

where

$$|\xi_{k,2m+1}|\leqq\frac{c}{km},\qquad 1\leqq k\leqq(1-\varepsilon)(2m+1).$$

Indeed, by [18, 8.21.18)],

$$\begin{aligned}
q_{2m+1}(x)&=\frac{1}{\sqrt{n+1}\sqrt{\pi}}\frac{1}{(\sin\frac{\theta}{2})^{\alpha+\frac12}(\cos\frac{\theta}{2})^{\beta+\frac12}}\times\\
&\times\left\{\cos\left[m+1+\frac{\alpha+\beta+1}{2}\theta-\left(\alpha+\frac12\right)\frac{\pi}{2}\right]+\frac{O(1)}{m\sin\theta}\right\}\times\\
&\times\frac{1}{\sqrt{m\pi}}\frac{1}{(\sin\frac{\theta}{2})^{\alpha+\frac32}(\cos\frac{\theta}{2})^{\beta+\frac32}}\times\\
&\times\left\{\cos\left[m+1+\frac{\alpha+\beta+1}{2}\theta-\left(\alpha+\frac12\right)\frac{\pi}{2}-\frac{\pi}{2}\right]+\frac{O(1)}{m\sin\theta}\right\}=\\
&=\sqrt{\frac{2m+1}{\pi m(m+1)}}\frac{1}{\sqrt{\pi}\sqrt{2m+1}}\frac{1}{(\sin\frac{\theta}{2})^{2\alpha+2}(\cos\frac{\theta}{2})^{2\beta+2}}\times\\
&\times\left\{\cos\left(2m+1+\frac{2\alpha+2\beta+4}{2}-(2\alpha+2)\frac{\pi}{2}\right)+\frac{O(1)}{m\sin\theta}\right\}=\\
&=\frac{\sqrt{2m+1}}{2\sqrt{\pi}\sqrt{m}\sqrt{m+1}}\left(p_{2m+1}^{2\alpha+\frac32,2\beta+\frac32}(x)+\frac{O(1)}{m^{\frac32}(\sin\frac{\theta}{2})^{2\alpha+3}(\cos\frac{\theta}{2})^{2\beta+3}}\right),
\end{aligned}\tag{3.10}$$

with $x=\cos\theta$ and $\frac{c}{m}\leqq\theta\leqq\pi-\frac{c}{m}$. From the third inequality in (3.10), by the argument of [22], since the roots of q_{2m+1} are different (see [7]), we get (3.9), for $K\leqq K_0$, with $K=\min(k,2m+2-k)$.

Note that if $k>(1-\varepsilon)(2m+1)$, we use symmetry: $p_m^{\alpha,\beta}(x)=$ $=(-1)^m p_m^{\beta,\alpha}(-x)$. Formula (3.9) shows that if $m_1=2m+2\alpha+\beta+3$, then

$$t_{k+1}-t_k \approx \frac{\pi}{m_1}, \qquad \text{if } K \geqq K_0. \tag{3.11}$$

From (3.9)

$$\left| t_k-\theta_{k,2m+1}^{2\alpha+\frac{3}{2},2\beta+\frac{3}{2}} \right| \leqq \frac{c}{Km}, \qquad k=1,\dots,2m+1,$$

if $\cos\theta_k^{2\alpha+\frac{3}{2},2\beta+\frac{3}{2}}$, $k=1,\dots,m+1$, are the zeros of $p_{2m+1}^{2\alpha+\frac{3}{2},2\beta+\frac{3}{2}}$.

Step 2. Now we prove that for the roots of q'_{2m+1} (denoted by $s_{k,2m}=$ $\cos\sigma_{k,2m}$, $k=1,\dots,2m$,) we have

$$\sigma_{k,2m}=\frac{t_k+t_{k+1}}{2}+\eta_{k,2m}, \qquad k=1,\dots,2m, \tag{3.12}$$

where $|\eta_{k,2m}|\leqq\frac{c}{Km}$; further $s_{k,2m}$ are different.

Indeed, they are different (see [7]). Further if we evaluate

$$q'_{2m+1}(x)=p'_{m+1}(w;x)p_m(w^*;x)+p_{m+1}(w;x)p'_m(w^*;x)$$

at $x_+=\cos\left(\frac{t_k+t_{k+1}}{2}+\frac{c_1}{Km}\right)$ and $x_-=\cos\left(\frac{t_k+t_{k+1}}{2}-\frac{c_1}{Km}\right)$, then

$$q'_{2m+1}(x_+)q'_{2m+1}(x_-)<0$$

whence we got (3.12). Here $K\geqq K_0$ and $c_1>0$ must be big, fixed.

Finally, from (3.9) and (3.12), following [25], we have the assertion. Further details are left to the reader. □

From the proof of Lemma 3.4, following [25, Corollary 4.3.1], we can deduce

Corollary 3.5. *Under the assumptions of Lemma 3.4, for $d\geqq 0$ and $s\leqq l$,*

$$\begin{aligned}\|[q_{2m+1}^q]^{(l)}1_m^d u\|_p \sim \|[q'_{2m+1}]^l q_{2m+1}^{q-l}1_m^d u\|_p \sim\\ \sim \|m^{l-s}v^{\frac{s-l}{2},\frac{s-l}{2}}[q_{2m+1}^q]^{(s)}1_m^d u\|_p.\end{aligned}$$

Later we use the next

STATEMENT 3.6. *The statements of Lemma 3.4 and Corollary 3.5 hold true replacing* $[-1,1]$ *by an arbitrary interval* Δ *from* $[-1,1]$ *and* $\sim$ *do not depend on* Δ.

The proof of this statement comes from the argument applied to get Lemma 3.4 and Corollary 3.5.

PROOF OF THEOREM 2.1. Assume that $r,s<q$. The case $r,s\geqq q$ is similar (see e.g. [25]). First we prove that (2.4) implies (2.3). Let $r_N=f-G_N$, where G_N is the polynomial of degree $N=(2m+1)q+r+s-1$ defined by Lemma 3.2 corresponding to the function f. For $0<p<\infty$ and $l=0,1,\dots,q-1$, we have

$$\left\|[f-H_{mq,r,s}(w;w^*;f)]^{(l)}u\right\|_p\leqq \mathrm{const}\left\{\left\|r_N^{(l)}u\right\|_p+\left\|H^{(l)}_{mq,r,s}(w;w^*;r_N)u\right\|_p\right\}.$$

By [15, Theorem 5, p. 242], there is a number $0<c^*\leqq 1$, such that

$$\begin{aligned}&\left\|[f-H_{mq,r,s}(w;w^*;f)]^{(l)}u\right\|_p\leqq\\ &\leqq \mathrm{const}\left\{\frac{E_{N-\rho}(f^{(\rho)})}{n^{\rho-l}}\|u\|_p+\left\|H_{mq,r,s}(w;w^*;r_N)v^{-\frac{l}{2},-\frac{l}{2}}u1_m^{c^*}\right\|_p m^l\right\},\end{aligned}\tag{3.13}$$

where, as above, $1_m^{c^*}$ denotes the characteristic function of the set $\Delta_m(c^*)$.

On the other hand $H_{mq,r,s}(w;w^*;f)$ can be written as

$$\begin{aligned}H_{mq,r,s}(w;w^*;f;x)=&\,v^{r,s}(x)p^q_{m+1}(w;x)H_{mq}\left(w^*;\frac{f}{v^{r,s}p^q_{m+1}(w)};x\right)+\\ &+v^{r,s}(x)p^q_m(w^*;x)H_{(m+1)q}\left(w;\frac{f}{v^{r,s}p^q_m(w^*)};x\right)+\\ &+v^{0,s}(x)q^q_{2m+1}(x)L_r\left(Z;\frac{f}{v^{0,s}q^q_{2m+1}};x\right)+\\ &+v^{r,0}(x)q^q_{2m+1}(x)L_s\left(Y;\frac{f}{v^{r,0}q^q_{2m+1}};x\right),\end{aligned}$$

where $H_{mq}\left(w^*;\frac{f}{v^{r,s}p^q_{m+1}(w)};x\right)$ is the Hermite polynomial of degree $mq-1$ interpolating the function $\frac{f}{v^{r,s}p^q_{m+1}(w)}$ and its derivatives up to the $(q-1)$-st one at the zeros of $p_m(w^*)$, $H_{(m+1)q}\left(w;\frac{f}{v^{r,s}p^q_m(w^*)};x\right)$ is the Hermite polynomial of degree $(m+1)q-1$ interpolating the function $\frac{f}{v^{r,s}p^q_m(w^*)}$ and its derivatives up to the $(q-1)$-st at the zeros of $p_{m+1}(w)$, $L_r\left(Z; \frac{f}{v^{0,s}q_{2m+1}^q}\right)$ is the "Lagrange" polynomial of degree $r-1$ interpolating the function $\frac{f}{v^{0,s}q_{2m+1}^q}$ and its derivatives up to the $(r-1)$-st one at the node -1 and $L_s\left(Y; \frac{f}{v^{r,0}q_{2m+1}^q}\right)$ is the "Lagrange" polynomial of degree $s-1$ interpolating the function $\frac{f}{v^{r,0}q_{2m+1}^q}$ and its derivatives up to the $(s-1)$-st one at the node 1. Hence

$$\|H_{mq,r,s}(w;w^*;r_N)v^{-\frac{l}{2},-\frac{l}{2}}u1_m^{c^*}\|_p \leqq$$

$$\leqq \text{const}\left[\left\|H_{mq}\left(w^*; \frac{r_N}{v^{r,s}p_{m+1}^q(w)}\right)v^{r-\frac{l}{2},s-\frac{l}{2}}p_{m+1}^q(w)u1_m^{c^*}\right\|_p + \right. \tag{3.14}$$

$$\left. + \left\|H_{(m+1)q}\left(w; \frac{r_N}{v^{r,s}p_m^q(w^*)}\right)v^{r-\frac{l}{2},s-\frac{l}{2}}p_m^q(w^*)u1_m^{c^*}\right\|\right] =: \text{const}\,[I_1+I_2].$$

To estimate I_1, first we assume $1<p<\infty$. Then, we recall that [24]

$$H_{mq}(w^*;f;x) = \sum_{i=0}^{q-1}\sum_{k=1}^{m} f^{(i)}(x_{k,m}^*)h_{i,k}(x),$$

where the polynomials $h_{i,k}(x)$ of degree exactly $mq-1$ are uniquely defined by

$$h_{i,k}^{(j)}(x_{l,m}^*) = \delta_{i,j}\delta_{k,l};$$

therefore

$$\left|H_{mq}\left(w^*; \frac{r_N}{v^{r,s}p_{m+1}^q(w)}; x\right)\right| \leqq \sum_{i=0}^{q-1}\sum_{k=1}^{m}\left|\left[\frac{r_N(x)}{v^{r,s}(x)p_{m+1}^q(w)}\right]_{x=x_{k,m}^*}^{(i)}\right| |h_{i,k}(x)| :=$$

$$:= \Sigma_0 + \Sigma_1 + \dots + \Sigma_{q-1}.$$

Hence, from (3.14),

$$I_1 \leqq \sum_{i=0}^{q-1}\left\|\Sigma_i p_{m+1}^q(w)uv^{r-\frac{l}{2},s-\frac{l}{2}}1_m^{c^*}\right\|_p.$$

Denoting by j the index corresponding to the closest knot to x, we have, for $i=0,\dots,q-1$,

$$\Sigma_i \leqq \left|\left[\frac{r_N(x)}{v^{r,s}(x)p_{m+1}^q(w;x)}\right]_{x=x_{j,m}^*}^{(i)}\right| |h_{i,j}(x)| +$$

$$+ \sum_{k=1,k\neq j}^{m}\left|\left[\frac{r_N(x)}{v^{r,s}(x)p_{m+1}^q(w;x)}\right]_{x=x_{k,m}^*}^{(i)}\right| |h_{i,k}(x)| := A_j + \overline{\Sigma}_i.$$

By Leibniz formula, from Lemma 3.2, [20, Lemma 3.3, p. 142] and [18, formula (4.21.7), p. 63], it follows, for $k=1,\dots,m$, and $N=(2m+1)q+r+s-1$,

$$\left|\left[\frac{r_N(x)}{v^{r,s}(x)p_{m+1}^q(w;x)}\right]^{(i)}_{x=x^*_{k,m}}\right| \leqq \tag{3.15}$$
$$\leqq \text{const}\left[\frac{\sqrt{1-x_k^{*2}}}{m}\right]^{\rho-i}\frac{E_{N-\rho}(f^{(\rho)})}{v^{r,s}(x_k^*)|p_{m+1}^q(w;x_k^*)|}.$$

On the other hand, recalling that ([14, Theorem 33, p. 171]) $l_{m,j}(x)\sim 1$, following [21, p. 374], we get

$$|h_{i,j}(x)|\leqq \text{const}\,\frac{(1-x^2)^{\frac{i}{2}}}{m^i}, \tag{3.16}$$

therefore, by (3.15) and (3.16), it follows, for $0\leqq l\leqq q-1$,

$$v^{r,s}(x)|p_{m+1}^q(w;x)|A_j\leqq \text{const}\,\frac{(\sqrt{1-x^2})^l E_{N-\rho}(f^{(\rho)})}{m^\rho}.$$

Hence

$$I_1\leqq \text{const}\,\frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}+\sum_{i=0}^{q-1}\left\|\overline{\Sigma}_i p_{m+1}^q(w)\overline{u}1_m^{c^*}\right\|_p := \tag{3.17}$$
$$:= \text{const}\,\frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}+\sum_{i=0}^{q-1}S_i,$$

where $\overline{u}=v^{r-\frac{l}{2},s-\frac{l}{2}}u$.

Now we recall ([21]) the useful inequality, for $k\neq j$, $q-i$ even, (similarly we work for $q-i$ odd ([21]))

(3.18)
$$|h_{i,k}(x)|\leqq \text{const}\,|l_{k,m}^q(x)|\,|x-x_k^*|^i\left[\frac{m|x-x_k^*|}{\sqrt{1-x_k^{*2}}}\right]^{q-2-i}\left(1+\frac{m|x-x_k^*|}{1-x_k^{*2}}\right).$$

Hence, by (3.15), (3.18), (3.2), (3.4), (3.5) and (3.7)

$$|p_{m+1}^q(w;x)\overline{\Sigma}_i|\leqq \text{const}\,|p_{m+1}^q(w;x)|E_{N-\rho}(f^{(\rho)})\times$$
$$\times\sum_{k\neq j}\left(\frac{\sqrt{1-x_{k,m}^{*2}}}{m}\right)^{\rho-i}\frac{|h_{i,k}(x)|}{v^{r,s}(x^*_{k,m})|p_{m+1}^q(w;x^*_{k,m})|}\leqq$$

$$\leqq \text{const}|q_{2m+1}^q(x)|E_{N-\rho}(f^{(\rho)})\times$$

(3.19)

$$\times\sum_{k\neq j}\left(\frac{\sqrt{1-x_{k,m}^{*2}}}{n}\right)^{\rho-i}\frac{\lambda_{m,k}^q(w^*)|p_{m-1}^q(w^*;x_{k,m}^*)|}{|p_{m+1}^q(w;x_{k,m}^*)|v^{r,s}(x_{k,m}^*)|x-x_{k,m}^*|^{q-i}}\times$$

$$\times\left[\frac{m|x-x_k^*|}{\sqrt{1-x_k^{*2}}}\right]^{q-2-i}\left[1+\frac{m|x-x_k^*|}{1-x_k^{*2}}\right]:=\text{const}\,[\Sigma_i'+\Sigma_i''],$$

with

$$\Sigma_i'\leqq\text{const}\,|q_{2m+1}^q(x)|\frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}\sum_{k\neq j}\frac{(1-x_{k,m}^*)^\gamma(1+x_{k,m}^*)^\delta}{m^2(x-x_{k,m}^*)^2},$$

$$\Sigma_i''\leqq\text{const}\,|q_{2m+1}^q(x)|\frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}\sum_{k\neq j}\frac{(1-x_{k,m}^*)^{\gamma-\frac12}(1+x_{k,m}^*)^{\delta-\frac12}}{m|x-x_{k,m}^*|},$$

and $\gamma=\dfrac{\rho}{2}+\alpha q-r+q+1,\delta=\dfrac{\rho}{2}+\alpha q-s+q-1.$

Now, from the assumptions $\gamma,\delta>0$, therefore, from [1, formula (4.21), p. 122] and [6, Lemmas 5.7 and 5.8, p. 164] we get

$$\Sigma_i'\leqq\text{const}\,|q_{2m+1}^q(w;x)|\frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}\left[v^{\gamma-1,\delta-1}(x)+\frac{1}{m}\right]\tag{3.20}$$

and

$$\Sigma_i''\leqq\text{const}\,|q_{2m+1}^q(w;x)|\frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}\left[v^{\gamma-1,\delta-1}(x)\log m+1\right].\tag{3.21}$$

Therefore, from (3.4), (3.13) and (3.19)-(3.21) it results

$$S_i\leqq\text{const}\,\frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}\left[\log m+\|uv^{r-\frac{l}{2}-\alpha q-q,s-\frac{l}{2}-\beta q-q}1_m^{c^*}\|_p\right].$$

Now, by routine calculations (see e.g. [25]) from (2.4) it follows

$$S_i\leqq\text{const}\,\frac{E_{N-\rho}(f^{(\rho)})}{m^{\rho-\tau}}.$$

Working similarly for I_2, by (3.6) and [20, Lemma 3.4, p. 142], the assertion follows, for $p>1$. In the case $0<p<1$, the inequality (2.3) can be proved following a procedure used in [16].

Now we prove that, if $u \in \mathrm{GPJ}$ and $\rho = q-1$, (2.5) implies (2.4). Following [9, 12, 24, 25], we define for $k = 0, \ldots, 2m+2$,

$$s_{k,m,q}(x) = \frac{1}{(q-1)!}\left(\frac{t_{k+1}-t_k}{\pi}\right)^{q-1}\left(\sin\frac{x-t_k}{t_{k+1}-t_k}\pi\right)^{q-1}\cos\frac{x-t_k}{t_{k+1}-t_k}\pi, \qquad x \in [t_{k+1}, t_k],$$

with t_k, $k = 1, \ldots, 2m+1$, the zeros of $q_{2m+1} = p_{m+1}(w)p_m(w^*)$ and $t_0 = +1$, $t_{2m+2} = -1$. Routine calculations show, for $k = 0, \ldots, 2m+1$,

$$\begin{gathered} s^{(t)}_{k,m,q}(t_k) = s^{(t)}_{k,m,q}(t_{k+1}) = 0, \qquad 0 \leqq t \leqq q-2, \\ s^{(q-1)}_{k,m,q}(t_k) = 1, \qquad s^{q-1}_{k,m,q}(t_{k+1}) = (-1)^q, \\ |s^{(t)}_{k,m,q}(x)| \leqq \frac{\text{const}}{m^{q-1-t}}, \qquad 0 \leqq t \leqq q-1, \qquad x \in [t_{k+1}, t_k]. \end{gathered}$$

Then we introduce the function

$$T_{m,q}(x) = \sum_{k=0}^{2m+2}(-1)^{km}s_{k,m,q}(x).$$

Obviously, $T_{m,q} \in C^{q-1}([-1,1])$ and

$$\begin{aligned} & T^{(t)}_{m,q}(t_k) = 0, && 0 \leqq k \leqq 2m+2, \qquad 0 \leqq t \leqq q-2, \\ (3.22) \qquad & T^{(q-1)}_{m,q}(t_k) = (-1)^{kq}, && 0 \leqq k \leqq 2m+2, \\ & \|T^{(l)}_{m,q}\| \leqq \text{const}\, m^{l+1-q}, && 0 \leqq l \leqq q-1. \end{aligned}$$

Since

$$\begin{aligned} p'_m(w^*; x^*_{k,m}) &= (-1)^{k+1}|p'_m(w^*; x^*_{k,m})|, \\ p'_{m+1}(w; x_k) &= (-1)^{k+1}|p'_{m+1}(w; x_k)|, \end{aligned}$$

from (3.22), for $f_1(x) = x$ we have

$$\begin{aligned} Z_m(x) &:= xH_{mq,r,s}(w; w^*; T_{m,q}; x) - H_{mq,r,s}(w; w^*; f_1T_{m,q}; x) = \\ &= v^{r,s}(x)p^q_{m+1}(w; x)\sum_{k=1}^{m}\frac{T^{(q-1)}_{m,q}(x^*_{k,m})(x-x^*_{k,m})}{v^{r,s}(x^*_{k,m})p^q_{m+1}(w; x^*_{k,m})}h_{q-1,k}(x) + \\ (3.23) \qquad & \\ &+ v^{r,s}(x)p^q_m(w^*; x)\sum_{k=1}^{m+1}\frac{T^{(q-1)}_{m,q}(x_k)(x-x_k)}{v^{r,s}(x_k)p^q_m(w^*; x_k)}h_{q-1,k}(x) = \end{aligned}$$

$$= \frac{q_{2m+1}^q(x)v^{r,s}(x)(-1)^q}{(q-1)!}\left[\sum_{k=1}^{m}\frac{1}{v^{r,s}(x_{k,m}^*)|p_{m+1}^q(w;x_{k,m}^*)p_m'^q(w^*;x_{k,m}^*)|}+\right.$$

$$\left.+\sum_{k=1}^{m+1}\frac{1}{v^{r,s}(x_k)|p_m^q(w^*;x_k)p_{m+1}'^q(w;x_k)|}\right]:=$$

$$:= \frac{q_{2m+1}^q(x)v^{r,s}(x)(-1)^q}{(q-1)!}W(x).$$

Now, using routine calculations (see e.g. [9, 12, 23, 25]), from the assumptions we get

$$W(x) \sim \text{const}\, m^{1-q}. \tag{3.24}$$

On the other hand, from Leibniz formula

$$[q_{2m+1}^q(x)v^{r,s}(x)]^{(l)} = \sum_{t=0}^{l}\binom{l}{t}[v^{r,s}(x)]^{(t)}[q_{2m+1}^q(x)]^{(l-t)},$$

and since for $x \in [0,1]$

$$[v^{r,s}(x)]^{(t)} \leqq C(1-x)^{r-t},$$

we get, for $x \in [0,1]$,

$$[q_{2m+1}^q(x)v^{r,s}(x)]^{(l)} \geqq \left|c_0(1-x)^r[q_{2m+1}^q]^{(l)}(x)\right| -$$

$$-\left|\sum_{t=1}^{l} c_t\frac{(1-x)^r}{m^t(1-x)^{\frac{t}{2}}}\frac{m^t}{(1-x)^{\frac{t}{2}}}[q_{2m+1}^q]^{(l-t)}(x)\right|.$$

Now, using that $m\sqrt{1-x} \geqq \sqrt{A}$ in $[0, 1-Am^{-2}]$, with A arbitrary positive real number, from Lemma 3.4 it follows

$$\left\|u[q_{2m+1}^q v^{r,s}]^{(l)}\chi_{\varepsilon_m(A)}\right\|_{p,[0,1]} \geqq c_0\left\|(1-x)^r[q_{2m+1}^q]^{(l)}(x)\chi_{\varepsilon_m(A)}u(x)\right\|_{p,[0,1]} -$$

$$-\sum_{t=1}^{l}\frac{c_t}{A^{\frac{t}{2}}}\left\|(1-x)^r[q_{2m+1}^q]^{(l)}(x)\chi_{\varepsilon_m(A)}u(x)\right\|_{p,[0,1]},$$

where $\|\cdot\|_{p,[0,1]}$ denotes the L_p-norm on $[0,1]$. Using similar arguments for $x \in [-1,0]$ we get for A sufficiently large

$$\left\|u[q_{2m+1}^q v^{r,s}]^{(l)}\chi_{\varepsilon_m(A)}\right\|_p \geqq C\left\|v^{r,s}[q_{2m+1}^q]^{(l)}\chi_{\varepsilon_m(A)}u\right\|_p. \tag{3.25}$$

Therefore, from (3.23)–(3.25) and Lemma 3.4

$$\begin{aligned}\|Z_m^{(l)}u\|_p &\geqq \text{const}\, m^{1-q}\left\|[v^{r,s}q_{2m+1}^q]^{(l)}u\right\|_p \geqq \\ &\geqq \text{const}\, m^{1-q+l}\left\|v^{r-\frac{l}{2},s-\frac{l}{2}}q_{2m+1}^q\chi_{\varepsilon_m(A)}u\right\|_p.\end{aligned} \tag{3.26}$$

On the other hand, from (3.22) we get

$$E_m(g) \leqq \text{const}, \quad \text{for} \quad g = T_{m,q}^{(q-1)} \quad \text{and} \quad g = T_{m,q}^{(q-1)}f_1.$$

Therefore, from (2.5), (3.8), (3.22) and (3.23) it follows

$$\begin{aligned}\|Z_m^{(l)}u\|_p &\leqq \|Z_m^{(l)}1_m^{c^*}u\|_p \leqq \\ &\leqq \text{const}\|[f_1(H_{mq,r,s}(w;w^*;T_{m,q})-T_{m,q})- \\ &\quad - H_{mq,r,s}(w;w^*;f_1T_{mq}) + f_1T_{m,q}]^{(l)}u1_m^{c^*}\|_p \leqq \\ &\leqq \text{const}\left[\left\|(H_{mq,r,s}(w;w^*;T_{m,q}) - T_{m,q})^{(l-1)}u1_m^{c^*}\right\|_p + \right. \\ &\quad + \left\|(H_{mq,r,s}(w;w^*;T_{m,q}) - T_{m,q})^{(l)}u1_m^{c^*}\right\|_p + \\ &\quad \left. + \left\|[H_{mq,r,s}(w;w^*;f_1T_{m,q}) + f_1T_{m,q}]^{(l)}u1_m^{c^*}\right\|_p\right] \leqq \\ &\leqq \text{const}\left[\left\|(H_{mq,r,s}(w;w^*;T_{m,q}) - T_{m,q})^{(l-1)}u1_m^{c^*}\right\|_p + m^{1-q+l+\tau}\right].\end{aligned} \tag{3.27}$$

Now let $G_m = [H_{mq,r,s}(w;w^*;T_{m,q}) - T_{m,q}]^{(l-1)}$. Following [24], using Hölder inequality two times, for $1 \leqq p < \infty$, we get

$$\|G_m1_m^{c^*}u\|_p \leqq \text{const}\, m^{-\frac{1}{q}}\|G_m'1_m^{c^*}u\|_p.$$

Hence, from (3.27)

$$\|Z_m^{(l)}u\|_p \leqq \text{const}\, m^{1-q+l+\tau}. \tag{3.28}$$

Now, if we put $e_m(A) = [-1 + Am^{-2}, -1 + 2Am^{-2}]$ or $[1 - 2Am^{-2}, 1 - Am^{-2}]$, from (3.26) and (3.28) we get

$$\begin{aligned}\text{const} &\geqq \text{const}\, m^{-\tau}\left\|v^{r-\frac{l}{2},s-\frac{l}{2}}q_{2m+1}^q\chi_{\varepsilon_m(A)}\right\|_p \geqq \\ &\geqq \text{const}\, m^{-\tau}\left\|v^{r-\frac{l}{2},s-\frac{l}{2}}q_{2m+1}^q\chi_{\varepsilon_m(A)}u\right\|_p\end{aligned}$$

from which by standard calculations (see e.g. [25]), we get (2.4). □

PROOF OF THEOREM 2.2. First we prove that (2.7) implies (2.6). Following the proof of Theorem 2.1, we have

$$S_i \leqq \left\| [\Sigma_i' + \Sigma_i''] \overline{u} 1_m^{c^*} \right\|_p := S_i' + S_i''. \tag{3.29}$$

Now set $\Psi = \operatorname{sgn} \Sigma_i''$. Then, working as in [1, 4, 5, 10, 12], by Lemma 3.2, (3.2) and (3.4), we can write

$$S_i''^p \leqq \text{const} \frac{E_{N-\rho}(f^{(\rho)})}{m^\rho} \sum_{i=1}^m \lambda_{i,m}(w^*) v^{\frac{\varrho}{2}-r+q\alpha-\alpha-1+q, \frac{\varrho}{2}-s+q\beta-\beta-1+q}(x_{i,m}^*) \times$$
$$\times \left| \int_{-1}^{1} \Psi(x) |\Sigma_i''(x) \overline{u}(x)|^{p-1} [\overline{u}(x) 1_m^{c^*}(x)] \frac{q_{2m+1}^q(w;x)}{x - x_{i,m}^*} dx \right|.$$

Thus, from the assumptions, by Lemma 3.3

$$S_i'' \leqq \text{const} \frac{E_{N-\rho}(f^{(\rho)})}{m^\rho}.$$

Working as in [1, 4, 5, 9, 11], we obtain

$$S_i' \leqq \text{const} \frac{E_{N-\rho}(f^{(\rho)})}{m^\rho},$$

and, from (3.25), (3.14) and (3.17) the assertion follows.

Finally we prove that, if $u \in \text{GPJ}$ and $\rho = q - 1$, (2.8) implies (2.7). Indeed, following the proof of Theorem 2.1, from (3.27) we have, for $q - 1 \geqq l \geqq 1$

$$\begin{aligned} \|Z_m^{(l)} u\|_p &\leqq \text{const} \, \|[f_1(H_{mq,r,s}(w; w^*; T_{m,q}) - T_{m,q}) - \\ &\quad - H_{mq,r,s}(w; w^*; f_1 T_{m,q}) + f_1 T_{m,q}]^{(l)} u 1_m^{c^*} \|_p \leqq \\ &\leqq \text{const} \left[\left\| (H_{mq,r,s}(w; w^*; T_{m,q}) - T_{m,q})^{(l-1)} u 1_m^{c^*} \right\|_p + \right. \\ &\quad + \left\| (H_{mq,r,s}(w; w^*; T_{m,q}) - T_{m,q})^{(l)} u 1_m^{c^*} \right\|_p + \\ &\quad \left. + \left\| [H_{mq,r,s}(w; w^*; f_1 T_{m,q}) + f_1 T_{m,q}]^{(l)} u 1_m^{c^*} \right\|_p \right]. \end{aligned} \tag{3.30}$$

Hence, from (2.8) and (3.22)

$$\begin{aligned} \|(H_{mq,r,s}(w; w^*; T_{m,q}) - T_{m,q})^{(l)} u 1_m^{c^*} \|_p &\leqq \text{const} \, m^{1+l-q}, \\ \|[H_{mq,r,s}(w; w^*; f_1 T_{m,q}) + f_1 T_{m,q}]^{(l)} u 1_m^{c^*} \|_p &\leqq \text{const} \, m^{1+l-q}. \end{aligned} \tag{3.31}$$

Moreover, let Δ be a fixed interval inside $(-1,1)$. Then working as in the proof of Theorem 2.1, from (3.22) we get (see also [25])

$$|H_{mq,r,s}(w;w^*;T_{m,q};x)| \leqq$$

$$\leqq \operatorname{const} m^{1-q}\left[v^{\frac{q-1}{2},\frac{q-1}{2}}(y_j)+\frac{\log m}{m}v^{r-\alpha q-q,s-\beta q-q}(y_j)\right]$$

where $2y_j = t_j + t_{j+1}$ and t_j, $1 \leqq j \leqq 2m+1$, denotes the closest knot to x. Then, by a theorem in [19, 4.8.72] it follows that

$$|H_{mq,r,s}(w;w^*;T_{m,q};x)^{(l-1)}| \leqq$$

$$\leqq \operatorname{const} m^{l-q} v^{-\frac{l-1}{2},-\frac{l-1}{2}}(y_j)\left[v^{\frac{q-1}{2},\frac{q-1}{2}}(y_j)+\frac{\log m}{m}v^{r-\alpha q-q,s-\beta q-q}(y_j)\right].$$

Therefore

$$|H_{mq,r,s}(w;w^*;T_{m,q};x)^{(l-1)}| \leqq \operatorname{const} m^{l-q}, \qquad x\in\Delta. \tag{3.32}$$

Hence, by (3.32) and (3.22)

$$\begin{aligned}\Big\|(H_{mq,r,s}(w;w^*;T_{m,q})-T_{m,q})^{(l-1)}u\chi_\Delta u\Big\|_p &\leqq \\ \leqq \operatorname{const}\Big\|(H_{mq,r,s}(w;w^*;T_{m,q})^{(l-1)}\chi_\Delta u\Big\|_p &+ \Big\|(T_{m,q})^{(l-1)}\chi_\Delta u\Big\|_p \leqq \\ \leqq \operatorname{const} m^{l-q}.\end{aligned} \tag{3.33}$$

Then from (3.30), (3.31) and (3.33) we have

$$\lim_{m\to\infty}\sup m^{q-l-1}\|Z_m^{(l)}\chi_\Delta u\|_p \leqq \operatorname{const},$$

and by (3.23)–(3.25)

$$\lim_{m\to\infty}\sup m^{-l}\|v^{r,s}[q_{2m+1}^q]^{(l)}\chi_\Delta u\|_p \leqq \operatorname{const}.$$

Now, recalling the argument of the proofs of Lemma 3.4 and Corollary 3.5

$$\lim_{m\to\infty}\sup m^{-l}\|v^{r,s}[q'_{2m+1}]^l q_{2m+1}^{q-l}u\chi_\Delta\|_p \leqq \operatorname{const},$$

with a constant independent of Δ, for every fixed $\Delta\in(-1,1)$. Now, if $l>0$, by Statement 3.6, from [17, Theorem 4] (see also [25, Lemma 4.7]), (2.7) follows.

If $l=0$, from the assumptions we deduce

$$\|f_1 H_{mq,r,s}(w;w^*;T_{m,q}) - H_{mq,r,s}(w;w^*;f_1 T_{m,q})u\|_p \leqq \operatorname{const} m^{1-q},$$

and working as above we have from (3.23) and (3.24)

$$\lim_{m\to\infty} \sup \|v^{r,s} q_{2m+1}^{q} u\|_p \leqq \text{const}\,.$$

Hence

$$\text{const} \geqq \|v^{r,s} q_{2m+1}^{q} u\|_p \geqq \|v^{r,s} q_{2m+1}^{q} u 1_m^{c^*}\|_p \sim \|v^{r-q,s-q} w^{-q} u\|_p$$

that is the assertion. □

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DIPARTIMENTO DI MATEMATICA
ISTITUTO G. CASTELNUOVO
UNIVERSITÀ DI ROMA LA SAPIENZA
P.LE ALDO MORO 2
I–00185 ROMA
ITALY

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DELLA BASILICATA
VIA N. SAURO 85
I–85100 POTENZA

and

ISTITUTO PER APPLICAZIONI DELLA MATEMATICA
CONSIGLIO NAZIONALE DELLE RICERCHE (CNR)
VIA P. CASTELLINO 111
I–80131 NAPOLI
ITALY

MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 323–347

SIMULTANEOUS EXTENSIONS

J. DEÁK

Abstract

This survey of simultaneous extensions of topological structures (meant for the general public rather than for specialists) is an amplified versions of a talk with the same title, presented on February 23rd 1994, in the course of a scientific session held to celebrate the 70th birthday of Professor Ákos Császár. Only a basic knowledge of topologies and metrics is presumed; anything else will be defined (and, in most cases, some motivation for the definitions will be given). Certain details, e.g. the probably new (but quite simple) results on simultaneous extensions of (semi)metrics, the tables summing up the basic results for a lot of structures, or the bibliography of simultaneous extensions, could be of interest to specialists, too.

§ 1. What is an extension?

Let $(X, \mathcal{T})$ be a metrizable topological space, $X_0 \subset X$, and denote by $\mathcal{T}_0$ the restriction $\mathcal{T} \,|\, X_0$ of $\mathcal{T}$ to X_0. Assume that we are given a metric d_0 on X_0 inducing $\mathcal{T}_0$ (the expression "d_0 is compatible with $\mathcal{T}_0$" will also be used in the same sense). Now can d_0 be extended to X, i.e. is there a metric d on X compatible with $\mathcal{T}$ such that $d_0 = d \,|\, X_0$? If yes then such a d is a *compatible extension*, or shortly an *extension*. Hausdorff [07] proved that if X_0 is closed then there are extensions. (This is a deep result.) In the general case, it is enough to know whether there is an extension to the closure of X_0, since if yes then Hausdorff's theorem yields an extension to X.

So there remains the case when X_0 is dense. The neighbourhood filter of each point $p \in X \setminus X_0$ has now a trace $\mathfrak{f}(p)$ on X_0, which will be called the *trace filter* of p. It is quite easy to prove that d_0 has an extension iff the trace filters are round and Cauchy, where a filter $\mathfrak{f}$ in X_0 is *round* if for each $S \in \mathfrak{f}$

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there are $T \in \mathfrak{f}$ and $\varepsilon > 0$ such that $x \in T$, $d_0(x, y) < \varepsilon$ imply $y \in S$, and *Cauchy* if it contains arbitrarily small sets, i.e. if for each $\varepsilon > 0$ there is an $S \in \mathfrak{f}$ with $d(x, y) < \varepsilon \quad (x, y \in S)$. There is in fact only one extension d from a dense subset, and (X, d) is a subspace of the completion of (X_0, d_0). (A point in the completion is usually given as an equivalence class of Cauchy sequences, see e.g. [06] 4.5.6, but there is a natural one-to-one correspondence between these equivalence classes and the round Cauchy filters = minimal Cauchy filters.)

There exist several (topological) structures other than topologies and metrics (some of them will be defined later), so the following more general question can be raised: Assume that σ is a structure on the set X, $\sigma_0 = \sigma \mid X_0$ its restriction to $X_0 \subset X$, Σ_0 a richer structure on X_0 compatible with σ_0 (= inducing σ_0). Now is there a compatible extension of Σ_0 (in the same sense as above)? If the answer is positive then we can look for extensions satisfying additional conditions, e.g. separation properties, completeness, upper bounds on cardinal functions, etc. (In fact, there are cases when no necessary and sufficient condition for the existence of an extension is known, but it can be proved that if there are extensions then there are "good" ones, too.)

The structure σ above is not necessarily a topology. E.g. let d be a *quasi-metric*, i.e. let us drop the symmetry condition $d(x, y) = d(y, x)$ from the definition of metric. Then d induces a topology in the usual way (the balls round a point form a neighbourhood base), but the conjugate d^{-1} of d defined by $d^{-1}(x, y) = d(y, x)$ also induces a topology, hence d induces a *bitopology*, i.e. an ordered pair of topologies. So extensions of quasi-metrics can be investigated in topological as well as bitopological spaces. Similarly to quasi-metrics, there are other non-symmetric structures whose extensions can be considered in topological and also in bitopological spaces; the results in the two cases are usually essentially different. [11] is a detailed survey of such problems, putting a special emphasis on extending uniformities in topological and quasi-uniformities in (bi)topological spaces. [11] also contains a fairly complete bibliography of extension problems, to which we can only add now [09] (dealing with quasi-metrics in bitopological spaces) and some recent papers on simultaneous extensions.

A metric in a topological space is continuous as a function on the square of the space iff it induces a topology coarser than the one of the space (*coarser* means that the collection of the open sets is smaller). More generally, the richer structure in the space (X, σ) is called *continuous* if it induces a structure coarser than σ (assuming that the relation finer/coarser is defined for structures of the type σ). It is often easier to find a continuous extension than a compatible one. Results on continuous extensions, however, can sometimes serve as stepping-stones when looking for compatible extensions. See [11] for details. (Caution! For quasi-metrics in a topological space, continuity as defined above is not equivalent to continuity on $X \times X$.)

The extension problem could also be considered for structures other than topological ones. Assume, e.g. that X is a group, X_0 a subgroup of X, and we are given a second operation on X_0 which makes it a ring (or a topology making it a topological group). Under what conditions can the operation (the topology) be extended to X such that X becomes a ring (a topological group)? In the case of topological groups, it could also be assumed that the topology instead of the group operation is prescribed on X. Should these particular problems turn out to be uninteresting, it is still possible that some similar questions, with different structures, are worth investigating. These algebraic and algebraic-topological extension problems (or their simultaneous versions, see § 2) may possibly have a literature; the author would be grateful for any information on this subject.

§ 2. What is a simultaneous extension?

Császár investigated in [1] § 5 extensions of proximities in symmetric closure spaces. (Both structures will be defined later; for the time being, it is immaterial what they exactly are.) He proved that if δ_0 is a compatible proximity on the subspace X_0 of a symmetric closure space (X,c) then δ_0 has a compatible extension δ such that $\delta \mid X \setminus X_0$ can also be prescribed. Generalizing the above result, we proved in [10] 4.3 (using different methods) that even more than two (possibly infinitely many) compatible proximities given on subspaces have a common extension, and the subspaces do not have to be disjoint, it is enough to assume that any two of the proximities coincide on the intersection of their fundamental sets. This leads to the following general simultaneous extension problem:

Let σ be a structure on X, $X_i \subset X \quad (i \in I)$, and Σ_i a richer structure on X_i for $i \in I$ (the structures Σ_i are assumed to be of the same kind, e.g. each of them a proximity). An *extension* of the (family of) structures Σ_i is a structure Σ on X compatible with σ such that $\Sigma \mid X_i = \Sigma_i$ $(i \in I)$. For $I = \emptyset$ we obtain the usually simple question whether there exists a compatible richer structure; $|I| = 1$ is the extension problem discussed in § 1. There are two conditions evidently necessary for the existence of an extension: (i) each Σ_i has to be compatible with $\sigma_i = \sigma \mid X_i$; (ii) $\Sigma_i \mid X_i \cap X_j = \Sigma_j \mid X_i \cap X_j \quad (i,j \in I)$. More precisely, these conditions are necessary if we consider structures for which (a) taking the induced structure commutes with restrictions; (b) $\Sigma \mid A = (\Sigma \mid B) \mid A$ if $A \subset B \subset X$ and Σ is a richer structure on X. When speaking about a *family of structures* (in the space (X,σ)), it will always be assumed that (i) and (ii) are satisfied; we shall use the expressions that the structures are *compatible* and *accordant*.

There is one more obvious necessary condition: each σ_i (taken individually) has to have an extension. Those cases are more interesting when this assumption is not sufficient: one has to search then for a set of necessary and sufficient conditions. Similarly to § 1, we can also ask whether there are extensions with some additional properties. When looking for an extension with property P, it has to be assumed that (X, σ) admits a compatible structure satisfying P, e.g. if we want to have a complete extension of metrics in a topological space then it has to be assumed that the topology is completely metrizable. Moreover, if P is a hereditary property (as most of the separation properties are) then there is yet another trivial necessary condition: each σ_i has to satisfy P.

§ 3. A family of metrics in a topological space

Let us return now to the case of metrics in a metrizable topological space, and assume first that $|I|=2$, i.e. we are given compatible accordant metrics d_i $(i=0,1)$ on subspaces X_i of a metrizable topological space X. If X_0 and X_1 are closed and $X_0 \cap X_1 \neq \emptyset$ then the following simple construction due to Bing [02] yields an extension d of d_0 and d_1 to $X_0 \cup X_1$:

$$d(x,y)=d(y,x)=\inf\{d(x,p)+d(p,y): p \in X_0 \cap X_1\}$$
$$(x \in X_0 \setminus X_1, y \in X_1 \setminus X_0).$$

If X_0 and X_1 are disjoint then pick points $p_i \in X_i$, and put:

$$d(x,y)=d(y,x)=d_0(x,p_0)+d_1(p_1,y)+1 \qquad (x \in X_0, y \in X_1).$$

In both cases, the distance of pairs of points not figuring in the formula is determined by $d \mid X_i = d_i$. Both the triangle inequality and the compatibility of d are easy to check. Applying then Hausdorff's theorem to d, we obtain an extension to X. The same constructions also work when X_0 and X_1 are open and $X_0 \cup X_1 = X$. But it can happen that metrics d_0 and d_1 given on disjoint open subspaces have no common extension, although they can be extended separately:

Let X be the plane with the usual Euclidean topology, d the Euclidean metric on X. Denote by X_0 the upper and X_1 the lower open half-plane, and let $d_0 = d \mid X_0$, $d_1 = 2d \mid X_1$. Both metrics have compatible extensions, e.g. d, respectively $2d$, but assuming the existence of a common extension leads, through the triangle inequality, to contradictory conditions on the line $X \setminus (X_0 \cup X_1)$.

In the case $|I|=3$, it is even possible that a family of metrics given on open-closed subspaces has no extensions, e.g. let $X=\{0,1,2\}$ with the discrete topology, $I=X, X_i = X \setminus \{i\}$, $d_0(1,2)=1=d_1(0,2)$, $d_2(0,1)=3$. Both examples suggest that we would perhaps get better positive results if the triangle inequality were dropped from the axioms of a metric. Therefore, following [03], we define:

A non-negative real function d on $X \times X$ is a *semimetric* on X if it satisfies the axioms

$$d(x,y)=0 \quad \text{iff } x=y \quad (x,y \in X);$$

$$d(x,y)=d(y,x) \quad (x,y \in X).$$

Balls are defined in the same way as in metric spaces:

$$B_\varepsilon(x)=B_\varepsilon^d(x)=\{y \in X : d(x,y)<\varepsilon\} \quad (x \in X, \varepsilon>0)$$

(the *ε-ball round x*). The *neighbourhood filter* $\mathfrak{n}(x)$ of the point x is the filter for which the balls round x form a base. In general, the neighbourhood filters do not generate a topology, but only a *neighbourhood structure*, which means that a filter $\mathfrak{n}(x)$ is assigned to each $x \in X$ such that $x \in \bigcap \mathfrak{n}(x)$. Nevertheless, the closure cA of a set $A \subset X$ can be defined just as in topological spaces:

$$x \in cA \text{ iff } A \cap S \neq \emptyset \quad (S \in \mathfrak{n}(x)).$$

This operation c satisfies the following conditions:

$$c\emptyset=\emptyset; \quad cA \supset A \quad (A \subset X);$$

$$c(A \cup B)=cA \cup cB \quad (A,B \subset X).$$

A function $c: \exp X \to \exp X$ satisfying the above axioms is a *closure* [03]; (X,c) is a *closure space*. There is a one-to-one correspondence between neighbourhood structures and closures, and this correspondence (at least in one direction) was described above. It will be more convenient to consider semimetrics and some other structures in closure spaces rather than neighbourhood spaces. It is well-known that a closure corresponds to a topology iff it is idempotent; such closures will be called *topological*. The closure $c=c(d)$ induced by a semimetric d can be described as follows:

$$x \in cA \quad \text{iff} \quad \forall \varepsilon>0 \ \exists y \in A, \quad d(x,y)<\varepsilon.$$

We do not deal here with the difficult question of giving conditions for the semimetrizability of a closure.

Let us consider a family of semimetrics $d_i \quad (i \in I)$ in a semimetrizable closure space. If I is finite then an extension d can be obtained in the following way: fix a compatible semimetric d^* on X, and let $d(x,y) = d_i(x,y)$ if $x,y \in X_i$ for some i (by the accordance, it makes no difference which $i \in I$ we pick), and $d(x,y)=d^*(x,y)$ otherwise. A straightforward reasoning, using the last axiom of a closure, yields that d is compatible. But there may not exist an extension if I is infinite:

Let (X,c) be a convergent sequence, i.e. $X=\mathbb{N} \cup \{0\}$ and $cA=A \cup \{0\}$ if A is infinite, $cA=A$ if A is finite. Take the (semi)metric d_i on $X_i = \{0,i\} \quad (i \in I=\mathbb{N})$ defined by $d_i(0,i)=1$. If d were an extension then we would have $d(0,x)=1 \quad (x \in \mathbb{N})$, contradicting $0 \in c\mathbb{N}$.

From topological point of view, not the actual values of a (semi)metric are of importance, but the uniform continuity they define. Thus there is essentially only one semimetric on a two-point set, i.e. d_i in the above example could be replaced by the equivalent (semi)metric d_i^* for which $d_i^*(0,i)=1/i$, and in this case there do exist extensions, even metric ones. Therefore we can perhaps hope to get better positive results if (semi)metrics are replaced by their equivalence classes; this question will be dealt with in the next section.

Our definition of $c(d)$ might need some justification, since some authors consider a topology induced by semimetrics and other structures: let $\mathcal{T}(d)$ be the topology for which a set $G \subset X$ is open iff for each $x \in G$, G contains a ball round x; in other words, F is closed iff $c(d)F = F$. Using this definition, the quite natural assumptions (i) and (a) from § 2 would not be satisfied, i.e. there would exist a non-compatible semimetric on a subspace of a topological space that has an extension:

Let $X_0 = \{(0,0)\} \cup \{(1/m, 1/n) : m, n \in \mathbb{N}\}$, $X = X_0 \cup \{(1/n, 0) : n \in \mathbb{N}\}$,

$$d((0,0),(1/n,0)) = d((1/m,0),(1/m,1/n)) = 1/n \quad (m, n \in \mathbb{N}),$$

the same when the order of the two points is reversed, and $d(x,y) = 1$ for other points $x \neq y$. Put $\mathcal{T} = \mathcal{T}(d)$. Then $d_0 = d \mid X_0$ has an extension, but it is not compatible, since $d_0(x,y) = 1$ for $x \neq y$, thus $\mathcal{T}(d_0)$ is discrete. $\mathcal{T} \mid X_0$ is, however, not discrete: if $G \ni (0,0)$ is $\mathcal{T}(d)$-open then it contains some point $(1/m, 0)$, hence also some point $(1/m, 1/n) \in X_0$, thus $(0,0)$ is not an isolated point in X_0.

§ 4. Equivalence classes of metrics

The semimetrics d and d^* on X are called (uniformly) *equivalent* if the identity map of X is uniformly continuous in both directions, i.e. if for each $\varepsilon > 0$ there is an $\eta > 0$ such that $d(x,y) < \varepsilon$ whenever $d^*(x,y) < \eta$, and also $d^*(x,y) < \varepsilon$ whenever $d(x,y) < \eta$ (clearly an equivalence relation). We shall write $d \sim d^*$. A semimetric equivalent to a metric is not necessarily a metric. By an equivalence class of metrics we mean the system of metrics equivalent to some metric; an equivalence class of bounded metrics is to be understood in the same way. An equivalence class of (semi)metrics could also be described as a (semi)metrizable (semi)uniformity. (Semiuniformities will be defined later.) Equivalent semimetrics induce the same closure (equivalent metrics the same topology), and if $\mathcal{E}$ is an equivalence class of semimetrics on X, $X_0 \subset X$ then $\{d \mid X_0 : d \in \mathcal{E}\}$ is an equivalence class of semimetrics on X_0 (straightforward).

Hence we can consider a family of equivalence classes of semimetrics in a closure space, and ask whether there is an extension. Equivalently: semimetrics instead of equivalence classes can be considered in a closure space, relaxing the accordance as follows:

$$(\sim) \qquad d_i \mid X_i \cap X_j \sim d_j \mid X_i \cap X_j \qquad (i, j \in I)$$

(compatibility has to left in its original form), and an extension is only required to satisfy $d \mid X_i \sim d_i$ instead of $d \mid X_i = d_i$.

In the same way, a family of equivalence classes of metrics in a topological space can be replaced by a family of metrics, assuming again $(\sim)$ instead of the accordance. But we have to be more careful now: it is not true that the trace of an equivalence class of metrics is an equivalence class (taking the usual metric d on the reals, one can easily define a metric d_0 on $\mathbb{N}$ equivalent to $d \mid \mathbb{N}$ such that $d(n, n+1) = n$, and then there is no metric $d^* \sim d$ with $d^* \mid \mathbb{N} = d_0$); it can only be proved that the trace of an equivalence class consists of all the bounded elements and of some of the unbounded elements of an equivalence class (see [06] 8.5.6). Consequently, the trace of an equivalence class of bounded metrics is also an equivalence class of bounded metrics, and the simultaneous extension problem can be formulated as follows:

Let us be given compatible bounded metrics d_i on subspaces X_i of a metrizable topological space; does there exist then an extension, i.e. a compatible (bounded) metric d on X such that $d \mid X_i \sim d_i \quad (i \in I)$? The positive results for $|I| = 2$ are the same as in § 3, see [11] 1.13, where (metrizable) uniformities are used instead of equivalence classes of metrics. (Or, assuming $X_0 \cap X_1 \neq \emptyset$, define d on $X_0 \cup X_1$ as in § 3, taking $d(x,y) = d_0(x,y)$ for $x, y \in X_0 \cap X_1$; this d is only a compatible semimetric, but, using the Metrization Lemma [08] 6.12, it can be replaced by an equivalent metric.) The counterexamples given in § 3 do not work now, but they can be easily modified:

Let X be the Euclidean plane, X_0 the upper and X_1 the lower open half-plane, d a bounded metric equivalent to the Euclidean metric on X, and d^* another bounded compatible metric such that the traces of d and d^* on $X \setminus (X_0 \cup X_1)$ are not equivalent. Then the bounded compatible metrics $d \mid X_0$ and $d^* \mid X_1$ given on disjoint open subspaces can be extended separately, but not simultaneously.

