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# ON BITOPOLOGICAL SPACES II

J. DEÁK

In this paper, we investigate the relations between multifunctions and bitopological separation properties. § 5 contains the definitions and gives, in terms of (families of) multifunctions into topological spaces, conditions guaranteeing that a bispace is $S_i$, respectively that one of its topologies is $S_i$ with respect to the other ($i=1, 2, 3$). (The results for $i=3$ are cited from Smithson [12].) Only a few of these conditions can be shown to be necessary and sufficient, so several problems remain open. § 6 gives a complete answer to the same question with $i=\pi$. § 7 contains some results on multifunctions between bispaces. § 8 deals with a special case of multifunctions into topological spaces, namely the decompositions of spaces. For §§ 0...4, see the first part of this series [5]; notions defined there will be used without explanation.

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## § 5. Bitopologies induced by families of multifunctions into topological spaces

**5.0** Let $X$ and $Y$ be sets. A function $\mathfrak{m}$ assigning to each $x\in X$ a subset of $Y$ is a *multifunction* (multivalued function, set-valued function) from $X$ into $Y$. If we are given a topology $\mathcal{T}$ on $Y$ [and also a topology $\mathcal{S}$ on $X$], we shall refer to $\mathfrak{m}$ as a *multifunction* [*from the space* $(X, \mathcal{S})$] *into the space* $(Y, \mathcal{T})$. A multifunction $\mathfrak{m}$ into a topological space will be called *compact valued*, respectively *closed valued* if for each $x\in X$, $\mathfrak{m}(x)$ is compact, respectively closed. For $A\subset Y$ and $B\subset X$, put

$$\mathfrak{m}^{-1}\langle A\rangle = \{x\in X\colon \mathfrak{m}(x)\cap A \neq \emptyset\}, \quad \mathfrak{m}\langle B\rangle = \bigcup \mathfrak{m}[B].$$

$\mathfrak{m}$ is *onto* if $\mathfrak{m}\langle X\rangle = Y$.

Remarks. a) The formula

$$x r_{\mathfrak{m}} y \Leftrightarrow y\in \mathfrak{m}(x)$$

establishes a one-to-one correspondence between multifunctions from $X$ into $Y$ and relations between elements of $X$ and $Y$, so we could just as well speak of relations instead of multifunctions. It is, however, more in keeping with the traditions to

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work with multifunctions, although some authors prefer the relations. In addition, if the definition below were formulated for relations, it would be in conflict with the usual definition of lower/upper semicontinuity in the special case when $(Y, \mathcal{T})=(\mathbf{R}, \mathcal{E})$ and $r_{\mathfrak{m}}$ happens to be a function.

b) The notations $\mathfrak{m}^{-1}\langle A\rangle$ and $\mathfrak{m}\langle B\rangle$ are motivated by the equalities $\mathfrak{m}^{-1}\langle A\rangle = r_{\mathfrak{m}}^{-1}[A]$ and $\mathfrak{m}\langle B\rangle = r_{\mathfrak{m}}[B]$.

c) Closed valued multifunctions are sometimes called point closed.

DEFINITION (Wilson [15], Kuratowski [8] and Bouligand [3]). A multifunction $\mathfrak{m}$ from a space $(X, \mathcal{S})$ into a space $(Y, \mathcal{T})$ is

a) *lower semicontinuous* if for each $\mathcal{T}$-open set $G$, $\mathfrak{m}^{-1}\langle G\rangle$ is $\mathcal{S}$-open;

b) *upper semicontinuous* if for each $\mathcal{T}$-closed set $F$, $\mathfrak{m}^{-1}\langle F\rangle$ is $\mathcal{S}$-closed.

REMARKS. d) For the motivation of the names, see e.g. [7] 1.7.17 (a).

e) Kuratowski gave these definitions only for closed valued multifunctions into compact metrizable spaces. (Wilson considered an even more special case.) The restriction on $Y$ was later dropped, but upper semicontinuous multifunctions are often closed valued or compact valued by definition; the same applies sometimes also to lower semicontinuity; see e.g. [13], [2], [6], [7], and also the references in [13].

f) It is often contained in the definition of lower/upper semicontinuity (or even in the definition of a multifunction) that $\mathfrak{m}(x)\neq\emptyset$ for each $x\in X$, cf. [2], the footnote on p. 114.

g) Sometimes other names are used instead of lower/upper semicontinuity, e.g. infra- and supra-continuity in [15]; cf. [13].

**5.1** DEFINITION (Smithson [12]). Let $X$ be a set and $\mathfrak{M}$ a family of multifunctions $\mathfrak{m}$ from $X$ into topological spaces $(Y_{\mathfrak{m}}, \mathcal{T}_{\mathfrak{m}})$. Let $\mathcal{P}_{\mathfrak{M}}$ and $\mathcal{Q}_{\mathfrak{M}}$ be the coarsest topologies on $X$ making each $\mathfrak{m}\in\mathfrak{M}$ lower, respectively upper semicontinuous. Then $(\mathcal{P}_{\mathfrak{M}}, \mathcal{Q}_{\mathfrak{M}})$ is the *bitopology induced by* $\mathfrak{M}$. For a single multifunction $\mathfrak{m}$, $(\mathcal{P}_{\{\mathfrak{m}\}}, \mathcal{Q}_{\{\mathfrak{m}\}})$ is called the *bitopology induced by* $\mathfrak{m}$.

NOTATIONS. $\mathcal{P}_{\mathfrak{m}}=\mathcal{P}_{\{\mathfrak{m}\}}$, $\mathcal{Q}_{\mathfrak{m}}=\mathcal{Q}_{\{\mathfrak{m}\}}$, $\mathsf{X}_{\mathfrak{M}}=(X;\ \mathcal{P}_{\mathfrak{M}}, \mathcal{Q}_{\mathfrak{M}})$, $\mathsf{X}_{\mathfrak{m}}=(X;\ \mathcal{P}_{\mathfrak{m}}, \mathcal{Q}_{\mathfrak{m}})$.

REMARKS. a) The systems

$$\{\mathfrak{m}^{-1}\langle G\rangle:\ G\in\mathcal{T}_{\mathfrak{m}},\ \mathfrak{m}\in\mathfrak{M}\}$$

and

$$\{\mathfrak{m}^{-1}\langle F\rangle:\ X\setminus F\in\mathcal{T}_{\mathfrak{m}},\ \mathfrak{m}\in\mathfrak{M}\}$$

form a subbase for $\mathcal{P}_{\mathfrak{M}}$, respectively a closed base for $\mathcal{Q}_{\mathfrak{M}}$. In the case of $\mathcal{P}_{\mathfrak{M}}$, it is enough to take inverse images of open bases (but usually not subbases); in the case of $\mathcal{Q}_{\mathfrak{M}}$, it is not enough to consider inverse images of closed bases.

b) $\mathcal{P}_{\mathfrak{M}} = \sup\{\mathcal{P}_{\mathfrak{m}}:\ \mathfrak{m}\in\mathfrak{M}\}$, $\mathcal{Q}_{\mathfrak{M}} = \sup\{\mathcal{Q}_{\mathfrak{m}}:\ \mathfrak{m}\in\mathfrak{M}\}$.

c) We have interchanged the role of $\mathcal{P}$ and $\mathcal{Q}$ in the above definition in order to adjust it to our other definitions,[1] e.g. R.

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[1] *Added in proof.* Cf. the footnote to 0.4.

d) In place of $\mathfrak{M}$, it would be more precise to take $\{(\mathfrak{m}_i, \mathcal{T}_i): i\in I\}$. We have chosen, however, the simpler terminology and assume instead that $\mathfrak{m}_i \neq \mathfrak{m}_j$ whenever $\mathcal{T}_i \neq \mathcal{T}_j$.

e) As the results of this paper are evidently valid for the empty bispace, we may always assume in the proofs that $X\neq\emptyset$.

**5.2** If arbitrary multifunctions into arbitrary spaces are allowed then each bitopology can be induced by a family of multifunctions. Indeed, put $Y=\{1,2\}$, $\mathcal{T} \triangleq \{\{1\}\}$ and

$$\mathfrak{m}_G(x) = \begin{cases} Y & \text{if} \quad x\in G \\ \{2\} & \text{if} \quad x\notin G \end{cases} \qquad \mathfrak{m}^F(x) = \begin{cases} Y & \text{if} \quad x\in F \\ \{1\} & \text{if} \quad x\notin F \end{cases}$$

$$\mathfrak{M} = \{\mathfrak{m}_G\colon \emptyset \neq G\in\mathcal{P}\}\cup\{\mathfrak{m}^F\colon \emptyset \neq F\in \text{co-}\mathcal{Q}\}.$$

Then $\mathsf{X}=\mathsf{X}_{\mathfrak{M}}$. The problem is more complicated if the multifunctions $\mathfrak{m}$ and/or the topologies $\mathcal{T}_\mathfrak{m}$ are supposed to satisfy some conditions. The cardinality of $\mathfrak{M}$ can also be restricted; in particular, it is an interesting question which bitopologies can be induced by a single multifunction (satisfying certain conditions). Let us start the investigation of such problems with some remarks:

a) One can usually assume without loss of generality that $\mathfrak{m}(x)\neq\emptyset$ $(x\in X)$ [equivalently: $\mathfrak{m}^{-1}\langle Y_\mathfrak{m}\rangle = X$]. Indeed, if this condition is not satisfied then take the topological sum of $Y$ and a one-point space $\{z\}$ and put $\mathfrak{n}(x)=\mathfrak{m}(x)\cup\{z\}$. Now $\mathfrak{n}(x)\neq\emptyset$ $(x\in X)$ and $\mathsf{X}_\mathfrak{n}=\mathsf{X}_\mathfrak{m}$. What is more, $\mathfrak{n}$ and $\mathcal{T}_\mathfrak{n}$ inherit from $\mathfrak{m}$ and $\mathcal{T}_\mathfrak{m}$ most of their good properties, in fact all the properties considered in what follows (multifunctions will be closed valued or compact valued, topological spaces will be compact and/or they will satisfy one of the usual separation axioms).

b) In contrast, supposing $\mathfrak{m}$ to be onto is a real restriction. We can make $\mathfrak{m}$ onto by substituting $\mathfrak{m}\langle X\rangle$ for $Y_\mathfrak{m}$, but in this case the non-hereditary properties of $\mathcal{T}_\mathfrak{m}$ (e.g. compactness) are lost.

c) Let $(Z, \mathcal{T}_\mathfrak{n})$ be the $T_0$-identification of $(Y_\mathfrak{m}, \mathcal{T}_\mathfrak{m})$ and

$$\mathfrak{n}(x) = \{z\in Z\colon \mathfrak{m}(x)\cap z \neq \emptyset\} \quad (z\in Z).$$

Then $\mathsf{X}_\mathfrak{n}=\mathsf{X}_\mathfrak{m}$ and $\mathfrak{n}$ is compact valued, closed valued, respectively onto if $\mathfrak{m}$ has the same property. Consequently, the topologies $\mathcal{T}_\mathfrak{m}$ can always be assumed $T_0$.

d) Each family of multifunctions can be replaced by a single multifunction inducing the same bitopology. Let $(Y, \mathcal{T})$ be the topological sum of the spaces $(Y_\mathfrak{m}, \mathcal{T}_\mathfrak{m})$ $(\mathfrak{m}\in\mathfrak{M})$ and $Z=Y\cup\{w\}$ where $w\in Y$. To simplify the notations, assume that the sets $Y_\mathfrak{m}$ are disjoint and $Y=\cup\{Y_\mathfrak{m}\colon \mathfrak{m}\in\mathfrak{M}\}$. Put

$$\mathfrak{n}(x) = \Big(\bigcup_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}(x)\Big)\cup\{w\} \quad (x\in X)$$

and let $A\in\mathcal{T}_\mathfrak{n}$ iff $A\setminus\{w\}\in\mathcal{T}$ and either $w\notin A$ or $A$ covers all but a finite number of the sets $Y_\mathfrak{m}$. Then $\mathsf{X}_\mathfrak{n}=\mathsf{X}_\mathfrak{M}$, furthermore, $\mathfrak{n}$ and $\mathcal{T}_\mathfrak{n}$ inherit many good properties of the multifunctions $\mathfrak{m}\in\mathfrak{M}$ and of the topologies $\mathcal{T}_\mathfrak{m}$.

In particular, c) and d) give:

THEOREM. *Each bitopology can be induced by a single compact valued multifunction onto a compact normal $T_0$-space.*

**5.3** We shall try to characterize certain classes of bispaces by the existence of some special (families of) multifunctions inducing their bitopologies.

THEOREM. a) *For a bispace* X, *the following conditions are equivalent:*
(i) $\mathscr{P}$ *is* $S_1$ *with respect to* $\mathscr{Q}$;
(ii) *its bitopology can be induced by a family of multifunctions into* $S_1$*-spaces;*
(iii) *its bitopology can be induced by a compact valued multifunction onto a compact* $T_1$*-space.*
b) *The following conditions are also equivalent for a bispace* X:
(iv) $\mathscr{Q}$ *is* $S_1$ *with respect to* $\mathscr{P}$;
(v) *its bitopology can be induced by a family of closed valued multifunctions into arbitrary spaces;*
(vi) *its bitopology can be induced by a closed valued multifunction onto a compact* $T_0$*-space.*

PROOF. (ii)⇒(i): Since the half $S_1$ property is preserved when taking the supremum of bitopologies, it is enough to show that for a multifunction $\mathfrak{m}$ into an $S_1$-space $(Y, \mathscr{T})$, $\mathscr{P}_\mathfrak{m}$ is $S_1$ with respect to $\mathscr{Q}_\mathfrak{m}$. Since $\mathscr{T}$ is $S_1$, each $H\in\mathscr{T}$ is the union of some $\mathscr{T}$-closed sets. Thus $\mathfrak{m}^{-1}\langle H\rangle$ is the union of the inverse images of these closed sets, and these inverse images are, by definition, $\mathscr{Q}_\mathfrak{m}$-closed. The sets $\mathfrak{m}^{-1}\langle H\rangle$ $(H\in\mathscr{T})$ form a subbase of $\mathscr{P}_\mathfrak{m}$, so $\mathscr{P}_\mathfrak{m}$ is indeed $S_1$ with respect to $\mathscr{Q}_\mathfrak{m}$.

(iii)⇒(ii): Evident.

(i)⇒(iii): Let $\emptyset\neq G\in\mathscr{P}$ be fixed and denote by $\mathscr{F}$ the family of the non-empty $\mathscr{Q}$-closed sets contained by $G$. Since $\mathscr{P}$ is $S_1$ with respect to $\mathscr{Q}$, we have $G=\bigcup\mathscr{F}$. Put $Y=(\mathscr{F}\times\omega)\cup\{\emptyset\}$,

$$\mathfrak{m}(x) = (\{F\in\mathscr{F}: x\in F\}\times\omega)\cup\{\emptyset\} \quad (x\in X)$$

and let $\mathscr{T}$ be the co-finite topology on $Y$. $\mathscr{T}$ is a hereditarily compact $T_1$-topology, therefore $\mathfrak{m}$ is a compact valued multifunction onto a compact $T_1$-space. $\mathfrak{m}^{-1}\langle\{\emptyset\}\rangle= =X$ and for each $(F, n)\in Y$, $\mathfrak{m}^{-1}\langle\{(F, n)\}\rangle=F$, thus the inverse image of any $\mathscr{T}$-closed set (i.e. finite set or $Y$) is $\mathscr{Q}$-closed; furthermore, we get each $F\in\mathscr{F}$ in this way. Any $\emptyset\neq H\in\mathscr{T}$ intersects all the sets $\{F\}\times\omega$ $(F\in\mathscr{F})$, so $\mathfrak{m}^{-1}\langle H\rangle=G$ if $\emptyset\notin H$ and $\mathfrak{m}^{-1}\langle H\rangle=X$ if $\emptyset\in H$. Take now such a multifunction $\mathfrak{m}_G$ for each $\emptyset\neq G\in\mathscr{P}$. These $\mathfrak{m}_G$ together clearly induce $(\mathscr{P}, \mathscr{Q})$ (in the case of $\mathscr{Q}$, recall that $G=X$ has not been excluded, thus we get each $F\in$co-$\mathscr{Q}$ as the inverse image of a closed set). To complete the proof, apply now Remark 5.2 d).

(v)⇒(iv); Let $\mathfrak{m}$ be a closed valued multifunction from $X$ into a space $(Y, \mathscr{T})$. Assume $x\in G=X\setminus\mathfrak{m}^{-1}\langle F\rangle$ where $F$ is $\mathscr{T}$-closed. $\mathfrak{m}(x)\cap F=\emptyset$, so with $H=Y\setminus \mathfrak{m}(x)\in\mathscr{T}$: $F\subset H$ and $X\setminus G=\mathfrak{m}^{-1}\langle F\rangle\subset\mathfrak{m}^{-1}\langle H\rangle\in\mathscr{P}_\mathfrak{m}$. Clearly, $x\notin\mathfrak{m}^{-1}\langle H\rangle$, i.e. $X\setminus\mathfrak{m}^{-1}\langle H\rangle\subset G$ is a $\mathscr{P}_\mathfrak{m}$-closed set containing $x$. Thus $G$ is the union of $\mathscr{P}_\mathfrak{m}$-closed sets, and the same holds for each element of a subbase for $\mathscr{Q}_\mathfrak{m}$. Consequently, $\mathscr{Q}_\mathfrak{m}$ is $S_1$ with respect to $\mathscr{P}_\mathfrak{m}$. According to the observation made in the first sentence of the proof, a similar statement holds for families of multifunctions.

(vi)⇒(v): Evident.

(iv)⇒(vi): Let $G\neq\emptyset$ be a $\mathscr{Q}$-open set. If $G$ is at the same time $\mathscr{P}$-closed, let $(Y, \mathscr{T})$ be a one-point space and $\mathfrak{m}^{-1}\langle Y\rangle=X\setminus G$. It is evident that $(\mathscr{P}_\mathfrak{m}, \mathscr{Q}_\mathfrak{m})$ is coarser than $(\mathscr{P}, \mathscr{Q})$ and $G$ is $\mathscr{Q}_\mathfrak{m}$-open. On the other hand, if $G$ is not $\mathscr{P}$-closed then let $\mathcal{F}$ denote the family of the $\mathcal{P}$-closed sets contained by $G$. Since $\mathcal{Q}$ is $S_1$ with respect to $\mathcal{P}$, we have $G=\bigcup\mathcal{F}$. Put $Y=\mathcal{F}\cup\{G\}$ (recall that $G\notin\mathcal{F}$),

$$\mathfrak{m}(x) = \{S\in Y : x\notin S\} \quad (x\in X)$$

and let the topology $\mathcal{T}$ on $Y$ consist of (i) the family $\Gamma$ of the ascending subcollections of $\mathcal{F}$ (i.e. $F\in\mathcal{A}\in\Gamma$, $F\subset F_1\in\mathcal{F}$ imply $F_1\in\mathcal{A}$) and (ii) the sets of the form $\mathcal{A}\cup\{G\}$ where $\emptyset\neq\mathcal{A}\in\Gamma$. It is easy to check that $\mathcal{T}$ is indeed a topology. If $\emptyset\in\mathcal{B}\in\mathcal{T}$ then $\mathcal{B}=Y$ or $\mathcal{B}=\mathcal{F}$, thus any $\mathcal{T}$-open covering of $Y$ has a subcovering of cardinality $\leqq 2$, hence $\mathcal{T}$ is compact. For $x\in G$, $\mathfrak{m}(x)$ is a proper descending subcollection of $\mathcal{F}$ (proper because $\mathrm{cl}_{\mathcal{P}}\{x\}\in\mathcal{F}\setminus\mathfrak{m}(x)$), so it is $\mathcal{T}$-closed. For $x\in X\setminus G$, $\mathfrak{m}(x)=Y$. So $\mathfrak{m}$ is closed valued. $G$ is not $\mathcal{P}$-closed, therefore $X\setminus G\neq\emptyset$, thus $\mathfrak{m}$ is onto, too. We claim that the inverse image of any $\mathcal{T}$-open set is $\mathcal{P}$-open. It is enough to show this for $\emptyset\neq A\in\Gamma$, since $\mathfrak{m}^{-1}\langle\{G\}\rangle\subset\mathfrak{m}^{-1}\langle\{F\}\rangle$ for any $F\in\mathcal{F}$, thus

$$\mathfrak{m}^{-1}\langle\mathcal{A}\cup\{G\}\rangle = \mathfrak{m}^{-1}\langle\mathcal{A}\rangle\cup\mathfrak{m}^{-1}\langle\{G\}\rangle = \mathfrak{m}^{-1}\langle\mathcal{A}\rangle.$$

But $\mathfrak{m}^{-1}\langle\{F\}\rangle=X\setminus F$ is $\mathcal{P}$-open for each $F\in\mathcal{F}$, consequently $\mathfrak{m}^{-1}\langle\mathcal{A}\rangle$ is $\mathcal{P}$-open, as it is the union of the $\mathcal{P}$-open sets $\mathfrak{m}^{-1}\langle\{F\}\rangle$ $(F\in\mathcal{A})$. Let us consider now the inverse images of the $\mathcal{T}$-closed sets. $\{G\}$ is $\mathcal{T}$-closed (this follows from $G\notin\mathcal{F}$) and $\mathfrak{m}^{-1}\langle\{G\}\rangle=X\setminus G$. Any other non-empty $\mathcal{T}$-closed set intersects $\mathcal{F}$ in a non-empty descending subcollection, i.e. it contains $\emptyset$ as an element, and its inverse image contains $\mathfrak{m}^{-1}\langle\{\emptyset\}\rangle$ as a subset; but $\mathfrak{m}^{-1}\langle\{\emptyset\}\rangle=X$. To sum it up: $\mathfrak{m}$ is again a closed valued multifunction onto a topological space such that $(\mathcal{P}_{\mathfrak{m}}, \mathcal{Q}_{\mathfrak{m}})$ is coarser than $(\mathcal{P}, \mathcal{Q})$ and $G$ is $\mathcal{Q}_{\mathfrak{m}}$-open

For ecah non-empty $G\in\mathcal{Q}$, let $\mathfrak{m}_G^*$ denote the multifunction we have just defined. Then

$$\{\mathfrak{m}_G^* : \emptyset\neq G\in\mathcal{Q}\}\cup\{\mathfrak{m}_G : \emptyset\neq G\in\mathcal{P}\}$$

(with $\mathfrak{m}_G$ taken from the beginning of 5.2) induces $(\mathcal{P}, \mathcal{Q})$. The proof can now be completed by applying Remark 5.2 d).

Corollary. *Any family of closed valued multifunctions into $S_1$-spaces induces an $S_1$-bitopology.*

Problem. Can each $S_1$-bitopology be induced by a family of closed valued multifunctions into $S_1$-spaces?

5.4 Proposition. *Any family of compact valued multifunctions into $S_2$-spaces induces an $S_2$-bitopology.*

Proof. Let $\mathfrak{m}$ be a compact valued multifunction from $X$ into the $S_2$-space $(Y, \mathcal{T})$. Assume $x\in G\in\mathcal{P}$, $z\in X\setminus G$. We may suppose without loss of generality that $G$ belongs to the subbase furnished by $\mathfrak{m}$, i.e. $G=\mathfrak{m}^{-1}\langle H\rangle$ for some $H\in\mathcal{T}$. Take a point $y\in\mathfrak{m}(x)\cap H$. Any point $t\in\mathfrak{m}(z)$ is outside the $\mathcal{T}$-neighbourhood $H$ of $y$, so they have disjoint neighbourhoods. As $\mathfrak{m}(z)$ is compact, there are disjoint $\mathcal{T}$-open sets $U$ and $V$ such that $y\in U$ and $\mathfrak{m}(z)\subset V$. Now

$$(1) \qquad x\in\mathfrak{m}^{-1}\langle U\rangle\in\mathcal{P}, \quad z\in X\setminus\mathfrak{m}^{-1}\langle Y\setminus V\rangle\in\mathcal{Q},$$

and these two sets are disjoint. Consequently, $\mathcal{P}_{\mathfrak{m}}$ is $S_2$ with respect to $\mathcal{Q}_{\mathfrak{m}}$. $(\mathcal{P}_{\mathfrak{m}}, \mathcal{Q}_{\mathfrak{m}})$

is $S_1$ by Corollary 5.3, thus it is $S_2$, too. To complete the proof, observe that the supremum of $S_2$-bitopologies is $S_2$.

PROBLEM. Is the converse to the above proposition true?

**5.5** On the analogy of Corollary 5.3, one could expect that Proposition 5.4 remains true if "compact" is replaced by the weaker condition "closed". The next example disproves this conjecture.

EXAMPLE. Let $X=Y=\mathbf{R}^2$, $S=]0, \to[\times\{0\}$, $t=(0,0)$,

$$\mathcal{T} \triangleq \mathcal{E}^2\cup\{Y\setminus S\}, \quad \mathfrak{m}(x) = \{x\}\cup S \quad (x\in X).$$

$\mathfrak{m}$ is a closed valued multifunction onto a $T_2$-space. It is easy to check that

$$\{A_c = ]-c, c[^2\setminus S\colon\ c > 0\}$$

is a $\mathcal{P}_\mathfrak{m}$-neighbourhood base of $t$. To get a closed subbase of $\mathcal{Q}_\mathfrak{m}$, it is enough to take the inverse images of $\mathcal{T}$-closed sets not meeting $S$ (since $\mathfrak{m}^{-1}\langle\{s\}\rangle=X$ for each $s\in S$). But for such a set $F$, $\mathfrak{m}^{-1}\langle F\rangle=F$. So the $\mathcal{T}$-closed sets disjoint from $S$, which are just the $\mathcal{E}^2$-closed sets disjoint from $S$, form a closed subbase for $\mathcal{Q}_\mathfrak{m}$, consequently they are the $\mathcal{Q}_\mathfrak{m}$-closed sets different from $X$ (since they satisfy the intersection and finite union axioms). Thus the non-empty $\mathcal{Q}_\mathfrak{m}$-open sets are $\mathcal{E}^2$-open sets containing $S$, so they intersect each $A_c$, i.e. for $s\in S$ and $t$, there are no disjoint $U\in\mathcal{P}_\mathfrak{m}$ and $V\in\mathcal{Q}_\mathfrak{m}$ with $t\in U$ and $s\in V$, although $t$ has a $\mathcal{P}_\mathfrak{m}$-neighbourhood not containing $s$. Therefore $\mathcal{P}_\mathfrak{m}$ is not $S_2$ with respect to $\mathcal{Q}_\mathfrak{m}$. Similarly, $\mathcal{Q}_\mathfrak{m}$ is not $S_2$ with respect to $\mathcal{P}_\mathfrak{m}$.

**5.6** THEOREM (Smithson [12]). a) *If $\mathfrak{M}$ is a family of multifunctions into $S_3$-spaces then $\mathcal{P}_\mathfrak{M}$ is $S_3$ with respect to $\mathcal{Q}_\mathfrak{M}$.*

b) *If $\mathfrak{M}$ is a family of closed valued multifunctions into normal spaces then $\mathcal{Q}_\mathfrak{M}$ is $S_3$ with respect to $\mathcal{P}_\mathfrak{M}$.*

c) *Any family of compact valued multifunctions into $S_3$-spaces induces an $S_3$-bitopology.*

PROBLEM. Can a) or c) in this theorem be reversed?

REMARKS. a) Part b) of the theorem cannot be reversed; this will follow from Theorem 6.5.

b) Part c) of the theorem does not remain valid if "closed" is substituted for "compact"; an example will be given in 6.9. Observe, however, that any family of closed valued multifunctions into $S_3$-spaces induces an $S_2$-bitopology [apply a) from the theorem and Corollary 5.3].

**5.7** The statement "if $\mathfrak{M}$ is a family of multifunctions into $S_i$-spaces then $\mathcal{P}_\mathfrak{M}$ is $S_i$ with respect to $\mathcal{Q}_\mathfrak{M}$" holds for $i=1$ (5.3), $i=3$ (5.6), and also for $i=\pi$ (6.4). Example 5.5 shows that this statement is false for $i=2$. What is more, we have:

THEOREM. *If $\mathcal{P}$ is $S_1$ with respect to $\mathcal{Q}$ then $(\mathcal{P}, \mathcal{Q})$ can be induced by a multifunction onto a $T_2$-space.*

REMARK. We cannot expect this multifunction to be compact valued (cf. Proposition 5.4), or even only closed valued (cf. Theorem 5.3 b)).

PROOF. For any cardinality $\varkappa>0$, let $(Z_\varkappa, \mathcal{S}_\varkappa)$ be a $T_2$-space such that there is an open set $H\subset Z_\varkappa$ satisfying the following conditions:

(i) there exist $\varkappa$ disjoint dense subsets of $H$;

(ii) any $\mathcal{S}_\varkappa$-closed subset of $H$ is finite.

We shall construct such a space at the end of the proof.

Let $\emptyset\neq G\in\mathcal{P}$ be fixed. Denote by $\mathcal{F}$ the family of the non-empty $\mathcal{Q}$-closed sets contained by $G$. Since $\mathcal{P}$ is $S_1$ with respect to $\mathcal{Q}$, we have $G=\cup\mathcal{F}$. Put $(Y_\mathfrak{m}, \mathcal{T}_\mathfrak{m})=$ $=(Z_\varkappa, \mathcal{S}_\varkappa)$ where $\varkappa=|\mathcal{F}|$. As there is no restriction on the values of $\mathfrak{m}$, it will be more convenient to define $\mathfrak{m}^{-1}$ instead of $\mathfrak{m}$. Let $H\subset Y_\mathfrak{m}$ be as given at the beginning of the proof. By (i), there are disjoint dense sets $A_F$ with $H=\bigcup_{F\in\mathcal{F}} A_F$. Let

$$\mathfrak{m}^{-1}\langle\{y\}\rangle=\begin{cases}F & \text{if } y\in A_F,\ F\in\mathcal{F},\\ X & \text{if } y\in Y_\mathfrak{m}\setminus H.\end{cases}$$

If $S\subset Y_\mathfrak{m}$ and $S\not\subset H$ then evidently $\mathfrak{m}^{-1}\langle S\rangle=X$. Let $S\subset H$ be $\mathcal{T}_\mathfrak{m}$-open; then $\mathfrak{m}^{-1}\langle S\rangle=\cup\mathcal{F}=G$, since $S$ intersects each $A_F$. On the other hand, if $S\subset H$ is $\mathcal{T}_\mathfrak{m}$-closed then $S$ is finite by (ii), i.e. $\mathfrak{m}^{-1}\langle S\rangle$ is $\mathcal{Q}$-closed, being a finite union of $\mathcal{Q}$-closed sets. In particular, we get each $F\in\mathcal{F}$ as the inverse image of a one-point set. Consequently: $P_\mathfrak{m}\triangleq\{G\}$ and $\mathcal{Q}_\mathfrak{m}\subset\mathcal{Q}$: in the special case $G=X$, we have $\mathcal{Q}_\mathfrak{m}=\mathcal{Q}$.

For each $\emptyset\neq G\in\mathcal{P}$, let $\mathfrak{m}_G$ denote the multifunction defined above. Then $\{\mathfrak{m}_G\colon \emptyset\neq G\in\mathcal{P}\}$ induces $(\mathcal{P}, \mathcal{Q})$. According to Remark 5.2 d), this family can be replaced by a single multifunction onto a $T_2$-space.

Now we get to constructing the spaces $(Z_\varkappa, \mathcal{S}_\varkappa)$. Let $\varkappa$ be fixed and assume $\varkappa>\omega$ (the space $(Z_{\omega_1}, \mathcal{S}_{\omega_1})$ will obviously do for countable cardinalities, too). First we define $H$ as the power set of $\varkappa$. Take the topology $\mathcal{R}$ on $H$ for which

$$\mathcal{U}(a)=\big\{U_c(a)=\{b\in H\colon a\cap c=b\cap c\}\colon c\in H,\ |c|\leqq\omega\big\}$$

is a neighbourhood base of $a$ $(a\in H)$. Let $d_i$ $(i\in\varkappa)$ be disjoint uncountable subsets of $\varkappa$. Put

$$D_i=\{b\in H\colon |b\setminus d_i|\leqq\omega,\ |d_i\setminus b|\leqq\omega\}\quad(i\in\varkappa).$$

These sets are disjoint and dense. In $(H, \mathcal{R})$, each $G_\delta$-set is open, therefore disjoint countable subsets of $H$ are contained by disjoint open sets. Take now a maximal almost disjoint collection $\mathcal{C}$ of countably infinite subsets of $H$, i.e.

$$C\in\mathcal{C}\Rightarrow C\subset H,\quad |C|=\omega; \tag{1}$$

$$C_1, C_2\in\mathcal{C},\ C_1\neq C_2\Rightarrow|C_1\cap C_2|<\omega; \tag{2}$$

$$B\subset H,\ |B|=\omega,\ B\notin\mathcal{C}\Rightarrow\exists C\in\mathcal{C},\ |B\cap C|=\omega. \tag{3}$$

(Zorn's Lemma guarantees the existence of such a collection.) For each $C\in\mathcal{C}$, take the open filter

$$\mathcal{O}_C=\{S\subset H\colon |C\setminus\mathrm{int}_\mathcal{R} S|<\omega\}.$$

If $C_1, C_2\in\mathcal{C}$, $C_1\neq C_2$ then $\mathcal{O}_{C_1}$ and $\mathcal{O}_{C_2}$ have disjoint elements (by (1) and (2)); if $a\in H$ and $C\in\mathcal{C}$ then $\mathcal{U}(a)$ and $\mathcal{O}_C$ have disjoint elements. Therefore any extension of $(H, \mathcal{R})$ with the trace filters $\mathcal{O}_C (C\in\mathcal{C})$ is Hausdorff. Let $(Z_\varkappa, \mathcal{S}_\varkappa)$ be the loose extension belonging to these trace filters, i.e.

$$Z_\varkappa = H \cup \{\mathcal{O}_C \colon C \in \mathcal{C}\},$$

$\mathcal{U}(a)$ is an $\mathcal{S}_\varkappa$-neighbourhood base of $a$ $(a \in H)$ and

$$\{\{\mathcal{O}_C\} \cup S \colon S \in \mathcal{O}_C\}$$

is an $\mathcal{S}_\varkappa$-neighbourhood base of $\mathcal{O}_C$ $(C \in \mathcal{C})$. $H$ is clearly an open subset satisfying (i). To show (ii), take an infinite set $I \subset H$. By (3), there is a $C \in \mathcal{C}$ with $|I \cap C| = \omega$, thus $\mathcal{O}_C \in \mathcal{S}_\varkappa$-cl $I$, i.e. $I$ is not closed. This completes the proof.

Problem. Characterize in terms of multifunctions the property "$\mathcal{P}$ is $S_2$ with respect to $\mathcal{Q}$".

## § 6. Multifunctions and complete regularity

**6.0** We begin with establishing a connexion between inducing bitopologies by multifunctions and by pseudo-directions, thus making Theorem 4.14 applicable.

Let $\boldsymbol{d}$ be an orderly pseudo-direction on the set $X$ and put

$$\mathfrak{n}(x) = [\chi_{\boldsymbol{d}}(x), \rightarrow[ \quad (x \in X).$$

Then $\mathfrak{n}$ is a closed valued multifunction from $X$ into $(\boldsymbol{d}, \mathcal{T}_{<\boldsymbol{d}})$ and $\mathsf{X}_\mathfrak{n} = \mathsf{X}_{\boldsymbol{d}}$. Indeed, observe that

$$\text{(1)} \qquad \mathfrak{n}^{-1}\langle\{(G, F)\}\rangle = F \quad (G\boldsymbol{d}F)$$

and

$$\mathfrak{n}^{-1}\langle]\leftarrow, (G, F)[\rangle = G \quad (G\boldsymbol{d}F).$$

Consequently, $\mathcal{P}_{\boldsymbol{d}} \subset \mathcal{P}_\mathfrak{n}$ and $\mathcal{Q}_{\boldsymbol{d}} \subset \mathcal{Q}_\mathfrak{n}$. On the other hand, to prove that $\mathcal{P}_\mathfrak{n} \subset \mathcal{P}_{\boldsymbol{d}}$, it is sufficient to show that $\mathfrak{n}^{-1}\langle \boldsymbol{a} \rangle$ is always $\mathcal{P}_{\boldsymbol{d}}$-open if $\boldsymbol{a}$ is an open interval or an open half-line (cf. Remark 5.1 a)); this is, however, evident since

$$\mathfrak{n}^{-1}\langle]y, (G, F)[\rangle = G \quad \text{if} \quad ]y, (G, F)[ \neq \emptyset$$

and

$$\mathfrak{n}^{-1}\langle]y, \rightarrow[\rangle = X \quad \text{if} \quad y \neq (X, X).$$

It remains to show that $\mathcal{Q}_\mathfrak{n} \subset \mathcal{Q}_{\boldsymbol{d}}$, i.e. that $\mathfrak{n}^{-1}\langle \boldsymbol{a} \rangle$ is $\mathcal{Q}_{\boldsymbol{d}}$-closed whenever $\boldsymbol{a}$ is closed. Two cases are to be considered.

a) If $\boldsymbol{a}$ has a last element $(G, F)$ then $\mathfrak{n}^{-1}\langle \boldsymbol{a} \rangle = F$, a $\mathcal{Q}_{\boldsymbol{d}}$-closed set.

b) If $\boldsymbol{a}$ does not have a last element then the set

$$\boldsymbol{b} = \{z \in \boldsymbol{d} \colon y \in \boldsymbol{a} \Rightarrow y <_{\boldsymbol{d}} z\}$$

cannot have a first element (otherwise $\boldsymbol{a}$ would not be closed), thus

$$\mathfrak{n}^{-1}\langle \boldsymbol{a} \rangle = \bigcup\{F \colon F \in \operatorname{ran} \boldsymbol{a}\} = \bigcap\{G \colon G \in \operatorname{dom} \boldsymbol{b}\} = \bigcap\{F \colon F \in \operatorname{ran} \boldsymbol{b}\} \in \text{co-}\mathcal{Q}_{\boldsymbol{d}}$$

by (1) and the orderliness of $\boldsymbol{d}$.

Now we define a closed valued multifunction *onto* $(\boldsymbol{d}, \mathcal{T}_{<\boldsymbol{d}})$:

$$\mathfrak{m}(x) = \{(\emptyset, \emptyset)\} \cup [\chi_{\boldsymbol{d}}(x), \rightarrow[ \quad (x \in X),$$

i.e. $\mathfrak{m}=\mathfrak{n}\cup\{(\emptyset,\emptyset)\}$. As $\mathfrak{m}^{-1}\langle a\rangle=\mathfrak{n}^{-1}\langle a\rangle$ or $\mathfrak{m}^{-1}\langle a\rangle=X$ $(a\subset d)$, we have $\mathscr{P}_{\mathfrak{m}}\subset\mathscr{P}_{\mathfrak{n}}$ and $\mathscr{Q}_{\mathfrak{m}}\subset\mathscr{Q}_{\mathfrak{n}}$. Further,

$$\mathfrak{m}^{-1}\langle\{(G,F)\}\rangle=F\quad(GdF,\ F\neq\emptyset)$$

and

$$\mathfrak{m}^{-1}\langle](\emptyset,\emptyset),(G,F)[\rangle=G\quad(GdF),$$

thus $\mathscr{P}_d\subset\mathscr{P}_{\mathfrak{m}}$ and $\mathscr{Q}_d\subset\mathscr{Q}_{\mathfrak{m}}$. Consequently, $\mathsf{X}_{\mathfrak{m}}=\mathsf{X}_d$. We shall denote the multifunctions $\mathfrak{n}$ and $\mathfrak{m}$ just defined by $\mathfrak{n}_d$ and $\mathfrak{m}_d$, respectively. If $d$ is in particular an orderly direction then $\mathfrak{m}_d$ is a compact valued multifunction onto a compact $T_2$-space.

Thus we have seen that inducing a bitopology by an orderly pseudo-direction can be regarded as a special case of inducing bitopologies by multifunctions. The next example shows that no similar statement holds if we allow arbitrary pseudo-directions (or only directions).

EXAMPLE. On $X=\{1,2\}$, take the direction

$$d=\{(\emptyset,\emptyset),(\emptyset,\{1\}),(X,X)\}.$$

Suppose that the bitopology of $\mathsf{X}_d$ can be induced by some multifunction $\mathfrak{m}$ into $(d,\mathscr{T}_{<d})$. As $\mathscr{T}_{<d}$ is discrete, $\{m^{-1}\langle a\rangle\colon a\subset d\}$ is an open subbase for $\mathscr{P}_d$ as well as a closed subbase for $\mathscr{Q}_d$; this is a contradiction since $\mathscr{P}_d$ is indiscrete while $\mathscr{Q}_d$ is not.

REMARK. Compare the definition of $\mathfrak{m}_d$ and $\mathfrak{n}_d$ with the following well-known fact: a real function $f$ is lower/upper semicontinuous iff the multifunction $\mathfrak{m}$ defined by $\mathfrak{m}(x)=]\leftarrow,f(x)]$ $(x\in\operatorname{dom}f)$ has the same property. (See e.g. [7] 1.7.17 (a).)

**6.1** DEFINITION. Let $\mathfrak{m}$ be a multifunction from $X$ into $Y$. For a pseudo-direction $d$ on $Y$,

$$\mathfrak{m}^{-1}d=\{(\mathfrak{m}^{-1}\langle G\rangle,\ \mathfrak{m}^{-1}\langle F\rangle)\colon GdF\}\cup\{(\mathfrak{m}^{-1}\langle Y\rangle,\ X),\ (X,\ X)\}.$$

LEMMA. a) *If $d$ is a pseudo-direction on $Y$ and $\mathfrak{m}$ is a multifunction from $X$ into $Y$ then $\mathfrak{m}^{-1}d$ is a pseudo-direction. If $d$ is normal then $\mathfrak{m}^{-1}d$ is normal, too.*

b) *If $d$ is a pseudo-direction of the space $(Y,\mathscr{T})$ and $\mathfrak{m}$ is a multifunction from $X$ into $(Y,\mathscr{T})$ then $\mathfrak{m}^{-1}d$ is a pseudo-direction of the bispace $\mathsf{X}_{\mathfrak{m}}$.*

PROOF. $\mathfrak{m}^{-1}d$ is a pseudo-direction because $\mathfrak{m}^{-1}$ preserves the ordering by inclusion. Assume now that $d$ is normal and take $\Phi\in\operatorname{ran}\mathfrak{m}^{-1}d$, $X\neq\Gamma\in\operatorname{dom}\mathfrak{m}^{-1}d$ with $\Phi\subsetneqq\Gamma$. Then there are $F\in\operatorname{ran}d$ and $G\in\operatorname{dom}d$ such that $\Phi=\mathfrak{m}^{-1}\langle F\rangle$ and $\Gamma=\mathfrak{m}^{-1}\langle G\rangle$. As $\operatorname{dom}d\cup\operatorname{ran}d$ is ordered by inclusion, $\Phi\subsetneqq\Gamma$ implies $F\subset G$; thus there are $G'\in\operatorname{dom}d$ and $F'\in\operatorname{ran}d$ with $F\subset G'\subset F'\subset G$. Therefore

$$\Phi\subset\mathfrak{m}^{-1}\langle G'\rangle\subset\mathfrak{m}^{-1}\langle F'\rangle\subset\Gamma,$$

$$\mathfrak{m}^{-1}\langle G'\rangle\in\operatorname{dom}\mathfrak{m}^{-1}d,\ \mathfrak{m}^{-1}\langle F'\rangle\in\operatorname{ran}\mathfrak{m}^{-1}d,$$

so $d$ is normal.

b) Evident.

**6.2** We intend to prove the theorems of this section without using real functions (cf. 4.13), therefore the obvious way of proving the next two lemmas (i.e. through Lemma 4.8) has to be avoided. Both lemmas could be stated in a stronger form, with orderly directions instead of normal pseudo-directions (cf. Lemma 4.9). On the other hand, these lemmas will be needed in this section only in the special case $\mathscr{P}=\mathscr{Q}$. (But in § 7 we shall use them in the bitopological form proved below.) Compare these lemmas with E. Deák [4].

LEMMA. *Let* X *be a bispace. If $\mathscr{P}$ is $S_\pi$ with respect to $\mathscr{Q}$, $A\cap B=\emptyset$, $A$ is $\mathscr{P}$-compact and $B$ is $\mathscr{P}$-closed then $A$ can be separated from $B$ by a normal pseudo-direction of* X.

PROOF. Straightforward from Definition 4.13 and Lemma 4.11 a).

**6.3** LEMMA. *If* X *is normal, $A\in$ co-$\mathscr{Q}$, $B\in$ co-$\mathscr{P}$, $A\cap B=\emptyset$ then $A$ can be separated from $B$ by a normal pseudo-direction of* X.

PROOF. Let $\boldsymbol{d}$ be a maximal pseudo-direction of X containing

$$\{(\emptyset,\emptyset),(\emptyset,A),(X\setminus B,X),(X,X)\}.$$

Clearly, $\boldsymbol{d}$ separates $A$ from $B$. If $(G_1,F_1)<_{\boldsymbol{d}}(G_2,F_2)$ are neighbours then choose $G\in\mathscr{P}$ and $F\in$co-$\mathscr{Q}$ with $F_1\subset G\subset F\subset G_2$ by the normality of X. Now $G\boldsymbol{d}F$, since $\boldsymbol{d}$ is maximal, therefore $G=F=G_i=F_i$ with $i=1$ or 2, thus $\boldsymbol{d}$ is normal.

**6.4** THEOREM. *For a bitopology $(\mathscr{P},\mathscr{Q})$, the following conditions are equivalent:*
(i) *$\mathscr{P}$ is $S_\pi$ with respect to $\mathscr{Q}$;*
(ii) *$(\mathscr{P},\mathscr{Q})$ can be induced by a family of multifunctions into $S_\pi$-spaces;*
(iii) *$(\mathscr{P},\mathscr{Q})$ can be induced by a multifunction onto a compact $T_2$-space.*

PROOF. (ii)⇒(i): Assume that $(\mathscr{P},\mathscr{Q})$ is induced by a family $\mathfrak{M}$ of multifunctions $\mathfrak{m}$ into $S_\pi$-spaces $(Y_\mathfrak{m},\mathscr{T}_\mathfrak{m})$. Take $x\in\mathfrak{m}^{-1}\langle H\rangle$ where $\mathfrak{m}\in\mathfrak{M}$ and $H\in\mathscr{T}_\mathfrak{m}$. Choose a $y\in\mathfrak{m}(x)\cap H$. By Definition 4.13, there is a normal pseudo-direction $\boldsymbol{d}$ of $(Y_\mathfrak{m},\mathscr{T}_\mathfrak{m})$ separating $\{y\}$ from $X\setminus G$. By Lemma 6.1, $\mathfrak{m}^{-1}\boldsymbol{d}$ is a normal pseudo-direction of X. If $F\in\operatorname{ran}\boldsymbol{d}$ and $G\in\operatorname{dom}\boldsymbol{d}$ with $y\in F\subset G\subset H$ then

$$x\in\mathfrak{m}^{-1}\langle F\rangle\subset\mathfrak{m}^{-1}\langle G\rangle\subset\mathfrak{m}^{-1}\langle H\rangle,$$

i.e. $\mathfrak{m}^{-1}\boldsymbol{d}$ separates $\{x\}$ from $X\setminus\mathfrak{m}^{-1}\langle H\rangle$. Hence $\mathscr{P}$ is $S_\pi$ with respect to $\mathscr{Q}$, by Lemma 4.11 b) and Definition 4.13.
(iii)⇒(ii): Evident.
(i)⇒(iii): Postponed until 6.7.

**6.5** THEOREM. *For a bitopology $(\mathscr{P},\mathscr{Q})$, the following conditions are equivalent:*
(i) *$\mathscr{Q}$ is $S_\pi$ with respect to $\mathscr{P}$;*
(ii) *$(\mathscr{P},\mathscr{Q})$ can be induced by a family of closed valued multifunctions into normal spaces;*
(iii) *$(\mathscr{P},\mathscr{Q})$ can be induced by a closed valued multifunction onto a compact normal $T_0$-space.*

PROOF. (ii)⇒(i): Assume that $(\mathscr{P},\mathscr{Q})$ is induced by a family $\mathfrak{M}$ of closed valued multifunctions $\mathfrak{m}$ into normal spaces $(Y_\mathfrak{m},\mathscr{T}_\mathfrak{m})$. Take $x\in X\setminus\mathfrak{m}^{-1}\langle A\rangle$ where $\mathfrak{m}\in\mathfrak{M}$ and $A$ is $\mathscr{T}_\mathfrak{m}$-closed. $\mathfrak{m}(x)$ is also $\mathscr{T}_\mathfrak{m}$-closed and $A\cap\mathfrak{m}(x)=\emptyset$, thus there is a normal pseudo-direction $\boldsymbol{d}$ of $(Y_\mathfrak{m},\mathscr{T}_\mathfrak{m})$ separating $A$ from $\mathfrak{m}(x)$ (Lemma 6.3). Now $\mathfrak{m}^{-1}\boldsymbol{d}$

is a normal pseudo-direction of X, by Lemma 6.1. If $F \in \operatorname{ran} \boldsymbol{d}$ and $G \in \operatorname{dom} \boldsymbol{d}$ with

$$A \subset F \subset G \subset Y_{\mathfrak{m}} \setminus \mathfrak{m}(x)$$

then

$$\mathfrak{m}^{-1}\langle A\rangle \subset \mathfrak{m}^{-1}\langle F\rangle \subset \mathfrak{m}^{-1}\langle G\rangle \subset X \setminus \{x\},$$

i.e. $\mathfrak{m}^{-1}\boldsymbol{d}$ separates $\mathfrak{m}^{-1}\langle A\rangle$ from $\{x\}$. Therefore $\mathscr{Q}$ is $S_\pi$ with respect to $\mathscr{P}$, by Lemma 4.11 a) and Definition 4.13 [cf. the observations made in the proof of Lemma 4.11 b)].

(iii)⇒(ii): Evident.

(i)⇒(iii): Postponed until 6.7.

**6.6** THEOREM. *For a bitopology $(\mathscr{P}, \mathscr{Q})$, the following conditions are equivalent:*

(i) *it is $S_\pi$;*

(ii) *it can be induced by a family of compact valued multifunctions into $S_\pi$-spaces;*

(iii) *it can be induced by a family of closed valued multifunctions into $S_4$-spaces;*

(iv) *it can be induced by a closed valued multifunction onto a compact $T_2$-space.*

PROOF. (iii)⇒(i): Apply (ii)⇒(i) of Theorem 6.4 and 6.5.

(ii)⇒(i): $\mathscr{P}$ is $S_\pi$ with respect to $\mathscr{Q}$, by Theorem 6.4. To show that $\mathscr{Q}$ is also $S_\pi$ with respect to $\mathscr{P}$, repeat the proof of (ii)⇒(i) in Theorem 6.5, using Lemma 6.2 instead of Lemma 6.3.

(iv)⇒(ii) and (iv)⇒(iii): Evident.

(i)⇒(iv): Take a compatible orderly directional structure $\boldsymbol{D}$ of X (Theorem 4.14). For each $\boldsymbol{d} \in \boldsymbol{D}$, let $\mathfrak{m}_{\boldsymbol{d}}$ be the multifunction defined in 6.0. Then $\{\mathfrak{m}_{\boldsymbol{d}}: \boldsymbol{d} \in \boldsymbol{D}\}$ is a family of closed valued multifunctions onto compact $T_2$-spaces and it induces $(\mathscr{P}, \mathscr{Q})$. To complete the proof, apply 5.2 d).

**6.7** PROOF of Theorems 6.4 and 6.5 continued.

a) (i)⇒(iii) of 6.4: By Corollary 4.14, there is a topology $\mathscr{Q}_1 \subset \mathscr{Q}$ such that $(\mathscr{P}, \mathscr{Q}_1)$ is $S_\pi$. By Theorem 6.6, $(\mathscr{P}, \mathscr{Q}_1)$ can be induced by a multifunction $\mathfrak{m}_1$ onto a compact $T_2$-space. Let $(Y, \mathscr{T})$ be $\omega+1$ with the order topology. For each non-empty $\mathscr{Q}$-closed set $F$, we define a multifunction $\mathfrak{m}_F$ onto the compact $T_2$-space $(Y, \mathscr{T})$ by

$$\mathfrak{m}_F(x) = \begin{cases} Y & \text{if } x \in F \\ \omega & \text{if } x \notin F. \end{cases}$$

Now $\mathfrak{m}_F^{-1}\langle\{\omega\}\rangle = F$; for any $\emptyset \neq A \subset Y$ different from $\{\omega\}$, we have $\mathfrak{m}_F^{-1}\langle A\rangle = X$, therefore $\mathscr{P}_{\mathfrak{m}_F}$ is indiscrete and $\mathscr{Q}_{\mathfrak{m}_F} \triangleq \{X \setminus F\}$. Thus

$$\mathfrak{M} = \{\mathfrak{m}_1\} \cup \{\mathfrak{m}_F: \emptyset \neq F \in \text{co-}\mathscr{Q}\}$$

induces $(\mathscr{P}, \mathscr{Q})$. According to 5.2 d), $\mathfrak{M}$ can be replaced by a single multifunction onto a compact $T_2$-space.

b) (i)⇒(iii) of 6.5: Take a closed valued multifunction onto a normal space such that $\mathscr{P}_{\mathfrak{m}_1} \subset \mathscr{P}$ and $\mathscr{Q}_{\mathfrak{m}_1} = \mathscr{Q}$ (Corollary 4.14 and Theorem 6.4). Now with $\mathfrak{m}_G$ from 5.2,

$$\{\mathfrak{m}_1\} \cup \{\mathfrak{m}_G: \emptyset \neq G \in \mathscr{P}\}$$

is a family of closed valued multifunctions onto normal spaces, inducing $(\mathscr{P}, \mathscr{Q})$. Finally, apply again 5.2 d).

REMARK. The proof of 6.4 is somewhat "less internal" than the other proofs in this section, or in § 4: it uses not only the natural numbers, but also the *set* of the natural numbers. To see that using $\omega$ cannot be avoided in the proof of 6.4 (i)⇒(iii), consider $X=\{1, 2\}$ with $\mathscr{P}$ indiscrete and $\mathscr{Q} \triangleq \{\{1\}\}$; here $\mathscr{P}$ is $S_\pi$ with respect to $\mathscr{Q}$, $X$ is finite, but $(\mathscr{P}, \mathscr{Q})$ cannot be induced by a multifunction into a finite $T_2$-space, (cf. Example 6.0).

**6.8** If using real functions is allowed, the lemma below may replace Lemmas 4.11, 6.1, 6.2 and 6.3 in the proof of Theorems 6.4, 6.5 and 6.6. In the proof of (i)⇒ ⇒(iv) in Theorem 6.6, 5.3 e) can be used instead of Theorem 4.14. The construction of 6.0 can be replaced by the following: a function $f\colon X\to[0, 1]$ is bi-continuous on $\mathsf{X}$ iff the multifunction $\mathfrak{m}$ defined by

$$\mathfrak{m}(x) = \{0\}\cup[f(x), 1] \quad (x\in X)$$

is $\mathscr{P}$-lower and $\mathscr{Q}$-upper semicontinuous; cf. Remark 6.0. Corollary 4.14 could also have been proved directly, without using directional structures.

LEMMA. *Let $\mathfrak{m}$ be a multifunction from a set $X$ into the space $(Y, \mathscr{T})$ such that $\mathfrak{m}^{-1}\langle Y\rangle=X$ and let $f$ be a bounded continuous real function on $(Y, \mathscr{T})$. If*

$$g(x) = \inf f[\mathfrak{m}(x)] \quad (x\in X)$$

*then $g$ is a bi-continuous real function on $\mathsf{X}_\mathfrak{m}$.*

PROOF. For any $t\in \mathbf{R}$, put

$$G_t = ]\leftarrow, t[, \quad F_t = ]\leftarrow, t].$$

It is easy to check that

$$g^{-1}[G_t] = \mathfrak{m}^{-1}\langle f^{-1}[G_t]\rangle\in\mathscr{P}_\mathfrak{m},$$

so $g$ is $\mathscr{P}_\mathfrak{m}$-upper semicontinuous. Further,

$$g^{-1}[F_t] = \bigcap_{s>t} g^{-1}[G_s] = \bigcap_{s>t} \mathfrak{m}^{-1}\langle f^{-1}[G_s]\rangle = \bigcap_{s>t} \mathfrak{m}^{-1}\langle f^{-1}[F_s]\rangle\in\text{co-}\mathscr{Q}_\mathfrak{m},$$

so $g$ is $\mathscr{Q}_\mathfrak{m}$-lower semicontinuous, too.

**6.9** EXAMPLE. [Showing that "compact" cannot be replaced by "closed" in Theorem 6.6 (ii) and Theorem 5.6 c).] Let $(Y, \mathscr{T})$ be a non-normal $T_\pi$-space (e.g. the Tikhonov plank). Take disjoint closed sets $A, B\subset Y$ which are not contained by disjoint open sets. Choose a $y\in A$ and put

$$X = (Y\setminus A)\cup\{y\}, \quad \mathfrak{m}(x) = \begin{cases} A & \text{if } x = y \\ \{x\} & \text{if } x\in Y\setminus A. \end{cases}$$

Now $\mathfrak{m}$ is a closed valued multifunction onto a $T_\pi$-space, but $\mathsf{X}_\mathfrak{m}$ is not even $S_3$. Indeed, assume that $\mathscr{Q}_\mathfrak{m}$ is $S_3$ with respect to $\mathscr{P}_\mathfrak{m}$; then there are $G\in\mathscr{P}_\mathfrak{m}$ and $F\in\text{co-}\mathscr{Q}_\mathfrak{m}$ such that

(1) $$B\subset G\subset F\subset Y\setminus A,$$

since $B=\mathfrak{m}^{-1}\langle B\rangle$ is $\mathscr{Q}_\mathfrak{m}$-closed. One can readily show that the $\mathscr{P}_\mathfrak{m}$-open (respectively, the $\mathscr{Q}_\mathfrak{m}$-closed) sets not containing $y$ are just the $\mathscr{T}$-open (respectively, $\mathscr{T}$-closed) sets not meeting $A$, i.e. (1) is a contradiction.

## § 7. Multifunctions between bitopological spaces

**7.0** Definition. A multifunction from a bispace X into a bispace $(Y; \mathscr{S}, \mathscr{T})$ is *semicontinuous* if it is $(\mathscr{P}, \mathscr{S})$-lower and $(\mathscr{Q}, \mathscr{T})$-upper semicontinuous. It *induces the bitopology* $(\mathscr{P}, \mathscr{Q})$ if $(\mathscr{P}, \mathscr{Q})$ is the coarsest bitopology on $X$ for which it is semicontinuous. A family of multifunctions into bispaces *induces the bitopology* $(\mathscr{P}, \mathscr{Q})$ if $(\mathscr{P}, \mathscr{Q})$ is the coarsest bitopology for which each of the multifunctions is semicontinuous.

**7.1** Those parts of the theorems in § 5 and § 6 which claim that the induced bitopologies satisfy certain separation axioms can be generalized for bitopologies induced by families of multifunctions into bispaces; the proofs are essentially the same, therefore we shall state only some of the results, and the proofs will be left to the reader. For example, the generalized form of Theorem 5.6 runs as follows:

Theorem. *Let $(\mathscr{P}, \mathscr{Q})$ be induced by a family $\mathfrak{M}$ of multifunctions* $\mathfrak{m}$ *into bispaces* $(Y_\mathfrak{m}; \mathscr{S}_\mathfrak{m}, \mathscr{T}_\mathfrak{m})$. *Then:*

a) *If $\mathscr{S}_\mathfrak{m}$ is $S_3$ with respect to $\mathscr{T}_\mathfrak{m}$* $(\mathfrak{m}\in\mathfrak{M})$ *then $\mathscr{P}$ is $S_3$ with respect to $\mathscr{Q}$.*

b) *If $(\mathscr{S}_\mathfrak{m}, \mathscr{T}_\mathfrak{m})$ is normal and* $\mathfrak{m}$ *is $\mathscr{S}_\mathfrak{m}$-closed valued* $(\mathfrak{m}\in\mathfrak{M})$ *then $\mathscr{Q}$ is $S_3$ with respect to $\mathscr{P}$.*

c) *If $\mathscr{T}_\mathfrak{m}$ is $S_3$ with respect to $\mathscr{S}_\mathfrak{m}$ and* $\mathfrak{m}$ *is $\mathscr{T}_\mathfrak{m}$-compact valued* $(\mathfrak{m}\in\mathfrak{M})$ *then $\mathscr{Q}$ is $S_3$ with respect to $\mathscr{P}$.*

d) *If $\mathscr{S}_\mathfrak{m}$ is $S_4$ with respect to $\mathscr{T}_\mathfrak{m}$ and* $\mathfrak{m}$ *is $\mathscr{S}_\mathfrak{m}$-closed valued* $(\mathfrak{m}\in\mathfrak{M})$ *then $(\mathscr{P}, \mathscr{Q})$ is $S_3$.*

e) *If $(\mathscr{S}_\mathfrak{m}, \mathscr{T}_\mathfrak{m})$ is $S_3$ and* $\mathfrak{m}$ *is $\mathscr{T}_\mathfrak{m}$-compact valued* $(\mathfrak{m}\in\mathfrak{M})$ *then $(\mathscr{P}, \mathscr{Q})$ is $S_3$.*

Remark. In b) and d), also $S_\pi$ could have been concluded, instead of $S_3$.

**7.2** The similar results for $S_1$ and $S_\pi$ can also be generalized in the same way: compact valued and closed valued are to be replaced by $\mathscr{T}_\mathfrak{m}$-compact valued and $\mathscr{S}_\mathfrak{m}$-closed valued, respectively. In the case of $S_\pi$, real functions are not necessary in the proofs; the lemmas in § 6 have been formulated in the generality needed here. Axiom $S_2$ deserves some attention, since it is not enough to copy the proof of Proposition 5.4.

Proposition. *Let $(\mathscr{P}, \mathscr{Q})$ be induced by a family $\mathfrak{M}$ of multifunctions* $\mathfrak{m}$ *into bispaces $(Y_\mathfrak{m}; \mathscr{S}_\mathfrak{m}, \mathscr{T}_\mathfrak{m})$. Assume that* $\mathfrak{m}$ *is $\mathscr{T}_\mathfrak{m}$-compact valued* $(\mathfrak{m}\in\mathfrak{M})$.

a) *If $\mathscr{S}_\mathfrak{m}$ is $S_2$ with respect to $\mathscr{T}_\mathfrak{m}$* $(\mathfrak{m}\in\mathfrak{M})$ *then $\mathscr{P}$ is $S_2$ with respect to $\mathscr{Q}$.*

b) *If $\mathscr{T}_\mathfrak{m}$ is $S_2$ with respect to* $S_\mathfrak{m}$ $(\mathfrak{m}\in\mathfrak{M})$ *then $\mathscr{Q}$ is $S_2$ with respect to $\mathscr{P}$.*

c) *If $(\mathscr{S}_\mathfrak{m}, \mathscr{T}_\mathfrak{m})$ is $S_2$* $(\mathfrak{m}\in\mathfrak{M})$ *then $(\mathscr{P}, \mathscr{Q})$ is $S_2$, too.*

Proof. a) Modify the proof of Proposition 5.4.

b) Take $z\in H\in\mathscr{Q}$ and $x\in X\setminus H$. Assume that $X\setminus H=\mathfrak{m}^{-1}\langle F\rangle$ where $\mathfrak{m}\in\mathfrak{M}$ and $F\in$co-$\mathscr{T}_\mathfrak{m}$. Choose a $y\in\mathfrak{m}(x)\cap F$. Now $Y_\mathfrak{m}\setminus F$ is a $\mathscr{T}_\mathfrak{m}$-neighbourhood of $\mathfrak{m}(z)$ not containing $y$, therefore there are disjoint sets $U\in\mathscr{S}_\mathfrak{m}$ and $V\in\mathscr{T}_\mathfrak{m}$ such that $y\in U$ and $\mathfrak{m}(x)\subset V$ [since $\mathfrak{m}(x)$ is $\mathscr{T}_\mathfrak{m}$-compact and $\mathscr{T}_\mathfrak{m}$ is $S_2$ with respect to $\mathscr{S}_\mathfrak{m}$]. So we have again 5.4 (1).

c) Combine a) and b).

Remark. Part b) of the above proof was not needed in the proof of Proposition 5.4 because it is enough to prove half of Axiom $S_2$ if the bispace is already known to satisfy $S_1$.

**7.3** The converse results are in some cases trivialities; e.g. each $S_2$-bitopology can be induced by a multifunction onto an $S_2$-bispace such that it is compact valued in the second topology: take $\mathfrak{m}(x)=\{x\}$ into the original bispace. The similar weakening of Problem 5.3 can also be answered by putting $\mathfrak{m}(x)=\mathrm{cl}_{\mathscr{P}}\{x\}$. In several other cases, the converse results were proved in § 5 and § 6 in a much stronger form.

Concerning multifunctions between bispaces, see also [9], [10], [11].

## § 8. Bitopologies induced by decompositions of topological spaces

**8.0** Let $\mathscr{D}$ be a decomposition of the topological space $(Y, \mathscr{T})$ (i.e. a family of non-empty disjoint sets with $\bigcup\mathscr{D}=Y$). We define the topologies $\mathscr{P}_{\mathscr{D}}$ and $\mathscr{Q}_{\mathscr{D}}$ on $\mathscr{D}$ as the coarsest topologies for which

$$\{D\in\mathscr{D}: D\cap S \neq \emptyset\}$$

is open, respectively closed, whenever $S$ is open, respectively closed. These topologies individually (but not together as a bitopology) have been considered by several authors, see e.g. [14] ("*g*-space" and "*c*-space"), [1] ("schwacher Zerlegungsraum"). Clearly, the bitopology $(\mathscr{P}_{\mathscr{D}}, \mathscr{Q}_{\mathscr{D}})$ is identical with $(\mathscr{P}_{\mathfrak{m}}, \mathscr{Q}_{\mathfrak{m}})$ where $\mathfrak{m}(D)=D$ $(D\in\mathscr{D})$. In this section, we shall investigate the problem whether or no bitopologies inducible by multifunctions into certain types of spaces can also be induced by decompositions of spaces with the same properties. As the points of a bispace $(\mathscr{D}; \mathscr{P}_{\mathscr{D}}, \mathscr{Q}_{\mathscr{D}})$ are always disjoint sets, the property of being inducible by a decomposition in the above sense is not an invariant under bi-homeomorphisms, so let us agree on the following definition, which will remove this inconvenience.

Definition. A multifunction $\mathfrak{m}$ from $X$ into $Y$ is a *decomposition of* $Y$ if

(i) $\mathfrak{m}^{-1}\langle Y\rangle = X$;

(ii) $\mathfrak{m}\langle X\rangle = Y$;

(iii) $x, z\in X,\ x \neq z \Rightarrow \mathfrak{m}(x)\cap\mathfrak{m}(z) = \emptyset$.

Clearly, a bitopology can be induced by a decomposition in this sense iff it is bi-homeomorphic to a bitopology inducible by a decomposition in the earlier sense.

A closed/compact valued decomposition will be called a *decomposition into closed/compact sets.*

**8.1** In the results of §§ 5 and 6, families of multifunctions can always be replaced by single multifunctions, furthermore, 8.0 (i) and (ii) can always be assumed without loss of generality [(i) by 5.2 a)]. Applying the well-known construction of resolving a relation into the composition of a function and the inverse of another function (see e.g. [12]), it is easy to define a decomposition inducing the same bitopology as the multifunction: let $\mathfrak{m}$ be a multifunction from $X$ into $(Y, \mathscr{T})$ satisfying 8.0 (i) and (ii), and denote by $R$ the relation $r_{\mathfrak{m}}$ introduced in Remark 5.0 a); define a function $f\colon R\to Y$ by

$$f((x, y)) = y \quad (y\in\mathfrak{m}(x))$$

and put

$$\mathfrak{d}(x) = \{(x, y) : y \in \mathfrak{m}(x)\} \quad (x \in X);$$

then $\mathfrak{d}$ is a decomposition of the space $(R, f^{-1}\mathcal{T})$ and $(\mathcal{P}_\mathfrak{d}, \mathcal{Q}_\mathfrak{d}) = (\mathcal{P}_\mathfrak{m}, \mathcal{Q}_\mathfrak{m})$.

If $\mathcal{T}$ is $S_i$ $(i=1, 2, 3, \pi, 4)$, compact, or normal then so is $f^{-1}\mathcal{T}$; if $\mathfrak{m}$ is a compact valued multifunction then $\mathfrak{d}$ is a decomposition into compact sets, since $\mathfrak{m}(x)$ and $\mathfrak{d}(x)$ are homeomorphic. On the other hand, closed valuedness and axiom $T_0$ are lost in this construction. Thus we have from Theorem 5.2:

THEOREM. *Each bitopology can be induced by a decomposition into compact sets of a compact normal space.*

**8.2** In the above theorem, we cannot assume that the space is $T_0$. Indeed, let $\mathfrak{d}$ be a decomposition of a compact $T_0$-space $(Y, \mathcal{T})$; then there is a closed one-point set in $Y$ [for each ordinal $\alpha$, take $\emptyset \neq F_\alpha \in$ co-$\mathcal{T}$ such that (i) if $\beta < \alpha$ then $F_\alpha \subset F_\beta$, (ii) if $|F_\alpha| > 1$ then $F_{\alpha+1} \neq F_\alpha$ (here we make use of $T_0$); the compactness of $\mathcal{T}$ guarantees that the recursion does not break off at limit ordinals; the sets $F_\alpha$ are all different as long as $|F_\alpha| > 1$, so $|F_\alpha| = 1$ if $\alpha$ is large enough]. Now if $\{y\}$ is closed then $\mathfrak{d}^{-1}\langle\{y\}\rangle$ is a $\mathcal{Q}_\mathfrak{d}$-closed one-point set.

**8.3** We have, however:

THEOREM. *Each bitopology can be induced by a decomposition of a normal $T_0$-space.*

PROOF. a) Let $S$ be a non-trivial subset of $X$; let $Y_S$ be a linearly ordered set with the normal $T_0$-topology $\mathcal{T}_S$ consisting of the descending subsets of $Y_S$. Fix an $A \in \mathcal{T}_S$ and define a decomposition $\mathfrak{d}_S$ of $(Y_S, \mathcal{T}_S)$ such that $\operatorname{dom} \mathfrak{d}_S = X$, $\mathfrak{d}_S(x) \subset A$ is coinitial and cofinal in $A$ if $x \in S$, $\mathfrak{d}_S(x)$ is coinitial and cofinal in $Y$ if $x \notin S$. Now $\mathcal{P}_{\mathfrak{d}_S}$ is indiscrete and $\mathcal{Q}_{\mathfrak{d}_S} \triangleq \{S\}$.

b) The same construction, with $A \in$ co-$\mathcal{T}_S$, gives a decomposition $\mathfrak{d}_S$ such that $\mathcal{P}_{\mathfrak{d}_S} \triangleq \{X \setminus S\}$ and $\mathcal{Q}_{\mathfrak{d}_S}$ is indiscrete.

c) Given an arbitrary bitopology $(\mathcal{P}, \mathcal{Q})$ on $X$, it can be induced by a family of decompositions of normal $T_0$-spaces, as described in a) and b).

d) To replace this family of decompositions by a single decomposition, we shall apply a modified form of the construction in 5.2 d). Let $\mathfrak{D}$ be the family of decompositions $\mathfrak{d}$ of spaces $(Y_\mathfrak{d}, \mathcal{T}_\mathfrak{d})$ obtained in c) and let $Z$ be a linearly ordered set. Assume that the sets $Y_\mathfrak{d}$ $(\mathfrak{d} \in \mathfrak{D})$ and $Z$ are disjoint and put

$$Y = Z \cup \bigcup_{\mathfrak{d} \in \mathfrak{D}} Y_\mathfrak{d}.$$

The topology $\mathcal{T}$ on $Y$ is defined as follows: for $y \in Y_\mathfrak{d}$, the $\mathcal{T}_\mathfrak{d}$-neighbourhoods of $y$ form a $\mathcal{T}$-neighbourhood base of $y$ $(\mathfrak{d} \in \mathfrak{D})$; for $z \in Z$, the sets

$$\{a \in Z : a \leqq z\} \cup \bigcup \{Y_\mathfrak{d} : \mathfrak{d} \in \mathfrak{D}_0\} \quad (\mathfrak{D}_0 \subset \mathfrak{D},\ |\mathfrak{D} \setminus \mathfrak{D}_0| < \omega)$$

form a neighbourhood base of $z$ ($\leqq$ is to be understood in the ordering of $Z$). Let $\mathfrak{e}$ be a decomposition of $Z$ such that $\operatorname{dom} \mathfrak{e} = X$ and for each $x \in X$, $\mathfrak{e}(x)$ is coinitial and cofinal in $Z$. Then

$$\mathfrak{c}(x)=\mathfrak{e}(x)\cup\bigcup_{\mathfrak{d}\in\mathfrak{D}}\mathfrak{d}(x)\quad(x\in X)$$

defines a decomposition $\mathfrak{c}$ of the normal $T_0$-space $(Y,\mathcal{T})$ such that $(\mathcal{P}_\mathfrak{c},\mathcal{Q}_\mathfrak{c})=(\mathcal{P},\mathcal{Q})$.

**8.4** It is not true that each bitopology can be induced by a decomposition into compact sets of a $T_0$-space. Indeed, if $X$ is finite and $\mathfrak{d}$ is a decomposition into compact sets of the $T_0$-space $(Y,\mathcal{T})$ then $\mathcal{T}$ is compact, thus there is a $\mathcal{Q}_\mathfrak{d}$-closed one-point set by 8.2.

**8.5** From Theorems 5.3 a) and 5.7 we have, through the construction in 8.1:

THEOREM. *For a bispace* $\mathsf{X}$, *the following conditions are equivalent;*
(i) $\mathcal{P}$ *is* $S_1$ *with respect to* $\mathcal{Q}$;
(ii) *its bitopology can be induced by a decomposition into compact sets of a compact* $S_1$*-space;*
(iii) *its bitopology can be induced by a decomposition of an* $S_2$*-space.*

REMARKS. a) $S_1$ and $S_2$ cannot be replaced by $T_1$ and $T_2$: if $\mathfrak{d}$ is a decomposition of a $T_1$-space then $\mathcal{Q}_\mathfrak{d}$ is $T_1$, too; thus $\mathsf{R}$ is an $S_4$-bispace which cannot be induced by a decomposition of a $T_1$-space.

b) Theorem 5.3 b) does not hold for decompositions: if a bitopology on a finite set $X$ can be induced by a decomposition $\mathfrak{d}$ into closed sets of a space then each $\mathfrak{d}(x)$ $(x\in X)$ is open-closed, therefore $\mathcal{P}_\mathfrak{d}$ and $\mathcal{Q}_\mathfrak{d}$ are both discrete. Thus the two-point sub-bispace of $\mathsf{R}$ (which is $S_4$) cannot be induced by a decomposition into closed sets of a space.

**8.6** Similarly, we have from Theorems 6.4 and 6.6:

THEOREM. a) *In a bispace* $\mathsf{X}$, $\mathcal{P}$ *is* $S_\pi$ *with respect to* $\mathcal{Q}$ *iff* $(\mathcal{P},\mathcal{Q})$ *can be induced by a decomposition of a compact* $S_2$*-space.*

b) *A bitopology is* $S_\pi$ *iff it can be induced by a decomposition into compact sets of a compact* $S_2$*-space.*

REMARK. Concerning Theorem 6.5, see Remark 8.5 b).

**8.7** CONJECTURE. A bitopology is $sT_\pi$ iff it can be induced by a decomposition into closed sets of a compact $T_2$-space.

## REFERENCES

[1] ALEXANDROFF, P. (Aleksandrov, P. S.) and HOPF, H., *Topologie I*, Springer, Berlin, 1935. *Zbl* **13**, 79.
[2] BERGE, C., *Espaces topologiques, fonctions multivoques*, Dunod, Paris, 1959. *MR* **21** #4401.
[3] BOULIGAND, *Ens. Math.* (1932), 14 (this incomplete reference has been taken from [2], p. 114).
[4] DEÁK, E., *Dimension und Konvexität* (in preparation).
[5] DEÁK, J., On bitopological spaces I, *Studia Sci. Math. Hungar.* **25** (1990), 457—481.
[6] ENGELKING, E., Quelques remarques concernant les opérations sur les fonctions semi-continues dans les espaces topologiques, *Bull. Acad. Pol. Sci. Sér. Math.* **11** (1963), No 12, 719—726. *MR* **28** #4505.
[7] ENGELKING, E., *General topology*, PWN, Warsaw, 1977. *MR* **58** #18316b.
[8] KURATOWSKI, K., Les fonctions semi-continues dans l'espace des ensembles fermés, *Fund. Math.* **18** (1932), 148—159. *Zbl.* **4**, 204.

[9] Popa, V., Multifunctions and bitopological spaces, *Bull. Math. Soc. Sci. Math. Roum. (N. S.)* **19** (67) (1975), No 1—2, 147—152. *MR* **54** # 8583.

[10] Popa, V., Aplicaţii conexe şi monotone intre spaţii bitopologice, *Stud. Cerc. Mat.* **32** (1980), No 2, 213—220 (French summary). *MR* **82j:** 54061.

[11] Popa, V., Multifuncţii slab continue definite pe spaţii bitopologice, *Stud. Cerc. Mat.* **34** (1982), 561—567 (English summary). *MR* **85f:** 54031.

[12] Smithson, R. E., Multifunctions and bitopological spaces, *J. Nat. Sci. Math.* **11** (1971), 191—198. *MR* **46** # 4467.

[13] Strother, W. L., Continuous multi-valued functions, *Bol. Soc. Mat. S. Paulo* **10** (1955), 87—120. *MR* **23** # A292.

[14] Vaidyanathaswamy, R., *Treatise on set topology I*, Indian Math. Soc., Madras, 1947. 2nd edition: *Set topology*, Chelsea Co., New York, 1960. *MR* **9**, 367; **22** # 5953.

[15] Wilson, W. A., On the structure of a continuum, limited and irreducible between two points, *Amer. J. Math.* **48** (1926), 147—168.

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# ON BITOPOLOGICAL SPACES III

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We shall consider the following questions: connexions between bitopologies and different kinds of asymmetrical proximity relations (§ 9); internal characterizations of complete regularity in bispaces (§ 10); compactifications of bispaces (§ 11). For notations and terminology, see Part I (§§ 0...4) [22] of this series; the present paper can be understood without being familiar with Part II (§§ 5...8) [23].

## § 9. Bitopologies induced by generalized proximities

**9.0** There are two approaches to proximities and their generalizations: as a primitive term, one can take either "near to", or "is a proximal neighbourhood of". We have chosen the second way; the definitions and the statements can be easily transcribed by putting: $A$ is near to $B$ ($A\delta B$) iff the complement of $B$ is not a proximal neighbourhood of $A$ ($A \sqsubset X \setminus B$), see e.g. [11] or [55].

DEFINITION. The relation $\sqsubset$ between subsets of a set $X$ is a *pseudo-proximity* on $X$ if

P1 $$X \sqsubset X;$$

P1$^c$ $$\emptyset \sqsubset \emptyset;$$

P2=P2$^c$ $$A \sqsubset B \Rightarrow A \subset B;$$

P3 $$A \subset B \sqsubset C \Rightarrow A \sqsubset C;$$

P3$^c$ $$A \sqsubset B \subset C \Rightarrow A \sqsubset C;$$

P4 $$A \sqsubset B,\ A \sqsubset C \Rightarrow A \sqsubset B \cap C;$$

P4$^c$ $$A \sqsubset C,\ B \sqsubset C \Rightarrow A \cup B \sqsubset C.$$

Defining

$$A \in \mathscr{T}_{\sqsubset} \Leftrightarrow \forall p \in A,\ \{p\} \sqsubset A,$$

we get the *topology* $\mathscr{T}_{\sqsubset}$ *induced by* $\sqsubset$. $(\mathscr{P}_{\sqsubset}, \mathscr{Q}_{\sqsubset}) = (\mathscr{T}_{\sqsubset}, \mathscr{T}_{\sqsubset^c})$ is the *bitopology induced* by $\sqsubset$. (See 0.1 for the definition of $\sqsubset^c$.)

REMARKS. a) To see that the above definition is correct, two trivialities are to be observed: $\mathscr{T}_{\sqsubset}$ is indeed a topology; if $\sqsubset$ is a pseudo-proximity then $\sqsubset^c$ is a pseudo-proximity, too, since P$i$ holds for $\sqsubset$ iff P$i^c$ holds for $\sqsubset^c$.

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b) A pseudo-proximity is called *symmetrical* if

S $$\sqsubset = \sqsubset^c,$$

i.e. if the relation "near to" associated with $\sqsubset$ is symmetrical in the usual sense, cf. [11].

c) Pseudo-proximities were introduced, as "topogenous orders", in [10].

d) Pseudo-proximities satisfying no further conditions provide an inadequate tool for inducing bitopologies: it does not depend only on the $\mathrm{wT}_0$-identification of the bitopology whether or no it can be induced by a pseudo-proximity. (The topology $\mathcal{T} \triangleq \{\omega\setminus\{0\}\}$ on $\omega$ can be induced by the symmetrical pseudo-proximity $\sqsubset$ where for $\emptyset \neq A \subset B \subsetneqq \omega$ we have

$$A \sqsubset B \Leftrightarrow (A = \{0\}, |\omega\setminus B| < \omega) \text{ or } (B = \omega\setminus\{0\}, |A| < \omega),$$

while the $\mathrm{wT}_0$-identification of $(\mathcal{T}, \mathcal{T})$ cannot be induced by a pseudo-proximity.) Therefore we shall consider some special classes of pseudo-proximities.

**9.1** For a pseudo-proximity $\sqsubset$,

$$\mathrm{int}_\sqsubset A = \{q\colon \{q\} \sqsubset A\}$$

defines an interior operation in the sense of [8] 14 A.10, and $\mathcal{T}_\sqsubset$ is the topology associated with $\mathrm{int}_\sqsubset$ ([8], 15 A).

DEFINITION. A pseudo-proximity $\sqsubset$ on $X$ is

a) a *quasi-T-proximity* if

T $$\{p\} \sqsubset B \Rightarrow \{p\} \sqsubset \mathrm{int}_\sqsubset B,$$

$\mathrm{T}^c$ $$\{p\} \sqsubset^c B \Rightarrow \{p\} \sqsubset^c \mathrm{int}_{\sqsubset^c} B;$$

b) a *quasi-L-proximity* if

L $$A \sqsubset B \Rightarrow A \sqsubset \mathrm{int}_\sqsubset B,$$

$\mathrm{L}^c$ $$A \sqsubset^c B \Rightarrow A \sqsubset^c \mathrm{int}_{\sqsubset^c} B;$$

c) a *quasi-$S_2$-proximity* if

$\mathrm{S}_2 = \mathrm{S}_2^c$ $$\{p\} \sqsubset X\setminus\{q\} \Rightarrow \exists C, \quad \{p\} \sqsubset C \sqsubset X\setminus\{q\};$$

d) a *quasi-R-proximity* if

R $$\{p\} \sqsubset B \Rightarrow \exists C, \quad \{p\} \sqsubset C \sqsubset B,$$

$\mathrm{R}^c$ $$\{p\} \sqsubset^c B \Rightarrow \exists C, \quad \{p\} \sqsubset^c C \sqsubset^c B;$$

e) a *quasi-E-proximity* or simply a *quasi-proximity* if

$\mathrm{E} = \mathrm{E}^c$ $$A \sqsubset B \Rightarrow \exists C, \quad A \sqsubset C \sqsubset B.$$

REMARKS. a) Quasi-proximities were introduced in [10], with the terminology "$\{\sqsubset\}$ is a topogenous structure".

b) Observe that if $\sqsubset$ is a quasi-A-proximity (A=T, L, $\mathrm{S}_2$, R, E) then so is $\sqsubset^c$.

c) Clearly, $E \Rightarrow R \Rightarrow S_2$

$$\begin{array}{ccc} E & \Rightarrow R & \Rightarrow S_2 \\ \Downarrow & \Downarrow & \\ L & \Rightarrow T & \end{array}$$

and the same holds with $^c$ added to each letter. It would be possible to define a number of similar conditions, see [15] for the symmetrical case.

d) If $\sqsubset$ is a quasi-T-proximity then $\mathrm{int}_{\sqsubset} = \mathscr{P}_{\sqsubset}$-int and $\{A: \{p\} \sqsubset A\}$ is a neighbourhood base for $\mathscr{P}_{\sqsubset}$ at $p$. A dual statement holds for $\sqsubset^c$ and $\mathscr{Q}_{\sqsubset}$.

e) For special kinds of symmetrical pseudo-proximities, we shall drop the prefix "quasi-" instead of using the adjective "symmetrical", e.g. a T-proximity is a symmetrical quasi-T-proximity.

f) There is no general agreement on the terminology concerning asymmetric (or even symmetrical) generalizations of proximities. Authors working in bitopologies clearly need Axioms A and $A^c$ together (A=P1, P3, P4, T, L, R), while those using such relations for inducing a single topology assume only one of the twin axioms, or both, according to their particular needs (see [31], [33], [41], [46], [53], [72], *[66]*, *[71]*, respectively [39], [47], [48], [55], [57], [58], [74], *[37]*. The term "quasi-proximity", however, seems to be universally accepted in the above sense, although it was used originally for a slightly more general notion, see [57].

**9.2** Let X be an $S_1$-bispace. Then

$$A \sqsubset_{\mathscr{P},\mathscr{Q}} B \Leftrightarrow \mathrm{cl}_{\mathscr{Q}} A \subset \mathrm{int}_{\mathscr{P}} B$$

defines a quasi-L-proximity on $X$, which induces $(\mathscr{P}, \mathscr{Q})$ [72]; if X is $S_2$ then $\sqsubset_{\mathscr{P},\mathscr{Q}}$ is a quasi-$S_2$-proximity (straightforward); if X is $S_3$ then $\sqsubset_{\mathscr{P},\mathscr{Q}}$ is a quasi-R-proximity [72]. Using Remark 9.1 d), it is easy to show that the bitopology induced by a quasi-A-proximity is $S_1$ (A=T), $S_2$ (A=$S_2$), respectively $S_3$ (A=R). So we have:

PROPOSITION. *For a bitopology $(\mathscr{P}, \mathscr{Q})$, the conditions within each of the following groups are equivalent.*

a) ([72] *for* (i)⇔(iii)) (i) *It is $S_1$;*

(ii) *it can be induced by a quasi-T-proximity;*

(iii) *it can be induced by a quasi-L-proximity.*

b) (i) *It is $S_2$;*

(ii) *it can be induced by a quasi-$S_2$-proximity;*

(iii) *it can be induced by a quasi-$S_2$-proximity, which is also a quasi-L-proximity.*

c) [72] (i) *It is $S_3$;*

(ii) *it can be induced by a quasi-R-proximity;*

(iii) *it can be induced by a quasi-R-proximity, which is also a quasi-L-proximity.*

**9.3** The following theorem is well-known; it is essentially the same as the theorem of Fletcher [29], [30] and Lane [44], [45] stating that a bitopology is $S_\pi$ iff it is quasi-uniformizable (apply the results from [11] concerning the connexions between quasi-uniformities and quasi-proximities; cf. [38], *[49]*, [13] and [31]); see also [46], *[66]* and [41].

THEOREM. *A bitopology is $S_\pi$ iff it can be induced by a quasi-proximity.*

We are going to re-prove this theorem, using the results of § 4. Our method is similar to that applied in the proof of [21] (1.1). To begin with, a connexion between pseudo-proximities and pseudo-directions has to be established.

**9.4** If $\boldsymbol{d}$ is a pseudo-direction on a set $X$ then

$$A \sqsubset^{\boldsymbol{d}} B \Leftrightarrow \exists F \in \operatorname{ran} \boldsymbol{d},\ \exists G \in \operatorname{dom} \boldsymbol{d},\ A \subset F \subset G \subset B \subset X$$

(i.e. $A \sqsubset^{\boldsymbol{d}} B$ iff $\boldsymbol{d}$ separates $A$ from $X \setminus B$, in the sense of 4.8) defines a pseudo-proximity. If $\boldsymbol{d}$ is normal then $(\mathscr{P}_{\sqsubset}, \mathscr{Q}_{\sqsubset}) = (\mathscr{P}^*_{\boldsymbol{d}}, \mathscr{Q}^*_{\boldsymbol{d}})$ where $\sqsubset = \sqsubset^{\boldsymbol{d}}$. In particular, $(\mathscr{P}_{\sqsubset}, \mathscr{Q}_{\sqsubset}) = (\mathscr{P}_{\boldsymbol{d}}, \mathscr{Q}_{\boldsymbol{d}})$ if $\boldsymbol{d}$ is orderly.

LEMMA. *If $\boldsymbol{d}$ is a normal pseudo-direction then $\sqsubset^{\boldsymbol{d}}$ is a quasi-proximity inducing $(\mathscr{P}^*_{\boldsymbol{d}}, \mathscr{Q}^*_{\boldsymbol{d}})$.*

REMARK. It is impossible to recover $\boldsymbol{d}$ from $\sqsubset^{\boldsymbol{d}}$, even if $\boldsymbol{d}$ is assumed to be an orderly direction.

**9.5** Let $\sqsubset_1$ and $\sqsubset_2$ be pseudo-proximities on the same set $X$. We say that $\sqsubset_2$ is finer than $\sqsubset_1$ if $\sqsubset_1 \subset \sqsubset_2$. For a family $\boldsymbol{R}$ of pseudo-proximities on $X$, there exists a unique pseudo-proximity $\sup \boldsymbol{R}$ (i.e. a pseudo-proximity coarser than any pseudo-proximity finer than each element of $\boldsymbol{R}$): in case $\boldsymbol{R} \neq \emptyset$, we have $\sup \boldsymbol{R} = (\cup \boldsymbol{R})^q$ ([11]; see 0.1 for the definition of $^q$). It is easy to show that if $\boldsymbol{R}$ is a family of quasi-T-proximities then $\sup \boldsymbol{R}$ is a quasi-T-proximity, too, and

$$\mathscr{T}_{\sup \boldsymbol{R}} = \sup \{\mathscr{T}_{\sqsubset} : \sqsubset \in \boldsymbol{R}\}.$$

If $\boldsymbol{R}$ consists of quasi-proximities then $\sup \boldsymbol{R}$ is also a quasi-proximity ([11] 8.24).

If $\boldsymbol{D}$ is a pseudo-directional structure, define

$$\sqsubset^{\boldsymbol{D}} = \sup \{\sqsubset^{\boldsymbol{d}} : \boldsymbol{d} \in \boldsymbol{D}\}.$$

PROPOSITION. *If $\boldsymbol{D}$ is a normal pseudo-directional structure then $\sqsubset^{\boldsymbol{D}}$ is a quasi-proximity inducing $(\mathscr{P}^*_{\boldsymbol{D}}, \mathscr{Q}^*_{\boldsymbol{D}})$. In particular, if $\boldsymbol{D}$ is a compatible orderly pseudo-directional structure of the bispace* X *then $\sqsubset^{\boldsymbol{D}}$ induces $(\mathscr{P}, \mathscr{Q})$.*

REMARK. In [21], the proximities $(\sqsubset^{\boldsymbol{d}})^s$ (cf. 0.1) and

$$(\sqsubset^{\boldsymbol{D}})^s = \sup \{(\sqsubset^{\boldsymbol{d}})^s : \boldsymbol{d} \in \boldsymbol{D}\}$$

are used, where $\boldsymbol{d}$ is an orderly direction and $\boldsymbol{D}$ an orderly directional structure.

**9.6** LEMMA. *Assume that $\boldsymbol{r}$ is a relation between subsets of $X$ satisfying*

(1) $$F \boldsymbol{r} G \Rightarrow F \subset G, \exists F', \exists G', F \boldsymbol{r} G' \subset F' \boldsymbol{r} G$$

*and $F_0 \boldsymbol{r} G_0$. Then there is a normal pseudo-direction $\boldsymbol{d}$ on $X$ such that*

(2) $$\emptyset \boldsymbol{d} F_0,\ G_0 \boldsymbol{d} X,$$

(3) $$\operatorname{dom} \boldsymbol{d} \subset \operatorname{ran} \boldsymbol{r} \cup \{\emptyset, X\},\ \operatorname{ran} \boldsymbol{d} \subset \operatorname{dom} \boldsymbol{r} \cup \{\emptyset, X\}.$$

PROOF. Let $\boldsymbol{d}$ be a maximal pseudo-direction on $X$, satisfying (2), (3) and

(4) $$(G_1, F_1) <_{\boldsymbol{d}} (G_2, F_2) \Rightarrow F_1 \boldsymbol{r} G_2.$$

We claim that $\boldsymbol{d}$ is normal. Indeed, assume that $(G_1, F_1)$ and $(G_2, F_2)$ are neighbours in $<_{\boldsymbol{d}}$ such that

$$G_1 \subsetneqq F_1 \subsetneqq G_2 \subsetneqq F_2. \tag{5}$$

Now $F_1 \boldsymbol{r} G_2$ by (4), so $\boldsymbol{e}=\boldsymbol{d}\cup\{(G', F')\}$ is a pseudo-direction on $X$ satisfying (2), (3) and (4), where $G'$ and $F'$ are chosen according to (1) (with $F=F_1$, $G=G_2$). (5) implies $\boldsymbol{e}\supsetneqq\boldsymbol{d}$, a contradiction.

Proof of Theorem 9.3. a) If X is $S_\pi$ then apply Theorem 4.14 and Proposition 9.5.

b) Conversely, assume that the bitopology of X is induced by a quasi-proximity $\sqsubset$.

1° Define

$$FrG \Leftrightarrow F \sqsubset G,\ F\in\text{co-}\mathcal{Q},\ G\in\mathcal{P}.$$

If $FrG$ then, by Axiom E, there is a $C$ with $F\sqsubset C\sqsubset G$, thus, by Axioms L and $L^c$,

$$F \sqsubset \operatorname{int}_{\mathcal{P}}C \subset \operatorname{cl}_{\mathcal{Q}}C \sqsubset G,$$

i.e. $\boldsymbol{r}$ satisfies (1). Take now $x\in G_0\in\mathcal{P}$. From Axiom $L^c$ we have $\operatorname{cl}_{\mathcal{Q}}\{x\}\boldsymbol{r}G_0$, therefore, according to the lemma, there is a normal pseudo-direction of X separating $\{x\}$ from $X\setminus G_0$. Hence $\mathcal{P}$ is $S_\pi$ with respect to $\mathcal{Q}$, by Definition 4.13 a).

2° To prove that $\mathcal{Q}$ is also $S_\pi$ with respect to $\mathcal{P}$, apply 1° to $(X;\ \mathcal{Q}, \mathcal{P})$ and $\sqsubset^c$.

## § 10. Some more characterizations of complete regularity

**10.0** There exist several internal characterizations of complete regularity in topological spaces, see e.g. [1], [12], [21], [32], [35], [36], [42], [73], [75], [82]. The bitopological counterparts of some of them can be found in [25], [27], [64], *[66]*. In this section, we shall obtain some characterizations of $S_\pi$; the proofs will be based on ideas due to E. Deák and Hamburger [21] and Hamburger [36].

**10.1** Lemma. *Let* X *be a bispace. If there is a closed subbase* $\mathcal{C}$ *for* $\mathcal{P}$ *such that* $\{x\}$ *can be separated from* $F$ *by a normal pseudo-direction of* X *whenever* $x\notin F\in\mathcal{C}$ *then* $\mathcal{P}$ *is* $S_\pi$ *with respect to* $\mathcal{Q}$.

Proof. Definition 4.13 a) and Lemma 4.11 b).

Theorem. *The bispace* X *is* $S_\pi$ *iff there is a relation* $f$ *between subsets of* $X$ *such that*

(i) $\operatorname{ran} f$ *is a closed subbase for* $\mathcal{P}$;

(ii) $\operatorname{dom} f$ *is a closed subbase for* $\mathcal{Q}$;

(iii) $AfB \Rightarrow A\cap B = \emptyset$;

(iv) $x\notin C\in\operatorname{ran} f \Rightarrow \exists A\exists B,\quad x\in AfB \supset C$;

(v) $x\notin C\in\operatorname{dom} f \Rightarrow \exists A\exists B,\quad C\subset AfB\ni x$;

(vi) $AfB \Rightarrow \exists A'\exists B',\ AfB',\ A'fB,\ A'\cup B' = X$.

PROOF. a) If X is $S_\pi$ then choose a compatible orderly pseudo-directional structure $\boldsymbol{D}$ and put

$$AfB \Leftrightarrow A\cap B=\emptyset,\ \exists \boldsymbol{d}\in \boldsymbol{D},\ A\in \operatorname{ran}\boldsymbol{d},\ B\in \text{co-dom}\,\boldsymbol{d}.$$

b) Conversely, assume the existence of an $f$. Put

$$FrG \Leftrightarrow Ff(X\setminus G).$$

(iii) and (vi) imply 9.6 (1), thus, by Lemma 9.6, there is, for each $AfB$, a normal pseudo-direction $\boldsymbol{d}_{A,B}$ of X separating $A$ from $B$ (it is a pseudo-direction of X by (i), (ii) and 9.6 (3); it separates $A$ from $B$ by 9.6 (2), where $F_0=A$, $G_0=X\setminus B$). If $x\notin C\in \operatorname{ran} f$ then $\boldsymbol{d}_{A,B}$ separates $\{x\}$ from $C$, where $A$ and $B$ are chosen according to (iv). Thus $\mathscr{P}$ is $S_\pi$ with respect to $\mathscr{Q}$, by (i) and the lemma. Analogously, (v) and (ii) imply that $\mathscr{Q}$ is $S_\pi$ with respect to $\mathscr{P}$.

**10.2** The conditions in Theorems 9.3 and 10.1 are internally equivalent to $S_\pi$ (see 4.13). It is an open question whether the same holds for Theorems 10.2 and 10.3, cf. [21] (3.3). (An internal proof for (i)⇒(ii) below ought to be found.)

THEOREM. *For a bispace* X, *the following conditions are equivalent:*
(i) *it is* $S_\pi$;
(ii) *there are a closed base* $\mathscr{E}$ *for* $\mathscr{P}$ *and a closed base* $\mathscr{F}$ *for* $\mathscr{Q}$ *such that* $\mathscr{E}$ *and* $\mathscr{F}$ *are lattices (i.e. closed for finite unions and finite intersections) and*

(1) $E\in\mathscr{E},\ F\in\mathscr{F},\ E\cap F=\emptyset \Rightarrow \exists E_1\in\mathscr{E}\,\exists F_1\in\mathscr{F},\ E_1\cup F_1=X,\ E\cap F_1=\emptyset=F\cap E_1,$

(2) $x\notin E\in\mathscr{E} \Rightarrow \exists F\in\mathscr{F},\ x\in F,\ E\cap F=\emptyset,$

(3) $x\notin F\in\mathscr{F} \Rightarrow \exists E\in\mathscr{E},\ x\in E,\ E\cap F=\emptyset;$

(iii) *there are a closed subbase* $\mathscr{E}$ *for* $\mathscr{P}$ *and a closed subbase* $\mathscr{F}$ *for* $\mathscr{Q}$ *satisfying* (1), (2) *and* (3).

PROOF. (i)⇒(ii) (cf. [64]): Take

$$\mathscr{E}=\{E_g=g^{-1}([0,\to[):\ g\in C(\mathsf{X})\},\quad \mathscr{F}=\{F_g=g^{-1}(]\leftarrow,0]):\ g\in C(\mathsf{X})\},$$

where $C(\mathsf{X})$ denotes the family of the bi-continuous real functions on X. $E_g\cup E_h=E_{\max(g,h)}$, $E_g\cap E_h=E_{\min(g,h)}$. If $E_g\cap E_h=\emptyset$ then $E_1=E_{g+h}$, $F_1=F_{g+h}$ satisfy (1).
(ii)⇒(iii): Evident.
(iii)⇒(i): Observe that $\tilde{\mathscr{E}}=\mathscr{E}\cup\{\emptyset,X\}$ and $\tilde{\mathscr{F}}=\mathscr{F}\cup\{\emptyset,X\}$ also satisfy (1)—(3), and then apply Theorem 10.1 to the relation $f$ defined by

$$AfB \Leftrightarrow A\in\tilde{\mathscr{F}},\ B\in\tilde{\mathscr{E}},\ A\cap B=\emptyset.$$

REMARKS. a) Saegrove [64] proved that (i) is equivalent to a condition lying between (ii) and (iii); see also [25], [27]. (The proof given in [27] does not work since the operations applied there to elements of $C(\mathsf{X})$ lead out from it.)

b) Theorem 10.3 will make (iii)⇒(i) in the above proof superfluous.

**10.3** NOTATION. For a set $S$, $[S]^f=\{A\subset S:\ |A|<\omega\}$.

THEOREM. *A bispace* X *is* $S_\pi$ *iff there are a closed subbase* $\mathcal{E}$ *for* $\mathcal{P}$ *and a closed subbase* $\mathcal{F}$ *for* $\mathcal{Q}$ *such that*

(1) $$E_0\in\mathcal{E},\ F_0\in\mathcal{F},\ E_0\cap F_0=\emptyset\Rightarrow\exists\mathcal{E}^*\in[\mathcal{E}]^f\exists\mathcal{F}^*\in[\mathcal{F}]^f,$$
$$E_0\subset\bigcup\mathcal{E}^*,\ F_0\subset\bigcup\mathcal{F}^*,\ \forall E\in\mathcal{E}^*\ \forall F\in\mathcal{F}^*\exists\mathcal{E}_{E,F}\in[\mathcal{E}]^f\exists\mathcal{F}_{E,F}\in[\mathcal{F}]^f,$$
$$\bigcup\mathcal{E}_{E,F}\cup\bigcup\mathcal{F}_{E,F}=X,\ F\cap\bigcup\mathcal{E}_{E,F}=\emptyset=E\cap\bigcup\mathcal{F}_{E,F};$$

(2) $x\notin E_0\in\mathcal{E}\Rightarrow\exists\mathcal{E}^*\in[\mathcal{E}]^f,\ E_0\subset\bigcup\mathcal{E}^*,\ \forall E\in\mathcal{E}^*\exists F_E\in\mathcal{F},\ x\in F_E,\ E\cap F_E=\emptyset;$

(3) $x\notin F_0\in\mathcal{F}\Rightarrow\exists\mathcal{F}^*\in[\mathcal{F}]^f,\ F_0\subset\bigcup\mathcal{F}^*,\ \forall F\in\mathcal{F}^*\exists E_F\in\mathcal{E},\ x\in E_F,\ E_F\cap F=\emptyset.$

We omit the proof, since a stronger statement can be deduced from the results of our paper "Preproximities and internal characterizations of complete regularity" (to appear[1] in Studia Sci. Math. Hungar.); cf. 5.34 c) of that paper.

COROLLARY. *A topological space* $(X, \mathcal{T})$ *is* $S_\pi$ *iff it has a closed subbase* $\mathcal{E}\cup\mathcal{F}$ *such that* $\mathcal{E}$ *and* $\mathcal{F}$ *satisfy* (1), (2) *and* (3).

PROOF. If $\mathcal{P}\triangleq$co-$\mathcal{E}$ and $\mathcal{Q}\triangleq$co-$\mathcal{F}$ then $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{Q}, \mathcal{P})$ are both $S_\pi$ by the theorem, thus $(\mathcal{T}, \mathcal{T})$ is $S_\pi$ by 3.3 b), i.e. $\mathcal{T}$ is $S_\pi$.

REMARK. This corollary can also be deduced from [36] Theorem 2.1: take the preproximity $(\mathcal{E}\cup\mathcal{F}\cup\{\emptyset, X\}, r\cup r^{-1})$, where

$$ArB \Leftrightarrow A\in\mathcal{E}\cup\{\emptyset, X\},\ B\in\mathcal{F}\cup\{\emptyset, X\},\ A\cap B=\emptyset.$$

Letting $\mathcal{E}=\mathcal{F}$ would weaken the corollary, see the example below.

EXAMPLE. Let $X=\mathbf{J}^2\setminus\{(1, 1)\}$ ($\mathbf{J}=[0, 1]$ with the Euclidean topology),

$$\mathcal{E}=\{\mathbf{J}\times[0, t],\ [0, t]\times\mathbf{J}:\ t\in\mathbf{J}\},\quad \mathcal{F}=\{\mathbf{J}\times[t, 1],\ [t, 1]\times\mathbf{J}:\ t\in\mathbf{J}\}|X.$$

The pair $(\mathcal{E}, \mathcal{F})$ satisfies the requirements in the corollary, but $(\mathcal{E}\cup\mathcal{F}, \mathcal{E}\cup\mathcal{F})$ does not so (consider $\{1\}\times[0, 1[$ and $[0, 1[\times\{1\}$).

## § 11. Compactifications

**11.0** In this section, we intend to deal with a bitopological generalization of the concept of a $T_2$-compactification (or, more generally, reduced $S_2$-compactification) of a topological space. (An extension $Y$ of a topological space $X$ is called *reduced* if distinct points $x\in Y$ and $y\in Y\setminus X$ have different neighbourhood filters in $Y$, see e.g. in [14].) First of all, we have to choose from the different bitopological generalizations of "$S_2$", "compact" and "dense". So let us consider the following bitopological notions, and examine which of them lend themselves to a reasonable theory of com-

[1] *Added in proof.* **24** (1989), No 2—3, 147—177. See also: AARTS, J. M. and MRŠEVIĆ, M., Pairwise complete regularity as a separation axiom, *J. Austral. Math. Soc. Ser. A* **48** (1990), No 2, 235—245. *MR* **90m**: 54029.

pactifications:

$$\begin{matrix} sS_2 \Rightarrow S_2 \Rightarrow wS_2 \Rightarrow \\ \text{bi-}S_2 \Rightarrow \end{matrix} qS_2 \Rightarrow \text{sup-}S_2, \qquad \text{pairwise } S_2$$

$$\text{sup-compact} = \text{quasi-compact} \begin{matrix} \Rightarrow \text{bi-compact} \\ \Rightarrow \text{pairwise compact} \end{matrix}$$

$$\text{sup-dense} \Rightarrow \text{quasi-dense} = \text{bi-dense} \Rightarrow \text{pairwise dense}$$

(cf. 0.9, 0.10, 1.5, 1.6).

Choosing $sS_2$, bi-$S_2$ or pairwise $S_2$ would exclude such good bispaces as the bitopological interval J; on the other hand, a compact (in whichever of the above senses) $wS_2$-bispace is not necessarily regular (Example 1.5 b)) or normal ($X=\{1, 2, 3\}$, $\mathscr{P} \triangleq \{\{1\}, \{2\}, \{2, 3\}\}$, $\mathscr{Q} \triangleq \{\{1, 2\}, \{2\}, \{3\}\}$). Take now the sub-bispaces A, B and C of J where $A=\{0, 1\}$, $B=]0, 1[$, $C=J\setminus\{1/2\}$. A and J are compact (in any sense) and A is bi-dense (therefore also pairwise dense) in J, an undesirable situation in a good theory of compactifications. On the other hand, B and C are dense in J (in any sense), B and J are pairwise compact, C and J are bi-compact.

Thus we are left with $S_2$, sup-compactness and sup-density, and there is in fact a theory of bitopological compactifications based on these notions, which shows a great analogy to the topological case (Császár [13], Salbany [65], *[66]*; see also [28], [69]). Another argumentation in favour of just this type of compactifications was given by Salbany [68].

We shall show (but not work out in detail) how to build up a theory of bitopological compactifications using directional structures.

For notions of bitopological compactness (including some not even mentioned in this paper), see also [2], [3], [4], [5], [6], [9], [16], [17], [24], [26], [30], [43], [50], [51], [52], [54], [59], [61], [64], [76], [80], [81], *[7]*, *[34]*, *[56]*, *[62]*, *[63]*, *[67]*, *[77]*, *[78]*, *[79]*[2]

**11.1** Notation. Let $Y=(Y; \mathscr{P}^*, \mathscr{Q}^*)$, $Z=(Z; \mathscr{P}^{**}, \mathscr{Q}^{**})$.

Definition. The bispace Y is a *compactification* of X if Y is $S_2$, X is a sub-bispace of Y, and $(Y, \mathscr{P}^* \vee \mathscr{Q}^*)$ is a reduced compactification of $(X, \mathscr{P} \vee \mathscr{Q})$.

Remarks. a) If Y is a compactification of X then Y is $S_4$ (Reilly [60]) and X is $S_\varkappa$ (Remark 1.7 f) and 3.3 d)).

b) The next examples show that there is no one-to-one correspondence between the compactifications of X and of $(X, \mathscr{P} \vee \mathscr{Q})$, i.e. that the results on bitopological compactifications are not direct consequences of the similar results on topological compactifications. Observe also that in the second example $\mathscr{P}=\mathscr{Q}$, but $\mathscr{P}^* \neq \mathscr{Q}^*$.

Examples. a) R is pairwise compact, so it has at most one compatible quasi-proximity (Jas and Baisnab [40]), therefore it has at most one compactification (Császár [13]). But it is clear that R has a compactification bi-homeomorphic to J. Thus R has neither a one-point-compactification, nor a compactification Y with $\mathscr{P}^* \vee \mathscr{Q}^* = \beta\mathscr{E}$.

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[2] *Added in proof.* See the footnote to 1.8.

b) Take

$$X = [0, 1[\times[0, 1], \quad Y = Z = [0, 1]^2,$$

$$\mathscr{P} = \mathscr{Q} = \mathscr{E}^2|X, \ \mathscr{P}^* \triangleq \mathscr{P}\cup\underline{\mathscr{E}}^2|Y, \ \mathscr{Q}^* \triangleq \mathscr{Q}\cup(\underline{\mathscr{E}}\times\bar{\mathscr{E}})|Y, \ \mathscr{P}^{**} = \mathscr{Q}^{**} = \mathscr{E}^2|Z.$$

Now Y and Z are both compactifications of X, and their sup-topologies coincide. Moreover, the bitopology of Z is strictly finer than that of Y.

c) The same construction starting from $Y=Z=(\omega_1+1)\times[0, 1]$ ($\omega_1+1$ taken with the order topology) shows that more than one compactification may correspond to the Stone—Čech compactification of the sup-topology.

**11.2** Definition (Császár [13]). Let Y and Z be compactifications of the bispace X. Y is said to be *finer* than Z if there is a bi-continuous mapping $f$: Y→Z such that $f(x)=x$ whenever $x\in X$. Y and Z are *equivalent* compactifications of X if Y is finer than Z and Z is finer than Y.

Remarks. a) The mapping $f$ in the above definition is onto ($f$ is sup-continuous, Y is sup-compact, therefore $f[Y]\supset X$ is sup-compact).

b) Y and Z are equivalent compactifications of X iff there is a bi-homeomorphism $f$ from Y onto Z such that $f(x)=x$ for $x\in X$.

**11.3** Definition. Let $\boldsymbol{D}$ be a compatible orderly directional structure of the $wT_0$ (hence $T_\pi$) bispace X. The ***D*-compactification** of X, denoted by $c_D$X, is the sup-closure in D of $\chi_D[X]$, equipped with the bitopology inherited from D (cf. 4.3 c) and Notations 4.4). We shall write

$$c_D X = (c_D X; \ \overset{1}{c}_D\mathscr{P}, \overset{2}{c}_D\mathscr{Q}).$$

Notation. Let $\chi_D$X denote $\chi_D[X]$ equipped with the bitopology inherited from D (i.e. from $c_D$X).

Remarks. a) The notations introduced in the definition contain redundant elements: as $\boldsymbol{D}$ determines X, it would be enough to write e.g. $c_D=(\overset{0}{c}_D; \overset{1}{c}_D, \overset{2}{c}_D)$.

b) As a matter of fact, $c_D$X is not a compactification of X, but of $\chi_D$X, which is bi-homeomorphic to X. In order to obtain a compactification of X bi-homeomorphic to $c_D$X, we can take e.g.

(1) $$\bar{c}_D X = X\cup(c_D X\setminus\chi_D[X])\times\{X\},$$

(2) $$\bar{c}_D X = (\bar{c}_D X; \ g^{-1}\overset{1}{c}_D\mathscr{P}, g^{-1}\overset{2}{c}_D\mathscr{Q})$$

where

(3) $$g(y) = \begin{cases}\chi_D^{-1}(y) & \text{if } \ y\in\chi_D[X], \\ (y, X) & \text{otherwise.}\end{cases}$$

c) If X is not $wT_0$ then $c_D$X is clearly not bi-homeomorphic to a compactification of X, but $\bar{c}_D$X defined by (1)—(3) is again a compactification of X. For the sake of simplicity, we shall state and prove the results to follow for $c_D$X (where X is $wT_0$), but everything remains true for $\bar{c}_D$X (with $wT_0$ dropped).

d) Definition 11.2 does not apply to $\boldsymbol{D}$-compactifications, so we shall say that $c_D$X is finer than $c_E$X if $\bar{c}_D$X is finer than $\bar{c}_E$X, i.e. if there is a bi-continuous mapping $f\colon \mathrm{c}_D X \to \mathrm{c}_E X$ such that $\chi_E = f \circ \chi_D$; similarly, a compactification Y of X is finer than $\mathrm{c}_E X$ if there is a bi-continuous mapping $g\colon Y \to \mathrm{c}_E X$ such that $g \restriction X = \chi_E$; Y and $\mathrm{c}_E X$ are equivalent if this $g$ is a bi-homeomorphism.

e) For $\boldsymbol{D}$-compactifications of topological spaces, see E. Deák [18], [19], [20].

Proposition. *If* $\boldsymbol{E} \subset \boldsymbol{D}$ *then* $\mathrm{c}_D X$ *is finer than* $\mathrm{c}_E X$.

Proof. Restrict to $\mathrm{c}_D X$ the projection from $\Pi \boldsymbol{D}$ onto $\Pi \boldsymbol{E}$.

**11.4** Proposition. *If* $\mathrm{c}_D X$ *is finer than* $\mathrm{c}_E X$ *then there is a* $\boldsymbol{D}_1 \supset \boldsymbol{E}$ *such that* $\mathrm{c}_{D_1} X$ *is equivalent to* $\mathrm{c}_D X$.

Proof. Take $\boldsymbol{D}_1 = \boldsymbol{E} \cup \boldsymbol{D}$. By Proposition 11.3, $\mathrm{c}_{D_1} X$ is finer than $\mathrm{c}_D X$. Conversely, let $f\colon \mathrm{c}_D X \to \mathrm{c}_E X$ be bi-continuous such that $\chi_D = f \circ \chi_E$ and let $h$ be the identity mapping of $\mathrm{c}_D X$; define the $\boldsymbol{D}$th coordinate of $g\colon \mathrm{c}_D X \to \Pi \boldsymbol{D}_1$ to be the $\boldsymbol{d}$th coordinate of $f$ if $g \in \boldsymbol{E}$ and of $h$ if $g \in \boldsymbol{D}$ (for $g \in \boldsymbol{D} \cap \boldsymbol{E}$, the two definitions give the same, since they are both equal to $\chi_d \circ \chi_D^{-1}$ on the sup-dense subset $\chi_D[X]$ and they are sup-continuous mappings into a sup-$T_2$ bispace). Now $g$ is a bi-continuous mapping into $\mathrm{c}_{D_1} X$ and $\chi_{D_1} = f \circ \chi_D$.

**11.5** Proposition. *If* $\boldsymbol{D}$ *is a compatible orderly directional structure of the* $wT_0$*-bispace* X *then there is a largest one among those directional structures* $\boldsymbol{C}$ *of* X *for which* $\mathrm{c}_D X$ *and* $\mathrm{c}_C X$ *are equivalent. Denoting this directional structure by* $\boldsymbol{D}_m$, *we have* $\boldsymbol{D}_m \supset \boldsymbol{E}_m$ *iff* $\mathrm{c}_D X$ *is finer than* $\mathrm{c}_E X$.

Proof. Let $\boldsymbol{D}_m$ be the union of all the $\boldsymbol{C}$s mentioned above. By Proposition 11.3, $\mathrm{c}_{D_m} X$ is finer than $\mathrm{c}_D X$; on the other hand, the argument in the foregoing proof (applied now to the $\boldsymbol{C}$s instead of $\boldsymbol{D}$ and $\boldsymbol{E}$) gives that $\mathrm{c}_D X$ is finer than $\mathrm{c}_{D_m} X$. The second statement follows from Propositions 11.3 and 11.4.

**11.6** Theorem. *Let* $\boldsymbol{D}$ *be a compatible orderly directional structure of the compact* $wT_0$*-bispace* Y *and let* X *be a sup-dense sub-bispace of* Y. *Then* Y *(regarded as a compactification of* X*) is finer than* $\mathrm{c}_{D|X} X$. Y *and* $\mathrm{c}_{D|X} X$ *are equivalent, assuming that* $\boldsymbol{D}$ *satisfies the following condition: whenever* $\emptyset \neq A \neq Y$ *and* $A \in \operatorname{dom} \boldsymbol{d} \cap \operatorname{ran} \boldsymbol{d}$ *for some* $\boldsymbol{d} \in \boldsymbol{D}$, *there are* $\boldsymbol{e} \in \boldsymbol{D}$ *and* $B \subset Y$ *such that either* $AeB$ *and* $(B \setminus A) \cap X \neq \emptyset$, *or* $BeA$ *and* $(A \setminus B) \cap X \neq \emptyset$.

Remarks. a) $\boldsymbol{D}$ clearly satisfies the condition in the theorem if it contains all the directions

$$\{(\emptyset, \emptyset), (\emptyset, A), (A, Y), (Y, Y)\} \quad (A \in \mathscr{P}^* \cap \text{co-}\mathscr{Q}^*).$$

b) The proof is analogous to that of [19] Theorem (3.1), but there is an important difference: in contrast to the topological case, it is not enough to prove the existence of a bi-continuous one-to-one mapping, since such a mapping from a sup-compact bispace onto a $T_2$-bispace is not necessarily a bi-homeomorphism (take the identity mapping from $(\mathbf{J}; \mathscr{E}|\mathbf{J}, \mathscr{E}|\mathbf{J})$ onto $\mathbf{J}$).

Proof. 1° Denote $\boldsymbol{d}|X$ by $\boldsymbol{d}''$ ($\boldsymbol{d} \in \boldsymbol{D}$). For each $\boldsymbol{d} \in \boldsymbol{D}$, define a mapping $\varrho_d$: $\mathrm{d} \to \mathrm{d}''$ by

$$\varrho_d((G, F)) = (G \cap X,\ F \cap X) \quad (GdF).$$

The sup-density of $X$ implies that if $\Gamma d'' \Phi$ then $\varrho_d^{-1}[\{(\Gamma, \Phi)\}]$ has a first and a last element in the ordering $<_d$ (in fact, it cannot have more than three elements, see E. Deák [19] Remark (2.3) (d)), therefore $\varrho_d$ is bi-continuous (copy the proof of [19] Lemma (2.4)).

For each $d \in D$, $\varphi_d = \varrho_d \circ \chi_d$: Y→d″ is bi-continuous and $\varphi_d \upharpoonright X = \chi_{d''}$, so $\varphi_{d_1} = \varphi_{d_2}$ whenever $d_1'' = d_2''$ (because they coincide on a sup-dense subset), hence it is admissible to define mappings $f_{d''}$: Y→$d''$ by $\varphi_d = f_{d''}$ $(d'' \in D|X)$ and $f$: Y→ →$\Pi(D|X)$ such that its $d''$th coordinate is $f_{d''}$. Now $f \upharpoonright X = \chi_{D|X}$ and $f$ is a bi-continuous mapping onto $\mathrm{c}_{D|X}$X. Therefore Y is finer than $\mathrm{c}_{D|X}$X.

2° Assume now that $D$ satisfies the additional condition, too. First we show that $f$ is one-to-one. Take $y, z \in Y$, $y \neq z$; then, by wT$_0$, there is a $d \in D$ such that $\chi_d(y) \neq \chi_d(z)$, say $\chi_d(y) <_d \chi_d(z)$.

If $F_d(y) \neq G_d(z)$, then, by the sup-density of $X$,

$$(G_d(z) \setminus F_d(y)) \cap X \neq \emptyset,$$

thus with an arbitrary point $x$ from this set we have

$$f_{d''}(y) <_{d''} f_{d''}(x) <_{d''} f_{d''}(z)$$

($f_{d''}(y) \leqq_{d''} f_{d''}(x)$ is evident, but if equality held then we would have $F_d(y) \cap X = F_d(x) \cap X = F_{d''}(x)$, implying $x \in F_d(y)$, a contradiction; the case of the second inequality is analogous); thus $f(y) \neq f(z)$.

On the other hand, if $F_d(y) = G_d(z)$ then choose $e$ and $B$ to $A = F_d(y)$. If $AeB$ and $(B \setminus A) \cap X \neq \emptyset$ then

$$f_{e''}(y) <_{e''} (A \cap X, B \cap X) \leqq_{e''} f_{e''}(z)$$

(the first inequality follows as above, the second one is a consequence of $z \notin G_d(z) = A$); i.e. $f(y) \neq f(z)$ again; the remaining case can be similarly dealt with.

3° To complete the proof, we have to show that $f^{-1}$ is bi-continuous, i.e. that $f$ is bi-open. For this purpose, it is enough to prove that $f[G] \in \mathrm{c}_{D|X}\mathscr{P}$ whenever $\emptyset \neq G \neq Y$, $G \in \operatorname{dom} d$ and $d \in D$ (the case of the second topologies is analogous). But this certainly holds if there are a $c \in D$ and an $F \subset Y$ such that $(G, F)$ is the $<_c$-smallest element of $\varrho_c^{-1}[\{\varrho_c((G, F))\}]$, because then a $p \in \mathrm{c}_{D|X}$X belongs to $f[G]$ iff the $c''$th coordinate of $p$ is $<_{c''}$-smaller than $\varrho_c((G, F))$. [Indeed, $y \in G$ iff $\chi_c(y) <_c (G, F)$, i.e. iff $f_{c''}(y) = \varrho_c(\chi_c(y)) <_{c''} \varrho_c((G, F))$.]

a) If $G \notin \operatorname{ran} d$ then there is a unique $F$ with $GdF$, and $(G, F)$ is the first element of $\varrho_d^{-1}[\{\varrho_d((G, F))\}]$. [Indeed, if $\varrho_d((G_1, F_1)) = \varrho_d((G, F))$ and $(G_1, F_1) <_d (G, F)$ then $F \cap X = F_1 \cap X \subset G \cap X \subset F \cap X$, so $(G \setminus F_1) \cap X = \emptyset$, i.e. $F_1 = G$ by the sup-density of $X$, a contradiction.]

b) If there are $e \in D$ and $B \subset Y$ such that $GeB$ and $(B \setminus G) \cap X \neq \emptyset$ then $(G, B)$ is the first (in fact, the only) element of $\varrho_e^{-1}[\{\varrho_e((G, B))\}]$.

c)[3] Finally, if there are $e \in D$ and $B \subset Y$ such that $BeG$ and $(G \setminus B) \cap X \neq \emptyset$ then let $b = ](B, G), \rightarrow[$ (in the ordering $<_e$). Now $b$ has a smallest element $(G_0, F_0)$ [otherwise $\{G\} \cup$co-ran $b$ would be a sup-open covering of $Y$ having no finite sub-

[3] This part of the proof is superfluous if $D$ is assumed to contain the directions mentioned in Remark a).

covering, contrary to the sup-compactness of $Y$]. By the orderliness of $e$, $G=G_0$, and $(G, F_0)$ is the first element of $\varrho_e^{-1}[\{\varrho_e((G, F_0))\}]$.

**11.7** The quasi-proximity $\sqsubset^D$ introduced in 9.5 is not suitable for establishing a connexion between the two theories of bitopological compactifications, namely the one presented in this paper and the one given by Császár [13]. Instead, we have to take

$$A \sqsubset_d B \Leftrightarrow \exists (G_1, F_1),\ (G_2, F_2) \in d,\ A \subset F_1 \subset G_2 \subset B,\ G_1 \neq F_2,$$

$$\sqsubset_D = \sup \{\sqsubset_d \colon d \in D\},$$

cf. E. Deák [20], where the proximities $(\sqsubset_d)^s$ and $(\sqsubset_D)^s$ are considered (see also Remark 9.5).

## REFERENCES

[1] Aarts, J. M. and Groot, J. de, Complete regularity as a separation axiom, *Canad. J. Math.* **21** (1969), No 1, 96—105. *MR* **38** # 5160.
[2] Bîrsan, T., Compacité dans les espaces bitopologiques, *An. Şti. Univ. «Al. I. Cuza» Iaşi Secţ. Ia Mat. (N. S.)* **15** (1969), No 2, 317—328. *MR* **41** # 4469.
[3] Bose, S., Weak Hausdorff axiom in bitopological spaces, *Bull. Calcutta Math. Soc.* **72** (1980), No 2, 95—106. *MR* **82j:** 54059.
[4] Brown, L. M., On extensions of bitopological spaces, *Topology* (Proc. Fourth Colloq. Budapest, 1978) Vol. I, Colloq. Math. Soc. János Bolyai **23**, North-Holland, Amsterdam, 1980, 181—213. *MR* **82a:** 54059.
[5] Brown, L. M., Confluence para-quasi-uniformities, *Topology, theory and applications I* (Proc. Fifth Colloq., Eger, 1983), Colloq. Math. Soc. János Bolyai **41**, North-Holland, Amsterdam, 1985, 125—151. *MR* **88e:** 54035.
[6] Brümmer, G. C. L., Two procedures in bitopology, *Categorical topology* (Proc. Internat. Conf., Free Univ. Berlin, Berlin, 1978), Lecture Notes in Math. **719**, Springer, Berlin, 1979, 35—43. *MR* **80k:** 54055.
[7] Byrne, C., On compactness of bitopological spaces, *Kyungpook Math. J.* **15** (1975), No 2, 159—162. *MR* **53** # 1534.
[8] Čech, E., *Topological spaces*, Revised by Z. Frolík and M. Katětov, Academia, Prague and Interscience, London, 1966. *MR* **35** # 2254.
[9] Cooke, I. E. and Reilly, I. L., On bitopological compactness, *J. London Math. Soc.* (2) **9** (1975), 518—522. *MR* **51** # 4178.
[10] Császár, Á. Sur une classe de structures topologiques générales, *Revue Math. Pures Appl.* **2** (1957), 399—407. *MR* **20** # 1289.
[11] Császár, Á., *Foundations of general topology*, Pergamon Press, Oxford, 1963. *MR* **28** # 575,
[12] Császár, Á., On the characterization of completely regular spaces, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **11** (1968), 79—82. *MR* **39** # 6247.
[13] Császár, Á., Doppeltkompakte bitopologische Räume, *Theory of sets and topology* (in honour of Felix Hausdorff, 1869—1942), VEB Deutscher Verlag Wissensch., Berlin, 1972, 59—67. *MR* **49** # 7990.
[14] Császár, Á., *General topology*, Akadémiai Kiadó, Budapest and Adam Hilger Ltd, Bristol, 1978. *MR* **57** # 13812.
[15] Császár, K., Separation axioms for proximity and closure spaces, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **30** (1987), 223—229. *MR* **89f:** 54056.
[16] Datta, M. C., Projective bitopological spaces, *J. Austral. Math. Soc.* **13** (1972), No 3, 327—334. *MR* **46** # 4496.
[17] Datta, M. C., Projective bitopological spaces II, *J. Austral. Math. Soc.* **14** (1972), No 1, 119—128. *MR* **47** # 2566.
[18] Deák, E., Theory and applications of directional structures, *Topics in topology* (Proc. Third Colloq., Keszthely, 1972), Colloq. Math. Soc. János Bolyai **8**, North-Holland, Amsterdam, 1974, 187—211. *MR* **52** # 11860.

[19] DEÁK, E., Untersuchungen über Richtungsstrukturen, III. Über die Reichweite des $\mathfrak{R}$-Kompaktifizierungsverfahrens, *Studia Sci. Math. Hungar.* **11** (1976), No 3—4, 437—449. *MR* **81k:** 54052.
[20] DEÁK, E., *Dimension und Konvexität* (in preparation).
[21] DEÁK, E. and HAMBURGER, P., Vollständig interne Charakterisierungen der $T_2$-kompaktifizierbaren Räume, *Period. Math. Hungar.* **4** (1973), No 2—3, 125—145. *MR* **50** # 14654.
[22] DEÁK, J., On bitopological spaces I, *Studia Sci. Math. Hungar.* **25** (1990), 457—481.
[23] DEÁK, J., On bitopological spaces II, *Studia Sci. Math. Hungar.* **26** (1991), 1—17.
[24] DIMITRIJEVIĆ, R., Pairwise compact bitopological quasiordered spaces, *Papers presented at the Fifth Balkan Math. Congress* (Belgrade, 1974), *Math. Balkanica* **4** (1974), 121—127. *MR* **54** # 8582.
[25] DVALISHVILI, B. P., Separation in bitopological spaces, *Sakharth. SSR Mecn. Akad. Moambe = Soobshch. Akad. Nauk Gruzin. SSR* **73** (1974), No 2, 285—288 (in Russian; English summary). *MR* **50** # 5757.
[26] DVALISHVILI, B. P., Certain types of compactness and separation axioms of bitopological spaces, *Sakharth. SSR Mecn. Akad. Moambe = Soobshch. Akad. Nauk Gruzin. SSR* **80** (1975), No 2, 289—292 (in Russian; English summary). *MR* **53** # 9163.
[27] DVALISHVILI, B. P., The dimension and certain other questions of the theory of bitopological spaces, *Collection of papers on topology I, Dimension theory and sheaf theory, Sakharth. SSR Mecn. Akad. Math. Inst. Shrom = Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **56** (1977), 15—51 (in Russian; English summary). *MR* **58** # 18366.
[28] DVALISHVILI, B. P., Bicompact extensions of bitopological spaces, *International Topology Conference* (Moscow State Univ., Moscow, 1979), *Uspekhi Mat. Nauk* **35** (1980), No 3 (213), 171—174 (in Russian). *MR* **81j:** 54050.
[29] FLETCHER, P., Pairwise uniform spaces, *Notices Amer. Math. Soc.* **12** (1965), No 5 (83), 612.
[30] FLETCHER, P., HOYLE, H. B. III and PATTY, C. W., The comparison of topologies, *Duke Math. J.* **36** (1969), No 2, 325—331. *MR* **39** # 3441.
[31] FLETCHER, P. and LINDGREN, W. F., Note on a result of E. P. Lane, *Portugal. Math.* **35** (1976), No 3—4, 211—212. *MR* **57** # 17597.
[32] FRINK, O., Compactifications and seminormal spaces, *Amer. J. Math.* **86** (1964), No 3, 602—607. *MR* **29** # 4028.
[33] GASTL, G. C., Bitopological spaces from quasiproximities, *Portugal. Math.* **33** (1974), No 3—4, 213—218. *MR* **52** # 6668.
[34] GOEL, C. K. and SINGHAL, A., Weaker forms of compactness in bitopological hyperspaces, *Indian J. Math.* **21** (1979), 169—176. *Zbl* **509**, 54029.
[35] HAMBURGER, P., Complete regularity as a separation axiom, *Period, Math. Hungar.* **4** (1973), No 1, 39—44. *MR* **49** # 6144.
[36] HAMBURGER, P., A general method to give internal characterizations of completely regular and Tychonoff spaces, *Acta Math. Acad. Sci. Hungar.* **23** (1972), No 3—4, 479—494. *MR* **47** # 5829.
[37] HAYASHI, E., A note on proximity spaces, *Bull. Aichi Gakugei Univ.* **14** (1965), 1—4.
[38] HUNSAKER, W. and LINDGREN, W., Construction of quasi-uniformities, *Math. Ann.* **188** (1970), No 1, 39—42. *MR* **42** # 1057.
[39] HUŠEK, M., Generalized proximity and uniform spaces I, *Comment. Math. Univ. Carolinae* **5** (1964), 247—266. *MR* **31** # 713.
[40] JAS, M. and BAISNAB, A. P., Bitopological spaces and associated $q$-proximity, *Indian J. Pure Appl. Math.* **13** (1982), No 10, 1142—1146. *MR* **83m:** 54048.
[41] JAS, M. and BANERJEE, C., Quasiproximity and associated bitopological spaces, *Indian J. Pure Appl. Math.* **12** (1981), No 8, 945—949. *MR* **82j:** 54060.
[42] JUHÁSZ, I. and MACHOVER, M., A note on non-standard topology, *Proc. Kon. Ned. Akad. Wetensch. A* **72** = *Indag. Math.* **31** (1969), No 5, 482—484. *MR* **40** # 6488.
[43] KIM, Y. W., Pairwise compactness, *Publ. Math. Debrecen* **15** (1968), 87—90. *MR* **38** # 6541.
[44] LANE, E. P., Concerning extensions of semicontinuous functions in bitopological spaces, *Notices Amer. Math. Soc.* **12** (1965), No 1 (79) Part I, 128.
[45] LANE, E. P., Bitopological spaces and quasi-uniform spaces, *Proc. London Math. Soc.* (3) **17** (1967), No 2, 241—256. *MR* **34** # 5054.
[46] LANE, E. P., Quasi-proximities and bitopological spaces, *Portugal. Math.* **28** (1969), No 3—4, 151—159. *MR* **44** # 3284.

[47] LEADER, S., On products of proximity spaces, *Math. Ann.* **154** (1964), No 2, 185—194. *MR* **28** # 5420.
[48] MATTSON, D. A., Separation relations and quasi-proximities, *Math. Ann.* **171** (1967), No 2, 87—92. *MR* **35** # 3628.
[49] METZGER, J. M., Quasi-proximities and quasi-uniformities, *Kyungpook Math. J.* **11** (1971), 123—138. *MR* **47** # 1028.
[50] MRŠEVIĆ, M., Local compactness in bitopological spaces, *Mat. Vesnik* **2** **(15)** **(30)** (1978), No 3, 265—272. *MR* **84 b**: 54063.
[51] MRŠEVIĆ, M., Local compactness in bitopological hyperspaces, *Mat. Vesnik* **3** **(16)** **(31)** (1979), No 1, 41—52. *MR* **82a**: 54017.
[52] MRŠEVIĆ, M., On bitopological local compactness, *Mat. Vesnik* **3** **(16)** **(31)** (1979), No 2, 187—196. *MR* **82f**: 54049.
[53] MRŠEVIĆ, M., On quasi-proximity spaces, *Indian J. Pure Appl. Math.* **14** (1983), No 4, 511—514. *MR* **85b**: 54043.
[54] MUKHERJEE, M. N., On pairwise almost compactness and pairwise $H$-closedness in a bitopological space, *Ann. Soc. Sci. Bruxelles Sér. I* **96** (1982), No 2, 98—106. *MR* **84b**: 54064.
[55] NAIMPALLY, S. A. and WARRACK, B. D., *Proximity spaces*, Cambridge Univ. Press, London, 1970. *MR* **43** # 3992.
[56] PAHK, D. H. and CHOI, B. D., Notes on pairwise compactness, *Kyungpook Math. J.* **11** (1971), 45—52. *MR* **46** # 2640.
[57] PERVIN, W. J., Quasi-proximities for topological spaces, *Math. Ann.* **150** (1963), 325—326. *MR* **27** # 2767.
[58] PERVIN, W. J., On separation and proximity spaces, *Amer. Math. Monthly* **71** (1964), No 2, 158—161. *MR* **29** # 2767.
[59] REILLY, I. L., Bitopological local compactness, *Proc. Kon. Ned. Akad. Wetensch. A* **75** = *Indag. Math.* **34** (1972), No 5, 407—411. *MR* **47** # 4222.
[60] REILLY, I. L., On essentially pairwise Hausdorff spaces, *Rend. Circ. Mat. Palermo* (2) **25** (1976), No 1—2, 47—52. *MR* **58** # 18367.
[61] RICHARDSON, G. D., $R_1$, pairwise compact, and pairwise complete spaces, *Canad. Math. Bull.* **15** (1972), No 1, 109—113. *MR* **47** # 4224.
[62] RIECKE, C. V., Products of pairwise compact spaces, *Kyungpook Math. J.* **15** (1975), 33—36. *MR* **52** # 1643.
[63] RIECKE, C. V., $H$-closure for bitopological spaces, *Kyungpook Math. J.* **18** (1978), No 2, 159—165. *MR* **80f**: 54024.
[64] SAEGROVE, M. J., Pairwise complete regularity and compactification in bitopological spaces, *J. London Math. Soc.* (2) **7** (1973), No 2, 286—290. *MR* **49** # 11483.
[65] SALBANY, S., Compactifications of bitopological spaces, *Math. Colloq. Univ. Cape Town* **7** (1971/72), 1—3. *MR* **55** # 6365.
[66] SALBANY, S., *Bitopological spaces, compactifications and completions*, Math. Monographs Univ. Cape Town **1**, Department Math., Univ. Cape Town, Cape Town, 1974. *MR* **54** # 13869.
[67] SALBANY, S., On compact bitopological spaces, *Proc. South Afr. Math. Soc.* **4** (1974), 219—223.
[68] SALBANY, S., Reflective subcategories and closure operators, *Categorical topology* (Proc. Conf., Mannheim, 1975), Lecture Notes in Math. **540**, Springer, Berlin, 1976, 548—565. *MR* **65** # 9473.
[69] SINGAL, M. K. and LAL, S., Biquasi-proximity spaces and compactification of a pairwise proximity space, *Math. Student* **41** (1973), No 1—2, 108.
[70] SINGAL, M. K. and LAL, S., Biquasi-proximity spaces and compactification of a pairwise proximity space, *Kyungpook Math. J.* **13** (1973), 41—49. *MR* **48** # 1178.
[71] SINGAL, M. K. and LAL, S., A note on proximities and pairwise regular spaces, *Kyungpook Math. J.* **15** (1975), 25—31. *MR* **51** # 6742.
[72] SINGAL, M. K. and LAL, S., On PR-proximities and regular spaces, *Glasnik Mat. Ser. III* **10** **(30)** (1975), No 2, 359—365. *MR* **56** # 1266.
[73] SMIRNOV, YU. M., On the theory of completely regular spaces, *Dokl. Akad. Nauk SSSR* **62** (1948), No 6, 749—752 (in Russian). *MR* **10**, 315.
[74] STEINER, E. F., The relation between quasi-proximities and topological spaces, *Math. Ann.* **155** (1964), 194—195. *MR* **29** # 581.
[75] STEINER, E. F., Normal families and completely regular spaces, *Duke Math. J.* **33** (1966), No 4, 743—745. *MR* **33** # 7975.
[76] SWART, J., Total disconnectedness in bitopological spaces and product bitopological spaces,

*Proc. Kon. Ned. Akad. Wetensch. A* **74** = *Indag. Math.* **33** (1971), No 2, 135—145. *MR* 46 # 4497.
[77] SWART, J., Pairwise C-compact spaces, *Kyungpook Math. J.* **19** (1979), No 1, 33—38. *MR* **81c:** 54049.
[78] SWART, J., Pairwise almost compact spaces, *Quaestiones Math.* **5** (1982/3), No 3, 277—287. *MR* **84b:** 54065.
[79] SWART, J., On bitopological C-compactness, *Kyungpook Math. J.* **22** (1982), No 1, 41—43. *MR* **83k:** 54038.
[80] VASUDEVAN, R. and GOEL, C. K., A note on *C*-compact bitopological spaces, *Mat. Vesnik* **1** **(14)** **(29)** (1977), No 2, 179—187. *MR* **57** # 17593.
[81] VASUDEVAN, R. and GOEL, C. K., Pairwise compactness in bitopological hyperspaces, *Glasnik Mat. Ser. III* **15** **(35)** (1980), No 1, 157—162. *MR* **81k:** 54009.
[82] ZAITSEV, V. I., On the theory of Tikhonov spaces, *Vestnik Moskov. Univ. Ser I Mat. Mekh.* **22** (1967), No 3, 48—57 (in Russian; English summary). *MR* **35** # 7286.

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# ON LOCAL FUNDAMENTAL SOLUTION OF COERCIVE LINEAR PARTIAL DIFFERENTIAL OPERATORS

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## Introduction

Let $G$ be an open set in $\mathbf{R}^n$ and let $L(x, D)=\sum_{|\sigma|\leqq r} a_\sigma(x)D^\sigma$ be a linear partial differential operator with $C^\infty(G)$-coefficients. Suppose that $L(x, D)$ is formally hypoelliptic, that is, $L(x, D)$ is of constant strength in $G$ and in addition $L(x_0, D)=\sum_{|\sigma|\leqq r} a_\sigma(x_0)D^\sigma$ is hypoelliptic. Let $\mathscr{B}_{p,k}$; $k\in\mathscr{K}$, $p\in[1, \infty]$ be the Hörmander space (cf. [2], p. 36). Furthermore, let $L_0^{\sim}\in\mathscr{K}$ be defined by

$$L_0^{\sim}(\xi)=\left(\sum_\alpha\left(\left|\frac{\partial^\alpha}{\partial\xi^\alpha}L(x_0, \xi)\right|^2\right)\right)^{1/2}.$$

Then for each $x\in G$ one can find an open neighbourhood $U_x\subset G$ of $x$ and an operator $E\colon C_0^\infty(U_x)\to\mathscr{B}_{p,kL_0^{\sim}}$ such that

$$E(L(x,D)\varphi)=\varphi \quad \text{in} \quad U_x \tag{1.1}$$

$$L(x,D)(E\varphi)=\varphi \quad \text{in} \quad U_x \tag{1.2}$$

and that

$$\|E\varphi\|_{p,kL_0^{\sim}}\leqq C_x\|\varphi\|_{p,k} \tag{1.3}$$

for all $\varphi\in C_0^\infty(U_x)$ (cf. [2], p. 174 and [9]). Especially, one gets the inequality

$$\|\varphi\|_{p,kL_0^{\sim}}\leqq C_x'\|L(x,D)\varphi\|_{p,k} \quad \text{for all} \quad \varphi\in C_0^\infty(U_x'), \tag{1.4}$$

where $U_x'$ is a relatively compact open set of $U_x$. In addition, one knows that $L_0^{\sim}(\xi)\to\infty$ with $|\xi|\to\infty$. The operator $E$ is so-called local fundamental solution of $L(x, D)$.

Let $\mathscr{B}_{p,k}(G)$ be the completion of $C_0^\infty(G)$ in $\mathscr{B}_{p,k}$. Furthermore, let $\mathbf{B}_{p,k}(G)$ be the subspace of $\mathscr{D}'(G)$ such that for each $u\in\mathbf{B}_{p,k}(G)$ there exists $f_u\in\mathscr{B}_{p,k}$ such that $u=f_u$ in $G$. We topologize the space $\mathbf{B}_{p,k}(G)$ in the obvious way (cf. Section 2.1).

Choose $k$ and $k^{\sim}$ from $\mathscr{K}$ such that $k^{\sim}(\xi)\to\infty$ with $|\xi|\to\infty$. We show a local existence result of distributional solution for the equation

$$L(x,D)u=f;\quad u\in\mathbf{B}_{p,kk^{\sim}}(G),\ f\in\mathbf{B}_{p,k}(G)$$

The basic assumption is that the formal transpose $L'(x, D)$ of $L(x, D)$ satisfies the

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a priori estimate

(1.5) $$\|L'(x,D)\varphi\|_{p',1/(kk^{\sim})^{\vee}} \geqq C_1\|\varphi\|_{p',1/k^{\vee}} - C_2\|\varphi\|_{p',1/(kk^{\sim})^{\vee}}.$$

Here $p$ lies in the interval $]1,\infty]$; $p'$ and $k\in\mathscr{K}$ are defined by $(1/p)+(1/p')=1$ and $k^{\vee}(\xi)=k(-\xi)$. For the sufficient algebraic conditions of the validity of this kind estimates we refer to [1], [5], [3] and [7], for example.

Supposing that

(1.6) $$\|L'(x,D)\varphi\|_{2,1/(kk^{\sim})^{\vee}} \geqq C_1\|\varphi\|_{2,1/k^{\vee}} - C_2\|\varphi\|_{2,1/(kk^{\sim})^{\vee}}$$

and

(1.7) $$\|L(x,D)\varphi\|_{2,k} \geqq C_1\|\varphi\|_{2,kk^{\sim}} - C_2\|\varphi\|_{2,k}$$

for all $\varphi\in C_0^\infty(G)$, we show that essentially for each $x\in G$ there exist a neighbourhood $U_x\subset G$ of $x$ and a continuous linear operator $E:\mathscr{B}_{2,k}(U_x)\to \mathbf{B}_{2,kk^{\sim}}^{\sim}(U_x)$ such that

(1.8) $$E(L_{\tilde{U}_x}u)=u \quad \text{in } U_x, \text{ for all } u\in D(L_{\tilde{U}_x})$$

and that

(1.9) $$L_{U_x}'^{\#}(Ef)=f \quad \text{in } U_x, \text{ for all } f\in\mathscr{B}_{2,k}(U_x).$$

Here $L_{\tilde{U}_x}$ ($L_{U_x}'^{\#}$) is the minimal (the maximal) realization of $L(x,D)$ in the spaces in question.

Finally, we consider the regularity of the Schwarz kernel $K\in\mathscr{D}'(U_x\times U_x)$ of $E$. A sufficient criterion under which $K$ is a $C^m$-function outside the diagonal $D$ of $U_x\times U_x$ is given.

## 2. Preliminaries

**2.1.** For the needed (unexplained) notations about the distribution theory and for the definition of spaces $\mathscr{B}_{p,k}$ and $\mathscr{B}_{p,k}^{\mathrm{loc}}(G)$; $p\in[1,\infty]$, $k\in\mathscr{K}$, we refer to the monograph [2], pp. 1—45. Here $G$ is an open set in $\mathbf{R}^n$. Let $\mathscr{B}_{p,k}(G)$ be the subspace of $\mathscr{B}_{p,k}$ such that for each $u\in\mathscr{B}_{p,k}(G)$ there exists a sequence $\{\varphi_n\}\subset C_0^\infty(G)$ with

$$\|\varphi_n-u\|_{p,k}\to 0 \quad \text{with} \quad n\to\infty.$$

Then one has

$$\varphi_n(\psi):=\int_{\mathbf{R}^n}\varphi_n(x)\psi(x)\,dx\to u(\psi) \quad \text{for all} \quad \psi\in C_0^\infty$$

with $n\to\infty$. The space $\mathscr{B}_{p,k}(G)$ is essentially the completion of $C_0^\infty(G)$ in $\mathscr{B}_{p,k}$. Furthermore, let $A$ be a closed set in $\mathbf{R}^n$. We denote by $\dot{\mathscr{B}}_{p,k}(A)$ the subspace of $\mathscr{B}_{p,k}$ such that for each $v\in\dot{\mathscr{B}}_{p,k}(A)$ one has

$$\operatorname{supp} v\subset A.$$

One sees that $\dot{\mathscr{B}}_{p,k}(A)$ is closed in $\mathscr{B}_{p,k}$ and that $\mathscr{B}_{p,k}(G)\subset\dot{\mathscr{B}}_{p,k}(\bar{G})$.

Let $\mathbf{B}_{p,k}^{\sim}(G)$ be the factor space

(2.1) $$\mathbf{B}_{p,k}^{\sim}(G)=\mathscr{B}_{p,k}/\dot{\mathscr{B}}_{p,k}(\mathbf{R}^n\setminus G).$$

We equip the space $\mathbf{B}_{p,k}^{\sim}(G)$ with the norm

$$|||T|||_{p,k}^{\sim}=\inf_{u\in T}\|u\|_{p,k}.$$

Then $\mathbf{B}_{p,k}^{\sim}(G)$ becomes a Banach space.

Assume that $T$ belongs to $\mathbf{B}^{\sim}_{p,k}(G)$ and that $u_T$ is a representative of $T$. Then the linear mapping $J: \mathbf{B}^{\sim}_{p,k}(G) \to \mathscr{D}'(G)$ defined by

$$J(T) = u_{T|G}$$

is injective (here $u_{T|G}$ denotes the restriction of $u_T$ on $G$). We define a linear subspace $\mathbf{B}_{p,k}(G)$ of $\mathscr{D}'(G)$ by

$$\mathbf{B}_{p,k}(G) = J\big(\mathbf{B}^{\sim}_{p,k}(G)\big). \tag{2.2}$$

The linear space $\mathbf{B}_{p,k}(G)$ is equipped with the norm

$$|||V|||_{p,k} = |||J^{-1}V|||^{\sim}_{p,k}, \tag{2.3}$$

and then $\mathbf{B}_{p,k}(G)$ becomes a Banach space. One sees easily that a distribution $V \in \mathscr{D}'(G)$ lies in $\mathbf{B}_{p,k}(G)$ if and only if there exists $f_V \in \mathscr{B}_{p,k}$ such that

$$V(\varphi) = f_V(\varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(G). \tag{2.4}$$

Let $C^\infty_{(0)}(G)$ be the subspace of $C^\infty(G)$ such that for each $\psi \in C^\infty_{(0)}(G)$ one finds an element $f_\psi \in C_0^\infty$ with

$$\psi(x) = f_\psi(x) \quad \text{for all} \quad x \in G. \tag{2.5}$$

When $p \in [1, \infty[$, $C^\infty_{(0)}(G)$ is dense in $\mathbf{B}_{p,k}(G)$, since $C_0^\infty$ is dense in $\mathscr{B}_{p,k}$. We also remark that $C_0^\infty$ is not dense in $\mathscr{B}_{\infty,k}$. Hence we have that $\mathscr{B}_{\infty,k}(\mathbf{R}^n) \neq \mathscr{B}_{\infty,k}$. When $p$ lies in the interval $[1, \infty[$ we however get $\mathscr{B}_{p,k}(\mathbf{R}^n) = \mathscr{B}_{p,k}$.

**2.2.** We establish some further properties of the space $\mathscr{B}_{p,k}(G), \dot{\mathscr{B}}_{p,k}(\bar{G})$ and $\mathbf{B}_{p,k}(G)$. Clearly for every open set $G$ of $\mathbf{R}^n$ the inclusion

$$\mathscr{B}_{p,k}(G) \subset \dot{\mathscr{B}}_{p,k}(\bar{G}) \tag{2.6}$$

holds. For the first instance we give a sufficient condition for the equality

$$\mathscr{B}_{p,k}(G) = \dot{\mathscr{B}}_{p,k}(\bar{G}), \tag{2.7}$$

where $p$ lies in the interval $[1, \infty[$.

Let $A$ be a closed set in $\mathbf{R}^n$ and let $\mathscr{E}'(A)$ denote the subspace of $\mathscr{E}'$ such that for each $u \in \mathscr{E}'(A)$ one has $\operatorname{supp} u \subset A$.

LEMMA 2.1. *The space* $\mathscr{B}_{p,k} \cap \mathscr{E}'(A)$ *is dense in the space* $\dot{\mathscr{B}}_{p,k}(A)$ *(when* $p \in [1, \infty[$). □

The proof of Lemma 2.1 follows from Theorem 2.2.11 of [2], p. 42.

We set the following property for the boundary $\partial G$ of the open set $G \subset \mathbf{R}^n$

CONDITION 2.2. For every $x \in \partial G$ there exist an open neighbourhood $U_x \subset \mathbf{R}^n$ of $x$, a vector $y_x \in \mathbf{R}^n$ and a positive number $\varepsilon_x > 0$ such that

$$\bar{G} \cap U_x + B(\varepsilon y_x, \varepsilon) \subset G \quad \text{for all} \quad \varepsilon \in ]0, \varepsilon_x[. \tag{2.8}$$

We show

THEOREM 2.3. *Suppose that the open set* $G \in \mathbf{R}^n$ *satisfies Condition 2.2. Then the relation* (2.7) *is valid (when* $p \in [1, \infty[$).

PROOF. Because of Condition 2.2 for every $x\in\bar G$ there exists an open neighbourhood $U_x\subset\mathbf{R}^n$ of $x$, $y_x\in\mathbf{R}^n$ and $\varepsilon_x>0$ such that

$$\bar G\cap U_x+B(\varepsilon y_x,\varepsilon)\subset G \quad \text{for all} \quad \varepsilon\in]0,\varepsilon_x[. \tag{2.9}$$

Choose a nonnegative function $\psi^x\in C_0^\infty(B(y_x,1/2))$ such that

$$(\mathcal{F}\psi^x)(0)=\int_{\mathbf{R}^n}\psi^x(y)\,dy=1.$$

Here $\mathcal{F}$ denotes the Fourier transform $\mathcal{S}'\to\mathcal{S}'$. Furthermore, define functions $\psi_j^x\in C_0^\infty$ by

$$\psi_j^x(y)=j^n\psi^x(jy),\quad j\in\mathbf{N}. \tag{2.10}$$

Assume that $u$ is in $\mathcal{B}_{p,k}\cap\mathcal{E}'(\bar G)$. Since one has

$$\operatorname{supp}u\subset\bigcup_{x\in\bar G}U_x$$

and since supp $u$ is compact in $\mathbf{R}^n$, one finds elements $x_l\in\bar G$, $l=1,\dots,N$ such that $\operatorname{supp}u\subset\bigcup_{l=1}^N U_{x_l}$. Take the $C^\infty$-partition of unity $\{\xi_l\}$ for the set $\operatorname{supp}u$ (with respect to the covering $\{U_{x_l}\}$). Then for each $j\in\mathbf{N}$ the distribution

$$\sum_{l=1}^N(\xi_l u)*\psi_j^{x_l} \tag{2.11}$$

belongs to $C^\infty(\mathbf{R}^n)$ and in addition one has

$$\begin{aligned}\operatorname{supp}\Big(\sum_{l=1}^N(\xi_l u)*\psi_j^{x_l}\Big)&\subset\bigcup_{l=1}^N\operatorname{supp}((\xi_l u)*\psi_j^{x_l})\subset\\ &\subset\bigcup_{l=1}^N(\operatorname{supp}(\xi_l u)+\operatorname{supp}\psi_j^{x_l})\subset\bigcup_{l=1}^N(\bar G\cap U_{x_l}+B(y_{x_l}/j,1/j)).\end{aligned} \tag{2.12}$$

Hence in virtue of (2.9) supp $\Big(\sum_{l=1}^N(\xi_l u)*\psi_j^{x_l}\Big)$ is a compact subset on $G$, when $j$ is large enough.

Furthermore, for every $\xi\in\mathbf{R}^n$

$$\begin{aligned}\mathcal{F}((\xi_l u)*\psi_j^{x_l})(\xi)&=\mathcal{F}(\xi_l u)(\xi)(\mathcal{F}\psi_j^{x_l})(\xi)=\\ &=\mathcal{F}(\xi_l u)(\xi)(\mathcal{F}\psi^{x_l})(\xi/j)\to\mathcal{F}(\xi_l u)(\xi)\end{aligned} \tag{2.13}$$

with $j\to\infty$ and in addition

$$|\mathcal{F}((\xi_l u)*\psi_j^{x_l})(\xi)|\leqq|\mathcal{F}(\xi_l u)(\xi)|\int_{\mathbf{R}^n}|\psi^{x_l}(y)|\,dy=|\mathcal{F}(\xi_l u)(\xi)| \tag{2.14}$$

for all $\xi\in\mathbf{R}^n$. Since $\xi_l u$ belongs to $\mathcal{B}_{p,k}$ for every $l\in\mathbf{N}$, the Lebesgue Dominated Convergence Theorem implies that

$$\|(\xi_l u)*\psi_j^{x_l}-\xi_l u\|_{p,k}\to 0 \quad\text{with}\quad j\to\infty$$

and then

$$\sum_{l=1}^{N} (\xi_l u) * \psi_{j'} \to \sum_{l=1}^{N} \xi_l u = u \quad \text{in} \quad \mathscr{B}_{p,k}.$$

Hence in virtue of Lemma 2.1 the proof is ready.

REMARK. A. Every open ball $B \subset \mathbf{R}^n$ and every interior of the complement of an open ball obeys the Condition 2.2. Hence for each open ball $B$

(2.15) $\quad \mathscr{B}_{p,k}(B) = \mathring{\mathscr{B}}_{p,k}(\bar{B}) \quad \text{and} \quad \mathscr{B}_{p,k}(\mathbf{R}^n \setminus \bar{B}) = \mathring{\mathscr{B}}_{p,k}(\mathbf{R}^n \setminus B)$

(when $p \in ]1, \infty[$).

B. Using the proof of Theorem 2.2.1 of [2] one sees that $\mathscr{B}_{\infty,k}(\mathbf{R}^n) \cap \mathscr{E}'(A)$ is dense in $\mathscr{B}_{\infty,k}(\mathbf{R}^n) \cap \mathring{\mathscr{B}}_{\infty,k}(A)$, where $A \subset \mathbf{R}^n$ is a closed set.

Let $k$ be in $\mathscr{K}$ and let $p \in [1, \infty]$. We define $k^{\check{}} \in \mathscr{K}$ and $p' \in [1, \infty]$ by $k^{\check{}}(\xi) = k(-\xi)$ and $(1/p') + (1/p) = 1$. Applying the Hahn—Banach Theorem and Theorem 2.2.9 of [2], p. 42, we can establish for the dual $\mathscr{B}^*_{p',1/k^{\check{}}}(G)$ of $\mathscr{B}_{p'1/k^{\check{}}}(G)$ the following characterization

THEOREM 2.4. *Let $G$ be an open set in $\mathbf{R}^n$ and let $p \in ]1, \infty]$. Then for every $T \in \mathscr{B}^*_{p',1/k^{\check{}}}(G)$ there exists a unique element $t \in \mathbf{B}_{p,k}(G)$ such that*

$$T\varphi = t(\varphi) \quad \textit{for all} \quad \varphi \in C_0^\infty(G).$$

*On the other hand, suppose that $t$ belongs to $\mathbf{B}_{p,k}(G)$. Then the linear form $T: C_0^\infty(G) \to C$ defined by*

$$T\varphi = t(\varphi)$$

*has a unique continuous extension onto the whole space $\mathscr{B}_{p',1/k^{\check{}}}(G)$. Furthermore, one has*

(2.16) $$\|T\| = |||t|||_{p,k}.$$

In addition we have

THEOREM 2.5. *Let $G$ be an open set in $\mathbf{R}^n$ and let $p \in ]1, \infty[$. Then for every $Q \in \mathbf{B}^*_{p,k}(G)$ there exists a unique element $q \in B_{p',1/k^{\check{}}}(G)$ such that*

$$Q(\varphi|_G) = q(\varphi) \quad \textit{for all} \quad \varphi \in C_0^\infty.$$

*On the other hand, suppose that $q$ belongs to $\mathscr{B}_{p',1/k^{\check{}}}(G)$. Then the linear form $Q: C_{(0)}^\infty(G) \to \mathbf{C}$ defined by*

$$Q\psi = q(f_\psi)$$

*has a unique continuous extension onto the whole space $\mathbf{B}_{p,k}(G)$. Furthermore one has*

(2.17) $$\|Q\| = \|q\|_{p',1/k^{\check{}}}.$$

PROOF. Since $p$ belongs to $]1, \infty[$, the spaces $\mathscr{B}_{p',1/k^{\check{}}}$ are reflexive. Hence the spaces $\mathscr{B}_{p',1/k^{\check{}}}(G)$ are (as the closed subspaces of $\mathscr{B}_{p',1/k^{\check{}}}$) reflexive, as well. Let $\varkappa_{p',1/k^{\check{}}}: \mathscr{B}_{p',1/k^{\check{}}}(G) \to \mathscr{B}^{**}_{p',1/k^{\check{}}}(G)$ be the canonical isomorphism. In virtue of Theorem 2.4 the linear mapping $J_{p,k}: \mathbf{B}_{p,k}(G) \to \mathscr{B}_{p',1/k^{\check{}}}(G)$ such that $J_{p,k}t =$

$=T$ is an isometrical isomorphism and it satisfies

$$(J_{p,k}t)(\varphi) = t(\varphi) \quad \text{for all} \quad \varphi\in C_0^\infty(G). \tag{2.18}$$

Hence one has for each $Q\in B^*_{p,k}(G)$ and $\psi\in C_0^\infty$

$$\begin{aligned} Q(\psi|_G) &= (J^*_{p,k}(J^{*-1}_{p,k}Q))(\psi|_G) = (J^{*-1}_{p,k}Q)(J_{p,k}(\psi|_G)) = \\ &= (J_{p,k}(\psi|_G))(\varkappa^{-1}_{p',1/k^\smile}(J^{*-1}_{p,k}Q)), \end{aligned} \tag{2.19}$$

where we used only the elementary properties of dual operators. Let $\{\varphi_n\}\subset C_0^\infty(G)$ be a sequence such that $\|\varphi_n-\varkappa^{-1}_{p',1/k^\smile}(J^{*-1}_{p,k}Q)\|_{p',1/k^\smile}\to 0$. Then by (2.19) and (2.18) one has

$$\begin{aligned} Q(\psi|_G) &= \lim_{n\to\infty}(J_{p,k}(\psi|_G))(\varphi_n) = \lim_{n\to\infty}\varphi_n(\psi) = \\ &= (\varkappa^{-1}_{p',1/k^\smile}(J^{*-1}_{p,k}Q))(\psi) =: q(\psi) \end{aligned} \tag{2.20}$$

and

$$\|Q\| = \|\varkappa_{p',1/k^\smile}(J^{*-1}_{p,k}Q)\|_{p',1/k^\smile}. \tag{2.21}$$

It is easy to prove the converse of the assertion and then the proof is complete. □

REMARK. A. In virtue of Theorems 2.4 and 2.5 there exist isometrical isomorphisms $J_{p,k}$: $\mathbf{B}_{p,k}(G)\to\mathscr{B}^*_{p',1/k^\smile}(G)$ and $j_{p',1/k^\smile}$: $\mathscr{B}_{p',1/k^\smile}(G)\to\mathbf{B}^*_{p,k}(G)$ such that

$$(J_{p,k}t)(\varphi) = t(\varphi), \quad \text{for all} \quad \varphi\in C_0^\infty(G) \quad (p\in]1,\infty]) \tag{2.18}$$

and

$$(j_{p',1/k^\smile}q)(\psi|_G) = q(\psi), \quad \text{for all } \psi\in C_0^\infty \quad (p\in]1,\infty[). \tag{2.22}$$

B. In the case when $p\in]1,\infty[$ the spaces $\mathscr{B}_{p',1/k^\smile}(G)$ and $\mathbf{B}_{p,k}(G)$ are reflexive.

C. Furthermore, we remark that the norm $|||V|||_{p,k}$ of $V\in\mathbf{B}_{p,k}(G)$ can be replaced with

$$|||V|||_{p,k} = \inf_{\psi\in C_0^\infty(\mathbf{R}^n\setminus G)} \|f_V-\psi\|_{p,k},$$

when $p\in[1,\infty[$ and when the relation (2.7) holds for $\mathbf{R}^n$-$\bar{G}$.

Finally we show for $\tilde{k}\geq 1$

COROLLARY 2.6. *Let $k$ and $k^\smile$ be in $K$ and let $p\in]1,\infty[$. Then the imbedding $\iota_1$: $\mathbf{B}_{p,kk^\smile}(G)\to\mathbf{B}_{p,k}(G)$ is compact if and only if the imbedding $\iota_2$: $\mathscr{B}_{p',1/k^\smile}(G)\to\to\mathscr{B}_{p',1/(kk^\smile)^\smile}(G)$ is compact.*

PROOF. Suppose that $\iota_1$ is compact. Then the dual operator $\iota_1^*$: $\mathbf{B}^*_{p,kk}(G)\to\to\mathbf{B}^*_{p,kk^\smile}(G)$ is compact, as well. One sees that

$$\iota_2 = j^{-1}_{p',1/(kk^\smile)^\smile}\circ\iota_1^*\circ j_{p',1/k^\smile}, \tag{2.23}$$

where $j_{p',1/k^\smile}$: $\mathscr{B}_{p',1/k^\smile}(G)\to\mathbf{B}^*_{p,k}(G)$ is the isometrical isomorphism (mentioned above). Hence $\iota_2$ is compact. Similarly one sees that

$$\iota_1 = J^{-1}_{p,k}\circ\iota_2^*\circ J_{p,kk^\smile} \tag{2.24}$$

and then the proof is ready. □

REMARK. A. One sees also that the compactness of the imbedding $\iota_2$: $\mathscr{B}_{1,1/k^\sim}(G)\to\mathscr{B}_{1,1/(kk^\sim)^\sim}(G)$ implies the compactness of the imbedding $\iota_1$: $\mathbf{B}_{\infty,kk^\sim}(G)\to\mathbf{B}_{\infty,k}(G)$ (cf. Theorem 2.4).

B. In the case when $G$ is a bounded open set and when $k^\sim(\xi)\to\infty$ with $|\xi|\to\infty$, the imbedding $\iota_2$: $\mathscr{B}_{p,kk^\sim}(G)\to\mathscr{B}_{p,k}(G)$ is compact (cf. [2], p. 38).

## 3. Existence results of solutions

**3.1.** Let $G$ be an open set in $\mathbf{R}^n$ and let $k, k^\sim\in K$ and $p\in[1,\infty]$. Furthermore, let $L(x, D)$ be a partial differential operator

$$L(x,D)=\sum_{|\sigma|\leq r} a_\sigma(x)D^\sigma \tag{3.1}$$

with $C^\infty(G)$-coefficients $a_\sigma$. The formal transpose

$$L'(x,D)(\cdot)=\sum_{|\sigma|\leq r}(-D)^\sigma\big(a_\sigma(x)(\cdot)\big) \tag{3.2}$$

satisfies the relation

$$\big(L(x,D)\varphi\big)(\psi)=\varphi\big(L'(x,D)\psi\big) \quad \text{for all} \quad \varphi,\psi\in C_0^\infty(G).$$

Define a linear operator $L'_G$: $\mathscr{B}_{p',1/k^\sim}(G)\to\mathscr{B}_{p',1/(kk^\sim)^\sim}(G)$ with

$$\begin{cases} D(L'_G)=C_0^\infty(G) \\ L'_G\varphi=L'(x,D)\varphi \quad \text{for} \quad \varphi\in C_0^\infty(G). \end{cases} \tag{3.3}$$

Then $L'_G$ is densily defined.

We need the following lemma

LEMMA 3.1. *Suppose that $G'$ is a relatively compact open subset of $G$, that is, $G'$ is open and $\bar{G}'$ is a compact subset of $G$. Then the operator $L'_{G'}$: $\mathscr{B}_{p',1/k^\sim}(G')\to\to\mathscr{B}_{p',1/(kk^\sim)^\sim}(G')$ is closable.*

PROOF. Let $\phi\in C_0^\infty(G)$ be such that $\phi(x)=1$ for all $x\in\bar{G}'$. Suppose that $\{\varphi_n\}\subset C_0^\infty(G')$ is a sequence such that $\lim_{n\to\infty}\|\varphi_n\|_{p',1/k^\sim}=0$ and

$$\lim_{n\to\infty}\|L'(x,D)\varphi_n-f\|_{p',1/(kk^\sim)^\sim}=0$$

with some $f\in\mathscr{B}_{p',1/(kk^\sim)^\sim}(G')$. Then for each $\varphi\in C_0^\infty$ one has

$$f(\varphi)=\lim_{n\to\infty}\big(L'(x,D)\varphi_n\big)(\varphi)=\lim_{n\to\infty}\varphi_n\big(L(x,D)\varphi\big)=$$

$$=\lim_{n\to\infty}\varphi_n\big(\phi L(x,D)\varphi\big)=0,$$

since for all $h\in\mathscr{B}_{p,k}$ and $\psi\in C_0^\infty$ the inequality

$$|h(\psi)|\leq\|h\|_{p,k}\|\psi\|_{p',1/k^\sim} \tag{3.4}$$

is valid. Hence $f=0$, which implies that $L'_{G'}$ is closable (cf. [10], p. 77). □

REMARK. Similarly one sees that

$$L'_G\colon \mathscr{B}_{p',1/k^{\vee}}(G) \to \mathscr{B}_{p',1/(kk^{\sim})^{\vee}}(G)$$

is closable, when $L(x, D)$ has $C^\infty(\mathbf{R}^n)$-coefficients and when $G$ is an arbitrary open set in $\mathbf{R}^n$. (In [7] it appears a misprint. The operator (2.4) of [7] must have $C^\infty(\mathbf{R}^n)$-coefficients.)

When $L'_G$ is closable, we denote by $L'^{\sim}_G$ its smallest closed extension

$$\mathscr{B}_{p',1/k^{\vee}}(G) \to \mathscr{B}_{p',1/(kk^{\sim})^{\vee}}(G).$$

Furthermore, we define a linear operator $L'^{\#}_G\colon \mathbf{B}_{p,kk^{\sim}}(G) \to \mathbf{B}_{p,k}(G)$ with the requirement

$$\text{(3.5)}\qquad \begin{cases} D(L'^{\#}_G) = \{v \in \mathbf{B}_{p,kk^{\sim}}(G) \mid \text{ there exists } g \in \mathbf{B}_{p,k}(G), \text{ such that } \\ \qquad v(L'(x,D)\varphi) = g(\varphi) \text{ for all } \varphi \in C_0^\infty(G)\} \\ L'^{\#}_G v = g. \end{cases}$$

Since for all $w \in \mathbf{B}_{p,k}(G)$ and $\psi \in C_0^\infty(G)$ one has

$$\text{(3.6)}\qquad |w(\psi)| \leqq |||w|||_{p,k} \|\psi\|_{p',1/k^{\vee}},$$

one sees that the operator $L'^{\#}_G$ is closed. In the case when $G = \mathbf{R}^n$, one sees that $L^{\sim}_{\mathbf{R}^n} \subset L'^{\#}_{\mathbf{R}^n}$. The equality $L^{\sim}_{\mathbf{R}^n} = L'^{\#}_{\mathbf{R}^n}$ holds when $L(x, D)$ has constant coefficients.

Let $L'^{*}_G\colon \mathscr{B}^{*}_{p',1/(kk^{\sim})^{\vee}}(G) \to \mathscr{B}^{*}_{p',1/k^{\vee}}(G)$ be the dual operator of $L'_G$. Then one sees easily

THEOREM 3.2. *Suppose that $p$ belongs to $]1, \infty]$. Then one has*

$$\text{(3.7)}\qquad L'^{\#}_G = J^{-1}_{p,k} \circ (L'^{*}_G) \circ J_{p,kk^{\sim}} = J^{-1}_{p,k} \circ (L'^{\sim *}_G) \circ J_{p,kk^{\sim}}. \quad \square$$

**3.2.** Let $x$ be in $G$. Choose $\varepsilon_x > 0$ such that the open ball $B(x, 2\varepsilon_x) \subset G$. In virtue of Lemma 3.1 the operator $L'_{B(x,\varepsilon_x)}\colon \mathscr{B}_{p',1/k^{\vee}}(B(x,\varepsilon_x)) \to \mathscr{B}_{p',1/(kk^{\sim})^{\vee}}(B(x,\varepsilon_x))$ is closable. Denote by $G_{p',1/k^{\vee}}$ the subset of $G$ defined by

$$G_{p',1/k^{\vee}} := \{x \in G \mid N(L'^{\sim}_{B(x,\varepsilon_x)}) \cap \mathscr{E}'(\{x\}) = \{0\}\}.$$

In the case when the principal part $\sum_{|\sigma|=r} a_\sigma(x)\xi^\sigma$ is different from zero for each $(x, \xi) \in G \times \mathbf{R}^n$, we know that $G_{p',1/k^{\vee}} = G$ (cf. [4], p. 469).

We have

THEOREM 3.3. *Suppose that there exist constants $C_1 > 0$ and $C_2 \geqq 0$ such that for all $\varphi \in C_0^\infty(G)$*

$$\text{(3.8)}\qquad \|L'(x,D)\varphi\|_{p',1/(kk^{\sim})^{\vee}} \geqq C_1\|\varphi\|_{p',1/k^{\vee}} - C_2\|\varphi\|_{p',1/(kk^{\sim})^{\vee}},$$

*where $p' \in [1, \infty]$, $k \in \mathscr{K}$ and $k^{\sim} \in \mathscr{K}$ such that $k^{\sim}(\xi) \to \infty$ with $|\xi| \to \infty$. Then $L'^{\sim}_{B(x,\varepsilon_x)}$, $x \in G$, is a semi-Fredholm operator with*

$$\text{(3.9)}\qquad \dim N(L'^{\sim}_{B(x,\varepsilon_x)}) < \infty. \quad \square$$

For the proof we refer to [7], p. 226, for example.

COROLLARY 3.4. *Suppose that* (3.8) *is valid for* $L'(x, D)$. *Then for each* $x \in G_{p',1/k^\vee}$ *there exist constants* $\delta \in ]0, \varepsilon_x]$ *and* $C > 0$ *such that*

$$\|u\|_{p',1/k^\vee} \leq C \|L'^{\sim}_{B(x,\delta)} u\|_{p',1/(kk^\sim)^\vee} \tag{3.10}$$

*for all* $u \in D(L'^{\sim}_{B(x,\delta)})$.

PROOF. Since $\dim N(L'^{\sim}_{B(x,\varepsilon_x)}) < \infty$ and since $x \in G_{p',1/k^\vee}$ one sees that there exists a constant $\delta \in ]0, \varepsilon_x]$ such that

$$N(L'^{\sim}_{B(x,\delta)}) = \{0\}. \tag{3.11}$$

Because the range $R(L'^{\sim}_{B(x,\delta)})$ is (by Theorem 3.3) closed, the Closed Graph Theorem implies the validity of (3.10). □

We can now show

THEOREM 3.5. *Suppose that* $p \in ]1, \infty]$ *and that* (3.8) *is valid for* $L'(x, D)$. *Then for each* $x \in G_{p',1/k^\vee}$ *there exists a constant* $\delta \in ]0, \varepsilon_x]$ *such that*

$$R(L'^{\#}_{B(x,\delta)}) = \mathbf{B}_{p,k}(B(x, \delta)). \tag{3.12}$$

PROOF. Let $\delta$ be as in Corollary 3.4. Then $L'^{\sim}_{B(x,\delta)}$ is (by (3.10)) correctly solvable. Thus one has

$$R(L'^{\sim *}_{B(x,\delta)}) = \mathscr{B}^*_{p',1/k^\vee}(B(x, \delta)), \tag{3.13}$$

and then (3.7) completes the proof. □

3.3. We now assume that $p = p' = 2$. Then the spaces $\mathscr{H}_k(G) := \mathscr{B}_{2,k}(G)$ and $H_k(G) := \mathbf{B}_{2,k}(G)$ are Hilbert spaces for each $k \in \mathscr{K}$. Hence we are able to show the following result:

THEOREM 3.6. *Suppose that there exist constants* $C_1 > 0$ *and* $C_2 \geq 0$ *such that for all* $\varphi \in C_0^\infty(G)$

$$\|L'(x, D)\varphi\|_{2,1/(kk^\sim)^\vee} \geq C_1 \|\varphi\|_{2,1/k^\vee} - C_2 \|\varphi\|_{2,1/(kk^\sim)^\vee}. \tag{3.14}$$

*Then for each* $x \in G_{2,1/k^\vee}$ *there exists a constant* $\delta \in ]0, \varepsilon_x]$ *and a continuous linear operator* $Q: H_k(B(x, \delta)) \to H_{kk^\sim}(B(x, \delta))$ *such that*

$$L'^{\#}_{B(x,\delta)} Qv = v, \quad \textit{for all} \quad v \in H_k(B(x, \delta)). \tag{3.15}$$

PROOF. Let $\delta \in ]0, \varepsilon_x]$ be as in Theorem 3.5. Since $N(L'^{\#}_{B(x,\delta)})$ is closed in $H_{kk^\sim}(B(x, \delta))$ there exists the orthogonal complement $N$ of $N(L'^{\#}_{B(x,\delta)})$. The linear operator $\mathscr{L} := L'^{\#}_{B(x,\delta)}|_N : N \cap D(L'^{\#}_{B(x,\delta)}) \to H_k(B(x, \delta))$ is closed, since $N$ is closed. Furthermore, one sees that $N(\mathscr{L}) = \{0\}$ and by Theorem 3.5, $R(\mathscr{L}) = H_k(B(x, \delta))$. The operator $Q := \mathscr{L}^{-1}$: $H_k(B(x, \delta)) \to N$ satisfies (3.15) and in view of the Closed Graph Theorem, $Q$ is continuous. □

A subset $G_{2,kk^\sim}$ is defined (as $G_{2,1/k^\vee}$) by

$$G_{2,kk^\sim} = \{x \in G \mid N(L^{\sim}_{B(x,\varepsilon_x)}) \cap \mathscr{E}'(\{x\}) = \{0\}\}.$$

Here $L_G$: $\mathscr{H}_{kk^\sim}(G)\to\mathscr{H}_k(G)$ is defined as $L'_G$ (cf. (3.3)) and $L_G^\sim$ is the smallest closed extension of $L_G$.

We show

THEOREM 3.7. *Suppose that there exist constants $C_1>0$ and $C_2\geqq 0$ such that (3.14) holds and that for all $\varphi\in C_0^\infty(G)$ the inequality*

$$\|L(x,D)\varphi\|_{2,k}\geqq C_1\|\varphi\|_{2,kk^\sim}-C_2\|\varphi\|_{2,k} \tag{3.16}$$

*holds. Then for each $x\in G_{2,1/k^\sim}\cap G_{2,kk^\sim}$ there exist a constant $\delta\in]0,\varepsilon_x]$ and a continuous linear operator*

$$E\colon \mathscr{H}_k(B(x,\delta))\to H_{kk^\sim}(B(x,\delta))$$

*such that*

$$E(L^\sim_{B(x,\delta)}u)=u|_{B(x,\delta)} \quad \textit{for all} \quad u\in D(L^\sim_{B(x,\delta)}) \tag{3.17}$$

*and*

$$L'^{\#}_{B(x,\delta)}(Ef)=f|_{B(x,\delta)} \quad \textit{for all} \quad f\in\mathscr{H}_k(B(x,\delta)). \tag{3.18}$$

PROOF. Let $\delta'\in]0,\varepsilon_x]$ such that the assertion of Theorem 3.6 holds (then the assertion of Theorem 3.6 holds for each $\delta\in]0,\delta']$, as well). For each $x\in G_{2,kk^\sim}$ we find a number $\delta\in]0,\delta']$ such that $R(L^\sim_{B(x,\delta)})$ is closed and that

$$\|u\|_{2,kk^\sim}\leqq C\|L^\sim_{B(x,\delta)}u\|_{2,k} \quad \text{for all} \quad u\in D(L^\sim_{B(x,\delta)}) \tag{3.19}$$

(cf. the proofs of Theorem 3.3 and Corollary 3.4).

Let $R$ be the orthogonal complement of $R(L^\sim_{B(x,\delta)})$ and let $\pi$: $\mathscr{H}_k(B(x,\delta))\to\to R(L^\sim_{B(x,\delta)})$ be the continuous projection. Since the restriction operator $r_k$: $\mathscr{H}_k(B(x,\delta))\to H_k(B(x,\delta))$ defined by

$$r_k u=u|_{B(x,\delta)}$$

is continuous, one sees that the operator

$$E\colon \mathscr{H}_k(B(x,\delta))\to H_{kk^\sim}(B(x,\delta))$$

defined by

$$Ef=r_{kk^\sim}(L^{\sim -1}_{B(x,\delta)}(\pi f))+Q(r_k((I-\pi)f)) \tag{3.20}$$

is continuous and satisfies (3.17)—(3.18). Here $Q$: $H_k(B(x,\delta))\to H_{kk^\sim}(B(x,\delta))$ is the operator as in the assertion of Theorem 3.6, and $I$ is the identical operator

$$\mathscr{H}_k(B(x,\delta))\to\mathscr{H}_k(B(x,\delta)).$$

Hence the proof is ready. □

## 4. On the regularity of the Schwarz kernel of $E$

**4.1.** In the previous chapter we gave a sufficient criterion under which for every $x\in G_{2,1/k^\sim}\cap G_{2,kk^\sim}$ there exist a number $\delta\in]0,\varepsilon_x]$ and a continuous linear operator $E$: $\mathscr{H}_k(B(x,\delta))\to H_{kk^\sim}(B(x,\delta))$ such that

$$E(L^\sim_{B(x,\delta)}u)=u|_{B(x,\delta)} \quad \text{for all} \quad u\in D(L^\sim_{B(x,\delta)}) \tag{4.1}$$

and

(4.2) $$L'^{*}_{B(x,\delta)}(Ef) = f|_{B(x,\delta)} \quad \text{for all} \quad f \in \mathscr{H}_k(B(x,\delta)).$$

We denote $U_x = B(x,\delta)$. Since the imbeddings $C_0^\infty(U_x) \to \mathscr{H}_k(U_x)$ and $H_{kk^\sim}(U_x) \to \mathscr{D}'(U_x)$ are continuous (here $C_0^\infty(U_x)$ is equipped with the standard inductive limit topology and $\mathscr{D}'(U_x)$ is equipped with the weak dual topology), one sees that the restriction $\bar{E}$ of $E$ on $C_0^\infty(U_x)$ is a continuous linear operator $C_0^\infty(U_x) \to \mathscr{D}'(U_x)$. In virtue of the Schwarz kernel theorem (cf. [8], p. 531) we find a distribution $K \in \mathscr{D}'(U_x \times U_x)$ such that

(4.3) $$(\bar{E}\psi)(\varphi) = K(\psi \otimes \varphi) \quad \text{for all} \quad \varphi, \psi \in C_0^\infty(U_x),$$

where $\psi \otimes \varphi \in C_0^\infty(U_x \times U_x)$ is defined by $(\psi \otimes \varphi)(x,y) = \psi(x)\varphi(y)$. In the sequel we shall study some regularity properties of the kernel $K$.

We assume further that $L(x,D)$ has $C^\infty(G)$-coefficients and that there exists constants $C_1 > 0$ and $C_2 \geqq 0$ such that for all $\varphi \in C_0^\infty(G)$ one has

(3.14) $$\|L'(x,D)\varphi\|_{1/(kk^\sim)^\vee} \geqq C_1 \|\varphi\|_{1/k^\vee} - C_2 \|\varphi\|_{1/(kk^\sim)^\vee}$$

and

(3.16) $$\|L(x,D)\varphi\|_k \geqq C_1 \|\varphi\|_{kk^\sim} - C_2 \|\varphi\|_k.$$

Here we denoted $\|\cdot\|_{2,k} = \|\cdot\|_k$ when $k \in \mathscr{K}$. Similarly we denote $|||\cdot|||_{2,k} = |||\cdot|||_k$. The open ball $B(x,\delta)$, where $\delta \in ]0, \varepsilon_x]$ is chosen so that the assertion of Theorem 3.6 holds, is denoted (as above) by $U_x$. Let $d_n$ be a positive number defined by

$$d_n = \inf\{d > 0 \,|\, \int_{\mathbf{R}^n} (1/(1+|\xi|^2))^d \, d\xi \quad \text{is finite}\}.$$

We show

THEOREM 4.1. *Assume that the inequalities* (3.14) *and* (3.16) *are valid with the weight functions $k$ and $k^\sim \in \mathscr{K}$ which satisfy*

(4.4) $$k_\gamma(\xi) := (1+|\xi|^2)^{\gamma/2} \leqq C(kk^\sim)(\xi) \quad \textit{for all} \quad \xi \in \mathbf{R}^n,$$

*where $\gamma > d_n + m$. Then there exists a mapping $q\colon U_x \to \mathscr{H}_k^*(U_x)$ such that for each $v \in \mathscr{H}_k(U_x)$ the function $z \to (q(z))(v)$ is in $C^m(U_x)$ and that*

(4.5) $$(Ev)(\varphi) = \int_{U_x} (q(z))(v)\varphi(z)\,dz \quad \textit{for all} \quad \varphi \in C_0^\infty(U_x),$$

*where $E$ is as in* (3.20). *Furthermore, one has*

(4.6) $$\sup_{z \in U_x} \|q(z)\| < \infty.$$

PROOF. A. For all $\varphi \in C_0^\infty$ and $|\alpha| \leqq m$ we get

$$(2\pi)^{-n} |(D^\alpha \varphi)(z)| = \Big| \int_{\mathbf{R}^n} (\mathscr{F}\varphi)(\xi)\xi^\alpha e^{i(\xi,z)}\,d\xi \Big| \leqq$$

$$\leqq \int_{\mathbf{R}^n} |k_m(\xi)(\mathscr{F}\varphi)(\xi)|\,d\xi \leqq \Big(\int_{\mathbf{R}^n} (1/k_{\gamma-m}(\xi))^2\,d\xi\Big)^{1/2} \|\varphi\|_{k_\gamma}$$

and then by (4.4)

$$\sup_{z\in\mathbf{R}^n} |(D^\alpha\varphi)(z)| \leqq C'\|\varphi\|_{kk^\sim} \tag{4.7}$$

with the suitable constant $C'>0$. Hence every $f\in\mathscr{H}_{kk^\sim}$ is in $C^m(\mathbf{R}^n)$ (more precisely, the density of $f$ is in $C^m(\mathbf{R}^n)$). The inequality (4.7) implies that for each $f\in\mathscr{H}_{kk^\sim}$ and for all $w\in\mathring{\mathscr{H}}_{kk^\sim}(\mathbf{R}^n\setminus U_x)$ one has

$$\sup_{z\in U_x} |(D^\alpha f)(z)| = \sup_{z\in U_x} \big|(D^\alpha(f+w))(z)\big| \leqq C'\|f+w\|_{kk^\sim} \tag{4.8}$$

and then

$$\sup_{z\in U_x} |(D^\alpha f)(z)| \leqq C'|||f|_{U_x}|||_{kk^\sim}. \tag{4.9}$$

Hence every $g\in H_{kk^\sim}(U_x)$ lies in $C^m(U_x)$ and

$$\sup_{z\in U_x} |(D^\alpha g)(z)| \leqq C'|||g|||_{kk^\sim} \quad \text{for all} \quad g\in H_{kk^\sim}(U_x). \tag{4.10}$$

B. Let $z$ be in $U_x$. Then the function $q_z\colon \mathscr{H}_k(U_x)\to\mathbf{C}$ defined by

$$q_z(v) = (Ev)(z)$$

is well-defined and in addition by (4.10)

$$|q_z(v)| = |(Ev)(z)| \leqq C'|||Ev|||_{kk^\sim} \leqq C'\|E\|\,\|v\|_k. \tag{4.11}$$

Hence $q_z$ lies in $\mathscr{H}_k^*(U_x)$. The function $q\colon U_x\to\mathscr{H}_k^*(U_x)$ defined by

$$q(z) = q_z$$

is well-defined and by (4.11)

$$\sup_{z\in U_x} \|q(z)\| \leqq C'\|E\| < \infty. \tag{4.12}$$

Since $\big(q(z)\big)(v)=(Ev)(z)$ and since $Ev\in C^m(U_x)$ one sees that the mapping $z\to$ $\to\big(q(z)\big)(v)$ is in $C^m(U_x)$. Finally, we see that

$$(Ev)(\varphi) = \int_{U_x}(Ev)(z)\varphi(z)\,dz = \int_{U_x}\big(q(z)\big)(v)\varphi(z)\,dz,$$

and so the proof is ready. □

**4.2.** Let $J_{1/k^\sim}:=J_{2,1/k^\sim}\colon H_{1/k^\sim}(U_x)\to\mathscr{H}_k^*(U_x)$ be the isometrical isomorphism given in Section 2. Define a mapping $e\colon U_x\to H_{1/k^\sim}(U_x)$ by

$$e(z) = J_{1/k^\sim}^{-1}\big(q(z)\big), \tag{4.13}$$

where $q\colon U_x\to\mathscr{H}_k^*(U_x)$ is as in Theorem 4.1.

LEMMA 4.2. *The function $e\colon U_x\to H_{1/k^\sim}(U_x)$ has the continuous partial derivatives $D^\alpha e$ up to the order $m$ (for the definition of partial derivatives in locally convex spaces cf.* [8], *p.* 285)).

PROOF. A. Let $z$ be in $U_x$. Then one sees that

$$|D^\alpha(Ev)(z)| \leqq C'|||Ev|||_{kk^\sim} \leqq C'\|E\|\,\|v\|_k \tag{4.14}$$

for all $v \in \mathscr{H}_k(U_x)$ and $|\alpha| \leqq m$ (cf. (4.10)). Thus by Theorem 2.4 there exists $e_{\alpha,z} \in H_{1/k^{\sim}}(U_x)$ such that

$$e_{\alpha,z}(\varphi) = \big(D^\alpha(E\varphi)\big)(z) \quad \text{for all} \quad \varphi \in C_0^\infty(U_x). \tag{4.15}$$

We show that $(D^\alpha e)(z) = e_{\alpha,z}$ and that the mapping $z \to e_{\alpha,z}$ is (well-defined) and continuous.

B. As in the proof of Theorem 4.1 we see that for all $z, y \in U_x$ and $\varphi \in C_0^\infty(U_x)$

$$\begin{aligned} \big|\big(e(z)-e(y)\big)(\varphi)\big| &= \big|\big(q(z)-q(y)\big)(\varphi)\big| = |(E\varphi)(z)-(E\varphi)(y)| \leqq \\ &\leqq \Big((2\pi)^{-n} \int_{\mathbf{R}^n} \big(|e^{i(z,\xi)}-e^{i(y,\xi)}|/k_\gamma(\xi)\big)^2\Big)^{1/2} |||E\varphi|||_{kk^{\sim}} \leqq \\ &\leqq \Big((2\pi)^{-n} \int_{\mathbf{R}^n} \big(|e^{i(z,\xi)}-e^{i(y,\xi)}|/k_\gamma(\xi)\big)^2\Big)^{1/2} \|E\| \, \|\varphi\|_k . \end{aligned} \tag{4.16}$$

Thus we obtain

(4.17)

$$|||e(z)-e(y)|||_{1/k^{\sim}} = \|q(z)-q(y)\| \leqq \Big((2\pi)^{-n} \int_{\mathbf{R}^n} \big(|e^{i(z,\xi)}-e^{i(y,\xi)}|/k_\gamma(\xi)\big)^2\Big)^{1/2} \|E\| .$$

According to the Lebesgue Dominated Convergence Theorem, the right-hand side of (4.17) is tending to zero with $y \to z$ (note that $\gamma > d_n + m \geqq d_n$). Hence $e$ is continuous.

Similarly one sees that

$$\begin{aligned} &|||-i\big((e(z+he_1)-e(z))/h\big)-e_{(1,0,\dots,0),z}|||_{1/k^{\sim}} \leqq \\ &\leqq \Big((2\pi)^{-n} \int_{\mathbf{R}^n} \big|\big(-i(e^{i(z+he_1,\xi)}-e^{i(z,\xi)})/h\big)-\xi_1 e^{i(z,\xi)}\big)\big|/k_\gamma(\xi)\big)^2\Big)^{1/2} \|E\|, \end{aligned} \tag{4.18}$$

(where $e_1 = (1, 0, \dots, 0)$) and then $D^{(1,0,\dots,0)}e$ exists and $(D^{(1,0,\dots,0)})e(z) = e_{(1,0,\dots,0),z}$. The continuity of $D^{(1,0,\dots,0)}e$ is seen as the continuity of $e$. In the same way we can show that $D^\alpha e$ exists and is continuous for each $|\alpha| \leqq m$. □

In addition we need

LEMMA 4.3. *Let* $e\colon U_x \to H_{1/k^{\sim}}(U_x)$ *be defined by* (4.13). *Then one has*

$$e(z)|_{U_x \setminus \{z\}} \in N(L^{\#}_{U_x \setminus \{z\}}) \quad \textit{for each} \quad z \in U_x, \tag{4.19}$$

*where* $L^{\#}_U$ *(for an open set* $U \subset U_x$*) is defined (as* $L'^{\#}_U$*) by*

$$\begin{cases} D(L^{\#}_U) = \{u \in H_{1/k^{\sim}}(U) \mid \; \text{there exists} \; f \in H_{1/(kk^{\sim})^{\sim}}(U) \; \text{such that} \\ \qquad\qquad u\big(L(x,D)\varphi\big) = f(\varphi) \; \text{for all} \; \varphi \in C_0^\infty(U)\}. \\ L^{\#}_U u = f. \end{cases} \tag{4.20}$$

PROOF. The distribution $e(z)|_{U_x \setminus \{z\}}$ is in $H_{1/k^{\sim}}(U_x \setminus \{z\})$. Furthermore we get for all $\psi \in C_0^\infty(U_x)$

$$e(z)\big(L(x,D)\psi\big) = \big(q(z)\big)\big(L(x,D)\psi\big) = \big(E(L(x,D)\psi)\big)(z) = \psi(z) = \delta_z(\psi),$$

where we used the relation (4.1) and the definition of $q$. $\delta_z$ is the Dirac measure at $z$. Hence for all $\varphi \in C_0^\infty(U_x \setminus \{z\})$

$$(e(z))(L(x, D)\varphi) = 0,$$

as desired. □

REMARK. The proof of Lemma 4.3 shows that the relation

$$L_{U_x}^{\#}(e(z)) = \delta_z \tag{4.21}$$

is valid.

**4.3.** We assume further that the assumptions of Theorem 4.1 are valid and that the mapping $e\colon U_x \to H_{1/k^\vee}(U_x)$ is defined by (4.13).

LEMMA 4.4. *Suppose that the inclusion*

$$N(L_U^{\#}) \subset \mathscr{H}_{k^\wedge}^{\mathrm{loc}}(U) := \mathscr{B}_{2,k^\wedge}^{\mathrm{loc}}(U) \tag{4.22}$$

*is valid, where $U$ is an open set of $U_x$ and $k^\wedge \in \mathscr{K}$. Then the function $\bar{e}\colon U_x \setminus \bar{U} \to \mathscr{H}_{k^\wedge}^{\mathrm{loc}}(U)$ defined by*

$$\bar{e}(z) = e(z)|_U \tag{4.23}$$

*has continuous partial derivatives up to the order $m$.*

PROOF. A. Since $L_U^{\#}$ is a closed operator, the kernel $N(L_U^{\#})$ is closed in $H_{1/k^\vee}(U)$. The imbedding $\imath\colon N(L_U^{\#}) \to \mathscr{H}_{k^\wedge}^{\mathrm{loc}}(U)$ is closed. Hence due to the Closed Graph Theorem $\imath$ is continuous, in other words, for each $\varphi \in C_0^\infty(U)$ one can find a constant $C > 0$ such that

$$\|\varphi u\|_{k^\wedge} \leqq C\,|||u|||_{1/k^\vee} \quad \text{for all} \quad u \in N(L_U^{\#}) \tag{4.24}$$

(cf. [10], p. 42 and note that the topology of $\mathscr{H}_{k^\wedge}^{\mathrm{loc}}(U)$ is defined by the semi-norms $u \to \|\varphi u\|_{k^\wedge}$, $\varphi \in C_0^\infty(U)$).

B. In virtue of (4.21) one sees that $\bar{e}(z) = e(z)|_U \in N(L_U^{\#})$ for each $z \in U_x \setminus \bar{U}$. Since the restriction mapping $R\colon H_{1/k^\vee}(U_x) \to H_{1/k^\vee}(U)$ defined by $Rv = v|_U$ is continuous and since

$$\bar{e}(z) = (\imath \circ R \circ e)(z) \quad \text{for all} \quad z \in U_x \setminus \bar{U}, \tag{4.25}$$

one sees by Lemma 4.2 that $\bar{e}$ has continuous partial derivatives up to the order $m$. □

Assume that $k^\wedge \in \mathscr{K}$ satisfies the inequality

$$k_\gamma(\xi) \leqq C k^\wedge(\xi) \quad \text{for all} \quad \xi \in \mathbf{R}^n, \tag{4.26}$$

where $\gamma > d_n + m$. Then for every $w \in \mathscr{H}_{k^\wedge}^{\mathrm{loc}}(U)$ there exists a function $f_w \in C^m(U)$ such that

$$w(\varphi) = \int_U f_w(y)\varphi(y)\,dy \quad \text{for all} \quad \varphi \in C_0^\infty(U).$$

Furthermore, the mapping $\lambda\colon \mathcal{H}^{\mathrm{loc}}_{k^{\sim}}(U)\to C^m(U)$ defined by

$$\lambda(w) = f_w$$

is continuous (for the definition of the topology in $C^m(U)$, cf. [8], p. 86). Hence the mapping $\tilde{e}\colon U_x\setminus\bar{U}\to C^m(U)$ defined by

$$\tilde{e} = \lambda\circ\bar{e}$$

has (by Lemma 4.4) continuous partial derivatives up to the order $m$.

Let $D$ be the diagonal of $U_x\times U_x$ (that is, $D=\{(z,y)\in U_x\times U_x | z=y\}$). We have

THEOREM 4.5. *Suppose that the assumptions of Theorem 4.1 are valid and that the inclusion*

$$N(L_U^{\#})\subset \mathcal{H}^{\mathrm{loc}}_{kk^{\sim}}(U) \tag{4.27}$$

*holds for each open set* $U\subset U_x$. *Then there exists a function* $h\in C^m(U_x\times U_x\setminus D)$ *such that*

$$K|_{U_x\times U_x\setminus D} = h. \tag{4.28}$$

PROOF. It suffices to show that for each $(z,y)\in U_x\times U_x\setminus D$ there exists a neighbourhood $V$ of $(z,y)$ such that $K|_V$ is a $C^m$-function. Suppose that $z\neq y$. Choose $\delta>0$ such that $B(z,\delta)\times B(y,\delta)\subset\subset U_x\times U_x\setminus D$ (that is, $B(z,\delta)\times B(y,\delta)$ is a relatively compact open subset of $U_x\times U_x\setminus D$). Then the mapping $\tilde{e}\colon B(z,\delta)\to C^m(U_x\setminus\overline{B(z,\delta)})$ is a $C^m$-function (as we above verified). Since $U_x\setminus\overline{B(z,\delta)}\supset B(y,\delta)$, one sees that also the mapping $\tilde{e}\colon B(z,\delta)\to C^m(B(y,\delta))$ is a $C^m$-function. Hence the function $h\colon B(z,\delta)\times B(y,\delta)\to \mathbf{C}$ defined by

$$h(t,s) = (\tilde{e}(t))(s)$$

is a $C^m$-function (cf. [8]). In addition, one sees that

$$\begin{aligned}\int\limits_{B(z,\delta)\times B(y,\delta)} h(t,s)\varphi(t)\psi(s)\,ds\,dt &= \int\limits_{B(z,\delta)} (\bar{e}(t))(\psi)\varphi(t)\,dt = \\ &= \int\limits_{B(z,\delta)} (q(t))(\psi)\varphi(t)\,dt = ((E\psi))(\varphi) = K(\psi\otimes\varphi).\end{aligned} \tag{4.29}$$

Since the linear hull of the subset $\{\psi\otimes\varphi | \psi\in C_0^\infty(B(z,\delta)),\ \varphi\in C_0^\infty(B(y,\delta))\}$ is dense in $C_0^\infty(B(z,\delta)\times B(y,\delta))$ we get the assertion from (4.29). □

REMARK. With the assumptions of Theorem 4.5 one sees that for each $z\in U_x$

$$(E\psi)(z) = \int\limits_{U_x\setminus\{z\}} h(z,y)\psi(y)\,dy \quad \text{for all} \quad \psi\in C_0^\infty(U_x\setminus\{z\}).$$

One sees immediately

COROLLARY 4.6. *Suppose that there exists a number* $m_0\in\mathbf{N}$ *such that*

1° *for each* $m\geqq m_0$ *there exist constants* $C_1>0$ *and* $C_2\geqq 0$ *such that*

$$\|L'(x,D)\varphi\|_{1/(k_m k^{\sim})^{\vee}} \geqq C_1\|\varphi\|_{1/k_m} - C_2\|\varphi\|_{1/(k_m k^{\sim})^{\vee}} \tag{4.30}$$

*and*

(4.31) $$\|L(x,D)\varphi\|_{k_m} \geqq C_1\|\varphi\|_{k_m k^\sim} - C_2\|\varphi\|_{k_m} \quad \textit{for all} \quad \varphi\in C_0^\infty(G),$$

*where* $k^\sim(\xi)\to\infty$ *when* $|\xi|\to\infty$,

2° *every solution* $u\in\mathcal{D}'(U)$ *of the distributional equation*

(4.32) $$L'(x,D)u = 0$$

*lies in* $C^\infty(U)$, *where* $U$ *is an open subset of* $G$,

3° $\sum_{|\sigma|=r} a_\sigma(x)\xi^\sigma \neq 0$ *for all* $(x,\xi)\in G\times\mathbf{R}^n$.

*Then for each* $x\in G$ *there exist a neighbourhood* $U_x\subset\subset G$ *and a continuous linear operator* $E\colon \mathcal{H}_{k_{m_0}}(U_x)\to\mathcal{H}_{k_{m_0}k^\sim}(U_x)$ *such that*

(4.33) $$E(\tilde{L}_{U_x}u) = u|_{U_x} \quad \textit{for all} \quad u\in D(\tilde{L}_{U_x})$$

*and*

(4.34) $$L'^{\#}_{U_x}(Ef) = f|_{U_x} \quad \textit{for all} \quad f\in\mathcal{H}_{k_{m_0}}(U_x).$$

*Here* $\tilde{L}_{U_x}\colon \mathcal{H}_{k_{m_0}k^\sim}(U_x)\to\mathcal{H}_{k_{m_0}}(U_x)$ *and* $L'^{\#}_{U_x}\colon H_{k_{m_0}k^\sim}(U_x)\to H_{k_{m_0}}(U_x)$ *are defined as above. Furthermore, the Schwarz kernel* $K$ *of* $E$ *satisfies*

$$K|_{U_x\times U_x\setminus D} = h,$$

*where* $h\colon U_x\times U_x\setminus D\to\mathbf{C}$ *is a* $C^\infty$*-function.* □

REMARK. A. The assumptions of Corollary 4.6 imply that one has for all

$$u\in\mathcal{H}_{k_{m_0}}(U_x)\cap\mathcal{E}'(U_x)$$

$$\operatorname{sing\,supp}\big(L(x,D)u\big) = \operatorname{sing\,supp} u$$

(cf. [6], p. 39).

B. Suppose that the assumptions of Theorem 4.1 are valid. Choose an open set $U$ of $U_x$ such that $\overline{U}$ is compact in $U_x$. Let $\Theta\in C_0^\infty(U_x)$ such that $\Theta(x)=1$ for all $x\in\overline{U}$. Then one has for all $\psi\in C_0^\infty(U)$

$$(E\psi)(z) = \big(q(z)\big)(\psi) = \big(e(z)\big)(\psi) = \big(\Theta e(z)\big)(\psi)$$

$$= (2\pi)^{-n}\int_{\mathbf{R}^n}\mathcal{F}\big(\Theta e(z)\big)(\xi)e^{-i(\xi,z)}(\mathcal{F}\psi)(\xi)e^{i(\xi,z)}\,d\xi,$$

where $E(z,\xi):=\mathcal{F}\big(\Theta e(z)\big)(\xi)e^{-i(\xi,z)}=\big(e(z)\big)(\Theta e^{-i(\xi,\cdot)})e^{-i(\xi,z)}=E(\Theta e^{-i(\xi,\cdot)})(z)e^{-i(\xi,z)}$ is a $C^m(U_x\times\mathbf{R}^n)$-function.

REFERENCES


[1] BEALS, R. and FEFFERMAN, C. L., Spatially inhomogenous pseudodifferential operators I, *Comm. Pure Appl. Math.* **27** (1974), 1—24, *MR* **50** #5234.

[2] HÖRMANDER, L., *Linear partial differential operators*, Die Grundlehren der mathematischen Wissenschaften, Bd. 116, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1963. *MR* **28** #4221.

[3] JACOB, N. and SCHOMBURG, B., On Gårding's inequality, *Aequationes Math.* **31** (1986), 7—17. *MR* **88e:** 35041.

[4] NIRENBERG, L. and TRÈVES, F., On local solvability of linear partial differential equations. II. Sufficient conditions, *Comm. Pure Appl. Math.* **23** (1970), 459—509. *MR* **41** #9064b.

[5] STUMMEL, F., *Rand- und Eigenwertaufgaben in Sobolewschen Räumen*, Lecture Notes in Mathematics, Vol. 102, Springer-Verlag, Berlin—New York, 1969. *MR* **57** #3845.

[6] TAYLOR, M. E., *Pseudodifferential operators*, Princeton Mathematical Series, 34, Princeton University Press, Princeton, N. J., 1981. *MR* **82i:** 35172.

[7] TERVO, J., On the properties of $k^{\sim}$-coercive linear partial differential operators, *Studia Sci. Math. Hungar.* **19** (1984), 221—229. *MR* **88a:** 35042.

[8] TRÈVES, F., *Topological vector spaces, distributions and kernels*, Academic Press, New York—London, 1967. *MR* **37** #726.

[9] WHITE, R. E., Interior regularity operators, *Anal. Math.* **7** (1981), 217—233. *MR* **83b:** 35029.

[10] YOSIDA, K., *Functional analysis*, Die Grundlehren der mathematischen Wissenschaften 123, Springer-Verlag, Berlin—Heidelberg—New York. Fourth edition, 1974. *MR* **50** # 2851; Fifth edition, 1978. *MR* **58** #17765.

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# РЕШЕТКИ ЭКВАЦИОНАЛЬНЫХ ТЕОРИЙ УНАРНЫХ АЛГЕБР

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Статья содержит развернутое изложение результатов из [1], полученных в 1982—83 г. В ней дается характеристика решеток эквациональных теорий унарных алгебр, проясняющая природу ряда известных фактов о таких решетках. В частности, доказывается следующая теорема: решетки эквациональных теорий унарных алгебр, имеющих постоянные термальные операции, — это (с точностью до изоморфизмов) все решетки конгруэнций моноидов с левыми нулями и только они. Из этой теоремы следует, что решетки эквациональных теорий алгебр, имеющих нульарные и разве лишь унарные операции, — это все решетки конгруэнций моноидов с левыми нулями и только они. Замечается, что всякая алгебраическая решетка представима как главный идеал решетки конгруэнций моноида с левыми нулями. Отсюда выводится, что всякая алгебраическая решетка представима как главный идеал решетки эквациональных теорий унарных алгебр. Отсюда же выводится, что всякая алгебраическая решетка представима как полный эндоморфный образ решетки конгруэнций моноида с левыми нулями. Первый из названных результатов был сообщен в 1983 г. на Международной алгебраической конференции в Сегеде.

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Пусть $\varkappa$ — некоторый кардинал. Будем рассматривать язык $L_\varkappa$ узкого исчисления предикатов с одноместными функциональными константами $f_\alpha(\alpha<\varkappa)$ и без индивидных и предикатных констант. Для всякого натурального $n>0$ будем обозначать через $T_n(I_n)$ множество всех термов (тождеств) языка $L_\varkappa$, в которые не входят иные переменные, кроме $x_0, \ldots, x_{n-1}$.

Замыканием в $I_n$ системы тождеств $\Sigma \subseteq I_n$ будем называть наименьшую из систем $\Sigma'$, включающих $\Sigma$ и удовлетворяющих условиям

(1) $$P = P \in \Sigma',$$

(2) $$P = Q \in \Sigma' \Rightarrow Q = P \in \Sigma',$$

(3) $$P = Q \in \Sigma' \wedge Q = R \in \Sigma' \Rightarrow P = R \in \Sigma',$$

(4) $$P = Q \in \Sigma' \Rightarrow R(P) = R(Q) \in \Sigma',$$

(5) $$P(x_i) = Q(x_j) \in \Sigma' \Rightarrow P(R_i) = Q(R_j) \in \Sigma'$$

для всяких $P, Q, R, R_i, R_j$ из $T_n$ $\big(S(T)$ обозначает терм, образуемый подстановкой терма $T$ вместо переменной в терм $S\big)$.

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Из известной теоремы Биркгофа (см. [4], § 64) следует, что замыкание в $I_n$ системы $\Sigma \subseteq I_n$ состоит из всех тождеств из $I_n$, являющихся следствиями системы $\Sigma$, и что эквациональные теории $\Sigma$ в $L_\varkappa$ взаимоопределимы с $\Sigma \cap I_2$. Поэтому всюду ниже мы будем рассматривать лишь тождества из $I_2$ и условимся замыкание в $I_2$ всякой системы $\Sigma \subseteq I_2$ называть эквациональной теорией, определяемой $\Sigma$, и обозначать через $CL(\Sigma)$.

Для всякой эквациональной теории $\Sigma$ будем обозначать через $\Lambda_\varkappa(\Sigma)$ или $\Lambda(\Sigma)$ решетку всех эквациональных теорий, включающих $\Sigma$. Решетку $\Lambda(\emptyset)$ будем обозначать через $\Lambda_\varkappa$ или $\Lambda$.

Замыкания (в $I_2$) систем регулярных тождеств, то есть тождеств вида $P(x_i)=Q(x_i)$ $(i \in \{0, 1\})$, будем называть регулярными эквациональными теориями. Для всякой регулярной эквациональной теории $\Sigma$ регулярные эквациональные теории, включающие её, образуют решетку, являющуюся главным идеалом $(CL(I_1)]$ решетки $\Lambda(\Sigma)$. Эту решетку будем обозначать через $\Lambda_\varkappa^R(\Sigma)$ или $\Lambda^R(\Sigma)$. Она изоморфна решетке замыканий в $I_1$ всевозможных систем тождеств из $I_1$, включающих $\Sigma \cap I_1$.

Пусть эквациональная теория содержит тождество $P(x_0)=Q(x_1)$. Тогда в силу (5), она содержит $P(x_1)=Q(x_1)$, а значит, и $P(x_0)=P(x_1)$.

Пусть $\Sigma_1$ и $\Sigma_2$ — нерегулярные эквациональные теории, $V_i(x_0)=V_i(x_1) \in \Sigma_i$ $(i=\{1, 2\})$. Тогда, в силу (5), нерегулярное тождество $V_1(V_2(x_0))=V_1(V_2(x_1))$ принадлежит $\Sigma_1$, а в силу (4) оно принадлежит $\Sigma_2$. Таким образом, нерегулярные эквациональные теории образуют подрешетку $\Lambda_\varkappa$ (этот факт приводится в [3]. Его доказательство, использующее суммы Плонки, дается в [5]). Эту подрешетку будем обозначать через $\Lambda_\varkappa^N$ или $\Lambda^N$.

Для всякого терма $P$ обозначим через $d(P)$ число вхождений в него функциональных констант. Для всякого тождества $\sigma\colon P=Q$ через $d(\sigma)$ будем обозначать $\min\{d(P), d(Q)\}$, а через $\bar{d}(\sigma)$ — $\max\{d(P), d(Q)\}$. Пусть правило вывода $\mathscr{F}$ таково, что для всякого кортежа тождеств $\langle \sigma_1, \ldots, \sigma_m \rangle$ тождество $\varphi$, являющееся результатом применения $\mathscr{F}$ к этому кортежу, удовлетворяет условию $d(\varphi) \geqq \min\{d(\sigma_1), \ldots, d(\sigma_m)\}$. Тогда будем говорить, что $\mathscr{F}$ монотонно.

(6) Пусть $\varkappa>0$, $\tau_0\colon P=Q$ — нетривиальное тождество и $V$-терм, для которого $d(V)>\bar{d}(\tau_0)$. Тогда тождество $\tau\colon V(P)=V(Q)$ таково, что $CL(\{\tau\})$ — собственная подсистема $CL(\{\tau_0\})$.

Действительно, пусть $CL^-(\{\tau\})$ — наименьшая из систем тождеств, содержащих $\tau$ и удовлетворяющих условиям (2)—(5). Так как правила вывода, выражаемые этими условиями, монотонны и $d(\tau)>\bar{d}(\tau_0)$, то $d(\varphi)>\bar{d}(\tau_0)$ для всякого $\varphi \in CL^-(\{\tau\})$. Следовательно, $\tau_0 \notin CL^-(\{\tau\})$, а так как рефлексивное замыкание $CL^-(\{\tau\})$ есть $CL(\{\tau\})$ и $\tau_0$ — нетривиальное тождество, то $\tau_0 \notin CL(\{\tau\})$.

Из (6) непосредственно следует

(7) Если $\varkappa>0$, то $\Lambda_\varkappa$ не имеет атомов (см. [6]).

Так как $\tau$ регулярно (не регулярно) в случае регулярности (нерегулярности) $\tau_0$, то из (6) следуют также

(8) Если $\varkappa>0$, то $\Lambda_\varkappa^R$ не имеет атомов (см. [3], предложение XV);

(9) Если $\varkappa>0$, то $\Lambda_\varkappa^N$ не имеет минимальных элементов (см. [3], теорема 1).

Легко видеть, что эти предложения справедливы и для эквациональных теорий алгебр, имеющих не нульарные сигнатурные операции.

Рассмотрим класс $\mathscr{M}$ всевозможных алгебраических систем $\mathbf{M}=\langle M;\ \circ, N, e, \omega\rangle$ таких, что $\circ$ — бинарная операция (на $M$), $N\subseteq M$, $\omega\in M$, и удовлетворяющих условиям

(a) $\langle M;\ \circ, e\rangle$ — моноид и $\langle N;\ \circ, e\rangle$ — его подмоноид,

(b) $\omega$ — левый ноль в $\langle M;\ \circ\rangle$,

(c) $N\setminus\{e\}$ — система образующих для $\mathbf{M}$ (и значит, всякий элемент $M$ либо принадлежит $N$ либо есть $P\circ\omega$ для некоторого $P\in N$),

(d) $P\circ\omega=Q\circ\omega\Rightarrow P=Q$ для всяких $P, Q\in N$.

Из этих условий следует, что $\omega\in N$ влечет одноэлементность $\mathbf{M}$ и что $P\circ\omega\in N$ влечет $P\circ\omega=P$ для всякого $P\in N$.

Всякая система $\langle M;\ \circ, N, e, \omega\rangle$, удовлетворяющая (a) и такая, что $\langle M;\ \circ\rangle$ образуется из $\langle N;\ \circ\rangle$ присоединением внешним образом левого нуля $\omega$, принадлежит $\mathscr{M}$. Класс всех таких систем будем обозначать через $\mathscr{N}$.

Конгруэнцию $E$ на $\mathscr{M}$-системе $\mathbf{M}$ будем называть $\mathscr{M}$-конгруэнцией, если $\mathbf{M}\setminus E\in\mathscr{M}$. Легко видеть, что $E$ в точности тогда есть $\mathscr{M}$-конгруэнция, когда она удовлетворяет условию

(10) $$\langle P\circ\omega,\ Q\circ\omega\rangle\in E\Rightarrow\langle P, Q\rangle\in E \quad \text{для всяких} \quad P, Q\in N.$$

Более того, для всякой $\mathscr{M}$-конгруэнции $E$ на $\mathbf{M}$ имеет место

(11) $$\langle P, Q\circ\omega\rangle\in E\Rightarrow\langle P\circ S_0,\ Q\circ S_1\rangle\in E \quad \text{для всяких} \quad P, Q\in N,\ S_0, S_1\in M.$$

Действительно, $\langle P, Q\circ\omega\rangle\in E$ влечет $\langle P\circ\omega, Q\circ\omega\rangle\in E$, а значит, $\langle P, Q\rangle\in E$ и $\langle Q\circ\omega, Q\rangle\in E$. Следовательно, для всяких $S_0, S_1\in M$ имеет место $\langle P\circ S_0, Q\circ\omega\rangle\in E$ и $\langle Q\circ\omega, Q\circ S_1\rangle\in E$, откуда $\langle P\circ S_0, Q\circ S_1\rangle\in E$.

$\mathscr{M}$-конгруенции на $\mathbf{M}\in\mathscr{M}$ образуют полную решетку, которую будем обозначать через $\mathrm{Con}_{\mathscr{M}}\,\mathbf{M}$.

Пусть $\mathbf{M}_\varkappa=\langle M_\varkappa;\ \circ, N_\varkappa, e, \omega\rangle$ — система из $\mathscr{M}$ такая, что $\mathbf{N}_\varkappa=\langle N_\varkappa;\ \circ, e\rangle$ — моноид со свободными образующими $\bar{f}_\alpha(\alpha<\varkappa)$, и значит, $\langle M_\varkappa;\ \circ\rangle$ образуется присоединением к $\langle N_\varkappa;\ \circ\rangle$ внешним образом левого нуля $\omega$. Обозначим через $\mathscr{T}$ функцию, определенную на $T_2$, со значениями в $M_\varkappa$, для которой

$$\mathscr{T}(f_k(\ldots(f_l(x_i))\ldots))=\begin{cases}\bar{f}_k\circ\ldots\circ\bar{f}_l, & \text{если} \quad i=0,\\ \bar{f}_k\circ\ldots\circ\bar{f}_l\circ\omega, & \text{если} \quad i=1.\end{cases}$$

Для всякого тождества $P=Q$ будем обозначать через $\mathscr{T}(P=Q)$ пару $\langle\mathscr{T}(P), \mathscr{T}(Q)\rangle$. Для всякой системы тождеств $\Sigma$ будем обозначать через $\mathscr{T}(\Sigma)$ множество $\{\mathscr{T}(\sigma)|\sigma\in\Sigma\}$.

Пусть $\Sigma$ — эквациональная теория, то есть $\Sigma$ удовлетворяет условиям (1)—(5). Тогда $\mathscr{T}(\Sigma)$ — конгруэнция на $\mathbf{M}_\varkappa$, удовлетворяющая (10), и значит, есть $\mathscr{M}$-конгруенция. Обратно, для всякой $\mathscr{M}$-конгруэнции $E$ на $\mathbf{M}_\varkappa$ множество $\mathscr{T}^{-1}(E)$ удовлетворяет (1)—(5) и, следовательно, есть эквациональная теория. Отсюда и из того, что $\mathscr{T}$ взаимно однозначно и изотонно вместе с обратным отображением, следует, что для всякой эквациональной теории $\Sigma$ решетка $\Lambda(\Sigma)$ изоморфна решетке всех $\mathscr{M}$-конгруэнций на $\mathbf{M}_\varkappa$, включающих $\mathscr{T}(\Sigma)$.

Так как последняя изоморфна $\mathrm{Con}_{\mathscr{M}}\, \mathbf{M}_\varkappa/\mathscr{T}(\Sigma)$, то $\Lambda(\Sigma)\cong \mathrm{Con}_{\mathscr{M}}\, \mathbf{M}_\varkappa/\mathscr{T}(\Sigma)$. В частности, $\Lambda\cong\mathrm{Con}_{\mathscr{M}}\, \mathbf{M}_\varkappa$. Таким образом, имеет место

Теорема 1. *Решетки эквациональных теорий в $L_\varkappa$—это все решетки $\mathrm{Con}_{\mathscr{M}}\, \mathbf{M}$ для $\mathscr{M}$-систем $\mathbf{M}$, имеющих $\leqq\varkappa$ образующих, и только они.*

Пусть $\Sigma$ — регулярная эквациональная теория. Тогда $\mathscr{T}(\Sigma)$ — конгруэнция на $\mathbf{M}_\varkappa$, натянутая на некоторую конгруэнцию на $\langle N_\varkappa;\circ\rangle$ и потому являющаяся $\mathscr{M}$-конгруэнцией. Очевидно и обратное: для всякой конгруэнции $E$ на $\langle N_\varkappa;\circ\rangle$ натянутая на неё конгруэнция $E'$ на $\mathbf{M}_\varkappa$ есть $\mathscr{M}$-конгруэнция такая, что $\mathscr{T}^{-1}(E')$ — регулярная эквациональная теория. $\langle M_\varkappa;\circ\rangle/E'$ образуется из $\langle N_\varkappa;\circ\rangle/E$ присоединением внешним образом левого нуля $\omega/E'$. Следовательно, справедлива

Теорема 2. *Для всякой регулярной эквациональной теории $\Sigma$ в $L_\varkappa$ существует система $\mathbf{M}\in\mathscr{N}$, имеющая $\leqq\varkappa$ образующих и такая, что $\Lambda_\varkappa(\Sigma)\cong\mathrm{Con}_{\mathscr{M}}\,\mathbf{M}$. Обратно, для всякой системы $\mathbf{M}\in\mathscr{N}$, имеющей $\leqq\varkappa$ образующих, существует регулярная эквациональная теория $\Sigma$ в $L_\varkappa$ такая, что $\mathrm{Con}_{\mathscr{M}}\,\mathbf{M}\cong\Lambda_\varkappa(\Sigma)$.*

Для всякого моноида $\mathbf{N}=\langle N;\circ,e\rangle$ будем обозначать через $\mathbf{N}^*$ моноид, образованный присоединением к $\mathbf{N}$ внешним образом левого нуля, $\omega$. Конгруэнцию $E$ на $\mathbf{N}^*$ будем называть $\mathscr{M}$-конгруэнцией, если она удовлетворяет (10). Из рассуждений, доказывающих теорему 2, следует

Теорема 2*. *Для всякой регулярной эквациональной теории $\Sigma$ в $L_\varkappa$ существует моноид $\mathbf{N}$, имеющий $\leqq\varkappa$ образующих и такой, что $\Lambda_\varkappa(\Sigma)$ изоморфна решетке $\mathrm{Con}_{\mathscr{M}}\,\mathbf{N}^*$ всех $\mathscr{M}$-конгруэнций на $\mathbf{N}^*$. Обратно, для всякого моноида $\mathbf{N}$, имеющего $\leqq\varkappa$ образующих, существует регулярная эквациональная теория $\Sigma$ в $L_\varkappa$ такая, что $\mathrm{Con}_{\mathscr{M}}\,\mathbf{N}^*\cong\Lambda_\varkappa(\Sigma)$. В частности, $\mathrm{Con}_{\mathscr{M}}\,\mathbf{M}_\varkappa\cong\Lambda_\varkappa$.*

Из теоремы 2* и из того, что решетка конгруэнций на $\mathbf{M}_\varkappa$, натянутых на конгруэнции на $\langle N_\varkappa;\circ\rangle$, изоморфна $\mathrm{Con}\,\langle N_\varkappa;\circ\rangle$, следует

Теорема 3 (А. И. Мальцев [7], § 13). *Для всякой регулярной эквациональной теории $\Sigma$ в $\Lambda_\varkappa$ решетка $\Lambda_\varkappa^R(\Sigma)$ представима как решетка конгруэнций моноида $\leqq\varkappa$ образующими. Обратно, для всякого моноида $\leqq\varkappa$ образующими решетка его конгруэнций представима как $\Lambda_\varkappa^R(\Sigma)$. В частности, $\mathrm{Con}\,\mathbf{N}_\varkappa\cong\Lambda_\varkappa^R$.*

Для всякой полугруппы $\mathbf{A}$ решетка $\mathrm{Con}\,\mathbf{A}$ представима как главный идеал $\mathrm{Con}\,\mathbf{A}^1$, где $\mathbf{A}^1$ — полугруппа, образованная присоединением к $\mathbf{A}$ единицы, $e$, а $\mathrm{Con}\,\mathbf{A}^1$ представима как главный идеал $\mathrm{Con}_{\mathscr{M}}\,\mathbf{M}$ для $\mathscr{M}$-системы $\mathbf{M}=\langle M;\circ,N,e,\omega\rangle$. Если $\mathbf{A}$ имеет $\leqq\varkappa$ образующих, то, согласно теореме 2, $\mathrm{Con}_{\mathscr{M}}\,\mathbf{M}\cong\Lambda_\varkappa(\Sigma)$ для некоторой регулярной эквациональной теории $\Sigma$. Отсюда следует, что для всякой полугруппы $\mathbf{A}$ с $\leqq\varkappa$ образующими $\mathrm{Con}\,\mathbf{A}$ представима как интервал $[\Sigma_1,\Sigma_2]$ в $\Lambda_\varkappa$ такой, что $\Sigma_1$ и $\Sigma_2$ — регулярные теории. Так что, в частности, справедлива следующая теорема Д. М. Смирнова [8]: для всякой полугруппы решетка её конгруэнций представима как главный идеал решетки эквациональных теорий унарных алгебр. Ниже эта теорема будет обобщена.

Заметим также, что из вложимости решетки $\mathrm{Part}(\omega)$ разбиений счетного множества в решетку конгруэнций полугруппы с двумя образующими и из вложимости последней в $\Lambda_2$ следует результат Ежека [9] и Барриса [10]: Part $(\omega)$ вложима в $\Lambda_2$.

Зафиксируем какое-нибудь нерегулярное тождество $\sigma$: $V(x_0)=V(x_1)$. $\sigma$-замыканием системы тождеств $\Sigma$ будем называть эквациональную теорию $CL_\sigma(\Sigma)=CL(\Sigma\cup\{\sigma\})$.

(12) Для всякого нерегулярного тождества $\tau$: $P(x_0)=Q(x_1)$ $\sigma$-замыкание системы $\{\tau\}$ совпадает с $\sigma$-замыканием системы, состоящей из тождеств $\tau_1$: $P(x_0)=P(V(x_0))$ и $\tau_2$: $P(x_0)=Q(x_0)$.

Действительно, пусть $\Sigma$ — какая-нибудь $\sigma$-замкнутая система. Если $\tau\in\Sigma$, то, в силу (5), $P(R)=Q(S)\in\Sigma$ для всяких $R, S$ из $T_2$, и значит, $\tau_2$ и $P(V(x_0))=Q(x_1)$ принадлежат $\Sigma$. Из $\tau\in\Sigma$ и $Q(x_1)=P(V(x_0))\in\Sigma$ следует $\tau_1\in\Sigma$. Обратно, пусть $\tau_1, \tau_2\in\Sigma$. Так как $\sigma\in\Sigma$, то, в силу (4), $P(V(x_0))=P(V(x_1))\in\Sigma$. Отсюда и из $\tau_1, \tau_2\in\Sigma$ следует, что тождества $Q(x_1)=P(x_1)=P(V(x_1))=P(V(x_0))=P(x_0)$ принадлежат $\Sigma$. Следовательно, $\tau\in\Sigma$.

Для всякой системы тождеств $\Sigma$ будем обозначать через $h(\Sigma)$ систему, образуемую из $\Sigma$ заменой всякого её тождества $P(x_1)=Q(x_1)$ на $P(x_0)=Q(x_0)$, а всякого тождества $P(x_0)=Q(x_1)$ или $P(x_1)=Q(x_0)$ — на пару тождеств $P(x_0)=P(V(x_0))$, $P(x_0)=Q(x_0)$. Из (12) следует

(13) $CL_\sigma(h(\Sigma))=CL_\sigma(\Sigma)$ для всякой системы тождеств $\Sigma$.

Пусть $\Sigma=h(\Sigma)$, то есть $\Sigma\subseteq I_1$. Через $\mathrm{cl}_\sigma(\Sigma)$ будем обозначать замыкание в $I_1$ системы $\Sigma$, пополненной всевозможными тождествами $V(x_0)=V(R(x_0))$. Это замыкание будем называть слабым $\sigma$-замыканием $\Sigma$.

Легко видеть, что $h(\Sigma)=\Sigma\cap I_1$ для всякой $\sigma$-замкнутой системы $\Sigma$. Следовательно, если $\Sigma$ $\sigma$-замкнута, то $h(\Sigma)$ слабо $\sigma$-замкнута. Иначе говоря, $h(CL_\sigma(\Sigma))=\mathrm{cl}_\sigma(h(CL_\sigma(\Sigma)))$ для всякой системы тождеств $\Sigma$. Более того, $h(CL_\sigma(\Sigma))=\mathrm{cl}_\sigma(h(\Sigma))$ для всякой системы $\Sigma$. Иначе говоря, имеет место

(14) $h(CL_\sigma(\Sigma))=\Sigma$ для всякой слабо $\sigma$-замкнутой системы $\Sigma$.

Действительно, пусть $\Sigma$ слабо $\sigma$-замкнута. Включение $\Sigma\subseteq h(CL_\sigma(\Sigma))$ очевидно. Всякое тождество из $CL_\sigma(\Sigma)$ выводится из системы $\Sigma^*$, образуемой добавлением к $\Sigma$ тождества $\sigma$ и всевозможных тождеств $P=P$, применением правил вывода, выражаемых условиями (2)—(5). Индукцией по длине вывода докажем, что всякое тождество из $h(CL_\sigma(\Sigma))$ принадлежит $\Sigma$.

Для всякого натурального $k$ через $\Sigma_k$ будем обозначать систему всех тождеств, выводимых из $\Sigma^*$ применением правил вывода (2)—(5) не более $k$ раз. Ясно, что $h(\Sigma_0)$, то есть $h(\Sigma^*)$, включается в $\Sigma$. Пусть $h(\Sigma_n)\subseteq\Sigma$. Покажем, что тогда $h(\Sigma_{n+1})\subseteq\Sigma$. Для этого покажем, что если тождество $\tau_0$: $P(x_i)=Q(x_j)$ принадлежит $\Sigma_{n+1}$, то тождество $\tau$: $P(x_0)=Q(S)$, где $S=x_0$ или $S=V(x_0)$ и $i\neq j$, принадлежит $\Sigma$.

1. Пусть $\tau_0$ образуется из принадлежащего $\Sigma_n$ тождества $\tau_1$ применением правила (2). Тогда $Q(x_0)=P(x_0)$ принадлежит $h(\Sigma_n)$, а значит, и $\Sigma$. Отсюда $P(x_0)=Q(x_0)\in\Sigma$. Если $i\neq j$, то и $Q(x_0)=Q(V(x_0))\in\Sigma$. Отсюда и из $P(x_0)=Q(x_0)\in\Sigma$ следует, в силу слабой $\sigma$-замкнутости $\Sigma$, $P(x_0)=Q(V(x_0))\in\Sigma$. Таким образом, в рассматриваемом случае $\tau\in\Sigma$.

2. Пусть $\tau_0$ образуется из принадлежащих $\Sigma_n$ тождеств $\tau_1$: $P(x_i)=R(x_k)$ и $\tau_2$: $R(x_k)=Q(x_j)$ применением правила (3). Так как тождества $P(x_0)=R(x_0)$ и $R(x_0)=Q(x_0)$ принадлежат $h(\Sigma_n)$, то они принадлежат и $\Sigma$. Но тогда $P(x_0)=Q(x_0)\in\Sigma$. Пусть $i\neq j$. Если при этом $k=i$, то $P(x_0)=R(x_0)$ и $R(x_0)=Q(V(x_0))$ принадлежат $\Sigma$. Если $k\neq i$, то $P(x_0)=R(V(x_0))$ и $R(V(x_0))=Q(V(x_0))$ принадлежат $\Sigma$. В обоих подслучаях $P(x_0)=Q(V(x_0))\in\Sigma$. Таким образом, в случае 2 $\tau\in\Sigma$.

3. Пусть $\tau_0$ образуется из принадлежащего $\Sigma_n$ тождества $\tau_1$: $T(x_i)=U(x_j)$ применением правила (4), а значит, $\tau_0$ есть $W(T(x_i))=W(U(x_j))$ для некоторого терма $W$. Из $\tau_1\in\Sigma_n$ следует $T(x_0)=U(x_0)\in\Sigma$. Но тогда, в силу слабой $\sigma$-замкнутости $\Sigma$, $W(T(x_0))=W(U(x_0))\in\Sigma$. Если $i\neq j$, то $T(x_0)=U(V(x_0))\in\Sigma$, откуда $W(T(x_0))=W(U(V(x_0)))\in\Sigma$. Таким образом, в рассматриваемом случае $\tau\in\Sigma$.

4. Пусть $\tau_0$ образуется из принадлежащего $\Sigma_n$ тождества $\tau_1$: $T(x_i)=U(x_j)$ применением правила (5), а значит, $\tau_0$ есть $T(R_i(x_{i_1}))=U(R_j(x_{j_1}))$ для некоторых термов $R_i, R_j$. Так как $T(x_0)=U(x_0)\in\Sigma$, то $T(R_i(x_0))=U(R_i(x_0))\in\Sigma$. Пусть $i_1\neq j_1$. Тогда $T(x_0)=U(V(x_0))\in\Sigma$. Отсюда и из $T(x_0)=U(x_0)\in\Sigma$ следует, что тождества $T(x_0)=T(V(x_0))=U(V(x_0))=U(x_0)$ принадлежат $\Sigma$. Но тогда, в силу слабой $\sigma$-замкнутости $\Sigma$, тождества $T(R_i(x_0))=T(V(R_i(x_0)))=T(V(x_0))=T(x_0)$ принадлежат $\Sigma$. В частности, $T(R_i(x_0))=T(x_0)\in\Sigma$. Из $U(x_0)=U(V(x_0))\in\Sigma$ следует, что тождества $U(R_j(V(x_0)))=U(V(R_j(V(x_0))))=U(V(x_0))=U(x_0)$ принадлежат $\Sigma$. Отсюда $T(R_i(x_0))=U(R_j(V(x_0)))\in\Sigma$ и $T(R_i(x_0))=U(R_j(x_0))\in\Sigma$, а значит, $\tau\in\Sigma$.

Для всякой эквациональной теории $\Sigma_0$ будем обозначать через $\Lambda^\sigma(\Sigma_0)$ решетку всех $\sigma$-замкнутых систем тождеств, включающих $\Sigma_0$, а через $\lambda^\sigma(\Sigma_0)$ решетку всех слабо $\sigma$-замкнутых систем тождеств, включающих $h(\Sigma_0)$. Отображение $h$ решетки $\Lambda^\sigma(\Sigma_0)$, переводящее всякую её систему $\Sigma$ в $h(\Sigma)$, изотонно. Так как $h(\Sigma)$ слабо $\sigma$-замкнута для всякой $\sigma$-замкнутой системы $\Sigma$, то $h$ отображает $\Lambda^\sigma(\Sigma_0)$ в $\lambda^\sigma(\Sigma_0)$. В силу (13) $h$ взаимно однозначно и $h^{-1}$ изотонно. Значит, $h$ — изоморфное вложение $\Lambda^\sigma(\Sigma_0)$ в $\lambda^\sigma(\Sigma_0)$. Отсюда и из (14) следует

$$\Lambda^\sigma(\Sigma_0)\cong\lambda^\sigma(\Sigma_0). \tag{15}$$

$\lambda^\sigma(\Sigma_0)$ изоморфна $\Lambda^R(CL(h(\Sigma_0)))$. Согласно теореме 3 последняя изоморфна решетке конгруэнций моноида $\mathbf{N}_\varkappa/\mathcal{T}(CL(h(\Sigma_0)))$, имеющего левые нули (таковым является $\mathcal{T}(V(x_0))$). Таким образом, $\Lambda^\sigma(\Sigma_0)$ изоморфна решетке конгруэнций моноида с левыми нулями, имеющего $\leqq\varkappa$ образующих.

Пусть $\mathbf{M}$ — моноид с левыми нулями, имеющий $\leqq\varkappa$ образующих. $\mathbf{M}\cong\mathbf{N}_\varkappa/E$ для подходящей конгруэнции $E$ такой, что для некоторого элемента $\overline{V}$ все пары $\langle\overline{V},\overline{V}\circ\overline{R}\rangle$ принадлежат $E$. Легко видеть, что $\mathcal{T}^{-1}(E)$ — слабо $\sigma$-замкнутая система тождеств, где $\sigma$ есть $V(x_0)=V(x_1)$, а $\mathcal{T}^{-1}(\overline{V})$ есть $V(x_0)$, и значит, $\operatorname{Con}\mathbf{M}\cong\lambda^\sigma(\mathcal{T}^{-1}(E))$. Отсюда и из (15) следует

Теорема 4. *Для всякой нерегулярной эквациональной теории $\Sigma$ в $L_\varkappa$ решетка $\Lambda_\varkappa(\Sigma)$ представима как решетка конгруенций моноида с левыми нулями, имеющего $\leqq\varkappa$ образующих. Обратно, для всякого моноида с левыми нулями,* *имеющего* $\leqq\varkappa$ *образующих, решетка его конгруэнций представима как* $\Lambda_\varkappa(\Sigma)$, *где* $\Sigma$ — *нерегулярная эквациональная теория.*

Из теоремы 4 очевидным образом выводится

Теорема 4*. *Для всякой эквациональной теории* $\Sigma$ *в языке* $L^*$, *относящемся к алгебрам, имеющим нульарные и разве лишь унарные операции, решетка* $L^*(\Sigma)$ *всех эквациональных теорий в* $L^*$, *включающих* $\Sigma$, *представима как решетка конгруэнций моноида с левыми нулями. Обратно, для всякого моноида с левыми нулями решетка его конгруэнций представима как* $L^*(\Sigma)$ *для некоторого языка* $L^*$, *относящегося к алгебрам, имеющим нульарные и разве лишь унарные операции.*

Из рассуждений, доказывающих теорему 4, легко выводится следующее предложение: преобразование $r$ решетки $\Lambda_\varkappa$, переводящее всякую эквациональную теорию $\Sigma$ в её регулярную часть $r(\Sigma)=CL(h(\Sigma))$, есть полный эндоморфизм. Действительно, равенство $r(\bigvee\Sigma_i)=\bigvee r(\Sigma_i)$ очевидно для регулярных $\Sigma_i$. Если же какое-нибудь из них не регулярно, то есть ему принадлежит некоторое тождество $\sigma: V(x_0)=V(x_1)$, то $\bigvee\Sigma_i=\bigvee CL_\sigma(\Sigma_i)$, а значит, $r(\bigvee\Sigma_i)=r(\bigvee CL_\sigma(\Sigma_i))$. В силу (15) $h(\bigvee CL_\sigma(\Sigma_i))=\bigvee h(CL_\sigma(\Sigma_i))=\bigvee h(\Sigma_i)$, а значит, $h(\bigvee\Sigma_i)=\bigvee h(\Sigma_i)$, откуда $r(\bigvee\Sigma_i)=\bigvee r(\Sigma_i)$. Равенство $r(\bigcap\Sigma_i)=\bigcap r(\Sigma_i)$ очевидно.

Пусть $\Sigma_1$ и $\Sigma_2$ — неравные нерегулярные эквациональные теории, $\sigma$: $V(x_0)=V(x_1)$ — тождество, принадлежащее $\Sigma_1\cap\Sigma_2$. Из (13) и из того, что $\Sigma_i=CL_\sigma(\Sigma_i)$ $(i\in\{1,2\})$, следует $h(\Sigma_1)\neq h(\Sigma_2)$, а значит, $r(\Sigma_1)\neq r(\Sigma_2)$. Таким образом, ограничение $r$ на $\Lambda_\varkappa^N$ взаимно однозначно. Следовательно, оно есть вложение $\Lambda_\varkappa^N$ в $\Lambda_\varkappa^r$. Отсюда и из (15) следует теорема 3 из [3]: ограничение $r$ на $\Lambda_\varkappa^N$ есть вложение, при котором всякий главный фильтр переходит в главный фильтр.

Из основного результата работы Лэмпа [11] следует, что всякая алгебраическая решетка представима как главный идеал решетки конгруэнций группоида. Более того, имеет место

Теорема 5. *Всякая алгебраическая решетка представима как главный идеал решетки конгруэнций моноида.*

Действительно, пусть $L$ — алгебраическая решетка, $\mathbf{A}=\langle A; F\rangle$ — унарная алгебра такая, что $L\cong\operatorname{Con}\mathbf{A}$ и $F$-клон. Рассмотрим группоид $\mathbf{M}=\langle A\cup F; \circ\rangle$ такой, что

a) элементы $A$ — левые нули $\mathbf{M}$;

b) для всяких $f\in F$ и $a\in A$ элемент $f\circ a$ есть $f(a)$ — результат операции $f$ над $a$ в $\mathbf{A}$;

c) для всяких $f, g$ из $F$ элемент $f\circ g$ есть композиция операций $f$ и $g$ в $\mathbf{A}$.

Очевидно, что $\mathbf{M}$ — полугруппа с единицей. Пусть $E$ — конгруэнция на $\mathbf{A}$. Тогда $E\cup\Delta_F$, где $\Delta_F=\{\langle f,f\rangle | f\in F\}$, есть конгруэнция на $\mathbf{M}$. Отсюда ясно, что $\operatorname{Con}\mathbf{A}$ изоморфна главному идеалу $(A^2\cup\Delta_F]$ решетки $\operatorname{Con}\mathbf{M}$.

Из теорем 4 и 5 следует, что всякая алгебраическая решетка представима как главный идеал $\Lambda_k(\Sigma)$ для некоторого $\varkappa$ и некоторой нерегулярной эквациональной теории $\Sigma$.

Заметим также, что с помощью теоремы 4 и конструкции, использованной в доказательстве теоремы 5, нетрудно, отправляясь от любой алгебраической решетки, имеющей пару компактных элементов с некомпактным пересечением, построить пару конечно базируемых эквациональных теорий унарных алгебр, пересечение которых не конечно базируемо (ср. [12]). Так, из существования пары компактных конгруэнций с некомпактным пересечением в полугруппе с двумя свободными образующими (таковы конгруэнция, натянутая на $\langle a, b\rangle$, и конгруэнция, натянутая на $\langle ba, a^2b\rangle$, где $a$ и $b$ — свободные образующие; этот пример принадлежит Д. И. Молдаванскому и Л. М. Шнеерсону) следует существование пары конечно базируемых (более того — однобазируемых) регулярных эквациональных теорий в $\Lambda_2$, пересечение которых не конечно базируемо.

Из рассуждений, доказывающих теорему 5, следует, что для всякой алгебраической решетки $L$ существует пара $\langle \mathbf{M}, \mathbf{A}\rangle$ такая, что $\mathbf{M}$ — моноид, $\mathbf{A}$ — подполугруппа $\mathbf{M}$ и $L$ изоморфна решетке тех конгруэнций на $\mathbf{A}$, которые индуцированы конгруэнциями на $\mathbf{M}$. Обыгрывая это обстоятельство, мы докажем следующую теорему:

Теорема 6. *Все алгебраические решетки (и только они) представимы как полные эндоморфные образы решеток конгруэнций моноидов (с левыми нулями).*

Пусть $L$ — алгебраическая решетка, $\mathbf{A}$ и $\mathbf{M}$ — те же, что и в доказательстве теоремы 5, $h$ — преобразование Con $\mathbf{M}$ такое, что $h(E)=(A^2\cap E)\cup \Delta_F$. Ясно, что $h(E)$ — конгруэнции на $\mathbf{M}$. Пусть $\{E_i | i\in I\}\subseteq \mathrm{Con}\,\mathbf{M}$. Очевидно, что $h(\bigcap E_i)=\bigcap h(E_i)$. Очевидно также, что $\bigvee h(E_i)\subseteq h(\bigvee E_i)$. Докажем обратное включение.

Пусть $\langle a, b\rangle\in h(\bigvee E_i)$. Тогда $\langle a, b\rangle\in \bigvee E_i$. Значит, существуют $g_0, g_1, \ldots, g_n$ такие, что $\langle a, g_0\rangle\in E_{i_0}$, $\langle g_0, g_1\rangle\in E_{i_1}, \ldots, \langle g_n, b\rangle\in E_{i_{n+1}}$ для некоторых $i_0, \ldots$ $\ldots, i_{n+1}\in I$. Но тогда для всякого $c\in A$ имеет место

$$\langle a\circ c, g_0\circ c\rangle\in E_{i_0},\ \langle g_0\circ c, g_1\circ c\rangle\in E_{i_1},\ \ldots,\ \langle g_n\circ c, b\circ c\rangle\in E_{i_{n+1}},$$

или:

$$\langle a, g_0\circ c\rangle\in E_{i_0},\ \langle g_0\circ c, g_1\circ c\rangle\in E_{i_1},\ \ldots,\ \langle g_n\circ c, b\rangle\in E_{i_{n+1}}.$$

Так как $g_0\circ c, \ldots, g_n\circ c$ принадлежат $A$, то

$$\langle a, g_0\circ c\rangle\in h(E_{i_0}),\ \langle g_0\circ c, g_1\circ c\rangle\in h(E_{i_1}),\ \ldots,\ \langle g_n\circ c, b\rangle\in h(E_{i_{n+1}}).$$

Отсюда $\langle a, b\rangle\in \bigvee h(E_i)$. Этим доказано, что $h$ — полный эндоморфизм решетки Con $\mathbf{M}$. Её $h$-образ есть её главный идеал, изоморфный Con $\mathbf{A}$, а значит, и $L$.

Из рассуждений, доказывающих теорему 5, нетрудно усмотреть, что для всякой конечной алгебры решетка её конгруэнций представима как главный идеал решетки конгруэнций конечного моноида. Отсюда и из рассуждений, доказывающих теорему 6, следует, что для всякой конечной алгебры решетка её конгруэнций представима как эндоморфный образ решетки конгруэнций конечного моноида. Естественен следующий вопрос: всякая ли конечная решетка представима как эндоморфный образ решетки конгруэнций конечного моноида? (Или, что то же, всякая ли конечная решетка представима как эндоморфный образ решетки конгруэнций конечной алгебры?) Положительное решение этого вопроса давало бы, как кажется, продуктивную характеристику конечных решеток. Из отрицательного решения следовало бы, что не всякая конечная решетка представима как решетка конгруэнций конечной алгебры.

В заключение — следующие открытые вопросы:

1. Существуют ли кардинал $\varkappa$ и регулярная (нерегулярная) эквациональная теория $\Sigma$ в $L_\varkappa$ такие, что $\Lambda_\varkappa(\Sigma)$ не изоморфна $\Lambda_\lambda(\Sigma')$ ни для каких кардинала $\lambda$ и нерегулярной (регулярной) эквациональной теории $\Sigma'$?

2. Существуют ли кардинал $\varkappa$ и регулярная эквациональная теория $\Sigma$ в $L_\varkappa$ такие, что $\Lambda_\varkappa(\Sigma)$ не представима как решетка конгруэнций полугруппы?

3. Существуют ли язык $L$, относящийся к универсальным алгебрам, и эквациональная теория $\Sigma$ в $L$ такие, что решетка всех включающих $\Sigma$ эквациональных теорий в $L$ не представима как решетка эквациональных теорий унарных алгебр?

С этим вопросом связана следующая задача:

4. Найти прозрачную абстрактную характеристику решеток эквациональных теорий унарных алгебр.

Вероятно, эта задача не менее трудна, чем задача абстрактной характеризации решеток эквациональных теорий универсальных алгебр. В [13] содержатся новые важные результаты, связанные с последней задачей. (Пользуюсь случаем сообщить, что мне не удалось восстановить доказательства заявленного мною результата о справедливости обращения леммы Маккензи, обсуждаемой в [13]).

Как показывает теорема 1, задача 4 близка к задаче абстрактной характеризации решеток конгруэнций полугрупп. Весьма вероятно, что $\Lambda_\varkappa^R$ определимы как подрешетки $\Lambda_\varkappa$ хорошо обозримым абстрактным свойством. Если это так, то задача 4 не менее трудна, чем задача абстрактной характеризации решеток конгруэнций моноидов.

## ЛИТЕРАТУРА

[1] Когаловский, С. Р., О решетках многообразий унарных алгебр, Депонировано в ВИНИТИ 23. VII. 1984 г., № 5280—84 Деп.

[2] Скорняков, Л. А., Шрайеровы и Холловы многообразия унарных алгебр, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **1983**, № 3, 24—28. *MR* **84g**: 08017.

[3] Płonka, J., On the lattice of varieties of unary algebras, *Simon Stevin* **59** (1985), No. 4, 353—364. *MR* **87k**: 08007.

[4] Grätzer, G., *Universal algebra*, 2nd edition, Springer-Verlag, New York—Heidelberg, 1979. *MR* **80g**: 08001.

[5] Płonka, J., On the sum of a direct system of universal algebras with nullary polynomials, *Algebra Universalis* **19** (1984), 197—207. *MR* **86b**: 08001.

[6] Tarski, A., Equational logic and equational theories of algebras, *Contributions to Math. Logic* (Colloquium, Hannover, 1966), H. A. Schmidt, K. Schutte, H. J. Thiele, eds., North-Holland, Amsterdam, 1968, 275—288. *MR* **38** # 5692.

[7] Мальцев, А. И., *Алгебраические системы (Algebraic systems)*, Наука, Москва, 1970 (in Russian). *MR* **44** # 142.

[8] Смирнов, Д. М., О регулярно определимых многообразиях алгебр, *Algebra i Logika* **15** (1976), № 3, 331—342. *MR* **55** # 12602.

[9] Ježek, J., On atoms in lattices of primitive classes, *Comment. Math. Univ. Carolinae* **11** (1970), 515—532. *MR* **42** # 4466.

[10] BURRIS, S., A note on varieties of unary algebras, *Colloq. Math.* **22** (1971), 195—196. *MR* **43** # 6147.
[11] LAMPE, W. A., Congruence lattices of algebras of fixed similarity type, II, *Pacific J. Math*, **103** (1982), No. 2, 475—508. *MR* **85c:** 08004.
[12] PIGOZZI, D., Amalgamation, congruence-extension, and interpolation properties in cylindric algebras, Ph. D. Thesis, Berkeley, 1970.
[13] LAMPE, W. A., A property of the lattice of equational theories, *Algebra Universalis* **23** (1986), 61—69. *MR* **88a:** 08014.

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ИВАНОВСКИЙ СЕЛЬСКОХОЗЯЙСТВЕННЫЙ ИНСТИТУТ
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# ON A PROBLEM CONCERNING ORTHOGONALITY IN NORMED LINEAR SPACES

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## Introduction

In [1], the following orthogonality concept in the real normed linear space $(X, \|\cdot\|)$ was introduced.

We call the vectors $x, y \in X$ orthogonal in the following cases.

Case I. $$\|x\|\cdot\|y\| = 0.$$

Case II. $$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \sqrt{2}.$$

We denote the orthogonality of $x$ and $y$ by $x \perp y$. In inner product spaces, this trivially coincides with the usual orthogonality concept.

We say that the above orthogonality relation is additive, if $x \perp y$ and $x \perp z$ imply $x \perp (y+z)$.

In [1], the following open problem is mentioned: Does additivity of the above-defined orthogonality imply that the space is an inner product space?

Clearly, this is not the situation in the case of two-dimensional spaces.

In this paper, we solve the problem affirmatively in the case $\dim X \geqq 3$.

## The result

THEOREM. *Let $(X, \|\cdot\|)$ be a real normed linear space with* $\dim X \geqq 3$. *Let us assume that the orthogonality relation is additive. Then $(X, \|\cdot\|)$ is an inner product space.*

PROOF. Let $(Y, \|\cdot\|)$ an arbitrary three-dimensional subspace of $(X, \|\cdot\|)$. We prove that $(Y, \|\cdot\|)$ is an inner product space.

First, we need some lemmas.

LEMMA 1. *Let $y \in Y$, $y \neq 0$. Then the set*

$$y^{\perp} \equiv \{x \in Y;\ x \perp y\}$$

*is a two-dimensional subspace of $Y$.*

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PROOF of Lemma 1. [1], 3.07 says that $x\perp y$ implies $x\perp \alpha y$ for arbitrary $\alpha\in\mathbf{R}$. So, $x_1, x_2\in y^\perp$ yields that $\alpha x_1+\beta x_2\in y^\perp$ for all $\alpha, \beta\in\mathbf{R}$, hence $y^\perp$ is a subspace of $Y$. Because $y\perp y$ is absurd, $y^\perp\neq Y$. On the other hand, elementary continuity reasoning shows that $\dim y^\perp\geqq 2$. Lemma 1 is proved.

LEMMA 2. *Let* $y\in Y$, $\|y\|=1$, $S=\{x\in Y;\ \|x\|=1\}$. *Then for all* $x\in y^\perp$, $\alpha\in\mathbf{R}$ *we have* $\|y+\alpha x\|\geqq 1$.

PROOF of Lemma 2. Without loss of generality, we can assume that $\|x\|=1$. Clearly, there exists $x^*\in y^\perp$, $\|x^*\|=1$ such that

$$\left\{\begin{array}{l}\|x^*+\alpha x\|\geqq 1\\ \|-x^*+\alpha x\|\geqq 1\end{array}\right.\quad\text{for all}\quad \alpha\in\mathbf{R}. \tag{1}$$

Let $S'=S\cap y^\perp$. If $z\in S'$, then $\|y-z\|=\sqrt{2}$. This implies that

$$\left\{\frac{-y+z}{\sqrt{2}};\ z\in S'\right\}\subset S,$$

where the left-hand side is a homothetic image of $S'$. Because of this, (1), and $x^*\in S'$, $x\in S'$

$$\left\|\frac{x^*+y}{\sqrt{2}}+\alpha x\right\|\geqq 1\quad\text{for all}\quad \alpha\in\mathbf{R}. \tag{2}$$

Clearly, $x$ is an element of the subspace generated by $x$ and $\dfrac{x^*+y}{\sqrt{2}}$. Now, applying the same argument again, an easy computation shows that

$$\left(\frac{x^*+y}{\sqrt{2}}\right)^\perp\ni\frac{x^*-y}{\sqrt{2}}. \tag{3}$$

Let

$$S''=S\cap\left(\frac{x^*+y}{\sqrt{2}}\right)^\perp.$$

For all $z\in S''$,

$$\left\|\frac{x^*+y}{\sqrt{2}}-z\right\|=\sqrt{2}.$$

This implies that

$$\left\{\frac{\left(\frac{x^*+y}{\sqrt{2}}\right)-z}{\sqrt{2}};\ z\in S''\right\}\subset S.$$

Because of this and (3),

$$\left\|\frac{\frac{x^*+y}{\sqrt{2}}-\frac{x^*-y}{\sqrt{2}}}{\sqrt{2}}+\alpha x\right\|\geqq 1\quad\text{for all}\quad \alpha\in\mathbf{R}.$$

So, $\|y+\alpha x\|\geqq 1$ for all $\alpha\in\mathbf{R}$. Lemma 2 is proved.

LEMMA 3. *Let* $M \subset Y$ *a two-dimensional subspace. Then there exists a* $y \in Y$, $\|y\|=1$ *such that* $y^{\perp}=M$.

PROOF of Lemma 3. Let $y_1, y_2 \in M$ be linearly independent. One can easily check that $L=y_1^{\perp} \cap y_2^{\perp}$ is a one-dimensional subspace. Let $y \in L$, $\|y\|=1$. Clearly, $y \perp y_1$, $y \perp y_2$, and so, $y \perp \alpha y_1+\beta y_2$ for all $\alpha, \beta \in \mathbf{R}$.

Using Lemma 1, Lemma 3 follows.

LEMMA 4. *Let* $M \subset Y$ *be a two-dimensional subspace. Then there exists a linear projection* $P: Y \to M$ *such that* $\|P\|=1$.

PROOF of Lemma 4. Let $\|y\|=1$ and $y^{\perp}=M$. (Here we have used Lemma 3.) Using Lemma 2, we have for all $m \in M$, $\alpha, \beta, \in \mathbf{R}$

$$\|\alpha m+\beta y\| \geqq |\alpha|\|m\|. \tag{4}$$

We define

$$P(\alpha m+\beta y) = \alpha m.$$

(4) easily implies that $\|P\|=1$. Lemma 4 is proved.

Applying now the Kakutani characterization of the inner product [2 p. 157], the Theorem follows.

## REFERENCES

[1] DIMINNIE, C. R., ANDALAFTE, E. Z. and FREESE, R. W., Angles in normed linear spaces and a characterization of real inner product spaces, *Math. Nachr.* **129** (1986), 197—204. *MR* **88h:** 46021.

[2] DAY, M. M., *Normed linear spaces*, Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 21, Springer-Verlag, New York—Heidelberg, 1973. *MR* **49** #9588.

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# STRONGNESS IN $J$-LATTICES

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**Abstract**

The investigations of strongness in lattices of finite length in [3], [12], [13] shall be continued for $J$-lattices of arbitrary length. In such lattices we investigate geometric exchange properties, the basis exchange property, join symmetry in the sense of Gaskill—Rival [4] and further properties as well as strong and strict elements.

## 1. Introduction

In [3] Faigle—Richter—Stern investigated some kinds of exchange properties and showed the equivalence of them with strongness in semimodular lattices of finite length. In [13] we extended in lattices of finite length the notion of a strong join-irreducible element to arbitrary lattice elements in two ways using two unary operations and we got the notion of a strong element and the notion of a strict element. Since lattices of finite length are special algebraic $J$-lattices we shall continue this investigations for $J$-lattices of arbitrary length. Some of the obtained results are improvements of unpublished results of [9].

In Section 2 we give some basic notions.

Further we investigate strongness and semimodularity in arbitrary $J$-lattices in Section 3 and strong semimodular $J$-lattices in Section 4. In Section 5 we restrict the investigated class of lattices to algebraic strong semimodular $J$-lattices. As a main result in this section we give a generalization of the Theorem of Kuroš-Ore for infinite join representations. In the foreground of Section 6 we are engaged in join symmetry and basis exchange.

## 2. Basic notions

Let $L$ be a complete lattice. An element $v \in L$ is called *join-irreducible,* if the implication

$$(*) \qquad T \subseteq L \quad \text{and} \quad v = \bigvee T \quad \text{imply} \quad v \in T$$

is satisfied for each finite subset $T$ of $L$. $v$ is called completely join-irreducible, if $(*)$ is satisfied for each subset $T$ of $L$.

Let $J = J(L)$ be the set of all completely join-irreducible elements of $L$. If an element $b \in L$ has a join representation $b = \bigvee U$ with $U \subseteq J$ we say $b$ has a *decomposition*. A decomposition $b = \bigvee U$ is *irredundant* if $b > \bigvee (U \setminus \{u\})$ holds for each $u \in U$.

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$L$ is a *$J$-lattice* if each element of $L$ has a decomposition. If $a \leqq x \leqq b$ $(a, b, x \in L)$ implies either $a=x$ or $b=x$ we say *$a$ is covered by $b$* or *$b$ covers $a$* and we write $a \prec b$. In a complete lattice each completely join-irreducible element $v$ covers exactly one element. This element will be denoted by $v'$.

An element $c \in L$ is called *compact* if $T \subseteqq L$ and $c \leqq \bigvee T$ imply $c \leqq \bigvee T'$ for any finite subset $T'$ of $T$.

Let $K=K(L)$ be the set of all compact elements of $L$. $L$ is an *algebraic* or a *compactly generated lattice* if each element $a \in L$ has a join representation $a=\bigvee T$ with $T \subseteqq K$.

An element $q \in L$ is called *precompact* (or *inaccessible from below*, cf. Birkhoff—Frink [1]) if $T \subseteqq L$ and $q=\bigvee T$ imply $q=\bigvee T'$ for any finite subset $T'$ of $T$.

Let $Q$ be the set of all precompact elements of $L$. $L$ is a *prealgebraic lattice* if each $a \in L$ has a join representation $a=\bigvee T$ with $T \subseteqq Q$.

It is obvious that $J \subseteqq Q$ and $K \subseteqq Q$ hold in each prealgebraic lattice. In an algebraic lattice $J \subseteqq K=Q$ holds.

For lattice elements $x$ and $y$ we define the *interval* $y/x$ to be the set of all $z \in L$ such that $x \leqq z \leqq y$ holds. If $y/x=\emptyset$ and $x/y=\emptyset$ hold then $x$ and $y$ are said to be *incomparable*.

For $E \in \{J, K, Q\}$ we denote by $E(y/x)$ the set of completely join-irreducible, compact and precompact elements, respectively, of the interval $y/x$.

A lattice $L$ is (upper) *semimodular* if, for $a, b \in L$, $a \wedge b \prec a$ implies $b \prec a \vee b$.

A $J$-lattice $L$ has

a) the derivation property (D) if $v \in J$, $T \subseteqq J$ and $v \leqq \bigvee T$ imply $v' \leqq \bigvee (t' : t \in T)$,

b) the derivation property (DO) if $a \in L$, $v \in J$ and $v \nleqq a$ imply $a \vee v' < a \vee v$,

c) the derivation property (D1) if $a \in L$, $v \in J$ and $v \nleqq a$ imply $a \vee v' \prec a \vee v$,

d) the exchange property (E1) if $a \in L$, $u, v \in J$, $v \leqq a \vee u$ and $v \nleqq a \vee u'$ imply $u \leqq a \vee u' \vee v$,

e) the exchange property (E2) if $a \in L$, $u, v \in J$, $v \leqq a \vee u$ and $v \nleqq a \vee u'$ imply $u \leqq a \vee v$,

f) the exchange property (E3) if $a \in L$, $u, v \in J$, $v \leqq a \vee u$ and $v \nleqq a \vee u'$ imply $r \leqq a \vee v$ and $u \nleqq a \vee v'$,

g) the hereditary property (HJ) if $a \in L$ and $v \in J$ imply $a \vee v \in J(a \vee v/a)$.

(E1), (E2) and (E3) are called geometric exchange properties (cf. [3]). It is obvious that a lattice with property (E$i$) also has property (E$i-1$) $(i=3, 2)$ and a lattice with property (D1) also has property (D0).

A $J$-lattice $L$ is *strong* if it has (D0).

For $a \in L$ let $U_a := \{u : u \in L, u \prec a\}$.

If $U_a = \emptyset$ then we define $a_+ := a$ otherwise let $a_+ := \bigwedge U_a$. Further let $a' := \bigvee (v' : v \in J(a/0))$ for each element $a$ of a $J$-lattice $L$.

An element $a$ of a $J$-lattice $L$ is *strong* if $x \in L$ and $a \leqq x \vee a'$ imply $a \leqq x$.

An element $a$ of a $J$-lattice $L$ is *strict* if $x \in L$ and $a \leqq x \vee a_+$ imply $a \leqq x$.

A *minimal pair* $(p, A)$ of $L$ is an antichain $A \subseteqq J$ together with an element $p \in J$ such that the following three conditions are fulfilled:

(Mp1) $p \notin A$

(Mp2) $p \leqq \bigvee A$

(Mp3) $p \nleqq \bigvee \tilde{A}$ for every antichain $\tilde{A} < A$.

$\tilde{A} \leqq A$ holds if to each $\tilde{a} \in \tilde{A}$ there exists an $a \in A$ such that $\tilde{a} \leqq a$ holds.

The lattice $L$ is *join symmetric* if it satisfies

(Js) $(p, A)$ minimal pair of $L$ and $q \in A$ imply that $(q, (A \setminus \{q\}) \cup \{p\})$ is a minimal pair of $L$.