Take $X = \mathbb{N} \times \{0,1,2\}$ with the discrete topology,

$$I = \{0,1,2\}, \quad X_i = X \setminus (\mathbb{N} \times \{i\}), \quad d_0(x,y) = 1 \ \text{ if } x \neq y,$$

$$d_1((n,0),(n,2)) = d_2((n,0),(n,1)) = 1/n \qquad (n \in \mathbb{N}),$$

the same if the order of the points is reversed, and $d_i(x,y) = 1$ for other pairs $x \neq y \quad (i = 1,2)$. $(\sim)$ is evident: in fact, the original form of the accordance holds. Assuming that d is a compatible metric on X with $d \mid X_i \sim d_i \quad (i \in I)$, a straightforward calculation (using the triangle inequality) leads to a contradiction.

§ 5. Equivalence classes of semimetrics in a closure space

Neither dropping the triangle inequality nor considering equivalence classes has led to positive results for infinite families, so we try now doing both at the same time: Let a family of semimetrics d_i on subspaces X_i

of a semimetrizable closure space X be given, requiring only the weaker form $(\sim)$ of the accordance; is there an extension d, in the sense that $d \mid X_i \sim d_i \quad (i \in I)$? As in § 3, the answer is yes again if I is finite:

Let $I = \{1, \ldots, n\}$, put $X_{n+1} = X$, take a compatible semimetric d_{n+1} on X, and define $d(x,y) = d_i(x,y)$ where i is the smallest one of the indices for which $x, y \in X_i$. $d \mid X_i \sim d_i$ is clear (for $\varepsilon > 0$ take η which is good in $(\sim)$ for each pair $i, j \in I$), while the compatibility of d can be proved in the same way as in § 3. It is, however, not true that any countable family has an extension. A counterexample is now somewhat more complicated than in § 3:

Let F denote the collection of all the functions from $\mathbb{N}$ into $\mathbb{N}$, $X = (\{0\} \cup \mathbb{N}) \times F$ with the discrete closure c (meaning that $cA = A$ for each $A \subset X$), $I = \mathbb{N}$, $X_i = \{0, i\} \times F$. Put

$$d_i((0,f),(i,f)) = d_i((i,f),(0,f)) = 1/f(i) \qquad (i \in I, f \in F),$$

and $d_i(x,y) = 1$ for other pairs $x \neq y$. We have defined a family of semimetrics (in fact, metrics), even in the original sense. Assume that d is an extension (in the sense used in the present section). For each $n \in \mathbb{N}$, take $\eta_n > 0$ such that $d(x,y) < 1/n$ whenever $x, y \in X_n$ and $d_n(x,y) < \eta_n$. Pick $f \in F$ with $1/f(n) < \eta_n \quad (n \in \mathbb{N})$. Now $d((0,f),(n,f)) < 1/n \quad (n \in \mathbb{N})$, contradicting the assumption that c is discrete.

We are going to consider now a generalization of the notion of an equivalence class of semimetrics for which the above example does not work. Given a semimetric d on X, put

$$U_{(\varepsilon)} = U_{(\varepsilon)}(d) = \{(x,y) : x, y \in X, d(x,y) < \varepsilon\} \qquad (\varepsilon > 0).$$

Each U_n is an *entourage* (=a reflexive relation) on X, and $\{U_{(\varepsilon)} : \varepsilon > 0\}$ is a base for a filter $\mathcal{U}(d)$ on $X \times X$ consisting of entourages. $\mathcal{U}(d) = \mathcal{U}(d^*)$ iff $d \sim d^*$. $\mathcal{U}(d)$ has a countable base, e.g. $\{U_{(1/n)} : n \in \mathbb{N}\}$; moreover, $\bigcap \mathcal{U}(d) = \Delta$, which denotes the diagonal of $X \times X$, and $\mathcal{U}(d)$ is *symmetric* in the sense that if $U \in \mathcal{U}(d)$ then so is U^{-1} (equivalently: $\mathcal{U}(d)$ has a base consisting of symmetric entourages). If a filter $\mathcal{U}$ of entourages on X satisfies the three conditions above then $\mathcal{U} = \mathcal{U}(d)$ for a suitable semimetric d on X : let $\{U_n : n \in \mathbb{N}\}$ be a countable base, take for each U_n a symmetric $V_n \in \mathcal{U}$ with $V_n \subset U_N$, and define

$$d(x,y) = \min\{1/n : n \in \mathbb{N}, \ x \, V_n \, y\}.$$

This means that families of semimetrics can be identified with symmetric filters of entourages satisfying two additional conditions. Dropping both conditions, we define a *semiuniformity* on X as a symmetric filter of entourages on X. (None of the notions defined in this paper is new; the reader interested in the sources might consult [6] § 1.) The essential part of the generalization consists in giving up countability; the separation property $\bigcap \mathcal{U} = \Delta$ might as well (but will not) be kept; a semiuniformity with this property is called T_1.

A semiuniformity $\mathcal{U}$ induces a symmetric closure $c = c(\mathcal{U})$ (*symmetric* means that $x \in c\{y\}$ implies $y \in c\{x\}$), for which

$$x \in cA \text{ iff for each } U \in \mathcal{U} \text{ there is a } y \in A \text{ with } x\, U\, y.$$

The restriction of a semiuniformity $\mathcal{U}$ on X to $X_0 \subset X$ is defined as follows:

$$U \mid X_0 = U \cap (X_0 \times X_0), \qquad \mathcal{U} \mid X_0 = \{U \mid X_0 : U \in \mathcal{U}\}.$$

These definitions are consistent with the ones used earlier for semimetrics.

Any family of semiuniformities in a symmetric closure space has extensions. We shall discuss the details later, after considering some other structures in a closure space, because these other structures behave even better than semiuniformities. Let us only see now why the example above cannot be modified for semiuniformities: assuming that $\mathcal{U}$ is an extension and taking arbitrary $U_n \in \mathcal{U}$ (instead of $U_{(1/n)}$ from the example), there may exist a $U \in \mathcal{U}$ (perhaps depending on $x \in X$) with $Ux \subset \bigcap_{n \in \mathbb{N}} U_n x$ if the existence of a countable base is not required; thus the discreteness of c does not lead to a contradiction. (We write here Ux for $\{y \in X : x\, U\, y\}$.)

§ 6. Proximities in a closure space

In a semimetric space (X, d), the sets A and B are *near* if for each $\varepsilon > 0$ there are $x \in A$ and $y \in B$ with $d(x, y) < \varepsilon$. The relation $\delta = \delta(d)$ of being near has the following properties ($\bar{\delta}$ denotes that δ does not hold):

$$\emptyset\, \bar{\delta}\, X;$$

$$\text{if } A\, \delta\, B \text{ then } B\, \delta\, A \qquad (A, B \subset X);$$

$$\text{if } A \cap B \neq \emptyset \text{ then } A\, \delta\, B \qquad (A, B \subset X);$$

$$A\, \delta\, B \cup C \text{ iff } A\, \delta\, B \text{ or } A\, \delta\, C \qquad (A, B, C \subset X).$$

($A\, \delta\, B \cup C$ stands for $A\, \delta\, (B \cup C)$). As a generalization of this notion of nearness of pairs of sets, let us call a relation δ between subsets of X a *proximity* on X if it satisfies the above axioms. The sets A and B are called *near*, respectively *far*, according as $A\, \delta\, B$ or $A\, \bar{\delta}\, B$. Note the following consequences of the axioms: if $X \supset A' \supset A\, \delta\, B \subset B' \subset X$ then $A'\, \delta\, B'$; $\emptyset\, \bar{\delta}\, A$ for $A \subset X$.

The restriction $\delta_0 = \delta \mid X_0$ is given by $A\, \delta_0\, B$ iff $A, B \subset X_0$ and $A\, \delta\, B$. For proximities δ and δ^* on X, δ is *finer* than δ^* (δ^* is *coarser* than δ) if $\delta \subset \delta^*$. We could have defined the (proximal) continuity of maps, and then obtain restrictions and the relation finer/coarser in the usual categorical way;

observe, however, that only the continuity of injective maps is of importance in extension problems, and the continuity of such maps can be described if we know the restrictions and the relation finer/coarser. (See [12] 1.2 for a more precise statement.)

In the more usual sense of the word (see e.g. [04] and [06], a proximity has to satisfy one more axiom:

$$(\triangle) \qquad \text{if } A\,\bar{\delta}\,B \text{ then there is a } C\subset X \text{ with } A\,\bar{\delta}\,C \text{ and } X\backslash C\,\bar{\delta}\,B.$$

$\delta(d)$ satisfies $(\triangle)$ if d is a metric, but not necessarily if d is only a semimetric.

A proximity δ on X induces a symmetric closure $c=c(\delta)$ on X:

$$x\in cA \text{ iff } \{x\}\,\delta\,A \qquad (x\in X,\ A\subset X).$$

This definition is consistent with the one given for semimetrics: $c(\delta(d)) = c(d)$. Let us take now a family of proximities δ_i on subspaces X_i $(i\in I)$ of a symmetric closure space (X,c); note that conditions (a) and (b) from § 2 are satisfied (as they will be in all the cases to be considered later). Then there are extensions; in fact, there are a coarsest and a finest one. The finest extension, denoted by δ^1, can be easily described: $A\,\delta^1\,B$ iff one of the following conditions holds:

$$(1) \qquad A\cap cB\neq\emptyset;$$

$$(2) \qquad cA\cap B\neq\emptyset;$$

$$(3) \qquad A\cap X_i\,\delta_i\,B\cap X_i \quad \text{for some } i\in I.$$

When necessary, the more precise notation $\delta^1(c,\delta_i)$, or even $\delta^1(c,\{\delta_i : i\in I\})$, can be used. (The same applies in any similar situation.) A formula for the coarsest extension δ^0 is more complicated; see [6] § 1A for details.

Assuming $(\triangle)$ would cause the same problems as the triangle inequality in the case of (equivalence classes of) metrics, see [11] 1.16. But we can obtain reasonable positive results with other axioms weaker than $(\triangle)$:

A proximity δ is *Riesz*, respectively *Lodato*, if $A\,\bar{\delta}\,B$ implies $cA\cap cB=\emptyset$, respectively $cA\,\bar{\delta}\,cB$, where $c=c(\delta)$. A proximity satisfying $(\triangle)$ is Lodato, while a Lodato proximity is evidently Riesz. A Lodato proximity induces a topological closure (which will be identified with the associated topology; symmetric topologies are also known as R_0 or S_1). If δ is Riesz then $c=c(\delta)$ is *weakly topological*, meaning that $x\in cA$ implies $c\{x\}\subset cA$. A closure c is symmetric and weakly topological iff $x\notin cA$ implies $c\{x\}\cap cA=\emptyset$ ("weakly separated" in [6]). A filter $\mathfrak{f}$ in a proximity space (X,δ) is *compressed* if for any $A,B\subset X$, $A\cap S\neq\emptyset\neq B\cap S$ $(S\in\mathfrak{f})$ implies $A\,\delta\,B$. A proximity δ is Riesz iff the neighbourhood filters are compressed; it is Lodato iff $c(\delta)$ is topological, and for any compressed filter $\mathfrak{f}$, the coarser filter generated by the open elements is also compressed. The Riesz and the Lodato properties are hereditary.

A family of Riesz proximities in a symmetric weakly topological closure space has Riesz extensions iff the trace filters are compressed (i.e. for each $x \in X$ and $i \in I$, $\mathfrak{n}(x) \mid X_i$ is δ_i-compressed where $\mathfrak{n}(x)$ denotes the c-neighbourhood filter of x; in case $x \notin cX_i$, we can either say that x has no trace filter on X_i, or $\exp X_i$ has to be allowed as a filter, which is evidently compressed). If so then δ^0 is Riesz, and there exists a finest Riesz extension δ^1_{R}, too. $A\, \delta^1_{\mathrm{R}}\, B$ iff either

$$(4) \qquad cA \cap cB \neq \emptyset,$$

or (3) holds ([6] § 1B). Consequently, a family of proximities in a closure space has Riesz extensions iff each of the proximities considered separately has one.

A family of Lodato proximities in a symmetric topological space has Lodato extensions iff the trace filters are compressed and

$$(5) \qquad A\, \overline{\delta}_i\, B \text{ implies } cA \cap X_j\, \overline{\delta}_j\, cB \cap X_j \qquad (i, j \in I)$$

(observe that this is a strengthening of the accordance; it follows from the other conditions that (5) holds for $i = j$, so we could write $i \neq j$ in (5)). If so then there is a finest Lodato extension δ^1_{L}, for which $A\, \delta^1_{\mathrm{L}}\, B$ iff either (4) holds or

$$(6) \qquad cA \cap X_i\, \delta_i\, cB \cap X_i \text{ for some } i \in I$$

(compare with (3)). Equivalently: $A\, \delta^1_{\mathrm{L}}\, B$ iff $cA\, \delta^1\, cB$. A formula for the coarsest Lodato extension $\delta^0_{(\mathrm{L})}$ is more complicated ([6] § 1C). Consequently, a family of proximities in a closure space has Lodato extensions iff each subfamily of cardinality $\leqq 2$ has one. All the proximities $\delta^0 \supset \delta^0_{(\mathrm{L})} \supset \delta^1_{\mathrm{L}} \supset \delta^1_{\mathrm{R}} \supset \delta^1$ can be different. Each of these proximities can be written as the supremum (for the first two), respectively infimum (for the others), of the extensions of the same type taken for the subfamilies of cardinality 1. (Supremum and infimum are to be understood with respect to the relation $\delta < \delta^*$ iff $\delta \supset \delta^*$. Infimum just means intersection, and a formula for the supremum could also be given.)

Some sufficient conditions: a family of Lodato proximities in a symmetric topological space has Lodato extensions assuming that (a) each X_i is closed, or (b) each X_i is open and the trace filters are compressed, or (c) $|\, I \,| = 1$ and the trace filters are compressed. As a common generalization of the above results, we also have the following sufficient condition:

$$(\mathrm{d}) \qquad c(X_i \backslash X_j) \cap (X_j \backslash X_i) = \emptyset \qquad (i, j \in I)$$

and the trace filters are compressed ([6] 1.13).

δ^1_{R} and δ^1_{L} can also be obtained in a different way based on a categorical consideration that works in several similar situations: Given a proximity δ, there is a finest one among the Riesz, respectively Lodato, proximities coarser than δ; this will be denoted by δ_{R}, respectively δ_{L}. Now a straightforward argument yields that if there are Riesz or Lodato extensions then $(\delta^1)_{\mathrm{R}}$, respectively $(\delta^1)_{\mathrm{L}}$, is the finest one; thus $\delta^1_{\mathrm{R}} = (\delta^1)_{\mathrm{R}}$ and $\delta^1_{\mathrm{L}} = (\delta^1)_{\mathrm{L}}$. The first equality remains in fact true even if there is no Riesz extension, assuming that δ^1_{R} is defined by the formulas (1) to (3) (if c is only symmetric but not weakly topological then it has to be replaced by the finest weakly topological closure coarser than c, see c_{R} in [13] 5.2, where c^* is *coarser* than c if $c^*A \supset cA$ for each A), cf. [14] 11.1. The analogous statement for $(\delta^1)_{\mathrm{L}}$ is false, see [14] Example 12.2. Note also that, even if there are Lodato extensions, $\delta^0_{(\mathrm{L})}$ cannot be always the same as $(\delta^0)_{\mathrm{L}}$, since it is finer (not coarser) than δ^0; this is why we have avoided the notation δ^0_{L} used in [6].

§ 7. Merotopies and contiguities in a closure space

Let us consider the covers $\mathfrak{c}$ in a semimetric space (X, d) for which there is an $\varepsilon > 0$ such that $B_\varepsilon(x)$ is a subset of some element of $\mathfrak{c}$ $(x \in X)$. (In a metric space, such covers are usually called *uniform.*) The collection $\mathfrak{M}$ of these covers satisfies the following conditions:

$$\text{if } \mathfrak{c} \in \mathfrak{M} \text{ and } \mathfrak{c} < \mathfrak{d} \text{ then } \mathfrak{d} \in \mathfrak{M};$$

$$\text{if } \mathfrak{c}, \mathfrak{d} \in \mathfrak{M} \text{ then there is a } \mathfrak{b} \in \mathfrak{M} \text{ with } \mathfrak{b} < \mathfrak{c},\ \mathfrak{b} < \mathfrak{d}.$$

Here $\mathfrak{c} < \mathfrak{d}$ means that $\mathfrak{c}$ *refines* $\mathfrak{d}$, i.e. for each $C \in \mathfrak{c}$ there is a $D \in \mathfrak{d}$ with $C \subset D$. Now a non-empty collection $\mathfrak{M}$ of covers of X is called a *merotopy* on X if it satisfies the two conditions above. A collection of finite covers with the same properties is a *contiguity* (in the first axiom, we have to add to the premise that $\mathfrak{d}$ is finite). Restrictions can be defined by restricting each cover: $\mathfrak{c} \mid X_0 = \{C \cap X_0 : C \in \mathfrak{c}\}$. $\mathfrak{M}^*$ is *finer* than $\mathfrak{M}$ if $\mathfrak{M}^* \supset \mathfrak{M}$.

A merotopy $\mathfrak{M}$ induces a contiguity $\Gamma(\mathfrak{M})$: take the finite elements of $\mathfrak{M}$. A contiguity Γ induces a proximity $\delta(\Gamma)$:

$$A \,\overline{\delta}\, B \ \text{ iff } \ \{X \setminus A,\ X \setminus B\} \in \Gamma.$$

Replacing Γ by $\mathfrak{M}$, a merotopy $\mathfrak{M}$ also induces a proximity $\delta(\mathfrak{M})$; clearly, $\delta(\Gamma(\mathfrak{M})) = \delta(\mathfrak{M})$. We define the induced closure by $c(\mathfrak{M}) = c(\delta(\mathfrak{M}))$, respectively $c(\Gamma) = c(\delta(\Gamma))$; this means that

$$x \in cA \ \text{ iff } \ \{X \setminus \{x\}, X \setminus A\} \notin \mathfrak{M}.$$

Equivalently, the $c(\mathfrak{M})$-neighbourhood filters can be given:

$$\mathfrak{n}(x) = \{\, \bigcup \{C \in \mathfrak{c} : x \in C\} : \mathfrak{c} \in \mathfrak{M}\}.$$

A merotopy $\mathfrak{M}$ is *Riesz* if $\mathfrak{c} \in \mathfrak{M}$ implies that int $\mathfrak{c}$ is a cover; it is *Lodato* if $\operatorname{int} \mathfrak{c} \in \mathfrak{M}$ whenever $\mathfrak{c} \in \mathfrak{M}$. Here

$$\operatorname{int} \mathfrak{c} = \{\operatorname{int} C : C \in \mathfrak{c}\}, \quad \operatorname{int} C = X \setminus c(X \setminus C),$$

with $c = c(\mathfrak{M})$. If $\mathfrak{M}$ is Riesz or Lodato then so is $\delta(\mathfrak{M})$. A filter $\mathfrak{M}$ in a merotopic space $(X, \mathfrak{M})$ is *Cauchy* if $\mathfrak{f} \cap \mathfrak{c} \neq \emptyset \quad (\mathfrak{c} \in \mathfrak{M})$. A merotopy is Riesz iff the neighbourhood filters are Cauchy; it is Lodato iff it induces a topology, and for each Cauchy filter $\mathfrak{f}$, the open elements of $\mathfrak{f}$ generate a Cauchy filter. A merotopy is Lodato iff it induces a topology and has a base consisting of open covers, where $\mathfrak{B} \subset \mathfrak{M}$ is a *base* for $\mathfrak{M}$ if any element of $\mathfrak{M}$ is refined by some element of $\mathfrak{B}$. The above definitions and statements can also be repeated for contiguities. If $\mathfrak{M}$ is Riesz or Lodato then so is $\Gamma(\mathfrak{M})$. Lodato merotopies are Riesz.

The same can be said about simultaneous extensions of merotopies or contiguities in a closure space as in the case of proximities in § 6. The formulas are, of course, different, and "compressed" has to be replaced by "Cauchy". Even the formulas can be made similar if we use another well-known approach to merotopies: $\mathfrak{a} \subset \exp \exp X$ is *far* if $\{X \setminus A : A \in \mathfrak{a}\} \in \mathfrak{M}$; otherwise, $\mathfrak{a}$ is called *near*. Now near or far pairs of sets from the formulas in § 6 have just to be replaced by near or far systems (only finite ones for contiguities).

§ 8. Semiuniformities in a closure space

Let us return now to the semiuniformities, which were defined in § 5. A *(sub)base* for a semiuniformity is to be understood in the usual sense of a filter (sub)base. For semiuniformities on the same set, $\mathcal{U}^*$ is *finer* than $\mathcal{U}$ if $\mathcal{U}^* \supset \mathcal{U}$. Similarly to the contiguities, the semiuniformities can be inserted between the proximities and the merotopies. A merotopy $\mathfrak{M}$ induces a semiuniformity $\mathcal{U}(\mathfrak{M})$, while a semiuniformity $\mathcal{U}$ induces a proximity $\delta(\mathcal{U})$:

$$\{\bigcup_{C \in \mathfrak{c}} C \times C : \mathfrak{c} \in \mathfrak{M}\} \text{ is a base for } \mathcal{U}(\mathfrak{M});$$

$$A\,\delta(\mathcal{U})\,B \text{ iff for each } U \in \mathcal{U} \text{ there are } x \in A,\ y \in B \text{ with } x\,U\,y.$$

Examining the case of structures induced by (semi)metrics, one can see that these definitions are quite natural; they are also consistent with the earlier definitions: $\delta(\mathfrak{M}) = \delta(\mathcal{U}(\mathfrak{M}))$ and $c(\mathcal{U}) = c(\delta(\mathcal{U}))$.

A semiuniformity $\mathcal{U}$ is *Riesz* if $\Delta \subset \operatorname{Int} U \quad (U \in \mathcal{U})$, and *Lodato* if $\operatorname{Int} U \in \mathcal{U} \quad (U \in \mathcal{U})$. Here Int denotes the $c(\mathcal{U}) \times c(\mathcal{U})$ -interior, and the product of two closures can be obtained, just like for topologies, with the products of the neighbourhood filters, i.e.

$$(x, y) \in \operatorname{Int} U \text{ iff there are } A \in \mathfrak{n}(x) \text{ and } B \in \mathfrak{n}(y) \text{ with } A \times B \subset U.$$

Lodato semiuniformities are Riesz; both properties are hereditary. A filter $\mathfrak{f}$ in a semiuniform space $(X,\mathcal{U})$ is *Cauchy* if for each $U \in \mathcal{U}$ there is an $S \in \mathfrak{f}$ with $S \times S \subset U$. A semiuniformity is Riesz iff the neighbourhood filters are Cauchy; it is Lodato iff it induces a topology and has a (sub)base consisting of open entourages. A Riesz semiuniformity induces a weakly topological closure.

The analogues of most of the results from §§ 6 and 7 hold for extensions of semiuniformities in a closure space. The notations $\mathcal{U}^0$, $\mathcal{U}^0_{\mathrm{L}}$, etc. will be used in the same way as the similar notations with δ. The entourages

$$(1) \qquad U_i^0 = U_i \cup (X \times X \setminus X_i \times X_i) \qquad (i \in I, U_i \in \mathcal{U}_i);$$

$$(2) \qquad (X\setminus\{x\}) \times (X\setminus\{x\}) \cup (X\setminus B) \times (X\setminus B) \qquad (x \in X,\ B \subset X,\ x \notin cB)$$

form a subbase for $\mathcal{U}^0$. $\mathcal{U}^1$ consists of the entourages satisfying the following two conditions:

$$Ux \in \mathrm{n}(x) \qquad (x \in X);$$

$$U \mid X_i \in \mathcal{U}_i \qquad (i \in I).$$

There are Riesz extensions iff the trace filters are Cauchy. For Lodato extensions, (5) from § 6 is replaced by the assumption

$$(\operatorname{Int} U_i^0) \mid X_j \in \mathcal{U}_j \qquad (i, j \in I,\ U_i \in \mathcal{U}_i).$$

Moreover, $\mathcal{U}^0$ is the coarsest Riesz extension;

$$\mathcal{U}^1_{\mathrm{R}} = \{U \in \mathcal{U}^1 : \Delta \subset \operatorname{Int} U\};$$

$$\mathcal{U}^1_{\mathrm{L}} = \{U \in \mathcal{U}^1 : \operatorname{Int} U \in \mathcal{U}^1\};$$

$$\{\operatorname{Int} U : U \in \mathcal{U}^0\} \text{ is a base for } \mathcal{U}^0_{\mathrm{L}}.$$

In other words, the $c \times c$-open elements of $\mathcal{U}^1$ form a base for $\mathcal{U}^1_{\mathrm{L}}$ (note that c is assumed to be a topology, hence $c \times c$ is a topology, too); the entourages $\operatorname{Int} U$ with U taken from the subbase (1) and (2) form a subbase for $\mathcal{U}^0_{\mathrm{L}}$.

There is, however, a slight difference: although the analogues of (a), (b) or (c) from § 6 (with "compressed" replaced by "Cauchy") are sufficient for the existence of a Lodato extension of a family of Lodato semiuniformities in a symmetric topological space, (d) is not sufficient. The deeper reason for this surprising phenomenon is not clear, therefore it seems to be worthwhile to reproduce the counterexample from [7] 2.9 (with a modification to make the semiuniformities easier to visualize):

Let $N = \{1/n : n \in \mathbb{N}\}$, $S_\varepsilon =]-\varepsilon, 0[$, $T_\varepsilon =]0, \varepsilon[$,

$$X = (\{0\} \cup N) \times (S_1 \cup T_1), \quad X_0 = X \setminus (\{0\} \times T_1), \quad X_1 = X \setminus (N \times S_1),$$

$V(\varepsilon)=U_{(\varepsilon)}(d)$ with the Euclidean metric d of the plane. Denote by c the Euclidean topology of the plane restricted to X. Writing $P\otimes Q$ for $P\times Q\cup Q\times P$, let

$$U_0(\varepsilon)=(V(\varepsilon)\mid X_0)\bigcup_{n\in\mathbb{N}}((\{1/n\}\times T_\varepsilon)\otimes((\{0\}\cup\{1/k:k>n\})\times S_\varepsilon)),$$

$$U_1(\varepsilon)=(V(\varepsilon)\mid X_1)\cup(((\{0\}\cup N)\times T_\varepsilon)\otimes(\{0\}\times S_\varepsilon)).$$

Take the semiuniformity $\mathcal{U}_i$ on X_i for which $\{U_i(\varepsilon):0<\varepsilon\leq 1\}$ is a base ($i=0,1$). We have defined a family of Lodato semiuniformities in the symmetric topological space (X,c) (the compatibility is easy to see; the accordance is evident; they are Lodato, since the basic entourages are open). (d) holds: even $c(X_0\setminus X_1)\cap c(X_1\setminus X_0)=\emptyset$; moreover, X_0 is open and X_1 is closed. To check that the trace filters are Cauchy, it is enough to consider the case $i=0$, $x\in\{0\}\times]0,1[$. Assuming that there is a Lodato extension,

$$U_1(\varepsilon)\subset(\operatorname{Int}(U_0(1))^0)\mid X_1$$

with some $\varepsilon>0$, a contradiction, since the pair $((0,-\varepsilon/2),\ (0,\varepsilon/2))$ belongs to the left-hand side, but not to the right-hand side (see the condition $k>n$ in the definition of $U_0(1)$).

§ 9. Screens in a closure space

Collections of filters arise quite naturally in topology: take e.g. the convergent filters in a topological space, the compressed filters in a proximity space, or the Cauchy filters in a merotopic (or other) space. Denoting one of these collections by $\mathfrak{S}$, the following simple conditions are always satisfied (although it makes no real difference, we allow now $\exp X$ as a filter on X):

$$\dot{x}=\{S\subset X: x\in S\}\in\mathfrak{S}\qquad(x\in X);$$

$$\text{if } \mathfrak{s}\in\mathfrak{S}, \mathfrak{s}\subset\mathfrak{s}'\in\operatorname{Fil}^+X \text{ then } \mathfrak{s}'\in\mathfrak{S}.$$

Here $\operatorname{Fil}^+X=\operatorname{Fil}X\cup\{\exp X\}$, and $\operatorname{Fil}X$ consists of the proper filters on X. $\mathfrak{S}\subset\operatorname{Fil}^+X$ satisfying the above axioms is called a *screen* (or filter merotopy); in view of the possibility $X=\emptyset$, $\mathfrak{S}\neq\emptyset$ has also to be assumed. $\mathfrak{S}\mid X_0=\{\mathfrak{s}\mid X_0:\mathfrak{s}\in\mathfrak{S}\}$; $\mathfrak{S}^*$ is *finer* than $\mathfrak{S}$ if $\mathfrak{S}^*\subset\mathfrak{S}$.

A screen $\mathfrak{S}$ induces a contiguity $\Gamma(\mathfrak{S})$ for which a finite system $\mathfrak{a}$ is near (see at the end of § 7) iff $\mathfrak{a}\,\Delta\,\mathfrak{s}$ for some $\mathfrak{s}\in\mathfrak{S}$, where

$$\mathfrak{a}\,\Delta\,\mathfrak{b}\ \text{ iff } A\cap B\neq\emptyset\qquad(A\in\mathfrak{a},\ B\in\mathfrak{b}).$$

Consequently, a screen $\mathfrak{S}$ induces a proximity $\delta(\mathfrak{S})=\delta(\Gamma(\mathfrak{S}))$, and also a symmetric closure $c(\mathfrak{S})=c(\Gamma(\mathfrak{S}))=c(\delta(\mathfrak{S}))$.

$$x\in c(S)A \text{ iff } x\in\bigcap\mathfrak{s} \text{ and } A\cap S\neq\emptyset\quad(S\in\mathfrak{s}) \text{ for some } \mathfrak{s}\in\mathfrak{S}.$$

In other words, $\mathfrak{n}(x)$ is the intersection of the elements of $\mathfrak{S}$ fixed at x ($\mathfrak{s}$ is *fixed at* x if $x \in \bigcap \mathfrak{s}$). A merotopy $\mathfrak{M}$ induces a screen $\mathfrak{S}(\mathfrak{M})$ consisting of the Cauchy filters. Each contiguity can be induced by screens, and each screen by merotopies. In general, $\Gamma(\mathfrak{M}) \neq \Gamma(\mathfrak{S}(\mathfrak{M}))$; even $c(\mathfrak{M})$ and $c(\mathfrak{S}(\mathfrak{M}))$ can be different.

A screen $\mathfrak{S}$ is *Riesz* if it contains the $c(\mathfrak{S})$-neighbourhood filters; it is *Lodato* if $c(\mathfrak{S})$ is a topology, and for any $\mathfrak{s} \in \mathfrak{S}$, the filter generated by the open elements of $\mathfrak{s}$ also belongs to $\mathfrak{S}$. Any Lodato screen is Riesz; both properties are hereditary. The contiguity and the proximity induced by a Riesz or Lodato screen have the same property. Similarly, a Riesz or Lodato merotopy induces a screen with the same property (the fact that the two structures may induce different closures causes no problem, since if $\mathfrak{M}$ is a Riesz merotopy then $c(\mathfrak{M}) = c(\mathfrak{S}(\mathfrak{M}))$, see [18] Lemma 2.1). A Riesz screen induces a weakly topological closure.

Let us consider now a family $\mathfrak{S}_i$ of screens in a symmetric closure space (X, c). The results (taken from [2]) are different from those valid for other structures: There is an extension iff

$$(1) \qquad i \in I,\ A \subset X_i,\ x \in cA \Longrightarrow \exists \mathfrak{s}_i \in \mathfrak{S}_i \setminus \{\exp X_i\}, \quad A \in \mathfrak{s}_i \to x,$$

where a filter base $\mathfrak{b}$ in X *converges* to x, denoted by $\mathfrak{b} \to x$, if it generates a filter finer than the c-neighbourhood filter $\mathfrak{n}(x)$ of x. It is easy to give an example where (1) is not satisfied ([2] 2.2). Thus even a single screen may fail to have an extension, but if each $\mathfrak{S}_i$ separately has an extension then so has the whole family. If there are extensions then there is a coarsest one, denoted by $\mathfrak{S}^0$, which consists of all $\mathfrak{s} \in \mathrm{Fil}^+ X$ satisfying

$$(2) \qquad \mathfrak{s} \text{ converges to the points it is fixed at;}$$

$$(3) \qquad \mathfrak{s} \,|\, X_i \in \mathfrak{S}_i \qquad (i \in I).$$

There is in general no finest extension (even in the case $I = \emptyset$, which means that there may be no finest compatible screen in a closure space, see [1] 3.15).

A family of Riesz screens in a symmetric weakly topological closure space has Riesz extensions iff for each $i \in I$, the trace filters on X_i belong to $\mathfrak{S}_i$ (observe that (1) follows from this condition). Hence it is again enough to know that each screen separately has a Riesz extension. If there are Riesz extensions then $\mathfrak{S}^0$ is the coarsest one, and there is also a finest Riesz extension $\mathfrak{S}^1_{\mathrm{R}}$, consisting of the c-convergent filters, and of the filters $\mathfrak{s}^1_i$ ($i \in I$, $\mathfrak{s}_i \in \mathfrak{S}_i$), where $\mathfrak{s}^1_i$ denotes the filter in X generated by the filter base $\mathfrak{s}_i$. (If there is a finest extension $\mathfrak{S}^1$ then $\mathfrak{S}^1_{\mathrm{R}}$ is the finest one among the Riesz screens coarser than $\mathfrak{S}^1$; but, as mentioned earlier, $\mathfrak{S}^1$ does not have to exist.)

A family of Lodato screens in a symmetric topological space has Lodato extensions if it has Riesz extensions (see above) and a condition similar to

(5) from § 6 holds; if so then there are a coarsest Lodato extension $\mathfrak{S}^0_{(L)}$ and a finest Lodato extension $\mathfrak{S}^1_L$. Explicit formulas for $\mathfrak{S}^0_{(L)}$ and $\mathfrak{S}^1_L$ can be given, see [2] § 2 for the details. $\mathfrak{S}^1_L$ is the finest one among the Lodato screens coarser than $\mathfrak{S}^1_R$. A family of screens in a closure space has Lodato extensions iff each subfamily of cardinality ≤ 2 has one. Differently from the case of semiuniformities, (d) from § 6 (including an assumption on the trace filters) is again sufficient for the existence of a Lodato extension.

§ 10. Simultaneous extensions in proximity and other spaces

The closure induced by a contiguity, semiuniformity, merotopy or screen can be obtained through a proximity. Therefore we can consider simultaneous extensions of these structures in a proximity space. There are also other variations: screens in a contiguity space; merotopies in a contiguity, semiuniform or screen space. These extension problems are more difficult than the ones in a closure space. For lack of space, we only mention here the basic results in a tabular form. Extensions in a closure space will be included in the table for easier comparison. Neither the necessary and sufficient conditions nor the formulas for particular extensions will be cited.

Notations in the table

First column: The structure to be extended (denoted by the letter typically used for the structure).

Second column: The structure on X (c stands for a symmetric closure).

Column E: Is there a simultaneous extension? (Assuming, of course, that the obvious necessary conditions are satisfied, i.e. that, in the case of Riesz or Lodato extensions, all the structures, including the one given on X, are Riesz or Lodato, respectively weakly topological or topological in the case of a closure.)

+: Yes, always.

n ($\in \mathbb{N}$): If each subfamily of cardinality $\leq n$ has an extension then so has the whole family, but the same is false with n replaced by $n-1$.

f: If each finite subfamily has an extension then so has the whole family, but the stronger statements above do not hold.

$-$: None of the above.

?: The answer is not known. It can be any $n \geq 2$ or f.

Column C: Assuming that there are extensions, is there a coarsest one? (+: Yes. $-$: No.)

Column F: The same with finest extensions.

Last column: References.

					Riesz			Lodato			
		E	C	F	E	C	F	E	C	F	
δ_i	c	+	+	+	1	+	+	2	+	+	[6]
Γ_i	c	+	+	+	1	+	+	2	+	+	[7]
$\mathfrak{M}_i$	c	+	+	+	1	+	+	2	+	+	[7]
$\mathcal{U}_i$	c	+	+	+	1	+	+	2	+	+	[7]
$\mathfrak{S}_i$	c	1	+	−	1	+	+	2	+	+	[2]
Γ_i	δ	+	+	+	1	+	+	2	+	+	[8]
$\mathfrak{M}_i$	δ	+	+	−	1	+	−	?	+	−	[8]
$\mathcal{U}_i$	δ	+	+	−	1	+	−	?	+	−	[9]
$\mathfrak{S}_i$	δ	2	+	−	2	+	−	−	+	−	[2], [19]
$\mathfrak{M}_i$	$\mathcal{U}$	+	+	+	1	+	+	2	+	+	[9]
$\mathfrak{M}_i$	Γ	f	+	−	f	+	−	f	+	−	[9]
$\mathfrak{S}_i$	Γ	f	+	−	f	+	−	−	+	−	[19]
$\mathfrak{M}_i$	$\mathfrak{S}$	+	+	+	+	+	+	2	+	+	[18]

Some remarks:

If a structure is Riesz then so is any coarser structure inducing the same closure. Hence if there are Riesz extensions then the coarsest extension is Riesz. This reasoning does not work, and the statement is in fact false, for extensions of merotopies in a screen space, because the extension and the screen can induce different closures. Instead, the following holds: if a family of merotopies in a screen space has Riesz extensions then the *finest* extension is Riesz.

When extending Lodato structures in a symmetric topological space, it was enough to know that each X_i is open and the trace filters satisfy a simple condition, or each X_i is closed. If a Lodato structure instead of a topology is given then it can occur that one of these conditions is sufficient, but not the other. E.g. a family of Lodato merotopies given on closed subsets in a Lodato proximity space has Lodato extensions ([8] 5.22); one can give, however, two merotopies on disjoint open subsets in a proximity space such that they have Lodato extensions separately, but not simultaneously ([8] Example 5.20). On the other hand, a family of Lodato merotopies given on open subsets in a Lodato screen space always has Lodato extensions (without any assumption on the trace filters), but the same is false for closed subsets ([18] 3.6).

§ 11. Cauchy structures

The following condition holds for the system $\mathfrak{S}$ of the Cauchy filters in a metric space, or for the compressed filters in a proximity space satisfying

($\triangle$), and also for the Cauchy filters in a uniform space (which was not defined in this paper):

$$\text{if } \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}, \quad \mathfrak{s} \,\Delta\, \mathfrak{t} \quad \text{then } \mathfrak{s} \cap \mathfrak{t} \in \mathfrak{S}$$

(see § 9 for Δ). A screen $\mathfrak{S}$ satisfying the above axiom is called a *Cauchy screen* or (in the more usual terminology) a *Cauchy structure*. Virtually nothing is known about simultaneous extensions of Cauchy screens in a closure space (if no additional properties are assumed); it is not even clear which closures can be induced by Cauchy screens, cf. [05]. Nevertheless, one might possibly look for a set of necessary and sufficient conditions, with the existence of a Cauchy screen compatible with c included as one of the conditions (similarly e.g. to Hausdorff's theorem, where it is assumed that the topology is metrizable).

[3] contains, however, results on Cauchy Riesz and Cauchy Lodato (shortly: CR and CL) extensions, including necessary and sufficient conditions. A closure can be induced by a CR (respectively CL) screen iff it satisfies the following condition:

$$\text{(1)} \qquad \mathfrak{n}(x) \,\Delta\, \mathfrak{n}(y) \text{ implies } \mathfrak{n}(x) = \mathfrak{n}(y)$$

(respectively it is a topology in addition). Closures satisfying (1) are known as R_1, S_2 or reciprocal. For Cauchy screens in a closure space, the following line can be added to the table in § 10:

–	–	–	f	–	+	f	–	+	[3]

Observe that this is the first case where there are no coarsest extensions. (Although there is a coarsest one among the CR or CL screens inducing a closure.)

One can obtain better results on extensions of Cauchy screens if they are considered in a convergence space rather than a closure space. More generally, extensions of screens can also be investigated in convergence spaces. Let us recall that a function $\lambda : X \to \exp \mathrm{Fil}^+ X$ is a *convergence* on X provided that

$$\dot{x} \in \lambda(x) \qquad (x \in X);$$

if $\mathfrak{s} \in \lambda(x)$ and $\mathfrak{s} \subset \mathfrak{t} \in \mathrm{Fil}^+ X$ then $\mathfrak{t} \in \lambda(x)$;

$$\text{if } \mathfrak{s} \in \lambda(x) \text{ then } \mathfrak{s} \cap \dot{x} \in \lambda(x).$$

The elements of $\lambda(x)$ are said to *converge* to x, and $\mathfrak{s} \xrightarrow{\lambda} x$ or $\mathfrak{s} \to x$ is also written for $\mathfrak{s} \in \lambda(x)$. The convergence λ is a *limitation* provided that the following stronger version of the last axiom holds:

$$\text{if } \mathfrak{s}, \mathfrak{t} \to x \text{ then } \mathfrak{s} \cap \mathfrak{t} \to x.$$

$\lambda_0 = \lambda \mid X_0$ is defined by $\lambda_0(x) = \{\mathfrak{s} \mid X_0 : \mathfrak{s} \in \lambda(x)\}$. λ is *symmetric* if $\mathfrak{s} \to x$ and $y \in \bigcap \mathfrak{s}$ imply $\mathfrak{s} \to y$, and *reciprocal* if $\lambda(x) = \lambda(y)$ whenever there is a proper filter converging to both points. A limitation λ is symmetric iff $\dot{x} \to y$ implies $\lambda(x) = \lambda(y)$; a convergence with this property will be called *pointwise reciprocal.* Reciprocal $\Rightarrow$ pointwise reciprocal $\Rightarrow$ symmetric. See e.g. the survey [01] for more on convergences.

A convergence λ induces a closure $c(\lambda)$ for which $\mathfrak{n}(x) = \bigcap \lambda(x) \quad (x \in X)$; note that more filters are $c(\lambda)$- convergent than λ-convergent. The convergence $\lambda(\mathfrak{S})$ induced by the screen $\mathfrak{S}$ is defined as follows:

$$\mathfrak{s} \to x \quad \text{iff} \quad \mathfrak{s} \in \operatorname{Fil}^+ X, \quad \mathfrak{s} \cap \dot{x} \in \mathfrak{S}.$$

$c(\mathfrak{S}) = c(\lambda(\mathfrak{S}))$. A convergence can be induced by a screen (a Cauchy screen) iff it is symmetric (it is a reciprocal limitation).

Simultaneous extensions of (Cauchy) screens in a convergence space were investigated in [16]. With the notations of the table, the results are 1 + + for screens in general, and f − + for Cauchy screens in a reciprocal limit space. Moreover, a family of two screens has a Cauchy extension iff the screens separately have one, but the same is false for a family of three screens (the analogous statement is also valid for CR extensions in a closure space, see [3] 2.14). Riesz, Lodato, CR or CL extensions in a convergence space do not have to be considered: these problems are equivalent to the same problems in a closure space, because if two Riesz screens induce the same closure then they also induce the same convergence.

Another type of structure can be inserted between convergences and closures: a function $\pi : X \to \exp \operatorname{Ult} X$ (where $\operatorname{Ult} X$ denotes the collection of the ultrafilters on X) is a *pseudotopology* on X if $\dot{x} \in \pi(x) (x \in X)$. Instead of $\mathfrak{u} \in \pi(x)$, we also write $\mathfrak{u} \xrightarrow{\pi} x$ or $\mathfrak{u} \to x$. Restrictions and the induced closure are defined just as for convergences. π is *symmetric* if $\dot{x} \to y$ implies $\dot{y} \to x$, *pointwise reciprocal* if $\dot{x} \to y$ implies $\pi(x) = \pi(y)$, and *reciprocal* if $\pi(x) = \pi(y)$ whenever $\pi(x) \cap \pi(y) \neq \emptyset$. A convergence λ induces a pseudotopology $\pi = \pi(\lambda)$ in an obvious way: $\mathfrak{u} \xrightarrow{\pi} x$ iff $\mathfrak{u} \in \operatorname{Ult} X$ and $\mathfrak{u} \xrightarrow{\lambda} x$. A screen $\mathfrak{S}$ induces a pseudotopology $\pi(\mathfrak{S}) = \pi(\lambda(\mathfrak{S}))$, i.e. $\mathfrak{u} \to x$ iff $\mathfrak{u} \in \operatorname{Ult} X$ and $\mathfrak{u} \cap \dot{x} \in \mathfrak{S}$. Most authors identify the pseudotopologies with certain convergences, which is all right as long as only the category of pseudotopologies is investigated (since it is replaced by an isomorphic category). But such an identification should be avoided when considering extensions of screens: an extension in a pseudotopological space is not necessarily compatible with the corresponding convergence.

The results on simultaneous extensions of (Cauchy) screens in a pseudotopological space are analogous to those valid in a convergence space (see [16]), but, oddly enough, some of the proofs are more complicated.

Extensions of convergences or limitations in a pseudotopological or closure space, and also of pseudotopologies in a closure space, were considered in [17]. The problems have been solved more or less completely, except in the case of reciprocal extensions in a closure space. The reason is that a closure can be induced by a reciprocal convergence or limitation or pseudotopology iff it can be induced by a Cauchy screen.

§ 12. Simultaneous extensions in a set

Let us consider now the following modification of the problem of simultaneous extensions: Given a set X (without any structure), $X_i \subset X \quad (i \in I)$, and structures Σ_i on X_i, is there an *extension*, i.e. a structure Σ on X such that $\Sigma \mid X_i = \Sigma_i \quad (i \in I)$? More precisely, we speak about an *extension in the set* X, as opposed to *extensions in a space.* Accordance is again a necessary condition, while compatibility is now meaningless; so a *family of structures (in a set)* is only assumed to be accordant.

A single structure always has extensions, at least for "good" structures (in particular, for all the structures mentioned in this paper): in categorical terminology, take the sum of the given structure and of the discrete structure on the remainder. Therefore it can be assumed without loss of generality that

$$X = \bigcup_{i \in I} X_i, \tag{1}$$

since, in the general case, there is an extension to X iff there is one to $\bigcup_{i \in I} X_i$. Assuming (1) makes no difference either when looking for finest or coarsest extensions:

Finest extensions do exist in each particular case, because the supremum commutes with restrictions (this holds in any topological category, while all the nontopological subcategories considered are closed for suprema). On the other hand, assuming that there is a coarsest extension whenever (1) holds (and there are extensions at all), the same will hold in the general case. (We have to know here that there is only one structure on a singleton. Then the unique structure on each singleton can be added to the family without spoiling the accordance. (1) holds now for the modified family, and the extensions of the two families are the same.)

The problem of extensions in a set can in fact be regarded as a special case (seemingly the easiest one) of the same problem in a space: assume that there is only one structure on X, say X itself (i.e. let Set be the category of the less rich structures). There are cases when, somewhat surprisingly, extensions in a set are more difficult to handle. E.g. the Riesz or Lodato property is defined in terms of the induced closure, therefore, when looking for such extensions, it is a disadvantage not to have the closure prescribed on X.

Similarly to § 10, we shall sum up the main results in a tabular form. In addition to the structures introduced so far, the following ones will also be included: A screen $\mathfrak{S}$ is *pointwise Cauchy* if $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$, $\bigcap \mathfrak{s} \cap \bigcap \mathfrak{t} \neq \emptyset$ imply $\mathfrak{s} \cap \mathfrak{t} \in \mathfrak{S}$; it is *fully Cauchy* if for any $\mathfrak{s} \in \mathfrak{S}$, the intersection of all the filters $\mathfrak{t} \in \mathfrak{S}$ satisfying $\mathfrak{s} \,\Delta\, \mathfrak{t}$ also belongs to $\mathfrak{S}$. (Fully Cauchy $\Rightarrow$ Cauchy

$\Rightarrow$ pointwise Cauchy; fully Cauchy $\Rightarrow$ Riesz $\Rightarrow$ pointwise Cauchy. See [4] and [5] for other related properties of screens.) A closure c on X is $\underline{T_0}$ if $x \in c\{y\}$ and $y \in c\{x\}$ imply $x = y$; it is $\underline{T_1}$ if $c\{x\} = \{x\}$ $(x \in X)$. (Obvious generalizations of the usual separation axioms for topologies. A closure is T_1 iff it is symmetric and T_0.) A proximity, semiuniformity, merotopy, contiguity or screen is $\underline{T_1}$ if the induced closure is T_1 (equivalently: T_0). For semiuniformities, this is equivalent to the definition in § 5.

The table will contain no Column F, since, as mentioned earlier, the existence of extensions always implies that there is a finest one. No result is indicated for weakly topological T_1 closures, since these are the same as the T_1 closures. The table also contains some other empty spaces for the same reason, e.g. fully Cauchy Riesz screen = fully Cauchy screen.

Notations in the table (additions to those in § 10)

Column 2: Does a family of two structures always have an extension?
In Column E:
f/−: f for countable families, — in general.
Riesz closure = weakly topological closure.
Lodato closure = topology.
FC = fully Cauchy.
PC = pointwise Cauchy.

				Riesz			Lodato			
	E	2	C	E	2	C	E	2	C	
closure	+	+	+	f	+	−	−	+	−	[13]
symm. closure	+	+	+	f	+	−	−	+	−	[13]
T_0 closure	+	+	−	f	+	−	−	−	−	[13]
T_1 closure	+	+	+				−	+	−	[13]
proximity	+	+	+	f	+	−	−	+	−	[14]
T_1 proximity	+	+	+	3	+	−	−	+	−	[14]
contiguity	+	+	+	f	+	−	−	+	−	[14], [15]
merotopy	+	+	+	−	+	−	−	+	−	[14], [15]
semiuniformity	+	+	+	−	+	−	−	−	−	[14], [15]
screen	+	+	+	−	+	−	−	+	−	[14], [15]
Cauchy screen	f	+	−	f/−	−	−	−	−	−	[14], [15]
T_1 Cauchy screen	f	−	−	f/−	−	−	−	−	−	[14], [15]
FC screen	f/−	+	−				−	−	−	[17]
T_1 FC screen	f/−	−	−				−	−	−	[17]
PC screen	f	+	−							[15]

				symm.			pw. rec.			
	E	2	C	E	2	C	E	2	C	
convergence	+	+	+	+	+	+	f	+	−	[17]
limitation	+	+	+	f	+	−				[17]
pseudotopology	+	+	+	+	+	+	f	+	−	[13], [17]

For topologies, "weakly topological and symmetric" corresponds to the property "pointwise reciprocal".

Thus closures could be included in the last table, too, with the same results as for convergences or pseudotopologies.

In the case of contiguities, semiuniformities, merotopies or (PC) screens, the results are the same with or without T_1. We did not consider T_1 extensions in a space (§ 10), because there are such extensions iff there are extensions at all, and the structure given on X is T_1; if so then each extension is T_1. (Since T_1 depends only on the induced closure. Concerning merotopies in a screen space, observe that, although $c(\mathfrak{M})$ and $c(\mathfrak{S}(\mathfrak{M}))$ can be different, the two closures of a singleton coincide.) It is, however, possible that stronger statements hold when the structure on X is T_1; only some insignificant results of this kind are known; e.g. if Lodato proximities are given on closed subspaces of a T_1 topological space then $\delta^0_{(\mathrm{L})} = \delta^0$ (and T_1 is essential here), see [6] 1.15.