An antichain $A \subseteqq J$ is called a *basis* of $x \in L$ if the following two conditions are satisfied:

(B1) $x = \bigvee A$

(B2) $x \neq \bigvee \tilde{A}$ for every antichain $\tilde{A} < A$.

A $J$-lattice $L$ has the *basis exchange property* if it satisfies

(Be) $x \in L$, $B_1$ and $B_2$ bases of $x$ and $b_1 \in B_1$ imply the existence of an element $b_2 \in B_2$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\}$ is a basis of $x$.

### 3. Strongness and semimodularity in arbitrary $J$- lattices

The first definition of strongness of a lattice was given by Faigle [2] by a property

(St′) $\quad p, q \in J,\ q < p,\ x \in L$ and $p \leqq q \vee x$ imply $p \leqq x$.

In lattices of finite length properties (D0), (St′) and

(St) $\quad p \in J,\ x \in L$ and $p \leqq x \vee p'$ imply $p \leqq x$

(given in Faigle—Richter—Stern [3]) are equivalent. It is obvious that (St) and (D0) are equivalent for arbitrary $J$-lattices. But there are $J$-lattices which have (St′) but not (D0) as is shown in Figure 1 (cf. also Richter [9], Figure 9.4).

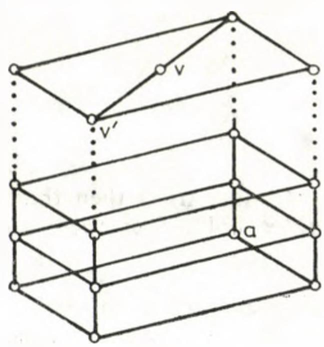


*Fig. 1*

Conversely, the following lemma holds.

LEMMA 1 (Richter [9], Satz 9.5). *Every strong lattice has property* (St′).

PROOF. Let $p, q \in J$, $q < p$, $x \in L$ and $p \leqq x \vee q$. Since $q < p$ we have $q \leqq p'$, i.e. $p \leqq x \vee q \leqq x \vee p' \leqq x \vee p$ and, therefore, $x \vee p' = x \vee p$ holds. If $L$ is strong, i.e. if $L$ has (D0), this is possible only in the case that $p \leqq x$ holds.

By the definitions of strong lattices, strong elements and strict elements one can simply verify the subsequent

LEMMA 2. *For a J-lattice L the following conditions are pairwise equivalent:*

(i) *L is strong.*
(ii) *Each element of $J(L)$ is strong.*
(iii) *Each element of $J(L)$ is strict.*

In the following we want to characterize strongness and strictness, respectively, of an element of a $J$-lattice by forbidden join-subsemilattices.

THEOREM 3. *Let L be a J-lattice. An element $a\in L$, $a\neq 0$, is strong (strict) iff $a\neq a'$ $(a\neq a_+)$ holds and L does not contain a join-subsemilattice of the form of Figure 2 or of Figure 3.*

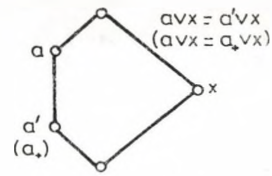


*Fig. 2*

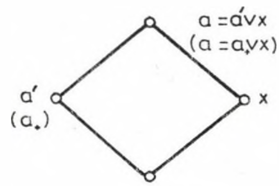


*Fig. 3*

PROOF. If $a\in L$ is not strong (strict) then there exists an $x\in L$ such that $a\leqq x\vee a'$ $(a\leqq x\vee a_+)$ and $a\not\leqq x$ hold. Then there are two possibilities:

(1) $a=a'$ $(a=a_+)$ or

(2) $a'<a$ $(a_+<a)$ and $a\vee x=a'\vee x$ $(a\vee x=a_+\vee x)$.

In the second case we have $x\not\leqq a'$ $(x\not\leqq a_+)$ since $x\leqq a'$ $(x\leqq a_+)$ and $a'<a$ $(a_+<a)$ yield $x\vee a'<a$ $(x\vee a_+<a)$ contradicting $a'\vee x=a\vee x$ $(a_+\vee x=a\vee x)$. Therefore, $x<a$ yields a join-subsemilattice of the form of Figure 3 and $x\not\leqq a$ yields a join-subsemilattice of the form of Figure 2.

If $L$ contains a join-subsemilattice of the form of Figure 2 or of Figure 3 then there is an $x\in L$ with $a\leqq x\vee a'$ $(a\leqq x\vee a_+)$ but $a\not\leqq x$ what means that $a\in L$ is not strong (strict).

REMARK. If $a\in L$ is a precompact element then there is no element $x\in L$ such that there exists a join-subsemilattice of the form

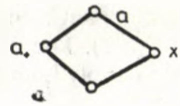


*Fig. 3a*

Since in the interval $a/x$ there would exist an element $c \in L$ such that $x \leqq c \prec a$ holds. Then $a_+ \leqq c$ and, therefore, $a \leqq a_+ \vee x \leqq c \prec a$ would hold, a contradiction.

In [13] (Proposition 3) we proved that a lattice of finite length is strong iff each of its elements is strong. In $J$-lattices of arbitrary length this does not hold, since there can exist, for instance, injective elements which are not strong. An element $a$ of $L$ is called injective if $a = a'$ holds. But we are able to prove that in a strong lattice each compact element is strong.

PROPOSITION 4. *Let L be a strong J-lattice. Then each compact element $c \in L$ $(c \neq 0)$ is strong.*

PROOF. Let $c \in K$, $c \neq 0$ and $x \in L$ with $c \leqq x \vee c' = x \vee c$ and $c \nleqq x$. Then there holds $c \leqq x \vee c' = x \vee \bigvee (v' \colon v \in J(c/0))$. Since $c$ is compact there are $v_1, \ldots, v_n \in J(c/0)$ such that $c \leqq x \vee v_1' \vee \ldots \vee v_n'$ holds. Without restriction of generality it is possible to assume that $b = x \vee v_1' \vee \ldots \vee v_n'$ is an irredundant join representation of $b$. Therefore, there exists an index $i$ $(1 \leqq i \leqq n)$ such that $v_i \leqq b = x \vee c$ and $v_i \nleqq x \vee v_1' \vee \ldots$ $\ldots \vee v_{i-1}' \vee v_{i+1}' \vee \ldots \vee v_n' < b$ hold.

This is a contradiction to our supposition that $L$ and, therefore, each completely join-irreducible element is strong.

In [13] (Theorem 6 and Corollary 8) it was shown that in a strong lattice of finite length $a' \leqq a_+$ holds and in a semimodular lattice of finite length $a_+ \leqq a'$ holds. In Theorem 5 and Theorem 7 these results will be generalized.

THEOREM 5 (cf. Richter—Stern [13], Theorem 6). *Let L be a strong lattice. Then for each $a \in L$ always $a' \leqq a_+$ holds.*

PROOF. Let $a \in L$. If $U_a = \emptyset$, i.e. $a_+ = a$, then it is obvious that $a' \leqq a_+ = a$ holds. Let $U_a \neq \emptyset$, $b \in U_a$ (i.e. $b \prec a$) and $v \in J(a/0)$. If $v' \nleqq b$ holds, it follows $a = b \vee v'$ and thus $v \leqq b \vee v'$. Since $v' \nleqq b$ also implies $v \nleqq b$ we obtain a contradiction to our supposition that $L$ is strong. Hence $v' \leqq b$ holds for each $v \in J(a/0)$ and, therefore, $a' = \bigvee (v' \colon v \in J(a/0)) \leqq b$ for each $b \in U_a$. This implies $a' \leqq \bigwedge U_a = a_+$.

The following lemma which is interesting in itself will be needed for the proof of Theorem 7. In [13] this lemma was called "Butterfly lemma" since in the proof was made use of a diagram in the form of a butterfly.

LEMMA 6 (cf. Richter—Stern [13], Theorem 7). *Let L be a semimodular lattice and for $a \in L$ let $a = u_1 \vee \ldots \vee u_n$ $(u_1, \ldots, u_n \in J(a/0))$ be an irredundant decomposition of a. Then $a_+ \leqq u_1' \vee \ldots \vee u_n'$ holds.*

The proof is the same as the proof of Theorem 7 in Richter—Stern [13], since in that proof was not made use of the assumption that $L$ is of finite length.

THEOREM 7 (cf. Richter—Stern [13], Corollary 8). *In a semimodular J-lattice L $a_+ \leqq a'$ holds for each precompact element $a \in L$.*

PROOF. In a $J$-lattice $L$ each precompact element $a$ has an irredundant decomposition $a=u_1\vee\ldots\vee u_n$ with $u_1, \ldots, u_n \in J(a/0)$. The definition of $a'$ yields $a'\geqq u_1'\vee\ldots\vee u_n'$. Since $L$ is semimodular it is possible to apply Lemma 6 and we obtain $a'\geqq u_1'\vee\ldots\vee u_n'\geqq \geqq a_+$.

In Faigle—Richter—Stern [3] (Theorem 1) it was shown that a lattice of finite length is strong semimodular iff it has the geometric exchange property (E2). We will show that in arbitrary $J$-lattices this is not so.

LEMMA 8 (cf. Richter [9], Hilfssatz 5.4). *Each semimodular $J$-lattice has the geometric exchange property* (E1).

PROOF. Let $a\in L$, $u, v\in J$, $u\leqq a\vee v$, $u\not\leqq a\vee v'$. Then $v\wedge(a\vee v')=v'\prec v$ holds. This implies $a\vee v'\vee v=a\vee v\succ a\vee v'$ by semimodularity of $L$. Therefore, we obtain $a\vee v=a\vee v'\vee u$, i.e. $v\leqq a\vee v'\vee u$.

The converse of Lemma 8 does not hold. It is even possible to show that a $J$-lattice with (E3) and, therefore, also with (E2) and (E1) must not be semimodular.

LEMMA 9 (cf. Richter [9], Satz 9.3). *A $J$-lattice $L$ with geometric exchange property* (E3) *is not necessarily semimodular.*

PROOF. In Figure 4 a $J$-lattice $L$ is shown, in which the conditions $u, v\in J$, $a\in L$, $u\leqq a\vee v$ and $u\not\leqq a\vee v'$ are satisfied only in the case $a\prec u=v$, i.e. it is obvious that $L$ has (E3). But $L$ is not semimodular, since $c\wedge b\prec c$ and $b\not\prec c\vee b$ hold.

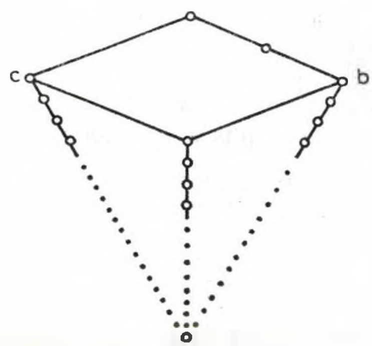


*Fig. 4*

In lattices of finite length (E1) and semimodularity are equivalent (cf. Stern [14], Theorem 2).

For strong lattices the following theorem holds.

THEOREM 10 (cf. Richter [9], Satz 5.5). *Let $L$ be a strong $J$-lattice. Then the following conditions are pairwise equivalent:*

(i) *$L$ has* (D1).
(ii) *$L$ has* (E1).
(iii) *$L$ has* (E2).
(iv) *$L$ has* (E3).
(v) *$L$ is semimodular.*

Proof. (i)⇒(v): Let $a, b \in L$ and $a \wedge b \prec a$. Then there is a $v \in J(a/0)$ with $v \nleqq a \wedge b$, i.e. $a=(a \wedge b) \vee v$ holds. By (D1) we obtain $a \wedge b \leqq (a \wedge b) \vee v' \prec (a \wedge b) \vee v = a$, i.e. $(a \wedge b) \vee v' = a \wedge b$ and, therefore, also $v' \leqq a \wedge b \leqq b$ hold.

Further $a \vee b = (a \wedge b) \vee v \vee b = b \vee v$ holds. By (D1) we obtain $a \vee b = v \vee b \succ v' \vee b = b$, i.e. $L$ is semimodular.

(v)⇒(ii): This holds by Lemma 8.

(ii)⇒(iii): Let $a \in L$, $u, v \in J$, $u \leqq a \vee v$ and $u \nleqq a \vee v'$. Then $v \leqq a \vee v' \vee u$ holds by (E1). Assume $v \nleqq a \vee u$. Then we obtain $a \vee v = (a \vee u) \vee v = (a \vee u) \vee v'$ in contradiction to the strongness of $L$. Therefore, $L$ has property (E2).

(iii)⇒(iv): Let $a \in L$, $u, v \in J$, $u \leqq a \vee v$ and $u \nleqq a \vee v'$. Then $v \leqq a \vee u$ holds by (E2), i.e. $a \vee v = a \vee u$. Further $u \nleqq a \vee v'$ implies $u \nleqq a$. Consequently, $a \vee u' < a \vee u$ holds since $L$ is strong. $v \leqq a \vee u'$ would yield $a \vee v \leqq a \vee u' < a \vee u = a \vee v$, a contradiction. Thus $v \nleqq a \vee u'$ holds, i.e. $L$ has (E3).

(iv)⇒(i): Let $a \in L$, $v \in J$, $v \nleqq a$. Then $a \vee v' < a \vee v$ holds since $L$ is strong. If there is an element $b$ with $a \vee v' < b < a \vee v$, then there is also an element $u \in J(b/0)$ such that $u \nleqq a \vee v'$ holds since otherwise $b \leqq a \vee v'$ would hold. $u \nleqq a \vee v'$ and $u \leqq b < a \vee v$ yield $v \leqq a \vee u$ by (E3), i.e. $a \vee v = a \vee u \leqq b$ holds in contradiction to $b < a \vee v$. Thus $L$ has (D1).

## 4. Strong semimodular J-lattices

As a consequence of Theorem 10 we obtain immediately the following corollary.

Corollary 11. *A J-lattice L has* (D1) *iff it is a strong semimodular lattice.*

Proof. Since a lattice with (D1) always is strong the assertion follows from Theorem 10.

Theorem 12 (Richter [9], Satz 5.6). *Every strong semimodular J-lattice L has the hereditary property (HJ).*

Proof. Let $a \in L$, $v \in J$, $v \nleqq a$. By (D1) $a \vee v' \prec a \vee v$ holds. If $a \vee v \notin J(a \vee v/a)$ holds then there exists an element $b$ with $a < b < a \vee v$ and $b \nleqq a \vee v'$. Therefore, there is a $u \in J(b/0)$ with $u \nleqq a \vee v'$ and $u \leqq a \vee v$ since $L$ is a J-lattice. By Theorem 10 $L$ has exchange property (E2). Thus we get $a \vee v = a \vee u \leqq b$ contradicting $b < a \vee v$. Therefore, $L$ has (HJ).

Remark. There are finite strong lattices with (HJ) which are not semimodular (Figure 5) and there are also finite strong lattices which have not the hereditary property (HJ) (Figure 6).

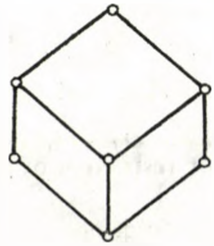

*Fig. 5*

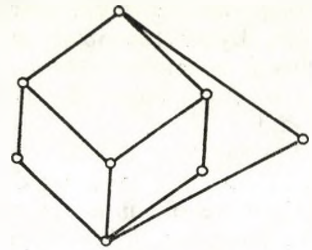

*Fig. 6*

As a corollary of Theorem 12 and results of the papers [7], [8] and [10] we obtain the following

Corollary 13. *Let $L$ be a strong semimodular lattice and let $a=u_1\vee\ldots\vee u_n=$ $=v_1\vee\ldots\vee v_m$ be two irredundant decompositions of $a$ with $u_1, \ldots, u_n, v_1, \ldots, v_m \in J$. Then the following conditions are satisfied:*

(1) *$n=m$ (Theorem of Kuroš-Ore).*

(2) *For each $u_i$ there exists a $v_j$ such that*

$$a = u_1\vee\ldots\vee u_{i-1}\vee v_j\vee u_{i+1}\vee\ldots\vee u_n$$
$$= v_1\vee\ldots\vee v_{j-1}\vee u_i\vee v_{j+1}\vee\ldots\vee v_n.$$

(3) *There exists a permutation $\pi$ of the numbers $1, \ldots, n$ such that for $i=1, \ldots, n$*

$$a = u_1\vee\ldots\vee u_{i-1}\vee v_{\pi(i)}\vee u_{i+1}\vee\ldots\vee u_n$$

*holds.*

Since $L$ has property (HJ) by Theorem 12 the assertion is proved by the Theorems 1, 4 and 7 of Richter [7] (Remark: In [7] property (HJ) is denoted by ($V_1$)). Analogously Theorem 7 of [7] yields the proof of the following corollary.

Corollary 14 (cf. Richter [7], Theorem 7). *Let $L$ be a strong semimodular lattice and let $a=\bigvee T=\bigvee R$ $(T, R\subseteq J)$ be two decompositions of $a$. Then for each $t\in T$ there exists an $r\in R$ such that $a=r\vee\bigvee(T\setminus\{t\})$. Moreover, this resulting decomposition is irredundant if the decomposition $a=\bigvee T$ is irredundant.*

In the subsequent lemma it will be shown that a strong semimodular $J$-lattice has the derivation property (D) restricted of finite sets $T\subseteq J$.

Lemma 15 (cf. Richter [9], Satz 5.8). *Let $L$ be a strong semimodular lattice and let*

$$v \leqq v_1\vee\ldots\vee v_n = a_0 \quad (v, v_1, \ldots, v_n\in J).$$

*Then $v'\leqq v_1'\vee\ldots\vee v_n'$ holds, i.e. $a_0'=v_1'\vee\ldots\vee v_n'$.*

Proof. Let $v_0=0$, $a_l=v_0'\vee\ldots\vee v_{l-1}'\vee v_{l+1}\vee\ldots\vee v_n$ and $b_l=v_1\vee\ldots\vee v_{l-1}\vee$ $\vee v_{l+1}\vee\ldots\vee v_n$ $(1\leqq l\leqq n)$. Without restriction of generality it is possible to assume that

(∗) $$v \nleqq b_l$$

holds for $l=1, \dots, n$. Further we have $v' \leqq a_0 = v'_0 \vee a_0 = v_1 \vee a_1$. Let us assume that $v' \leqq v_m \vee a_m$ and $v' \not\leqq v'_m \vee a_m$ $(1 \leqq m \leqq n)$ hold. Then there exists a $u \in J(v'/0)$ with $u \leqq a_m \vee v_m$ and $u \not\leqq a_m \vee v'_m$. Since $L$ has exchange property (E3) by Theorem 10, $v_m \leqq a_m \vee u$ and, therefore, $a_m \vee v_m = a_m \vee u = a_m \vee v'$ hold. Let $c_m = v_1 \vee \dots \vee v_{m-1}$. Then $a_0 = c_m \vee (a_m \vee v_m)$ and $b_m \vee v' = c_m \vee (a_m \vee v')$ hold, i.e. $a_0 = b_m \vee v'$ since $a_m \vee v' = a_m \vee v_m$ holds. Thus $v \leqq a_0 = b_m \vee v' \leqq b_m \vee v$ holds, i.e. we obtain $b_m \vee v' = b_m \vee v$ in contradiction to the strongness of $L$ since $v \not\leqq b_m$ holds by (∗). Consequently, $v' \leqq a_m \vee v'_m$ and also $v' \leqq a_n \vee v'_n = v'_1 \vee \dots \vee v'_n$ hold.

An important class of $J$-lattices is the class of AC-lattices (cf. for instance Maeda-Maeda [6]). An AC-lattice is an atomistic lattice, that is a $J$-lattice in which each completely join-irreducible element $v$ is an atom (i.e. $v \succ 0$), which has the covering property (C) $a, p \in L$, $p$ atom and $a \wedge p = 0$ imply $a \prec a \vee p$.

In the subsequent we investigate connections between strong semimodular $J$-lattices and AC-lattices.

THEOREM 16 (Richter [9], Satz 5. 11). *Let $L$ be a strong semimodular $J$-lattice and let $a \in L$. Then the interval $a/a'$ is an* AC-*lattice.*

PROOF. Strongness, semimodularity and $v' \leqq a'$ yield either $v \leqq a'$ or $a' \prec a' \vee v$ $(v \in J(a/0))$. Let $u \in J(a/a')$ and $u \neq a'$. Since $L$ is a $J$-lattice $u = \bigvee (v\colon v \in J(u/0)) = \bigvee (a' \vee v\colon v \in J(u/0))$ holds. Since $u$ is completely join-irreducible in the interval $a/a'$ there is a $v_0 \in J(u/0)$ such that $u = a' \vee v_0$, i.e. $u = a' \vee v_0 \succ a'$ holds. Therefore, each element $u$ of $J(a/a')$ with $u \neq a'$ is an atom in $a/a'$, i.e. $L$ is atomistic. In addition to that we have only to show that $L$ has the covering property (C).

Let $b \in a/a'$ and $q$ an atom in $a/a'$ with $q \not\leqq b$. Then there is an $v \in J(a/0)$ with $q = a' \vee v$, $v \not\leqq b$ and $v' \leqq a' \leqq b$. Consequently, $b = b \vee v' \prec b \vee v = b \vee q$ holds, since $L$ has property (D1).

At last in this section we intend to compare $a'$ and $a_+$ for precompact elements $a$ of a strong semimodular lattice $L$.

THEOREM 17 (cf. Richter—Stern [13], Theorem 9). *Let $L$ be a strong semimodular $J$-lattice. Then $a_+ = a'$ holds for each precompact element $a \in L$.*

PROOF. Theorem 5 and Theorem 7 yield the assertion.

## 5. Algebraic $J$-lattices

In algebraic $J$-lattices we are able to sharpen some of the above mentioned results.

THEOREM 18 (cf. Richter—Stern [13], Proposition 3). *Let $L$ be an algebraic $J$-lattice. $L$ is strong iff each compact element is strong.*

PROOF. In a strong $J$-lattice each compact element is strong (Proposition 4). Let $L$ be an algebraic $J$-lattice in which each compact element is strong. Since in an algebraic lattice each completely join-irreducible element is compact, each completely join-irreducible element is strong, i.e. $L$ is strong.

PROPOSITION 19 (cf. Richter—Stern [13], Proposition 10). *Let $L$ be an algebraic semimodular $J$-lattice. Each compact element of $L$ is strong iff it is strict.*

Proof. If each compact element is strong then $L$ is strong by Theorem 18. Thus $c'=c_+$ holds for each compact element $c\in L$ by Theorem 17. Hence $c\leqq x\vee c_+$ $(=x\vee c')$ yields $c\leqq x$, i.e. each compact $c\in L$ is strict.

Conversely, if each compact element of $L$ is strict, then also each completely join-irreducible element is strict and, therefore, strong, i.e. $L$ is strong. Thus $c_+=c'$ holds for each compact $c\in L$ by Theorem 17. Consequently, $c\leqq x\vee c'$ $(=x\vee c_+)$ yields $c\leqq x$, i.e. each compact $c\in L$ is strong.

Corollary 20 (cf. Richter—Stern [13], Corollary 11). *Let $L$ be an algebraic semimodular $J$-lattice. Then the following three conditions are equivalent:*

(i) *$L$ is strong.*

(ii) *Each compact element of $L$ is strong.*

(iii) *Each compact element of $L$ is strict.*

Proof. The equivalence of (i) and (ii) follows from Theorem 18. Proposition 19 yields the equivalence of (ii) and (iii).

Theorem 21 (cf. Richter [9], Folgerung 5.9). *Let $L$ be an algebraic strong semimodular lattice. Then $L$ has the derivation property (D).*

Proof. Let $v\in J$, $T\subseteqq J$ and $v\leqq\vee T$. Since in every algebraic lattice $J\subseteqq K$ holds, there exists a finite subset $S$ of $T$ with $v\leqq\vee S$. By Lemma 15 $v'\leqq\vee(s'\colon s\in S\subseteqq T)\leqq\vee(t'\colon t\in T)$ holds.

Proposition 22 (cf. Richter [9], Hilfssatz 5.10). *In each algebraic $J$-lattice which has derivation property (D) and, therefore, also in each algebraic strong semimodular $J$-lattice for any subset $T$ of $J$ always $(\vee T)'=\vee(t'\colon t\in T)$ holds.*

Proof. By definition $(\vee T)'=\vee(v'\colon v\in J(\vee T/0))\geqq\vee(t'\colon t\in T)$ holds since $T\subseteqq J(\vee T/0)$. For each $v\in J(\vee T/0)$ we get $v'\leqq\vee(t'\colon t\in T)$ since $v\leqq\vee T$ holds and $L$ has property (D), i.e. $(\vee T)'\leqq\vee(t'\colon t\in T)\leqq(\vee T)'$.

An algebraic AC-lattice is called a geometric lattice. As a consequence of Theorem 16 we obtain

Corollary 23 (cf. Richter [9], Folgerung 5.12). *Let $L$ be an algebraic strong semimodular $J$-lattice. Then for each $a\in L$ the interval $a/a'$ is a geometric lattice.*

Proof. Head [5], (Lemma 1), proved that each interval $b/a$ of an algebraic lattice is also algebraic. Thus Theorem 16 yields the assertion.

In Corollary 14 we proved that if $a=\vee T=\vee R$ $(T, R\subseteqq J)$ are two decompositions of $a$ in a strong semimodular $J$-lattice $L$ then any element $t$ of $T$ can be replaced by an element $r\in R$. But if these decompositions are irredundant there are no statements about the cardinality of $T$ and $R$ except in the finite case (Corollary 13). For algebraic strong semimodular $J$-lattices we are able to generalize the Theorem of Kuroš-Ore for infinite decompositions and to give a statement about the cardinality of $T$ and $R$. The proof of the following Theorem 24 will be published in the subsequent paper [11], since for this proof it is necessary to investigate an abstract independence relation in a special subset of $J$ in prealgebraic lattices. One can find the proof also in [9] (Satz 5.15).

THEOREM 24 (cf. Richter [9], Satz 5.15, and [11]). *Let L be an algebraic strong semimodular J-lattice. If* $a=\bigvee T=\bigvee R$ $(T, R\subseteq J)$ *are two irredundant decompositions of a then T and R have the same cardinality.*

## 6. Join symmetry and basis exchange

In [3] Faigle—Richter—Stern investigated in connection with strong semimodular lattices of finite length also join symmetric lattices and lattices with basis exchange property. In lattices of finite length the equivalence of the following three conditions holds:

(i) $L$ is strong and semimodular.
(ii) $L$ is join symmetric.
(iii) $L$ has the basis exchange property.

In infinite $J$-lattices this does not hold as it will be shown in the subsequent results. At first we prove that a finite strong lattice is not necessarily join symmetric.

PROPOSITION 25 (cf. Richter [9], Satz 9.8). *Let L be a strong finite lattice. Then L must not be join symmetric.*

PROOF. Figure 5 shows a finite strong lattice which is not semimodular. Therefore, this lattice is not join symmetric by Theorem 4 of Faigle—Richter—Stern [3].

PROPOSITION 26 (cf. Richter [9], Satz 9.9). *Let L be a strong semimodular J-lattice. Then L is not necessarily join symmetric.*

PROOF. Figure 7 shows a strong semimodular $J$-lattice, since $a\in L$, $v\in J=\{x, y_1, \ldots, z, q\}$ and $v\not\leq a$ imply $a\vee v' \prec a\vee v$. The pair $(q, \{x, z\})$ is not minimal but the pair $(z, \{x, q\})$ is minimal. Thus $L$ is not join symmetric.

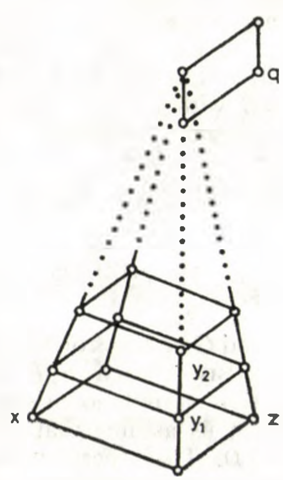


*Fig. 7*

If in a strong semimodular $J$-lattice each element has an irredundant decomposition what is satisfied, for instance, in each lattice of finite length, then $L$ is join symmetric but the converse does not hold.

THEOREM 27 (cf. Richter [9], Satz 9.10). *Let $L$ be a strong semimodular $J$-lattice and let each $q'$ with $q \in J$ have an irredundant decomposition. Then $L$ is join symmetric.*

PROOF. Let $(p, C)$ be a minimal pair and let $q \in C$ and $x = \bigvee(C \setminus \{q\})$. Then by the supposition $q'$ has an irredundant decomposition $q' = \bigvee B$ and $B$ is an antichain. Since $b < q$ holds for each $b \in B$ we obtain $D = (C \setminus \{q\}) \cup (B \setminus (x/0)) < C$, where $D$ is also an antichain. Thus $p \leqq x \vee q' = \bigvee D \prec \bigvee C = x \vee q$ would be a contradiction to the minimality of $(p, C)$, i.e. $p \leqq x \vee q$ and $p \not\leqq x \vee q'$ hold. By Theorem 10 $L$ has exchange property (E3), i.e. $q \leqq x \vee p$ and $q \not\leqq x \vee p'$ hold. Let $M = (C \setminus \{q\}) \cup \{p\}$. Then the pair $(q, M)$ satisfies conditions (Mp1) and (Mp2). If $(q, M)$ violates condition (Mp3) then there exists because of $q \not\leqq x \vee p'$ an antichain $E < C \setminus \{q\}$ such that $q \leqq \bigvee E \vee p$ and $q \not\leqq \bigvee E \vee p' \leqq x \vee p'$ hold. By (E3) we obtain $p \leqq \bigvee E \vee q$, i.e. $p \leqq \bigvee F$ where $F = (E \setminus (q/0)) \cup \{q\}$ is an antichain of $J$ with $F < C$. This is a contradiction to the minimality of $(p, C)$. Thus $(q, M)$ satisfies condition (Mp3), i.e. $L$ is join symmetric.

THEOREM 28 (cf. Richter [9], Satz 9.11). *Let $L$ be a join symmetric $J$-lattice which has the basis exchange property and in which each element has an irredundant decomposition. Then $L$ must not be strong or semimodular.*

PROOF. In the lattice of Figure 8

$$J = \{v_i, x_j, y_i, z_i \colon i = 0, 1, -1, 2, -2, \ldots, j = 0, -1, -2, \ldots\}$$

holds. Each element has an irredundant decomposition with at most two elements of $J$. Let $a \in L$ and $u_n \in \{v_n, y_n, z_n\}$ $(n \in \{0, 1, -1, \ldots\})$ or $u_n = x_n$ $(n \in \{0, -1, -2, \ldots\})$. Then $u_n' = u_{n-1}$ and, for $a \not\leqq u_n$, $a \vee u_n = a \vee u_{n-1} = a \vee u_n'$ hold especially also if $u_n \not\leqq a$ holds, i.e. $L$ is not strong.

Further $x_0 = v \wedge z \prec v$ and $z \not\prec v \vee z = 1$ imply that $L$ is not semimodular.

Since to each antichain $B \in J$ and to each $u \in L$ with $u \leqq \bigvee B$ and $|B| \geqq 2$ there is an antichain $B' = \{b' \colon b \in B\} < B$ with $u \leqq \bigvee B'$, there exists no minimal pair in $L$, i.e. $L$ is join symmetric. The same reason yields that except the elements of $J$ no element has a basis, i.e. $L$ has the basis exchange property.

The following theorem shows that each strong semimodular lattice has the basis exchange property.

THEOREM 29 (cf. Richter [9], Satz 9.12). *Each strong semimodular $J$-lattice $L$ has the basis exchange property.*

PROOF. Let $x \in L$ and let $B$ and $C$ be bases of $x$. Since by Theorem 12 $L$ has the hereditary property (HJ) there exists to each $b \in B$ an element $c \in C$ with $x = \bigvee(B \setminus \{b\}) \vee c$. This decomposition is irredundant since $x = \bigvee B$ is an irredundant decomposition (Corollary 14). Let us assume that there is an antichain $D$ with $D < (B \setminus \{b\}) \cup \{c\} = B'$ and $x = \bigvee D$. Then there are an element $d \in D$ and an element $e \in B'$ with $d < e$, i.e. $d \leqq e'$. Let $y = \bigvee(B' \setminus \{e\})$. Then $x = y \vee e = \bigvee C = \bigvee D \leqq \bigvee(D \setminus \{d\}) \vee e' \leqq y \vee e'$ holds, i.e. $e \not\leqq y$ and $y \vee e = y \vee e'$ hold contradicting that $L$ is

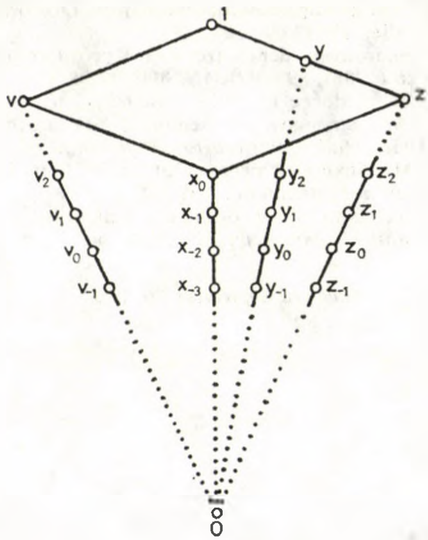


*Fig. 8*

strong and semimodular. Therefore, there is no antichain $D$ with $D \ll B'$. Since $B'$ is an antichain of $J$, $B'$ is a basis of $x$, i.e. $L$ has the basis exchange property.

Theorem 29 and Proposition 26 yield that a $J$-lattice which has the basis exchange property must not be join-symmetric. There are even finite lattices which have the basis exchange property but which are not join-symmetric (cf. Figure 5). By Theorems 4 and 5 of [3] each join symmetric lattice of finite length has the basis exchange property.

PROBLEM. Does there exist an infinite join-symmetric $J$-lattice which does not have the basis exchange property?

## REFERENCES

[1] BIRKHOFF, G. and FRINK, O., Representations of lattices by sets, *Trans. Amer. Math. Soc.* **64** (1948), 299—316. *MR* **10**—279.

[2] FAIGLE, U., Geometries on partially ordered sets, *J. Combin. Theory Ser. B* **28** (1980), 26—51. *MR* **81m:** 05054.

[3] FAIGLE, U., RICHTER, G. and STERN, M., Geometric exchange properties in lattices of finite length, *Algebra Universalis* **19** (1984), 355—365. *MR* **86h:** 06020.

[4] GASKILL, H. S. and RIVAL, I., An exchange property for modular lattices, *Algebra Universalis* **8** (1978), 354—356. *MR* **57** # 12317.

[5] HEAD, T. J., Purity in compactly generated modular lattices, *Acta Math. Acad. Sci. Hungar.* **17** (1966), 55—59. *MR* **32** # 7644.

[6] MAEDA, F. and MAEDA, S., *Theory of symmetric lattices*, Die Grundlehren der mathematischen Wissenschaften, Bd. 173, Springer-Verlag, New York—Berlin, 1970. *MR* **44** # 123.

[7] RICHTER, G., The Kuroš-Ore theorem, finite and infinite decompositions, *Studia Sci. Math. Hungar.* **17** (1982), 243—250. *MR* **86c:** 06015.

[8] RICHTER, G., Applications of some lattice theoretic results in group theory, *Contributions to General Algebra 2* (Proceedings of the Klagenfurt Conference, June 10—13, 1982), Wien, 1983, 305—317.

[9] RICHTER, G., Zur Theorie der präalgebraischen Verbände, Doctoral Thesis (Diss. B), Martin-Luther-Univ., Halle, 1984.
[10] RICHTER, G., Verallgemeinerungen der Sätze von Kuroš-Ore und Schmidt-Ore, *Beiträge Algebra Geom.* **26** (1988), 89—100. *MR* **89i:** 06008.
[11] RICHTER, G., Abstract independence in lattices, *Beiträge Algebra Geom.* **31** (1990), 57—67.
[12] RICHTER, G. and STERN, M., Strongness in (semimodular) lattices of finite length, *Wiss. Z. Martin-Luther-Univ. Halle—Wittenberg Math.-Natur. Reihe* **33** (1984), 73—77.
[13] RICHTER, G. and STERN, M., Strong elements in lattices of finite length, Martin-Luther-Univ. Preprint der Sektion Mathematik, Nr. 106.
[14] STERN, M., Exchange properties in lattices of finite length, *Wiss. Z. Martin-Luther-Univ. Halle—Wittenberg Math.-Natur. Reihe* **31** (1982), no. 3, 15—26. *MR* **84f:** 06011.

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# IRREGULARITIES OF DISTRIBUTION AND CATEGORY THEOREM

JÓZSEF BECK

## 1. Introduction

Suppose that we have a configuration $P=P(N)$ of $N$ points, not necessarily distinct, in the unit square $U^2=[0, 1)^2$. Let $\mathbf{Z}^2$ denote, as usual, the set of integer lattice points in the plane $\mathbf{R}^2$. Denote by $P^*$ the set of points $\mathbf{p}+\mathbf{n}$ where $\mathbf{p}\in P$ and $\mathbf{n}\in \mathbf{Z}^2$. Thus $P^*$ is a periodic set.

Let $B\subset \mathbf{R}^2$ be an arbitrary compact set with usual Lebesgue measure (i.e. area) $\mu(B)$. Write $Z[P^*; B]$ for the number of points of $P^*$ in $B$, and

$$D^{\text{tor}}[P; B] = Z[P^*; B]-N\mu(B).$$

The quantity $D^{\text{tor}}[P; B]$ tells us how far $Z[P^*; B]$ deviates from the expected number $N\mu(B)$ of points of $P^*$ in $B$.

Let $A$ be an arbitrary compact and convex set in the plane. For arbitrary real number $\alpha$ and two-dimensional vector $\mathbf{v}\in \mathbf{R}^2$, set

$$A(\alpha, \mathbf{v}) = \{\alpha\mathbf{x}+\mathbf{v} : \mathbf{x}\in A\}.$$

Clearly $A(\alpha, \mathbf{v})$ is a homothetic image of $A$. Let

$$\Delta^{\text{tor}}[P;\ A] = \sup_{\alpha, \mathbf{v}} |D^{\text{tor}}[P;\ A(\alpha, \mathbf{v})]|$$

and

$$\Delta_N^{\text{tor}}[A] = \inf_P \Delta^{\text{tor}}[P;\ A]$$

where the supremum is extended over all contractions $-1\leqq\alpha\leqq 1$ and translations $\mathbf{v}\in\mathbf{R}^2$, and the infimum is extended over all $N$-element sets $P$ in the unit square $U^2$.

We say that $\Delta_N^{\text{tor}}[A]$ is the "torus discrepancy" of the homothetic family $A(\alpha, \mathbf{v})$, $-1\leqq\alpha\leqq 1$, $\mathbf{v}\in\mathbf{R}^2$ (note that reflection across the origin is allowed).

We recall the following two results from Beck (1987) (see Corollary 1B and Corollary 4C, respectively).

THEOREM A. *Let $A\subset\mathbf{R}^2$ be a compact convex region such that the boundary curve of $A$ is twice continuously differentiable and has strictly positive curvature. Then*

$$\liminf_{n\to\infty} \frac{\Delta_N^{\text{tor}}[A]}{N^{1/4}(\log N)^{-1/2}} > 0.$$

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THEOREM B. *Let* $A \subset \mathbf{R}^2$ *be a convex polygon. Then for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\Delta_N^{\mathrm{tor}}[A]}{(\log N)^{4+\varepsilon}} = 0.$$

Comparing Theorems A and B, we see that the torus discrepancy $\Delta_N^{\mathrm{tor}}[A]$ is "large" or "small" according as $A$ is "smooth" or "cornered", respectively. The next result demonstrates the existence of a compact convex region with very irregular discrepancy behaviour. We shall actually prove that "most" (in the sense of category) convex regions have this property.

Let CONV(2) be the metric space of all compact convex regions in $\mathbf{R}^2$ endowed with the Hausdorff metric, defined by

$$\mathrm{dist}_H(A, B) = \max\left(\max_{\mathbf{x}\in A}\min_{\mathbf{y}\in B} |\mathbf{x}-\mathbf{y}|, \max_{\mathbf{y}\in B}\min_{\mathbf{x}\in A} |\mathbf{x}-\mathbf{y}|\right)$$

for $A, B \in \mathrm{CONV}(2)$. Here $|\mathbf{x}-\mathbf{y}|$ denotes, as usual, the euclidean distance. By the Blaschke selection theorem (see, for example, Hadwiger (1957), p. 154 and p. 201), CONV(2) is locally compact. It follows from the Baire category theorem that the sets of first category in CONV(2) are "small" compared to their complements.

We have

THEOREM 1.1. *Let* $f\colon \mathbf{N}\to\mathbf{R}^+$ *and* $g\colon \mathbf{N}\to\mathbf{R}^+$ *satisfy* $(N \geqq 2)$

$$\lim_{N\to\infty} \frac{f(N)}{(\log N)^{4+\varepsilon}} > 0$$

*for some* $\varepsilon > 0$ *and*

$$\lim_{N\to\infty} \frac{g(N)}{N^{1/4}(\log N)^{-1/2}} = 0.$$

*Then for all* $A \in \mathrm{CONV}(2)$, *except those in a set of first category,*

(i) $\Delta_N^{\mathrm{tor}}[A] < f(N)$ *for infinitely many* $N$, *and*

(ii) $\Delta_N^{\mathrm{tor}}[A] > g(N)$ *for infinitely many* $N$.

Note that Theorem 1.1 was motivated by Gruber and Kenderov [2] (see especially Theorem 2).

## 2. Proof of Theorem 1.1

The simple underlying idea of the proof of Theorem 1.1 is as follows: Let $P^1$ be a convex polygon. By "smoothing" the corners of $P^1$ slightly one can obtain a convex region $B^1$ of differentiability class two. In $B^1$ one can inscribe a convex polygon $P^2$ which approximates $B^1$ very closely. By "smoothing" the corners of $P^2$ slightly one can obtain a convex region $B^2$ of differentiability class two. And so on. If in this process, $P^1, B^1, P^2, B^2, \ldots$ differ by only very little, then $A = P^1 \cap B^1 \cap \cap P^2 \cap B^2 \cap \ldots$ satisfies the requirements (i) and (ii) of Theorem 1.1.

After the heuristics, we begin the proof of (i). We shall actually deduce it from Theorem C below (see Theorem 4B in Beck [1]).

THEOREM C. *Let* $A \subset R^2$ *be a compact convex region. Given any integer* $l \geqq 3$, *let* $A_l \subset A$ *denote an inscribed l-gon (i.e. polygon with l sides) of largest area. Denote* *by* $\xi_N(A)$ *the smallest integer* $l \geqq 3$ *such that* $\mu(A \setminus A_l) \leqq l^2 N^{-1}$. *Then for any* $N \geqq 2$ *and* $\varepsilon > 0$,

$$\Delta_N^{\mathrm{tor}}[A] < c_1(A, \varepsilon) \cdot \xi_N(A) \cdot (\log N)^{4+\varepsilon}.$$

Let POL$(n)$ denote the subspace of CONV(2) consisting of all convex polygons of at most $n$ vertices. For every $A \in \mathrm{CONV}(2)$, let

$$\nu(A, n) = \inf \mu(A \setminus P),$$

where the infimum is extended over all $P \in \mathrm{POL}(n)$ satisfying $P \subset A$.

The following lemma was independently proved by Schneider and Wieacker [4] and by Gruber and Kenderov [2].

LEMMA 2.1. *Let* $h: \mathbf{N} \to \mathbf{R}^+$ *satisfy* $h(n) \to 0$ *as* $n \to \infty$. *Then for all* $A \in \mathrm{CONV}(2)$, *except those in a set of first category,* $\nu(A, n) < h(n)$ *for infinitely many* $n$.

PROOF. The function $A \to \nu(A, n)$ is clearly continuous for each $n \in \mathbf{N}$. Thus the sets $\{A \in \mathrm{CONV}(2) \colon \nu(A, n) \geqq h(n)\}$ are closed for each $n \in \mathbf{N}$. It follows that the set

$$\text{(1)} \quad A_k = \{A \in \mathrm{CONV}(2) \colon \nu(A, n) \geqq h(n) \text{ for all } n \geqq k\} = \\ = \bigcap_{n=k}^{\infty} \{A \in \mathrm{CONV}(2) \colon \nu(A, n) \geqq h(n)\}$$

is again closed for each $k \in \mathbf{N}$. We shall show that

(2) $\qquad A_k$ is nowhere dense in CONV(2) for each $k \in \mathbf{N}$.

In view of (1), it is sufficient to show that $A_k$ has empty interior. Suppose that for some $k \in \mathbf{N}$, the interior of $A_k$ is non-empty. Since the set of convex polygons is dense in CONV(2), there exists a convex polygon $P \in A_k$. Then $\nu(P, n) = 0$ for all sufficiently large $n$. This contradicts the definition of $A_k$. Hence (2) is established, and Lemma 2.1 follows. ∎

Let $h(n) = 2^{-2^n}$, and write

$$\mathscr{A} = \{A \in \mathrm{CONV}(2) \colon \nu(A, n) < h(n) \text{ for infinitely many } n\}.$$

Then for every $A \in \mathscr{A}$ and $N \geqq 3$,

$$\liminf_{N \to \infty} \frac{\xi_N(A)}{\log \log N} < \infty,$$

and so, by Theorem C,

$$\Delta_N^{\mathrm{tor}}[A] < f(N) \quad \text{for infinitely many } N$$

(here $f(N)$ is a function satisfying the hypothesis of Theorem 1.1). Theorem 1.1 (i) follows, since by Lemma 2.1, $\mathrm{CONV}(2) \setminus \mathscr{A}$ forms a set of first category.

Next we prove (ii). The proof is based on the following result, which is implicitly contained in the proof of Theorem A (see Beck [1]).

THEOREM A*. *Let $A\subset \mathbf{R}^2$ be a compact convex region. If the boundary curve $\Gamma$ of $A$ is twice continuously differentiable and if, for some real number $\gamma>0$, the ratio*

$$\frac{\text{minimum curvature of } \Gamma}{\text{maximum curvature of } \Gamma}$$

*is greater than $\gamma$, then for arbitrary $0<\delta<1$ and for arbitrary n-element set $P\subset U^2$, we have*

$$\delta^{-1}\int_{-\infty}^{\infty}\Big(\int_{U^2}D^{\text{tor}}[P;\ A(\lambda,\mathbf{x})]|^2\,d\mathbf{x}\Big)e^{-\lambda^2/\delta^2}\,d\lambda \gg_\gamma \delta\cdot(n\mu(A))^{1/2}.$$

REMARK. The Vinogradov's notation $\gg_\gamma$ means that the implicit positive constant may depend on the value of $\gamma$.

For every $A\in\text{CONV}(2)$, let

$$\varrho(A,n)=\inf_{\mathcal{P}} l(A,n)\int_{-\infty}^{\infty}\Big(\int_{U^2}|D^{\text{tor}}[\mathcal{P};\ A(\lambda,\mathbf{x})]|^2\,d\mathbf{x}\Big)e^{-\lambda^2 l^2(A,n)}\,d\lambda,$$

where $l(A,n)=\log\big(2+n\mu(A)\big)$ and the infimum is taken over all $n$-element subsets $\mathcal{P}$ of $U^2$.

The function $A\to\varrho(A,n)$ is continuous for each $n\in\mathbf{N}$. We indicate this as follows: Let $A,B\in\text{CONV}(2)$ and $\text{dist}_{\mathscr{H}}(A,B)<\delta$. Observe that if $\delta<\delta(A,\varepsilon)$, then

$$(1-\varepsilon)B+\mathbf{u}\subset A\subset(1+\varepsilon)B+\mathbf{v} \tag{3}$$

for some $\mathbf{u},\mathbf{v}\in\mathbf{R}^2$. For the sake of brevity, let

$$D_A(\lambda,\mathbf{y})=D^{\text{tor}}[\mathcal{P};\ A(\lambda,\mathbf{y})].$$

Then by (3), for arbitrary $\mathbf{x}$,

$$\begin{aligned}
&\min\{|D_B((1-\varepsilon)\lambda,\lambda\mathbf{u}+\mathbf{x})|-(\mu(A)-(1-\varepsilon)^2\mu(B)),\\
&\big|D_B((1+\varepsilon)\lambda,\lambda\mathbf{v}+\mathbf{x})\big|-((1+\varepsilon)^2\mu(B)-\mu(A))\}\le|D_A(\lambda,\mathbf{x})|\\
&\le\max\{\big|D_B((1-\varepsilon)\lambda,\lambda\mathbf{u}+\mathbf{x})\big|+(\mu(A)-(1-\varepsilon)^2\mu(B)),\\
&\big|D_B((1+\varepsilon)\lambda,\lambda\mathbf{v}+\mathbf{x})\big|+((1+\varepsilon)^2\mu(B)-\mu(A))\}.
\end{aligned}\tag{4}$$

Moreover, let

$$U_B(\lambda)=\int_{U^2}|D_B(\lambda,\mathbf{y})|^2\,d\mathbf{y}.$$

Clearly (let $d(B)$ stand for the diameter of $B$)

$$|D_B(\lambda,\mathbf{y})|\ll n\lambda\,d(B),$$

and so we have (note that $n=\#\mathcal{P}$ is fixed)

$$\left|\int_{-\infty}^{\infty} U_B((1+\varepsilon)\lambda)e^{-\lambda^2}d\lambda-\int_{-\infty}^{\infty} U_B((1-\varepsilon)\lambda)e^{-\lambda^2}d\lambda\right|=$$

$$(5)\qquad =\left|\int_{-\infty}^{\infty} U_B(\lambda)\left(\frac{1}{1+\varepsilon}e^{-\lambda^2/(1+\varepsilon)^2}-\frac{1}{1-\varepsilon}e^{-\lambda^2/(1-\varepsilon)^2}\right)d\lambda\right|\ll$$

$$\ll n^2(d(B))^2\int_{-\infty}^{\infty}\lambda^2\left|\frac{1}{1-\varepsilon}e^{-\lambda^2/(1-\varepsilon)^2}-\frac{1}{1+\varepsilon}e^{-\lambda^2/(1+\varepsilon)^2}\right|d\lambda.$$

Since the right-hand side of (5) tends to 0 as $B\to A$ and $\varepsilon\to 0$, the continuity of $A\to\varrho(A,n)$ for fixed $n$ easily follows from (4) and (5).

We need

LEMMA 2.2. *Let* $G\colon \mathbf{N}\to\mathbf{R}^+$ *satisfy*

$$\lim_{n\to\infty}\frac{G(n)}{n^{1/2}(\log n)^{-1}}=0.$$

*Then for all* $A\in\mathrm{CONV}(2)$, *except those in a set of first category,* $\varrho(A,n)>G(n)$ *for infinitely many* $n$.

PROOF. We repeat the argument of the previous lemma. Since the function $A\to\varrho(A,n)$ is continuous, the sets $\{A\in\mathrm{CONV}(2)\colon \varrho(A,n)\leqq G(n)\}$ are closed for each $n\in\mathbf{N}$. It follows that the set

$$\mathcal{B}_k=\{A\in\mathrm{CONV}(2)\colon\varrho(A,n)\leqq G(n)\text{ for all } n\geqq k\}=$$

$$=\bigcap_{n=k}^{\infty}\{A\in\mathrm{CONV}(2)\colon\varrho(A,n)\leqq G(n)\}$$

is again closed for each $k\in\mathbf{N}$. We shall show that

(6) $\mathcal{B}_k$ is nowhere dense in CONV(2) for each $k\in\mathbf{N}$.

Suppose on the contrary that for some $k\in\mathbf{N}$, the interior of $\mathcal{B}_k$ is nonempty. Since the analytic convex sets are dense in CONV(2), there exists an analytic set $B\in\mathcal{B}_k$.

We now recall Theorem A* with $\delta^{-1}=l(A,n)=\log(2+n\mu(A))$: If $A$ satisfies the hypotheses of Theorem A*, then

$$(7)\qquad l(A,n)\int_{-\infty}^{\infty}\left(\int_{U^2}|D^{\mathrm{tor}}[\mathcal{P};A(\lambda,\mathbf{x})]|^2\,d\mathbf{x}\right)e^{-\lambda^2 l^2(A,n)}\,d\lambda\gg_{\gamma}\frac{(n\mu(A))^{1/2}}{\log(2+n\mu(A))},$$

where $\mathcal{P}$ is an arbitrary $n$-element subset of $U^2$

Applying (7) to the analytic set $B\in\mathscr{B}_k$, we get

$$\liminf_{n\to\infty}\frac{\varrho(B, n)}{n^{1/2}(\log n)^{-1}} > 0.$$

This contradicts the definition of $\mathscr{B}_k$, and Lemma 2.2 follows. ∎

Let $\mathscr{P}\subset U^2$, where $\#\mathscr{P}=n$, be arbitrary but fixed. Clearly

$$|D^{\rm tor}[\mathscr{P};\ A(\lambda, \mathbf{x})]| \ll n\lambda\, d(A), \quad \text{where } d(A) \text{ is the diameter.}$$

Thus we have

$$l(A, n) \int_{|\lambda|>1} \Big(\int_{U^2} |D^{\rm tor}[\mathscr{P};\ A(\lambda, \mathbf{x})]|^2\, d\mathbf{x}\Big)\, e^{-\lambda^2 l^2(A,n)}\, d\lambda \ll$$

$$\ll l(A, n) \int_{|\lambda|>1} (n\lambda\, d(A))^2 (l(A, n))^{-\lambda^2 l(A,n)}\, d\lambda \leqq 1$$

if $n$ is sufficiently large depending on $A$.

It follows from Lemma 2.2 that for all $A\in\mathrm{CONV}(2)$, except those in a set of first category,

$$l(A, n) \int_{-1}^{1} \Big(\int_{U^2} |D^{\rm tor}[\mathscr{P};\ A(\lambda, \mathbf{x})]|^2\, d\mathbf{x}\Big)\, e^{-\lambda^2 l^2(A,n)}\, d\lambda > G(n)-O(1) \tag{8}$$

for infinitely many $n$. Since

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \pi^{1/2},$$

we obtain

$$\pi^{1/2}|\Delta^{\rm tor}[\mathscr{P};\ A]|^2 \geqq l(A, n) \int_{-1}^{1} \Big(\int_{U^2} |D^{\rm tor}[\mathscr{P};\ A(\lambda, \mathbf{x})]|^2\, d\mathbf{x}\Big)\, e^{-\lambda^2 l^2(A,n)}\, d\lambda. \tag{9}$$

Theorem 1.1 (ii) now follows from (8) and (9). ∎

## REFERENCES

[1] Beck, J., Irregularities of distribution. II, *Proc. London Math. Soc.* (3) **56** (1988), 1—50. *MR* **89c:** 11118.

[2] Gruber, P. M. and Kenderov, P. S., Approximation of convex bodies by polytopes, *Rend. Circ. Mat. Palermo* (2) **31** (1982), 195—225. *MR* **84d:** 52004.

[3] Hadwiger, H., *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1957. *MR* **21** # 1561.

[4] Schneider, R. and Wieacker, J. A., Approximation of convex bodies by polytopes, *Bull. London Math. Soc.* **13** (1981), 149—156. *MR* **82g:** 52008.

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# SOME FACTORIZATION THEOREMS FOR HILBERT SPACE OPERATORS

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The present paper contains two theorems about factorization type problems with respect to not necessarily bounded operators in Hilbert space. We give necessary and sufficient conditions for the existence of a positive (resp. self-adjoint) operator $B$ on $H$ satisfying the identity

(1) $$Ax = BCx \quad (x \in D(A))$$

where $H$ is a (complex) Hilbert space, $A$ and $C$ are given densely defined operators with domains $D(A)$ and $D(C)$, respectively.

Of course, it is assumed once and for all that the following relation holds true

(2) $$D(A) \subset D(C).$$

As a reference we use a theorem of Z. Sebestyén [1] concerning positive extendibility of an operator on a subspace of a Hilbert space.

THEOREM 1. *Let $A$ and $C$ be densely defined operators in the Hilbert space $H$, with $D(A) \subset D(C)$. Then there exists a positive operator $B$ on $H$ satisfying* (1) *if and only if there exists $m \geqq 0$ such that*

(3) $$\langle Ax, (mC - A)x \rangle \geqq 0 \quad (x \in D(A)).$$

PROOF. Assume first that such an operator $B$ exists. Since $B$ is positive, $B$ satisfies the Schwarz inequality:

$$\|By\|^2 \leqq \|B\| \langle By, y \rangle \quad (y \in H).$$

Using this we have by (1), for any $x \in D(A)$

$$\|Ax\|^2 = \|B(Cx)\|^2 \leqq \|B\| \langle B(Cx), Cx \rangle = \|B\| \langle Ax, Cx \rangle.$$

Therefore (3) is true with $m = \|B\|$.

Assume now (3), let $b: C(D(A)) \to H$ given by

(4) $$b(Cx) = Ax.$$

Now $b$ is well defined because if $Cx = 0$, then (3) gives $Ax = 0$. Moreover, $b$ is linear because $A$ and $C$ are linear operators. We have thus $b$ is a linear map satisfying

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the Schwarz inequality

$$\|by\|^2 \leqq M\langle by, y\rangle \quad (y\in C(D(A))). \tag{5}$$

Indeed, for $y=Cx$ we have by (3)

$$\|b(Cx)\|^2 = \|Ax\|^2 \leqq m\langle Ax, Cx\rangle = m\langle b(Cx), Cx\rangle$$
$$(x\in D(A)).$$

Therefore by Theorem of [1] there exists a positive extension $B$ of $b$ to the space $H$. The theorem is proved. (For another proof of this Theorem see [2].)

THEOREM 2. *Let $A$ and $C$ be densely defined operators in a Hilbert space $H$, with $D(A)\subset D(C)$. Then there exists a self-adjoint operator $B$ on $H$ satisfying* (1) *if and only if there exists $m\geqq 0$ such that*

$$\langle (mC+A)x, (mC-A)x\rangle \geqq 0 \quad (x\in D(A)). \tag{3)'$$

PROOF. Assume first (1), with a self-adjoint operator $B$. Then, since $B$ is bounded, we have

$$\|Ax\| = \|B(Cx)\| \leqq \|B\|\|Cx\| \quad (x\in D(A)).$$

Therefore

$$\langle(\|B\|C+A)x, (\|B\|C-A)x\rangle =$$
$$= \|B\|^2\|Cx\|^2+\|B\|\langle Ax, Cx\rangle-\|B\|\langle Cx, Ax\rangle-\|Ax\|^2 =$$
$$= \|B\|^2\|Cx\|^2+\|B\|\langle BCx, Cx\rangle-\|B\|\langle Cx, B(x)-\|Ax\|^2 =$$
$$= \|B\|^2\|Cx\|^2+\|B\|\langle BCx, Cx\rangle-\|B\|\langle BCx, Cx\rangle-\|A\|^2 =$$
$$= \|B\|^2\|Cx\|^2-\|Ax\|^2 \geqq 0$$

((3)' is true, with $m=\|B\|$).

Assume now (3)', let $b\colon C(D(A))\to H$ defined by

$$b(Cx) = Ax \tag{4)'$$

Clearly:

a) $b$ is well defined since if $Cx=0$, then by (3)' $Ax=0$.
b) $b$ is linear because of the same property of $A$ and $C$.
c) $b$ is bounded since by (3)'

$$\langle Ax, Cx\rangle-\langle Cx, Ax\rangle\in\mathbf{R} \quad (x\in D(A)) \Rightarrow \operatorname{Im}\langle Ax, Cx\rangle = 0 \quad (x\in D(A)) \tag{$*$}$$
$$\Rightarrow \langle Ax, Cx\rangle\in\mathbf{R} \quad (x\in D(A)).$$

Now we have by (3)', (4)' and ($*$)

$$m^2\|Cx\|^2-\|b(Cx)\|^2 = m^2\|Cx\|^2-\|Ax\|^2 =$$
$$= m^2\|Cx\|^2-\|Ax\|^2+m(\langle Ax, Cx\rangle-\langle Cx, Ax\rangle) =$$
$$= \langle (mC+A)x, (mC-A)x\rangle \geqq 0 \qquad (x\in D(A))$$

and therefore $b$ is bounded with $\|b\|\leqq m$, moreover,

d) $b$ is symmetric in $C(D(A))$ since in view of $(*)$

$$\langle b(Cx), Cx\rangle = \langle Ax, Cx\rangle = \langle Cx, Ax\rangle = \langle Cx, b(Cx)\rangle \qquad (x\in D(A)).$$

Now, by Krein's Theorem (see [1], Cor. 3) there exists a self-adjoint extension $B$ of $b$ defined on the whole space, with the same bound.

## REFERENCES

[1] Sebestyén, Z., Restrictions of positive operators, *Acta Sci. Math. (Szeged)* **46** (1983), 299—301. *MR* **85i:** 47003b.

[2] Sebestyén, Z., On the existence of certain semibounded selfadjoint operators in Hilbert space, *Acta Math. Hungar.* **47** (1986), 29—32. *MR* **88d:** 47033.

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# REFINEMENTS FOR FINITE DIRECT DECOMPOSITIONS IN MODULAR LATTICES

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## 1. Introduction

In the present paper we investigate finite direct decompositions of the unit element of a modular lattice. We give here a generalization of some results of papers [2], [3] and [5].

Let $L$ be a bounded lattice. Then $L$ has a least and a largest element, these elements will be denoted by 0 or 1, respectively. Denote by $\vee$ or $\wedge$ the join or meet in $L$, respectively.

If $a$ is an element of $L$, then we say that $a$ is a direct join of the elements $a_1, a_2, \ldots$ $\ldots, a_n$, and we write

$$a = a_1 \dot{\vee} a_2 \dot{\vee} \ldots \dot{\vee} a_n,$$

if $a = a_1 \vee a_2 \vee \ldots \vee a_n$ and for each $i = 1, 2, \ldots, n$,

$$a_i \wedge (a_1 \vee \ldots \vee a_{i-1} \vee a_{i+1} \vee \ldots \vee a_n) = 0.$$

The direct join of the elements $a_i$, $i \in I$ is also written $\dot{\vee}(a_i\colon i \in I)$.

An element $b$ is called a direct summand of $a$ if and only if $a = b \dot{\vee} x$ for some element $x$. We denote by $S(L)$ the set of all direct summands of the unit element of $L$.

Let $a \in S(L)$. An element $b$ is called a complement of $a$ iff $1 = a \dot{\vee} b$. For $a \in S(L)$, if $b$ is a complement of $a$, then the function $\alpha$ of $L$ defined by

$$x\alpha = a \wedge (x \vee b) \quad \text{for every} \quad x \in L, \tag{1}$$

is called a decomposition endomorphism of $a$.

Let

$$1 = a \dot{\vee} b, \tag{2}$$

and let $\alpha$ be the function of $L$ defined by the formula (1). Define the function $\beta$ on $L$ by $x\beta = b \wedge (x \vee a)$. The maps $\alpha$, $\beta$ are called the decomposition endomorphisms related to decomposition (2). We say also that $\alpha, \beta$ is a pair of complementary decomposition endomorphisms of $L$. Any member of the pair will be called a decomposition endomorphism of $L$.

Let $E(L)$ denote the smallest set satisfying 1) and 2):

1) if $\varphi$ is a decomposition endomorphism of $L$, then $\varphi \in E(L)$,

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2) if $\varphi, \psi \in E(L)$, then $\varphi\psi \in E(L)$[1].
The elements of the set $E(L)$ are called the endomorphisms of $L$.

Let $\varphi \in E(L)$ and $a, b \in L$. If $a\varphi = b$ and from $c \leqq a$, $c\varphi = 0$ follows $c = 0$, then we say that $\varphi$ induces an isomorphism of $a$ onto $b$. In particular, if $\varphi$ induces an isomorphism of $a$ onto $a$, then we say that $\varphi$ induces an automorphism of $a$.

DEFINITION 1. Let $a \in S(L)$. We say that $a$ satisfies the B-condition in the lattice $L$ iff for every decomposition endomorphism $\alpha$ of $a$ and for every pair $\delta$, $\varepsilon$ of complementary decomposition endomorphisms of $L$, there exists a direct decomposition $a = a_1 \dot{\vee} a_2$ such that an automorphism is induced in $a_1$ by $\alpha\delta\alpha$ and in $a_2$ by $\alpha\varepsilon\alpha$.

We shall say that a direct summand $a$ of 1 satisfies the $B^*$-condition (in $L$) if for every $a' \in L$ such that $a'$ is a direct summand of $a$, $a'$ satisfies the B-condition (in $L$).

The main result of our paper is expressed in the following refinement theorem.

THEOREM 1. *Let $L$ be a bounded modular lattice. If the unit element of $L$ has two direct decompositions*

$$1 = a_1 \dot{\vee} \dots \dot{\vee} a_m = b_1 \dot{\vee} \dots \dot{\vee} b_n \tag{3}$$

*such that, for each $1 \leqq j \leqq n-1$, $b_j$ satisfies the $B^*$-condition, then there exist direct decompositions*

$$\begin{aligned} a_i &= a_{i1} \dot{\vee} \dots \dot{\vee} a_{in} \quad (i = 1, 2, \dots, n) \\ b_j &= b_{j1} \dot{\vee} \dots \dot{\vee} b_{jm} \quad (j = 1, 2, \dots, m) \end{aligned} \tag{4}$$

*with the property:*

*if $1 \leqq k \leqq m$, and if $J$ is a subset of $\{1, 2, \dots, n\}$, then*

$$1 = \dot{\bigvee}(a_i\colon i \neq k) \dot{\vee} \dot{\bigvee}(a_{kj}\colon j \in J) \dot{\vee} \dot{\bigvee}(b_{jk}\colon j \notin J) \tag{$*$}$$

*holds.*

In Section 4 we give some consequences of this theorem. Throughout this paper $L$ will denote a bounded modular lattice.

## 2. Properties of decomposition endomorphisms

The following lemmas state elementary properties of endomorphisms of lattices.

LEMMA 1 (cf. [3], Lemma 2). *If $\varphi$ is an endomorphism of a lattice $L$ and $T$ is a subset of $L$ such that $\bigvee T$ exists, then*

$$(\bigvee T)\varphi = \bigvee(t\varphi\colon t \in T)$$

*holds.*

LEMMA 2 (cf. [3], Lemma 3). *Let $\varphi \in E(L)$, $x, y \in L$, and $x\varphi = y$. Then for any $y' \in L$ with $y' \leqq y$, there is an $x' \in L$ satisfying $x' \leqq x$ and $x'\varphi = y'$.*

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[1] $\varphi\psi$ is the map of $L$ defined by $x(\varphi\psi) = (x\varphi)\psi$, $x \in L$.

LEMMA 3 (cf. [3], p. 92). *Let* $\varphi\in E(L)$, *and let* $a, b\in L$. *If* $\varphi$ *induces an isomorphism of a onto b, then* $\varphi$ *is an isomorphism between* $a/0$[2] *and* $b/0$.

LEMMA 4 (cf. [4], p. 453). *If* $\varphi$ *induces an automorphism of elements* $x_1$ *and* $x_2$, *and* $x_1\wedge x_2=0$, *then* $\varphi$ *induces an automorphism of* $x_1\dot\vee x_2$.

LEMMA 5 (cf. [4], Theorem 20). *Let* $\alpha$ *be a decomposition endomorphism of L and let* $\delta, \varepsilon$ *be a pair of complementary decomposition endomorphisms of L. We put*

$$\eta = \alpha\delta\alpha\varepsilon\alpha, \quad \eta_1 = \alpha\delta\alpha, \quad \eta_2 = \alpha\varepsilon\alpha.$$

*For every natural number i,* $\eta$ *induces an automorphism of* $1\eta^i$ *iff an automorphism of* $1\eta_k^i$ *is induced by* $\eta_k$, *for* $k=1, 2$.

LEMMA 6 (cf. [1], Lemma 1). *Let* $\eta_1$ *and* $\eta_2$ *be given as in Lemma 5. Then for every* $x\in L$,

$$x\eta_1\eta_2 = x\eta_2\eta_1.$$

LEMMA 7. *Let a be a direct summand of* 1 *and let* $\alpha$ *be a decomposition endomorphism of a. Then for every* $x\in L$,

$$x \leqq a \quad \textit{implies} \quad x\alpha = x.$$

PROOF. Let $b$ be a complement of $a$ such that $x\alpha=a\wedge(x\vee b)$ for every $x\in L$. Hence by modularity we obtain

$$x\alpha = x\vee(a\wedge b) = x\vee 0 = x.$$

LEMMA 8 (cf. [6], Lemma 3.1). *Let* $1=a\dot\vee b$ *and*

$$a = a_1\dot\vee a_2. \tag{5}$$

*Then*

$$1 = a_1\dot\vee(a_2\vee b). \tag{6}$$

*If* $\alpha_1, \alpha_2$ *and* $\alpha_1', \alpha_2'$ *are the pairs of decomposition endomorphisms related to decompositions* (5) *and* (6), *respectively, then for every* $x\leqq a$ *we obtain*

$$x\alpha_1 = x\alpha_1' \quad \textit{and} \quad x\alpha_2 = x\alpha_2'.$$

Let $\varphi$ be an endomorphism of a lattice $L$. We denote by $k(\varphi)$ the join of all $x\in L$ such that $x\varphi=0$[3].

By Lemma 1 we have

$$k(\varphi)\varphi = 0. \tag{7}$$

LEMMA 9 (cf. [4], Lemma 6). *If* $\varphi\in E(L)$ *and* $i$ *is a natural number, then*

$$k(\varphi^i) = k(\varphi^{i+1}) \quad \textit{if and only if} \quad 1\varphi^i\wedge k(\varphi^i) = 0.$$

---

[2] For two elements $x, y\in L$, $y/x=\{u\in L\colon x\leqq u\leqq y\}$.
[3] By Lemma 4 from [3], $\bigvee(x\colon x\varphi=0)$ exists.

Lemma 10. *For every endomorphism* $\varphi$ *of* $L$ *the following conditions are equivalent:*

(i) *there exists a natural number* $i$ *such that* $1\varphi^i=1\varphi^{i+1}$ *and* $k(\varphi^i)=k(\varphi^{i+1})$,

(ii) *there exists a natural number* $i$ *such that an automorphism of* $1\varphi^i$ *is induced by* $\varphi$.

Proof. The proof of the implication (i)⇒(ii) is given in [5], p. 746.

(ii) implies (i). Suppose that an automorphism of $1\varphi^i$ is induced by $\varphi$. Consequently,

$$1\varphi^i = 1\varphi^{i+1}. \tag{8}$$

Now we will prove that $1\varphi^i\wedge k(\varphi^i)=0$. Suppose on the contrary that $1\varphi^i\wedge k(\varphi^i)\neq 0$. Then there exists an element $x\neq 0$ such that $x\leqq 1\varphi^i$ and $x\leqq k(\varphi^i)$. By (7), we have $x\varphi^i=0$. Hence $x=0$ because $\varphi^i$ induces an automorphism of $1\varphi^i$. Thus we obtain a contradiction and therefore we must have $1\varphi^i\wedge k(\varphi^i)=0$. By Lemma 9 we conclude that $k(\varphi^i)=k(\varphi^{i+1})$. From this and (8) we obtain (i).

## 3. Proof of Theorem 1

The following lemmas are required for the proof of refinement theorem.

Lemma 11. *Let* $a_1$ *be a direct summand of the direct summand* $a$ *of* 1. *If* $a_1$ *satisfies the B-condition in* $L$, *then* $a_1$ *satisfies the B-condition in the lattice* $a/0$.

Proof. Since $a_1$ is a direct summand of $a$, there exists an element $a_2$ such that $a=a_1\dot\vee a_2$. Let $\alpha_1, \alpha_2$ be the decomposition endomorphisms related to this decomposition of $a$. Let $\beta_1, \beta_2$ be a pair of complementary decomposition endomorphisms of $a/0$ — for example — induced by a direct decomposition $a=b_1\dot\vee b_2$. Since $a$ is a direct summand of 1, there exists an element $b$ such that $1=a\dot\vee b$. Therefore,

$$1 = a_1\dot\vee(a_2\vee b) = b_1\dot\vee(b_2\vee b).$$

Let $\alpha_1', \alpha_2'$ and $\beta_1', \beta_2'$ be the decomposition endomorphisms related to these decompositions of 1. Since $a_1$ satisfies the B-condition in $L$, there exists a direct decomposition $a_1=a_1'\dot\vee a_1''$ such that an automorphism is induced in $a_1'$ by $\alpha_1'\beta_1'\alpha_1'$ and in $a_1''$ by $\alpha_1'\beta_2'\alpha_1'$. By Lemma 8, for every $x\leqq a$ we have $x\alpha_1\beta_i\alpha_1=x\alpha_1'\beta_i'\alpha_1'$, $i=1, 2$. Therefore, $\alpha_1\beta_1\alpha_1$ induces an automorphism of $a_1'$ and $\alpha_1\beta_2\alpha_1$ induces an automorphism of $a_1''$. Then, by Definition 1, $a_1$ satisfies the B-condition in $a/0$.

The next lemma is an immediate consequence of Lemma 11.

Lemma 12. *Let* $a_1$ *be a direct summand of the direct summand* $a$ *of* 1. *If* $a_1$ *satisfies the* $B^*$*-condition in* $L$, *then* $a_1$ *satisfies the* $B^*$*-condition in* $a/0$.

Lemma 13. *Let the unit element of* $L$ *have two direct decompositions*

$$1 = a\dot\vee b = d\dot\vee e, \tag{9}$$

*and let at least one of the summands of these decompositions of* 1 *satisfy the B-condition* *in L. Then there exist direct decompositions*

(10) $$a = a'\dot{\vee} a'', \quad b = b'\dot{\vee} b'', \quad d = d'\dot{\vee} d'', \quad e = e'\dot{\vee} e''$$

*such that*

(11) $$a'\dot{\vee} b' = b'\dot{\vee} d' = d'\dot{\vee} e' = e'\dot{\vee} a',$$
$$a''\dot{\vee} d'' = d''\dot{\vee} e'' = e''\dot{\vee} b'' = b''\dot{\vee} a''.$$

PROOF. The proof of the lemma is given in [5] (cf. Theorem 1).

We need the following

LEMMA 14. *If an element* $a \in L$ *has two direct decompositions*

$$a = d\dot{\vee} a_1 \dot{\vee} \ldots \dot{\vee} a_n = d\dot{\vee} b,$$

*then* $b = b_1 \dot{\vee} \ldots \dot{\vee} b_n$, *where*

$$b_i = b \wedge (d \vee a_i), \qquad i = 1, 2, \ldots, n,$$

*and*

$$d\dot{\vee} a_i = d\dot{\vee} b_i \quad \textit{for} \quad i = 1, 2, \ldots, n.$$

PROOF. The proof of the lemma is just the same as the proof of Lemma 3 from [5].

LEMMA 15. *Let* $1 = a\dot{\vee} b$ *and let* $a_1$ *be a direct summand of* $a$. *If* $a_0$ *is a direct summand of* 1 *such that*

(12) $$b\dot{\vee} a_0 = b\dot{\vee} a_1,$$

*and if* $a_0$ *satisfies the B-condition in L, then* $a_1$ *satisfies the B-condition in* $a/0$.

PROOF. Since $a_1$ is direct summand of $a$, there exists an element $a_2$ such that $a = a_1 \dot{\vee} a_2$. Let $\alpha_1, \alpha_2$ be the decomposition endomorphisms related to this decomposition of $a$. Let $\beta_1, \beta_2$ be a pair of complementary decomposition endomorphisms of $a/0$. Then there exists a direct decomposition $a = b_1 \dot{\vee} b_2$ of $a$ such that $\beta_1, \beta_2$ are the decomposition endomorphisms related to this decomposition. Clearly,

$$1 = a_1 \dot{\vee} (a_2 \vee b) = b_1 \dot{\vee} (b_2 \vee b) = a_0 \dot{\vee} (a_2 \vee b).$$

Denote by $\alpha_1', \alpha_2', \beta_1', \beta_2'$ and $\alpha', \beta'$ the pairs of decomposition endomorphisms related to these decompositions of 1. Since $a_0$ satisfies the B-condition, there exists a direct decomosition $a_0 = a_0' \dot{\vee} a_0''$ such that an automorphism is induced in $a_0'$ by $\alpha' \beta_1' \alpha'$ and in $a_0''$ by $\alpha' \beta_2' \alpha'$. From (12) we conclude, by Lemma 14, that $a_1 = a_1' \dot{\vee} a_1''$, where $a_1' = a_1 \wedge (b \vee a_0')$, $a_1'' = a_1 \wedge (b \vee a_0'')$ and

(13) $$b\dot{\vee} a_0' = b\dot{\vee} a_1', \quad b\dot{\vee} a_0'' = b\dot{\vee} a_1''.$$

It is obvious that for every $x_1, x_2 \in L$,

(14) $$\text{if} \quad x_1 \vee b = x_2 \vee b, \quad \text{then} \quad x_1 \alpha_1' = x_2 \alpha_1' \quad \text{and} \quad x_1 \beta_1' = x_2 \beta_1'.$$

Hence in view of (13) we obtain

(15) $$a_0'\alpha_1' = a_1'\alpha_1' \quad \text{and} \quad a_0'\beta_1' = a_1'\beta_1'.$$

Now we shall prove that for every $x \in L$

(16) $$x\alpha' \vee b = x\alpha_1' \vee b.$$

Indeed, using (12) and by modularity we have

$$x\alpha' \vee b = [a_0 \wedge (x \vee a_2 \vee b)] \vee b = (a_0 \vee b) \wedge (x \vee a_2 \vee b) =$$
$$= (a_1 \vee b) \wedge (x \vee a_2 \vee b) = [a_1 \wedge (x \vee a_2 \vee b)] \vee b = x\alpha_1' \vee b.$$

By (14) and (16) we conclude that for every $x \in L$,

(17) $$x\alpha'\alpha_1' = x\alpha_1'\alpha_1' = x\alpha_1' \quad \text{and} \quad x\alpha'\beta_1' = x\alpha_1'\beta_1'.$$

By Lemmas 8 and 7, and from (15) and (17) we obtain

$$a_1'\alpha_1\beta_1\alpha_1 = a_1'\alpha_1'\beta_1'\alpha_1' = a_1'\beta_1'\alpha_1' = a_0'\beta_1'\alpha_1' = a_0'\beta_1'\alpha'\alpha_1' = (a_0'\alpha'\beta_1'\alpha')\alpha_1'.$$

Then, since $\alpha'\beta_1'\alpha'$ induces an automorphism of $a_0'$ we have $a_1'\alpha_1\beta_1\alpha_1 = a_0'\alpha_1'$, and hence, by (15) and Lemma 7 we obtain

(18) $$a_1'\alpha_1\beta_1\alpha_1 = a_1'.$$

Suppose now that $x \leqq a_1'$ and $x\alpha_1\beta_1\alpha_1 = 0$. Since $a_0'\alpha_1' = a_1'$ and $x \leqq a_1'$, by Lemma 2 we conclude that there exists an element $y \in L$ such that $y \leqq a_0'$ and $y\alpha_1' = x$. Compute:

$$y\alpha'\beta_1'\alpha' = (\text{using (17) and } y\alpha_1' = x) =$$
$$= y\alpha_1'\beta_1'\alpha' = x\beta_1'\alpha' = (\text{apply Lemmas 8 and 7}) =$$
$$= x\beta_1\alpha' = x\alpha_1\beta_1\alpha' \leqq x\alpha_1\beta_1\alpha' \vee b = (\text{by (16) and Lemma 8}) =$$
$$= x\alpha_1\beta_1\alpha_1' \vee b = x\alpha_1\beta_1\alpha_1 \vee b.$$

Hence, since $x\alpha_1\beta_1\alpha_1 = 0$, we get $y\alpha'\beta_1'\alpha' \leqq b$. Furthermore, $y\alpha'\beta_1'\alpha' \leqq a_0$. Consequently, $y\alpha'\beta_1'\alpha' \leqq b \wedge a_0 = 0$, that is $y\alpha'\beta_1'\alpha' = 0$. Therefore, since $\alpha'\beta_1'\alpha'$ induces an automorphism of $a_0'$ we conclude that $y = 0$. Then $x = y\alpha_1' = 0$. Thus, from $x \leqq a_1'$ and $x\alpha_1\beta_1\alpha_1 = 0$ follows $x = 0$. From this and (18) we obtain that $\alpha_1\beta_1\alpha_1$ induces an automorphism of $a_1'$.

Now observe that

(19) $$a_0''\alpha_1' = a_1''\alpha_1', \quad a_0''\beta_2'\alpha_1' = a_1''\beta_2'\alpha_1', \quad \text{and for every} \quad x \in L,\ x\alpha'\beta_2'\alpha' = x\alpha_1'\beta_2'\alpha'.$$

The first equality holds because $a_0'' \vee b = a_1'' \vee b$. We prove the second. By (13) and Lemma 1 we obtain

$$a_0''\beta_2' \vee b\beta_2' = a_1''\beta_2' \vee b\beta_2'.$$

Since $b\beta_2' = b$, we have $a_0''\beta_2' \vee b = a_1''\beta_2' \vee b$. Hence in view of (14) we get the second equality of (19). Similarly, by (16) we conclude that the last equality of (19) holds for every $x \in L$.

Applying (19) it is easy to verify that $\alpha_1\beta_2\alpha_1$ induces an automorphism of $a_1''$. As an immediate consequence of Lemma 15 we get

LEMMA 16. *Let* $1=a\dot{\vee}b$ *and let* $a_1$ *be a direct summand of* $a$. *Let* $a_0$ *be a summand of* 1 *such that* (12) *holds. If* $a_0$ *satisfies the* $B^*$*-condition in* $L$, *then* $a_1$ *satisfies the* $B^*$*-condition in* $a/0$.

Finally, we prove the following

LEMMA 17. *Let*

$$1 = d\dot{\vee}e = b_1\dot{\vee}b_2\dot{\vee}\dots\dot{\vee}b_n \tag{20}$$

*be two decompositions of* 1. *If for every* $j\in\{2, \dots, n\}$, $b_j$ *satisfies the B-condition, then there exist direct decompositions*

$$\begin{gathered} b_j = b_j'\dot{\vee}b_j'', \quad j = 1, 2, \dots, n \\ d = d_1\dot{\vee}\dots\dot{\vee}d_n, \quad e = e_1\dot{\vee}\dots\dot{\vee}e_n \end{gathered} \tag{21}$$

*such that*

$$1 = d\dot{\vee}\dot{\bigvee}(e_j\colon j\in J)\dot{\vee}\dot{\bigvee}(b_j''\colon j\notin J), \tag{22}$$

*and*

$$1 = e\dot{\vee}\dot{\bigvee}(d_j\colon j\in J)\dot{\vee}\dot{\bigvee}(b_j'\colon j\notin J), \tag{23}$$

*hold for every subset* $J$ *of* $\{1, 2, \dots, n\}$.

PROOF. We prove Lemma 17 by induction on $n$. Let $n=2$. By Lemma 13, two direct decompositions $1=d\dot{\vee}e=b_1\dot{\vee}b_2$ have refinements:

$$d = d_1\dot{\vee}d_2,\ e = e_2\dot{\vee}e_1,\ b_1 = b_1'\dot{\vee}b_1'',\ b_2 = b_2''\dot{\vee}b_2'$$

such that

$$d_1\dot{\vee}e_2 = e_2\dot{\vee}b_1' = b_1'\dot{\vee}b_2'' = b_2''\dot{\vee}d_1,$$

$$d_2\dot{\vee}b_1'' = b_1''\dot{\vee}b_2' = b_2'\dot{\vee}e_1 = e_1\dot{\vee}d_2.$$

Therefore,

$$\begin{aligned} 1 &= d\dot{\vee}e_1\dot{\vee}b_2'' = d\dot{\vee}e_2\dot{\vee}b_1'' = e\dot{\vee}d_1\dot{\vee}b_2' = \\ &= e\dot{\vee}d_2\dot{\vee}b_1' = d\dot{\vee}b_1''\dot{\vee}b_2'' = e\dot{\vee}b_1'\dot{\vee}b_2', \end{aligned}$$

and hence the lemma holds for $n=2$.

Let $n>2$. We set

$$b_1\dot{\vee}\dots\dot{\vee}b_{n-1} = a,\quad b_n = b.$$

Then we obtain direct decompositions (9). By Lemma 13 these decompositions of 1 have refinements (10) such that (11) holds.

We consider the following decompositions of $a$:

$$a = d'\dot{\vee}a'' = b_1\dot{\vee}\dots\dot{\vee}b_{n-1}. \tag{24}$$

From Lemma 11 we conclude that each $b_j$, $j=2, \ldots, n-1$ satisfies the B-condition in $a/0$. Therefore, by the induction hypothesis we obtain refinements:

(25)
$$b_j = b'_j \dot{\vee} b''_j, \quad j = 1, 2, \ldots, n-1,$$
$$a' = a'_1 \dot{\vee} \ldots \dot{\vee} a'_{n-1}, \quad a'' = a''_1 \dot{\vee} \ldots \dot{\vee} a''_{n-1}$$

such that

(26)
$$a = d' \dot{\vee} \dot{\bigvee}(a''_j\colon j\in J) \dot{\vee} \dot{\bigvee}(b''_j\colon j\notin J),$$
$$a = a'' \dot{\vee} \dot{\bigvee}(a'_j\colon j\in J) \dot{\vee} \dot{\bigvee}(b'_j\colon j\notin J),$$

where $J$ is a subset of $\{1, \ldots, n-1\}$. Since $e' \dot{\vee} a' = e' \dot{\vee} d'$ and $a' = a'_1 \dot{\vee} \ldots \dot{\vee} a'_{n-1}$, so $e' \dot{\vee} a'_1 \dot{\vee} \ldots \dot{\vee} a'_{n-1} = e' \dot{\vee} d'$. Hence, by Lemma 14, there exists the following direct decomposition of $d'$:

(27)
$$d' = d_1 \dot{\vee} \ldots \dot{\vee} d_{n-1},$$

where $d_j = d' \wedge (e' \vee a'_j)$ for $j<n$, and

$$e' \dot{\vee} a'_j = e' \dot{\vee} d_j, \quad j = 1, 2, \ldots, n-1.$$

We set $e_j = e'' \wedge (d'' \vee a''_j)$ for $j<n$. Then it follows from $e'' \dot{\vee} d'' = d'' \dot{\vee} a''$, $a'' = a''_1 \dot{\vee} \ldots \dot{\vee} a''_{n-1}$ and from Lemma 14, that

(28)
$$e'' = e_1 \dot{\vee} \ldots \dot{\vee} e_{n-1}$$

and

(29)
$$d'' \dot{\vee} a''_j = d'' \dot{\vee} e_j, \quad \text{for} \quad j < n.$$

Finally, we let

$$b' = b''_n, \quad b'' = b'_n, \quad d'' = d_n, \quad e' = e_n.$$

Then, from (10), (25), (27) and (28) we obtain (21).

We have to show the validity of two contentions (22) and (23). For reasons of symmetry it suffices to verify one of these properties, say (22). We distinguish two cases.

*Case 1.* $n$ does belong to $J$. Then we may assume without loss of generality that $J = \{n, 1, 2, \ldots, i\}$ for some $i<n$. From (29), and from equalities $d' \dot{\vee} e' = e' \dot{\vee} a'$ and $a'' = a''_1 \dot{\vee} \ldots \dot{\vee} a''_{n-1}$ we obtain

$$1 = d \dot{\vee} e = d' \dot{\vee} d'' \dot{\vee} e_1 \dot{\vee} \ldots \dot{\vee} e_{n-1} \dot{\vee} e' = d' \dot{\vee} e' \dot{\vee} d'' \dot{\vee} a''_1 \dot{\vee} \ldots \dot{\vee} a''_{n-1} =$$
$$= e' \dot{\vee} a' \dot{\vee} d'' \dot{\vee} a'' = e' \dot{\vee} d'' \dot{\vee} a.$$

Hence, by (26) and (29) we have

$$1 = e' \dot{\vee} d'' \dot{\vee} a = e' \dot{\vee} d'' \dot{\vee} a' \dot{\vee} a''_1 \dot{\vee} \ldots \dot{\vee} a''_i \dot{\vee} b''_{i+1} \dot{\vee} \ldots \dot{\vee} b''_{n-1} =$$
$$= d' \dot{\vee} e' \dot{\vee} d'' \dot{\vee} e_1 \dot{\vee} \ldots \dot{\vee} e_i \dot{\vee} b''_{i+1} \dot{\vee} \ldots \dot{\vee} b''_{n-1} = d \dot{\vee} \dot{\bigvee}(e_j\colon j\in J) \dot{\vee} \dot{\bigvee}(b''_j\colon j\notin J).$$

*Case 2.* $n$ does not belong to $J$. Then we may assume without loss of generality that $J=\{1, 2, \dots, i\}$ for some $i<n$. Then

$$1 = d'\dot{\vee} e'\dot{\vee} d''\dot{\vee} e_1\dot{\vee}\dots\dot{\vee} e_i\dot{\vee} b''_{i+1}\dot{\vee}\dots\dot{\vee} b''_{n-1} =$$
$$= d'\dot{\vee} b'\dot{\vee} d''\dot{\vee} e_1\dot{\vee}\dots\dot{\vee} e_i\dot{\vee} b''_{i+1}\dot{\vee}\dots\dot{\vee} b''_{n-1} = d\dot{\vee}\dot{\bigvee}(e_j\colon j\in J)\dot{\vee}\dot{\bigvee}(b''_j\colon j\notin J).$$

This completing the verification of (22) and completing the inductive proof of Lemma for every $n$.

Now we are able to prove our fundamental refinement theorem.

For $m=2$ and for every $n$ the statement follows from Lemma 17. Now let us assume the statement to hold for $m-1$ and for every $n$. Let

$$d = a_1\dot{\vee}\dots\dot{\vee} a_{m-1},\ e = a_m. \tag{30}$$

Then we obtain direct decompositions (20). Applying Lemma 17 upon these decompositions we obtain refinements (21) meeting the requirements (22) and (23). From (23) we infer in particular that

$$1 = e\dot{\vee} b'_1\dot{\vee}\dots\dot{\vee} b'_n. \tag{31}$$

Let $v_j=d\wedge(e\vee b'_j)$. Then, from $1=e\dot{\vee}d$, (31) and Lemma 14 it follows that $d=$ $d=v_1\dot{\vee}\dots\dot{\vee} v_n$ and

$$e\dot{\vee} b'_j = e\dot{\vee} v_j \qquad \text{for} \quad j = 1, 2, \dots, n. \tag{32}$$

For every $j=1, 2, \dots, n-1$, $b'_j$ satisfies the $B^*$-condition in $L$, and therefore, by Lemma 16 and (32), $v_j$ $(j=1, 2, \dots, n-1)$ satisfies the $B^*$-condition in $d/0$. Thus we may apply the induction hypothesis upon the direct decompositions

$$d = a_1\dot{\vee}\dots\dot{\vee} a_{m-1} = v_1\dot{\vee}\dots\dot{\vee} v_n.$$

Consequently there exist direct decompositions

$$\begin{aligned} a_i &= a_{i1}\dot{\vee}\dots\dot{\vee} a_{in} \qquad (i = 1, 2, \dots, m-1)\\ v_j &= v_{j1}\dot{\vee}\dots\dot{\vee} v_{j,m-1} \qquad (j = 1, 2, \dots, n)\end{aligned} \tag{33}$$

with the property

if $0<k<m$, and if $J$ is a subset of $\{1, 2, \dots, n\}$, then

$$d = \dot{\bigvee}(a_i\colon i\neq k)\dot{\vee}\dot{\bigvee}(a_{kj}\colon j\in J)\dot{\vee}\dot{\bigvee}(v_{jk}\colon j\notin J) \tag{34}$$

holds.

Let $b_{ji}=b'_j\wedge(e\vee v_{ji})$, $j=1, \dots, n$, $i=1, \dots, m-1$. In view of (32) and (33), from Lemma 14 we infer that

$$\begin{aligned} b'_j &= b_{j1}\dot{\vee}\dots\dot{\vee} b_{j,m-1},\\ e\dot{\vee} v_{ji} &= e\dot{\vee} b_{ji},\end{aligned} \tag{35}$$

for $j=1, 2, \ldots, n$ and $i=1, 2, \ldots, m-1$. Finally, we let

(36) $$b_{jm} = b_j'' \quad \text{and} \quad a_{mj} = e_j \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Then it is clear that

$$a_i = a_{i1}\dot\vee\ldots\dot\vee a_{in}, \qquad b_j = b_{j1}\dot\vee\ldots\dot\vee b_{jm}$$

for $i=1, 2, \ldots, m$, and $j=1, 2, \ldots, n$. Now we shall prove that these direct decompositions have the property $(*)$.

Let $J$ be a subset of $\{1, 2, \ldots, n\}$ and let $1 \leqq k \leqq m$. We shall consider two cases.

*Case 1.* $k<m$. We may assume without loss of generality that $k=1$. By (34) and (35) we obtain

$$1 = d\dot\vee e = a_2\dot\vee\ldots\dot\vee a_{m-1}\dot\vee\dot\bigvee(a_{1j}\colon j\in J)\dot\vee\dot\bigvee(v_{j1}\colon j\notin J)\dot\vee e =$$

$$= a_2\dot\vee\ldots\dot\vee a_{m-1}\dot\vee\dot\bigvee(a_{1j}\colon j\in J)\dot\vee\dot\bigvee(b_{j1}\colon j\notin J)\dot\vee e =$$

$$= \dot\bigvee(a_i\colon i \neq 1)\dot\vee\dot\bigvee(a_{1j}\colon j\in J)\dot\vee\dot\bigvee(b_{j1}\colon j\notin J).$$

*Case 2.* $k=m$. From (22), (30) and (36) we have

$$1 = d\dot\vee\dot\bigvee(e_j\colon j\in J)\dot\vee\dot\bigvee(b_j''\colon j\notin J) = a_1\dot\vee\ldots\dot\vee a_{m-1}\dot\vee\dot\bigvee(a_{mj}\colon j\in J)\dot\vee\dot\bigvee(b_{jm}\colon j\notin J).$$

Thus Theorem 1 is true for every $m$ and $n$.

## 4. Some consequences of refinement theorem

Let $L$ be a modular lattice with 0 and 1. If every direct summand of the unit element of $L$ satisfies the $B$-condition in $L$, then we shall call $L$ a $B$-lattice.

From Theorem 1 we obtain now

THEOREM 2. *Let L be a B-lattice and let the unit element of L have two direct decompositions* (3). *Then there exist direct decompositions* (4) *with the property* $(*)$.

REMARK 1. By Lemma 10 from [2] (p. 489), Theorem 2 implies Theorem 2 [2].

REMARK 2. By Lemma 14 [3] (p. 98) and Theorem 2 we obtain Theorem 5 [3].

DEFINITION 2 (see [3], p. 95). We say that an endomorphism $\varphi$ of a lattice $L$ is distinguished if there is a decomposition endomorphism $\alpha$ of $L$ and a pair $\delta, \varepsilon$ of complementary decomposition endomorphisms of $L$ such that $\varphi=\alpha\delta\alpha\varepsilon\alpha$.

LEMMA 18. *Let every distinguished endomorphism $\varphi$ of a lattice L (L is modular with* 0 *and* 1) *satisfy condition* (ii) *of Lemma 10. Then L is a B-lattice.*

PROOF. Let $a$ be a direct summand of 1, and let $\alpha$ be a decomposition endomorphism of $a$. Let $\delta, \varepsilon$ be a pair of complementary decomposition endomorphisms of $L$. By assumption, endomorphism $\eta=\alpha\delta\alpha\varepsilon\alpha$ satisfies (ii), that is, there exists a natural number $i$ such that an automorphism of $1\eta^i$ is induced by $\eta$. By the proof of Theorem 11 from [5] we obtain

$$a = [a\wedge k(\eta_1^i)]\dot\vee[a\wedge k(\eta_2^i)]\dot\vee 1\eta^i,$$

where $\eta_1=\alpha\delta\alpha$ and $\eta_2=\alpha\varepsilon\alpha$. We put

$$a_1 = a\wedge k(\eta_2^i), \quad a_2 = [a\wedge k(\eta_1^i)]\dot{\vee} 1\eta^i.$$

By the proof of Lemma 12 from [3] we conclude that $\eta_1$ induces an automorphism of $a\wedge k(\eta_2^i)$ and $\eta_2$ induces an automorphism of $a\wedge k(\eta_1^i)$.

Now we will prove that an automorphism of $1\eta^i$ is induced by $\eta_2$. In view of Lemma 6, we have

$$(1\eta^i)\eta_2 = 1(\eta_1\eta_2)^i\eta_2 = (1\eta_2^{i+1})\eta_1^i.$$

By Lemma 5, $\eta_2$ induces an automorphism of $1\eta_2^i$, and therefore $1\eta_2^{i+1}=1\eta_2^i$. Hence, $(1\eta^i)\eta_2=1\eta_2^i\eta_1^i=1\eta^i$. Moreover, if $x\leqq 1\eta^i$ and $x\eta_2=0$, then $x=0$. Thus $\eta_2$ induces an automorphism of $1\eta^i$, and by Lemma 4 we conclude that an automorphism of $a_2$ is induced by $\eta_2$. Therefore, $a$ satisfies the B-condition and the proof is complete.

Definition 3 (see [3], p. 101). We say that an element $x\in L$ has property $(A_1)$ iff, for every $y, z\in L$ with $z\leqq y\leqq x$, if the lattices $y/0$ and $y/z$ are isomorphic, then $z=0$. We say that an element $x\in L$ has property $(A_2)$ iff, for every $y, z\in L$ with $z\leqq y\leqq x$, if the lattices $y/0$ and $x/0$ are isomorphic, then $y=z$.

Lemma 19. *Let for every distinguished endomorphism $\varphi$ of a lattice L the following condition* (iii) *or* (iv) *be satisfied:*

(iii) $1\varphi$ *has property $(A_1)$ and the lattice $1\varphi/0$ satisfies the descending chain condition;*

(iv) $1\varphi$ *has property $(A_2)$ and the lattice $1\varphi/0$ satisfies the ascending chain condition.*

*Then L is a B-lattice.*

Proof. By Lemmas 18 and 10, it suffices to prove that (iii) implies condition (i) of Lemma 10 and (iv) implies (ii).

(iii) implies (i). Since $1\varphi\geqq 1\varphi^2\geqq\ldots\geqq 1\varphi^n\geqq\ldots$, and the lattice $1\varphi/0$ satisfies the descending chain condition, there exists a natural number $i$ such that $1\varphi^i=1\varphi^{i+1}$. Moreover, by the proof of Corollary 1 from [5] (p. 747) we conclude that $k(\varphi^i)=k(\varphi^{i+1})$. Therefore $\varphi$ satisfies condition (i).

(iv) implies (ii). Since

$$1\varphi\wedge k(\varphi) \leqq 1\varphi\wedge k(\varphi^2) \leqq \ldots \leqq 1\varphi\wedge k(\varphi^n) \leqq \ldots,$$

and $1\varphi/0$ satisfies the ascending chain condition, there exists a natural number $i$ such that $1\varphi\wedge k(\varphi^{i-1})=1\varphi\wedge k(\varphi^i)$. Clearly, $(1\varphi^i)\varphi=1\varphi^{i+1}$. Furthermore,

$$\text{(37)} \qquad \text{if } x\leqq 1\varphi^i \text{ and } x\varphi=0, \text{ then } x=0.$$

Indeed, since $x\leqq(1\varphi)\varphi^{i-1}$, by Lemma 2, there exists an element $y$ such that $y\leqq 1\varphi$ and $y\varphi^{i-1}=x$. Hence $y\varphi^i=0$, and so $y\leqq k(\varphi^i)$. Then $y\leqq 1\varphi\wedge k(\varphi^i)=1\varphi\wedge k(\varphi^{i-1})$ and therefore $x=y\varphi^{i-1}\leqq(1\varphi\wedge k(\varphi^{i-1}))\varphi^{i-1}\leqq k(\varphi^{i-1})\varphi^{i-1}$ (by (7))$=0$, i.e., $x=0$. Thus, $\varphi$ induces an isomorphism of $1\varphi^i$ onto $1\varphi^{i+1}$. Then, by Lemma 3, the lattices $1\varphi^i/0$ and $1\varphi^{i+1}/0$ are isomorphic. Since $1\varphi^{i+1}\leqq 1\varphi^i\leqq 1\varphi$, and $1\varphi$ has the property $(A_2)$, we conclude that $1\varphi^i=1\varphi^{i+1}$. This and (37) implies that $\varphi$ induces an automorphism of $1\varphi^i$. Then condition (ii) is satisfied. This ends the proof of Lemma 19.

By Lemmas 18, 10 and 19 we obtain the following

THEOREM 3. *Let L be a bounded modular lattice. If for every distinguished endomorphism $\varphi$ of L at least one of the conditions* (i)—(iv) *is satisfied, then L is a B-lattice.*

From this theorem we get at once

COROLLARY. *Every modular lattice of finite length is a B-lattice.*

DEFINITION 4. Let

$$1 = a_1 \dot{\vee} a_2 \dot{\vee} \dots \dot{\vee} a_n = b_1 \dot{\vee} b_2 \dot{\vee} \dots \dot{\vee} b_n. \tag{38}$$

We say that these decompositions are exchange isomorphic, if there is a permutation $\pi$ of the set $I=\{1, 2, \dots, n\}$ such that

$$1 = a_1 \dot{\vee} \dots \dot{\vee} a_{i-1} \dot{\vee} b_{\pi(i)} \dot{\vee} a_{i+1} \dot{\vee} \dots \dot{\vee} a_n,$$

for all $i \in I$.

Direct decompositions (38) are said to be directly similar if there is a permutation $\pi$ of $I$ such that for each $i \in I$ there exists an element $c_i$ such that

$$1 = a_i \dot{\vee} c_i = b_{\pi(i)} \dot{\vee} c_i.$$

Theorem 2 give the following

THEOREM 4. *If L is a B-lattice, then any two finite direct decompositions of the unit element of L have exchange isomorphic refinements (and so also directly similar refinements).*

Combining Theorems 3 and 4 we obtain

THEOREM 5. *Let L be a bounded modular lattice. If for every distinguished endomorphism $\varphi$ of L at least one of the conditions* (i)—(iv) *is satisfied, then any two finite direct decompositions of* 1 *have directly similar refinements.*

REMARK 3. From this theorem we have immediately Theorems 3 and 4 of paper [2] and Theorems 11 and 12 of [5].

## REFERENCES


[1] KUROŠ, A. G., Isomorphisms of direct decompositions, II, *Izv. Akad. Nauk SSSR Ser. Mat.* **10** (1946), 47—72 (in Russian, English summary). *MR* **8**—309.

[2] LIVŠIC, A. H., Direct decompositions of complete Dedekind structures, *Mat. Sbornik (N. S.)* **28** (70) (1951), 481—502 (in Russian). *MR* **13**—100.

[3] MOČUL'SKIĬ, E. N., Direct decompositions of Dedekind lattices, *Mat. Sbornik (N. S.)* **37 (79)** (1955), 89—102 (in Russian). *MR* **17**—120.

[4] MOČUL'SKIĬ, E. N., Splitting endomorphisms of lattices, *Izv. Akad. Nauk SSSR Ser. Mat.* **25** (1961), 443—476 (in Russian). *MR* **24** # A 3102.

[5] Močul'skiĭ, E. N., Direct decompositions in lattices, *Izv. Akad. Nauk SSSR Ser. Mat.* **25** (1961), 717—748 (in Russian). *MR* **25** # 2987.
[6] Močul'skiĭ, E. N., Isomorphisms of direct decompositions in lattices, *Sibirsk. Mat. Ž.* **9** (1968), 1360—1385 (in Russian). *MR* **40** # 68.

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# ОБ ОДНОМ ЭКСТРЕМАЛЬНОМ ПОЛИНОМИАЛЬНОМ ОПЕРАТОРЕ ДИФФЕРЕНЦИРОВАНИЯ

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**1.** Ради простоты результаты формулируются и доказываются для пространства $C$ $2\pi$-периодических непрерывных функций $f(x)$ с нормой $\|f\|=$ $=\operatorname{Max}|f(x)|$. Обозначим через $\Omega^{(r)}_{n,n+m}$ множество всевозможных линейных операторов $U_{n,n+m}(f,x)$ из $C$ в $C$, обладающих свойствами: для любой $f\in C$ $U_{n,n+m}(f,x)$ есть тригонометрический полином порядка не выше $n+m$, и если $T(x)$ тригонометрический полином порядка не выше $n$, то $U_{n,n+m}(T,x)=$ $=T^{(r)}(x)$. Положим $\varrho^{(r)}_{n,n+m}=\inf\limits_{U_{n,n+m}\in\Omega^{(r)}_{n,n+m}}\|U_{n,n+m}\|$. Пусть оператор $\bar{U}$ из $\Omega^{(r)}_{n,n+m}$. Будем говорить, что он экстремальный, если выполняется равенство $\varrho^{(r)}_{n,n+m}=$ $=\|\bar{U}\|$. Возникает естественный вопрос о нахождении в множестве $\Omega^{(r)}_{n,n+m}$ экстремального оператора и о вычислении $\varrho^{(r)}_{n,n+m}$. Эта задача была поставлена в [1]. Она, как нам кажется, еще не решена. В [1] она была решена для $r=1$ и $m=n-1$. В настоящей заметке дается полное решение этой задачи для любого натурального $r$ и $m=n-1$.

**2.** Обозначим через $\Pi_n$ множество всех тригонометрических полиномов порядка не выше $n$.

Теорема 1. *Для любого натурального $r$ и любого $T\in\Pi_n$ имеет место тождество*

$$T^{(r)}(x)=\frac{1}{\pi}\int\limits_0^{2\pi}T(x+t)\cos\left(nt+\frac{r\pi}{2}\right)\left[n^r+2\sum_{k=1}^{n-1}k^r\cos(n-k)t\right]dt. \tag{1}$$

Доказательство. Для любого $T\in\Pi_n$ имеем

$$T(x)=\frac{1}{\pi}\int\limits_0^{2\pi}T(t)D_n(x-t)\,dt$$

где

$$D_n(t)=\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}.$$

Продифференцируем это равенство $r$ раз, тогда получим, что

$$T^{(r)}(x)=\frac{1}{\pi}\int\limits_0^{2\pi}T(x+t)D_n^{(r)}(t)\,dt,\quad D_n^{(r)}(t)=\sum_{k=1}^{n}k^r\cos\left(kt+\frac{r\pi}{2}\right). \tag{2}$$

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С другой стороны, для любого $T \in \Pi_n$ выполняется равенство

$$\frac{1}{\pi}\int_0^{2\pi} T(x+t)\sum_{k=1}^{n-1} k^r \cos\left[(2n-k)t+\frac{r\pi}{2}\right]dt = 0. \tag{3}$$

Из (2) и (3) следует, что

$$\begin{aligned} T^{(r)}(x) = \frac{1}{\pi}\int_0^{2\pi} T(x+t)\Bigg[n^r\cos\left(nt+\frac{r\pi}{2}\right)+\sum_{k=1}^{n-1}k^r\Bigg(\cos\left(kt+\frac{r\pi}{2}\right)+ \\ +\cos\left((2n-k)t+\frac{r\pi}{2}\right)\Bigg)\Bigg]dt. \end{aligned} \tag{4}$$

Так как

$$\cos\left(kt+\frac{r\pi}{2}\right)+\cos\left((2n-k)t+\frac{r\pi}{2}\right) = 2\cos\left(nt+\frac{r\pi}{2}\right)\cos(n-k)t,$$

то из (4) вытекает (1).

**3.** Построим теперь экстремальный оператор.

Теорема 2. *Оператор*

$$U(f,x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\cos\left(nt+\frac{r\pi}{2}\right)\left[n^r+2\sum_{k=1}^{n-1}k^r\cos(n-k)t\right]dt \tag{5}$$

*принадлежит классу* $\Omega^{(r)}_{n,2n-1}$.

Доказательство. Из равенства (4) следует, что $\overline{U}(f, x)$ можно записать в виде

$$\overline{U}(f,x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\left[D_n^{(r)}(t)+\sum_{k=1}^{n-1}k^r\cos\left((2n-k)t+\frac{r\pi}{2}\right)\right]dt.$$

Поэтому ясно, что оператор $\overline{U}(f, x)$ переводит функции из $C$ в тригонометрические полиномы порядка $2n-1$. Далее, в силу теоремы 1 для $T \in \Pi_n$ $\overline{U}(T, x) = T^{(r)}(x)$. Итак, $\overline{U} \in \Omega_{n,2n-1}$.

**4.** Для дальнейшего важна

Теорема 3. *Для всех* $t \in (-\infty, \infty)$ *выполняется неравенство*

$$F_n(t) \equiv n^r+2\sum_{k=1}^{n-1}K^r\cos(n-k)t \geqq 0.$$

Доказательство. Л. Фейер [2] доказал теорему: Пусть полином

$$T_n(x) = a_0+2a_1\cos t+\ldots+2a_n\cos nt$$

удовлетворяет условиям

$$a_\nu - 2a_{\nu+1} + a_{\nu+2} \geqq 0, \quad \nu = 0, 1, \dots, (n-2), \tag{6}$$

$$a_{n-1} - 2a_n \geqq 0, \quad a_n \geqq 0, \tag{7}$$

тогда $T_n(x) \geqq 0, \quad -\infty < x < \infty$.

Это утверждение доказывается весьма просто. Достаточно к правой части применить два раза преобразование Абеля и затем воспользоваться неравенствами (6) и (7) и тем фактом, что ядро Фейера неотрицательно на всей числовой оси. Проверим, что полином $F_n(t)$ удовлетворяет неравенствам (6) и (7). Ясно, что неравенство (7) выполняется. Известно, что при $a \geqq 0$, $b \geqq 0$ и любом вещественном $r \geqq 1$ выполняются неравенства

$$\left(\frac{a+b}{2}\right)^r \leqq \frac{a^r + b^r}{2}.$$

Поэтому $(n-\nu)^r - 2(n-\nu+1)^r + (n-\nu+2)^r \geqq 0$, $\nu = 0, 1, \dots, (n-2)$. Стало быть, выполняется неравенства (6).

Следствие 1. *При всех* $t \in (-\infty, \infty)$ *выполняется равенство*

$$\operatorname{Sign} F_n(t) \cos\left(nt + \frac{r\pi}{2}\right) = \operatorname{Sign} \cos\left(nt + \frac{r\pi}{2}\right), \tag{8}$$

*за исключением корней полинома* $F_n(t)$, *где левая часть равенства* (8) *равна нулю.*

**5.** Для дальнейшего нужна теорема из [1].

Теорема 4. *Справедливо равенство*

$$\varrho^{(r)}_{n,2n-1} = \inf_{\alpha_k, \beta_k} \mathscr{I}(\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}),$$

*где*

$$\mathscr{I} = \mathscr{I}(\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}) =$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left| D_n^{(r)}(t) + \sum_{j=1}^{n-1} (\alpha_j \cos(n+j)t + \beta_j \sin(n+j)t) \right| dt.$$

*Если интеграл* $\mathscr{I}$ *достигает наименьшего значения при* $\alpha_j = \alpha_j^{(1)}$, $\beta_j = \beta_j^{(1)}$, $j = 1, 2, \dots, (n-1)$ *то экстремальной является операция*

$$\tilde{U}(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) \left[ D_n^{(r)}(t) + \sum_{j=1}^{n-1} (\alpha_j^{(1)} \cos(n+j)t + \beta_j^{(1)} \sin(n+j)t) \right] dt.$$

Заметим, что при четном $r$, $r = 2s$, $D_n^{(2s)}(x)$ — четная функция, а при нечетном $r$, $r=2s+1$, $D_n^{(2s+1)}(x)$ — нечетная функция. Поэтому справедливы равенства

$$\inf_{\alpha_k,\beta_k}\int_0^{2\pi}\left|D_n^{(2s)}(t)+\sum_{j=1}^{n-1}(\alpha_j\cos(n+j)t+\beta_j\sin(n+j)t)\right|dt= \tag{9}$$
$$=\inf_{\gamma_k}\int_0^{2\pi}\left|D_n^{(2s)}(t)+\sum_{j=1}^{n-1}\gamma_j\cos(n+j)t\right|dt,$$

$$\inf_{\alpha_k,\beta_k}\int_0^{2\pi}\left|D_n^{(2s+1)}(t)+\sum_{j=1}^{n-1}(\alpha_j\cos(n+j)t+\beta_j\sin(n+j)t)\right|dt= \tag{10}$$
$$=\inf_{\delta_k}\int_0^{2\pi}\left|D_n^{(2s+1)}(t)+\sum_{j=1}^{n-1}\delta_j\sin(n+j)t\right|dt.$$

Воспользуемся теперь известными фактами теории наилучших приближений в метрике $L$ [3]. Тогда из теоремы 4 и равенств (9), (10) получим

Теорема 5. 1) *Для того чтобы оператор $\tilde{U}(f,x)$ обладал наименьшей нормой в классе операторов $\Omega_{n,2n-1}^{(2s)}$ необходимо и достаточно, чтобы числа $\{\gamma_i\}_{i=1}^{n-1}$ из формулы* (9) *удовлетворяли условиям*

$$\int_0^{\pi}\operatorname{Sign}\left[D_n^{(2s)}(t)+\sum_{j=1}^{n-1}\gamma_j\cos(n+j)t\right]\cos(n+i)t\,dt=0,\quad i=1,2,\ldots,(n-1).$$

2) *Для того чтобы оператор $\tilde{U}(f,x)$ обладал наименьшей нормой в классе операторов $\Omega_{n,2n-1}^{(2s+1)}$ необходимо и достаточно, чтобы числа $\{\delta_i\}_{i=1}^{n-1}$ из формулы* (10) *удовлетворяли условиям*

$$\int_0^{\pi}\operatorname{Sign}\left[D_n^{(2s+1)}(t)+\sum_{j=1}^{n-1}\delta_j\sin(n+j)t\right]\sin(n+i)t\,dt=0,\quad i=1,2,\ldots,(n-1).$$

Поэтому из следствия 1 вытекает

Следствие 2. 1) *Для того чтобы оператор $\tilde{U}(f,x)$ обладал наименьшей нормой в классе операторов $\Omega_{n,2n-1}^{(2s)}$ необходимо и достаточно, чтобы выполнялись равенства*

$$\int_0^{\pi}\operatorname{Sign}\cos nt\cos jt\,dt=0,\qquad j=n+1,\ldots,(2n-1). \tag{11}$$

2) *Для того чтобы оператор $\tilde{U}(f,x)$ обладал наименьшей нормой в классе операторов $\Omega_{n,2n-1}^{(2s+1)}$ необходимо и достаточно, чтобы выполнялись равенств*

$$\int_0^{\pi}\operatorname{Sign}\sin nt\sin jt\,dt=0,\qquad j=n+1,\ldots,(2n-1). \tag{12}$$

Теорема 6. *Среди всех линейных операторов $U_{n,2n-1}(f,x)$ из $C$ в $C$, переводящих функции из $C$ в полиномы порядка $2n-1$ и обладающих тем свойст-*

*вом, что для любого полинома $T$ порядка не выше $n$ имеет место равенство $U_{n,2n-1}(T,x)=T^{(r)}(x)$, оператор (5) обладает наименьшей нормой. Таким образом*

$$\|U\| = \varrho^{(r)}_{n,2n-1} = \frac{4}{\pi} n^r, \quad r = 1, 2, \ldots .$$

Доказательство. В [3], стр. 99—100, доказано следующее утверждение. Пусть интегрируемая функция $\Phi(x)$ удовлетворяет условию $\Phi(x+\pi) = -\Phi(x)$. Пусть $m, n$ — целые числа и отношение $\frac{m}{n}$ не есть нечетное число, тогда

$$\int_{-\pi}^{\pi} e^{imx}\Phi(nx)\,dx = 0. \tag{13}$$

В частности, беря $\Phi(x) = \operatorname{sign}\cos x$ и $\Phi(x) = \operatorname{Sign}\sin x$ получаем из (13) равенства (11) и (12). Для вычисления $\varrho^{(r)}_{n,2n-1}$ заметим, что из формулы для экстремального операторе (5) следует, что

$$\varrho^{(r)}_{n,2n-1} = \frac{1}{\pi}\int_0^{2\pi} \left|\cos\left(nt + \frac{r\pi}{2}\right)\right| \left[n^r + 2\sum_{k=1}^{n-1} k^r \cos(n-k)t\right] dt.$$

При четном $r$, $r=2s$ получим

$$\varrho^{(2s)}_{n,2n-1} = \frac{1}{\pi}\int_0^{2\pi} |\cos nt| \left[n^r + 2\sum_{k=1}^{n-1} k^r \cos(n-k)t\right] dt. \tag{14}$$

При нечетном $r$, $r=2s+1$, получим

$$\varrho^{(2s+1)}_{n,2n-1} = \frac{1}{\pi}\int_0^{2\pi} |\sin nt| \left[n^r + 2\sum_{k=1}^{n-1} k^r \cos(n-k)t\right] dt. \tag{15}$$

К равенствам (14) и (15) опять применяем равенство (13), тогда выводим, что

$$\varrho^{(2s)}_{n,2n-1} = \frac{n^r}{\pi}\int_0^{2\pi} |\cos nt|\,dt, \quad \varrho^{(2s+1)}_{n,2n-1} = \frac{n^r}{\pi}\int_0^{2\pi} |\sin nt|\,dt. \tag{16}$$

Легко видеть, что

$$\int_0^{2\pi} |\cos nt|\,dt = \int_0^{2\pi} |\sin nt|\,dt = 4.$$

Поэтому из (16) получим, что

$$\varrho^{(r)}_{n,2n-1} = \frac{4}{\pi} n^r .$$

## ЛИТЕРАТУРА

[1] Берман, Д. Л., Экстремальные задачи теории полиномиальных операторов, *Mat. Sbornik* **60** (1963), № 3, 354—365. *MR* **27** # 516.

[2] Fejér, L., Über die Positivität von Summen, die nach trigonometrischen oder Legendreschen Funktionen fortschreiten, I, *Acta Szeged* **2** (1925), 75—86. *J. Fortschritte Math.* **51**, 219.

[3] Ахиезер, Н. И., *Лекции по теории аппроксимации*, Наука, Москва, 1965.

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# KREISUNTERDECKUNGEN AUF EINEM HYPERZYKELBEREICH IN DER HYPERBOLISCHEN EBENE

I. VERMES

K. Böröczky hat in [3] gezeigt, daß die Dichte eines Systems von kongruenten Kreisen in der hyperbolischen Ebene keine eindeutige Erklärung hat. Er konstruierte zwei Zellenzerlegungen zu demselben Kreissystem so, daß die Dichten bezüglich der ersten und der zweiten voneinander verschiedene Werte geben.

In dieser Arbeit wollen wir uns mit den Kreisunterdeckungen auf einem Hyperzykelbereich und mit ihren Dichten beschäftigen.

Man versteht unter einem Hyperzykel (oder einer Abstandslinie bzw. Äquidistante) die Gesamtheit derjenigen Punkte der Ebene, die von einer Geraden (Grundlinie) gleichen Abstand $l$ haben, und alle auf derselben Seite von ihr gelegen sind. Die beiden kongruenten Äquidistanten, die auf verschiedenen Seiten der Grundlinie sind, begrenzen einen Teil der Ebene, der als Hyperzykelbereich vom Abstand $l$ heißt.

Unsere Untersuchungen gründen sich auf einem Satz von K. Bezdek [1]:

Sind $n\geqq 2$ einander nicht überdeckende Kreise $K_1, \ldots, K_n$ vom Radius $r>0$ in der hyperbolischen Ebene, so gilt

$$\frac{\sum_{i=1}^{n} K_i}{T_r} < \frac{\pi}{\sqrt{12}}$$

wo $T$ die konvexe Hülle der Mittelpunkte von $K_1, \ldots, K_n$ und $T_r$ die äußere $r$-Parallelmenge (oder Äquidistantmenge) von $T$ im Abstand $r$ bedeutet. ($T_r$ ist die Vereinigungsmenge der Kreise vom Radius $r$, deren Mittelpunkte zu $T$ gehören. Wir bezeichnen den Flächeninhalt eines Bereiches ebenso mit demselben Symbol wie den Bereich.)

Vor allem definieren wir den Begriff des $H$-Bereiches vom Abstand $l$. Unter einem $H$-Bereich vom Abstand $l$ verstehen wir denjenigen Teil der hyperbolischen Ebene, der auf folgende Weise begrenzt ist: durch zwei Äquidistantbogen vom Abstand $l$, die zu einer Strecke von Größe $h>0$ gehören, und durch zwei Halbkreise vom Radius $l$, die diese obigen Bogen berühren (Fig. 1).

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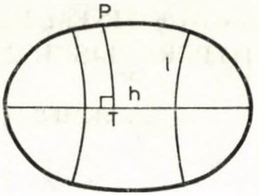


*Fig. 1*

Die große Achse des $H$-Bereiches ist der Durchmesser, der auf der Grundlinie der Äquidistanten liegt, und seine Länge $h+2l$ ist. Die Radien dieses $H$-Bereiches sind die gerichteten Strecken $(\overrightarrow{PT})$, die aus den Punkten der Äquidistanten auf die Grundlinie senkrecht stehen, bzw. die die Radien der oben erwähnten zwei Halbkreise sind.

Wir beschäftigen uns mit den gesättigten Kreissystemen in den $H$-Bereichen vom Abstand $l$. Es ist klar, daß die Ungleichung $r \leqq l$ für die Radien der kongruenten Kreise gilt. Wir betrachten die Dichten der Unterdeckungen der kongruenten geschlossenen Kreise vom Radius $r$ in den geschlossenen $H$-Bereichen. Dazu muß man das folgende Lemma beweisen:

LEMMA. *Es seien zwei Kreise vom Radius $r$ $(r<l)$ in einem geschlossenen $H$-Bereich vom Abstand $l$ gegeben. Betrachten wir die Abstandslinie vom Abstand $r$, deren Grundlinie auf der Gerade liegt, die die zwei Mittelpunkte verbindet, und so berührt die Abstandslinie beide Kreise. Wir behaupten, daß der Bogen dieser Abstandslinie zwischen den Berührungspunkten zum $H$-Bereich gehört.*

Zum Beweis ist es genügend nur untersuchen, falls die beiden Kreise die Grenze des $H$-Bereiches berühren. Insofern man einen oder beiden von Kreisen in das Innere des $H$-Bereiches bewegt, so verbindet dieser Abstandsbogen vom Abstand $r$ zwei innere Punkte, und hat er — wegen $r<l$ — eine kleinere Krümmung, als die Grenze von $H$, also liegt dieser Bogen im Inneren von $H$.

Falls die beiden Kreise $K_1$, $K_2$ die Grenze von $H$ in den Punkten $S_1$, $S_2$ berühren, und $S_1$, $S_2$ auf einer Abstandslinie oder einem Kreisbogen liegen (Fig. 2), so schließt

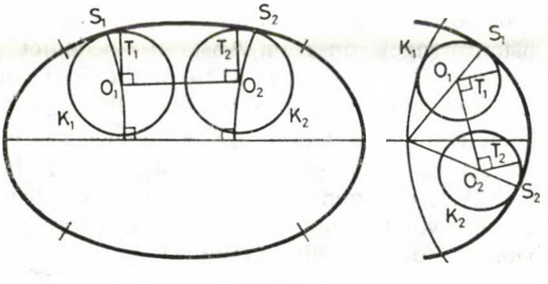


*Fig. 2*

die Verbindungsstrecke der Mittelpunkte $O_1$ und $O_2$ wie mit dem Radius $\overrightarrow{S_1O_1}$ als auch mit dem Radius $\overrightarrow{S_2O_2}$ des H-Bereiches einen spitzen Winkel. Folglich schneiden die auf $O_1O_2$ senkrechten Geraden diese Kreise in den Berührungspunkten $T_1$, $T_2$

der Äquidistante vom Abstand $r$. Die Punkte $T_1$, $T_2$ und folglich der Bogen $\widehat{T_1T_2}$ liegen im Inneren des $H$-Bereiches.

Ebenso kann man das Lemma beweisen, falls der erste Kreis $K_1$ mit dem Kreisbogen von $H$ den Berührungspunkt $S_1$ und gleichzeitig der zweite $K_2$ mit dem Abstandsbogen von $H$ den Berührungspunkt $S_2$ haben (Fig. 3) und falls die Radien $\overrightarrow{S_1O_1}$ und $\overrightarrow{S_2O_2}$ auf derselben Seite von $O_1O_2$ gerichtet sind.

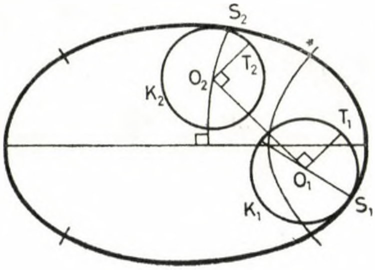


*Fig. 3*

Falls $K_1$ den Kreisbogen von $H$ in dem Punkt $S_1$ berührt, und $K_2$ mit dem Abstandsbogen von $H$ den Berührungspunkt $S_2$ hat, ferner die $\overrightarrow{S_1O_1}$ und $\overrightarrow{S_2O_2}$ Radien in verschiedene Halbebene bezüglich $O_1O_2$ zeigen (Fig. 4), so bilden diese Radien mit den Strahlen $O_1O_2$ bzw. $O_2O_1$ je einen spitzen Winkel. Daraus folgt, daß die auf $O_1O_2$ senkrecht stehenden Geraden $O_1T_1$ bzw. $O_2T_2$ die Kreise $K_1$ bzw. $K_2$ in den Punkten $T_1$ bzw. $T_2$ schneiden. Die Punkte $T_1$, $T_2$ und auch der Abstandsbogen $\widehat{T_1T_2}$ vom Abstand $r$ sollen im Inneren von $H$ liegen.

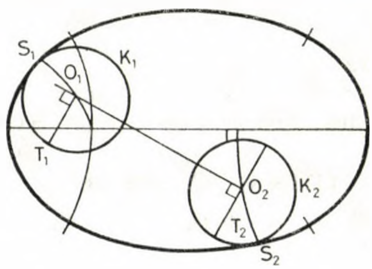


*Fig. 4*

Auf ähnliche Weise kann man die Fälle — inklusive auch die Grenzfälle — erledigen, wenn $K_1$ und $K_2$ zwei verschiedene Kreis- bzw. Abstandsbogen von $H$ berühren.

Damit ist unser Lemma bewiesen.

Ein unmittelbaren Korollar unseres Lemmas ist die folgende

BEHAUPTUNG. *Es seien $K_1, \ldots, K_n$ die Elemente eines gesättigten Systems von kongruenten Kreisen auf einem H-Bereich, und P die konvexe Hülle der Kreismittel-*

*punkten $O_1, \ldots, O_n$. Die äußere r-Äquidistantmenge (oder Parallelmenge) $P_r$ von P gehört zum H-Bereich.*

Nach dem Satz von Bezdek ist die Dichte dieses Kreissystems kleiner im Bereich $P_r$ als $\frac{\pi}{\sqrt{12}}$, folglich besteht die Ungleichung auch für die Dichte $d_H$ bezüglich $H$:

$$d_H < \frac{\pi}{\sqrt{12}}.$$

Falls man die Größe der großen Achse des $H$-Bereiches unbegrenzt steigert, so erhält man eine willkürliche Folge $H_1 \subset H_2 \subset H_3 \subset \ldots \subset H_k \subset \ldots$ von $H$-Bereichen. Auf Grund des obenerwähnten Lemmas gilt die Ungleichung

$$d_i < \frac{\pi}{\sqrt{12}}$$

für alle Bereiche $H_i$, in denen $d_i$ die Dichte eines gesättigten Kreissystems vom Radius $r$ bedeutet $(i=1, 2, 3, \ldots)$.

Ein Hyperzykelbereich vom Abstand $l$ kann durch die obigen $\{H_i\}$ Folgen willkürlich approximiert werden. Dieses Verfahren zeigt, daß die Dichte eines Systems kongruenter Kreise auf einem Hyperzykelbereich erklärt werden kann, und diese Dichte kann nicht größer als $\frac{\pi}{\sqrt{12}}$ sein.

ANMERKUNG. Man kann — wie obenan — eine ebensolche Ungleichung für die Dichte eines gesättigtes Systems kongruenter Kreise bezüglich eines Horozykelbereiches auf Grund der folgenden zwei Tatsachen erhalten.

Erstens hat K. Bezdek in [2] bewiesen: Sind in der hyperbolischen Ebene in einem Kreis mindestens zwei kongruente Kreise eingelagert, so ist die Kreisdichte in dem Kreis kleiner als $\frac{\pi}{\sqrt{12}}$.

Andererseits kann jeder Horozykelbereich durch die Folge der Kreise von unbegrenzt-zunehmenden Radien willkürlich approximiert werden.

## LITERATURVERZEICHNIS

[1] BEZDEK, K., Ausfüllungen in der hyperbolischen Ebene durch endliche Anzahl kongruenter Kreise, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **27** (1984), 113—124. *MR* **87d:** 52023.

[2] BEZDEK, K., Ausfüllung eines Kreises durch kongruente Kreise in der hyperbolischen Ebene, *Studia Sci. Math. Hungar.* **17** (1982), 353—366. *MR* **85i:** 52006.

[3] BÖRÖCZKY, K., Sphere packing in spaces of constant curvature I, *Mat. Lapok* **25** (3—4) (1974), 265—306 (in Hungarian).

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# TAUBERIAN THEOREMS FOR POWER SERIES OF TWO REAL VARIABLES, II

L. ALPÁR

*Dedicated to Professor Paul Erdős on his 75-th birthday*

## § 1. Introduction

This paper concerns the generalization of some theorems from [1] and is the continuation of [2] in the sense that we apply here the same methods as in [2].

NOTATION. a) $\alpha, \beta, \gamma, \delta, \xi$ are positive constants, $r$ and $s$ signify positive finite parameters and $\lambda > 0$ is a variable tending to infinity, so that if

(1.1) $$x \sim e^{-1/r\lambda}, \quad y \sim e^{-1/s\lambda}$$

then $x \to 1$, $y \to 1$ for $\lambda \to \infty$ and $0 < r$, $s < \infty$ means that when the point $(x, y)$ approaches to (1,1) on an arbitrary continuous curve it is not tangent to the straight lines $x=1$, $y=1$. These curves will be called admissible.

b) Let us denote by $L$ an open, two dimensional, bounded, Jordan measurable set in the first quadrant of the plane ($x \geqq 0$, $y \geqq 0$) with closure $\bar{L}$, boundary $\partial L$ and measure $|L| > 0$. We shall use the same notation for other sets of similar nature. We derive from $\bar{L}$ the following sets: if $(x, y) \in \bar{L}$ then

(1.2) $$\text{(i)} \quad (\lambda x, \lambda y) \in \bar{L}_\lambda, \quad \bar{L}_1 = \bar{L},$$
$$\text{(ii)} \quad (x^\xi, y^\xi) \in \bar{L}^\xi, \quad \bar{L}^1 = \bar{L}.$$

c) $R_{u,v}$ is called a basic rectangle if it has a vertex at the origin and two sides of length $u$ and $v$ on the axes $Ox$ and $Oy$, respectively. $R^\xi_{u,v}$ is the basic rectangle obtained from $R_{uv}$ by the mapping (1.2) (ii).

d) Finally we use the square

$$Q = \{(x, y)\colon 0 \leqq x < 1,\ 0 \leqq y < 1\}.$$

Our starting point is the following result of Turán [5].