Some of the results deserve special attention:

T_0 closure, C. The reason is that there is no coarsest T_0 closure on a set of cardinality > 1. Thus we can take the (unique and T_0) closures on the one-point subsets.

T_0 topology, 2. [13] Example 6.5.

T_1 Riesz proximity, E. The necessary and sufficient condition for the existence of Riesz extensions of proximities ([14] Theorem 11.5) contains three of the proximities, and a sequence of points $x_0, x_1, \dots, x_n$ such that $\{x_{k-1}\}\delta_{i_k}\{x_k\}$ with $i_k \in I$; if each proximity is T_1 then $x_0 = \dots = x_n$, so the condition contains now only three proximities.

Lodato semiuniformity, 2. Take the example from § 8, but without the topology on X; see [15] 19.6 for details.

(T_1) CR screen, E. [16] Example 17.3.

(T_1) CR screen, 2. [16] Example 17.2.

T_1 (fully) Cauchy screen, 2. This is very easy: Let $X = \mathbb{N}$, $X_i = X \setminus \{i\}$ $(i = 1, 2)$, $\mathfrak{u}$ a free ultrafilter on X, $\{\dot{x} : x \in X\} \cup \{\dot{2} \cap (\mathfrak{u} \mid X_1)\}$ a base for $\mathfrak{S}_1$, and define $\mathfrak{S}_2$ analogously. Any Cauchy extension has to contain $\dot{1} \cap \dot{2}$, so it cannot be T_1.

(T_1) FC screen, E. [17] Example 7.1 a).

(T_1) FC Lodato screen, 2. In [15] Example 20.1, given for CL screens, the two screens are actually FC.

Topology, E. It is possible that an infinite family has no extension, although each proper subfamily has one ([13] Example 6.9). It should be investigated systematically whether all the negative results in Column E (also in § 10) can be strengthened analogously. For Lodato screens in a proximity or contiguity space, see [19] § 3.

REMARK. The extension problem in sets can also be investigated in a categorical setting, in topological categories satisfying some additional conditions, but over mSet rather than Set (where mSet is the category of the sets with the injections as morphisms); see [12] for details. A categorical approach to the extension problem in spaces seems to be more difficult; only some trivialities are known (e.g. [7] 2.2).

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 349–353

AN ILLUMINATION PROBLEM FOR CONVEX POLYHEDRA

L. SZABÓ and I. TALATA

Abstract

Consider a convex polytope P in the d-dimensional Euclidean space. We say that a vertex v of P illuminates a point $u \in E^d$ lying outside P if the line segment $\overline{uv}$ does not intersect the interior of P. Furthermore, we say that the vertices $v_1, v_2, \dots, v_l$ of P illuminate the entire exterior of P if every point of E^d lying outside P is illuminated by at least one of the vertices $v_1, v_2, \dots, v_l$. In this paper we consider the three-dimensional situation and we prove that $\lfloor \frac{n}{2} \rfloor$ ($\lfloor \frac{n}{2} \rfloor - 1$ resp.) vertices are always sufficient and sometimes necessary to illuminate the exterior of a convex polyhedron of $n \neq 4m+2$ ($n = 4m+2$ resp.) vertices. We give lower bounds in even dimensions as well.

Introduction

Let $K \subseteqq E^d$ be a convex body, i.e. a compact convex set with a non-empty interior. We say that a point $q \in E^d$ exterior to the convex body K illuminates a boundary point p of K if the open ray emanating from q having direction vector $\overrightarrow{pq}$ has a non-empty intersection with the interior of K. Furthermore, we say that the points $q_1, q_2, \dots, q_l \in E^d$ exterior to the convex body K illuminate K if every boundary point of K is illuminated by at least one of the points $q_1, q_2, \dots, q_l$. It is a challenging open problem to determine the smallest number of points lying outside K which illuminate K for $d \geqq 3$. For further information we refer the reader to the survey paper of Károly Bezdek [2].

In this note we deal with a related problem which was inspired by Károly Bezdek, too. Consider a convex polytope P in the d-dimensional Euclidean space. We say that a vertex v of P illuminates a point $u \in E^d$ lying outside P if the line segment $\overline{uv}$ does not intersect the interior of P. Furthermore, we say that the vertices $v_1, v_2, \dots, v_l$ of P illuminate the entire exterior of P if every point of E^d lying outside P is illuminated by at least one of the vertices $v_1, v_2, \dots, v_l$. Now, our problem can be formulated as follows. What

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is the smallest number of vertices of a convex polytope P of n vertices in the d-dimensional Euclidean space which illuminate the entire exterior of P.

During the preparation of this paper Godfried Toussaint pointed out to us that the problem was considered independently in [3]. The authors of this technical report dealt only with the three-dimensional version of the problem and their bounds are not tight for all vertex numbers.

The planar case is trivial, $\lceil \frac{n}{2} \rceil$ vertices are always sufficient and necessary to illuminate the exterior of a convex polygon of n vertices. But in higher dimensions the situation is not as straightforward. We prove the following

THEOREM. $\lfloor \frac{n}{2} \rfloor$ ($\lfloor \frac{n}{2} \rfloor - 1$ *resp.*) *vertices are always sufficient and sometimes necessary to illuminate the exterior of a convex polyhedron of* $n \neq 4m+2$ ($n = 4m+2$ *resp.*) *vertices.*

We will give lower bounds in even dimensions as well.

Proof of Theorem

First, we note that the vertices $v_1, v_2, \dots, v_l$ illuminate the exterior of a convex polyhedron if and only if the vertices $v_1, v_2, \dots, v_l$ cover the faces of the polyhedron, i.e. each face of the polyhedron contains at least one of these vertices on its boundary.

Now, let us see the sufficiency proof. Triangulate the faces of the polyhedron. Then four colour the vertices of this triangulated polyhedron via the *Four Colour Theorem* [1] and take those two colours, say black and white, which together occur at most $\lfloor \frac{n}{2} \rfloor$ times ($\lfloor \frac{n}{2} \rfloor - 1$ times if $n = 4m+2$ resp.). As at each triangle of our triangulation we have three different colours the vertices coloured black and white cover all faces of the original polyhedron.

Observe that the above proof immediately leads to an $O(n^2)$ algorithm since the *Four Colour Algorithm* [1] runs in $O(n^2)$ time. To find linear time covering algorithm with at most $\lfloor \frac{n}{2} \rfloor$ vertices seems to be an extremely difficult task. However, if we are satisfied with a covering system of cardinality at most $\lfloor \frac{3n}{5} \rfloor$, such linear time algorithm exists since any planar graph can be five coloured in $O(n)$ time (see [5], p. 87).

To finish the proof we have to construct convex polyhedra of n vertices with the property that $\lfloor \frac{n}{2} \rfloor$ vertices ($\lfloor \frac{n}{2} \rfloor - 1$ vertices if $n = 4m+2$ resp.) are necessary to cover their faces. Obviously, using a simple duality argument (see [4], p. 46), it is enough to construct convex polyhedra of n faces with the property that $\lfloor \frac{n}{2} \rfloor$ faces ($\lfloor \frac{n}{2} \rfloor - 1$ faces if $n = 4m+2$ resp.) are necessary to cover their vertices. Furthermore, by a remarkable theorem of Steinitz (see [4], p. 235) we can restrict ourselves to 3-connected planar graphs of n regions.

For the sake of simplicity let $n = 4m$ and let us consider the 3-connected planar graph of n regions in Figure 1. Suppose that the vertices are covered

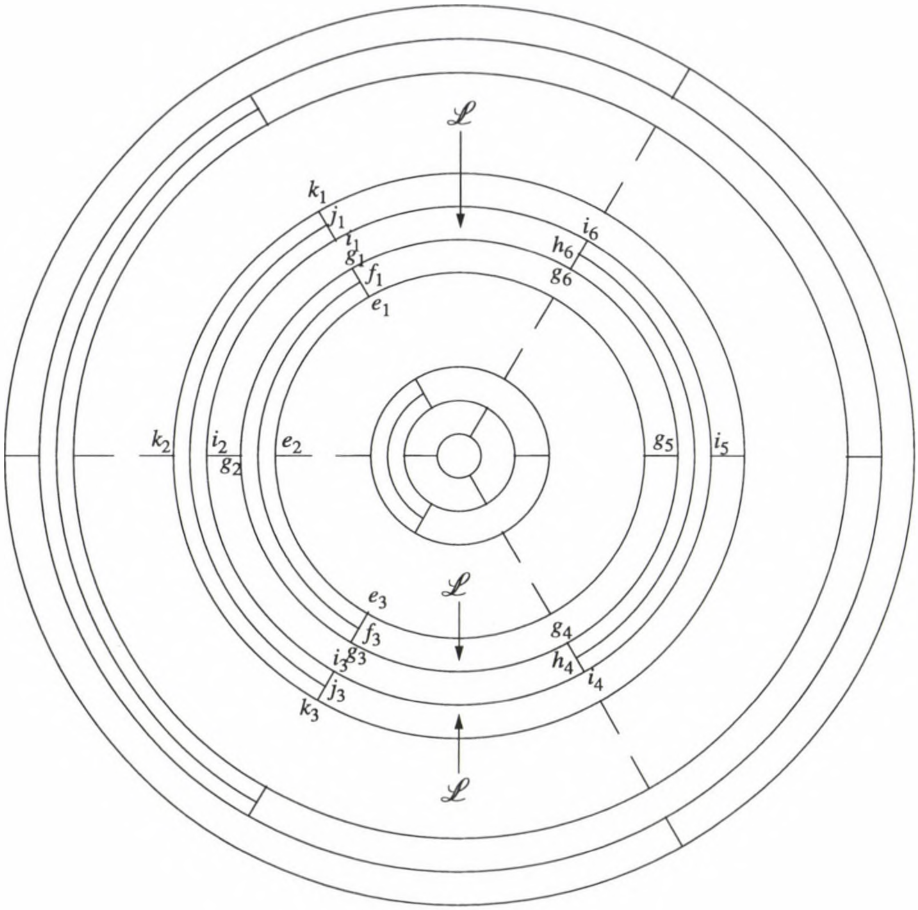

Fig. 1

by a suitable subset of the regions of this graph. Beside the "inner" and the "outer" four-four regions, our graph consists of identical layers. Consider one of these layers, say the layer $\mathcal{L}$ (the heptagons $g_2g_1g_6h_6i_6i_1i_2$ and $g_2g_3g_4h_4i_4i_3i_2$ and the pentagons $g_4g_5g_6h_6h_4$ and $i_4i_5i_6h_6h_4$ form this layer). Since the vertices h_4 and h_6 are covered, at least one region is marked in $\mathcal{L}$. Clearly, the only interesting case is when exactly one region is marked in $\mathcal{L}$. Obviously, if this region is $g_4g_5g_6h_6h_4$ then the "next" layer $\mathcal{L}''$ must contain at least three marked regions and if this region is $i_4i_5i_6h_6h_4$ then the "previous" layer $\mathcal{L}'$ must contain at least three marked regions. We have to deal with one more case when only the regions $e_1e_2e_3f_3f_1$ and $k_1k_2k_3j_3j_1$ are marked in $\mathcal{L}'$ and $\mathcal{L}''$, respectively. But in this case all the four regions must be marked in the layer $\mathcal{L}$. These observations show that the number of the marked regions in all the layers is not smaller than half times the number of all regions in these layers. It would be still left to prove that there is no confusion about the "inner" and the "outer" parts of the graph but this can be settled similarly and the details we leave to the reader. This completes the proof.

Obviously, by a slight modification of the above construction (in fact, it is enough to modify the "inner" four regions only) we can obtain the required lower bounds for the remaining cases.

Higher dimensional lower bounds

In the d-dimensional Euclidean space consider the moment curve M_d defined parametrically by $x(t) = (t, t^2, \dots, t^d)$. Then a cyclic d-polytope $C(n, d)$ is the convex hull of $n \geqq d+1$ points $x(t_i)$ on M_d, with $t_1 < t_2 < \dots < t_n$. It is well-known that $C(n, d)$ is a simplicial d-polytope of n vertices (see [4], p. 61).

Now, using "Gale's evenness condition" which says that a d-tuple V_d of points of $V \subseteqq M_d$ determines a facet $\operatorname{conv} V_d$ of $\operatorname{conv} V = C(n, d)$ if and only if every two points of $V \setminus V_d$ are separated on M_d by an even number of points of V_d (see [4], p. 62), one can prove easily that any covering of the facets of $C(n, d)$ requires at least $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$ vertices, if d is even.

First we consider the case when n is even. Then the vertices $x(t_d)$, $x(t_{d+2}), \dots, x(t_n)$ cover the facets of $C(n, d)$ and the number of these vertices is $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$. On the other hand, if we have only $\lceil \frac{n}{2} \rceil - \lceil \frac{d}{2} \rceil$ marked vertices, then at least $\lceil \frac{d}{2} \rceil$ of the pairs $\{x(t_1), x(t_2)\}$, $\{x(t_3), x(t_4)\}$, $\dots$, $\{x(t_{n-1}), x(t_n)\}$ of consecutive vertices are not covered. But $\lceil \frac{d}{2} \rceil$ of these pairs represent a facet of $C(n, d)$ which is not covered.

Next we consider the case when n is odd. Now the vertices $x(t_{d-1})$, $x(t_{d+1}), \dots, x(t_n)$ cover the facets of $C(n, d)$. The number of these vertices is again $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$. To prove that this is the smallest number of vertices with the required property suppose that we have only $\lceil \frac{n}{2} \rceil - \lceil \frac{d}{2} \rceil$ marked vertices. Without loss of generality we may assume that the vertex $x(t_1)$ is marked (indeed, a cyclic shift of the vertices of $C(n, d)$ on the moment curve M_d does not affect the facet structure of $C(n, d)$). Then among the pairs $\{x(t_2), x(t_3)\}, \{x(t_4), x(t_5)\}, \dots, \{x(t_{n-1}), x(t_n)\}$ of consecutive vertices at least $\lceil \frac{d}{2} \rceil$ are not covered. But $\lceil \frac{d}{2} \rceil$ of these pairs determine again a non-covered facet of $C(n, d)$.

Therefore $\lceil \frac{n}{2} \rceil - \lceil \frac{d-2}{2} \rceil$ vertices are necessary and sufficient to illuminate the exterior of $C(n, d)$, if d is even.

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EÖTVÖS LORÁND TUDOMÁNYEGYETEM
TERMÉSZETTUDOMÁNYI KAR
GEOMETRIA TANSZÉK
RÁKÓCZI ÚT 5
H–1088 BUDAPEST
HUNGARY

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COMMUTATIVITY OF RINGS WITH CONSTRAINTS INVOLVING POTENT AND NILPOTENT ELEMENTS

M. HASANALI* and A. YAQUB

A well-known theorem of Jacobson [3; p. 217] asserts that a ring R with the property that for each x in R there exists an integer $n(x)>1$ such that $x^{n(x)}=x$ is necessarily commutative. Our objective is to generalize Jacobson's Theorem by imposing some *weaker* conditions which are implied by the above "$x^{n(x)}=x$" hypothesis and which force R to be commutative. Let P be the set of all elements x in R such that $x^{n(x)}=x$ for some $n(x)>1$. Such elements are called *potent*. Our main result is the following

THEOREM 1. *Suppose R is a ring with center C and N is the set of nilpotent elements of R. Let*

$$P=\{x \mid x\in R,\ x^{n(x)}=x \text{ for some } n(x)>1\} \tag{1}$$

denote the set of potent elements of R. Suppose that:

(i) *Every x in $R\backslash C$ can be written (not necessarily uniquely) in the form $x=a+b$ for some a in N and some b in P.*

(ii) *$[a,b]=ab-ba$ is potent for all a in N and b in N.*

(iii) *$[xy,yx]$ is potent for all $x\in R\backslash N$, $y\in R\backslash N$.*

Then R is commutative (and conversely).

Observe that in Jacobson's Theorem (quoted above), every element of R is potent. Thus, $P=R$ and hence $N=\{0\}$. Therefore, Theorem 1 readily implies Jacobson's Theorem.

In preparation for the proof of Theorem 1, we first establish the following lemmas.

LEMMA 1. *Any semisimple ring R which satisfies hypotheses* (i) *and* (iii) *of Theorem 1 is commutative.*

PROOF. As is well known,

$$R\cong \text{a subdirect sum of rings } R_i, \text{ each } R_i \text{ primitive } (i\in\Gamma). \tag{2}$$

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Observe that each R_i inherits the hypotheses (i) and (iii) of Theorem 1. Now suppose, for the moment, that R_i is a division ring. Then, by hypothesis (i), for every x in R_i, $x \in C_i$ or $x \in P_i$, where C_i and P_i denote the center and the set of potent elements of R_i, respectively. Thus, $x - x^{n(x)} \in C_i$ for some $n(x) > 1$, and hence by Herstein's Theorem [2], R_i is commutative. Next, suppose R_i is a primitive ring which is *not* a division ring. Observe that hypothesis (iii) is inherited by every subring and by every homomorphic image of R_i, and hence by Jacobson's Density Theorem, for some $m \geqq 2$ and some division ring D, D_m satisfies hypothesis (iii). Let $x = E_{11} + E_{12} \in D_m$, $y = E_{11} \in D_m$. Then, by hypothesis (iii), $[xy, yx] = E_{12}$ is potent, contradiction. This contradiction shows that a primitive ring which satisfies hypotheses (iii), (i) must be a division ring and thus is commutative (as shown above). In view of (2), we see that R itself is commutative.

COROLLARY 1. *Let R be a ring which satisfies hypotheses* (i) *and* (iii) *of Theorem 1. Let $C(R)$ and J denote the commutator ideal and Jacobson radical of R, respectively. Then*

$$C(R) \subseteqq J. \tag{3}$$

LEMMA 2. *Suppose R is a ring which satisfies hypotheses* (i) *and* (iii) *of Theorem 1. Then,*

(a) $C(R) \subseteqq J \subseteqq N \cup C$;

(b) $N \subseteqq J$.

PROOF of (a). In view of (3), it suffices to show that

$$J \subseteqq N \cup C. \tag{4}$$

Suppose $x \in J$, $x \notin C$. Then, by hypothesis (i),

$$x = a + b, \ a \in N, \ b^m = b \ \text{ for some } m > 1. \tag{5}$$

Let $a^q = 0$ (since $a \in N$). Then, since $m \geqq 2$, $x \in J$, $a^q = 0$, we have

$$(x - a)^{(m-1)q+1} \in J. \tag{6}$$

Moreover, since $b^m = b$, by re-iterating we see that (see (5), (6))

$$b = b^m = b^{(m-1)q+1} = (x - a)^{(m-1)q+1} \in J,$$

and hence b^{m-1} is an *idempotent element of* J. Therefore, $b^{m-1} = 0$, and thus $b = b^m = 0$. Hence, by (5), $x = a \in N$, which proves (4) and completes the proof of (a).

PROOF of (b). Let $\bar{R} = R/C(R)$, $\bar{a} = a + C(R)$. Suppose $a \in N$. Then $a^k = 0$ for some positive integer k. Let $x \in R$. Since $\bar{R} = R/C(R)$ is commutative, therefore $\bar{a}\bar{x}$ is nilpotent in $\bar{R}$, say $(\bar{a}\bar{x})^m = \bar{0}$. Hence, $(ax)^m \in C(R) \subseteqq$

$N \cup C$ (by part (a)). Thus, $(ax)^m \in N$ or $(ax)^m \in C$. Now, if $(ax)^m \in C$, then

$$((ax)^m)^k = (ax)^m(ax)^m \cdots (ax)^m = a^k y \quad \text{for some } y \text{ in } R,$$

and hence $(ax)^{mk} = 0$ (since $a^k = 0$). Therefore, in any case, $ax \in N$, which implies ax is right-quasi-regular for all x in R, and hence $a \in J$. This proves (b).

LEMMA 3. *Suppose R is a ring which satisfies hypotheses* (i) *and* (iii) *of Theorem 1. Then,*

(a) *N is an ideal of R;*
(b) *For all $x \in R$, we have $x \in C$ or $x^n - x \in N$ for some integer $n > 1$;*
(c) *If $f: R \to R^*$ is a homomorphism of R onto R^*, then the set N^* of nilpotents of R^* is contained in $f(N) \cup C^*$, where C^* is the center of R^*.*

PROOF of (a). By Lemma 2, we have

$$N \subseteqq J \subseteqq N \cup C. \tag{7}$$

Suppose $a \in N$, $b \in N$. Then $a \in J$, $b \in J$, and hence $a - b \in J \subseteqq N \cup C$. Thus, $a - b \in N$ or $a - b \in C$. Now, if $a - b \in C$, then $ab = ba$ and hence $a - b \in N$ (since $a \in N$, $b \in N$). Thus, in any case, $a - b \in N$. Next, suppose $a \in N$, $x \in R$. Then $a \in J$, $x \in R$, and hence $ax \in J \subseteqq N \cup C$. Thus, $ax \in N$ or $ax \in C$. Now, if $ax \in C$, then by induction, $(ax)^k = a^k x^k$ for all positive integers k, which implies $ax \in N$ (since $a \in N$). Hence, in any case, $ax \in N$. Similarly, $a \in N$, $x \in R$ imply $xa \in N$, and thus N is an ideal of R.

PROOF of (b). Let $x \in R$, $x \notin C$. Then, by hypothesis (i),

$$x = a + b, \ a \in N, \ b^n = b, \ n > 1.$$

Hence,

$$x - a = b = b^n = (x - a)^n, \ a \in N, \ n > 1. \tag{8}$$

By part (a), N is an *ideal*, and since $a \in N$, therefore by (8), we see that

$$x^n - x \in N, \ n > 1.$$

PROOF of (c). Suppose $d^* \in N^*$, $d^* \notin C^*$. Then, $(d^*)^k = 0$ for some positive integer k. Let $d \in R$ be such that $f(d) = d^*$. Since $d^* \notin C^*$, therefore $d \notin C$, and hence by part (b),

$$d - d^q \in N \text{ for some integer } q > 1. \tag{9}$$

It is readily verified that

$$d - d^{k+1} d^{k(q-2)} = (d - d^q) + d^{q-1}(d - d^q) + \cdots + (d^{q-1})^{k-1}(d - d^q). \tag{10}$$

Combining (9) and (10), we conclude that

$$d - d^{k+1}d^{k(q-2)} \in N, \qquad (q \geqq 2),$$

and hence $f(d - d^{k+1}d^{k(q-2)}) \in f(N)$, which implies $d^* \in f(N)$, since $(d^*)^k = 0$. Thus, $d^* \in f(N) \cup C^*$ for all $d^* \in N^*$, which proves (c).

LEMMA 4. *Suppose R is a ring which satisfies hypothesis* (iii) *of Theorem 1. Then all idempotents of R are contained in the center C of R.*

PROOF. Let $e^2 = e \in R$, $x \in R$, and let $a = ex - exe$. We claim that $a = 0$. Suppose not. Let $f = e + a$. Then,

$$f^2 = f, \; ef = f, \; fe = e. \tag{11}$$

Moreover, since, by hypothesis, $a \neq 0$, therefore $e \neq 0$, $f \neq 0$, and hence by hypothesis (iii) of Theorem 1, $[fe, ef]$ is potent; that is,

$$[fe, ef]^m = [fe, ef] \quad \text{for some integer } m > 1. \tag{12}$$

It is readily verified that (11) and (12) imply $a = 0$ (since $m > 1$), contradiction. This contradiction shows that $a = 0$, and hence $ex = exe$. Similarly, $xe = exe$.

LEMMA 5. *Suppose that R is a ring with center C and N is the set of nilpotent elements of R. Suppose that* (a) *N is commutative;* (b) *for all $a \in N$ and $b \in R$, $ab - ba$ commutes with b;* (c) *for all $b \in R$, we have $b \in C$ or $b^n - b \in N$ for some $n = n(b) > 1$. Then R is commutative.*

This lemma was proved in [1].

We are now in a position to prove Theorem 1, which is the main result of this paper.

PROOF of Theorem 1. By Lemma 3 (a), N is an ideal of R, and hence

$$ab - ba \in N \quad \text{for all } a \in N, \quad b \in N. \tag{13}$$

Moreover, by hypothesis (ii), $ab - ba$ is *potent*, and thus $ab - ba = 0$ (since $ab - ba$ is also *nilpotent*, by (13)). Thus,

(14) The set N of nilpotents of R is *commutative.*

As is well known,

(15) $R \cong$ a subdirect sum of subdirectly irreducible rings R_i $(i \in \Gamma)$.

We now distinguish two cases.

Case 1. R_i does not have an identity.

First, *observe that* R_i *satisfies both hypotheses* (i), (iii) *of Theorem 1 and all the conclusions of Lemma 3.* Let $x_i \in R_i$, $x_i \notin C_i$ (center of R_i), and let $f(x) = x_i$, $x \in R$, where $f: R \to R_i$ is the natural homomorphism of R onto R_i (see (15)). Then $x \notin C$, and thus by Lemma 3 (b), $x^n - x \in N$ for some integer $n > 1$. Hence $x^m = x^{m+1}g(x)$ for some positive integer m and some $g(\lambda) \in \mathbb{Z}[\lambda]$. This implies, by re-iterating, that

$$x^m = x^m(xg(x))^m = x^m e; \quad e = (xg(x))^m \text{ is } \textit{idempotent.}$$

Combining this with Lemma 4, we conclude that

$$x^m = x^m e, \quad e = e^2, \quad e \textit{ central.}$$

Letting $x_i = f(x)$, $e_i = f(e)$, we see that this implies, in R_i,

$$x_i^m = x_i^m e_i, \quad e_i = e_i^2, \quad e_i \text{ central.} \tag{16}$$

Since e_i is a central idempotent in the subdirectly irreducible ring R_i, and since R_i does not have an identity in the present case, therefore $e_i = 0$, and hence by (16), $x_i^m = 0$; that is, x_i is nilpotent in R_i. We have thus shown that

$$R_i = N_i \cup C_i; \quad N_i = \{\text{nilpotents of } R_i\}; \quad C_i = \text{Center of } R_i. \tag{17}$$

Moreover, by Lemma 3 (c), $N_i \subseteq f(N) \cup C_i$. Also, by (14), $f(N)$ is commutative, and hence N_i is *commutative.* Combining this with (17), we conclude that R_i is commutative.

Case 2. R_i has an identity 1_i.

Recall that R_i *satisfies hypotheses* (i), (iii) *of Theorem 1 and all the conclusions of Lemma 3.* Recall also that N_i and C_i denote the set of nilpotents and the center of R_i, respectively. Let $a_i \in N_i$, $b_i \in R_i \backslash N_i$. Then, $1_i + a_i \notin N_i$, $b_i \notin N_i$, and hence by hypothesis (iii)

$$[(1_i + a_i)b_i, b_i(1_i + a_i)] \quad \text{is potent,}$$

which implies

$$[b_i, b_i a_i] + [a_i b_i, b_i] + [a_i b_i, b_i a_i] \text{ is potent.} \tag{18}$$

As we saw in the proof in Case 1, N_i is *commutative*, and N_i is an *ideal* of R_i, too (Lemma 3, (a)). Therefore, the commutator $[a_i b_i, b_i a_i]$ in (18) is zero, and hence (18) now reduces to

$$[b_i, b_i a_i] + [a_i b_i, b_i] \text{ is potent,}$$

which is equivalent to

$$[a_i b_i - b_i a_i, b_i] \text{ is potent, } (a_i \in N_i, b_i \notin N_i). \tag{19}$$

Since N_i is commutative, (19) is trivially satisfied if $b_i \in N_i$, and hence

$$(20) \qquad [a_ib_i - b_ia_i, b_i] \text{ is } \textit{potent} \text{ for all } a_i \in N_i, b_i \in R_i.$$

Note that, since N_i is an *ideal* of R_i, therefore the *potent* commutator in (20) is also *nilpotent*, and hence it must be zero. Thus,

$$(21) \qquad a_ib_i - b_ia_i \text{ commutes with } b_i \text{ for all } a_i \in N_i, b_i \in R_i.$$

Combining our results, we see that R_i satisfies all the hypotheses of Lemma 5 (see (21), Lemma 3 (b), and the above proof that N_i is *commutative*), and hence by Lemma 5, R_i is commutative. This completes the proof of Theorem 1.

As remarked in the introduction, Theorem 1 generalizes Jacobson's "$x^{n(x)} = x$" theorem. We conclude with the following

COROLLARY 2. *Let R be a ring such that for every $x \in R \backslash C, x = a + b$ for some $a \in N, b \in P$. Suppose, further, that the nilpotents of R commute and, moreover, $[xy, yx] = 0$ for all x, y in R. Then R is commutative.*

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DEPARTMENT OF MATHEMATICS
AL-QUDS UNIVERSITY
P.O. BOX 20002
JERUSALEM
ABU-DEIS
ISRAEL

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CA 93106
U.S.A.

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REFINEMENT OF A CARLEMAN-TYPE INEQUALITY

H. ALZER

Abstract

We prove: If $k \geqq 0, p \geqq 1$ and $0 < x_m \leqq 1$ $(m = 1, 2, \dots)$, then

$$\sum_{m=1}^{\infty} m^k \left[\prod_{n=1}^{m} x_n{}^{n^p-(n-1)^p} \right]^{(k+1)/m^p} \leqq e^{\psi(p)+\gamma+1/p} \sum_{m=1}^{\infty} m^k x_m,$$

where ψ is the logarithmic derivative of the gamma function and γ is Euler's constant. The above inequality sharpens a result of E. R. Love.

In 1923 T. Carleman [5] proved the following inequality. If x_m $(m = 1, 2, \dots)$ are non-negative real numbers, then

$$\sum_{m=1}^{\infty} \left[\prod_{n=1}^{m} x_n \right]^{1/m} \leqq e \sum_{m=1}^{\infty} x_m. \tag{1}$$

The constant e is best possible.

Inequality (1) has evoked the interest of several mathematicians and many papers have been published providing new proofs, sharpenings and extensions of Carleman's theorem; see [1], [2], [4], [6], [8-11], [14-17].

In 1984 J. A. Cochran and C.-S. Lee [7] established the following companion of inequality (1). If $k \geqq 0$, $p \geqq 1$ and $0 \leqq x_m \leqq 1$ $(m = 1, 2, \dots)$, then

$$\sum_{m=1}^{\infty} m^k \left[\prod_{n=1}^{m} x_n{}^{pn^{p-1}} \right]^{1/m^p} \leqq e^{(k+1)/p} \sum_{m=1}^{\infty} m^k x_m. \tag{2}$$

The constant $e^{(k+1)/p}$ is best possible.

We note that the special case $k = 0$, $p = 1$ yields Carleman's inequality with the restricted assumption that $0 \leqq x_m \leqq 1$. In 1991 E. R. Love [13]

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established that in (2) the expression $\left[\prod_{n=1}^{m} x_n{}^{pn^{p-1}}\right]^{1/m^p}$ can be replaced by the weighted geometric mean of the numbers x_n $(n=1,\dots,m)$ with weights $n^p-(n-1)^p$ $(n=1,\dots,m)$. More precisely Love proved: If $k \geqq 0$, $p>0$ and $x_m \geqq 0$ $(m=1,2,\dots)$, then

$$\sum_{m=1}^{\infty} m^k \left[\prod_{n=1}^{m} x_n{}^{n^p-(n-1)^p}\right]^{1/m^p} \leqq e^{(k+1)/p} \sum_{m=1}^{\infty} m^k x_m. \tag{3}$$

The constant $e^{(k+1)/p}$ is best possible.

If we assume that $p \geqq 1$ and $0 \leqq x_n \leqq 1$, then we have

$$n^p-(n-1)^p \leqq pn^{p-1}$$

and

$$x_n{}^{pn^{p-1}} \leqq x_n{}^{n^p-(n-1)^p},$$

so that (3) provides a sharpening of inequality (2).

The subject of this paper is the following inequality, proved by Love [12] in 1986, which is closely related to (3).

PROPOSITION. *If $k \geqq 0$, $p \geqq 1$ and $0 < x_m \leqq 1$ $(m=1,2\dots)$, then*

$$\sum_{m=1}^{\infty} m^k \left[\prod_{n=1}^{m} x_n{}^{pn^{p-1}}\right]^{(k+1)/m^p} \leqq e^{\psi(p)+\gamma+1/p} \sum_{m=1}^{\infty} m^k x_m, \tag{4}$$

where ψ is the logarithmic derivative of the gamma function and γ is Euler's constant. If (x_m) is decreasing, then in (4) $\psi(p)+\gamma$ can be replaced by 0.

In view of the fact that inequality (3) refines (2), it is natural to ask whether (4) remains valid if on the left-hand side of (4) the exponent pn^{p-1} will be replaced by $n^p-(n-1)^p$. It is our aim to show that this is indeed true.

First we formulate a lemma, due to Love [12], which we need to establish our main result.

LEMMA. *Let α be a non-negative and integrable function on $(0,1)$ with integral non-zero, and let $\bar{\alpha}$ be a decreasing rearrangement of α on $(0,1)$. Further, let*

$$\lambda_n > 0, \qquad \Lambda_m = \sum_{n=1}^{m} \lambda_n, \qquad w_{mn} = \int_{\Lambda_{n-1}/\Lambda_m}^{\Lambda_n/\Lambda_m} \alpha(t)\,dt.$$

If (x_m) $(m=1,2,\dots)$ *is a positive sequence, then*

$$\begin{aligned}&\sum_{m=1}^{\infty}\lambda_m\exp\left(\sum_{n=1}^{m}w_{mn}\log(x_n)\Big/\sum_{n=1}^{m}w_{mn}\right)\leqq\\ &\exp\left(\int_0^1\overline{\alpha}(t)\log(1/t)dt\Big/\int_0^1\overline{\alpha}(t)dt\right)\sum_{m=1}^{\infty}\lambda_m x_m.\end{aligned}\tag{5}$$

If (x_m) *is decreasing, then in* (5) $\overline{\alpha}$ *can be replaced by* α.

We are now in a position to prove the following refinement of inequality (4).

THEOREM. *If* k,p *and* x_m $(m=1,2,\dots)$ *are real numbers with* $k\geqq 0$, $p\geqq 1$ *and* $0<x_m\leqq 1$ $(m=1,2,\dots)$, *then*

$$\sum_{m=1}^{\infty}m^k\left[\prod_{n=1}^{m}x_n^{n^p-(n-1)^p}\right]^{(k+1)/m^p}\leqq e^{\psi(p)+\gamma+1/p}\sum_{m=1}^{\infty}m^k x_m,\tag{6}$$

where $\psi=\Gamma'/\Gamma$ *is the logarithmic derivative of the gamma function and* γ *is Euler's constant. If* (x_m) *is decreasing, then* $\psi(p)+\gamma$ *can be replaced by* 0.

PROOF. We follow the method of proof given in [12]. We set

$$\alpha(t)=t^{p-1},\qquad \lambda_n=n^k,\qquad \Lambda_m=\sum_{n=1}^{m}n^k,$$

$$w_{mn}=\int_{\Lambda_{n-1}/\Lambda_m}^{\Lambda_n/\Lambda_m}t^{p-1}dt=\frac{1}{p}\left[(\Lambda_n/\Lambda_m)^p-(\Lambda_{n-1}/\Lambda_m)^p\right].$$

Then we have

$$\sum_{n=1}^{m}w_{mn}=\frac{1}{p},$$

$$\int_0^1\alpha(t)\log(1/t)dt=-\int_0^1 t^{p-1}\log(t)dt=1/p^2,$$

and, since $\overline{\alpha}(t)=(1-t)^{p-1}$, we get

$$\begin{aligned}\int_0^1 \overline{\alpha}(t)\log(1/t)dt &= -\int_0^1 x^{p-1}\log(1-x)dx \\ &= \int_0^1 \sum_{i=1}^{\infty} \frac{1}{i} x^{i+p-1}dx \\ &= \sum_{i=1}^{\infty} \frac{1}{i(i+p)} \\ &= \frac{1}{p}\left[\psi(p)+\gamma+\frac{1}{p}\right].\end{aligned}$$

Applying the Lemma we obtain

$$\sum_{m=1}^{\infty} m^k \exp\left(p\sum_{n=1}^{m} w_{mn}\log(x_n)\right) \leqq \exp(\psi(p)+\gamma+1/p)\sum_{m=1}^{\infty} m^k x_m,$$

where the factor $\exp(\psi(p)+\gamma+1/p)$ can be replaced by $\exp(1/p)$, if (x_m) is decreasing.

To prove inequality (6) it remains to show that

$$\frac{k+1}{m^p}\left[n^p-(n-1)^p\right]\log(x_n) \leqq p w_{mn}\log(x_n) \qquad (1 \leqq n \leqq m).$$

Since $0 < x_n \leqq 1$, the last inequality is equivalent to

$$(7) \qquad (\Lambda_n)^p-(\Lambda_{n-1})^p \leqq (\Lambda_m/m)^p(k+1)\left[n^p-(n-1)^p\right] \quad (1 \leqq n \leqq m).$$

From

$$\begin{aligned}\Lambda_{n+1}-\frac{n+1}{n}\Lambda_n &= (n+1)^k-\frac{1}{n}\sum_{i=1}^{n} i^k \\ &\geqq (n+1)^k-n^k \geqq 0,\end{aligned}$$

we conclude that (Λ_n/n) is increasing . Hence, it suffices to prove that (7) holds for $m=n$.

Next we make use of the inequality

$$(8) \qquad \Lambda_{n-1}/\Lambda_n \geqq ((n-1)/n)^{k+1}.$$

(A proof for (8) is given in [3].) From (8) we conclude

$$(9) \qquad \begin{aligned}1-(\Lambda_{n-1}/\Lambda_n)^p &\leqq 1-((n-1)/n)^{p(k+1)} \\ &\leqq (k+1)\left[1-((n-1)/n)^p\right],\end{aligned}$$

where the second inequality in (9) is an immediate consequence of the simple inequality

$$(k+1)x \leqq k + x^{k+1} \qquad (0 \leqq x \leqq 1)$$

with $x = ((n-1)/n)^p$. Multiplying (9) by $(\Lambda_n)^p$ we obtain (7) with $m = n$. This completes the proof of the Theorem.

REMARK. It remains an open problem to determine the best possible constant in inequality (6). We only know that for $k = 0$ the best possible constant is given by $e^{1/p}$. This follows from Love's result that the optimal constant in (3) is $e^{(k+1)/p}$, even if $0 \leqq x_m \leqq 1$ for all $m \geqq 1$.

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MORSBACHER STR. 10
D–51545 WALDBRÖL
GERMANY

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ON EXISTENCE AND UNIQUENESS CONDITIONS FOR ORDINARY DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

G. HERZOG

Abstract

We will establish existence and uniqueness conditions for ordinary differential equations in Fréchet spaces which are formulated as Lipschitz and one-sided Lipschitz conditions with the aid of a generalized distance and row-finite matrices.

1. Introduction

Let F be a real or complex Fréchet space, $\emptyset \neq D \subseteqq F$, $f\colon [0,T]\times D \to F$ a function and $u_0 \in F$. We consider the initial value problem

$$u'(t) = f(t, u(t)), \quad u(0) = u_0, \tag{1}$$

and we say that (1) has at most one solution if for every $\tau \in (0,T]$ there is at most one function in $C^1([0,\tau],F)$ solving (1). We will establish conditions on f such that (1) has at most one solution using a generalized distance in the sense of Schröder [12]. We will further prove an existence and uniqueness theorem in the case $D = F$.

2. Polynorms and row-finite matrices

Let $\mathbb{R}^{\mathbb{N}}$ be the Fréchet space of all real sequences endowed with the product topology. A continuous mapping $\|\cdot\|\colon F \to [0,\infty)^{\mathbb{N}}$ is called a *polynorm* on F if $\{\|\cdot\|_k : k \in \mathbb{N}\}$ is a separating family of seminorms inducing the Fréchet space topology of F. Inequalities between elements of $\mathbb{R}^{\mathbb{N}}$ are intended componentwise. For $x, y \in F$ and $\lambda \in \mathbb{R}$, resp. $\lambda \in \mathbb{C}$ we have $\|x\| \geqq 0$, $\|x\| = 0 \Leftrightarrow x = 0$, $\|\lambda x\| = |\lambda|\,\|x\|$ and $\|x+y\| \leqq \|x\| + \|y\|$. Following the description in [10], we define $m_+\colon F\times F \to \mathbb{R}^{\mathbb{N}}$ by

$$m_+[x,y] = \lim_{h\to 0+} \frac{1}{h}(\|x+hy\| - \|x\|).$$

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The existence of this limit as well as the following properties are consequences of the properties of convex functions on linear spaces, see [10], p. 36. For $x, y, z \in F$ we have, for example,

$$m_+[x,x] = \|x\|, \qquad m_+[x,y] \leqq \|y\|,$$
$$m_+[x,y+z] \leqq m_+[x,y] + m_+[x,z],$$
$$|m_+[x,y] - m_+[x,z]| \leqq \|y-z\|.$$

If $u\colon [0,\tau) \to F$ is differentiable from the right-hand side, then $\|u\|$ is differentiable from the right-hand side, and $\|u\|'_+(t) = m_+[u(t), u'_+(t)], t \in [0,\tau)$.

A matrix $L = (l_{ij})_{i,j\in\mathbb{N}}$, $l_{ij} \in \mathbb{C}$, is called *row-finite* if $\{j \in \mathbb{N} : l_{ij} \neq 0\}$ is a finite set for every $i \in \mathbb{N}$. The row-finite matrices are exactly the continuous endomorphisms of the Fréchet space $(\mathbb{C}^{\mathbb{N}}, \|\cdot\|), \|(|x_k|)_{k=1}^{\infty}\| = (|x_k|)_{k=1}^{\infty}$. We call L *monotone* if all entries are real nonnegative numbers, and we call L *quasimonotone* (in the sense of Volkmann [14]) if $l_{ij} \in \mathbb{R}$, $i,j \in \mathbb{N}$, and $l_{ij} \geqq 0$, $i \neq j$.

Now let $\emptyset \neq D \subseteqq F$ and $f\colon [0,T] \times D \to F$ be a function. We consider the following conditions on f:

(L) There is a monotone matrix L such that

$$\|f(t,x) - f(t,y)\| \leqq L\,\|x-y\|, \quad (t,x),(t,y) \in [0,T] \times D.$$

(L_+) There is a quasimonotone matrix L such that

$$m_+[x-y, f(t,x) - f(t,y)] \leqq L\,\|x-y\|, \quad (t,x),(t,y) \in [0,T] \times D.$$

Remark that (L) implies (L_+), since $m_+[x,y] \leqq \|y\|$, $x,y \in F$. In spite of the fact that (L) and (L_+) look like classical uniqueness and existence conditions for the initial value problem (1), they in general imply neither uniqueness nor existence without further assumptions. For example, see [4], [5], [9] and [11].

3. Existence and uniqueness theorems

Let L be a row-finite matrix. The spectrum $\sigma(L) = \{\lambda \in \mathbb{C} : L - \lambda I$ is not invertible$\}$ is either at most countable or has an at most countable complement $\varrho(L) = \mathbb{C} \setminus \sigma(L)$. For this and other properties of row-finite matrices, see [4], [5], [6], [13] and [15].

According to Lemmert [8], the following theorem holds.

THEOREM 1. *Let $f\colon [0,T] \times F \to F$ be a function.*

a) *If f is satisfying* (L_+) *and* $\sigma(\mathrm{L})$ *is at most countable, then* (1) *has at most one solution.*

b) *If f is continuous and is satisfying* (L) *and $\sigma(L)$ is at most countable, then* (1) *is uniquely solvable on* $[0,T]$.

We will show that we can omit the condition that $\sigma(L)$ is at most countable if we have a priori bounds of a certain kind for possible solutions of (1)

Let L be a monotone row-finite matrix, and let $b\in[0,\infty)^{\mathbb{N}}$. We consider the following growth condition:

$$(G(L,b))\qquad \text{There exist } \alpha\in[0,\infty)^{\mathbb{N}} \text{ and } \beta\geqq 0 \text{ such that} \quad L^n b\leqq \beta^n n^n \alpha,\quad n\in\mathbb{N}.$$

REMARKS. 1) If $\sigma(L)$ is at most countable, then $(G(L,b))$ holds for every $b\in[0,\infty)^{\mathbb{N}}$, cf. [6], [15].

2) If $G(L,b)$ holds, then $G(\lambda L,\mu b)$ holds for every $\lambda,\mu\in[0,\infty)$.

THEOREM 2. *Let $\emptyset\neq D\subseteqq F$ and $f\colon [0,T]\times D\to F$ be a function satisfying (L_+) with $L=A-B$ such that A is monotone and B is a monotone diagonal matrix. For $\tau\in(0,T]$ let $x_1,x_2\in C^1([0,\tau],F)$ be solutions of* (1). *If $\|x_1(t)-x_2(t)\|\leqq b, t\in[0,\tau]$, for some $b\in[0,\infty)^{\mathbb{N}}$, then $x_1=x_2$ provided that $G(A,b)$ or $G(B^{-1}A^2,b)$ holds.*

THEOREM 3. *Let $f\colon [0,T]\times F\to F$ be continuous, satisfying* (L) *and*

$$\|f(t,x)-f(t,y)\|\leqq b,\quad (t,x),(t,y)\in[0,T]\times F,$$

for some $b\in[0,\infty)^{\mathbb{N}}$ such that $G(L,b)$ holds. Then (1) *is uniquely solvable on* $[0,T]$.

Remark that, if $f(t,x)=g(t,x)+h(t), (t,x)\in[0,T]\times F, h\in C([0,T],F)$ and $\|g(t,x)\|\leqq b, (t,x)\in[0,T]\times F$, then (1) is uniquely solvable if g satisfies (L) and $G(L,b)$ holds.

To prove Theorem 2 we first need the following proposition.

PROPOSITION 1. *Let $L=A-B$ be a row-finite matrix with A and B as in Theorem 2. Let $v\colon [0,\tau]\to[0,\infty)^{\mathbb{N}}$ be continuous and differentiable from the right-hand side on $[0,\tau)$ satisfying $v'_+(t)\leqq Lv(t), t\in[0,\tau), v(0)=0$. If $v(t)\leqq b, t\in[0,\tau]$ for some $b\in[0,\infty)^{\mathbb{N}}$, then $v=0$ provided that $G(A,b)$ or $G(B^{-1}A^2,b)$ holds.*

PROOF. a) Let $G(A,b)$ be true. We assume $v\neq 0$ and $t_0:=\inf\{t\in[0,\tau): v(t)\neq 0\}$. Since $0\leqq v\leqq b$, it holds that

$$v'_+(t)\leqq Lv(t)\leqq Av(t)\leqq Ab,\quad t\in[t_0,\tau).$$

Since $v(t_0)=0$, we have $v(t)\leqq(t-t_0)Ab, t\in[t_0,\tau]$ which implies

$$v'_+(t)\leqq Av(t)\leqq(t-t_0)A^2 b,\qquad t\in[t_0,\tau).$$

Therefore, $v(t) \leqq \frac{1}{2}(t-t_0)^2 A^2 b, t \in [t_0, \tau]$. Successive application of this step leads to

$$v(t) \leqq \frac{1}{n!}(t-t_0)^n A^n b \leqq \frac{n^n \beta^n (t-t_0)^n}{n!}\alpha, \quad t \in [t_0, \tau], \quad n \in \mathbb{N}.$$

According to Stirling's formula

$$\lim_{n\to\infty} \frac{n^n \beta^n (t-t_0)^n}{n!} = 0 \quad \text{for all } t \in (t_0, \tau] \text{ with } (t-t_0)\beta < \frac{1}{e},$$

this contradicts the definition of t_0.

b) Let B be invertible and $G(B^{-1}A^2, b)$ be true. We assume $v \neq 0$ and set t_0 as in part a). We get $v(t) \leqq (t-t_0)Ab$, $t \in [t_0, \tau]$ as well as in part a). Therefore,

$$v'_+(t) + Bv(t) \leqq (t-t_0)A^2 b, \qquad t \in [t_0, \tau).$$

Since B is a diagonal matrix, e^{tB} exists and is a monotone matrix for every $t \in \mathbb{R}$. Hence

$$v(t) \leqq \int_{t_0}^{t} (s-t_0)e^{B(s-t)} A^2 b\,ds \leqq (t-t_0)B^{-1}A^2 b, \qquad t \in [t_0, \tau],$$

according to the inequality

$$\int_{t_0}^{t} (s-t_0)^n e^{\gamma(s-t)} ds \leqq \frac{(t-t_0)^n}{\gamma}, \quad \gamma > 0, \quad t \geqq t_0, \quad n \in \mathbb{N}_0.$$

Therefore, $v(t) \leqq \frac{1}{2}(t-t_0)^2 AB^{-1}A^2 b, t \in [t_0, \tau]$, which implies

$$v(t) \leqq \int_{t_0}^{t} \frac{1}{2}(s-t_0)^2 e^{B(s-t)} A^2 B^{-1} A^2 b\,ds \leqq \frac{1}{2}(t-t_0)^2 (B^{-1}A^2)^2 b,$$

$t \in [t_0, \tau]$. By induction we get

$$v(t) \leqq \frac{1}{n!}(t-t_0)^n (B^{-1}A^2)^n b \leqq \frac{n^n \beta^n (t-t_0)^n}{n!}\alpha, \qquad t \in [t_0, \tau], \quad n \in \mathbb{N},$$

which leads to a contradiction as in part a). □

PROOF OF THEOREM 2. According to the results in Section 2, we have

$$\begin{aligned} \|x_1 - x_2\|'_+(t) &= m_+[x_1(t) - x_2(t), x'_1(t) - x'_2(t)] \\ &= m_+[x_1(t) - x_2(t), f(t, x_1(t)) - f(t, x_2(t))] \\ &\leqq L\|x_1(t) - x_2(t)\|, \end{aligned}$$

$t \in [0, \tau)$. Application of Proposition 1 completes the proof. □

We now will prove Theorem 3. Since (L) implies (L_+), the presuppositions of Theorem 3 imply, according to Theorem 2, that (1) has at most one solution. So we will prove that (1) has a solution on $[0, T]$.

PROOF OF THEOREM 3. We consider the Fréchet space $(C([0,T],F), |||\cdot|||)$ with

$$|||u||| := \left(\max_{t\in[0,T]} \|u(t)\|_k \right)_{k=1}^{\infty}.$$

We further assume $\beta T < e^{-1}$. The result for arbitrary T follows by continuation of the solution.

Starting from $v_0(t) = u_0, t \in [0, T]$, we consider the sequence of successive approximations

$$v_{n+1}(t) = u_0 + \int_0^t f(s, v_n(s))ds, \quad n \geqq 0.$$

As usual, it holds that

$$\|v_{n+1}(t) - v_n(t)\| \leqq \frac{T^{n-1}}{(n-1)!} L^{n-1} \|v_2(t) - v_1(t)\|, \qquad t \in [0, T], \quad n \geqq 1$$

and

$$\|v_2(t) - v_1(t)\| \leqq \int_0^t \|f(s, v_1(s)) - f(s, v_0(s))\| ds \leqq Tb, \qquad t \in [0, T].$$

Therefore,

$$|||v_{n+1} - v_n||| \leqq \frac{T^n}{(n-1)!} L^{n-1} b, \qquad n \geqq 1.$$

Since $G(L, b)$ holds, we have

$$|||v_{n+1} - v_n||| \leqq T \frac{(\beta T)^{n-1}(n-1)^{n-1}}{(n-1)!} \alpha, \qquad n \geqq 2.$$

Now, $\beta T < e^{-1}$ implies the convergence of $\sum \dfrac{(\beta T)^n n^n}{n!}$, according to Stirling's formula. Therefore $(v_n)_{n=0}^{\infty}$ is convergent in $C([0, T], F)$ and $u := \lim\limits_{n\to\infty} v_n$ is a solution of (1) on $[0, T]$, since

$$\left\| u(t) - u_0 - \int_0^t f(s, u(s))ds \right\| \leqq \|u(t) - v_{n+1}(t)\| + \left\| \int_0^t f(s, v_n(s)) - f(s, u(s))ds \right\|$$
$$\leqq |||u - v_{n+1}||| + TL|||u - v_n|||,$$

$t \in [0, T]$, $n \geqq 0$. □

4. Examples

1) Let $(\gamma_k)_{k=1}^{\infty}$ be a sequence of positive numbers. We consider the infinite system

$$\begin{cases} u_k'(t) = \gamma_k \arctan u_{k+1}(t) \\ u_k(0) = 0 \end{cases}, \quad k \in \mathbb{N}, \quad t \in [0,T]. \tag{2}$$

With $f\colon [0,T] \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, $f(t,x) = (\gamma_k \arctan x_{k+1})_{k=1}^{\infty}$, (2) is an initial value problem of the form (1) in $(\mathbb{R}^{\mathbb{N}}, \|\cdot\|)$, $\|(x_k)_{k=1}^{\infty}\| = (|x_k|)_{k=1}^{\infty}$, with $u_0 = 0$.

Of course, $u(t) = 0$, $t \in [0,T]$, is a solution of (2), but it is in general not the only one, although f is satisfying (L) with

$$L = \begin{pmatrix} 0 & \gamma_1 & 0 & 0 & \cdots \\ 0 & 0 & \gamma_2 & 0 & \cdots \\ 0 & 0 & 0 & \gamma_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Remark that $\sigma(L) = \mathbb{C}$. Let

$$v_1(t) := \begin{cases} e^{-1/t}, & t \in (0,T], \\ 0, & t = 0, \end{cases}$$

and define recursively

$$v_{k+1}(t) = \tan \frac{v_k'(t)}{\gamma_k}, \qquad t \in [0,T],$$

with $\gamma_k > 0$ such that

$$\frac{v_k'(t)}{\gamma_k} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad t \in [0,T].$$

For such a sequence $(\gamma_k)_{k=1}^{\infty}$ we have $u = (v_k)_{k=1}^{\infty}$ as a solution of (2). Now consider (2) for $(\gamma_k)_{k=1}^{\infty} = (k)_{k=1}^{\infty}$. We have

$$\|f(t,x)\| \leqq \frac{\pi}{2}(k)_{k=1}^{\infty}, \quad (t,x) \in [0,T] \times \mathbb{R}^{\mathbb{N}}$$

and therefore

$$L^n b = \frac{\pi}{2}\left(\prod_{j=k}^{k+n} j\right)_{k=1}^{\infty} = \frac{\pi}{2}\left(\frac{(k+n)!}{(k-1)!}\right)_{k=1}^{\infty}, \qquad n \geqq 1.$$

Since $\displaystyle\lim_{n\to\infty} \frac{(k+n)!}{n^n} = 0$ for every $k \in \mathbb{N}$, there is a sequence $\alpha \in [0,\infty)^{\mathbb{N}}$ such that $L^n b \leqq \alpha n^n$, $n \in \mathbb{N}$, and, according to Theorem 3, the only solution of

(2) is $u=0$. Remark that, according to Theorem 3, also the initial value problem

$$\begin{cases} u_k'(t) = k \arctan u_{k+1}(t) \\ u_k(0) = u_{k0} \end{cases}, \quad k \in \mathbb{N}, \quad t \in [0,T],$$

is uniquely solvable for every $u_0 = (u_{k0})_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$.

2) Let $(\gamma_k)_{k=1}^{\infty}$ be a sequence with $\gamma_k \geqq 1, k \in \mathbb{N}$, and consider the initial value problem (1) with

$$f\colon [0,T] \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}, \qquad f(t,x) = (-\gamma_k \gamma_{k+1} x_k + \gamma_k \arctan x_{k+1})_{k=1}^{\infty}$$

and $u_0 = 0$.

Again $u=0$ is a solution of (1), and we claim that it is the only one, no matter how fast $(\gamma_k)_{k=1}^{\infty}$ is growing:

Let $\tau \in (0,T]$ and $v \in C^1([0,\tau], \mathbb{R}^{\mathbb{N}})$ be a solution of (1). Since

$$-\gamma_k \gamma_{k+1} v_k(t) - \frac{\pi}{2}\gamma_k \leqq v_k'(t) \leqq -\gamma_k \gamma_{k+1} v_k(t) + \frac{\pi}{2}\gamma_k,$$

$t \in [0,\tau], k \in \mathbb{N}$, we have

$$\|v(t)\| \leqq \frac{\pi}{2}\left(\frac{1}{\gamma_{k+1}}\right)_{k=1}^{\infty} \leqq b := \frac{\pi}{2}(1)_{k=1}^{\infty}, \quad t \in [0,\tau].$$

Now f is satisfying (L_+) with

$$L = \begin{pmatrix} -\gamma_1\gamma_2 & \gamma_1 & 0 & 0 & \dots \\ 0 & -\gamma_2\gamma_3 & \gamma_2 & 0 & \dots \\ 0 & 0 & -\gamma_3\gamma_4 & \gamma_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Choosing

$$A = \begin{pmatrix} 0 & \gamma_1 & 0 & 0 & \dots \\ 0 & 0 & \gamma_2 & 0 & \dots \\ 0 & 0 & 0 & \gamma_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and $B = A - L$, we find

$$B^{-1}A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

and $(B^{-1}A^2)^n b = b$, $n \geqq 1$. According to Theorem 2, we have $v = 0$.

3) Let F be the Fréchet space $C(\mathbb{R},\mathbb{R})$,

$$\|x\| = \left(\max_{s\in[-k,k]} |x(s)| \right)_{k=1}^{\infty}$$

and $K\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ be a continuous function with

(a) $$\max_{s\in[-k,k]} |K(s,x,\nu) - K(s,y,\nu)| \leqq \gamma_k |x-y|,$$

$$(x,\nu),(y,\nu) \in \mathbb{R}\times[0,1],$$

$k \in \mathbb{N}$, for a sequence $(\gamma_k)_{k=1}^{\infty}$ of nonnegative numbers and

(b) $$b_k := \sup_{(s,x,\nu)\in[-k,k]\times\mathbb{R}\times[0,1]} |K(s,x,\nu)| < \infty, \quad k \in \mathbb{N}.$$

We consider the initial value problem (1) with $f\colon [0,1]\times F \to F$ defined by

$$(f(t,x))(s) = \int_0^t K(s, x(s+\nu), \nu)\,d\nu,$$

$$(s,t) \in \mathbb{R}\times[0,1] \text{ and } u_0 \in F.$$

The function f is continuous and is satisfying (L) with

$$L = \begin{pmatrix} 0 & \gamma_1 & 0 & 0 & \dots \\ 0 & 0 & \gamma_2 & 0 & \dots \\ 0 & 0 & 0 & \gamma_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and $\|f(t,x)\| \leqq b := (b_k)_{k=1}^{\infty}$, $(t,x) \in [0,1]\times F$.

So if $G(L,b)$ holds, (1) is uniquely solvable on $[0,1]$, according to Theorem 3. Let, for example, $K(s,x,\nu) = s\sin(x+\nu)$. Then $b_k = k$, $k \in \mathbb{N}$ and $\gamma_k = k$, $k \in \mathbb{N}$, can be chosen, and $G(L,b)$ is true (cf. Example 1).

FINAL REMARK. We want to draw the reader's attention to some papers where related concepts are used to study differential equations in locally convex spaces: [1], [2], [3], [11].

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MATHEMATISCHES INSTITUT I
UNIVERSITÄT KARLSRUHE
ENGLERSTRASSE 2
POSTFACH 6980
D–76128 KARLSRUHE
GERMANY

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TOTAL-SEPARABLE KREISSYSTEME UND MOSAIKE IN DER HYPERBOLISCHEN EBENE

I. VERMES

Dem Andenken von Herrn Professor J. Strommer gewidmet

Eine Menge der kongruenten Bereiche in der Ebene wird nach P. Erdős separabel genannt (Vgl. [4], [5]), wenn eine Gerade existiert, die keine innere Punkte von Bereichen hat, und die Bereiche auf ihren beiden Seiten liegen. Die Definition der total-separablen Menge von Bereichen haben G. Fejes Tóth und L. Fejes Tóth in [2] gegeben. Eine Menge von Bereichen ist total-separabel genannt, wenn jede zwei Bereiche durch eine Gerade so separiert werden können, daß alle Bereichen damit auch separiert werden. G. Fejes Tóth [3] und G. Kertész [6] haben sich mit total-separablen Kreissystemen bzw. Kugelsystemen in der euklidischen Geometrie beschäftigt.

In dieser Arbeit wollen wir total-separable Mosaike und Kreissysteme der hyperbolischen Ebene — elementar wie möglich — konstruieren, und diesbezüglich extremale Eigenschaften in Sätzen 1–4 beschreiben.

Zunächst beweisen wir den folgenden

HILFSSATZ. *Falls die Geraden g_1 und g_2 eine gemeinsame Lotstrecke n haben, deren Größe $d>0$ beliebig vorgegeben ist, so existieren solche Punkte A und B auf g_1 bzw. g_2 (auf einer festen Seite von n), daß die Gerade AB mit g_1 bzw. g_2 die vorgegebenen Winkel $\alpha \leqq \frac{\pi}{2}$ bzw. $\beta < \frac{\pi}{2}$ einschließen (Abb. 1).*

BEWEIS. Es ist genügend nur eine Seite von der Geraden n betrachten. Bezeichne A^1 bzw. B^1 die Fußpunkte von g_1 bzw. g_2 auf n.

Falls $\alpha = \frac{\pi}{2}$ und $\beta < \frac{\pi}{2}$ bestehen, so wird A^1B^1BA ein Lambertsches Viereck sein, das durch d und β auf Grund der trigonometrischen Formel (siehe z. B. in [10] S. 76–82)

$$\cos\beta = \sinh\frac{d}{k}\sinh\frac{AA^1}{k}$$

eindeutig bestimmt ist.

Falls $\alpha < \frac{\pi}{2}$ und $\beta < \frac{\pi}{2}$ bestehen, so müssen die Geraden AB und n ein gemeinsames Lot TT^1 haben, wo die Fußpunkte T bzw. T^1 auf den

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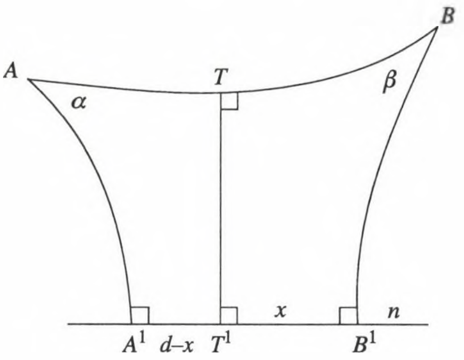

Abb. 1

Strecken AB bzw. A^1B^1 liegen. Bezeichne $x = T^1B^1$ und $d - x = A^1T^1$ die zwei Teile der Strecke $A^1B^1 = d$. Die Größe x kann durch α, β, d eindeutig mit der Eigenschaft $x < d$ bestimmt werden, so wird auch die Existenz der benachbarten Lambertschen Vierecken TT^1B^1B und TT^1A^1A bewiesen.

Sei die Strecke TT^1 mit m bezeichnet. Auf den Lambertschen Vierecken TT^1B^1B und TT^1A^1A gelten die trigonometrischen Formeln:

$$\cos\beta = \sinh\frac{m}{k}\sinh\frac{x}{k}$$

bzw.

$$\cos\alpha = \sinh\frac{m}{k}\sinh\frac{d-x}{k}.$$

Man verwendet die Additionsidentitäten für die hyperbolischen Funktionen, und bekommt die folgende Gleichung mit ihrer Abschätzung:

$$\tanh\frac{x}{k} = \frac{\cos\beta\sinh\frac{d}{k}}{\cos\alpha + \cos\beta\cosh\frac{d}{k}} < \tanh\frac{d}{k} < 1.$$

Daraus folgt, daß α, β, d die Größen x und $TT^1 = m$ eindeutig bestimmen, ferner gilt die Ungleichung $x < d$. Auf Grund dieses Hilfssatzes werden wir den folgenden Satz beweisen.

SATZ 1. *Wenn die Winkel* $\alpha_1 = \frac{\pi}{n_1}, \alpha_2 = \frac{\pi}{n_2}, \dots, \alpha_p = \frac{\pi}{n_p}$ *(mit* $p \geqq 3$, $n_i \geqq 2$ $i = 1, 2, \dots, p$*) gegeben sind und die Ungleichung* $\alpha_1 + \alpha_2 + \ldots + \alpha_p < (p-2)\pi$, *d.h.* $\sum_{i=1}^{p} \frac{1}{n_i} < p - 2$ *gilt, so existiert solches p-Eck in der hyperbolischen Ebene, dessen Winkel in einer festen zyklischen Reihenfolge* $\alpha_1, \alpha_2, \dots, \alpha_p$ *sind.*

BEWEIS. Ein Dreieck ($p = 3$) wird durch seine Winkel eindeutig bestimmt. Der Satz gilt für die Vierecke mit der Anwendung des Hilfssatzes:

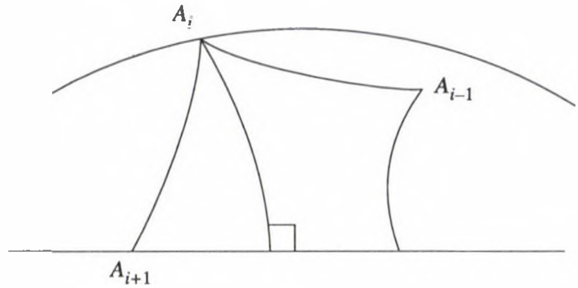

Abb. 2

man betrachtet zwei beliebige Geraden, die ein gemeinsames Lot haben, und die gegebenen Winkel können auf beiden Seiten des gemeinsamen Lotes so eingelagert werden, daß je zwei Ecken des Viereckes auf den zwei Geraden seien.

Die folgende Abschätzung zeigt, daß die Vielecke — im Falle $p > 4$ — bereits auch voll-rechteckige Vielecke sein können:

$$(p-2)\pi > p\frac{\pi}{2} \geqq \sum_{i=1}^{p} \frac{\pi}{n_i}$$

besteht wegen $n_i \geqq 2$, wenn $p > 4$ gilt.

Auf Grund dieser Bemerkung ist es leicht zu sehen, wie elementar ein Fünfeck gegebener Winkel aus einem mit ihm in vier oder drei Winkeln übereinstimmten Viereck konstruiert werden kann. Bezeichne A_i die Ecke des geeigneten Viereckes, deren Abstand von seinen Seiten maximal ist. Betrachte man das auf diese Seite gefällte Lot durch A_i, das das Viereck in zwei starre Teile zerlegt. Zu dieser Seite — als Grundlinie — gehört eine Abstandslinie durch den Punkt A_i (Abb. 2).

Entferne man diese zwei Teile voneinander entlang der Grundlinie so, daß die Tangenten der Abstandslinie in den verschobenen Punkten A_i' bzw. A_i'' zueinander parallel seien (Abb. 3). In dieser Lage haben die Halbgeraden $A_{i+1}'A_i'$ und $A_{i-1}''A_i''$ ein gemeinsames Lot, und die vierte bzw. fünfte Ecke können auf dieser Halbgeraden nach dem Hilfssatz eingelagert werden.

Zur vollen Allgemeinheit ($p > 5$) kann der Beweis auf zwei Wegen fortgesetzt werden.

I. Der Beweis führt durch eine vollständige Induktion im Bezug der Eckenzahlen der Vielecke. Der Induktionsschritt geht ähnlicher Weise wie vorher von vier auf fünf.

II. Falls $p > 5$ ist, so unterscheidet man zwei Fällen: 1) p ist eine gerade Zahl, 2) p ist eine ungerade Zahl.

Im Falle 1) zieht man $\frac{p}{2}$ Halbgeraden aus einem Punkt O so, daß das durch ihren Enden bestimmte $\frac{p}{2}$-Eck ein asymptotisches Vieleck ist, das den

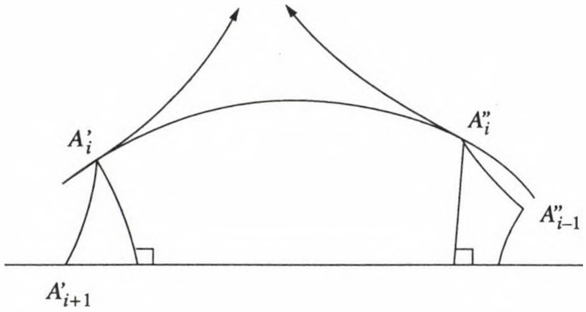

Abb. 3

Punkt O im Inneren enthält. Fällen wir die Lote aus dem Punkt O auf die Seitengeraden, und tragen wir je eine (z.B. gleiche), willkürliche Strecke auswärts auf diesen Loten von den Fußpunkten auf. Verschieben wir die Seitengeraden entlang der Lote zu diesen Punkten, so haben die verschobenen benachbarten Seitengeraden je ein gemeinsames Lot. Jetzt können wir den Hilfssatz zu jeden Geradenpaaren verwenden und so die gewünschten Winkel $\alpha_1, \alpha_2, \dots, \alpha_p$ an den Ecken des p-Eckes gewinnen.

Im Falle 2) zieht man zunächst $\frac{p-1}{2}+1$ Halbgeraden aus einem Punkt O, daß die $\frac{p-1}{2}$ Enden ein asymptotisches Vieleck bestimmen, das den Punkt O im Inneren enthält. Nehmen wir ein solches $\frac{p-1}{2}$-Eck, ferner liege das $(\frac{p-1}{2}+1)$-te Ende bezüglich einer Seitengeraden in der entgegengesetzten Halbebene, wie der Punkt O. In dieser Halbebene, mit der Seitengeraden als Grundlinie, läuft eine Abstandslinie vom Abstand $\Delta(\frac{\alpha_1}{2})$ (d.h. vom Parallellot zum Parallelwinkel $\frac{\alpha_1}{2}$), die die zur betrachteten Ende gehörige Halbgerade in einem Punkt P_1 schneidet. Die aus P_1 gezogenen, zur Grundlinie parallelen Halbgeraden und die gebliebenen und nach 1) verschobenen $(\frac{p-1}{2}-1)$ Seitengeraden haben — in der nacheinanderfolgenden Reihe — je ein gemeinsames Lot, und dem Hilfssatz gemäß können die weiteren Winkel $\alpha_2, \alpha_3, \dots, \alpha_p$ eingelagert werden. Damit haben wir unseren Satz 1 vollständig bewiesen.

Bezeichne Π_p *ein oben konstruiertes Vieleck.* Es ist leicht zu sehen, daß jede Ecke des Vieleckes Π_p durch die nacheinanderen Spiegelungen an den in diesem Eckpunkt sich getroffenen Seiten lückenlos umgelegt werden kann. Die gespiegelten Exemplare bilden einen Gürtel um das ursprüngliche Vieleck Π_p, daraus folgt der Satz 2 auf Grund des Alexandrow–Poincaréschen Satzes (vgl. [7] und [8]):

SATZ 2. *In der hyperbolischen Ebene können die Mosaike aus der kongruenten Exemplaren eines Vieleckes* Π_p *— durch die Seitenspiegelungen — aufgebaut werden.*

Wenn wir ein solches Mosaik untersuchen, so können wir leicht feststellen, daß jede Seite eines Elements des Mosaiks auf ihrer Geraden durch die Seite eines benachbarten Elements fortgesetzt werden kann. Diese zueinander sich verknüpfenden Seiten bilden je eine vollständige Gerade im Mosaik. So haben wir ein total-separables Mosaik nach einer diskreten Spiegelungsgruppe konstruiert.

Es ist leicht zu sehen, daß je ein Inkreis zu jedem inneren Punkt eines Vieleckes Π_p gehört, denn die Lote aus einem willkürlichen inneren Punkt zu den Seiten laufen im Inneren vom Vieleck Π_p. Die minimale Entfernung kann als der zu diesem Punkt gehörige maximale Inkreisradius betrachtet werden. Folglich gilt der folgende

SATZ 3. *Die durch Seitenspiegelungen aus Π_p aufgebauten Mosaike sind total-separabel. Die entsprechenden gespiegelten Inkreise um einen beliebigen inneren Punkt der Mosaikelemente bilden ein total-separables Kreissystem kongruenter Kreise in der hyperbolischen Ebene.*

Ein solches Vieleck Π_p, mit gegebenen, in einer bestimmten zyklischen Reihenfolge geordneten Winkeln kann sehr vielfältige, einander nicht kongruente Formen annehmen. Denn wir haben bei der vorigen Konstruktion gewisse Freiheitsparameter, deren Anzahl mit $p > 3$ (in linearer Ordnung) zunimmt. Nach einem Lemma von Makarow (vgl. [1] und [9]) gilt das folgende: Wenn ein Vieleck (p-Eck) mit den Winkeln $\alpha_1, \alpha_2, \ldots, \alpha_p$ in der hyperbolischen Ebene gibt, so existiert immer ein solches p-Eck, dessen Seiten einen Kreis berühren. Es ist leicht zu sehen, daß der eindeutige Radius dieses Kreises — unter den Inkreisradien solches p-Eckes — maximal ist. Da der Flächeninhalt eines solchen Vieleckes

$$I = k^2 \left\{ (p-2)\pi - \sum_{i=1}^{p} \alpha_i \right\}$$

konstant ist, folglich gilt der

SATZ 4. *Unter den separablen Mosaiken, die zur obigen Klasse der Spiegelbilder eines Vielecks Π_p mit den Winkeln $\alpha_1, \alpha_2, \ldots, \alpha_p$ gehören, hat das Mosaik die maximale Dichte, bei dem Π_p um einen Kreis umgeschrieben ist.*

Wir nennen diese zu den dichtesten Inkreispackungen gehörigen Mosaike *Makarowsche Mosaike.*

Man sieht sofort, daß noch vielfaltige inkongruente Makarow-Mosaikelemente zu denselben Winkelgrößen $\alpha_1, \alpha_2, \ldots, \alpha_p$ — abhängig von ihren verschiedenen zyklischen Reihenfolgen — gehören.

BEMERKUNG. Zu jedem Makarowschen Mosaik, dessen Elemente p-Ecke mit den Winkeln $\frac{\pi}{n_1}, \frac{\pi}{n_2}, \ldots, \frac{\pi}{n_p}$ sind, gehört je ein duales, Archimedisches Mosaik mit regulären $2n_1$-, $2n_2$-, $\ldots, 2n_p$-Ecken. Die Eckpunkte des dualen Mosaiks stimmen mit den Inkreismittelpunkten des ursprünglichen Mosaiks überein.

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BUDAPESTI MŰSZAKI EGYETEM
GÉPÉSZMÉRNÖKI KAR
GEOMETRIA TANSZÉK
EGRY JÓZSF U. 1
H–1521 BUDAPEST
HUNGARY

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ZWANGLÄUFIG BEWEGLICHE POLYEDERMODELLE. II

O. RÖSCHEL

Abstract

In this paper there is shown how to construct overconstrained mechanisms with systems linked by spherical 2R-links. Given a tetrahedron with faces tangent to a common sphere we cut the vertices of this polyhedron with planes tangent to the sphere. In the faces of this new polyhedron we define plane equiform euclidean motions with common parametrization and common time-depending scaling factor $f(t)$. The motions in different faces are linked by spherical links. 'Blowing up' the tetrahedron with factor $1/f(t)$ then gives an overconstrained kinematic chain consisting of 8 systems linked by spherical 2R-links. It has to be remarked that this procedure may be used to gain a very great variety of overconstrained mechanisms: the given algorithm just has to work on other polyhedra with faces tangent to a common sphere. Further parts of this paper will show that fact.

In [1] ist es gelungen, aus einem Würfel durch Eckenstutzen ein zwangläufig bewegliches Polyedermodell mit sphärischen Doppelscharnieren zu konstruieren. Die eckenstutzenden Ebenen mußten dabei Tangentialebenen der Inkugel des Würfels sein. Daß dahinter ein allgemeiner Sachverhalt verborgen ist, soll in dieser Arbeit gezeigt werden. Die so gefundenen Resultate werden dann zur Konstruktion von zwangläufig beweglichen (überbestimmten) Polyedermodellen verwendet, die aus (nicht notwendig regulären) Tetraedern hervorgehen. Als Spezialfall stellt sich dabei das bekannte Modell des HEUREKA-Polyeders ein.

1. Wir studieren vorerst ebene äquiforme Zwangläufe $\xi := \varepsilon/\varepsilon^*$ einer Gangebene ε gegenüber einer fest gedachten Rastebene ε^* mit folgenden Eigenschaften: ξ besitze einen globalen Fixpunkt $A^* \in \varepsilon^*$ (bzw. $A \in \varepsilon$) und führe einen gangfesten Punkt $P \neq A$ auf einer A^* nicht enthaltenden Bahngeraden $b^*(P)$. Wenn wir wie üblich komplexe Zahlen zur Beschreibung dieser Zwangläufe verwenden, empfiehlt es sich, in ε und ε^* kartesische Normalkoordinatensysteme $\{0^* = A^*; x^*, y^*\}$ und $\{0 = A; x, y\}$ so einzuführen, daß P in ε den Einheitspunkt der x-Achse bezeichnet, und die beiden Koordinatensysteme für den Ausgangszeitpunkt bei ξ zur Deckung gelangen. Eine Parametrisierung von ξ ist dann etwa durch

$$(1) \qquad \xi: \quad \vec{z} := x + iy \to \vec{z}^*(t, \vec{z}) := \vec{z}(1 + it) \quad (t \in R)$$

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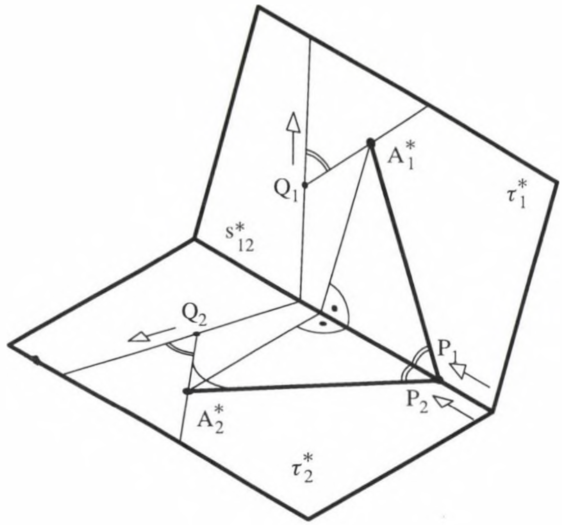

Abb. 1

gegeben. Dabei wurde vorausgesetzt, daß sich der Punkt P für $t = 0$ auf seiner Bahngeraden $b^*(P)$ gerade in jenem dem Zentrum $A^* = 0^*$ nächsten Punkt befindet. Aus (1) ist ersichtlich, daß der so definierte Zwanglauf ξ alle Punkte $X \in (\varepsilon - A)$ auf Bahngeraden $b^*(X)$ führt. Wir wollen diesen Zwanglauf ξ daher als *linearen ebenen äquiformen Zwanglauf mit globalem Fixpunkt* $A^*(A)$ ansprechen.

2. Nun geben wir im euklidischen Dreiraum eine feste Kugel κ^* (mit Mittelpunkt M^*) sowie zwei *nichtparallele Kugeltangentialebenen* τ_1^*, τ_2^* mit Berührpunkten A_1^*, A_2^* vor. Die Schnittgerade $\tau_1^* \cap \tau_2^*$ bezeichnen wir mit s_{12}^* und parametrisieren sie mit einem Parameter $t \in R$ (vgl. Abbildung 1 — die Kugel κ^* ist nicht eingetragen).

Wir denken uns die beiden Ebenen τ_1^*, τ_2^* als Rastebenen bzw. Gangebenen τ_1, τ_2 doppelt ausgeführt und definieren lineare ebene äquiforme Zwangläufe $\xi_i := \tau_i/\tau_i^*$ $(i = 1, 2)$ mit globalen Fixpunkten A_i^* durch Vorgabe der parametrisierten Bahngeradan s_{12}^* für zwei gangfeste Punkte $P_2 \in \tau_2$, $P_1 \in \tau_1$, die sich für alle $t \in R$ an *derselben Stelle auf* s_{12}^* *befinden sollen*. Da es bekanntlich eine Drehung um die Achse s_{12}^* gibt, die (τ_1^*, A_1^*) mit (τ_2^*, A_2^*) zur Deckung bringt, *sind die beiden äquiformen Zwangläufe* ξ_1 *und* ξ_2 *euklidisch kongruent* und über die *"Schleppbahnen"* der Punkte P_1 und P_2 sogar kongruent parametrisiert (vgl. Abbildung 1 — dort sind auch die Bahngeraden je eines weiteren Punktes eingetragen).

In Abbildung 1 erkennen wir sehr schön, *daß sich* ξ_1 *und* ξ_2 *zu jedem Zeitpunkt* t *durch Spiegelung an der Symmetrieebene* $\sigma_{12}^* := [s_{12}^*, M^*]$ *zur Deckung bringen lassen*.

3. Hilfsüberlegungen. Für die weiteren Abschnitte benötigen wir fol-

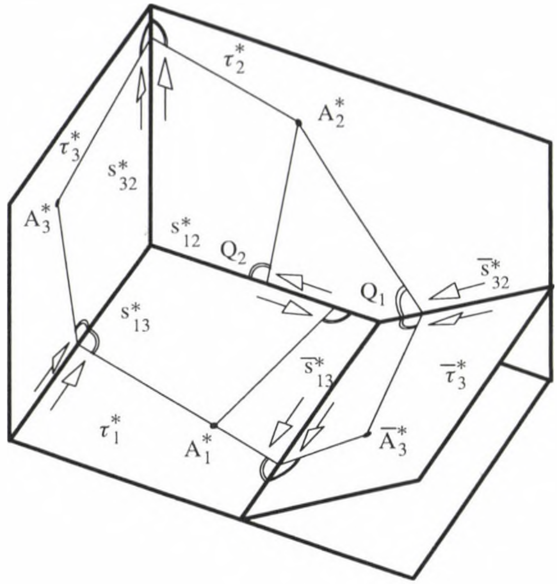

Abb. 2

gende *Überlegungen:*

A. Wir betrachten 3 Tangentialebenen $\tau_1^*, \tau_2^*, \tau_3^*$ der Kugel κ^* mit Berührpunkten $A_i^*(i=1,2,3)$, wobei τ_3^* weder zu τ_1^* noch τ_2^* parallel sei (vgl. Abbildung 2). Gebén wir nun in τ_1^* einen linearen äquiformen Zwanglauf ζ_1 mit globalem Fixpunkt A_1^* vor, so läßt sich dieser nach Abschnitt 2 über die Schleppbahn $s_{13}^* := \tau_1^* \cap \tau_3^*$ eines geeigneten gangfesten Punktes kongruent in die Ebene τ_3^* übertragen — der entstehende Zwanglauf werde mit ζ_3 bezeichnet. Über die Schleppbahn $s_{32}^* := \tau_3^* \cap \tau_2^*$ kann in τ_2^* analog ein zu ζ_1 kongruenter linearer äquiformen Zwanglauf ζ_2 definiert werden. Dabei ist auf drei Besonderheiten hinzuweisen:

1) Aus Abbildung 2 ist ersichtlich, daß *diese Koppelung der Zwangläufe über die Schleppbahnen zwischen den kongruenten Zwangläufen* ζ_1 *und* ζ_2 *nicht möglich* ist: ζ_1 führe den gangfesten Punkt Q_1 auf der Schnittgeraden $s_{12}^* = s_{21}^*$. Auch im Gangsystem des Zwanglaufes ζ_2 gibt es einen Punkt Q_2, der auf dieser Schnittgeraden geführt wird. Allerdings wird diese Gerade von Q_1 und Q_2 in verschiedener Richtung durchlaufen, was die oben erwähnte Koppelung unmöglich macht (vgl. die Pfeile in Abbildung 2).

2) Überraschend ist der so in τ_2^* definierte Zwanglauf ζ_2 *nicht von der Wahl der zur Übertragung verwendeten Kugeltangentialebene* τ_3^* *abhängig!* Dies sieht man mit Abbildung 2 wie folgt ein: Wählen wir statt τ_3^* zur Übertragung eine andere (weder zu τ_1^* noch zu τ_2^* parallele) Kugeltangentialebene $\bar{\tau}_3^*$ mit Berührpunkt $\bar{A}_3^*$. Der dann über die Schleppbahnen und

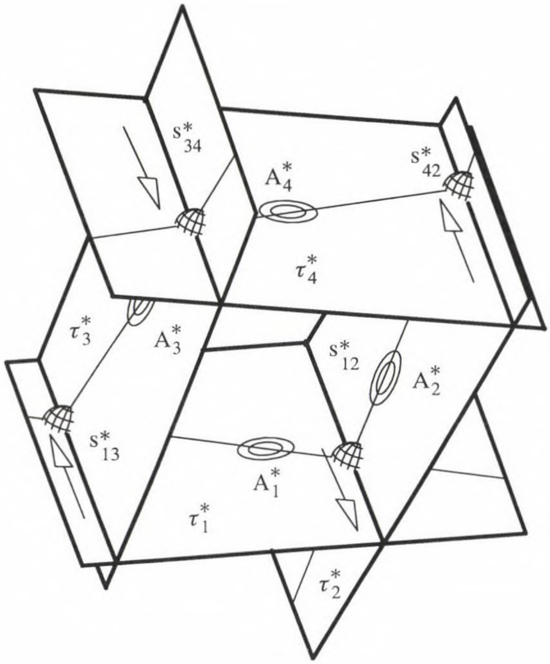

Abb. 3

$\bar{s}^*_{13} := \tau^*_1 \cap \bar{\tau}^*_3$ und $\bar{s}^*_{32} := \bar{\tau}^*_3 \cap \tau^*_2$ in τ^*_2 definierte Zwanglauf $\bar{\zeta}_2$ ist zu ζ_2 kongruent und besitzt denselben globalen Fixpunkt. Auch die Parametrisierungen sind dieselben. Da nach Abbildung 3 auch der Drehsinn von $\bar{\zeta}_2$ mit jenem von ζ_2 übereinstimmt, sind ζ_2 und $\bar{\zeta}_2$ sogar identisch.

3) Wir haben in Abschnitt 2 festgestellt, daß sich die Zwangläufe ζ_1 und ζ_3 bzw. ζ_3 und ζ_2 jeweils durch Spiegelung an den Ebenen $\sigma^*_{13} := [s^*_{13}, M^*]$ bzw. $\sigma^*_{32} := [s^*_{32}, M^*]$ ineinander überführen lassen. *Der Übergang von ζ_1 zu ζ_2 kann daher i.a. als Drehung um die Schnittgerade dieser beiden Symmetrieebenen gewonnen werden.*

B. Gegeben seien zwei verschiedene Tangentialebenen τ^*_1, τ^*_2 einer Kugel κ^* (Mitte M^*) mit Berührpunkten $A^*_i (i = 1, 2)$. Nun versuchen wir, einen in τ^*_1 vorgelegten linearen äquiformen Zwanglauf ζ_1 mit globalem Fixpunkt A^*_1 *über zwei Zwischenglieder* in Tangentialebenen τ^*_3, τ^*_4 der Kugel κ^* (Berührpunkte A^*_3, A^*_4) wie in Abbildung 3 in die Ebene τ^*_2 zu übertragen. Im folgenden sollen die Schnittgeraden $s^*_{13} := \tau^*_1 \cap \tau^*_3$, $s^*_{34} := \tau^*_3 \cap \tau^*_4$ und $s^*_{42} := \tau^*_4 \cap \tau^*_2$ eigentliche Geraden sein. Die Übertragung von ζ_1 in die Ebenen τ^*_3, τ^*_4 und τ^*_2 soll sukzessiv über die Schleppbahnen s^*_{13}, s^*_{34} und s^*_{42} erfolgen. Die so in τ^*_3, τ^*_4 bzw. τ^*_2 induzierten linearen äquiformen Zwangläufe ζ_3, ζ_4 bzw. ζ_2 besitzen die globalen Fixpunkte A^*_3, A^*_4 bzw. A^*_2 und sind alle zu ζ_1 kongruent. Wir haben unter **A** bemerkt, daß der in τ^*_4 induzierte Zwanglauf ζ_4 unabhängig von der Wahl der Übertragungsebene τ_3^* ist (solange wir eine Kugeltangentialebene verwenden). Daher dürfen wir o.B.d.A. τ_3^* parallel zur Schnittgerade $s_{12}^* := \tau_1^* \cap \tau_2^*$ wählen — sollten τ_1^* und τ_2^* zueinander parallel liegen, verändern wir die Lage von τ_3^* nicht. Analog verfahren wir für die Ebene τ_4^*, sodaß schließlich die Schnittgeraden s_{13}^*, s_{34}^* und s_{42}^* sowie s_{12}^* (falls eigentlich) zueinander parallel liegen. Diese Situation ist in Abbildung 3 dargestellt. Insgesamt gilt dann nach Abschnitt 2: Der Zwanglauf ζ_2 entsteht aus ζ_1 durch fortgesetzte Spiegelung an den Ebenen $\sigma_{13}^* := [s_{13}^*, M^*]$, $\sigma_{34}^* := [s_{34}^*, M^*]$ und $\sigma_{42}^* := [s_{42}^*, M^*]$. Die Spiegelungsebenen gehören einem Büschel an; daher ist die Zusammensetzung dieser drei Spiegelungen insgesamt eine Spiegelung an einer weiteren Ebene σ^* dieses Büschels. Falls τ_1^* und τ_2^* nicht zueinander parallel sind, gilt $\sigma^* = \sigma_{12}^* := [s_{12}^*, M^*]$, anderfalls ist σ^* Mittenebene der parallelen Ebenen τ_1^* und τ_2^*. Abschnitt 2 lehrt dann, daß *im Fall nichtparalleler Ebenen* τ_1^* *und* τ_2^* *der Zwanglauf* ζ_2 *auch über die Schleppbahn auf der Schnittgeraden* s_{12}^* *direkt aus* ζ_1 *gewonnen werden kann* (vgl. Abbildung 3). Sind dagegen τ_1^* und τ_2^* parallel, so geht ζ_2 durch Schiebung aus ζ_1 hervor — Schiebvektor ist der Vektor $\overrightarrow{A_1^* A_2^*}$.

C. Zuletzt betrachten wir 4 Tangentialebenen $\tau_1^*, \tau_2^*, \tau_3^*$ und τ_5^* der Kugel κ^* mit Berührpunkten $A_i^* (i = 1, 2, 3, 5)$. Dabei sei vorausgesetzt, daß die Tangentialebene τ_5^* mit den Ebenen $\tau_1^*, \tau_2^*, \tau_3^*$ *eigentliche Schnittgeraden* $s_{15}^* = s_{51}^*$, $s_{25}^* = s_{52}^*, s_{35}^* = s_{53}^*$ besitzt (vgl. Abbildung 4). Wie unter **A** definieren wir in einer dieser Ebenen (etwa in τ_1^*) einen linearen äquiformen Zwanglauf ζ_1 mit globalem Fixpunkt A_1^* durch Vorgabe der geeignet parametrisierten Geraden $s_{15}^* = s_{51}^*$ als Bahn eines gangfesten Punktes P_{15}. Wieder wird dieser Zwanglauf über die *"Schleppbahn"* auf der Schnittgeraden $s_{15}^* = s_{51}^*$ kongruent in die Ebene τ_5^* übertragen. Über in der Gangebene dieses Zwanglaufes ζ_5 feste Punkte P_{52} und P_{53}, deren Bahnen als Schleppbahnen nach $s_{25}^* = s_{52}^*$ bzw. $s_{35}^* = s_{53}^*$ fallen (die Punkte sind mit Hilfe des Peripheriewinkelsatzes leicht zu ermitteln), definieren wir in den Ebenen τ_2^* und τ_3^* Zwangläufe ζ_2 und ζ_3 mit globalen Fixpunkten A_2^* und A_3^*. So wie für **A** läßt sich wieder nachweisen, daß statt der Kugeltangentialebene τ_5^* jede beliebige andere Kugeltangentialebene zur Übertragung verwendet werden könnte, ohne daß sich die resultierenden Zwangläufe ζ_2 und ζ_3 ändern. Dabei ist bloß darauf zu achten, daß die neue Übertragungsebene zu τ_1^*, τ_2^* und τ_3^* nicht parallel ist.

4. Diese Sachverhalte ermöglichen es, in den 4 Seitenflächen $\tau_1^*, \tau_2^*, \tau_3^*, \tau_4^*$ eines (*nicht notwendig regulären!*) Tetraeders Δ^* mit Hilfe von 4 geeigneten Tangentialebenen $\tau_5^*, \tau_6^*, \tau_7^*, \tau_8^*$ der Inkugel κ^* von Δ^* (auch eine der Ankugeln könnte verwendet werden) kongruente lineare ebene äquiforme Zwangläufe $\zeta_i := \tau_i/\tau_i^*$ $(i = 1, \dots, 8)$ mit globalen Fixpunkten in den Berührpunkten A_i^* mit der Kugel κ zu definieren (vgl. Abbildung 5).

Die Tangentialebenen $\tau_5^*, \tau_6^*, \tau_7^*, \tau_8^*$ seien so gewählt, daß sie zum Stutzen

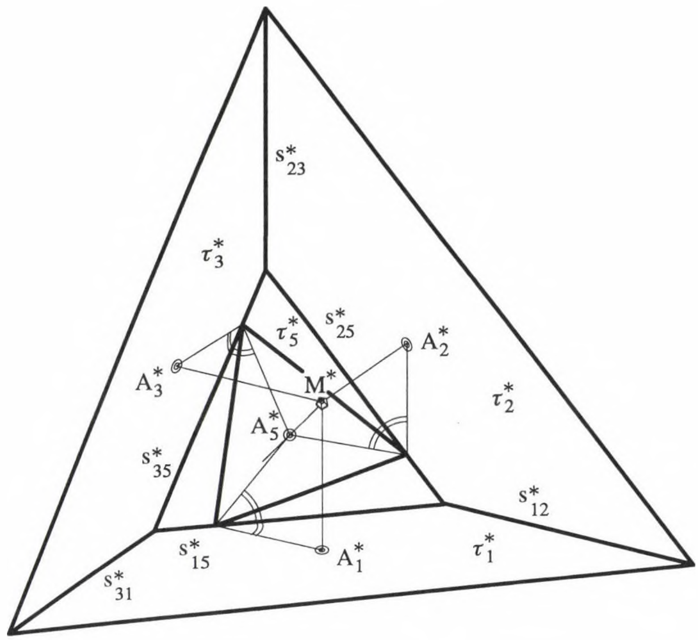

Abb. 4

je einer Ecke des Tetraeders verwendet werden können. Wir verfahren dann wie oben und erhalten eine sich schließende Konfiguration aus 8 kongruenten linearen äquiformen Zwangläufen, die dann durch Schleppbahnen gekoppelt sind, wenn sie in einer Tetraederebene und einer der benachbarten zum Stutzen verwendeten Tangentialebenen $\tau_5^*, \tau_6^*, \tau_7^*, \tau_8^*$ ablaufen. Von diesen 8 kongruenten äquiformen Zwangläufen ξ_i werden die Ecken jeweils gangfester Dreiecke auf den Schnittgeraden der entsprechenden Ebenen geführt, die zu jedem Zeitpunkt $\tau \in R$ mit einem Eckpunkt in einer Ecke eines Nachbardreieckes zu liegen kommen.

Abbildung 5 zeigt die Situation für jenen Zeitpunkt $\tau \in R$, für den alle Zwangläufe ξ_i $(i = 1, \dots, 8)$ gleichzeitig eine Momentandrehung besitzen. Dies wäre in (1) für $t = 0$ der Fall gewesen. Links oben ist das Ausgangstetraeder samt Inkugel unter der Annahme abgebildet, daß die vordere Tetraederfacette weggeschnitten ist. Rechts danaben ist ein durch Eckenstutzen entstehende Restkörper dargestellt. Links unten sind die Lote auf die Schnittkanten aus den Berührpunkten mit der Inkugel sowie die entstehenden Fußpunktedreiecke eingetragen. Die Figur rechts unten zeigt schließlich ein Rohmodell des dadurch bestimmten übergeschlossenen Mechanismus. Aus den Hilfsüberlegungen von oben ist klar, daß auf den ursprünglichen Tetraederkanten keine Punktkoppelung zwischen den in den benachbarten Tetraederseitenflächen ablaufenden äquiformen Zwangläufen möglich ist.

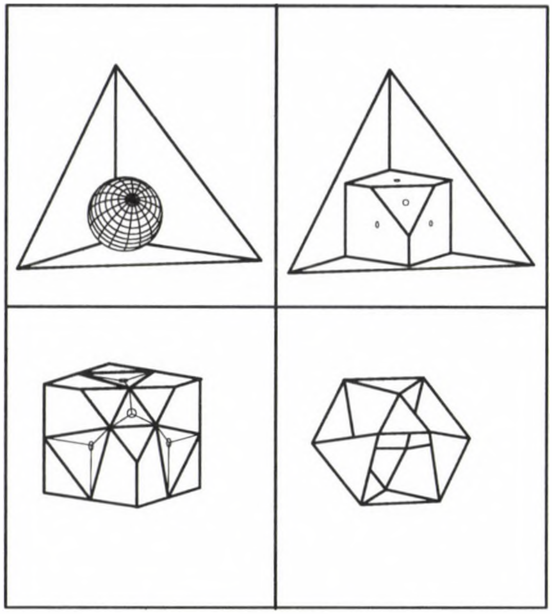

Abb. 5

Wird nun die gesamte Figur aus dem Mittelpunkt der Kugel κ^* in Abhängigkeit von $t \in R$ so gestreckt oder gestaucht, daß der Zwanglauf, den τ_1 gegenüber dem Gesamtraum vollführt, zu einem euklidischen wird, so gilt dies für alle unsere Teilzwangläufe (vgl. die Idee in [I, 15]). Unsere Figur aus 8 Dreiecken bildet dann einen in der Bewegungsgruppe des E_3 zumindest zwangläufigen Mechanismus aus 8 starren Dreiecken, die in den Ecken sphärisch miteinander gekoppelt sind. Da die Dreiecksebenen vor Ausüben dieser Streckungen im E_3 fixiert waren, halten sie nun festen Winkel zueinander und lassen sich daher in den Ecken sogar durch sphärische Doppelscharniere (sphärische 2R-Gelenke mit Drehachsen in den Normalen der Dreiecksebenen) koppeln. Der entstehende Mechanismus besteht daher aus 8 Dreiecken und 12 sphärischen 2R-Gelenken. Er besitzt den theoretischen Freiheitsgrad

(2) $$F = 7.6 - 12.4 = -6$$

und ist daher eine *übergeschlossene kinematische Kette.* Die so entstehenden Beispiele sind im Gegensatz zu den bislang bekannten (auch die in [1] beschriebenen besitzen ja als Grundstruktur noch die des Würfels) i.a. vollkommen frei von regulären Teilsystemen. Die 8 Dreiecke werden i.a. weder kongruent noch ähnlich sein. Auch die Konfiguration ihrer Trägerebenen ist bis auf ihre Kugeltangentialebeneneigenschaft nur dadurch eingeschränkt, daß noch Schnittdreiecke mit den Nachbartangentialebenen entstehen sollen.

Die so gefundene kinematische (übergeschlossene) Kette ließe sich mit Abbildung 5 durch folgenden Algorithmus **(A)** aus einem Tetraeder Δ^* (Ebenen $\tau_1^* \dots \tau_4^*$) herstellen:

A1) Bestimmung der Inkugel κ^* des Tetraeders Δ^* (theoretisch wäre auch die Verwendung einer der Ankugeln möglich) und ihrer Berührpunkte $A_1^* \dots A_4^*$.

A2) Eckenstutzen des Tetraeders mittels 4 Tangentialebenen $\tau_5^* \dots \tau_8^*$ der Kugel κ^* und Bestimmung der Berührpunkte $A_1^* \dots A_8^*$ mit κ^*.

A3) Konstruktion der Lotfußpunkte auf den neu entstandenen Schnittkanten ($\neq$ Tetraederkanten) aus den Berührpunkten $A_1^* \dots A_8^*$ und Ermittlung der 8 Fußpunktsdreiecke in den Ebenen $\tau_1^* \dots \tau_8^*$.

A4) Verbindung je zweier benachbarter Dreiecke durch sphärische 2R-Gelenke, wobei der Winkel der Scharnierachsen aus dem Winkel der Dreiecksebenen am eckengestutzten Tetraeder abgelesen werden kann.

Wir fassen zusammen in folgendem

Satz 1. *Wird ein allgemeines Tetraeder Δ^* mit Hilfe von 4 Tangentialebenen der Inkugel (bzw. einer Ankugel) von Δ^* eckengestutzt, so liefert oben beschriebene Konstruktionsvorschrift* (**A**) *einen zumindest zwangläufig beweglichen übergeschlossenen Mechanismus aus 8 starren Dreiecken und 12 sphärischen 2R-Gelenken.*

Bemerkungen. 1. Auch einige übergeschlossene Modelle aus der Arbeit [1] könnten so interpretiert werden, daß dieser allgemeine Algorithmus beim Eckenstutzen des Würfels W^* verwendet wird. Nach dem hier vorgestellten Sachverhalt ist klar, daß statt des Würfels W^* auch eine Konfiguration bestehend aus 6 Tangentialebenen einer gemeinsamen Kugel κ^* für das Ausgangsobjekt verwendet werden könnte. Zum Eckenstutzen lassen sich dann 8 weitere Tangentialebenen dieser Kugel κ^* heranziehen. Obiger Algorithmus führt in diesem Fall auf übergeschlossene Mechanismen aus 6 starren Vierecken (nicht notwendig kongruent und auch nicht notwendig in paarweise orthogonalen Ebenen!), 8 starren Dreiecken (ebenfalls nicht notwendig kongruent) und 24 sphärischen 2R-Gelenken. Da die Formenvielfalt hier kaum zu überblicken ist, sollen neben in [1] vorgestellten Beispiele vorerst keine neuen gestellt werden.

2. Ganz allgemein läßt sich der Algorithmus (**A**) immer dann einsetzen, wenn das Grundobjekt aus Facetten besteht, die eine gemeinsame In- oder Ankugel berühren, und hinsichtlich der Kanten- und Eckenfiguren die grobe Struktur eines regulären Polyeders besitzen.

3. Zusammenhang mit dem HEUREKA-Polyeder: Jedes reguläre Oktaeder kann als geeignet eckengestutztes reguläres Tetraeder Δ^* angesehen werden. Die Seitenflächen von Δ^* sind z.B. so aus den Oktaederseitenflächen

auszuwählen, daß sich je zwei nicht in einer Kante des entstehenden Oktaeders schneiden. Wird der Algorithmus (**A**) für diese Konfiguration nachvollzogen, so entsteht der Mechanismus des HEUREKA-Polyeders.