THEOREM T. *Let the series*

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n, \quad a_{m,n} \geqq 0$$

*be convergent in $Q$ and assume that*

$$\lim f(x, y)(1-x)(1-y) = 1$$

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*as* $(x, y) \to (1, 1)$ *on an arbitrary admissible curve, then*

(1.3) $$\lim_{\lambda \to \infty} \lambda^{-2} \sum_{(m,n) \in \bar{L}_\lambda} a_{m,n} = |L|.$$

REMARK 1. Since in (1.3) the pairs of indices $(m, n)$ are taken on the closed set $\bar{L}_\lambda$, we may neglect the cases in which a subset $h$ of $\partial L$ consists only of inner points of $\bar{L}$ (see Fig. 1).

*Fig. 1*

For in such a case $h_\lambda$ consists merely of inner points of $\bar{L}_\lambda$ and the two-dimensional Jordan measure of $h$ is equal to zero, so that $|L \setminus h| = |L|$.

## § 2. Statement and proof of results

**1.** To formulate the first theorem we still need the notation

$$M^\xi = Ox \cap \bar{L}^\xi, \quad N^\xi = Oy \cap \bar{L}^\xi.$$

THEOREM 1. *Suppose that the series*

$$f(x,y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n, \quad a_{mn} \geqq 0$$

*is convergent in* $Q$ *and if* $(x, y) \to (1, 1)$ *on an arbitrary admissible curve, we have*

(2.1) $$f(x, y) \sim (1-x^\alpha)^{-\xi} + (1-y^\beta)^{-\xi}.$$

*Moreover if the linear sets* $M^\xi$ *and* $N^\xi$ *are Jordan measurable, then*

(2.2) $$A_1(L) = \lim_{\lambda \to \infty} \lambda^{-\xi} \sum_{(m,n) \in \bar{L}_\lambda} a_{mn} = (|M^\xi| \alpha^{-\xi} + |N^\xi| \beta^{-\xi}) / \Gamma(\xi+1).$$

PROOF. We have by (1.1) and (2.1)

$$\sum_{m,n=0}^{\infty} a_{mn} x^m y^n \sim \lambda^\xi \left[ \left(\frac{r}{\alpha}\right)^\xi + \left(\frac{s}{\beta}\right)^\xi \right]$$

or

(2.3) $$\lim_{\lambda \to \infty} \lambda^{-\xi} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n = \left(\frac{r}{\alpha}\right)^\xi + \left(\frac{s}{\beta}\right)^\xi.$$

Let $p \geqq 0$, $q \geqq 0$ integers. Replacing $x$ and $y$ by $x^{p+1}$ and $y^{q+1}$, respectively, then (2.3) yields

$$\begin{aligned}(2.4)\qquad \lim_{x\to\infty} \lambda^{-\xi} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n x^{mp} y^{nq} &= \left(\frac{r}{\alpha}\right)^{\xi} \frac{1}{(p+1)^{\xi}} + \left(\frac{s}{\beta}\right)^{\xi} \frac{1}{(q+1)^{\xi}} = \\ &= \frac{1}{\Gamma(\xi)} \left[\left(\frac{r}{\alpha}\right)^{\xi} \int_0^{\infty} e^{-x} e^{-px} x^{\xi-1}\, dx + \left(\frac{s}{\beta}\right)^{\xi} \int_0^{\infty} e^{-y} e^{-qy} y^{\xi-1}\, dy\right],\end{aligned}$$

Therefore if $g(x, y)$ is any real polynomial, it follows from (2.4) that

$$\begin{aligned}(2.5)\qquad \mathcal{F}^{r,s}[g] &= \lim_{\lambda\to\infty} \lambda^{-\xi} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n g(x^m, y^n) = \\ &= \frac{1}{\Gamma(\xi)} \left[\left(\frac{r}{\alpha}\right)^{\xi} \int_0^{\infty} e^{-x} g(e^{-x},1) x^{\xi-1}\, dx + \left(\frac{s}{\beta}\right)^{\xi} \int_0^{\infty} e^{-y} g(1, e^{-y}) y^{\xi-1}\, dy\right].\end{aligned}$$

It is easy to see as in [2] (p. 170) that for fixed $r$ and $s$, $\mathcal{F}^{r,s}$ is a positive linear functional on the normed linear space of real polynomials $g(x, y)$ where $\|g\| = \max_{(x,y)\in \bar{Q}} |g(x, y)|$. Obviously $\mathcal{F}^{r,s}$ is positive, additive and homogeneous and it results from (2.3) that it is bounded, too, namely

$$|\mathcal{F}^{r,s}[g]| \leqq \Big(\lim_{\lambda\to\infty} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n\Big) \max_{(x,y)\in \bar{Q}} |g(x, y)| = \left[\left(\frac{r}{\alpha}\right)^{\xi} + \left(\frac{s}{\beta}\right)^{\xi}\right] \|g\|,$$

that is

$$\|\mathcal{F}^{r,s}\| = \left(\frac{r}{\alpha}\right)^{\xi} + \left(\frac{s}{\beta}\right)^{\xi}.$$

Hence it is natural according to the theorem of F. Riesz that $\mathcal{F}^{r,s}$ has the integral representation (2.5).

The polynomials form an everywhere dense subspace of the normed linear space of continuous functions on $\bar{Q}$ with maximum norm. Consequently, by the Hahn—Banach theorem $\mathcal{F}^{r,s}$ is extendable uniquely to the entire space without changing the norm. Furthermore, the representation theorem of F. Riesz [3], [4, pp. 250—261] states that $\mathcal{F}^{r,s}$ can be extended to the class of functions which are limits (everywhere) of sequences of continuous, increasing, bounded functions. This larger class contains the discontinuous function

$$(2.6)\qquad w(x, y) = \begin{cases} 1/xy & \text{if} \quad e^{-1} \leqq x \leqq 1 \quad \text{and} \quad e^{-1} \leqq y \leqq 1 \\ 0 & \text{if} \quad x \notin [e^{-1}, 1], \quad \text{or} \quad y \notin [e^{-1}, 1]. \end{cases}$$

That is we may write $w(x, y)$ instead of $g(x, y)$ in (2.5):

$$\begin{aligned}(2.7)\qquad \mathcal{F}^{r,s}[w] &= \lim_{\lambda\to\infty} \lambda^{-\xi} \sum_{m,n=0}^{\infty} a_{mn} x^m y^x w(x^m, y^n) = \\ &= \frac{1}{\Gamma(\xi)} \left[\left(\frac{r}{\alpha}\right)^{\xi} \int_0^{1} x^{\xi-1}\, dx + \left(\frac{s}{\beta}\right)^{\xi} \int_0^{1} y^{\xi-1}\, dy\right].\end{aligned}$$

In virtue of (2.6) and (1.1) we have to consider the values $x^m \sim e^{-m/r\lambda} \geqq e^{-1}$ and $y^n \sim e^{-n/s\lambda} \geqq e^{-1}$ merely, such that $m \leqq r\lambda$, $n \leqq s\lambda$ and so (2.7) gives

$$(2.8)\qquad \lim_{\lambda\to\infty} \lambda^{-\xi} \sum_{\substack{m\leqq r\lambda\\ n\leqq s\lambda}} a_{mn} = \frac{1}{\Gamma(\xi+1)}\left[\left(\frac{r}{\alpha}\right)^{\xi} + \left(\frac{s}{\beta}\right)^{\xi}\right] = \Lambda_1(R_{rs}).$$

Hence if $\bar{L}$ is the basic rectangle $R_{rs}$ (see notation c)) then $|M^\xi| = r^\xi$, $|N^\xi| = s^\xi$, further $m \leqq r\lambda$, $n \leqq s\lambda$ means that $(m, n) \in \bar{L}_\lambda$. In other words, (2.8) proves (2.2) in this particular case.

Moreover, let $\bar{L} = \bar{R}_x$ be the rectangle with the side $(a, b)$ on $Ox$, $0 \leqq a < b$ and the sides parallel to $Oy$ of arbitrary positive length, then we have by (2.8)

$$(2.9)\quad \Lambda_1(R_x) = \lim_{\lambda\to\infty} \lambda^{-\xi} \sum_{\substack{a\lambda\leqq m\leqq b\lambda\\ n\leqq s\lambda}} a_{mn} = (b^\xi - a^\xi)/\Gamma(\xi+1)\alpha^\xi = |M^\xi|/\Gamma(\xi+1)\alpha^\xi$$

for any $s > 0$. Likewise we have for a rectangle $\bar{R}_y = \bar{L}$ defined as $\bar{R}_x$ with the side $(c, d)$ on $Oy$, $0 \leqq c < d$

$$(2.10)\quad \Lambda_1(R_y) = \lim_{\lambda\to\infty} \lambda^{-\xi} \sum_{\substack{m\leqq r\lambda\\ c\lambda\leqq n\leqq d\lambda}} a_{mn} = (d^\xi - c^\xi)/\Gamma(\xi+1)\beta^\xi = |N^\xi|/\Gamma(\xi+1)\beta^\xi$$

for any $r > 0$. We infer from (2.9) and (2.10) that if $a > 0$ and $c > 0$ and $\bar{R}_0$ is the rectangle with vertices $(a, c)$, $(b, c)$, $(a, d)$, $(b, d)$ then $\Lambda_1(R_0) = 0$. Thus if the set $\bar{L}' \subset R_0$ then $0 \leqq \Lambda_1(\bar{L}') \leqq \Lambda_1(R_0) = 0$. Therefore $\Lambda_1(\bar{L})$ does not change if we complete or truncate $\bar{L}$ by any set having no common points with the axes. This fact enables us to prove Theorem 1 in full generality.

Assume first that $N^\xi = \emptyset$ and $M^\xi$ is the unique interval $(a^\xi, b^\xi)$. Consider two rectangles of type $\bar{R}_x$. One of them $\bar{R}_-$ with the side $[(a+\varepsilon)^\xi, (b-\varepsilon)^\xi]$, $\varepsilon > 0$ and the other one $\bar{R}_+$ with the side $[(a-\varepsilon)^\xi, (b+\varepsilon)^\xi]$ if $a > 0$ and $[0, (b+\varepsilon)^\xi]$ if $a = 0$. Their sides parallel to $Oy$ are of the same length $c > 0$. If $c$ and $\varepsilon$ are small enough then $\bar{R}_- \subset \bar{L}$ and by (2.9)

$$\Lambda_1(R_-) = [(b-\varepsilon)^\xi - (a+\varepsilon)^\xi]/\Gamma(\xi+1)\alpha^\xi \leqq \Lambda_1(L).$$

On the other hand $(\bar{L} - \bar{R}_+) \cap Ox = \emptyset$ so that $\Lambda_1(\bar{L} - \bar{R}_+) = 0$ and $\Lambda_1(\bar{L})$ being an additive set function we have

$$\Lambda_1(\bar{L}) \leqq \Lambda_1(\bar{L} - \bar{R}_+) + \Lambda_1(\bar{R}_+) = [(b+\varepsilon)^\xi - (a-\varepsilon)^\xi]/\Gamma(\xi+1)\alpha^\xi,$$

whence for $\varepsilon \to 0$

$$\Lambda_1(\bar{L}) = (b^\xi - a^\xi)/\Gamma(\xi+1)\alpha^\xi = |M^\xi|/\Gamma(\xi+1)\alpha^\xi.$$

This is the desired result. This proof remains valid even if $M^\xi$ and $N^\xi$ are the union of a finite number of pairwise disjoint intervals. Finally, let $M^\xi$ be a general Jordan measurable set, then to any $\varepsilon > 0$ we can find two sets $H$ and $K$ each of them generated by a finite number of pairwise disjoint intervals and such that

$$H \subset M^\xi \subset K,\quad |M^\xi| - \varepsilon \leqq |H|,\quad |M^\xi| + \varepsilon \geqq |K|$$

and $\varepsilon \to 0$ completes the proof.

**2.** To announce the next theorem we introduce some new notation. Let $l_{\alpha\beta}$ be the half straight line $y=(\beta/\alpha)x$, $x\geqq 0$ and denote by $l^{\xi}_{\alpha\beta}$ the image of $l_{\alpha\beta}$ obtained by the mapping (1.2) (ii), this is the half straight line $Y=(\beta/\alpha)^{\xi}X$ where $Y=y^{\xi}$ and $X=x^{\xi}$. Put

$$F^{\xi}_{\alpha\beta}=l^{\xi}_{\alpha\beta}\cap L^{\xi},\quad \bar{G}^{\xi}_{\alpha\beta}=l^{\xi}_{\alpha\beta}\cap \bar{L}^{\xi}.$$

THEOREM 2. *Assume that the series*

$$f(x,y)=\sum_{m,n=0}^{\infty} a_{mn}x^m x^n,\qquad a_{mn}\geqq 0$$

*is convergent in $Q$ and if $(x,y)\to(1,1)$ on an admissible curve we have*

$$f(x,y)\sim(1-x^{\alpha}y^{\beta})^{-\xi}. \tag{2.11}$$

*Moreover, if the linear sets $F^{\xi}_{\alpha\beta}$, $\bar{G}^{\xi}_{\alpha\beta}$ are Jordan measurable and*

$$|\bar{G}^{\xi}_{\alpha\beta}-F^{\xi}_{\alpha\beta}|=|l^{\xi}_{\alpha\beta}\cap\partial L^{\xi}|=0, \tag{2.12}$$

*then*

$$A_2(L)=\lim_{\lambda\to\infty}\lambda^{-\xi}\sum_{(m,n)\in L_\lambda} a_{mn}=|F^{\xi}_{\alpha\beta}|/\Gamma(\xi+1)(\alpha^{2\xi}+\beta^{2\xi})^{1/2}. \tag{2.13}$$

PROOF. Using (1.1) and (2.11) we obtain

$$\lim_{\lambda\to\infty}\lambda^{-\xi}\sum_{m,n=0}^{\infty}a_{mn}x^m y^n=\left(\frac{rs}{\alpha s+\beta s}\right)^{\xi}.$$

Replacing $x$ and $y$ again by $x^{p+1}$ and $y^{q+1}$, respectively, we have

$$\lim_{\lambda\to\infty}\lambda^{-\xi}\sum_{m,n=0}^{\infty}a_{mn}x^m y^n x^{mp}y^{nq}=\left[\frac{rs}{\alpha s(p+1)+\beta r(q+1)}\right]^{\xi}= \tag{2.14}$$
$$=\frac{r^{\xi}s^{\xi}}{\Gamma(\xi)}\int_0^{\infty}e^{-(\alpha s+\beta r)t}e^{-(\alpha sp+\beta rq)t}t^{\xi-1}\,dt.$$

It follows from (2.14) as previously that for fixed $r$ and $s$ and any polynomial $g(x,y)$ we may write

$$\mathcal{F}^{r,s}[g]=\lim_{\lambda\to\infty}\lambda^{-\xi}\sum_{m,n=0}^{\infty}a_{mn}x^m y^n g(x^m,y^n)= \tag{2.15}$$
$$=\frac{r^{\xi}s^{\xi}}{\Gamma(\xi)}\int_0^{\infty}e^{-(\alpha s+\beta r)t}g(e^{-\alpha st},e^{-\beta rt})t^{\xi-1}\,dt.$$

$\mathcal{F}^{r,s}$ is a positive linear functional on the normed linear space of polynomials in $\bar{Q}$ like in the proof of the Theorem 1 with

$$\|\mathcal{F}^{r,s}\|=\left(\frac{rs}{\alpha s+\beta r}\right)^{\xi}.$$

Thus we may write once more $w(x, y)$ instead of $g(x, y)$ and taking (2.6) and (2.15) into account we have for $\beta r \geqq \alpha s$

$$\mathscr{F}^{r,s}[w] = \lim_{\lambda\to\infty} \lambda^{-\xi} \sum_{m,n=0}^{\infty} a_{mn} x^m y^n w(x^m, y^n) = \frac{r^\xi s^\xi}{\Gamma(\xi)} \int_0^{1/\beta r} t^{\xi-1}\, dt, \tag{2.16}$$

i.e. for $L^\xi = R^\xi_{rs}$ we conclude from (2.16)

$$\Lambda_2(L) = \lim_{\lambda\to\infty} \lambda^{-\xi} \sum_{\substack{m\leqq r\lambda \\ n\leqq s\lambda}} a_{mn} = \frac{s^\xi}{\Gamma(\xi+1)\beta^\xi}. \tag{2.17}$$

(2.17) exhibits that $\Lambda_2(L)$ is independent of $r$ and $\alpha$ and $F^\xi_{\alpha\beta}$ is a unique interval in this case (see Fig. 2) further

$$s^\xi = |F^\xi_{\alpha\beta}|\, \beta^\xi/(\alpha^{2\xi}+\beta^{2\xi})^{1/2}. \tag{2.18}$$

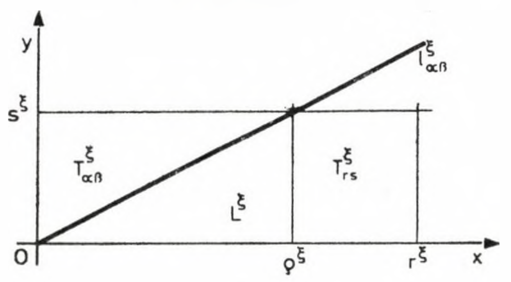


*Fig. 2*

(2.17) and (2.18) yield (2.13) in this special case.

Hereupon we may draw another conclusion from (2.17). Figure 2 shows that $L^\xi$ is the union of the basic rectangle $R^\xi_{\varrho s}$ where $\varrho^\xi = (\alpha/\beta)^\xi s^\xi$ and the rectangle $T^\xi_{rs}$ which is the difference of $R^\xi_{rs}$ and $R^\xi_{\varrho s}$. In view of (2.17)

$$\Lambda_2(L) = \Lambda_2(R_{\varrho s} + T_{rs}) = \Lambda_2(R_{\varrho s}) + \Lambda_2(T_{rs}) = \Lambda_2(R_{\varrho s}) \tag{2.19}$$

where $T_{rs}$ is the inverse image of $T^\xi_{rs}$ by (1.2) (ii). Hence (2.19) yields

$$\Lambda_2(T_{rs}) = 0 \tag{2.20}$$

for any $r$ if $\beta r \geqq \alpha s$. Likewise $\Lambda_2(T_{sr}) = 0$ for any $s$ if $\beta r \leqq \alpha s$. $T_{sr}$ has a similar meaning as $T_{rs}$. These rectangles have a vertex on $l_{\alpha\beta}$ and a side on $Ox$ or on $Oy$. We shall call them rectangles of type $T$. Consequently, if $S$ is a set contained in a family of rectangles of type $T$, then $\Lambda_2(S) = 0$.

Now we can prove Theorem 2 when $L$ is a rectangle $R$ with sides parallel to the axes and with a diagonal on $l_{\alpha\beta}$. The rectangle of this kind will be called of type $R^*$. Let us denote the co-ordinates of the vertices of $R$ by $(a, b)$, $(c, b)$, $(c, d)$, $(a, d)$ such that $b = (\beta/\alpha)a$, $d = (\beta/\alpha)c$ (see Fig. 3). Clearly,

$$R_{cd} = R_{ab} \cup T_{cb} \cup R \cup T_{da}. \tag{2.21}$$

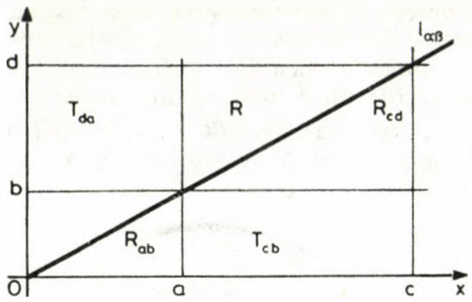


*Fig. 3*

Taking into account (2.20) we obtain from (2.21)

$$\Lambda_2(R) = \Lambda_2(R_{cd}) - \Lambda_2(R_{ab}). \tag{2.22}$$

$F^{\xi}_{\alpha\beta}$ being the diagonal of $R^{\xi}$ (2.17) and (2.22) imply

$$\Lambda_2(R) = \frac{d^{\xi} - b^{\xi}}{\Gamma(\xi+1)\beta^{\xi}} = |F^{\xi}_{\alpha\beta}|/\Gamma(\xi+1)(\alpha^{2\xi}+\beta^{2\xi})^{1/2}. \tag{2.23}$$

Thus the theorem is proved for $L = R$.

At present we are in position to show the statement in the general case. Assume first that $F^{\xi}_{\alpha\beta}$ is a single interval, then its inverse image by (1.2) (ii) is an interval $PQ \subset l_{\alpha\beta}$. Let $R$ be a rectangle with the diagonal $PQ$ of type $R^*$. If $L-R$ can be covered with rectangles of type $T$, so $\Lambda_2(\bar{L}-\bar{R})=0$ and $\Lambda_2(L)=\Lambda_2(R)$ given under (2.23). If some subset of $R$ is not contained in $L$ we make a subdivision of $PQ = \bigcup_{j=1}^{k} I_j$ where the intervals $I_j = A_{j-1}A_j$, $j=1, 2, \ldots, k$, $A_0 = P$, $A_k = Q$ are such that each rectangle $R_j$ with the diagonal $I_j$ and of type $R^*$ lies in $L$.

This is possible in virtue of (2.17) except perhaps at the endpoints $P$ and $Q$. If no difficulty arises at the endpoints, then $(\bar{L} - \bigcup_{j=1}^{k} \bar{R}_j)$ is included in a family of rectangles of type $T$ and $\Lambda_2(L - \bigcup_{j=1}^{k} R_j) = 0$. So it follows from (2.23) that

$$\Lambda_2(L) = \Lambda_2\Big(\bigcup_{j=1}^{k} R_j\Big) = F^{\xi}_{\alpha\beta}/\Gamma(\xi+1)(\alpha^{2\xi}+\beta^{2\xi})^{1/2},$$

which furnishes the proof. If $R_1$ or $R_k$ is not entirely in $L$, say this is the case of $R_k$, then we choose $A_{k-1}$ and $A_k$ on the two sides of $Q$ such that $|A_{k-1}Q| = |QA_k| = \varepsilon$ and we may write

$$\Lambda_2\Big(\bigcup_{j=1}^{k-1} R_j\Big) \leqq \Lambda_2(L) \leqq \Lambda_2\Big(\bigcup_{j=1}^{k} R_j\Big)$$

and $\varepsilon \to 0$ gives the result. Similar procedure may be applied in the neighbourhood of $P = A_0$. A simple modification is required if $P$ is at the origin. This proof holds if $F^{\xi}_{\alpha\beta}$

is the union of a finite number of pairwise disjoint intervals. Lastly, if $F^{\xi}_{\alpha\beta}$ is a Jordan measurable set we may use the same reasoning as in the proof of Theorem 1.

Consider still the eventuality when the condition (2.12) is not satisfied, that is if $|l^{\xi}_{\alpha\beta}\cap\partial L^{\xi}|>0$, in other words when $\partial L^{\xi}$ contains at least an interval of $l^{\xi}_{\alpha,\beta}$, see Figure 4 where $AB=F^{\xi}_{\alpha\beta}$, $\overline{AC}=\overline{G}^{\xi}_{\alpha\beta}$ and $BC\subset l^{\xi}_{\alpha\beta}$. In such a case there are no rectangles of type $R^*$ with diagonals on $BC$. Consequently, our preceding argumentation is not valid in this place.

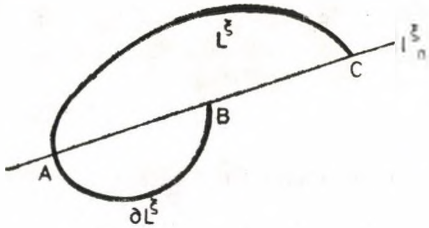


*Fig. 4*

**3.** The third result is a simple corollary of Theorems 1 and 2.

THEOREM 3. *Let the series*

$$f(x, y)=\sum_{m,n=0}^{\infty} a_{mn}x^m y^n,\quad a_{mn}\geqq 0$$

*be convergent in $Q$. Assume that we have*

$$f(x, y)\sim(1-x^{\alpha})^{-\xi}+(1-x^{\gamma}y^{\delta})^{-\xi}$$

*when $(x, y)\to(1, 1)$ on an admissible curve and in addition that the other conditions of Theorems 1 and 2 are simultaneously satisfied, then*

$$A_3(L)=\lim_{\lambda\to\infty}\lambda^{-\xi}\sum_{(m,n)\in L_{\lambda}} a_{mn}=[|M^{\xi}|\alpha^{-\xi}+F^{\xi}_{\gamma\delta}(\gamma^{2\xi}+\delta^{2\xi})^{-1/2}]/\Gamma(\xi+1).$$

REFERENCES

[1] ALPÁR, L., Tauberian theorems for power series of several variables I, *Fourier analysis and approximation theory* (Proc. Colloq., Budapest, 1976), Vol. I, Colloq. Math. Soc. J. Bolyai **19**, North-Holland, Amsterdam, 1978, 51—87. *MR* **80j:** 40006.

[2] ALPÁR, L., Tauberian theorems for power series of two variables, *Studia Sci. Math. Hungar.* **19** (1984), 165—176. *MR* **86j:** 40009.

[3] RIESZ, F., Démonstration nouvelle d'un théorème concernant les opérations fonctionnelles linéaires, *Ann. École Norm. Sup.* **31** (1914), 9—14.

[4] SZ.-NAGY, B., *Introduction to real functions and orthogonal expansions*, Akadémiai Kiadó, Budapest, 1964; Oxford University Press, New York, 1965. *MR* **31** # 5938.

[5] TURÁN, P., Problem 168, *Mat. Lapok* **21** (1970), 165—167 (in Hungarian).

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# MULTIPLICATIVE FUNCTIONS SATISFYING A CONGRUENCE PROPERTY

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Let $\mathcal{M}$ and $\mathcal{M}^*$ be the set of integer-valued multiplicative and completely multiplicative functions, respectively.

In 1966 M. V. Subbarao [4] proved that if $f \in \mathcal{M}$ satisfies the relation

$$f(n+m) \equiv f(m) \pmod{n} \tag{1}$$

for every positive integer $n$ and $m$, then $f(n)$ is a power of $n$ with a non-negative integer exponent. In 1972 A. Iványi [2] showed that if $f \in \mathcal{M}^*$ and (1) holds for a fixed $m$ and every $n$, then

$$f(n) = n^a, \tag{2}$$

where $a$ is a non-negative integer; furthermore if $f \in \mathcal{M}$ satisfies the relation

$$f(n+m) \equiv f(n)+f(m) \pmod{n} \tag{3}$$

for every positive integer $n$ and $m$, then (2) holds with a positive integer exponent $a$.

In this note we extend the above mentioned results of Subbarao and Iványi for cases, when (1) and (3) hold for every positive integer $n$ and every prime $m$, or for every prime $n$ and every positive integer $m$.

We first prove the following result.

THEOREM 1. *Let $f \in \mathcal{M}$. If one of the following assertions holds*

$$f(n+p) \equiv f(p) \pmod{n} \tag{a}$$

$$f(m+p) \equiv f(m) \pmod{p} \tag{b}$$

*for every positive integer $n, m$ and every prime $p$, then*

$$f(n) = n^a,$$

*where $a$ is a non-negative integer.*

PROOF. 1. Assume that $f \in \mathcal{M}$ satisfies the relation

$$f(n+p) \equiv f(p) \pmod{n} \tag{4}$$

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for every positive integer $n$ and every prime $p$. We first prove that for a prime $q$ and integer $n$

$$q|f(n) \quad \text{implies} \quad q|n. \tag{5}$$

Since $f\in\mathcal{M}$, we have to show it only for the case, when $n$ is a power of a prime number.

Let $q$ be a prime and let $q^b\|f(q)$, where $b\geqq 0$ is an integer. By Dirichlet's theorem there exist infinitely many primes $p(q)\neq q$ of the form $q^bk+1$. For such a prime $p(q)$, using (4), we have

$$f(p(q))f(q) = f(p(q)\cdot q) = f(q^{b+1}k+q) \equiv f(q) \pmod{q^{b+1}}$$

and so

$$f(p(q)) \equiv 1 \pmod{q}. \tag{6}$$

Let $p$ be a given arbitrary prime, $\alpha\geqq 1$ be an integer. If $q$ is prime divisor of $f(p^\alpha)$ and $q\neq p$, then there is a prime $p(q)$ satisfying (6) and there exist positive integers $x, y$ such that

$$(x, p) = 1 \quad \text{and} \quad p^\alpha x = p(q)+qy. \tag{7}$$

Using (4), (6) and (7), we have

$$0 \equiv f(p^\alpha)f(x) = f(p^\alpha x) = f(qy+p(q)) = f(p(q)) \equiv 1 \pmod{q},$$

which is a contradiction, since $q$ is a prime number. So we proved (5).

We now prove that $f\in\mathcal{M}$ with (4) implies $f\in\mathcal{M}^*$. We shall show it by proving that for each prime $p$ and each positive integer $\alpha$

$$f(p^\alpha) = (f(p))^\alpha. \tag{8}$$

We prove (8) by induction on $\alpha$. Obviously, (8) holds for $\alpha=1$. We assume that (8) holds for $\alpha$. Let $q>p$ be an arbitrary prime. Then there exist positive integers $u$ and $v$ such that

$$(u, p) = 1 \quad \text{and} \quad p^\alpha u = qv+1. \tag{9}$$

Let $p(q)$ be a prime, which satisfies (6) and

$$p(q) > p^\alpha u. \tag{10}$$

Using (4), (6), (9) and (10), we have

$$f(p^\alpha u) \equiv f(p(q))f(p^\alpha u) = f(p(q)p^\alpha u) = f(p(q)qv+p(q)) \equiv$$
$$\equiv f(p(q)) \equiv 1 \pmod{q}$$

and

$$f(p^{\alpha+1}u) = f(pqv+p) \equiv f(p) \pmod{q}.$$

From these

$$f(p^{\alpha+1}u) \equiv f(p)f(p^\alpha u) \pmod{q}$$

and so, using (5) and (9),

$$f(p^{\alpha+1}) \equiv f(p)f(p^\alpha) \pmod{q} \tag{11}$$

follows. Since $q>p$ is an arbitrary prime, by (8) and (11) we have

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) = (f(p))^{\alpha+1}.$$

Thus (8) holds for every positive integer $\alpha \geqq 1$, and so $f \in \mathcal{M}^*$.

Since $f \in \mathcal{M}^*$ and (4) holds, by the result of Iványi mentioned above the proof of the first part is finished.

2. Assume that $f \in \mathcal{M}$ satisfies the relation

$$f(m+p) \equiv f(m) \pmod p \tag{12}$$

for every positive integer $m$ and every prime $p$. We note that by (12) it follows that

$$f(m+kp) \equiv f(m) \pmod p$$

for every positive integer $k$, and so

$$f(m+n) \equiv f(m) \pmod{n^*} \tag{13}$$

holds for every positive integer $n$ and $m$, where $n^*$ denotes the product of all distinct prime divisors of $n$.

In this case, using (12), (13) and a little modification of the first part's argument, we also have $f \in \mathcal{M}^*$ and for a prime $q$, a positive integer $n$

$$q \mid f(n) \quad \text{implies} \quad q \mid n. \tag{14}$$

From this for a prime $p$

$$f(p) = \pm p^a \tag{15}$$

follows, where $a \geqq 0$ is an integer. If $f(p) = -p^a$, then applying (13) with $m=1$ and $n=p^t-1$, where $t$ is an odd integer, we have

$$f(p^t) = (f(p))^t = (-p^a)^t = -p^{at} \equiv 1 \quad (\operatorname{mod}(p^t-1)^*)$$

and so it follows that

$$2 \equiv 0 \quad (\operatorname{mod}(p^t-1)^*)$$

for any odd integer $t$. This is a contradiction, since by the result of G. D. Birkhoff and H. S. Vandiver [1] we have

$$(p^t-1)^* \geqq t+1$$

for any integer $t>6$. Thus $f(p)=p^a$.

For distinct primes $p$ and $q$, let

$$f(p) = p^a \quad \text{and} \quad f(q) = q^b,$$

where $a \geqq b \geqq 0$ are integers. Then using (13) with $m=1$ and $n=pq^s-1$, we get

$$f(pq^s) = f(p)f(q^s) = p^a q^{sb} \equiv 1 \quad (\operatorname{mod}(pq^s-1)^*).$$

From this it follows that

$$p^{a-b} \equiv 1 \quad (\operatorname{mod}(pq^s-1)^*) \tag{16}$$

for every positive integer $s$. But it is well-known that

$$(pq^s-1)^* \to \infty \quad \text{as} \quad s \to \infty$$

(see, e.g. [3]). Thus (16) implies $a=b$, and so

$$f(p) = p^a \tag{17}$$

for every prime $p$, where $a \geqq 0$ is an integer.

Finally, $f \in \mathcal{M}^*$ and (17) imply $f(n)=n^a$ for every $n$. It completes the proof of the second part.

From Theorem 1 we deduce the following

THEOREM 2. *Let $f \in \mathcal{M}$. If one of the following assertions holds:*

(a′) $$f(n+p) \equiv f(n)+f(p) \pmod n$$

(b′) $$f(m+p) \equiv f(m)+f(p) \pmod p$$

*for every positive integer $n, m$ and every prime $p$, then*

$$f(n) = n^a,$$

*where $a$ is a positive integer.*

PROOF. 3. Assume that $f \in \mathcal{M}$ satisfies the relation

$$f(n+p) \equiv f(n)+f(p) \pmod n \tag{18}$$

for every positive integer $n$ and every prime $p$. Using Theorem 1(a), we prove Theorem 2 (a′) by showing that

$$f(n) \equiv 0 \pmod n \tag{19}$$

for every positive integer $n$.

Using (18), by induction on $k$, we have

$$f(kq) \equiv kf(q) \pmod q \tag{20}$$

for every positive integer $k$ and every prime $q$. If $p$ is a prime and $p \neq q$, then by (18) and (20) we obtain

$$f\big(q(p+q)\big) = f(q)f(p+q) \equiv f^2(q)+f(p)f(q) = f^2(q)+f(pq) =$$
$$\equiv f^2(q)+pf(q) \pmod q$$

and

$$f\big(q(p+q)\big) \equiv (p+q)f(q) \equiv pf(q) \pmod q.$$

From these

$$f(q) \equiv 0 \pmod q$$

and so, by (20)

$$f(kq) \equiv 0 \pmod q \tag{21}$$

follows.

We shall prove that $f \in \mathcal{M}^*$. First we show that for every fixed prime $q$ there exist infinitely many primes $p(q) \neq q$ such that

$$f\big(p(q)\big) \equiv 1 \pmod q. \tag{22}$$

Let $q$ be a prime. We choose the prime $p_0 \neq q$ and the integer $s$ such that

$$s \geqq e+1 \quad \text{and} \quad p_0 \nmid q^s+1,$$

where $q^e \| f(p_0)$. By Dirichlet's theorem there are infinitely many primes $p(q) \neq p_0$ of the form $q^{e+1}k+1$ with $(k, p_0)=1$. We can choose $k$ in the form

$$q^{s-(e+1)}(p_0 t+1).$$

For such a prime $p(q)$, using (18) and (21), we have

$$f(p_0)f(p(q)) = f(p_0 p(q)) = f(p_0 q^{e+1}k+p_0) \equiv f(p_0 q^{e+1}k)+f(p_0) =$$
$$= f(p_0)f(q^{e+1}k)+f(p_0) \equiv f(p_0) \pmod{q^{e+1}},$$

and so (22) follows.

Using (18), (21) and (22), the fact that

$$\text{(23)} \qquad q|f(n) \quad \text{implies} \quad q|n,$$

where $q$ is a prime and $n$ is a positive integer, may be proved similarly to the proof of Theorem 1 (see (5)).

Finally, using (18), (21) and (23), similarly to the proof of Theorem 1 (a) we also obtain that $f \in \mathcal{M}^*$. Thus (21) implies (19). This completes the proof of the first part.

4. Assume that $f \in \mathcal{M}$ satisfies the relation

$$\text{(24)} \qquad f(m+p) \equiv f(m)+f(p) \pmod{p}$$

for every positive integer $m$ and every prime $p$. Using Theorem 1 (b), we have to prove that

$$\text{(25)} \qquad f(p) \equiv 0 \pmod{p}$$

for every prime $p$.

By (24) it is easily seen that

$$f(p^2) \equiv pf(p) \equiv 0 \pmod{p}$$

for every prime $p$. Applying (24) with $m=p^2$, we have

$$f(p^2+p) \equiv f(p^2)+f(p) \equiv f(p) \pmod{p}$$

and

$$f(p^2+p) = f(p)f(p+1) \equiv f(p)+f^2(p) \pmod{p}.$$

From these

$$f^2(p) \equiv 0 \quad \text{and} \quad f(p) \equiv 0 \pmod{p}$$

follow. Thus we proved (25) and the second part of Theorem 2.

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## REFERENCES

[1] BIRKHOFF, G. D. and VANDIVER, H. S., On the integral divisors of $a^n - b^n$, *Ann. of Math.* (2) **5** (1904), 173—180. *J. Fortschritte Math.* **35**, 205.
[2] IVÁNYI, A., On multiplicative functions with congruence property, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **15** (1972), 133—137. *MR* **48** # 8364.
[3] STEWART, C. L., On the greatest prime factor of terms of a linear recurrence sequence, *Rocky Mountain J. Math.* **15** (1985), 599—608. *MR* **87h:** 11017.
[4] SUBBARAO, M. V., Arithmetic functions satisfying a congruence property, *Canad. Math. Bull.* **9** (1966), 143—146. *MR* **33** # 3993.

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# AN ERROR ESTIMATE FOR INTERPOLATION BY MARKOV SYSTEMS

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## 1. Results

The system $\{\varphi_j\}_{j=1}^m$ of functions $\varphi_j$, continuous on $(a, b)$, is called a Markov system if for every natural number $n \leqq m$ the system $\{\varphi_j\}_{j=1}^n$ is a Čebyšev system which means that every 'polynomial of degree $n-1$'

$$p_{n-1}(x) = \alpha_1\varphi_1(x)+\alpha_2\varphi_2(x)+\ldots+\alpha_n\varphi_n(x), \quad x\in(a, b),$$

$\alpha_j$ real, $|\alpha_1|+|\alpha_2|+\ldots+|\alpha_n|>0$, has at most $n-1$ zeros.

Associating with each Čebyšev system $\{\varphi_j\}_{j=1}^n$ a set of $n$ interpolation nodes $\{x_i^{(n)}\}_{i=1}^n$

(2) $$a < x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} < b,$$

then given a function $f$ on $(a, b)$ there exists a polynomial $P_{n-1}[f]$ of degree $n-1$ possessing the interpolation property, i.e.

(3) $$P_{n-1}[f](x_i^{(n)}) = f(x_i^{(n)}), \qquad i = 1, 2, \ldots, n.$$

This interpolation polynomial can be written in the form

(4) $$P_{n-1}[f](x) = \sum_{k=1}^{n} f(x_k^{(n)})s_k^{(n)}(x),$$

where the polynomials $s_k^{(n)}$ of degree $n-1$ are uniquely determined by the property

(5) $$s_k^{(n)}(x_i^{(n)}) = \delta_{ki}, \qquad 1 \leqq i, k \leqq n.$$

The so called "Lebesgue-function" $L_n$ is defined by

(6) $$L_n(x) = \sum_{k=1}^{n} |s_k^{(n)}(x)|.$$

Now for every fixed $x\neq x_i^{(n)}$, $(i=1, 2, \ldots, n)$, let us suppose the set of $n+1$ numbers

$$\{x, x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}\} = \{y_i^{(n)}\}_{i=1}^{n+1}$$

to be ordered by size, i.e.

(7) $$a < y_1^{(n)} < y_2^{(n)} < \ldots < y_{n+1}^{(n)} < b.$$

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We then define

$$\alpha_n = \max_{1 \leq i \leq n} |y_{i+1}^{(n)} - y_i^{(n)}|. \tag{8}$$

Finally if $\omega(f, \cdot)$ denotes the modulus of continuity of $f$, $f$ being continuous and bounded on $(a, b)$, we prove the following

THEOREM 1. *If* $\varphi_1(x) \equiv 1$ *in the Markov system* $\{\varphi_j\}_{j=1}^{\infty}$ *then*

$$|P_{n-1}[f](x) - f(x)| \leq \frac{1}{2}(L_n(x)+1)\,\omega(f, \alpha_n). \tag{9}$$

Now let

$$\beta_n = \max_{0 \leq i \leq n} |x_{i+1}^{(n)} - x_i^{(n)}|, \tag{10}$$

where $x_0^{(n)} = a$, $x_{n+1}^{(n)} = b$. Of course we have $\alpha_n \leq \beta_n$ thus (9) remains true if $\alpha_n$ is substituted by $\beta_n$ the latter not depending on $x$. The following theorem shows that the estimate given in (9) is optimal.

THEOREM 2. *Given a Čebyšev system* $\{\varphi_j\}_{j=1}^{n}$ *and a point* $x \in (a, b)$, $x \neq x_i^{(n)}$, $(i = 1, 2, \ldots, n)$, *then there exists a nonconstant function f, continuous and bounded on* $(a, b)$, *such that*

$$|P_{n-1}[f](x) - f(x)| \geq \frac{1}{2}(L_n(x)+1)\,\omega(f, \beta_n). \tag{11}$$

Of course in (11) $\beta_n$ may be replaced by $\alpha_n$.

## 2. Proofs

As $\varphi_1(x) \equiv 1$ in the Markov system $\{\varphi_j\}_{j=1}^{\infty}$ and because of (5) we must have

$$\sum_{k=1}^{n} s_k^{(n)}(x) \equiv 1, \tag{12}$$

therefore

$$P_{n-1}[f](x) - f(x) = \sum_{k=1}^{n} \left(f(x_k^{(n)}) - f(x)\right) s_k^{(n)}(x). \tag{13}$$

In the following we omit the superscript $(n)$. Now let

$$x_i < x < x_{i+1}, \qquad i = 0, 1, 2, \ldots, n, \tag{14}$$

$(x_0 = a,\ x_{n+1} = b)$. From (13) we obtain by Abel's transformation

$$\begin{aligned} P_n[f](x) - f(x) = {} & \sum_{k=1}^{i-1} \left(f(x_k) - f(x_{k+1})\right) a_k + \left(f(x_i) - f(x)\right) a_i + \\ & + \sum_{k=i+2}^{n} \left(f(x_k) - f(x_{k-1})\right) b_k + \left(f(x_{i+1}) - f(x)\right) b_{i+1}, \end{aligned} \tag{15}$$

where

$$(16)\qquad \begin{aligned} a_k &= a_k(x) = \sum_{j=1}^{k} s_j(x), \\ b_k &= b_k(x) = \sum_{j=k}^{n} s_j(x), \end{aligned} \qquad k = 1, 2, \ldots, n,$$

(as usual, empty sums have to be replaced by zero).

LEMMA. *For* $x \in (x_i, x_{i+1})$ *and* $i=0, 1, 2, \ldots, n$ *we have*

$$(17)\qquad \begin{aligned} (-1)^{i+k} a_k(x) &\geqq 0, \quad k \leqq i, \\ (-1)^{i+k+1} b_k(x) &\geqq 0, \quad i < k. \end{aligned}$$

PROOF. Let us show that

$$(-1)^{i+k} a_k(x) > 0, \quad x \in (x_i, x_{i+1}),\ k \leqq i \leqq n,$$

which in view of (12) is obvious if $k=i=n$. Now let $k<n$. From

$$a_k(x_j) = \begin{cases} 1, & j = 1, 2, \ldots, k \\ 0, & j = k+1, \ldots, n \end{cases}$$

it follows that in each of the $k-1$ intervals $(x_1, x_2), (x_2, x_3), \ldots, (x_{k-1}, x_k)$ the polynomial $a_k$ of degree $n-1$ has at least one extremum and the same must happen in each of the $n-k-1$ intervals $(x_{k+1}, x_{k+2}), \ldots, (x_{n-1}, x_n)$ which amounts to at least $n-2$ extrema. We note that the number of extrema of such a polynomial cannot exceed $n-2$ (cf. [3] or [4]). Apart from $x_{k+1}, x_{k+2}, \ldots, x_n$ there are no other zeros of $a_k$ in $(x_k, x_{n+1})$ otherwise this would yield further extrema. But now $a_k$ does change sign at each of the zeros $x_{k+1}, x_{k+2}, \ldots, x_n$, otherwise this would lead to further extrema at these points, and as $a_k(x_k)=1$, $a_k(x_{k+1})=0$, the function $a_k$ starts with positive sign on the interval $(x_k, x_{k+1})$. The proof of the second part is quite similar therefore being omitted. This completes the proof of (17).

To prove Theorem 1 we start from (15) which yields

$$(18)\qquad |P_n[f](x)-f(x)| \leqq \omega(f, \alpha_n)\Big(\sum_{k=1}^{i} |a_k(x)| + \sum_{k=i+1}^{n} |b_k(x)|\Big).$$

But now using (16) and the lemma we get

$$(19)\qquad \begin{aligned} r(x) &= \sum_{k=1}^{i} |a_k(x)| + \sum_{k=i+1}^{n} |b_k(x)| = \sum_{k=1}^{i} \Big[(-1)^{i+k} \sum_{j=1}^{k} s_j(x)\Big] + \\ &+ \sum_{k=i+1}^{n} \Big[(-1)^{i+k+1} \sum_{j=k}^{n} s_j(x)\Big] = \big(s_i(x)+s_{i-2}(x)+s_{i-4}(x)+\ldots\big) + \\ &+ \big(s_{i+1}(x)+s_{i+3}(x)+s_{i+5}(x)+\ldots\big). \end{aligned}$$

For reasons of simplicity the subindices are assumed to range only from 1 to $n$, (remember that for certain values of $i$ one of these sums may be empty, which of course does not upset the proof). Bearing this in mind we conclude from (19)

$$r(x) = \frac{1}{2}(s_i(x)-s_{i-1}(x)+s_{i-2}(x)-s_{i-3}(x)+s_{i-4}(x)-+\ldots)+$$

$$+\frac{1}{2}(s_{i+1}(x)-s_{i+2}(x)+s_{i+3}(x)-s_{i+4}(x)+s_{i+5}(x)-+\ldots)+$$

$$+\frac{1}{2}(s_1(x)+s_2(x)+\ldots+s_i(x)+s_{i+1}(x)+\ldots+s_n(x)) \leqq \frac{1}{2}\sum_{k=1}^{n}|s_k(x)|+\frac{1}{2}\sum_{k=1}^{n}s_k(x) =$$

$$= \frac{1}{2}L_n(x)+\frac{1}{2},$$

which in view of (18) and (19) completes the proof of Theorem 1.

To prove Theorem 2 it is easy to construct a function $f$ such that for fixed $x \neq x_k$

$$f(x) = -1, \quad f(x_k) = \operatorname{sgn} s_k(x), \quad (k = 1, 2, \ldots, n), \tag{20}$$

$f$ being continuous on $(a, b)$ and not exceeding 1 in absolute value. Of course $2 \geqq \omega(f, \beta_n) > 0$ thus we conclude from (4), (20) and (6)

$$P_{n-1}[f](x)-f(x) = L_n(x)+1 \geqq \frac{1}{2}\omega(f, \beta_n)(L_n(x)+1),$$

which proves Theorem 2.

## 3. Remarks

The estimate given by Theorem 1 has first been suggested in case of ordinary interpolation by Kis [2] then proved by Brass—Günttner [1]. To infer algebraic interpolation from Theorem 1 on the closed interval $[-1, 1]$ we take $\varphi_j(x) = x^{j-1}$ and $(a, b) = (-1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$ sufficiently small, but of course we simply can choose $(a, b) = (-1, 1)$, if $-1$ or 1 are no points of interpolation.

In [1] it is made use of Rolle's theorem whereas no differentiability is needed here.

## REFERENCES

[1] Brass, H. and Günttner, R., Eine Fehlerabschätzung zur Interpolation stetiger Funktionen, *Studia Sci. Math. Hungar.* **8** (1973), 363—367. *MR* **50** # 5329.

[2] Kis, O., Замечания об оценке погрешности интерполирования (Remarks on the estimation of interpolation error), *Studia Sci. Math. Hungar.* **4** (1969), 279—289 (in Russian). *MR* **40** # 5107.

[3] Рабкин, Е. Л., О числе экстремумов полиномов по системам Чебышева (The number of extrema of polynomials in Čebyšev systems), *Izv. Vysš. Učebn. Zaved, Matematika* **1970**, no. 2, 70—75 (in Russian). *MR* **43** # 3704.

[4] Zielke, R., *Discontinuous Čebyšev systems*, Lecture Notes in Mathematics, Vol. 707, Springer-Verlag, Berlin—Heidelberg—New York, 1979. *MR* **81a:** 41001.

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# ON EXTENSIONS OF SYNTOPOGENOUS STRUCTURES I

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**Abstract**

We investigate the possibility of extending a syntopogenous structure compatible on a subspace of a topological or bitopological space to a compatible structure of the whole space.

We shall follow the terminology and notations of Császár's book [3], except that (i) $^c$ will be replaced by $^{-1}$ (or just $^-$ before another operation), and $^i$ by $^1$; (ii) a topology in the sense of [3] will be identified with the associated "classical topology", cf. [7]; (iii) a "quasi-uniformity" in the sense of [3] will be called (in conformity with current terminology) a *quasi-uniform base*; the *quasi-uniformity induced by* $\mathscr{S}$ is the one for which the "quasi-uniformity" associated with $\mathscr{S}$ is a base; (iv) if $\mathscr{T}=\mathscr{S}^{tp}$ then we say that $\mathscr{S}$ *induces the topology* $\mathscr{T}$, or that it is *compatible with* $\mathscr{T}$.

$A\subset_X B$ means that $A\subset B\subset X$ (i.e. $\mathscr{D}_X=\{\subset_X\}$). $\mathfrak{p}(X)$ denotes the power set of $X$. For $\mathfrak{a}\subset\mathfrak{p}(X)$, $\sec\mathfrak{a}=\sec_X\mathfrak{a}$ consists of those subsets of $X$ that meet each element of $\mathfrak{a}$.

$(\mathscr{S}^{-tp}, \mathscr{S}^{tp})$ is the *bitopology induced by* the syntopogenous structure $\mathscr{S}$. (The two topologies are taken in reverse order in [5].) We shall also say that $\mathscr{S}$ is a *compatible* syntopogenous structure in the bitopological space $(X; \mathscr{S}^{-tp}, \mathscr{S}^{tp})$.

## § 0. The problem

**0.1** Let us be given a topological space $(X, \mathscr{T})$, and a compatible syntopogenous structure $\mathscr{S}_0$ on the non-empty subspace $(S, \mathscr{T}|\mathscr{S})$. We are looking for a compatible extension $\mathscr{S}$ of $\mathscr{S}_0$, which means that $\mathscr{S}^{tp}=\mathscr{T}$ and $\mathscr{S}|S\sim\mathscr{S}_0$. (So density is not required when we speak about an extension of a syntopogenous structure. The terminology concerning extensions of bitopologies will be different.) This question was investigated in [7], and we have very little to add. (In [7], $S$ is assumed to be $\mathscr{T}$-dense, but, as we shall see, this restriction does not make much difference.)

The analogous problem in bitopological spaces seems to be unexplored: let $\mathscr{S}_0$ be a compatible syntopogenous structure on the non-empty subspace $(S; \mathscr{T}_{-1}|S, \mathscr{T}_1|S)$ of the bitopological space $(X; \mathscr{T}_{-1}, \mathscr{T}_1)$; find extensions $\mathscr{S}$ of $\mathscr{S}_0$ compatible with $(\mathscr{T}_{-1}, \mathscr{T}_1)$.

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We shall describe in § 1 a general construction, which will be applied to the topological case in § 2, and to the bitopological case in § 3.

**0.2** The appropriateness of the definition of an extension is open to question: why do we not demand $\mathscr{S}|S=\mathscr{S}_0$ instead of $\mathscr{S}|S\sim\mathscr{S}_0$? One can argue in favour of the weaker requirement that equivalence classes of syntopogenous structures are more important than the structures themselves (similarly to the case of uniformities v. uniform bases). Should this reasoning not carry conviction, the doubts will be settled by the following lemma, in consequence of which the two possible definitions are essentially the same: if there is a compatible extension then there exists one in the stronger sense as well (because equivalent structures induce the same (bi)topology).

LEMMA. *Let $\mathscr{S}_0$ and $\mathscr{S}$ be (perfect/biperfect/symmetrical) syntopogenous structures on $S$, respectively on $X$, and $\mathscr{S}|S\sim\mathscr{S}_0$. Then there exists a (perfect/biperfect/symmetrical) syntopogenous structure $\mathscr{S}'$ such that $\mathscr{S}'\sim\mathscr{S}$, $\mathscr{S}'|S=\mathscr{S}_0$ and $|\mathscr{S}'|\leqq|\mathscr{S}|\cdot|\mathscr{S}_0|$.*

PROOF. 1° Assign to each $<_0\in\mathscr{S}_0$ and $<\in\mathscr{S}$ satisfying the condition

$$<_0\subset <|S \tag{1}$$

a relation $<'=<'(<_0,<)$ as follows:

$$A<'B \quad\text{iff}\quad A<B \quad\text{and}\quad A\cap S<_0 B\cap S. \tag{2}$$

$<'$ is a topogenous order on $X$, (bi)perfect or symmetrical if $<_0$ and $<$ are so (the simple proof will be given for a more general construction in 1.1). Take the order family

$$\mathscr{S}'=\{<'(<_0,<)\colon\ <_0\in\mathscr{S}_0,\ <\in\mathscr{S},\ <_0\subset<|S\}.$$

We do not have to prove that $\mathscr{S}'$ is a syntopogenous structure, since this will follow from $\mathscr{S}'\sim\mathscr{S}$. The statement about $|\mathscr{S}'|$ is evident.

2° Let us prove that $\mathscr{S}'\sim\mathscr{S}$. Each $<'\in\mathscr{S}'$ is of the form $<'(<_0,<)$; now $<'\subset<\in\mathscr{S}$, and so $\mathscr{S}'\prec\mathscr{S}$.

Conversely, take $<_1\in\mathscr{S}$. In consequence of $\mathscr{S}|S\sim\mathscr{S}_0$, there are $<_0\in\mathscr{S}_0$ such that

$$<_1|S\subset<_0 \tag{3}$$

and $<_2\in\mathscr{S}$ with $<_0\subset<_2|S$. As $\mathscr{S}$ is directed, we can take $<\in\mathscr{S}$ satisfying $<_1\cup<_2\subset<$. Now $<_0\subset<_2|S\subset<|S$, thus $<'=<'(<_0,<)\in\mathscr{S}'$, and so $\mathscr{S}\prec\mathscr{S}'$ will follow from $<_1\subset<'$.

Assume $A<_1B$. Then $A<B$, and also $A\cap S<_0B\cap S$ by (3), so $A<'B$ indeed.

3° It is clear from (2) and (1) that for any $<'\in\mathscr{S}'$,

$$<'|S=<'(<_0,<)|S=<_0, \tag{4}$$

so $\mathscr{S}'|S\subset\mathscr{S}_0$. To prove the converse, take an order $<_0\in\mathscr{S}_0$. By $\mathscr{S}|S\sim\mathscr{S}_0$, we can choose $<\in\mathscr{S}$ satisfying (1), and then, according to (4), $<_0$ is the restriction of some order from $\mathscr{S}'$, hence $\mathscr{S}_0\subset\mathscr{S}'|S$. □

REMARKS. a) If $\mathscr{S}_0$ and $\mathscr{S}$ are finite then $|\mathscr{S}'|=|\mathscr{S}_0|$ can be required: with the notations of the proof, let

$$\mathscr{S}' = \{<'(<_0, <): <_0 \in \mathscr{S}_0\},$$

where $<$ is the finest element of $\mathscr{S}$.

b) It is clear from the proof that the lemma remains valid for directed order families.

## § 1. A general construction

**1.1** Let $<_{-1}$, $<_1$ and $<_2$ be semi-topogenous orders on $X$, and $<_0$ a semi-topogenous order on $S \subset X$. Define a relation $< = [<_{-1}, <_0, <_1, <_2]$ between subset of $X$ as follows:

$$A<B \textit{ iff } A<_2 B \textit{ and there exist sets } A', B' \subset X \textit{ such that}$$

$$A <_1 A', \quad A' \cap S <_0 B' \cap S, \quad B' <_{-1} B.$$

(Such sets $A'$ and $B'$ will be referred to as "*showing* $A<B$".) Equivalently: $A<B$ iff $A<_2 B$ and there exist $A''$, $B'' \subset S$ such that $A<_1 A'' \cup (X \setminus S)$, $A'' <_0 B''$ and $B'' <_{-1} B$. (To prove the equivalence, take $A''=A' \cap S$, $B''=B' \cap S$, respectively $A'=A'' \cup (X \setminus S)$, $B'=B''$. The second version of the definition will look more symmetrical if we write $A<_1 A'' \cup (X \setminus S)$ in the form $S \setminus A'' <_1^{-1} X \setminus A$.)

LEMMA. *Let* $< = [<_{-1}, <_0, <_1, <_2]$.

a) $<$ *is a semi-topogenous order on* $X$. $< \subset <_2$ *and* $<|S \subset <_0$.

b) *If each* $<_j$ $(-1 \leqq j \leqq 2)$ *is topogenous, perfect or biperfect then so is* $<$.

c) $<^{-1} = [<_1^{-1}, <_0^{-1}, <_{-1}^{-1}, <_2^{-1}]$. *Consequently, if* $<_0$ *and* $<_2$ *are symmetrical and* $<_1^{-1} = <_{-1}$ *then* $<$ *is symmetrical, too.*

PROOF. a) $\emptyset < \emptyset$ and $X<X$ are shown by $A'=B'=\emptyset$, respectively $A'=B'=X$. If $A<B$ then $A<_2 B$, and therefore $A \subset B$. If $A_0 \subset A < B \subset B_0$, and $A'$, $B'$ are chosen to show $A<B$ then they show $A_0 < B_0$ as well.

If $C(<|S)D$ then there are sets $A, B$ with $A<B$, $A \cap S = C$, $B \cap S = D$. Take $A'$, $B'$ showing $A<B$; then $C \subset A' \cap S <_0 B' \cap S \subset D$, hence $<|S \subset <_0$.

b) Let each $<_j$ be a topogenous order, and assume $A_k < B_k$ $(k=1, 2)$. We have to show that $A_1 \square A_2 < B_1 \square B_2$ holds for $\square = \cap$ and $\square = \cup$. $A_k < B_k$ implies $A_k <_2 B_k$, therefore $A_1 \square A_2 <_2 B_1 \square B_2$. Choosing sets $A_k'$, $B_k'$, with $A_k <_1 A_k'$, $A_k' \cap S <_0 B_k' \cap S$ and $B_k' <_{-1} B_k$, we take $A' = A_1' \square A_2'$ and $B' = B_1' \square B_2'$; now these sets show that $A_1 \square A_2 < B_1 \square B_2$ (e.g. $A_1 \square A_2 <_1 A'$ because $<_1$ is topogenous). To prove the statement about (bi)perfectness, repeat the above reasoning with arbitrary collections instead of pairs.

c) An easy calculation. □

REMARK. With the notation from the proof of Lemma 0.2,

$$<'(<_0, <) = [\subset_X, <_0, \subset_X, <].$$

**1.2** LEMMA. *If* $<_j \subset <_j'$ $(-1 \leqq j \leqq 2)$ *then*

$$[<_{-1}, <_0, <_1, <_2] \subset [<_{-1}', <_0', <_1', <_2']. \quad \square$$

**1.3** Let $\mathscr{A}_{-1}, \mathscr{A}_1$ and $\mathscr{A}_2$ be order families on $X$, and $\mathscr{A}_0$ an order family on $S$. We consider the following order family on $X$:

$$[\mathscr{A}_{-1}, \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2] = \{[<_{-1}, <_0, <_1, <_2]\colon \ <_j \in \mathscr{A}_j \quad (-1 \leqq j \leqq 2)\}.$$

LEMMA. *If each $\mathscr{A}_j$ is directed $(-1 \leqq j \leqq 2)$ then so is $[\mathscr{A}_{-1}, \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2]$.*

PROOF. Lemma 1.2. □

**1.4** LEMMA. *If $\mathscr{A}_j < \mathscr{B}_j$ $(-1 \leqq j \leqq 2)$ then*

$$[\mathscr{A}_{-1}, \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2] < [\mathscr{B}_{-1}, \mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2].$$

PROOF. Lemma 1.2. □

**1.5** LEMMA. *If each $\mathscr{A}_j$ is directed $(-1 \leqq j \leqq 2)$ then*

$$[\mathscr{A}_{-1}, \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2]^t = [\mathscr{A}_{-1}^t, \mathscr{A}_0^t, \mathscr{A}_1^t, \mathscr{A}_2^t].$$

PROOF. Let $\mathscr{A} = [\mathscr{A}_{-1}, \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2]$, $\mathscr{B} = [\mathscr{A}_{-1}^t, \mathscr{A}_0^t, \mathscr{A}_1^t, \mathscr{A}_2^t]$. From $\mathscr{A}_j < \mathscr{A}_j^t$ and Lemma 1.4 we have $\mathscr{A} < \mathscr{B}$, which implies $\mathscr{A}^t < \mathscr{B}$, because $\mathscr{B}$ is a simple order family.

To prove $\mathscr{B} < \mathscr{A}^t$, we have to show that $< \subset <'$ where $\mathscr{B} = \{<\}$ and $\mathscr{A}^t = \{<'\}$. Assume $A < B$. As $\mathscr{A}_j$ is directed, we have $\mathscr{A}_j^t = \{\cup \mathscr{A}_j\}$ ([3] (8.38)). $A < B$ means now that there are $A'$ and $B'$ with $A(\cup\mathscr{A}_2)B$, $A(\cup\mathscr{A}_1)A'$, $A' \cap S(\cup\mathscr{A}_0)B' \cap S$ and $B'(\cup\mathscr{A}_1)B$. Hence there are $<_j \in \mathscr{A}_j$ for which $A'$ and $B'$ show that $A <'' B$ where $<'' = [<_{-1}, <_0, <_1, <_2]$. $<'' \in \mathscr{A}$, therefore $<'' \subset <'$, and so $A <' B$.

$\mathscr{A}^t = \mathscr{B}$ follows now from $\mathscr{A}^t \sim \mathscr{B}$, since both order families are simple. □

**1.6.** LEMMA. *Let* $\mathscr{A} = [\mathscr{A}_{-1}, \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2]$.

a) $\mathscr{A} < \mathscr{A}_2$, $\mathscr{A}|S < \mathscr{A}_0$.

b) *If $\mathscr{A}_0 < \mathscr{A}_j|S$ $(j = \pm 1, 2)$ and $\mathscr{A}_0$ is a syntopogenous structure then $\mathscr{A}|S \sim \mathscr{A}_0$.*

PROOF. a) Lemma 1.1 a).

b) Given $<_{00} \in \mathscr{A}_0$, we need $<_j \in \mathscr{A}_j$ such that

$$<_{00} \subset [<_{-1}, <_0, <_1, <_2]|S. \tag{1}$$

Take $<_0 \in \mathscr{A}_0$ such that $<_{00} \subset <_0^3$. As $\mathscr{A}_0 < \mathscr{A}_j|S$, there are $<_j \in \mathscr{A}_j$ with

$$<_0 \subset <_j|S \qquad (j = \pm 1, 2). \tag{2}$$

These orders $<_j$ will do in (1).

Indeed, assume $C <_{00} D$, and let $< = [<_{-1}, <_0, <_1, <_2]$. Pick $C', D'$ with $C <_0 C' <_0 D' <_0 D$. By (2), $C(<_1|S)C'$, $D'(<_{-1}|S)D$ and $C(<_2|S)D$. Thus the sets $C' \cup (X \setminus S)$ and $D'$ show that $C < D \cup (X \setminus S)$, and so $C(<|S)D$. □

**1.7** In the most important special case of the general construction, the three order families on $X$ coincide: if $\mathscr{A}$ is an order family on $X$, and $\mathscr{A}_0$ on $S$ then define

$$\mathscr{A} + \mathscr{A}_0 = \{< + <_0\colon \ < \in \mathscr{A}, \ <_0 \in \mathscr{A}_0\}$$

where

$$< + <_0 = [<, <_0, <, <].$$

LEMMA. *If* $\mathscr{A}$ *is directed then* $\mathscr{A}+\mathscr{A}_0\sim[\mathscr{A},\mathscr{A}_0,\mathscr{A},\mathscr{A}]$. □

**1.8** LEMMA. *If* $\mathscr{S}$ *is a syntopogenous structure on* $X$ *and* $\mathscr{A}_0$ *an order family on* $S$ *then*

$$\mathscr{S}+\mathscr{A}_0\sim\{<+^*<_0:\ <\in\mathscr{S},\ <_0\in\mathscr{A}\},\tag{1}$$

*where, for* $A,B\subset X$, $A(<+^*<_0)B$ *iff there are* $A',B'$ *such that* $A<A'<B'<B$ *and* $A'\cap S<_0B'\cap S$.

PROOF. $<+^*<_0\subset<+<_0$ is evident. Conversely, we are going to show that if $<\subset<'^3$ and $<'\in\mathscr{S}$ then $<+<_0\subset<'+^*<_0$.

Assume $A(<+<_0)B$. Then there are $A',B'$ such that $A<B$, $A<A'$, $B'<B$ and $A'\cap S<_0B'\cap S$. As $A<B$, we may pick $A'',B''$ with $A<'A''<'B''<'B$. Now if $A'''=A'\cap A''$ and $B'''=B'\cup B''$ then $A<'A'''<'B'''<'B$ and $A'''\cap S<_0$ $<_0B'''\cap S$, implying $A(<'+^*<_0)B$. □

REMARKS. a) The lemma remains valid if we replace $A'<B'$ by $A'\subset B'$. b) If $\mathscr{S}$ is a topology then equality holds in (1).

**1.9** LEMMA. *If* $\mathscr{S}$ *and* $\mathscr{S}_0$ *are syntopogenous structures, and* $\mathscr{S}_0<\mathscr{S}|S$ *then* $\mathscr{S}+\mathscr{S}_0$ *is a syntopogenous structure, too. Moreover,* $\mathscr{S}+\mathscr{S}_0<\mathscr{S}$ *and* $(\mathscr{S}+\mathscr{S}_0)|S\sim\mathscr{S}_0$.

PROOF. The statements in the second sentence follow from Lemmas 1.7 and 1.6. $\mathscr{S}+\mathscr{S}_0$ is directed by Lemmas 1.7 and 1.3. To prove that $\mathscr{S}+\mathscr{S}_0$ is a syntopogenous structure, it is enough to show (taking Lemma 1.8 into account) that if $<\in\mathscr{S}$ and $<_0\in\mathscr{S}_0$ then there are $<'\in\mathscr{S}$ and $<_0'\in\mathscr{S}_0$ such that

$$<+^*<_0\subset(<'+<_0')^2.\tag{1}$$

This will hold if we choose $<_0'$ and $<'$ satisfying the following conditions:

$$<_0\subset<_0'^3,\quad <_0'\subset<'^2|S,\quad <\subset<'^2.\tag{2}$$

In order to check (1), assume $A(<+^*<_0)B$, and take $A'$, $B'$ with

$$A<A'<B'<B\tag{3}$$

and $A'\cap S<_0B'\cap S$. According to (2), there are $C_0$, $D_0$, $E$, $A''$, $B''$ such that

$$A'\cap S<_0'C_0<_0'D_0<_0'B'\cap S,\tag{4}$$

$$C_0<'E<'D_0\cup(X\setminus S),\tag{5}$$

$$A<'A''<'A',\quad B'<'B''<'B.\tag{6}$$

We claim that if

$$F=(E\cup A'')\cap B''\tag{7}$$

then

$$A(<'+<_0')F(<'+<_0')B.\tag{8}$$

The first part of (8) is shown by the sets $A'$ and $C_0$: it follows from (3) and (6) that $A''\subset B''$, thus $A<'F$ by (7) and the first part of (6); we have $A<'A'$ from (6);

$A'\cap S<'C_0\cap S=C_0$ is contained in (4); finally, $C_0<'F$, because $C_0<'E$ is known from (5), and $C_0\subset B'<'B''$ follows from (4) and the second part of (6).

An analogous reasoning gives that the second part of (8) is shown by the sets $D_0\cup(X\setminus S)$ and $B'$. Thus (1) holds indeed. □

**1.10** Lemma. *If $\mathcal{S}$ is a syntopogenous structure on $X$ then $\mathcal{S}+(\mathcal{S}|S)\sim\mathcal{S}$.*

Proof. By Lemma 1.9, it is enough to check that $\mathcal{S}<\mathcal{S}+(\mathcal{S}|S)$. Given $<\in\mathcal{S}$, take $<'\in\mathcal{S}$ with $<\subset<'^3$; now $<\subset<'+(<'|S)$, since if $A<'A'<'B'<'B$ then the sets $A', B'$ show that $A(<'+(<'|S))B$. □

**1.11** For a syntopogenous structure $\mathcal{S}$ on $X$, the *$\mathcal{S}$-trace filter* $\mathfrak{f}(x)$ of the point $x\in X$ is the trace on $S$ of the $\mathcal{S}^{tp}$-neighbourhood filter $\mathfrak{n}(x)$ of $x$, i.e.

$$\mathfrak{f}(x)=\mathfrak{n}(x)|S=\{F\cap S\colon F\in\mathfrak{n}(X)\}.$$

Let $\mathcal{A}$ be a directed order family on $X$. The filter $\mathfrak{f}$ on $X$ is *$\mathcal{A}$-round* [2, 4] if for any $F\in\mathfrak{f}$ there are $E\in\mathfrak{f}$ and $<\in\mathcal{A}$ such that $E<F$. $\mathfrak{f}$ is $\mathcal{A}$-round iff it is $\mathcal{A}^t$-round.

Lemma. *If $\mathcal{S}$ is a syntopogenous structure on $X$, and $\mathcal{A}_0$ a directed order family on $S$ then $(\mathcal{S}+\mathcal{A}_0)^{tp}=\mathcal{S}^{tp}$ iff each $\mathcal{S}$-trace filter is $\mathcal{A}_0$-round.*

Proof. Let $\mathcal{S}^{tp}=\{<''\}$ and $(\mathcal{S}+\mathcal{A}_0)^t=\{<'''\}$.

1° *Necessity*. Fix a point $x\in X$ and take an $F\in\mathfrak{f}(x)$. Now $F\cup(X\setminus S)$ is an $\mathcal{S}$-neighbourhood of $x$, so $\{x\}<''F\cup(X\setminus S)$. From the assumption $(\mathcal{S}+\mathcal{A}_0)^{tp}=\{<''\}$ it follows that $\{x\}<'''F\cup(X\setminus S)$. $\mathcal{S}+\mathcal{A}_0$ is directed by Lemmas 1.3 and 1.7, so, according to [3] (8.38), there are $<\in\mathcal{S}$ and $<_0\in\mathcal{A}_0$ with

$$\{x\}(<+<_0)F\cup(X\setminus S).$$

Hence there are $A', B'$ such that $\{x\}<A'$, $B'<F\cup(X\setminus S)$ and $A'\cap S<_0B'\cap S$. $A'$ is an $\mathcal{S}$-neighbourhood of $x$, so $\mathfrak{f}(x)\in A'\cap S<_0B'\cap S\subset F$, i.e. $\mathfrak{f}(x)$ is $\mathcal{A}_0$-round.

2° *Sufficiency*. Assume that the trace filters are round. We have to show that $\{x\}<''B$ *iff* $\{x\}<'''B$. If $\{x\}<'''B$ then $\{x\}(<+<_0)B$ with suitable $<\in\mathcal{S}$ and $<_0\in\mathcal{A}_0$, thus $\{x\}<B$, and therefore $\{x\}<''B$.

Conversely, assume that $\{x\}<''B$. Then $\{x\}<'B$ with some $<'\in\mathcal{S}$. We need $<\in\mathcal{S}$ and $<_0\in\mathcal{A}_0$ such that

$$\{x\}(<+<_0)B. \tag{1}$$

a) If $x$ lies outside the $\mathcal{S}^{tp}$-closure of $S$ then there is a $<\in\mathcal{S}$ such that $\{x\}<X\setminus S$ and $<'\subset<$; now an arbitrary $<_0\in\mathcal{A}_0$ will do: $A'=X\setminus S$ and $B'=\{x\}$ show (1).

b) If $x$ is in the closure then take $<_1\in\mathcal{S}$ and $B'\subset X$ with $<'\subset<_1^2$ and $\{x\}<_1B'<_1B$. $B'\cap S\in\mathfrak{f}(x)$, so (as $\mathfrak{f}(x)$ is round) there are $A'\subset X$, $<_2\in\mathcal{S}$ and $<_0\in\mathcal{A}_0$ such that $\{x\}<_2A'$ and $A'\cap S<_0B'\cap S$. Choosing $<\in\mathcal{S}$ with $<_1\cup<_2\subset<$, (1) is shown by the sets $A'$ and $B'$. □

Remark. In this lemma (and in some other statements to follow), $\mathfrak{p}(S)$ is to be counted among the filters on $S$ (which is in this case evidently a round filter), or we have to use the convention that the points outside the closure of $S$ have no trace filter.

## § 2. Syntopogenous extensions in topological spaces

**2.1** Let $\mathcal{S}_0$ be a compatible syntopogenous structure on the subspace $S$ of the topological space $(X, \mathcal{T})$. If $\mathcal{S}_0$ has a compatible extension then each trace filter is $\mathcal{S}_0$-round ([15] 2.1 or [7] 1.5).

Theorem. *If the $\mathcal{T}$-trace filters are $\mathcal{S}_0$-round then there exist compatible extensions of $\mathcal{S}_0$; $\mathcal{T}+\mathcal{S}_0$ is the finest compatible extension.*

Remark. The statement of this theorem is contained by [7] 2.1 and 2.2 for the case when $S$ is dense; $\mathcal{T}+\mathcal{S}_0$ does not appear in the theorems, but the construction given in the proof is in fact the version of $\mathcal{T}+\mathcal{S}_0$ described in Lemma 1.8 (cf. Remark 1.8 b)).

Proof. $\mathcal{T}+\mathcal{S}_0$ is a compatible extension by Lemmas 1.9 and 1.11. Let $\mathcal{S}'$ be another compatible extension. Then we have $\mathcal{S}' \sim \mathcal{S}'+\mathcal{S}_0$ from Lemma 1.10, thus Lemmas 1.4 and 1.7 imply that $\mathcal{S}' < \mathcal{T}+\mathcal{S}_0$ (since $\mathcal{S}' < \mathcal{T}$). □

**2.2** Let $\delta$ be a proximity on $S \subset X$, and $\varepsilon$ a proximity on $X$. The following definition can be obtained through the associated topogenous structures:

$$A\,\overline{\varepsilon+\delta}\,B \text{ iff } A\bar{\varepsilon}B \text{ and there are } A', B' \text{ such that}$$

$$A\bar{\varepsilon}(X\setminus A'),\ (X\setminus B')\bar{\varepsilon}B,\ A'\cap S\bar{\delta}B'\cap S.$$

One can arrive at this definition of $\varepsilon+\delta$ in another way, too, see [11] Remark 2.2 a).

Take now a closed subspace $S$ of a normal regular space $(X, \mathcal{T})$, and let $\delta$ be a compatible proximity on $S$. We are going to re-prove the non-trivial part of the theorem in [1], stating that $\delta$ has a compatible extension. (See also [12], where totally bounded uniformities are used.) The explicit form of $\varepsilon+\delta$ will not be needed.

For $A, B \subset X$, let $A\varepsilon B$ iff the closures of $A$ and $B$ meet. This $\varepsilon$ is a compatible proximity (in fact the finest one) on $X$ (e.g. [6] (3.1.13)). If $A, B \subset S$ and $A\varepsilon B$ then clearly $A\delta B$, thus $\varepsilon|S$ is finer than $\delta$, and it follows now from the lemmas of § 1 that $\varepsilon+\delta$ is a compatible extension of $\delta$.

Remarks. a) If $\mathcal{S}_0$ and $\mathcal{S}$ are biperfect, $\mathcal{U}$ and $\mathcal{V}$ denote the quasi-uniformities induced by $\mathcal{S}_0$, respectively by $\mathcal{S}$, then, with the notation of [10] 1.1, $\mathcal{V}+\mathcal{U}$ is the same as the quasi-semiuniformity induced by the biperfect order family $\mathcal{S}+\mathcal{S}_0$. Thus the operation $+$ introduced in the present paper can be regarded as a generalization of the one defined in [10].

b) See [7] for further results on syntopogenous extensions in topological spaces.

## § 3. Syntopogenous extensions in bitopological spaces

**3.1** Let now $(X; \mathcal{T}_{-1}, \mathcal{T}_1)$ be a bitopological space, and $\mathcal{S}_0$ a compatible syntopogenous structure on $S \subset X$; we are looking for extensions of $\mathcal{S}_0$ compatible with $(\mathcal{T}_{-1}, \mathcal{T}_1)$. If there is such an extension then $(\mathcal{T}_{-1}, \mathcal{T}_1)$ has to be completely regular ([5] (1.1); for the sake of simplicity, let "completely regular" mean "quasi-uniformizable" by definition; see [14] for the original definition given in terms of semicontinuous functions). The $\mathcal{T}_i$-trace filters are necessarily $\mathcal{S}_0^i$-round $(i=\pm 1)$, because if $\mathcal{S}$ is an extension of $\mathcal{S}_0$ compatible with $(\mathcal{T}_{-1}, \mathcal{T}_1)$ then $\mathcal{S}^i$ is an extension of $\mathcal{S}_0^i$ compatible with $\mathcal{T}_i$.

We shall say that the bitopological space $(X; \mathcal{T}_{-1}, \mathcal{T}_1)$ is an *extension* of $(S; \mathcal{T}'_{-1}, \mathcal{T}'_1)$ if $\mathcal{T}_i | S = \mathcal{T}'_i$ and $S$ is dense in both topologies. $(\mathfrak{f}^{-1}(x), \mathfrak{f}^1(x))$ will denote the trace filter pair of the point $x \in X$ (i.e. $\mathfrak{f}^i(x)$ is the $\mathcal{T}_i$-trace filter of $x$). The filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ in the syntopogenous space $(X, \mathcal{S})$ is called *round* if $\mathfrak{f}^i$ is $\mathcal{S}^i$-round $(i=\pm 1)$. Using this terminology, we can say that if a syntopogenous structure has a compatible extension to an extension of the induced bitopology then the trace filter pairs are round.

To formulate a third necessary condition, we need the following

Definition. The filter pair $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ in the syntopogenous space $(X, \mathcal{S})$ is *compressed* if $A < B$ with some $< \in \mathcal{S}$ and $A \in \sec \mathfrak{f}^{-1}$ imply $B \in \mathfrak{f}^1$.

Remark. $(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is $\mathcal{S}$-compressed iff $(\mathfrak{f}^1, \mathfrak{f}^{-1})$ is $\mathcal{S}^{-1}$-compressed. This is clear from:

Lemma. *$(\mathfrak{f}^{-1}, \mathfrak{f}^1)$ is $\mathcal{S}$-compressed iff for each $< \in \mathcal{S}$, $C_i \in \sec \mathfrak{f}^i$ $(i=\pm 1)$ implies $C_{-1} \not< X \setminus C_1$.* □

Being $\mathcal{S}$-compressed depends only on $\mathcal{S}^t$. It follows from the above lemma that a filter pair is $\mathcal{S}$-compressed iff it is compressed (in the sense of [8] Definition 5.1) with respect to the quasi-proximity associated with $\mathcal{S}^t$.[1] An analogous statement holds for round filter pairs.

Proposition. *If a syntopogenous structure $\mathcal{S}_0$ has an extension compatible with an extension $(\mathcal{T}_{-1}, \mathcal{T}_1)$ of the induced bitopology then each $(\mathcal{T}_{-1}, \mathcal{T}_1)$-trace filter pair is $\mathcal{S}_0$-compressed.*

Proof. Let $(X, \mathcal{S})$ be a compatible extension. Assume $A \in \sec \mathfrak{f}^{-1}(x)$, $<_0 \in \mathcal{S}_0$ and $A <_0 B$. Choose $< \in \mathcal{S}$ with $<_0 \subset <^2 | S$. The assumption $A \in \sec \mathfrak{f}^{-1}(x)$ means that each $\mathcal{T}_{-1} = \mathcal{S}^{-tp}$-neighbourhood of $x$ meets $A$, hence $\{x\} \not<^{-1} X \setminus A$, and so

$$A \not< X \setminus \{x\}. \tag{1}$$

On the other hand, $A <_0 B$ and the choice of $<$ imply that there exists a set $C$ with

$$A < C < B \cup (X \setminus S).$$

[1] The notion of a proximity associated with a symmetrical topogenous structure has to be extended here to the non-symmetrical case in the obvious way.

Now $x \in C$ follows from (1), so $B \cup (X \setminus S)$ is a $\mathcal{T}_1 = \mathcal{S}^{tp}$-neighbourhood of $x$, i.e. $B \in \mathfrak{f}^1(x)$. □

**3.2** Our next aim is to prove that the necessary conditions given in 3.1 are sufficient in a special case. (The example in [8] 5.4 shows that they are not sufficient in general.) For this purpose, let us recall some facts from [8].

A bitopological space $(X; \mathcal{T}_{-1}, \mathcal{T}_1)$ is *regular* ([13]; a condition weaker than complete regularity) if each point has a $\mathcal{T}_i$-neighbourhood base consisting of $\mathcal{T}_{-i}$-closed sets $(i = \pm 1)$. We gave a necessary and sufficient condition for the existence of regular bitopological extensions inducing prescribed trace filter pairs ([8], 2.1 (1)). Assuming this condition to hold, there is a finest one among such extensions, called *fine regular*, which can be constructed as follows ([8] Theorem 2.2): for $x \in X$, $i = \pm 1$ and $F \in \mathfrak{f}^i(x)$, define

$$N_F^i(x) = \{x\} \cup \{y\colon\ F \in \sec \mathfrak{f}^{-i}(y)\};$$

now a base for the $\mathcal{T}_i$-neighbourhood filter of $x$ (where $(\mathcal{T}_{-1}, \mathcal{T}_1)$ is the fine regular extension associated with the trace filter pairs $(\mathfrak{f}^{-1}(x), \mathfrak{f}^1(x))$) is given by

$$\mathfrak{b}^i(x) = \{N_F^i(x)\colon\ F \in \mathfrak{f}^i(x)\}.$$

THEOREM. *If* $(S, \mathcal{S}_0)$ *is a syntopogenous space,* $(X; \mathcal{T}_{-1}, \mathcal{T}_1)$ *is a fine regular extension of* $(S; \mathcal{S}_0^{-tp}, \mathcal{S}_0^{tp})$, *and the trace filter pairs are round and compressed then* $\mathcal{S}_0$ *has extensions compatible with* $(\mathcal{T}_{-1}, \mathcal{T}_1)$.

$$\mathcal{S} = [\mathcal{T}_{-1}^{-1}, \mathcal{S}_0, \mathcal{T}_1^{-1}, \mathcal{D}_X]$$

*is one of the finest compatible extensions.*

REMARK. If round and compressed trace filter pairs are prescribed in a syntopogenous space in such a way that the neighbourhood filter pairs are assigned to the original points then there does exist the fine regular extension. (Check [8] 2.1 (1) directly, or apply [8] Lemma 2.3 to the totally bounded quasi-uniformity compatible with the quasi-proximity associated with $\mathcal{S}_0^t$.)

PROOF. 1° $\mathcal{S}$ is a directed order family by Lemma 1.3. $\mathcal{S}_0 < \mathcal{S}_0^{tp} = \mathcal{T}_1|S$, $\mathcal{S}_0 < (\mathcal{S}_0^{-tp})^{-1} = \mathcal{T}_{-1}^{-1}|S$, and $\mathcal{S}_0 < \mathcal{D}_S = \mathcal{D}_X|S$, so it follows from Lemma 1.6 b) that $\mathcal{S}|S \sim \mathcal{S}_0$.

2° $\mathcal{S}$ *is a syntopogenous structure*. Let $\mathcal{T}_i = \{<_i\}$. It is enough to show that if $<_0, <_0' \in \mathcal{S}_0$ such that $<_0 \subset <_0'^3$ then $< \subset <'^2$ where

$$< = [<_{-1}^{-1}, <_0, <_1, \subset_X], \quad <' = [<_{-1}^{-1}, <_0', <_1, \subset_X].$$

Assume $A < B$. Then there are $A', B'$ such that

$$A <_1 A', \quad A' \cap S <_0 B' \cap S, \quad B' <_{-1}^{-1} B.$$

Pick $D, E$ satisfying

$$A' \cap S <_0' D <_0' E <_0' B' \cap S.$$

We claim that $A <' C <' B$ holds with

$$C = A \cup \{x \in X\colon\ D \in \sec \mathfrak{f}^{-1}(x)\}.$$

$D$ being a subset of $S$, $D \in \sec \mathfrak{f}^{-1}(x)$ is equivalent to[2] $D \in \sec \mathfrak{n}^{-1}(x)$, hence $C = A \cup \mathrm{Cl}^{-1} D$.

a) $A <' C$ is shown by the sets $A'$ and $D$: $A \subset C$ by the definition of $C$; $A <_1 A'$ was assumed; $D <_{-1}^{-1} C$ is equivalent to the statement that $\mathrm{Cl}^{-1} D \subset C$; finally, $A' \cap S <_0' D = D \cap S$.

b) $C <' B$ is shown by the sets $E' = E \cup (X \setminus S)$ and $B'$: $B' <_{-1}^{-1} B$ means that $\mathrm{Cl}^{-1} B' \subset B$, so $C \subset B$ follows from $D \subset B'$ and $A \subset B$; if $x \in C \setminus A$ then $D \in \sec \mathfrak{f}^{-1}(x)$, therefore $E \in \mathfrak{f}^1(x)$ (because $D <_0' E$, and $(\mathfrak{f}^{-1}(x), \mathfrak{f}^1(x))$ is compressed), and so $E' \in \mathfrak{n}^1(x)$, implying $C \setminus A <_1 E'$, whence $C <_1 E'$, since $A <_1 A' \subset E'$; $B' <_{-1}^{-1} B$ was assumed; finally, $E' \cap S = E <_0' B' \cap S$.

3° *$\mathscr{S}$ is compatible.* We only show that $\mathscr{S}^{tp} = \mathscr{T}_1$; the proof of $\mathscr{S}^{-tp} = \mathscr{T}_{-1}$ is analogous. (Or Lemma 1.1 c) can be applied.)

a) Let us first check that $\mathscr{S}^{tp} < \mathscr{T}_1$. If $\{x\} < B$ for some

$$(1) \qquad < = [<_{-1}^{-1}, <_0, <_1, \subset_X] \in \mathscr{S}$$

then take $A', B'$ with $\{x\} <_1 A'$, $B' <_{-1}^{-1} B$ and $A' \cap S <_0 B' \cap S$. Now with $F = A' \cap S \in \mathfrak{f}^1(x)$ we have $F \subset B'$, thus $\mathrm{Cl}^{-1} F \subset B$. This means that $B$ is a $\mathscr{T}_1$-neighbourhood of $x$, because $N_F^1(x)$ can be written as $\{x\} \cup \mathrm{Cl}^{-1} F$, and $x \in B$.

b) Conversely, assume $\{x\} <_1 B$. As $(\mathscr{T}_{-1}, \mathscr{T}_1)$ is regular, there is a $\mathscr{T}_{-1}$-closed set $B'$ with $\{x\} <_1 B' \subset B$. Now $B' \cap S \in \mathfrak{f}^1(x)$, which is $\mathscr{S}_0$-round, so there are $G \in \mathfrak{f}^1(x)$ and $<_0 \in \mathscr{S}_0$ with $G <_0 B' \cap S$. Defining $<$ by (1), the sets $A' = G \cup (X \setminus S)$ and $B'$ show that $\{x\} < B$ ($B' <_{-1}^{-1} B$ holds because $B'$ is a $\mathscr{T}_{-1}$-closed subset of $B$).

4° *$\mathscr{S}$ is the finest extension.* Let $\mathscr{S}'$ be another compatible extension; we have to show that $\mathscr{S}' < \mathscr{S}$. Given $<' \in \mathscr{S}'$, take $<'' \in \mathscr{S}'$ with $<' \subset <''^3$. As $\mathscr{S}'|S \sim \mathscr{S}_0$, there is $<_0 \in \mathscr{S}_0$ such that $<''|S \subset <_0$. Define now $<$ by (1). Then $<' \subset <$.

Indeed, if $A <' B$ then take $A', B'$ such that $A <'' A' <'' B' <'' B$. These sets $A'$, $B'$ show that $A < B$: $<'' \in \mathscr{S}'$ and $\mathscr{S}'^{tp} = \{<_1\}$, thus $<'' \subset <_1$, and so $A <_1 A'$; similarly, $<''^{-1} \subset <_{-1}$, i.e. $<'' \subset <_{-1}^{-1}$, and therefore $B' <_{-1}^{-1} B$; $A' \cap S <_0 B' \cap S$ follows from $<''|S \subset <_0$. □

**3.3** Concerning extensions from arbitrary subspaces, we have only:

THEOREM. *The following conditions are equivalent for a syntopogenous structure $\mathscr{S}_0$ compatible on a subspace of a bitopological space:*

(i) *$\mathscr{S}_0$ has a compatible extension;*

(ii) *$\mathscr{S}_0^t$ has a compatible extension;*

(iii) *$\mathscr{S}_0^t$ has a compatible extension $\mathscr{S}$ with $|\mathscr{S}| = 1$.*

PROOF. (i)⇒(ii) and (ii)⇒(iii). If $\mathscr{S}$ is an extension of $\mathscr{S}_0$ or of $\mathscr{S}_0^t$ then $\mathscr{S}^t$ is an extension of $\mathscr{S}_0^t$.

(iii)⇒(i). Let $\mathscr{S} = \mathscr{S}^t$ be an extension of $\mathscr{S}_0^t$. Take $\mathscr{S}' = \mathscr{S} + \mathscr{S}_0$. Now $\mathscr{S}_0 < \mathscr{S}_0^t = \mathscr{S}|S$, thus it follows from Lemma 1.9 that $\mathscr{S}'$ is a syntopogenous structure

[2] $\mathfrak{n}^i(x)$ denotes the $\mathscr{T}_i$-neighbourhood filter of $x$; $\mathrm{Cl}^i$ is the $\mathscr{T}_i$-closure.

and $\mathcal{S}'|S \sim \mathcal{S}_0$. By Lemmas 1.7 and 1.5, $\mathcal{S}'^t = \mathcal{S}^t + \mathcal{S}_0^t = \mathcal{S} + (\mathcal{S}|S)$, thus Lemma 1.10 gives $\mathcal{S}'^t = \mathcal{S}$, and therefore $\mathcal{S}'$ and $\mathcal{S}$ induce the same bitopology. □

REMARKS. a) When proving that $\mathcal{S}'$ is compatible, we could have used Lemma 1.11.

b) Theorem 3.2 (without its last statement) can be deduced from [8] Theorem 5.3 and (iii)⇒(i).

## REFERENCES


[1] BOGNÁR, M., Extending compatible proximities, *Acta Math. Hungar.* **45** (1985), No 3—4, 377—378. *MR* **86j:** 54050.

[2] CSÁSZÁR, Á., Complétion et compactification d'espaces syntopogènes, *General topology and its relations to modern analysis and algebra* (Proc. Sympos., Prague, 1961), Publ. House Czechoslovak Acad. Sci., Prague, 1962, 133—137. *MR* **33** # 1834.

[3] CSÁSZÁR, Á., *Foundations of general topology*, Pergamon Press, Oxford, 1963. *MR* **28** # 575.

[4] CSÁSZÁR, Á., *Grundlagen der allgemeinen Topologie*, Akadémiai Kiadó, Budapest, 1963. *MR* **26** # 6917.

[5] CSÁSZÁR, Á., Doppeltkompakte bitopologische Räume, *Theory of sets and topology* (in honour of Felix Hausdorff), VEB Deutscher Verlag Wissensch., Berlin, 1972, 59—67. *MR* **49** # 7990.

[6] CSÁSZÁR, Á., *General topology*, Akadémiai Kiadó, Budapest and Adam Hilger Ltd, Bristol, 1978. *MR* **57** # 13812.

[7] CSÁSZÁR, Á. and MATOLCSY, K., Syntopogenous extensions for prescribed topologies, *Acta Math. Acad. Sci. Hungar.* **37** (1981), No 1—3, 59—75. *MR* **82i:** 54006.

[8] DEÁK, J., Extensions of quasi-uniformities for prescribed bitopologies I, *Studia Sci. Math. Hungar.* **25** (1990), 45—67.

[9] DEÁK, J., Extensions of quasi-uniformities for prescribed bitopologies II, *Studia Sci. Math. Hungar.* **25** (1990), 69—91.

[10] DEÁK, J., Quasi-uniform extensions for finer topologies, *Studia Sci. Math. Hungar.* **25** (1990), 97—105.

[11] DEÁK, J., Uniform and proximal extensions with cardinality limitations, *Studia Sci. Math. Hungar.* **25** (1990), 343—352.

[12] GANTNER, T. E., Extensions of uniform structures, *Fund. Math.* **66** (1970), No 3, 263—281. *MR* **42** # 5220.

[13] KELLY, J. C., Bitopological spaces, *Proc. London Math. Soc.* (3) **13** (1963), No 49, 71—89. *MR* **26** # 729.

[14] LANE, E. P., Bitopological spaces and quasi-uniform spaces, *Proc. London Math. Soc.* (3) **17** (1967), No 2, 241—256. *MR* **34** # 5054.

[15] MATOLCSY, K., On extensions of syntopogenous spaces, *Publ. Math. Debrecen* **28** (1981), No 1—2, 103—119. *MR* **82i:** 54007.

[16] MATOLCSY, K., Refined extensions of syntopogenous structures and quasi-uniformities, *Acta Math. Hungar.* **42** (1983), No 1—2, 111—119. *MR* **84j:** 54018.


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# PREFACE

The Mathematical Institute of the Hungarian Academy of Sciences at March 18-19, 1991 celebrated the 70-th birthday of late Professor A. Rényi, the founder of the Institute. The Organizing Committee of this Memorial Meeting invited a few of his friends and a few younger mathematicians whose works are strongly connected to, or influenced by Rényi's research.

The Editorial Board of the Studia Sci. Math. Hung. decided to devote a special Issue of the Journal to the memory of Rényi, the founder of the Journal. This Issue consists of some of the lectures held at the Meeting and some other papers devoted to the memory of Rényi.

The Organizing Committee is indebted to Prof. D. Kosáry, the President of the Hungarian Academy of Sciences for his valuable help. We also acknowledge the support provided by the Hungarian National Foundation for Scientific Research Grant No. 1808 and 1905.

The Organizing Committee

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# PROGRAM OF THE MEMORIAL MEETING

**March 18, 1991**

10.00 D. KOSÁRY President of the Hungarian Academy of Science: Opening

10.15 J. SZENTÁGOTHAI (Hungary): An early probabilistic model of synaptic transmission

11.00 P. ERDŐS (Hungary): My work with Rényi

11.45 V. T. SÓS (Hungary): Quasirandom graphs

14.30 N. H. BINGHAM (UK): The work of Alfréd Rényi: Some aspects in probability and number theory

15.15 I. CSISZÁR (Hungary): Axiomatic justification of the methods of least squares and maximum entropy

16.00 G. O. H. KATONA (Hungary): Combinatorial search problems

**March 19, 1991**

10.00 P. DEHEUVELS (France): Functional Erdős-Rényi-type laws

10.45 M. CSÖRGŐ (Canada): A note on local and global functions of a Wiener process and some Rényi-type statistics

11.30 E. CSÁKI (Hungary): Erdős-Rényi laws for local times

12.00 I. BERKES (Hungary): Limit theorems related to the a.s. central limit theorem

12.30 I. VINCZE (Hungary): A few words on A. Rényi

14.30 J. L. TEUGELS (Belgium): The region of convergence of the Laplace transform: almost sure estimation

15.15 P. MAJOR (Hungary): Poissonian limit law for the number of lattice points in a random stripe with finite volume

16.00 P. RÉVÉSZ (Hungary-Austria): On the coverage of Strassen's set

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# ALFRÉD RÉNYI: A TRUE EUROPEAN

DAVID KENDALL

I think I must first have seen Rényi at the International Congress of Mathematicians in Amsterdam in 1954, a delightful meeting taking its colour from the fact that the activities centred (perhaps appropriately) around the Zoo. He startled everyone by reading a paper announcing his recent discovery of conditional probability spaces.

Our first personal encounter followed later. I remember asking him why all the references to genetics had been deleted from the Russian language edition of Feller's book, and he teasingly replied with another question: why were the references to dialectical materialism deleted from the English language edition of Gnedenko's book? After that, we got on famously.

Rényi often visited us, at our home in Oxford, and later, at our home in Cambridge. We saw a lot of him and his daughter when they spent a long period in Cambridge during his tenure of a Fellowship at Churchill College.

Of course we also met frequently at conferences. I remember inviting him to a NATO sponsored conference, and getting a reply saying: 'Of course I want to come, and I think I will indeed be able to come if you will kindly send me a slightly modified invitation: just replace 'NATO' by 'North Atlantic Treaty Organisation' - there will then be no problem'.

About this time I started a series of Rényi-type visits on my own account, attending meetings in Eastern Europe every year for a long period, until I knew all the countries really well, except for Albania - somehow I never managed to get to that country. I know that Rényi and I were at one in believing such exchanges to be of the highest importance, and not just for mathematical reasons.

My knowledge of the Hungarian language never advanced very far. In a journey into the Hungarian countryside I noticed in a small village a sign that I thought said *Matematikai Kutato Intezete*, and I expressed surprise that it too should have a mathematical research institute. 'Ah,' said Rényi, 'this time it means 'Beware of the dog' '.

Later I had another encounter with this mysterious language. I had been invited to write an account of Rényi's life and work for an international scientific encyclopaedia. I managed to get together a suitable selection of references from the enormous corpus of his writings, with the titles in English.

*Akadémiai Kiadó, Budapest*

But the publishers would not have this: they demanded that they all be translated back intc Hungarian.

Rényi told me that in the fighting in Budapest during the war he became worried about the fate of his mathematical books. These were hidden in packing cases in the basement of a house in the battle zone. So he stole a German uniform and a wheel-barrow and pushed them to safety.

I am sorry to have no mathematical reminiscences to grace this occasion, and even sadder that I cannot be with you. When I wrote an Obituary Notice for Rényi in one of the Applied Probability journals I remarked on his passionate belief in the basic unity of Europe, both East and West. How happy he would have been, to know that there is no longer any need to insist upon this. One dream, at least, came true.

*Studia Scientiarum Mathematicarum Hungarica 26 (1991), 153—164*

# COUNTEREXAMPLES RELATED TO THE A.S. CENTRAL LIMIT THEOREM

I. BERKES*, H. DEHLING, T. F. MÓRI*

## 1. Introduction

Let $(X_n)$ be a sequence of r.v.'s, $(a_n)$ a numerical sequence and $G$ a distribution function. We say that $(X_n)$ satisfies the a.s. central limit theorem with norming $(a_n)$ and limit distribution $G$ if setting $S_n = X_1 + \cdots + X_n$ we have

$$(1)\qquad \lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} I\left\{\frac{S_k}{a_k} < x\right\} = G(x) \quad \text{a.s. for all } x \in SC_G.$$

(Here $SC_G$ denotes the set of continuity points of $G$.) In recent years several papers dealt with limit theorems of the type (1) and related asymptotic results. Fisher (1989) and Lacey-Philipp (1990) proved that if $X_n$ are i.i.d. with $EX_1 = 0$, $EX_1^2 = 1$ then

$$(2)\qquad \lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} I\left\{\frac{S_k}{\sqrt{k}} < x\right\} = \Phi(x) \quad \text{a.s. for all } x.$$

(Under the existence of finite $(2+\delta)$-th moments this was proved earlier by Brosamler (1988) and Schatte (1988).) A more general result was obtained by Berkes and Dehling (1991) who proved the following

THEOREM. *Let $X_1, X_2, \ldots$ be a sequence of independent r.v.'s and put $S_n = X_1 + \cdots + X_n$; let $a_n > 0$ be a numerical sequence such that*

$$(3)\qquad a_n/n^\gamma \quad \textit{is nondecreasing for some } \gamma > 0$$

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*Key words and phrases.* Weak and strong central limit theorem, logarithmic density, domain of attraction.

*and either*

$$E|S_n/a_n|^p \le K\exp((\log n)^{1-\varepsilon}) \text{ for some } p>0,\ \varepsilon>0,\ K>0 \tag{4}$$

*or*

$$E((\log\log|S_n/a_n|)^{1+\varepsilon}) \le K \text{ for some } \varepsilon>0,\ K>0. \tag{5}$$

*Then for any distribution function $G$ the a.s. central limit theorem* (1) *holds iff*

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{k\le N}\frac{1}{k}P\left(\frac{S_k}{a_k}<x\right)=G(x) \text{ for all } x\in SC_G. \tag{6}$$

The above theorem shows that under mild technical assumptions the a.s. CLT (1) is a consequence of the weak convergence relation

$$S_n/a_n \xrightarrow{d} G. \tag{7}$$

Thus, despite its pointwise character, the a.s. central limit theorem (1) is actually a weaker statement than the distributional result (7). It is natural to ask if any converse relationship between (1) and (7) holds i.e. if the a.s. CLT (1) yields any information on the weak convergence properties of $S_n/a_n$ or the class of its weak (distributional) limits. The purpose of this paper is to show that, except the case of a limiting normal distribution $G$, the answer is negative even in the simplest case of i.i.d.r.v.'s $(X_n)$. In fact, we are going to construct (see Examples 1,2) i.i.d. sequences $(X_n)$ such that (1) holds where $G$ is a stable or mixed stable distribution but the distribution of $S_n/a_n$ fluctuates irregularly without a limit as $n\to\infty$ and the tails $P(X_1>t)$ behave, as $t\to\infty$, in a very erratic way. Example 1 also shows that the class of all possible limit distributions $G$ in (1) for i.i.d. sequences $(X_n)$ is larger than the class of limit distributions in (7) (i.e. stable and normal laws) and also that under the validity of (1) $S_n/a_n$ can have limit distributions along subsequences which do not even resemble the limit $G$ in (1). The case of a normal limit distribution in (1) is an exception: from a concentration function inequality of Esseen (1968) it follows easily that if an i.i.d. sequence $(X_n)$ satisfies (2) then we have $EX_1=0$, $EX_1^2=1$ i.e. automatically $n^{-1/2}S_n \xrightarrow{d} N(0,1)$. However, Example 3 below gives an independent, nearly identically distributed sequence $(X_n)$ satisfying (2) but with $S_n/\sqrt{n}$ having no limit distribution (in fact the weak limit set of $S_n/\sqrt{n}$ being the class of all normal laws $N(\mu,1/2)$, $\mu$ real).

In conclusion we note that by the examples of our paper there is a fundamental difference between classical domains of attraction and their 'logarithmic' counterparts defined by means of (1). By the standard definition, a distribution $F$ belongs to the domain of attraction of a nondegenerate distribution $G$ if there exist numerical sequences $(a_n), (b_n)$ such that for any i.i.d. sequence $(X_n)$ with distribution $F$ we have, setting $S_n = X_1 + \cdots + X_n$,

$$a_N^{-1} S_N - b_N \xrightarrow{d} G \qquad (N \to \infty).$$

In analogy with this, let us say that $F$ belongs to the *domain of* log *attraction* of $G$ if there exist numerical sequences $(a_n), (b_n)$ such that for any i.i.d. sequence $(X_n)$ with distribution $F$ we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} I\left\{\frac{S_k}{a_k} - b_k < x\right\} = G(x) \text{ a.s. for all } x \in SC_G.$$

As is well known, a distribution $G$ has a nonempty domain of attraction iff $G$ is normal or $\alpha$-stable ($0 < \alpha < 2$) and the corresponding domains of attraction consist of distributions having regularly behaving tails and truncated variances at infinity. As our examples show, the situation is completely different in the log case: there exist nonstable distributions with nonempty domain of log attraction and the domain of log attraction of any nonnormal stable law contains many 'pathological' distributions with irregular tail behavior. Hence any characterization (if exists) of the domain of log attraction of a stable law $G_\alpha$ ($0 < \alpha < 2$) must be of an entirely different nature than that of the ordinary domain of attraction.

## 2. Examples

In the constructions that follow, we shall make repeated use of the fact that if relation (1) holds along a subsequence $N_k$ such that $\log N_{k+1}/\log N_k \to 1$, then (1) holds for all $N$. This remark is immediate from the fact that denoting the sum on the left hand side of (1) by $T_N$, we have $|T_M - T_N| \le$ $\le \log N - \log M$ for $1 \le M < N$.

EXAMPLE 1. Let $\alpha$ and $\beta$ be positive numbers, $1 \ge \alpha > \beta > 0$. Let $\psi : \mathbf{R} \to \mathbf{R}$ be a continuous even function, increasing and concave on the non-negative halfline, in addition, $\psi(t) \sim t^\alpha$ as $t \to 0$ and $\psi(t) \sim t^\beta$ as $t \to \infty$. Then $\exp(-\psi(t))$ is a Pólya type characteristic function. Let

$$a_{k!} = (k!)^{2/(\alpha+\beta)}, \quad k \ge 1,$$

while for $k! < n < (k+1)!$ let

$$a_n = (a_{(k+1)!})^{f(s)}(a_{k!})^{1-f(s)},$$

where $s = (\log n - \log k!)/\log(k+1)$ (that is, $n = ((k+1)!)^s(k!)^{1-s}$) and

$$f(s) = \begin{cases} \dfrac{\alpha+\beta}{2\alpha}s & \text{if} \quad s \le \dfrac{\alpha}{\alpha+\beta} \\ \dfrac{\alpha+\beta}{2\beta}s - \dfrac{\alpha-\beta}{2\beta} & \text{if} \quad s > \dfrac{\alpha}{\alpha+\beta}. \end{cases}$$

For the sake of convenience let us introduce $c_k = (k!)^{(\alpha-\beta)/(\alpha+\beta)}$, then

(8) $$\lim_{k\to\infty} c_k \sum_{i=k+1}^{\infty} \frac{1}{c_i} = 0$$

and

(9) $$\lim_{k\to\infty} \frac{1}{c_k} \sum_{i=1}^{k-1} c_i = 0.$$

Clearly, $k!/a_{k!}^{\alpha} = 1/c_k$ and $k!/a_{k!}^{\beta} = c_k$. It is easy to see that $n/a_n^{\alpha}$ is decreasing and $n/a_n^{\beta}$ is increasing.

Finally, let

$$\lambda(t) = \sum_{k=1}^{\infty} \frac{1}{k!}\psi(a_{k!}t), \quad \varphi(t) = \exp(-\lambda(t)).$$

The above sum is convergent since $\psi(a_{k!}t)/k! \sim |t|^{\beta}/c_k$ as $k \to \infty$, and $\varphi$ is again a Pólya type characteristic function. We suppose $X_1, X_2, \ldots$ i.i.d. with characteristic function $\varphi$.

We shall prove that $\varphi^n(t/a_n)$ does not converge for $t \ne 0$, but at the same time

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{i\le N} \frac{1}{i}\varphi^i(t/a_i) = \frac{\alpha}{\alpha+\beta}\exp(-|t|^{\alpha}) + \frac{\beta}{\alpha+\beta}\exp(-|t|^{\beta})$$

from which (6) will follow immediately with $G$ being a mixture of symmetric stable distributions. Finally, we show that the conditions of the Theorem in Section 1 are met, completing the counterexample.

First we show that for $k! \le n \le (k+1)!$ the series

$$n\lambda(t/a_n) = \sum_{i=1}^{\infty} \frac{n}{i!}\psi\left(\frac{a_{i!}}{a_n}t\right)$$

is asymptotically equal to the sum of its $k$-th and $(k+1)$-st terms, as $n \to \infty$. Indeed,

$$(10) \qquad \sum_{i=1}^{k-1} \frac{n}{i!}\psi\left(\frac{a_{i!}}{a_n}t\right) \sim \frac{n|t|^\alpha}{a_n^\alpha}\sum_{i=1}^{k-1}\frac{a_{i!}^\alpha}{i!} \le \frac{k!|t|^\alpha}{a_{k!}^\alpha}\sum_{i=1}^{k-1}\frac{a_{i!}^\alpha}{i!} = \frac{|t|^\alpha}{c_k}\sum_{i=1}^{k-1} c_i$$

which tends to 0 by (9). Similarly,

$$(11) \qquad \sum_{i=k+2}^{\infty} \frac{n}{i!}\psi\left(\frac{a_{i!}}{a_n}t\right) \sim \frac{n|t|^\beta}{a_n^\beta}\sum_{i=k+2}^{\infty}\frac{a_{i!}^\beta}{i!}$$

$$\le \frac{(k+1)!|t|^\beta}{a_{(k+1)!}^\beta}\sum_{i=k+2}^{\infty}\frac{a_{i!}^\beta}{i!} = |t|^\beta c_{k+1}\sum_{i=k+2}^{\infty}\frac{1}{c_i},$$

also converging to 0 by (8) as $n \to \infty$.

Let us deal with the remaining two terms. Suppose $n \to \infty$ in such a way that $(\log n - \log k!)/\log(k+1) \to s$, $0 < s < 1$, where $k! < n < (k+1)!$. Then

$$(12) \qquad \frac{n}{k!}\psi\left(\frac{a_{k!}}{a_n}t\right) + \frac{n}{(k+1)!}\psi\left(\frac{a_{(k+1)!}}{a_n}t\right)$$

$$\sim (k+1)^{s-\frac{2\alpha}{\alpha+\beta}f(s)}|t|^\alpha + (k+1)^{\frac{2\beta}{\alpha+\beta}(1-f(s))-(1-s)}|t|^\beta$$

$$\sim \begin{cases} |t|^\alpha & \text{if} \quad s < \dfrac{\alpha}{\alpha+\beta}, \\ |t|^\beta & \text{if} \quad s > \dfrac{\alpha}{\alpha+\beta}. \end{cases}$$

Thus $\varphi^n(t/a_n) = \exp(-n\lambda(t/a_n))$ does not converge for $t \neq 0$. However, letting $n(s)$ denote the integer following $((k+1)!)^s(k!)^{1-s}$ $(0 < s < 1)$ we have

$$T_k := \frac{1}{\log(k+1)}\sum_{n=k!+1}^{(k+1)!}\frac{1}{n}\varphi^n(t/a_n)$$

$$\sim \frac{1}{\log(k+1)}\sum_{n=k!+1}^{(k+1)!}(\log n - \log(n-1))\varphi^n(t/a_n) = \int_0^1 \varphi^{n(s)}(t/a_{n(s)})ds$$

and since by (12) and $(\log n(s) - \log k!)/\log(k+1) \to s$ the last integrand converges, as $k \to \infty$, to $\exp(-|t|^\alpha)$ or $\exp(-|t|^\beta)$ according as $s < \alpha/(\alpha+\beta)$ or $s > \alpha/(\alpha+\beta)$, the dominated convergence theorem yields

$$\lim_{k\to\infty} T_k = \frac{\alpha}{\alpha+\beta}\exp(-|t|^\alpha) + \frac{\beta}{\alpha+\beta}\exp(-|t|^\beta).$$

Consequently, using the remark at the beginning of this section we get

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{i\le N} \frac{1}{i} \varphi^i(t/a_i) = \lim_{k\to\infty} \frac{1}{\log(k+1)!} \sum_{i\le k} \log(i+1) T_i$$
$$= \frac{\alpha}{\alpha+\beta} \exp(-|t|^\alpha) + \frac{\beta}{\alpha+\beta} \exp(-|t|^\beta).$$

The only thing left is to check the moment type condition (4) for $S_n/a_n$. This can be done on the basis of the following well-known fact (Zolotarev, 1957): Let $\xi$ be a random variable with characteristic function $\chi$. Suppose $\mathrm{Re}(1-\chi(t)) = O(|t|^\beta)$ as $t\to 0$, $0<\beta<1$. Then $\xi$ has finite moments of order less than $\beta$. Furthermore, for $0<\gamma<\beta$ $\mathbf{E}(|\xi|^\gamma)$ can be estimated in terms of $\beta, \gamma$ and $\sup_{0<t<1} |t|^{-\beta}\mathrm{Re}(1-\chi(t))$.

Thus, we are going to estimate $\sup_{0<t<1} |t|^{-\beta}(1-\varphi^n(t/a_n))$ uniformly in $n$. Clearly, $\psi(t) \le K_\alpha |t|^\alpha$ and $\psi(t) \le K_\beta |t|^\beta$ for all $t$ and thus for $k! \le n \le (k+1)!$ we have

$$n\lambda(t/a_n) = \sum_{i=1}^{k} \frac{n}{i!} \psi\left(\frac{a_{i!}}{a_n} t\right) + \sum_{i=k+1}^{\infty} \frac{n}{i!} \psi\left(\frac{a_{i!}}{a_n} t\right)$$
$$\le \frac{k!}{a_{k!}^\alpha} \sum_{i=1}^{k} \frac{a_{i!}^\alpha}{i!} K_\alpha |t|^\alpha + \frac{(k+1)!}{a_{(k+1)!}^\beta} \sum_{i=k+1}^{\infty} \frac{a_{i!}^\beta}{i!} K_\beta |t|^\beta$$
$$= K_\alpha |t|^\alpha \frac{1}{c_k} \sum_{i=1}^{k} c_i + K_\beta |t|^\beta c_{k+1} \sum_{i=k+1}^{\infty} \frac{1}{c_i} \le K|t|^\beta \qquad (0 \le t \le 1),$$

where $K$ does not depend on $n$ or $k$. Since $1-\varphi^n(t/a_n) \le n\lambda(t/a_n)$, this implies that $\sup_n \mathbf{E}(|S_n/a_n|^\gamma) < \infty$ for all $0<\gamma<\beta$.

In conclusion we note that for $n = k!$ the first sum in (12) is

$$\psi(t) + (k+1)^{-1}\psi((k+1)^{2/(\alpha+\beta)} t) = \psi(t) + o(1) \quad \text{as} \quad k\to\infty$$

for any real $t$ by the choice of $\psi$ and $0<\beta<\alpha$. Thus along the sequence $n = k!$ we have $n\lambda(t/a_n) \to \psi(t)$ for all $t$, i.e. along this sequence $S_n/a_n$ converges weakly to the distribution with characteristic function $\exp(-\psi(t))$. Hence the validity of (1) with a simple $G$ allows that along suitable subsequences $S_n/a_n$ has limit distributions which do not even resemble $G$.

EXAMPLE 2. Let $0<\alpha<2$ and let $X$ be a symmetric r.v. taking the values $\pm 3, \pm 4, \dots$ such that

$$P(|X| = i) = Ci^{-\alpha-1}(\log i)^{-1} \qquad i = 3, 4, \dots$$

for some (uniquely determined) $C > 0$. Let $F$ denote the distribution function of $X$. By a simple calculation,

$$P(|X| \geq t) \sim C_1 t^{-\alpha} (\log t)^{-1} \quad (t \to \infty), \tag{13}$$

$$n \int_{|x| \geq n^{1/\alpha}} dF(x) \to 0 \quad (n \to \infty), \tag{14}$$

$$n^{1-2/\alpha} \int_{|x| \leq n^{1/\alpha}} x^2 dF(x) \to 0 \quad (n \to \infty). \tag{15}$$

(Here, and in the sequel, $C_1, C_2, \ldots$ denote positive constants.) Let

$$c_k = 2^{k^4}, \quad d_k = 2^{k^4 + k^2}, \quad I_k = [c_k, d_k]$$

and define $t_k$ by

$$t_k^2 = \int_{I_k} x^2 dF(x) \Big/ \int_{I_k} dF(x) \,.$$

Clearly $t_k \in I_k$; a simple calculation shows actually that

$$t_k \sim C_2 2^{k^4 + (1-\alpha/2)k^2}. \tag{16}$$

Let $\mu$ be the atomic measure on $\mathbf{N}$ defined by $\mu(\{n\}) = P(|X| = n)$ $(n = 3, 4, \ldots)$ and construct a new probability measure $\mu'$ from $\mu$ by concentrating, for each $k \geq 1$, the total mass of $\mu$ on the interval $[c_k, d_k]$ into the single point $t_k$. Let $X'$ be a symmetric r.v. such that the distribution of $|X'|$ is $\mu'$; let $F$ denote the distribution function of $X'$. Set also

$$H = \{n \in \mathbf{N} : n^{1/\alpha} \in \bigcup_{k \geq 1} I_k\}$$

It is easy to see that

$$H \text{ is of log density } 0, \tag{17}$$

$$n \int_{|x| \geq n^{1/\alpha}} dF'(x) \to 0 \quad (n \to \infty,\ n \notin H), \tag{18}$$

$$n^{1-2/\alpha} \int_{|x| \leq n^{1/\alpha}} x^2 dF'(x) \to 0 \quad (n \to \infty,\ n \notin H), \tag{19}$$

$$t_k^\alpha \int_{|x| \geq t_k} dF'(x) \to \infty \quad (k \to \infty), \tag{20}$$

$$n \int_{|x| \geq b_n} dF'(x) \to 0 \quad (n \to \infty), \tag{21}$$

$$nb_n^{-2}\int_{|x|\le b_n} x^2 dF'(x) \to 0 \quad (n\to\infty), \tag{22}$$

where

$$b_n = n^{1/\alpha}\exp\left(\frac{4}{\alpha^{3/2}}\sqrt{\log n}\right). \tag{23}$$

To verify (17) note that $H = \mathbf{N}\cap\bigcup_{k\ge 1} I_k^*$ where $I_k^* = [2^{\alpha k^4}, 2^{\alpha(k^4+k^2)}]$. Given $n\in\mathbf{N}$, let $k=k(n)$ be the largest integer such that $I_k^*\subset[0,n]$. Clearly $k\sim C_3(\log n)^{1/4}$ and

$$\sum_{i\in I_r^*} i^{-1}\sim \alpha r^2\log 2 \quad \text{as} \quad r\to\infty.$$

Thus

$$\sum_{\substack{i\le n\\ i\in H}} i^{-1}\le \sum_{r\le k+1}\sum_{i\in I_r^*} i^{-1} = O(k^3) = O((\log n)^{3/4})$$

proving (17). (18) and (19) follow immediately from (14), (15) and the fact that by the definition of $t_k$ and $\mu'$ we have

$$\int_{|x|\ge n^{1/\alpha}} dF'(x) = \int_{|x|\ge n^{1/\alpha}} dF(x)$$

and

$$\int_{|x|\le n^{1/\alpha}} x^2 dF'(x) = \int_{|x|\le n^{1/\alpha}} x^2 dF(x)$$

for $n\notin H$. To prove (20) note that using (13) and the definition of $\mu'$ we get $P(|X'|\ge t_k) = P(|X|\ge c_k)\sim \text{const.}\,2^{-\alpha k^4}k^{-4}$ whence (20) follows in view of (16). For the proof of (21), (22) we first note that for $t\ge t_0$

$$P(|X'|\ge t)\le e^{2\sqrt{\log t}}P(|X|\ge t) \tag{24}$$

and

$$\int_0^t x^2 dF'(x)\le e^{2\sqrt{\log t}}\int_0^t x^2 dF(x). \tag{25}$$

Let $H^* = \bigcup_{k\ge 1} I_k$; it suffices to prove (24) for $t\in H^*$ since $P(|X'|\ge t) = P(|X|\ge t)$ for $t\notin H^*$. Given $t\in H^*$, let $k=k(t)$ denote the integer such that $t\in I_k$; obviously $k\sim(\log t/\log 2)^{1/4}$ as $t\to\infty$. Setting

$$\beta_k = P(|X|\ge c_k)/P(|X|\ge d_k)$$

we have for $t\in I_k$

$$P(|X'|\ge t)\le P(|X|\ge c_k) = \beta_k P(|X|\ge d_k)\le \beta_k P(|X|\ge t).$$

By (13) we have $\beta_k \sim 2^{\alpha k^2} \leq \exp(2(\log t)^{1/2})$ for $t \geq t_0$ and thus (24) is valid. The same argument yields (25); now using (24) and (13) it follows that for $n \geq n_0$ the left hand side of (21) is $\leq 2C_1 n b_n^{-\alpha} \exp(2(\log b_n)^{1/2})$ which is $o(1)$ by (23). Relation (22) is verified similarly.

Let now $G_\alpha$ denote the distribution function with characteristic function $\exp(-|t|^\alpha)$ and define the i.i.d. sequence $(X_n)$ by $X_n = Y_n + Z_n$ where $(Y_n), (Z_n)$ are i.i.d. sequences, independent of each other, with respective distribution functions $G_\alpha$ and $F'$. Set $S_n = X_1 + \cdots + X_n$; we claim that

$$n^{-1/\alpha} S_n \xrightarrow{d} G_\alpha \quad \text{as} \quad n \to \infty,\ n \notin H, \tag{26}$$

$$n^{-1/\alpha} S_n \text{ has no limit distribution as } n \to \infty, \tag{27}$$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\leq N} \frac{1}{k} I\left\{\frac{S_k}{k^{1/\alpha}} < x\right\} = G_\alpha(x) \quad \text{a.s. for all } x. \tag{28}$$

(26) is immediate from $n^{-1/\alpha}\sum_{i\leq n} Y_i \stackrel{d}{=} G_\alpha$ and the relation

$$n^{-1/\alpha} \sum_{i\leq n} Z_i \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty,\ n \notin H$$

which follows from (18) and (19) by truncating the r.v.'s $Z_i$, $1 \leq i \leq n$ at $n^{1/\alpha}$ and applying the Chebisev inequality. To prove (27) note that setting $r_k = t_k - \sqrt{t_k}$ we have by (20) for $k \geq k_0$

$$\begin{aligned} P(|X_1| \geq r_k) &\geq P(|Z_1| \geq t_k) P(|Y_1| < \sqrt{t_k}) \\ &\geq \frac{1}{2} P(|Z_1| \geq t_k) \geq \omega_k t_k^{-\alpha} \geq \frac{1}{2}\omega_k r_k^{-\alpha} \end{aligned}$$

where $\omega_k \to \infty$. Thus $\overline{\lim}_{t\to\infty} t^\alpha P(|X_1| \geq t) = +\infty$ and thus by a classical criterion (see e.g. Feller (1966) p. 547) $n^{-1/\alpha} S_n \stackrel{d}{\not\to} G_\alpha$ for $n \to \infty$. Together with (26) this implies (27). Finally, to verify (28) we note that

$$b_n^{-1} \sum_{i\leq n} Y_i \xrightarrow{P} 0, \quad b_n^{-1} \sum_{i\leq n} Z_i \xrightarrow{P} 0 \quad (n \to \infty). \tag{29}$$

The first relation of (29) is obvious from $Y_i \stackrel{d}{=} G_\alpha$ and $n^{-1/\alpha} b_n \to \infty$ while the second follows from (21) and (22) by truncating the r.v.'s $Z_i$, $1 \leq i \leq n$ at $b_n$ and applying the Chebisev inequality. (29) shows that $b_n^{-1} S_n \xrightarrow{P} 0$ as $n \to \infty$ and since by (23) we have

$$b_{mn}/b_n \leq m^{1/\alpha + \tau(n)}$$

where $\tau(n) = 2\alpha^{-3/2}(\log n)^{-1/2} \to 0$, a trivial modification of the proof of Theorem (6.1) of de Acosta-Giné (1979) shows that

$$E|S_n/b_n|^p = O(1) \quad \text{for each } p < \alpha$$

i.e.

$$E|n^{-1/\alpha}S_n|^{\alpha/2} \le K_1 \exp\{K_2\sqrt{\log n}\} \text{ for some } K_1 > 0,\ K_2 > 0 \tag{30}$$

We also note that since $H$ is of log density 0, (26) implies

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} P\left(\frac{S_k}{k^{1/\alpha}} < x\right) = G_\alpha(x) \text{ for all } x. \tag{31}$$

Now (28) follows from (30), (31) and the Theorem in Section 1.

EXAMPLE 3. We first construct a *numerical* sequence $(X_n,\ n \ge 1)$ satisfying the a.s. central limit theorem (2). Let $(c_n,\ n \ge 1)$ be a uniformly distributed sequence of real numbers in the interval $(0,1)$, that is,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\le N} I(c_k < t) = t \text{ for all } 0 < t < 1.$$

Define $X_n = \Phi^{-1}(c_n)\sqrt{n} - \Phi^{-1}(c_{n-1})\sqrt{n-1},\ n \ge 1$; clearly for the sums $S_n = X_1 + \cdots + X_n$ we have

$$S_n/\sqrt{n} = \Phi^{-1}(c_n)$$

whence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\le N} I(S_k/\sqrt{k} < t) = \lim_{N\to\infty} \frac{1}{N} \sum_{k\le N} I(c_k < \Phi(t)) = \Phi(t)$$

for all real $t$. This is stronger than (2).

Being degenerate themselves, the random variables $S_n/\sqrt{n}$ can only have degenerate weak limits. But the set of limit points of the sequence $(\Phi^{-1}(c_n),\ n \ge 1)$ is clearly the whole real line and thus $S_n/\sqrt{n}$ does not converge in distribution.

Note that $(X_n,\ n \ge 1)$ can be constructed in such a way that $\lim_{n\to\infty} X_n = 0$. For example, set

$$d_k = \left(1 - \frac{1}{2k} - \frac{1}{2(k+1)}\right)((k+1)^5 - k^5)^{-1}$$

and for $k^5 \le n < (k+1)^5$ let

$$c_n = \begin{cases} \frac{1}{2k} + (n - k^5)d_k & \text{if } k \text{ is odd,} \\ 1 - \frac{1}{2k} - (n - k^5)d_k & \text{if } k \text{ is even.} \end{cases}$$

Thus, for $k^5 \le n \le (k+1)^5$ $c_n$ wanders from one end of the interval $(0,1)$ towards the other one by equal steps. Hence $c_n$ is uniformly distributed in $(0,1)$ and defining $X_n$ as above, it is not hard to see that for $(k-1)^5 < n \le k^5$

$$\begin{aligned} |X_n| &\le k^{5/2}\left((\Phi^{-1}(\frac{1}{2k} + d_{k-1}) - \Phi^{-1}(\frac{1}{2k})\right) + O(\Phi^{-1}(\frac{1}{2k})k^{-5/2}) \\ &= O((k \log k)^{-1/2}). \end{aligned}$$

Of course, our degenerate sequence $(X_n,\ n \ge 1)$ can be smoothed by adding i.i.d. random variables to the terms. For example, let $(Y_n,\ n \ge 1)$ be a sequence of i.i.d. $N(0,1)$ random variables, and let us define $\tilde{X}_n = (X_n + Y_n)/\sqrt{2}$. Then $(\tilde{X}_n,\ n \ge 1)$ is a sequence of independent random variables, not identically distributed but converging in distribution to $N(0,1/2)$. As to the corresponding sums $\tilde{S}_n$, the set of weak limits of $\tilde{S}_n/\sqrt{n}$ coincides with the translation family $\{N(\mu, 1/2) : \mu \in \mathbf{R}\}$, while using again $S_n/\sqrt{n} = \Phi^{-1}(c_n)$ and the equidistribution of $c_n$ we get for all $t$

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{k \le N} P(\tilde{S}_k/\sqrt{k} < t) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k \le N} P((Y_1 + \cdots + Y_k)/\sqrt{k} < \sqrt{2}t - \Phi^{-1}(c_k)) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k \le N} \Phi(\sqrt{2}t - \Phi^{-1}(c_k)) \\ &= \int_0^1 \Phi(\sqrt{2}t - \Phi^{-1}(u))du \\ &= \int_{-\infty}^{\infty} \Phi(\sqrt{2}t - z)d\Phi(z) = \Phi(t). \end{aligned}$$

Consequently

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k \le N} \frac{1}{k} P(\tilde{S}_k/\sqrt{k} < t) = \Phi(t).$$

The conditions of the Theorem in Section 1 are trivially met since

$\mathbf{E}|\tilde{S}_n/\sqrt{n}| \le |\Phi^{-1}(c_n)| + \mathbf{E}|(Y_1 + \cdots + Y_n)/\sqrt{n}| \le |\Phi^{-1}(c_n)| + 1 = O(\sqrt{\log n})$.

Thus the a.s. central limit theorem (2) holds.

## REFERENCES

[1] deACOSTA, A. and GINÉ, E. (1979) Convergence of moments and related functionals in the general central limit theorem in Banach spaces. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **48**, 213—231. *MR* **80h**: 60011

[2] BERKES, I. and DEHLING, H. (1991). Strong and weak limit theorems involving log density. *Annals of Probability* (to appear)

[3] BROSAMLER, G. A. (1988) An almost everywhere central limit theorem. *Math. Proc. Camb. Phil. Soc.* **104**, 561—574. *MR* **89i**: 60045

[4] ESSEEN, C. G. (1968) On the concentration function of a sum of independent random variables. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **9**, 290—308. *MR* **37** # 6974

[5] FELLER, W. (1966) *An introduction to probability theory and its application*, Vol. II. Wiley: New York, London. *MR* **35** # 1048

[6] FISHER, A. (1989) A pathwise central limit theorem for random walks. *Annals of Probability* (to appear)

[7] LACEY, M. T. and PHILIPP, W. (1990) A note on the almost everywhere central limit theorem. *Stat. Probab. Letters* **9**, 201—205. *MR* **91e**: 60100

[8] SCHATTE, P. (1988) On strong versions of the central limit theorem. *Math. Nachr.* **137**, 249—256. *MR* **89i**: 60070

[9] ZOLOTAREV, V. M. (1957) The Mellin-Stieltjes transform in probability theory. *Probab. Theor. Appl.* **2**, 444—469. *MR* **21** # 7555

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# THE WORK OF ALFRÉD RÉNYI: SOME ASPECTS IN PROBABILITY AND NUMBER THEORY

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The mathematical output of Alfréd Rényi was so extensive and so diverse that no one speaker at this Memorial Meeting can hope to touch on more than a few themes. I shall concentrate here on those aspects of Rényi's work which have particularly interested me, and British mathematics generally. Rényi's personal influence on British probability was strong; see e.g. the obituary acticle Kendall (1970).

We write [n] for the nth paper in Rényi's Selected Works, and refer to works of others by author and year.