5. Aus der Fülle der nach Satz 1 konstruierbaren Modelle sei abschließend ein *bemerkenswertes Beispiel* vorgestellt, das überraschend auch als bewegliches Würfelmodell zu deuten wäre: Das Ausgangstetraeder Δ^* bestehe aus Ursprung und den Einheitspunkten auf den Achsen eines kartesischen Normalkoordinatensystems. Zum Eckenstutzen verwenden wir jene Tangentialebenen der Inkugel κ^* von Δ^*, die zu den Tetraederebenen parallel sind. Dieses eckengestutzte Tetraeder, das auch durch geeignetes Stutzen zweier gegenüberliegender Ecken eines Würfels entsteht, zeigt Abbildung 5. Der Restkörper werde mit P^* bezeichnet. Wir verfahren nach Algorithmus (**A**) und erhalten überraschend in allen Ebenen $\tau_i \quad (i = 1, \ldots, 8)$ *gleichseitige Fußpunktsdreiecke*. Dies ist wie folgt einzusehen:

a. Das in der geneigten Seitenebene τ_4^* mit der Gleichung $x^* + y^* + z^* = 1$ des Tetraeders Δ^* liegende Dreieck ist gleichseitig mit der Mitte im Berührpunkt A_4^* mit der Inkugel κ^*. Durch unser Eckenstutzen ist auch das am Restkörper P^* in τ_4^* entstehende Dreieck gleichseitig mit Mitte A_4^*. Schritt (A3) des Algorithmus liefert daher in dieser Tetraederseitenebene sicher ein gleichseitiges Dreieck. In der zu τ_4^* parallelen Tangentialebene τ_8^* erzeugt Schritt (A3) auf dem Restkörper P^* ein dazu kongruentes gleichseitiges Dreieck, da P^* wie erwähnt auch durch Eckenstutzen eines Würfels mit parallelen Tangentialebenen seiner Inkugel entsteht. Für die restlichen Facetten gehen wir analytisch vor:

b. Die anderen Facetten auf dem Restkörper P^* und ihre Berührpunkte A_1^*, A_2^*, A_3^*, A_5^*, A_6^*, A_7^* sind kongruent. Wir untersuchen o.B.d.A. das in der Ebene $\tau_1^* \ldots z^* = 0$ entstehende Dreieck: Die Kugel κ^* besitzt den Mittelpunkt $M^*(\frac{3-\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6})$, und die zu τ_4^* parallele Tangentialebene τ_8^* die Gleichung $x^* + y^* + z^* = 2 - \sqrt{3}$. Die Tangentialebenen τ_5^* und τ_6^* werden durch $y^* = \frac{3-\sqrt{3}}{3}$ bzw. $x^* = \frac{3-\sqrt{3}}{3}$ erfaßt. Die auf den Spuren von τ_8^*, τ_5^* und τ_6^* in der Ebene τ_1^* bezüglich des Kugelberührpunktes $A_1^*(\frac{3-\sqrt{3}}{3}, \frac{3-\sqrt{3}}{3}, 0)$ interessanten Lotfußpunkte besitzen der Reihe nach die Koordinaten $(\frac{3-\sqrt{3}}{6}, \frac{3-\sqrt{3}}{3}, 0), (\frac{3-\sqrt{3}}{3}, \frac{3-\sqrt{3}}{6}, 0)$ und $(\frac{2-\sqrt{3}}{2}, \frac{2-\sqrt{3}}{2}, 0)$. Unschwer läßt sich nachprüfen, daß diese Punkte in τ_1^* ein gleichseitiges Dreieck bestimmen. Wegen der Symmetrie unserer Aufstellung ist damit die Behauptung bewiesen.

Abbildung 6 zeigt eine Ansicht des so enstehenden Mechanismus, bei dem die Facetten als Prismen ausgebildet sind.

Bemerkung. Aus Überlegung **B** in Abschnitt 3 folgt, daß bei diesem speziellen Mechanismus *parallelen Facetten reine Schiebungen längs Geraden als Relativzwangläufe bestimmen*: Vor der beschriebenen Streckung aus dem Kugelmittelpunkt M^* laufen in diesen parallelen Ebenen ja schiebungsgleiche

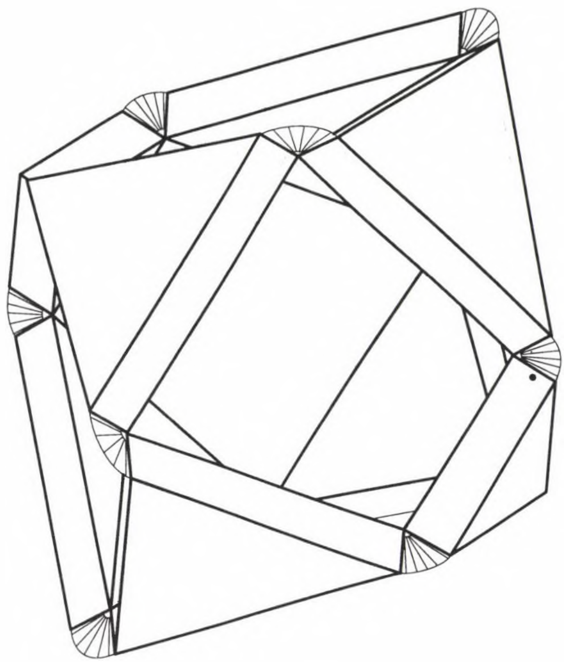

Abb. 6

lineare äquiforme Zwangläufe ab. Der Schiebvektor dieser Schiebung ist normal zu den Ebenen. Vom Zeitparameter t abhängige Streckungen aus M^* ändern bloß die Länge des Schiebvektors, nicht aber seine Richtung — der Relativzwanglauf ist damit als Schiebung längs Geraden erkannt.

6. Selbstverständlich lassen sich auch die Seiten der durch unser Eckenstutzen entstehenden neuen Dreiecke als starre Stäbe zwischen den Tetraederebenen τ_1^*, τ_2^*, τ_3^*, τ_4^* materialisieren, ohne daß dadurch die Beweglichkeit des Mechanismus gestört wird. Diese Stäbe müssen dann sphärisch an den Tetraederebenen angelenkt werden. Es ist bloß darauf zu achten, daß einerseits ausreichend Stäbe zur Verfügung gestellt werden und diese sich während des Bewegungsvorganges nicht stören. Es entstehen so aus dem Ausgangstetraeder zumindest zwangläufig bewegliche übergeschlossene Stabwerke. Dies wäre als Analogie zu den beweglichen Würfelstabwerken [1] anzusehen. Der dort gegebene hohe Grad an Symmetrie muß hier im allgemeinen nicht mehr auftreten. Beispiele für solche Stabwerke sollen in einer eigenen Arbeit dieser Serie angegeben werden.

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INSTITUT FÜR GEOMETRIE
TU GRAZ
KOPERNIKUSGASSE 24
A–8010 GRAZ
AUSTRIA

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HELLY-TYPE THEOREMS ON TRANSVERSALITY FOR SET-SYSTEM

S. REVENKO and V. SOLTAN

Abstract

There are studied some Helly-type theorems on set-system as generalizations of Helly-type results on the existence of a common supporting line for a family of disjoint convex bodies in the plane.

1. Introduction

R. Dawson (see [1]) has proved several Helly-type theorems on the existence of a common supporting line for a finite family of disjoint convex bodies in the plane. In what follows, we mean by a plane convex body any compact convex set in the plane, with interior points. Recall that a line l supports a plane convex body B provided l contains at least one boundary point of B and the whole B lies in a closed halfplane determined by l.

Following [1], we say that a family $\mathcal{F}$ of convex bodies in the plane has the (support) property S if there is a line l supporting every member of $\mathcal{F}$. Similarly, $\mathcal{F}$ has the property $S(k)$ if any subfamily of k members in $\mathcal{F}$ has the property S. Now the results from [1] can be summarized as follows. Let $\mathcal{F}$ be a finite family of n (pairwise) disjoint convex bodies in the plane. Then: 1) $S(5) \Rightarrow S$, 2) $S(4) \Rightarrow S$ if $n \geqq 7$, 3) $S(3) \Rightarrow S$ if $n \geqq 237$.

The methods of proofs in [1] are based on the considerations of some set-systems and on further investigations whether these systems can be realized geometrically. We develop here this method and complete some results from [1] on set-systems. In particular, we show that the number 237 in assertion 3) above can be lowered to 143.

2. Definitions and main results

Let $\mathcal{A}$ be a set (finite or infinite) consisting of distinct elements $A, B, C, \ldots$, called by us *letters*. A *set-system* $\mathcal{L}$ on $\mathcal{A}$ consists of *words*, each being a subset of $\mathcal{A}$. For simplicity of notation, each word of $\mathcal{L}$ will be written as an alphabetically ordered sequence of letters, e.g. $ABD, DEFH$, etc.

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Our geometric interpretation of a set-system is based on a family $\mathcal{F}$ of disjoint convex bodies in the plane, corresponding to distinct letters of the set-system. In this way a subfamily of $\mathcal{F}$ forms a word if and only if there is a line supporting all bodies of the subfamily such that no other body of $\mathcal{F}$ is supported by the line.

As it is shown in [1], any family of disjoint convex bodies in the plane has the following properties:

(1) any two bodies have exactly four common supporting lines;
(2) any three bodies have at most three common supporting lines;
(3) any five bodies have at most two common supporting lines.

Basing on these and following [1], we introduce two definitions.

DEFINITION 1. A set-system $\mathcal{L}$ on the set $\mathcal{A}$ is called *special* provided it has the following properties:

(P1) no pair of letters in $\mathcal{A}$ are contained in more than four words;
(P2) no triple of letters in $\mathcal{A}$ are contained in more than three words;
(P3) no quintuple of letters in $\mathcal{A}$ are contained in more than two words.

DEFINITION 2. A set-system $\mathcal{L}$ on the set $\mathcal{A}$ has the (transversal) property T if all the letters of $\mathcal{A}$ belong to a word. $\mathcal{A}$ has the property $T(k)$, where k is a given positive integer, if any k (pairwise) distinct letters in $\mathcal{A}$ belong to a word.

In our interpretation, the property T (respectively, $T(k)$) of set-systems corresponds to the property S (respectively, $S(k)$) of families of disjoint convex bodies in the plane. Now we are able to formulate the main results of the paper.

THEOREM 1. *$T(6) \Rightarrow T$ for any special set-system.*

THEOREM 2. *$T(5) \Rightarrow T$ for any special set-system on at least* 7 *letters.*

THEOREM 3. *$T(4) \Rightarrow T$ for any special set-system on at least* 11 *letters.*

THEOREM 4. *$T(3) \Rightarrow T$ for any special set-system on at least* 143 *letters.*

As a consequence of Theorem 4 we obtain

COROLLARY. *$S(3) \Rightarrow S$ for any collection of at least* 143 *disjoint convex bodies in the plane.*

The following problem still remains open.

PROBLEM. What is the minimum positive integer n such that $S(3) \Rightarrow S$ for any collection of at least n disjoint convex bodies in the plane?

Paper [1] contains an example of 10 disjoint convex bodies in the plane with the property $S(3)$ but not the property S. Figure 1 shows a similar arrangement of 16 convex bodies in the plane (with the respective set-system shown in Table 4). Hence the number n in the Problem is between 17 and 143.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	T	U
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+	+
+	+	+	+	+	−	−	−	−	−	−	−	−	−	−	−	−	+	+	+
+	−	−	−	−	+	+	+	+	−	−	−	−	−	−	−	−	+	+	+
+	−	−	−	−	−	−	−	−	+	+	+	+	−	−	−	−	+	−	−
+	−	−	−	−	−	−	−	−	−	−	−	−	+	+	+	+	+	−	−
−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−	−
−	−	−	−	−	−	−	−	−	+	+	−	−	+	+	−	−	+	+	−
−	−	−	−	−	−	−	−	−	−	−	+	+	−	−	+	+	+	+	−
−	−	−	−	−	−	−	−	−	+	−	+	−	+	−	+	−	+	−	+
−	−	−	−	−	−	−	−	−	−	+	−	+	−	+	−	+	+	−	+

Table 1

Examples below show that assertions of Theorems 2 and 3 are tight. The problem on the minimum number n of letters in a special set-system $\mathcal{A}$, which guarantees the implication $T(3) \Rightarrow T$, remains open. Table 1 shows a special set-system of 10 words on 20 letters with the property $T(3)$ but not the property T. Hence this number is at least 21. Here and subsequently each row of a table with "+" and "−" means a word of the set-system. Thus the second row in Table 1 means the word $ABCDERTU$.

A	B	C	D	E	F
+	+	+	+	+	−
+	+	+	+	−	+
+	+	+	−	+	+
+	+	−	+	+	+
+	−	+	+	+	+
−	+	+	+	+	+

Table 2

EXAMPLE 1. Table 2 shows a special set-system of 6 words on 6 letters A, B, C, D, E, F, with the property $T(5)$ but not the property T.

EXAMPLE 2. Table 3 shows a special set-system of 5 words on 10 letters with the property $T(4)$ but not the property T.

The following Table 4 gives the set-system of 6 words on 16 letters corresponding to the family of disjoint plane convex bodies from Figure 1.

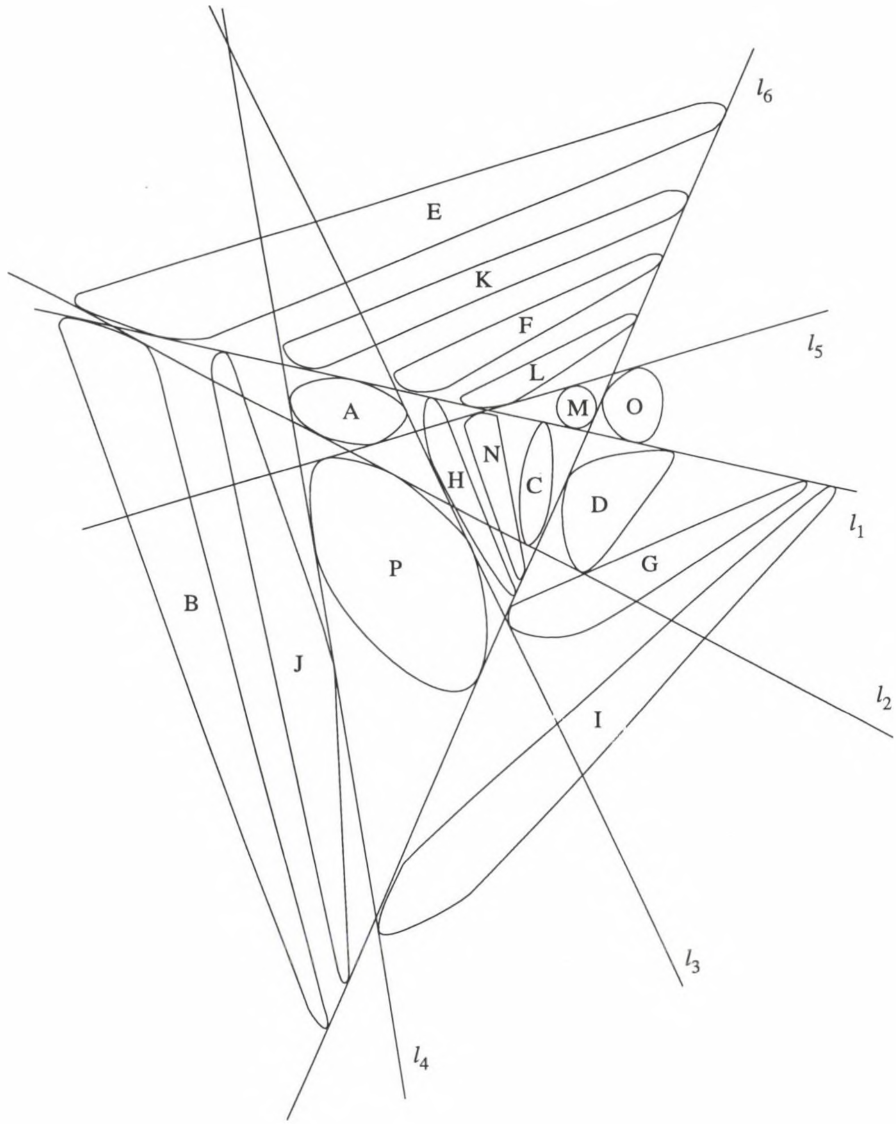

Fig. 1

3. Proofs of Theorems

Proof of Theorem 1. Assume, for contradiction, the existence of a special set-system $\mathcal{L}$ on a set $\mathcal{A}$, with the property $T(6)$ but not the property T. Clearly, $\mathcal{A}$ has more than 6 letters and every word of $\mathcal{L}$ is distinct from $\mathcal{A}$.

Choose a pair AB of letters in $\mathcal{A}$. Due to $T(6)$, there is at least one

A	B	C	D	E	F	G	H	I	J
+	+	+	+	+	+	+	+	−	−
+	+	+	+	+	+	−	−	+	+
+	+	+	+	−	−	+	+	+	+
+	+	−	−	+	+	+	+	+	+
−	−	+	+	+	+	+	+	+	+

Table 3

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−
+	+	+	+	+	−	−	−	−	−	−	−	−	−	−	+
+	−	−	−	−	+	+	+	−	−	−	−	−	−	−	+
+	−	−	−	−	−	−	−	+	+	+	−	−	−	−	+
+	−	−	−	−	−	−	−	−	−	−	+	+	+	+	+
−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

Table 4

word containing AB, and, by (P1), the are at most 4 such words. Denote by $W_1, \ldots, W_i, 1 \leqq i \leqq 4$ all the words in $\mathcal{L}$ containing both letters A, B. Since all W_j are distinct from $\mathcal{A}$, one can find in $\mathcal{A}$ some letters $C_j \notin W_j, 1 \leqq j \leqq i$. Again by $T(6)$, there is a word $W \in \mathcal{L}$ containing all of $A, B, C_1, \ldots, C_i$. Since W is distinct from any of $W_1, \ldots, W_i$, we obtain a contradiction. Hence the property T holds.

Proof of Theorem 2. Due to Theorem 1, it is sufficient to show that $T(5) \Rightarrow T(6)$ provided $|\mathcal{A}| \geqq 7$. In fact, we will prove below that $T(5) \Rightarrow T(7)$ holds.

Let $\mathcal{P} = \{A, B, C, D, E, F, G\}$ be any subset of 7 letters in $\mathcal{A}$. Our purpose is to show that all these letters belong to a word.

Claim 1. For any 4 letters in $\mathcal{P}$ there are 2 more letters from $\mathcal{P}$ such that all the 6 letters belong to a word.

Indeed, choose in $\mathcal{P}$ any 4 letters, say A, B, C, D, and consider the sets

$$\{A, B, C, D, E\}, \{A, B, C, D, F\}, \{A, B, C, D, G\}, \{A, B, C, E, F\}.$$

Due to (P2) for ABC, at least 2 of these sets belong to the same word. The union of these 2 sets is a set of at least 6 letters from $\mathcal{P}$, containing $ABCD$.

We continue the proof of Theorem 2. Assume, in order to obtain a contradiction, that $\mathcal{P}$ is not in a word. By Claim 1, for the letters A, B, C, D there are 2 more letters, say E, F, such that A, B, C, D, E, F belong to a word, W_1. By the assumption, $G \notin W_1$. Similarly, for the letters A, B, C, G there are 2 more letters, say D, E, such that all A, B, C, D, E, G belong to a word W_2 and $F \notin W_2$. Now for the letters A, B, F, G one can find 2 more letters, say C, D, such that all A, B, C, D, F, G belong to a word W_3 and $E \notin W_3$. In the same way, for A, E, F, G two more letters, say B, C, exist such that A, B, C, E, F, G belong to a word W_4 and $D \notin W_4$. Finally, for D, E, F, G there are, say A, B, such that A, B, D, E, F, G belong to a word W_5, with $C \notin W_5$.

Summing up, we have obtained 5 distinct words $W_1, \dots, W_5$ each containing AB. By (P1), at least 2 of these words coincide. If, for example, $W_1 = W_2$, then $F \in W_2$, a contradiction. A similar contradiction appears in any of the cases $W_i = W_j, i \neq j$.

Proof of Theorem 3. Let $\mathcal{L}$ be a special set-system with the property $T(4)$ on the set $\mathcal{A}$ having at least 11 letters. It suffices to show that $T(4) \Rightarrow T(11)$.

Let $\mathcal{Q} = \{A, B, C, D, E, F, G, H, I, J, K\}$ be any set of 11 letters in $\mathcal{A}$. So, given $\mathcal{Q}$, every 4 letters of which belong to a word, we wish to show that all of them do. We prove this in some steps.

Claim 2. Any 2 letters in $\mathcal{Q}$ belong to a word with at least 7 letters from $\mathcal{Q}$.

Assume, for contradiction, that the letters A, B do not belong to a word with at least 7 letters from $\mathcal{Q}$. According to (P2), there are at most 3 words W_1, W_2, W_3 each containing A, B, C, and by $T(4)$, these words cover the quadruples $\{ABCX : X \in \mathcal{Q}\}$. Since there are 8 such quadruples, the number of W_i's is exactly three (otherwise one of W_i's would contain at least 7 letters from $\mathcal{Q}$). Moreover, 2 of these words, say W_1 and W_2, contain exactly 6 letters from $\mathcal{Q}$, and W_3 has either 5 or 6 such letters. Without loss of generality, we may suppose that W_1, W_2, and W_3 are as shown on Table 5 (here blank spaces in the row corresponding to a word mean that we do not know whether a respective letter belongs or does not belong to a word).

Now consider the quadruples

$$(1) \qquad \{ABXY : X \in \{D, E, F\}, Y \in \{G, H, I, J, K\}\}.$$

By the above and according to (P1) for AB, some of these quadruples may be in W_3, and the others belong to a new word W_4. It is easily seen that at most 2 of quadruples (1) can be in W_3, and if 2 of them are in W_3 then they are of the form $ABXJ, ABXK$ for the same X. All the other quadruples (1) belong to W_4, and hence $ABDEFGHIJK$ lies in W_4, a contradiction.

Claim 3. Any 2 letters in $\mathcal{Q}$ belong to a word with at least 8 letters from $\mathcal{Q}$.

Assume, for contradiction, that the letters A, B do not belong to a word with at least 8 letters from $\mathcal{Q}$. By Claim 2, there is a word W containing

A	B	C	D	E	F	G	H	I	J	K	
+	+	+	+	+	+	−	−	−	−	−	W_1
+	+	+	−	−	−	+	+	+	−	−	W_2
+	+	+							+	+	W_3
+	+		+			+					
+	+		+				+				
+	+		+					+			
+	+		+						+		
+	+		+							+	
+	+			+		+					
+	+			+			+				
+	+			+				+			
+	+			+					+		
+	+			+						+	
+	+				+	+					
+	+				+		+				
+	+				+			+			
+	+				+				+		
+	+				+					+	

Table 5

exactly 7 letters in Q, $ABCDEFG$, say. Consider the quadruples

$$\{ABXY : X \in \{C, D, E, F, G\}, Y \in \{H, I, J, K\}\}. \tag{2}$$

According to (P1) for AB, all these 20 quadruples (see Table 6) are covered by at most 3 words. Hence at least one of these words contains at least 7 quadruples of the form (2). It is easily seen that the union of any such 7 quadruples contains at least 8 letters from Q (including AB), a contradiction.

Claim 4. Any 2 letters in Q belong to a word with at least 9 letters from Q.

Assume that a pair AB does not belong to a word with at least 9 letters from Q. Due to Claim 3, there exists a word W containing $ABCDEFGH$, say. First, we will show that for any fixed letter Y from $Q \backslash W = \{I, J, K\}$ the quadruples $\{ABXY : X \in W \setminus \{A, B\}\}$ are covered by only 2 words permitted for A, B, Y by (P2).

Indeed, fix any letter, say I, from $\{I, J, K\}$, and suppose that the quadruples $\{ABXI : X \in W \backslash \{A, B\}\}$ are covered by 3 distinct words W_1, W_2, and W_3. It means that for any $W_i, i = 1, 2, 3$ there is a letter $X_i \in (W_i \cap W) \backslash \{A\}$ such that $X_i \notin W_j$ for $i \neq j, i, j = 1, 2, 3$. Together with W they constitute exactly 4 words containing AB (according to (P1)). The quadruples

$\{ABXJ : X \in W\backslash\{A,B\}\}$ are covered by some words, each contains AB and distinct from W; so they are covered by some of W_1, W_2, and W_3. Moreover, due to the choice of X_1, X_2, and X_3, these quadruples are covered by exactly 3 of these words. Thus we conclude that each of W_1, W_2, W_3 contains the pair IJ. Considering the quadruples $\{ABXK : X \in W\}$ and repeating the same argument, we conclude that each of W_1, W_2, W_3 contains the triple IJK. As a result, we obtain the quintuple $ABIJK$ contained in 3 distinct words, contradicting (P3).

So, the quadruples $\{ABXI : X \in W\backslash\{A,B\}\}$ are covered by 2 distinct words, say W_1 and W_2 (if $W_1 = W_2$ then AB would lie in $ABCDEFGHI$). Similarly, each family of quadruples $\{ABXJ : X \in W\backslash\{A,B\}\}$ and $\{ABXK : X \in W\backslash\{A,B\}\}$ is covered by exactly 2 words. Since all these quadruples cover AB, there are at most 3 distinct words W_1, W_2, W_3 (besides W) covering all of them. We will distinguish here two different cases.

A	B	C	D	E	F	G	H	I	J	K	
+	+	+	+	+	+	+	−	−	−	−	W
+	+	+					+				1
+	+	+						+			2
+	+	+							+		3
+	+	+								+	4
+	+		+				+				5
+	+		+					+			6
+	+		+						+		7
+	+		+							+	8
+	+			+			+				9
+	+			+				+			10
+	+			+					+		11
+	+			+						+	12
+	+				+		+				13
+	+				+			+			14
+	+				+				+		15
+	+				+					+	16
+	+					+	+				17
+	+					+		+			18
+	+					+			+		19
+	+					+				+	20

Table 6

(i) Two of the words W_1, W_2, W_3, say W_1, W_2, cover the quadruples

$$\{ABXY : X \in \{C,D,E,F,G,H\}, Y \in \{I,J,K\}\} \tag{3}$$

(see Table 7). Then, as above, IJK is in both W_1, W_2. Since each of W_1, W_2 contains at most 8 letters, we have, up to symmetry on C, D, E, F, G, H, that W_1 and W_2 contain in Q exactly $ABCDEIJK$ and $ABFGHIJK$, respectively (see Table 7, case (a)).

Now consider the quadruples $\{ACXY : X \in \{F, G, H\}, Y \in \{I, J, K\}\}$. According to (P1) for AC, these quadruples are covered by at most 2 words different from W, W_1, W_2. Since the number of these quadruples is 9, one of the words contains at least 5 of the letters F, G, H, I, J, K.

Similarly, for any of the pairs $AD, AE, BC, BD, BE, CD, CE, DE$, there is a word containing this pair and at least 5 of the letters F, G, H, I, J, K. The total number of such (not necessarily different) words is 9. Hence at least 5 of them contain either FGH or IJK. Let FGH belong to these at least 5 words (the case of IJK is similar). Since $FGH \subset W \cap W_2$, there is only one new word containing FGH (according to (P2)). Hence all these at least 5 words coincide, i.e. a new word W' contains FGH, at least 4 of the letters A, B, C, D, E, and at least 2 letters from $\{I, J, K\}$. By assumption on AB, at least one of A, B is not in W'. Let $B \notin W'$. Then considering the above words containing BC, BD, BE (these do not contain FGH, since already contained by W_1, W_2, W'), we obtain a new word W'' which contains $BCDE, FG$ say, and IJK. But then $CDEFG$ is in 3 distinct words W, W', W'', contradicting (P3).

(ii) Quadruples (3) are covered by exactly 3 distinct words W_1, W_2, W_3. We claim that in this case, each of W_1, W_2, W_3 contains at most 4 letters from $\{C, D, E, F, G, H\}$, and at most 2 letters from $\{I, J, K\}$.

Indeed, first assume that W_1, say, contains the letters C, D, E, F, G. Since W_1 has at most 8 letters (by assumption on AB), W_1 contains exactly one of I, J, K. Let $I \in W_1$ (the cases $J \in W_1, K \in W_2$ are similar). Then $W_1 = ABCDEFGI$. Now consider the quadruples

$$\{ABXY : X \in \{C, D, E, F, G\}, Y \in \{J, K\}\}. \tag{4}$$

Clearly, all of them are covered by W_2 or W_3. Hence one of W_2, W_3, say W_2, covers at least 5 quadruples from (4). This implies that W_2 contains 3 letters from $\{C, D, E, F, G\}$, and hence some 5 letters from $\{A, B, C, D, E, F, G\}$ belong to 3 distinct words, contradicting (P3). Thus none of W_1, W_2, W_3 contains 5 letters from $\{C, D, E, F, G, H\}$.

Now let us assume that W_1, say, contains IJK. Since W_1 has at most 8 letters, it does not contain more than 3 letters from $\{C, D, E, F, G, H\}$. Let, for example, F, G, H be outside W_1. If both W_2 and W_3 contained FGH, then $ABFGH$ would be in 3 distinct words W, W_2, W_3, contradicting (P3). Hence at least one of the letters F, G, H is not in one of the words W_2, W_3. Let $H \notin W_2$. Then $H \notin W_1 \cup W_2$, and therefore $HIJK \subset W_3$. If there were a letter $X \in \{C, D, E, F, G\} \backslash (W_1 \cup W_3)$, then, by the same reason, $XIJK \subset W_2$ and $ABIJK$ would be in each of W_1, W_2, W_3. Hence $\{C, D, E, F, G, H\} \subset W_1 \cup W_3$. In this case W_1 and W_3 cover all quadruples (3), contradicting the assumption. Thus, none of W_1, W_2, W_3 contains IJK.

The just shown two facts imply, like above at H, that any $X \in \{B, C, D, E, F, G\}$ and any $Y \in \{I, J, K\}$ are in at least two of the words W_1, W_2, W_3. According to these observations, we conclude that, up to symmetry on C, D, E, F, G, H, the words W_1, W_2, W_3 are of the form shown on Table 7, case (b).

A	B	C	D	E	F	G	H	I	J	K		
+	+	+	+	+	+	+	+	−	−	−	W	
+	+	+						+				
+	+	+							+			
+	+	+								+		
+	+		+					+				
+	+		+						+			
+	+		+							+		
+	+			+				+				
+	+			+					+			
+	+			+						+		
+	+				+			+				
+	+				+				+			
+	+				+					+		
+	+					+		+				
+	+					+			+			
+	+					+				+		
+	+						+	+				
+	+						+		+			
+	+						+			+		
+	+	+	+	+	+	+	+	−	−	−	W	(a)
+	+	+	+	+	−	−	−	+	+	+	W_1	
+	+	−	−	−	+	+	+	+	+	+	W_2	
+	+	+	+	+	+	+	+	−	−	−	W	(b)
+	+	+	+	+	+	−	+	−	+	−	W_1	
+	+	+	+	−	−	+	+	+	−	+	W_2	
+	+	−	−	+	+	+	+	−	+	+	W_3	

Table 7

Now, considering the quadruples $ACEK, ACFK$ and applying (P1) for AC, we obtain that $ACEFK$ lies in a new word. Similarly, each of $ADEFK$, $BCEFK$ lies in a word different from W, W_1, W_2, W_3. Now, according to (P1) for EF, we have that $ABCDEFK$ lies in a word. In this case $ABCDEF$ is in 3 distinct words, contradicting (P3).

Claim 5. All the letters from $\mathcal{Q}$ belong to a word.

Assume, for contradiction, the absence of a word in $\mathcal{L}$ covering the letters of $\mathcal{Q}$. By Claim 4, there is a word, say W_1, containing $ABCDEFGHI$, say. At least one of J, K, say K, is not in W_1. The pair AK belongs to a new word W_2 containing at least 9 letters in $\mathcal{Q}$, and whence containing at least 7 letters in $\{A, B, C, D, E, F, G, H, I\}$. Let $ABCDEFG \subset W_2$. If $Y \in \mathcal{Q}$ is a letter missing in W_2, then for the pair YK there is a new word W_3 containing at least 9 letters in $\mathcal{Q}$. Clearly, W_3 contains 5 letters from $\{A, B, C, D, E, F, G\}$, and these 5 letters lie in 3 distinct words W_1, W_2, W_3, contradicting (P3).

Proof of Theorem 4. Let $\mathcal{L}$ be a special set-system with the property $T(3)$ on a set $\mathcal{A}$ with $|\mathcal{A}| \geqq 143$.

Assume, in order to obtain a contradiction, that $\mathcal{A}$ is not a word. First we will show that every triple would have to be contained in at least 2 words of $\mathcal{L}$. For, otherwise, let some triple ABC be contained in exactly one word $W_1 \neq \mathcal{A}$.

First consider the case when W_1 contains at least 62 letters. Choose a letter $D \notin W_1$. The triples $\{ADX : X \in W_1 \setminus \{A\}\}$ are covered, due to (P1), by at most 4 words each containing AD. Thus one of these words, say W_2, contains A and at least 16 other letters of W_1, i.e. $|W_1 \cap W_2| \geqq 17$. By the hypothesis, W_2 does not contain ABC. Without loss of generality, we may assume that $B \notin W_2$. Then at most 4 words each containing BD cover all the triples $\{BDX : X \in W_1 \cap W_2\}$. Hence one of these words, W_3, contains at least 5 letters common to W_1 and W_2, contradicting (P3).

Now consider the case when W_1 contains at most 31 letters. Then, due to (P1), at most 3 other words T_1, T_2, T_3 each containing AB cover all the triples $\{ABX : X \notin W_1\}$. Similarly, at most 3 new words U_1, U_2, U_3 cover the triples $\{ACX : X \notin W_1\}$, and at most 3 new words V_1, V_2, V_3 cover the triples $\{BCX : X \notin W_1\}$. The at most 27 intersections $T_i \cap U_j \cap V_k$ cover the set $\mathcal{A} \setminus W_1$, which has at least 112 letters. Thus the largest of the intersections contains at least 5 letters, again contradicting (P3).

Finally, consider the case $32 \leqq |W_1| \leqq 61$. Due to (P1), at most 3 other words each containing AB cover all the triples $\{ABX : X \notin W_1\}$. Thus one of these words, say W_2, contains AB and at least 28 other letters of $\mathcal{A} \setminus W_1$. At most 3 other words each containing AC cover all the triples $\{ACX : X \in W_2 \setminus W_1\}$, and we obtain another word, say W_3, which contains AC and at least 10 other letters in $W_2 \setminus W_1$.

We are going to prove that $W_1 \setminus \{A, B, C\} \subset W_2 \cup W_3$. Indeed, assume for a moment the existence of a letter $D \in W_1 \setminus \{A, B, C\}$ which is not in $W_2 \cup W_3$. Then at most 3 new words each containing AD cover all the triples $\{ADX : X \in (W_2 \cap W_3) \setminus W_1\}$. Thus one of these words contains A and at least 4 other letters common to $(W_2 \cap W_3) \setminus W_1$, i.e. at least 5 letters are contained in 3 words, contradicting (P3). Thus $W_1 \setminus \{A, B, C\} \subset W_2 \cup W_3$.

Since the set $W_1 \setminus \{A, B, C\}$ contains at least 29 letters, one of the words W_2, W_3 contains AB or AC, and at least 15 other letters of W_1. We will assume that it is W_2, which contains AB, and hence C is not in W_2 (the other case is similar). Choose a letter $Z \notin W_1$, and consider the triples $\{CZX : X \in W_1 \cap W_2\}$. Due to (P1), they are covered by at most 4 words different from W_1 and W_2. Since the number of the triples is at least 17, one of these words contains at least 5 letters common to both W_1 and W_2, a contradiction with (P3).

We thus conclude that every triple of letters in $\mathcal{A}$ is in at least 2 words of $\mathcal{L}$. Then pairwise intersections of at most 4 words, say S_1, S_2, S_3, S_4, each containing AB, cover all of $\mathcal{A}$. There are at most 6 such intersections; so, the largest of them, say $S_1 \cap S_2$, contains at least 26 letters. Let C be a letter not in $S_1 \cap S_2$; then at most 4 words different from S_1, S_2, cover $\{ACX : X \in S_1 \cap S_2 \setminus \{A\}\}$. One of these words contains at least 8 letters of $S_1 \cap S_2$, contradicting (P3).

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MATHEMATICAL INSTITUTE OF THE
ACADEMY OF SCIENCES OF MOLDOVA
STR. ACADEMIEI 5
MD–2028 CHIŞINĂU
REPUBLICA MOLDOVA

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 407–414

A NOTE ON MATRIC-EXTENSIBILITY AND THE ADS CONDITION

W. G. LEAVITT

1. Introduction

In the following we will assume, unless otherwise stated, that all rings will be from the class of all not necessarily associative rings. For a class M of rings we will, as usual, write $UM=\{R\mid \text{ every } 0\neq R/I\notin M\}$ and $SM=\{R\mid \text{ if } 0\neq I\triangleleft R \text{ then } I\notin M\}$. We will also write LM for the lower radical generated by M, using the construction $LM=\cup_\alpha M_\alpha$ where M_1 is the homomorphic closure of M and (inductively) for α any ordinal, $M_\alpha=\{R\mid \text{ every } 0\neq R/I \text{ has some non-zero ideal in } M_\beta \text{ for some } \beta<\alpha\}$. Remark that for $I\triangleleft R$ we will define I^2 as the set of all sums of words in R each containing at least two members of I, and I^3 all sums of words each containing at least three members of I. Also remark that for any ring R we will write R^0 for the ring with trivial (zero) multiplication on the additive group of R.

In [6] a class was called "matric-extensible" (me-class) if for all n we have $R\in M$ if and only if $R_n\in M$ where R_n is the ring of all n by n matrices over R. In [2] and [6] me-radicals were considered, particularly radicals P which satisfy what in [7] was called the "matrix equation", namely

$$P(R_n)=(P(R))_n. \tag{1}$$

In the following section we give a number of results on radicals satisfying the matrix equation, showing in particular that a hereditary me-radical satisfies the matrix equation in all rings. We also complete the proof of [2; Theorem 1.5 and Corollary 1.7], namely we show that if M is a homomorphically closed hereditary me-class, then LM is a hereditary me-radical (therefore satisfying the matrix equation in all rings).

In the third section we note that many results, such as those of [2] and [7], are valid in classes more general than the associative (or alternative) rings, namely those in which radicals P satisfy the so-called "ADS" condition: that $P(I)\triangleleft R$ whenever $I\triangleleft R$. We also include some remarks relative to such classes (ADS classes) and give a construction for the largest ADS class contained in any given class.

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In the final section we consider the relationship between a class M and the associated classes UM and SM. For a regular class M (that is one in which $M \subseteqq SUM$) several conditions are given under which M an me-class implies UM an me-radical satisfying the matrix equation in all rings. In a similar way we also give conditions such that M an me-class implies SM an me-class.

2. The lower radical and the matrix equation

In [6; Theorem 1] we gave a condition on an me-radical sufficient for it to satisfy the matrix equation and, by a similar proof, this can be extended to

PROPOSITION 1. *If P is a radical either containing all the nilpotent rings or none, then P is an me-radical if and only if SP is an me-class, and in either case P satisfies the matrix equation in all rings R.*

In any case it is easy to show (as we remarked in [6]) that

PROPOSITION 2. *A radical P satisfies the matrix equation in all rings R if and only if both P and SP are me-classes.*

Remark that Snider in [7] showed that in the class of all associative rings $P(R_n) = I_n$ for some $I \triangleleft R$, from which follows the equivalence of: (1) P is an me-radical, (2) SP is an me-class, and (3) P satisfies the matrix equation in all rings (associative).

Also note that in a ring R with unit we also have $P(R_n) = I_n$ for some $I \triangleleft R$ and therefore an me-radical will satisfy the matrix equation in all rings with unit. We also can show that

PROPOSITION 3 [see 2; Lemma 1.6]. *Let P be a hereditary radical either containing all the nilpotent rings or none. If P satisfies the matrix equation in rings with unit then it satisfies the matrix equation in all rings.*

PROOF. Let R be an arbitrary ring and let $H = P(R_n)$. By [6; Lemma] there exist ideals J, I of R such that $J_n \subseteqq H \subseteqq I_n$ with I_n/J_n nilpotent. But then H/J_n and I_n/H are nilpotent so $H/J_n \neq 0$ would be a contradiction if P contains no nilpotent rings, whereas $I_n/H \neq 0$ would be a contradiction if P contains all the nilpotent rings. Thus $H = I_n$ for some $I \triangleleft R$. Now imbed R in a ring S with unit in the usual way. It is well-known that then $I \triangleleft S$ so $I_n \triangleleft S_n$, and of course also $P(R) \triangleleft S$. Thus the argument of [2; Lemma 1.6] applies.

Remark that unfortunately the induction argument for [2; Lemma 1.4] is not valid, so [2; Theorem 1.5 and Corollary 1.7] have not been proved. However, see Theorem 7 below.

We can also extend Proposition 1 in the following way: First we have

LEMMA 4. *If P is an me-radical which fails to satisfy the matrix equation in at least one ring R, then both P and SP contain zero rings.*

PROOF. If $Q = P(R)$ then $Q_n \in P$ so $Q_n \subseteq H = P(R_n)$ with $Q_n \neq H$. If $H = I_n$ for any $I \triangleleft R$ then $0 \neq I_n/Q_n \in P$ implies $0 \neq I/Q \in P$ contradicting $Q = P(R)$. Thus the inequalities are proper in [6; Lemma] $J_n \subseteq H \subseteq I_n$ with $I_n^3 \subseteq J_n$. Since $H^3 \subseteq J_n$ we have that $0 \neq H/H^2$ is a zero ring in P. Also $0 \neq I_n/H$ and either $I_n^2 \subseteq H$ or $I_n^2 \nsubseteq H$ but $(I_n^2)^2 \subseteq H$. In either case R_n/H contains an ideal which is a zero ring and since $R_n/H \in SP = SUSP$, this ideal has a non-zero image in SP.

COROLLARY 5. *If an me-radical P contains Z^0 then P satisfies the matrix equation in all rings.*

PROOF. It is well-known that $Z^0 \in P$ implies P contains all zero rings and so all nilpotent rings. Thus the result follows from Proposition 1 or Lemma 4.

THEOREM 6. *If P is a hereditary me-radical then it satisfies the matrix equation in all rings.*

PROOF. Suppose not, then if $Q = P(R)$ we have $Q_n \in P$ so that $Q_n \subseteq H = P(R_n)$ with $Q_n \neq H$. Now $H \neq I_n$ for any $I \triangleleft R$ since $0 \neq I_n/Q_n$ would imply the contradiction $0 \neq I/Q \in P$. If $Q_n \subseteq I_n \subseteq H$, then by heredity $I_n/Q_n \in P$ so $I/Q \in P$, a contradiction unless $I = Q$, that is Q is already maximal relative to $Q_n \subseteq H$. Since by [6; Lemma] we have $J_n \subseteq H \subseteq I_n$ with $I_n^3 \subseteq J_n$ then from $J_n \subseteq Q_n \subseteq H$ we have $H^3 \subseteq Q_n$. Thus, dividing out Q, we can assume without loss of generality that R is P-semisimple, and $H \triangleleft R_n$ with $H \in P$ and $H^3 = 0$.

Now let $S = \{x \in R \mid x$ is an element of some $A \in H$ if $H^2 = 0$, otherwise of some $A \in H^2\}$. (Recall that we are defining H^2 to be all sums of products each of which contains at least two matrices from H.) We will write RxR to mean all sums of words in R in which x is an interior element, and assume first that $RxR \neq 0$ for some $x \in S$. Thus, say $t = (yx)z \neq 0$, and for any such t, since $x \in s$ there is some $A \in H$ or H^2 with x in, say, the i, j position. Let $[t]$ denote the matrix with t in the upper left corner, 0 everywhere else. It is easy to see that by multiplying A first at the left then at the right, we can obtain $[t] \in H$ or H^2, and so further multiplication, associating properly, yields $[u] \in H$ or H^2 for any word u with x as an interior element. But then by adding we have all $[RxR] \subseteq H$ or H^2. But since either $H^2 = 0$, or in any case $H^3 = 0$ it follows that $[RxR] \triangleleft H$ and so $[RxR] \in P$. But the zero ring $RxR \triangleleft R$ and since $RxR \cong [RxR]$ we have a contradiction.

Now suppose that $RxR = 0$ for all $x \in S$, but that $Rx \neq 0$ for some $x \in S$, so there is some $t = yx \neq 0$. Again x is an element of some $A \in H$ or H^2, and we can apply left multiplication to get a matrix $V_1 = [t_1, t_2, \ldots, t_n] \in H$ or H^2 with $t_j = t \neq 0$, where $[t_1, \ldots, t_n]$ means a matrix with this as its first row, zero everywhere else. Write ft to mean some sum of words with t in the final position (agreeing that ft' shall mean the same sum of words except that t' replaces t), and suppose there should be some f such that $ft_1 = \ldots = ft_{i-1} = 0$, but that $ft_i \neq 0$. Then carrying out the multiplications matrixwise, and adding we have $V_2 = [0, \ldots, 0, ft_i, \ldots, ft_n] \in H$ or H^2. Continuing the process we end with a matrix $V = [0, \ldots, 0, v_k, \ldots, v_n] \in H$ or H^2 with $v_k \neq 0$ and such that whenever we have any $fv_k = 0$ then $fv_r = 0$ for all $r \geqq k$. Writing RV to mean all matrices of form $[0, \ldots, 0, fv_k, \ldots, fv_n]$, then again $RV \triangleleft H$ so $RV \in P$. Now recalling that $RzR = 0$ for all $z \in S$, so that $v_k R = 0$, we have $Rv_k \triangleleft R$. Hence the same contradiction that $Rv_k \cong RV \in P$. Now it could happen that $Rv_k = 0$ so all $Rv_r = 0$ for all $r \geq k$. If V has additive order ∞ or some m, then at least one v_i has the same order. Then the additive group $\{nv_i\} \cong \{nV\}$ so again the same contradiction $\{nv_i\} \triangleleft R$ with $\{nv_i\} \in P$.

Clearly the case all $RxR = 0$ and some $xR \neq 0$ can be handled similarly, on the right, so finally suppose $Rx = xR = 0$ for all $x \in S$. Then choosing any $A \in H$ or H^2 we have $\{nA\} \triangleleft H$ so is in P. In a similar way if A has additive order ∞ or m then it has at least one element x with the same order. Thus the same contradiction $\{nx\} \triangleleft R$ and $\{nx\} \cong \{nA\} \in P$. We conclude, in fact, that $Q = P(R)$ implies $Q_n = P(R_n)$ so that P satisfies the matrix equation in all rings.

THEOREM 7. *If M is a homomorphically closed hereditary me-class then LM is an me-radical which satisfies the matrix equation in all rings.*

PROOF. By [6; Theorem 2] if $R \in M$ implies $R_n \in M$, then the same is true for LM, so it suffices to prove that $R_n \in LM$ implies $R \in LM$. As we remarked earlier, we will use the construction $LM = \cup_\alpha M_\alpha$ where $M_1 = M$ and $M_\alpha = \{R$ every $0 \neq R/I$ has an ideal $0 \neq J/I \in M_\beta$ for some $\beta < \alpha\}$. We assume that M_β is an me-class for all $\beta < \alpha$ and it is well-known that all M_α are hereditary [see 8; Lemma 13.2 p. 59]. Suppose there could be some $R_n \in M_\alpha$ but with $R \notin M_\alpha$. Then R would have some image R/Q with no non-zero ideal in any M_β with $\beta < \alpha$, whereas R_n/Q_n would have an ideal $H/Q_n \in M_\beta$ for some $\beta < \alpha$.

The proof is then similar to that of Theorem 6, namely that $H \neq I_n$ for and $I \triangleleft R$ since if $I_n/Q_n \in M_\beta$ then by the induction hypothesis we would have the contradiction $I/Q \in M_\beta$. Also the hereditary property of M_β rules out any $Q_n \subseteqq I_n \subseteqq H$ unless Q_n is already maximal relative to $Q_n \subseteqq H$. Thus again $H^3 \subseteqq Q_n$ so dividing out Q we have a ring R with no non-zero ideals in any M_β, but $H \triangleleft R_n$ with $H \in M_\beta$ and $H^3 = 0$. The proof is then exactly as for Theorem 6, using the hereditary property of M_β to find ideals in M_β isomorphic to ideals of R, a contradiction. Thus LM is an me-radical which by Theorem 6 satisfies the matrix equation in all rings.

Note that this now provides a proof of [2; Theorem 1.5 and Corollary 1.7]. Also remark that [6; Theorem 3] provides one further M to LM result, namely: If M is a homomorphically closed me-class which contains no nilpotent rings then LM is an me-radical which satisfies the matrix equation in all rings. This radical will of course contain only idempotent rings.

3. The ADS condition

In [1] it was shown that, in any associative or alternative ring, any radical P has the so-called "ADS property", namely that for any ring R

$$P(I) \triangleleft R \text{ for any } I \triangleleft R, \tag{2}$$

from which it follows that if P is hereditary then $P(I) = I \cap P(R)$. We will say that a radical P satisfies the "ADS condition" in a class M if it has the ADS property for all rings $R \in M$, and call M an "ADS class" if the ADS condition is satisfied in M for all radicals P defined in M.

Thus any class of associative or alternative rings is an ADS class, but the following example shows that there exist non-alternative ADS classes:

EXAMPLE 8. Let R be generated over Z_2 by the (non-associative) symbols x, y where $x^2 = x, xy = yx = x,$ and $y^2 = 0$. Let A be the universal class consisting of R together with Z_2 and Z_2^0. Since $xy^2 = 0 \neq (xy)y = x$, the class is non-alternative. But $I = (x)$ is the only proper ideal of R, so that $P(I) = I$ or 0 for all radicals P. Thus (2) is satisfied (trivially).

This example may seem somewhat contrived but it is contained in many larger ADS classes. Indeed we may show that

THEOREM 9. *If P is a radical defined in some universal class A then every subclass $M \subseteqq A$ contains a largest subclass in which P satisfies the ADS condition.*

PROOF. For a class $M \subseteqq A$ define the class function

$$FM = \{R \in M \mid P(I) \triangleleft R \text{ for all } I \triangleleft R\}. \tag{3}$$

Clearly P satisfies the ADS condition in a class M if and only if $FM = M$, and it is easy to see that F has the properties $M \supseteqq FM$ for all $M \subseteqq A$ and $M \supseteqq N$ implies $FM \supseteqq FN$, that is F is what in [5; p.168] we called a "dn-admissible" function. The result then follows from [5; Proposition 2] which states that if F is dn-admissible in A then any $M \subseteqq A$ contains a largest F-invariant class. In fact it is clear that $FFM = FM$ so FM is already the largest subclass of M in which P satisfies the ADS condition.

COROLLARY 10. *Every subclass $M \subseteqq A$ contains a largest ADS class.*

PROOF. If $M(P)$ is the largest subclass of M in which P satisfies the ADS condition, then simply take $\cap M(P)$ over all radicals P defined in A, where $\cap M(P) = 0$ simply means that M contains no non-trivial ADS class.

Note that FM need not be a universal class and for some applications that would not matter. However, one could be constructed by combining F with some function, say G, which throws away rings with an ideal or a homomorphic image not in M. For some of the applications the ADS property is only needed in rings with unit and again this could be constructed by starting with a sufficiently large class of rings with unit.

COROLLARY 11. *The ring R of Example 1 is contained in a largest ADS class of not necessarily associative rings.*

Remark that one such class would be R adjoined to the class of all associative rings.

We now note that for the results of Snider cited above and for the results of [2] only the ADS property is needed. Thus they remain valid in any ADS class, as for example the class of all alternative rings. Indeed for the arguments of [2] the ADS property in rings with unit is all that would be required.

4. The classes UM and SM

It is well-known [3; Theorem 1] that UM is radical if and only if every $R \in M$ has some $0 \neq R/I \in SUM$ (in particular UM is radical if M is regular, that is $M \subseteqq SUM$). Whether UM is a radical or not it was shown in [6] (and as part of the proof of [2; Theorem 2.1]) that

LEMMA 12. *If $R \in M$ implies $R_n \in M$ then $R_n \in UM$ implies $R \in UM$.*

And (also whether or not UM is radical) by [6; Theorem 4].

THEOREM 13. *If M is a homomorphically closed me-class either containing all the nilpotent rings or none, then UM satisfies the matrix equation in all rings.*

It is also clear that the proof of [2; Theorem 2.1] applies equally well to the class of all not necessarily associative rings, so we have

THEOREM 14. *If M is a regular me-class then UM satisfies the matrix equation in all rings with unit.*

Thus, using Theorem 14 with Proposition 3, and combining [6; Theorem 5] with [2; Corollary 2.3] we obtain

THEOREM 15. *Let M be a regular me-class. If either* (1) *M contains no nilpotent rings, or* (2) *UM is hereditary and M contains all the nilpotent rings, or* (3) *UM is hereditary and contained in a class in which rings with unit satisfy the ADS condition, then UM is an me-class which satisfies the matrix equation in all rings.*

It is also true in general that the class SM is not semisimple (in fact SM is semisimple if and only if $SM = SUSM$ [see 4; p. 312 for the equivalent conditions (2) and (3)]. In general we can say less about SM then we did for its dual UM. However, again whether or not SM is semisimple, we have:

LEMMA 16. *If* $R \in M$ *implies* $R_n \in M$ *then* $R_n \in SM$ *implies* $R \in SM$.