1. Geometric probability
2. Information theory
3. Records
4. Erdős-Rényi laws
5. Exponentiality
6. Expansions of real numbers
7. Summability methods
8. Divisors
9. The large sieve; Goldbach's conjecture
10. Probabilistic number theory

Postscript

References


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## 1. Geometric probability

In geometric probability, one deals with a measure $m$ (defined by, e.g., a suitable invariance requirement); typically $m$ will be unbounded, and determined uniquely to within constant multiples (as with, e.g., invariant measures for Markov chains). This indeterminacy vanishes on taking quotients; this quotient operation allows an interpretation in terms of conditional probability. Thus one obtains formulae of the type

$$P(X \in A \mid X \in B) = m(A)/m(B), \quad A \subset B.$$

This point of view was advocated by Rényi, in his book *Wahrscheinlichkeitsrechnung* (§§II. 10, 11) and in his papers [110], [120]. The resulting theory, which has roots going back to Buffon and Crofton, finds its textbook synthesis in Santaló (1976). The subject was dear to Rényi's heart, and for good reason: recall the closing paragraph of Mark Kac's foreword to Santaló's book:

"Above all the book should remind all of us that Probability Theory is measure theory with a "soul", which in this case is provided not by Physics or by games of chance or by Economics but by the most ancient and noble of all mathematical disciplines, namely Geometry."

On a personal level, I cherish the memory of Rényi giving (in Cambridge, on 28 May 1969) what I regarded then and regard now as the best mathematical talk I have ever heard ('Conditional probability spaces defined by unbounded measures').

Rényi's parking problem - or packing problem in one dimension - concerns $\lim_{x\to\infty} EN(x)/x$, where $N(x)$ is the number of unit line-segments ('cars') which can be packed ('parked') in $[0, x]$ ([149]). For a textbook treatment, see Hall (1988), §1.10. For background and further developments, see Palásti (1960), Ney (1962), Mannion (1964), (1976), (1979), (1983).

Rényi and Sulanke [208], [223], [302] also worked on convex hulls of random points. For further developments on this important topic, see e.g. Fisher (1969), Eddy (1980), Jewell and Romano (1982), Brozius and de Haan (1987), Davis, Mulrow and Resnick (1987).

For a survey of developments in geometric probability, see Baddeley (1977).

## 2. Information theory

Rényi wrote extensively on this subject. In particular, his book *Wahrscheinlichkeitsrechnung* is distinguished for the weight it gives to information theory (*Anhang über Informationstheorie*, 435—498) and for its proof of the limit theorem for Markov chains by information-theoretic methods (ibid., §9).

One particularly important contribution is [180]. Here Rényi considers the question of characterization of Shannon's entropy by functional equations, as well as generalizations of it. For an account of current progress, see e.g. Aczél and Dhombres (1989). Here also (§5) is given the proof of the limit theorem for Markov chains via information theory; cf. Csiszár (1963), Kendall (1964), Fritz (1973). Rényi raises the question of simplifying the information-theoretic proof of the central limit theorem in Linnik (1959). He did not return to this, but see recent work by Barron (1986) and Takano (1987).

In [160] Rényi gives a theory of $\varepsilon$-entropy related to that of Kolmogorov. See Csiszár's comments to [160] (Selected Papers, Volume 2, 342), and - for the Kolmogorov theory - Cover et al. (1989), §2, Shiryaev (1989), 'The fifties' (910—920).

In his later work, Rényi considered the application of information theory to statistics - principally to parametric statistics by Bayesian methods, but to non-parametric statistics also. See in particular [285] (Bayesian version of the Neyman-Pearson lemma), [288] (Kakutani's dichotomy) and [328] (large-deviation theory; commentary by Csiszár, 574—6). For background on information theory in statistics, see e.g. the paper of Csiszár (1975) and the books of Kullback (1959), Kullback et al. (1987), Liese and Vajda (1987), Vajda (1989); cf. Vajda (1990).

The information-theoretic approach to statistics is connected with the idea of stochastic complexity. Again, this involves Kolmogorov's work; see e.g. the obituary article Kendall (1990), 44—5, the works of Rissanen and others cited there, and Chentsov (1990).

## 3. Records

Suppose that $X_1, X_2, \dots$ are independent and identically distributed (iid) with continuous law $F$. Call $X_n$ a *record* if $X_n > \max(X_1, \dots, X_{n-1})$; write $(L(n))$ for the *record times*, defined inductively by $L(1) := 1$, $L(n) := \min\{k > L(n-1) : X_k > X_{L(n-1)}\}$, $(X_{L(n)})$ for the *record values.*

So far as the record times are concerned, matters do not depend on $F$

(are 'distribution-free'). For if $H := -\log(1-F)$ is the hazard function of $F$, $U_i := H(X_i)$ are iid with the unit exponential law $E(1)$; the record times of $(X_i)$ and $(U_i)$ are the same, and $R_n := U_{L(n)}$ are the record values of $(U_i)$.

This distribution-free property is due to Rényi [194], who also showed that $(L(n))$ is a non-homogeneous Markov chain, with transition probabilities

$$P(L(n) = k \mid L(n-1) = j) = \frac{j}{k(k-1)} \quad (n \le j \le k-1).$$

Rényi also gave a law of large numbers (LLN), a central limit theorem (CLT) and law of the iterated logarithm (LIL) for $(L(n))$. He considered the number of records up to time $n$, identified its law with that of the number of cycles in a random permutation, and found it in terms of Stirling numbers.

Rényi's ideas have done much to fertilize the extensive subsequent work on records. For instance, Resnick (1973) observed that $(R(n))$ is a unit Poisson process ('structure lemma for records'), whence the LLN, CLT and LIL for $(R(n))$. Resnick also finds the limit laws for $(X_{L(n)})$ under the appropriate condition on $F$; the almost-sure behaviour is in de Haan and Resnick (1973). For a synthesis of record-value theory, see Chapter 4 of Resnick (1987) (cf. Bingham et al. (1987/89), §8.14).

The method of strong approximation may be combined very fruitfully with the above structure theory for records. This idea goes back to Williams (1973), and has been developed by Deheuvels and others; see e.g. Deheuvels (1983) and the references cited there.

In his work on the combinatorial aspects of records, Rényi was partially anticipated by Foster and Stuart (1954). For subsequent work, see e.g. Barton and Mallows (1965), Imhof (1983), Bingham (1988), §4.2, Goldie (1989).

## 4. Erdős-Rényi laws

Laws of large numbers have traditionally been used to average out the distribution to leave the mean. Thus for $X, X_1, X_2$, iid, with law $F$, mean $\mu$ and variance $\sigma^2$, the strong law tells us that $\mu$ is all that survives averaging in the a.s. limit, while the central limit theorem tells us that $\mu$ and $\sigma$ are all that survive in the limit distribution.

It has been realised over the years that a less harsh averaging, in which more of $F$ survives when we take the limit, is often more appropriate. A flexible framework is provided by taking *moving averages*, of the form

$$\frac{1}{a_n} \sum_{n \le k < n+a_n} X_k,$$

in which the length of the averaging block $a_n$ is accurately tied to the properties (e.g., integrability) of $F$; for a recent survey see Bingham (1989). The results there illustrate the general principle that the more integrability $F$ has, the less averaging is needed.

Suppose now that $F$ has extremely good integrability properties: that its characteristic function is analytic in a neighbourhood of the origin. Then we may use the moment-generating function instead or - more conveniently - its logarithm, the cumulant-generating function:

$$k(t) := \log E \exp\{tX\}.$$

This is convex, so we may form its Fenchel dual $k^*$:

$$k^*(\alpha) := \sup_t \{\alpha t - k(t)\}, \qquad k(t) = \sup_\alpha \{\alpha t - k^*(\alpha)\}.$$

In their 'new law of large numbers' of 1970, Erdős and Rényi [342] considered moving averages with $a_n = c \log n$, where $c = c(\alpha)$ and $\alpha = \alpha(c)$ are linked by

$$1/c = k^*(\alpha).$$

They showed that then

$$\max_{0 \leq i \leq n - c \log n} \frac{1}{c \log n} \sum_{i \leq k < i + c \log n} X_k \to \alpha \quad \text{a.s.} \quad (n \to \infty).$$

Here the a.s. limit $\alpha$ is the $\alpha(c)$ above. So by varying $c$ we can determine successively $\alpha(\cdot)$; its inverse function $c(\cdot)$; then $k^*(\cdot) = 1/c(\cdot)$; then its dual $k(\cdot)$; then $F$. Thus *the entire distribution survives the passage to the limit.*

This very interesting phenomenon (called 'almost-sure non-invariance', in contrast to the more usual a.s. invariance principles) goes back even earlier, to the work of Shepp (1964). For a good account, see de Acosta and Kuelbs (1983), who consider the three cases $a_n / \log n \to \infty, \to c \in (0, \infty), \to 0$, for random vectors (taking values in a Banach space). The critical growth-rate $a_n = O(\log n)$ is of great interest, and has been studied intensively by, e.g., Deheuvels and Steinebach (1986), Deheuvels, Devroye and Lynch (1986), Deheuvels and Devroye (1987).

For recent developments, motivated by problems involving DNA, see Arratia and Waterman (1989), Arratia et al. (1990).

## 5. Exponentiality

The special properties of the exponential and uniform distributions, and their ramifications in the theory of Poisson processes, empirical processes, order statistics, spacings and many other fields, have attracted many authors; see e.g. Chapter I of Feller (1971), or Chapter 8 of Shorack and Wellner (1986). Rényi's interest in this area is reflected in his papers [84], [86], [127], [246], [284]. Related to this are Rényi's ideas on thinning of point processes, (see e.g. [84]), which have been developed at length by Mogyoródi and others.

We note in particular one result (whose short proof we include for convenience); see [84] and, e.g., Kakosyan et al. (1984), Th. 2.4.10. If $(X_n)$ are iid with mean $\mu > 0$, and $\nu$ is independent of $(X_n)$ and geometrically distributed with parameter $p$, then

$$p \sum_{k=1}^{\nu} X_k \to E(\mu) \quad (p \to 0) \quad \text{in distribution,}$$

where $E(\mu)$ denotes the exponential law with mean $\mu$. For, if the $X_n$ have characteristic function $\phi$, then that of $p \sum_1^{\nu} X_k$ is

$$\begin{aligned} E\exp\{itp \sum_1^{\nu} X_k\} &= \sum_k E[\exp\{itp \sum_1^{k} X_j\} \mid \nu = k] \\ &= \sum_k pq^{k-1}\phi(tp)^k \qquad (q := 1-p) \\ &= \frac{p/q}{1 - q\phi(tp)}. \end{aligned}$$

Now $\phi(t) = 1 + i\mu t + o(t)$ for small $t$, so as $p \to 0$ the right is

$$\frac{p(1+o(1))}{1-(1-p)(1+i\mu tp + o(p))} \sim \frac{p}{p - i\mu tp + o(p)} \to \frac{1}{1 - i\mu t},$$

the characteristic function of $E(\mu)$.

This simple result has extensive and important consequences. The monograph of Keilson (1979) develops the reliability theory of Markovian systems with many degrees of freedom, in which limiting exponential laws are ubiquitous in view of the results above. Similar ideas have been developed by Aldous (1982), (1983) in his theory of rapidly mixing Markov chains. A good illustration of these ideas is provided by the classical Ehrenfest urn, which may be viewed as a random walk on the group $\mathbf{Z}_2^d$ (the $d$-cube), for $d$ large. For detail, we refer to the survey by Bingham (1991).

## 6. Expansions of real numbers

The representation of rationals $x \in (0,1)$ preferred by the Egyptians was a sum of unit fractions; thus $\frac{2}{3}$ was written as $\frac{1}{2}+\frac{1}{6}$, etc. Such representations can be extended to irrationals as series expansions. A number of variants and generalizations (Engel, Cantor, Sylvester, Oppenheim,...) are known; for background, see e.g. Perron (1921/39), Galambos (1976). Rényi, with his deep sense of history, was no doubt attracted to this area for historical as well as mathematical interest.

Just as one may apply probabilistic methods to obtain LLN, CLT and LIL for ordinary decimal (or dyadic,...) expansions, one may do the same for expansions of the above type. A detailed treatment was given by Erdős, Rényi and Szüsz [150]; cf. [199], [151], §§8-10. There are interesting connections with the theory of records; see Williams (1973).

Consider, for instance, the Sylvester series $\sum 1/d_n(t)$ for $t \in (0,1]$, where $d_1(t)$ is the least integer with $1/d_1(t) < t$, and then $d_n(t)$ is the least integer with $1/d_n(t) < t - \sum_{k=1}^{n-1} 1/d_k(t)$. The rate of growth of the denominators $d_n(t)$ is of particular interest. In [150] it was shown that

$$2^{-n}\log d_n(t) \to L(t) < \infty \quad (n \to \infty) \quad \text{for almost all } t \in (0,1].$$

A full account of the limit function $L$ is given by Goldie and Smith (1987). In particular, $L$ is the first explicit, deterministic example known of a function having a jointly continuous occupation density (previous examples involved sample paths of stochastic processes: the theory of 'local time').

## 7. Summability methods

Links between probability theory and summability theory are to be expected, since a summability method is essentially a (limit of) a weighted average, and weighted averages are extensively used in probability theory (expectations, etc.) and statistics (sample means, etc.).

The classic book by Hardy (1949) provides a rich harvest for anyone with a background in both probability and analysis. Rényi was such a person *par excellence*, and addressed himself to the probability-summability interface in [168]. He considered matrix methods $A = (a_{nk})$, mapping a sequence $s = (s_n)$ to $t = (t_n)$, where

$$t_n := \sum_k a_{nk} s_k;$$

the case of greatest probabilistic interst is that of $A$ stochastic,

$$a_{nk} = P(\nu_n = k), \qquad \text{say.}$$

The Hausdorff methods are those with

$$a_{nk} = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k} dF(x)$$

with $F$ a probability law on $[0,1]$ (Hardy (1949), Ch. IX); Rényi (§2) studied their composition properties, using the probabilistic interpretation in terms of binomial mixing. Replacing the binomial by the Poisson, one obtains the Henriksson methods, which Rényi (§3) interpreted similarly. He also considered (§4) limit distributions, where one has

$$f(s_n) \to \int f(y) dS(y) \qquad \text{(A)}$$

(convergence in the sense of the summability method A) for all bounded continuous $f$. The case with $S$ degenerate reduces to the concept of almost convergence (Zygmund (1979), Vol. II, 181); cf. statistical convergence (see e.g. Fast (1951)).

We note that Schmetterer (1963) also discussed probabilistic interpretations of summability theory.

A rather different application of probabilistic methods to summability theory has been given by the present author, in a series of papers (see e.g. Bingham (1984a), (1984b), (1988), Bingham and Rogers (1991)). The motivating examples, as in Rényi's work, are the methods of Euler and Borel (see e.g. Hardy (1949), VIII, IX). Instead of being regarded as, respectively, degenerate Hausdorff and degenerate Henriksson methods, these are handled together as instances of the circle methods or Kreisverfahren, for the theory of which see Meyer-König (1949).

In [168] §5, Rényi proved a gap ('high-indices') theorem, under a (Tauberian) condition later proved superfluous by Halász (1967). Rényi worked on gap theorems elsewhere ([198], with Erdős); see Turán (1984), 222.

## 8. Divisors

Write $\Omega(n)$, $\omega(n)$ for the number of prime divisors of $n \in \mathbf{N}$, counted with and without multiplicities. If $B_k(x)$ denotes the number of $n \le x$ with $\Omega(n) - \omega(n) = k$, Rényi [112] showed that

$$B_k(x)/x \to \rho_k \quad (x \to \infty),$$

where the generating function of the densities $\rho_k$ is given by

$$\sum_{k=0}^{\infty} \rho_k z^k = \prod_p (1 - \frac{1}{p})(1 + \frac{1}{p - z})$$

($\prod_p$ denoting a product over primes). (Thus in particular, taking $z = 0$, we get $d_0 = 6/\pi^2$, giving the well-known density of the square-free integers.) A simpler proof was later given by Rényi and Turán [144], §4. This elegant formula clearly fascinated Mark Kac, who proved it in the following paper (Kac (1955)), and again in both his books (Kac (1959a), §I.4, Kac (1959b)). The result generalises, to the context of Beurling's theory of generalized primes; see Bateman and Diamond (1969).

The rate of convergence of the density can be estimated:

$$B_k(x)/x = d_k + O(x^{-1/2}(\log\log x)^{k-1}/\log^2 x)$$

(Delange (1965), (1968), (1973); cf. Ivić (1985), Ch. 14). For generalizations - local theorems for additive functions - see Kubilius (1964), Ch. IV, Elliott (1980), Ch. 21.

If $E$ is a set of primes (with $\sum_{p\in E} 1/p = \infty$), one may consider the divisor functions $\Omega_E$, $\omega_E$ with divisors restricted to lie in $E$. Delange (1956) showed that if $A$ is an arithmetic progression, $\Omega_E^{-1}(A)$ and $\omega_E^{-1}(A)$ both have a natural density, equal to that of $A$. This was extended to more general $A$ by Halász (1971). Tenenbaum (1980) showed, by an intersting Tauberian argument adapted from work of hardy and Littlewood, that the necessary and sufficient condition on $A$ for existence of a natural density for $\Omega_E^{-1}(A)$, $\omega_E^{-1}(A)$ is existence of a density for $A$ in a sense (Valiron, Euler, Borel, ...) stronger than the usual Cesàro sense. For further developments see Bingham (1984a), Bingham and Tenenbaum (1986).

For the number $\tau(n)$ of unrestricted (instead of prime) divisors, for which the theory goes back to Hardy and Ramanujan in 1917, see Elliott (1980), Ch. 15, Hall and Tenenbaum (1988).

## 9. The large sieve; Goldbach's conjecture

The large sieve originated in the work of Linnik (1941); his ideas were developed by Rényi in a series of papers [17], [20], [23], [29], [151], [155], [161] and (with Erdős) [315]. In particular, [315] contains both a commentary on the probabilistic aspects of the large sieve (the distributions mod$p$ and mod$q$ of integers $n \leq N$ are, for distinct primes $p, q \leq N^{1/3}$, almost independent:

532) and an account of the development of the large sieve by Roth, Bombieri, Gallagher and others. In addition to the references cited there, we mention the accounts in Elliott (1979) (Ch. 4 gives the link between the large sieve inequality and the Turán-Kubilius inequality, while pp. 183—4 outlines the history), and Montgomery (1971), (1977). Hildebrand (1986a) gives a proof of the prime number theorem via the large sieve.

The Goldbach conjecture of 1742 - that every even integer $\geq 6$ is the sum of two primes - was attacked by Rényi [9], [200] using his large sieve. He showed that every even $N \geq 6$ is the sum of a prime $p$ and an 'almost-prime', $p_k$ (a number with at most $k$ prime factors, for some absolute constant $k$). A detailed commentary on [200], with references to subsequent developments, is given by Turán. The best result known is that every sufficiently large even $N$ is of the form $p + p_2$ (Chen (1973); see e.g. Halberstam and Richert (1974), Ch. 11).

## 10. Probabilistic number theory

This subject may be said to begin with the Erdős-Kac central limit theorem (Erdős and Kac (1939), (1940)) for the divisor functions:

$$\frac{1}{x}\sum_{n\leq x} I(\omega(n) - \log\log x \leq t\sqrt{\log\log x}) \to \Phi(t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-y^2/2}dy \qquad (x\to\infty) \quad \forall t \in \mathbb{R},$$

and similarly with $\omega$ replaced by $\Omega$. This may be loosely paraphrased in probabilistic language: '$\omega(n)$ is asymptotically normally distributed with mean $\log\log n$ and variance $\log\log n$'. For the origins and early history of this result, see Elliott (1980), 24; for generalizations, see Kubilius (1964), Elliott (1980), Ch. 12. The rate of convergence has been considered (by Le Veque and others); it is in fact $O((\log\log x)^{1/2})$ (Rényi and Turán [144]; cf. Elliott (1980), Ch. 20 and Ch. 12, 18—24, Kubilius (1964), Ch. IX).

The basic idea is to use suitably defined finite probability spaces, and independent random variables defined thereon, to bring to bear the powerful machinery of probability theory on number-theoretic problems such as the above. This is due to Kac; see for instance his books Kac (1959a), (1959b), the monographs of Kubilius (1964) and Elliott (1979), (1980) and the survey of Billingsley (1974). See in particular [151], §5 for a nice account of the theory above, and Billingsley (1979), 349—351 for a very short proof of the Erdős-Kac CLT.

Probabilistic methods may also be applied to diophantine problems; see [135], [151] §6.

Rényi [251] applied the Turán-Kubilius inequality to give a simple proof of a theorem of Delange (1961) on mean values of multiplicative functions $g$ with $|g(\cdot)| \leq 1$. For subsequent developments, see e.g. Halász's commentary to [251], Elliott (1979), Ch. 6, (1980), Ch. 19, Daboussi and Delange (1982), Delange (1983), Hildebrand (1984), (1986b), (1987). For $g \geq 0$, see Erdős and Rényi [250].

Related to [251] is Rényi's paper [210] on the Erdős-Wintner theorem; see Halász's commentary on [210], Elliott (1979), Chs. 5, 6, Galambos (1970).

To close, we consider a succinct application of one of Rényi's interests, entropy (specifically, its role in the theory of large deviations) to another, the prime divisor functions $\omega, \Omega$ of §8. If $t$ in the Erdős-Kac theorem is $O(\sqrt{\log\log x})$ (for $t = o(\sqrt{\log\log x})$, see Kubilius (1964), Ch. IX), the centring and scaling coalesce, and one considers

$$\frac{1}{x}\sum_{n\leq x} I(\omega(n) < a\log\log x) \qquad (0 < a < 1),$$

$$\frac{1}{x}\sum_{n\leq x} I(\omega(n) > a\log\log x) \qquad (a > 1).$$

By Selberg's formula (Selberg (1954)),

$$\frac{1}{n}\sum_{m=1}^{n} z^{\omega(m)} = (F(z) + \frac{O(1)}{\log n})(\log n)^{\operatorname{Re} z - 1}$$

with $F(\cdot)$ entire.

Writing $P_n$, $E_n$ for probability and expectation on $\{1, 2, \ldots, n\}$ with probability $1/n$ on each point, and replacing $z$ by $e^t$, the left is $E_n \exp\{t\omega\}$. Selberg's formula gives

$$\frac{\log E_n \exp\{t\omega\}}{\log\log n} \to e^t - 1 \quad (n \to \infty).$$

In the language of large-deviation theory, the limit, $c(t) := e^t - 1$, is the *free-energy function*. It is convex; form its Fenchel dual

$$I(z) := \sup_t \{tz - c(t)\},$$

the *entropy function*. Thus

$$I(z) = z\log z - z + 1 \quad (z \geq 0), \quad +\infty (z < 0).$$

Application of the large-deviation theorem of Ellis (1984) (or (1985), VII) yields

$$\log \frac{1}{n} \sum_{1}^{n} I(\omega(\cdot) > a \log\log n) \sim -\log\log n(a \log a - a + 1) \quad (n \to \infty)\ (a > 1),$$

and similarly for $0 < a < 1$. With more work, much stronger results may be obtained: an asymptotic formula for $\frac{1}{n}\sum_1^n I(\omega > a \log\log n)$ rather than its logarithm - due to Delange - is given by Erdős and Nicolas (1981), Balazard et al. (1991); cf. Norton (1976), (1979), (1982). The situation for $\Omega$ is more complicated, as the value $a = 2$ then becomes critical; for details, see the references above, and Tenenbaum (1990), II. 6.1.

## POSTSCRIPT

'A wise man has remarked that every mathematician has his own personal view of Kolmogorov' (Kendall (1990), 45). Each of us too has his own view of Rényi; here in brief outline in mine.

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## REFERENCES

ACOSTA de, A. and KUELBS, J. (1983), Limit theorems for moving averages of independent random vectors, *Z. Wahrsch. Verw. Gebiete* **64** (1983), 67—123. *MR* **84i**: 60011

ACZÉL, J. and DHOMBRES, J. (1989), *Functional equations in several variables.* With applications to mathematics, information theory and to the natural and social sciences, Encycl. Math. Appl. **31**, Cambridge Univ. Press, Cambridge - New York, 1989. *MR* **90h**: 39001

ALDOUS, D. (1982), Markov chains with almost exponential hitting times, *Stochastic Process. Appl.* **13** (1982), 305—310. *MR* **84b**: 60096a

ALDOUS, D. (1983), Random walks on finite groups and rapidly mixing Markov chains, *Seminar on probability, XVII*, Lecture Notes in Math., Vol. 986, Springer-Verlag, Berlin - New York, 1983, 243—297. *MR* **86j**: 60156

ARRATIA, R., GORDON, L. and WATERMAN, M. S. (1990), The Erdős-Rényi law in distribution, for coin-tossing and sequence matching, *Ann. Statist.* **18** (1990), 539—570.

ARRATIA, R. and WATERMAN, M. S. (1989), The Erdős-Rényi strong law for pattern matching with a given proportion of mismatches, *Ann. Probab.* **17** (1989), 9—25.

BADDELEY, A. (1977), A fourth note on recent research in geometrical probability, *Advances in Appl. Probability* **9** (1977), 131—860. *MR* **57** # 7724

BARRON, A. R. (1986), Entropy and the central limit theorem, *Ann. Probab.* **14** (1986), 336-342. *MR* **87h**: 60048

BARTON, D. E. and MALLOWS, C. L. (1965), Some aspects of the random sequence, *Ann. Math. Statist.* **36** (1965), 236—260. *MR* **31** # 2745

BATEMAN, P. T. and DIAMOND, H. G. (1969), Asymptotic distribution of Beurling's generalized prime numbers, *Studies in Number Theory* (ed. W. J. Le Veque), Math. Assoc. Amer., Prentice-Hall, Englewood Cliffs, N. J., 1969, 152—210. *MR* **39** # 4105

BILLINGSLEY, P. (1974), The probability theory of additive arithmetic functions, *Ann. Probability* **2** (1974), 749—791. *MR* **57** # 5938

BINGHAM, N. H. (1984a), On Borel and Euler summability, *J. London Math. Soc.* (2) **29** (1984), 141—146. *MR* **85k**: 40010

BINGHAM, N. H. (1984b), On Valiron and circle convergence, *Math. Z.* **186** (1984), 273—286. *MR* **86g**: 40008

BINGHAM, N. H. (1988), Tauberian theorems for Jakimovski and Karamata-Stirling methods, *Mathematika* **35** (1988), 216—224. *MR* **90f**: 40006

BINGHAM, N. H. (1989), Moving averages, *Almost everywhere convergence* ed. by G. A. Edgar and L. Sucheston, Academic Press, New York, 1989, 131—144.

BINGHAM, N. H. (1991), Fluctuation theory for the Ehrenfest urn, *Adv. Appl. Probab.* **23** (1991).

BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987), *Regular variation*, 2nd edition, Encycl. Math. Appl. **27**, Cambridge - New York, 1989. *MR* **90i**: 26003

BINGHAM, N. H. and ROGERS, L. C. G. (1991), Summability methods and almost-sure convergence, *Almost sure convergence* II, ed. by A. Bellow and R. L. Jones, Academic Press, New York, 1991.

BINGHAM, N. H. and TENENBAUM, G. (1986), Riesz and Valiron means and fractional moments, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 143—149. *MR* **86m**: 40010

BROZIUS, H. A. and HAAN de, L. (1987), On limiting laws for the convex hull of a sample, *J. Appl. Probab.* **24** (1987), 852—862. *MR* **89h**: 60036

CHEN, J. R. (1973), On the representation of a large even integer as the sum of a prime and the product of at most two primes, *Sci. Sinica* **16** (1973), 157—176. *MR* **55** # 7959

CHENTSOV, N. N. (1990), The unfathomable influence of Kolmogorov, *Ann. Statist.* **18** (1990), 987—998.

COVER, T. M., GÁCS, P. and GRAY, R. M. (1989), Kolmogorov's contributions to information theory and algorithmic complexity, *Ann. Probab.* **17** (1989), 840—865.

CSISZÁR, I. (1963), Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 85—108. *MR* **29** # 1671

CSISZÁR, I. (1975), *I*-divergence geometry of probability distributions and minimization problems, *Ann. Probability* **3** (1975), 146—158. *MR* **51** # 2050

DABOUSSI, H. and DELANGE, H. (1982), On multiplicative arithmetical functions whose modulus does not exceed one, *J. London Math. Soc.* (2) **26** (1982), 245—264. *MR* **83k**: 10082

DAVIS, R., MULROW, E. and RESNICK, S. I. (1987), The convex hull of a random sample in $IR^2$, *Comm. Statist. Stochastic Models* **3** (1987), 1—27. *MR* **88h**: 60020

DEHEUVELS, P. (1983), The complete characterization of the upper and lower classes of the record and inter-record times of an i.i.d. sequence, *Z. Wahrsch. Verw. Gebiete* **62** (1983), 1—6. *MR* **84f**: 600033

DEHEUVELS, P. and DEVROYE, L. (1987), Limit laws of Erdős-Rényi-Shepp type, *Ann. Probab.* **15** (1987), 1363—1386. *MR* **88f**: 600055

DEHEUVELS, P., DEVROYE, L. and LYNCH, J. (1986), Exact convergence rate in the limit theorems of Erdős-Rényi and Shepp, *Ann. Probab.* **14** (1986), 209—223. *MR* **87d**: 60032

DEHEUVELS, P. and STEINEBACH, J. (1986), Exact convergence rate of an Erdős-Rényi strong law for moving quantiles, *J. Appl. Probab.* **23** (1986), 355—369. *MR* **87i**: 60033

DELANGE, H. (1956), Sur la distribution des entiers ayant certaines propriétés, *Ann. Sci. École Norm. Sup.* (3) **73** (1956), 15—74. *MR* **18**—720

DELANGE, H. (1961), Sur les fonctions arithmétiques multiplicatives, *Ann. Sci. École Norm. Sup.* (3) **78** (1961), 273—304. *MR* **30** # 72

DELANGE, H. (1965), (1968), (1973), Sur un théorème de Rényi, I: *Acta Arith.* **11** (1965), 241—252; II: *ibid.* **13** (1967/68), 339—362; III: *ibid.* **23** (1973), 152—182. *MR* **34** # 1282; *MR* **37** # 4037; *MR* **48** # 8414

DELANGE, H. (1983), Sur les fonctions arithmétiques multiplicatives de module $\leq 1$, *Acta Arith.* **42** (1983), 121—151. *MR* **84m**: 10037

EDDY, W. F. (1980), The distribution of the convex hull of a Gaussian sample, *J. Appl. Probab.* **17** (1980), 686—695. *MR* **81k**: 60030

ELLIOTT, P. D. T. A. (1979), (1980), *Probabilistic number theory. I: Mean-value theorems, II: Central limit theorems,* Grundlehren der Math. Wiss. **239, 240**, Springer-Verlag, Berlin - New York, 1979, 1980. *MR* **82h**: 10002a, 10002b

ELLIOTT, P. D. T. A. (1987), Functional analysis and additive arithmetic functions, *Bull. Amer. Math. Soc.* (N. S.) **16** (1987), 179—223. *MR* **88i**: 11052

ERDŐS, P. and KAC, M. (1939), On the Gaussian law of errors in the theory of additive functions, *Proc. Nat. Acad. Sci.* **25** (1939), 206—207. *Zbl* **21**, 207

ERDŐS, P. and KAC, M. (1940), The Gaussian law of errors in the theory of additive number-theoretic functions, *Amer. J. Math.* **62** (1940), 738—742. *MR* **2**—42

FAST, H. (1951), Sur la convergence statistique, *Colloquium Math.* **2** (1951), 241—244. *MR* **14**—29

FELLER, W. (1971), *An introduction to probability theory and its applications*, Vol. II, 2nd ed., John Wiley, New York - London, 1971. *MR* **42** # 5292

FISHER, L. (1969), Limiting sets and convex hulls of samples from product measures, *Ann. Math. Statist.* **40** (1969), 1824—1832. *MR* **40** # 6606

FOSTER, F. G. and STUART, A. (1954), Distribution-free tests in time-series based on the breaking of records, *J. Roy. Statist. Soc. Ser. B.* **16** (1954), 1—22. *MR* **16**—385

FRITZ, J. (1973), An information-theoretical proof of limit theorems for reversible Markov processes, *Transactions of the Sixth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes* (Tech. Univ., Prague, 1971), Academia, Prague, 1973, 183—197. *MR* **51** # 6992

GALAMBOS, J. (1970), A probabilistic approach to mean values of multiplicative functions, *J. London Math. Soc.* (2) **2** (1970), 405—419. *MR* **43** # 176

GALAMBOS, J. (1971), Limit distribution of sums of (dependent) random variables with applications to arithmetical functions, *Z. Wahrsch. Verw. Gebiete* **18** (1971), 261—270. *MR* **45** # 5101

GALAMBOS, J. (1976), *Representations of real numbers by infinite series*, Lecture Notes in Mathematics, Vol. 502, Springer-Verlag, Berlin -New York, 1976. *MR* **58** # 27873

GOLDIE, C. M. (1989), Records, permutations and greatest convex minorants, *Math. Proc. Cambridge Philos. Soc.* **106** (1989), 169—177. *MR* **91a**: 60138

GOLDIE, C. M. and SMITH, R. L. (1987), On the denominators in Sylvester's series, *Proc. London Math. Soc.* (3) **54** (1987), 445—476. *MR* **88f**: 11073

HAAN de, L. and RESNICK, S. I. (1973), Almost sure limit points of record values, *J. Appl. Probability* **10** (1973), 528—542. *MR* **51** # 9171

HALÁSZ, G. (1967), On the sequence of generalized partial sums of a series, *Studia Sci. Math. Hungar.* **2** (1967), 435—439. *MR* **36** # 4203

HALÁSZ, G. (1971), On the distribution of additive and the mean values of multiplicative arithmetic functions, *Studia Sci. Math. Hungar.* **6** (1971), 211—233. *MR* **47** # 8471

HALBERSTAM, H. and RICHERT, H. -E. (1974), *Sieve methods*, London Mathematical Society Monographs, No. 4, Academic Press, London - New York, 1974. *MR* **54** # 12689

HALL, P. (1988), *Introduction to the theory of coverage processes*, John Wiley, New York, 1988. *MR* **90f**: 60024

HALL, R. R. and TENENBAUM, G. (1988), *Divisors*, Cambridge Tracts in Mathematics, **90**, Cambridge Univ. Press, Cambridge - New York, 1988. *MR* **90a**: 11107

HILDEBRAND, A. (1984), Quantitative mean value theorems for nonnegative multiplicative functions. I., *J. London Math. Soc.* (2) **30** (1984), 394—406. *MR* **87g**: 11118

HILDEBRAND, A. (1986a), On Wirsing's mean value theorem for multiplicative functions, *Bull. London Math. Soc.* **18** (1986), 147—152. *MR* **87f**: 11075

HILDEBRAND, A. (1986b), The prime number theorem via the large sieve, *Mathematika* **33** (1986), 23—30. *MR* **88a**: 11085

HILDEBRAND, A. (1987), Quantitative mean value theorems for nonnegative multiplicative functions II, *Acta Arith.* **48** (1987), 209—260. *MR* **89b**: 11078

HARDY, G. H. (1949), *Divergent series.* Oxford, at the Clarendon Press, 1949. *MR* **11**—25

IMHOF, J. -P. (1983), Stirling numbers and records, *J. Combin. Theory Ser. A* **34** (1983), 252—254. *MR* **84j**: 05011

IVIĆ, A. (1985), *The Riemann zeta-function. The theory of the Riemann zeta-function with applications*, John Wiley and Sons, New York, 1985. *MR* **87d**: 11062

JEWELL, N. P. and ROMANO, J. P. (1985), Evaluating inclusion functionals for random convex hulls, *Z. Wahrsch. Verw. Gebiete* **68** (1985), 415—424. *MR* **86c**: 60023

KAC, M. (1955), A remark on the preceding paper by A. Rényi, *Acad. Serbe Sci. Publ. Inst. Math.* **8** (1955), 163—165. *MR* **17**—944

KAC, M. (1959a), *Probability and related topics in physical sciences*, Proceedings of the Summer Seminar, Boulder, Colo., 1957, Vol.1, Interscience Publishers, London - New York, 1959. *MR* **21** # 1635

KAC, M. (1959b), *Statistical independence in probability, analysis and number theory*, Carus Math. Monographs **12**, Math. Assoc. of America, Distributed by John Wiley, New York, 1959. *MR* **22** # 996

KAKOSYAN, A. V., KLEBANOV, L. B. and MELAMED, J. A. (1984), *Characterization of distributions by the method of intensively monotone operators*, Lecture Notes in Math. Vol. 1088, Springer-Verlag, Berlin - New York, 1984. *MR* **86g**: 62022

KEILSON, J. (1979), *Markov chain models - rarity and exponentiality*, Applied Mathematical Sciences, Vol. 28, Springer-Verlag, New York - Berlin, 1979. *MR* **80f**: 60061

KENDALL, D. G. (1964), Information theory and the limit-theorem for Markov chains and processes with a countable infinity of states, *Ann. Inst. Statist. Math.* **15** (1963), 137—143. *MR* **30** # 1548

KENDALL, D. G. (1970), Obituary: Alfréd Rényi, *J. Appl. Probability* **7** (1970), 509—522. *MR* **41** # 5174

KENDALL, D. G. (1990), Obituary: Andrei Nikolaevich Kolmogorov, *Bull. London Math. Soc.* **22** (1990), 31-100.

KUBILIUS, J. (1964), *Probabilistic methods in the theory of numbers*, Transl. Math. Monographs **11**, Amer. Math. Soc., Providence, RI, 1964. *MR* **28** # 3956

KULLBACK, S. (1959), *Information theory and statistics*, John Wiley, New York/ Chapman and Hall, London, 1959. *MR* **21** # 2325

KULLBACK, S., KEEGEL, J. C. and KULLBACK, J. H. (1987), *Topics in statistical information theory*, Lecture notes in statistics, Vol. 42, Springer-Verlag, Berlin - New York, 1987. *MR* **89b**: 62015

LIESE, F. and VAJDA, I. (1987), *Convex statistical distances*, Teubner-Texte zur Mathematik, **95**, Teubner, Leipzig, 1987. *MR* **89g**: 62006

LINNIK, Yu. V. (1941), " The large sieve" *C. R. Doklady Acad. Sci. USSR (N. S.)* **30** (1941), 292—294 (in Russian). *MR* **2**—349

LINNIK, Yu. V. (1959), An information-theoretic proof of the central limit theorem with Lindeberg condition, *Teor. Verojatnost. i Primenen.* **4** (1959), 311—321 (in Russian). *MR* **23** # A1397

MANNION, D. (1964), Random space-filling in one dimension, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* A **9** (1964), 143—154. *MR* **31** # 1698

MANNION, D. (1976), (1979), Random packing of an interval, *Advances in Appl. Probability* **8** (1976), 477—501; II: *ibid.* **11** (1979), 591—602. *MR* **80i**: 60014

MANNION, D. (1983), Sequential random displacements of points in an interval, *J. Appl. Probab.* **20** (1983), 251—263. *MR* **84g**: 60021

MEYER-KÖNIG, W. (1949), Untersuchungen über einige verwandte Limitierungsverfahren, *Math. Z.* **52** (1949), 257—304. *MR* **11**—242

MONTGOMERY, H. L. (1971), *Topics in multiplicative number theory*, Lecture Notes in Mathematics, Vol. 227, Springer-Verlag, Berlin - New York, 1971. *MR* **49** # 2616

MONTGOMERY, H. L. (1977), The analytic principle of the large sieve, *Bull. Amer. Math. Soc.* **84** (1978), 547—567. *MR* **57** # 5931

NEY, P. E. (1962), A random interval filling problem, *Ann. Math. Statist.* **33** (1962), 702—718. *MR* **25** # 1561

PALÁSTI, I. (1960), On some random space filling problems, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5** (1960), 353—360. *MR* **26** # 4466

PERRON, O. (1939), *Irrationalzahlen*, 2nd ed. (1st ed. 1921), Walter de Gruyter, Berlin, 1939. *Zbl* **22**, 123

RESNICK, S. I. (1973), Limit laws for record values, *Stochastic Processes Appl.* **1** (1973), 67—82. *MR* **50** # 14895

RESNICK, S. I. (1987), *Extreme values, regular variation and point processes*, Applied Probability Series, Vol. 4, Springer-Verlag, New York - Berlin, 1987. *MR* **89b**: 60241

SANTALÓ, L. (1976), *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications, Vol. 1, Addison-Wesley Publ. Co., Reading, Mass. - London, 1976. *MR* **55** # 6340

SCHMETTERER, L. (1963), Wahrscheinlichkeitstheoretische Bemerkungen zur Theorie der Reihen, *Arch. Math.* **14** (1963), 311—316. *MR* **27** # 3971

SHEPP, L. A. (1964), A limit law concerning moving averages, *Ann. Math. Statist.* **35** (1964), 424—428. *MR* **29** # 4091

SHIRYAEV, A. N. (1989), Kolmogorov: life and creative activities, *Ann. Probab.* **17** (1989), 866—944. *MR* **90j**: 01106

SHORACK, G. R. and WELLNER, J. A. (1986), *Empirical processes with applications to statistics*, John Wiley, New York, 1986. *MR* **88e**: 60002

TAKANO, S. (1987), Convergence of entropy in the central limit theorem, *Yokohama Math. J.* **35** (1987), 143—148. *MR* **89f**: 600032

TENENBAUM, G. (1980), Sur le procédé de sommation de Borel et la répartition du nombre des facteurs premiers des entiers, *Enseign. Math.* (2) **26** (1980), 225—245. *MR* **82k**: 10057

TURÁN, P. (1984), *On a new method of analysis and its applications*, With the assistance of G. Halász and J. Pintz, With a foreword by Vera T. Sós, Pure and Applied Mathematics, John Wiley, New York, 1984. *MR* **86b**: 11059

VAJDA, I. (1989), *Theory of statistical inference and information*, Kluwer, Dordrecht, 1989.

VAJDA, I. (1990), Distances and discrimination rates for stochastic processes, *Stochastic Proc. Appl.* **35** (1990), 47—57.

WILLIAMS, D. (1973), On Rényi's "record" problem and Engel's series, *Bull. London Math. Soc.* **5** (1973), 235—237. *MR* **48** # 12636

ZYGMUND, A. (1979), *Trigonometric series*, Volumes I, II, Reprint of the 1979 edition, Cambridge Univ. Press, Cambridge - New York, 1988. *MR* **89c**: 42001

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# ON THE INCREMENTS OF ADDITIVE FUNCTIONALS

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## 1. Introduction

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables such that $\{S_i\}_{i=0}^{\infty}$, $S_0 = 0$, $S_i = X_1 + \cdots + X_i$, $i = 1, 2, \ldots$ is a recurrent aperiodic random walk on integer lattice $\mathbf{Z}$. Let $f(z)$, $z \in \mathbf{Z}$ be a real valued function vanishing outside a finite set. Define the additive functional by

$$A_N = \sum_{i=0}^{N-1} f(S_i), \qquad N = 1, 2, \ldots \tag{1.1}$$

Assume that $0 < a_N \leq N$ is a nondecreasing sequence of integers and let

$$A_N^* = \max_{0 \leq j \leq N - a_N} (A_{j+a_N} - A_j). \tag{1.2}$$

The local time $\xi(x, N)$ of the random walk $\{S_i\}_{i=1}^{\infty}$ is defined by

$$\xi(x, N) = \sum_{i=0}^{N-1} \mathbf{1}_{\{x\}}(S_i), \tag{1.3}$$

where $\mathbf{1}_A(z)$ denotes the indicator function of $A$. (1.3) is obviously a particular case of (1.1). Put

$$\xi_N^* = \max_{0 \leq j \leq N - a_N} (\xi(0, j + a_N) - \xi(0, j)). \tag{1.4}$$

It was shown in [2] that under the condition $\sigma^2 = \mathrm{var} X_1 < \infty$ we have

$$\lim_{N \to \infty} \frac{\xi_N^*}{\beta_N} = 1 \qquad \text{a.s.} \tag{1.5}$$

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provided that

$$\lim_{N\to\infty} \frac{\log(N/a_N)}{\log\log N} = \infty \tag{1.6}$$

and

$$\lim_{N\to\infty} \frac{a_N}{\log N} = c \tag{1.7}$$

with $0 < c \le \infty$, where

$$\beta_N = (a_N \log(N/a_N))^{1/2}/\sigma \quad \text{if} \quad c = \infty \tag{1.8}$$

and

$$\beta_N = bc \log N \quad \text{if} \quad 0 < c < \infty \tag{1.9}$$

with certain positive constant $b = b(c)$.

The result (1.5) may be regarded as an Erdős-Rényi law (see [4]) for local times. (1.5) was extended in [7] under certain conditions on $X_i$, more general than being in the domain of attraction of a stable law of index $\alpha > 1$ and also, the conditions (1.6) and (1.7) were relaxed. In general, however, when (1.6) does not hold, the lim in (1.5) should be replaced by limsup. In the case when $X_i$ is in the domain of attraction of a stable law of index $\alpha$, $1 < \alpha \le 2$, it was shown in [7] that

$$\beta_N \sim C_\alpha \frac{a_N^{1-1/\alpha}\left(\left(1-\frac{1}{\alpha}\right)\log(N/a_N) + \log\log N\right)^{1/\alpha}}{\ell_1\left(\frac{a_N}{\alpha-1}\left(\left(1-\frac{1}{\alpha}\right)\log\frac{N}{a_N} + \log\log N\right)^{-1}\right)} \tag{1.10}$$

with certain (explicitly given) constant $C_\alpha$ and slowly varying function $\ell_1(\cdot)$, provided that (1.7) holds true with $c = \infty$. More precisely in this case

$$\limsup_{N\to\infty} \frac{\xi_N^*}{\beta_N} = 1 \qquad \text{a.s.} \tag{1.11}$$

and if (1.6) holds true, then

$$\lim_{N\to\infty} \frac{\xi_N^*}{\beta_N} = 1 \qquad \text{a.s.} \tag{1.12}$$

In this paper we study similar problems for $A_N^*$ and show that in the case when $c = \infty$ in (1.7), the limit behaviour of $A_N^*$ is the same as that of $\xi_N^*$. We treat also the case when $0 < c < \infty$ in (1.7) and show that an Erdős-Rényi-type law holds for $A_N^*$ provided that a large deviation result is valid for $A_N$. We do not know, however, any general condition under which such a large deviation holds. It certainly holds in the cases treated in [2] and [7].

Similar problems can be investigated also for additive functionals of a Wiener process. Let $\{W(t),\ t \geq 0\}$ be a standard Wiener process and let $L(x,t),\ x \in \mathbf{R},\ t \geq 0$ be its jointly continuous local time. Assume that $g(x)$ is an integrable function having compact support. The additive functional is defined by

$$B_t = \int_0^t g(W(s))ds = \int_{-\infty}^{\infty} g(x)L(x,t)dx. \tag{1.13}$$

Let $a_t$ be a nondecreasing function of $t$. Then we consider

$$B_T^* = \sup_{0 \leq t \leq T - a_T} (B_{t+a_T} - B_t). \tag{1.14}$$

Put

$$L_T^* = \sup_{0 \leq t \leq T - a_T} (L(0, t + a_T) - L(0,t)). \tag{1.15}$$

It was shown in [1] that

$$\limsup_{T \to \infty} \frac{L_T^*}{\gamma_T} = 1 \qquad \text{a.s.,} \tag{1.16}$$

where

$$\gamma_T = \left( a_T \left( \log \frac{T}{a_T} + 2 \log\log T \right) \right)^{1/2}. \tag{1.17}$$

If we also assume that

$$\lim_{T \to \infty} \frac{\log(T/a_T)}{\log\log T} = \infty, \tag{1.18}$$

then limsup in (1.16) can be replaced by lim.

Here we show that the same result is true for $B_T^*$ provided that $a_T \gg \log T$. We investigate also the case when $a_T = c \log T$.

## 2. The random walk case

THEOREM 2.1. *Assume that* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of i.i.d. integer valued random variables such that* $S_0 = 0,\ S_1, S_2, \ldots$ *is a recurrent aperiodic random walk on* $\mathbf{Z}$. *Let* $A_N$ *be defined by* (1.1) *where* $f(z),\ z \in \mathbf{Z}$ *is a real* *valued function having finite support. Let* $\{a_N\}_{N=1}^{\infty}$ *be a nondecreasing sequence of integers. Let furthermore* $\xi(x,N)$ *be defined by* (1.3) *and* $\xi_N^*$ *by* (1.4). *Assume that*

$$\xi_N^* = O(\beta_N) \qquad a.s. \quad N \to \infty \tag{2.1}$$

*where* $\{\beta_N\}_{N=1}^{\infty}$ *is a non-random sequence of positive numbers. Then we have*

$$\max_{0\le j\le N-a_N} |A_{j+a_N} - A_j - \overline{f}(\xi(0,j+a_N) - \xi(0,j))| = \tag{2.2}$$

$$= O((\beta_N \log N)^{1/2} + \log N) \quad a.s., \quad N \to \infty,$$

*where* $\overline{f} = \sum_{x\in\mathbf{Z}} f(x)$.

PROOF. Since $f(\cdot)$ has finite support, there exist $x_1$ and $x_2$ $-\infty < x_1 \le x_2 < \infty$ such that $f(x) = 0$ if $x < x_1$ or $x > x_2$. Moreover, since the random walk is recurrent, we can define a.s. an infinite sequence $\{\rho_i\}_{i=0}^{\infty}$ such that $\rho_0 = 0$ and

$$\rho_i = \min\{k : S_k = 0, \rho_{i-1} < k\}, \qquad i = 1,2,\dots \tag{2.3}$$

Put $\rho = \rho_1$ and let

$$p = p(x) = P(\xi(x,\rho) > 0). \tag{2.4}$$

Then we have for $x = \pm 1, \pm 2, \dots$

$$P(\xi(x,\rho) = 0) = 1 - p \tag{2.5}$$

$$P(\xi(x,\rho) = k) = p^2(1-p)^{k-1}, \qquad k = 1,2,\dots \tag{2.6}$$

It is then easy to see that

$$E(\xi(x,\rho)) = 1 \qquad x \in \mathbf{Z} \tag{2.7}$$

and

$$E(e^{t(\xi(x,\rho)-1)}) = 1 + \frac{2(1-p)}{1-(1-p)e^t}(\cosh t - 1), \quad t < \log\frac{1}{1-p}. \tag{2.8}$$

It follows from the aperiodicity of the random walk that $0 < p(x) < 1$ and therefore there exist constants $K > 0$ and $t_0 > 0$ such that for $x_1 \le x \le x_2$

$$E(e^{t(\xi(x,\rho)-1)}) \le e^{Kt^2}, \quad |t| \le t_0. \tag{2.9}$$

Let

$$Z=\sum_{x=x_1}^{x_2} f(x)\xi(x,\rho). \tag{2.10}$$

From Jensen's inequality and (2.9) we conclude that

$$E(e^{t(Z-\overline{f})})=E\left(e^{t(x_2-x_1+1)\frac{\sum_{x=x_1}^{x_2} f(x)(\xi(x,\rho)-1)}{x_2-x_1+1}}\right)\le \tag{2.11}$$
$$\le \frac{1}{x_2-x_1+1}\sum_{x=x_1}^{x_2} E(e^{t(x_2-x_1+1)f(x)(\xi(x,\rho)-1)})\le$$
$$\le \frac{1}{x_2-x_1+1}\sum_{x=x_1}^{x_2} e^{Kt^2(x_2-x_1+1)^2f^2(x)}\le e^{K_1t^2},\quad |t|\le t_0$$

with certain $K_1>0$.

Define

$$Z_i=\sum_{k=\rho_{i-1}+1}^{\rho_i} f(S_k)=\sum_{x=x_1}^{x_2} f(x)(\xi(x,\rho_i)-\xi(x,\rho_{i-1})),\quad i=1,2,\ldots \tag{2.12}$$

Then $\{Z_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables. From exponential Markov inequality and (2.11) we get

$$P(\max_{1\le k\le r}|\sum_{i=1}^{k}(Z_i-\overline{f})|\ge u\sqrt{r})\le \tag{2.13}$$
$$\le e^{-tu\sqrt{r}}\left(E\left(e^{t\sum_{i=1}^{r}(Z_i-\overline{f})}\right)+E\left(e^{-t\sum_{i=1}^{r}(Z_i-\overline{f})}\right)\right)\le$$
$$\le 2e^{-tu\sqrt{r}+K_1rt^2}=2e^{-\frac{u^2}{4K_1}}$$

with the choice $t=u/(2K_1\sqrt{n})$, provided that $0<u\le 2K_1t_0\sqrt{n}$. From (2.13) we get

$$\max_{0\le j\le n-r_n}\max_{1\le k\le r_n}\left|\sum_{i=j+1}^{j+k}(Z_i-\overline{f})\right|=O(\sqrt{r_n\log n})\quad \text{a.s.}\quad n\to\infty, \tag{2.14}$$

where $r_n$ is a nondecreasing sequence of integers. Let

$$Z_i^*=\sum_{k=\rho_{i-1}+1}^{\rho_i}|f(S_k)|=\sum_{x=x_1}^{x_2}|f(x)|(\xi(x,\rho_i)-\xi(x,\rho_{i-1})). \tag{2.15}$$

Then $Z_i^*$, $i = 1, 2, \ldots$ are also i.i.d. random variables having finite moment generating function in some neighbourhood of zero. Therefore

$$\max_{1 \le i \le n} |Z_i^*| = O(\log n) \quad \text{a.s.}, \quad n \to \infty. \tag{2.16}$$

It is easily seen that the estimation

$$\begin{aligned} &|A_{j+a_N} - A_j - \overline{f}(\xi(0, j + a_N) - \xi(0, j))| \le \\ &\le \max_{0 \le \ell \le \xi(0,N) - \xi_N^*} \max_{1 \le k \le \xi_N^*} \left| \sum_{i=\ell+1}^{\ell+k} (Z_i - \overline{f}) \right| + 2 \max_{1 \le \ell \le \xi(0,N)} Z_\ell^* \le \\ &\le \max_{0 \le \ell \le N - \xi_N^*} \max_{1 \le k \le \xi_N^*} \left| \sum_{i=\ell+1}^{\ell+k} (Z_i - \overline{f}) \right| + 2 \max_{1 \le \ell \le N} Z_\ell^* \end{aligned} \tag{2.17}$$

holds true for $0 \le j \le N - a_N$. Hence (2.2) follows from (2.1), (2.14), (2.16) and (2.17).

Theorem 2.1 and the result of [7], mentioned in the Introduction, imply the following

COROLLARY 2.1. *Assume that $X_i$, $i = 1, 2, \ldots$ are i.i.d. integer valued random variables being in the domain of attraction of a stable law of index $\alpha$ $(1 < \alpha \le 2)$, $EX_i = 0$, the random walk $S_0 = 0, S_i = X_1 + \ldots + X_i$, $i = 1, 2, \ldots$ is aperiodic and $f(z)$, $z \in \mathbf{Z}$ is a real valued function having finite support. Let $a_N$ $(0 < a_N \le N)$ be a nondecreasing function of integers such that $N/a_N$ is also nondecreasing and (1.7) holds with $c = \infty$. Then*

$$\limsup_{N \to \infty} \frac{A_N^*}{\beta_N} = \overline{f} \qquad a.s. \tag{2.18}$$

*where $A_N^*$ is defined by (1.2) and $\beta_N$ is given by (1.10). Moreover, if (1.6) holds, then*

$$\lim_{N \to \infty} \frac{A_N^*}{\beta_N} = \overline{f} \qquad a.s. \tag{2.19}$$

*If (1.7) holds with $0 < c < \infty$, then*

$$A_N^* = O(\log N) \qquad a.s., \quad N \to \infty. \tag{2.20}$$

REMARK 2.1. In the case when $\overline{f} = 0$, it is an open problem to find the right normalizers and constants in (2.18) and (2.19). This is equivalent to finding the sharp limsup (or lim) in (2.2).

In case when (1.7) holds with finite $c$, (2.20) and 0-1 law imply that

$$\limsup_{N\to\infty} \frac{A_N^*}{\log N} = \text{constant} \qquad \text{a.s.} \tag{2.21}$$

The Erdős-Rényi law for local times (cf. [2] and [7]) suggests that in fact, we should have lim in (2.21). To prove such a result we would need large deviations for $A_N$. In the next theorem we show that an Erdős-Rényi law holds for $A_N^*$ assuming large deviations for $A_N$ and also, the extra condition $f(x) \geq 0$.

THEOREM 2.2. *Assume that* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of i.i.d. integer valued random variables being in the domain of attraction of a stable law of index* $\alpha$ $(1 < \alpha \leq 2)$, $EX_i = 0$, *the random walk* $S_0 = 0$, $S_i = X_1 + \cdots + X_i$, $i = 1, 2, \ldots$ *is aperiodic. Let* $f(z)$, $z \in \mathbf{Z}$ *be a nonnegative function having finite support. Suppose that for fixed* $x$ *we have*

$$-\lim_{N\to\infty} \frac{1}{N} \log P_x(A_N \geq Ny) = \psi(y) \tag{2.22}$$

*where* $P_x(\cdot)$ *denotes the probability under the condition that the random walk starts at* $x$ *and* $\psi(y)$ *is a decreasing continuous function which does not depend on* $x$. *Let* $a_N = [c \log N]$ *and* $A_N^*$ *be defined by* (1.2) *with this* $a_N$. *Then*

$$\lim_{N\to\infty} \frac{A_N^*}{c \log N} = y \qquad a.s. \tag{2.23}$$

*where* $y$ *is defined by*

$$\psi(y) = \exp\left\{-\frac{\alpha - 1}{\alpha c}\right\}. \tag{2.24}$$

PROOF. First we prove the upper bound in (2.23), i.e.

$$\limsup_{N\to\infty} \frac{A_N^*}{c \log N} \leq y \qquad \text{a.s.} \tag{2.25}$$

We follow [2] and [7]. Let

$$\kappa_j^{(x)} = \min\{k : S_{k+j} = x,\ k \geq 0\}, \quad x \in \mathbf{Z} \tag{2.26}$$

and

$$\kappa_j = \min_{x_1 \leq x \leq x_2} \kappa_j^{(x)}, \tag{2.27}$$

where $[x_1, x_2]$ is an interval containing the support of $f(x)$. Then

$$P(\sum_{i=j}^{j+a-1} f(S_i) \geq ay) =$$

$$= \sum_{k=0}^{a-1} \sum_{x=x_1}^{x_2} P(\sum_{i=j}^{j+a-1} f(S_i) \geq ay \mid \kappa_j = k,\ S_{j+k} = x) P(\kappa_j = k,\ S_{j+k} = x) =$$

$$= \sum_{k=0}^{a-1} \sum_{x=x_1}^{x_2} P_x(\sum_{i=0}^{a-1-k} f(S_i) \geq ay) P(\kappa_j = k,\ S_{j+k} = x) \leq$$

$$\leq \sum_{k=0}^{a-1} \sum_{x=x_1}^{x_2} P_x(\sum_{i=0}^{a-1} f(S_i) \geq ay) P(\kappa_j = k,\ S_{j+k} = x) \leq$$

$$\leq \sum_{k=0}^{a-1} \sum_{x=x_1}^{x_2} (\psi(y(1-\varepsilon)))^a P(\kappa_j = k,\ S_{j+k} = x) \leq$$

$$\leq (\psi(y(1-\varepsilon)))^a P(\kappa_j \leq a-1)$$

for $\varepsilon > 0$ provided that $a$ is large enough. But

$$P(\kappa_j \leq a-1) \leq \sum_{x=x_1}^{x_2} P(\kappa_j^{(x)} \leq a-1)$$

and this can be estimated as in [2]:

$$P(\kappa_j^{(x)} \leq a-1) = \sum_z P(\kappa_j^{(x)} \leq a-1 \mid S_j = z) P(S_j = z).$$

It follows (see [5]) that

$$P(S_j = z) \leq \frac{C}{j^{1/\alpha} \ell_1(j)}, \qquad z \in \mathbf{Z} \tag{2.28}$$

where $\ell_1(\cdot)$ is a slowly varying function, and by considering the reverse random walk one can see that

$$P(\kappa_j^{(x)} \leq a-1 \mid S_j = z) = P_z(\kappa_0^{(x)} \leq a-1) =$$

$$= \sum_{k=0}^{a-1} P_z(\kappa_0^{(x)} = k) = \sum_{k=0}^{a-1} P(\rho \geq k,\ S_k = z-x),$$

hence from [7], Proposition 6.1

$$P(\kappa_j^{(x)} \leq a-1) \leq \frac{C}{j^{1/\alpha} \ell_1(j)} \sum_{k=0}^{a-1} P(\rho \geq k) \leq$$

$$\le \frac{C_1}{j^{1/\alpha}\ell_1(j)}\left(1+\sum_{k=1}^{a} k^{1-1/\alpha}\ell_1(k)\right) \le \frac{C_2 a^{1/\alpha}\ell_1(a)}{j^{1/\alpha}\ell_1(j)}.$$

Here we obtained the following estimation:

$$P\left(\sum_{i=j}^{j+a-1} f(S_i) \ge ay\right) \le C_2(\psi(y(1-\varepsilon)))^a \frac{a^{1/\alpha}\ell_1(a)}{j^{1/\alpha}\ell_1(j)}.$$

From this we get

$$P(A_N^* \ge a_N u) \le C_2(\psi(y(1-\varepsilon)))^{a_N} \sum_{j=1}^{N} \frac{a_N^{1/\alpha}\ell_1(a_N)}{j^{1/\alpha}\ell_1(j)} \le$$

$$\le C_3(\psi(y(1-\varepsilon)))^{a_N} \frac{a_N^{1/\alpha}\ell_1(a_N)N^{1-1/\alpha}}{\ell_1(N)}$$

and we can complete the proof of (2.25) by the usual way of taking subsequence and applying Borel-Cantelli lemma (cf. [2] or [7]).

To show

$$\liminf_{N\to\infty} \frac{A_N^*}{c\log N} \ge y \qquad \text{a.s.} \tag{2.29}$$

define

$$\eta_0 = 0, \quad \eta_k = \min\{j : j \ge \eta_{k-1} + a_N, S_j = 0\} \tag{2.30}$$

and

$$\nu_N = \max\{r : \eta_r \le N\}. \tag{2.31}$$

$\eta_1, \eta_2 - \eta_1, \dots$ are i.i.d. random variables with the properties (cf. [7])

$$P(\eta_1 \ge N) \sim \left(\frac{a_N}{N}\right)^{1-1/\alpha} \frac{\ell_1(N)}{\ell_1(a_N)} \tag{2.32}$$

and

$$E(\eta_1 \wedge N) \sim N\left(\frac{a_N}{N}\right)^{1-1/\alpha} \frac{\ell_1(N)}{\ell_1(a_N)}. \tag{2.33}$$

Hence

$$P(\nu_N \le K) = P(\eta_K \ge N) \le$$

$$\le \frac{1}{N}\sum_{i=1}^{K} E((\eta_i - \eta_{i-1})\wedge N) + \sum_{i=1}^{K} P(\eta_i - \eta_{i-1} \ge N) \le$$

$$\le CK\left(\frac{a_N}{N}\right)^{1-1/\alpha} \frac{\ell_1(N)}{\ell_1(a_N)}$$

with certain constant $C$.

Since

$$A_N^* \ge \max_{0\le j\le \nu_N-1} \sum_{i=\eta_j}^{\eta_j+a_N-1} f(S_i) \tag{2.34}$$

and

$$\sum_{i=\eta_j}^{\eta_j+a_N-1} f(S_i), \qquad j=0,1,2,\ldots$$

are i.i.d. random variables, we have

$$\begin{aligned}
&P(A_N^* \le a_N y) \le \\
&\le P(A_N^* \le a_N y, \nu_N > K) + P(\nu_N \le K) \le \\
&\le P(\max_{0\le j\le K} \sum_{i=\eta_j}^{\eta_j+a_N-1} f(S_i) \le a_N y) + P(\nu_N \le K) \le \\
&\le (1-P(\sum_{i=0}^{a_N-1} f(S_i) \ge a_N y))^K + CK\left(\frac{a_N}{N}\right)^{1-1/\alpha} \frac{\ell_1(N)}{\ell_1(a_N)} \le \\
&\le e^{-K(\psi((1+\varepsilon)y))^{a_N}} + CK\left(\frac{a_N}{N}\right)^{1-1/\alpha} \frac{\ell_1(N)}{\ell_1(a_N)}.
\end{aligned}$$

Now using (2.24) and choosing

$$K = K_N = \left(\frac{N}{a_N}\right)^{1-1/\alpha-\varepsilon_1} \tag{2.35}$$

with some $\varepsilon_1 > 0$, one can complete the proof of (2.29) as in [2] or [7].

## 3. The Wiener process case

We have the analogue of Theorem 2.1.

THEOREM 3.1. *Let $\{W(t),\ t\ge 0\}$ be a standard Wiener process and let $\{L(x,t),\ -\infty < x < \infty,\ t \ge 0\}$ be its jointly continuous local time. Assume that $g(x)$, $x\in\mathbf{R}$ is an integrable function vanishing outside a finite interval $[x_1,x_2]$. Let $B_t$ and $B_T^*$ be defined by* (1.13) *and* (1.14), *resp. Let $a_T$, $T\ge 0$ be a nondecreasing function of $T$ such that $T/a_T$ is also nondecreasing. Then*

$$\sup_{0\le t\le T-a_T} |B_{t+a_T} - B_t - \overline{g}(L(0,t+a_T) - L(0,t))| = \tag{3.1}$$

$$= O((\gamma_T \log T)^{1/2} + \log T) \quad a.s., \quad T \to \infty,$$

*where* $\gamma_T$ *is defined by* (1.17) *and*

$$\bar{g} = \int_{-\infty}^{\infty} g(x)dx. \tag{3.2}$$

For the proof of this theorem we note that (cf. [6])

$$E(\exp\{\eta L(x, T_r)\}) = \exp\left\{\frac{r\eta}{1 - 2x\eta}\right\}, \quad \eta < 1/(2x) \tag{3.3}$$

where $T_r$ is the inverse local time process defined by

$$T_r = \inf\{t : t \geq 0, L(0,t) \geq r\}, \quad 0 \leq r. \tag{3.4}$$

Moreover, $T_r$ is a process with independent increments, so the proof can be given along the same lines as that of Theorem 2.1, with the obvious modification that sums, at times, should be replaced by integrals. Here $Z$ defined by (2.10) should be replaced by

$$\tilde{Z} = \int_{x_1}^{x_2} g(x) L(x, T_1) dx. \tag{3.5}$$

Note that (3.3) implies that $E(L(x, T_1)) = 1$ for all $x \in \mathbf{R}$. Thus

$$E\tilde{Z} = \int_{x_1}^{x_2} g(x)dx = \bar{g}. \tag{3.6}$$

We omit further details.

Similarly, to the random walk case, we have

COROLLARY 3.1. *Under the conditions of Theorem 3.1. and, in addition*

$$\lim_{T\to\infty} \frac{a_T}{\log T} = \infty, \tag{3.7}$$

*we have*

$$\limsup_{T\to\infty} \frac{B_T^*}{\gamma_T} = 1 \qquad a.s. \tag{3.8}$$

*If* (1.18) *holds, then*

$$\lim_{T\to\infty} \frac{B_T^*}{\gamma_T} = 1 \qquad a.s. \tag{3.9}$$

*If*

$$\lim_{T\to\infty} \frac{a_T}{\log T} < \infty, \tag{3.10}$$

*then*

$$B_T^* = O(\log T) \quad a.s., \quad T \to \infty. \tag{3.11}$$

Here again, we consider the case, when $a_T = c\log T$ with some positive finite $c$.

THEOREM 3.2. *Let $\{W(t),\ t \geq 0\}$ be a standard Wiener process and let $g(x)$, $x \in \mathbf{R}$ be a nonnegative function vanishing outside a finite interval $[x_1, x_2]$. Let $B_t$ and $B_T^*$ be defined by (1.13) and (1.14), resp. with $a_T = c\log T$ $(0 < c < \infty)$. Assume that*

$$-\lim_{t\to\infty} \frac{1}{t} \log P_x(B_t \geq ty) = \psi(y), \tag{3.12}$$

*uniformly in $x \in [x_1, x_2]$, where $P_x(\cdot)$ denotes the probability under the condition that the Wiener process starts at $x$ and $\psi(y)$ is a decreasing continuous function which does not depend on $x$. Then*

$$\lim_{T\to\infty} \frac{B_T^*}{c\log T} = y \qquad a.s. \tag{3.13}$$

*where $y$ is defined by*

$$\psi(y) = \exp\left\{-\frac{1}{2c}\right\}. \tag{3.14}$$

PROOF. First we prove

$$\limsup_{T\to\infty} \frac{B_T^*}{c\log T} \leq y \qquad \text{a.s.} \tag{3.15}$$

Put $a = a_t$ and estimate the probability

$$P(B_{t+a} - B_t > ay) = \int_t^{t+a} P(B_{t+a} - B_t > ay \mid \tau_t = s) d_s P(\tau_t \leq s),$$

where $\tau_t = \inf\{u : t \leq u,\ W(u) \in [x_1, x_2]\}$. But

$$\begin{aligned} &P(B_{t+a} - B_t > ay \mid \tau_t = s,\ W(s) = x) = \\ &= P_x(B_{t+a} - B_s > ay) \leq P_x(B_{t+a} - B_t > ay) \leq (\psi((1-\varepsilon)y))^a \end{aligned}$$

if $a$ is large enough. Hence

$$P(B_{t+a} - B_t > ay) \leq (\psi((1-\varepsilon)y))^a P(\tau_t \leq t + a) \tag{3.16}$$

An easy calculation shows that

$$\begin{aligned} P(\tau_t \leq t+a) = P(x_1 \leq W(t) \leq x_2) + 2\int_{x_2}^{\infty} (1 - \phi(\frac{x - x_2}{\sqrt{a}})) \frac{1}{\sqrt{t}} \varphi(\frac{x}{\sqrt{t}}) dx + \\ + 2\int_{-\infty}^{x_1} (1 - \phi(\frac{x_1 - x}{\sqrt{a}})) \frac{1}{\sqrt{t}} \varphi(\frac{x}{\sqrt{t}}) dx \leq C\sqrt{\frac{a}{t+a}}, \quad t \geq 0 \end{aligned}$$

with certain positive constant $C$.

Hence we obtained

$$P(B_{t+a} - B_t > ay) \le C\sqrt{\frac{a}{t+a}}(\psi((1-\varepsilon)y))^a \tag{3.17}$$

for $0 \le t$ and large enough $a$.

Since

$$\sup_{0\le t\le T-a}(B_{t+a} - B_t) \le \sup_{0\le i\le \frac{T-a}{a\varepsilon}}(B_{t_i+a(1+\varepsilon)} - B_{t_i}), \tag{3.18}$$

where

$$t_i = ia\varepsilon, \tag{3.19}$$

we have from (3.16),

$$P(\sup_{0\le t\le T-a}(B_{t+a} - B_t) > ay) \le \tag{3.20}$$
$$\le C\sum_{i=0}^{\frac{T-a}{a\varepsilon}}\sqrt{\frac{1+\varepsilon}{i\varepsilon+1+\varepsilon}}\left(\psi\left(\frac{1-\varepsilon}{1+\varepsilon}y\right)\right)^a \le C_1\sqrt{\frac{T}{a}}\left(\psi\left(\frac{1-\varepsilon}{1+\varepsilon}y\right)\right)^a$$

with certain constant $C_1$. Using this estimation, one can complete the proof of (3.14), as in [1], proof of Lemma 2.

We show next

$$\liminf_{T\to\infty}\frac{B_T^*}{c\log T} \ge y \qquad \text{a.s.} \tag{3.21}$$

For $a > 0$, define

$$\eta_0 = 0, \quad \eta_k = \inf\{t : t > \eta_{k-1} + a,\ W(t) = 0\}, \quad k = 1, 2, \ldots \tag{3.22}$$

and

$$\nu_T = \max\{k : \eta_k \le T - a\}. \tag{3.23}$$

As shown in [3], we have

$$P\left(\nu_T < \left(\frac{T}{a}\right)^{\frac{1-\varepsilon}{2}}\right) \le 3\sqrt{2}\left(\frac{a}{T}\right)^{\varepsilon/2}. \tag{3.24}$$

Since

$$\sup_{0\le t\le T-a}(B_{t+a} - B_t) \ge \sup_{0\le k\le \nu_T}(B_{\eta_k+a} - B_{\eta_k}) \tag{3.25}$$

and $\{B_{\eta_k+a} - B_{\eta_k}\}_{k=1}^{\infty}$ are i.i.d. random variables, we get

$$P(\sup_{0\le t\le T-a}(B_{t+a}-B_t)\le ay)\le$$

$$\le P\left(\sup_{0\le t\le T-a}(B_{t+a}-B_t)\le ay,\ \nu_t\ge\left(\frac{T}{a}\right)^{(1-\varepsilon)/2}\right)+$$

$$+P\left(\nu_T<\left(\frac{T}{a}\right)^{(1-\varepsilon)/2}\right)\le$$

$$\le P(\sup_{0\le k\le(\frac{T}{a})^{(1-\varepsilon)/2}}(B_{\eta_k+a}-B_{\eta_k})\le ay)+3\sqrt{2}\left(\frac{a}{T}\right)^{\varepsilon/2}=$$

$$=(1-P(B_a>ay))^{(\frac{T}{a})^{(1-\varepsilon)/2}}+3\sqrt{2}\left(\frac{a}{T}\right)^{\varepsilon/2}\le$$

$$\le\exp\left\{-\left(\frac{T}{a}\right)^{(1-\varepsilon)/2}(\psi(y(1+\varepsilon)))^a\right\}+3\sqrt{2}\left(\frac{a}{T}\right)^{\varepsilon/2}$$

and one can complete the proof of (3.2) as in [1], proof of Lemma 1.

Now (3.12) follows from (3.14) and (3.20).

## REFERENCES

[1] CSÁKI, E., CSÖRGŐ, M., FÖLDES, A. and RÉVÉSZ, P., How big are the increments of the local time of a Wiener process? *Ann. Probab.* **11** (1983), 593—608. *MR* **84i**: 60104.

[2] CSÁKI, E. and FÖLDES, A., How big are the increments of the local time of a recurrent random walk? *Z. Wahrsch. Verw. Gebiete* **65** (1983), 307—322. *MR* **85b**: 60027.

[3] CSÁKI, E. and FÖLDES, A., On the narrowest tube of a Wiener process, *Limit Theorems in Probability and Statistics* (Veszprém, Hungary, 1982), Colloq. Math. Soc. J. Bolyai, Vol. 36, North-Holland, Amsterdam, 1984, 173—197. *MR* **87h**: 60146.

[4] ERDŐS, P. and RÉNYI, A., On a new law of large numbers, *J. Analyse Math.* **23** (1970), 103—111. *MR* **42** # 6907.

[5] GRIFFIN, P. S., JAIN, N. C. and PRUITT, W. E., Approximate local limit theorems for laws outside domains of attraction, *Ann. Probab.* **12** (1984), 45—63. *MR* **85f**: 60037.

[6] ITÔ, K. and McKEAN, H. P., Jr., *Diffusion processes and their sample paths*, Die Grundlehren der math. Wissenschaften, Bd. 125, Springer, Berlin, 1965. *MR* **33** # 8031.

[7] JAIN, N. C. and PRUITT, W. E., Maximal increments of local time of a random walk, *Ann. Probab.* **15** (1987), 1461—1490. *MR* **88j**: 60060.

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# A NOTE ON "THREE PROBLEMS ON THE RANDOM WALK IN $Z^d$" BY P. ERDŐS AND P. RÉVÉSZ

ENDRE CSÁKI *

Consider a simple symmetric random walk on the line, i.e.

(1) $$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad n \geq 1$$

where $\{X_i\}_{i=1}^{\infty}$ is an i.i.d. sequence with

(2) $$P(X_1 = 1) = P(X_1 = -1) = 1/2.$$

In [1] the waiting time needed to meet a new point is defined by

(3) $$\nu_n = \min\{k : k > 0, \ S_{k+n} \neq S_j \ (j = 0, 1, \ldots, n)\}.$$

One of the results for $d = 1$ in [1] reads as follows:

(4) $$\frac{1}{4\pi^2} \leq \limsup_{n\to\infty} \frac{\nu_n}{n(\log\log n)^2} \leq \frac{16}{\pi^2} \quad \text{a.s.}$$

In this note our aim is to show that the exact constant in (4) is in fact equal to $1/\pi^2$.

THEOREM. *For $d = 1$ we have*

(5) $$\limsup_{n\to\infty} \frac{\nu_n}{n(\log\log n)^2} = \frac{1}{\pi^2} \quad a.s.$$

PROOF. First define the following quantities:

(6) $$m_n = -\min_{0\leq j\leq n} S_j$$

(7) $$M_n = \max_{0\leq j\leq n} S_j$$

(8) $$R_n = M_n - m_n.$$

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It is then readily seen that

$$P(\nu_n > N) = \tag{9}$$
$$= \sum_{c=1}^{n} \sum_{a=0}^{c} \sum_{k=-a}^{c-a} P(m_n = -a,\ R_n = c,\ S_n = k;\ -a \le S_j \le c-a,\ j = n+1, \dots, n+N)$$
$$= \sum_{c=1}^{n} \sum_{a=0}^{c} \sum_{k=-a}^{c-a} P(m_n = -a,\ R_n = c,\ S_n = k) \times P(-a-k \le S_j \le c-a-k,\ j = 1, \dots, N).$$

These probabilities can be obtained from the following formulae (cf. [2]):

$$P(-a \le S_j \le c-a,\ j = 1,2,\dots,n;\ S_n = k) = \tag{10}$$
$$= \frac{1}{2^n} \sum_{r=-\infty}^{\infty} \left( \binom{n}{\frac{1}{2}(n+k) + r(c+2)} - \binom{n}{\frac{1}{2}(n+k) + a + 1 + r(c+2)} \right)$$

where the binomial coefficients $\binom{a}{b}$ are understood to be 0 if $b > a$ or if $b$ is not a nonnegative integer. (10) has also an equivalent expression (cf. [2]):

$$P(-a \le S_j \le c-a,\ j = 1,2,\dots,n;\ S_n = k) = \tag{11}$$
$$= \frac{2}{c+2} \sum_{r=0}^{c+2} \left(\cos \frac{r\pi}{c+2}\right)^n \sin\left(\frac{r\pi(c-a+1)}{c+2}\right) \sin\left(\frac{r\pi(c-a+1-k)}{c+2}\right).$$

Summing up for the possible values of $k$, we obtain

$$P(-a \le S_j \le c-a,\ j = 1,2,\dots,n) = \tag{12}$$
$$= \frac{2}{c+2} \sum_{r=1}^{c+1} \left(\cos \frac{r\pi}{c+2}\right)^n \sin\left(\frac{r\pi(c-a+1)}{c+2}\right) \frac{1+\cos\frac{r\pi}{c+2}}{\sin\frac{r\pi}{c+2}} \left(\frac{1-(-1)^r}{2}\right).$$

Now we prove the upper bound, i.e.

$$\limsup_{n\to\infty} \frac{\nu_n}{n(\log\log n)^2} \le \frac{1}{\pi^2} \quad \text{a.s.} \tag{13}$$

It follows from (12) that there exists an absolute constant $\overline{K}_1$ such that

$$P(-a-k \le S_j \le c-a-k,\ j = 1,\dots,N) \le \overline{K}_1 \left(\cos \frac{\pi}{c+2}\right)^N \tag{14}$$

for all $1 \le N$, $1 \le c \le n$, $0 \le a \le c$ and $-a \le k \le c-a$. Hence

$$P(\nu_n > N) \le \overline{K}_1 \sum_{c=1}^{n} P(R_n = c) \left(\cos \frac{\pi}{c+2}\right)^N . \tag{15}$$

One can obtain from (10) that there exists an absolute constant $\overline{K}_2$ such that

$$P(R_n = c) \le \frac{\overline{K}_2}{\sqrt{n}} e^{-\frac{(c+2)^2}{2n}} \quad \text{for} \quad 1 \le c \le n. \tag{16}$$

Therefore by using the inequality

$$\log \cos x \le -\frac{x^2}{2} \qquad 0 \le x \le \frac{\pi}{2} \tag{17}$$

we get

$$\begin{aligned} P(\nu_n > N) &\le \overline{K}_1 \overline{K}_2 \sum_{c=1}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{(c+2)^2}{2n} - \frac{\pi^2}{2(c+2)^2} N} \\ &\le \frac{\overline{K}_3}{\sqrt{n}} \int_0^\infty e^{-\frac{u^2}{2n} - \frac{\pi^2}{2u^2} N} du = \overline{K}_4 e^{-\pi \sqrt{\frac{N}{n}}}, \end{aligned} \tag{18}$$

where $\overline{K}_3$ and $\overline{K}_4$ are appropriate constants. For the evaluation of the integral in (18) see [3]. By putting $N = ((1+\varepsilon)/\pi^2) n (\log\log n)^2$ and using that $\nu_n + n$ is increasing, one can complete the proof of (13) by the usual Borel-Cantelli argument.

To show

$$\limsup_{n\to\infty} \frac{\nu_n}{n(\log\log n)^2} \ge \frac{1}{\pi^2} \quad \text{a.s.} \tag{19}$$

we estimate the following probability in (9) from below.

$$\begin{aligned} P(\nu_n > N) &\ge \\ &\ge \sum_{c=\left[\frac{\sqrt{n}}{\log n}\right]}^{[\sqrt{n}\log n]} \sum_{a=0}^{c} \sum_{k=-a+\left[\frac{c}{4}\right]}^{-a+\left[\frac{3c}{4}\right]} P(m_n = -a,\ R_n = c,\ S_n = k) \\ &\quad \times P(-a-k \le S_j \le c-a-k,\ j = 1, \dots, N). \end{aligned} \tag{20}$$

For these values of $c$, $a$ and $k$ one can see from (10) and (12) that there exist absolute positive constants $\underline{K}_1$ and $\underline{K}_2$ such that

$$P(-a-k \le S_j \le c-a-k,\ j = 1, \dots, N) \ge \underline{K}_1 \left(\cos \frac{\pi}{c+2}\right)^N \tag{21}$$

and

$$\sum_{a=0}^{c}\sum_{k=-a+\left[\frac{c}{4}\right]}^{-a+\left[\frac{3c}{4}\right]} P(m_n=-a,\ R_n=c,\ S_n=k)\geq \frac{K_2}{\sqrt{n}}e^{-\frac{(c+2)^2}{2n}}. \tag{22}$$

Hence

$$P(\nu_n>N)\geq K_1K_2\sum_{c=\left[\frac{\sqrt{n}}{\log n}\right]}^{[\sqrt{n}\log n]}\frac{1}{\sqrt{n}}e^{-\frac{(c+2)^2}{2n}}\left(\cos\frac{\pi}{c+2}\right)^N. \tag{23}$$

Now for $n$ large enough and $n\leq N\leq n\log n$

$$\left(\cos\frac{\pi}{c+2}\right)^N\geq\left(1-\frac{\pi^2}{2(c+2)^2}\right)^N\geq K_3e^{-\frac{\pi^2N}{2(c+2)^2}}, \tag{24}$$

provided $\sqrt{n}/\log n\leq c\leq\sqrt{n}\log n$, thus

$$\begin{aligned}P(\nu_n>N)&\geq K_4\int_{\frac{\sqrt{n}}{\log n}}^{\sqrt{n}\log n}\frac{1}{\sqrt{n}}e^{-\frac{u^2}{2n}-\frac{\pi^2}{2u^2}N}du\geq\\
&\geq K_4\left(\int_0^\infty\frac{1}{\sqrt{n}}e^{-\frac{u^2}{2n}-\frac{\pi^2}{2u^2}N}du-\int_0^{\frac{\sqrt{n}}{\log n}}\frac{1}{\sqrt{n}}e^{-\frac{\pi^2}{2u^2}N}du\right.\\
&\qquad\left.-\int_{\sqrt{n}\log n}^\infty\frac{1}{\sqrt{n}}e^{-\frac{u^2}{2n}}du\right)\\
&\geq K_5e^{-\pi\sqrt{\frac{N}{n}}}-\frac{K_6}{n^2}.\end{aligned} \tag{25}$$

Now we are ready to show (19). To this end define the events:

$$A_k=\{\min_{T_{k-1}\leq i\leq T_{k-1}+n_k}S_i\leq S_j\leq\max_{T_{k-1}\leq i\leq T_{k-1}+n_k}S_i,\quad j=T_{k-1}+n_k+1,\dots,T_k\} \tag{26}$$

where

$$n_k=e^{k(\log(k+1))^2}, \tag{27}$$

$$N_k=\frac{1}{\pi^2}n_k(1-\varepsilon)^2(\log\log n_k)^2, \tag{28}$$

$$T_k=\sum_{\ell=1}^{k}(n_\ell+N_\ell). \tag{29}$$

Then the events $\{A_k\}_{k=1}^{\infty}$ are independent and (25) implies that

$$\sum_{k=1}^{\infty} P(A_k) = \infty, \tag{30}$$

hence

$$P(A_k \text{ i.o.}) = 1. \tag{31}$$

This implies that

$$P(\nu_{T_{k-1}+n_k} \geq N_k \text{ i.o.}) = 1. \tag{32}$$

Since $T_{k-1} \ll n_k$, (19) follows, completing the proof of the Theorem.

## REFERENCES

[1] ERDŐS, P. and RÉVÉSZ, P., Three problems on the random walk in $\mathbf{Z}^d$, *Studia Sci. Math. Hungar.* **26** (1991),

[2] FELLER, W., *An introduction to probability theory and its applications*, Vol. 1, 3rd ed., J. Wiley, New York, 1968. *MR* **37** # 3604.

[3] GRADSHTEĬN, I. S. and RYZHIK, I. M., *Table of integrals, series, and products*, 4th ed., Academic Press, New York, 1980. *MR* **81g**: 33001.

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# NEW AXIOMATIC RESULTS ON INFERENCE FOR INVERSE PROBLEMS

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**Abstract**

The axiomatic approach to inference for linear inverse problems developed by Csiszár [3] has led, among others, to intuitively appealing characterizations of the methods of least squares and maximum entropy. Here some further results with this approach are presented, including an axiomatic characterization of the method of minimizing an $\ell^p$-distance $(1 < p < \infty)$. We also study selection and projection rules for a larger resp. smaller class of possible feasible sets than in the reference above. In particular, certain non-linear inverse problems are also covered.

## 1. Introduction

Often, an unknown function $f$ has to be inferred from known values of certain linear functionals $R_i f$, $i = 1, \dots, k$. A typical example of such "linear inverse problems" is image reconstruction, e.g., in computerized tomography. In X-ray tomography, the unknown X-ray attenuation function $f$ is inferred from its line integrals $R_i f$ along the pathes of the rays. Another example is when a probability density or mass function $f$ has to be inferred from the knowledge of certain moments; then $R_i f$ represents the expectation of some known function of the underlying random variable.

Since the known constrains typically do not determine $f$ uniquely, the inference problem is solved by using a more or less arbitrary *selection rule* to pick one element of the set of feasible functions. If this selection depends also on a "prior guess" of the unknown $f$, the selected function is considered an abstract projection of the prior guess onto the feasible set, and we speak of a *projection rule*. The obvious question of what selection (projection) rule is best is hard to give a mathematical meaning. Csiszár [3] suggested an axiomatic approach, based on the intuitive idea that a "good" rule, when

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applied to a class of problems, should lead to "logically consistent" inferences. The latter is interpreted to mean that the selection or projection rule has to satisfy certain intuitively appealing postulates. It should be emphasized that this approach does not involve stochastics. In the above paper, selection and projection rules satisfying various postulates were characterized. In particular, the "least squares" and "minimum I-divergence" (or maximum entropy) rules were arrived at from small sets of natural axioms.

Here, this axiomatic investigation will be continued. In order to keep the paper self-contained, the basic definitions and some results of Csiszár [3] will be briefly reviewed in Section 2. As there, for technical convenience we concentrate on the discrete case (i.e., the functions to be inferred are represented by finite-dimensional vectors)and we consider in parallel three different choices for the basic set $S$ of all potentially permissible vectors; these three cases represent inverse problems where the unknown function can be any real-valued function, or any non-negative function, or a probability mass function, respectively.

In the above reference, it was assumed that the available information consisted in linear equality constrains, and any set of vectors defined by such constrains could arise as a feasible set. In Section 3 we prove some further results under the same basic assumptions, including an axiomatic characterization of the selection and projection rules defined by minimizing an $\ell^p$-distance $(1 < p < \infty)$. We also show how all such results extend to the case when the closed convex sets are the possible feasible sets, whereby some non-linear inverse problems are also covered.

In Section 4 we study selection and projection rules for a much smaller class of possible feasible sets, corresponding to particularly simple linear inverse problems, and also prove results about the possibility of extending them to "good" selection (projection) rules for the class of all linear inverse problems.

Finally, in Section 5 we indicate some problems for future research.

Our axiomatic approach to inverse problems has been motivated mainly by Shore and Johnson [11]. For further relevant references cf. Csiszár [3]. It is appropriate to add that, indirectly, Rényi's work on information measures also provided motivation for this investigation. Indeed, Rényi's views on the "pragmatic" (operational) and axiomatic approaches to the problem of measuring information substantially influenced the author's thinking. Rényi [10] wrote: "These two points of view are according to the opinion of the author of this paper not as opposed to each other as they seem to be; they are compatible and even complement each other and therefore both deserve attention. Both of the mentioned approaches may and should be used as a control to the other". Rényi also emphasized that the operational significance of information measures need not be restricted to communication and coding. As inverse problems represent one of the fields where information measures are being successfully applied (even though no intrinsic relationship of this field to information theory is apparent), it was clearly desirable to complement this "pragmatic" side by an axiomatic approach.

It may be interesting to point out that our original goal was to give an improved axiomatic "justification" of the method of minimum I-divergence and it was a welcome result of this effort that axiomatic characterizations of other standard (such as "least squares") or potentially useful methods could also be obtained. Needless to say, I-divergence was not distinguished by its being an information measure, rather, it emerged as a measure of distance whose minimization gave rise to a particularly attractive projection rule. Some new families of distances obtained by our approach may also be useful in certain applications, cf. the remarks after Theorems 2.3 and 3.2. Of course, not all selection and projection rules we have axiomatically characterized are expected to withstand a "pragmatic control". It is likely that many of our results will remain of mathematical interest only.

## 2. Review of definitions and results from Csiszár [3]

The real line and the positive half-line are denoted by $R$ and $R_+$, respectively; the latter does not contain zero. The vectors in $R^n$ whose components are all zero or all one are denoted by $\mathbf{0}$ or $\mathbf{1}$, respectively. All vectors are column vectors. The set of $n$-dimensional vectors with positive components of sum 1 is denoted by $\Delta_n$, i.e.,

(2.1) $$\Delta_n = \{\mathbf{v} : \mathbf{v} \in R_+^n,\ \mathbf{1}^T\mathbf{v} = 1\}.$$

We consider three cases in parallel, namely our basic set $S$ of all potentially permissible vectors is either of $R^n$, $R_+^n$ or $\Delta_n$ where $n \geq 3$ or, if $S = \Delta_n$, $n \geq 5$. According to the three cases, $V$ will denote $R, R_+$ or the open interval $(0,1)$, respectively. Unless stated otherwise, $u, v$ and $w$ will always denote elements of $V$, and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $V^n$. Further, $\mathcal{L}$ denotes the family of non-void subsets of $S$ defined by linear constraints. Thus $L \in \mathcal{L}$ iff

(2.2) $$L = \{\mathbf{v} : \mathbf{A}\mathbf{v} = \mathbf{b}\} \neq \phi$$

for some $k \times n$ matrix $\mathbf{A}$ and some $\mathbf{b} \in R^k$; in the case $S = \Delta_n$ it is assumed that $\mathbf{A}\mathbf{v} = \mathbf{b}$ implies $\mathbf{1}^T\mathbf{v} = 1$.

A *selection rule* (with basic set $S$) is a mapping $\Pi : \mathcal{L} \to S$ such that $\Pi(L) \in L$ for every $L \in \mathcal{L}$. A *projection rule* is a family of selection rules $\Pi(\cdot|\mathbf{u})$, $\mathbf{u} \in S$, such that $\mathbf{u} \in L$ implies $\Pi(L|\mathbf{u}) = \mathbf{u}$.

A selection rule $\Pi$ is *generated* by a function $F(\mathbf{v})$, $\mathbf{v} \in S$, if for every $L \in \mathcal{L}$, $\Pi(L)$ is the unique element of $L$ where $F(\mathbf{v})$ is minimized subject to $\mathbf{v} \in L$. A projection rule is generated by a function $F(\mathbf{v}|\mathbf{u})$, $\mathbf{u} \in S$, $\mathbf{v} \in S$, if its component selection rules are generated by the functions $F(\cdot|\mathbf{u})$.

If a projection rule is generated by some function, it is also generated by a *measure of distance* on $S$, i.e., by a function with the property $F(\mathbf{v}|\mathbf{u}) \geq 0$, with equality iff $\mathbf{v} = \mathbf{u}$.

For any set of indices $J = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ and any vector $\mathbf{a} \in R^n$, we denote by $\mathbf{a}_J$ the vector in $R^k$ defined by

$$\mathbf{a}_J = (a_{j_1}, \dots, a_{j_k})^T. \tag{2.3}$$

For a selection rule $\Pi$, we denote $(\Pi(L))_J$ briefly by $\Pi_J(L)$.

Our basic axioms on selection rules are the following:

(1) (consistency) if $L' \subset L$ and $\Pi(L) \in L'$ then $\Pi(L') = \Pi(L)$;

(2) (distinctness) if $L \neq L'$ are both $n-1$-dimensional (or $n-2$-dimensional if $S = \Delta_n$) then $\Pi(L) \neq \Pi(L')$ unless both $L$ and $L'$ contain $\mathbf{v}^\circ = \Pi(S)$;

(3) (continuity) the restriction of $\Pi$ to any subclass of $\mathcal{L}$ consisting of sets of equal dimension is continuous;

(4) (locality) if $L \in \mathcal{L}$ is defined by a matrix $\mathbf{A}$ in (2.2) such that for some $I \subset \{1, \dots, k\}$, $J \subset \{1, \dots, n\}$ we have $a_{ij} = 0$ whenever $(i, j) \in (I \times J^c) \cup (I^c \times J)$, then $\Pi_J(L)$ depends on $\mathbf{A}$ and $\mathbf{b}$ through $(a_{ij})_{i \in I, j \in J}$ and $\mathbf{b}_I$ only.

For projection rules, these postulates are required to hold for all component selection rules $\Pi(\cdot|\mathbf{u})$, $\mathbf{u} \in S$ (notice that in (2), $\Pi(S|\mathbf{u}) = \mathbf{u}$ by definition), and in (4) it is additionally required that $\Pi_J(L|\mathbf{u})$ depends on $\mathbf{u}$ through $\mathbf{u}_J$ only.

As explained in Csiszár [3], axioms (1) and (4) are intuitively compelling (though not necessarily for all kinds of inference problems), and axiom (3) is an obvious regularity condition. Axiom (2), needed for a technical reason, is intuitively less compelling than the others. It would be desirable if it could be dispensed with.

The key result of Csiszár [3] was the following theorem. In that theorem, the term *standard $n$-tuple* with zero at $\mathbf{v}^\circ$ means an $n$-tuple of functions $(f_1, \dots, f_n)$ defined on $V$ such that

(i) each $f_i$ is continuously differentiable and

$$f_i(v_i^\circ) = f_i'(v_i^\circ) = 0, \quad i = 1, \dots, n;$$

(ii) in the cases $S = R_+^n$ or $\Delta_n$, $f_i'(v) \to -\infty$ as $v \to 0$;

(iii) $F(\mathbf{v}) = \sum_{i=1}^n f_i(v_i)$ is non-negative and strictly quasi-convex on $S$, i.e., for any $\mathbf{v}$ and $\mathbf{v}'$ in $S$

$$F(\alpha\mathbf{v} + (1-\alpha)\mathbf{v}') < \max(F(\mathbf{v}), F(\mathbf{v}')), \quad 0 < \alpha < 1. \tag{2.4}$$

THEOREM 2.1. (a) *If a selection rule $\Pi : \mathcal{L} \to S$ satisfies the basic axioms* (1)-(4) *then it is generated by a function*

$$F(\mathbf{v}) = \sum_{i=1}^n f_i(v_i) \tag{2.5}$$

*where $(f_1, \dots, f_n)$ is a standard n-tuple with zero at $\mathbf{v}^\circ = \Pi(S)$. Conversely, if $(f_1, \dots, f_n)$ is a standard n-tuple with zero at $\mathbf{v}^\circ$ then* (2.5) *generates a selection rule with $\Pi(S) = \mathbf{v}^\circ$ that satisfies the basic axioms.*

(b) *If a projection rule satisfies the basic axioms then it is generated by a measure of distance*

$$F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^n f_i(v_i|u_i) \tag{2.6}$$

*where the functions $f_1(\cdot|u_1), \dots, f_n(\cdot|u_n)$ form a standard n-tuple with zero at* $\mathbf{u}$. *Conversely, any such measure of distance generates a projection rule satisfying the basic axioms.*

(c) *Two functions $F$ and $\tilde{F}$ as in* (a) *or* (b) *generate the same selection or projection rule iff their terms $f_i$ and $\tilde{f}_i$ satisfy $f_i = c\tilde{f}_i$, $i = 1, \dots, n$, for some constant $c > 0$.*

REMARK. If a selection rule $\Pi : \mathcal{L} \to S$ satisfies the basic axioms, its generating function (2.5) also has the following property:

(iv) $\operatorname{grad} F(\mathbf{v}) \neq \mathbf{0}$, and in the case $S = \Delta_n$ also $\operatorname{grad} F(\mathbf{v}) \neq \lambda\mathbf{1}$, for all $\mathbf{v} \in S$ with $\mathbf{v} \neq \mathbf{v}^\circ$.

Indeed, this has been established in the mentioned reference, cf. eq. (5.20). On the other hand, for the converse assertion in Theorem 2(a) property (iv) is needed to check axiom (2). Thus, unless the properties (i)-(iii) in the definition of a standard $n$-tuple already imply (iv), the latter has to be explicitely added to fill a minor gap in Theorem 2.1. It remains open whether (i)-(iii) imply (iv) if $S = \Delta_n$. In the cases $S = R^n$ and $R^n_+$, however, it is an immediate consequence of Theorem 3.1 in the next Section that (i)-(iii) do imply (iv).

The class of functions $F$ occurring in Theorem 2.1 can be restricted by imposing some further postulates. We recall the intuitively appealing postulates of invariance and transitivity, for projection rules. Another highly intuitive postulate, called composition consistency, that lead to the perhaps most interesting result of Csiszár [3], will not be recalled here because it will not be used in this paper.

(5a) (scale invariance, for $S = R^n$ or $R^n_+$)

$$\Pi(\lambda L|\lambda \mathbf{u}) = \lambda \Pi(L|\mathbf{u}) \text{ for every } L \in \mathcal{L},\ \lambda > 0,\ \mathbf{u} \in S$$

(5b) (translation invariance, for $S = R^n$)

$$\Pi(L + \mu \mathbf{1}|\mathbf{u} + \mu \mathbf{1}) = \Pi(L|\mathbf{u}) + \mu \mathbf{1} \text{ for every } L \in \mathcal{L},\ \mu \in R,\ \mathbf{u} \in S.$$

REMARKS. Selection rules can also be scale invariant, i.e., satisfy $\Pi(\lambda L) = \lambda \Pi(L)$ for every $L \in \mathcal{L}$ and $\lambda > 0$, but only in the case $S = R^n$ (because $L = S$ yields $\Pi(S) = \mathbf{0}$). Translation invariance is not possible for selection rules.

It should be mentioned that in the case $S = R^n$, postulate (5a) and its analog for selection rules could be imposed also in a stronger form, for $\lambda < 0$ as well. This stronger postulate, called ***strong scale invariance***, will be used in Section 3.

(6a) (subspace transitivity) for every $L' \subset L$ and $\mathbf{u} \in S$

$$\Pi(L'|\mathbf{u}) = \Pi(L'|\Pi(L|\mathbf{u}))$$

(6b) (parallel transitivity) for every $L$ and $L'$ defined as in (2.2) with the same matrix $\mathbf{A}$, and for every $\mathbf{u} \in S$

$$\Pi(L'|\mathbf{u}) = \Pi(L'|\Pi(L|\mathbf{u})).$$

The main results invoving these postulates were:

THEOREM 2.2. *A projection rule satisfies the basic axioms* (1)-(4) *and the transitivity postulate* (6a) *iff it is generated by*

$$F(\mathbf{v}|\mathbf{u}) = \Phi(\mathbf{v}) - \Phi(\mathbf{u}) - (\mathrm{grad}\Phi(\mathbf{u}))^T(\mathbf{v}-\mathbf{u}) \tag{2.7}$$

*where* $\Phi(\mathbf{v}) = \sum_{i=1}^n \varphi_i(v_i)$, *the functions* $\varphi_i$ *defined on* $V$ *are continuously differentiable,* $\Phi(\mathbf{v})$ *is strictly convex on* $S$, *and in the cases* $S = R_+^n$ *or* $\Delta_n$, $\lim_{v\to 0}\varphi_i'(v) = -\infty$. *Further, subject to the basic axioms* (1)-(4), *the transitivity postulates* (6a) *and* (6b) *are equivalent.*

REMARK. Measures of distance associated with strictly convex functions $\Phi$ as in (2.7) were introduced by Bregman [2].

THEOREM 2.3. (a) *In the case* $S = R^n$, *a projection rule as in Theorem* 2.2 *is location and scale invariant iff it is generated by*

$$F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^n a_i(v_i - u_i)^2, \tag{2.8}$$

*for certain positive constants* $a_1, \ldots, a_n$.

(b) *In the case* $S = R_+^n$, *a projection rule as in Theorem* 2.2 *is scale invariant iff it is generated by*

$$F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^n a_i h_\alpha(v_i|u_i), \qquad \alpha \le 1 \tag{2.9}$$

*where*

$$h_\alpha(v|u) = \begin{cases} v\log\frac{v}{u} - v + u & \text{if } \alpha = 1 \\ \log\frac{u}{v} + \frac{v}{u} - 1 & \text{if } \alpha = 0 \\ \frac{1}{\alpha}(u^\alpha - v^\alpha) + u^{\alpha-1}(v-u) & \text{otherwise} \end{cases} \tag{2.10}$$

*and* $a_1, \ldots, a_n$ *are positive constants.*

REMARKS. Part (a) is an axiomatic characterization of ordinary orthogonal projections for weighted $\ell^2$-norms. The distences (2.9) with $a_1 = \ldots = a_n = 1$ resemble the $\alpha$-divergences of Rényi [9]; still, a closer relationship of the two families is not apparent, except that both contain (for $\alpha = 1$) Kullback's I-divergence or "information for discrimination" (Kullback [8]). The family (2.9) contains the Itakura-Saito [5] distance, as well ($\alpha = 0$), and there are indications that other members of this family may also be practically useful, cf. Jones and Trutzer [7].

## 3. Some further results

In this Section, we first show that for selection rules $\Pi : \mathcal{L} \to S$ satisfying the basic axioms (1)-(4), the generating function (2.4) is necessarily convex if $S$ equals $R^n$ or $R^n_+$; in Csiszár [3] this was mentioned without proof. Then we determine how the invariance postulates (5a), (5b) restrict the class of possible generating functions when – unlike in Theorem 2.3 – a transitivity postulate is not imposed. Finally, we show that a simple additional axiom permits to uniquely extend any selection or projection rule, satisfying the basic axioms, from $\mathcal{L}$ to the class $\mathcal{C}$ of all closed convex subsets of $S$.

THEOREM 3.1. *Let $f_1, \ldots, f_n$ be continuously differentiable functions on $V = R$ or $R_+$ having the properties* (i) *and* (iii) *in the definition of a standard $n$-tuple. Then each $f_i$ is convex and there can be at most one $i$ for which $f_i$ is not strictly convex.*

PROOF. Since $F(\mathbf{v}) = \sum_{i=1}^n f_i(v_i)$ satisfies (2.4), for $i \neq j$ there can not exist intervals $I_i$ and $I_j$ such that $f_i$ is linear in $I_i$ and $f_j$ is linear in $I_j$. Hence it suffices to prove that each $f_i$ is convex and, clearly, this will be done if we show that $f_1$ is convex.

From the non-negativity and strict quasi-convexity of $F(\mathbf{v})$ together with the assumption that 0 is a possible value of $f_i$ for each $i$, it follows that $\tilde{F}(u,v) = f_1(u) + f_2(v)$ is strictly quasi-convex on $V^2$, and $\tilde{F}(u,v) > 0$ except for the point $(u_0, v_0)$ where $f_1(u_0) = f_2(v_0) = 0$. It follows, in particular, that $f_1(u)$ and $f_2(v)$ are strictly increasing for $u > u_0$ and $v > v_0$, and they are strictly decreasing for $u < u_0$ and $v < v_0$. Further, the level sets

$$A_c = \{(u,v) : f_1(u) + f_2(v) \leq c\}, \quad c > 0 \tag{3.1}$$

of $\tilde{F}$ are strictly convex, i.e., any convex combination of any two points of $A_c$ are in the interior of $A_c$. We denote

$$I_c = \{u : f_1(u) \leq c\}, \quad c > 0. \tag{3.2}$$

First we show that

$$\lim_{v \to \infty} f_2(v) = \infty. \tag{3.3}$$

Supposing, indirectly, that $\lim_{v\to\infty} f_2(v) = c < \infty$, we would have $(u_0, v) \in A_c$ for all $v \geq v_0$, and also we could find $u_1 > u_0, v_1 > v_0$ such that $f_1(u_1) + f_2(v_1) = c$. But then $(u_1, v) \notin A_c$ if $v > v_1$, and this gives the desired contradiction. Indeed, if $(u_0, v) \in A_c$ for all $v \geq v_0$, the convexity of $A_c$

implies that also $(u_1, v) \in A_c$ for all $v \geq v_0$, because $(u, v_0) \in A_c$ for all $u \in I_c$.

Consider now the function $v = v_c(u)$ defined in the interval $I_c$ by

$$f_1(u) + f_2(v) = c, \quad v \geq v_0; \tag{3.4}$$

on account of (3.3), this function is well-defined for any $c > 0$, and its graph is the upper boundary of the set $A_c$. Thus $v = v_c(u)$ is a strictly concave function, in particular, its derivative can vanish only at its unique maximum, i.e., at $u = u_0$. As

$$\frac{dv_c}{du} = -\frac{f_1'(u)}{f_2'(v_c)} \tag{3.5}$$

if $f_2'(v_c) \neq 0$, this proves that $f_1'(u) \neq 0$ if $u \neq u_0$ (because, given $u \neq u_0$, picking any $v > v_0$ with $f_2'(v) \neq 0$, (3.5) certainly applies with the choice $c = f_1(u) + f_2(v)$). Since $f_1(u)$ is strictly increasing (decreasing) for $u > u_0$ ($u < u_0$), it follows that $f_1'(u) \gtrless 0$ as $u \gtrless u_0$. By symmetry, we thus also have $f_2'(v) > 0$ if $v > v_0$, and it follows that (3.5) holds for all $u$ in the interior of $I_c$.

Now let $u$ and $\overline{u}$ be arbitrary such that $\overline{u} < u < u_0$ or $u_0 < u < \overline{u}$. The convexity of $f_1(u)$ will be proved if we show that in both cases

$$\frac{f_1'(u)}{f_1'(\overline{u})} \leq 1. \tag{3.6}$$

Let $v$ and $\overline{v}$ satisfy

$$f_1(u) + f_2(v) = f_1(\overline{u}) + f_2(\overline{v}), \quad v > v_0, \overline{v} > v_0. \tag{3.7}$$

Taking for $c$ the common value of both sides of (3.7), the strict concavity of the function $v_c(u)$ and (3.5) imply that

$$\frac{f_1'(\overline{u})}{f_2'(\overline{v})} < \frac{f_1'(u)}{f_2'(v)} < 0 \quad \text{if} \quad \overline{u} < u < u_0$$

and the reversed inequalities hold if $u_0 < u < \overline{u}$. It follows that in both cases

$$\frac{f_1'(u)}{f_1'(\overline{u})} < \frac{f_2'(v)}{f_2'(\overline{v})}. \tag{3.8}$$

The right hand side of (3.8) equals the derivative of the function $\overline{v} = \overline{v}(v)$ defined by (3.7) (on account of (3.3), this function is well defined for any fixed $u$ and $\overline{u}$ if $v$ is sufficiently large). Since (3.7) implies $\overline{v} < v$ in both cases $\bar{u} < u < u_0$ and $u_0 < u < \bar{u}$, this derivative can not have a lower bound greater than 1. Thus, (3.8) proves (3.6), and this completes the proof of Theorem 3.1.

We recall from Csiszár [3] the notation

$$(3.9)\quad L_{ij}(t) = \begin{cases} \{\mathbf{v} : v_i + v_j = t\} & \text{if} \quad S = R^n \text{ or } R^n_+ \\ \{\mathbf{v} : v_i + v_j = t, \sum_{\ell \neq i,j} v_\ell = 1 - t\} & \text{if} \quad S = \Delta_n. \end{cases}$$

For a selection or projection rule generated by $F(\mathbf{v})$ resp. $F(\mathbf{v}|\mathbf{u})$ as in (2.5), (2.6), the $i$-th and $j$-th components $v_i$ and $v_j$ of $\Pi(L_{ij}(t))$ resp. $\Pi(L_{ij}(t)|\mathbf{u})$ are determined by the equations

$$(3.10)\qquad f_i'(v_i) = f_j'(v_j), \quad v_i + v_j = t$$

$$(3.11)\qquad f_i'(v_i|u_i) = f_j'(v_j|u_j), \quad v_i + v_t = t.$$

A projection rule will be called *smooth* if for every $i \neq j$ and $t \in V$, the $i$-th and $j$-th components of $\Pi(L_{ij}(t)|\mathbf{u})$ depend continuously on $u_i$ and $u_j$. While smoothness is a natural regularity postulate, it was not needed in Csiszár [3]. It will be used here, in assertions (b) and (d) of Theorem 3.2 below. In that Theorem, by "selection rule" or "projection rule" we mean selection or projection rule satisfying the basic axioms (1)-(4).

THEOREM 3.2. (a) *A selection rule with basic set $S = R^n$ is scale invariant iff it is generated by*

$$(3.12)\qquad F(\mathbf{v}) = \sum_{i=1}^{n} c_i(\mathrm{sign} v_i)|v_i|^p, \quad p > 1,$$

*where $c_i(\mathrm{sign} v_i)$ denotes a positive coefficient depending on $i$ and the sign of $v_i$. Further, this selection rule is strongly scale invariant iff the coefficients do not depend on the sign of $v_i$, i.e., iff*

$$(3.13)\qquad F(\mathbf{v}) = \sum_{i=1}^{n} a_i|v_i|^p, \quad p > 1.$$

(b) *A projection rule with $S = R^n_+$ is smooth and scale invariant iff it is generated by*

$$(3.14)\qquad F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^{n} u_i^\alpha f_i\left(\frac{v_i}{u_i}\right), \quad \alpha \in R,$$

*where $(f_1, \ldots, f_n)$ is a standard $n$-tuple with zero at* **1**.

(c) *A projection rule with $S = R^n$ is scale invariant iff it is generated by*

$$(3.15)\quad F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^{n} f_i(v_i|u_i); \quad f_i(v|u) = \begin{cases} u^p f_i^+(\frac{v}{u}) & \text{if } u > 0 \\ |u|^p f_i^-(\frac{v}{u}) & \text{if } u < 0 \\ c_i(\text{sign} v_i)|v_i|^p & \text{if } u = 0 \end{cases}$$

*where $(f_1^+, \ldots, f_n^+)$ and $(f_1^-, \ldots, f_n^-)$ are standard n-tuples with zero at* $\mathbf{1}$, *the* $c_i(\text{sign} v_i)$ *are as in part* (a), *and* $p > 1$. *This projection rule is strongly scale invariant iff here $f_i^+ = f_i^-$ and the coefficients* $c_i(\text{sign} v_i)$ *do not depend on* $\text{sign} v_i$.

(d) *A projection rule with $S = R^n$ is smooth and translation invariant iff it is generated by*

$$(3.16)\qquad F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^{n} e^{\beta u_i} f_i(v_i - u_i), \quad \beta \in R$$

*where $(f_1, \ldots, f_n)$ is a standard n-tuple with zero at* $\mathbf{0}$.

(e) *A projection rule with $S = R^n$ is both scale and translation invariant iff it is generated by*

$$(3.17)\qquad F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^{n} c_i(\text{sign}(v_i - u_i))|v_i - u_i|^p, \quad p > 1.$$

*It is also strongly scale invariant iff here the coefficients do not depend on the sign of $v_i - u_i$, i.e., iff*

$$(3.18)\qquad F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^{n} a_i|v_i - u_i|^p, \quad p > 1.$$

REMARKS. Assertions (a) and (e) provide axiomatic characterizations of the families of (weighted) $\ell^p$-norms and $\ell^p$-distances, respectively. We notice that the projection rules characterized in (c) are not necessarily smooth because the functions $f_i(v|u)$ in (3.15) may be discontinuous in $u$ at $u = 0$. The straightforward conditions needed for smoothness are omitted.

Among the families of projection rules characterized in Theorem 3.2, only those generated by $\ell^p$-distances, viz. (3.18), look familiar. It appears to this author that the distances (3.17), whose $p$-th root may be called a "*skewed $\ell^p$-distance*", also deserve interest. They might turn out useful in such problems where positive and negative discrepancies from the "prior guess" $\mathbf{u}$ are not equally significant.

PROOF OF THEOREM 3.2. (a) Consider a scale-invariant selection rule $\Pi$ with basic set $S = R^n$, generated as in Theorem 2.1 by $F(\mathbf{v}) = \sum_{i=1}^n f_i(v_i)$, where $(f_1, \ldots, f_n)$ is a standard $n$-tuple. The scale-invariance $\Pi(\lambda L) = \lambda\Pi(L)$ means that $F(\mathbf{v})$ is minimized subject to $\mathbf{v} \in \lambda L$ by $\hat{\mathbf{v}} = \lambda \mathbf{v}^*$ iff $\min_{\mathbf{v}\in L} F(\mathbf{v})$ is attained at $\mathbf{v} = \mathbf{v}^*$. Thus $F_\lambda(\mathbf{v}) = F(\lambda\mathbf{v})$ also generates $\Pi$. Since the scale invariance of $\Pi$ implies $\Pi(S) = \mathbf{0}$, the standard $n$-tuple $(f_1, \ldots, f_n)$ has zero at $\mathbf{0}$, and therefore the functions $f_1(\lambda v), \ldots, f_n(\lambda v)$ also form a standard $n$-tuple with zero at $\mathbf{0}$. It follows by Theorem 2.1 (c) that

$$f_i(\lambda v) = c(\lambda) f_i(v), \quad i = 1, \ldots, n. \tag{3.19}$$

Clearly, this implies that the factors $c(\lambda)$ satisfy

$$c(\lambda_1\lambda_2) = c(\lambda_1)c(\lambda_2) \tag{3.20}$$

for all $\lambda_1 > 0$, $\lambda_2 > 0$, and $c(\lambda)$ is a continuous function of $\lambda$. Hence (cf. Aczél [1], Section 2.1.2)

$$c(\lambda) = \lambda^\alpha, \quad \alpha \in R. \tag{3.21}$$

Substituting in (3.19) $\lambda = |v|^{-1}$, it follows with (3.21) that $f_i(v)$ equals $v^\alpha f_i(1)$ if $v > 0$ and $|v|^\alpha f_i(-1)$ if $v < 0$. Writing $p = \alpha$, this establishes (3.12); the condition $p > 1$ comes from the property of a standard $n$-tuple that its component functions are continuously differentiable. Conversely, it is clear that the function (3.12) generates a scale-invariant selection rule, and the latter is strongly scale invariant iff $c_i(\text{sign} v_i)$ does not depend on $\text{sign} v_i$.

(b) Consider a scale invariant projection rule with basic set $S = R^n_+$, generated as in Theorem 2.1 (b) by $F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^n f_i(v_i|u_i)$. It follows as in part (a) that $F_\lambda(\mathbf{v}|\mathbf{u}) = F(\lambda\mathbf{v}|\lambda\mathbf{u})$ also generates this projection rule, hence by Theorem 2.1 (c)

$$f_i(\lambda v|\lambda u) = c(\lambda) f_i(v|u), \quad i = 1, \ldots, n \tag{3.22}$$

for every $\lambda > 0$.

Again, (3.22) implies the functional equation (3.20) for $c(\lambda)$ but unlike in part (a), the continuity of $c(\lambda)$ does not follow from (3.22). We claim, however, that the smoothness postulate already implies the continuity of $c(\lambda)$.

To verify this, fix $i \neq j$, $u$, $\overline{u}$ and $t$, and denote the $i$-th component of $\Pi(L_{ij}(t)|\mathbf{u})$ by $v_\lambda$ if the $i$-th and $j$-th components of $\mathbf{u}$ are $\lambda u$ and $\overline{u}$, respectively. Then by (3.11)

$$f_i'(v_\lambda|\lambda u) = f_j'(t - v_\lambda|\overline{u}). \tag{3.23}$$

Next, differentiate (3.22) by $v$ and then substitute $v = \lambda^{-1}v_\lambda$ to obtain

$$\lambda f_i'(v_\lambda|\lambda u) = c(\lambda) f_i'(\lambda^{-1}v_\lambda|u). \tag{3.24}$$

Since $v_\lambda$ is a continuous function of $\lambda$, by smoothness, and $f_i'(v|u)$ is continuous in $v$ for every fixed $u$, it follows from (3.23) and (3.24) that $c(\lambda)$ is continuous, as claimed.

Having established the continuity of $c(\lambda)$, we again have (3.21). Then, substituting $\lambda = u^{-1}$, (3.22) gives that

$$f_i(v|u) = u^\alpha f_i(\frac{v}{u}|1).$$

Denoting $f_i(t|1)$ by $f_i(t)$, this completes the proof of (3.14).

(c) Changing the setup of (b) to $S = R^n$, the proof remains almost identical except that now we get for free (without requiring smoothness) that $c(\lambda) = \lambda^p$ for some $p > 1$, namely by applying the result of part (a) to the scale invariant selection rule $\Pi(\cdot|\mathbf{0})$.

(d) Consider a translation invariant projection rule with basic set $S = R^n$ generated, as in Theorem 2.1 (b), by $F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^n f_i(v_i|u_i)$. It follows as in (a) and (b) that $F_\mu(\mathbf{v}|\mathbf{u}) = F(\mathbf{v}+\mu\mathbf{1}|\mathbf{u}+\mu\mathbf{1})$ also generates this projection rule (for any $\mu \in R$), and therefore by Theorem 2.1 (c)

$$f_i(v+\mu|u+\mu) = c(\mu) f_i(v|u), \quad i = 1, \dots, n \tag{3.25}$$

This implies that $c(\mu)$ satisfies the functional equation

$$c(\mu_1 + \mu_2) = c(\mu_1)c(\mu_2), \tag{3.26}$$

and as in part (b), smoothness implies the continuity of $c(\mu)$. Hence (cf. Aczél (1966), Section 2.1.2) we have $c(\mu) = e^{\beta\mu}$.

Finally, substituting $\mu = -u$ in (3.25) results in

$$f_i(v|u) = e^{\beta u} f_i(v-u|0).$$

Writing $f_i(t) = f_i(t|0)$, this establishes (3.16).

(e) If a projection rule with basic set $S = R^n$ is both scale and translation invariant then, by (c) and (d), its generating function $F(\mathbf{v}|\mathbf{u})$ can be represented both as in (3.15) and as in (3.16). For the latter, the smoothness postulate is now not needed, because (3.15) implies that the terms $f_i(v|u)$ of $F(\mathbf{v}|\mathbf{u})$ are continuous in $u$ except possibly at $u = 0$, whence the continuity of $c(\mu)$ in (3.25) follows. Since from (3.15) $f_i(v|0) = c_i(\text{sign})|v|^p$, we obtain that in (3.16)

$$f_i(v|u) = e^{\beta u} c_i(\text{sign}(v-u))|v-u|^p. \tag{3.27}$$

Comparing (3.27) with (3.15) for $v = 2u > 0$ yields that $e^{\beta u} c_i(+) = f_i^+(2)$ for all $u > 0$. Thus $\beta = 0$ and (3.27) gives (3.17). Finally, it is clear that $F(\mathbf{v}|\mathbf{u})$ as in (3.17) generates a scale and translation invariant projection rule, and this is strongly scale invariant iff the coefficients $c_i(\text{sign}(v_i - u_i))$ do not depend on $\text{sign}(v_i - u_i)$.

The proof of Theorem 3.2 is complete.

We conclude this section by a result about the possibility of extending the domain $\mathcal{L}$ of selection and projection rules to the class $\mathcal{C}$ of all (non-void) closed convex subsets of $S$ (when $S$ equals $R_+^n$ or $\Delta_n$, the sets $C \in \mathcal{C}$ are closed in the relative topology of $S$). Selection and projection rules with domain $\mathcal{C}$ are defined analogously to those with domain $\mathcal{L}$ considered so far; when needed to avoid ambiguity, we will speak about $\mathcal{C}$- and $\mathcal{L}$-selection (projection) rules, respectively. The need for considering the larger domain $\mathcal{C}$ arises, e.g., in inverse problems where the available information consists in linear inequality constraints; then the feasible set consists of those $\mathbf{v} \in S$ that satisfy the given inequality constraints. More generally, non-linear inequality constraints also often lead to convex feasible sets.

For $\mathcal{C}$-selection rules $\Pi : \mathcal{C} \to S$, we adopt the following modification of the consistency axiom (1).

(1') If $C' \subset C$ are in $\mathcal{C}$ and $\Pi(C) \in C'$ then $\Pi(C') = \Pi(C)$. In addition, if $\tilde{C}$ is determined by one linear inequality constraint, i.e.,

$$\tilde{C} = \begin{cases} \{\mathbf{v} : \mathbf{a}^T\mathbf{v} \geq b\} & \text{if} \quad S = R^n \text{ or } R_+^n \\ \{\mathbf{v} : \mathbf{a}^T\mathbf{v} \geq b, \mathbf{1}^T\mathbf{v} = 1\} & \text{if} \quad S = \Delta_n \end{cases} \tag{3.28}$$

and $\tilde{C}$ does not contain $\mathbf{v}^\circ = \Pi(S)$ then $\Pi(\tilde{C}) = \Pi(L)$, where $L \in \mathcal{L}$ is the boundary of $\tilde{C}$ defined by changing $\geq$ to $=$ in (3.28).

THEOREM 3.3. *Every $\mathcal{L}$-selection rule satisfying the basic axioms* (1)-(4) *can be uniquely extended to a $\mathcal{C}$-selection rule satisfying axiom* (1'). *This extension is still generated by the function $F(\mathbf{v})$ of Theorem* 2.1 (a), *i.e., $\Pi(C)$ for $C \in \mathcal{C}$ is that element of $C$ where $F(\mathbf{v})$ attains its minimum on $C$.*

PROOF. The function $F(\mathbf{v})$ generating $\Pi : \mathcal{L} \to S$ as in Theorem 2.1 (a) attains its minimum on each $C \in \mathcal{C}$, at a unique point $\mathbf{v}^* \in C$; this follows from properties (i)-(iii) of a standard $n$-tuple in the same way as it was shown in the proof of Theorem 2.1 (in Csiszár [3]) that $F(\mathbf{v})$ attains a unique minimum on each $L \in \mathcal{L}$. If $\mathbf{v}^\circ \in C$ then $\mathbf{v}^* = \mathbf{v}^\circ$. Otherwise $F(\mathbf{v}^*) = c > 0$ and $A = \{\mathbf{v} : F(\mathbf{v}) \leq c\}$ (or $A = \{\mathbf{v} : F(\mathbf{v}) \leq c,\ \mathbf{1}^T\mathbf{v} = 1\}$ if $S = \Delta_n$) is a convex subset of $S$ such that $A \cap C = \{\mathbf{v}^*\}$. Hence there exists

$$L = \begin{cases} \{\mathbf{v} : \mathbf{a}^T\mathbf{v} = b\} & \text{if} \quad S = R^n \text{ or } R^n_+ \\ \{\mathbf{v} : \mathbf{a}^T\mathbf{v} = b,\ \mathbf{1}^T\mathbf{v} = 1\} & \text{if} \quad S = \Delta_n \end{cases}$$

containing $\mathbf{v}^*$ that separates $C$ and $A$, i.e., $C \subset \tilde{C}$, $A \subset \tilde{C}'$, where $\tilde{C}$ is defined by (3.28) and $\tilde{C}'$ is defined by reversing the inequality in (3.28). Then the minimum of $F(\mathbf{v})$ on $\tilde{C}$ or on $L$ is also attained at $\mathbf{v} = \mathbf{v}^*$.

It follows that if $\Pi : \mathcal{C} \to S$ is an extension of $\Pi : \mathcal{L} \to S$ satisfying postulate (1') then necessarily $\Pi(C) = \mathbf{v}^*$. On the other hand, it is obvious that letting $\Pi(C) = \mathbf{v}^*$ gives rise to an extension of $\Pi : \mathcal{L} \to S$ to $\mathcal{C}$ satisfying (1').

Of course, a similar result holds for the extension of $\mathcal{L}$-projection rules to $\mathcal{C}$-projection rules, subject to the obvious analogue of postulate (1') for projection rules. Further, if an $\mathcal{L}$-projection rule is scale and/or translation invariant then so will be also the $\mathcal{C}$-projection rule obtained as its (unique) extension. A somewhat weaker assertion holds for transitivity, as well. Namely, if an $\mathcal{L}$-projection rule satisfies the transitivity postulate (6a) then for its unique extension to $\mathcal{C}$ satisfying the analogue of (1') the following holds: For any $L \in \mathcal{L}$ and $C \in \mathcal{C}$ with $C \subset L$, we have for every $\mathbf{u} \in S$

$$\Pi(C|\mathbf{u}) = \Pi(C|\Pi(L|\mathbf{u})).$$

This is an immediate consequence of the fact that every measure of distance as in (2.7) has the "Pythagorean property"

$$F(\mathbf{v}|\mathbf{u}) + F(\mathbf{w}|\mathbf{v}) = F(\mathbf{w}|\mathbf{u}) \quad \text{if} \quad \Pi(L|\mathbf{u}) = \mathbf{v},\ \mathbf{w} \in L,\ L \in \mathcal{L}. \tag{3.29}$$

Notice that for $F(\mathbf{v}|\mathbf{u}) = \|\mathbf{v} - \mathbf{u}\|^2$, (3.29) reduces to the Pythagorean theorem. Recently, Jones and Byrne [6] showed that the Pythagorean property (3.29) was equivalent to an intuitively desirable postulate called projectivity (unlike us, they used a continuous framework).

## 4. Selection and projection rules with a restricted domain

Perhaps the simplest special case of linear inverse problems is to infer an unknown function $f$ from its mean values on the atoms of a given partition of the domain of $f$. For probability density or mass functions this means that the probabilities of certain mutually disjoint sets are known. The solution offered in the literature for this case is "Jeffrey's rule", cf. Diaconis and Zabell [4], which means the choice $g_i(v|u) = \frac{v}{u}$ in (4.22) below.

Formally – restricting attention to the discrete case as before – the class of inverse problems we have in mind is determined by the following family $\mathcal{P}$ of possible feasible sets.

Let $\mathcal{P}$ denote the family of all those non-void sets $E \subset S$ (where $S$ is either of $R^n, R^n_+$ and $\Delta_n$) which are of form

$$E = \left\{ \mathbf{v} : \sum_{h \in J_\ell} v_h = b_\ell, \ \ell = 1, \dots, m \right\} \tag{4.1}$$

where $(J_1, \dots, J_m)$ is a partition of $\{1, \dots, n\}$ and $b_1, \dots, b_m$ are constants. A *$\mathcal{P}$-selection rule* is a mapping $\Pi : \mathcal{P} \to S$ such that $\Pi(E) \in E$ for every $E \in \mathcal{P}$. A *$\mathcal{P}$-projection rule* is a family $\Pi(\cdot|\mathbf{u})$, $\mathbf{u} \in S$, of $\mathcal{P}$-selection rules such that $\Pi(E|\mathbf{u}) = \mathbf{u}$ whenever $E$ contains $\mathbf{u}$.

Selection (projection) rules with domain $\mathcal{L}$ as before will be referred to as $\mathcal{L}$-selection (projection) rules.

In this Section we study $\mathcal{P}$-selection rules and $\mathcal{P}$-projection rules. We adopt as basic postulates consistency (same as postulate (1) in Section 2, the family $\mathcal{L}$ being replaced by $\mathcal{P}$) and locality (viz. postulate (4) in Section 2) which now means that for $E$ as in (4.1), $\Pi_{J_\ell}(E)$ depends only on $b_\ell$ and, for projection rules, $\Pi_{J_\ell}(E|\mathbf{u})$ depends only on $b_\ell$ and $\mathbf{u}_{J_\ell}$.

Recall that $V$ denotes $R, R_+$ or $(0,1)$ according as $S$ equals $R^n, R^n_+$ or $\Delta_n$.

Given a $\mathcal{P}$-selection (projection) rule that satisfies the locality postulate, for any $i \neq j$ and $t \in V$ we denote by $q(i,j,t)$ and $q(i,j,t|u,u')$ the $i$-th component of $\Pi(E)$ and $\Pi(E|\mathbf{u})$, respectively, for $E$ as in (4.1) such that $J_\ell = \{i,j\}$, $b_\ell = t$ for some $\ell$, and for any $\mathbf{u} \in S$ whose $i$-th and $j$-th components are equal to $u$ and $u'$. By the locality postulate, this definition of $q(i,j,t)$ and $q(i,j,t|u,u')$ is unambiguous. Of course,

$$q(i,j,t) + q(j,i,t) = t, \qquad q(i,j,t|u,u') + q(j,i,t|u',u) = t. \tag{4.2}$$

$$q(i,j,t|u,u') = u \quad \text{if} \quad t = u + u'. \tag{4.3}$$

If the given $\mathcal{P}$-selection or $\mathcal{P}$-projection rule is the restriction to $\mathcal{P}$ of an $\mathcal{L}$-selection or $\mathcal{L}$-projection rule then $q(i,j,t)$ equals the $i$-th component of $\Pi(L_{ij}(t))$, and $q(i,j,t|u,u')$ equals the $i$-th component of $\Pi(L_{ij}(t)|\mathbf{u})$ if

$u_i = u$, $u_j = u'$, where $L_{ij}(t) \in \mathcal{L}$ is defined by (3.9) (notice that $L_{ij}(t) \notin \mathcal{P}$ except when $S = \Delta_n$). Having this in mind, we extend the notation $\overset{ij}{\leftrightarrow}$ of Csiszár [3] to arbitrary $\mathcal{P}$-selection and $\mathcal{P}$-projection rules that satisfy the locality postulate, writing

$$\begin{aligned} v \overset{ij}{\leftrightarrow} v' \quad &\text{iff} \quad v = q(i,j,t), \quad v' = q(j,i,t), \quad t = v + v' \\ (v|u) \overset{ij}{\leftrightarrow} (v'|u') \quad &\text{iff} \quad v = q(i,j,t|u,u'), \quad v' = q(j,i,t|u',u), \quad t = v + v'. \end{aligned} \tag{4.4}$$

The third basic postulate we adopt in this Section is

(M) (monotonicity) for every $i \neq j$, $q(i,j,t)$ is a non-decreasing function of $t$, resp. $q(i,j,t|u,u')$ is a non-decreasing function of $t$ for every fixed $u$ and $u'$.

On account of (4.2), (M) implies that the functions in question are continuous.

Clearly, postulate (M) is intuitively very appealing, and so is also the stronger postulate

(SM) (strict monotonicity) the functions above are strictly increasing.

For the main results of this Section, the "natural" mathematical conditions will be intermediate between (M) and (SM). In particular, the following (intuitively not too suggestive) postulates will be useful:

(QM) (quasi-strict monotonicity) in addition to (M), to any $i \in \{1, \ldots, n\}$ and $v \in V$ (and $u \in V$) there exists at most one $j$ such that for some $t' \in V$ (and $u' \in V$) satisfying $q(i,j,t') = v$ resp. $q(i,j,t'|u,u') = v$, the function $q(i,j,t)$ resp. $q(i,j,t|u,u')$ is not strictly increasing at $t = t'$;

(RM) (restricted strict monotonicity) in addition to (M), the functions $q(i,j,t)$ resp. $q(i,j,t|u,u')$ are strictly increasing except possibly for a fixed $j \in \{1, \ldots, n\}$; for projection rules, $q(i,j,t|u,u')$ is strictly increasing at $t = u + u'$ even for the exceptional $j$.

With the notation (4.4), postulate (M) means that for any $v > \overline{v}$, the relations $v \overset{ij}{\leftrightarrow} v'$, $\overline{v} \overset{ij}{\leftrightarrow} \overline{v}'$ (or $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$ and $(\overline{v}|u) \overset{ij}{\leftrightarrow} (\overline{v}'|u')$) imply $v' \geq \overline{v}'$. Postulate (SM) is satisfied iff, in addition, $v \overset{ij}{\leftrightarrow} v'$ and $v \overset{ij}{\leftrightarrow} \overline{v}'$ (or $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$ and $(v|u) \overset{ij}{\leftrightarrow} (\overline{v}'|u')$) imply that $v' = \overline{v}'$. Postulate (QM) requires the same except possibly for one index $j$ depending on $i$ and $v$ (and $u$), whereas (RM) means that this exceptional index $j$ (if any) must be fixed (and, for projection rules, $(u|u) \overset{ij}{\leftrightarrow} (v'|u')$ implies $v' = u'$ even for the exceptional $j$, cf. (4.3)).

We notice that the restriction to $\mathcal{P}$ of an $\mathcal{L}$-selection or $\mathcal{L}$-projection rule satisfying the basic axioms (1)-(4) in Section 2 necessarily satisfies (RM) if the basic set is equal to $R^n$ or $R^n_+$; indeed this follows from (3.10), (3.11) by Theorem 3.1. On the other hand, in the case $S = \Delta_n$ not even (M) is necessarily satisfied by such a restriction.

Another kind of monotonicity postulate, intuitively very natural, would be to require the limit relations

$$\lim_{t\to\pm\infty} q(i,j,t) = \pm\infty, \qquad \lim_{t\to\pm\infty} q(i,j,t|u,u') = \pm\infty \tag{4.5}$$

if the basic set is $S = R^n$, and the same for $t \to +\infty$ if $S = R^n$. These would make sure, for any $i \neq j$ and $v \in V$ (and $u \in V$, $u' \in V$), the existence of $v' \in V$ satisfying $v \overset{ij}{\leftrightarrow} v'$ or $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$, respectively (in the cases when $S$ equals $R^n$ or $R^n_+$). The limit relations (4.5) will not be used as postulates in our main results. The following "non-degeneracy" property, however, which is substantially weaker than the above consequence of (4.5), will enter some of our results as a postulate:

(N) (non-degeneracy) given any $v \overset{ij}{\leftrightarrow} v'$ or $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$, there exist $k \neq j$ and $v''$ (and $u''$) such that $v \overset{ik}{\leftrightarrow} v''$ or $(v|u) \overset{ik}{\leftrightarrow} (v''|u'')$, respectively.

LEMMA 4.1. *Given a $\mathcal{P}$-selection rule $\Pi : \mathcal{P} \to S$ satisfying the postulates of consistency, locality and monotonicity, and a set $E$ as in* (4.1), *write* $\Pi(E) = \mathbf{v}^*$. *Then*

(a) *for every $i \neq j$ in the same $J_\ell$ in* (4.1), *we have $v^*_i \overset{ij}{\leftrightarrow} v^*_j$*

(b) *if certain $v_j$'s satisfy $v_i \overset{ij}{\leftrightarrow} v_j$ for some $i \in J_\ell$ and all $j \in J_\ell \setminus \{i\}$, and also $\sum_{j\in J_\ell} v_j = \sum_{j \in J_\ell} v^*_j$, then necessarily $v_i = v^*_i$.*

COROLLARY. *To any distinct $i, j, k$ and any $s \in V$, there exists a unique triple $(v_i, v_j, v_k)$ such that*

$$v_i \overset{ij}{\leftrightarrow} v_j, \quad v_i \overset{ik}{\leftrightarrow} v_k, \quad v_j \overset{jk}{\leftrightarrow} v_k, \quad v_i + v_j + v_k = s. \tag{4.6}$$

*If $v, v'$ and $v''$ with $v \overset{ij}{\leftrightarrow} v'$, $v \overset{ij}{\leftrightarrow} v''$ are given then for the triple $(v_i, v_j, v_k)$ satisfying* (4.6) *with $s = v + v' + v''$ we have $v_i = v$. In particular, if* (QM) *holds then $v \overset{ij}{\leftrightarrow} v'$, $v \overset{ik}{\leftrightarrow} v''$ imply $v' \overset{jk}{\leftrightarrow} v''$ provided, in the case $S = \Delta_n$, that $v + v' + v'' < 1$.*

PROOF. (a) Let $E$ be as in (4.1), and $\Pi(E) = \mathbf{v}^*$. For given $i$ and $j$ both in $J_1$, say, let $J' = \{i, j\}$, $t = v^*_i + v^*_j$, and consider

$$E' = \{\mathbf{v} : v_i + v_j = t,\ \sum_{h\in J_1\setminus J'} v_h = b_1 - t,\ \sum_{h\in J_\ell} v_h = b_\ell,\ \ell = 2, \dots, m\}.$$

Then $E' \subset E$, $\mathbf{v}^* \in E'$, thus $\Pi(E') = \mathbf{v}^*$ by consistency. This means, by definition, that $q(i,j,t) = v_i^*$, $q(j,i,t) = v_j^*$, proving that $v_i^* \overset{ij}{\leftrightarrow} v_j^*$.

(b) Comparing the hypothesis that $v_i \overset{ij}{\leftrightarrow} v_j$ for all $j \in J_\ell \setminus \{i\}$ with the relations $v_i^* \overset{ij}{\leftrightarrow} v_j^*$ just proved, it follows by (M) that if we had $v_i > v_i^*$, this would imply $v_j \geq v_j^*$ for each $j \in J_\ell \setminus \{i\}$. This contradicts the hypothesis that $\sum_{j\in J_\ell} v_j = \sum_{j \in J_\ell} v_j^*$. The assumption $v_i < v_i^*$ leads to a similar contradiction, hence necessarily $v_i = v_i^*$ as claimed.

The Corollary follows by applying the Lemma to $E$ as in (4.1) with $m = 2$, $J_1 = \{i,j,k\}$, $J_2 = J_1^c$, $b_1 = s$, $b_2$ arbitrary if $S = R^n$ or $R_+^n$ and $b_2 = 1 - s$ if $S = \Delta_n$. To obtain the last assertion of the Corollary from the previous one, notice that since $v \overset{ij}{\leftrightarrow} v'$ and $v \overset{ik}{\leftrightarrow} v''$ imply that (4.6) holds with $v_i = v$, it follows by (QM) that at least one of $v' = v_i$ and $v'' = v_k$ must hold. Either of these inequalities implies the other, as $v_i + v_j + v_k = s = v + v' + v''$, and then (4.6) gives that $v' \overset{jk}{\leftrightarrow} v''$.

On account of Lemma 4.1, $\mathcal{P}$-selection rules satisfying our three basic postulates are uniquely determined by the corresponding relations $\overset{ij}{\leftrightarrow}$; of course, so are $\mathcal{P}$-projection rules, too. We henceforth concentrate on characterizing these relations.

THEOREM 4.1. *In the cases $S = R^n$ and $R_+^n$, for a $\mathcal{P}$-selection rule $\Pi$ satisfying the postulates of consistency, locality and monotonicity, the following are true:*

(a) *There exist strictly increasing, left continuous functions $g_1, \ldots, g_n$ such that*

$$v \overset{ij}{\leftrightarrow} v' \quad \textit{iff} \quad g_i(v) \leq g_j(v'+0),\ g_i(v+0) \geq g_j(v') \tag{4.7}$$

(b) *if $\Pi$ satisfies* (QM) *and, in addition, for every $i \neq j$ and $t'$ at least one of the functions $q(i,j,t)$ and $q(j,i,t)$ is strictly increasing at $t = t'$, then there exist continuous non-decreasing functions $g_1, \ldots, g_n$ such that*

$$v \overset{ij}{\leftrightarrow} v' \quad \textit{iff} \quad g_i(v) = g_j(v); \tag{4.8}$$

*these functions $g_i$ are strictly increasing iff $\Pi$ satisfies* (SM)

(c) *if $\Pi$ satisfies* (RM), *it can be extended to an $\mathcal{L}$-selection rule satisfying the basic axioms* (1)-(4) *in Section 2, provided in the case $S = R^n$ that neither component of the vector $\Pi(E(t))$ is independent of $t$, where $E(t) = \{\mathbf{v} : \sum_{h=1}^n v_h = t\}$.*

REMARKS. (i) The functions $g_i$ appearing in Theorem 4.1 are not unique; clearly, if certain $g_i$ are suitable then so are also $\bar{g}_i = \psi(g_i)$, for any strictly increasing continuous function $\psi$.

(ii) In part (a), it may be assumed (by (i) above) that $g_i(v_i^\circ) \le 0 \le g_i(v_i^\circ + 0)$ where $\mathbf{v}^\circ = \Pi(E(t))$ for some fixed $t$, $E(t)$ being defined in part (c). Then it follows that $\Pi$ can be extended to the $\mathcal{L}$-selection rule generated by $F(\mathbf{v}) = \sum_{i=1}^n f_i(v)$, $f_i(v) = \int_{v_i^\circ}^{v} g_i(t)dt$, where the functions $f_i$ are non-negative and strictly convex with $f_i(v_i^\circ) = 0$, but not necessarily differentiable; thus the $\mathcal{L}$-selection rule generated by $F(\mathbf{v})$ need not satisfy axiom (2) in Section 2.

(iii) The sufficient condition for extendability to an $\mathcal{L}$-selection rule satisfying axioms (1)-(4), given in part (c), is necessary, as well (the necessity of (RM) has already been established). Notice that the extra condition imposed in the case $S = R^n$ is automatically satisfied if $S = R_+^n$.

(iv) For the assertion of part (b), the sufficient conditions given there are easily seen to be necessary, as well. It can be shown by an example that the condition imposed in addition to (QM) can not be dropped, i.e., it is not implied by (QM). On the other hand, (QM) and (N) already imply that additional condition. Notice also that the assertion of (b) does not imply an extendability result, because if the functions $g_i$ in (4.7) were used to define $F(\mathbf{v})$ as in (ii) above, this $F(\mathbf{v})$ would not necessarily generate an $\mathcal{L}$-selection rule.

(v) No similar results hold in the case $S = \Delta_n$. On the other hand, our results on projection rules – to which some additional conditions will be needed – will cover also that case.

PROOF. (a) Given $\Pi$, define

$$a_{ik}(v) = \begin{cases} \inf\{q(k,i,t) : q(i,k,t) \ge v\} & \text{if} \quad k \ne i \\ v & \text{if} \quad k = i \end{cases} \tag{4.9}$$

(with the understanding that the inf of the empty set is $+\infty$). Further, let $\varphi(x)$ be any strictly increasing, continuous and bounded function defined on the extended real line, e.g., $\varphi(x) = \operatorname{arctg} x$. We claim that the functions

$$g_i(v) = \sum_{k=1}^n \varphi(a_{ik}(v)) \tag{4.10}$$

satisfy the assertion.

Notice first that the functions $a_{ik}(v)$ are obviously non-increasing and left continuous and $a_{ii}(v) = v$ is strictly increasing. Thus $g_i(v)$ defined by (4.10) is strictly increasing and left continuous.

Observe next that the sets

$$(4.11)\qquad I_{ik}(v)=\{w: v \overset{ik}{\leftrightarrow} w\}=\{q(k,i,t): q(i,k,t)=v\}\quad (k\neq i)$$

are either void or singletons or (possibly infinite) intervals that contain their finite endpoints. It follows from (4.9) that in the case $I_{ik}(v)\neq 0$ the left and right endpoints of $I_{ik}(v)$ are $a_{ik}(v)$ and $a_{ik}(v+0)$, respectively (with some abuse of terminology, if $I_{ik}(v)$ is a singleton $\{w\}$ then its "endpoints" are meant to equal $w$). In the case $I_{ik}(v)=\emptyset$, $a_{ik}(v)$ and $a_{ik}(v+0)$ are equal and infinite.

We will show that for every $k$

$$(4.12)\qquad a_{ik}(v)\leq a_{jk}(v'+0)\quad \text{if}\quad v \overset{ij}{\leftrightarrow} v'.$$

This and the similar inequality with $i$ and $j$ exchanged will prove that the functions defined by (4.10) satisfy the inequalities in (4.7) if $v \overset{ij}{\leftrightarrow} v'$.

Now, (4.12) clearly holds for $k=i$ and $k=j$. Suppose therefore that $k\neq i$, $k\neq j$ and $a_{ik}(v)>-\infty$. Then either $q(i,k,t)<v$ for every $t$ (thus $a_{ik}(v)=+\infty$) or $I_{ik}(v)\neq\emptyset$ and $a_{ik}(v)$ is the (finite) left endpoint of $I_{ik}(v)$.

In the latter case, set $v''=a_{ik}(v)$ and apply the Corollary of Lemma 4.1 to obtain $v_i, v_j, v_k$ satisfying (4.6) with $s=v+v'+v''$ and $v_i=v$. Then $v_k\in I_{ik}(v)$ implies that $v_k\geq v''=a_{ik}(v)$, hence $v_i+v_j+v_k=v+v'+v''$ with $v_i=v$ yields $v_j\leq v'$. Thus, in this case, there exist $v_j$ and $v_k$ satisfying

$$(4.13)\qquad v_j \overset{jk}{\leftrightarrow} v_k,\quad v_j\leq v',\quad v_k\geq a_{ik}(v).$$

In the remaining case when $q(i,k,t)<v$ for every $t$, use the Corollary of Lemma 4.1 to obtain $(v_i,v_j,v_k)$ satisfying (4.6) with $s$ arbitrarily large. Then $v_i \overset{ik}{\leftrightarrow} v_k$ implies by definition, cf. (4.3), that $v_i<v$. This and $v_i \overset{ij}{\leftrightarrow} v_j$ together with $v \overset{ij}{\leftrightarrow} v'$ imply by monotonicity that $v_j\leq v'$. Since $v_i+v_j+v_k=s$ can be arbitrarily large, we have shown that in this case there exist $v_j$ and $v_k$ satisfying

$$(4.14)\qquad v_j \overset{jk}{\leftrightarrow} v_k,\quad v_j\leq v',\quad v_k \text{ arbitrarily large.}$$

Since $v_j \overset{jk}{\leftrightarrow} v_k$, i.e., $v_k\in I_{jk}(v_j)$ means that $v_k\leq a_{jk}(v_j+0)$, cf. the passage containing (4.11), by the monotonicity of the functions $a_{jk}(v)$ it follows from (4.13) that $a_{jk}(v'+0)\geq a_{ik}(v)$ and from (4.14) that $a_{jk}(v'+0)=+\infty$. This completes the proof of (4.12).

We still have to show that if $v \overset{ij}{\leftrightarrow} v'$ does not hold then either $g_i(v)>g_j(v'+0)$ or $g_i(v+0)<g_j(v')$. If $v \overset{ij}{\leftrightarrow} v'$ does not hold, it may be assumed

by symmetry that for $t = v + v'$, $w = q(i,j,t)$, $w' = q(j,i,t)$ we have $w < v$. We claim that in this case $g_j(v) > g_j(v'+0)$. This will be proved by showing that

$$w \overset{ij}{\leftrightarrow} w', \quad v > w, \quad v' < w' \tag{4.15}$$

implies

$$a_{ik}(v) \geq a_{jk}(v'+0) \tag{4.16}$$

for all $k$, with the strict inequality for $k = i$ and $k = j$. Since the assertion for $k = i$ and $k = j$ is obvious, suppose that $k \neq i$, $k \neq j$ and apply (4.12) to $w'$ and $w$ in the role of $v$ and $v'$ (interchanging $i$ and $j$). It follows that

$$a_{jk}(w') \leq a_{ik}(w+0); \tag{4.17}$$

since the functions $a_{ik}(v)$ are non-decreasing, (4.15) and (4.17) imply (4.16).

The proof of part (a) is complete.

(b) Given a $\mathcal{P}$-selection rule satisfying the hypothesis of part (b), we can define an equivalence relation on $\{1,\ldots,n\} \times V$ by letting $(i,v) \sim (j,v)$ iff either $i \neq j$ and $v \overset{ij}{\leftrightarrow} v'$ or $i = j$ and there exist $\ell$ and $w$ such that $v \overset{i\ell}{\leftrightarrow} w$, $v' \overset{i\ell}{\leftrightarrow} w$ or $i = j$, $v = v'$. To show that then $(i,v) \sim (j,v')$ and $(i,v) \sim (k,v'')$ imply $(j,v') \sim (k,v'')$, first use (QM) to cover the case $i \neq j$, $i \neq k$ (by the last assertion of the Corollary of Lemma 4.1) and notice that of the remaining cases it suffices to deal with $i = j$, $i \neq k$; for that case use the previous result if $v \overset{i\ell}{\leftrightarrow} w$, $v' \overset{i\ell}{\leftrightarrow} w$ hold with $\ell \neq k$ and use the "additional condition" in the statement of the Theorem to check that if the last relations hold with $\ell = k$ then $v \overset{ik}{\leftrightarrow} v''$ and $v \neq v'$ imply that $v'' = w$.

Now, for any $k \in \{1,\ldots,n\}$, a subset of $V$ is the $k$-section of an equivalence class of the relation $\sim$ iff it either equals $I_{ik}(v)$, cf. (4.11), for some $i \neq k$ and $v \in V$, or it is a singleton $\{w\}$ such that no $i$ and $v$ exist with $v \overset{ik}{\leftrightarrow} w$. Let $\varphi_k(\cdot)$ be a continuous and bounded fuction defined on the extended real line, constant on those intervals which are $k$-sections of equivalence classes of the relation $\sim$, and strictly increasing outside these intervals.

For $i \neq k$, define $h_{ik}(v)$ as the value of $\varphi_k$ on $I_{ik}(v)$ if $I_{ik}(v) \neq \emptyset$, and otherwise let $h_{ik}(v)$ equal $\varphi_k(\pm\infty)$ according as $q(i,k,t) < v$ for all $t$ or $q(i,k,t) > v$ for all $t$ (of course, the latter is possible only if $S = R^n$). Further, let $h_{ii}(v) = \varphi_i(v)$. Then, clearly, the functions $h_{ik}$ are continuous, non-decreasing, $v \overset{ij}{\leftrightarrow} v'$ implies $h_{ik}(v) = h_{jk}(v')$ for all $k$, and (4.15) implies $h_{ik}(v) \geq h_{jk}(v')$ for all $k$, with strict inequality whenever the $k$-section of the equivalence class containing $(i, \overline{v})$ and $(j, \overline{v}')$ is non-void, thus at least for $k = i$ and $k = j$. It follows that the functions

$$g_i(v) = \sum_{k=1}^{n} h_{ik}(v) \tag{4.18}$$

satisfy the assertion of Theorem 4.1 (b).

(c) Suppose that $\Pi$ satisfies the hypothesis of part (c) and let $j$ denote the exceptional index of postulate (RM), or any fixed index if there is no such exceptional one (i.e., if (SM) holds). Consider the functions $g_i$ as in part (b). Then these are strictly increasing except possibly for $i = j$.

We need to find $\mathbf{v}^\circ \in S$ such that $v_i^\circ \overset{ij}{\leftrightarrow} v_j^\circ$, i.e.,

$$g_i(v_i^\circ) = g_j(v_j^\circ) \quad \text{for each } i \tag{4.19}$$

and $g_j(v)$ is strictly increasing at $v = v_j^\circ$. Clearly, these properties hold for $\mathbf{v}^\circ = \Pi(E(t_0))$ if the $j$-th component of $\Pi(E(t))$ as a function of $t$ is strictly increasing at $t = t_0$ (cf. Lemma 4.1). The existence of such a $t_0$ is obvious if $S = R_+^n$ and has been assumed if $S = R^n$.

Now, by remark (i), we may assume without any loss of generality that in (4.19) actually $g_i(v_i^\circ) = 0$ for each $i$. Further, in the case $S = R_+^n$ the limit of each $g_i(v)$ as $v \to 0$ must be the same, on account of (4.5). Again, this common limit may be assumed to be $-\infty$. Then

$$f_i(v) = \int_{v_i^\circ}^{v} g_i(t)dt$$

defines a standard $n$-tuple $(f_1, \ldots, f_n)$ and it is clear that the given $\Pi$ is the restriction to $\mathcal{P}$ of the $\mathcal{L}$-selection rule generated by $F(\mathbf{v}) = \sum_{i=1}^{n} f_i(v_i)$.

This completes the proof of Theorem 4.1.

For $\mathcal{P}$-projection rules we do not have available an exact analogue of Theorem 4.1. Still, we will prove results similar to parts (b) and (c) of Theorem 4.1, under some additional hypotheses, that represent, in a practical sense, hardly any restrictions. Unlike in Theorem 4.1, also the case $S = \Delta_n$ will be covered. We will need the following postulates for $\mathcal{P}$-projection rules.

(S) (separability) there exists a sequence $\{u^{(m)}\} \subset V$ such that for every fixed $i \in \{1, \ldots, n\}$ and $u \in V$, the supremum and infimum of $q(i, j, t|u, u')$ for all permissible $j, t$ and $u'$ remain unchanged if $u'$ is restricted to be from the sequence $\{u^{(m)}\}$.

Here the infimum has to be considered in the case $S = R^n$ only because otherwise always $\inf_t q(i,j,t|u,u') = 0$.

(P) (prior regularity, for the cases $S = R^n_+$ or $\Delta_n$) for every $i \neq j$, $t \in V$, $u \in V$

$$\lim_{u' \to 0} q(j,i,t|u',u) = 0. \tag{4.20}$$

The separability postulate is technical but $\mathcal{P}$-projection rules not satisfying (S) can hardly have practical interest. Obvious sufficient conditions for (S) are, e.g., the limit relations (4.5) or the continuous dependence of $q(i,j,t|u,u')$ on $u'$; postulate (P) also implies (S), cf. below. The intuitive meaning of postulate (P) is clear and it certainly appears desirable.

LEMMA 4.2. *Given a $\mathcal{P}$-projection rule satisfying* (P) *(with basic set $S = S^n_+$ or $\Delta_n$), to any $i \neq j$, $u \in V$, $v \in V$, $\varepsilon > 0$, a $\delta > 0$ can be found such that to every $u' \in (0,\delta)$ there exists $v' \in V$ satisfying*

$$(v|u) \overset{ij}{\leftrightarrow} (v'|u'), \quad v' < \varepsilon. \tag{4.21}$$

COROLLARY. *Postulate* (P) *implies both* (N) *and* (S).

PROOF. Apply (4.20) to $t = v + \varepsilon$ (in the case $S = \Delta_n$, we assume without any loss of generality that $\varepsilon < 1 - v$). It follows that for a suitable $\delta > 0$, for $u \in (0,\delta)$ we have $q(j,i,v+\varepsilon|u',u) < \varepsilon$. On account of (4.2), this means that $q(i,j,v+\varepsilon|u,u') > v$. Hence, by continuity, there exists $t < v+\varepsilon$ with $q(i,j,t|u,u') = v$, and then $v' = q(j,i,t|u',u)$ satisfies (4.21).

The Corollary is obvious.

THEOREM 4.2. *Let us be given a $\mathcal{P}$-projection rule satisfying consistency, locality and* (QM). *In addition, in the cases $S = R^n$ or $R^n_+$ we assume* (N) *and* (S), *and in the case $S = \Delta_n$ we assume* (P). *Then there exist functions $g_i(v|u)$, $v \in S$, $u \in S$, continuous and non-decreasing in $v$ for every fixed $u$, with $g_i(u|u) = 0$, such that for every $i \neq j$*

$$(v|u) \overset{ij}{\leftrightarrow} (v'|u') \quad \textit{iff} \quad g_i(v_i|u_i) = g_j(v_j|u_j), \tag{4.22}$$

*provided in the case $S = \Delta_n$ that $u+u' < 1$, $v+v' < 1$. Further, if also* (RM) *is satisfied then this $\mathcal{P}$-projection rule can be extended to an $\mathcal{L}$-projection rule satisfying the basic axioms* (1)-(4) *in Section* 2.

REMARK. It is easy to see that (QM) and (S) are also necessary for (4.22), and in the cases $S = R^n$ or $R^n_+$, (RM) is necessary for extendability. In particular, if $S = R^n$ or $S = R^n_+$ then consistency, locality, (RM) and (S) represent the necessary and sufficient conditions for extendability within the class of $\mathcal{P}$-projection rules satisfying (N). A general necessary and sufficient condition for extendability remains elusive, particularly in the case $S = \Delta_n$.

PROOF. Our first claim is that under the hypothesis of Theorem 4.2 we can define an equivalence relation on $\{1,\ldots,n\} \times V \times V$ as follows (actually, here the separability hypothesis is not needed). Let us say that two triples $(i, v, u)$ and $(j, v', u')$ are comparable if they are either identical or else $i \neq j$ and in the case $S = \Delta_n$ also $u + u' < 1$, $v + v' < 1$. Write $(i, v, u) \sim (j, v', u')$ if these triples are identical or $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$ or the two triples are incomparable but there exists $(\ell, \overline{v}, \overline{u})$ such that

$$(4.23) \qquad (v|u) \overset{i\ell}{\leftrightarrow} (\overline{v}|\overline{u}), \quad (v'|u') \overset{j\ell}{\leftrightarrow} (\overline{v}|\overline{u}).$$

We have to show that if $(i, v, u) \sim (j, v', u')$ and $(i, v, u) \sim (k, v'', u'')$ then also $(j, v', u') \sim (k, v'', u'')$. The following consequence of the Corollary of Lemma 4.1 will be used: If postulate (QM) holds then for any distinct $i, j, k$

$$(4.24) \quad (v|u) \overset{ij}{\leftrightarrow} (v'|u') \text{ and } (v|u) \overset{ik}{\leftrightarrow} (v''|u'') \text{ imply } (v'|u') \overset{jk}{\leftrightarrow} (v''|u'')$$

provided in the case $S = \Delta_n$ that

$$(4.25) \qquad u + u' + u'' < 1, \quad v + v' + v'' < 1.$$

Now, we distinguish several cases.

(i) $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$, $(v|u) \overset{ik}{\leftrightarrow} (v''|u'')$. In this case $(j, v', u') \sim (k, v'', u'')$ holds by definition if these triples are incomparable, and (4.24) applies if $j \neq k$ provided in the case $S = \Delta_n$ that (4.25) holds. In the remaining subcase ($S = \Delta_n$, the triples $(j, v', u')$ and $(k, v'', u'')$ are comparable, but (4.25) fails) pick any $\ell$ distinct from $i, j, k$; by Lemma 4.2, there exist $\overline{u}, \overline{v}$ such that $(v|u) \overset{i\ell}{\leftrightarrow} (\overline{v}|\overline{u})$ and $\overline{u}, \overline{v}$ are sufficiently small to make sure that replacing in (4.25) either of $u, u', u''$ by $\overline{u}$ and either of $v, v', v''$ by $\overline{v}$, the inequalities will hold. Then, by the result already proved, we first obtain $(\overline{v}|\overline{u}) \overset{\ell j}{\leftrightarrow} (v'|u')$, $(\overline{v}|\overline{u}) \overset{\ell k}{\leftrightarrow} (v''|u'')$ and hence, in turn, that $(v'|u') \overset{jk}{\leftrightarrow} (v''|u'')$.

(ii) $(v|u) \overset{ik}{\leftrightarrow} (v''|u'')$, while $(i,v,u)$ and $(j,v',u')$ are incomparable and (4.23) holds for some $\ell,\overline{v},\overline{u}$. Suppose first that $S=R^n$ or $R^n_+$, then the incomparability assumption means that $i=j$. If $\ell\neq k$ in (4.23), it follows as above first that $(\overline{v}|\overline{u}) \overset{\ell k}{\leftrightarrow} (v''|u'')$ and then that $(v'|u') \overset{jk}{\leftrightarrow} (v''|u'')$. If $\ell=k$ in (4.23), we use postulate (N) to pick $\ell'$ (distinct from $\ell=k$ and from $i=j$) and $\overline{u}',\overline{v}'$ such that $(\overline{v}|\overline{u}) \overset{\ell\ell'}{\leftrightarrow} (\overline{v}'|\overline{u}')$. Then by (4.24), also $(\ell',\overline{v}',\overline{u}')$ will satisfy (4.23), and $(v'|u') \overset{jk}{\leftrightarrow} (v''|u'')$ holds by the previous result. Next, if $S=\Delta_n$, pick any $\ell'$ distinct from $i,j,k,\ell$, and use Lemma 4.2 to find $\overline{u}',\overline{v}'$ such that $(\overline{v}|\overline{u}) \overset{\ell\ell'}{\leftrightarrow} (\overline{v}'|\overline{u}')$ and $(\ell',\overline{v}',\overline{u}')$ is comparable to each of $(i,v,u)$, $(j,v',u')$, $(k,v'',u'')$. Then, applying the result of (i) repeatedly, we first obtain that $(\ell',\overline{v}',\overline{u}')$ also satisfies (4.23), hence that $(\overline{v}'|\overline{u}') \overset{\ell' k}{\leftrightarrow} (v''|u'')$, and finally that $(j,v',u')\sim(k,v'',u'')$.

(iii) $(i,v,u)$ and $(j,v',u')$ are incomparable and so are $(i,v,u)$ and $(k,v'',u'')$ but both pairs of triples are related by $\sim$. In this case, consider $(\ell,\overline{v},\overline{u})$ satisfying (4.23). By the result of (ii), it follows that $(\ell,\overline{v},\overline{u})\sim(k,v'',u'')$. This and (4.23) imply by (i) or (ii) that $(i,v,u)\sim(k,v'',u'')$.

Having established our first claim, we notice some easily checked facts about the equivalence classes of the equivalence relation introduced above.

(a) All triples of form $(i,u,u)$ belong to the same equivalence class, say $A_0$ (it is not claimed that these triples are the only members of $A_0$).

(b) For any equivalence class $A\neq A_0$, if $(i,v,u)$ and $(j,v',u')$ are both in $A$ (where $i$ and $j$ are not necessarily distinct) then $v \gtrless u$ according as $v' \gtrless u'$. If, in addition, $w$ satisfies $u<w<v$ or $v<w<u$ and the equivalence class of $(i,w,u)$ is distinct from $A$ and $A_0$ then there exists $w'$ with $u'<w'<v'$ resp. $v'<w'<u'$ such that $(i,w,u)\sim(j,w',u')$.

(c) For any equivalence class $A$, every section of form

$$A(i,u)=\{v:(i,v,u)\in A\} \tag{4.26}$$

is either void or a singleton or an interval that contains its endpoints except for the eventual $\pm\infty$, or 1 if $S=\Delta_n$ (in the case $S=\Delta_n$, Lemma 4.2 can be used for checking this).

We will also need the following property that depends on the separability postulate (S); recall the Corollary of Lemma 4.2, by which the hypothesis of Theorem 4.2 implicitly include (S) and (N) also in the case $S=\Delta_n$.

(d) If an equivalence class $A$ is not a singleton then there exist $k\in\{1,\dots,n\}$ and some $u^{(m)}$ from the sequence appearing in postulate (S) such that $A(k, u^{(m)})$ is non-void and it is not an infinite interval, resp. not an interval with right endpoint 1 if $S = \Delta_n$.

To check (d), suppose that $(i, v, u) \in A$ with $v > u$, say (the case $v < u$ is similar, and that of $v = u$ is trivial by (a)). Since $A$ is not a singleton, it follows by postulate (N) that there exist $(j, v', u')$ and $(\ell, v'', u'')$ such that $i, j, \ell$ are distinct and $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$, $(v|u) \overset{i\ell}{\leftrightarrow} (v''|u'')$. Then for $t' > v + v'$, $t'' > v + v''$, at least one of $q(i, j, t'|u, u')$ and $q(i, \ell, t''|u, u'')$ must be larger than $v$, by (QM). Thus, by postulate (S), there exist $u^{(m)}$, $k$ and $t$ such that $q(i, k, t|u, u^{(m)}) > v$. Hence, by monotonicity, there exists $\bar{t}$ with $q(i, k, \bar{t}|u, u^{(m)}) = v$ and any such $\bar{t}$ satisfies $u + u^{(m)} < \bar{t} < t$. Then by (4.26) we have $v \in A(k, u^{(m)}) \subset (u^{(m)}, q(k, i, t|u^{(m)}, u))$, proving (d).

Now, we proceed similarly as in the proof of Theorem 4.1 (b). For every $k \in \{1, \ldots, n\}$ and $\overline{u} \in V$, let $\varphi_{k,\overline{u}}(\cdot)$ be a function defined and continuous on $V$, constant on each interval (if any) which is the section $A(k, \overline{u})$ of some equivalence class $A$, cf. (4.26), strictly increasing elsewhere, and satisfying

$$(4.27) \qquad \varphi_{k,\overline{u}}(\overline{u}) = 0, \quad \inf_{\overline{v} \in V} \varphi_{k,\overline{u}}(\overline{v}) = -1, \quad \sup_{\overline{v} \in V} \varphi_{k,\overline{u}}(\overline{v}) = 1.$$

Further, for every $i \in \{1, \ldots, n\}$, define the function $h_{i,k,\overline{u}}(v|u)$ as the value of $\varphi_{k,\overline{u}}(\overline{v})$ for $\overline{v}$ such that $(i, v, u) \sim (k, \overline{v}, \overline{u})$ if such $\overline{v}$ exists, and as $\pm 1$ otherwise according as $v \gtrless u$.

By (b) above, $h_{i,k,\overline{u}}(v|u)$ is a continuous, non-decreasing function of $v$ (for any fixed $u \in V$), and if $(i, v, u)$ and $(i, w, u)$ are not in the same equivalence class then

$$(4.28) \qquad h_{i,k,\overline{u}}(v|u) \gtrless h_{i,k,\overline{u}}(w|u) \quad \text{according as} \quad v \gtrless w,$$

providing not both $h_{i,k,\overline{u}}(v|u)$ and $h_{i,k,\overline{u}}(w|u)$ are equal to $\pm 1$. Notice that $h_{i,k,\overline{u}}(w|u) = \pm 1$ holds iff for the equivalence class $A$ containing $(i, w, u)$, its section $A(k, \overline{u})$, cf. (4.26), is either void or it is an infinite interval or, in the case $S = \Delta_n$, an interval whose right endpoint is 1.

We claim that the functions

$$(4.29) \qquad g_i(v|u) = \sum_{k=1}^{n} \sum_{m=1}^{\infty} 2^{-m} h_{i,k,u^{(m)}}(v|u)$$

satisfy the assertions of Theorem 4.2, if $\{u^{(m)}\}$ is a sequence as in postulate (S).

The function (4.29) is continuous and non-incresing in $v$ because each term of (4.29) is, and the series in (4.29) converges uniformly. Also, $g_i(u|u) = 0$, by (4.27) and (a) above. Further, $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$ implies $g_i(v|u) =$

$g_j(v'|u')$ since each term of (4.29) depends on $(i,v,u)$ through its equivalence class only. To show that $g_i(v|u) \neq g_j(v'|u')$ if $(i,v,u)$ and $(j,v',u')$ are comparable but $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$ does not hold, consider $w = q(i,j,t|u,u')$, $w' = q(j,i,t|u',u)$ with $t = v + v'$. By (4.2) we have $w + w' = v + v'$ and by symmetry we may assume that $v > w$, $v' < w'$. Then

$$(4.30) \qquad h_{i,k,\overline{u}}(v|u) \geq h_{i,k,\overline{u}}(w|u) = h_{j,k,\overline{u}}(w'|u') \geq h_{j,k,\overline{u}}(v'|u')$$

for every $k \in \{1,\dots,n\}$ and $\overline{u} \in V$, and we have to show that for at least one $k$ and $\overline{u} = u^{(m)}$, at least one of the inequalities in (4.30) is strict. By (4.28), a $k$ and $\overline{u} = u^{(m)}$ will be suitable if the common value of the middle terms in (4.30) is different from $\pm 1$, because the assumption that $(v|u) \overset{ij}{\leftrightarrow} (v'|u')$ does not hold implies that at least one of $(i,v,u)$ and $(j,v',u')$ does not belong to the equivalence class containing $(i,w,u)$ and $(j,w',u')$. Denoting the latter equivalence class by $A$, we have by the last observation in the passage containing (4.28) that a $k$ and $\overline{u} = u^{(m)}$ will be suitable if the section $A(k,u^{(m)})$ of $A$ is non-void and it is not an infinite interval or, in the case $S = \Delta_n$, an interval whose right endpoint is 1. Since such $k$ and $u^{(m)}$ do exist by (d) above, the proof of (4.22) is complete.

Finally, if (RM) holds then the functions $g_j(v|u)$ satisfying (4.22) must be strictly increasing in $v$ except possibly for one index $j$ (not depending on $u$), and even for that exceptional index (if any), $g_j(v|u)$ must be strictly increasing at $v = u$. Further, in the cases $S = R_+^n$ or $\Delta_n$ it can be attained by a suitable transformation that $g_j(v|u) \to -\infty$ as $v \to 0$ (as in the proof of Theorem 4.1 (c)). Then the desired $\mathcal{L}$-selection rule that extends the given $\mathcal{P}$-projection rule is the one generated by $F(\mathbf{v}|\mathbf{u}) = \sum_{i=1}^n f_i(v_i|u_i)$, with $f_i(v|u) = \int_u^v g_i(t|u)dt$, cf. (3.11).

## 5. Open problems

The axiomatic theory of inference for inverse problems is still in its beginning stage, and there are so many open questions that it is hard to select only a few. Below we hint to some general directions for further research.

Undoubtedly the most important would be to extend the axiomatic approach to inverse problems involving errors, at least to the extent of covering the methods used in practice for this kind of inverse problems (more common than those without errors), and possibly to arrive at new methods, as well. For some comments on this problem cf. Csiszár [3], Section 1. Even if the possibility of errors is discounted, it may still be desirable to permit "solutions" that are not "data consistent", i.e., do not necessarily belong to the feasible set determined by the available constraints. At present, it is not clear how this situation could be treated axiomatically.

Within the framework adopted here, there are at least three natural directions for further research. First, other choices of the basic set $S$ could be considered. Optimistically, one might try a "general" $S$ (say any convex subset of $R^n$), but already the natural modifications of our choices $S = R^n_+$ and $S = \Delta_n$ by adding the boundary to $S$ lead to substantial new mathematical problems.

Second, one should study selection (projection) rules whose domain is not the whole $\mathcal{L}$ (or $\mathcal{C}$) but a perhaps small subfamily thereof. If an inference method is "good" for a particular class of inverse problems where the possible feasible sets are of some restricted form (such as in X-ray tomography, cf. the Introduction) but it cannot be extended to a "good" selection (projection) rule with domain $\mathcal{L}$, it remains elusive in an axiomatic study dealing only with the latter. A problem whose positive solution would significantly enhance the power of our results is to show, for a possibly large class of subfamilies of $\mathcal{L}$, that "good" selection (projection) rules whose domain is such a subfamily of $\mathcal{L}$ must be restrictions of "good" selection (projection) rules with domain $\mathcal{L}$. Our study of $\mathcal{P}$-selection (projection) rules in Section 4 represents a first step in this direction. Of course, further study of $\mathcal{P}$-projection rules would be desirable, in particular to specialize the permissible functions $g_i(v|u)$ in Theorem 4.2 by imposing intuitively appealing further postulates. We recall from Csiszár [3], Section 1, that inverse problems involving errors can be interpreted within our framework by letting the errors be part of the vector to be inferred; this interpretation, however, leads to feasible sets of a restricted type. Thus, a sufficiently general positive answer to the above problem might also permit coverage of inverse problems involving errors.

Third, even for the present choices of the basic set and domain, the basic axioms might be challenged. Since axiom (2) ("distinctness") is intuitively less compelling than the others, the consequences of dropping it (and perhaps introducing some other axiom) should be considered. It may be conjectured that Theorem 2.1 would still remain valid, except that the "generating function" were not necessarily differentiable. One hint in this direction is provided by Theorem 4.1, cf. Remark (ii) to that Theorem. Another natural question is whether if axiom (4) ("locality") were dropped, could still a result like Theorem 2.1 be proved, with a "generating function" not necessarily of a sum form as in (2.5). Even if without axiom (2) or (4) a meaningful result could not be obtained in general, this might become possible when imposing some other intuitively attractive postulates, such as invariance and/or transitivity.

The author intends to return to some of these questions elsewhere.

## REFERENCES

[1] ACZÉL, J., *Lectures on functional equations and their applications*, Academic Press, New York, 1966. *MR* **34** # 8020

[2] BREGMAN, L. M., The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming, *USSR Computational Mathematics and Mathematical Physics* **7** (1967), 200—217. See *MR* **35** # 6457

[3] CSISZÁR, I., Why least squares and maximum entropy? An axiomatic approach to inference for linear inverse problems, *Ann. Statist.* **19** (1991) (to appear).

[4] DIACONIS, P. and ZABELL, S. L., Updating subjective probability, *J. Amer. Statist. Assoc.* **77** (1982), 822—830. *MR* **84g**: 62002

[5] ITAKURA, F. and SAITO, S., Analysis synthesis telephony based on the maximum likelihood method, *Proc. 6-th Int. Conf. Acoust. C*, Tokyo, 1968, 17—20.

[6] JONES,L. K. and BYRNE, C. L., General entropy criteria for inverse problems, with applications to data compression, pattern classification and cluster analysis, *IEEE Trans. Inform. Theory*, **36** (1990), 23—30.

[7] JONES, L. K. and TRUTZER, V., Computationally feasible high-resolution minimum distance procedures which extend the maximum entropy method, *Inverse Problems* **5** (1989), 749—766.

[8] KULLBACK, S., *Information theory and statistics*, Wiley, New York, 1959. *MR* **21** # 2325

[9] RÉNYI, A., On measures of entropy and information, *Proc. Fourth Berkeley Sympos. Math. Statist. Probab.*, Vol. **1**, Univ. California Press, Berkeley, Calif., 1961, 547—561. *MR* **24** # A2410

[10] RÉNYI, A., On the foundations of information theory, *Rev. Inst. Internat. Statist.* **33** (1965), 1—14. *MR* **31** # 5712

[11] SHORE, J. E. and JOHNSON, R. W., Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy, *IEEE Trans. Inform. Theory* **26** (1980), 26—37. *MR* **80m**: 94029. Corrections: *ibid.* **29** (1983), 942—943. *MR* **85b**: 94008

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# A NOTE ON LOCAL AND GLOBAL FUNCTIONS OF A WIENER PROCESS AND SOME RÉNYI-TYPE STATISTICS

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We give an improved single characterization for a class of local and global functions of a standard Brownian motion starting at zero. We also detail an application to Rényi-type statistics.

## 1. Introduction and review of results

Here we give some definitions and discuss a few known, as well as new results.

A *function* $q$ defined on $(0,1]$ will be called *positive* if $\inf_{\delta \leq s \leq 1} q(s) > 0$ for all $0 < \delta < 1$.

Let $Q$ be the *class of those positive functions* on $(0,1]$ which are *non-decreasing in a neighbourhood of zero.*

Given a standard Wiener process $\{W(t),\ t \geq 0\}$ and a function $q \in Q$, by Blumenthal's 0-1 (cf. Itô and McKean [14]) or by the direct 0 - 1 law for Brownian motion as in Doob [12], we have

$$P\{\limsup_{t \downarrow 0} |W(t)|/q(t) < \infty\} = 0 \text{ or } 1. \tag{1.1}$$

The class of functions $q \in Q$ for which the latter probability is 1 was called the Erdős-Feller-Kolmogorov-Petrovski (EFKP) upper-class of $W$ in M. Csörgő, S. Csörgő, Horváth and Mason [8]. Here we will call these functions simply *local* $(t \downarrow 0)$ *functions of* $W$. This is due to one of the aims of this paper, which is to discuss the latter functions of $W$, as well as their global $(t \to \infty)$ duals, in terms of the classical definitions of upper and lower-classes of $W$.

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Given a function $r \in Q$, by Blumenthal's 0-1 law we have $P\{|W(t)| < r(t),\ t \downarrow 0\} = 0$ or $1$; $r$ is said to belong to the *local upper-class* if this probability is 1 and to the *local lower-class* otherwise.

We note in passing that excluding from $Q$ all positive functions which are *not* nondecreasing near zero, i.e. the decreasing ones, constitutes no loss of generality in the above definitions and statements when $t \downarrow 0$, since we start $W$ at zero with probability one.

As in Itô and McKean ([14], p. 33), *Kolmogorov's test* states that *if $r$ is a positive function in $C(0,1]$, non-decreasing near zero and if $r(t)/t^{1/2}$ is non-increasing for small $t > 0$, then $r$ belongs to the local upper or to the local lower-class according as*

$$\int_0^1 t^{-3/2} r(t) \exp(-r^2(t)/(2t))dt$$

*converges or diverges.*

In agreement with our just mentioned convention, a *function $q \in Q$ will be called a local function of a standard Wiener process $\{W(t),\ t \geq 0\}$ if*

$$\limsup_{t \downarrow 0} |W(t)|/q(t) < \infty \quad \text{a.s.} \tag{1.2}$$

An application of Blumenthal's 0-1 law shows that (1.2) holds true if and only if there exists a constant $0 \leq \beta < \infty$ such that

$$\limsup_{t \downarrow 0} |W(t)|/q(t) = \beta \quad \text{a.s.} \tag{1.3}$$

i.e., *a positive function $q$ on $(0,1]$ which is non-decreasing in a neighbourhood of zero, is a local function of a standard Brownian motion starting at zero if and only if* (1.3) *obtains.*

*A local function will be called a Chibisov-O'Reilly local function if $\beta = 0$ in* (1.3).

The statement (1.3) with $\beta > 0$ is equivalent to saying that for any $\varepsilon > 0$ we have

$$P\{|W(t)| < (\beta + \varepsilon)q(t),\ t \downarrow 0\} = 1 \tag{1.3a}$$

and

$$P\{|W(t)| < (\beta - \varepsilon)q(t),\ t \downarrow 0\} = 0, \tag{1.3b}$$

i.e., *$q \in Q$ is a local function for $W$ if and only if $(\beta + \varepsilon)q$ belongs to its local upper-class and $(\beta - \varepsilon)q$ belongs to its local lower-class.*

We introduce the following integrals:

$$E_0(q,c) := \int_0^1 t^{-3/2} q(t) \exp(-ct^{-1}q^2(t))dt \tag{1.4}$$

and

$$I_0(q,c) := \int_0^1 t^{-1} \exp(-ct^{-1}q^2(t))dt \tag{1.5}$$

with some constant $0 < c < \infty$.

The integral $E_0(q,c)$ appeared in the works of Kolmogorov, Petrovski, Erdős and Feller. For details we refer to Itô and McKean ([14], Section 1.8).

The integral $I_0(q,c)$ appeared in the works of Chibisov [3] and O'Reilly [21].

For further comments on these two integrals, as well as for the proof of the next three theorems we refer to [8]. We have (cf. Proposition 3.1, and Theorems 3.3 and 3.4, respectively, of [8]):

THEOREM A. (i) *Whenever the integral* $I_0(q,c) < \infty$ *for* $q \in Q$, *then* $E_0(q, c+\varepsilon) < \infty$ *for every* $\varepsilon > 0$ *and* $q(t)/t^{1/2} \to \infty$ *as* $t \downarrow 0$.

(ii) *Whenever* $E_0(q,c) < \infty$ *and* $q(t)/t^{1/2} \to \infty$ *as* $t \downarrow 0$ *for* $q \in Q$, *then* $I_0(q,c) < \infty$.

THEOREM B. *A function* $q \in Q$ *is a local function of a standard Wiener process starting at zero if and only if the integral* $I_0(q,c) < \infty$ *for some* $c > 0$ *or, equivalently, if and only if the integral* $E_0(q,c) < \infty$ *for some* $c > 0$ *and* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

THEOREM C. *A function* $q \in Q$ *is a Chibisov-O'Reilly local function if and only if the integral* $I_0(q,c) < \infty$ *for all* $c > 0$ *or, equivalently, if and only if the integral* $E_0(q,c) < \infty$ *for all* $c > 0$ *and* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

By making the connection between the two integrals, Theorem A enabled the authors of [8] to prove Theorem B and C, which amount to saying that there can be only one characterization of local functions.

We note that in O'Reilly [21], as well as in [8], a Chibisov-O'Reilly *local* function $q$ on (0,1) is defined by requiring $q(t)$ and $q(1-t)$ to be Chibisov-O'Reilly local functions on $(0, 1/2]$. The definition of a *local* function is extended to $(0,1)$ in a similar way in [8]. This convention in the latter paper is only for the sake of enabling ourselves to talk about *local* functions of a Brownian bridge as well.

While proving Theorem B (cf. Proof of Theorem 3.3 of [8]), [8] actually established more than what the second part of the latter theorem claims. Namely they proved also

THEOREM D. *Let $q \in Q$, and $W(\cdot)$ be a standard Wiener process starting at zero. Then $I_0(q,c) < \infty$ implies*

$$\limsup_{t\downarrow 0} |W(t)|/q(t) \le (2c)^{1/2} \quad a.s., \tag{1.6}$$

*and if* (1.3) *holds true for some $0 \le \beta < \infty$, then $I_0(q,c) < \infty$ for any $c > 8\beta^2$.*

Hence [8] conjectured that (1.3) holding true for some $\beta \ge 0$ should imply $I_0(q,c) < \infty$ for any $c > \beta^2/2$, for then the latter combined with (1.6) and Theorem A would amount to saying that when testing for (1.3a) and (1.3b), the assumptions inherited from the classical EFKP local upper-lower functions integral test that $q$ be continuous, and be such that $q(t)/t^{1/2}$ is non-increasing for small $t > 0$, could be dropped.

We prove in our Section 2 here that this conjecture is true. Namely we have

THEOREM 1.1. *Let $q \in Q$, and $W(\cdot)$ be a standard Wiener process starting at zero. Then* (1.3) *holds true with some $\beta \ge 0$ if and only if*

$$I_0(q,c) < \infty \quad \text{for any} \quad c > \beta^2/2 \tag{1.7}$$

*and*

$$I_0(q,c) = \infty \quad \text{for any} \quad c < \beta^2/2, \tag{1.8}$$

*or, equivalently, if and only if*

$$E_0(q,c) < \infty \quad \text{for any} \quad c > \beta^2/2. \tag{1.9}$$

$$E_0(q,c) = \infty \quad \text{for any} \quad c < \beta^2/2, \tag{1.10}$$

*while* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

Another, somewhat more convenient version of Theorem 1.1 reads as follows.

THEOREM 1.1*. *Let $q \in Q$ and $W(\cdot)$ be a standard Wiener process starting at zero. Then with some $c \ge 0$ we have*

$$\limsup_{t\downarrow 0} |W(t)|/q(t) = (2c)^{1/2} \quad a.s. \tag{1.11}$$

*if and only if for any $\varepsilon > 0$ we have $I_0(q,c+\varepsilon) < \infty$ and $I_0(q,c-\varepsilon) = \infty$ or, equivalently, if and only if $E_0(q,c+\varepsilon) < \infty$ and $E_0(q,c-\varepsilon) = \infty$ with* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

Consequently, *requiring only that* $q \in Q$, *with* $\varepsilon > 0$ *we conclude that* $((2c)^{1/2} + \varepsilon)q$ *belongs to the local upper-class (resp.* $((2c)^{1/2} - \varepsilon)q$ *belongs to the local lower-class) for* $W$ *if and only if* $I_0(q, c+\varepsilon) < \infty$ *(resp.* $I_0(q, c-\varepsilon) = \infty$*) or, equivalently, if and only if* $E_0(q, c+\varepsilon) < \infty$ *(resp.* $E_0(q, c-\varepsilon) = \infty$*) with* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

REMARKS. Theorem B constitutes a test for the two statements in (1.1) by saying that with $q \in Q$

$$(1.12) \qquad P\{\limsup_{t\downarrow 0} |W(t)|/q(t) < \infty\} = 1,$$

i.e. *$q \in Q$ is a local function of $W$*, if and only if there is a $c > 0$ such that $I_0(q, c) < \infty$ (or, $E_0(q, c) < \infty$ and $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$), while we have

$$(1.13) \qquad P\{\limsup_{t\downarrow 0} |W(t)|/q(t) < \infty\} = 0,$$

i.e. $q \in Q$ is not a local function of $W$, if and only if $I_0(q, c) = \infty$ (or $E_0(q, c) = \infty$) for all $c > 0$.

Theorem C is an extension of O'Reilly's [21] Proposition 2.1 for possibly discontinuous functions in $Q$.

Naturally, if with $q \in Q$ we have

$$(1.14) \qquad P\{|W(t)| < q(t),\ t \downarrow 0\} = 1,$$

then we have also (1.12), i.e. *all EFKP local upper-class functions $q \in Q$ of $W$ are local functions of $W$* and hence, for any local upper-class function $q \in Q$ of $W$ there is a $c > 0$ such that $I_0(q, c) < \infty$ (or, $E_0(q, c) < \infty$ and $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$). Conversely, however, we can only conclude (cf. Theorem B) that $q \in Q$ is a local function of $W$, i.e. that we have (1.12) but not necessarily also (1.14) at the same time. Thus, such a $q \in Q$ may not be an EFKP local upper-class function of $W$. If it is, however, then it does not have to be in $C(0, 1]$, nor does it have to be such that $q(t)/t^{1/2}$ is non-decreasing for small $t > 0$, as required by the classical EFKP test as quoted above from Itô and McKean [14]. What Theorem 1.1 achieves is to test for $(\beta + \varepsilon)q(t) \in Q$, respectively for $(\beta - \varepsilon)q(t) \in Q$, with $\varepsilon > 0$ and $\beta$ as in (1.3), being a local upper-class, respectively local lower-class function of $W$, i.e. that *$q \in Q$ is a local function of $W$, without requiring the just mentioned continuity and monotonicity assumptions of the classical EFKP test.* Whether such a test is feasible along the lines of Theorem 1.1 also for

$$(1.15) \qquad P\{|W(t)| < q(t),\ t \downarrow 0\} = 0 \text{ or } 1,$$

assuming only that $q \in Q$, remains an open question. In the light of Theorem 1.1 one should like to believe that the answer to this question is in the affirmative.

In Section 3 we translate these local ($t \downarrow 0$) form results to their global ($t \to \infty$) forms and obtain a single characterization for both of these forms.

## 2. Proof of Theorem 1.1.

First we show that (1.3) implies (1.7). From the second part of Theorem D we find that it suffices to show that (1.3) implies (1.7) when $\beta > 0$. For arbitrary but fixed $c > \beta^2/2$, let $\varepsilon > 0$ be such that

$$(1+\varepsilon)^4\beta^2/(2(1-\varepsilon)) < c. \tag{2.1}$$

We have

$$\begin{aligned} &P\{W(t) > q(t)(1+\varepsilon)\beta \text{ for some } t \text{ in } (0,b]\} \\ &\geq P\{W(b) > (1+\varepsilon)\beta q(b)\} = 1 - \Phi((1+\varepsilon)\beta q(b)b^{-1/2}), \end{aligned}$$

where $\Phi(\cdot)$ is the unit-normal distribution function. The left-hand side of the latter inequality tends to zero as $b \downarrow 0$ by (1.3). Hence we have

$$\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty \tag{2.2}$$

for any local function $q$ of $W$. Let us assume that $b$ is so small that we have

$$P\{\sup_{0<t\leq b} |W(t)|/q(t) \leq (1+\varepsilon)\beta\} \geq 1/2, \tag{2.3}$$

and that $q$ is non-decreasing on $(0,b]$. We introduce the following notations:

$$\begin{aligned} b_{j,k} &= b\varepsilon^j(1+\varepsilon)^k, \quad j = 0,1,\ldots, \; k = 0,1,\ldots,k_0, \\ k_0 &= 1 + [\log(1/\varepsilon)/\log(1+\varepsilon)]. \end{aligned}$$

Then, for each $0 \leq k \leq k_0$

$$\begin{aligned} &P\{|W(t)| \geq (1+\varepsilon)\beta q(t) \text{ for some } t \text{ in } (0,b]\} \\ &\geq P\left\{\bigcup_{j=2}^{\infty}(|W(b_{j,k})| \geq (1+\varepsilon)\beta q(b_{j,k}))\right\} \\ &\geq \lim_{N\to\infty}\sum_{j=2}^{N-1} P\{|W(b_{j,k})| \geq (1+\varepsilon)\beta q(b_{j,k}) \text{ and} \end{aligned} \tag{2.4}$$

$$|W(b_{i,k})| < (1+\varepsilon)\beta q(b_{i,k}) \text{ for all } j < i \le N\}$$

$$= \lim_{N\to\infty} \sum_{j=2}^{N-1} \underset{\substack{|x_i|<(1+\varepsilon)\beta q(b_{i,k}) \\ j<i\le N}}{\int \cdots \int} P\{|W(b_{j,k}) - W(b_{j+1,k}) + x_{j+1}|$$

$$\ge (1+\varepsilon)\beta q(b_{j,k})\} dP\{W(b_{i,k}) < x_i,\ i = j+1,\dots,N\},$$

where the last equality holds because $W(b_{j,k}) - W(b_{j+1,k})$ is independent of $\{W(b_{i,k}),\ j < i \le N\}$. Noting that $1 - (\Phi(a+x) - \Phi(-a+x))$ as a function of $x$ with $a > 0$ has minimal value at $x = 0$, we obtain

$$\begin{aligned}
&P\{|W(b_{j,k}) - W(b_{j+1,k}) + x_{j+1}| \ge (1+\varepsilon)\beta q(b_{j,k})\} \\
&\ge P\{|W(b_{j,k}) - W(b_{j+1,k})| \ge (1+\varepsilon)\beta q(b_{j,k})\} \\
&= 2(1 - \Phi((1+\varepsilon)\beta q(b_{j,k})/(b_{j,k} - b_{j+1,k})^{1/2})) \\
&\ge \frac{2}{(2\pi)^{1/2}} \left( \frac{(b_{j,k} - b_{j+1,k})^{1/2}}{(1+\varepsilon)\beta q(b_{j,k})} - \left( \frac{(b_{j,k} - b_{j,k})^{1/2}}{(1+\varepsilon)\beta q(b_{j,k})} \right)^3 \right) \\
&\quad \times \exp\left( - \frac{(1+\varepsilon)^2\beta^2 q^2(b_{j,k})}{2(b_{j,k} - b_{j+1,k})} \right) \\
&\ge \exp\left( - \frac{(1+\varepsilon)^3\beta^2 q^2(b_{j,k})}{2(b_{j,k} - b_{j+1,k})} \right),
\end{aligned} \tag{2.5}$$

where the first before the last inequality is the well known lower estimation of the function $1 - \Phi(\cdot)$ (cf., e.g, Feller [13], p. 175), while the last one is obtained via (2.2) by taking $b$ small enough, after adding and subtracting the exponent of the exponential function of the last inequality to the exponent of the exponential function in the first before the last inequality. Consequently, from (2.4), (2.5) and (2.3) we get

$$\begin{aligned}
&P\{|W(t)| \ge (1+\varepsilon)\beta q(t) \text{ for some } t \text{ in } (0,b]\} \\
&\ge \lim_{N\to\infty} \sum_{j=2}^{N-1} \exp\left( - \frac{(1+\varepsilon)^3\beta^2 q^2(b_{j,k})}{2(b_{j,k} - b_{j+1,k})} \right) \\
&\quad \times P\{|W(b_{i,k})| < (1+\varepsilon)\beta q(b_{i,k}) \text{ for all } j < i \le N\} \\
&\ge \lim_{N\to\infty} \sum_{j=2}^{N-1} \exp\left( - \frac{(1+\varepsilon)^3\beta^2 q^2(b_{j,k})}{2b_{j,k}(1-\varepsilon)} \right) \\
&\quad \times P\{\max_{0<b_{i,k}\le b_{3,k_0}} |W(b_{i,k})|/q(b_{i,k}) < (1+\varepsilon)\beta\}
\end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&\geq \lim_{N\to\infty} \sum_{j=2}^{N-1} \exp\left(-\frac{(1+\varepsilon)^3\beta^2q^2(b_{j,k})}{2b_{j,k}(1-\varepsilon)}\right)\\
&\quad \times P\{\sup_{0<t\leq b} |W(t)|/q(t) \leq (1+\varepsilon)\beta\}\\
&\geq \frac{1}{2}\sum_{j=2}^{\infty} \exp\left(\frac{(1+\varepsilon)^3\beta^2q^2(b_{j,k})}{2b_{j,k}(1-\varepsilon)}\right)\\
&\geq \frac{1}{2\log(1+\varepsilon)}\sum_{j=2}^{\infty}\int_{b_{j,k}}^{b_{j,k+1}} t^{-1}\exp\left(\frac{(1+\varepsilon)^4\beta^2q^2(t)}{2(1-\varepsilon)t}\right)dt\\
&\geq \frac{1}{2\varepsilon}\sum_{j=2}^{\infty}\int_{b_{j,k}}^{b_{j,k+1}} t^{-1}\exp(-cq^2(t)/t)dt,
\end{aligned}
$$

where in the last inequality we made use of (2.1). On summing now in $k$, from 0 to $k_0$, on both sides of (2.6) we obtain

$$
\begin{aligned}
k_0+1 &\geq \frac{1}{2\varepsilon}\sum_{j=2}^{\infty}\sum_{k=0}^{k_0}\int_{b_{j,k}}^{b_{j,k+1}} t^{-1}\exp(-cq^2(t)/t)dt\\
&\geq \frac{1}{2\varepsilon}\sum_{j=2}^{\infty}\int_{b_{j,0}}^{b_{j-1,0}} t^{-1}\exp(-cq^2(t)/t)dt\\
&= \frac{1}{2\varepsilon}\int_0^{b\varepsilon} t^{-1}\exp(-cq^2(t)/t)dt,
\end{aligned}
$$

which, in turn, implies that $I_0(q,c) < \infty$ for any $c > \beta^2/2$.

Assuming (1.3), by (1.6) we conclude also (1.8), and using now Theorem A, we obtain (1.9) and (1.10) as well. From here the converse direction is obvious, and hence the proof of Theorem 1.1 is now complete.

## 3. Global forms of local functions

In this section we show that *global* functions for a standard Brownian motion are equivalent to *local* functions, and give a single test for these under conditions which are duals of those of Theorem 1.1.

A *function* $h$ defined on $[1,\infty)$ will be called *positive* if $\inf_{1\leq t\leq 1/\delta} h(t) > 0$ for all $0 < \delta < 1$.

Let $\mathcal{H}$ be the *class of those positive functions* $h$ *on* $[1,\infty)$ *for which* $h(t)/t$ *is non-increasing in a neighbourhood of infinity.*

REMARK 3.1. If $q \in Q$ then $q(1/t)$ is well defined for $t \in [1, \infty)$, positive and non-increasing in $t$ as $t \to \infty$. Hence $tq(1/t) \in \mathcal{H}$.

Given a standard Wiener process $\{W(t),\ t \geq 0\}$ and a function $h \in \mathcal{H}$, by the 0-1 law as earlier, we have that

$$P\{\limsup_{t\to\infty} |W(t)|/h(t) < \infty\} = 0 \text{ or } 1. \tag{3.1}$$

The functions $h \in \mathcal{H}$ for which the latter probability is 1 will be called *global* $(t \to \infty)$ *functions of* $W$, i.e. if for $h \in \mathcal{H}$ we have

$$\limsup_{t\to\infty} |W(t)|/h(t) < \infty \quad \text{a.s.}, \tag{3.2}$$

then $h$ is said to belong to the latter class.

Applying again 0-1 law for Brownian motion, *we have* (3.2) *if and only if* there exists a constant $0 \leq \gamma < \infty$ such that

$$\limsup_{t\to\infty} |W(t)|/h(t) = \gamma \quad \text{a.s.} \tag{3.3}$$

Those *global functions of* $W$ *for which we have* $\gamma = 0$ *in* (3.3) *will be called Chibisov-O'Reilly global functions.*

Using the time-inversion property of the Wiener process, namely that $\{W(t),\ t \geq 0\}$ being a standard Wiener process implies that

$$W'(t) = \begin{cases} tW(1/t), & t > 0, \\ 0, & t = 0, \end{cases}$$

is again a standard Wiener process, we get

PROPOSITION 3.1. *Let* $q \in Q$. *Then* $q$ *is a local function if and only if* $tq(1/t)$ *is a global function of a standard Wiener process. Also,* $q \in Q$ *is a Chibisov-O'Reilly local function if and only if* $tq(1/t)$ *is a Chibisov-O'Reilly global function.*

OBSERVATION. Obviously, $h \in \mathcal{H}$ being a global function of $W$ is equivalent to saying that $th(1/t)$ is a local function for $W$.

REMARK 3.2. We noted that for $q \in Q$ which is a local function of $\{W(t),\ t \geq 0\}$, the fact that for some $0 \leq \beta < \infty$ we have

$$\limsup_{t\downarrow 0} |W(t)|/q(t) = \beta \quad \text{a.s.}$$

implies not only that $tq(1/t)$ is a global function for $W$ but also that

$$\limsup_{t\to\infty} |W(t)|/(tq(1/t)) = \beta \quad \text{a.s.}$$

The most well-known example of a local function is

$$q(t) = (2t\log\log(1/t))^{1/2}.$$

The famous Khinchin [15] local law of the iterated logarithm (LIL) states

$$\limsup_{t\downarrow 0} |W(t)|/(2t\log\log(1/t))^{1/2} = 1 \quad \text{a.s.}$$

Combining Proposition 3.1 and Remark 3.2 we obtain that $tq(1/t)$, namely $(2t\log\log t)^{1/2}$, is a global function and hence

$$\limsup_{t\to\infty} |W(t)|/(2t\log\log t)^{1/2} = 1 \quad \text{a.s.}$$

The latter is known as Khinchin's global LIL (cf. also Lévy [17], [18], and, for further considerations along these lines, Révész [23] and Bingham [1]).

Our Proposition 3.1 and Remark 3.2 amount to saying that the duality of Khinchin's laws is valid in a more general context for standard Brownian motion starting at zero.

Since any global function can be viewed as a simple transformation of a local function, the characterization of local functions of $W$ as given in Theorems B and C is the characterization of global functions as well. Namely we have

COROLLARY 3.1. *A function $tq(1/t)$, where $q \in Q$, is a global function of a standard Wiener process if and only if the integral $I_0(q,c) < \infty$ for some $c > 0$ or, equivalently, if and only if the integral $E_0(q,c) < \infty$ for some $c > 0$ and* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

COROLLARY 3.2. *A function $tq(1/t)$, where $q \in Q$, is a Chibisov-O'Reilly global function if and only if the integral $I_0(q,c) < \infty$ for all $c > 0$ or, equivalently, if and only if the integral $E_0(q,c) < \infty$ for all $c > 0$ and* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

In terms of a function $h(t) = tq(1/t)$, $q \in Q$, the integrals $E_0(q,c)$ and $I_0(q,c)$ will be transformed into

$$E_\infty(h,c) = \int_1^\infty t^{-3/2} h(t)\exp(-ct^{-1}h^2(t))dt \tag{3.4}$$

and

$$I_\infty(h,c) = \int_1^\infty t^{-1}\exp(-ct^{-1}h^2(t))dt \tag{3.5}$$

respectively.

Consequently, corresponding to Corollaries 3.1 and 3.2, we have the following equivalent ways of characterizing global functions.

COROLLARY 3.1*. *A function $h \in \mathcal{H}$ is a global function for $W$ if and only if the integral $I_\infty(h,c) < \infty$ for some $c > 0$ or, equivalently, if and only if, the integral $E_\infty(h,c) < \infty$ for some $c > 0$ and $\lim_{t\to\infty} h(t)/t^{1/2} = \infty$.*

COROLLARY 3.2*. *A function $h \in \mathcal{H}$ is a Chibisov-O'Reilly global function for $W$ if and only if the integral $I_\infty(h,c) < \infty$ for all $c > 0$ or, equivalently, if and only if, the integral $E_\infty(h,c) < \infty$ for all $c > 0$ and $\lim_{t\to\infty} h(t)/t^{1/2} = \infty$.*

Given a function $r \in \mathcal{H}$, by the $0-1$ law as earlier, we have $P\{|W(t)| < r(t),\ t \uparrow \infty\} = 0$ or 1, and we will say that $r$ belongs to the *global upper-class* if this probability is 1 and to the *global lower-class* otherwise.

We note in passing that Remark 3.2 also implies that any global upper (resp. lower) class function $h \in \mathcal{H}$ can be transformed into a local upper (resp. lower) class function and vice versa.

Since, for $q \in \mathcal{Q}$,

$$\limsup_{t\downarrow 0} |W(t)|/q(t) = \beta \quad \text{a.s.}$$

is equivalent to

$$\limsup_{t\to\infty} |W(t)|/(tq(1/t)) = \beta \quad \text{a.s.},$$

Theorem 1.1 translates into

THEOREM 3.1. *Let $q \in \mathcal{Q}$. Then we have*

$$\limsup_{t\to\infty} |W(t)|/(tq(1/t)) = \beta \quad \textit{a.s.}$$

*for some $0 \le \beta < \infty$ if and only if* (1.7) *and* (1.8) *hold true, or equivalently, if and only if* (1.9) *and* (1.10) *hold true.*

Theorem 1.1* has the following translated form.

THEOREM 3.1*. *Let $q \in \mathcal{Q}$. Then with some $c \geq 0$ we have*

$$\limsup_{t\to\infty} |W(t)|/(tq(1/t)) = (2c)^{1/2} \quad a.s.,$$

*i.e.* $((2c)^{1/2}+\varepsilon)(tq(1/t))$ *belongs to the global upper-class (resp.* $((2c)^{1/2}-\varepsilon)(tq(1/t))$ *belongs to the global lower-class) for $W$, if and only if, for any $\varepsilon > 0$ we have $I_0(q, c+\varepsilon) < \infty$ (resp. $I_0(q, c-\varepsilon) = \infty$) or, equivalently, if and only if $E_0(q, c+\varepsilon) < \infty$ (resp. $E_0(q, c-\varepsilon) = \infty$) with* $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$.

In terms of $h(t) = tq(1/t)$, $q \in \mathcal{Q}$, these theorems read as follows.

THEOREM 3.2. *Let $h \in \mathcal{H}$. Then*

$$\limsup_{t\to\infty} |W(t)|/h(t) = \beta \quad a.s.$$

*if and only if*

$$I_\infty(h,c) < \infty \quad \textit{for any} \quad c > \beta^2/2$$

*and*

$$I_\infty(h,c) = \infty \quad \textit{for any} \quad c < \beta^2/2$$

*or, equivalently, if and only if*

$$E_\infty(h,c) < \infty \quad \textit{for any} \quad c > \beta^2/2,$$

$$E_\infty(h,c) = \infty \quad \textit{for any} \quad c < \beta^2/2$$

*and* $\lim_{t\to\infty} h(t)/t^{1/2} = \infty$.

THEOREM 3.3. *Let $h \in \mathcal{H}$. Then with $c \geq 0$ we have*

$$\limsup_{t\to\infty} |W(t)|/h(t) = (2c)^{1/2} \quad a.s.,$$

*i.e. for any $\varepsilon > 0$, $((2c)^{1/2}+\varepsilon)h(t)$ belongs to the global upper-class (resp. $((2c)^{1/2}-\varepsilon)h(t)$ belongs to the global lower-class) for $W$ if and only if for any $\varepsilon > 0$ we have $I_\infty(h, c+\varepsilon) < \infty$ (resp. $I_\infty(h, c-\varepsilon) = \infty$) or, equivalently, if and only if $E_\infty(h, c+\varepsilon) < \infty$ (resp. $E_\infty(h, c-\varepsilon) = \infty$) with* $\lim_{t\to\infty} h(t)/t^{1/2} = \infty$.

We note that Remarks of Section 1 can be repeated also here, relating now our notion of global functions to that of the classical EFKP global upper-class and lower-class functions of a Wiener process.

Sometimes it is more convenient (cf., e.g., Révész [23]) to talk about upper and lower functions for the process $\{W(t)/t^{1/2},\ t > 0\}$ where $W$ is a standard Wiener process starting from zero with probability one.

Let $\mathcal{G}$ be the *class of those positive functions on* $(0, 1]$ *which are such that* $t^{1/2}g(t)$ *is non-decreasing near zero,* i.e. $t^{1/2}g(t) \in \mathcal{Q}$.

Let $\mathcal{F}$ be the *class of those positive functions on* $[1, \infty)$ *which are such that* $f(t)/t^{1/2}$ *is non-increasing in a neighbourhood of infinity,* i.e. $t^{1/2}f(t) \in \mathcal{H}$.

Introducing exactly the same way as before the notions of global and local functions for the process $\{W(t)/t^{1/2},\ t > 0\}$, as well as those of EFKP global (resp. local) upper and lower-classes, we arrive at the following conclusions.

REMARK 3.3. If $g \in \mathcal{G}$ then $g(1/t)$ is well defined for $t \in [1, \infty)$ and $g(1/t)/t^{1/2}$ is non-increasing in a neighbourhood of zero, and hence $g(1/t) \in \mathcal{F}$.

PROPOSITION 3.2. *Let $g \in \mathcal{G}$. Then $g$ is a local function for $\{W(t)/t^{1/2},\ t > 0\}$ if and only if $g(1/t)$ is a global function for the same process.*

The first example one should have in mind is $g(t) = (2 \log\log(1/t))^{1/2}$, which is a local function for $W(t)/t^{1/2}$, and hence $g(1/t) = (2 \log\log t)^{1/2}$ is a global function for the same process.

Exactly the same way as before, we have a single characterization for all these functions of $W(t)/t^{1/2}$. We introduce the following integrals:

$$\overline{I}_0(g, c) = \int_0^1 t^{-1} \exp(-cg^2(t))dt,$$

$$\overline{E}_0(g, c) = \int_0^1 t^{-1} g(t) \exp(-cg^2(t))dt,$$

$$\overline{I}_\infty(f, c) = \int_1^\infty t^{-1} \exp(-cf^2(t))dt,$$

$$\overline{E}_\infty(f, c) = \int_1^\infty t^{-1} f(t) \exp(-cf^2(t))dt.$$

THEOREM 3.4. *For local and global functions of the process $\{W(t)/t^{1/2},\ t > 0\}$ we have the following characterizations.*

(i) *Let $g \in \mathcal{G}$. A function $g(t)$ is a local function and $g(1/t)$ is a global function if and only if $\overline{I}_0(g, c) < \infty$ for some $c > 0$ or, equivalently, if and only if $\overline{E}_0(g, c) < \infty$ for some $c > 0$ and $\lim_{t\downarrow 0} g(t) = \infty$.*

(ii) *Let $f \in \mathcal{F}$. A function $f(t)$ is a global function and $f(1/t)$ is a local function if and only if $\overline{I}_\infty(f, c) < \infty$ for some $c > 0$ or, equivalently, if and only if $\overline{E}_\infty(f, c) < \infty$ for some $c > 0$ and $\lim_{t\to\infty} f(t) = \infty$.*

(iii) *Let* $g \in \mathcal{G}$. *A function* $g(t)$ *is a Chibisov-O'Reilly local function and* $g(1/t)$ *is a Chibisov-O'Reilly global function if and only if* $\overline{I}_0(g,c) < \infty$ *for all* $c > 0$ *or, equivalently,* $\overline{E}_0(g,c) < \infty$ *for all* $c > 0$ *and* $\lim_{t\downarrow 0} g(t) = \infty$.

(iv) *Let* $f \in \mathcal{F}$. *A function* $f(t)$ *is a Chibisov-O'Reilly global function and* $f(1/t)$ *is a Chibisov-O'Reilly local function if and only if* $\overline{I}_\infty(f,c) < \infty$ *for all* $c > 0$ *or, equivalently,* $\overline{E}_\infty(f,c) < \infty$ *for all* $c > 0$ *and* $\lim_{t\to\infty} f(t) = \infty$.

For the process $\{W(t)/t^{1/2},\ t > 0\}$ Theorems 1.1*, 3.1* and 3.3 read as follows.

THEOREM 3.5.

(i) *Let* $g \in \mathcal{G}$. *Then for some* $c \geq 0$ *we have*

$$\limsup_{t\downarrow 0} \frac{|W(t)|}{t^{1/2} g(t)} = (2c)^{1/2} \quad a.s.,$$

*i.e. for any* $\varepsilon > 0$, $((2c)^{1/2} + \varepsilon)g(t)$ *is a local upper-class function and* $((2c)^{1/2} - \varepsilon)g(t)$ *is a local lower-class function, and*

$$\limsup_{t\to\infty} \frac{|W(t)|}{t^{1/2} g(1/t)} = (2c)^{1/2} \quad a.s.,$$

*i.e. for any* $\varepsilon > 0$, $((2c)^{1/2} + \varepsilon)g(1/t)$ *is a global upper-class function and* $((2c)^{1/2} - \varepsilon)g(1/t)$ *is a global lower-class function, if and only if, for any* $\varepsilon > 0$ *we have* $\overline{I}_0(g, c+\varepsilon) < \infty$ *and* $\overline{I}_0(g, c-\varepsilon) = \infty$ *or, equivalently,* $\overline{E}_0(g, c+\varepsilon) < \infty$ *and* $\overline{E}_0(g, c-\varepsilon) = \infty$ *with* $\lim_{t\downarrow 0} g(t) = \infty$.

(ii) *Let* $f \in \mathcal{F}$. *Then for some* $c \geq 0$ *we have*

$$\limsup_{t\to\infty} \frac{|W(t)|}{t^{1/2} f(t)} = (2c)^{1/2} \quad a.s.,$$

*i.e. for any* $\varepsilon > 0$, $((2c)^{1/2} + \varepsilon)f(t)$ *is a global upper-class function and* $((2c)^{1/2} - \varepsilon)f(t)$ *is a global lower-class function, and*

$$\limsup_{t\downarrow 0} \frac{|W(t)|}{t^{1/2} f(1/t)} = (2c)^{1/2} \quad a.s.,$$

*i.e. for any* $\varepsilon > 0$, $((2c)^{1/2} + \varepsilon)f(1/t)$ *is a local upper-class function and* $((2c)^{1/2} - \varepsilon)f(1/t)$ *is a local lower-class function, if and only if, for any* $\varepsilon > 0$ *we have* $\overline{I}_\infty(f, c+\varepsilon) < \infty$ *and* $\overline{I}_\infty(f, c-\varepsilon) = \infty$ *or, equivalently,* $\overline{E}_\infty(f, c+\varepsilon) < \infty$ *and* $\overline{E}_\infty(f, c-\varepsilon) = \infty$ *with* $\lim_{t\to\infty} f(t) = \infty$.

## 4. Applications to Rényi-type statistics

Let $U_1, U_2, \ldots$ be independent uniform-$(0,1)$ random variables and, based on the first $n \geq 1$ of these random variables, we define the ***uniform empirical process*** $\alpha_n$ by

$$\alpha_n(s) := n^{1/2}(E_n(s) - s), \quad 0 \leq s \leq 1,$$

where $E_n$ is the uniform empirical distribution function, i.e. with $\mathbf{1}\{A\}$ being the indicator function of the set $A$, $E_n(s) := n^{-1}\sum_{i=1}^{n} \mathbf{1}\{U_i \leq s\}$. Rényi [22] studied the asymptotic behaviour of statistics like

$$\sup_{a \leq s \leq 1} \alpha_n(s)/s \quad \text{and} \quad \sup_{0 \leq s \leq b} \alpha_n(s)/(1-s),$$

as well as that of their two-sided versions. His idea of introducing these modifications of the classical Kolmogorov-Smirnov statistics was to make them more sensitive to detecting deviations on the tails from a hypothesized distribution.

We quote here Theorem 2.8 of Csáki [2], where he proves several further results, which were also inspired by Rényi [22].

THEOREM E. *Let $a_n$ be any sequence of positive constants such that, as $n \to \infty$,*

$$a_n \to 0 \quad \text{and} \quad na_n \to \infty. \tag{4.1}$$

*Then, as $n \to \infty$,*

$$\left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{a_n \leq s \leq 1} \alpha_n(s)/s \xrightarrow{\mathcal{D}} \sup_{0 \leq t \leq 1} W(t) \tag{4.2}$$

*and*

$$\left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{a_n \leq s \leq 1} |\alpha_n(s)|/s \xrightarrow{\mathcal{D}} \sup_{0 \leq t \leq 1} |W(t)|, \tag{4.3}$$

*where $W$ is a standard Wiener process.*

In the special case when $a_n = a$, a positive constant, then (4.2) and (4.3) are due to Rényi [22]. [See also M. Csörgő [4] and page 165 of M. Csörgő and Révész [11].] A slight generalization of Theorem E, requiring only that, instead of $a_n \to 0$, we have $0 < a_n < a$, for all large enough $n$ for some $0 < a < 1$, is Theorem 4.5.1 of [8]. For further results along these lines we refer to Mason [19], M. Csörgő and Mason [9], and M. Csörgő and Horváth [7]. Here we prove the following weighted version of Theorem E.

THEOREM 4.1. *Let $a_n$ be a sequence of positive constants as in* (4.1). *Let $q \in Q$ be a Chibisov-O'Reilly local function of a standard Wiener process. Then, as $n \to \infty$, we have*

$$(4.4)\qquad \left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{a_n \le s \le 1-a_n} \frac{\alpha_n(s)}{sq\left(\dfrac{a_n}{1-a_n}\dfrac{1-s}{s}\right)} \xrightarrow{\mathcal{D}} \sup_{0<t<1} W(t)/q(t)$$

*and*

$$(4.5)\qquad \left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{a_n \le s \le 1-a_n} \frac{|\alpha_n(s)|}{sq\left(\dfrac{a_n}{1-a_n}\dfrac{1-s}{s}\right)} \xrightarrow{\mathcal{D}} \sup_{0<t<1} |W(t)|/q(t).$$

*Also, if $a_n$ of* (4.1) *is decreasing in $n = 1, 2, \ldots$, then* (4.4) *and* (4.5) *hold true for any local function $q \in Q$ of a standard Wiener process.*

The right-sided versions of (4.4) and (4.5) for

$$\left(\frac{a_n}{1-a_n}\right)^{1/2} \alpha_n(s) \Big/ \left\{(1-s)q\left(\frac{a_n}{1-a_n}\frac{s}{1-s}\right)\right\}$$

can be easily formulated. Similar results hold true for the uniform quantile process, as well as for general quantile process under further conditions on its underlying distribution function like in M. Csörgő and Révész [10], and M. Csörgő and Horváth [7].

As an example for Theorem 4.1, with the local function

$$q(t) = (t \log\log(1/t))^{1/2} \in Q$$

we have

$$\sup_{a_n \le s \le 1-a_n} |\alpha_n(s)| \Big/ \left(s(1-s)\log\log\frac{s(1-a_n)}{a_n(1-s)}\right)^{1/2}$$

$$\xrightarrow{\mathcal{D}} \sup_{0<t<1} |W(t)| \Big/ \left(t \log\log\frac{1}{t}\right)^{1/2}.$$

For the sake of proving Theorem 4.1 we first state some known results.

LEMMA A. *We can define a sequence of Brownian bridges $\{B_n(s),\ 0 \le s \le 1\}$ such that, as $n \to \infty$,*

$$(4.6)\qquad \sup_{0<s<1} \frac{|\alpha_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = \begin{cases} O_P(n^{-1/2}\log n), & \text{if} \quad \nu = 1/2 \\ O_P(n^{-\nu}), & \text{if} \quad 0 < \nu < 1/2, \end{cases}$$

*and*

$$\text{(4.7)} \qquad \sup_{\lambda/n \le s \le 1-\lambda/n} \frac{|\alpha_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = O_P(n^{-\nu}), \quad \text{if} \quad 0 \le \nu < 1/2,$$

*for all* $\lambda > 0$.

The statement of (4.6) is Lemma 7 in M. Csörgő and Horváth [6]. In the case of $\nu = 1/2$, this result follows from Komlós, Major and Tusnády [16]. For $0 \le \nu < 1/4$, (4.7) is by the [8] inequality, and a new direct proof in this case is given by M. Csörgő and Horváth [5]. Mason and van Zwet [20] established (4.7) for $0 \le \nu < 1/2$ (cf. also Remark 2.1 in [8]). We note also that the [16] inequality gives (4.7) immediately with $O_P(n^{-\nu}\log n)$, $0 \le \nu \le 1/2$.

PROOF OF THEOREM 4.1. With the Brownian bridges of Lemma A we consider

$$\begin{aligned}
&\left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{a_n \le s \le 1-a_n} \frac{|\alpha_n(s) - B_n(s)|}{sq\left(\dfrac{a_n}{1-a_n}\dfrac{1-s}{s}\right)} \\
&\le \left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{a_n \le s \le \delta} \frac{|\alpha_n(s) - B_n(s)|}{sq\left(\dfrac{a_n}{1-a_n}\dfrac{1-s}{s}\right)} \\
\text{(4.8)} \qquad &\quad + \left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{\delta \le s \le 1-\delta} \frac{|\alpha_n(s) - B_n(s)|}{sq\left(\dfrac{a_n}{1-a_n}\dfrac{1-s}{s}\right)} \\
&\quad + \left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{1-\delta \le s \le 1-a_n} \frac{|\alpha_n(s) - B_n(s)|}{sq\left(\dfrac{a_n(1-s)}{s(1-a_n)}\right)} \\
&= I_1(n) + I_2(n) + I_3(n).
\end{aligned}$$

Let $n$ be so large, and hence $a_n$ so small, that $q(\cdot)$ is already non-decreasing. Then

$$\sup_{\delta \le s \le 1-\delta} \frac{1}{sq\left(\dfrac{a_n(1-s)}{s(1-a_n)}\right)} \le \frac{1}{\delta q\left(\dfrac{a_n\delta}{(1-\delta)(1-a_n)}\right)}.$$

Consequently, by (4.6) and (2.2), as $n \to \infty$ we obtain

$$\text{(4.9)} \qquad I_2(n) \le O_P(n^{-1/2}\log n)\frac{(1-\delta)^{1/2}}{\delta^{3/2}} \frac{\left(\dfrac{a_n\delta}{(1-\delta)(1-a_n)}\right)^{1/2}}{q\left(\dfrac{a_n\delta}{(1-\delta)(1-a_n)}\right)}$$

$$= O_P(n^{-1/2}\log n)o(1).$$

Next, with $0 < \nu < 1/2$, we have

$$I_1(n) = \sup_{a_n \le s \le \delta} \frac{|\alpha_n(s) - B_n(s)|}{(s(1-s))^{1/2} q\left(\frac{a_n(1-s)}{s(1-a_n)}\right)} \left(\frac{a_n(1-s)}{s(1-a_n)}\right)^{1/2}$$

$$\le (1-\delta)^{-1/2} \sup_{a_n \le s \le \delta} \frac{|\alpha_n(s) - B_n(s)|}{s^{\nu} s^{1/2-\nu}} \sup_{a_n \le s \le \delta} \frac{\left(\frac{a_n(1-s)}{s(1-a_n)}\right)^{1/2}}{q\left(\frac{a_n(1-s)}{s(1-a_n)}\right)}$$

$$\le (1-\delta)^{-1/2} a_n^{-\nu} O_P(n^{-\nu}) \sup_{\frac{a_n(1-\delta)}{\delta(1-a_n)} \le t < 1} t^{1/2}/q(t) \tag{4.10}$$

$$\le O_P((na_n)^{-\nu}) \max\left( \sup_{\frac{a_n(1-\delta)}{\delta(1-a_n)} \le t < \varepsilon} \frac{t^{1/2}}{q(t)}, \sup_{\varepsilon \le t \le 1} \frac{t^{1/2}}{q(t)} \right)$$

$$= O_P((na_n)^{-\nu}) O(1) = o_P(1),$$

on account of (4.1), (4.6) and (2.2).

We have also

$$I_3(n) \le \sup_{1-\delta \le s \le 1-a_n} \frac{|\alpha_n(s) - B_n(s)|}{(s(1-s))^{1/2}} \sup_{\left(\frac{a_n}{1-a_n}\right)^2 \le t \le \frac{a_n\delta}{(1-\delta)(1-a_n)}} t^{1/2}/q(t)$$

$$= O_P(1)o(1) = o_P(1), \tag{4.11}$$

by (4.7) and (2.2).

A combination of (4.8), (4.9), (4.10) and (4.11) yields

$$\left(\frac{a_n}{1-a_n}\right)^{1/2} \sup_{a_n \le s \le 1-a_n} \frac{|\alpha_n(s) - B_n(s)|}{sq\left(\frac{a_n(1-s)}{s(1-a_n)}\right)} = o_P(1), \quad n \to \infty. \tag{4.12}$$

We note that, by Doob's transformation (cf., e.g., (1.4.5) in M. Csörgő and Révész [11]), we have for each $n = 1, 2, \ldots$

$$\left\{ \left(\frac{a_n}{1-a_n}\right)^{1/2} \frac{B_n(s)}{s},\ a_n \le s \le 1 - a_n \right\}$$

$$\stackrel{\mathcal{D}}{=} \left\{ W\left(\frac{a_n(1-s)}{(1-a_n)s}\right),\ a_n \le s \le 1-a_n \right\},$$

and, therefore, for each $n = 1, 2, \ldots$

$$\sup_{a_n \le s \le 1-a_n} \left(\frac{a_n}{1-a_n}\right)^{1/2} \frac{B_n(s)}{s} \stackrel{\mathcal{D}}{=} \sup_{a_n \le s \le 1-a_n} W\left(\frac{a_n(1-s)}{(1-a_n)s}\right) \tag{4.13}$$

$$\stackrel{\mathcal{D}}{=} \sup_{\left(\frac{a_n}{1-a_n}\right)^2 \le t \le 1} W(t).$$

Hence, in order to obtain (4.4) and (4.5), by (4.12) and (4.13) it suffices to show that we have

$$\sup_{\left(\frac{a_n}{1-a_n}\right)^2 \le t < 1} W(t)/q(t) \xrightarrow{\mathcal{D}} \sup_{0<t<1} W(t)/q(t) \tag{4.14}$$

and

$$\sup_{\left(\frac{a_n}{1-a_n}\right)^2 \le t < 1} |W(t)|/q(t) \xrightarrow{\mathcal{D}} \sup_{0<t<1} |W(t)|/q(t). \tag{4.15}$$

Now if $q \in Q$ is a Chibisov-O'Reilly local function of $W$ then (4.14) and (4.15) are immediate, for in the latter case we have by definition

$$\lim_{n\to\infty} \sup_{0<t\le\left(\frac{a_n}{1-a_n}\right)^2} |W(t)|/q(t) \stackrel{\text{a.s.}}{=} 0. \tag{4.16}$$

On the other hand, if $q \in Q$ is a local function of $W$, then, on assuming that $a_n$ of (4.1) is decreasing in $n = 1, 2, \ldots$ to zero, the proofs of (4.14) and (4.15) can be easily established along the lines of the proof of Lemma 4.2.2 of [8].

## REFERENCES


[1] BINGHAM, N. H., Variants on the law of the iterated logarithm, *Bull. London Math. Soc.* **18** (1986), 433—467. *MR* **87k**: 60087.

[2] CSÁKI, E., On tests based on empirical distribution functions, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **23** (1977), 239—327. *MR* **57** # 4415. (English translation in *Selected Transl. Math. Statist. Probab.* **15** (1981), 229—317.)

[3] CHIBISOV, D. M., Some theorems on the limiting behaviour of an empirical distribution function, *Trudy Mat. Inst. Steklov* **71** (1964), 104—112. *MR* **30** # 2543. (English translation in *Selected Transl. Math. Statist. Probab.* **6** (1964), 147—156.)

[4] CSÖRGŐ, M., A new proof of some results of Rényi and the asymptotic distribution of the range of his Kolmogorov-Smirnov type random variables, *Canad. J. Math.* **19** (1967), 550—558. *MR* **35** # 3712

[5] CSÖRGŐ, M. and HORVÁTH, L., Approximations of weighted empirical and quantile processes, *Stat. Probab. Lett.* **4** (1986), 275—280. *MR* **88m**: 60095

[6] CSÖRGŐ, M. and HORVÁTH, L., Central limit theorems for $L^p$-norms of density estimators, *Probab. Theory Related Fields* **80** (1988), 269—291. *MR* **89i**: 60046

[7] CSÖRGŐ, M. and HORVÁTH, L., On the distributions of the supremum of weighted quantile processes, *Studia Sci. Math. Hungar.* **25** (1990), 353—375.

[8] CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M., Weighted empirical and quantile processes, *Ann. Probab.* **14** (1986), 31—85. *MR* **87e**: 60041

[9] CSÖRGŐ, M. and MASON, D. M., On the asymptotic distribution of weighted empirical and quantile processes in the middle and on the tails, *Stoch. Process. Appl.* **21** (1985), 119—132. *MR* **87j**: 60036.

[10] CSÖRGŐ, M. and RÉVÉSZ, P., Strong approximations of the quantile process, *Ann. Statist.* **6** (1978), 882—894. *MR* **58** # 18683

[11] CSÖRGŐ, M. and RÉVÉSZ, P., *Strong approximations in probability and statistics*, Academic Press, New York, 1981. *MR* **84d**: 600050

[12] DOOB, J. L., *Classical potential theory and its probabilistic counterpart*, Grundlehren der mathematischen Wissenschaften, Bd. 262, Springer-Verlag, New York, 1984. *MR* **85k**: 31001

[13] FELLER, W., *An introduction to probability theory and its applications*, Vol I, Third edition, Wiley, New York, 1968. *MR* **37** # 3604

[14] ITÔ, K. and McKEAN, H. P. Jr., *Diffusion processes and their sample paths*, Die Grundlehren der mathematischen Wissenschaften, Bd. 125, Springer-Verlag, Berlin, 1965. *MR* **33** # 8031

[15] KHINTCHINE, A., *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse der Math. und ihrer Grenzgebiete, Bd. 2, H. 4, Springer-Verlag, Berlin, 1933. *Zbl* **7**, 216. (Reprinted by Chelsea, New York, 1948.)

[16] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G., An approximation of partial sums of independent RV's, and the sample DF I, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **32** (1975), 111—131. *MR* **51** # 11605b

[17] LÉVY, P., *Processus stochastiques et mouvement brownien*, Gauthier-Villars, Paris, 1948. *MR* **10** - 551

[18] LÉVY, P., Le caractère universel de la courbe du mouvement brownien et la loi du logarithme itéré, *Rend. Circ. Mat. Palermo* (2) **4** (1955), 337—366. *MR* **17** - 1096

[19] MASON, D. M., The asymptotic distribution of generalized Rényi statistics, *Acta Sci. Math. (Szeged)* **48** (1985), 315—323. *MR* **87e**: 60042

[20] MASON, D. M. and van ZWET, W., A refinement of the KMT inequality for the uniform empirical process, *Ann. Probability* **15** (1987), 871—884. *MR* **88k**: 60068

[21] O'REILLY, N., On the weak convergence of empirical processes in sup-norm metrics, *Ann. Probability* **2** (1974), 642—651. *MR* **52** # 4367

[22] RÉNYI, A., On the theory of order statistics, *Acta Math. Acad. Sci. Hungar.* **4** (1953), 191—231. *MR* **15** - 885

[23] RÉVÉSZ, P., On the increments of Wiener and related processes, *Ann. Probability* **10** (1982), 613—622. *MR* **83i**: 60048

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# FUNCTIONAL ERDŐS-RÉNYI LAWS

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**Abstract**

We present the functional form of the Erdős-Rényi " new law of large numbers".

## 1. Introduction

Let $X, X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Let $\phi(t) = E(\exp(tX))$ be the moment-generating function of $X$, and let $t_0 = \sup\{t : \phi(t) < \infty\}$ and $t_1 = \inf\{t : \phi(t) < \infty\}$. We will assume, at times, that the following conditions hold:

(A1) $E(X) =: \mu \in (-\infty, \infty)$;

(A2) $X$ is nondegenerate, i.e. $P(X = x) < 1$ for all $x$;

(A3) $t_0 > 0$;

(A4) $t_1 < 0$;

(A5) $\mathrm{Var}(X) =: \sigma^2 \in [0, \infty)$.

Note for further use that (A3)-(A4) $\Rightarrow$ (A1)-(A5). On the other hand, (A3) (resp. (A4)) alone is not sufficient to ensure (A1) or (A5).

Denote by $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ the partial sums of $X_1, \ldots, X_n$ and set $S(t) = S_{[t]}$ for $t \geq 0$, where $[t] \leq t < [t] + 1$ denotes the integer part of $t$. In this paper, we are concerned with the limiting behavior as $T \to \infty$ of the standardized increment functions, defined for $x \geq 0$ by

$$\eta_{x,T}(s) = a_T^{-1}(S(x + sa_T) - S(x)) \quad \text{for} \quad 0 \leq s \leq 1, \tag{1.1}$$

where $0 < a_T \leq T$ is a function of $T > 0$. We will use the notation

$$\mathcal{F}_T = \{\eta_{x,T} : 0 \leq x \leq T - a_T\}, \tag{1.2}$$

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and assume, at times, that the following conditions are imposed upon $\{a_T,\ T>0\}$:

(K1) $a_T \uparrow$ and $T^{-1}a_T \downarrow$;

(K2) $a_T/\log T \to c \in (0,\infty]$ as $T \to \infty$;

(K3) $(\log(T/a_T))/\log_2 T \to d \in (0,\infty]$ as $T \to \infty$.

Here, and in the sequel, we set $\log_2 T = \log(\log(\max(T,3)))$. We will make use of the following notation. $B(0,1)$ (resp. $C(0,1)$) denotes the set of all bounded (resp. continuous) functions on $[0,1]$. These spaces will be, at times, endowed with the topology $U$ of uniform convergence on $[0,1]$ defined by the sup-norm $\|f\| := \sup_{0\leq s\leq 1}|f(s)|$. In general, we will denote by $(\mathcal{E},\mathcal{T})$ the set $\mathcal{E}$, endowed with the topology $\mathcal{T}$. For any $\delta>0$ and $f \in A \subset B(0,1)$, we will set

$$N_\varepsilon(f) = \{g \in B(0,1) : \|f-g\| < \varepsilon\}, \tag{1.3}$$

and

$$A^\varepsilon = \{h \in B(0,1) : \|h-g\| < \varepsilon \text{ for some } g \in A\} = \bigcup_{g\in A} N_\varepsilon(g). \tag{1.4}$$

Whenever (A1-2-3-4-5) and (K1-2) hold with $c=\infty$, the description of the limiting behavior of $\mathcal{F}_T$ follows from the corresponding results for the Wiener process. Namely, the strong invariance principle for partial sums due to Komlós, Major and Tusnády (1976) (see also Einmahl (1989)) shows the existence of a probability space on which is defined a standard Wiener process $\{W(t),\ t\geq 0\}$ such that, under (A1-3-4-5)

$$\sup_{0\leq t\leq T}|S(t)-\mu t-\sigma W(t)| = O(\log T) \text{ a.s. as } T\to\infty. \tag{1.5}$$

By combining (1.5) with the results of Révész (1979) and of Deheuvels and Révész (1991), one obtains readily the following Theorem 1.1. For the statement of this theorem, we need more notation. It will be convenient to set (under (A1-2-5))

$$\mathcal{H}_T = b_T^{-1}(\mathcal{F}_T-\mu I) = \{b_T^{-1}(\eta_{x,T}-\mu I) :\ 0\leq x\leq T-a_T\}, \tag{1.6}$$

where $I$ is the identity function on $[0,1]$, and $b_T := \sigma a_T^{-1/2}\{2(\log(T/a_T)+\log_2 T)\}^{1/2}$.

Let further, for $f \in B(0,1)$,

$$(1.7)\quad \begin{aligned} &J(f) = \int_0^1 \dot{f}^2(s)ds && \text{if } f \text{ is absolutely continuous on } [0,1] \\ &&& \text{with Lebesgue derivative } \dot{f} = df/ds; \\ &J(f) = \infty && \text{if } f \text{ is not absolutely continuous on } [0,1]. \end{aligned}$$

For any $\lambda \geq 0$, consider the Strassen-type sets (see e.g. Strassen (1964) for $\lambda = 1$):

$$(1.8)\qquad S_\lambda = \{f \in C(0,1) : f(0) = 0 \text{ and } J(f) \leq \lambda\}.$$

Note for further use that $S_\lambda$ is a compact subset of $(C(0,1), U)$ (see e.g. Varadhan (1966) and Lemma 2.1 and Example 2.1 in Deheuvels and Mason (1990)).

THEOREM 1.1. *Assume that* (A1-2-3-4-5) *and* (K1-2-3) *hold with* $c = \infty$. *Then:*

1°) *Whenever* $d = \infty$, *for any* $\varepsilon > 0$, *there exists almost surely a* $T(\varepsilon) < \infty$ *such that for all* $T \geq T(\varepsilon)$

$$(1.9)\qquad S_1 \subset \mathcal{H}_T^\varepsilon \subset S_1^{2\varepsilon}.$$

2°) *Whenever* $d < \infty$, *for any* $\varepsilon > 0$, *there exists almost surely a* $T(\varepsilon) < \infty$ *such that for all* $T \geq T(\varepsilon)$

$$(1.10)\qquad S_{(\frac{d}{d+1})} \subset \mathcal{H}_T^\varepsilon \subset S_1^{2\varepsilon}.$$

*Moreover, we have*

$$(1.11)\qquad \lim_{T\to\infty} P(S_{(\frac{d}{d+1})} \subset \mathcal{H}_T^\varepsilon \subset S^{2\varepsilon}_{(\frac{d}{d+1})}) = 1,$$

*and for any* $f \in S_1$, *there exists almost surely a sequence* $T(n,f) \to \infty$ *as* $n \to \infty$, *such that* $f \in \mathcal{H}^\varepsilon_{T(n,f)}$ *for all* $n$.

PROOF. By (1.5) and the assumptions imposed upon $\{a_T,\ T > 0\}$ the proof of (1°) is a consequence of the results of Révész (1979), while (2°) follows from the results of Deheuvels and Révész (1991). □

REMARK 1.1. Let $K(\cdot)$ denote a continuous functional on $(B(0,1), U)$. A direct application of Theorem 1.1 is that, under the assumptions of this theorem, we have:

1°) For $d = \infty$, almost surely

$$\lim_{T\to\infty}\{\sup_{f\in\mathcal{K}_T} K(f)\} = \sup_{f\in S_1} K(f) \text{ a.s.}; \tag{1.12}$$

2°) For $d < \infty$, almost surely

$$\begin{aligned}\liminf_{T\to\infty}\{\sup_{f\in\mathcal{K}_T} K(f)\} &= \sup_{f\in S_{(\frac{d}{d+1})}} K(f) \\ &\le \limsup_{T\to\infty}\{\sup_{f\in\mathcal{K}_T} K(f)\} = \sup_{f\in S_1} K(f),\end{aligned} \tag{1.13}$$

and

$$\lim_{T\to\infty}\{\sup_{f\in\mathcal{K}_T} K(f)\} = \sup_{f\in S_{(\frac{d}{d+1})}} K(f) \text{ in probability.} \tag{1.14}$$

Typical examples of functionals $K(\cdot)$ are $K(f) = \pm f(1)$, $|f(1)|$, $\sup_{0\le s\le 1} \pm f(s)$ and $\sup_{0\le s\le 1} |f(s)|$, in which case (1.12)-(1.13) yield well-known results due to Csörgő and Révész (1979, 1981) and Book and Shore (1978), for increments of Wiener processes (and partial sums via (1.5)).

Whenever (A1-2-3) and (K2) hold with $c < \infty$, the fact that Theorem 1.1 does not hold in general is easily deduced from Remark 1.1 and the famous Erdős-Rényi law of large numbers which we will now discuss. We refer to Csörgő (1979), de Acosta and Kuelbs (1983), Deheuvels, Devroye and Lynch (1986), and Deheuvels and Devroye (1987) for general references and refinements of the original statement of Erdős-Rényi (1970). We will need the following notation. Denote by $\psi(\cdot)$ the so-called Chernoff function (see e.g. Chernoff (1952)) of $X$, defined for all $-\infty < \alpha < \infty$ by

$$\psi(\alpha) = \sup_t\{\alpha t - \log\phi(t)\}, \tag{1.15}$$

where the supremum in (1.15) is evaluated over all $t$'s such that $\phi(t) < \infty$ (i.e. over an interval with end-points $t_1$ and $t_0$). It is noteworthy (see Lemma 2.1 in the sequel) that $\psi(\cdot)$ is a convex function of $(-\infty, \infty)$ onto $[0, \infty]$, satisfying (under either (A1-3) or (A1-4)) $\psi(\mu) = 0$. Therefore, for any $v > 0$, there exist $-\infty \le \alpha^-(v) \le \mu \le \alpha^+(v) \le \infty$, defined by

$$\alpha^- = \begin{cases} \sup\{\alpha < \mu : \psi(\alpha) \ge 1/v\} & \text{if } t_1 < 0, \\ -\infty & \text{if } t_1 = 0, \end{cases} \tag{1.16}$$

and

$$\alpha^+ = \begin{cases} \inf\{\alpha > \mu : \psi(\alpha) \ge 1/v\} & \text{if } t_0 > 0, \\ \infty & \text{if } t_0 = 0. \end{cases} \tag{1.17}$$

The following theorem gives a general version of the Erdős-Rényi theorem for partial sums.

THEOREM 1.2. *Assume that (A1) and (K2) hold with* $c < \infty$. *Then:*

1°) *Whenever (A3) (resp. (A4)) holds (i.e.* $t_0 > 0$ *(resp.* $t_1 < 0$*)), we have, almost surely,*

$$\lim_{T\to\infty}\{\sup_{0\le x\le T-a_T}\eta_{x,T}(1)\} = \alpha^+(c) \tag{1.18}$$
$$(resp.\ \lim_{T\to\infty}\{\inf_{0\le x\le T-a_T}\eta_{x,T}(1)\} = \alpha^-(c)).$$

2°) *Whenever (A3) (resp. (A4)) does not hold (i.e.* $t_0 = 0$ *(resp.* $t_1 = 0$*)), we have, almost surely,*

$$\limsup_{T\to\infty}\{\sup_{0\le x\le T-a_T}\eta_{x,T}(1)\} = \infty \tag{1.19}$$
$$(resp.\ \liminf_{T\to\infty}\{\inf_{0\le x\le T-a_T}\eta_{x,T}(1)\} = -\infty).$$

PROOF. (1.19) is due to Steinebach (1978) (see also Lynch (1983)), while (1.18) is the so-called "full form" of the Erdős-Rényi theorem due to Deheuvels and Devroye (1987) (see also Csörgő (1979)). □

REMARK 1.2. In the original statement of their "new law of large numbers", Erdős and Rényi (1970) assumed that (A3-4) (and consequently (A1-3-4-5)) hold. Moreover, they also assumed that $\alpha^{\pm}(c)$ in (1.18) is restricted to vary within the set $\{m(t) := \phi'(t)/\phi(t) : t_1 < t < t_0\}$. Deheuvels, Devroye and Lynch (1986) showed that this condition entails that $c > c_0 := \int_0^{t_0} tm'(t)dt$ in the "+" case, and $c > c_1 := \int_0^{t_1} tm'(t)dt$ in the "-" case, and characterized the distributions for which $c_0 > 0$ (resp. $c_1 > 0$). Deheuvels and Devroye (1987) proved that (1.18) holds for all $c > 0$. Following an observation of Steinebach (1978), it may be verified that the assumption (A1) that $-\infty < \mu < \infty$ may also be relaxed in Theorem 1.2.

REMARK 1.3. In the degenerate case where $P(X = \mu) = 1$, we have $\psi(\alpha) = \infty$ for $\alpha \neq \mu$ and $\psi(\mu) = 0$. Thus, by (1.18)-(1.19) $\alpha^{\pm}(c) = \mu$ for all $c > 0$. Theorem 1.2 is then a trivial consequence of the fact that $\|\eta_{x,T} - \mu I\| \to 0$ a.s. as $T \to \infty$, uniformly over $0 \le x \le T - a_T$. We will not consider this case further and assume throughout the sequel that (A2) holds.

REMARK 1.4. In view of Remark 1.1 taken with $K(f) = \pm f(1)$, and Theorem 1.2, we see that the conclusion of Theorem 1.1 is invalid for $c < \infty$ when $X$ is not normally distributed. This is essentially the solution of the so-called "stochastic geyser problem" proved originally by Bártfai (1966) (see e.g. §2.4 in Csörgő and Révész (1981), Erdős-Rényi (1970) and Grill (1989)).

In view of Theorem 1.1 and 1.2, it is natural to seek functional versions of the Erdős-Rényi Theorem 1.2. We will obtain these results in Theorems 3.1 and 3.2, stated in Section 3, and which constitute the main contributions of this paper. Before stating these theorems, we will need several technical results, concerning large deviations and functional spaces, which will be proved in Section 2. In Section 4, we mention briefly some applications of these functional laws.

We conclude this introduction by mentioning the following relevant references on the topic of Erdős-Rényi theorems. The list is far from exhaustive, but shows evidently that this topic has received a continuous interest in the last decades. We refer to Arratia and Waterman (1989), Arratia, Gordon and Waterman (1990), Book (1973), Book and Truax (1976), Csáki, Földes and Komlós (1987), Csörgő and Steinebach (1981), Deheuvels (1985), Deheuvels, Erdős, Grill and Révész (1987), Erdős and Révész (1975), Földes (1979), Guibas and Odlyzko (1980), Révész (1983), and the references therein.

## 2. Large deviations and moment-generating functions

### 2.1. The Chernoff function

We inherit the notation of Section 1, and let $\psi(\cdot)$ be as in (1.15). The following lemma describes some useful general properties of $\psi(\cdot)$.

LEMMA 2.1. *Under* (A1), *the Chernoff function* $\psi(\cdot)$ *is a (possibly infinite) non-negative convex function on* $(-\infty,\infty)$, *such that* $\psi(\mu) = 0$,

$$\lim_{\alpha\to\infty}(\psi(\alpha)/\alpha) = t_0 \quad \textit{and} \quad \lim_{\alpha\to-\infty}(\psi(\alpha)/\alpha) = t_1. \tag{2.1}$$

PROOF. The case where $t_1 = t_0 = 0$ is trivial for then $\psi(\alpha) = 0$ for all $\alpha \in (-\infty,\infty)$, and (2.1) is satisfied. For $t_1 < t_0$, the convexity of $\psi(\cdot)$ is straightforward since $\psi(\cdot)$ is the supremum of the linear functions $\alpha \to \alpha t - \log\phi(t)$, when $t$ varies in the interval (with end-points $t_1$ and $t_0$) in which $\phi(t) < \infty$. If $t_1 = 0$ (resp. $t_0 = 0$), then $\psi(\alpha) = 0$ for all $\alpha < \mu$

(resp. $\alpha > \mu$). It remains to consider the cases $t_1 < 0 \leq t_0$ and $t_1 \leq 0 < t_0$, which are equivalent after the formal change of $X$ into $-X$. Thus, all we need for (2.1) is to show that, whenever $t_1 \leq 0 < t_0$, we have

$$\lim_{\alpha \to \infty} (\psi(\alpha)/\alpha) = t_0. \tag{2.1a}$$

To prove (2.1a), it is convenient to set $c(\alpha) = 1/\psi(\alpha)$, or equivalently

$$\exp(-1/c(\alpha)) = \inf_t \{\phi(t)e^{-t\alpha}\} = \exp(-\psi(\alpha)). \tag{2.1b}$$

The study of the functional relation (2.1b) is made in Deheuvels, Devroye and Lynch (1986). By setting $A = \lim_{t \uparrow t_0} m(t)$, where $m(t) = \phi'(t)/\phi(t)$, they show (see their §2 and Theorem 2) that, under (A1), $c(\alpha)$ varies between $\infty$ and $c_0 = 1/\int_0^{t_0} tm'(t)dt$ when $\alpha$ varies between $\mu$ and $A$. Moreover, if $t^*(\alpha)$ is the solution of the equation $m(t) = \alpha$, then $t^*(\alpha) = \psi'(\alpha)$ and varies between 0 and $t_0$ when $\alpha$ varies between $\mu$ and $A$. Finally, $c_0 = 0$ except in the following two cases:

(i) $A < \infty$, $t_0 < \infty$, in which case $\operatorname{ess\,sup} X_1 = \infty$ and $c_0 = 1/(At_0 - \log \phi(t_0))$;

(ii) $A < \infty$, $t_0 = \infty$, in which case $\operatorname{ess\,sup} X_1 = A$, $P(X = A) > 0$ and $c_0 = -1/\log P(X = A)$.

Consider the remainder case:

(iii) $A = \infty$, in which case $\psi(\alpha) = 1/c(\alpha)$ varies between 0 and $\infty$ as $\alpha$ varies between $\mu$ and $\infty$. Here, $\lim_{\alpha \to \infty} (\psi(\alpha)/\alpha) = \lim_{\alpha \to \infty} \psi'(\alpha) = \lim_{\alpha \to \infty} t^*(\alpha) = t_0$, yielding (2.1a) as sought.

In case (i), $\phi(t_0) < \infty$ and $\phi(t) = \infty$ for all $t > t_0$. It is readily verified here that $\psi(\alpha) = \alpha t_0 - \log \phi(t_0)$ for all $\alpha \geq A$, so that again (2.1a) holds.

Finally, in case (ii), $\log \phi(t) = (1 + o(1))At$ as $t \to \infty$, so that, by (1.15), $\psi(\alpha) = \infty$ for $\alpha > A$, $\psi(A) = -\log P(X = A)$. Since then $\lim_{\alpha \to \infty} (\psi(\alpha)/\alpha) = \infty$ and $t_0 = \infty$, we have (2.1a).

Combining the preceding three cases, we obtain readily the proof of (2.2). This completes the proof of Lemma 2.1. □

LEMMA 2.2. *Under* (A1-2), *the Chernoff function* $\psi(\cdot)$ *is always finite on a non-empty sub-interval of* $(-\infty, \infty)$. *This sub-interval has upper (resp. lower) end-point equal to a (resp. b) if and only if* $a = \operatorname{ess\,sup} X < \infty$ *(resp.* $b = \operatorname{ess\,inf} X > -\infty$*), and an infinite upper (resp. lower) end-point otherwise.*

PROOF. Excluding the trivial case where $t_1 = t_0 = 0$ and $\psi(\alpha) = 0$ for all $\alpha$, we see that for $t_1 < t_0$, the function $m(t) = \phi'(t)/\phi(t)$ is strictly convex and increasing on $(t_1, t_0)$ (see e.g. Deheuvels, Devroye and Lynch (1986)). Let, accordingly,

$$B = \lim_{t \downarrow t_1} m(t) \quad \text{and} \quad A = \lim_{t \uparrow t_0} m(t). \tag{2.1c}$$

Denoting by $t^*(\alpha)$ the solution of $m(t) = \alpha$ for $B < \alpha < A$, we see that

$$\psi(\alpha) = \alpha t^*(\alpha) - \log \phi(t^*(\alpha)) < \infty \quad \text{for} \quad B < \alpha < A. \tag{2.1d}$$

By (2.1d), $\psi(\cdot)$ is finite on a non-empty sub-interval of $(-\infty, \infty)$. The conclusion of Lemma 2.2 corresponds to the fact that, by the just-given proof of Lemma 2.1, $\psi(\alpha) = \infty$ for $\alpha > A$ in case (ii), while $\psi(\alpha) < \infty$ for all $\alpha \geq \mu$ in the other two cases. □

## 2.2. Functional spaces

Let $BV(0,1)$ denote the space of all right-continuous distribution functions of bounded Lebesgue-Stieltjes measures on $[0,1]$. Namely, any $f \in BV(0,1)$ is of the form

$$f(s) = \nu([0,s]) \quad \text{for} \quad -\infty < \nu < \infty,$$

where $\nu = \nu_1 - \nu_2$ is the difference of positive measures on $(-\infty, \infty)$, satisfying $\nu_i([0,1]) \geq 0$ and $\nu_i((-\infty,-1) \cup (1,\infty)) = 0$ for $i = 1,2$. In general, such a decomposition is not unique, and we are led to choose a specific representative. For this sake, we set

$$f_\pm(s) = \sup\{\sum_{i=1}^{k} (f(\tau_i) - f(\tau_{i-1}))^{(\pm)} : \tau_0 = 0- < \tau_1 < \ldots < \tau_k = s\} \text{ for } s > 0, \tag{2.2}$$

where the sup is taken over all $k \geq 2$ and $0 < \tau_1 < \ldots < \tau_{k-1} < s$, $(u)^{(\pm)} := \max(\pm u, 0)$, and $0-$ denotes an arbitrary value of $t < 0$ (for which $f(t) = 0$). By letting further

$$f_\pm(s) = 0 \text{ for } s < 0, \text{ and } f_\pm(0) = (f(0))^{(\pm)}, \tag{2.3}$$

it is readily verified that $f = f_+ - f_-$, and that both $f_+$ and $f_-$ belong to the subset $IR(0,1)$ of $BV(0,1)$ which consists of all right-continuous distribution functions of non-negative and bounded measures on $[0,1]$. Moreover, $f_\pm(s) = \nu_\pm([0,s])$, where $df = \nu_+ - \nu_-$ is the Hahn-Jordan decomposition of $df$ (see e.g. Rudin (1979) p. 173). This decomposition is such that there exists measurable sets $A_+$ and $A_-$ with

$$A_+ \cap A_- = \emptyset, \; A_+ \cup A_- = [0,1], \; \nu_+(A_-) = \nu_-(A_+) = 0. \tag{2.4}$$

A direct consequence of (2.3)-(2.4) is stated in the following lemma.

LEMMA 2.3. *For any decomposition* $f = f_{(+)} - f_{(-)}$ *of* $f \in BV(0,1)$, *with* $f_{(\pm)} \in IR(0,1)$, *we have, letting* $\nu_\pm = df_\pm$ *and* $\nu_{(\pm)} = df_{(\pm)}$, *for each measurable subset* $B$ *of* $[0,1]$,

$$\nu_\pm(B) \leq \nu_{(\pm)}(B). \tag{2.5}$$

PROOF. By (2.4), if $\nu = df$, we have $\nu_\pm(B) = \nu_\pm(B \cap A_\pm) = \pm\nu(B \cap A_\pm) \leq \nu_{(\pm)}(B \cap A_\pm) = \nu_{(\pm)}(B)$ for $B \subset A_\pm$, and $\nu_\pm(B) = 0 \leq \nu_{(\pm)}(B)$ for $B \subset A_\mp$, hence result. □

Write the Lebesgue decomposition of the non-negative measure $df$ as follows. Let

$$f_\pm(s) = \int_0^s \dot{f}_\pm(u)du + f_\pm^{(S)}(s) \text{ for } -\infty < s < \infty, \tag{2.6}$$

where $\dot{f}_\pm = \frac{d}{ds} f_\pm$ denotes the Lebesgue derivative of $f_\pm$, defined uniquely up to an a.e. equivalence, and $f^{(S)}$ denotes the distribution function of a measure orthogonal to the Lebesgue measure on $[0,1]$. In view of (2.4), we will assume, without loss of generality, that $\dot{f}_\pm$ is a finite measurable function such that

$$\begin{aligned} \dot{f}_\pm(s) &= 0 \quad \text{for} \quad s \notin [0,1], \\ \dot{f}_-(s) &= 0 \quad \text{for} \quad s \in A_+, \\ \dot{f}_+(s) &= 0 \quad \text{for} \quad s \in A_-. \end{aligned} \tag{2.7}$$

Following (2.6), we will set for $f \in BV(0,1)$,

$$f(s) = \int_0^s \dot{f}(u)du + f^{(S)}(s) \text{ for } -\infty < s < \infty, \tag{2.8}$$

where $\dot{f} = \dot{f}_+ - \dot{f}_-$ and $f^{(S)} = f_+^{(S)} - f_-^{(S)}$.

It will be convenient to denote by $|f|_v(s) := f_+(s) + f_-(s)$ for $-\infty < s < \infty$ the total variation of $f$ in the interval $[0,s]$.

We will consider the following topologies defined on $BV(0,1)$ and $IR(0,1)$. We denote by $W$ the weak (abbreviated from weak*) topology on either $BV(0,1)$ or $IR(0,1)$. On $IR(0,1)$, the weak topology may be metricised by the Lévy metric $d_L$ defined as follows. For any $f \in IR(0,1)$ and $g \in IR(0,1)$, set

$$d_L(f,g) = \tag{2.9}$$

$$= \inf\{y > 0 : f(x-y) - y \le g(x) \le f(x+y) + y \text{ for } -\infty < x < \infty\}.$$

The weak convergence of $f_n \in IR(0,1)$ to $f \in IR(0,1)$ is equivalent to the (pointwise) convergence of $f_n(s)$ to $f(s)$ for each $s$, continuity point of $f$. Such a simple characterization does not hold for the weak topology on $BV(0,1)$. Given a net $f_\alpha \in BV(0,1)$, the weak convergence of $f_\alpha$ to $f \in BV(0,1)$ is equivalent to the convergence of $\int_{-\infty}^{\infty} \gamma(s) df_\alpha(s)$ to $\int_{-\infty}^{\infty} \gamma(s) df(s)$ for each continuous function $\gamma$ (recall that $df_\alpha$ and $df$ have support in $[0,1]$ so that no restriction on $\gamma$ is needed). The following result, due to Högnäs (1977), gives a simple characterization of this convergence. Introduce the metric

$$(2.10) \quad d_W(f,g) = \int_0^1 |f(u) - g(u)|du + |f(1) - g(1)|,$$
$$\text{for } f \in BV(0,1) \text{ and } g \in BV(0,1).$$

We will denote by $BV_C(0,1)$ the set of all $f \in BV(0,1)$ such that $|f|_v(1) \le C$.

LEMMA 2.4. *A net $f_\alpha \in BV(0,1)$ is weakly convergent to $f \in BV(0,1)$ if and only if:*

(i) *There exists a constant $0 \le C < \infty$ such that $f_\alpha$ is ultimately in $BV_C(0,1)$;*

(ii) $d_W(f_\alpha, f) \to 0$.

PROOF. See e.g. Högnäs (1977). □

The nice properties of the weak topology on IR(0,1) are not always satisfied on BV(0,1) as shown in Example 2.1 below. Denote by $\mathcal{O}$ the null function, and set "$\to_{\mathcal{T}}$" for the convergence in the topological space $(\mathcal{E}, \mathcal{T})$.

EXAMPLE 2.1. Let $\{u_n\}$ be a bounded positive sequence such that $u_n \not\to 0$. If $f_n$ denotes the distribution function of the measure with mass $u_n$ and $\frac{1}{2} - \frac{1}{n}$ and $-u_n$ at $\frac{1}{2} + \frac{1}{n}$, we see that $f_n \to_W \mathcal{O}$. On the other hand, $\frac{1}{2}$ is a continuity point of $\mathcal{O}$, and $f_n(\frac{1}{2}) = u_n \not\to \mathcal{O}(\frac{1}{2}) = 0$.

LEMMA 2.5. *For any $0 \le C < \infty$, $(BV_C(0,1), W)$ is a compact metric space (with the specific choice of $d_W$). Moreover, from any sequence $\{f_n\} \subset BV_C(0,1)$, one can extract a subsequence $\{f_{n_j}\}$ such that $\{(f_{n_j})_+\}$ and $\{(f_{n_j})_-\}$ are convergent in both $(IR(0,1), W)$ and $(BV_C(0,1), W)$.*

PROOF. The result immediately follows from the Helly selection lemma and the observation that, if $(f_{n_j})_\pm \to_W f_{(\pm)}$, then $f_{n_j} \to_W f_{(+)} - f_{(-)}$. □

The following lemma gives a useful inequality.

LEMMA 2.6. *For any $f \in BV(0,1)$ and $g \in BV(0,1)$, we have*

$$f_\pm - g_\pm \leq (f-g)_\pm. \tag{2.11}$$

PROOF. For $\tau_{i-1} < \tau_i$, we have

$$(f(\tau_i)-f(\tau_{i-1}))^\pm-(g(\tau_i)-g(\tau_{i-1}))^\pm \leq \{(f(\tau_i)-f(\tau_{i-1}))-(g(\tau_i)-g(\tau_{i-1}))\}^\pm,$$

so that (2.11) follows readily from (2.2). □

REMARK 2.1. For $0 \leq C < \infty$, denote by $W_1$ (resp. $W_2$) the topologies defined on $BV_C(0,1)$ as follows. A net $f_\alpha \in BV_C(0,1)$ satisfies $f_\alpha \to_{W_1} f$ (resp. $f_\alpha \to_{W_2} f$) if and only if $(f_\alpha)_\pm \to_W f_{(\pm)}$ and $f = f_{(+)} - f_{(-)}$ (resp. $(f_\alpha)_\pm \to_W f_\pm$). Note here that Lemma 2.3 implies that $f_{(\pm)} - f_\pm \in BV_C(0,1) \cap IR(0,1) = IR_C(0,1) := \{f \in IR(0,1) : f(1) \leq C\}$. Since a compact metric space is totally bounded (or precompact), Lemma 2.5 implies the existence of a sequence $\{h_n,\ n \geq 1\} \subset IR_C(0,1)$ whose closure in $(IR(0,1), W)$ is equal to $IR_C(0,1)$. Moreover, for any $\varepsilon > 0$, there exists an $N(\varepsilon) < \infty$ such that, for any $k \in IR_C(0,1)$, we have $\min_{1\leq n\leq N(\varepsilon)} d_L(h_n,k) < \varepsilon$. Let now, for any $f \in BV_C(0,1)$ and $g \in BV_C(0,1)$,

$$D_C(f,g) = \inf_{m\geq1,n\geq1,q\geq1}\{d_L(f_+ + qh_m, g_+ + qh_n) + d_L(f_- + qh_m, g_- + qh_n)\}.$$

It is readily verified that $(f_\alpha)_\pm \to_W f_{(\pm)} \Leftrightarrow D_C(f_\alpha, f) \to 0$. Moreover, for $f, g, \ell \in BV_C(0,1)$ and $\varepsilon > 0$, there exist $m_1, n_1, q_1$ and $m_2, n_2, q_2$ such that

$$\begin{aligned}
D_C(f,g) + D_C(g,\ell) \geq\ & d_L(f_+ + q_1h_{m_1}, g_+ + q_1h_{n_1}) \\
&+ d_L(g_+ + q_2h_{m_2}, \ell_+ + q_2h_{n_2}) \\
&+ d_L(f_- + q_1h_{m_1}, g_- + q_1h_{n_1}) \\
&+ d_L(g_- + q_2h_{m_2}, \ell_- + q_2h_{n_2}) - \varepsilon \\
\geq\ & d_L(f_+ + q_1h_{m_1} + q_2h_{m_2}, \ell_+ + q_1h_{n_1} + q_2h_{n_2}) \\
&+ d_L(f_- + q_1h_{m_1} + q_2h_{m_2}, \ell_- + q_1h_{n_1} + q_2h_{n_2}) - \varepsilon.
\end{aligned}$$

where we have used the triangle inequality and the fact that $d_L(f,g) = d_L(f+k, g+k)$.

Since our assumptions imply that there exist $m_3, n_3$ and $q_3$ (we may choose $q_3 = q_1 + q_2$) such that $d_L(q_1h_{m_1} + q_2h_{m_2}, q_3h_{m_3}) + d_L(q_1h_{n_1} +$

$q_2h_{n_2}, q_3h_{n_3}) < \varepsilon$, it follows from the triangle inequality that

$$\begin{aligned} D_C(f,g) + D_C(g,\ell) &\geq d_L(f_+ + q_3h_{m_3}, \ell_+ + q_3h_{n_3}) \\ &\quad + d_L(f_- + q_3h_{m_3}, \ell_- + q_3h_{n_3}) - 2\varepsilon \\ &\geq D_C(f,\ell) - 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ may be chosen arbitrarily small, we see that $D_C$ satisfies the triangle inequality. By all this, $BV_C(0,1)$ is, when endowed with the topology defined by $W_1$, a metric space. Since Lemma 2.5 implies that $(BV_C(0,1), W_1)$ is sequentially compact, it follows that $(BV_C(0,1), W_1)$ is a compact metric space.

Likewise, we may define on $BV_C(0,1)$ the topology $W_2$ by means of the metric

$$D'_C(f,g) = d_L(f_+, g_+) + d_L(f_-, g_-).$$

Obviously, $W_2$ is stronger than $W_1$, which is in turn stronger than $W$. Even though $(BV_C(0,1), W)$ and $(BV_C(0,1), W_1)$ are compact metric spaces, it is not the case for $(BV_C(0,1), W_2)$ as follows from Example 2.1.

In view of the characterization given in Lemma 2.4, it is convenient to extend the definition of $W_1$ (resp. $W_2$) to $BV(0,1)$ as follows. A net $f_\alpha \in BV(0,1)$ will be said to satisfy $f_\alpha \to_{W_1} f$ (resp. $f_\alpha \to_{W_2} f$) if and only if there exists a $0 \leq C < \infty$ such that $f_\alpha$ is ultimately in $BV_C(0,1)$, and $f_\alpha \to_{W_1} f$ (resp. $f_\alpha \to_{W_2} f$) in $BV_C(0,1)$.

In the sequel, we will consider a function $\Psi(\cdot)$, defined on $(-\infty, \infty)$ and taking values in $[0, \infty]$, which satisfies the following properties.

(C1) $\Psi$ is convex and non-negative;

(C2) $\Psi < \infty$ on some nondegenerate interval, and $\Psi(0) = 0$;

(C3) $T_0 := \lim_{\alpha \to \infty}(\Psi(\alpha)/\alpha) > 0$ and $T_1 := \lim_{\alpha \to -\infty}(\Psi(\alpha)/\alpha) < 0$.