PROOF. If $R_n \in SM$ but $R \notin SM$ then there is some $0 \neq I \triangleleft R$ with $I \in M$ so the contradiction $I_n \in M$ with $I_n \triangleleft R_n$.

Also we have:

THEOREM 17. *If* M *is an me-class then* SM *is an me-class if either:* (1) M *contains no nilpotent rings and is homomorphically closed,* (2) M *contains no nilpotent rings and is co-regular (that is* $M \subseteq USM$*), or* (3) M *either contains all the nilpotent rings or none and is hereditary.*

PROOF. By Lemma 16 we need only show that $R \in SM$ implies $R_n \in SM$, so suppose not. Then there exists some $0 \neq H \triangleleft R_n$ with $H \in M$. If $H = I_n$ for any $I \triangleleft R$ there would be the contradiction $I \in M$. Thus the containments are proper in $J_n \subseteq H \subseteq I_n$ with I_n/J_n nilpotent [from 6; Lemma]. For (1) we would have the contradiction that the nilpotent $0 \neq H/J_n \in M$. For (2) we will have the same contradiction since $M \subseteq USM$ says that nilpotent $0 \neq H/J_n$ has an ideal in M. For (3) since M is hereditary we would have $J_n \in M$ which would be a contradiction unless $J = 0$. But then H would be nilpotent which would contradict if M has no nilpotents, and I_n would be nilpotent which would contradict if M contains all nilpotents.

Since $H = I_n$ for some $I \triangleleft R$ whenever R is a ring with unit, we also have:

COROLLARY 18. *If* M *is any me-class then* SM *has the me-property in any ring with unit.*

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEBRASKA
LINCOLN, NE 68588–0323
U.S.A.

wleavitt@unl.edu

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HOW TO GENERATE THE INVOLUTION LATTICE OF QUASIORDERS?

Dedicated to E. Tamás Schmidt on his 60th birthday

I. CHAJDA and G. CZÉDLI

Abstract

Given a set A, let $\mathrm{Quord}(A)$ denote the set of all quasiorders (i.e., reflexive and transitive relations) on A. Equipped with meet (intersection), join (transitive hull of union) and involution ($\rho \mapsto \{\langle x,y\rangle : \langle y,x\rangle \in \rho\}$), $\mathrm{Quord}(A)$ is an involution lattice. When A is infinite, $\mathrm{Quord}(A)$ is considered a complete involution lattice. Let $\kappa_0 = \aleph_0$, the smallest infinite cardinal, and define $\kappa_{n+1} = 2^{\kappa_n}$. It is shown that if $|A| \leqq \kappa_n$ for some integer n, then $\mathrm{Quord}(A)$ has a three-element generating set.

Given a set A, let $\mathrm{Quord}(A)$ denote the set of all quasiorders (i.e., reflexive and transitive relations) on A. Similarly, the set of equivalences on A will be denoted by $\mathrm{Equ}(A)$. Both $\mathrm{Quord}(A)$ and $\mathrm{Equ}(A)$ are algebraic lattices if we define meet and join as intersection and transitive hull of union, respectively. According to the following table, which was partly produced by a computer program, $\mathrm{Equ}(A)$ and especially $\mathrm{Quord}(A)$ have quite many elements:

$\lvert A\rvert$	1	2	3	4	5	6	7
$\lvert\mathrm{Equ}(A)\rvert$	1	2	5	15	52	203	877
$\lvert\mathrm{Quord}(A)\rvert$	1	4	29	355	6942	?	?

It was proved by Strietz [8] (cf. also Zádori [10]) that the lattice $\mathrm{Equ}(A)$, $4 < |A| < \infty$, has a four-element generating set, but cannot be generated by three elements.

By an *involution lattice* we mean a lattice L equipped with an additional unary operation * such that * is an involutory automorphism of the lattice reduct. I.e., $L = \langle L; \vee, \wedge, ^*\rangle$ is an involution lattice if $\langle L; \vee, \wedge\rangle$ is a lattice and $(x \vee y)^* = x^* \vee y^*$, $(x \wedge y)^* = x^* \wedge y^*$ and $x^{**} = x$ hold for all $x, y \in L$. If the lattice reduct of L is a complete lattice, then L is called a *complete involution lattice*. The most typical example is $\mathrm{Quord}(A)$ where α^* for $\alpha \in \mathrm{Quord}(A)$ is defined to be $\{\langle x,y\rangle \in A^2 : \langle y,x\rangle \in \alpha\}$. From now on, $\mathrm{Quord}(A)$ will always

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be considered a complete involution lattice. Involution lattices and Quord(A) have recently been studied in [1], [2], [4], [5] and [6]. The relation between involution lattices and Quord(A) is similar to but not quite the same as that between lattices and Equ(A). E.g., while each lattice can be embedded in some Equ(A) by Whitman [9], there are involution lattices that can be embedded in no Quord(A), cf. [4].

Now for any ordinal number ν we define a cardinal number κ_ν via induction. Set $\kappa_0 = \aleph_0$, the cardinality of $\mathbf{N}_0 = \{0, 1, 2, 3, \dots\}$. If κ_ν is defined, then let $\kappa_{\nu+1} = 2^{\kappa_\nu}$. If ν is a limit ordinal, then let κ_ν be the sum of all κ_μ, $\mu < \nu$. For example, κ_ω is the sum of all cardinals κ_n, $n \in \mathbf{N}_0$. The goal of the present paper is to prove the following

THEOREM 1. *Let A be a set with $3 \leqq |A| < \kappa_\omega$. Then* Quord($A$)*, as a complete involution lattice, has a 3-element generating set. In fact,* Quord(A) *can be generated by three partial orders.*

Before proving this theorem, some remarks are worth formulating.

If A is finite then Theorem 1 holds for Quord(A) as an involution lattice in the usual sense (when the operations are the *binary* join and meet, and the unary involution).

The proof of Theorem 1 will (more or less) give the right feeling that there are many countable ordinals $\nu > \omega$ such that Quord(A) is 3-generated for $|A| < \kappa_\nu$. But proving this stronger statement would require a much more complicated proof without proving the result for all sets A; therefore the present paper is restricted to $\nu = \omega$.

If $\{\alpha, \beta, \gamma\}$ generates Quord(A) as a (complete) involution lattice, then $\{\alpha, \beta, \gamma, \alpha^*, \beta^*, \gamma^*\}$ generates it as a complete lattice. Thus Theorem 1 offers a six-element generating set for the lattice reduct of Quord(A).

If $|A| \in \{3, 4\}$, then a straightforward computer program shows that Quord(A) cannot be generated by two elements. This encourages us to conjecture that Theorem 1 is sharp in the sense that Quord(A) has no two-element generating set for $|A| \geqq 3$.

Besides the mentioned computer program, there is manual proof of the fact that no $\{\alpha, \beta\} \subseteq$ Quord(A) generates Quord(A) for $|A| = 3$. We can list all possible $\{\alpha, \beta\}$, apart from symmetries and duality, and we can associate a nontrivial unary operation $f_{\{\alpha,\beta\}} : A \to A$ with $\{\alpha, \beta\}$ such that α and β are compatible with $f_{\{\alpha,\beta\}}$. Then all elements of $[\{\alpha, \beta\}]$, the involution sublattice generated by α and β, are compatible with $f_{\{\alpha,\beta\}}$. Hence $[\{\alpha, \beta\}] \neq$ Quord(A), for all members of Quord(A) (or Equ(A)) are simultaneously compatible only with trivial (unary) operations (i.e., projections and constants). The long but easy details of this argument will not be presented here.

Unfortunately, the above idea, borrowed from Zádori [10], does not seem to work for $|A| \geqq 4$. By Demetrovics and Rónyai [7], for $|A| \geqq 4$ there are $\alpha, \beta \in$ Quord(A) such that they are simultaneously compatible only with trivial $A^n \to A$ operations. At present, there is no good description of these $\{\alpha, \beta\}$. E.g., both α and β can be a three-element chain (cf. [7]), but (as it is not too hard to check) the choice $\alpha = \{\langle 1,2\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 5,4\rangle, \langle 5,6\rangle\} \cup \Delta$ and $\beta = \{\langle 1,3\rangle, \langle 1,5\rangle, \langle 1,6\rangle, \langle 2,6\rangle, \langle 4,6\rangle\} \cup \Delta$ for $A = \{1,2,3,4,5,6\}$ is also possible. (Here and in the sequel Δ stands for the diagonal relation $\{\langle x,x\rangle : x \in A\}$; since this is the smallest element of Quord(A), it will also be denoted by 0.)

PROOF of Theorem 1. For a relation $\mu \subseteq A^2$, let μ^{qo} denote the smallest quasiorder including μ, i.e., the transitive hull of $\mu \cup \Delta$. As usual, $P(X)$ will stand for the set of all subsets of X, and let $P^+(X) = P(X) \setminus \{\emptyset\}$. First we deal with the infinite case.

For each nonnegative integer n we will define an *n-scheme*

$$S_n = \langle A_n; e_n^n, e_n^{n+1}, \dots ; D_n^{(n)}, D_n^{(n+1)}, \dots ; \alpha_n \beta^{(n)}, \gamma_n \rangle$$

via induction on n. (The meaning of its components will be given soon.) This n-scheme will depend only on n. Further, associated with S_n and $U \in P(D_n^{(n)})$, we will define a (unique) *n-box*

$$B_n = B_n(U) = \langle A_n; e_n^n, e_n^{n+1}, \dots ; U; D_n^{(n)}, D_n^{(n+1)}, \dots ; \alpha_n, \beta_n, \gamma_n \rangle.$$

In danger of confusion, the more accurate notation

$$\begin{aligned} B_n(U) = \langle A_n(U); e_n^n(U), e_n^{n+1}(U), \dots ; U; \\ D_n^{(n)}(U), D_n^{(n+1)}(U), \dots ; \alpha_n(U), \beta_n(U), \gamma_n(U) \rangle \end{aligned}$$

will be used, even if most of the components do not depend on U. For $m < n$, we will also define the *sub-m-boxes* of $B_n(U)$ or S_n; and for $m = n$, $B_n(U)$ will be considered the only sub-n-box of itself. After the necessary definitions and preliminaries we will show that A_n is a set with power κ_n, and $\{\alpha_n, \beta_n, \gamma_n\}$, no matter which $U \in P(D_n^{(n)})$ is considered, is a generating set of Quord(A_n).

Now we define the 0-scheme S_0, cf. Figure 1. Let $A_0 = \{a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, \dots\}$. We define three partial orders on A_0:

$$\begin{aligned} \alpha_0 = \{\langle a_i, a_j\rangle : 0 \leqq i \leqq j\} \cup \{\langle b_i, b_j\rangle : 0 \leqq j \leqq i\} \cup \{\langle c_1, c_0\rangle\} \cup \\ \{\langle c_i, c_j\rangle : 1 \leqq i \leqq j\} \cup \{\langle d_i, d_j\rangle : 0 \leqq j \leqq i\} \cup \Delta, \\ \gamma_0 = \{\langle b_i, a_{i+1}\rangle : 0 \leqq i\} \cup \{\langle d_i, c_{i+1}\rangle : 0 \leqq i\} \cup \Delta, \\ \beta^{(0)} = \Big(\{\langle a_0, b_0\rangle, \langle b_0, c_0\rangle, \langle c_0, d_0\rangle, \langle c_3, d_3\rangle, \langle c_6, b_6\rangle\} \cup \\ \{\langle b_i, a_i\rangle : 1 \leqq i\} \cup \{\langle d_i, c_i\rangle : 1 \leqq i\}\Big)^{\mathrm{qo}}. \end{aligned}$$

These quasiorders are represented by horizontal, southwest $\to$ northeast, and (solid) vertical directed edges, respectively. For $k > 1$, $e = \langle\langle b_{9k}, c_{9k}\rangle,$

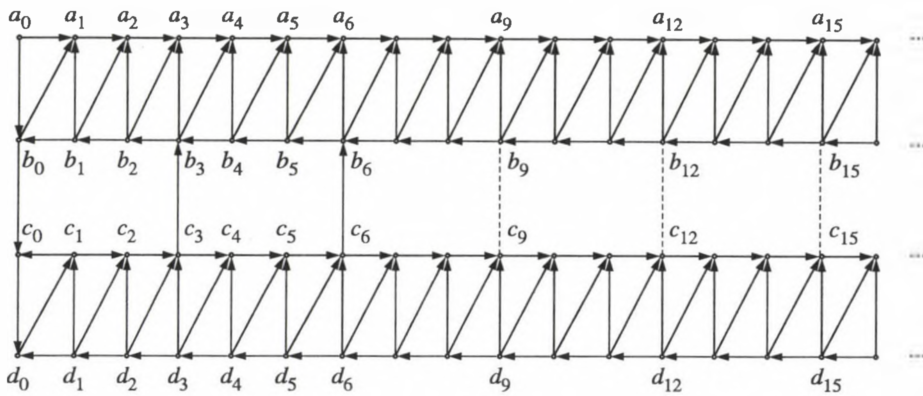

Fig. 1

$\langle b_{9k+3}, c_{9k+3}\rangle, \langle b_{9k+6}, c_{9k+6}\rangle\rangle$ will be called an *edge triplet.* (For $k=1$, this is represented by three dotted vertical lines on Figure 1.) Associated with this e we will use the notation

$$e = \langle\langle b_e, c_e\rangle, \langle b'_e, c'_e\rangle, \langle b''_e, c''_e\rangle\rangle.$$

The binary relations

$$\delta(e) = \{\langle b_e, c_e\rangle, \langle c'_e, b'_e\rangle, \langle c''_e, b''_e\rangle\} \quad \text{and}$$
$$\delta^*(e) = (\delta(e))^*$$

will have special role. (Sometimes we use the notation $\rho^* = \{\langle y, x\rangle : \langle x, y\rangle \in \rho\}$ even when ρ is not a quasiorder.) Let $\{D_0^{(-1)}, D_0^{(0)}, D_0^{(1)}, D_0^{(2)}, \dots\}$ be a fixed partition on the set of edge triplets of S_0 such that all the classes $D_0^{(i)}$ are infinite. Let $e_0^0, e_0^1, e_0^2, e_0^3, \dots$ be a fixed enumeration of the elements in $D_0^{(-1)}$. We have defined S_0, and clearly $|A_0| = |D_0^{(0)}| = |D_0^{(1)}| = |D_0^{(2)}| = \dots = \kappa_0$.

Now let $U \in P(D_0^{(0)})$, and define

$$\beta_0 = \beta_0(U) = \Big(\beta^{(0)} \cup \bigcup_{e\in U} \delta(e) \cup \bigcup_{e\in D_0^{(0)}\setminus U} \delta^*(e)\Big)^{\mathrm{qo}}.$$

Thus we obtain the 0-box

$$B_0 = B_0(U) = \langle A_0; e_0^0, e_0^1, \dots; U; D_0^{(0)}, D_0^{(1)}, \dots; \alpha_0, \beta_0, \gamma_0\rangle.$$

Now let us assume that S_n, $B_n(U)$ for $U \in P(D_n^{(n)})$ and their sub-m-boxes for $m < n$ are already defined. We may assume that $A_n(U) \cap A_n(V) = \emptyset$ for distinct $U, V \in P(D_n^{(n)})$. Let

$$A_{n+1} = \bigcup_{U\in P(D_n^{(n)})} A_n(U),$$

$$\alpha_{n+1} = \bigcup_{U \in P(D_n^{(n)})} \alpha_n(U),$$
$$\gamma_{n+1} = \bigcup_{U \in P(D_n^{(n)})} \gamma_n(U), \quad \text{and}$$
$$D_{n+1}^{(i)} = \bigcup_{U \in P(D_n^{(n)})} D_n^{(i)}(U), \quad \text{for } i \geqq n+1.$$

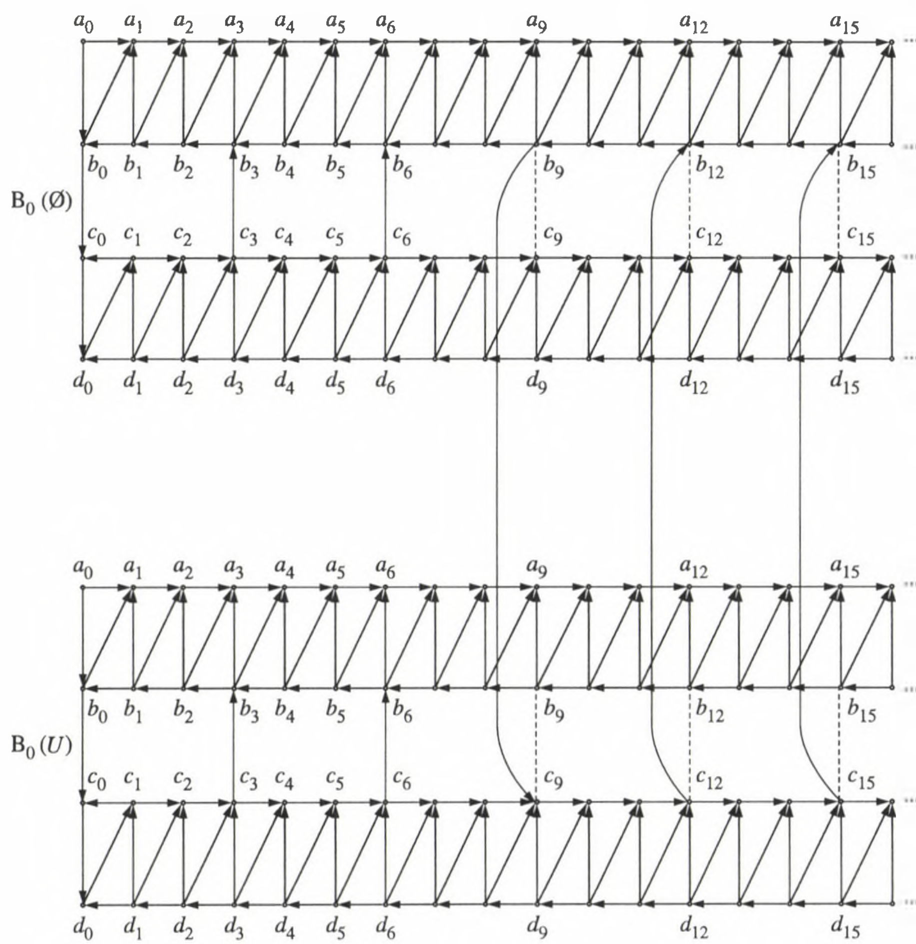

Fig. 2

(Of course, all these unions are unions of pairwise disjoint sets. For $n=0$, the situation is outlined on Figure 2, where the three dotted lines stand for the edge triplet e_0^0, and only one of $B_0(U)$, $U \neq \emptyset$, is indicated.) Define

$e^i_{n+1} = e^i_n(\emptyset)$ for $i \geqq n+1$ and

$$\varepsilon(n,\emptyset,U) = \left\{ \langle b_{e^n_n(\emptyset)}, c_{e^n_n(U)} \rangle, \langle c'_{e^n_n(U)}, b'_{e^n_n(\emptyset)} \rangle, \langle c''_{e^n_n(U)}, b''_{e^n_n(\emptyset)} \rangle \right\}$$

for $U \in P^+(D^{(n)}_n)$. Set

$$\beta^{(n+1)} = \Big(\bigcup_{U \in P(D^{(n)}_n)} \beta_n(U) \cup \bigcup_{U \in P^+(D^{(n)}_n)} \varepsilon(n,\emptyset,U) \Big)^{\mathrm{qo}}.$$

This way we have defined

$$S_{n+1} = \left\langle A_{n+1}; e^{n+1}_{n+1}, e^{n+2}_{n+1}, \dots ; D^{(n+1)}_{n+1}, D^{(n+2)}_{n+1}, \dots ; \alpha_{n+1}, \beta^{(n+1)}, \gamma_{n+1} \right\rangle .$$

Clearly, $|A_{n+1}| = |D^{(n+1)}_{n+1}| = |D^{(n+2)}_{n+1}| = \ldots = \kappa_{n+1}$. The sub-$n$-boxes of S_{n+1} are just the $B_n(U)$, $U \in P(D^{(n)}_n)$. For $m < n$, the sub-m-boxes of S_{n+1} are the sub-m-boxes of its sub-n-boxes. Now let $U \in P(D^{(n+1)}_{n+1})$, and define

$$\beta_{n+1} = \Big(\beta^{(n+1)} \cup \bigcup_{e \in U} \delta(e) \cup \bigcup_{e \in D^{(n+1)}_{n+1} \setminus U} \delta^*(e) \Big)^{\mathrm{qo}}.$$

Thus we obtain the $(n+1)$-box

$$B_{n+1} = B_{n+1}(U) = \\ \left\langle A_{n+1}; e^{n+1}_{n+1}, e^{n+2}_{n+1}, \dots ; U; D^{(n+1)}_{n+1}, D^{(n+2)}_{n+1}, \dots ; \alpha_{n+1}, \beta_{n+1}, \gamma_{n+1} \right\rangle .$$

For $m \leqq n$, the sub-m-boxes of $B_{n+1}(U)$ are the same as that of S_{n+1}.

In order to show that $\alpha_n, \beta_n = \beta_n(U)$ and γ_n generate $\mathrm{Quord}(A_n)$ (no matter which $U \in P(D^{(n)}_n)$ is considered), we introduce certain binary terms $f^n_{p,q} = f^n_{p,q}(x,y,z)$ ($n \in \mathbf{N}_0, p, q \in A_n$). While the $f^0_{p,q}$ will be involution lattice terms in the usual sense, for $n > 0$ the $f^n_{p,q}$ will contain the infinitary join and/or meet operations as well. Instead of developing the exact definition of "terms" (like in [3, Chapter 2]) prior to their usage, we only note that all complete involution sublattices are closed with respect to the "term functions" they induce, and we will not make a distinction between two terms if they induce the same term function on each complete involution lattice. Set $f^n_{p,q} = x \wedge y \wedge z \wedge x^* \wedge y^* \wedge z^*$, and notice that $f^n_{p,p}(\alpha_n, \beta_n, \gamma_n) = 0$ in $\mathrm{Quord}(A_n)$. (This follows from $\alpha_n \wedge \alpha^*_n = \beta_n \wedge \beta^*_n = \gamma_n \wedge \gamma^*_n = 0$.) When we define the $f^n_{p,q}$ in the sequel, we implicitly always assume on $\langle n,p,q \rangle$ that neither $f^n_{p,q}$ nor $f^n_{q,p}$ has previously been defined. Further, for $p \neq q$, $f^n_{q,p} = (f^n_{p,q})^*$. (Remember, we do not make a distinction between $f^n_{p,q}$ and $(f^n_{p,q})^{**}$.) When defining our terms, we keep in mind that the final purpose is to show

$$f^n_{p,q}(\alpha_n, \beta_n, \gamma_n) = \{\langle p,q \rangle\}^{\mathrm{qo}}. \tag{1}$$

Then α_n, β_n and γ_n will evidently generate $\mathrm{Quord}(A)$, for any element μ of $\mathrm{Quord}(A)$ is the join of all $\{\langle p,q\rangle\}^{\mathrm{qo}}$ below μ. However, (1) is not appropriate to be an induction hypothesis; something stronger is necessary. For $p,q\in A_n$ and $n\leqq m$, let H be the set of all sub-n-boxes of $B_m=B_m(U)$. These sub-n-boxes are pairwise disjoint, of course. For $h\in H$, let p_h and q_h denote (the elements corresponding to) p and q in the h-th copy of A_n (i.e., in the base set of the h-th sub-n-box). Define

$$\langle p,q\rangle^{(n,m)}=\Big(\bigcup_{h\in H}\{\langle p_h,q_h\rangle\}\Big)^{\mathrm{qo}}\in \mathrm{Quord}(A_m).$$

Note that $\{\langle p,q\rangle\}^{\mathrm{qo}}=\langle p,q\rangle^{(n,n)}$ in $\mathrm{Quord}(A_n)$ and $\langle p,q\rangle^{(n,m)}=\Delta\cup \cup\bigcup_{h\in H}\{\langle p_h,q_h\rangle\}$. We will define terms $f^n_{p,q}$ such that

$$f^n_{p,q}(\alpha_m,\beta_m,\gamma_m)=\langle p,q\rangle^{(n,m)}\quad \text{in}\ \ \mathrm{Quord}(A_m) \tag{2}$$

holds for all $0\leqq n\leqq m$ and $p,q\in A_n$. Note that (2) implies (1), and therefore it implies Theorem 1 for $|A|=|A_n|=\kappa_n$.

The verification of (2) will be based on the geometric arrangement of elements in A_m. These elements are in κ_m rows and $\kappa_0=\aleph_0$ columns. The subset $\{a_0,a_1,a_2,\dots\}$, $\{b_0,b_1,b_2,\dots\}$, $\{c_0,c_1,c_2,\dots\}$, and $\{d_0,d_1,d_2,\dots\}$ of sub-0-boxes of S_m are called *rows* (a-row, b-row, c-row and d-row), while a_j, b_j, c_j and d_j of sub-0-boxes belong to the j-the *column*. For $u\in A_m$ we introduce the notation $\mathrm{col}(u)=j$ to express the fact that u is in the j-th column. For an edge-triplet e, let $\mathrm{col}(e)$ denote $\{\mathrm{col}(b_e),\mathrm{col}(b'_e),\mathrm{col}(b''_e)\}$. It is worth mentioning that for $\tau\in\{\beta_m,\beta^*_m\}$, $\rho\in\{\gamma_m,\gamma^*_m\}$ and $p,q\in A_m$

$$\langle p,q\rangle\in\tau\vee\rho\Longrightarrow|\mathrm{col}(p)-\mathrm{col}(q)|<3. \tag{3}$$

This explains why the "column distance" of edges in an edge triplet is chosen to be three in the construction. Some other, more or less self-explaining, terminology induced by the "geometry" of A_m will also be used. For example, α_m is *row preserving* and β_m is *column preserving*. If $\tau\in\mathrm{Quord}(A_m)$ and $X\subseteq A_m$ has the property that $u\in X$ and $\langle u,v\rangle\in\tau$ imply $v\in X$, then X is said to be *closed with respect to* τ. E.g., columns are closed with respect to β_m and rows are closed with respect to α^*_m. If $\langle u,v\rangle\in\tau$ and $u\neq v$ imply $\mathrm{col}(u)\neq\mathrm{col}(v)$ resp. $\mathrm{col}(u)<\mathrm{col}(v)$, then τ is said to be *column changing* resp. *column increasing*. If, for some $i\neq j$, $\langle u,v\rangle\in\tau$ and $u\neq v$ imply $\mathrm{col}(u)=i$ and $\mathrm{col}(v)=j$, then we say that τ *changes the column* from i to j. Associated with a 0-box or 0-scheme we may speak of its *halves*; the a-row and b-row form the *upper half* while c-row and d-row constitute the *lower half*.

Now define

$$f^0_{a_0,b_0}=y\wedge(x\vee z^*)\quad\text{and}\quad f^0_{c_0,d_0}=y\wedge(x^*\vee z^*).$$

In order to show (2) for $f^0_{a_0,b_0}$, suppose $u,v\in A_m$ are distinct elements and $\langle u,v\rangle\in f^0_{a_0,b_0}(\alpha_m,\beta_m,\gamma_m)=\beta_m\wedge(\alpha_m\vee\gamma_m^*)$. From $\langle u,v\rangle\in\beta_m$ we conclude $\mathrm{col}(u)=\mathrm{col}(v)$, whence u and v are in distinct rows. Since $\langle u,v\rangle\in\alpha_m\vee\gamma_m^*$, u and v belong to the same sub-0-box B_0, and even to the same half of B_0. Since the γ-arrows "go up" (cf. Figure 1), either u is in the a-row and v is in the b-row or u is in the c-row and v is in the d-row of B_0. Therefore $\mathrm{col}(u)=0$, for otherwise the β-arrow would go up between the rows of u and v. Thus $\langle u,v\rangle\in\{\langle a_0,b_0\rangle,\langle c_0,d_0\rangle\}$. But $\langle c_0,d_0\rangle\notin\alpha_m\vee\gamma_m^*$, for c_0 is a maximal element with respect to α_m and it is isolated with respect to γ_m. So $\langle u,v\rangle\in\langle a_0,b_0\rangle\in\langle a_0,b_0\rangle^{(0,m)}$. The inclusion $\langle a_0,b_0\rangle^{(0,m)}\subseteq f^0_{a_0,b_0}(\alpha_m,\beta_m,\gamma_m)$ is evident, hence we have shown that (2) holds for $f^0_{a_0,b_0}$. The treatment for $f^0_{c_0,d_0}$ is very similar.

Simple considerations like the above for $f^0_{a_0,b_0}$ will not be detailed usually. Moreover, when we define a term $f^n_{p,q}$ in the sequel without further reasoning, this definition should be understood also as a statement claiming (2) for the term in question; the proof of this implicit assertion is left to the reader.

Now we assume that $f^0_{a_i,b_i}$ and $f^0_{c_i,d_i}$ satisfying (2) are already defined. Let

$$\begin{aligned}
f^0_{a_i,a_{i+1}}&=x\wedge(f^0_{a_i,b_i}\vee z),\\
f^0_{c_i,c_{i+1}}&=\begin{cases} x^*\wedge(f^0_{c_0,d_0}\vee z), & \text{if } i=0\\ x\wedge(f^0_{c_i,d_i}\vee z), & \text{if } i>0,\end{cases}\\
f^0_{b_i,a_{i+1}}&=z\wedge(f^0_{b_i,a_i}\vee f^0_{a_i,a_{i+1}}),\\
f^0_{d_i,c_{i+1}}&=z\wedge(f^0_{d_i,c_i}\vee f^0_{c_i,c_{i+1}}),\\
f^0_{b_i,b_{i+1}}&=x^*\wedge(f^0_{b_i,a_{i+1}}\vee y^*),\\
f^0_{d_i,d_{i+1}}&=x^*\wedge(f^0_{d_i,c_{i+1}}\vee y^*),\\
f^0_{a_{i+1},b_{i+1}}&=y^*\wedge(f^0_{a_{i+1},b_i}\vee f^0_{b_i,b_{i+1}}),\\
f^0_{c_{i+1},d_{i+1}}&=y^*\wedge(f^0_{c_{i+1},d_i}\vee f^0_{d_i,d_{i+1}}).
\end{aligned}$$

For example, the argument proving (2) for $f^0_{b_i,b_{i+1}}$ runs as follows. Suppose $u,v\in A_m$ are distinct elements and $\langle u,v\rangle\in f^0_{b_i,b_{i+1}}(\alpha_m,\beta_m,\gamma_m)=\alpha_m^*\wedge(\langle b_i,a_{i+1}\rangle^{(0,m)}\vee\beta_m^*)$. Since α_m^* is row preserving, u and v are in the same row (and in the same sub-0-box) but in distinct columns. There are distinct elements $w_0=u,w_1,\dots,w_t=v$ in A_m such that $\langle w_{j-1},w_j\rangle\in\langle b_i,a_{i+1}\rangle^{(0,m)}\cup\beta_m^*$ for all j. Since β_m^* is column preserving and $\langle b_i,a_{i+1}\rangle^{(0,m)}$ changes the column from i to $i+1$, there is a k such that $\langle w_{k-1},w_k\rangle\in\langle b_i,a_{i+1}\rangle^{(0,m)}$ and $\langle w_{j-1},w_j\rangle\in\beta_m^*$ for all $j\neq k$. Hence $\langle u,w_{k-1}\rangle$, $\langle w_k,v\rangle\in\beta_m^*$, $\mathrm{col}(u)=\mathrm{col}(w_{k-1})=i$ and $\mathrm{col}(v)=\mathrm{col}(w_k)=i+1$. Suppose i is not a multiple of 3 (the other case, when 3 does not divide $i+1$, is similar). The intersection of the i-th column with an arbitrary sub-0-box (and the upper half of this sub-0-box) is closed with respect to β_m^* and $\langle b_i, a_{i+1}\rangle^{(0,m)}$. Hence u, w_{k-1}, w_k and v belong to the upper half of the same sub-0-box B_0 of S_m, and $w_{k-1} = b_i$, $w_k = a_{i+1}$ in B_0. Therefore (2) for $f^0_{b_i,b_{i+1}}$ follows easily from Figure 1.

Now let $p \neq q$ belong to the same half (upper or lower) of S_0, and consider the smallest circle in the undirected variant of the graph on Figure 1 which contains p, q and consists of

$$\text{vertical and horizontal} \tag{4}$$

edges only, and goes within the same half of S_0 that contains p and q. Let $\{p = r_0, r_1, r_2, \dots, r_i = q, r_{i+1}, \dots, r_{k-1}, r_k = r_0 = p\}$ be this circle (which is uniquely determined, the elements are listed anti-clockwise); the elements $r_0, r_1, \dots, r_{k-1}$ are pairwise distinct. Define

$$f^0_{p,q} = (f^0_{r_0,r_1} \vee f^0_{r_1,r_2} \vee \dots \vee f^0_{r_{i-1},r_i}) \wedge (f^0_{r_k,r_{k-1}} \vee f^0_{r_{k-1},r_{k-2}} \vee \dots \vee f^0_{r_{i+1},r_i}).$$

Now we can set

$$f^0_{b_0,c_0} = y \wedge (f^0_{b_0,b_3} \vee y^* \vee f^0_{c_3,c_0}) \wedge (f^0_{b_0,b_6} \vee y^* \vee f^0_{c_6,c_0}) \quad \text{and}$$
$$f^0_{b_3,c_3} = y^* \wedge (f^0_{b_3,b_0} \vee f^0_{b_0,c_0} \vee f^0_{c_0,c_3}).$$

Now suppose that p is in the upper half and q is in the lower half of S_0, and define

$$f^0_{p,q} = (f^0_{p,b_0} \vee f^0_{b_0,c_0} \vee f^0_{c_0,q}) \wedge (f^0_{p,b_3} \vee f^0_{b_3,c_3} \vee f^0_{c_3,q}).$$

We have defined all the f^0 terms, and these terms satisfy (2).

Now let us assume that appropriate ternary terms $f^n_{p,q}$ ($p, q \in A_n$) are already defined (and they satisfy (2)); we start defining the f^{n+1} terms.

First assume that $p, q \in A_{n+1}$ belong to the same sub-n-box $B_n(U)$ of S_{n+1} such that $\mathrm{col}(p) \neq \mathrm{col}(q)$ and neither $\mathrm{col}(p)$ nor $\mathrm{col}(q)$ is divisible by 3. Here $U \in P(D_n^{(n)})$. Let

$$f^{n+1}_{p,q} = f^n_{p,q} \wedge \bigwedge_{e \in U} (f^n_{p,b_e} \vee y \vee f^n_{c_e,q}) \wedge \bigwedge_{e \in D_n^{(n)} \setminus U} (f^n_{p,b_e} \vee y^* \vee f^n_{c_e,q}).$$

To show that this term satisfies (2), let $m \geqq n+1$, and consider an m-box B_m. By definitions and the validity of (2) for f^n-terms we obtain

$$\langle p, q\rangle^{(n+1,m)} \subseteqq f^{n+1}_{p,q}(\alpha_m, \beta_m, \gamma_m) \subseteqq f^n_{p,q}(\alpha_m, \beta_m, \gamma_m) = \langle p, q\rangle^{(n,m)}. \tag{5}$$

To show that the first inclusion in (5) is in fact an equality, suppose there is a pair $\langle u,v\rangle\in f^{n+1}_{p,q}(\alpha_m,\beta_m,\gamma_m)\setminus\langle p,q\rangle^{(n+1,m)}$. It follows from (5) that $u=p(V)$ and $v=q(V)$ in some sub-n-box $B_n(V)$ of B_m. Here $B_n(V)$ belongs to a unique sub-$(n+1)$-box of B_m, $V\in P(D^{(n)}_n)$, and $p(V)$, $q(V)$ are the elements of $B_n(V)$ that correspond to $p,q\in B_n(U)$. From $\langle u,v\rangle\notin\langle p,q\rangle^{(n+1,m)}$ we conclude that $U\neq V$. Let e be an edge triplet in $(U\setminus V)\cup(V\setminus U)$. Since $\mathrm{col}(b_e)$ is divisible by 3, the elements u, v and b_e belong to distinct columns.

Suppose first that $e\in U\setminus V$; we claim that $\langle u,v\rangle=\langle p(V),q(V)\rangle$ does not belong to $f^n_{p,b_e}(\alpha_m,\beta_m,\gamma_m)\vee\beta_m\vee f^n_{c_e,q}(\alpha_m,\beta_m,\gamma_m)=\langle p,b_e\rangle^{(n,m)}\vee\beta_m\vee\langle c_e,q\rangle^{(n,m)}$. Indeed, let us assume the opposite. Then there is a shortest sequence (of distinct elements) $w_0=p(V)$, $w_1,w_2,\dots,w_t=q(V)$ such that $\langle w_{i-1},w_i\rangle\in\langle p,b_e\rangle^{(n,m)}\cup\beta_m\cup\langle c_e,q\rangle^{(n,m)}$ for all i. Since β_m is column preserving, $\langle p,b_e\rangle^{(n,m)}$ changes the column (only) from $\mathrm{col}(p)$ to $\mathrm{col}(b_e)=\mathrm{col}(c_e)$ and $\langle c_e,q\rangle^{(n,m)}$ changes the column from $\mathrm{col}(c_e)$ to $\mathrm{col}(q)$, all the w_i belong to the $\mathrm{col}(p)$-th, $\mathrm{col}(q)$-th and $\mathrm{col}(c_e)$-th columns. By the construction, no e^l_k has an element in these three columns, whence the intersection of these columns with $B_n(V)$ (or even with any sub-0-box) is closed with respect to β_m. Consequently, all the w_i belong to the same sub-n-box, i.e., to $B_n(V)$. Our present information on the columns $\mathrm{col}(w_i)$ imply that, within $B_n(V)$,

$$p(V)=w_0\ \langle p,b_e\rangle^{(n,m)}\ w_1=b_e(V)\ \beta_m\ w_2=c_e(V)\ \langle c_e,q\rangle^{(n,m)}\ w_3=q(V)$$

is the only possibility. But this is a contradiction, for $\langle b_e(V),c_e(V)\rangle$ is not in β_m (in fact, it is in β^*_m) by the construction. For $e\in V\setminus U$, $\langle u,v\rangle\notin f^n_{p,b_e}(\alpha_m,\beta_m,\gamma_m)\vee\beta^*_m\vee f^n_{c_e,q}(\alpha_m,\beta_m,\gamma_m)$ follows similarly. Thus (2) holds for $f^{n+1}_{p,q}$.

Now let us assume that $p,q\in A_{n+1}$, $p\neq q$, still belong to the same sub-n-box $B_n(U)$ of S_{n+1} but the previous additional assumption does not hold (i.e., $\mathrm{col}(p)=\mathrm{col}(q)$ or $3\mid\mathrm{col}(p)$ or $3\mid\mathrm{col}(q)$). Choose elements $p',p'',q',q''\in A_{n+1}$ such that p,p',p'' are in the same row, q,q',q'' are in the same (possibly another) row, none of $\mathrm{col}(p')$, $\mathrm{col}(p'')$, $\mathrm{col}(q')$, $\mathrm{col}(q'')$ is divisible by 3, $|\{\mathrm{col}(p'),\mathrm{col}(p''),\mathrm{col}(q'),\mathrm{col}(q'')\}|=4$ and $\{\mathrm{col}(p'),\mathrm{col}(p''),\mathrm{col}(q'),\mathrm{col}(q'')\}\cap\{\mathrm{col}(p),\mathrm{col}(q)\}=\emptyset$. Note that this choice can be made unique by fixing an appropriate $\mathbf{N}_0^2\to\mathbf{N}_0^4$ map and requiring $\langle\mathrm{col}(p),\mathrm{col}(q)\rangle\mapsto$ $\mapsto\langle\mathrm{col}(p'),\mathrm{col}(p''),\mathrm{col}(q'),\mathrm{col}(q'')\rangle$, but the explicit knowledge of this map is unimportant for us. Now we can define

$$f^{n+1}_{p,q}=(f^n_{p,p'}\vee f^{n+1}_{p',q'}\vee f^n_{q',q})\wedge(f^n_{p,p''}\vee f^{n+1}_{p'',q''}\vee f^n_{q'',q}).$$

This way we have defined all $f^{n+1}_{p,q}$ when p and q belong to the same sub-n-box of S_{n+1}. Now let us consider the sub-n-boxes $B_n(\emptyset)$ and $B_n(U)$ of S_{n+1}, $U\in P^+(D^{(n)}_n)$. The e^n_n edge triplets in these sub-n-boxes will be denoted by

$$e^n_n(\emptyset)=\big\langle\langle b_\emptyset,c_\emptyset\rangle,\langle b'_\emptyset,c'_\emptyset\rangle,\langle b''_\emptyset,c''_\emptyset\rangle\big\rangle\quad\text{and}\quad e^n_n(U)=\big\langle\langle b_U,c_U\rangle,\langle b'_U,c'_U\rangle,\langle b''_U,c''_U\rangle\big\rangle,$$

respectively. Let

$$f^{n+1}_{b_\emptyset,c_U} = y \wedge \left(f^{n+1}_{b_\emptyset,b'_\emptyset} \vee y^* \vee f^{n+1}_{c'_U,c_U}\right) \wedge \left(f^{n+1}_{b_\emptyset,b''_\emptyset} \vee y^* \vee f^{n+1}_{c''_U,c_U}\right).$$

To show that this term satisfies (2), suppose that B_{n+1} is a sub-$(n+1)$-box of some m-box B_m and, for distinct $u, v \in A_m$, $\langle u, v\rangle \in f^{n+1}_{b_\emptyset,c_U}(\alpha_m, \beta_m, \gamma_m) = \beta_m \wedge (\langle b_\emptyset, b'_\emptyset\rangle^{(n+1,m)} \vee \beta^*_m \vee \langle c'_U, c_U\rangle^{(n+1,m)}) \wedge (\langle b_\emptyset, b''_\emptyset\rangle^{(n+1,m)} \vee \beta^*_m \vee$ $\vee \langle c''_U, c_U\rangle^{(n+1,m)})$. Since $\langle u, v\rangle \in \beta_m$, $\mathrm{col}(u) = \mathrm{col}(v)$ and $\langle u, v\rangle \notin \beta^*_m$. Hence in any sequence $w_0 = u$, $w_1, \dots , w_t = v$ in A_m such that $\langle w_{i-1}, w_i\rangle \in \langle b_\emptyset, b'_\emptyset\rangle^{(n+1,m)} \cup \beta^*_m \cup \langle c'_U, c_U\rangle^{(n+1,m)}$ for all i not all the $\langle w_{i-1}, w_i\rangle$ belong to β^*_m. Therefore $\{\mathrm{col}(w_0), \mathrm{col}(w_1), \dots , \mathrm{col}(w_t)\} = \{\mathrm{col}(b_\emptyset), \mathrm{col}(b'_\emptyset)\}$ (which is the same as $\{\mathrm{col}(c_\emptyset), \mathrm{col}(c'_\emptyset)\}$). In particular, $\mathrm{col}(u) = \mathrm{col}(v) \in \{\mathrm{col}(b_\emptyset), \mathrm{col}(b'_\emptyset)\}$. Since $\mathrm{col}(u) \in \{\mathrm{col}(b_\emptyset), \mathrm{col}(b''_\emptyset)\}$ comes similarly, we obtain $\mathrm{col}(u) = \mathrm{col}(v) = \mathrm{col}(b_\emptyset)$. We can assume on the sequence that $\{\langle w_{i-1}, w_i\rangle, \langle w_i, w_{i+1}\rangle\} \subseteqq \beta^*_m$ holds for no i. Now the only possibility concerning the elements w_i is the following:

$$u \; \beta^*_m \; w_1 = b_\emptyset \; \langle b_\emptyset, b'_\emptyset\rangle^{(n+1,m)} \; w_2 = b'_\emptyset \; \beta^*_m \; w_3 = c'_U \; \langle c'_U, c_U\rangle^{(n+1,m)} \; w_4 = c_U \; \beta^*_m \; v.$$

Since $\mathrm{col}(e^n_n) \cap \mathrm{col}(e^k_k) = \emptyset$ for $k > n$, the intersection of a sub-$(n+1)$-box with the $\mathrm{col}(b_\emptyset)$-th or the $\mathrm{col}(b'_\emptyset)$-th column is closed with respect to β^*_m. Hence all the w_i, including u and v, belong to the same sub-$(n+1)$-box. Working within this sub-$(n+1)$-box, from

$$b_\emptyset \; \beta_m \; u \; \beta_m \; v \; \beta_m \; c_U,$$

$u \neq v$ and $b_\emptyset \prec c_U$ (with respect to the partial order β_m) we conclude $u = b_\emptyset$ and $v = c_U$. Thus $\langle u, v\rangle \in \langle b_\emptyset, c_U\rangle^{(n+1,m)}$, proving (2) for $f^{n+1}_{b_\emptyset,c_U}$.

We define

$$f^{n+1}_{b'_\emptyset,c'_U} = y^* \wedge \left(f^{n+1}_{b'_\emptyset,b_\emptyset} \vee f^{n+1}_{b_\emptyset,c_U} \vee f^{n+1}_{c_U,c'_U}\right).$$

Suppose now that $p \in B_n(\emptyset)$ and $q \in B_n(U)$ for $U \in P^+(D^{(n)}_n)$. (The previous cases, $\langle p, q\rangle = \langle b_\emptyset, c_U\rangle$ or $\langle p, q\rangle = \langle b'_\emptyset, c'_U\rangle$, are excluded, of course.) We can define

$$f^{n+1}_{p,q} = \left(f^{n+1}_{p,b_\emptyset} \vee f^{n+1}_{b_\emptyset,c_U} \vee f^{n+1}_{c_U,q}\right) \wedge \left(f^{n+1}_{p,b'_\emptyset} \vee f^{n+1}_{b'_\emptyset,c'_U} \vee f^{n+1}_{c'_U,q}\right).$$

Finally, let $p \in B_n(U_1)$ and $q \in B_n(U_2)$ for distinct $U_1, U_2 \in P^+(D^{(n)}_n)$, and let $b_\emptyset, c_\emptyset \in B_n(\emptyset)$ be as before. We define

$$f^{n+1}_{p,q} = \left(f^{n+1}_{p,b_\emptyset} \vee f^{n+1}_{b_\emptyset,q}\right) \wedge \left(f^{n+1}_{p,c_\emptyset} \vee f^{n+1}_{c_\emptyset,q}\right).$$

We have defined $f_{p,q}^{n+1}$ satisfying (2) for all $p, q \in A_{n+1}$. The induction is complete. So Theorem 1 is proved for all Quord(A_n), i.e., for Quord(A) with $|A| = \kappa_n$. Now if A is infinite and $|A| < \kappa_\omega$, then $\kappa_n \leqq |A| < \kappa_{n+1}$ for some n. We may suppose $\kappa_n < |A| < \kappa_{n+1}$ (note that this case would not occur if we assumed the generalized continuum hypothesis). Then simply modifying the construction of A_{n+1} so that we replace $P(D_n^{(n)})$ by a subset $\hat{P}(D_n^{(n)})$ of it such that $|\hat{P}(D_n^{(n)})| = |A|$ and $\emptyset \in \hat{P}(D_n^{(n)})$ we easily obtain the result for Quord(A).

Now let us deal with the finite case. If A consists of three elements a, b and c, then Quord(A) is clearly generated by $\langle a, b\rangle^{\mathrm{qo}}$, $\langle b, c\rangle^{\mathrm{qo}}$ and $\langle c, a\rangle^{\mathrm{qo}}$. If $|A| = 2k \geqq 4$, then we can restrict α_0, β_0 and γ_0 to $A = \{a_0, a_1, \dots, a_{k-1}, b_0, b_1, \dots, b_{k-1}\}$; the terms $f_{p,q}^0$ for $p, q \in A$ still satisfy (1). The odd case, $A = \{a_0, a_1, \dots, a_k, b_0, b_1, \dots, b_{k-1}\}$ ($k \geqq 2$), is essentially the same, but instead of (4) we have to say

(4') vertical, horizontal and $\{b_{k-1}, b_k\}$.

The proof of Theorem 1 is complete.

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DEPARTMENT OF ALGEBRA AND GEOMETRY
PALACKÝ UNIVERSITY OF OLOMOUC
TOMKOVA 40
CZ–779 00 OLOMOUC
CZECH REPUBLIC

chajda@risc.upol.cz

JÓZSEF ATTILA TUDOMÁNYEGYETEM
BOLYAI INTÉZETE
ARADI VÉRTANÚK TERE 1
H–6720 SZEGED
HUNGARY

czedli@math.u-szeged.hu

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THE DEALER'S RANDOM BITS IN PERFECT SECRET SHARING SCHEMES

L. CSIRMAZ

Abstract

A secret sharing scheme permits a secret to be shared among participants of an n-element group in such a way that only qualified subsets of participants can recover the secret. If any non-qualified subset has absolutely no information on the secret, then the scheme is called *perfect*. The *share* in a scheme is the information what a participant must remember. It was known that in any perfect secret sharing scheme realizing a certain collection of qualified sets over n participant, at least one participant must use at least $O(n/\log n)$ random bits for each bit in the secret. Here we present a collection of qualified sets so that the total number of random bits used by all the participants, i.e. the *dealer's random bits* is at least $O(n^2/\log n)$ for each bit in the secret.

1. Introduction

An important issue in secret sharing systems is the size of the shares distributed among the participants which has received considerable attention in the last few years, see e.g. [16], [5], [6], [9] etc. The reason is practical on one hand: the more information must be kept secret the less secure the system is since human being are not too good at remembering even medium size random data. On the other hand the problem is theoretically intriguing, too. All the known general constructions which work for arbitrary access structures assign exponentially large shares. For a long time even it was not known whether the size of the shares should tend to infinity. The first results in this direction were [7] and [9] where an almost linear lower bound was given. In [9] the question for access structures based on graphs was settled: here the lower and upper bounds agree. For other access structures still there is a gap, and as it was remarked in [7], any better lower bound would yield an affirmative answer to a long standing question in information theory: are there more (linear) inequalities among the joint entropies of random variables which are not consequences of the known ones [8]?

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In this paper we construct an access structure on which any perfect secret sharing scheme must use $n/4\log n$ random bits for each secret bit *on average* for each participant, i.e. total $n^2/4\log n$ bits. The construction in [7] gave an access structure where *some* (in fact, at least $\log n$) participant must use $O(n/\log n)$ random bits.

The paper is arranged as follows. First we give some definitions, and cite notions and facts from information theory which we shall use. Then we present the construction and prove that it is good. Finally we outline a conjecture about the entropy function.

2. Prerequisites

In this section we review the technical concepts both from information theory and from secret sharing which will be used in this paper. For a more complete treatment of information theory the reader is referred to [8]; its application to secret sharing is explained in [5].

2.1. Information theoretic notions. Given a probability distribution $\{p(x)\}_{x\in X}$ in a finite set X, define the *entropy* of X as

$$H(X) = -\sum_{x\in X} p(x)\log_2 p(x).$$

The entropy $H(X)$ is a measure of the average information content of the elements in X. By definition, the entropy is always non-negative.

Given two sets X and Y and a joint probability distribution $\{p(x,y)\}_{x\in X, y\in Y}$ on the Cartesian product of X and Y, the *conditional entropy* $H(X|Y)$ of X assuming Y is defined as

$$H(X|Y) = \sum_{y\in Y} p(y)H(X|Y=y), \tag{1}$$

where "$X|Y=y$" is the probability distribution got from p by fixing the value $y\in Y$. The conditional entropy can also be given in the form

$$H(X|Y) = H(XY) - H(Y) \tag{2}$$

where Y is the marginal distribution. From definition (1) it is easy to see that $H(X|Y)\geqq 0$.

The *mutual information* between X and Y is defined by

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X) \\ &= H(X) + H(Y) - H(XY) \end{aligned}$$

and is always non-negative: $I(X;Y)\geqq 0$. This inequality expresses the intuitive fact that the knowledge of Y, on average, can only decrease the uncertainty one has on X.

Similarly to the conditional entropy, the *conditional mutual information* between X and Y given Z is defined as

$$I(X;Y|Z) = H(X|Z) - H(X|YZ) = \\ = H(XZ) + H(YZ) - H(XYZ) - H(Z), \tag{3}$$

and is also non-negative: $I(X;Y|Z) \geqq 0$. In fact, the only known (linear) inequalities for the entropy function are $H(X) \geqq 0$, $H(X|Y) \geqq 0$ and $I(x;Y|Z) \geqq 0$ and their algebraic consequences. One of the open questions in information theory is to find more, or to show that there are none. We shall say more about it in the last section.

2.2. Secret sharing schemes. In the following individuals will be denoted by small letters: a, b, x, y, etc., sets (groups) of individuals by capital letters A, B, X, Y, etc., finally collection of groups by script letters $\mathcal{A}$, $\mathcal{B}$. We use P to denote the set of *participants* who will share the secret.

An *access structure* on an n-element set P of participants is a collection $\mathcal{A}$ of subsets of P: exactly the qualified groups are collected into $\mathcal{A}$. We shall denote a group simply by listing its members, so x denotes both a member of P and the group which consists solely of x.

A *secret sharing scheme* permits a secret to be shared among n participants in such a way that only qualified subsets of them can recover the secret. Secret sharing schemes satisfying the additional property that unqualified subsets can gain absolutely no information about the secret are called *perfect* as opposed to schemes where unqualified groups may have some information on the secret.

A natural property of access structures is *monotonicity*, i.e. $A \in \mathcal{A}$ and $A \subseteqq B \subseteqq P$ implies that $B \in \mathcal{A}$. This property expresses the fact that if any subset can recover the secret, then the whole group can also recover the secret. Also, a natural requirement is that the empty set should not be in $\mathcal{A}$, i.e. there must be some secret at all. Thus we may concentrate on *minimal* qualified subsets, no members of which can be dismissed without changing the subset into an unqualified one. We say that the access structure is *generated* by its minimal elements,

Let P be the set of participants, $\mathcal{A}$ be an access structure, and S be the set of possible secrets. A *secret sharing scheme*, given a secret $s \in S$, assigns to each member $x \in p$ a random *share* from some domain. The shares are thus random variables with some disjoint distribution determined by the value of the secret s. Thus a scheme can be regarded as a collection of random variables, one for the secret, and one for each $x \in P$. The scheme determines the joint distribution of these $n+1$ random variables. For $x \in P$ the x's share, which is (the value of) a random variable, will also be denoted by x. For a subset $A \subseteqq P$ of participants, A also denotes the joint (marginal) distribution of the shares assigned to the participants in A.

Following [5] we call the scheme *perfect* if the following hold:

1. Any qualified subset can reconstruct the secret, that is, the shares got by the participants in A determine uniquely the secret. This means $H(s|A)=0$ for all $A\in\mathcal{A}$.
2. Any non qualified subset has absolutely no information on the secret, i.e. s and the shares got by members of A are statistically independent: knowing the shares in A, the conditional distribution of s is exactly the same as its a priori distribution. Translated to information theoretic notions this gives $H(s|A)=H(s)$ for all $A\notin\mathcal{A}$.

By the above description the entropy of the secret, $H(s)$ can be considered as the *length* of the secret. Any lower bound on the entropy of $x\in P$ gives immediately a lower bound on the size of x's share, and any lower bound on any subset $X\subseteqq P$ of participants gives a lower bound on the *total* amount of random bits the dealer must have when distributing the shares among the participants.

2.3. Polymatroid structure. Let Q be any finite set, and $\mathcal{B}=2^Q$ be the collection of the subsets of Q. Let $f:\mathcal{B}\to\mathbf{R}$ be a function assigning real numbers to subsets of Q and suppose f satisfies the following conditions:

(i) $f(A)\geqq 0$ for all $A\subseteqq Q$, $f(\emptyset)=0$,

(ii) f is monotone, i.e. if $A\subseteqq B\subseteqq Q$ then $f(A)\leqq f(B)$,

(iii) f is submodular, i.e. if A and B are different subsets of Q then

$$f(A)+f(B)\geqq f(A\cap B)+f(A\cup B).$$

The system (Q,f) is called *polymatroid*. If, in addition, f takes only integer values and $f(x)\leqq 1$ for one-element subsets, then the system is a *matroid*.

S. Fujishige in [10] observed that having a finite collection of random variables, we will get a polymatroid by assigning the entropy to each subset. The proof of the following proposition can also be found in [14].

PROPOSITION 2.1. *By defining $f(A)=H(A)/H(s)$ for each $A\subseteqq P\cup\{s\}$ we get a polymatroid.*

In our case the random variable s, the "secret" plays a special role. By our extra assumptions on the conditional entropies containing s we can calculate the value of $f(As)$ from $f(A)$ for any $A\subseteqq P$, see [5], [14].

PROPOSITION 2.2. *The secret sharing scheme is perfect if and only if for any $A\subseteqq P$ we have*
if $A\in\mathcal{A}$ then $f(As)=f(A)$;
if $A\notin\mathcal{A}$ then $f(As)=f(A)+1$.

Now let us consider the function f defined in Proposition 2.1 restricted to the subsets of P. From this restriction we can calculate easily the whole function; and since the extension is also a polymatroid, the restriction will satisfy some additional inequalities.

PROPOSITION 2.3. *The function f defined in Proposition 2.1 satisfies the following additional inequalities:*
(i) *if $A \subseteq B$, $A \notin \mathcal{A}$ and $B \in \mathcal{A}$ then $f(B) \geq f(A) + 1$;*
(ii) *if $A \in \mathcal{A}$, $B \in \mathcal{A}$ but $A \cap B \notin \mathcal{A}$ then $f(A) + f(B) \geq f(A \cap B) + f(A \cup B) + 1$.*

The method can be outlined as follows. We define an access structure on the n-element set P, an $A \subseteq P$, and show that for *any* polymatroid (P, f) satisfying (i) and (ii) above we have $f(A) \geq n^2/4 \log n$. By the discussion at the beginning of this section this implies that for any perfect secret sharing scheme $H(A)/H(s) \geq n^2/4 \log n$. This means that members of A have to remember $n/4 \log n$ bits for every secret bit on the average, and also that the dealer must use at least $n^2/4 \log n$ random bits for each secret bit for distributing the shares to the members of $\mathcal{A}$.

3. The construction

The first lemma expresses a trivial fact about qualified and unqualified subsets.

LEMMA 3.1. *Let $A_1, \ldots, A_k$, and $B_1, \ldots, B_l$ be subsets of P. There exists an access structure $\mathcal{A}$ on P for which all A_i are qualified and all B_j are unqualified if and only if no A_i is a subset of B_j.*

The next lemma is also the main lemma in [7]. Let $k > 1$ and $t < 2^k - 1$; X be a k-element set, $X_0 = X$, $X_1, \ldots, X_{2^k-1} = \emptyset$ all the subsets of X in such an order that if $i < j$ then $X_i \nsubseteq X_j$. (Reverse order, for example, by the size of the subsets.) Let $b_1, \ldots, b_t$ be individuals, not in X, and $B_0 = \emptyset$, $B_1 = \{b_1\}$, in general $B_j = \{b_1, \ldots, b_j\}$ for $j \leq t$.

LEMMA 3.2. *Let $\mathcal{A}$ be an access structure on P, (P, f) be a polymatroid satisfying* (i) *and* (ii) *of Proposition 2.3; $Y \subseteq P$, X_j and B_j as above. Suppose that for each $j \leq t$, $Y \cup B_j \cup X_j \in \mathcal{A}$ and $Y \cup B_j \cup X_{j+1} \notin \mathcal{A}$. Then*

$$f(X \cup Y) - f(Y) \geq t + 1.$$

PROOF. Observe that $Y \cup B_j \notin \mathcal{A}$ since it has a superset not in $\mathcal{A}$, and $Y \cup B_j \cup X \in \mathcal{A}$ since it has a subset in $\mathcal{A}$. Thus (i) of Proposition 2.3 gives immediately

$$f(Y \cup B_j \cup X) - f(Y \cup B_j) \geq 1. \tag{4}$$

Similarly, for each $0 \leq j < t$, (ii) of Proposition 2.3 gives

$$f(Y \cup B_{j+1} \cup X_{j+1}) + f(Y \cup B_j \cup X) \geq f(Y \cup B_j \cup X_{j+1}) + f(Y \cup B_{j+1} \cup X) + 1.$$

The submodular inequality applied to $Y \cup B_{j+1}$ and $Y \cup B_j \cup X_{j+1}$ yields

$$f(Y \cup B_{j+1}) + f(Y \cup B_j \cup X_{j+1}) \geq f(Y \cup B_j) + f(Y \cup B_{j+1} \cup X_{j+1}).$$

Adding up the last two inequalities, after rearranging we get

(5) $$[f(Y \cup B_j \cup X) - f(Y \cup B_j)] - [f(Y \cup B_{j+1} \cup X) - f(Y \cup B_{j+1})] \geqq 1.$$

This holds for $j = 0, \ldots, j = t-1$. Since $B_0 = \emptyset$, adding (5) for all of these values to (4) gives the claim of the lemma. □

THEOREM 3.3. *Let $k > 1$, $t < 2^k - 1$, $s \geqq 1$. There is an access structure $\mathcal{A}$ on an $n = t + sk + \lceil \log_2 s \rceil$ element set P so that for any polymatroid (P, f) satisfying the conditions of Proposition 2.3, $f(P) \geqq s(t+1)$.*

PROOF. Let $B_t = \{b_1, \ldots, b_t\}$; have $X^{(i)}$ exactly k elements for $1 \leqq i \leqq s$, finally let Z be a $\lceil \log_2 s \rceil$ element set, and $Z_1, Z_2, \ldots, Z_s$ be subsets of Z such that if $i < j$ then $Z_i \not\subseteqq Z_j$. The set of participants P will be just the union of the disjoint sets B_t, $X^{(i)}$ and Z, obviously $|P| = n$. Let moreover $W^{(1)} = \emptyset$, $W^{(2)} = X^{(1)}$, $W^{(3)} = X^{(1)} \cup X^{(2)}$, $\ldots$, $W^{(s+1)} = X^{(1)} \cup \ldots \cup X^{(s)}$, and $Y^{(i)} = Z_i \cup W^{(i)}$ for $1 \leqq i \leqq s+1$.

Applying Lemma 3.2 to the sets $Y^{(i)}$, $B_1, \ldots, B_t$, and $X^{(i)}$ we get

$$f(X^{(i)} \cup Y^{(i)}) - f(Y^{(i)}) \geqq t+1,$$

i.e.

$$f(Z_i \cup W^{(i+1)}) - f(Z_i \cup W^{(i)}) \geqq t+1.$$

The submodularity applied to $W^{(i+1)}$ and $Z_i \cup W^{(i)}$ gives

$$[f(W^{(i+1)}) - f(W^{(i)})] - [f(Z_i \cup W^{(i+1)}) - f(Z_i \cup W^{(i)})] \geqq 0,$$

from where we get

$$f(W^{(i+1)}) - f(W^{(i)}) \geqq t+1.$$

Since $f(W^{(1)}) = f(\emptyset) = 0$ and $f(P) \geqq f(W^{(s+1)})$ the claim of the theorem follows. We still have to check that there is an access structure so that conditions of Lemma 3.2 hold. Let the subsets of $X^{(i)}$ be $X_j^{(i)}$ as in the lemma; picking any i and j we must have $Y^{(i)} \cup B_j \cup X_j^{(i)} \in \mathcal{A}$, and $Y^{(k)} \cup B_l \cup X_{l+1}^{(k)} \notin \mathcal{A}$. By our observation 3.1 such an access structure exists if for no two different pairs (i, j) and (k, l)

$$Y^{(i)} \cup B_j \cup X_j^{(i)} \subseteqq Y^{(k)} \cup B_l \cup X_{l+1}^{(k)}.$$

Suppose on the contrary that this is the case. Replacing Y's with their definitions this means

$$Z_i \cup W^{(i)} \cup X_j^{(i)} \cup B_j \subseteqq Z_k \cup W^{(k)} \cup X_{l+1}^{(k)} \cup B_l.$$

Since Z, B, and $W^{(s+1)}$ are pairwise disjoint, this inclusion means that $Z_i \subseteqq Z_k$, $W^{(i)} \cup X_j^{(i)} \subseteqq W^{(k)} \cup X_{l+1}^{(k)}$, and $B_j \subseteqq B_l$. Now, if $i < k$ then by the choice of the Z's, $Z_i \not\subseteq Z_k$, and if $i > k$ then $W^{(i)}$ is a proper superset (and not a subset) of $W^{(k)} \cup X_{l+1}^{(k)}$. Therefore we must have $i = k$. Similarly, if $j < l+1$ then $X_j^{(i)}$ is not a subset of $X_{l+1}^{(k)} = X_{l+1}^{(i)}$, finally if $j \geqq l+1$ then B_j is a proper superset of B_l. No cases left, the claim is proved. □

To get the result announced in the Introduction, choose $k = \log(n/2)$, $t = n/2$, and $s = n/(2\log n)$, this gives $f(P) \geqq n^2/(4\log n)$, as claimed. The following table summarizes the best values for k, t and s, and the coefficient λ so that $f = n^2/\lambda_n \log_2 n$. It is not hard to see that λ_n converges to 4 as n tends to infinity.

n	f	λ_n	t	s	k
3	2	2.839184	1	1	2
4	2	4.000000	1	1	3
5	3	3.588971	2	1	3
10	8	3.762875	3	2	3
20	24	3.856304	11	2	4
30	52	3.527222	12	4	4
50	116	3.818617	28	4	5
100	400	3.762875	49	8	6
200	1386	3.775585	98	14	7
400	4900	3.777603	195	25	8
800	17556	3.780103	398	44	9
1600	63520	3.786435	793	80	10
3200	231710	3.795407	1597	145	11
6400	851200	3.805825	3199	266	12

4. Conclusion

There are several general methods for generating shares, see [2], [16]. These usually work well on "structured" access structures, but assign exponentially large shares on the worst case. We have constructed an access structure on an n-element group so that in any perfect secret sharing scheme the dealer must use at least $n^2/4\log n$ random bits for each bit in the secret. This shows that any method must assign almost linear shares on the average in some cases.

Karchmer and Wigderson in [13] showed that there is a strong connection between the so-called (monotone) span programs and certain secret sharing schemes. Thus our result also gives immediately a lower bound on the size of monotone span programs. Beimel, Gál, and Paterson in [1] gave general lower bounds for the size of monotone span programs, which implies that for some access structure on n participants, if the scheme is of Karchmer–Wigderson type the dealer must use at least cn^2 random bits for each secret bit.

Given any, say random, access structure $\mathcal{A}$ on a set P of n participants, all perfect secret sharing schemes can be generated as follows.

(i) devise a polymatroid (P, f) satisfying the conditions of Proposition 2.2, and then

(ii) *realize* f by assigning random variables to each participant so that for each $A \subseteq P$, $f(A) = \lambda H(A)$ for some constant λ.

The total number of random bits used by the dealer will then be $\lambda H(P)$.

The lower bound proved in the present paper comes from (i). We showed that for a particular access structure every feasible polymatroid must satisfy $f(P) \geqq O(n^2/\log n)$. We cannot push it higher since for any access structure there exists a polymatroid with $f(P) \leqq n^2$. Thus we have to concentrate on (ii) and consider only *representable* polymatroids. Unfortunately very little is known along this line. For $n \leqq 3$ all polymatroids are representable. For $n = 4$ F. Matuš in [15] gives a non representable polymatroid. In fact, he proves that if $P = \{a, b, c, d\}$ and (P, f) is a polymatroid then

$$f(ac) + f(bc) + f(ad) + f(bd) + f(cd) - f(c) - f(d) - f(acd) - f(bcd) - f(ab) \geqq -\frac{1}{4} f(abcd),$$

and for representable polymatroids equality holds only if $f(abcd) = 0$. There are polymatroids for which equality holds here, thus they are not representable. It is interesting to note that the left-hand side also appears in matroid theory: it cannot be negative for matroids representable over fields [11]. For those more familiar with the entropy function the above inequality can be written as

$$I(a; b) + I(c; d|a) + I(c; d|b) - I(c; d) \geqq -\frac{1}{4} H(abcd),$$

and is the consequence of the usual entropy inequalities. We conjecture that for representable polymatroids (i.e. for random variables) the constant 1/4 can be replaced by a much smaller value. Showing that it holds with any value less than 1/4 would also give a new linear inequality for the entropy thus settling an important open problem of information theory.

Conjecture 4.1. If a, b, c, and d are random variables, then

$$I(a; b) + I(c; d|a) + I(c; d|b) - I(c; d) \geqq -0.09876... H(abcd),$$

and equality attained, for example, if all variables take only 0-1 values, $c = \min(a, b)$, $d = \max(a, b)$, and

$$\mathrm{Prob}(a = 0, b = 0) = \mathrm{Prob}(a = 1, b = 1),$$
$$\mathrm{Prob}(a = 0, b = 1) = \mathrm{Prob}(a = 1, b = 0).$$

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

Studia Scientiarum Mathematicarum Hungarica 32 (1996), 439–444

SEPARATION OF POINTS BY CONGRUENT DOMAINS

T. ÓDOR

Abstract

In this article we investigate how we can divide balls or spheres into congruent domains (by scissor) if we want to separate a given finite point set. We formulate conjectures about the characterization of divisions of balls into congruent domains (by scissor).

1. Introduction

As usual $D^d = \{x \in R^d : |x| \leqq 1\}$ and $S^{d-1} = \{x \in R^d : |x| = 1\}$ denote the unit ball and the unit sphere, respectively, and O is the origin in the d-dimensional real vector space R^d, where $|\,.\,|$ is the norm.

We define the notion of the so called scissor division of domains of R^d having piecewisely smooth boundary [1].

Let M be the unit ball B^d or the unit sphere S^{d-1}.

Let $M_i \subset M$ be connected closed sets for $i \in I$ which we call later domains. We say that the system of domains $\{M_i : i \in I\}$ is the scissor division, (or division by scissor) of M if and only if their union is M, their interiors are pairwise disjoint and pairwise congruent, their (relative) boundary is a closed surface in R^d. We call the union of the boundaries of the domains of the division the boundary of the division.

It is clear that the boundary of the division (by scissor) determine the division itself. This essentially means that we divide M into domains by surfaces, which explains the term "scissor division".

The main results are as follows:

THEOREM 1. *Let $\mathcal{P} = \{ P_0, \ldots, P_{n-1} \in \operatorname{int} D^2\}$ be a set of distinct points which differ from the origin O. Assume that if the points P_i and P_j have the same distances from the origin, then the angle $\sphericalangle P_iOP_j$ is not equal to*

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$2\pi m/n$ for any positive integer m. Then we can divide D^2 (by scissor) into n congruent domains such that each of them contains exactly one point of $\mathcal{P}$.

THEOREM 2. *Let $\mathcal{P} = \{ P_0, \dots, P_{n-1} \in \operatorname{int} D^d$ (or S^{d-1})$\}$ be a set of distinct points which differ from the origin, where $d \geqq 3$. Then we can divide D^d (or S^{d-1}) into n congruent domains (by scissor) such that each of them contains exactly one point of $\mathcal{P}$.*

These results are special cases of a more general statement [3], which is presented in terms of fundamental domains of discrete isometry groups.

It would be interesting to characterize the possible divisions of D^d. The following two conjectures give the characterization in the two-dimensional case and give an interesting property in higher dimensions.

Let γ_0 be a simple topological arc in D^2 with endpoint O, the other endpoint lying in the boundary of D^2 and the relative interior points of γ_0 are in the interior of D^2. Denote the rotated copy of γ_0 around O through the angle $2\pi m/n$ by γ_n for $m = 0, 1, \dots, n-1$. Assume that these simple arcs do not have common points except O. Then these curves determine a division of D^2 in a natural way.

Let us call this type of divisions of D^2 the first type division.

Let $Q_1, \dots, Q_6 \in S^1$, where the angle $\sphericalangle Q_i O Q_{i+1}$ is equal to $\pi/3$. Let I be a subset of the integers from 1 to 6 and let I^c be its complement.

Consider the circles with centre Q_{i+1} and radius 1 for $i \in I$. The points S_{ij} divide the circular arcs $Q_i O$ into $k+1$ equal parts, where k is a fixed positive integer and $1 \leqq j \leqq k$. We connect the points S_{ij} with the points Q_{i+1} with circular arcs of radius 1 in a way so that they are the rotated copies of OQ_{i+1} The points S_{ij} divide the circular arcs $Q_i Q_{i+1}$ into $k+1$ equal part for $i \in I^c$. Connect the points S_{ij} with O by circular arcs which are the rotated copies of the circular arc OQ_i through the origin.

The system of circular arcs divides D^2 into $6(k+1)$ congruent domains (by scissor). We call these type of divisions the second type divisions.

Consider again the circles with centre Q_{i+1} and radius 1 for $i \in I$. The points T_i are the midpoints of the circular arc $Q_i O$. The union of the segments $Q_{i-1} T_i$ and the circular arcs $Q_i O$ divide D^2 into 12 congruent domains (by scissor).

We call this type of division the third type division.

Now we can formulate our conjectures:

CONJECTURE 1. There exist only first, second and third type divisions (by scissor) of D^2 into congruent domains.

CONJECTURE 2. There is no division (by scissor) of D^d for $d \geqq 2$ into congruent domains whose boundary does not contain the origin.

CONJECTURE 3. There exist only first second and third type divisions (by scissor) of D^2 into affine equivalent or Möbius equivalent domains.

CONJECTURE 4. There is no division (by scissor) of D^d for $d \geqq 2$ into affine equivalent or Möbius equivalent domains whose boundary does not contain the origin.

Let $\mathcal{M}$ be a division of S^{d-1} into congruent domains (by scissor). Connecting O and the points of $\partial\mathcal{M}$ by segments, we get a division $\mathcal{M}'$ of D^d into congruent domains (by scissor) in a natural way, which we call the natural extension of $\mathcal{M}$.

CONJECTURE 5. Assume that $d \geqq 3$ and $\mathcal{N}$ is a division of D^d into congruent domains (by scissor). Then there exist a division $\mathcal{M}$ of S^{d-1} and a homotopy $F:[0,1]\times\partial\mathcal{M}' \to D^d$ which leaves the origin O and the boundary S^{d-1} of D^d fixed, homeomorphism for every $t \in [0,1]$, the mapping F_0 is the identity, the image of F_t generates a division of D^d into congruent domains (by scissor) for every fixed $t \in [0,1]$ and the image of F_1 generates the division $\mathcal{M}'$, which is the natural extension of $\mathcal{M}$.

The proofs of the theorems

PROOF OF THEOREM 1. We use only first type divisions of D^2 in our proof . Fix an orientation on the plane. Let S_i be the image of P_i under the rotation about O through the angle $2\pi i/n$. As $P_iOP_j \neq 2\pi m/n$ for positive integers m, it is clear that the points S_i and S_j are different if i and j are different. Let $\mathcal{S} = \{S_0, \dots, S_{n-1}\}$ be the set of these rotated points. If we can find an arc γ_0 which determines a first type division $\mathcal{D}(\gamma_0) = \{D_i : 0 \leqq i \leqq n-1\}$ of D^2 and $S_i \in \text{int } D_0$, then the division $\mathcal{D}$ satisfies our assumptions, that is $P_i \in D_i$ for $0 \leqq i \leqq n-1$.

Let $Q \in S^1$ and assume that $\sphericalangle QOP_i/\pi$ is not a rational number. We produce γ_0 as a suitable deformation of the segment OQ. Without restricting the generality we can assume that the line OQ is the first coordinate axis in R^2.

Let $C_1, \dots, C_s$ be circles with centre O passing through the points $P_0, \dots, P_{n-1}$.

We labelled our circles C_i in a strictly decreasing order of the radii $\rho_1, \dots, \rho_s$. Let $\rho_0 = 1$ and $\rho_{s+1} = 0$ and $C_0 = S^1$ and $C_{s+1} = O$. Let Q_i denote the intersection of the circle C_i and the segment OQ.

Let μ be the minimum of the difference of the consecutive radii of our circles, that is $\mu = \min\limits_{0 \leqq i \leqq s} (\rho_i - \rho_{i+1})$. Let θ be a fixed positive real number which is smaller than $\mu/4$.

Let $\mathcal{S}_i = \{S_{j_1}, \dots, S_{j_{t_i}}\} = \mathcal{P} \cap C_i$ be the points in the point set $\mathcal{P}$ which lie on the circle C_h. Let relabel the points of $\mathcal{S}_i$ by the symbols $S_{i1}, \dots, S_{it_i}$ in such a way that the angles $\sphericalangle QOS_{iu}$ for $1 \leqq u \leqq t_i$ are strictly decreasing.

To simplify notations, let $v_\rho(\psi)$ and $w_\rho(\psi)$ be simple curves in a polar coordinate system with centre in the origin on R^1, given by

$$v_\rho(\psi) = (\rho + \theta) - \theta\psi/2\pi \quad \text{and} \quad w_\rho(\psi) = (\rho - \theta) + \theta\psi/2\pi.$$

Let $v_i(\psi) = v_{\rho_i}(\psi)$ and $w_i(\psi) = w_{\rho_i}(\psi)$ for $1 \leqq i \leqq s$ and denote the image of them $v_i(\psi)$ and $w_i(\psi)$, respectively.

Let $V_i = v_i(0)$ and $W_i = w_i(0)$ be the starting point of the curves v_i and w_i. It is clear that V_i and W_i lie on the segment OQ and their distance from the point Q_i is θ.

It is clear that the simple arc

$$\Gamma = QV_1 \cup \bigcup_{i=1}^{s-1} (v_i \cup w_i \cup W_i V_{i+1}) \cup W_s O$$

determines a division of the first type because its rotated copies have exactly one common point pairwise, namely O.

We deform this curve Γ into a simple arc γ_0 in a way, that the division $\mathcal{D}(\gamma_0) = \{D_i : 0 \leqq i \leqq n-1\}$ determined by it separate the points of $\mathcal{P}$. It is clear that this holds if D_0 contains the point set $\mathcal{S}$.

Let S_{iu}^l be the image of S_{iu} under the rotation about O through the angle $2\pi l/n$, where $0 \leqq u \leqq l_i$, $1 \leqq i \leqq s$ and $0 \leqq l \leqq n-1$ and let W_{iu}^l be the intersection points of the curve w_i and the halfline OS_{iu}^l.

Consider the segments $S_{iu}^l W_{iu}^l$. "Blowing up" (in increasing magnitude in the index l) these segments for $1 \leqq l \leqq n-1$, we deformate the curves $v_i \cup w_i$ in a way that the deformated curves "leave out" the points $S_{iu} = S_{iu}^0$ and the rotated copies (about O through the angle $2m\pi/n$ for $m \in Z$) of the "blowed up segments" does not intersect each other.

Now we present this method more thoroughly. Let

$$\nu_0 = \min_{1 \leqq i \leqq s} \min_{1 \leqq u < v \leqq t_i} \min_{1 \leqq k < l \leqq n-1} \left| \sphericalangle S_{iu}^k S_{iv}^l \right|$$

be the minimum of the absolute value of the angles $\sphericalangle S_{iu}^k S_{iv}^l$, where we note that the points S_{i0}^l are rotated copies of Q_i, as defined earlier.

It is clear that ν_0 is positive. Let

$$\nu = \frac{\min\{\nu_0, \theta\}}{16n^2}.$$

Let F_{iu}^l and $H_{iu}^l \in w_i$ be points for $0 \leqq l \leqq n-1$ and $1 \leqq u \leqq t_i$ (but not for $u = 0$) in this order on the curve w_i, assuming that $\sphericalangle F_{iu}^l O S_{iu}^l = (l+1)\nu = \sphericalangle S_{iu}^l O H_{iu}^l$.

Let T_{iu}^l and $U_{iu}^l \in w_i$ be the intersection points of the halflines OF_{iu}^l and OH_{iu}^l (with starting point O) and the curve v_i.

Let ξ_0 denote the minimum of the distances of the points T_{iu}^l, U_{iu}^l and the circle C_i. Because of our choice of v_i, ξ_0 is positive. Let $\xi=\xi_0/16n$ be a fixed number.

Let A_{iu}^l and $B_{iu}^l \in w_i$ be points on the halflines OF_{iu}^l and OH_{iu}^l, assuming that their distance from O is $\rho_i+(l+1)\xi$.

Denote w_{iu}^l the arcs between the points F_{iu}^l and $H_{iu}^l \in w_i$ on the curve w_i. Let the curve

$$\tau_{iu}^l = F_{iu}^l A_{iu}^l \cup A_{iu}^l B_{iu}^l \cup B_{iu}^l H_{iu}^l$$

be the union of three segments and let

$$\sigma_i = v_i \cup \left(w_i \setminus \bigcup_{l=0}^{n-1} \bigcup_{u=1}^{t_i} w_{iu}^l \right) \cup \bigcup_{l=0}^{n-1} \bigcup_{u=1}^{t_i} \tau_{iu}^l$$

be the deformated copy of the curve $v_i \cup w_i$.

It is clear that the rotated copies of the curves τ_{iu}^l about O through the angle $2m\pi/n$ do not intersect each other for $1 \leqq m \leqq n-1$. Furthermore, if $k-l=m$ then the triangle $OA_{iu}^k B_{iu}^k$ contains the rotated copy of τ_{iu}^l and if $u \neq v$, then the triangles $OA_{iu}^k B_{iu}^k$ and the rotated copy of $OA_{iv}^l B_{iv}^l$ have exactly one common point for every $0 \leqq k, l \leqq n-1$ and $1 \leqq u \neq v \leqq t_i$.

Let

$$\gamma_0 = \bigcup_{i=1}^{s-1} (\sigma_i \cup W_i V_{i+1}) \cup QV_1 \cup W_s O$$

and let γ_1 be its rotated copy about O through the angle $2\pi/n$.

Using our previous remark it is easy to see that the domain D_0 bounded by the closed arc $\gamma_0 \cup QQ^1 \cup \gamma_1$ contains $\mathcal{S}$ because the points $S_{iu}=S_{iu}^0$ lie on the left side of the directed curve γ_0 (with starting point O) and lie on the left side of the directed curve γ_1 (with starting point Q^1, where Q^1 is the image of Q).

This proves our statement. □

Proof of Theorem 2. We will prove our theorem by induction simultaneously in the two cases. Let $d=3$. If $\omega \in S^2$ then let $H(\omega)$ be the two-plane passing through the origin with normal vector ω. Let π_ω denote the orthogonal projection to the two-plane $H(\omega)$. Let $\mathcal{P}^\omega$ be the image of the point set $\mathcal{P}$ under the projection π_ω. It is easy to see that there exists an $\omega \in S^2$ such that the distances of the points of $\mathcal{P}^\omega$ from the origin O are different. Apply the previous construction in the plane $H(\omega)$ for the disc $H(\omega) \cap D^3$ and the point set $\mathcal{P}^\omega$.

The boundary of the division arises as the restriction of the inverse image under π_ω of the boundary of the division in the plane $H(\omega)$ to the sets D^3 (or S^2).

We can apply this method also for larger values of d, too.

So the statement is proved. □

THEOREM 3. *Let three distinct points be given in the unit disc D^2 and suppose that they differ from the origin. Then we can divide D^2 into three congruent domains (by scissor), such that each of them contains exactly one of the given points.*

PROOF. We can label the three points by 0, 1 and 2 in such a way that if the distances of P_i and P_j from the origin are equal then $S_i = S_j$ and we can apply our method applied in the proof of the first theorem. □

COUNTEREXAMPLE 4. *There are four distinct points in D^2 such that there does not exist a first type division of D^2 in which every domain contains exactly one of the given points.*

PROOF. Let $Q_1, \dots, Q_4$ be the vertices of a square in S^1 in this order. Let $P_1 = Q_1/2$, $P_2 = Q_3/2$, $P_3 = Q_1/3$ and $P_4 = Q_2/3$.

We cannot number this set in such a way that the image of P_1, P_2 and the P_3, P_4 become equal after the rotation of the i-th point. So we cannot divide D^2 in the desired way.

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MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

odor@math-inst.hu

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WHEN IS $\frac{|S_n|^p}{n^p}$ AN AMART?

B. HEINKEL

Abstract

Let (X_k) be a sequence of real-valued random variables (r.v.), which are independent, centered and such that

$$\exists p>2: \sum_{k \geqq 1} \frac{\mathbf{E}|X_k|^p}{k^p} < +\infty.$$

For every integer n, S_n will denote the partial sum $X_1 + \ldots + X_n$ and $\mathcal{F}_n$ will be the σ-field generated by $X_1, \ldots, X_n$. Fuk and Nagaev [3] have shown that if furthermore the Prohorov exponential series converges then (X_k) fulfils the strong law of large numbers (SLLN). Here their result is completed by showing the equivalence of the following three properties:

(i) The SLLN holds for (X_k).

(ii) $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ is an amart.

(iii) The Prohorov exponential series converges.

1. The SLLN in the Kolmogorov setting

The following problem is very classical in probability theory:

A sequence (X_k) of independent, centered, real-valued r.v. being given, under what assumptions – involving the individual laws of the X_k – does the strong law of large numbers (SLLN) hold, in other words, when does $\frac{S_n}{n} = \frac{1}{n}(X_1 + \ldots + X_n)$ converge almost surely (a.s.) to 0?

It is easy to see that, when the SLLN holds, then:

(a) $\frac{S_n}{n} \to 0$ in probability (the weak law of large numbers – WLLN – holds),

(b) $\frac{X_k}{k} \to 0$ a.s..

It is therefore natural to look for sufficient conditions for the SLLN which express in a convenient way that the a.s. convergence in (b) is "fast enough" to imply the SLLN.

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A famous hypothesis – ensuring (b) – under which the SLLN has been studied for a sequence (X_k) of independent, centered, real-valued r.v., is that there exists $p \geqq 1$ for which

$$\sum_{k \geqq 1} \frac{\mathbf{E}|X_k|^p}{k^p} < +\infty. \tag{1.1}$$

The knowledge on the SLLN under that classical "Kolmogorov restriction (1.1)" is summarized in the following statement:

THEOREM 1.1. *Let (X_k) be a sequence of real-valued r.v., which are centered, independent and fulfil* (1.1) *for a $p \geqq 1$.*

(1) *If $p \in [1, 2]$, then the SLLN holds.*

(2) *If $p > 2$ and furthermore*

$$\forall \varepsilon > 0, \qquad \sum_{n \geqq 1} \exp\left\{-\frac{\varepsilon}{s_n^2}\right\} < +\infty, \tag{1.2}$$

where

$$s_n^2 = \frac{1}{2^{2n}} \sum_{k \in I(n)} \mathbf{E}(X_k^2),$$

with $I(n) = \{2^n + 1, \dots, 2^{n+1}\}$, then the SLLN holds.

REMARK. The above statement contains results due to several authors: the case $p = 2$ is the classical Kolmogorov result [6], the case $1 \leqq p < 2$ goes back to Petrov [11] and the remainder situation, $p > 2$, has been obtained by Fuk and Nagaev [3].

The natural observation that $\dfrac{S_n}{n}$ is not "too far away" from the martingale $\sum_{1 \leqq k \leqq n} \dfrac{X_k}{k}$ brought several authors to look for a generalized martingale behaviour of $\dfrac{S_n}{n}$ when the SLLN holds. Such a behaviour occurs in many cases (see Peligrad [10], Krengel and Sucheston [7], Dam [1] and the numerous references given in Gut and Schmidt [4]).

Before to make more precise this generalized martingale behaviour in the Kolmogorov setting (1.1), we will recall some definitions and results concerning some useful types of asymptotic martingales.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $(\mathcal{F}_n)$ be an increasing sequence of sub-σ-fields of $\mathcal{F}$. The set of stopping times associated to $(\mathcal{F}_n)$ will be denoted by $\mathcal{T}^*$; the set $\mathcal{T}$ will be the one of those which are $\mathbb{N}$-valued and take only a finite number of values. The set $\mathcal{T}^*$ is obviously partially ordered:

$$\forall \sigma, \tau \in \mathcal{T}^*, \sigma \leqq \tau \Leftrightarrow \forall \omega \in \Omega,\ \sigma(\omega) \leqq \tau(\omega).$$

DEFINITION 1.2. Let (ξ_k) be a sequence of integrable r.v., which is adapted to $(\mathcal{F}_k)$.

(1) One says that $(\xi_k, \mathcal{F}_k)$ is an amart if $\lim_{\tau \in T} \mathbf{E}(\xi_\tau)$ exists.

(2) One says that $(\xi_k, \mathcal{F}_k)$ is a quasimartingale if

$$\sum_{n \geq 1} \mathbf{E}|\mathbf{E}(\xi_{n+1} \mid \mathcal{F}_n) - \xi_n| < +\infty.$$

REMARK. Of course, a quasimartingale is an amart, but the converse is not true in general.

These generalized matringales have nice convergence properties; the ones that we will need later are summarized in the following statement (see Edgar and Sucheston [2], § 1.2, and Rao [13], § 4.5, for details):

PROPOSITION 1.3. *Let (ξ_k) be a sequence of integrable r.v., which is adapted to $(\mathcal{F}_k)$.*

(1) *If there exists $\xi \in L^1(\Omega, \mathcal{F}, \mathbf{P})$, such that $\forall k, |\xi_k| \leq \xi$ a.s., then $(\xi_k, \mathcal{F}_k)$ is an amart if and only if (ξ_k) is a.s. convergent.*

(2) *If $(\xi_k, \mathcal{F}_k)$ is an L^1-bounded amart, then (ξ_k) converges a.s..*

Now we come back to the SLLN.

In the whole sequel $\mathcal{F}_n$ will denote the σ-field generated by $X_1, \dots, X_n$. Dam [1] has studied the generalized martingale behaviour of $\left(\frac{S_n}{n}, \mathcal{F}_n\right)$ under the restriction (1.1), when $p \in [1, 2]$. He proved the following

PROPOSITION 1.4. *Let (X_k) be a sequence of independent r.v., which are centered and which fulfil (1.1) for a $p \in [1, 2]$. Then $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ is an amart.*

In fact, an elementary computation allows to improve Dam's result:

PROPOSITION 1.5. *Let (X_k) be a sequence of independent r.v., which are centered and which fulfil (1.1) for a $p \in [1, 2]$. Then $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ is a quasimartingale.*

When $p > 2$ also, the SLLN in the Kolmogorov setting (1.1) is a nice generalized martingale property of the sequence $\frac{S_n}{n}$. Precisely, the part (2) of the above Theorem 1.1 can be completed in the following way:

THEOREM 1.6. *Let (X_k) be a sequence of independent r.v., which are centered and which fulfil (1.1) for a $p > 2$. Then the following are equivalent:*

(1) *The SLLN holds.*

(2) *Property* (1.2) *holds.*

(3) $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ *is an amart.*

In Section 2, we will prove this result; with the help of Theorem 1.1 the proof reduces to check the implication (1) $\Rightarrow$ (2) and the equivalence of (1) and (3). In Section 3, we will give an example showing that property (3) in Theorem 1.6 is optimal in the sense that usually the amart $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ is not a quasimartingale. This example shows that the SLLN under (1.1) is not the same kind of asymptotic behaviour according as $p \in [1,2]$ or $p > 2$.

2. The proof of Theorem 1.6

We first show that (1) implies (3).

According to (1.1), one has

$$\mathbf{E} \sup_{k \geqq 1} \frac{|X_k|^p}{k^p} < +\infty. \tag{2.1}$$

As the SLLN holds, a well-known result of Hoffmann-Jørgensen (Corollary 3.4 in [5]), allows to deduce from (2.1) that

$$\mathbf{E} \sup_{n} \frac{|S_n|^p}{n^p} < +\infty. \tag{2.2}$$

By the first part of Proposition 1.3, property (2.2) implies immediately that $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ is an amart.

Now we check the implication (3) $\Rightarrow$ (1).

By positivity, the amart $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ is L^1-bounded. Therefore $\left(\frac{|S_n|^p}{n^p}\right)$ converges a.s. and of course $\left(\frac{|S_n|}{n}\right)$ does also. Furthermore, by Kolmogorov's 0–1 law, there exists a positive number $a \geqq 0$ such that: $\frac{|S_n|}{n} \to a$ a.s..

If $a = 0$, (1) holds. Let us suppose that $a > 0$.

By (1.1), $\left(\frac{X_k}{k}\right)$ converges a.s. to 0. So almost all $\omega \in \Omega$ are such that

$$\frac{X_k(\omega)}{k} \to 0 \quad \text{and} \quad \frac{|S_n(\omega)|}{n} \to a.$$

Fix an ω of this type. If the sequence of real numbers $\left(\frac{S_n(\omega)}{n}\right)$ would not converge, the positivity of a would imply:

$$\forall n_0, \exists k \geqq n_0 : \frac{|X_k(\omega)|}{k} > \frac{a}{2},$$

which would contradict the convergence of $\left(\frac{X_k(\omega)}{k}\right)$ to 0! So the sequence $\left(\frac{S_n}{n}\right)$ is a.s. convergent. By the Kolmogorov 0–1 law, the a.s. limit is a degenerate r.v., which can only be a or $-a$. Suppose that it is a (the argument would be the same for $-a$).

From the L^1-boundedness of the sequence $\left(\frac{|S_n|^p}{n^p}\right)$, it follows that $\left(\frac{S_n}{n}\right)$ is uniformly integrable. This implies: $\lim_{n\to\infty} \mathbf{E}\left(\frac{S_n}{n}\right) = a$. The r.v. X_k being centered, a has necessarily to be equal to 0. This concludes the proof of the implication (3) $\Rightarrow$ (1).

Finally, let us check that (1) $\Rightarrow$ (2).

By an elementary symmetrization argument, one sees that there is no loss of generality in assuming that the X_k are symmetrically distributed. So we will assume it. The key argument of the proof will be a recent necessary condition for the SLLN, which applies to a large class of sequences of r.v. This condition is a simple corollary of the necessary part of the rather general Theorem 7.5 in Ledoux and Talagrand [8]. Before stating that corollary, we need to recall a notation.

NOTATION. A sequence (U_k) of r.v. being given, one defines, for every integer n: $U^{(r)}_{I(n)} = |U_j|$ whenever $|U_j|$ is the r-th maximum of the sample $(|U_i|)_{i\in I(n)}$ (breaking ties by priority of index), and setting: $U^{(r)}_{I(n)} = 0$ if $r > 2^n$.

The announced necessary condition for the SLLN is as follows:

LEMMA 2.1. *Let (U_k) be a sequence of independent and symmetric r.v.. Assume that there exists $q \geqq 2$ and a sequence (k_n) of integers such that the following hold:*

$$\sum_{n\geqq 1} q^{-k_n} < +\infty, \tag{2.3}$$

$$\exists \varepsilon > 0 : \sum_{n\geqq 1} \mathbf{P}\left(\sum_{1\leqq r\leqq k_n} U^{(r)}_{I(n)} > \varepsilon 2^n\right) < +\infty. \tag{2.4}$$

If (U_k) fulfils the SLLN, then:

$$\forall \delta > 0, \quad \sum_{n \geqq 1} \exp\left(-\frac{\delta 2^{2n}}{\sigma_n^2}\right) < +\infty, \tag{2.5}$$

where

$$\sigma_n^2 = \sum_{k \in I(n)} \mathbf{E}(U_k^2 I_{(|U_k| \leqq \varepsilon 2^n / k_n)}).$$

REMARK. In the contrary to Nagaev's theorem [9], this Lemma 2.1 is not an universal necessary condition for the SLLN. But its range of application is rather large, and its assumptions are more handy than the ones of Nagaev's result.

For applying this lemma in our context we will start by checking that under hypothesis (1.1) there exists a sequence (k_n) such that (2.3) and (2.4) hold. Define, for every integer n

$$\Lambda(n) = 2^{-np} \sum_{k \in I(n)} \mathbf{E}|X_k|^p$$

and put

$$k_n = 1 + [\Lambda(n)^{-\frac{1}{3p}}], \tag{2.6}$$

where [ ] denotes the integer part of a real number. From (1.1) it follows that, for every $q \geqq 2$, (2.3) is realized for that sequence (k_n).

Now we will use a trick due to J. Zinn (see the beginning of the proof of Lemma 4.11 in [12]); for the reader's convenience, we detail his argument:

$$\forall r \in \{1, \ldots, k_n\}, \forall h > 0, \mathbf{P}\left(\frac{X_{I(n)}^{(r)}}{2^n} > \Lambda(n)^{\frac{1}{2p}}\right) \leqq \mathbf{P}\left(\sum_{k \in I(n)} I_{(|X_k| > 2^n \Lambda^{\frac{1}{2p}})} \geqq r\right)$$

$$\leqq \exp(-hr)\mathbf{E}\exp\left(h \sum_{k \in I(n)} I_{(|X_k| > 2^n \Lambda(n)^{\frac{1}{2p}})}\right)$$

$$\leqq \exp\left(-hr + (e^h - 1) \sum_{k \in I(n)} \mathbf{P}(|X_k| > 2^n \Lambda(n)^{\frac{1}{2p}})\right).$$

Taking $h = \ln\left(1 + \Lambda(n)^{-\frac{1}{2}}\right)$ in the above inequalities, one gets

$$\mathbf{P}(X_{I(n)}^{(r)} > 2^n \Lambda(n)^{\frac{1}{2p}}) \leqq e\Lambda(n)^{\frac{r}{2}},$$

so that for n large enough

$$\alpha_n = \sum_{4 \leqq r \leqq k_n} \mathbf{P}(X_{I(n)}^{(r)} > 2^n \Lambda(n)^{\frac{1}{2p}}) \leqq \Lambda(n). \tag{2.7}$$

Consider now the rough estimate

$$\mathbf{P}\left(\sum_{1 \leqq r \leqq k_n} X_{I(n)}^{(r)} > 2^n\right) \leqq \mathbf{P}\Big(\sup_{j \in I(n)} |X_j| > \frac{2^{n-1}}{3}\Big) + \mathbf{P}\left(\sum_{4 \leqq r \leqq k_n} X_{I(n)}^{(r)} > 2^{n-1}\right). \tag{2.8}$$

As by definition of $\Lambda(n), \lim k_n \Lambda(n)^{\frac{1}{2p}} = 0$, it follows from (2.7) and (2.8) that

$$\mathbf{P}\left(\sum_{1 \leqq r \leqq k_n} X_{I(n)}^{(r)} > 2^n\right) \leqq \alpha_n + \mathbf{P}\Big(\sup_{j \in I(n)} |X_j| > \frac{2^{n-1}}{3}\Big) \leqq \tag{2.9}$$
$$\leqq \alpha_n + 6^p \Lambda(n) \leqq (1+6^p)\Lambda(n).$$

From (2.9), one sees finally that assumption (2.4) of Lemma 2.1 is fulfilled (with $\varepsilon = 1$). By Hölder's and Tchebycheff's inequalities, one observes that for n large enough

$$2^{-2n} \sum_{k \in I(n)} \mathbf{E}|X_k|^2 I_{(|X_k| > 2^n/k_n)} \leqq \Lambda(n)^{\frac{2}{p}} \left(\sum_{k \in I(n)} \mathbf{P}(|X_k| > 2^n/k_n)\right)^{1-\frac{2}{p}}$$
$$\leqq 2^p \Lambda(n)^{\frac{2}{p}} \Lambda(n)^{\frac{2}{3}(1-\frac{2}{p})} = 2^p \Lambda(n)^{\frac{2}{3} - \frac{1}{3p}},$$

and so one deduces from (2.5) and (1.1)

$$\forall \delta > 0, \qquad \sum_{n \geqq 1} \exp\left(-\frac{\delta}{s_n^2}\right) < +\infty,$$

which is nothing else than property (1.2). This concludes the proof of the implication (1) $\Rightarrow$ (2).

3. About the optimality of Property 3 in Theorem 1.6

Now we will construct the example announced at the end of Section 1. Let (θ_n) be a sequence of independent standard normal r.v. Define:

$$X_1 = 0;$$
$$\forall n \geqq 2, \quad X_n = \sqrt{n}(\ln n)^{-\frac{1}{4}}\theta_n.$$

It is clear that the sequence (X_n) fulfils (1.1) for $p = 4$ and fulfils also (1.2). So $\left(\frac{|S_n|^4}{n^4}, \mathcal{F}_n\right)$ is an amart. But:

PROPOSITION 3.1. $\left(\frac{|S_n|^4}{n^4}, \mathcal{F}_n\right)$ *is not a quasimartingale.*

PROOF. By definition $\left(\frac{|S_n|^p}{n^p}, \mathcal{F}_n\right)$ is a quasimartingale if and only if the series having general term $\mathbf{E}|u_n|$ converges, u_n being defined as

$$\forall n \geqq 1, \qquad u_n = \mathbf{E}\left(\frac{|S_{n+1}|^4}{(n+1)^4} \,\Big|\, \mathcal{F}_n\right) - \frac{|S_n|^4}{n^4}.$$

By the independence of the r.v. X_k, u_n can be written as
(3.1)

$$u_n = -\frac{(4n^3+6n^2+4n+1)}{(n+1)^4 n^4} S_n^4 + \frac{6}{(n+1)^3(\ln(n+1))^{\frac{1}{2}}} S_n^2 + \frac{3}{(\ln(n+1))(n+1)^2}.$$

It is easy to see that, for all $n \geqq 2, S_n$ has an $N(0, \beta_n^2)$ distribution, where

$$\lim_{n\to+\infty} \beta_n \frac{(4\ln(n+1))^{\frac{1}{4}}}{(n(n+1))^{\frac{1}{2}}} = 1. \tag{3.2}$$

So, if one defines now the auxiliary r.v.

$$Z_n = \frac{(4\ln(n+1))^{\frac{1}{4}}}{(n(n+1))^{\frac{1}{2}}} S_n,$$

one sees from (3.1) and (3.2) that the series having general term $\mathbf{E}|u_n|$ converges if and only if

$$\sum_{n\geqq 2} \frac{1}{\ln(n+1)} \mathbf{E}\left|\frac{3n}{(n+1)^2} Z_n^2 - \frac{n}{(n+1)^2} Z_n^4\right| < +\infty. \tag{3.3}$$

This property (3.3) can even be written simpler as

$$\sum_{n\geqq 2} \frac{1}{n\ln(n+1)} \mathbf{E}|3Z_n^2 - Z_n^4| < +\infty. \tag{3.4}$$

It follows from (3.2) that

$$\liminf_{n\to+\infty} \mathbf{E}(Z_n^2 I_{(|Z_n|\leqq 1)}) > 0,$$

so from the elementary inequality

$$\mathbf{E}|3Z_n^2 - Z_n^4| \geqq 2\mathbf{E}Z_n^2 I_{(|Z_n|\leqq 1)}$$

one deduces that (3.4) cannot hold. Therefore $\left(\frac{|S_n|^4}{n^4}, \mathcal{F}_n\right)$ is not a quasi-martingale.

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DÉPARTEMENT DE MATHÉMATIQUE
UNIVERSITÉ LOUIS PASTEUR
7, RUE RENÉ DESCARTES
F–67084 STRASBOURG
FRANCE

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ON *-SIMPLE INVOLUTION RINGS WITH MINIMAL *-BIIDEALS

U. A. ABURAWASH

Abstract

Let A be an involution ring. Then A is *-simple with a minimal *-biideal if and only if A is isomorphic either to a Rees matrix ring over a division ring or to a direct sum of such a Rees matrix ring and its opposite ring.

The involutive version of the celebrated Litoff–Ánh Theorem, which gives the local characterization of nontrivial simple rings having minimal right ideals (cf. [2]), is given in [1]. The purpose of this paper is to give the involutive version of Theorem 79.1 in [3] (cf. [5] and [6]), which characterizes completely nontrivial simple rings with minimal right ideals.