For any $\Psi(\cdot)$ satisfying (C1-2-3) and $f \in BV(0,1)$ being as in (2.6)-(2.8), let

(2.12) $$J_\Psi(f) = T_0 f_+^{(S)}(1) - T_1 f_-^{(S)}(1) + \int_0^1 \Psi(\dot{f}(u))du,$$

with the convention that $T_0 f_+^{(S)}(1) = 0$ if $T_0 = \infty$ and $f_+^{(S)}(1) = 0$ (resp. $T_1 f_-^{(S)}(1) = 0$ if $T_1 = -\infty$ and $f_-^{(S)}(1) = 0$). Set further, for any $v > 0$,

(2.13) $$\begin{aligned} J_{\Psi,v}(f) &= vJ_\Psi(f/v) \\ &= T_0 f_+^{(S)}(1) - T_1 f_-^{(S)}(1) + \int_0^1 v\Psi(\dot{f}(u)/v)du. \end{aligned}$$

It is readily verified that the definitions (2.12)-(2.13) of $J_\Psi(f)$ and $J_{\Psi,v}(f)$ do not depend of the specific representative of $\dot{f}$ (recall that $\dot{f}$ is defined uniquely up to an a.e. equivalence, with respect to the Lebesgue measure).

LEMMA 2.7. *Under* (C1-2-3), *we have, for any* $v > 0$ *and* $f \in BV(0,1)$,

$$\begin{aligned} J_\Psi(f) &= J_\Psi(f_+) + J_\Psi(-f_-) \quad \textit{and} \\ J_{\Psi,v}(f) &= J_{\Psi,v}(f_+) + J_{\Psi,v}(-f_-). \end{aligned} \tag{2.14}$$

PROOF. Consider a Hahn-Jordan decomposition of $df$ as in (2.4) with $df_\pm = \nu_\pm$. We have, by (2.6) and (2.8),

$$\begin{aligned} \int_0^1 \Psi(\dot{f}(u))du &= \int_{A_+} \Psi(\dot{f}(u))du + \int_{A_-} \Psi(\dot{f}(u))du \\ &= \int_0^1 \Psi(\dot{f}_+(u))du + \int_0^1 \Psi(-\dot{f}_-(u))du, \end{aligned}$$

which, in view of (2.12) and (2.13), yields readily (2.14). □

For any $v > 0$ and $0 \le \lambda < \infty$, consider the sets

$$\begin{aligned} \Delta_\lambda &= \{f \in BV(0,1) : J_\Psi(f) \le \lambda\} \quad \text{and} \\ \Delta_{\lambda,v} &= \{f \in BV(0,1) : J_{\Psi,v}(f) \le \lambda\}. \end{aligned} \tag{2.15}$$

LEMMA 2.8. *Under* (C1-2-3), *for all* $\lambda \ge 0$ *and* $v > 0$, *the sets* $\Delta_\lambda$ *and* $\Delta_{\lambda,v}$ *are compact in* $(BV(0,1), W_1)$. *Moreover, if* $T_0 = \infty$ *and* $T_1 = -\infty$, *these sets are also compact in* $(BV(0,1), U)$.

PROOF. By convexity of $\Psi(\cdot)$ and making use of the assumption that $T_1 < 0 < T_0$, we obtain readily from (2.12) that there exists a $C < \infty$ such that $\Delta_\lambda \subset BV_C(0,1)$ (resp. $\Delta_{\lambda,v} \subset BV_C(0,1)$).

In the remainder of our proof, we will make use of the following facts.

FACT 1. The function $f \in IR(0,1) \to J_\Psi(\pm f)$ is a lower semi-continuous mapping of $(IR(0,1), W)$ onto $[0,\infty]$. Moreover, for any $0 \le \lambda < \infty$, the sets $\Delta_\lambda^\pm = \{f \in IR(0,1) : J_\Psi(\pm f) \le \lambda\}$ are compact in $(IR(0,1), W)$.

The proof of Fact 1 is due to Lynch and Sethuraman (1987) (see e.g. Lemmas 3.3 and 3.4). Recall that when $(\mathcal{E}, \mathcal{T})$ is a metric space, a mapping $\Theta : \mathcal{E} \to [0,\infty]$ is lower semi-continuous whenever, for each sequence

$e_n \to_T e$ in $\mathcal{E}$, we have $\Theta(e) \le \liminf_{n\to\infty} \Theta(e_n)$. Denote by $J_1$ the classical Skorohod topology (see e.g. Skorohod (1956) and Billingsley (1968), pp. 111-123). This topology is metrizable and such that, whenever $f$ is continuous, $f_n \to_{J_1} f$ is equivalent to $f_n \to_U f$. Note in general that $f_n \to_U f \Rightarrow f_n \to_{J_1} f \Rightarrow f_n \to_W f$.

FACT 2. When $T_0 = \infty$ and $T_1 = -\infty$, $f \to J_\Psi(f)$ is a lower semi-continuous mapping of $(BV(0,1), J_1)$ onto $[0,\infty]$. Moreover, for any $0 \le \lambda < \infty$, the set $\Delta_\lambda = \{f \in BV(0,1) : J_\Psi(f) \le \lambda\}$ is compact in $(C(0,1), U)$.

The proof of Fact 2 is due to Varadhan (1966). It follows readily from (2.12) that, for $T_0 = \infty$ and $T_1 = -\infty$, $\Delta_\lambda$ is composed of absolutely continuous functions on $[0,1]$. Thus, the compactness of $\Delta_\lambda$ in $(BV(0,1), J_1)$ is equivalent to that of $\Delta_\lambda$ in $(BV(0,1), U)$.

Consider now a sequence $f_n = (f_n)_+ - (f_n)_- \in BV(0,1)$, and assume that $f_n \to_{W_1} f$, or equivalently, that $(f_n)_\pm \to_W f_{(\pm)}$ and $f = f_{(+)} - f_{(-)}$. Applying Fact 1 to $(f_n)_+$ and $(f_n)_-$, we obtain, by semi-continuity of $J_\Psi$ on $(IR(0,1), W)$ and by (2.14),

$$\text{(2.16)}\quad \begin{aligned} J_\Psi(f_{(+)}) + J_\Psi(-f_{(-)}) &\le \liminf_{n\to\infty} J_\Psi((f_n)_+) + \liminf_{n\to\infty} J_\Psi(-(f_n)_-) \\ &\le \liminf_{n\to\infty} \{J_\Psi((f_n)_+) + J_\Psi(-(f_n)_-)\} \\ &= \liminf_{n\to\infty} J_\Psi(f_n). \end{aligned}$$

Consider now the decomposition (2.6) applied to $f_{(\pm)}$. We have namely

$$\text{(2.17)}\qquad f_{(\pm)}(s) = \int_0^s \dot{f}_{(\pm)}(u)du + f_{(\pm)}^{(S)}(s) \quad \text{for} \quad -\infty < s < \infty.$$

By (2.4)-(2.7), we see that, without loss of generality, we can choose the representatives of $\dot{f}_{(\pm)}$ and $\dot{f}_\pm$ in such a way that the following inequalities hold:

$$\text{(2.18)}\qquad \dot{f}_\pm(u) \le \dot{f}_{(\pm)}(u) \quad \text{for} \quad -\infty < u < \infty.$$

[Notice that if $B_\pm = \{u \in [0,1] : \dot{f}_\pm(u) > \dot{f}_{(\pm)}(u)\}$, $B_\pm$ is measurable and such that, by (2.4)-(2.5), $\nu_\pm(B_\pm) \le \nu_{(\pm)}(B_\pm)$. By eventually extracting from $B_\pm$ a set of Lebesgue measure 0 corresponding to the singular components of $\nu_\pm$ and $\nu_{(\pm)}$, we see that this implies that the Lebesgue measure of $B_\pm$ is 0.]

It follows from (C1-2-3) and (2.18) that

$$\text{(2.19)}\qquad \int_0^1 \Psi(\pm\dot{f}_\pm(u))du \le \int_0^1 \Psi(\pm\dot{f}_{(\pm)})(u)du.$$

By (2.12), (2.14), (2.19) and the obvious fact that $f^{(S)} = f_+^{(S)} - f_-^{(S)} = f_{(+)}^{(S)} - f_{(-)}^{(S)}$, we have

$$J_\Psi(f) = J_\Psi(f_+) + J_\Psi(-f_-) \le J_\Psi(f_{(+)}) + J_\Psi(-f_{(-)}). \tag{2.20}$$

Combining (2.16) and (2.20), we obtain

$$J_\Psi(f) \le \liminf_{n\to\infty} J_\Psi(f_n), \tag{2.21}$$

so that $J_\Psi$ is lower semi-continuous on $(BV(0,1), W_1)$, which, in turn, implies that $\Delta_\lambda$ and $\Delta_{\lambda,v}$ are closed in $(BV_C(0,1), W_1)$ [recall that $\Theta : \mathcal{E} \to [0,\infty]$ is lower semi-continuous on a metric space $(\mathcal{E}, \mathcal{T})$ iff the set $\{e \in \mathcal{E} : \Theta(e) \le \lambda\}$ is closed for each $\lambda \ge 0$].

Since by (2.14),

$$\begin{aligned}\Delta_\lambda &= \{f \in BV(0,1) : J_\Psi(f_+) + J_\Psi(-f_-) \le \lambda\} \\ &\subset \{f \in BV(0,1) : J_\Psi(f_+) \le \lambda \quad \text{and} \quad J_\Psi(-f_-) \le \lambda\}.\end{aligned} \tag{2.22}$$

Fact 1 implies that from any sequence $\{f_n\}$ we can extract a subsequence $\{f_{n_j}\}$ such that $(f_{n_j})_\pm \to_W f_{(\pm)} \in IR(0,1)$. Therefore $f_{n_j} \to_{W_1} f := f_{(+)} - f_{(-)}$ and $\Delta_\lambda$ is relatively compact in $(BV_C(0,1), W_1)$. Since $\Delta_\lambda$ is closed in $(BV_C(0,1), W_1)$, it follows that $\Delta_\lambda$ is compact. A similar argument proves that $\Delta_{\lambda,v}$ is compact in $(BV(0,1), W_1)$.

The fact that $\Delta_\lambda$ and $\Delta_{\lambda,v}$ are compact in $(BV(0,1), U)$ when $T_0 = \infty$ and $T_1 = -\infty$, is a direct application of Fact 2. The proof of Lemma 2.8 is now complete. □

LEMMA 2.9. *Under* (C1-2-3), *the mapping* $J_\Psi : f \in BV(0,1) \to J_\Psi(f)$ *is lower semi-continuous on* $(BV(0,1), W)$.

PROOF. Observe that, the topology $W_1$ being stronger than $W$, a set may be closed (or open) with respect to $W_1$ without being closed (or open) with respect to $W$. Thus, the just proven fact that $J_\Psi$ is lower semi-continuous on $(BV(0,1), W_1)$ is not sufficient to prove that the same holds on $(BV(0,1), W)$.

Fix a $\lambda \ge 0$. By convexity of $\Psi$ and since $T_0 > 0$ and $T_1 < 0$, we obtain readily from (2.12) that there exists a $C = C_\lambda < \infty$ such that $\Delta_\lambda \subset BV_C(0,1)$, where $BV_C(0,1)$ is as in Lemma 2.5. By this lemma, $(BV_C(0,1), W)$ is a compact metric space. Therefore, all we need is to show that $\Delta_\lambda$ is complete in $(BV_C(0,1), W)$, with respect to a metric defining $W$. Consider therefore a sequence $\{f_n\} \subset \Delta_\lambda$ such that

$$\lim_{N\to\infty} \sup_{m\geq N,\ n\geq N} d_W(f_m, f_n) = 0.$$

Since, by Lemma 2.8, $\Delta_\lambda$ is compact in $(BV(0,1), W_1)$, the set $\mathcal{L}$ of all possible $W_1$-limits of subsequences of $\{f_n\}$ is non-void. Moreover, if $f_{n'_j} \to_{W_1} f'$ and $f_{n''_j} \to_{W_1} f''$, then we have also $f_{n'_j} \to_W f'$ and $f_{n''_j} \to_W f''$. Since $d_W(f_{n'_j}, f_{n''_j}) \to 0$, it follows that $f' = f''$, and that $\mathcal{L} = \{f\}$ consists of one single function $f \in BV(0,1)$. If $d_W(f_n, f) \not\to 0$, then, there exists a subsequence $\{f_{n_j}\}$ such that $\liminf_{j\to\infty} d_W(f_{n_j}, f) > 0$. However, since $\Delta_\lambda$ is compact in $(BV(0,1), W_1)$, one can then extract a further subsequence which converges to $f$ in $(BV(0,1), W)$. Since this is impossible, we see that $f_n \to_W f$. This proves that $\Delta_\lambda$ is complete (and hence closed), as sought. □

In the sequel, we will use the following notation. Let $\mathcal{P} = \{\tau_0 < 0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = 1\}$ define a partition of $[0,1]$, and set $d(\mathcal{P}) = \max\{\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}\}$. We will denote by $d(\mathcal{P}) \to 0$ the fact that $\mathcal{P}$ belongs to a directed net $\mathcal{N}$, under the partial order defined by inclusion, and that $d(\mathcal{P}) \to 0$ (in the usual sense) along $\mathcal{N}$. Let further

$$J^{\mathcal{P}}_{\Psi,v}(f) = v\left\{\Psi\left(\frac{f(\tau_1)}{v\tau_1}\right)\tau_1 + \sum_{i=2}^{n}\Psi\left(\frac{f(\tau_i) - f(\tau_{i-1})}{v(\tau_i - \tau_{i-1})}\right)(\tau_i - \tau_{i-1})\right\}, \tag{2.23}$$

and

$$J^{\mathcal{P}\pm}_{\Psi,v}(f) = v\left\{\Psi\left(\frac{\pm f(\tau_1)^{\pm}}{v\tau_1}\right)\tau_1 + \sum_{i=2}^{n}\Psi\left(\frac{\pm(f(\tau_i) - f(\tau_{i-1}))^{\pm}}{v(\tau_i - \tau_{i-1})}\right)(\tau_i - \tau_{i-1})\right\}. \tag{2.24}$$

LEMMA 2.10. *Under* (C1-2-3), *we have, for any* $v > 0$, $\mathcal{P}$, *partition of* $[0,1]$, *and* $f \in BV(0,1)$,

$$\begin{aligned} J^{\mathcal{P}}_{\Psi,v}(f) &= J^{\mathcal{P}+}_{\Psi,v}(f) + J^{\mathcal{P}-}_{\Psi,v}(f) \leq J^{\mathcal{P}}_{\Psi,v}(f_+) + J^{\mathcal{P}}_{\Psi,v}(-f_-) \\ &\leq J_{\Psi,v}(f_+) + J_{\Psi,v}(-f_-) = J_{\Psi,v}(f). \end{aligned} \tag{2.25}$$

PROOF. Since $\Psi(0) = 0$ by (C2), we see that for any $-\infty < x < \infty$, $\Psi(x) = \Psi(x^+) + \Psi(-x^-)$. Therefore, the first equality in (2.25) follows directly from (2.23)-(2.24). For the next inequality, observe that $f = f_+ - f_-$, so that for $\tau_{i-1} < \tau_i$ we have

$$(2.26)\quad (f(\tau_i) - f(\tau_{i-1}))^{\pm} = ((f_+(\tau_i) - f_+(\tau_{i-1})) - (f_-(\tau_i) - f_-(\tau_{i-1})))^{\pm} \\ \leq f_{\pm}(\tau_i) - f_{\pm}(\tau_{i-1}).$$

By (C1-2), $\Psi(\cdot)$ is non-increasing on $(-\infty, 0]$ and non-decreasing on $[0, \infty)$. Thus, the first inequality in (2.25) follows from (2.23), (2.24) and (2.26).

Since the last equality in (2.25) is stated in (2.14), all we need is to prove the inequalities

$$(2.27)\qquad J^{\mathcal{P}}_{\Psi,\upsilon}(\pm f) \leq J_{\Psi,\upsilon}(\pm f) \quad \text{for all} \quad f \in IR(0,1).$$

The proof of (2.27) is due to Lynch and Sethuraman (1987), who showed (see e.g. their Remark following Theorem 3.2) that

$$(2.28)\qquad \sup_{\mathcal{P}} J^{\mathcal{P}}_{\Psi,\upsilon}(\pm f) = J_{\Psi,\upsilon}(\pm f) \quad \text{for all} \quad f \in IR(0,1).$$

□

LEMMA 2.11. *Under* (C1-2-3), *for each* $f \in BV(0,1)$, *there exists a directed net* $\mathcal{N}$ *of partitions of* $[0,1]$ *such that* $d(\mathcal{P}) \to 0$ *along* $\mathcal{N}$, *and*

$$(2.29)\qquad J^{\mathcal{P}}_{\Psi,\upsilon}(f) \to J_{\Psi,\upsilon}(f) \quad \text{as} \quad d(\mathcal{P}) \to 0.$$

PROOF. For any $\mathcal{P}$ as given above, define

$$(2.30)\quad f^{\mathcal{P}}_{\pm}(s) = \sum_{i=1}^{k-1} (f(\tau_i) - f(\tau_{i-1}))^{\pm} + \frac{s - \tau_{k-1}}{\tau_k - \tau_{k-1}} (f(\tau_k) - f(\tau_{k-1}))^{\pm} \\ \text{for } \tau_{k-1} \leq s \leq \tau_k,\ 2 \leq k \leq n, \\ f^{\mathcal{P}}_{\pm}(s) = \frac{s}{\tau_1} f(\tau_1)^{\pm} \text{ for } 0 \leq s \leq \tau_1.$$

It follows readily from (2.2) and (2.30) that one may define a net $\mathcal{N}$ such that

$$(2.31)\qquad f^{\mathcal{P}}_{\pm} \to_W f_{\pm} \quad \text{as} \quad d(\mathcal{P}) \to 0.$$

A weak neighborhood of $f_{\pm}$ in $IR(0,1)$ is defined by a finite set $\{\tau_j,\ 1 \leq j \leq K\}$ of continuity points of $f_{\pm}$ and $\varepsilon > 0$ as $N(f_{\pm}) = \{g \in IR(0,1) : |g(\tau_j) - f_{\pm}(\tau_j)| < \varepsilon,\ 1 \leq j \leq K\}$. Since the weak topology on $IR(0,1)$ is metrizable, for each $f$, there exists a countable basis of such neighborhoods, which may be defined by an increasing sequence of continuity points $\{\tau_j\}$ defining partitions $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots$, with $d(\mathcal{P}_n) \to 0$, and a sequence $\varepsilon = \varepsilon_n \downarrow 0$. Obviously, the net defined by $\mathcal{P}_1, \mathcal{P}_2, \ldots$, satisfies (2.31).

Since $J_\Psi$ is lower semi-continuous on $(BV(0,1)W)$ by Lemma 2.9, it follows readily from (2.31) that, along $\mathcal{N}$,

$$J_{\Psi,v}(\pm f_\pm) \le \liminf_{d(\mathcal{P})\to 0} J_{\Psi,v}(\pm f_\pm^{\mathcal{P}}) = \liminf_{d(\mathcal{P})\to 0} J_{\Psi,v}^{\mathcal{P}}(\pm f_\pm). \tag{2.32}$$

Combining (2.27) and (2.32), we obtain that $J_{\Psi,v}^{\mathcal{P}}(\pm f_\pm) \to J_{\Psi,v}(\pm f_\pm)$ as $d(\mathcal{P}) \to 0$. In view of (2.14), this proves (2.29) and completes the proof of the lemma. □

## 2.3. Large deviations

We now assume that $\Psi = \psi$, so that, by Lemma 2.1, (C1-2-3) hold if (A1-2) are satisfied with $\mu = 0$. We will assume here that these conditions hold.

Let $\mathcal{P} = \{\tau_0 < 0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = 1\}$ define a partition of $[0,1]$. Define, for any subset $B$ of $BV(0,1)$,

$$J_{\psi,v}(B) = \inf_{f\in B} J_{\psi,v}(f), \quad J_{\psi,v}^{\mathcal{P}}(B) = \inf_{f\in B} J_{\psi,v}^{\mathcal{P}}(f), \tag{2.33}$$

and

$$F_B^{\mathcal{P}} = \tag{2.34}$$

$$\{(x_1,\ldots,x_n) \in \mathbf{R}^n : v\{\psi(\frac{x_1}{v\tau_1})\tau_1 + \sum_{i=2}^n \psi(\frac{x_i}{v(\tau_i - \tau_{i-1})})(\tau_i - \tau_{i-1})\} \ge J_{\psi,v}^{\mathcal{P}}(B)\}.$$

Note for further use that for $f \in BV(0,1)$ and $B \subset BV(0,1)$,

$$J_{\psi,v}^{\mathcal{P}}(f) \ge J_{\psi,v}^{\mathcal{P}}(B) \tag{2.35}$$
$$\Leftrightarrow (f(\tau_1), f(\tau_2) - f(\tau_1), \ldots, f(\tau_n) - f(\tau_{n-1})) \in F_B^{\mathcal{P}}.$$

LEMMA 2.12. *Assume that* (A1-3-4) *hold. Then, for any closed subset* $F$ *of* $\mathbf{R}$, *we have*

$$\limsup_{\lambda\to\infty} \lambda^{-1} \log P(\lambda^{-1} S(\lambda) \in F) \le -I_\psi(F), \tag{2.36}$$

*and for any open subset* $G$ *of* $\mathbf{R}$, *we have*

$$\liminf_{\lambda\to\infty} \lambda^{-1} \log P(\lambda^{-1} S(\lambda) \in G) \ge -I_\psi(G), \tag{2.37}$$

*where we set, for each* $H \subset \mathbf{R}$,

$$I_\psi(H) = \inf_{x \in H} \psi(x). \tag{2.38}$$

PROOF. Cramér (1937) and Chernoff (1952) (see e.g. Theorem 3.1 of Lynch and Sethuraman (1987)) have shown that, under the assumptions of Lemma 2.12, for $F$, closed subset of $\mathbf{R}$,

$$\limsup_{n\to\infty} n^{-1} \log P(n^{-1}S_n \in F) \le -I_\psi(F), \tag{2.39}$$

and for $G$, open subset of $\mathbf{R}$,

$$\liminf_{n\to\infty} n^{-1} \log P(n^{-1}S_n \in G) \ge -I_\psi(G), \tag{2.40}$$

where, in (2.39) - (2.40), $n$ is integer. For each $H \subset \mathbf{R}$, set $\omega_\pm(H) = \pm\inf\{\pm y : y \in H\}$ and $\overline{\overline{H}} = ]-\infty, \omega_-(H)] \cup [\omega_+(H), \infty[$. Observe that, when $F \subset \mathbf{R}$ is closed, $F \subset \overline{\overline{F}}$ and $I_\psi(F) = I_\psi(\overline{\overline{F}}) = \min\{\psi(\omega_+(F)), \psi(\omega_-(F))\}$. Thus, we have the inequalities for $n \le \lambda < n+1$ and $\omega_-(F) < 0 < \omega_+(F)$

$$\begin{aligned} P(\lambda^{-1}S(\lambda) \in F) &\le P(\lambda^{-1}S_n \ge \omega_+(F)) + P(\lambda^{-1}S_n \le \omega_-(F)) \\ &\le P(n^{-1}S_n \ge \omega_+(F)) + P(n^{-1}S_n \le \omega_-(F)) \\ &= P(n^{-1}S_n \in \overline{\overline{F}}). \end{aligned} \tag{2.41}$$

Thus, by combining (2.39) and (2.41), we obtain (2.36). The proof follows along the same lines when $\omega_-(F) = 0$ (resp. $\omega_+(F) = 0$).

Assuming now that $G$ is an open subset of $\mathbf{R}$, fix any $\varepsilon > 0$ and choose $F = [a, b] \subset G$ with $-\infty < a < b < 0 < \infty$ or $-\infty < 0 < a < b < \infty$, and $I_\psi(F) \le I_\psi(G) + \varepsilon$ ($I_\psi(F) = \infty$ if $I_\psi(G) = \infty$). It is readily verified from the Cramér (1937) and Chernoff (1952) theorems that in this case we have

$$\lim_{n\to\infty} n^{-1} \log P(n^{-1}S_n \in F) = -I_\psi(F) \in [-I_\psi(G) - \varepsilon, -I_\psi(G)]. \tag{2.42}$$

Since, uniformly over $n \le \lambda < n+1$, we have ultimately as $n \to \infty$

$$P(\lambda^{-1}S(\lambda) \in G) \ge P(n^{-1}S_n \in F), \tag{2.43}$$

we obtain (2.37) by combining (2.40), (2.42) and (2.43) and by choosing $\varepsilon > 0$ arbitrarily small. □

REMARK 2.2. The Cramér (1937) and Chernoff (1952) theorems show that for any $\omega_+ > 0$ (resp. $\omega_- < 0$), we have

$$(2.44)\qquad \lim_{n\to\infty} n^{-1}\log P(n^{-1}S_n \geq \omega_+) = -\psi(\omega_+)$$
$$(\text{resp. } \lim_{n\to\infty} n^{-1}\log P(n^{-1}S_n \leq \omega_-) = -\psi(\omega_-)).$$

Since $\psi(\cdot)$ is convex, it follows readily from (2.44) that for any sequence $u_n$ such that $u_n \geq 0$ and $u_n \to 0$ as $n \to \infty$, we also have

$$(2.45)\qquad \lim_{n\to\infty} n^{-1}\log P(n^{-1}S_n \geq \omega_+ - u_n) = -\psi(\omega_+)$$
$$(\text{resp. } \lim_{n\to\infty} n^{-1}\log P(n^{-1}S_n \leq \omega_- + u_n) = -\psi(\omega_-)).$$

On the other hand, (2.45) is not true in general if $u_n$ is negative [consider the example where $P(X = 1) = P(X = -1) = 1/2$; we have $P(n^{-1}S_n \geq 1) = 2^{-n}$ while $P(n^{-1}S_n > 1) = 0$].

By combining (2.45) with the first inequality in (2.41), we obtain the following extended version of (2.36). Under the assumptions of Lemma 2.12, for any closed subset $F$ of $\mathbf{R}$, we have

$$(2.46)\qquad \limsup_{\lambda\to\infty} \lambda^{-1}\log P(r(\lambda)^{-1}S(\lambda) \in F) \leq -I_\psi(F),$$

where $r(\lambda)$ is any function of $\lambda$ such that $\lambda^{-1}r(\lambda) \to 1$ as $\lambda \to \infty$.

Likewise, the arguments used for the proof of Lemma 2.12 show that, under the assumptions of this lemma, for any open subset $G$ of R, we have, for any such function $r(\cdot)$,

$$(2.47)\qquad \liminf_{\lambda\to\infty} \lambda^{-1}\log P(r(\lambda)^{-1}S(\lambda) \in G) \geq -I_\psi(G).$$

LEMMA 2.13. *Assume that* (A1-3-4) *hold. Fix $n \geq 1$ and $w_1 > 0, \ldots, w_n > 0$, and set for $\lambda > 0$*

$$(2.48)\qquad V_{\lambda,i} = \lambda^{-1}\{S(\sum_{j=1}^{i}\lambda w_j) - S(\sum_{j=1}^{i-1}\lambda w_j)\} \quad \textit{for} \quad 1 \leq i \leq n,$$

*with the convention that $\sum_\emptyset(\cdot) = 0$. Then, for any $v > 0$, and $F$, closed subset of $\mathbf{R}^n$, we have*

$$(2.49)\quad \limsup_{\lambda\to\infty} \lambda^{-1}\log P((vV_{\lambda,1},\ldots,vV_{\lambda,n}) \in F) \leq -v^{-1}I_{\psi;vw_1,\ldots,vw_n}(F),$$

*and for any open subset $G$ of $\mathbf{R}^n$, we have*

$$\liminf_{\lambda\to\infty} \lambda^{-1} \log P((vV_{\lambda,1},\dots,vV_{\lambda,n}) \in G) \geq -v^{-1} I_{\psi;vw_1,\dots,vw_n}(G), \tag{2.50}$$

*where we set, for each $H \subset \mathbf{R}^n$ and $\omega_1 > 0,\dots,\omega_n > 0$,*

$$I_{\psi;\omega_1,\dots,\omega_n}(H) = \inf\{\sum_{i=1}^{n} \omega_i \psi(\frac{x_i}{\omega_i}) : (x_1,\dots,x_n) \in H\}. \tag{2.51}$$

PROOF. Following Lynch and Sethuraman [LS] (1987), we will say that the measures $\{P_\lambda\}$ satisfy the large deviation principle [LDP] with rate function $I(\cdot)$ if the following conditions hold, with $I(H) := \inf_{x\in H} I(x)$:

(i) $I(\cdot)$ is positive, lower semi-continuous, and such that for each $c < \infty$, $\{x : I(x) \leq c\}$ is compact;

(ii) For each closed set $F$, $\limsup_{\lambda\to\infty} \lambda^{-1} \log P_\lambda(F) \leq -I(F)$;

(iii) For each open set $G$, $\liminf_{\lambda\to\infty} \lambda^{-1} \log P_\lambda(G) \geq -I(G)$.

We will say also that the measures $\{P_\lambda\}$ are large deviation tight [LD tight] if for each $M < \infty$, there exists a compact set $K_M$ such that the complement $K_M^c$ of $K_M$ satisfies

$$\limsup_{\lambda\to\infty} \lambda^{-1} \log P_\lambda(K_M^c) \leq -M.$$

We will consider first the case where $P_\lambda = P_\lambda^i$ is the probability distribution of the random variable $vV_{\lambda,i}$. In view of (2.48), an application of Lemma 2.12 and Remark 2.2 shows that $P_\lambda^i$ satisfies the LDP with rate function $I(x) = w_i\psi(\frac{x}{vw_i})$. The fact that the above conditions (i) are satisfied for this choice of $I(\cdot)$ is a consequence of Lemma 2.1. Moreover, Theorem 3.1 of LS (1987) shows that $P_\lambda^i$ is also LD tight.

The proof of Lemma 2.13 now follows from Lemmas 2.7 and 2.8 of LS (1987). By these lemmas, if $P_\lambda := P_\lambda^1 \times \dots \times P_\lambda^n$, and if the $P_\lambda^i$ are LD tight and satisfy the LDP with rate functions $I^i(\cdot)$ for $i = 1,\dots,n$, then, so does $P_\lambda$ with rate function $I^1(x_1) + \cdots + I^n(x_n)$. The conclusion follows from the observation that $V_{\lambda,1},\dots,V_{\lambda,n}$ are independent. □

LEMMA 2.14. *Assume that* (A-1-3-4) *hold. Let $v > 0$ be fixed, and denote by $Z_\lambda \in BV(0,1)$ the function defined by $Z_\lambda(t) = \lambda^{-1}S(\lambda t)$ for*

*$0 \le t \le 1$, and $\lambda > 0$. Then, for any $C > 0$ and for any closed subset $\mathcal{F}$ of $(BV_C(0,1), W)$, we have*

$$\limsup_{\lambda\to\infty} \lambda^{-1} \log P(vZ_\lambda \in \mathcal{F}) \le -v^{-1} J_{\psi,v}(\mathcal{F}), \tag{2.52}$$

*and for any open subset $\mathcal{G}$ of $(BV_C(0,1), W)$, we have*

$$\liminf_{\lambda\to\infty} \lambda^{-1} \log P(vZ_\lambda \in \mathcal{G}) \ge -v^{-1} J_{\psi,v}(\mathcal{G}). \tag{2.53}$$

PROOF. For any $\mathcal{P} = \{\tau_0 < 0 < \tau_1 < \ldots < \tau_n = 1\}$, set $w_i = \tau_i - \tau_{i-1}$ for $i = 2, \ldots, n$, and (see e.g. (2.48)) $V_{\lambda,i} = Z_\lambda(\sum_{j=1}^{i} w_j) - Z_\lambda(\sum_{j=1}^{i-1} w_j)$ for $i = 1, \ldots, n$. In view of (2.33)-(2.35), we have

$$P(vZ_\lambda \in \mathcal{F}) \le P(J_{\psi,v}^{\mathcal{P}}(Z_\lambda) \ge J_{\psi,v}^{\mathcal{P}}(\mathcal{F})) = P((V_{\lambda,1}, \ldots, V_{\lambda,n}) \in F_{\mathcal{F}}^{\mathcal{P}}). \tag{2.54}$$

Since $F_{\mathcal{F}}^{\mathcal{P}}$, as defined in (2.34), is closed in $\mathbf{R}^n$, it follows from (2.23), (2.51) and (2.54) that

$$\limsup_{\lambda\to\infty} \lambda^{-1} \log P(vZ_\lambda \in \mathcal{F}) \le -v^{-1} I_{\psi;vw_1,\ldots,vw_n}(F_{\mathcal{F}}^{\mathcal{P}}) = -v^{-1} J_{\psi,v}^{\mathcal{P}}(\mathcal{F}). \tag{2.55}$$

We now make use of the assumption that $\mathcal{F}$ is closed, which, by Lemma 2.5, ensures that $\mathcal{F}$ is compact in $(BV_C(0,1), W)$. Consider a directed net $\mathcal{N}$ of partitions $\mathcal{P}$ such that $d(\mathcal{P}) \to 0$ along $\mathcal{N}$. Noting that $\psi$ is a convex function and that $\mathcal{F}$ is compact, for any $\mathcal{P} \in \mathcal{N}$, there exists an $f_{\mathcal{P}}$ such that $J_{\psi,v}^{\mathcal{P}}(f_{\mathcal{P}}) = J_{\psi,v}^{\mathcal{P}}(\mathcal{F})$. Recalling the notation (2.30), set $f^{\mathcal{P}} = f_+^{\mathcal{P}} - f_-^{\mathcal{P}}$ for any partition $\mathcal{P}$ and $f \in BV(0,1)$. Recalling that the total variation of $f \in BV(0,1)$ on $[0,t]$ is $|f|_v(t) = f_+(t) + f_-(t)$, we have the following inequality, where $d_W$ is as in (2.10).

$$d_W(f, f^{\mathcal{P}}) \le \frac{d(\mathcal{P})}{2} |f|_v(1). \tag{2.56}$$

For the proof of (2.56), we observe that, by (2.10) and (2.30),

$$\begin{aligned} d_W(f, f^{\mathcal{P}}) &= \int_0^1 |f(s) - f^{\mathcal{P}}(s)| ds = \int_0^{\tau_1} |f(s) - \frac{s}{\tau_1} f(\tau_1)| ds \\ &\quad + \sum_{i=2}^{n} \int_{\tau_{i-1}}^{\tau_i} |f(s) - f(\tau_{i-1}) - \frac{s - \tau_{i-1}}{\tau_i - \tau_{i-1}} (f(\tau_i) - f(\tau_{i-1}))| ds \\ &\le \frac{1}{2} \{|f|_v(\tau_1)\tau_1 + \sum_{i=2}^{n} (|f|_v(\tau_i) - |f|_v(\tau_{i-1}))(\tau_i - \tau_{i-1})\} \\ &\le \frac{d(\mathcal{P})}{2} |f|_v(1). \end{aligned}$$

It follows from (2.56) and from the definition of $f^{\mathcal{P}}$, that $d_W(f, f_{\mathcal{P}}^{\mathcal{P}}) \leq \frac{d(\mathcal{P})}{2}C$ for each $f \in \mathcal{F} \subset BV_C(0,1)$. Moreover, we have

$$J_{\psi,v}^{\mathcal{P}}(\mathcal{F}) = J_{\psi,v}^{\mathcal{P}}(f_{\mathcal{P}}) = J_{\psi,v}(f_{\mathcal{P}}^{\mathcal{P}}). \tag{2.57}$$

Since $\mathcal{F}$ is compact in $(BV_C(0,1), W)$, there exists a sub-net $\mathcal{M}$ of $\mathcal{N}$, and and $f \in \mathcal{F}$, such that $f_{\mathcal{P}} \to_W f$, or equivalently by Lemma 2.4, $d_W(f, f_{\mathcal{P}}) \to 0$ along $\mathcal{M}$. Since $d(\mathcal{P}) \to 0$ along $\mathcal{M}$, we have by the triangle inequality $d_W(f, f_{\mathcal{P}}^{\mathcal{P}}) \leq d_W(f, f^{\mathcal{P}}) + \frac{d(\mathcal{P})}{2}C \to 0$ along $\mathcal{M}$. Since, by Lemma 2.9, $J_{\psi,v}$ is lower semi-continuous, this, in combination with (2.57), implies that, along $\mathcal{M}$,

$$\liminf_{d(\mathcal{P})\to 0} J_{\psi,v}^{\mathcal{P}}(\mathcal{F}) = \liminf_{d(\mathcal{P})\to 0} J_{\psi,v}(f_{\mathcal{P}}^{\mathcal{P}}) \geq J_{\psi,v}(f) \geq J_{\psi,v}(\mathcal{F}). \tag{2.58}$$

On the other hand, (2.25) readily implies that $J_{\psi,v}^{\mathcal{P}}(\mathcal{F}) \leq J_{\psi,v}(\mathcal{F})$, so that, by (2.58), we have, along $\mathcal{M}$,

$$\lim_{d(\mathcal{P})\to 0} J_{\psi,v}^{\mathcal{P}}(\mathcal{F}) = J_{\psi,v}(\mathcal{F}). \tag{2.59}$$

Combining (2.55) and (2.59), we obtain readily (2.52) by choosing $\mathcal{P}$ such that $J_{\psi,v}^{\mathcal{P}}(\mathcal{F})$ is arbitrarily close to $J_{\psi,v}(\mathcal{F})$.

Consider now an open subset $G$ of $(BV_C(0,1), W)$. Since (2.53) is obviously true when $J_{\psi,v}(\mathcal{G}) = \infty$, we may limit ourselves to the case where $J_{\psi,v}(\mathcal{G}) < \infty$. Fix an arbitrary $\varepsilon > 0$, and select $g \in \mathcal{G}$ such that $J_{\psi,v}(g) \leq J_{\varepsilon,v}(\mathcal{G}) + \varepsilon$. Since $\mathcal{G}$ is open, by Lemma 2.4, there exists a $\rho > 0$ such that $V_\rho' := \{f \in BV_C(0,1) : d_W(f,g) < \rho\} \subset \mathcal{G}$. Let now $\mathcal{P}$ be an arbitrary partition of $[0,1]$ such that $d(\mathcal{P}) \leq \frac{\rho}{C}$, and set $V_\rho'' := \{f^{\mathcal{P}} \in BV_C(0,1) : d_W(f^{\mathcal{P}}, g^{\mathcal{P}}) < \frac{1}{2}\rho\}$. By (2.56), for any $f^{\mathcal{P}} \in V''$, we have

$$d_W(f^{\mathcal{P}}, g) \leq d_W(f^{\mathcal{P}}, g^{\mathcal{P}}) + d_W(g^{\mathcal{P}}, g) < \frac{1}{2}\rho + \frac{1}{2}\frac{\rho C}{C} = \rho, \tag{2.60}$$

so that $V_\rho'' \subset V_\rho' \subset \mathcal{G}$. Observe that, for any $f \in BV_C(0,1)$ and $h \in BV_C(0.1)$, we have

$$\begin{aligned} &d_W(f^{\mathcal{P}}, g^{\mathcal{P}}) \\ &= \frac{1}{2}\{\tau_1|f(\tau_1) - g(\tau_1)| + \sum_{i=2}^{n}(\tau_{i+1} - \tau_i)K(f(\tau_i) - g(\tau_i), f(\tau_{i-1}) - g(\tau_{i-1}))\} \\ &\quad + |f(1) - g(1)|, \end{aligned} \tag{2.61}$$

where

$$K(u,v) = \frac{(|u| + |v|)^2 + (u+v)^2}{2(|u| + |v|)} \text{ if } |u| + |v| > 0, \text{ and } K(u,v) = 0 \text{ otherwise.}$$

Set now $x_1 = f(\tau_1)$, $x_i = f(\tau_i) - f(\tau_{i-1})$, $y_1 = g(\tau_1)$, $y_i = g(\tau_i) - g(\tau_{i-1})$, and observe by (2.61) that

$$d_W(f^P, g^P) \leq 2n \max_{1\leq i\leq n} |x_i - y_i|. \tag{2.62}$$

Moreover, we have

$$\begin{aligned} J_{\psi,v}(f^P) &= v\{\psi\left(\frac{x_1}{v\tau_1}\right)\tau_1 + \sum_{i=2}^{n} \psi\left(\frac{x_i}{v(\tau_i - \tau_{i-1})}\right)(\tau_i - \tau_{i-1})\} \\ &=: H(x_1, \ldots, x_n). \end{aligned} \tag{2.63}$$

Since $H(y_1, \ldots, y_n) = J_{\psi,v}(g^P) = J^P_{\psi,v}(g) \leq J_{\psi,v}(g) < \infty$ by (2.25), we see that $H(\cdot)$ is continuous in the neighborhood of $(y_1, \ldots, y_n)$. Therefore, we may select a $\delta < \frac{\rho}{2n}$ such that, if $G_\delta = \{(z_1, \ldots, z_n) \in \mathbf{R}^n : \max_{1\leq i\leq n} |z_i - y_i| < \delta\}$, the following conditions hold:

(i) $G_\delta$ is an open subset of $\mathbf{R}^n$;

(ii) $(x_1, \ldots, x_n) \in G_\delta \Rightarrow f^P \in V''_\rho$

(iii) $I_{\psi;vw_1,\ldots,vw_n}(G_\delta)$
$= \inf\{\sum_{i=1}^n vw_i\psi(\frac{z_i}{vw_i}) : (z_1, \ldots, z_n) \in G_\delta\} \in [J^P_{\psi,v}(g) - \varepsilon, J^P_{\psi,v}(g)]$,

where $w_1 = \tau_1$ and $w_i = \tau_i - \tau_{i-1}$ for $i = 2, \ldots, n$.

Next, using the notation of Lemma 2.13, we see that

$$P(\lambda^{-1}Z_\lambda \in \mathcal{G}) \geq P((\lambda^{-1}Z_\lambda)^P \in V''_\rho) \geq P((V_{\lambda,1}, \ldots, V_{\lambda,n}) \in G_\delta),$$

which, when combined with (2.50)) and (i-ii-iii) above, yields

$$\begin{aligned} \liminf_{\lambda\to\infty} \lambda^{-1} \log P(\lambda^{-1}Z_\lambda \in \mathcal{G}) &\geq -v^{-1} I_{\psi;vw_i,\ldots,vw_n}(G_\delta) \\ &\geq -v^{-1}J^P_{\psi,v}(g) \geq -v^{-1}J_{\psi,v}(g). \end{aligned} \tag{2.64}$$

Since we have chosen $g$ in such a way that $-J_{\psi,v}(g) \geq -J_{\psi,v}(\mathcal{G}) - \varepsilon$, (2.53) follows from (2.64) by letting $\varepsilon > 0$ become arbitrarily small. □

## 3. Functional Erdős-Rényi theorems

Throughout this section, we will assume that (A1-2-3-4-5) hold with $\mu = 0$. We will let $\eta_{x,T}$ and $\mathcal{F}_T$ be as in (1.1) and (1.2), and assume that $\{a_T, T > 0\}$

satisfies $0 < a_T < T$, together with (K2) with $0 < c < \infty$. We assume specifically that

$$a_T / \log T \to c \in (0, \infty) \quad \text{as} \quad T \to \infty. \tag{3.1}$$

For each $c > 0$, we will set $\mathcal{L}_T = c\mathcal{F}_T = \{cf : f \in \mathcal{F}_T\}$, and

$$\mathcal{D}_{\psi,c} = \{f \in BV(0,1) : J_{\psi,c}(f) \leq 1\}, \tag{3.2}$$

where $J_{\psi,c}$ is as in (2.13), with $\Psi = \psi$, $v = c$, $T_0 = t_0$ and $T_1 = t_1$.

It follows from Lemma 2.8 and Facts 1-2 in Section 2 that $\mathcal{D}_{\psi,v}$ is a compact subset of $(BV(0,1), W)$. Moreover, if in addition $t_0 = \infty$ and $t_1 = -\infty$, $\mathcal{D}_{\psi,v}$ is also a compact subset of $(BV(0,1), U)$. It will be convenient to use the following notation, with $d_W$ as in (2.10). For any $B \subset BV(0,1)$ and $\varepsilon > 0$, set

$$\hat{B}^\varepsilon = \{f \in BV(0,1) : d_W(f,g) < \varepsilon \text{ for some } g \in B\}. \tag{3.3}$$

THEOREM 3.1. *Assume that* (A1-2-3-4-5) *hold with* $\mu = 0$, *and that* (K1-2) *hold with* $0 < c < \infty$. *Then, for any* $\varepsilon > 0$, *there exist almost surely a* $C < \infty$ *and a* $T(\varepsilon) < \infty$ *such that, for all* $T \geq T(\varepsilon)$,

$$\mathcal{D}_{\psi,c} \subset \hat{\mathcal{L}}_T^\varepsilon \subset \hat{\mathcal{D}}_{\psi,c}^{2\varepsilon} \quad \textit{and} \quad \mathcal{L}_T \subset BV_C(0,1). \tag{3.4}$$

PROOF. The proof will be achieved through the following successive steps.

STEP 1. Set $A_n = [c \log n]$, and $\gamma_{m,n}(s) = cA_n^{-1}(S(m + A_n s) - S(m))$ for $0 \leq s \leq 1$ and $m = 0, 1, \ldots, n$. Let further $K_n = \{\gamma_{m,n} : 0 \leq m \leq n - e_n\}$, for $e_n = A_n$ or $e_n = 1$. For any $\mathcal{E} \subset BV(0,1)$,

$$P(K_n \not\subset \mathcal{E}) = P\Big(\bigcup_{0 \leq m \leq n - e_n} \{\gamma_{m,n} \notin \mathcal{E}\}\Big) \leq nP(cZ_{A_n} \notin \mathcal{E}), \tag{3.5}$$

where $Z_\lambda$ is as in Lemma 2.14, and $n \geq e^{1/c}$.

STEP 2. Observe that $|\gamma_{m,n}|_v(1) = cA_n^{-1} \sum_{i=1}^{A_n} |X_{m+i}|$. Set $\Theta(t) = E(e^{t|X|})$. Obviously,

$$\begin{aligned} \Theta(t) &= \int_{-\infty}^{0-} e^{-tx} dP(X \leq x) + \int_0^\infty e^{tx} dP(X \leq x) \\ &\leq \phi(t) + \phi(-t) < \infty \text{ for } |t| < \min(t_0, -t_1). \end{aligned} \tag{3.6}$$

Thus, by applying Theorem 1.2 to the partial sums of the sequence $\{|X_n|,\ n \geq 1\}$, we readily obtain that there exists a constant $C = C(c) < \infty$ such that

$$\lim_{n\to\infty}\{\max_{0\leq m\leq n-e_n} |\gamma_{m,n}|_v(1)\} = \frac{1}{2}C \quad \text{a.s.},$$

which in turn implies that, almost surely for all $n$ sufficiently large

$$K_n \subset BV_C(0,1). \tag{3.7}$$

Throughout the sequel, $C$ will denote a generic positive constant such that (3.7) holds. We will make use of the following variant of (3.3). Let, for $B \in BV_C(0,1)$,

$$\tilde{B}^\varepsilon = \{f \in BV_C(0,1) : d_W(f,g) < \varepsilon \text{ for some } g \in B\}. \tag{3.8}$$

Note that $\tilde{B}^\varepsilon$ as defined in (3.8) depends upon $C$. However, and in view of (3.7), this dependence will not affect our arguments in the sequel. We will choose without loss of generality $C < \infty$ (this is possible by Lemma 2.8 and Fact 1) so large that

$$\mathcal{D}_{\psi,c} \subset BV_{C/2}(0,1). \tag{3.9}$$

STEP 3. With the conventions of Step 2, and by assuming $C$ sufficiently large, it is readily verified by the triangle inequality that $\hat{\mathcal{K}}_n^\varepsilon$ is almost surely ultimately included in $\hat{\mathcal{D}}_{\psi,c}^{2\varepsilon}$ whenever

$$P(K_n \not\subset \tilde{\mathcal{D}}_{\psi,c}^\varepsilon \text{ i.o.}) = 0. \tag{3.10}$$

Denote by $\mathcal{F}$ the complement of $\tilde{\mathcal{D}}_{\psi,c}^\varepsilon$ in $BV_C(0,1)$. Obviously $\mathcal{F}$ is closed, and therefore compact, in $(BV_C(0,1), W)$. Moreover, $\mathcal{F}$ satisfies $J_{\psi,c}(\mathcal{F}) =: 1+\delta > 1$. An application of (3.5) with $\mathcal{E} = \tilde{\mathcal{D}}_{\psi,c}^\varepsilon$ and of (2.52) with $v = c$ and $\lambda = A_n$ now shows that for all $n$ sufficiently large

$$P(K_n \not\subset \tilde{\mathcal{D}}_{\psi,c}^\varepsilon) \leq n\exp(-A_n c^{-1}(1+\frac{3\delta}{4})) \leq n^{-\delta/2}. \tag{3.11}$$

Introduce now the sequence of integers $n_k = \max\{n : A_n = k\}$. Recalling that $A_n = [c\log n]$, we see that $n_k$ is properly defined for $k \geq k_0$, where $k_0$ is large enough. Moreover, $e^{k/c} \leq n_k < e^{(k+1)/c}$, and $K_n \subset K_{n_k}$ for $n_{k-1} < n \leq n_k$ and $k \geq k_0+1$. Thus, (3.11) is equivalent to

$$P(K_{n_k} \not\subset \tilde{\mathcal{D}}_{\psi,c}^\varepsilon \text{ i.o. (in } k)) = 0. \tag{3.12}$$

By (3.11), we obtain that, for $M$ sufficiently large,

$$\sum_{k=M}^{\infty} P(K_{n_k} \not\subset \tilde{D}^{\varepsilon}_{\psi,c}) \leq \sum_{k=M}^{\infty} n_k^{-\delta/2} \leq \sum_{k=M}^{\infty} \exp(-\frac{k\delta}{2c}) < \infty, \tag{3.13}$$

so that (3.12) follows by Borel-Cantelli. This, in turn, proves that (3.10) holds.

STEP 4. Let $R_n = [(n - A_n)/A_n]$, and set $\mathcal{M}_n = \{\gamma_{rA_n,n} : 1 \leq r \leq R_n\}$. Observe that $\mathcal{M}_n \subset K_n$. Select now any $f \in \mathcal{D}_{\psi,c}$ and $\varepsilon > 0$, and set $\mathcal{N}_\varepsilon(f) = \{g \in BV_C(0,1) : d_W(f,g) < \varepsilon\}$. Since $\mathcal{G} = \mathcal{N}_\varepsilon(f)$ is open in $(BV_C(0,1), W)$ and $J_{\psi,c}(\mathcal{N}_\varepsilon(f)) =: 1 - \delta < 1$, by (2.53) and making use of the independence of the $\gamma_{rA_n,n}$ for $r = 1, \ldots, R_n$, we obtain

$$\begin{aligned} P(f \notin \tilde{\mathcal{M}}^{\varepsilon}_n) &= P(\bigcap_{1\leq r\leq R_n} \{\gamma_{rA_n,n} \notin \mathcal{N}_\varepsilon(f)\}) \\ &= (1 - P(cZ_{A_n} \in \mathcal{N}_\varepsilon(f)))^{R_n} \\ &\leq \exp(-R_n P(cZ_{A_n} \in \mathcal{N}_\varepsilon(f))) \\ &\leq \exp(-R_n \exp(-c^{-1} A_n(1 - \frac{3\delta}{4}))) \\ &\leq \exp(-n^{\delta/2}) \end{aligned} \tag{3.14}$$

for all $n$ sufficiently large. Since by (3.14) $\sum_n P(f \notin \tilde{\mathcal{M}}^{\varepsilon}_n) < \infty$, the Borel-Cantelli lemma, when combined with $\mathcal{M}_n \subset K_n$, shows that

$$P(f \notin \tilde{K}^{\varepsilon}_n \text{ i.o.}) = 0. \tag{3.15}$$

STEP 5. Since $\mathcal{D}_{\psi,c}$ is a compact subset of $(BV_C(0,1), W)$ by (3.9), for any $\varepsilon > 0$, there exists a finite sequence $f_1, \ldots, f_m$ with $f_i \in \mathcal{D}_{\psi,c}$ for $i = 1, \ldots, m$, such that $\mathcal{D}_{\psi,c} \subset \bigcup_{i=1}^m \mathcal{N}_{\varepsilon/2}(f_i)$. By applying repeatedly (3.15) with the formal replacements of $f$ by $f_i$ and $\varepsilon$ by $\varepsilon/2$, for $i = 1, \ldots, m$, we obtain by the triangle inequality that

$$P(\mathcal{D}_{\psi,c} \not\subset \tilde{K}^{\varepsilon}_n \text{ i.o.}) = 0. \tag{3.16}$$

STEP 6. By combining (3.7)-(3.9) with (3.10)-(3.16), we obtain a version of (3.4) corresponding to the formal replacements of $a_T$, $T$ and $\mathcal{L}_T$ by $A_n$, $n$ and $K_n$. To obtain the original statement (3.4), some more work is necessary. In the first place, we observe that the arguments given in Steps 1,2,3,4,5 remain valid when we assume that $A_n = [u_n]$, where $u_T$ is a function of $T > 0$ such that

$$(3.17) \qquad u_T/\log T \downarrow c \quad \text{and} \quad u_T \uparrow \infty \quad \text{as} \quad T \uparrow \infty.$$

We now choose $u_T$ to be any function of $T > 0$ such that $u_T \geq a_T + 3$ for $T > 0$ and satisfying (3.17). To prove that such a function exists is routine analysis. Consider now $g = c\eta_{x,T} = ca_T^{-1}(S(x + Ia_T) - S(x))$ for $0 \leq x \leq T - a_T$, an arbitrary element of $\mathcal{L}_T$. Let $n = [T]$, $\tilde{m} = [x]$, and $\tilde{g} = cA_n^{-1}(S(m + IA_n) - S(m)) = \gamma_{m,n}$. It is readily verified that

$$(3.18) \qquad g(s) = (A_n/a_T)\tilde{g}(A_n^{-1}([x + a_n s] - [x])) \quad \text{for} \quad 0 \leq s \leq 1.$$

For $0 \leq \rho \leq 1$, $\lambda \geq 0$ and $f \in BV_C(0,1)$, let

$$(3.19) \qquad H(\rho, \lambda; f) = d_W(\lambda f(\rho I), f).$$

If $f \in \tilde{D}_{\psi,c}^{2\varepsilon}$, then, we obtain by (3.19) and (2.10)

$$(3.20) \qquad H(\rho, \lambda; f) \leq 2(\lambda + 1)\varepsilon + \sup_{h \in D_{\psi,c}} d_W(\lambda h(\rho I), h).$$

Since for each $h \in D_{\psi,c}$, the mapping $(\rho, \lambda) \to d_W(\lambda h(\rho I), h)$ is continuous, the compactness of $D_{\psi,c}$ entails that the RHS of (3.20) can be rendered less than an arbitrary $\Theta > 0$ for all $0 < \varepsilon \leq \varepsilon_0$, $\rho_0 \leq \rho \leq 1$ and $\lambda_0 \leq \lambda \leq 1$, with $\varepsilon_0 > 0$ sufficiently small and $\rho_0 < 1$, $\lambda_0 < 1$ sufficiently large.

It follows readily from (3.18) that, for all $n$ sufficiently large, we have

$$(3.21) \qquad \sup_{g \in \mathcal{L}_T} d_W(g, \tilde{g}) < \Theta.$$

We now apply (3.10) (with $e_n = 1$) and (3.16) (with $e_n = A_n$), combined with (3.21) and the fact that $\varepsilon > 0$ and $\Theta > 0$ can be chosen as small as desired to conclude that (3.4) holds. □

THEOREM 3.2. *Assume that the assumptions of Theorem* 3.1 *hold, and that* $t_0 = \infty$ *and* $t_1 = -\infty$. *Then, for any* $\varepsilon > 0$, *there exist almost surely a* $C < \infty$ *and a* $T(\varepsilon) < \infty$ *such that, for all* $T \geq T(\varepsilon)$,

$$(3.22) \qquad D_{\psi,c} \subset \mathcal{L}_T^{\varepsilon} \subset D_{\psi,c}^{2\varepsilon}.$$

PROOF. We will show that Theorem 3.2 is the consequence of Theorem 3.1 when combined with the following two statements.

STATEMENT 1. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that, uniformly over $0 \le s' \le 1$, $0 \le s'' \le 1$, $|s' - s''| \le \delta$ and $g \in \mathcal{D}_{\psi,c}$, we have

$$|g(s') - g(s'')| \le \varepsilon. \tag{3.23}$$

STATEMENT 2. For any $\varepsilon > 0$, there exist almost surely a $\delta > 0$ and a $T_\varepsilon < \infty$ such that, uniformly over $0 \le s' \le 1$, $0 \le s'' \le 1$, $|s' - s''| \le \delta$, $T \ge T_\varepsilon$ and $f \in \mathcal{L}_T$, we have

$$|f(s') - f(s'')| \le \varepsilon. \tag{3.24}$$

We conclude the proof in three steps.

STEP 1. We show that Statements 1 and 2, when combined with Theorem 3.1, imply Theorem 3.2. Let $\varepsilon > 0$ be fixed, and let $\delta > 0$ be such that both (3.23) and (3.24) hold for $g \in \mathcal{D}_{\psi,c}$ and $f \in \mathcal{L}_T$, $T \ge T_\varepsilon$. We will show that these conditions imply that, uniformly over $g \in \mathcal{D}_{\psi,c}$ and $f \in \mathcal{L}_T$, $T \ge T_\varepsilon$,

$$d_W(f,g) \le \varepsilon\delta \Rightarrow \|f - g\| < 6\varepsilon. \tag{3.25}$$

To prove (3.25), set $c = \|f - g\|$. There is nothing to prove if $c = 0$. When $c > 0$, let $0 \le s \le 1$ be such that $|f(s) - g(s)| \ge c/2$, and let $[a,b]$ be an interval of length $\delta$ with $0 \le a \le s \le b \le 1$. By (2.10), we have

$$d_W(f,g) \ge \int_a^b |f(u) - g(u)|du \ge \delta|f(s) - g(s)| - 2\delta\varepsilon \ge \delta(\frac{c}{2} - 2\varepsilon),$$

which in turn readily implies (3.25).

Assume now that (3.4) holds with the formal replacement of $\varepsilon$ by $\varepsilon\delta$. By (3.25), we have then

$$\mathcal{D}_{\psi,c} \subset \mathcal{L}_T^{6\varepsilon} \subset \mathcal{D}_{\psi,c}^{12\varepsilon},$$

which, since $\varepsilon > 0$ may be chosen as small as desired, suffices for (3.22).

STEP 2. To show that Statement 1 holds, it suffices to combine Fact 2 with the fact that the mapping $(f,\delta) \to \sup\{|f(s') - f(s'')| : 0 \le s' \le 1, 0 \le s'' \le 1, |s' - s''| \le \delta\}$ is continuous on $(C(0,1), U) \times \mathbf{R}$.

STEP 3. By Section 2 and Theorem 5 of Deheuvels and Devroye (1987), we have

$$\max_{0\leq i\leq n-k}\max_{1\leq j\leq k}\{S_{i+j}-S_i\}/(\log n)\to v\alpha^+(v)\quad \text{a.s.}\quad \text{as } n\to\infty, \tag{3.26}$$

when $k = k_n = [v\log n]$ and $v > 0$ is arbitrary. Under the assumptions of Theorem 3.2, it is readily verified from (2.1), (2.2) and (1.17) that $t_0 = \infty \Rightarrow v\alpha^+(v)\to 0$ as $v\downarrow 0$. Repeating a similar argument with the formal replacements of max by min and of $\alpha^+(v)$ by $\alpha^-(v)$ in (3.26), we see that Statement 3 is a direct consequence of the fact that $v\alpha^{\pm}(v)\to 0$ as $v\downarrow 0$, and (3.26).

The proof of Theorem 3.2 is now completed. □

REMARK 3.1. In view of (1.19), there is no possible functional form of the Erdős-Rényi (1970) theorem with the topologies $U$ and $W$ when $t_1 = 0 < t_0$ or $t_1 < 0 = t_0$. The problem of finding a topology which would render such a law possible is open.

## 4. Applications

Let $\mathcal{K}(\cdot)$ be a functional defined on $BV(0,1)$ and continuous on $(BV(0,1),\mathcal{T})$ where $\mathcal{T}$ will denote either the weak topology $W$, or the topology $U$ of uniform convergence. A direct application of Theorems 3.1 and 3.2 is stated in the following corollary.

COROLLARY 4.1. *Under the assumptions of Theorem* 3.1, *if* $\mathcal{T} = W$, *then*

$$\lim_{T\to\infty}\{\sup_{0\leq x\leq T-a_T}\mathcal{K}(ca_T^{-1}(S(x+Ia_T)-S(x)))\} = \sup_{g\in D_{\psi,c}}\mathcal{K}(g)\quad a.s. \tag{4.1}$$

*Likewise,* (4.1) *holds under the assumptions of Theorem* 3.2 *when* $\mathcal{T} = U$.

PROOF. Straightforward (see for instance Section 5 in Deheuvels and Mason (1991a)). A typical example of functional $\mathcal{K}$ is given by $\mathcal{K}(f) = \pm f(1)$. An application of Corollary 4.1 in this case yields directly Theorem 1.2. Interestingly, by using the convexity of $\psi$ and (2.13), it is readily verified that the functions $g$ which maximize the right-hand-side of (4.1) are linear in this case, and of the form

$$g(s) = sc\alpha^{\pm}(c)\quad \text{for}\quad 0\leq s\leq 1. \tag{4.2}$$

The computation of the constant on the right-hand-side of (4.1) leads to a variational problem in an Orlicz-type space. Deheuvels and Mason (1991) have treated this problem in the special cases of the Poisson and of the exponential distributions, the latter being of great impotance for applications to empirical processes. However, their methods can be adapted to the general case considered here.

Extensions of these results to other processes which may be derived from partial sums, such as renewal and cumulated renewal processes will be considered elsewhere.

## 5. Discussion and further comments

The essential part of this paper originated from discussions I have had with David M. Mason in 1987. We were working on tail processes (see e.g. Mason (1988), Einmahl and Mason (1988), and Deheuvels and Mason (1990)), and made use of the remarkable large deviations results of Lynch and Sethuraman (1987), which were instrumental in the study of quantile processes. Because of the similarities of empirical and partial sum processes, it was obvious that one could obtain a functional form of the Erdős-Rényi theorem of the form given in Theorem 3.1 by the same arguments. In fact, the large deviation principles of Lynch and Sethuraman (1987) are sufficient to obtain such a result for partial sums of nonnegative random variables. Likewise, one may use Varadhan (1966), but here the restriction that $t_0 = \infty$ and $t_1 = -\infty$ is needed. In 1989, I attempted to join efforts with James Lynch and Jayaram Sethuraman, and to write a joint paper concerning these Erdős-Rényi laws in the general case. However, we ended up in shifting our interests to different aspects of this problem, James Lynch and Jayaram Sethuraman being more interested by large deviations for the positive and negative parts of the partial sums, while my own concern was limited to the study of the fluctuations of partial sum processes. It is likely that their results will enable in the future to obtain refined versions of Theorem 3.1, as far as the topological aspects are concerned. By all this, it turns out that these two authors have in hands, to my best knowledge, unpublished results which come close to Theorem 3.1. I wish also to mention that, after I had submitted this paper, I learned from Endre Csáki that K. A. Borovkov had made in June-July 1989 a communication at the Fifth International Vilnius Conference, in which he obtained a result essentially identical to Theorem 3.2 for $\mathbf{R}^d$-valued random vectors (see e.g. Borovkov (1989)). The list of authors who may have good reasons to share parts of Theorem 3.1 and 3.2

is very likely not complete. To whom must go the credit of the discovery of these functional laws will be judged by history and is not very important to my point of view. The Erdős-Rényi theorem in the case of partial sums is only a piece of a very large puzzle in a much wider setting. What is however extremely important to point out is that the first paper which motivated all these developments is the "new law of large numbers" discovered by Pál Erdős and Alfréd Rényi. Interestingly, a version of the Erdős-Rényi law had been given by Shepp (1964) quite a few years before was published the afore-mentioned Erdős-Rényi (1970) paper. However, the interest into these results really began in 1970, which fully explains why these strong laws are named after these two scientists.

It is a great priviledge for me to dedicate this paper to the memory of Alfréd Rényi whose example has greatly stimulated my own work in the field of probability and statistics.

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## REFERENCES

ACOSTA, A. de and KUELBS, J. (1983), Limit theorems for moving averages of independent random vectors, *Z. Wahrsch. Verw. Gebiete* **64** (1983), 67—123. *MR* **84i**: 60011

ARRATIA, R. and WATERMAN, M. S. (1989), The Erdős-Rényi strong law for pattern matching with a given proportion of mismatches, *Ann. Probab.* **17** (1989), 1152—1169.

ARRATIA, R., GORDON, L. and WATERMAN, M. S. (1990), The Erdős-Rényi law in distribution for coin tossing and sequence matching, *Ann. Statist.* **18** (1990), 539—570.

BÁRTFAI, P. (1966), Die Bestimmung der zu einem wiederkehrenden Prozess gehörenden Verteilungsfunktion aus den mit Fehlern behafteten Daten einer einzigen Realisation, *Studia Sci. Math. Hungar.* **1** (1966), 161—168. *MR* **35** # 6218

BILLINGSLEY, P. (1968), *Convergence of probability measures*, John Wiley, New York - London, 1968. *MR* **38** # 1718

BOOK, S. A. (1973), The Erdős-Rényi new law of large numbers for weighted sums, *Proc. Amer. Math. Soc.* **38** (1973), 165—171. *MR* **46** # 10044

BOOK, S. A. and TRUAX, D. R. (1976), An Erdős-Rényi strong law for sample quantiles, *J. Appl. Probability* **13** (1976), 578—583. *MR* **54** # 6243

BOOK, S. A. and SHORE, T. R. (1978), On large intervals in the Csörgő-Révész theorem on increments of a Wiener process, *Z. Wahrsch. Verw. Gebiete* **46** (1978/79), 1—11. *MR* **80h**: 60052

CHERNOFF, H. (1952), A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Statistics* **23** (1952), 493—507. *MR* **15** — 241

CRAMÉR, H. (1938), *Sur un nouveau théorème-limite de la théorie des probabilitiés*, In: Cramér, H., Lévy, P. et R. de Mises, *Les sommes et les fonctions de variables aléatoires*, Actualités Scientifiques et Industrielles **736**, Hermann, Paris, 1938, 5—23. *Zbl* **22**, 241—242

CSÁKI, E., FÖLDES, A. and KOMLÓS, J. (1987), Limit theorems for Erdős-Rényi type problems, *Studia Sci. Math. Hungar.* **22** (1987), 321—332. *MR* **89b**: 60052

CSÖRGŐ, M. and RÉVÉSZ, P. (1979), How big are the increments of a Wiener process?, *Ann. Probab.* **7** (1979), 731—737. *MR* **80g**: 60025

CSÖRGŐ, M. and RÉVÉSZ, P. (1981), *Strong approximations in probability and statistics*, Academic Press, New York - London, 1981. *MR* **84d**: 60050

CSÖRGŐ, M. and STEINEBACH, J. (1981), Improved Erdős-Rényi and strong approximation laws for increments of partial sums, *Ann. Probab.* **9** (1981), 988—996. *MR* **82k**: 60063

CSÖRGŐ, S. (1979), Erdős-Rényi laws, *Ann. Statist.* **7** (1979), 772—787. *MR* **80g**: 60027

DEHEUVELS, P. (1985), On the Erdős-Rényi theorem for random fields and sequences and its relationships with the theory of runs and spacings, *Z. Wahrsch. Verw. Gebiete* **70** (1985), 91—115. *MR* **86k**: 60045

DEHEUVELS, P. and STEINEBACH, J. (1986), Exact convergence rate of an Erdős-Rényi strong law for moving quantiles, *J. Appl. Probab.* **23** (1986), 355—369. *MR* **87i**: 60033

DEHEUVELS, P., DEVROYE, L. and LYNCH, J. D. (1986), Exact convergence rate in the limit theorems of Erdős-Rényi and Shepp, *Ann. Probab.* **14** (1986), 209—223. *MR* **87d**: 60032

DEHEUVELS, P. and DEVROYE, L. (1987), Limit laws of Erdős-Rényi-Shepp type, *Ann. Probab.* **15** (1987), 1363—1386. *MR* **88f**: 60055

DEHEUVELS, P., ERDŐS, P., GRILL, K. and RÉVÉSZ, P. (1987), Many heads in a short block, *Mathematical statistics and probability theory*, Vol. A. (Bad Tatzmannsdorf, 1986), Puri, M. L., Révész P. and Wertz, W. eds., Reidel, Dordrecht - Boston, MA - London, 1987, 53—67. *MR* **89b**: 60073

DEHEUVELS, P. and MASON, D. M. (1990), Nonstandard functional laws of the iterated logarithm for tail empirical and quantile processes, *Ann. Probab.* **18** (1990), 1693—1722.

DEHEUVELS, P. and MASON, D. M. (1991), A tail empirical process approach to some nonstandard laws of the iterated logarithm, *J. Theoretical Probab.* **4** (1991), 53—85.

DEHEUVELS, P. and RÉVÉSZ, P. (1991), On the coverage of Strassen-type sets by sequences of Wiener processes (submitted).

EINMAHL, J. H. J. and MASON, D. M. (1990), Strong limit theorems for weighted quantile processes, *Ann. Probab.* **16** (1988), 1623—1643. *MR* **90d**: 62063

EINMAHL, U. (1989), Extensions of results of Komlós, Major and Tusnády to the multivariate case, *J. Multivariate Anal.* **28** (1989), 20—68. *MR* **90g**: 60032

ERDŐS, P. and RÉNYI, A. (1970), On a new law of large numbers, *J. Analyse Math.* **23** (1970), 103—111. *MR* **42** # 6907

ERDŐS, P. and RÉVÉSZ, P. (1975), On the length of the longest head-run, *Topics in Information Theory*, (Second Colloq., Keszthely, 1975) Csiszár, I. and Elias, P. eds., Colloq. Math. Soc. J. Bolyai, Vol. **16**, North-Holland, Amsterdam, 1977, 219—228. *MR* **57** # 17788

FÖLDES, A. (1979), The limit distribution of the length of the longest head-run, *Period. Math. Hungar.* **10** (1979), 301—310. *MR* **81b**: 60022b

GUIBAS, L. J. and ODLYZKO, A. M. (1980), Long repetitive patterns in random sequences, *Z. Wahrsch. Verw. Gebiete* **53** (1980), 241—262. *MR* **81m**: 60047

HÖGNÄS, G. (1977), Characterization of weak convergence of signed measures on [0,1], *Math. Scand.* **41** (1977), 175—184. *MR* **58** # 2948

KOMLÓS, J. and TUSNÁDY, G. (1975), On sequences of "pure heads", *Ann. Probability* **3** (1975), 608—617. *MR* **55** # 11353

KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976), An approximation of partial sums of independent RV's and the sample DF, II, *Z. Wahrsch. Verw. Gebiete* **34** (1976), 33—58. *MR* **53** # 6697

LYNCH, J. D. (1983), Some comments on the Erdős-Rényi law and a theorem of Shepp, *Ann. Probab.* **11** (1983), 801—802. *MR* **84m**: 60040

LYNCH, J. D. and SETHURAMAN, J. (1987), Large deviations for processes with independent increments, *Ann. Probab.* **15** (1987), 610—627. *MR* **88m**: 60076

MASON, D. M. (1988), A strong invariance theorem for the tail empirical process, *Ann. Inst. H. Poincaré Probab. Statist.* **24** (1988), 491—506. *MR* **89m**: 60076

RÉVÉSZ, P. (1979), A generalization of Strassen's functional law of the iterated logarithm, *Z. Wahrsch. Verw. Gebiete* **50** (1979), 257—264. *MR* **82c**: 60057

RÉVÉSZ, P. (1983), Three problems on the lengths of increasing runs, *Stochastic Process. Appl.* **15** (1983), 169—179. *MR* **84h**: 60064

RUDIN, W. (1979), *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987. *MR* **88k**: 00002

SHEPP, L. A. (1964), A limit law concerning moving averages, *Ann. Math. Statist.* **35** (1964), 424—428. *MR* **29** # 4091

SKOROHOD, A. V. (1956), Limit theorems for stochastic processes, *Teor. Veroyatnost. i Primenen.* **1** (1956), 289—319 (in Russian). *MR* **18** - 943

STEINEBACH, J. (1978), On a necessary condition for the Erdős-Rényi law of large numbers, *Proc. Amer. Math. Soc.* **68** (1978), 97—100. *MR* **57** # 1622

STRASSEN, V. (1964), An invariance principle for the law of the iterated logarithm, *Z. Wahrsch. Verw. Gebiete* **3** (1964), 211—226. *MR* **30** # 5379

VARADHAN, S. R. S. (1966), Asymptotic probabilities and differential equations, *Comm. Pure Appl. Math.* **19** (1966), 261—286. *MR* **34** # 3083

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# THE ISLAMIC MEAN: A PECULIAR L-STATISTIC

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**Abstract**

We study the limit behavior of the islamic mean $T_n$, a certain L-statistic with moving weights. Under some regularity conditions, $T_n$ is a consistent and asymptotically normal estimator of the median. Nonstandard limit results are obtained in case the regularity conditions are violated.

## 1. Introduction

Let $X_1, \dots, X_n$ be a sequence of independent random variables, each having the same distribution function $F(x) = P(X \leq x)$. Let $X_{[n:1]}, \dots, X_{[n:n]}$ denote the order statistics of the first $n$ observations. If $n = 2^k$ then the arithmetic mean can be computed by taking pairwise averages. The first run results in $\frac{1}{2}(X_{[n:1]} + X_{[n:2]}), \frac{1}{2}(X_{[n:3]} + X_{[n:4]}), \dots, \frac{1}{2}(X_{[n:n-1]} + X_{[n:n]})$. The second run is executed by applying the same procedure as above to the $2^{k-1}$ pairwise averages in the first run. In this way one obtains an increasing sequence of $2^{k-2}$ numbers and one can continue the procedure until after $k$ runs a single number is obtained, which is the arithmetic mean.

The islamic mean is obtained if the above-described procedure is modified by allowing overlap. More precisely, we define inductively

$$M_{0,1}, \dots, M_{0,n} = X_{[n:1]}, \dots, X_{[n:n]}$$

and $M_{h+1,i} = \frac{1}{2}(M_{h,i} + M_{h,i+1})$. It is easy to see that

$$M_{h,i} = 2^{-h} \sum_{j=0}^{h} \binom{h}{j} X_{[n:i+j]}$$

for $i = 1, \dots, n-h$. The final result $M_{n-1,1}$ is the islamic mean. Henceforth it is denoted as

$$T_n = \sum_{i=1}^{n} w_{ni} X_{[n:i]}$$

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where the weights are given by

$$w_{ni} = 2^{-n+1}\binom{n-1}{i}.$$

The islamic mean was brought to our attention by S. A. Mahmoud (1987). He gave a theological motivation, derived from the fact that the prophet prays at one third, one half and two thirds of the night. He also noted the representation of the islamic mean as a linear combination of order statistics.

Note that the weights $w_{ni}$ correspond with the probabilities of a binomial distribution with parameters $(n-1, \frac{1}{2})$. As this distribution centers around $\frac{n-1}{2}$ we are led to conjecture that $T_n$ is close to the sample median

$$T'_n = \begin{cases} X_{[n:\frac{n+1}{2}]} & n \text{ odd} \\ \frac{1}{2}(X_{[n:\frac{n}{2}]} + X_{[n:\frac{n}{2}+1]}) & n \text{ even} \end{cases}$$

We thus try to establish asymptotic properties of $T_n$ by studying the difference $T_n - T'_n$ and by using the well-known asymptotic properties of $T'_n$. Under some regularity conditions we can indeed prove that

$$n^{1/2}(T_n - T'_n) \to 0 \text{ in probability}$$

with as a consequence that

$$n^{1/2}(T_n - \xi_{1/2}) \to N(0, \frac{1}{4(f(\xi_{1/2}))^2})$$

in distribution. Obviously, we have to assume that the median $\xi_{1/2} = F^{-1}(1/2)$ is well-defined and that $f = F'$ exists and is continuous in a neighborhood of $\xi_{1/2}$. In addition a weak moment bound is required. We will prove these results first for the special case of a uniform distribution on $[0,1]$. This is needed for some of the proofs in the general case. In the final section we study the limit behavior in some cases where the regularity conditions are violated.

REMARK. The extensive literature about L-statistics is surveyed in Shorack-Wellner (1986). However, all known results are either concerned with a single quantile or with linear combinations of the form

$$\frac{1}{n}\sum_{i=1}^{n} J_n(\frac{i}{n+1})X_{[n:i]}$$

where the score functions $J_n : [0,1] \to R$ converge pointwise to a limit function $J(x)$. The islamic mean is peculiar in the sense that its weight generating function $J_n$ converges to a Dirac delta function.

## 2. The case of a uniform distribution on $[0,1]$

Suppose now that the random variables $X_i$ are uniformly distributed on $[0,1]$. The Glivenko-Cantelli theorem implies that $\sup_{0\leq x\leq 1}|F_n(x)-x|$ tends to 0 with probability 1. Here $F_n$ denotes the empirical distribution. It follows immediately that almost surely

$$\max_i |X_{[n:i]}-\frac{i}{n+1}|\to 0$$

and hence

$$\sum_{i=1}^{n} w_{ni}(X_{[n:i]}-\frac{i}{n+1})\to 0$$

which is equivalent to

$$T_n\to\frac{1}{2}$$

almost surely.

PROPOSITION 1. *If the underlying distribution is uniform on* $[0,1]$ *then*

$$n^{3/4}(T_n-T_n')\to N(0,\frac{\sqrt{2}-1}{2\sqrt{\pi}})$$

*in distribution.*

REMARK. Since the sample median is known to be asymptotically normal, we can infer from Proposition 1 the asymptotic normality of $T_n$. This can also be deduced from a general result of Hecker (1976). However, the statement of Proposition 1 is strictly stronger than asymptotic normality and we will need later that $n^{1/2}(T_n-T_n')=o_P(1)$.

PROOF. Let $Y_1,\ldots,Y_{n+1}$ be independent random variables with an exponential distribution with probability density $f(y)=e^{-y}1_{[0,\infty)}(y)$. Let $Z_j=Y_1+\ldots+Y_j$ be their partial sums. It is well-known (see e.g. Breiman (1968)) that the vector $(\frac{Z_1}{Z_{n+1}},\ldots,\frac{Z_n}{Z_{n+1}})$ has the same joint distribution as the uniform order statistic $(X_{[n:1]},\ldots,X_{[n:n]})$. For notational convenience we restrict the attention to odd $n$ so that we can write

$$n^{3/4}(T_n-T_n')\stackrel{\mathcal{L}}{=}\frac{n}{Z_{n+1}}n^{-1/4}\sum_{i=1}^{n}w_{ni}(Z_i-Z_{\frac{n+1}{2}}).$$

As $nZ_{n+1}^{-1} \to 1$ in probability, it suffices to show that

$$D_n := n^{-1/4} \sum_{i=1}^{n} w_{ni}(Z_i - Z_{\frac{n+1}{2}}) \to N(0, \frac{\sqrt{2}-1}{2\sqrt{\pi}})$$

in distribution. As $Z_i = \sum_{k=1}^{i} Y_k$ we get

$$\sum_{i=1}^{n} w_{ni} Z_i = \sum_{i=1}^{n}(w_{ni} \sum_{k=1}^{i} Y_k) = \sum_{k=1}^{n}(\sum_{i=k}^{n} w_{ni})Y_k$$

and thus

$$\begin{aligned} D_n &= n^{-1/4} \left\{ \sum_{k=1}^{\frac{n+1}{2}} \left( \sum_{i=k}^{n} w_{ni} - 1 \right) Y_k + \sum_{k=\frac{n+3}{2}}^{n} \left( \sum_{i=k}^{n} w_{ni} \right) Y_k \right\} = \\ &= n^{-1/4} \left\{ \sum_{k=1}^{\frac{n+1}{2}} \left( -\sum_{i=1}^{k-1} w_{ni} \right) Y_k + \sum_{k=\frac{n+3}{2}}^{n} \left( \sum_{i=k}^{n} w_{ni} \right) Y_k \right\}. \end{aligned}$$

To this sum of independent random variables we can apply the Lindeberg central limit theorem for triangular arrays. The Lindeberg condition is easily checked, it remains to compute the variance. By symmetry we have

$$\mathrm{var}(D_n) = \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{n+1}{2}} \left( \sum_{i=1}^{k-1} w_{ni} \right)^2 .$$

Now let $B_{n-1}$ and $B'_{n-1}$ be two independent binomial $(n-1, \frac{1}{2})$ random variables and define $M_{n-1} = \max(B_{n-1}, B'_{n-1})$. Then we can write $(\sum_{i=1}^{k-1} w_{ni})^2 = P(M_{n-1} \le k-2)$ and hence

$$\begin{aligned} \mathrm{var}(D_n) &= \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{n+1}{2}} P(M_{n-1} \le k-2) = \\ &= \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{n+1}{2}} \sum_{j=0}^{k-2} P(M_{n-1} = j) = \\ &= \frac{2}{\sqrt{n}} \sum_{j=0}^{\frac{n-3}{2}} \sum_{k=j+2}^{\frac{n+1}{2}} P(M_{n-1} = j) = \end{aligned}$$

$$= \frac{2}{\sqrt{n}} \sum_{j=0}^{\frac{n-3}{2}} (\frac{n+1}{2} - (j+2) + 1) P(M_{n-1} = j) =$$
$$= \frac{2}{\sqrt{n}} E\{\frac{n-1}{2} - M_{n-1}\} 1_{\{M_{n-1} \leq \frac{n-3}{2}\}} =$$
$$= -\frac{\sqrt{n-1}}{\sqrt{n}} E\{2\sqrt{n-1}(\frac{M_{n-1}}{n-1} - \frac{1}{2})\} 1_{\{2(\frac{M_{n-1}}{n-1} - \frac{1}{2})\sqrt{n-1} \leq \frac{2}{\sqrt{n-1}}\}}.$$

Observe that

$$2\sqrt{n-1}(\frac{M_{n-1}}{n-1} - \frac{1}{2}) = \max(2\sqrt{n-1}(\frac{B_{n-1}}{n-1} - \frac{1}{2}), 2\sqrt{n-1}(\frac{B'_{n-1}}{n-1} - \frac{1}{2}))$$

converges in distribution to $V = \max(W, W')$, where $W$ and $W'$ are two independent standard normal random variables. So finally we get

$$\mathrm{var}(D_n) \to -EV1_{\{V \leq 0\}}.$$

The density of $V$ being $2\phi(x)\Phi(x)$, where $\phi$ and $\Phi$ are the standard normal density and distribution function, respectively, we find

$$-EV1_{\{V\leq 0\}} = -2\int_{-\infty}^{0} x\phi(x)\Phi(x)dx = 2\int_{-\infty}^{0} \phi'(x)\Phi(x)dx =$$
$$= 2\phi(x)\Phi(x)|_{-\infty}^{0} - 2\int_{-\infty}^{0} \phi^2(x)dx = \frac{1}{\sqrt{2\pi}} - \frac{2}{2\pi}\int_{-\infty}^{0} e^{-x^2}dx =$$
$$= \frac{1}{\sqrt{2\pi}} - \frac{\sqrt{\pi}}{2\pi} = \frac{\sqrt{2}-1}{2\sqrt{\pi}}.$$

A SMALL SAMPLE COMPARISON. If one postulates a location model and tries to estimate the point of symmetry of a uniform distribution on $[\theta, \theta+1]$ from a sample of size 3, then various estimators can be considered and their exact variances can be calculated. The variance of the median $T'$ is equal to the variance of the $Beta(2,2)$ distribution, i.e. $\frac{1}{20}$. The variance of the islamic mean $T = \frac{1}{4}(X_{[1]} + 2X_{[2]} + X_{[3]})$ is equal to $\frac{1}{32}$, that of the ordinary mean is $\frac{1}{36}$, while that of the mid-range $\frac{1}{2}(X_{[1]} + X_{[3]})$ is smallest, namely $\frac{1}{40}$.

## 3. The general case; consistency

To prove convergence of the islamic mean towards the population median we have to assume that the median is uniquely defined in the sense that

$\xi_{\frac{1}{2}} = \xi^{+}_{\frac{1}{2}}$ (*Assumption A*), where

$$\xi_{\frac{1}{2}} = \inf\{x : P(X \leq x) \geq \frac{1}{2}\},$$
$$\xi^{+}_{\frac{1}{2}} = \inf\{x : P(X \leq x) > \frac{1}{2}\}.$$

THEOREM 1. *Suppose that Assumption A holds. Then*
(a) *$T_n$ converges to $\xi_{\frac{1}{2}}$ in probability if and only if*

$$P(|X| > x) = o(\frac{1}{\log x}) \quad as \quad x \to \infty.$$

(b) *If for some $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} nP(|X| > e^{n\varepsilon}) < \infty$$

*then $T_n$ converges to $\xi_{\frac{1}{2}}$ almost surely.*

REMARK. It is easy to see that the assumption in (b) holds for all $\varepsilon > 0$ if and only if it holds for some $\varepsilon > 0$. Moreover the assumption is equivalent to the moment condition $E(\log(X \vee 1))^2 < \infty$.

The proof of the theorem requires an estimate of the tails of the weights $w_{ni}$, a sharper version of which is needed in the following sections. We therefore state this result as a lemma for future reference. Let for $0 \leq x \leq 1$

$$I(x) = x \log x + (1 - x) \log(1 - x) + \log 2$$

denote the large deviation rate function of a symmetric Bernoulli variable. Note that $I(x)$ is strictly convex with minimum at $x = \frac{1}{2}$ and $I(\frac{1}{2}) = 0$.

LEMMA 1. *For $r < \frac{n-1}{2} + 1$ we have*

$$-I(\frac{r-1}{n-1}) - \frac{1}{n-1}(\frac{1}{2}\log 2\pi +$$
$$+ \frac{1}{2}\log\{\frac{(r-1)(n-r)}{n-1}\} + \frac{n-1}{12(r-1)(n-r)} - \frac{1}{12n-11}) \leq$$
$$\leq \frac{1}{n-1}\log\sum_{i=1}^{r} w_{ni} \leq -I(\frac{r-1}{n-1}).$$

PROOF. The right-hand side is a well-known basic inequality in large deviation theory. It can be proved by observing that

$$\begin{aligned}\sum_{i=1}^{r} w_{ni} &= P(B_{n-1} \leq r-1) = \\ &= P\big(e^{tB_{n-1}} \geq e^{t(r-1)}\big) \leq \\ &\leq e^{-t(r-1)} E e^{tB_{n-1}} = \\ &= \exp[-(n-1)\{t\frac{r-1}{n-1} - \log(\frac{1}{2} + \frac{1}{2}e^t)\}]\end{aligned}$$

holds for all $t \leq 0$. Thus $\frac{1}{n-1}\log(\sum_{i=1}^{r} w_{ni})$ is bounded from above by $-\sup_{t\leq 0}\{t\frac{r-1}{n-1} - \log(\frac{1}{2} + \frac{1}{2}e^t)\}$ which using elementary calculus is seen to be equal to $-I(\frac{r-1}{n-1})$.

The left-hand inequality is a consequence of

$$\sum_{i=1}^{r} w_{ni} \geq w_{nr} = 2^{-n+1}\frac{(n-1)!}{(r-1)!(n-r)!}$$

and a sharpened version of Stirling's formula, which can e.g. be found in Feller (1968):

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi}\, n^{n+1/2} e^{-n} e^{\frac{1}{12n}}.$$

PROOF OF THEOREM 1. Given $\varepsilon > 0$, let $\delta_1 = F(\xi_{\frac{1}{2}} + \varepsilon) - 1/2 > 0$ and $\delta_2 = 1/2 - F(\xi_{\frac{1}{2}} - \varepsilon) > 0$. Then by the law of large numbers $F_n(\xi_{\frac{1}{2}} + \varepsilon) \to \frac{1}{2} + \delta_1$ and $F_n(\xi_{\frac{1}{2}} - \varepsilon) \to \frac{1}{2} - \delta_2$ almost surely. Hence

$$\begin{aligned}\limsup_{n\to\infty} X_{[n:n\frac{1+\delta_1}{2}]} &\leq \xi_{\frac{1}{2}} + \varepsilon \\ \liminf_{n\to\infty} X_{[n:n\frac{1-\delta_2}{2}]} &\geq \xi_{\frac{1}{2}} - \varepsilon.\end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Since by Lemma 1 we know that $\sum_{i=n\frac{1-\delta}{2}}^{n\frac{1+\delta}{2}} w_{ni} \to 1$ as $n \to \infty$, these inequalities imply

$$\xi_{\frac{1}{2}} - \varepsilon \leq \liminf_{n\to\infty} \sum_{i=n\frac{1-\delta}{2}}^{n\frac{1+\delta}{2}} w_{ni} X_{[n:i]} \leq \limsup_{n\to\infty} \sum_{i=n\frac{1-\delta}{2}}^{n\frac{1+\delta}{2}} w_{ni} X_{[n:i]} \leq \xi_{\frac{1}{2}} + \varepsilon.$$

The sufficiency part of the proof can be finished by showing convergence of the tails of the sum to zero in probability under the conditions of part (a) and almost surely under the conditions of part (b). By Lemma 1 we have with $a = I(\frac{1+\delta}{2}) > 0$:

$$P\left(\sum_{i=1}^{n^{\frac{1-\delta}{2}}} w_{ni}X_{[n:i]} + \sum_{i=n^{\frac{1+\delta}{2}}}^{n} w_{ni}X_{[n:i]} > \varepsilon\right)$$
$$\leq P(\max_{i=1,\ldots,n} |X_i| > \varepsilon e^{na}) = 1 - \left(1 - \frac{nP(|X_1| > \varepsilon e^{na})}{n}\right)^n \leq$$
$$\leq nP(|X_1| > \varepsilon e^{na})$$

Under the assumption of part (a) we have $nP(|X_1| > \varepsilon e^{na}) \to 0$, which proves convergence in probability. Under the assumptions of part (b), we have that

$$\sum_{n=1}^{\infty} nP(|X_1| > \varepsilon e^{na}) < \infty$$

and hence almost sure convergence follows from the Borel-Cantelli lemma.

The proof of necessity in part (a) is more delicate. We will give the proof for non-negative random variables $X_i$. In this case $T_n \geq 2^{-n}X_{[n:n]}$ and hence

$$P(T_n \geq c) \geq P(\max_{i=1,\ldots,n} X_i \geq c2^{-n})$$
$$= 1 - \left(1 - \frac{nP(X_i > c2^{-n})}{n}\right)^n.$$

Thus, if $T_n$ converges to $\xi_{\frac{1}{2}}$ in probability, then $nP(X_1 > c2^n) \to 0$ for some $c > 0$. But this is equivalent with $P(X_1 > x) = o(\frac{1}{\log x})$. The proof for random variables that take both positive and negative values uses the same idea, combined with the asymptotic independence of the left and right extreme order statistics. Details are left to the reader.

## 4. The general case: asymptotic normality

To prove asymptotic normality of the islamic mean, we need a stronger assumption than assumption A, namely that the density $f = F'$ exists and is positive and continuous in a neighborhood of $\xi_{\frac{1}{2}}$ (*Assumption A'*).

THEOREM 2. *Under assumption A' the following are equivalent:*
**(i)** $n^{1/2}(T_n - T_n') \to 0$ *in probability.*
**(ii)** $n^{1/2}(T_n - \xi_{\frac{1}{2}}) \xrightarrow{\mathcal{L}} N(0, \frac{1}{4(f(\xi_{\frac{1}{2}}))^2})$.
**(iii)** $P(|X_1| > x) = o(\frac{1}{\log x})$.

PROOF. It is well-known that Assumption A' implies asymptotic normality of the sample median, i.e. $n^{1/2}(T_n' - \xi_{\frac{1}{2}}) \xrightarrow{\mathcal{L}} N(0, \frac{1}{4(f(\xi_{\frac{1}{2}}))^2})$. This proves that (i) $\Rightarrow$ (ii). If (ii) holds, $T_n$ converges to $\xi_{\frac{1}{2}}$ in probability and hence by part (a) of Theorem 1 we have (iii). For the remaining part (iii) $\Rightarrow$ (i) note that under assumption A' the quantile function

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}$$

is continuously differentiable at $u = 1/2$ with $\frac{d}{du}F^{-1}(1/2) = \frac{1}{f(\xi_{\frac{1}{2}})}$. Hence given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|F^{-1}(v) - F^{-1}(u) - \frac{1}{f(\xi_{\frac{1}{2}})}(v - u)| \leq \varepsilon^2 |v - u|$$

if $|v - \frac{1}{2}| \leq \delta$ and $|u - \frac{1}{2}| \leq \delta$. We represent the underlying random variables $X_i$ as $F^{-1}(U_i)$ where the $U_i$ are independent random variables, uniformly distributed on $[0,1]$. We assume again for notational convenience that the sample size is odd, so that we may write

$$n^{1/2}(T_n - T_n') = n^{1/2} \sum_{i=1}^{n} w_{ni}(X_{[n:i]} - X_{[n:\frac{n+1}{2}]}).$$

We split the right-hand sum into its central part

$$n^{1/2} \sum_{i=n\frac{1-\delta}{2}}^{n\frac{1+\delta}{2}} w_{ni}(X_{[n:i]} - X_{[n:\frac{n+1}{2}]})$$

and the two tail parts. That both tails converge to zero in probability can be established as in the proof of Theorem 1 (the extra factor of $n^{1/2}$ gets swallowed by the negative exponential). In the central part we write

$$X_{[n:i]} - X_{[n:\frac{n+1}{2}]} = \frac{1}{f(\xi_{\frac{1}{2}})}(U_{[n:i]} - U_{[n:\frac{n+1}{2}]}) + R_{ni}$$

where $|R_{ni}| \leq \varepsilon^2 |U_{[n:i]} - U_{[n:\frac{n+1}{2}]}|$ if $U_{[n:i]}$ and $U_{[n:\frac{n+1}{2}]}$ are in $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$. Note that by the law of large numbers

$$P\left(\frac{1}{2} - \delta \leq U_{[n:n\frac{1-\delta}{2}]} \leq U_{[n:n\frac{1+\delta}{2}]} \leq \frac{1}{2} + \delta\right)$$

converges to 1. Thus we get

$$P(n^{1/2}\sum_{i=n\frac{1-\delta}{2}}^{n\frac{1+\delta}{2}} w_{ni}|R_{ni}| > \varepsilon) \leq$$
$$\leq P(n^{1/2}\varepsilon^2\sum_{i=1}^{n} w_{ni}|U_{[n:i]} - U_{[n:\frac{n+1}{2}]}| > \varepsilon) + o(1) \leq$$
$$\leq \frac{1}{\varepsilon}E(\varepsilon^2 n^{1/2}\sum_{i=1}^{n} w_{ni}|U_{[n:i]} - U_{[n:\frac{n+1}{2}]}|) + o(1) \leq$$
$$\leq \varepsilon n^{1/2}\sum_{i=1}^{n} w_{ni}|\frac{i}{n+1} - \frac{1}{2}| + o(1) \leq$$
$$\leq \varepsilon n^{-1/2}E|B_{n-1} - \frac{n-1}{2}| + o(1) \leq$$
$$\leq \varepsilon n^{-1/2}(E(B_{n-1} - \frac{n-1}{2})^2)^{1/2} + o(1) \leq$$
$$\leq \varepsilon/2 + o(1).$$

From Proposition 1 we get that the linear part

$$\frac{1}{f(\xi_{1/2})}\sum_{i=n\frac{1-\delta}{2}}^{n\frac{1+\delta}{2}} (U_{[n:i]} - U_{[n:\frac{n+1}{2}]})$$

converges to 0 in probability and thus the proof is finished.

## 5. Some nonstandard limit theorems

In the previous sections we have shown consistency and asymptotic normality of the islamic mean under regularity conditions. What happens if these conditions are violated? We will give an answer to this question in two cases. First assume that the median is not uniquely defined, i.e. that there is a non-degenerate interval of medians $[\xi_{1/2}, \xi^+_{1/2}]$. In this case the sample median has a limit distribution that attaches mass 1/2 to each of the endpoints of the interval.

THEOREM 3. *If $\xi_{1/2} < \xi^+_{1/2}$, and if the tail probability assumption*

$$P(|X| \leq x) = o(\frac{1}{\log x})$$

*is satisfied, then $T_n$ converges weakly to the uniform distribution on the interval of medians.*

PROOF. We begin by showing this for a symmetric Bernoulli sequence. Note that in this case

$$T_n = \sum_{j=n-S_n+1}^{n} w_{nj} = \sum_{j=1}^{S_n} w_{nj}$$

where $S_n = \sum_{i=1}^{n} X_i$ has a binomial $(n, \frac{1}{2})$ distribution. By the de Moivre-Laplace limit theorem we know that

$$\sum_{j=1}^{k} w_{nj} - \Phi(\frac{k - \frac{n}{2}}{\frac{1}{2}(n-1)^{1/2}})$$

converges to zero, uniformly in $k$, as $n \to \infty$. Hence

$$T_n = \Phi\left(\frac{S_n - \frac{n}{2}}{\frac{1}{2}(n-1)^{1/2}}\right) + o(1)$$

almost surely. The distribution of the random variables in brackets converges to $N(0,1)$. Hence that of $T_n$ converges to the uniform distribution on $[0,1]$.

The proof of the general case follows now by approximating $X_i$ by

$$X_i' = \xi_{1/2} 1_{\{X_i \le \xi_{1/2}\}} + \xi_{1/2}^{+} 1_{\{X_i \ge \xi_{1/2}^{+}\}}$$

and by proving that the difference between $T_n$ and the islamic mean of the new variables tends to 0 in probability.

EXAMPLE. Finally, we want to treat Bernoulli random variables with $P(X_i = 1) = p < 1/2$ as an example where $\xi_{1/2} = \xi_{1/2}^{+}$, but $F$ is discontinuous at $\xi_{1/2}$. In this case of course $T_n$ converges to 0 quicker than any power of $n$ and it seems appropriate to look for the exponential rate of convergence. Precisely, we obtain the following result:

$$n^{1/2}(\frac{1}{n}\log T_n + I(p)) \xrightarrow{\mathcal{L}} N(0, [\log\frac{p}{1-p}]^2 p(1-p))$$

where $I(p)$ is the large deviation rate function introduced in section 3.

To prove this assertion, we make use of the result of Lemma 1, which shows that

$$n^{1/2}\left(\frac{1}{n-1}\log\sum_{j=1}^{r_n} w_{nj} + I(\frac{r_n - 1}{n-1})\right)$$

converges to 0 provided that

$$0 < \liminf r_n/n \le \limsup r_n/n < \frac{1}{2}.$$

Thus it suffices to show that

$$n^{1/2}(I(\frac{S_n}{n}) - I(p)) \xrightarrow{\mathcal{L}} N(0, [\log \frac{p}{1-p}]^2 p(1-p)).$$

This can be established using the familiar delta method.

NOTE ADDED IN PROOF. The statistic $T_n$ was also introduced by N. L. Hjort (*Ann. of Statistics* **14**, (1986), p. 54) in the context of Bayesian estimation of the median based on the Dirichlet prior process. Hjort states (without proof) that $n^{1/2}(T_n - T'_n) = o_P(1)$ under regularity conditions on the tails of the distribution of $X$. We thank R. Helmers for this remark.

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REFERENCES

[1] BREIMAN, L., *Probability*, Addison-Wesley Publ. Co., Reading, Mass. - London, 1968. *MR* **37** # 4841

[2] FELLER, W., *An introduction to probability theory and its applications*, Vol 1, Third edition, J. Wiley, New York - London, 1968. *MR* **37** # 3604

[3] HECKER, H., A characterization of the asymptotic normality of linear combinations of order statistics from the uniform distribution, *Ann. Statist.* **4** (1976), 1244—1246. *MR* **54** # 11617

[4] MAHMOUD, S. A., Islamic statistics science (personal communication, 1987).

[5] SHORACK, G. R. and WELLNER, J. A., *Empirical processes with applications to statistics*, J. Wiley, New York, 1986. *MR* **88e**: 60002

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# THREE PROBLEMS ON THE RANDOM WALK IN $Z^d$

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**Abstract**

Let $\{S_n;\ n = 0, 1, 2, \ldots\}$ be the simple, symmetric random walk in $Z^d$ and let

$$\nu(n) = \min\{k:\ k > 0, S_{k+n} \neq S_j\ (j = 0, 1, 2, \ldots, n)\}$$

be the waiting time needed to meet a new point. The limit behaviour of $\nu(n)$ is investigated. Two further similar problems are also treated.

## 1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random vectors taking values from $Z^d$ with distribution

$$\mathbf{P}\{X_1 = e_i\} = \mathbf{P}\{X_1 = -e_i\} = \frac{1}{2d} \quad (i = 1, 2, \ldots, d)$$

where $\{e_1, e_2, \ldots, e_d\}$ is a system of orthogonal unit vectors of $Z^d$. Let

$$S_0 = 0 = \{0, 0, \ldots, 0\} \text{ and } S(n) = S_n = X_1 + X_2 + \cdots X_n\ (n = 1, 2, \ldots)$$

i.e. $\{S_n\}$ is the simple symmetric random walk in $Z^d$. Further let

$$\xi(x, n) = \#\{k : 0 < k \leq n,\ S_k = x\}$$

$(n = 1, 2, \ldots;\ x = (x_1, x_2, \ldots, x_d);\ x_j = 0, \pm 1, \pm 2, \ldots;\ j = 1, 2, \ldots, d)$ be the local time of the random walk. We say that the ball

$$Q(N, u; d) = \left\{x = (x_1, x_2, \ldots, x_d) : \|x - u\| = \left(\sum_{i=1}^{d}(x_i - u_i)^2\right)^{1/2} \leq N\right\}$$

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(where $u = (u_1, u_2, \dots, u_d)$) is covered by the random walk in time $n$ if

$$\xi(x,n) > 0 \text{ for every } x \in Q(N,u;d).$$

Let $R_d(n)$ be the largest integer for which there exists a random variable $u = u(n) \in Z^d$ such that $Q(R_d(n), u; d)$ is covered by the random walk in time $n$ i.e.

$$\xi(x,n) > 0 \text{ for every } x \in Q(R_d(n),u;d).$$

Révész [5], [6] proved

THEOREM A. (i) *For any* $\varepsilon > 0$ *and* $d \geq 3$

$$(\log n)^{\frac{1}{d-1}-\varepsilon} \leq R_d(n) \leq (\log n)^{\frac{1}{d-2}+\varepsilon} \quad a.s.$$

*for all but finitely many* $n$,

(ii) *Let*

$$\psi_0 = \frac{1}{50} \qquad and \qquad \chi_0 = 0,42.$$

*Then for any* $0 < \psi < \psi_0$ *and* $\chi > \chi_0$ *we have*

$$n^{\psi} \leq R_2(n) \leq n^{\chi} \quad a.s.$$

*for all but finitely many* $n$.

Here we prove

THEOREM 1. *For any* $\varepsilon > 0$ *and* $d \geq 3$

$$R_d(n) \geq (\log n)^{\frac{1}{d-2}-\varepsilon} \quad a.s.$$

*for all but finitely many* $n$.

Theorems A and 1 combined imply

THEOREM 1*. *For any* $d \geq 3$ *we have*

$$\lim_{n\to\infty} \frac{\log R_d(n)}{\log\log n} = \frac{1}{d-2} \quad a.s.$$

(ii) of Theorem A suggests the following

CONJECTURE. There exists a $\psi_0 < \alpha < \chi_0$ such that

$$\lim_{n\to\infty} \frac{\log R_2(n)}{\log n} = \alpha \quad \text{a.s.}$$

Our second problem is to investigate the time needed to meet a new point. In order to formulate this question introduce the following notation:

$$\nu(n) = \nu(n,d) = \min\{k : k > 0, S_{k+n} \neq S_j \ (j = 0,1,2,\dots,n)\}.$$

Since clearly

$$\liminf_{n\to\infty} \nu(n,d) = 1 \quad \text{a.s.} \quad (d = 1,2,\dots)$$

we are interested in the lim sup of $\nu(n)$. This question seems to be very hard and we can present only partial results. In fact we have

THEOREM 2.

(1) $$\frac{1}{4\pi^2} \leq \limsup_{n\to\infty} \frac{\nu(n,1)}{n(\log\log n)^2} \leq \frac{16}{\pi^2} \quad a.s.$$

(2) $$\limsup_{n\to\infty} \frac{\log \nu(n,2)}{\log n} \geq \frac{2}{50} \quad a.s.$$

(3) $$\limsup_{n\to\infty} \frac{\nu(n,2)}{n} = 0 \quad a.s.$$

(4) $$\limsup_{n\to\infty} \frac{\nu(n,d)}{n^\varepsilon} = 0 \quad a.s. \quad for\ any\ \varepsilon > 0,\ d = 3,4,\dots$$

(5) $$\limsup_{n\to\infty} \frac{\nu(n,d)}{(\log n)^{2/(d-2)-\varepsilon}} = \infty \quad a.s. \quad for\ any\ \varepsilon > 0\ d = 3,4,\dots$$

Our last problem is concerned with the favourite values of a random walk. We say that $x_n$ is a favourite value of $\{S_1, S_2, \dots, S_n\}$ if

$$\xi(x_n, n) = \max_{x\in Z^d} \xi(x,n) = \xi(n).$$

Let $\mathcal{F}_n$ be the set of favourite values i.e.

$$\mathcal{F}_n = \{x : x \in Z^d,\ \xi(x,n) = \max_{x\in Z^d} \xi(x,n)\}$$

and let $f_n = |\mathcal{F}_n|$ be the cardinality of $\mathcal{F}_n$. In our paper [2] we proposed to study the properties of the sequence $\{f_n\}$. It is easy to see that

$$f_n = 1 \quad \text{i.o. a.s.} \quad d = 1,2,\dots$$

$$f_n = 2 \quad \text{i.o. a.s.} \quad d = 1,2,\dots$$

Hence we proposed the question

(6) $$\mathbf{P}\{f_n = 3 \text{ i.o.}\} = ? \quad d = 1.$$

This question is still open. Now we prove

THEOREM 3. *If $d \geq 3$ then*

$$\limsup_{n\to\infty} f_n = \infty \quad a.s. \tag{7}$$

Hence beside the question (6) we also propose the questions

$$\mathbf{P}\{f_n = 3 \text{ i.o.}\} = ? \quad d = 2.$$

## 2. Proof of Theorem 1

In the proof the following lemmas will be used

LEMMA A. ([4]). *For any $d \geq 3$ there exists a positive constant $C_d$ such that*

$$\mathbf{P}\{S_n = x \textit{ for some } n = 1,2,\dots\} = \mathbf{P}(J(x) = 1) = \frac{C_d + o(1)}{R^{d-2}} \ (R \to \infty)$$

*where $R = \|x\|$ and*

$$J(x) = \begin{cases} 0 & \textit{if } \xi(x,n) = 0 \textit{ for every } n = 1,2,\dots, \\ 1 & \textit{otherwise.} \end{cases}$$

LEMMA B. ([5]). *There exists a constant $K > 0$ such that*

$$\mathbf{P}\{\xi(x,n) > 0\} \geq \frac{C_d}{2} R^{2-d} \quad R = \|x\|$$

*if $n \geq KR^2$ where $C_d$ is the constant of Lemma A.*

Let $L = L(n) = [(\log n)^{1-\epsilon}]$ and define

$$\tau_1 = \tau_1(n) < \psi_1 = \psi_1(n) < \tau_2 = \tau_2(n) < \psi_2 = \psi_2(n) < \dots < \tau_L = \tau_L(n)$$

by

$$\begin{aligned} \tau_1 &= n + [(\log n)^{\frac{2}{d-2}}], \\ \psi_1 &= \inf\{k : k > \tau_1,\ S_n = S_k\}, \\ \tau_2 &= \psi_1 + [(\log n)^{\frac{2}{d-2}}], \\ \psi_2 &= \inf\{k : k > \tau_2,\ S_n = S_k\}, \dots \end{aligned}$$

Clearly with a positive probability (depending on $n$) $\psi_1$ is not defined. However we have

LEMMA 1. *For any $\delta > 0$ there exists a constant $M = M(\delta) > 0$ such that*

$$\mathbf{P}\{\max_{1\le i\le L} \psi_i - \tau_i \le M(\log n)^{\frac{2}{d-2}}\} \ge n^{-\delta}. \tag{8}$$

PROOF. Clearly for any $N > 0$ there exists a constant $0 < p = p(N) < 1$ such that

$$\mathbf{P}\{\|S_{\tau_1} - S_n\| \le N(\log n)^{\frac{1}{d-2}}\} \ge p > 0.$$

Observe that by Lemma B we have

$$\begin{aligned}
&\mathbf{P}\{\psi_1 \le M(\log n)^{\frac{2}{d-2}}\} \\
&\ge \mathbf{P}\{\psi_1 \le M(\log n)^{\frac{2}{d-2}} \mid \|S_{\tau_1} - S_n\| \le N(\log n)^{\frac{1}{d-2}}\} \times \\
&\quad \mathbf{P}\{\|S_{\tau_1} - S_n\| \le N(\log n)^{\frac{1}{d-2}}\} \\
&\ge \frac{C_d}{2}\left(N(\log n)^{\frac{1}{d-2}}\right)^{2-d} p = \frac{C_d p}{2N^{d-2}} \frac{1}{\log n}
\end{aligned}$$

provided that $M > KN^2$ where $C_d$ and $K$ are the constants of Lemma B. Since $\psi_1, \psi_2, \ldots$ are i.i.d.r.v.'s we get

$$\mathbf{P}\{\max_{1\le i\le L} \psi_i \le K(\log n)^{\frac{2}{d-2}}\} \ge \left(\frac{C_d p}{2N^{d-2}\log n}\right)^{(\log n)^{1-\epsilon}} \ge n^{-\delta}$$

which implies (8).

Let

$$A_n = A(n) = \{\max_{1\le i\le L} \psi_i \le M(\log n)^{\frac{2}{d-2}}\}$$

and $x$ be an arbitrary element of $Z^d$ for which

$$\|x - S_n\| \le (\log n)^{\frac{1}{d-2} - 2\epsilon}. \tag{9}$$

Then applying again Lemma B we have

$$\begin{aligned}
\mathbf{P}\{\xi(x, n + [(\log n)^{\frac{2}{d-2}}]) - \xi(x,n) = 0\} &\le 1 - \frac{C_d}{2((\log n)^{1/(d-2)-2\epsilon})^{(d-2)}} \\
&= 1 - \frac{C_d}{2(\log n)^{1-2\epsilon(d-2)}}.
\end{aligned}$$

Hence the conditional probability (given $A(n)$) that $x$ is not covered is less than or equal to

$$\left(1 - \frac{C_d}{2(\log n)^{1-2\epsilon(d-2)}}\right)^{(\log n)^{1-\epsilon}} \le \exp\left(-\frac{C_d}{2}(\log n)^{2\epsilon(d-2)-\epsilon}\right).$$

Consequently the conditional probability that there exists a point for which (9) is satisfied and which is not covered is less than or equal to

$$O\left((\log n)^{d/(d-2)-2\varepsilon d}\right)\exp\left(-\frac{C_d}{2}(\log n)^{2\varepsilon(d-2)-\varepsilon}\right).$$

Let $\alpha > 1$ then by Lemma 1

$$\mathbf{P}\left\{\left(\sum_{k=n^\alpha}^{(n+1)^\alpha} A_k\right)^c\right\} \le \left(1-\frac{1}{n^\delta}\right)^{T(n)} \le \exp\left(-\frac{T(n)}{n^\delta}\right) \le \exp(-n^{\alpha-1-\delta-\varepsilon})$$

for any $\varepsilon > 0$ where $T(n) = n^{\alpha-1}(\log n)^{-(\frac{2}{d-2}+1-\varepsilon)}$.

Consequently with probability 1 for all but finitely many $k$ between $2^k$ and $2^{k+1}$ there exists an $n$ for which $A_n$ holds. Given this $n$ the conditional probability that $R_d(n) \le (\log n)^{\frac{1}{d-2}-2\varepsilon}$ is less than or equal to

$$O\left((\log n)^{d/(d-2)-2\varepsilon d}\right)\exp\left(-\frac{C_d}{2}(\log n)^{2\varepsilon(d-2)-\varepsilon}\right).$$

Hence among these $n$'s there are only finitely many for which $R_d(n) \le (\log n)^{1/(d-2)-2\varepsilon}$ i.e. between $2^k$ and $2^{k+1}$ there exists an $n$ (if $k$ is big enough) for which $R_d(n) > (\log n)^{1/(d-2)-2\varepsilon}$. This implies Theorem 1.

## 3. Proof of Theorem 2

(I) THE PROOF OF THE UPPER PART OF (1). Let the range of $\{S(0), S(1), \ldots, S(n)\}$ be

$$R(n) = \max_{0\le k\le n} S(k) - \min_{0\le k\le n} S(k).$$

It is well-known that for any $\varepsilon > 0$

$$R(n) \le ((1+\varepsilon)2n\log\log n)^{1/2} \quad \text{a.s.} \tag{10}$$

for all but finitely many $n$. It is also well-known that

$$\max_{n\le k\le N}|S_k - S_n| \ge \left((1-\varepsilon)\frac{N}{\log\log N}\right)^{1/2}\frac{\pi}{\sqrt{8}} \quad \text{a.s.} \tag{11}$$

for all but finitely many $N$. Choose $N$ such that

$$\frac{\pi}{\sqrt{8}}\left((1-\varepsilon)\frac{N}{\log\log N}\right)^{1/2} \ge ((1+\varepsilon)2n\log\log n)^{1/2}$$

i.e.

$$N \geq \frac{1+\varepsilon}{1-\varepsilon}\frac{16}{\pi^2} n(\log\log n)^2.$$

Then (10) and (11) combined imply the upper part of (1).

(II) THE PROOF OF THE LOWER PART OF (1). Consider a path with the following properties

$$\max_{0\leq k\leq n} S_k \geq (2\alpha n \log\log n)^{1/2} \quad (0<\alpha<1) \tag{12}$$

$$\xi(0,2n) - \xi(0,n) > 0 \tag{13}$$

$$\min_{2n\leq k\leq 3n} S_k \leq -(2\alpha n \log\log n)^{1/2} \tag{14}$$

$$\xi(0,4n) - \xi(0,3n) > 0 \tag{15}$$

$$\max_{4n\leq k\leq Qn(\log\log n)^2} |S_k| \leq (2\alpha n \log\log n)^{1/2} \quad (Q>0). \tag{16}$$

A simple calculation implies that the probability to get such a path is

$$O\left((\log n)^{-4\alpha-\frac{\pi^2 Q}{16\alpha}}\right).$$

Choose $\alpha = 1/8$ and $0 < Q < 1/\pi^2$. Then

$$4\alpha + \frac{\pi^2 Q}{16\alpha} < 1$$

and by the usual way one can see that there exist infinitely many $n$ for which conditions (12)-(16) are satisfied.

Let $N = 4n$. Then

$$\nu(N,1) \geq Qn(\log\log n)^2 \geq \frac{(1-\varepsilon)}{4\pi^2} N(\log\log N)^2$$

i.e. the lower part of (1) is proved.

(III) PROOF OF (2). In order to prove (2) we have to present the stronger version of (ii) of Theorem A.

THEOREM B. ([6]). *Let $\psi_0 = 1/50$. Then for any $0 < \psi < \psi_0$ there exist a sequence of random vectors $u = u(n) \in Z^2$ $(n = 1, 2, \ldots)$ and an $\varepsilon > 0$ such that*

$$\limsup_{n\to\infty} \sup_{\|x-u\|\leq n^{\psi}} \left|\frac{\xi(x,n)}{\xi(u,n)} - 1\right| \leq 1-\varepsilon \quad a.s.$$

*and*

$$\liminf_{n\to\infty} \frac{\xi(u,n)}{\log^2 n} \geq \varepsilon \quad a.s.$$

Following the proof of Theorem B one can easily prove

THEOREM C. *For any $0 < \psi < \psi_0$ there exists a sequence of random vectors $u = u(n) \in Z^2$ $(n = 1, 2, \ldots)$ and an $\varepsilon > 0$ such that*

$$\liminf_{n\to\infty} \frac{\xi(u,n)}{\log^2 n} \geq \varepsilon \quad a.s.$$

*and*

$$\xi(x, N) > 0$$

*for every $x \in Z^2$ with $\|x - u\| \leq n^{\psi}$ where*

$$N = \inf\{k : \xi(u,k) \geq \frac{\varepsilon}{2} \log^2 n\}.$$

Clearly Theorems B and C tell us that if we are waiting till the time when the point $u$ will be visited $(\varepsilon/2)\log^2 n$ times then the disc $Q(n^{\psi}, u; 2)$ will be covered and if after that we wait till $n$ then $u$ will be visited at least $\varepsilon \log^2 n$ times.

Having Theorem C it is clear that $\nu(N, 2) \geq n^{2\psi}$ with positive probability. This proves (2).

(IV) PROOF OF (3). The proof of (3) is based on the following

THEOREM D. ([1]). *Let $L(n)$ be the number of different vectors among $S(0), S(1), \ldots, S(n)$ $(d = 2)$. Then*

$$\lim_{n\to\infty} \frac{(\log n)L(n)}{\pi n} = 1 \quad a.s.$$

This result implies that for any $\varepsilon > 0$

$$L((1+\varepsilon)n) > L(n) \quad \text{a.s.}$$

for all but finitely many $n$. This, in turn, implies (3).

(V) PROOF OF (4). At first we prove

LEMMA 2. *Let*

$$\Xi(Q(R,u;d),\infty) = \sum_{x\in Q(R,u;d)} \xi(x,\infty) \quad (d \geq 3)$$

*be the occupation time of* $Q(R,u;d)$ $(R = 1,2,\ldots)$. *Then for any* $\varepsilon > 0$ *and* $T > 0$

$$\sup_{\|u\|\leq R^T} \Xi(Q(R,u;d),\infty) < R^{2+\varepsilon} \quad a.s.$$

*for all but finitely many R.*

PROOF. Define the following r.v.'s

$$\begin{aligned}
n_1 &= n_1(R,u) = \min\{k : k \geq 0,\ S(k) \in Q(R,u;d)\},\\
n_2 &= n_2(R,u) = \min\{k : k \geq n_1,\ \|S(k) - S(n_1)\| \geq 4R\},\\
n_3 &= n_3(R,u) = \min\{k : k \geq n_2,\ S(k) \in Q(2R,S(n_1);d)\},\\
n_4 &= n_4(R,u) = \min\{k : k \geq n_3,\ \|S(k) - S(n_1)\| \geq 4R\},\\
n_5 &= n_5(R,u) = \min\{k : k \geq n_4,\ S(k) \in Q(2R,S(n_1);d)\},\\
n_6 &= n_6(R,u) = \min\{k : k \geq n_5,\ \|S(k) - S(n_1)\| \geq 4R)\},\ldots
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathbf{P}\{n_2 - n_1 \geq R^{2+\varepsilon}\} &\leq \exp(-O(R^\varepsilon)),\\
\mathbf{P}\{n_3 < \infty\} &\leq p < 1,\\
\mathbf{P}\{n_4 - n_3 \geq R^{2+\varepsilon}\} &\leq \exp(-O(R^\varepsilon)),\\
\mathbf{P}\{n_5 < \infty\} &\leq p < 1,\\
\mathbf{P}\{n_6 - n_5 \geq R^{2+\varepsilon}\} &\leq \exp(-O(R^\varepsilon)),\ldots
\end{aligned}$$

Consequently

$$\mathbf{P}\{\Xi(Q(R,u;d),\infty) \geq R^{2+\varepsilon}\} \leq \exp(-O(R^{\varepsilon/2}))$$

and

$$\mathbf{P}\{\sup_{\|u\|\leq R^T} \Xi(Q(R,u;d),\infty) \geq R^{2+\varepsilon}\} \leq \exp(-O(R^{\varepsilon/4})).$$

Hence we have Lemma 2.

Consider the ball

$$Q_1 = Q(n^\varepsilon, S(n); d).$$

By Lemma 2 it consists of at most $n^{5\varepsilon/2}$ elements of the sequence $S(0), S(1), \ldots, S(n)$ (with probability 1 for all but finitely many $n$). Let

$$\tau_1(n) = \inf\{k : k > 0,\ S(k+n) \notin Q_1\}.$$

Then

$$\tau_1(n) \le n^{5\varepsilon/2} \quad \text{a.s.}$$

if $n$ is big enough. Clearly the probability that the sequence $S(n+1)$, $S(n+2), \ldots, S(n+\tau_1(n))$ does not contain any new points is less than or equal to $n^{5\varepsilon/2-\varepsilon d}$.

Now we consider the ball

$$Q_2 = Q(n^\varepsilon,\ S(n+\tau_1(n)); d)$$

and define

$$\tau_2(n) = \inf\{k : k > 0,\ S(n+\tau_1(n)+k) \notin Q_2\}.$$

Similarly as above one can see that the probability that the sequence $S(n+\tau_1(n)+1), S(n+\tau_1(n)+2), \ldots, S(n+\tau_1(n)+\tau_2(n))$ does not contain any new points is less than or equal to $n^{5\varepsilon/2-\varepsilon d}$. Repeat this procedure $n^\varepsilon$ times we obtain that the probability that we do not get any new points is less than or equal to $n^{(5\varepsilon/2-\varepsilon d)n^\varepsilon}$. Since during the procedure the number of steps made by the random walk is less than $n^{5\varepsilon/2}n^\varepsilon = n^{7\varepsilon/2}$ we obtain (4).

(VI) PROOF OF (5). Repeating the proof of (2) but using Theorem 1 instead of (ii) of Theorem A we get (5).

## 4. Proof of Theorem 3

Let $A \subset Z^d$ and $x \in Z^d$. We say that $A$ is blocking $x$ if any path going from $x$ to infinity hits $A$. For example

$$A = \{(0,1),(1,0),(0,-1),(-1,0)\} \subset Z^2$$

is blocking $x = (0,0)$.

Let $A \subset Z^d$ with $|A| = l < \infty$. Assume that $A$ is not blocking $S_n$ ($n$ is fixed). Then there exists an $\alpha_l > 0$ such that

$$\mathbf{P}\{\Xi(A,\infty) = \Xi(A,n)\} \ge \alpha_l > 0 \quad (d \ge 3) \tag{17}$$

where

$$\Xi(A,n) = \#\{k:\ 0 \le k \le n,\ S_k \in A\}.$$

By (17) it is also easy to see that

$$\mathbf{P}\{\max_{n<k} f_k > j \text{ and } \mathcal{F}_\kappa \text{ is not blocking } S_\kappa \mid \mathcal{B}\} > 0 \tag{18}$$

where $\kappa > n$ is the smallest integer for which $f_\kappa = j+1$ and

$$\mathcal{B} = \{f_n = j \text{ and } \mathcal{F}_n \text{ is not blocking } S_n\}.$$

(18) clearly implies that for any $\ell = 1, 2, \ldots$

$$\mathbf{P}\{\limsup_{n\to\infty} f_n \ge \ell\} > 0.$$

This, in turn, by the zero-one law implies (7).

## 5. Two questions

We recall the following

THEOREM E. ([3]). *Let $d = 2$. Then we have*

$$\frac{1}{4\pi} \le \liminf_{n\to\infty} \frac{\xi(n)}{(\log n)^2} \le \limsup_{n\to\infty} \frac{\xi(n)}{(\log n)^2} \le \frac{1}{\pi} \quad a.s.$$

Theorems D and E combined easily imply

THEOREM 4. *Let $A(n,d) \subset Z^d$ be the largest set on which $\xi(\cdot,n)$ is a positive constant i.e.*

$$\xi(x,n) = \xi(y,n) \ne 0 \quad \textit{if} \quad x \in A(n,d) \textit{ and } y \in A(n,d).$$

*Then for any $\varepsilon > 0$*

$$|A(n,2)| \ge (1-\varepsilon)\frac{\pi^2 n}{(\log n)^3} \quad a.s.$$

*for all but finitely many $n$.*

*Similarly one can see that for any $d \ge 3$ there exists a $C_d > 0$ such that*

$$|A(n,d)| \ge \frac{C_d n}{\log n} \quad a.s.$$

*for all but finitely many $n$.*

The analogous result for $d = 1$ is not clear. We present the following

CONJECTURE.

$$\lim_{n\to\infty} |A(n,1)| = \infty \quad \text{a.s.}$$

and

$$\mathbf{P}\{\limsup_{n\to\infty} \min_{\{x:\xi(x,n)>0\}} \xi(x,n) < \infty\} = 1 \quad (d = 1).$$

Note that it is easy to see that for $d \geq 2$

$$\min_{\{x:\xi(x,n)>0\}} \xi(x,n) = 1 \quad \text{a.s.}$$

for all but finitely many $n$.

## References

[1] DVORETZKY, A. and ERDŐS, P., Some problems on random walk in space, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (1950) University of California Press, Berkeley and Los Angeles, 1951, 353-367, *MR* **13**—852

[2] ERDŐS, P. and RÉVÉSZ, P, On the favourite points of a random walk, *Mathematical structure - Computational mathematics - Mathematical Modelling*, **2**, Bulgar. Acad. Sci., Sofia, 1984, 152—157. *MR* **86 k**: 60126

[3] ERDŐS, P. and TAYLOR, S. J., Some problems concerning the structure of random walk paths, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 137—162. *MR* **22** # 12599

[4] RÉVÉSZ, P., Simple symmetric random walk in $Z^d$, *Almost everywhere convergence*, Ed. by G. A. Edgar and L. Sucheston, Academic Press, Inc., New York, 1989, 369—392.

[5] RÉVÉSZ, P., On the volume of the spheres covered by a random walk, *A Tribute to Paul Erdős*, Ed. by A. Baker, B. Bollobás and A. Hajnal, Cambridge University Press, Cambridge, 1990, 341—347.

[6] RÉVÉSZ, P., Clusters of a random walk on the plane (to appear).

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# THE EXTENSION OF WILLIAMS' METHOD TO THE METRIC THEORY OF GENERAL OPPENHEIM EXPANSIONS

JANOS GALAMBOS

## 1. Introduction

There has much been written on the metric theory of series expansions of real numbers. One usually chooses, as we shall do here, Lebesgue measure on the Borel subsets of the interval $(0,1]$ as the underlying probability space, and asks about metric (distributional) properties of digits in an expansion by a specified algorithm. For a variety of results on this line, see the monograph Galambos (1976). In this paper we return to the Oppenheim algorithm (see Oppenheim (1972) and Galambos (1970)), and extend an "imbedding" method of Williams (1973), developed for a particular case known as Engel series, to the metric theory of digits in Oppenheim expansions. This will simplify proofs of known results as well as lead to new ones.

The Oppenheim algorithm is as follows. Let $h_n(j)$, $n \geq 1$, be a sequence of rational valued functions on integers $j \geq 2$. We assume that $h_n(j) \geq 1$ for all $n$ and $j$. Let $0 < x \leq 1$, and define $x_1 = x$, and, for $n \geq 1$,

$$x_{n+1} = \left(x_n - \frac{1}{d_n}\right) \frac{d_n(d_n - 1)}{h_n(d_n)}, \tag{1}$$

where the digits $d_n = d_n(x)$ are defined as positive integers satisfying

$$\frac{1}{d_n} < x_n \leq \frac{1}{d_n - 1}. \tag{2}$$

The algorithm (1) and (2) leads to the convergent infinite series representation

$$y(x) = \frac{1}{d_1} + \frac{1}{d_1(d_1 - 1)} \frac{1}{d_2} + \ldots + \prod_{s=1}^{n-1} \frac{h_s(d_s)}{d_s(d_s - 1)} \frac{1}{d_n} + \ldots \tag{3}$$

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and, in fact,

$$y(x) = x \text{ for all } x. \tag{4}$$

In order to see this last claim, put $y_n(x)$ for the first $n$ terms of $y(x)$ in (3). Then the algorithm (1) and (2) entails

$$x - y_n(x) = x_{n+1} \prod_{s=1}^{n} \frac{h_s(d_s)}{d_s(d_s - 1)}, \tag{5}$$

and thus, by the right hand side's being positive, we get $0 < y_n(x) < x$, yielding

$$y(x) = \lim y_n(x) \leq x \qquad (n \to \infty).$$

But then from the convergence of $y(x)$ we have that the general term in (3) converges to zero which, on account of (2), is of the same magnitude as the right hand side of (5). This proves (4).

Upon applying the upper bound of $x_n$ in (2) to the right hand side of (1), we get $x_{n+1} \leq 1/h_n(d_n)$, while by the lower bound in (2) $x_{n+1} > 1/d_{n+1}$. Consequently, for every $n$,

$$d_{n+1} > h_n(d_n). \tag{6}$$

Therefore, by the choice of a fast growing sequence $h_n(j)$, one can automatically guarantee a fast growth for $d_n(x)$ for all $x$. This is satisfied in several classical cases, out of which we mention (i) Engel series corresponding to $h_n(j) = h(j) = j - 1$ for all $n$, (ii) Sylvester series obtained by choosing $h_n(j) = h(j) = j(j-1)$ and (iii) Lüroth expansions which we obtain via $h_n(j) = h(j) = 1$ for all $n$ and $j$. Note that, in terms of (6), Lüroth series stands out as an expansion in which the value of $d_n$ does not impose a condition on $d_{n+1}$, while in the cases of Engel and Sylvester $d_n$ and $d_{n+1}$ are strongly dependent. Therefore, if one can expect a unified approach to all Oppenheim series from a probabilistic (metric) point of view, one cannot start with the direct investigation of the sequence $d_n$ but rather with some relation between $d_{n+1}$ and $h_n(d_n)$. The present author's previous tool was an integer approximation to the ratio $d_{n+1}/h_n(d_n)$, which reduced every Oppenheim series to that of Lüroth (see Chapter 6 in Galambos (1976)). However, while this approximation proved to be very powerful for proving a variety of strong laws of large numbers and central limit types of theorems, it is not sufficiently fine to be comparable with imbedding methods first proposed by Williams for the Engel series which, in a modified form, is fully exploited in a paper of Deheuvels (1982), and more recently, adopted to the Sylvester algorithm by Goldie and Smith (1987). The present paper is devoted to the extension of the method of Williams to the general case of Oppenheim expansions with the single restriction of assuming that $h_n(j)$ is integer valued for all $n$.