All rings considered are associative. A subring B of a ring A is called a *biideal* of A if $BAB \subseteq B$. A ring A is said to be *simple* if $A^2 \neq 0$ and the only ideals of A are 0 and A. A is called *semiprime* if $I^2 = 0$ and $I \triangleleft A$ imply $I = 0$. Simple rings are obviously semiprime rings. It is well-known that *every minimal right ideal R of a semiprime ring A contains an idempotent element e such that $R = eA$.*

To prove the main theorem of this note, we need the following results.

PROPOSITION 1 ([3], Theorem 31.6 and [7], Theorem 1). *Let A be a semiprime ring with an idempotent $e \neq 0$. Then the following conditions are equivalent*

(i) *eA is a minimal right ideal in A;*

(ii) *Ae is a minimal left ideal in A;*

(iii) *eAe is a division ring and is a minimal biideal in A.*

PROPOSITION 2 ([7], Theorem 4). *If R is a minimal right ideal and L a minimal left ideal of a ring A, then either $RL = 0$ or RL is a minimal biideal of A.*

PROPOSITION 3 ([7], Theorem 5). *Any minimal biideal B of a semiprime ring A can be represented in the form $B = RL$ with a minimal right ideal R and a minimal left ideal L of A.*

Following the terminology of [3], let $M(I, K, \Lambda, P)$ denote the Rees matrix ring over a division ring K and the sandwich matrix $P = (p_{\lambda i})$ $(\lambda \in \Lambda,$

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$i \in I$, $p_{\lambda i} \in K$) is left row- and right row-independent. The elements of M are all matrices of order $I \times \Lambda$ over K and have only finitely many nonzero entries. Addition is defined in the usual way while multiplication is defined for all $X = (x_{i\lambda}), Y = (y_{i\lambda})$ in M by $X \circ Y = XPY$.

PROPOSITION 4 ([3], Theorem 79.1). *A is a simple ring having a minimal right ideal if and only if A is isomorphic to a Rees matrix ring $M(I, K, \Lambda, P)$ over a division ring K.*

A ring A is called an *involution ring* if a unary operation $*$, called *involution*, is defined on A such that

$$(a+b)^* = a^* + b^*, \qquad (ab)^* = b^* a^*, \qquad (a^*)^* = a$$

for all $a, b \in A$. A **-ideal (*-biideal)* I of A, denoted by $I \triangleleft^* A$, will indicate an ideal (biideal) I of A which is closed under involution, that is

$$I^{(*)} = \{a^* \in A \mid a \in I\} \subseteq I.$$

Let A be a ring and A^{op} its opposite ring. On the direct sum

$$R = A \bigoplus A^{\mathrm{op}}$$

we may define an involution $*$, called the *exchange involution*, by

$$(a, b)^* = (b, a) \ \text{ for every } \ (a, b) \in R.$$

An involution ring A is said to be $*$-simple if $A^2 \neq 0$ and A has no nontrivial $*$-ideals. By the above example, a $*$-simple ring need not be simple although the converse is trivially true. A is called $*$-semiprime if $I \triangleleft^* A$ and $I^2 = 0$ imply $I = 0$. One can easily prove that an involution ring A is $*$-semiprime if and only if it is semiprime.

Specializing Corollary 3 of [4] for simple rings, we get

PROPOSITION 5. *For any nonempty subset $M \neq 0$ of a simple involution ring A, $MAM^{(*)} \neq 0$ and $M^{(*)}AM \neq 0$.*

The following result due to Loi [4] asserts that if a semiprime involution ring A possesses a minimal $*$-biideal, then A has also a minimal biideal.

PROPOSITION 6 ([4], Proposition 4). *Let A be a semiprime involution ring. If B is a minimal $*$-biideal of A, then either B is a minimal biideal of A or $B = C \oplus C^{(*)}$, where C is a minimal biideal of A.*

$*$-simple rings can be characterized as in the following lemma.

LEMMA 1. *An involution ring A is $*$-simple if and only if either A is simple or $A = I \oplus I^{\mathrm{op}}$ with $I \triangleleft A$ and I is a simple ring and the involution is the exchange involution.*

PROOF. If A is $*$-simple then it is either simple or not. If A is not simple, then there is an ideal I of $A, I \neq 0$ and $I \neq A$. Now $I \cap I^{(*)}$ is a $*$-ideal of A

which is *-simple and not equal to A, whence $I \cap I^{(*)} = 0$. Moreover, $I \oplus I^{(*)}$ is a nonzero *-ideal of A, with the exchange involution $(a, b^*)^* = (b, a^*)$ for all $a, b \in I$, hence $A = I \oplus I^{(*)} \cong I \oplus I^{\mathrm{op}}$ since $I^{(*)} \cong I^{\mathrm{op}}$. To show that I is a simple ring, let $K \triangleleft I$ and $K \neq I$, hence $K \oplus K^{\mathrm{op}} \triangleleft^* I \oplus I^{\mathrm{op}} = A$ and consequently $K \oplus K^{\mathrm{op}} = 0$. That is $K = 0$ and I is simple. The sufficiency is obvious. □

Concerning Lemma 1, it is interesting to point out that if A is a simple zero-ring ($A^2 = 0$), then $A \oplus A^{\mathrm{op}}$ with the exchange involution is not *-simple, because the diagonal

$$D = \{(a, a) \mid a \in A\}$$

is a nonzero *-ideal in $A \oplus A^{\mathrm{op}}$ (being a zero-ring, $A = A^{\mathrm{op}}$).

Now, the involutive version of Proposition 4 is given by

THEOREM 1. *An involution ring A is a *-simple ring having a minimal *-biideal if and only if A is isomorphic to a Rees matrix ring $M = M(I, K, \Lambda, P)$ over a division ring K when A is simple or A is isomorphic to $M \oplus M^{\mathrm{op}}$ with exchange involution whenever A is not simple.*

PROOF. Let A be a *-simple ring having a minimal *-biideal. Hence A is semiprime and by Proposition 6, A has a minimal biideal. Applying Proposition 3, we see that A contains also a minimal right ideal R. Using Lemma 1, we distinguish two cases:

(i) If A is simple, then by Proposition 4, A is isomorphic to a Rees matrix ring $M(I, K, \Lambda, P)$ over a division ring K.

(ii) If A is not simple, then $A = J \oplus J^{\mathrm{op}}$, $J \triangleleft A$ and J is a simple ring. Since $A = \operatorname{Soc} A$, we have also $J = \operatorname{Soc} J$ and J has a minimal right ideal. Applying Proposition 4 once again, we get

$$J \cong M(I, K, \Lambda, P).$$

Therefore, $A = J \oplus J^{\mathrm{op}} \cong M \oplus M^{\mathrm{op}}$, and the involution is the exchange involution.

To prove the sufficiency, let first $A \cong M(I, K, \Lambda, P)$. By Proposition 4, A is a simple ring having a minimal right ideal R and $R = eA$ with idempotent $e \in R$. Hence $R^* = Ae^*$ is a minimal left ideal of A. Using Propositions 2 and 5, we get

$$RR^* = (eA)(Ae^*) = eAe^* \neq 0$$

is a minimal biideal of A, which is closed under involution, whence it is a minimal *-biideal of A. That is, A is *-simple with minimal *-biideal. Secondly, assume that $A \cong M \oplus M^{\mathrm{op}}$, with exchange involution, hence A is *-simple, by Lemma 1. Since M is a simple ring having a minimal right ideal $R = eA$ with an idempotent $e \in R$, it follows from Proposition 1 that $eAe \neq 0$ is a minimal biideal of M. Putting $C = eAe$, we get C^{op} a minimal biideal of M^{op}. Finally, $B = C \oplus C^{\mathrm{op}}$ is a minimal *-biideal of $A \cong M \oplus M^{\mathrm{op}}$. That is A is a *-simple ring with a minimal *-biideal. □

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ALEXANDRIA UNIVERSITY
P.O. BOX 21511
ALEXANDRIA
EGYPT

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DISTRIBUTIONS OF RANK ORDER STATISTICS RELATED TO A GENERALIZED RANDOM WALK

J. SARAN and S. RANI

Abstract

This paper deals with the null joint and marginal distributions of some two-sample rank order statistics when one sample size is an integer multiple of the other (i.e., $m=\mu n$) by using the extended Dwass technique evolved by Mohanty and Handa [3] based on a generalized random walk with steps $+1$ and $-\mu$. The rank order statistics considered include the total length of all sojourns above height r, the number of crossings of height r, the number of reflections at height r, and the number of positive reflections at height r (where $r>0$).

1. Introduction

This paper is a continuation of an earlier paper by the authors [4] and deals with the derivation of joint and marginal distributions of certain two-sample rank order statistics when one sample size is an integer multiple of the other. Let $X_1, X_2, \dots, X_{\mu n}$ and $Y_1, Y_2, \dots, Y_n$ be two independent random samples of sizes μn and n (where μ is a positive integer) from the same population having continuous distribution function. Let $F_{\mu n}(x)$ and $G_n(x)$ be the corresponding empirical distribution functions of the two samples. Define the rank order indicator of $\{X_1, X_2, \dots, X_{\mu n}, Y_1, Y_2, \dots, Y_n\}$ as a vector $(Z_1, Z_2, \dots, Z_{(\mu+1)n})$ such that

$$Z_j=\begin{cases} +1, & \text{if the } j\text{th minimun among } \{X_1,\dots,X_{\mu n},Y_1,\dots,Y_n\} \\ & \text{is } X_t \text{ for some } t\in\{1,2,\dots,\mu n\} \\ -\mu, & \text{if the } j\text{th minimum among } \{X_1,\dots,X_{\mu n},Y_1,\dots,Y_n\} \\ & \text{is } Y_t \text{ for some } t\in\{1,2,\dots,n\}, \end{cases}$$

$j=1,2,\dots,(\mu+1)n$. Obviously, $(Z_1, Z_2, \dots, Z_{(\mu+1)n})$ is a sequence of μn $(+1)$'s and n $(-\mu)$'s which we call a sequence of rank order indicators. Under the assumption, the $\binom{(\mu+1)n}{n}$ possible sequences of rank order indicators are equally likely. Any random variable defined on the rank order

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indicator $(Z_1, Z_2, \ldots, Z_{(\mu+1)n})$ is called a rank order statistic. Defining

$$H_{\mu,n}(u) = n\mu[F_{\mu n}(u) - G_n(u)], \qquad -\infty < u < \infty,$$

we note that statistics defined through $H_{\mu,n}(u)$ can be treated as rank order statistics.

Dwass [2] developed a new technique (other than the combinatorial one) based on the simple random walk with independent steps, in order to derive the distributions of some rank order statistics for the case of equal sample sizes (i.e., for $\mu = 1$) which are defined on $H_{1,n}(u)$. Mohanty and Handa [3] extended the technique of Dwass [2] to the case when one sample size is a multiple of the other, so as to cover the case of general μ and derived the distributions of quite a few rank order statistics. For this purpose, they considered the generalized random walk $\left\{S_j : S_j = \sum_{i=1}^{j} W_i, S_0 = W_0 = 0\right\}$ generated by a sequence $\{W_i\}$ of independent random variables with common probability distribution

$$\mathbf{P}(W_i = +1) = p, \qquad \mathbf{P}(W_i = -\mu) = q = 1 - p, \qquad 1 \leqq i < \infty.$$

Further, Saran and Sen [6], Sen and Saran [8], Sen and Kaul [7] and Saran and Rani [4], [5] have derived the joint and marginal distributions of some rank order statistics related to the foregoing generalized random walk $\{S_j\}$ with steps +1 and $-\mu$. Saran and Rani [4] derived, for this random walk with steps +1 and $-\mu$, the joint and marginal distributions of

$L_{\mu,n}(r) =$ the total length of all sojourns above height r,

$N_{\mu,n}(r) =$ the total number of sojourns at height r,

$N^{+}_{\mu,n}(r) =$ the number of sojourns at height r from above, and

$N^{*}_{\mu,n}(r) =$ the number of crossings of height r $(r > 0)$.

In this paper we consider the above mentioned generalized random walk with steps +1 and $-\mu$ and derive, for $r > 0$, the joint and marginal distributions of $L_{\mu,n}(r), R_{\mu,n}(r), R^{+}_{\mu,n}(r)$ and $N^{*}_{\mu,n}(r)$, where

$R_{\mu,n}(r) =$ the total number of reflections at height r,

$=$ the number of subscripts i for which either $H_{\mu,n}(Z_i) = r$

$H_{\mu,n}(Z_{i-1}) = r + \mu,\ H_{\mu,n}(Z_{i+1}) = r + 1$ holds or $H_{\mu,n}(Z_i) = r$

$H_{\mu,n}(Z_{i-1}) = r - 1,\ H_{\mu,n}(Z_{i+1}) = r - \mu$ holds

and

$R^{+}_{\mu,n}(r) =$ the number of positive reflections at height r,

$=$ the number of reflections on the upper side of height r,

$$= \text{the number of subscripts } i \text{ for which } H_{\mu,n}(Z_i) = r,$$
$$H_{\mu,n}(Z_{i-1}) = r + \mu \text{ and } H_{\mu,n}(Z_{i+1}) = r + 1, \; i = 1, 2, \ldots,$$

by employing the extended Dwass technique evolved by Mohanty and Handa [3]. These distributions for the special case $r = 0$ have been obtained by Sen and Kaul [7].

2. Some basic results

Some basic results needed in the sequel are quoted from [3] and [4]. The main theorem of [3] which plays vital role for finding the distributions of rank order statistics is the following:

THEOREM 1. *Suppose $V_{\mu,n}$ is a rank order statistic for every n and V_μ is the corresponding function defined on the random walk which is completely determined by $W_1, W_2, \ldots, W_T$ and does not depend on $W_{T+1}, W_{T+2}, \ldots$, whenever $T > 0$ (where T is the time for the last return to zero in the random walk). Define*

$$h(p) = \mathbf{E}(V_\mu), \qquad p < \mu/(\mu+1). \tag{1}$$

Then we have the following power series (in powers of $p^\mu q$) expansion

$$\frac{h(p)}{1 - (\mu+1)p^\mu q \;\; y^\mu} = \sum_{n=0}^{\infty} \mathbf{E}(V_{\mu,n}) \binom{(\mu+1)n}{n} (p^\mu q)^n, \tag{2}$$

where for y see (4) *and* (5) *below.*

REMARK. The usefulness of Theorem 1 in deriving the distributions of rank order statistics depends on the ease with which one can explicitly evaluate $h(p)$ and then determine the power series expansion

$$h(p)/(1 - (\mu+1)p^\mu q y^\mu) = \sum_{n=0}^{\infty} a_{\mu,n} (p^\mu q)^n.$$

Once such a power series expansion is available, then since the $a_{\mu,n}$'s are uniquely determined, we immediately read off the relationship

$$\mathbf{E}(V_{\mu,n}) = a_{\mu,n} \Big/ \binom{(\mu+1)n}{n}.$$

Further the results (3), (4), (5), (15), (12), (13) and (14), respectively, of [4] are quoted below:

(i) For any a and b,

$$\sum_{k=0}^{\infty} A_k(a,b)\theta^k = x^a, \tag{3}$$

where

$$A_k(a,b) = \frac{a}{a+kb}\binom{a+kb}{k}, \quad \theta = (x-1)/x^b \text{ and } |\theta| < |(b-1)^{b-1}/b^b|,$$

the last inequality assuring the convergence of the series.

(ii) The probability generating function (PGF) for the first return to the origin in a generalized random walk with steps $+1$ and $-\mu$ is

$$F(t) = (\mu+1)p^\mu q x^\mu t^{\mu+1} \tag{4}$$
$$\text{where } p^\mu q t^{\mu+1} = (x-1)/x^{\mu+1} \text{ and } |t|^{\mu+1}p^\mu q < \mu^\mu/(\mu+1)^{\mu+1}.$$

(iii) The probability of never returning to the origin is

$$\delta = 1 - F(1) = 1 - (\mu+1)p^\mu q y^\mu, \tag{5}$$

where y is the value of x when $t=1$.

(iv) Let $\mathbf{E}(t^{L_\mu(r)};\ N_\mu(r)=a,\ \overset{+}{N_\mu}(r)=b,\ N_\mu^*(r)=2c)$ denote the PGF of the length $L_\mu(r)$ of all sojourns above height r when the number $N_\mu(r)$ of total sojourns at height r is a, the number $\overset{+}{N_\mu}(r)$ of positive sojourns at height r from above is b and the number $N_\mu^*(r)$ of crossings of height r is $2c$ in the generalized random walk $\{S_j, 0 \leq j < \infty\}$ with steps $+1$ and $-\mu$. Then for $r > 0$

$$\begin{aligned}
&\mathbf{E}(t^{L_\mu(r)};\ N_\mu(r)=a,\ \overset{+}{N_\mu}(r)=b,\ N_\mu^*(r)=2c) = \\
&\qquad = \sum_g \mathbf{P}(L_\mu(r)=g,\ N_\mu(r)=a,\ \overset{+}{N_\mu}(r)=b,\ N_\mu^*(r)=2c)t^g = \\
&\qquad = (py)^r\{A\alpha^b\beta^{a-b-1}\gamma^{c-1}q(1-(py)^\mu) + \\
&\qquad\quad + B\alpha^b\beta^{a-b-1}\gamma^c q(1-(py)^\mu) + \\
&\qquad\quad + A\alpha^{b-1}\beta^{a-b-1}\gamma^{c-1}qt\sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s})\}
\end{aligned} \tag{6}$$

where

$$A = \binom{b-1}{c-1}\binom{a-b-1}{c-1}, \qquad B = \binom{b-1}{c-1}\binom{a-b-1}{c},$$

$$\alpha = p^\mu q x^\mu t^{\mu+1}, \quad \beta = p^\mu q y^\mu \text{ and } \gamma = 1 + \frac{t}{\alpha}\sum_{j=1}^{\mu-1}(xt/y)^j.$$

(v) Some useful power series expansions are the following:

$$p^k/\delta = \sum_{n=\langle k/\mu\rangle}^{\infty} \binom{(\mu+1)n-k}{n} (p^\mu q)^n, \tag{7}$$

where $\langle z\rangle$ is the smallest integer greater than or equal to z. Further

$$\gamma^c = \sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}(-1)^k\binom{c}{j}\binom{j}{k}\binom{j+i-1}{i}(t/\alpha)^j(xt/y)^{j+i+k(\mu-1)}, \tag{8}$$

whence when $t=1$, $x=y$ and $\alpha=\beta$ and we have

$$\begin{aligned}[\gamma^c]_{t=1} &= (1/\beta y)^c\sum_{r=0}^{c}(-1)^r\binom{c}{r}(\mu y)^{c-r} = \\ &= (1/\beta y)^c(\mu y-1)^c.\end{aligned} \tag{9}$$

3. Joint distribution of $L_\mu(r), R_\mu(r), R_\mu^+(r)$ and $N_\mu^*(r)$, $r>0$

We shall use here the same symbols as used by the authors in [4]. It is easy to observe the following relationships in a generalized random walk with steps $+1$ and $-\mu$:

$$N_{\mu,n}(r) = N^*_{\mu,n}(r) + R_{\mu,n}(r) \tag{10}$$

and

$$N^+_{\mu,n}(r) = \frac{1}{2}N^*_{\mu,n}(r) + R^+_{\mu,n}(r). \tag{11}$$

Let $\mathbf{E}(t^{L_\mu(r)}; R_\mu(r)=d, R_\mu^+(r)=h, N_\mu^*(r)=2c)$ denote the probability generating function (PGF) of $L_\mu(r)$ when $R_\mu(r)=d$, $R_\mu^+(r)=h$ and $N_\mu^*(r)=2c$. Noting that $\mathbf{E}(t^{L_\mu(r)}; R_\mu(r)=d,\ R_\mu^+(r)=h,\ N_\mu^*(r)=2c)$ is to be interpreted as

$$\sum_g \mathbf{P}(L_\mu(r)=g,\ R_\mu(r)=d,\ R_\mu^+(r)=h,\ N_\mu^*(r)=2c)t^g,$$

its expression is given by

$$\begin{aligned}&\mathbf{E}(t^{L_\mu(r)}; R_\mu(r)=d,\ R_\mu^+(r)=h,\ N_\mu^*(r)=2c) = \\ &= (py)^r\binom{c+h-1}{c-1}\Bigg\{\binom{c+d-h-1}{c-1}\alpha^{c+h}\beta^{c+d-h-1}\gamma^{c-1}q(1-(py)^\mu)+ \\ &+\binom{c+d-h-1}{c}\alpha^{c+h}\beta^{c+d-h-1}\gamma^{c}q(1-(py)^\mu)+ \\ &+\binom{c+d-h-1}{c-1}\alpha^{c+h-1}\beta^{c+d-h-1}\gamma^{c-1}qt\sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s})\Bigg\},\end{aligned} \tag{12}$$

where α, β and γ are the same as given in (6).

The result (12) follows immediately on using relations (6), (10) and (11).

4. Deductions

(i) Summing (12) over d, we get

$$\begin{aligned}
&\mathbf{E}(t^{L_\mu(r)}; R_\mu^+(r)=h,\ N_\mu^*(r)=2c)= \\
&=(py)^r\binom{c+h-1}{c-1}\Big\{\alpha^{c+h}\beta^{c-1}\gamma^{c-1}qy^c(1-(py)^\mu)+ \\
&\quad+\alpha^{c+h}\beta^c\gamma^c y^{c+1}q(1-(py)^\mu)+\alpha^{c+h-1}\beta^{c-1}\gamma^{c-1}y^c qt\times \\
&\quad\times\sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s})\Big\}.
\end{aligned}\tag{13}$$

(ii) Summing (13) over h, we get

$$\begin{aligned}
&\mathbf{E}(t^{L_\mu(r)}; N_\mu^*(r)=2c)= \\
&=(py)^r\Big\{(\alpha x)^c\beta^{c-1}\gamma^{c-1}qy^c(1-(py)^\mu)(1+\beta\gamma y)+\alpha^{c-1}x^c\beta^{c-1}\times \\
&\quad\times\gamma^{c-1}y^c qt\sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s})\Big\}.
\end{aligned}\tag{14}$$

(iii) Summing (12) over c, we get

$$\begin{aligned}
&\mathbf{E}(t^{L_\mu(r)}; R_\mu(r)=d,\ R_\mu^+(r)=h)= \\
&=(py)^r\sum_c\binom{c+h-1}{c-1}\Big[\alpha^{c+h}\beta^{c+d-h-1}\gamma^{c-1}q(1-(py)^\mu)\times \\
&\quad\times\Big\{\binom{c+d-h-1}{c-1}+\binom{c+d-h-1}{c}\gamma\Big\}+\alpha^{c+h-1}\beta^{c+d-h-1}\gamma^{c-1} \\
&\quad\times qt\binom{c+d-h-1}{c-1}\sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s})\Big].
\end{aligned}\tag{15}$$

(iv) Summing (13) over c, we get

$$\mathbf{E}(t^{L_\mu(r)}; R_\mu^+(r)=h)=$$

$$= (py)^r \sum_c \binom{c+h-1}{c-1} \Big\{ \alpha^{h+c}\beta^{c-1}\gamma^{c-1}y^c q(1-(py)^\mu)(1+\beta\gamma y)+ \tag{16}$$
$$+ \alpha^{c+h-1}\beta^{c-1}\gamma^{c-1}y^c qt \sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s}) \Big\}.$$

(v) Summing (15) over h, we get

$$\mathbf{E}(t^{L_\mu(r)}; R_\mu(r)=d) =$$
$$= (py)^r \sum_h \sum_c \binom{c+h-1}{c-1} \Big\{ \binom{c+d-h-1}{c-1} \alpha^{c+h}\beta^{d+c-h-1}\gamma^{c-1}q(1-(py)^\mu)+ \tag{17}$$
$$+ \binom{c+d-h-1}{c} \alpha^{c+h}\beta^{c+d-h-1}\gamma^c q(1-(py)^\mu)+$$
$$+ \binom{c+d-h-1}{c-1} \alpha^{c+h-1}\beta^{c+d-h-1}\gamma^{c-1}qt\times$$
$$\times \sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s}) \Big\}.$$

When $t=1$, $x=y$, $\alpha=\beta$ and $\gamma^c=\gamma_1^c$, say, then the results (12) to (17) reduce, respectively, to the results (18) to (23) given below.

(vi) $\mathbf{P}(R_\mu(r)=d,\ R_\mu^+(r)=h,\ N_\mu^*(r)=2c) =$

$$= (py)^r \binom{c+h-1}{c-1} \Big\{ \binom{c+d-h-1}{c-1} \beta^{2c+d-1}\gamma_1^{c-1}q(1-(py)^\mu)+ \tag{18}$$
$$+ \binom{c+d-h-1}{c} \beta^{2c+d-1}\gamma_1^c q(1-(py)^\mu)+$$
$$+ \binom{c+d-h-1}{c-1} \beta^{2c+d-2}\gamma_1^{c-1} q \sum_{s=1}^{\mu-1}(py)^s(1-(py)^{\mu-s}) \Big\}.$$

(vii) $\mathbf{P}(R_\mu^+(r)=h,\ N_\mu^*(r)=2c) =$

$$= (py)^r \binom{c+h-1}{c-1} \Big\{ \beta^{2c+h-1}\gamma_1^{c-1}qy^c(1-(py)^\mu)(1+\beta\gamma_1 y)+ \tag{19}$$
$$+ \beta^{2c+h-2}\gamma_1^{c-1}y^c q \sum_{s=1}^{\mu-1}(py)^s(1-(py)^{\mu-s}) \Big\}.$$

(viii) $\mathbf{P}(N_\mu^*(r)=2c)=$

$$=(py)^r(\mu y-1)^{c-1}\beta^{c-1}y^{c+1}q\left[\sum_{s=0}^{\mu-1}(py)^{\mu-s}-\mu(py)^{\mu+1}\right] \tag{20}$$

(equivalent to (26) in [8]).

(ix) $\mathbf{P}(R_\mu(r)=d,\ R_\mu^+(r)=h)=$

$$\begin{aligned}&=(py)^r\sum_c\binom{c+h-1}{c-1}\Bigg[\beta^{2c+d-1}\gamma_1^{c-1}q(1-(py)^\mu)\times\\ &\times\left\{\binom{c+d-h-1}{c-1}+\binom{c+d-h-1}{c}\gamma\right\}+\beta^{2c+d-2}\gamma_1^{c-1}q\binom{c+d-h-1}{c-1}\times\\ &\times\sum_{s=1}^{\mu-1}(py)^s(1-(py)^{\mu-s})\Bigg].\end{aligned} \tag{21}$$

(x) $\mathbf{P}(R_\mu^+(r)=h)=$

$$\begin{aligned}&=(py)^r\sum_c\binom{c+h-1}{c-1}\Bigg\{\beta^{2c+h-1}\gamma_1^{c-1}y^cq(1-(py)^\mu)(1+\beta\gamma_1y)+\\ &+\beta^{2c+h-2}\gamma_1^{c-1}y^cq\sum_{s=1}^{\mu-1}(py)^s(1-(py)^{\mu-s})\Bigg\}.\end{aligned} \tag{22}$$

(xi) $\mathbf{P}(R_\mu(r)=d)=$

$$\begin{aligned}&=(py)^r\sum_h\sum_c\binom{c+h-1}{c-1}\Bigg\{\binom{c+d-h-1}{c-1}\beta^{2c+d-1}\gamma_1^{c-1}q(1-(py)^\mu)+\\ &+\binom{c+d-h-1}{c}\beta^{2c+d-1}\gamma_1^{c}q(1-(py)^\mu)+\\ &+\binom{c+d-h-1}{c-1}\beta^{2c+d-2}\gamma_1^{c-1}q\sum_{s=1}^{\mu-1}(py)^s(1-(py)^{\mu-s})\Bigg\}.\end{aligned} \tag{23}$$

When $\mu=1$, results (12) to (23) reduce, respectively, to the corresponding results of Aneja [1], Ch. II.

5. Probability distributions

Identifying $h(p)$ as $\mathbf{E}(t^{L_\mu(r)}; R_\mu(r)=d,\ R_\mu^+(r)=h,\ N_\mu^*(r)=2c)$, we have from (12)

$$h(p)/\delta = (py)^r \binom{c+h-1}{c-1}\Bigg\{\binom{c+d-h-1}{c-1}\alpha^{c+h}\beta^{c+d-h-1}\gamma^{c-1}q(1-(py)^\mu)+$$

$$+\binom{c+d-h-1}{c}\alpha^{c+h}\beta^{c+d-h-1}\gamma^{c}q(1-(py)^\mu)+$$

(24)

$$+\binom{c+d-h-1}{c-1}\alpha^{c+h-1}\beta^{c+d-h-1}\gamma^{c-1}qt\sum_{s=1}^{\mu-1}(pxt)^s(1-(py)^{\mu-s})\Bigg\}/\delta$$

$$=I_1+I_2+I_3,\quad \text{say.}$$

Expanding the expression in I_1 as a power series in powers of $p^\mu q$ with the help of (3), (7) and (8), we get

$$I_1=\binom{c+h-1}{c-1}\binom{c+d-h-1}{c-1}\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}\times$$

$$\times(p^\mu q x^\mu t^{\mu+1})^{c+h}(p^\mu q y^\mu)^{c+d-h-1}(t/\alpha)^j(xt/y)^{j+i+k(\mu-1)}\times$$

$$\times(py)^r q(1-(py)^\mu)/\delta=$$

$$=\binom{c+h-1}{c-1}\binom{c+d-h-1}{c-1}\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}\times$$

$$\times x^{\mu(k+c+h-j)+i+j-k}y^{\mu(c+d-h-1-k)-i-j+k+r}(p^\mu q)^{2c+d-1-j}\times$$

(25)

$$\times t^{\mu(c+h-j+k)+c+h+j+i-k}p^r(1-p-(p^\mu q)y^\mu)/\delta=$$

$$=\binom{c+h-1}{c-1}\binom{c+d-h-1}{c-1}\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}\times$$

$$\times\sum_\lambda A_\lambda(\mu(k+c+h-j)+i+j-k,\mu+1)\times$$

$$\times(p^\mu q t^{\mu+1})^\lambda t^{\mu(c+h-j+k)+c+h+i+j-k}\times$$

$$\times\Bigg[(p^\mu q)^{2c+d-1-j}\sum_{\nu_1}A_{\nu_1}(\mu(c+d-h-1-k)-i-j+k+r,\mu+1)(p^\mu q)^{\nu_1}\times$$

$$\times\sum_{m=\langle r/\mu\rangle}^{\infty}\left\{\binom{(\mu+1)m-r}{m}-\binom{(\mu+1)m-r-1}{m}\right\}(p^\mu q)^m-(p^\mu q)^{2c+d-j}\times$$

$$\times \sum_{\nu_2} A_{\nu_2}(\mu(c+d-h-k)-i-j+k+r,\mu+1)(p^\mu q)^{\nu_2}\times$$

$$\times \sum_{m=\langle r/\mu\rangle}^{\infty} \binom{(\mu+1)m-r}{m}(p^\mu q)^m\Bigg].$$

Likewise I_2 and I_3 can be expanded. Upon substituting the values of I_1, I_2 and I_3, so obtained, in (24), and then comparing the coefficient of $t^g(p^\mu q)^n\times$ $\times\binom{(\mu+1)n}{n}$ on both sides of (24), we get the following joint distribution (with the help of Theorem 1):

$$\binom{(\mu+1)n}{n}\mathbf{P}(L_{\mu,n}(r)=g,\ R_{\mu,n}(r)=d,\ R^+_{\mu,n}(r)=h,\ N^*_{\mu,n}(r)=2c)=$$

$$=\binom{c+h-1}{c-1}\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^k\binom{j}{k}\binom{j+i-1}{i}A_\lambda(J,\mu+1)\times$$

$$\times\left[\binom{c+d-h-1}{c-1}\binom{c-1}{j}+\binom{c+d-h-1}{c}\binom{c}{j}\right]\left[\binom{(\mu+1)m-r-1}{m-1}\times\right.$$

$$\times A_{\nu_1}(r-J+\mu(2c+d-j-1),\mu+1)-\binom{(\mu+1)m-r}{m}\times$$

(26)

$$\left.\times A_{\nu_2}(r-J+\mu(2c+d-j),\mu+1)\right]+$$

$$+\binom{c+h-1}{c-1}\binom{c+d-h-1}{c-1}\sum_{j=0}^{c-1}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{s=1}^{\mu-1}(-1)^k\binom{c-1}{j}\binom{j}{k}\times$$

$$\times\binom{j+i-1}{i}A_{\lambda_1}(J+s-\mu,\mu+1)\left[\sum_{m_1=\langle (r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}\times\right.$$

$$\times A_{\nu_1}(r-J+\mu(2c+d-j-1),\mu+1)-$$

$$-\sum_{m_2=\langle (r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}\times$$

$$\left.\times A_{\nu_3}(r-s-J+\mu(2c+d-j),\mu+1)\right],$$

where

$$J=j+i-k+\mu(k+c+h-j),$$

$$J+c+h+(\mu+1)\lambda=J+c+h+(\mu+1)\lambda_1+s-\mu=g,$$

$$\lambda+\nu_1+2c+d-j-1+m=\lambda+\nu_2+2c+d-j+m=$$

$$= \lambda_1 + \nu_1 + m_1 + 2c + d - j - 2 =$$
$$= \lambda_1 + \nu_3 + m_2 + 2c + d - j - 2 = n.$$

Likewise the following distributions can easily be derived from the corresponding PGF's derived in Section 4 by using results (3), (7), (8), (9) and Theorem 1.

$$\binom{(\mu+1)n}{n}\mathbf{P}(L_{\mu,n}(r)=g,\ R^+_{\mu,n}(r)=h,\ N^*_{\mu,n}(r)=2c)=$$
$$=\binom{c+h-1}{c-1}\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^k\binom{j}{k}\binom{j+i-1}{i}A_\lambda(J,\mu+1)\times$$
$$\times\left\{\binom{(\mu+1)m-r-1}{m-1}\left[\binom{c-1}{j}A_{\nu_1}(r-J+c+\mu(2c+h-j-1),\mu+1)+\right.\right.$$
$$+\binom{c}{j}A_{\nu_3}(r-J+c+1+\mu(2c+h-j),\mu+1)-$$
$$-\binom{(\mu+1)m-r}{m}\left[\binom{c-1}{j}A_{\nu_2}(r-J+c+\mu(2c+h-j),\mu+1)+\right.$$
$$\left.+\binom{c}{j}A_{\nu_4}(r-J+c+1+\mu(2c+h-j+1),\mu+1)\right]+\tag{27}$$
$$+\binom{c+h-1}{c-1}\sum_{j=0}^{c-1}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{s=1}^{\mu-1}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}\times$$
$$\times A_{\lambda_1}(J+s-\mu,\mu+1)\times$$
$$\times\left[\sum_{m_1=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}A_{\nu_1}(r+c-J+\mu(2c+h-j-1),\mu+1)-\right.$$
$$\left.-\sum_{m_2=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}A_{\nu_5}(r+c-J-s+\mu(2c+h-j),\mu+1)\right],$$

where

$$j+i-k+\mu(k+c+h-j)=J,$$
$$J+c+h+(\mu+1)\lambda=J+c+h+(\mu+1)\lambda_1+s-\mu=g,$$
$$\lambda+\nu_1+2c+h-j+m-1=\lambda+\nu_2+2c+h-j+m=$$
$$=\lambda+\nu_3+2c+h-j+m=$$
$$=\lambda+\nu_4+2c+h-j+m+1=$$
$$=2c+h-j-2+\lambda_1+\nu_1+m_1=$$
$$=\lambda+\nu_5+2c+h-j-2+m_2=n.$$

$$\binom{(\mu+1)n}{n}\mathbf{P}(L_{\mu,n}(r)=g,\ N_\mu^*(r)=2c)=$$
$$=\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^k\binom{j}{k}\binom{j+i-1}{i}A_\lambda(I,\mu+1)\times$$
$$\times\left\{\binom{(\mu+1)m-r-1}{m-1}\left[\binom{c-1}{j}A_{\nu_1}(J,\mu+1)+\binom{c}{j}A_{\nu_3}(J+1+\mu,\mu+1)\right]-\right.$$
(28)
$$\left.-\binom{(\mu+1)m-r}{m}\left[\binom{c-1}{j}A_{\nu_2}(J+\mu,\mu+1)+\binom{c}{j}A_{\nu_4}(J+1+2\mu,\mu+1)\right]\right\}+$$
$$+\sum_{j=0}^{c-1}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{s=1}^{\mu-1}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}\times$$
$$\times A_{\lambda_1}(I+s-\mu,\mu+1)\left[\sum_{m_1=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}A_{\nu_1}(J,\mu+1)-\right.$$
$$\left.-\sum_{m_2=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}A_{\nu_5}(J-s+\mu,\mu+1)\right],$$

where

$$\begin{aligned}I&=c+j+i-k+\mu(k+c-j),\\ J&=c+r-j-i+k+\mu(c-1-k),\\ I+(\mu+1)\lambda&=I+s-\mu+(\mu+1)\lambda_1=g,\\ \lambda_1+\nu_1+2c-j-1+m&=\lambda+\nu_2+2c-j+m=\\ &=\lambda+\nu_3+2c-j+m=\\ &=\lambda+\nu_4+2c+1-j+m=\\ &=\lambda_1+\nu_1+2c-j-2+m_1=\\ &=\lambda_1+\nu_5+2c-j-2+m_2=n.\end{aligned}$$

$$\binom{(\mu+1)n}{n}\mathbf{P}(L_{\mu,n}(r)=g,\ R_{\mu,n}(r)=d,\ R^+_{\mu,n}(r)=h)=$$
$$=\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\left\{\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^k\binom{j}{k}\binom{j+i-1}{i}\times\right.$$
$$\times A_\lambda(J,\mu+1)\left[\binom{c+d-h-1}{c-1}\binom{c-1}{j}+\binom{c+d-h-1}{c}\binom{c}{j}\right]\times$$

$$
\begin{aligned}
&\cdot \left[\binom{(\mu+1)m-r-1}{m-1} A_{\nu_1}(r-J+\mu(2c+d-j-1),\mu+1)-\right.\\
(29)\quad &\left.-\binom{(\mu+1)m-r}{m} A_{\nu_2}(r-J+\mu(2c+d-j),\mu+1)\right]+\\
&+\sum_{j=0}^{c-1}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{s=1}^{\mu-1}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}A_{\lambda_1}(J+s-\mu,\mu+1)\times\\
&\times\left[\sum_{m_1=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}A_{\nu_1}(r-J+\mu(2c+d-j-1),\mu+1)-\right.\\
&\left.-\sum_{m_2=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}A_{\nu_3}(r-s-J+\mu(2c+d-j),\mu+1)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
J&=j+i-k+\mu(k+c+h-j),\\
J+c+h+(\mu+1)\lambda&=J+c+h+(\mu+1)\lambda_1+s-\mu=g,\\
\lambda+\nu_1+2c+d-j-1+m&=\lambda+\nu_2+2c+d-j+m=\\
&=\lambda_1+\nu_1+m_1+2c+d-j-2=\\
&=\lambda_1+\nu_3+m_2+2c+d-j-2=n.
\end{aligned}
$$

$$
\begin{aligned}
&\binom{(\mu+1)n}{n}\mathbf{P}(L_{\mu,n}(r)=g,R^+_{\mu,n}(r)=h)=\\
&=\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^k\binom{j}{k}\binom{j+i-1}{i}\times\\
&\times A_{\lambda}(J,\mu+1)\left\{\binom{(\mu+1)m-r-1}{m-1}\times\right.\\
&\times\left[\binom{c-1}{j}A_{\nu_1}(r-J+c+\mu(2c+h-j-1),\mu+1)+\right.\\
&\left.+\binom{c}{j}A_{\nu_3}(r-J+c+1+\mu(2c+h-j),\mu+1)\right]-\binom{(\mu+1)m-r}{m}\times\\
(30)\quad &\times\left[\binom{c-1}{j}A_{\nu_2}(r-J+c+\mu(2c+h-j),\mu+1)+\binom{c}{j}\times\right.\\
&\left.\left.\times A_{\nu_4}(r-J+c+1+\mu(2c+h-j+1),\mu+1)\right]\right\}+
\end{aligned}
$$

$$+\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\sum_{j=0}^{c-1}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{s=1}^{\mu-1}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}\times$$
$$\times A_{\lambda_1}(J+s-\mu,\mu+1)\Bigg[\sum_{m_1=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}\times$$
$$\times A_{\nu_1}(r+c-J+\mu(2c+h-j-1),\mu+1)-$$
$$-\sum_{m_2=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}\times$$
$$\times A_{\nu_5}(r+c-J-s+\mu(2c+h-j),\mu+1)\Bigg],$$

where

$$\begin{aligned}
&j+i-k+\mu(k+c+h-j)=J,\\
&\qquad J+c+h+(\mu+1)\lambda=J+c+h+(\mu+1)\lambda_1+s-\mu=g,\\
&\lambda+\nu_1+2c+h-j+m-1=\lambda+\nu_2+2c+h-j+m=\\
&\qquad=\lambda+\nu_3+2c+h-j+m=\\
&\qquad=\lambda+\nu_4+2c+h-j+m+1=\\
&\qquad=2c+h-j-2+\lambda_1+\nu_1+m_1=\\
&\qquad=\lambda+\nu_5+2c+h-j-2+m_2=n.
\end{aligned}$$

$$\binom{(\mu+1)n}{n}\mathbf{P}(L_{\mu,n}(r)=g,\ R_{\mu,n}(r)=d)=$$
$$=\sum_{h=0}^{d}\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\Bigg\{\sum_{j=0}^{c}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^k\binom{j}{k}\times$$
$$\times\binom{j+i-1}{i}A_\lambda(J,\mu+1)\left[\binom{c+d-h-1}{c-1}\binom{c-1}{j}+\binom{c+d-h-1}{c}\binom{c}{j}\right]\times$$
$$\times\Bigg[\binom{(\mu+1)m-r-1}{m-1}A_{\nu_1}(r-J+\mu(2c+d-j-1),\mu+1)-$$
$$-\binom{(\mu+1)m-r}{m}A_{\nu_2}(r-J+\mu(2c+d-j),\mu+1)\Bigg]+\tag{31}$$
$$+\sum_{j=0}^{c-1}\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{s=1}^{\mu-1}(-1)^k\binom{c-1}{j}\binom{j}{k}\binom{j+i-1}{i}\times$$

$$\times A_{\lambda_1}(J+s-\mu,\mu+1)\Bigg[\sum_{m_1=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}\times$$

$$\times A_{\nu_1}(r-J+\mu(2c+d-j-1),\mu+1)-$$

$$-\sum_{m_2=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}\times$$

$$\times A_{\nu_3}(r-s-J+\mu(2c+d-j),\mu+1)\Bigg]\Bigg\},$$

where

$$\begin{aligned}J&=j+i-k+\mu(k+c+h-j),\\ J+c+h+(\mu+1)\lambda&=J+c+h+(\mu+1)\lambda_1+s-\mu=g,\\ \lambda+\nu_1+2c+d-j-1+m&=\lambda+\nu_2+2c+d-j+m=\\ &=\lambda_1+\nu_1+m_1+2c+d-j-2=\\ &=\lambda_1+\nu_3+m_2+2c+d-j-2=n.\end{aligned}$$

$$\binom{(\mu+1)n}{n}\mathbf{P}(R_{\mu,n}(r)=d,\ \overset{+}{R}_{\mu,n}(r)=h,\ N^*_{\mu,n}(r)=2c)=$$

$$=\binom{c+h-1}{c-1}\binom{c+d-h-1}{c-1}\sum_{j=0}^{c-1}(-1)^j\binom{c-1}{j}\mu^{c-1-j}\times$$

$$\times\Bigg\{\sum_{m=\langle r/\mu\rangle}^{\infty}\Bigg[\binom{(\mu+1)m-r-1}{m-1}A_\lambda(r-j+\mu(c+d),\mu+1)-$$

$$-\binom{(\mu+1)m-r}{m}A_{\lambda_1}(r-j+\mu(c+d+1),\mu+1)\Bigg]+$$

(32)

$$+\sum_{s=1}^{\mu-1}\Bigg[\sum_{m_1=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}A_{\lambda_3}(r-j+s+\mu(c+d-1),\mu+1)-$$

$$-\sum_{m_2=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}A_\lambda(r-j+\mu(c+d),\mu+1)\Bigg]\Bigg\}+$$

$$+\binom{c+h-1}{c-1}\binom{c+d-h-1}{c}\sum_{j=0}^{c}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^j\binom{c}{j}\mu^{c-j}\times$$

$$\times\Bigg[\binom{(\mu+1)m-r-1}{m-1}A_{\lambda_2}(r-j+\mu(c+d-1),\mu+1)-$$

$$-\binom{(\mu+1)m-r}{m}A_\lambda(r-j+\mu(c+d),\mu+1)\Bigg],$$

where

$$\lambda+c+d+m=\lambda_1+c+d+1+m=\lambda_2+c+d-1+m$$
$$=\lambda_3+c+d-1+m_1=\lambda+c+d-1+m_2=n.$$

$$\binom{(\mu+1)n}{n}\mathbf{P}(R^+_{\mu,n}(r)=h,\ N^*_{\mu,n}(r)=2c)=$$
$$=\binom{c+h-1}{c-1}\sum_{j=0}^{c-1}(-1)^j\binom{c-1}{j}\mu^{c-1-j}\times$$
$$\times\Bigg[\sum_{s=0}^{\mu-1}\sum_{m=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m-r-s-1}{m-1}\times \tag{33}$$
$$\times A_\lambda(c-j+r+s+\mu(c+h-1),\mu+1)-$$
$$-\mu\sum_{m_1=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-\mu-1}{m_1-1}\times$$
$$\times A_{\lambda_1}(c+r-j+1+\mu(c+h),\mu+1)\Bigg],$$

where

$$\lambda+c+h-1+m=\lambda_1+m_1+c+h-1=n.$$

$$\binom{(\mu+1)n}{n}\mathbf{P}(N^*_{\mu,n}(r)=2c)=$$
$$=\sum_{i=0}^{c-1}(-1)^i\binom{c-1}{i}\mu^{c-1-i}\Bigg[\sum_{s=0}^{\mu-1}\sum_{k=\langle(r-s)/\mu\rangle}^{n-c}\binom{(\mu+1)k-r+s}{k}\times$$
$$\times A_{n-k-c}((\mu+2)c+r-i-s,\mu+1)- \tag{34}$$
$$-\mu\sum_{k=\langle(r+1)/\mu\rangle}^{n-c}\binom{(\mu+1)k-r-1}{k}\times$$
$$\times A_{n-k-c}((\mu+2)c+r-i+1,\mu+1)\Bigg]$$

(equivalent to (25) in [8]).

$$\begin{aligned}
&\binom{(\mu+1)n}{n}\mathbf{P}(R_{\mu,n}(r)=d,\ R^+_{\mu,n}(r)=h)=\\
&=\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\binom{c+d-h-1}{c-1}\sum_{j=0}^{c-1}(-1)^j\binom{c-1}{j}\mu^{c-1-j}\times\\
&\quad\times\Bigg\{\sum_{m=\langle r/\mu\rangle}^{\infty}\Bigg[\binom{(\mu+1)m-r-1}{m-1}A_{\lambda}(r-j+\mu(c+d),\mu+1)-\\
&\quad-\binom{(\mu+1)m-r}{m}A_{\lambda_1}(r-j+\mu(c+d+1),\mu+1)\Bigg]+\\
&\quad+\sum_{s=1}^{\mu-1}\Bigg[\sum_{m_1=\langle (r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}\times\\
&\quad\times A_{\lambda_3}(r-j+s+\mu(c+d-1),\mu+1)-\\
&\quad-\sum_{m_2=\langle (r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}\times\\
&\quad\times A_{\lambda}(r-j+\mu(c+d),\mu+1)\Bigg]\Bigg\}+\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\binom{c+d-h-1}{c}\times\\
&\quad\times\sum_{j=0}^{c}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^j\binom{c}{j}\mu^{c-j}\Bigg[\binom{(\mu+1)m-r-1}{m-1}\times\\
&\quad\times A_{\lambda_2}(r-j+\mu(c-d+1),\mu+1)-\binom{(\mu+1)m-r}{m}\times\\
&\quad\times A_{\lambda}(r-j+\mu(c+d),\mu+1)\Bigg],
\end{aligned}\tag{35}$$

where

$$\begin{aligned}
\lambda+c+d+m&=\lambda_1+c+d+1+m=\lambda_2+c+d-1+m=\\
&=\lambda_3+c+d-1+m_1=\lambda+c+d-1+m_2=n.
\end{aligned}$$

$$\begin{aligned}
&\binom{(\mu+1)n}{n}\mathbf{P}(R^+_{\mu,n}(r)=h)=\\
&=\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\sum_{j=0}^{c-1}(-1)^j\binom{c-1}{j}\mu^{c-1-j}\times
\end{aligned}$$

$$\times\left[\sum_{s=0}^{\mu-1}\sum_{m=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m-r-s-1}{m-1}\times\right.$$

(36)
$$\times A_{\lambda}(c-j+r+s+\mu(c+h-1),\mu+1)-$$

$$-\mu\sum_{m_1=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-\mu-1}{m_1-1}\times$$

$$\left.\times A_{\lambda_1}(r+c-j+1+\mu(c+h),\mu+1)\right],$$

where

$$\lambda+c+h-1+m=\lambda_1+m_1+c+h-1=n.$$

$$\binom{(\mu+1)n}{n}\mathbf{P}(R_{\mu,n}(r)=d)=$$

$$=\sum_{h=0}^{d}\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\binom{c+d-h-1}{c-1}\sum_{j=0}^{c-1}(-1)^j\binom{c-1}{j}\mu^{c-1-j}\times$$

$$\times\left\{\sum_{m=\langle r/\mu\rangle}^{\infty}\left[\binom{(\mu+1)m-r-1}{m-1}\times\right.\right.$$

$$\times A_{\lambda}(r-j+\mu(c+d),\mu+1)-\binom{(\mu+1)m-r}{m}\times$$

$$\left.\times A_{\lambda_1}(r-j+\mu(c+d+1),\mu+1)\right]+$$

$$+\sum_{s=1}^{\mu-1}\left[\sum_{m_1=\langle(r+s)/\mu\rangle}^{\infty}\binom{(\mu+1)m_1-r-s-1}{m_1-1}\times\right.$$

(37)
$$\times A_{\lambda_3}(r-j+s+\mu(c+d-1),\mu+1)-$$

$$-\sum_{m_2=\langle(r+\mu)/\mu\rangle}^{\infty}\binom{(\mu+1)m_2-r-\mu-1}{m_2-1}\times$$

$$\left.\left.\times A_{\lambda}(r-j+\mu(c+d),\mu+1)\right]\right\}+$$

$$+\sum_{h=0}^{d}\sum_{c=1}^{\infty}\binom{c+h-1}{c-1}\binom{c+d-h-1}{c}\sum_{j=0}^{c}\sum_{m=\langle r/\mu\rangle}^{\infty}(-1)^j\binom{c}{j}\times$$

$$\times\mu^{c-j}\left[\binom{(\mu+1)m-r-1}{m-1}A_{\lambda_2}(r-j+\mu(c-d+1),\mu+1)-\right.$$

$$-\binom{(\mu+1)m-r}{m}A_\lambda(r-j+\mu(c+d),\mu+1)\bigg],$$

where

$$\begin{aligned}\lambda+c+d+m&=\lambda_1+c+d+1+m=\lambda_2+c+d-1+m=\\&=\lambda_3+c+d-1+m_1=\lambda+c+d-1+m_2=n.\end{aligned}$$

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DEPARTMENT OF STATISTICS
FACULTY OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELHI
IND–110007 DELHI
INDIA

DEPARTMENT OF STATISTICS
P.G.D.A.V. COLLEGE
NEHRU NAGAR
IND–110065 NEW DELHI
INDIA

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