## 2. A new method for studying the metric theory

As alluded to earlier, we use Lebesgue measure as the underlying probability over Borel sets. In other words, we view the initial value $x$ as a uniformly distributed random variable over the interval $(0,1]$. Then, by looking at cylinder sets of the form

$$\{x : d_1 = j_1, d_2 = j_2, \ldots, d_n = j_n\},$$

which are intervals whenever (6) is not violated, the algorithm (1) and (2) immediately yields that the sequence $d_1, d_2, \ldots$ forms a Markov chain with initial distribution

$$P(d_1 = j) = \frac{1}{j(j-1)}, \qquad j \geq 2, \tag{7}$$

and with transition probabilities

$$P(d_n = k \mid d_{n-1} = j) = \begin{cases} \dfrac{h_{n-1}(j)}{k(k-1)} & \text{if } k > h_{n-1}(j) \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Now, since the distributional properties of a Markov chain are uniquely determined by its initial distribution and its one step transition probabilities, one can study the metric theory of the sequence $d_n$ through any Markov chain $D_n$ to which (7) and (8) apply when $d_n$ is replaced by $D_n$. Upon utilizing an idea of Williams (1973), we shall introduce the following Markov chain $D_n$ in place of $d_n$.

Let $X_1, X_2, \ldots$ be independent random variables on some probability space with common distribution function $F(x) = 1 - e^{-x}$, $x \geq 0$. We shall refer to this distribution as unit exponential. Set

$$D_{n+1} = [h_n(D_n) \exp(X_{n+1})] + 1, \qquad n \geq 0, \tag{9}$$

where $D_0 = 2$, $h_0(2) = 1$, and $[y]$ signifies the integer part of $y$. Clearly, the sequence $D_n$ is a Markov chain, and since, for $k > h_n(j)$,

$$P([h_n(j) \exp(X_{n+1})] + 1 = k) = P(k - 1 \leq h_n(j) \exp(X_{n+1}) < k)$$

$$= F\left(\log \frac{k}{h_n(j)}\right) - F\left(\log \frac{k-1}{h_n(j)}\right)$$
$$= \frac{h_n(j)}{k(k-1)},$$

while this same probability equals 0 if $h_n(j)/k \le 1$, we have that both (7) and (8) apply to $D_n$, $n \ge 1$. Hence, our method for proving metric results on $d_n$ is that we prove a similar statement for the Markov chain $D_n$, $n \ge 1$. The term $D_0$ plays no rule other then defining the first term $D_1$.

The following two properties of the sequence $D_n$ will repeatedly be utilized. First, note that if $h_n(D_n) \to +\infty$ with $n$ then (9) entails

$$\left|\frac{D_{n+1}}{h_n(D_n)} - e^{X_{n+1}}\right| \le \frac{1}{h_n(D_n)}. \tag{10}$$

Furthermore, the elementary inequality $\log(1+z) \le z$, $z \ge 0$, yields from (10),

$$R_n = \log \frac{D_{n+1}}{h_n(D_n)} = X_{n+1} + \frac{c_n \exp(-X_{n+1})}{h_n(D_n)}, \quad 0 \le c_n \le 1. \tag{11}$$

Another consequence of our new method for proof is that it gives a new light to the earlier approximation method of the present author, which has already been mentioned in connection with (6). Define the positive integers $T_n$ by

$$T_n < \frac{D_{n+1}}{h_n(D_n)} \le T_n + 1 \tag{12}$$

and let

$$U_n = [\exp(X_{n+1})]. \tag{13}$$

The present authors's previous tool was to utilize that the $T_n$ are independent and identically distributed random variables whose common distribution is the one given at (7). Since (8) reduces to (7) for Lüroth expansions, that is, the Lüroth digits are independent, one actually has that the approximation by $T_n$ to the ratios in (12) is a comparison of these ratios to the Lüroth digits. We can now understand this so far unexplainable phenomenon: it turns out that $T_n = U_n$ for every $n \ge 1$. Indeed, if we write

$$\exp(X_{n+1}) = U_n + r_n, \quad 0 \le r_n < 1,$$

and if $0 \le g_n < h_n(D_n)$ is an integer such that

$$g_n \le r_n h_n(D_n) < (g_n + 1),$$

that is, $g_n = [r_n h_n(D_n)]$, then

$$\frac{D_{n+1}}{h_n(D_n)} = \frac{[h_n(D_n)\exp(X_{n+1})] + 1}{h_n(D_n)} = U_n + \frac{g_n + 1}{h_n(D_n)}.$$

Since the last fraction on the extreme right hand side is positive and does not exceed one, we got that $U_n$ does satisfy the inequalities of (12), that is, $U_n = T_n$.

## 3. Applications of the new method for proof

An essential part of estimating speed of convergence is the establishment of a strong law of large numbers for $R_n$ of (11). From our new method we immediately have that if $h_n(D_n) \to +\infty$ with $n$ almost surely then the arithmetical mean of $R_n$ of (11) converges almost surely to one. Simply apply (11) and the classical strong law of large numbers to $X_{n+1}$. This short argument now replaces the proof of Theorem 6.17 in Galambos (1976), which theorem in turn plays a fundamental role in several other estimates, including those for speed of convergence.

In an equally simple manner can one get several new results concerning the sequence $R_n$ of (11). Before we state such results we assume that $h_n(j) \geq j - 1$, and conclude that in such a case $h_n(D_n)$, and thus $D_n$ itself (recall (6)), diverges to infinity with $n$, for almost all $x$. We just have to observe that for all $n$, with the exception of perhaps of a finite number of its values,

$$D_{n+1} \geq h_n(D_n) + 2 \geq D_n + 1. \tag{14}$$

Indeed, by (6), the violation of (14) means that $D_{n+1} = h_n(D_n) + 1$, which, on account of (9), is equivalent to

$$\exp(X_{n+1}) < 1 + \frac{1}{h_n(D_n)}.$$

By one more appeal to (9) we see that $X_{n+1}$ and $D_n$ are independent, and thus by the exponentiality of $X_{n+1}$ and by the total probability rule we get

$$P((14)\text{ fails}) = \sum_{k=2}^{+\infty} \frac{P(D_n = k)}{h_n(k) + 1}$$

However, by an induction argument, using the Markov property only, it is shown on pp. 101 - 102 in Galambos (1976) that the right hand side above is bounded by $(5/6)^n$. Hence, its sum over $n$ is finite, entailing, via the Borel-Cantelli lemma, that, for almost all $x$, (14) fails at most for a finite number of $n$. Consequently, when we choose such $h_n(j)$ which satisfies $h_n(j) \geq j-1$ for all $n$, we can ignore the error terms in (10) and (11) in most asymptotic results. The following theorem summarizes a selective list of results utilizing the just mentioned possibility.

THEOREM 1. *Let $h_n(j) \geq j-1$. Then, putting $S_n = d_{n+1}/h_n(d_n)$ and $R_n = \log S_n$, we have almost surely*

(i) $(1/n)(S_1 + S_2 + \ldots + S_n) \to +\infty$;

(ii) *if $b_n \to +\infty$ and such that $(S_1 + S_2 + \ldots + S_n)/b_n$ does not converge to zero then its* limsup *equals $+\infty$;*

(iii) $(1/n)(R_1 + R_2 + \ldots + R_n) \to 1$;

(iv) $R_n \geq \log n + \log\log n$ *infinitely often;* $R_n < \log n + (1+a)\log\log n$ *for all but a finite number of $n$, whatever $a > 0$.*

One could go further and include the iterated logarithm theorem in order to extend (iii), or even finer results are obtainable by approximating the sum of $R_k$ by a Wiener process. But the power of the new method for proving results on the sequence $d_n$ is adequately demonstrated by the collection in Theorem 1. Details of proof are not needed; one has to appeal to (14) and then to (10) and (11) without their error terms. Well known results on independent and identically distributed random variables then entail Theorem 1.

Weak convergence results also follow from (10) and (11).

THEOREM 2. *Assuming that $h_n(j) \geq j-1$ for all $n$, we have*

(i) $(S_1 + S_2 + \ldots + S_n)/n\log n \to 1$ *in probability;*

(ii) $(R_1 + R_2 + \ldots + R_n - n)/n^{1/2}$ *is asymptotically standard normal;*

(iii) $\max(R_1, R_2, \ldots, R_n) - \log n$ *has the asymptotic law* $\exp(-\exp(-x))$.

While there is a proof for (ii) in Galambos (1976), it is quite lengthy; by our new method it is immediate. Part (iii) is new, and indeed, it was attempted by previous methods but the errors were always too disturbing. By our new method it follows from (14) and (11) upon observing that the maximum of the set $R_1, R_2, \ldots, R_n$ has the same asymptotic behaviour as the maximum of $R_m, R_{m+1}, \ldots, R_n$, where $m$ is an arbitrary large, but fixed number.

In concluding we wish to point out that both Williams's approximation method and that of Deheuvels heavily rely on Rényi's (1962) work. These authors restrict their investigation to the Engel series. In addition, much of the early research on series expansions was influenced by the paper of Erdős, Rényi and Szüsz (1958).

## REFERENCES


[1] DEHEUVELS, P., L'encadrement asymptotique des éléments de la série d'Engel d'un nombre réel, *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982), 21—24. *MR* **83h**: 10095

[2] ERDŐS, P., RÉNYI, A. and SZÜSZ, P., On Engel's and Sylvester's series, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **1** (1958), 7—32. *MR* **21** # 1288

[3] GALAMBOS, J., The ergodic properties of the denominators in the Oppenheim expansion of real numbers into infinite series of rationals, *Quart. J. Math. Oxford Ser.* (2) **21** (1970), 177—191. *MR* **41** # 3423

[4] GALAMBOS, J., *Representations of real numbers by infinite series*, Lecture Notes in Math., Vol. 502, Springer - Verlag, Berlin - New York, 1976. *MR* **58** # 27873

[5] GOLDIE, C. M. and SMITH, R. L., On the denominators in Sylvester's series, *Proc. London Math. Soc.* (3) **54** (1987), 445—476. *MR* **88f**: 11073

[6] OPPENHEIM, A., The representation of real numbers by infinite series of rationals, *Acta Arith.* **21** (1972), 391—398. *MR* **46** # 8982

[7] RÉNYI, A., A new approach to the theory of Engel's series, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **5** (1962), 25—32. *MR* **27** # 126

[8] WILLIAMS, D., On Rényi's "record" problem and Engel's series, *Bull. London Math. Soc.* **5** (1973), 235—237. *MR* **48** # 12636


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# ON THE INCREMENTS OF THE WIENER PROCESS

KARL GRILL

**Abstract**

Let $W(t)$ be a standard Wiener process and $a(t)$ a positive nondecreasing function with $a(t) \leq t$. We investigate the a.s. asymptotic behaviour of some increment processes defined in terms of $W(\cdot)$ and $a(\cdot)$ (the "large" increments in the terminology of Ortega and Wschebor [20]), obtaining an integral test for the upper classes and also some results on the lower classes.

## 1. Introduction

Let $(W(t), t \geq 0)$ be a standard Wiener process. Furthermore, let $a(t)$ be a positive nondecreasing function that is regularly varying as $t$ tends to infinity. Define the following types of increments:

$$Y_1(t, a(t)) = a(t)^{-1/2} \sup_{0 \leq u \leq t - a(t)} (W(u + a(t)) - W(u)),$$
$$Y_2(t, a(t)) = a(t)^{-1/2} \sup_{0 \leq u \leq t - a(t)} |W(u + a(t)) - W(u)|,$$
$$Y_3(t, a(t)) = a(t)^{-1/2} \sup_{0 \leq s \leq a(t)} \sup_{0 \leq u \leq t - s} (W(u + s) - W(u)),$$
$$Y_4(t, a(t)) = a(t)^{-1/2} \sup_{0 \leq s \leq a(t)} \sup_{0 \leq u \leq t - s} |W(u + s) - W(u)|.$$

We are concerned with the strong limiting behaviour of these quantities as $t \to \infty$. We shall make use of the following definitions (see, e.g. Révész [21, 22]):

Let $Z(t)$ be a stochastic process. Then we formulate:

DEFINITION 1. *The function $f(t)$ belongs to the upper-upper class of $Z(t)$ ($f \in \mathcal{UUC}(Z(t))$) if for almost every $\omega \in \Omega$ there is a $t_0 = t_0(\omega)$ such that $Z(t) \leq f(t)$ a.s. for all $t > t_0$.*

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DEFINITION 2. *The function $f(t)$ belongs to the upper-lower class of $Z(t)$ ($f \in \mathcal{ULC}(Z(t))$) if for almost every $\omega \in \Omega$ there is an increasing sequence $t_n = t_n(\omega)$ tending to infinity such that $Z(t_n) > f(t_n)$ for all $n$.*

DEFINITION 3. *The function $f(t)$ belongs to the lower-upper class of $Z(t)$ ($f \in \mathcal{LUC}(Z(t))$) if for almost every $\omega \in \Omega$ there is an increasing sequence $t_n = t_n'(\omega)$ tending to infinity such that $Z(t_n) < f(t_n)$ for all $n$.*

DEFINITION 4. *The function $f(t)$ belongs to the lower-lower class of $Z(t)$ ($f \in \mathcal{LLC}(Z(t))$) if for almost every $\omega \in \Omega$ there is a $t_0 = t_0(\omega)$ such that $Z(t) \geq f(t)$ a.s. for all $t > t_0$.*

The limiting classes of $Y_i(t, a(t))$ have been investigated extensively since the pioneering work of Lai [17, 18], Taylor [26], Csörgő and Révész [4] and Csáki and Révész [3]. A study of the weak limit behaviour of these statistics is given in Deheuvels and Révész [8], while related increments have been investigated by Hanson and Russo [12, 13]. Among the most sophisticated results on the subject are those of Révész [22] and Ortega and Wschebor [20], the latter of whom having proved the following Theorems A and B, close to a complete characterization of $\mathcal{UUC}(Y_i(t, a(t)))$, $i = 1, 2, 3, 4$.

THEOREM A. *Let $\phi$ be a positive, continuous, nondecreasing function such that $\phi(t) \to \infty$ as $t \to \infty$. Then for $i = 1, 2, 3, 4$*

$$(1)\quad I_3(\phi) = \int_1^\infty \phi^3(t) a^{-1}(t) \exp(-\phi^2(t)/2)\, dt < \infty \Rightarrow \phi(t) \in \mathcal{UUC}(Y_i).$$

*Suppose in addition that*

$$a(t) = C_0 \exp\left(\int_{C_1}^t \frac{\eta(y)}{y} dy\right),$$

*where $\eta(\cdot)$ is a continuous function such that $\phi^2(t)\eta(t)$ is ultimately bounded as $t \to \infty$. Then for $i = 1, 2, 3, 4$*

$$(2)\quad I_1(\phi) = \int_1^\infty \phi(t) a^{-1}(t) \exp(-\phi^2(t)/2)\, dt < \infty \Rightarrow \phi(t) \in \mathcal{UUC}(Y_i).$$

*On the other hand, if $\eta(t) \downarrow 0$ and $\phi^2(t)\eta(t) \uparrow$ then for $i = 1, 2, 3, 4$*

$$(3) J(\phi) = \int_1^\infty \eta^2(t) \phi^3(t) a^{-1}(t) \exp(-\phi^2(t)/2)\, dt < \infty \Rightarrow \phi(t) \in \mathcal{UUC}(Y_i).$$

THEOREM B. *Let $\phi$ be a positive, continuous, nondecreasing function such that $\phi(t) \to \infty$ as $t \to \infty$. Then for $i = 1,2,3,4$*

$$I_1(\phi) = \infty \Rightarrow \phi \in \mathcal{ULC}(Y_i(t, a(t))). \tag{4}$$

Motivated by (2), (4), and the similarity of $I_1(\phi)$ with the Erdős-Kolmogorov-Petrowski upper class test for the Wiener process (see, e.g., Ortega and Wschebor [20], Theorem A), Ortega and Wschebor [20, p. 332] conjectured that the finiteness of $I_1(\phi)$ might be a necessary (as well as a sufficient) condition for $\phi \in \mathcal{UUC}(Y_i(t, a(t)))$, $i = 1,2,3,4$. It is the purpose of Section 2 of the present paper to give an integral characterization of $\mathcal{UUC}(Y_i(t, a(t)))$ which will show that this conjecture is not true in general, and the following statement holds instead:

THEOREM 1. *Let $\phi$ be nondecreasing and positive. Let also*

$$a(t) = C_0 \exp\left(\int_{C_1}^{t} \frac{\eta(y)}{y} dy\right)$$

*be as in Theorem A, where $\eta(y)$ is slowly varying as $y \to \infty$ and $a(t) < \gamma t$ with some $\gamma < 1$. Then for $i = 1,2,3,4$*

$$K(\phi) = \int_1^\infty (1 + \eta(t)\phi^2(t)) \frac{\phi(t)}{a(t)} \exp\left(-\frac{\phi^2(t)}{2}\right) dt < \infty \Leftrightarrow \phi \in \mathcal{UUC}(Y_i(t, a(t))). \tag{5}$$

REMARK 1.

1. Theorem 1 is in agreement with Theorems A and B, and shows that the range of the above-mentioned conjecture of Ortega and Wschebor corresponds to the case where $\phi^2(t)\eta(t)$ is ultimately bounded.
2. It is easy to construct a positive slowly varying function $\eta$ such that $\eta(y) \to 0$ as $y \to \infty$, $\int_1^\infty \eta(y) y^{-1} dy = \infty$, and $\eta(y) \log y \to a$ as $y \to \infty$, where $0 \le a \le \infty$ is arbitrary. For $a = 0$, we see that $J(\phi) < \infty \not\Rightarrow \phi \in \mathcal{UUC}(Y_i(t, a(t)))$ while if $a = \infty$,we have likewise $I_1(\phi) < \infty \not\Rightarrow \phi \in \mathcal{UUC}(Y_i(t, a(t)))$.
3. The form of the function $a$ as demanded in the statement of Theorem 1 may seem very restrictive. However, most functions that one quickly thinks of happen to fit this description. A few examples follow, giving also "typical" upper and lower class functions (We use the usual notation $\log_n$ for the $n$-th iterate of log).

(a) $a(t) = (\log t)^\alpha$, $(\alpha > 0)$. Here $\eta(t) = \alpha/\log t$. The function

$$\phi^{(a)}_{n,\varepsilon}(t) =$$

$$= (2\log t + (3-2\alpha)\log\log t + 2\log_3 t + 2\log_4 t + \ldots + (2+\varepsilon)\log_n t)^{1/2}$$

belongs to the upper-upper class of $Y_i(t, a(t))$ iff $\varepsilon > 0$. This can also be obtained from Theorem A.

(b) $a(t) = \exp((\log t)^\alpha)$, $(0 < \alpha < 1)$. Here $\eta(t) = \alpha(\log t)^{\alpha-1}$. The function

$$\phi^{(b)}_{n,\varepsilon}(t) =$$

$$= (2\log t - 2(logt)^\alpha + (3+2\alpha)\log\log t + 2\log_3 t + \ldots + (2+\varepsilon)\log_n t)^{1/2}$$

belongs to the upper-upper class of $Y_i(t, a(t))$ iff $\varepsilon > 0$.

(c) $a(t) = t^\alpha$, $(0 < \alpha < 1)$. Here $\eta(t) = \alpha$. The function

$$\phi^{(c)}_{n,\varepsilon}(t) =$$

$$= (2(1-\alpha)\log t + 5\log\log t + 2\log_3 t + 2\log_4 t + \ldots + (2+\varepsilon)\log_n t)^{1/2}$$

belongs to the upper-upper class of $Y_i(t, a(t))$ iff $\varepsilon > 0$.

(d) $a(t) = \alpha t$, $(0 < \alpha < 1)$. Here $\eta(t) = 1$ The function

$$\phi^{(d)}_{n,\varepsilon}(t) =$$

$$= (2\log\log t + 5\log_3 t + 2\log_4 t + \ldots + (2+\varepsilon)\log_n t)^{1/2}$$

belongs to the upper-upper class of $Y_i(t, a(t))$ iff $\varepsilon > 0$.

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We now turn our attention to the lower class for which the following theorem has been stated by Révész [22, Theorem 2.1].

THEOREM C. *Let* $a(t)$ *be such that* $(\log(t/a(t))/\log\log t) \to \infty$ *as* $t \to \infty$. *Set* $\Delta(t) = t/(a(t)\log\log t)$. *Then for* $i = 1, 2, 3, 4$

*1. For* $C < \log(\pi/51^2)$

$$(2\log\Delta(t) + \log\log\Delta(t) - C)^{1/2} \in \mathcal{LUC}(Y_i(t, a(t))),$$

*and*

*2. For* $C > \log(\pi)$

$$(2\log\Delta(t) + \log\log\Delta(t) - C)^{1/2} \in \mathcal{LLC}(Y_i(t, a(t))).$$

Révész' proof of the lower class result, however, contains a slight error. Namely, for the sequence $t_k$ used in his proof of formula (2.6) in that paper, the use of his formula (2.3) yields an error term that is too large to give the desired result. This can be mended by using the same argument with the sequence $t_k = \exp(k^{1/3-\delta})$, resulting, however, in a larger constant ($\log(9\pi)$ instead of $\log\pi$) in the final result.

In Section 3 we precise this result, namely by showing that there exist constants $C_i$, $i = 1,2,3,4$ such that, under suitable assumptions on $a(t)$,

$$C < C_i \Rightarrow (2\log\Delta(t) + \log\log\Delta(t) - C)^{1/2} \in \mathcal{LUC}(Y_i(t, a(t))),\ i = 1,2,3,4,$$

and

$$C > C_i \Rightarrow (2\log\Delta(t) + \log\log\Delta(t) - C)^{1/2} \in \mathcal{LLC}(Y_i(t, a(t))),\ i = 1,2,3,4.$$

For a restricted range of $a(t)$, we find the explicit values of $C_1 = \log\pi$ and $C_2 = \log(\pi/4)$, and bounds for $C_3$ and $C_4$. In view of the limiting weak laws of Deheuvels and Révész [8], we may only conjecture at present that $C_3$ and $C_4$ could be also equal to $\log(\pi/4)$ which is in agreement with our bounds stated as follows

$$\log\frac{\pi}{4} \le C_3 \le \log 4\pi \quad \text{and} \quad \log\frac{\pi}{16} \le C_4 \le \log\pi.$$

Our methods may be applied to cover increments of partial sum processes which can be treated by invariance principles for large increments, and by direct evaluation for increments of the order of some power of $\log n$, where $n$ is the length of the observed series. This problem has been studied recently by Deheuvels, Devroye, and Lynch [6], Deheuvels and Devroye [5], Deheuvels and Steinebach [9, 10] and Mason [19] among others. This will appear elsewhere.

In the sequel $C$, with or without index, will denote an absolute constant whose actual value may differ from one occurrence to the next, so that notations like $C = C + 1$ are possible. As usual, $f(t) \sim g(t)$ will denote asymptotic equality and $f(t) \asymp g(t)$ will mean that the ratio $f(t)/g(t)$ is ultimately bounded away from both zero and infinity as $t \to \infty$.

## 2. The upper classes

The proof of Theorem 1 rests on the following

LEMMA 1. *Let* $0 < a < b < \theta t$ *and* $x > 1$ *be given and assume that* $b < Ka$, *where* $0 < \theta < 1$ *and* $K$ *are some constants. Then there are positive constants* $K_1, K_2$ *(depending on* $K$ *and* $\theta$*) such that*

$$\mathbf{P}(\sup_{a\le s\le b}\sup_{0\le v\le s}\sup_{0\le u\le t-v}|\frac{W(u+v)-W(u)}{\sqrt{s}}|\ge x)\le K_1p,$$

$$K_2(p-p^2)\le \mathbf{P}(\sup_{a\le s\le b}\sup_{0\le u\le t-s}\frac{W(u+s)-W(u)}{\sqrt{s}}\ge x),$$

*where*

$$p=p(a,b,x,t)=t\left(1+\frac{b-a}{a}x^2\right)\frac{x}{a}\exp(-x^2/2).$$

PROOF. First observe that, by a standard reflection argument,

$$\mathbf{P}(\sup_{a\le s\le b}\sup_{0\le v\le s}\sup_{0\le u\le t-v}|\frac{W(u+v)-W(u)}{\sqrt{s}}|\ge x)\le$$

$$\le 2\mathbf{P}(\sup_{a\le s\le b}\sup_{0\le u\le t-s}\frac{|W(u+s)-W(u)|}{\sqrt{s}}\ge x),$$

so it is sufficient to prove the upper inequality for the double sup.

We shall only prove the lemma for the case $(b-a)x^2 > a$ since otherwise it would be a trivial consequence of Lemma 1 in Ortega and Wschebor [20]. In this case, define

$$A_{jk}=\{\sqrt{a(k-j)}\le W(\frac{ka}{x^2})-W(\frac{ja}{x^2})\le \sqrt{a(k-j)}(1+\frac{\varepsilon}{x^2})\}$$

and

$$B_{jk}=\{\sup_{jax^{-2}\le u\le v\le kax^{-2}}|W(v)-W(u)|\ge\sqrt{a(k-j-2)}\}.$$

Furthermore, let

$$S=\{(j,k):0\le j<k\le\frac{x^2t}{a},x^2\le k-j\le\frac{x^2b}{a}\}$$

and

$$S'=\{(j,k):0\le j<k\le\frac{x^2t}{a}+1,x^2\le k-j\le\frac{x^2b}{a}+2\}.$$

It is readily obtained that, for $(j,k) \in S$ resp. $S'$,

$$C_1 x^{-1} \exp(-x^2/2) \le \mathbf{P}(A_{jk}) \le \mathbf{P}(B_{jk}) \le C_2 x^{-1} \exp(-x^2/2),$$

(the latter inequality follows from (3.1) of Ortega and Wschebor [20]), and since

$$\bigcup_{(j,k)\in S} A_{jk} \subset \{ \sup_{a\le s\le b} \sup_{0\le u\le t-s} \frac{W(u+s)-W(u)}{\sqrt{s}} \ge x\} \subset \bigcup_{(j,k)\in S'} B_{jk},$$

we get

$$\mathbf{P}( \sup_{a\le s\le b} \sup_{0\le u\le t-s} \frac{W(u+s)-W(u)}{\sqrt{s}} \ge x) \le \sum_{(j,k)\in S'} \mathbf{P}(B_{jk})$$

$$\le C_2 \frac{tx^3(b-a)}{a^2} \exp(-x^2/2)$$

which proves the first part of our lemma. In order to prove the second part, we use the inclusion-exclusion formula to obtain

$$\mathbf{P}( \sup_{a\le s\le b} \sup_{0\le u\le t-s} \frac{W(u+s)-W(u)}{\sqrt{s}} \ge x)$$

$$\ge \sum_{(j,k)\in S} \mathbf{P}(A_{jk}) - \frac{1}{2} \sum_{\substack{(j,k)\in S \\ (l,m)\in S \\ (j,k)\ne(l,m)}} \mathbf{P}(A_{jk}A_{lm}).$$

We now estimate the second sum on the right-hand side. To this end, observe that the correlation coefficient between $W(ka/x^2) - W(ja/x^2)$ and $W(ma/x^2) - W(la/x^2)$ is just

$$\rho = \rho(j,k,l,m) = \frac{\min(k,m) - \max(j,l)}{\sqrt{(k-j)(m-l)}}$$

if the intervals $[j,k]$ and $[l,m]$ intersect, and 0 otherwise.

In the first case, if $(j,k)$ and $(l,m)$ are both in $S$, this can be upper estimated by

$$1 - K\frac{|j-l| + |k-m|}{x^2}, \tag{6}$$

where $K$ is a suitable constant.

Namely, let us assume w.l.o.g. that $j < l$. Then, as the intervals $[j,k]$ and $[l,m]$ intersect, we must have $l < k$. If now $m < k$ then $m - l = k - j - |j-l| - |k-m|$ and

$$\rho = \left(\frac{m-l}{k-j}\right)^{1/2} = \left(1 - \frac{|j-l| + |k-m|}{k-j}\right)^{1/2} \leq 1 - \frac{|j-l| + |k-m|}{2(k-j)}.$$

On the other hand, if $m > k$,

$$\rho = \frac{k-l}{\sqrt{(k-j)(m-l)}} = \frac{m-j-|k-m|-|j-l|}{\sqrt{(m-j-|k-m|)(m-j-|j-l|)}}$$
$$\leq 1 - \frac{|k-m| + |j-l|}{2(m-j)}.$$

In both of the above inequalities the denominator of the last fraction is bounded above by a fixed multiple of $x^2$ which proves assertion (6).

Now, as $W(kax^{-2}) - W(jax^{-2})$ and $W(max^{-2}) - W(lax^{-2})$ are jointly normally distributed, we have

$$\mathbf{P}(A_{jk}A_{lm}) = \int_x^{x+\varepsilon/x}\int_x^{x+\varepsilon/x} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right) ds\,dt.$$

substituting $s = x + u$, $t = x + v$, the fraction in the argument of the exponential equals

$$\frac{(2x^2 + 2x(u+v))(1-\rho) + u^2 - 2\rho uv + v^2}{2(1-\rho^2)} =$$

$$= \left(\frac{x^2}{1+\rho} + \frac{x(u+v)}{1+\rho} + \frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right).$$

As both $u$ and $v$ are bounded above by $\epsilon/x$, and by our estimate (6), we see that the second and third terms of the last sum are bounded by constants, so, using again (6) to estimate $\sqrt{1-\rho^2}$, we obtain the estimate

$$\mathbf{P}(A_{jk}A_{lm}) \leq \frac{C\varepsilon^2}{x} \exp\left(-\frac{x^2}{1+\rho}\right).$$

Observing that $\mathbf{P}(A_{jk}) \asymp \varepsilon x^{-1} \exp(-x^2/2)$ and

$$\frac{x^2}{1+\rho} \geq \frac{x^2}{2} + \frac{(1-\rho)x^2}{4},$$

we finally obtain by another application of (6),

$$\mathbf{P}(A_{jk}A_{lm}) \leq C\varepsilon \exp(-c(|j-l|+|k-m|))\mathbf{P}(A_{jk}),$$

where $c$ and $C$ are suitable positive constants.

If the intervals $[j,k]$ and $[l,m]$ do not intersect, clearly

$$\mathbf{P}(A_{jk}A_{lm}) = \mathbf{P}(A_{jk})\mathbf{P}(A_{lm}).$$

If we now fix $(j,k) \in S$, we get

$$\begin{aligned}
&\sum_{\substack{(l,m)\in S\\ (j,k)\neq(l,m)}} \mathbf{P}(A_{jk}A_{lm}) \\
&\leq \mathbf{P}(A_{jk}) \sum_{\substack{(l,m)\in S\\ (j,k)\neq(l,m)}} (C\varepsilon \exp(-c(|j-l|+|k-m|)) + \mathbf{P}(A_{lm})) \\
&\leq \mathbf{P}(A_{jk})(C\varepsilon + \sum_{(l,m)\in S} \mathbf{P}(A_{lm})).
\end{aligned}$$

Applying this result to (2), we obtain,

$$\begin{aligned}
&\mathbf{P}(\sup_{a\leq s\leq b}\ \sup_{0\leq u\leq t-s} \frac{W(u+s)-W(u)}{\sqrt{s}} \geq x) \\
&\geq (1-C\varepsilon) \sum_{(j,k)\in S} \mathbf{P}(A_{jk}) - \frac{1}{2}(\sum_{(j,k)\in S} \mathbf{P}(A_{jk}))^2.
\end{aligned}$$

By choosing $\varepsilon$ small enough, we finally obtain the second part of our lemma.

Let us now proceed to the proof of Theorem 1. Define $t_r = M^r$ with some $M > (1-\gamma)^{-1}$ and let

$$\begin{aligned}
D_r &= \{ \sup_{a(t_r)\leq s\leq a(t_{r+1})}\ \sup_{0\leq v\leq s}\ \sup_{0\leq u\leq t_{r+1}-v} \frac{W(u+v)-W(u)}{\sqrt{s}} \geq \phi(t_r)\}, \\
E_r &= \{ \sup_{a(t_r)\leq s\leq a((1+\varepsilon)t_r)}\ \sup_{t_{r-1}\leq u\leq t_r - s} \frac{W(u+s)-W(u)}{\sqrt{s}} \geq \phi(t_{r+1})\},
\end{aligned}$$

with $\varepsilon < 1-\gamma-M^{-1}$.

By Lemma 1 and the trivial relations $t_r \asymp t_{r+1}$, $a(t_r) \asymp a(t_{r+1})$, we get the estimates

$$\mathbf{P}(D_r) \asymp (1+\eta(t_r)\phi^2(t_r))\frac{t_r}{a(t_r)}\phi(t_r)\exp(-\phi^2(t_r)/2)$$

and

$$\mathbf{P}(E_r) \asymp (1+\eta(t_{r+1})\phi^2(t_{r+1}))\frac{t_{r+1}}{a(t_{r+1})}\phi(t_{r+1})\exp(-\phi^2(t_{r+1})/2).$$

Thus, it is seen that both $\sum \mathbf{P}(D_r)$ and $\sum \mathbf{P}(E_r)$ converge or diverge as $K(\phi)$ converges or diverges, respectively.

Furthermore, the events $E_r$ are independent.

Now, if $K(\phi)$ is finite, the Borel-Cantelli lemma implies that only finitely many of the events $D_r$ occur with probability one. We are going to show that this implies that $\phi(t) \in \mathcal{UUC}(Y_4(t, a(t))$. To this end, let us look at a $t$ for which $Y_4(t, a(t)) > \phi(t)$. There is an $r$ such that $t_r \leq t < t_{r+1}$. As $\phi$ is nondecreasing, we have $Y_4(t, a(t)) > \phi(t_r)$, and, since also $a(t_r) \leq a(t) \leq a(t_{r+1})$, we see that the event $D_r$ occurs. Now, as only finitely many of the events $D_r$ occur with probability one, one half of our theorem is proved.

If $K(\phi)$ is infinite, infinitely many of the events $E_r$ occur with probability one by the second Borel-Cantelli lemma.

Let $r$ be such that $E_r$ occurs. This means that we can find an $s$ between $a(t_r)$ and $a((1+\varepsilon)t_r)$ and a $u \leq t_r - s$ such that $W(u+s) - W(u) > s^{1/2}\phi(t_{r+1})$. By the continuity of $a(t)$, we can find a $t$ between $t_r$ and $t_{r+1}$ such that $s = a(t)$. For this $t$ we have $Y_1(t) > \phi(t_{r+1}) \geq \phi(t)$. Thus, Theorem 1 is proved.

A case of special interest that is not covered by Theorem 1 is $a(t)/t \to 1$. For that case we state the following theorem which can be proved by the methods used in proving Theorem 1.

THEOREM 2. *Let $a(t) = t(1-b(t))$, where $b(t)$ is decreasing to 0 and slowly varying as $t \to \infty$, and $\phi(\cdot)$ be positive, nondecreasing and continuous. Then $\phi(t) \in \mathcal{UUC}(Y_i(t, a(t))$, $i = 1,2,3,4$ iff*

$$\int_1^\infty (1+b(t)\phi^2(t))\frac{\phi(t)}{t}\exp(-\phi^2(t)/2)dt < \infty.$$

## 3. The lower classes

Regarding the lower classes, the picture currently is far less complete than for the upper classes. For an account on the subject, see, e.g., Deheuvels, Erdős, Grill, and Révész [7].

Our investigation branches into two cases. First, we shall consider the case where $\Delta(t) = a(t)\log\log t/t$ tends to zero, then the other extreme, where $\Delta(t)$ tends to infinity. The intermediate case where $\Delta(t)$ remains bounded poses too many problems to be handled by our crude methods.

Before going on to the theorems, let us prove some lemmas which will be useful in both cases. To this end, first recall the following

LEMMA 2. *Let* $M^+(t) = \max\{W(u) : 0 \le u \le t\}$ *and* $M^-(t) = \min\{W(u) : 0 \le u \le t\}$. *Then it holds for* $a, b \ge 0$ *and* $-b \le c \le d \le a$:

$$\begin{aligned}&\mathbf{P}(M^+(t) \le a\sqrt{t}, M^-(t) \ge -b\sqrt{t}, c\sqrt{t} \le W(t) \le d\sqrt{t})\\ &= \sum_{k\in\mathbf{Z}} (\Phi(d + 2k(a+b)) - \Phi(c + 2k(a+b)))\\ &\quad - \sum_{k\in\mathbf{Z}} (\Phi(2a - c + 2k(a+b)) - \Phi(2a - d + 2k(a+b))).\end{aligned}$$

For a proof, see, e.g., Billingsley [1], p.79.

LEMMA 3. *For* $0 < t \le 1$ *it holds true:*

$$\begin{aligned}&\mathbf{P}(Y_1(1+t,1) \le x | W(1) = y, W(t+1) - W(t) = z)\\ &\qquad = \begin{cases} 0 & if x \le y \vee z\\ 1 - \exp(-\frac{(x-y)(x-z)}{t}) & if x > y \vee z, \end{cases}\\ &\mathbf{P}(Y_2(1+t,1) \le x | W(1) = y, W(t+1) - W(t) = z)\\ &\qquad = \begin{cases} 0 & if |x| \le |y| \vee |z|\\ G_t(x|y,z) & if |x| > |y| \vee |z|, \end{cases}\end{aligned}$$

*where*

$$\begin{aligned}G_t(x|y,z) &= \sum_{k\in\mathbf{Z}} \exp(-\frac{4k^2x^2 + 2kx(z-y)}{t})\\ &\quad - \sum_{k\in\mathbf{Z}} \exp(-\frac{yz - (2k+1)x(y+z) + (2k+1)^2x^2}{t})\\ &= \sum_{n=1}^{\infty} \frac{2\sqrt{\pi t}}{x} \exp(\frac{(y-z)^2}{4t} - \frac{n^2\pi^2 t}{4x^2}) \sin\frac{n\pi(x-y)}{2x} \sin\frac{n\pi(x-z)}{2x}.\end{aligned}$$

PROOF. We shall only prove the second assertion here; the first one can be proved in a similar way.

To this end, let $Y(u) = W(u+1) - W(u)$ and observe that for $0 \le t, u \le 1$

$$B(u) = (2t)^{-1/2}(Y(ut) - uY(t) - (1-u)Y(0))$$

is a Brownian bridge independent of $Y(0)$ and $Y(t)$.

This implies that

$$\begin{aligned}
&\mathbf{P}(Y_2(1+t,1) \le x | W(1) = y, W(1+t) - W(t) = z) \\
&= \mathbf{P}(\sup_{0\le u\le 1} |Y(ut)| \le x | Y(0) = y, Y(t) = z) \\
&= \mathbf{P}(\sup_{0\le u\le 1} |(1-u)y + uz + (2t)^{1/2}B(u)| \le x | Y(0) = y, Y(t) = z) \\
&= \mathbf{P}(\sup_{0\le u\le 1} |(1-u)y + uz + (2t)^{1/2}B(u)| \le x) \\
&= \mathbf{P}(\sup_{0\le u\le 1} |W(u) + (2t)^{-1/2}y| \le (2t)^{-1/2}x | W(1) = (2t)^{-1/2}(z-y)).
\end{aligned}$$

The last equality follows from the fact that $W(u) - uW(1)$ is a Brownian bridge independent of $W(1)$. This conditional probability is readily obtained using Lemma 2 to calculate

$$\lim_{\delta\to 0} \frac{\mathbf{P}(M^+(1) \le \frac{x-y}{\sqrt{2t}}, M^-(1) \ge -\frac{y+x}{\sqrt{2t}}, \frac{z-y}{\sqrt{2t}} \le W(1) \le \frac{z-y}{\sqrt{2t}} + \delta)}{\mathbf{P}((2t)^{-1/2}(z-y) \le W(1) \le (2t)^{-1/2}(z-y) + \delta)},$$

yielding the first expression for $G_t(x|y,z)$ given above. The second form is obtained from the first by an application of Poisson's formula.

Integrating the formulas of Lemma 3 with respect to the joint distribution of $Y(0)$ and $Y(t)$, we obtain the following two lemmas:

LEMMA 4. *If $t \le a$ and $x^2t/a \to \infty$, then the following asymptotic relations are true for all $a > 0$:*

$$\mathbf{P}(Y_1(t+a,a) > x) \sim \frac{t}{a\sqrt{2\pi}} x \exp(-x^2/2)$$

*and*

$$\mathbf{P}(Y_2(t+a,a) > x) \sim \frac{2t}{a\sqrt{2\pi}} x \exp(-x^2/2).$$

The first of these assertions can also be obtained from Slepian's [24] result (see also Shepp [23], pp. 348–350).

We shall again only proof the second assertion. It suffices to carry out the proof for $a = 1$ as by the scale-change property of the Wiener process the distributions of $Y_i(t,a)$ and $Y_i(t/a,1)$ agree. So, as $W(1)$ and $W(t+1)-W(t)$ are jointly normally distributed with correlation coefficient $\rho = 1-t$, the probability in question can be calculated as

$$\int_{-x}^{x}\int_{-x}^{x} (1 - G_t(x|y,z)) \frac{1}{2\pi\sqrt{t(2-t)}} \exp(-\frac{y^2 - 2(1-t)yz + z^2}{2t(2-t)}) dy dz.$$

Using the first form of $G_t(x|y,z)$ given in Lemma 2, we see that in the first series the summand for $k = 0$ cancels against 1. In the same series, all terms with $|k| \geq 2$ are readily seen to be negligible. The contribution of the summand with $k = 1$ can be estimated by extending the inner integral over the whole real axis, yielding $\Phi(3x) - \Phi(x)$ which, too, is of smaller order than our asymptotic formula. In the second sum, the only summands that are not negligible are those for $k = 0$ and $k = -1$.

The integral of the term with $k = 0$ evaluates as

$$\int_{-x}^{x}\int_{-x}^{x} \frac{1}{2\pi\sqrt{t(2-t)}}$$
$$\times \exp\left(-\frac{y^2 - 2(1-t)yz + z^2 + 2(2-t)(yz - x(y+z) + x^2)}{2t(2-t)}\right)dydz$$
$$= \exp(-x^2/2)\int_{-2x}^{2x} \frac{1}{2\pi\sqrt{t(2-t)}}(2x - |u|)\exp\left(-\frac{(u-(2-t)x)^2}{2t(2-t)}\right)du,$$

where we have used the substitution $u = y + z$. In the latter integral, the part for $u > 0$ is dominating, yielding

$$\frac{t}{\sqrt{2\pi}}x\exp(-x^2/2)(1 + o(1)).$$

The integral for $k = -1$ can be obtained from the one for $k = 0$ by substituting $y$ for $-y$ and $z$ for $-z$. Summing up, we finally get the assertion of our lemma.

REMARK 2. It is readily seen from our proof of Lemma 4 and from Lemma 1 that if we only assume that $x^2t/a$ is bounded away from zero, we still have

$$\mathbf{P}(Y_i(1+t,1) > x) = O\left(\frac{tx}{a}\exp(-x^2/2)\right)$$

for $i = 1,2,3,4$ (for $i = 3,4$ use a reflection argument).

LEMMA 5. *For $t \leq 1$ and $x^2/t \to 0$*

$$\mathbf{P}(Y_2(1+t,1) \leq x) \sim \frac{16x}{\sqrt{\pi^5(2-t)}}\exp\left(-\frac{\pi^2 t}{4x^2}\right).$$

PROOF. We use the second form of $G_t(x,y)$. It is obvious that the dominating term of the series is the one for $n = 1$. This yields the integral

$$\frac{1}{x\sqrt{\pi(2-t)}}\exp\left(-\frac{\pi^2 t}{4x^2}\right)\int_{-x}^{x}\int_{-x}^{x}\exp\left(-\frac{(y+z)^2}{4(2-t)}\right)\cos\frac{\pi y}{2x}\cos\frac{\pi z}{2x}dydz.$$

By our assumptions on $x$, the exponential within the integral is $1+o(1)$, so we get

$$\begin{aligned}&\int_{-x}^{x}\int_{-x}^{x}\exp(-\frac{(y+z)^2}{4(2-t)})\cos\frac{\pi y}{2x}\cos\frac{\pi z}{2x}dydz\\&=(1+o(1))(\int_{-x}^{x}\cos\frac{\pi y}{2x}dy)^2=(1+o(1))\frac{16x^2}{\pi^2},\end{aligned}$$

so our assertion is proved.

From Lemma 4 we obtain the following

LEMMA 6. *If $x\to\infty$ , $t$ and $a<t$ vary in such a way that*

$$a^{-1}tx\exp(-x^2/2)\to\infty$$

*then*

$$\begin{aligned}\mathbf{P}(Y_1(t,a)\le x)&=\exp(-(1+o(1))\frac{tx}{a\sqrt{2\pi}}\exp(-x^2/2)),\\\mathbf{P}(Y_2(t,a)\le x)&=\exp(-(1+o(1))\frac{2tx}{a\sqrt{2\pi}}\exp(-x^2/2)),\\\mathbf{P}(Y_3(t,a)\le x)&\ge\exp(-(1+o(1))\frac{2tx}{a\sqrt{2\pi}}\exp(-x^2/2)),\\\mathbf{P}(Y_4(t,a)\le x)&\ge\exp(-(1+o(1))\frac{4tx}{a\sqrt{2\pi}}\exp(-x^2/2)).\end{aligned}$$

*If, in addition,* $0<\beta<1$ *and* $0<d<1$ *then*

$$\begin{aligned}&\mathbf{P}(\inf_{0\le s\le d}Y_1(t,a(1+s))\le x)\\&\le\exp(-(1+o(1))(1-\beta)\frac{tx}{a\sqrt{2\pi}}\exp(-x^2(1+d)/2)(1-\frac{1}{2}\exp(-\frac{\beta^2}{2x^2d})))\end{aligned}$$

*and*

$$\begin{aligned}&\mathbf{P}(\inf_{0\le s\le d}Y_2(t,a(1+s))\le x)\\&\le\exp(-(1+o(1))(1-\beta)\frac{2tx}{a\sqrt{2\pi}}\exp(-x^2(1+d)/2)(1-\frac{1}{2}\exp(-\frac{\beta^2}{2x^2d}))).\end{aligned}$$

Again, by the scale-change property of the Wiener process, we may assume without loss of generality that $a=1$. We shall only prove the assertions for $Y_1$ in detail; the remaining can be proved in a quite analogous way.

As the upper half of the first assertion is a consequence of the second, we only have to prove the lower half. In order to do so, let $G(t) = W(t+1) - W(t)$ and $G'(k+t) = G_k(t)$ for $0 \le t < 1$ and $k \in \mathbb{N}$, where $G_k(\cdot)$ is a sequence of independent copies of $G(\cdot)$. The one-dimensional marginals of both $G(t)$ and $G'(t)$ are standard normal. Their correlation functions $\rho(t_1, t_2)$ and $\rho'(t_1, t_2)$ agree if $[t_1] = [t_2]$, and otherwise we have that $\rho'(t_1, t_2) = 0 \le \rho(t_1, t_2)$. Thus Slepian's lemma [25, Theorem 1] is applicable, yielding

$$\mathbf{P}(Y_1(t,1) \le x) = \mathbf{P}(\sup_{0 \le u \le t-1} G(u) \le x) \ge \mathbf{P}(\sup_{0 \le u \le t-1} G'(u) \le x)$$
$$\ge \prod_{k=1}^{[t]} \mathbf{P}(\sup_{0 \le u \le 1} G(u) \le x) \ge \exp(-(1+o(1))\frac{tx}{\sqrt{2\pi}} \exp(-x^2/2)),$$

which proves this assertion. For the other one, choose $m$ in such a way that $m \to \infty$ and $mx \exp(-x^2/2) \to 0$. Define

$$B_k = \{\inf_{0 \le s \le d} \sup_{(k-1)m \le t \le km-1-d} W(t+1+s) - W(t) \le x\sqrt{1+d}\}.$$

The events $B_k$ are clearly independent, and

$$\mathbf{P}(\inf_{0 \le s \le d} Y_1(t, 1+s) \le x) \le \mathbf{P}(\bigcap_{k=1}^{[t/m]-1} B_k) \tag{7}$$
$$= \prod_{k=1}^{[t/m]-1} \mathbf{P}(B_k) = (\mathbf{P}(B_1))^{[t/m]-1}.$$

Now, in order to estimate $\mathbf{P}(B_1)$, we proceed as follows: First, we want to get rid of the infimum. To this end, observe that

$$\{\sup_{0 \le t \le m-1} W(t+1) - W(t) \ge x\sqrt{1+d} + \frac{b}{x}\}$$
$$\subset B_1^C \cup \{\inf_{0 \le s \le d} \sup_{t \le m-1-d} W(t+1+s) - W(t) \le x\sqrt{1+d},$$
$$\sup_{t \le m-1-d} W(t+1) - W(t) \ge x\sqrt{1+d} + \frac{b}{x}\},$$

and by a simple reflection principle argument the probability of the last event is overestimated by

$$\mathbf{P}(\sup_{t \le m-1-d} W(t+1) - W(t) \ge x\sqrt{1+d} + \frac{b}{x})\mathbf{P}(\sup_{0 \le s \le d} W(s) > \frac{b}{x}),$$

so it follows that

$$\begin{aligned}
&\mathbf{P}(B_1^C)\\
&\geq \mathbf{P}(\sup_{t\leq m-1-d} W(t+1)-W(t)\geq x\sqrt{1+d}+\frac{b}{x})(1-\mathbf{P}(\sup_{0\leq s\leq d} W(s)>\frac{b}{x}))\\
&\geq \mathbf{P}(\sup_{t\leq m-1-d} W(t+1)-W(t)\geq x\sqrt{1+d}+\frac{b}{x})(1-\frac{1}{2}\exp(-\frac{b^2}{2x^2d})),
\end{aligned}$$

where we have used the inequality $1-\Phi(u)\leq \frac{1}{2}\exp(-u^2/2)$ that holds for all $u\geq 0$.

In order to achieve somewhat simpler notations, let us define $y=x\sqrt{1+d}+b/x$ and $\hat{B}=\{\sup_{t\leq m-1-d} W(t+1)-W(t)\geq y\}$. Furthermore, let

$$\begin{aligned}
D_j=\{&\sup_{\max(0,j-1)/2\leq u\leq j/2} W(u+1)-W(u)\leq y,\\
&\sup_{j/2<u\leq (j+1)/2} W(u+1)-W(u)>y\}.
\end{aligned}$$

Clearly,

$$\hat{B}\supset \bigcup_{j=0}^{[2m-2d-3]} D_j,$$

and it follows from the inclusion-exclusion formula that

$$\mathbf{P}(\hat{B})\geq \sum_{0\leq j\leq 2m-2d-3}\mathbf{P}(D_j)-\sum_{0\leq j<r\leq 2m-2d-3}\mathbf{P}(D_jD_r). \tag{8}$$

Now, for $j<r$, the events $D_j$ and $D_r$ are disjoint if $r=j+1$, and independent if $r\geq j+4$. In the remaining cases we have

$$\begin{aligned}
\mathbf{P}(D_jD_{j+3})&\leq \mathbf{P}(D_j\{\sup_{r/2\leq u\leq (r+1)/2} W(u+1)-W(u)>x\})\\
&=\mathbf{P}(D_j)\mathbf{P}(D_0)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}(D_jD_{j+2})&\leq \mathbf{P}(\{\sup_{0\leq u\leq 1/2}|W(\frac{j+3}{2})-W(\frac{j+3}{2}-u)|\geq\frac{3x}{4}\})\\
&\qquad+\mathbf{P}(D_j)\mathbf{P}(\{\sup_{0\leq u\leq 1}|W(\frac{j+3}{2}+u)-W(\frac{j+3}{2})\geq\frac{x}{4}\})\\
&=o(\mathbf{P}(D_j)).
\end{aligned}$$

This yields

$$\sum_{r=j+1}^{[2m-2d-3]} \mathbf{P}(D_j D_r) \leq \mathbf{P}(D_j)\Big(\sum_{r=0}^{[2m-2d-3]} \mathbf{P}(D_r) + o(1)\Big)$$

and

$$\sum_{0\leq j<r\leq[2m-2d-3]} \mathbf{P}(D_j D_r) \leq \Big(\sum_{j=0}^{[2m-2d-3]} \mathbf{P}(D_j)\Big)^2 + o\Big(\sum_{j=0}^{[2m-2d-3]} \mathbf{P}(D_j)\Big).$$

Now, in order to estimate $\sum_{j=0}^{[2m-2d-3]} \mathbf{P}(D_j)$, observe that

$$\mathbf{P}(D_0) + \mathbf{P}(D_1) = \mathbf{P}(D_0 \cup D_1) = \mathbf{P}(Y_1(2,1) > y) \sim \frac{y}{\sqrt{2\pi}} \exp(-y^2/2),$$

by Lemma 4, whereas for $j \geq 2$ we have

$$\mathbf{P}(D_j) = \mathbf{P}(Y_1(2,1) > y) - \mathbf{P}(Y_1(\frac{3}{2},1) > y) \sim \frac{y}{2\sqrt{2\pi}} \exp(-y^2/2).$$

Putting everything together, we obtain

$$\mathbf{P}(\hat{B}) \geq (1 + o(1))\frac{my}{\sqrt{2\pi}} \exp(-y^2/2).$$

This implies

$$\begin{aligned}\mathbf{P}(B_1^C) &\geq (1+o(1))\frac{my}{\sqrt{2\pi}}\exp(-y^2/2)\Big(1-\frac{1}{2}\exp\Big(-\frac{b^2}{2x^2d}\Big)\Big)\\ &\geq (1+o(1))(1-2b)\frac{mx}{\sqrt{2\pi}}\exp(-x^2(1+d)/2)\Big(1-\frac{1}{2}\exp\Big(-\frac{b^2}{2x^2d}\Big)\Big)\end{aligned}$$

and inserting this estimate into (7) completes the proof.

We are now able to state

THEOREM 3. *Let $a(t)$ be chosen in such a way that $a(t) \uparrow \infty$ and $\Delta(t) = t/(a(t)\log\log t) \uparrow \infty$. Then for $i = 1,2,3,4$ there are constants $K_i$ which may still depend on $a$, such that*

$$\phi_K(t) = (2\log\Delta(t) + \log\log\Delta(t) - K)^{1/2} \in \begin{cases} \mathcal{LLC}(Y_i(t,a(t))) & \text{if } K > K_i \\ \mathcal{LUC}(Y_i(t,a(t))) & \text{if } K < K_i \end{cases}$$

*and*

$$\begin{aligned}\log \pi \le K_1 &\le \log 4\pi,\\ \log \frac{\pi}{4} \le K_2 &\le \log \pi,\\ \log \frac{\pi}{4} \le K_3 &\le \log 4\pi,\\ \log \frac{\pi}{16} \le K_4 &\le \log \pi.\end{aligned}$$

*If, in addition, either* $a(t)$ *is of the form*

$$a(t) = C_0 \exp\left(\int_{C_1}^{t} \frac{\eta(y)}{y}\,dy\right) \tag{9}$$

*with* $\limsup \log \eta(t)/\log\log t \le -1$ *(this is roughly equivalent to* $a(t) = o(\exp((\log t)^{\varepsilon})$ *for all* $\varepsilon > 0$*), or*

$$\log\log(t/a(t))/\log\log t \to 0, \tag{10}$$

*then*

$$\begin{aligned}K_1 &= \log \pi,\\ K_2 &= \log \frac{\pi}{4},\\ \log \frac{\pi}{4} \le K_3 &\le \log \pi,\\ \log \frac{\pi}{16} \le K_4 &\le \log \frac{\pi}{4}.\end{aligned}$$

We shall again only carry out the proof for $i = 1$, the other cases are similar.

First observe that if we find both an upper class function and a lower class function of the form $\phi_K(\cdot)$ then the existence of the constants $K_j$ already follows from the Hewitt-Savage zero-one law.

Assume now that $K = \log(4\pi) + \theta$ with $\theta > 0$. We are going to prove that $\phi_K(t)$ is in the lower-lower class of $Y_1(t, a(t))$. To this end, let $t_k = \exp(k^{1/2-\delta})$ and

$$D_k = \{\inf_{a(t_k)\le u \le a(t_{k+1})} Y_1(t_k, u) \le \phi_K(t_{k+1}).\}$$

By Lemma 6, we have

$$\mathbf{P}(D_k) \le \exp(-(1+o(1))(1-2\beta)\frac{t_k \phi_K(t_{k+1})}{a(t_k)\sqrt{2\pi}} \times$$
$$\times \exp(-\phi_K^2(t_{k+1})(1+d)/2)(1 - C\exp(-\frac{\beta^2}{2\phi_K^2(t_{k+1})d}))),$$

for any $b > 0$, where we have put $d = a(t_{k+1})/a(t_k) - 1$. Using that

$$\frac{a(t_{k+1})}{a(t_k)} \leq \frac{t_{k+1}}{t_k} \leq 1 + Ck^{-\frac{1}{2}-\delta}$$

and some calculation, we obtain that

$$\mathbf{P}(D_k) \leq \exp(-(1+o(1))(1-2\beta)e^{\theta/2}(1-2\delta)\log k),$$

so, by choosing both $\delta$ and $\beta$ small enough, we can achieve that

$$\sum_{k=1}^{\infty} \mathbf{P}(D_k) < \infty,$$

and the Borel-Cantelli lemma implies that $\phi_K(t) \in \mathcal{LLC}(Y_1(t, a(t)))$.

If $a$ satisfies condition (9) then, for any $\varepsilon > 0$ and $t \leq u \leq 2t$ we have $a(u)/a(t) \leq 1 + c(\log t)^{\varepsilon-1}(u-t)/t$, whereas if condition (10) is satisfied, we have $\phi_K(t) \leq (\log t)^{\varepsilon}$. In either case, we can use the sequence $t_k = \exp(k^{1-\delta})$ instead of $\exp(k^{1/2-\delta})$ in the above proof, yielding the sharper estimates on $K_i$.

It remains to prove the upper class result. So, let now $K = \log \pi - \theta$, $t_k = \exp(k^{1+\delta})$, and

$$E_k = \{Y_1(t_k, a(t_k)) < \phi_K(t_k)\}.$$

By Lemma 6, it is readily verified that the series

$$\sum_{k=1}^{\infty} \mathbf{P}(E_k)$$

is divergent if $\delta$ is small enough.

In order to show that with probability one infinitely many of the events $E_k$ occur, we shall use the following Borel-Cantelli

LEMMA 7. *Let $(A_k,\ k \in IN)$ be a sequence of events satisfying the following conditions*

$$\text{(i)} \qquad \sum_{k=1}^{\infty} \mathbf{P}(A_k) = \infty$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{r=1}^{n} \mathbf{P}(A_k A_r)}{\left(\sum_{k=1}^{n} \mathbf{P}(A_k)\right)^2} \leq 1$$

*Then it holds:*

$$\mathbf{P}(A_k\ i.o.) = 1.$$

So, now we are going to estimate the sum

$$\sum_{1\le k<r\le N} \mathbf{P}(E_kE_r).$$

For sake of simplicity, in the following few lines, we omit the argument $t_r$ in both $a(t_r)$ and $\phi_K(t_r)$. It holds:

$$\begin{aligned}\mathbf{P}(E_kE_r) &\le \mathbf{P}(E_k\{\sup_{t_k\le u\le t_r-a} W(u+a)-W(u)\le \phi_K\sqrt{a}\})\\ &= \mathbf{P}(E_k)\mathbf{P}(\{\sup_{t_k\le u\le t_r-a} W(u+a)-W(u)\le \phi_K\sqrt{a}\}).\end{aligned}$$

The latter probability can be estimated in the following way:

$$\begin{aligned}&\mathbf{P}(\{\sup_{t_k\le u\le t_r-a} W(u+a)-W(u)\le \phi_K\sqrt{a}\})\\ &\le \mathbf{P}(\{\sup_{0\le u\le t_r-a} W(u+a)-W(u)\le \phi_K\sqrt{a}\})\\ &\quad+\mathbf{P}(\{\sup_{0\le u\le t_k} W(u+a)-W(u)>\phi_K\sqrt{a}, W(t_k+a)-W(t_k)\le \phi_K\sqrt{a}\}).\end{aligned}$$

Here, the first probability is just $\mathbf{P}(E_r)$. If $t_k > a/\phi_K^2$ then, with the help of Lemma 4, the second probability can be overestimated by

$$\begin{aligned}C\frac{t_k}{a}\phi_K\exp(-\phi_K^2/2) &\le C\frac{t_k}{t_r}\log\log t_r\\ &\le C\log r\exp(k^{1+\delta}-r^{1+\delta}),\end{aligned}$$

and it is readily verified that the sum over all $r>k$ is bounded by a constant not depending on $k$.

On the other hand, if $t_k < a/\phi_K^2$, then we have for all $z$

$$\begin{aligned}&\mathbf{P}(\sup_{0\le u\le t_k} W(u+a)-W(u)>\phi_K\sqrt{a},\\ &\qquad W(t_k+a)-W(t_k)\le\phi_K\sqrt{a}|W(a)-W(t_k)=z)\\ &\le 2\exp(-\frac{(z-\phi_K\sqrt{a})^2}{8t_k}).\end{aligned}$$

Namely, if $z>\phi_K a^{1/2}$, then we have to assume that $W(t_k+a)-W(a)$ be less than $\phi_K a^{1/2}-z$, whereas in the opposite case there must be a $u\le t_k$

such that either $W(a+u)-W(a)$ or $W(t_k)-W(u)$ exceeds $(z-\phi_k a^{1/2})/2$. Using the reflection principle and the estimate $1-\Phi(x) \leq \frac{1}{2}\exp(-x^2/2)$, the stated bound is obtained. Integrating this conditional probability with respect to the distribution of $W(a)-W(t_k)$, we obtain

$$\mathbf{P}(\{\sup_{0\leq u\leq t_k} W(u+a)-W(u) > \phi_K a^{1/2}, W(t_k+a)-W(t_k) \leq \phi_K a^{1/2}\})$$

$$\leq C\sqrt{\frac{t_k}{a}}\exp(-\phi_K^2/2).$$

It is readily verified that the sum of these expressions over all $r > k$ is bounded by a constant not depending on $k$. Summing up, we see that

$$\sum_{\substack{1\leq k,r\leq N\\ k\neq r}} \mathbf{P}(E_k E_r) \leq (\sum_{1\leq k\leq N} \mathbf{P}(E_k))^2 + C(\sum_{1\leq k\leq N} \mathbf{P}(E_k)).$$

Thus the hypotheses of Lemma 7 are satisfied, so we can conclude that infinitely many of the events $E_k$ occur, which clearly implies that $\phi_K(t) \in \mathcal{LUC}(Y_1(t,a(t)))$.

Now, let us turn to the case where $a(t)$ is large, namely that $\Delta(t) \to 0$. In this case, it seems more convenient to consider $J_i(t,a(t)) = a^{1/2}(t)Y_i(t,a(t))$ instead of $Y_i(t,a(t))$. For these $a(t)$, even the first-order asymptotic behaviour is known only for $J_1$; Namely, it has been proved in Csáki-Révész [3, Theorem 2]:

THEOREM D. *If* $\Delta(t) \to 0$ *then*

$$\liminf_{t\to\infty} \frac{J_1(t)}{\sqrt{2t\log\log t}} = -\beta(a(t)/t),$$

*where*

$$\beta(x) = \left(\frac{(2r+1)x-1}{r(r+1)}\right)^{1/2}$$

*and*

$$r = [1/x].$$

REMARK 3. The original proof is given only for the case that $a(t)/t = \alpha > 0$. However, it is easily extended to the general case.

For the remaining increments, we shall need some additional lemmas:

LEMMA 8. *As in Lemma 2, let*

$$M^+(t) = \sup_{0\le u\le t} W(u),$$
$$M^-(t) = \inf_{0\le u\le t} W(u),$$
$$M(t) = \sup_{0\le u\le t} |W(u)|.$$

*It holds:*

(11)
$$\begin{aligned}&\mathbf{P}(\sup_{0\le u\le v\le t} |W(v) - W(u)| \le x\sqrt{t})\\ &= \mathbf{P}(M^+(t) - M^-(t) \le x\sqrt{t})\\ &= 2\sum_{n\in\mathbf{Z}} ((-1)^n(n+1)(\Phi((n+1)x) - \Phi(nx)))\\ &= 8\sum_{n=0}^{\infty} \left(\frac{1}{x^2} + \frac{1}{(2n+1)^2\pi^2}\right) \exp(-\frac{(2n+1)^2\pi^2}{2x^2}),\end{aligned}$$

*and*

(12)
$$\begin{aligned}&\mathbf{P}(\sup_{0\le u\le v\le t} W(v) - W(u) \le x\sqrt{t})\\ &= \mathbf{P}(\sup_{0\le u\le t} W(u) - M^-(u) \le x\sqrt{t}) = \mathbf{P}(M(t) \le x\sqrt{t})\\ &= \sum_{n\in\mathbf{Z}} ((-1)^n(\Phi((2k+1)x) - \Phi((2k-1)x)))\\ &= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \exp(-\frac{(2n+1)^2\pi^2}{8x^2}).\end{aligned}$$

PROOF. To obtain (11), observe that by putting $c = -b$ and $d = a$ in Lemma 2, we get the joint distribution of $M^+(t)$ and $M^-(t)$. By differentiating this distribution function we get its density, and integrating this density over the domain $M^- \le 0 \le M^+ \le M^- + x\sqrt{t}$ yields the first part of (11). The second half of this equation follows by another application of Poisson's formula.

For (12), we use the fact that by a well-known theorem of P. Lévy the processes $W(t) - M^-(t)$ and $|W(t)|$ are identical in distribution. Thus, in particular, the distributions of $\sup_{0\le u\le t} W(u) - M^-(u)$ and of $M(t)$ are the same. The latter is well-known to be of the form given in (12), cf., e.g., Feller [11], pp. 340–343.

From Lemmas 5 and 8 we obtain

LEMMA 9. *If* $x/\sqrt{a} \to 0$ *then it holds:*

$$\mathbf{P}(J_3(t,a) \le x) = \exp(-\frac{\pi^2 t}{8x^2}(1+o(1)))$$

*and*

$$\mathbf{P}(J_4(t,a) \le x) = \exp(-\frac{\pi^2 t}{2x^2}(1+o(1))).$$

*If, in addition* $x/\sqrt{t-a} \to 0$ *then*

$$\mathbf{P}(J_2(t,a) \le x) \le \exp(-\frac{\pi^2 (t-a)}{8x^2}(1+o(1)).$$

PROOF. To prove the first statement, observe that

$$J_3(t,a) \le \sup_{0 \le u \le v \le t} (W(v) - W(u)).$$

By Lemma 8 this implies that

$$\mathbf{P}(J_3(t,a) \le x) \ge \exp(-\frac{\pi^2 t}{8x^2}(1+o(1))).$$

For the other inequality, let $n = [t/a] + 1$, $a' = t/n$ and

$$D_j = \{ \sup_{(j-1)a' \le u \le v \le ja'} (W(v) - W(u)) \le x \}.$$

The events $D_j$ are independent, and by Lemma 8

$$\mathbf{P}(D_j) = \exp(-\frac{\pi^2 a'}{8x^2}(1+o(1))).$$

So,

$$\mathbf{P}(J_3(t,a) \le x) \ge \mathbf{P}(\bigcap_{j=1}^{n} D_j) = \mathbf{P}(D_1)^n = \exp(-\frac{\pi^2 t}{8x^2}(1+o(1))).$$

The second assertion is proved along the same lines as the first one. In order to prove the last assertion, let $n = [t/a]$ and

$$E_j = \begin{cases} \{\sup_{(j-1)a \le u \le ja} |W(u+a) - W(u)| \le x\} & \text{if } 1 \le j < n, \\ \{\sup_{(n-1)a \le u \le t-a} |W(u+a) - W(u)| \le x\} & \text{if } j = n. \end{cases}$$

We have

$$\{J_2(t,a) \le x\} = \bigcap_{j=1}^{n} E_j \subset \bigcap_{j=1}^{[(n+1)/2]} E_{2j-1}.$$

So using Lemma 5 and independence, we get that

$$\mathbf{P}(J_2(t,a) \le x) \le \prod_{j=1}^{[(n+1)/2]} \mathrm{P}(E_{2j-1}) \le \exp(-\frac{\pi^2(t-a)}{8x^2}(1+o(1))).$$

Thus, Lemma 9 is proved.

By another standard Borel-Cantelli argument, we can derive the following theorem from Lemmas 5 and 9.

THEOREM 4. *Assume that $\Delta(t) \downarrow 0$ and that $a(t)/t$ is nonincreasing. Then*

$$\liminf_{t\to\infty} J_3(t,a(t))\sqrt{\frac{8\log\log t}{\pi^2 t}} = 1,$$

$$\liminf_{t\to\infty} J_4(t,a(t))\sqrt{\frac{2\log\log t}{\pi^2 t}} = 1,$$

*and if $a(t) < t$ then*

$$\frac{1}{\sqrt{2}} \le \liminf_{t\to\infty} J_2(t,a(t))\sqrt{\frac{4\log\log t}{\pi^2(t-a(t))}} \le 2.$$

*If, in addition, $a(t) \ge t/2$ then*

$$\liminf_{t\to\infty} J_2(t,a(t))\sqrt{\frac{4\log\log t}{\pi^2(t-a(t))}} = 1.$$

REMARK 4. We conjecture that the last statement is true in the general case.

## REFERENCES

[1] BILLINGSLEY, P., *Convergence of Probability Measures*, Wiley, New York, 1968. *MR* **38** # 1718

[2] CHUNG, K.L. and ERDŐS, P., On the application of the Borel-Cantelli lemma, *Trans. Amer. Math. Soc.* **72** (1952), 179—186. *MR* **13** - 567

[3] CSÁKI, E. and RÉVÉSZ, P., How big must be the increments of a Wiener process?, *Acta Math. Acad. Sci. Hungar.* **33** (1979), 37—49. *MR* **80d**: 60102

[4] CSÖRGŐ, M. and RÉVÉSZ, P., How big are the increments of a Wiener process?, *Ann. Probab.* **7** (1979), 731—737. *MR* **80g**: 60025

[5] DEHEUVELS, P. and DEVROYE, L., Limit laws of Erdős-Rényi-Shepp type, *Ann. Probab.* **15** (1987), 1363—1386. *MR* **88f**: 60055

[6] DEHEUVELS, P., DEVROYE, L. and LYNCH, J., Exact convergence rate in the limit theorems of Erdős-Rényi and Shepp, *Ann. Probab.* **14** (1986), 209—223. *MR* **87d**: 60032

[7] DEHEUVELS, P., ERDŐS, P., GRILL, K. and RÉVÉSZ, P., Many heads in a short block, *Mathematical Statistics and Probability Theory*, Vol. A (Bad Tatzmannsdorf, 1986), ed. by M. L. Puri et al., D. Reidel, Dordrecht, 1987, 53—67. *MR* **89b**: 60073

[8] DEHEUVELS, P. and RÉVÉSZ, P., Weak laws for the increments of Wiener processes, Brownian bridges, empirical processes and partial sums of i.i.d.r.v.'s, *Mathematical Statistics and Probability Theory*, Vol. A (Bad Tatzmannsdorf. 1986), ed. by M. L. Puri et al., D. Reidel, Dordrecht, 1987, 69—88. *MR* **89a**: 60057

[9] DEHEUVELS, P. and STEINEBACH, J., Exact convergence rates in strong approximation laws for large increments of partial sums, *Probab. Theory Related Fields* **76** (1987), 369–393. *MR* **88m**: 60081

[10] DEHEUVELS, P. and STEINEBACH, J., Limit laws for the modulus of continuity of the partial sum process and for the Shepp statistic, *Stochastic Process. Appl.* **29** (1988), 223—245. *MR* **89i**: 60066

[11] FELLER, W., *An Introduction to Probability Theory and its Application*, Vol. II, Wiley, New York, 1966. *MR* **35** # 1048

[12] HANSON, D. L. and RUSSO, R. P., Some results on increments of the Wiener process with applications to lag sums of i.i.d. random variables, *Ann. Probab.* **11** (1983), 609—623. *MR* **85c**: 60127

[13] HANSON, D. L. and RUSSO, R. P., Some more results on increments of the Wiener process, *Ann. Probab.* **11** (1983), 1009–1015. *MR* **85c**: 60040

[14] HÖGLUND, T., A unified formulation of the central limit theorem for small and large deviations from the mean, *Z. Wahrsch. Verw. Gebiete* **49** (1979), 105—117. *MR* **81j**: 60026

[15] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G., An approximation of partial sums of independent RV'-s and the sample DF.II, *Z. Wahrsch. Verw. Gebiete* **34** (1976), 33—58. *MR* **53** # 6697

[16] KŌNO, N., On the asymptotic behavior of increments of a Wiener process, *J. Math. Kyōto Univ.* **31** (1991), 289—306.

[17] LAI, T. L., On Strassen-type laws of the iterated logarithm for delayed averages of the Wiener process, *Bull. Inst. Math. Acad. Sinica* **1** (1973), 29—39. *MR* **47** # 9694

[18] LAI, T. L., Limit theorems for delayed sums, *Ann. Probab.* **3** (1974), 432—440. *MR* **50** # 8664

[19] MASON, D. M., An extended version of the Erdős-Rényi strong law of large numbers, *Ann. Probab.* **17** (1989), 257—265.

[20] ORTEGA, J. and WSCHEBOR, M., On the increments of the Wiener process, *Z. Wahrsch. Verw. Gebiete* **65** (1984), 329—339. *MR* **86c**: 60044

[21] RÉVÉSZ, P., A note to the Chung-Erdős-Sirao theorem, *Asymptotic Theory of Statistical Tests and Estimation* (Proc. Adv. Internat. Sympos., Univ. North Carolina, Chapel Hill, N. C., 1979), ed. by L. M. Chakravarty, Academic Press, New York, 147—158. *MR* **81g**: 60082

[22] RÉVÉSZ, P., On the increments of Wiener and related processes, *Ann. Probab.* **10** (1982), 613—622. *MR* **83i**: 60048

[23] SHEPP, L. A., Radon-Nikodym derivatives of Gaussian measures, *Ann. Math. Statist.* **37** (1966), 321—354. *MR* **32** # 8408

[24] SLEPIAN, D., First passage time for a particular Gaussian process, *Ann. Math. Statist.* **32** (1961), 610—612. *MR* **23** # A2918

[25] SLEPIAN, D., The one-sided barrier problem for Gaussian noise, *Bell System Tech. J.* **41** (1962), 463—501. *MR* **24** # A3017

[26] TAYLOR, S. J., Regularity of irregularities of a Brownian path, *Ann. Inst. Fourier (Grenoble)* **24** (1974), 195–203. *MR* **53** # 14699

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# ON SOME LARGE DEVIATION PROBABILITIES FOR FINITE MARKOV PROCESSES

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**Abstract**

For a continuous-time, finite state space Markov process, we investigate the tail behavior of the numbers of jumps between different states before time $T$ and the amounts of time spent in states before time $T$ as $T$ tends to infinity.


## 1. Introduction

Let $\xi = (\xi(t), t \geq 0)$ be a continuous-time Markov process with finite state space $X = \{1, \ldots, n\}$ and infinitesimal generator $\gamma$. For $T > 0$, let $\hat{z}(T) = [\hat{z}_{ij}(T), i \neq j]$ and $\hat{p}(T) = [\hat{p}_i(T), i \in X]'$, where

$$\begin{aligned} \hat{z}_{ij}(T) &= T^{-1} \text{[number of jumps from } i \text{ to } j \text{ before time } T], \\ \hat{p}_i(T) &= T^{-1} \text{[total amount of time spent in } i \text{ before time } T], \end{aligned}$$

and $'$ denotes transpose. The statistics $\hat{z}(T)$ and $\hat{p}(T)$ are of basic interest in both theory and applications of finite Markov processes; in particular, they compactly summarize the whole information which the sample $(\xi(t), 0 \leq t \leq T)$ contains about the underlying generator.

If for simplicity we assume that all off-diagonal entries of $\gamma$ are (strictly) positive, then (Albert [1, theorem 6.9])

$$\lim_{T\to\infty} \hat{z}_{ij}(T) = \pi_i(\gamma)\gamma_{ij} \quad \text{a.s.}, \qquad \lim_{T\to\infty} \hat{p}_i(T) = \pi_i(\gamma) \quad \text{a.s.},$$

where $\pi(\gamma) = [\pi_1(\gamma), \ldots, \pi_n(\gamma)]'$ is the stationary distribution of $\gamma$ (i.e., the unique probability vector $p$ on $X$ which satisfies $\gamma' p = 0$), and it is very natural to ask how "fast" the above convergence occurs, in the sense of exponential rates of convergence. Of course, the tail behavior of the empirical measure $\hat{p}(T)$ is easily identified using the elegant Donsker-Varadhan theory

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(see e.g. Donsker and Varadhan [2] and Varadhan [8]), which however cannot be applied to the jump-type statistic $\hat{z}(T)$.

In this paper, we show that the joint tail behavior of $\hat{z}(T)$ and $\hat{p}(T)$ can be identified by means of more or less standard large deviation-theoretic tools. We explicitly compute the rate function and show that it is intimately related to the (Kullback-Leibler) information distance between generators. The main result is given in section 2; section 3 discusses three particular cases of special interest.

## 2. The Main Result

Let $\mathcal{Z}$ be the set of all arrays $z = [z_{ij}, i \neq j]$ which satisfy $z_{ij} \geq 0$ for all $i \neq j$, and let $\mathcal{P}$ be the set of all probability vectors on $X$. Then for all $T > 0$, $(\hat{z}(T), \hat{p}(T)) \in \mathcal{Z} \times \mathcal{P}$. For $z \in \mathcal{Z}$, it will also be convenient to write $z_{i\cdot} = \sum_{j:j\neq i} z_{ij}$ and similarly $z_{\cdot i} = \sum_{j:j\neq i} z_{ji}$. Finally, for $u \geq 0$ and $v > 0$, let $H(u,v) = u\log(u/v) - u + v$; then $H(u,v) \geq 0$ with strict inequality for $u \neq v$.

We have the following main result.

THEOREM. *$(\hat{z}(T), \hat{p}(T))$ satisfies a large deviation principle with rate function*

$$S(z,p) = \begin{cases} \sum_{i:p_i>0} \sum_{j:j\neq i} p_i H(z_{ij}/p_i, \gamma_{ij}), & \text{if } (z,p) \in \mathcal{R}, \\ \infty, & \text{otherwise,} \end{cases}$$

*where $\mathcal{R}$ is the set of all $(z,p) \in \mathcal{Z} \times \mathcal{P}$ for which $z_{i\cdot} = z_{\cdot i}$ for all $i \in X$ and $z_{ij} = 0$ whenever $p_i = 0$, i.e., for all open subsets $U$ of $\mathcal{Z} \times \mathcal{P}$,*

$$\liminf_{T\to\infty} T^{-1} \log P\{(\hat{z}(T), \hat{p}(T)) \in U\} \geq -\inf_U S(z,p),$$

*and for all closed subsets $C$ of $\mathcal{Z} \times \mathcal{P}$,*

$$\limsup_{T\to\infty} T^{-1} \log P\{(\hat{z}(T), \hat{p}(T)) \in C\} \leq -\inf_C S(z,p).$$

PROOF. Let $\mathcal{W}$ be the set of all arrays $w = [w_{ij}, i \neq j]$, for $w, \tilde{w} \in \mathcal{W}$, let $\langle w, \tilde{w}\rangle = \sum_{i\neq j} w_{ij}\tilde{w}_{ij}$, and let

$$Q_{w,b}(z,p) = \langle w, z\rangle + b'p.$$

By using the method in Albert [1], cf. also Grenander [5, theorem 2, p. 311],

$$E_{T,\gamma} \exp[TQ_{w,b}(\hat{z}(T), \hat{p}(T))] = \mu' \exp(TA(w,b))\mathbf{1},$$

where $\mathbf{1} = [1, \ldots, 1]'$, $\mu$ is the initial distribution of $\xi$, and $A(w,b) = [a_{ij}(w,b)]$ is an $n \times n$ matrix with entries given by

$$a_{ij}(w,b) = \begin{cases} \gamma_{ij} e^{w_{ij}}, & \text{if } i \neq j, \\ \gamma_{ii} + b_i, & \text{if } i = j. \end{cases}$$

Clearly, for all $(w,b) \in \mathcal{W} \times I\!R^n$, $A(w,b) \in \mathcal{A}$, the set of all $n \times n$ matrices with (strictly) positive off-diagonal entries. By the celebrated Perron-Frobenius theorem (cf. e.g. Ellis [3]), for every $A \in \mathcal{A}$ there exists a unique real and simple eigenvalue $\lambda(A)$ of $A$, the so-called *principal eigenvalue* of $A$, such that $\Re(\lambda) < \lambda(A)$ for all other eigenvalues of $A$ and

$$\lim_{T\to\infty} T^{-1} \log(\exp(TA))_{ij} = \lambda(A) \qquad \forall i,j.$$

To $\lambda(A)$ there corresponds a unique right eigenvector $r(A) \in \mathcal{P}$ with strictly positive entries. In particular, if $A\mathbf{1} = 0$, then $\lambda(A) = 0$ and $r(A) = n^{-1}\mathbf{1}$. Furthermore, if for some $s$, $A = A(v)$ is an $\mathcal{A}$-valued $C^\infty$ function of $v \in I\!R^s$, then $\lambda(A(v))$ and $r(A(v))$ are $C^\infty$ in $v$. Hence,

$$\lim_{T\to\infty} T^{-1} \log E_{T,\gamma} \exp\left[T Q_{w,b}(\hat{z}(T), \hat{p}(T))\right] = \lambda(A(w,b))$$

is $C^\infty$ in $(w,b) \in \mathcal{W} \times I\!R^n$. By lemma 1.1 and 1.2 in Gärtner [4], it follows that $(\hat{z}(T), \hat{p}(T))$ satisfies a large deviation principle with rate function

$$S(z,p) = \sup_{\mathcal{W}\times I\!R^n} \Phi_{w,b}(z,p), \qquad \Phi_{w,b}(z,p) = Q_{w,b}(z,p) - \lambda(A(w,b)),$$

and it remains to compute $S(z,p)$.

To start with, suppose that $(z,p) \in \mathcal{R}$. Denote the $i$-th component of $r(A(w,b))$ by $r_i(w,b)$ and let

$$\Psi_{w,b}(z,p) := \sum_{i:p_i>0} \sum_{j:j\neq i} p_i \; H\left(\frac{z_{ij}}{p_i}, a_{ij}(w,b)\frac{r_j(w,b)}{r_i(w,b)}\right)$$

and

$$V(z,p) := \sum_{i:p_i>0} \sum_{j:j\neq i} p_i \; H(z_{ij}/p_i, \gamma_{ij}).$$

Then, as $\sum_{i:p_i>0} \sum_{j:j\neq i} z_{ij} \log(x_j/x_i) = \sum_{i:p_i>0} (z_{\cdot i} - z_{i\cdot}) \log x_i = 0$ for arbitrary $x_1, \ldots, x_n > 0$ and

$$\sum_{j:j\neq i} a_{ij}(w,b) r_j(w,b) = (\lambda(A(w,b)) - (\gamma_{ii} + b_i)) r_i(w,b),$$

we have

$$\begin{aligned} &V(z,p) - \Psi_{w,b}(z,p) \\ &= \sum_{i:p_i>0} \sum_{j:j\neq i} \left( z_{ij} \log \left( e^{w_{ij}} \frac{r_j(w,b)}{r_i(w,b)} \right) + p_i \gamma_{ij} - p_i a_{ij}(w,b) \frac{r_j(w,b)}{r_i(w,b)} \right) \\ &= \langle w, z \rangle + b'p - \lambda(A(w,b)) \\ &= \Phi_{w,b}(z,p). \end{aligned}$$

Hence, as $\Psi_{w,b}(z,p) \geq 0$, we immediately deduce that $S(z,p) \leq V(z,p)$. On the other hand, if for $\epsilon > 0$, $w(\epsilon)$ is such that $w_{ij}(\epsilon) = \log(\max(z_{ij}, \epsilon)/p_i\gamma_{ij})$ whenever $p_i > 0$ (and e.g. zero otherwise), and we choose $b(\epsilon)$ in a way that $A(w(\epsilon), b(\epsilon))\mathbf{1} = 0$, then $r(A(w(\epsilon), b(\epsilon))) = n^{-1}\mathbf{1}$ and $\lim_{\epsilon\to 0} \Psi_{w(\epsilon),b(\epsilon)}(z,p) = 0$, such that we actually have $S(z,p) = V(z,p)$.

Next, suppose that for some $i$, $z_{i\cdot} \neq z_{\cdot i}$. Then, for $\sigma \in \mathbb{R}$, construct $w(\sigma) \in \mathcal{W}$ as follows. For $j \neq i$, let $w_{ij}(\sigma) = \sigma$ and $w_{ji}(\sigma) = -\sigma$, and let all other entries of $w(\sigma)$ be zero. Then $A(w(\sigma), 0) = D(\sigma)\gamma D(\sigma)^{-1}$, where $D(\sigma)$ is the diagonal matrix with $i$-th entry equal to $e^\sigma$ and all other diagonal entries equal to one. Hence, for all $\sigma$, $A(w(\sigma), 0)$ has the same eigenvalues as $\gamma$, in particular, $\lambda(A(w(\sigma), 0)) = 0$ and thus

$$\Phi_{w(\sigma),0}(z,p) = \sigma(z_{i\cdot} - z_{\cdot i}).$$

Hence, by either letting $\sigma \to \infty$ if $z_{i\cdot} > z_{\cdot i}$ or $\sigma \to -\infty$ if $z_{i\cdot} < z_{\cdot i}$, we conclude that $S(z,p) = \infty$ if for some $i$, $z_{i\cdot} \neq z_{\cdot i}$.

Finally, suppose that there exists two distinct states $k$ and $l$ such that $z_{kl} > 0$ and $p_k = 0$. In this case, define $w(\sigma) \in \mathcal{W}$ and $b(\sigma) \in \mathbb{R}^n$ by

$$w_{ij}(\sigma) = \begin{cases} \sigma, & \text{if } (i,j) = (k,l), \\ 0, & \text{otherwise}, \end{cases} \qquad b_i(\sigma) = \begin{cases} (1 - e^\sigma)\gamma_{kl}, & \text{if } i = k, \\ 0, & \text{otherwise}. \end{cases}$$

Then $A(w(\sigma), b(\sigma))\mathbf{1} = 0$, hence $\lambda(A(w(\sigma), b(\sigma))) = 0$ and $\Phi_{w(\sigma),b(\sigma)}(z,p) = \sigma z_{kl}$. By letting $\sigma \to \infty$, we conclude that $S(z,p) = \infty$ if $z_{kl} > 0$ and $p_k = 0$ for some pair $(k,l)$, and the proof of the theorem is complete.

## 3. Three Special Cases

Let $\Theta$ respectively $\Theta^0$ be the set of all generators on $X$ with nonnegative respectively positive off-diagonal entries. For $\theta \in \Theta^0$, the (Kullback-Leibler) information distance between $\theta$ and $\gamma$ is given by (see Hornik [6])

$$K(\theta, \gamma) = \sum_{i,j:i\neq j} \pi_i(\theta)\, H(\theta_{ij}, \gamma_{ij}),$$

where $\pi(\theta)$ is the (uniquely determined) stationary distribution of $\theta$. .

Let the mapping $\ell : \mathcal{Z} \times \mathcal{P} \to \Theta$ be defined by letting the $(i,j)$-th entry, $i \neq j$, of $\ell(z,p)$ be $z_{ij}/p_i$ if $p_i > 0$ and zero otherwise (and letting $\ell_{ii}(z,p) = -z_{i\cdot}/p_i$ if $p_i > 0$ and zero otherwise to make $\ell(z,p)$ a generator). Then $\hat{\theta}(T) = \ell(\hat{z}(T), \hat{p}(T))$ is the so-called standard maximum likelihood-estimator of the underlying generator from a sample $(\xi(t), 0 \leq t \leq T)$. If $\mathcal{R}^0$ denotes the interior of $\mathcal{R}$, i.e. the set of all pairs $(z,p) \in \mathcal{Z} \times \mathcal{P}$ such that all $z_{ij}$ and all $p_i$ are positive, then $\ell$ is a bijection between $\mathcal{R}^0$ and $\Theta^0$, and, as clearly

$$\sum_j p_j \ell_{ji}(z,p) = \sum_{j:j\neq i} z_{ji} - z_{i\cdot} = z_{\cdot i} - z_{i\cdot} = 0,$$

$\pi(\ell(z,p)) = p$ and thus

$$S(z,p) = K(\ell(z,p), \gamma).$$

Unfortunately, $\ell$ is not continuous (or lower semicontinuous) at the boundary of $\mathcal{R}$; hence, with the aid of our theorem, we can only describe the asymptotic behavior of $\mathbf{P}\{\hat{\theta}(T) \in \Delta\}$ for subsets $\Delta$ of $\Theta^0$. Using different methods, the tail behavior of $\hat{\theta}(T)$ has completely been described in Hornik [7] where it was shown that $\hat{\theta}(T)$ "basically" satisfies a large deviation principle with a rate function which is given by

$$K(\theta, \gamma) = \min_{p:\theta' p = 0} \sum_{i,j:i\neq j} p_i \, H(\theta_{ij}, \gamma_{ij})$$

and is the unique lower semicontinuous extension of $K(\theta,\gamma) = S(\ell^{-1}(\theta))$ from $\Theta^0$ to $\Theta$. It is quite remarkable to mention that the level sets of this rate function are not necessarily bounded.

Next, let us consider linear combinations of the form $Q_{w,b}(\hat{z}(T), \hat{p}(T))$. The statistics $Q_{w,b}$ play an important role in statistical inference for finite Markov processes. Choosing e.g. $w_{ij} = 1$ and $b = 0$ we get $T^{-1}$ [total number of jumps before time $T$], and, more general, choosing $b = 0$ and $w_{ij} = 1$ if $i \in Y$, $j \in Z$ (where $Y$ and $Z$ are subsets of $X$) and 0 otherwise, we get $T^{-1}$ [total number of jumps out of $Y$ into $Z$ before time $T$]. The choice $w_{ij} = \log(\theta_{ij}/\delta_{ij})$, $b_i = \theta_{ii} - \delta_{ii}$ gives $T^{-1}$ [log-likelihood ratio between $P_{T,\theta}$ and $P_{T,\delta}$], where $P_{T,\theta}$ denotes the probability law of $(\xi(t), 0 \leq t \leq T)$ with generator $\theta$. Finally, taking $w = 0$ gives

$$Q_{0,b}(\hat{z}(T), \hat{p}(T)) = \sum_{i=1}^{n} b_i \hat{p}_i(T) = T^{-1} \int_0^T b(\xi(t))dt,$$

which is the empirical mean of the function $b = [b_i, i \in X]'$ based on the sample $(\xi(t), 0 \le t \le T)$.

COROLLARY 1. *$Q_{w,b}(\hat{z}(T), \hat{p}(T))$ satisfies a large deviation principle with rate function*

$$S_{w,b}(u) = \inf_{Q_{w,b}(z,p)=u} S(z,p).$$

*If $\mathcal{U}_{w,b}$ denotes the image of $\Theta^0$ under the mapping $Q_{w,b} \circ \ell^{-1}$, then for all $u \notin \partial \mathcal{U}_{w,b}$ (and thus in particular for all $u$ if $\mathcal{U}_{w,b} = I\!R$),*

$$S_{w,b}(u) = \inf \left\{ K(\theta, \gamma) : \theta \in \Theta^0, Q_{w,b}(\ell^{-1}(\theta)) = u \right\}.$$

PROOF. The first assertion follows immediately by an application of the contraction principle (see e.g. Varadhan [8]), and it remains to establish the formula for the rate function.

To start with, let $(z,p) \in \mathcal{R}$ and $\theta \in \Theta^0$ be such that $Q_{w,b}(z,p) = u$ and $Q_{w,b}(\ell^{-1}(\theta)) = v$. For $0 \le \epsilon \le 1$, define $z(\epsilon) \in \mathcal{Z}$ and $p(\epsilon) \in \mathcal{P}$ by means of

$$\begin{aligned} z_{ij}(\epsilon) &= (1-\epsilon)\, z_{ij} + \epsilon\, \pi_i(\theta)\theta_{ij} \\ p_i(\epsilon) &= (1-\epsilon)\, p_i + \epsilon\, \pi_i(\theta), \end{aligned}$$

and let $\delta(\epsilon) = \ell(z(\epsilon), p(\epsilon))$. Then for all $\epsilon \in (0,1]$, $\delta(\epsilon) \in \Theta^0$, $\pi(\delta(\epsilon)) = p(\epsilon)$, $Q_{w,b}(\ell^{-1}(\delta(\epsilon))) = Q_{w,b}(z(\epsilon), p(\epsilon)) = (1-\epsilon)u + \epsilon v$, and $\lim_{\epsilon \to 0} K(\delta(\epsilon), \gamma) = S(z,p)$.

Let us write $R_{w,b}(u) = \inf\{K(\theta,\gamma) : \theta \in \Theta^0, Q_{w,b}(\ell^{-1}(\theta)) = u\}$. If $u \in \mathcal{U}_{w,b}$, we may take $v = u$ in the above construction to conclude that for all $(z,p) \in \mathcal{R}$ such that $Q_{w,b}(z,p) = u$, $S(z,p) \ge R_{w,b}(u)$ and thus $S_{w,b}(u) \ge R_{w,b}(u)$; on the other hand, it is trivial that $R_{w,b}(u) \ge S_{w,b}(u)$, hence $S_{w,b}(u) = R_{w,b}(u)$ for all $u \in \mathcal{U}_{w,b}$. If $u$ is in the complement of the closure of $\mathcal{U}_{w,b}$, then by definition no $\theta \in \Theta^0$ exists such that $Q_{w,b}(\ell^{-1}(\theta)) = u$, and the above construction shows that there is also no $(z,p) \in \mathcal{R}$ such that $Q_{w,b}(z,p) = u$, hence, in this case, $S_{w,b}(u) = \infty = R_{w,b}(u)$.

Finally, we have the following result. For vectors $u = [u_i, i \in X]'$ let us write $u > 0$ iff $u_i > 0$ for all $i$.

COROLLARY 2. *$\hat{p}(T)$ satisfies a large deviation principle with rate function*

$$I(p) = \inf_{z \in \mathcal{Z}} S(z,p) = -\inf_{u>0} \sum_i p_i \frac{(\gamma u)_i}{u_i}.$$

*If $p > 0$, then $I(p)$ is the information distance between the set of all generators in $\Theta^0$ with stationary distribution $p$ and the underlying generator $\gamma$.*

PROOF. Again by an application of the contraction principle, it is immediate that $\hat{p}(T)$ satisfies a large deviation principle with rate function $I(p) = \inf_{z \in Z} S(z,p)$. That $I(p)$ equals $J(p) := -\inf_{u>0} \sum_i p_i(\gamma u)_i/u_i$ follows from results in Donsker and Varadhan [2], but can also very easily be established directly. On the one hand, for arbitrary $u > 0$ and $(z,p) \in \mathcal{R}$,

$$\begin{aligned} S(z,p) &+ \sum_i p_i \frac{(\gamma u)_i}{u_i} \\ &= \sum_{i:p_i>0} \sum_{j:j\neq i} \left( z_{ij} \log \frac{z_{ij}}{p_i \gamma_{ij}} - z_{ij} + p_i \gamma_{ij} \frac{u_j}{u_i} \right) \\ &= \sum_{i:p_i>0} \sum_{j:j\neq i} \left( z_{ij} \log \frac{z_{ij} u_i}{p_i \gamma_{ij} u_j} - z_{ij} + p_i \gamma_{ij} \frac{u_j}{u_i} \right) \\ &= \sum_{i:p_i>0} \sum_{j:j\neq i} p_i H(z_{ij}/p_i, \gamma_{ij} u_j/u_i) \\ &\geq 0 \end{aligned}$$

and thus $I(p) \geq J(p)$. On the other hand, as (cf. the proof of the theorem)

$$b'p - \lambda(A(0,b)) = -\sum_{i,j} p_i \gamma_{ij} \frac{r_j(0,b)}{r_i(0,b)} \leq J(p)$$

and clearly $I(p) = \sup_b [b'p - \lambda(A(0,b))]$, we have $I(p) = J(p)$ as asserted.

Finally, let $p > 0$ and consider an arbitrary $z \in Z$ such that $z_{i\cdot} = z_{\cdot i}$ for all $i$. If we choose some $\theta \in \Theta^0$ with $\pi(\theta) = p$ and apply the construction in the proof of corollary 1, we obtain a sequence $\delta(\epsilon) \in \Theta^0$ such that $\pi(\delta(\epsilon)) = p$ and $S(p,z) = \lim_{\epsilon \to 0} K(\delta(\epsilon), \gamma)$, whence $I(p) = \inf_{z \in Z} S(z,p) = \inf_{\delta \in \Theta^0 : \pi(\delta) = p} K(\delta, \gamma)$.

CONCLUDING REMARKS. If the underlying generator $\gamma$ is not in $\Theta^0$, but ergodic in the sense that all states in $X$ communicate, the theorem continues to hold, with $\mathcal{R}$ now the set of all pairs $(z,p) \in Z \times \mathcal{P}$ such that $z_{i\cdot} = z_{\cdot i}$ for all $i$ and $z_{ij} = 0$ whenever $p_i \gamma_{ij} = 0$.

Finally, as pointed out by Pál Révész, it should be possible to generalize the results of this paper to the case of a *countable* state space $X$, at least when $\|\gamma\| := \sum_{i \neq j} \gamma_{ij} < \infty$. This question will be investigated elsewhere.

## REFERENCES


[1] ALBERT, A. (1962). Estimating the infinitesimal generator of a continuous time, finite state space Markov process. *Ann. Math. Statist.*, **33**, 727—753.

[2] DONSKER, M. D., and VARADHAN, S. R. S. (1975). Asymptotic evaluation of certain Markov process expectations for large time, I. *Comm. Pure Appl. Math.*, **28**, 1—47.

[3] ELLIS, R. S. (1985). *Entropy, large deviations and statistical mechanics.* Grundlehren der mathematischen Wissenschaften **271**. Springer, Berlin–New York.

[4] GÄRTNER J. (1977). On large deviations from the empirical measure. *Theory Prob. Applications*, **22**, 24—39.

[5] GRENANDER, U. (1981). *Abstract inference.* Wiley, New York.

[6] HORNIK, K. (1988). On asymptotically optimal tests. *J. Statist. Plann. Inference*, **20**, 131—154.

[7] HORNIK, K. (1990). Exponential rate optimal estimation of the generator of a finite Markov process. *Statistics and Decisions*, **8**, 101—113.

[8] VARADHAN, S. R. S. (1984). *Large deviations and applications.* Regional Conference Series in Applied Mathematics **46**. SIAM, Philadelphia, PA.


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# RÉNYI AND THE COMBINATORIAL SEARCH PROBLEMS

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## 1. Introduction

The whole story has started with the Hillman, Rényi's car. It was a kind of a member of his family. That time, in the early sixties, very few people had cars in Hungary, the gasoline was cheap, there was no parking problem. Once he gave me a ride from the Institute to the cafeteria, less than 200 meters. One day, however, the Hillman stopped its smooth services. Obviously, there was some electrical problem with it. The electrician, however, was unable to find the source of the trouble. Rényi had to find it, himself. He has found it, and in the mean-time he has developed a general mathematical model for the situation.

The car can be considered as a finite set of its parts. The car does not work since one (hopefully exactly one, so we suppose it) of its parts does not work properly. When trying to find it, tests are performed. One test tries to function a subset of the parts. If it does not work then the defective part is in this subset, otherwise it is not contained in it. After performing several such tests we have to determine the defective part.

Let us formulate it a little more mathematically. A finite set $X$ of $n$ elements is given. A distinguished element $x$ of $X$ is given but it is unknown by us. Furthermore, a family $\mathcal{A}$ of subsets of $X$ is given. We can ask the questions if "$x$ is in $A$" or not, for members $A$ of $\mathcal{A}$. We have to identify $x$ on the basis of the answers for the above questions. We call the members of $\mathcal{A}$ ***question sets***. They are the potential questions.

There are two basically different models. If the sequence $A_1, \ldots, A_m$ of actual questions (a subfamily of $\mathcal{A}$) is fixed in advance then we say that this is a ***linear search***. The obvious mathematical aim is to minimize the number $m$ of questions. On the other hand, the choice of the next question may depend on the answers to the previous questions. The first question set is

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$A \in \mathcal{A}$. If the answer is $x \notin A$ then the second question set is $A_0 \in \mathcal{A}$ while in the case of the answer $x \in A$ the second question set is $A_1 \in \mathcal{A}$. The third question set is $A_{00} \in \mathcal{A}$, $A_{01} \in \mathcal{A}$, $A_{10} \in \mathcal{A}$ and $A_{11} \in \mathcal{A}$, resp. depending on the two previous answers, etc. This is called a *tree search.* In this case the maximum number of questions (the length of the longest path of the tree) should be minimized. The tree search seems to be more practical, however the linear search is much simpler to organize, so in the era of fast computers it might be equally important.

With this model Rényi [27] initiated an area, the combinatorial search problems. He and his followers have written many papers. However his work was not the only source of these investigations. Now we show some other sources.

Let $X$ be a set of soldiers in World War II. A sample of blood is drawn from every person. The ones containing *syphilitic antigen* should be found using the *Wasserman test.* It was an original idea that the blood samples could be poured and tested together. In this way it can be decided if a certain subset of soldiers contains an infected person or not. This model is basicly identical with the previous one, the only difference is that the number of elements to be identified is not known, in advance. (See [10] and [31].) This kind of models are called group testing and considered to belong to Mathematical Statistics.

An even older question was raised by Steinhaus [30]. A set of $n$ table tennis players is given. Suppose that their abilities are constant, it can be described with a real number, there is no randomness, so the better one always defeats the weaker one. The aim is to determine their total order by pairwise comparisons, that is, table tennis matches. Although it is not clear at the first sight, this problem is also covered by the above model. Let $X$ be the set of $n!$ permutations of the players. One of these permutations is searched. One match determines if this unknown permutation belongs to the set of $n!/2$ permutations where player $a$ is better than player $b$.

So, one can say that the area has three sources (Fig. 1). The present author wrote a survey paper [16] containing 66 references, the book by Ahlswede and Wegener [1] has 166 references and finally the most recent summarization of the area, the book of Aigner [2] quotes 198 papers. These numbers show that the area became quite large, a small paper cannot survey it. Therefore the aim of the present paper is to survey those papers which were written (mostly by Hungarian authors) under a direct or an indirect influence of Rényi.

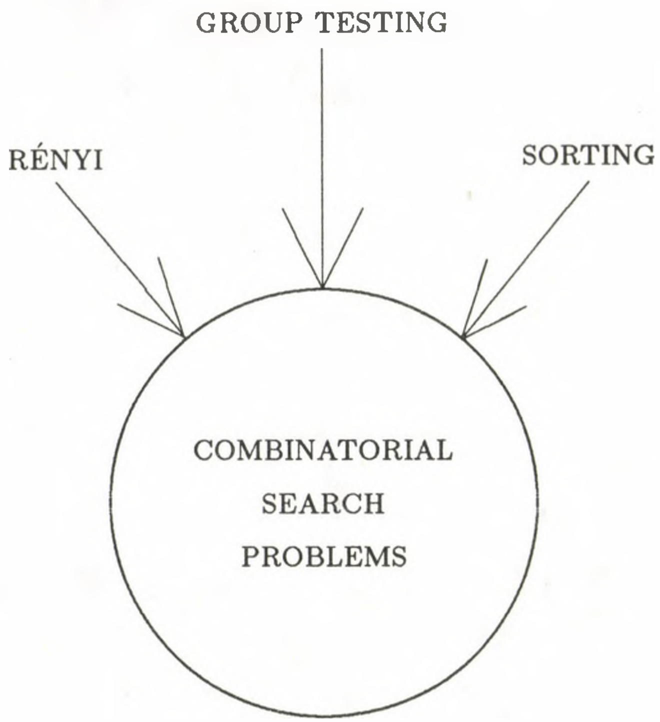


Fig. 1

## 2. Qualitatively independent sets and partitions

Let $A, B \subset X$ be two question sets. Suppose that they are disjoint. First ask if the unknown $x$ is in $A$ or not. If the answer is no, we have to ask $B$, as well. However, if the answer is yes then there is no need to ask $B$, we know that $x$ is not in $B$. If one of

$$A \cup B, \quad A \cup \overline{B}, \quad \overline{A} \cup B, \quad \overline{A} \cup \overline{B} \tag{1}$$

is empty, the situation is similar, that is, one of the possible answers to the first question makes the second question superfluous. We say that $A$ and $B$ are ***qualitatively independent*** if none of the sets in (1) is empty.

Rényi [28] raised the question what is the maximum of $|\mathcal{A}|$ on an $n$-element set if any two different members of $\mathcal{A}$ are qualitatively independent.

The answer is very easy for even $n$-s. It is easy to see that the sets in (1) are all non-empty iff there is no inclusion among the sets $A, \overline{A}, B, \overline{B}$.
A family $S$ of subsets is called a *Sperner family* iff there is no inclusion in it, that is, $C \not\subseteq D$ holds for any two distinct members of $S$. Using this notion, we can state that any two members of $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ are qualitatively independent if and only if $\mathcal{A}^* = \{A_1, A_2, \dots, A_m, \overline{A}_1, \overline{A}_2, \dots, \overline{A}_m\}$ is a Sperner family. However, the maximum size of a Sperner family is known:

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Therefore, if $\mathcal{A}$ is a family of pairwise qualitatively independent sets then $\mathcal{A}^*$ is a Sperner family and $2m$ is less than equal to the above binomial coefficient, so

$$m \leq \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This inequality is true for any $n$ but it is also sharp for even $n$-s, due to the following construction. Take all $n/2$-element subsets containing a fixed element $f$.

The odd case is non-trivial, but easy. It was independently discovered by many authors [17], [6], [7], [20].

THEOREM 1. *The maximum size of a family of pairwise qualitatively independent sets on $n$ elements is*

$$\binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}.$$

The construction coincides with the even case.

[20] also contains good estimates on a more general problem. We say that $r \geq 2$ sets are qualitatively independent, if they divide $X$ into $2^r$ non-empty sets. The maximum size of a family in which any $r$ sets are qualitatively independent is estimated.

One may consider a more general condition. If all the sets in (1) are of size at least $d$ then we say that $A$ and $B$ are *qualitatively independent of depth $d$*.

PROBLEM 1. What is the maximum size of a family of pairwise qualitatively independent sets of depth $p$ on $n$ elements?

It might be true that the obvious generalization of Theorem 1 holds for fixed $d$ and large $n$. The case when $d = cn$ seems to be hard.

In what follows, we will consider another generalization. Before that a further motivation will be presented. It can be considered as the fourth source of the area of combinatorial search problems. Given $n$ coins, one of them is counterfeit. It is known that the counterfeit coin is lighter than the good ones and it should be found by the minimum number of weighings using an equal arm balance. (No additional weights can be used.) The novelty in this problem is that one question (weighing) devides the groundset into three parts: the set of coins in the left arm, the set of coins in the right arm and the rest. In our earlier model the groundset was devided into two sets: the question set and its complement. This example of the equal arm balance suggests to introduce the notion of the *question partition*: $P = \{A_1, \ldots, A_r\}$, where $A_1, \ldots, A_r$ is a partition of the groundset $X$. The answer to this question determines the unique $i$ satisfying $x \in A_i$. In this case a family $\mathcal{P}$ of partitions is given and the partitions for a linear search or tree search are chosen from this $\mathcal{P}$. If the number of parts in a partition does not exceed $r$ we call it an *r-partition.*

The notion of the qualitatively independent partitions is straightforward. $\mathcal{P}_1$ and $\mathcal{P}_2$ are *qualitatively independent r-partitions* if they divide $X$ into $r^2$ non-empty subsets. One cannot expect that the exact maximum number $m(n,r)$ of the pairwise qualitatively independent $r$-partitions could be determined, only its exponent. Poljak and Tuza [25] proved that

$$\limsup \frac{1}{n} \log m(n,r) \leq \frac{2}{r}.$$

A recent great achievement is

THEOREM 2 (Gargano, Körner, Vaccaro [13]).

$$\limsup \frac{1}{n} \log m(n,r) = \frac{2}{r}.$$

One should mention the preliminary work of Körner and Simonyi [22].

The following result does not belong to this section, but it is a surprising development in the area of the counterfeit coin problem and this is the best point to mention it. Suppose that $n$ coins are given, $m \leq n$ of them are of weight $1-\varepsilon$ (counterfeit coins) the rest of them are of weight 1 (good coins). Find the shortest tree search determining all the counterfeit coins. As the number of possibilities is $\binom{n}{m}$ and one question has three different answers, the minimum number of questions (in the worst case) is at least $\log_3 \binom{n}{m}$. This lower estimate is not the best because we cannot always divide the possibilities into 3 equal parts. The exact result, in general, expected to be quite complicated as it depends on number theoretical properties of $n$ and $m$. The cases $m = 1, 2, 3, 4, 5$ are considered in [8], [32], [33], [34], [35].

It is surprising that the above lower estimate is still almost sharp in the following sense.

THEOREM 3 (Pyber [26]). *If exactly $m$ (lighter) conterfeit coins are to be found among $n$ coins then it can be done in at most*

$$\log_3 \binom{n}{m} + 15m$$

*steps in all cases.*

## 3. Optimal search with constraints on the sizes of question sets

Let us turn back to the original question. Suppose, again, that there is exactly one unknown element $x \in X$, the family of question sets is the family of all subsets in $X$ and a tree search is used. Let $k$ denote the length of the longest branch of the tree, that is, the number of questions in the worst case. Then the number of sequences of answers is at most $2^k$ since a sequence cannot be an extension of another one. For different $x$ the sequence must be different. Hence $2^k \geq n$ holds. This implies $k \geq \lceil \log_2 n \rceil$. Now we show a construction of a (very special) tree search whose length is $\lceil \log_2 n \rceil$. It will be a linear search. Label the elements of $X$ by $1, 2, \dots, n$. These labels can be written with $\lceil \log_2 n \rceil$ binary digits. Let $A_j$ consist of the elements of $X$ whose label's $j$th digit is 1. The answers to these questions determine all digits of the label of the unknown $x$. Thus we can formulate the following theorem.

THEOREM 4 (Folklore). *The minimum number of questions in a linear (tree) search to find the only unknown element in an $n$-element set is*

$$\lceil \log_2 n \rceil.$$

Rényi asked what the situation is if $\mathcal{A} = \{A : A \in X, |A| \leq k\}$. The situation is considerably different here. *E.g.* the results for the tree search and the linear search do not coincide any more. The case of the tree search is easier both to describe and to prove.

Observe that any question set can be replaced by its complement. Thus, $k \leq n/2$ can be supposed. Furthermore, if after some questions in a tree search it is known that the unknown $x$ is in a subset of size $s \leq 2k$ then $x$ can be found by $\lceil \log_2 s \rceil$ further questions, using Theorem 4. Now let us give an optimal tree search. Write $n$ in the form $n = qk + s$ where $k < s \leq 2k$ and take a partition $B_1, B_2, \ldots, B_q, B_{q+1}$ where $|B_1| = \ldots = |B_q| = k$, $|B_{q+1}| = s$. Ask $B_1, B_2, \ldots$ until the answer is "yes", $x \in B_i$ $(i \leq q)$. Then $x$ can be found by $\lceil \log_2 k \rceil$ further questions. If the first such case is $i = q + 1$ then we need $\lceil \log_2 s \rceil$ more questions. It is not hard to prove that this is (one of) the best tree search [16].

THEOREM 5. *Suppose that the question sets are the subsets of size at most $k \leq n$. Then the shortest tree search needs*

$$\left\lceil \frac{n}{k} \right\rceil - 2 + \left\lceil \log_2 \left( n - k \left\lceil \frac{n}{k} \right\rceil + 2k \right) \right\rceil .$$

When, in his seminar, Rényi asked what the minimum number of questions in a tree search is, many students (B. Bollobás, J. Galambos, T. Nemetz and D. Szász) brought the solution for the next seminar for the case $\frac{k(k+1)}{2} + 1 \leq n$. Then the result is $\lceil 2\frac{n-1}{k+1} \rceil$. The present author has constructed ([15]) the best linear search for all $k$. This leads to the following estimates.

THEOREM 6. *Denote by $l(n, k)$ the minimum length of a linear search finding an unknown element in an $n$-element set using question sets of size at most $k$. Then*

$$\frac{\log n}{\frac{k}{n} \log \frac{n}{k} + \frac{n-k}{n} \log \frac{n}{n-k}} \leq l(n, k) \leq \left\lceil \frac{\log 2n}{\log \frac{n}{k}} \right\rceil \cdot \frac{n}{k}.$$

(See also [23] and [37].) As it is pointed out by Dyachkov, the lower estimate is asymptotically sharp when $k = cn$.

Baranyai [5] generalized the construction for $r$-partitions whose parts (except the last one) are bounded in size. Proving this result he proved a "small lemma" which turned out to be a 120 year old conjecture of Sylvester:

THEOREM 7 (Baranyai [4]). *Suppose that $k$ divides $n$. Then the set of all $k$-element subsets of the $n$-element set $X$ can be partitioned into such classes that each class forms a partition of $X$.*

Although this famous result does not belong to the Combinatorial Search Theory, it was created under an indirect influence of Rényi. It is obvious to ask what can be said if $k$ does not divide $n$. To formulate a conjecture concerning this general case, a new definition is needed. Suppose that the elements of the groundset $X$ are ordered: $X = \{x_1, x_2, \dots, x_n\}$. Define $A_i = \{x_{(i-1)k+1}, x_{(i-1)k+2}, \dots, x_{(i-1)k+k}\}$ where the indices are considered mod $n$. The family $A_1, A_2, \dots, A_w$ where $w = n/\gcd(n,k)$ is called a ***wreath***. (Neighboring $k$-element subsets are taken after each other until the end of one fits to the beginning of the first set.)

CONJECTURE (Baranyai and Katona). There are permutations of the groundset in such a way that these permutations of the wreath give all $k$-element sets exactly once.

It seems to be hard to settle this conjecture. Sylvester's conjecture was earlier attacked by algebraic methods and an algebraic way of thinking. Baranyai's brilliant idea was to use matrices and flows in networks. This conjecture, however, seems to be too algebraic. One does not expect to solve it without algebra. (Unless it is not true.)

Let us turn back to the search problems with restrictions on $\mathcal{A}$. We will use the problem of Steinhaus to obtain motivations. The problem actually became an important problem of computer science (with numbers rather than table tennis players). It is the simplest one of the so called sorting problems (see [21]). It is obvious by Theorem 4 that a tree search to find the actual permutation needs at least $\lceil \log_2 n! \rceil$ pairwise comparisons. This is $n \log_2 n + c_1 n + o(n)$. The tree search given by Ford and Johnson [12] (which is belived to be the best) needs $n \log_2 n + c_2 n + o(n)$ steps. That is, the first term is determined, but not the second one. Let us see the reason why the lower estimate $\lceil \log_2 n! \rceil$ cannot be realized by a tree search. To reach this bound we have to halve at each stage the set of possible cases, therefore the question sets should divide the groundset (of permutations) into 4 equal parts. This is, however not always possible. Consider the comparisons $a <?< b$ and $b <?< c$. Denote by $A$ and $B$ the set of permutations giving positive answer to the first and the second question, resp. Then two of the sets in (1) have size $n!/3$ and the other two have size $n!/6$. This is the motivation to the following investigations.

THEOREM 8 ([18].) *The minimum length of a linear search using question sets satisfying*

$$|A \cap B| \leq 1$$

*is*

$$\left\lceil \frac{\sqrt{8n-7}-1}{2} \right\rceil.$$

Let us mention that this strange formula is equal to the smallest $m$ such that

$$n \le 1 + m + \binom{m}{2}.$$

[18] also gives the exact minimum up to an additional constant 1 for the case $|A \cap B| \le 2$. Fairly good estimates are given for the cases $|A \cap B| \le k$ where $k \le c\sqrt[3]{n}$.

For the case of tree search let us start with an observation of Sebő [29]. If the first question set is $A$ and the answer is $x \in A$ and the second question set is $A_1$ then it can be replaced by $A \cap A_1$. On the other hand, when $x \notin A$ then $A_0$ can be replaced by $A \cap A_0$. Continuing in this way we obtain a modified tree search where the question sets on different branches of the tree are disjoint while the ones along the same branch are in inclusion. Of course the lengths of the branches are unchanged. Thus, when looking for the shortest tree search, this strong property may be supposed. *E. g.* if $k = 1$ then the original condition becomes simply the condition that all question sets, with exception of the very first one, are of size 1.

Sebő [29] has determined the length of the shortest tree search under the condition $|A \cap B| \le k$ for all $k < n/4$, however the formula is rather complicated so we give only the case $k = 1$ here.

THEOREM 9 (Sebő [29]). *The minimum length of a tree search using question sets satisfying*

$$|A \cap B| \le 1$$

*is*

$$\left\lceil \frac{\sqrt{8n-7}-1}{2} \right\rceil.$$

Compare it with Theorem 8. The best linear search is not longer than the best tree search, in this case. For $k = 2$, however, the former one is about $\sqrt{3/2}$-times larger than the latter one.

[29] contains good estimates also for the case when the intersection of any $m$ question sets is bounded.

The combination of the previous two types of constraints has not been studied, yet:

PROBLEM 2. Determine the length of the shortest linear and tree searchs, resp., under the conditions

$$|A| \leq k, \ |A \cap B| \leq l \quad \text{for all} \quad A, B \in \mathcal{A}.$$

Let us see the situation at the search of a permutation by binary comparisons. We observed that the comparisons $a <? < b$ and $b <? < c$ imply the existence of two question sets dividing the set of permutations into four parts containing one third, one third, one sixth, one sixth of the whole set, instead of the "good" proportion one fourth, one fourth, one fourth, one fourth. However, this is not true for all pairs of questions! What we can say is that among any $\frac{n}{2} + 1$ questions there is one such pair. One way of expressing the fact that two question sets are not intersecting each other in a "good" proportion is the use of the *entropy function* of the Information Theory. The *entropy of the partition* $A_1 \cup A_2 \cup A_3 \cup A_4$ of $X$ is

$$\sum_{i=1}^{4} \frac{|A_i|}{|X|} \log_2 \frac{|X|}{|A_i|}.$$

This expression is 2 for the case when $A_i = \frac{1}{4}|X|$. It is known from Information Theory that it is smaller in all other cases. This suggests the following problem.

PROBLEM 3. Determine the length $f(t, E)$ of the shortest linear search under the condition that among any $t$ question sets there is a pair such that the 4-partition induced by them has an entropy at most $E$.

We have only estimates. To formulate them some more definitions are needed. Put $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$. The inverse of $h$ is defined using the interval $0 \leq x \leq 1/2$ where it is monotone.

THEOREM 10.

$$\frac{2}{E} \log_2 n - (t-1)\frac{2-E}{E} \leq f(t, E) \leq \frac{2}{E} \log_2 n + O(\log_2 n).$$

PROOF. Start with the lower estimate. Let $\xi$ be a random variable taking on values from $X$. Define the probabilities to be equal: $P(\xi = x) = 1/n$. Denote the question sets by $A_1, \ldots, A_m$. They define some further random variables:

$$\xi_i = \begin{cases} 0 & \text{if} \quad \xi \notin A_i \\ 1 & \text{if} \quad \xi \in A_i. \end{cases}$$

Now define the *entropy* of a random variable $\eta$ taking on its different values with probabilities $p_1, \ldots, p_n$:

$$H(\eta) = -\sum_{i=1}^{n} p_i \log p_i.$$

Observe that $\xi$ determines the random vector $(\xi_1, \ldots, \xi_m)$. On the other hand, as the answers to the questions "$x \in A_i$" determine $x$, therefore the converse is also true, $(\xi_1, \ldots, \xi_m)$ determines $\xi$. Consequently the distribution of the two random variables are identical and

$$H(\xi) = H((\xi_1, \ldots, \xi_m)) = \log_2 n. \tag{2}$$

Here we need an elementary lemma from Information Theory (see any textbook on Information Theory, *e.g.* [11], [9]):

$$H((\eta_1, \eta_2)) \leq H(\eta_1) + H(\eta_2). \tag{3}$$

(2) and (3) imply

$$\log_2 n \leq H((\xi_1, \xi_2)) + H(\xi_3) + H((\xi_4, \xi_7)) + \ldots \tag{4}$$

for any partition of the set $\{\xi_1, \ldots, \xi_m\}$ into one and two-element subsets. If they are all one-element sets then (4) leads to $\log_2 n \leq m$, only, since the entropy of one $\xi_i$ is bounded by one. However if we find $M$ such disjoint pairs that $H((\xi_i, \xi_j)) \leq E(< 2)$ then (4) results in

$$\log_2 n + M(2 - E) \leq m. \tag{5}$$

To find the best $M$, the problem will be reformulated for graphs in an obvious way. Define a graph whose vertices are $\xi_i$ and two vertices are connected iff $H((\xi_i, \xi_j)) \leq E$. The following graph theoretic lemma is needed:

LEMMA. *Given a simple graph $G$ on $m$ vertices, there is at least one edge among any $t$ vertices. Then it contains at least*

$$\left\lceil \frac{m - t + 1}{2} \right\rceil \tag{6}$$

*vertex-disjoint edges in $G$. This result is sharp.*

PROOF. Take the largest set $L$ of vertex-disjoint edges. Let $|L| = l$. If $m - 2l \geq t$ then there are $t$ vertices not contained in any member of $L$. By the conditions of the lemma there is an edge among these $t$ edges which is vertex-disjoint to the members of $L$. This contradicts the maximality of $L$. The contradiction proves $m - 2l < t$ and the first part of the lemma.

Now consider the graph $G$ of $m$ vertices consisting of a complete graph on $m - t + 2$ vertices and isolated vertices. This graph obviously satisfies the conditions of the lemma and cannot contain more vertex-disjoint edges than (6). □

The lemma and (5) imply

$$\log_2 n + \frac{m - t + 1}{2}(2 - E) \leq m$$

and the lower estimate in Theorem 10.

The upper estimate will be proved by a simple construction. Define $k = \lfloor nh^{-1}(E/2) \rfloor$. Question sets of size at most $k$ will be used. Then, by the monotonity of $h(x)$, we have $H(\xi_i) \leq h(k/n) \leq h(h^{-1}(E/2)) = E/2$. (3) implies $H(\xi_i, \xi_j) \leq E$, as needed.

Use the construction mentioned after Theorem 6 as $k/n$ is a constant. Then the lower estimate is sharp in Theorem 6. It gives the upper estimate

$$\frac{\log n}{h(h^{-1}(E/2))} + O(\log n)$$

which coincides with the one given in the Theorem. □

Let us have some remarks concerning this Theorem.

1. It gives an approximate solution to the problem of Theorem 8 in a new case.

2. Problem 3 and Theorem 10 are not intended to help finding the shortest linear search for a permutation by pairwise comparisons. It is a trivial problem, one has to compare all $\binom{n}{2}$ pairs. However the solution of the analogous problem for the tree search might give a better lower estimate on the permutation problem.

## 4. Miscellany

As it was mentioned earlier, a tree search needs $n \log_2 n + O(n)$ steps to find the proper permutation of $n$ objects (numbers) by pairwise comparisons. Modern computers have complex hardwares able to execute many operations simultanously. This is called parallel computation. As one object can be used only in one comparison at each moment, not more than $n/2$ parallel operations are possible. Therefore at least $O(\log n)$ steps are needed even if parallel steps are allowed. Ajtai, Komlós and Szemerédi [3] proved that this can really be done in this many steps.

Á. Varecza has proved many interesting results of sorting type. Let us mention only one of them here. Let $x_1, \ldots, x_{n-2}, y_1, y_2$ be distinct integers. It should be decided by pairwise comparisons if $y_1$ are $y_2$ neighboring in the natural order of all these numbers. It is easy to see that $2(n-2)$ steps are enough. However it is not at all trivial to prove that there is no shorter tree algorithm. It is proved in [36].

If there are more unknown elements in $X$ then one question set $A$ may give different answers. One of the natural models is when there are two possible answers: either a) there is one unknown element in $A$ or b) there is none. Hwang and Vera Sós [14] gave good estimates for the minimum length of a linear search when the number $d$ of unknown elements is known in advance.

J. N. Srivastava's following idea connected search theory with the theory of statistical factor analysis. The usual aim of factor analysis is (roughly speaking) to determine the weights of the influence of different factors for the investigated quantity. Now suppose that there are many possible factors, very few of them have a real influence the other ones have no influence (or are negligable). However it is not known which ones are the non-negligable. Thus in this model two things are done simultanously: a) determination of the non-negligable factors, b) determination of the weights of them. [19] contains results on the tools used to solve this problem, the so called *search designs*. These investigations led to the problem of finding the largest family $S$ closed under the operation symmetric difference and such that $S - \{\emptyset\}$ is a Sperner family [19]. Generalizations can be found in [24].

## REFERENCES

[1] AHLSWEDE, R. and WEGENER, I., *Suchproblemen*, Teubner Studienbücher, Mathematik, Teubner, Stuttgart, 1979. *MR* **81d**: 94022; and *Search Problems*, John Wiley, Chichester, 1987. *MR* **89c**: 68019

[2] AIGNER, M., *Combinatorial Search*, John Wiley, Chichester and Teubner, Stuttgart, 1988. *MR* **89k**: 68021

[3] AJTAI, M., KOMLÓS, J. and SZEMERÉDI, E., Sorting in $c \log n$ parallel steps, *Combinatorica* **3** (1983), 1—19. *MR* **85d**: 68017

[4] BARANYAI, ZS., On the factorization of the complete uniform hypergraph, *Infinite and finite sets* (Colloq., Keszthely, 1973), Vol. I, Colloq. Math. Soc. J. Bolyai **10**, North-Holland, Amsterdam, 1975, 91—108. *MR* **54** # 5047

[5] BARANYAI, ZS., On a search problem of G. O. H. Katona, *Combinatorics* (Proc. Fifth Hvngarian Colloq., Keszthely, 1976), Vol. I, Colloq. Math. Soc. J. Bolyai **18**, North-Holland, Amsterdam-New York, 1978, 61—76. *MR* **80d**: 05002

[6] BOLLOBÁS, B., Sperner systems consisting of pairs of complementary subsets, *J. Combinatorial Theory Ser. A* **15** (1973), 363—366. *MR* **48** # 5867

[7] BRACE, B. A. and DAYKIN, D. E., Sperner type theorems for finite sets, *Combinatorics* (Proc. Conf. Combinatorial Math., Oxford, 1972), Inst. Math. Appl., Southend-on-Sea, 1972, 18—37. *MR* **49** # 10561

[8] CAIRNS, S. S., Balance scale sorting, *Amer. Math. Monthly* **70** (1963), 136—148. *MR* **26** # 4929

[9] CSISZÁR, I. and KÖRNER, J., *Information theory. Coding theorems for discrete memoryless systems*, Akadémiai Kiadó, Budapest, 1981. Academic Press, New York-London, 1981. *MR* **84e**: 94007

[10] DORFMAN, R., The detection of defective members of large populations, *Ann. Math. Statist.* **14** (1943), 436—440.

[11] FEINSTEIN, A., *Foundations of information theory*, McGraw-Hill, New York, 1958. *MR* **20** # 1594

[12] FORD, L. R. and JOHNSON, S. M., A tournament problem, *Amer. Math. Monthly* **66** (1959), 387—389. *MR* **21** # 1942

[13] GARGANO, L., KÖRNER, J. and VACCARO, U., Sperner capacities, *Graphs Combin.*, (to appear).

[14] HWANG, F. K. and SÓS, V. T., Nonadaptive hypergeometric group testing, *Studia Sci. Math. Hungar.* **22** (1987), 257—263. *MR* **89c**: 62133

[15] KATONA, G., On separating systems of a finite set, *J. Combinatorial Theory* **1** (1966), 174—194. *MR* **33** # 3947

[16] KATONA, G. O. H., Combinatorial search problems, *Survey of Combinatorial Theory* (Proc. Internat. Sympos. Colorado State Univ., Fort Collins, Colo., 1970), ed. by J. N. Srivastava et al., North - Holland, Amsterdam, 1973, 285—308. *MR* **51** # 4976

[17] KATONA, G. O. H., Two applications (for search theory and truth functions) of Sperner type theorems, *Period. Math. Hungar.* **3** (1973), 19—26. *MR* **49** # 53

[18] KATONA, G. O. H., Search using sets with small intersection, *Théorie de l'information* (Proc. Internat. CNRS Colloq., Cachan, 1977), Colloq. Internat. CNRS, No. 276, CNRS, Paris, 1978, 381—388. *MR* **82b**: 94017

[19] KATONA, G. O. H. and SRIVASTAVA, J. N., Minimal 2-coverings of a finite affine space based on GF(2), *J. Statist. Plann. Inference* **8** (1983), 375—388. *MR* **85i**: 05056

[20] KLEITMAN, D. J. and SPENCER, J., Families of $k$-independent sets, *Discrete Math.* **6** (1973), 255—262. *MR* **48** # 1934

[21] KNUTH, D. E., *The art of computer programming*, Vol. 3, Addison-Wesley, Reading, Mass. Menlo Park, Calif.-London-Don Mills, Ont., 1973. *MR* **56** # 4281

[22] KÖRNER, J. and SIMONYI, G., Sperner type theorems and qualitative independence, *J. Combin. Theory*, (to appear).

[23] LUZGIN, V. N., Separeting systems of partitions of a finite set (in Russian), *Combinatorial analysis*, No 5, Moskov. Gos. Univ., Moscow, 1980, 39—45. *MR* **85h**: 05011

[24] MIKLÓS, D., Linear binary codes with intersection properties, *Discrete Appl. Math.* **9** (1984), 187—196. *MR* **85m**: 94003

[25] POLJAK, S. and TUZA, ZS., On the maximum number of qualitatively independent partitions, *J. Combin. Theory Ser. A* **51** (1989), 111—116. *MR* **90c**: 05007

[26] PYBER, L., How to find many counterfeit coins?, *Graphs Combin.* **2** (1986), 173—177. *MR* **89c**: 05003

[27] RÉNYI, A., On the theory of random search, *Bull. Amer. Math. Soc.* **71** (1965), 809—828. *MR* **32** # 3941

[28] RÉNYI, A., *Foundations of Probability*, Holden-Day, San Francisco, 1970. *MR* **41** # 9314

[29] SEBŐ, A., Sequential search using question-sets with bounded intersections, *J. Statist. Plann. Inference* **7** (1982/83), 139—150. *MR* **84m**: 94021

[30] STEINHAUS, H., *Mathematical Snapshots*, Oxford University Press, New York, 1950. *MR* **12** - 44

[31] STERRETT, A., On the detection of defective members of large populations, *Ann. Math. Statist.* **28** (1957), 1033—1036. *Zbl* **79**, 355

[32] TOŠIĆ, R., Two counterfeit coins, *Discrete Math.* **46** (1983), 295—298. *MR* **85d**: 05011

[33] TOŠIĆ, R., Three counterfeit coins, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.* **15** (1985), 225—233. *MR* **87h**: 90257

[34] TOŠIĆ, R., Four counterfeit coins, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.* **14** (1984), 99—108. *MR* **86e**: 90064

[35] TOŠIĆ, R., Five counterfeit coins, *J. Statist. Plann. Inference* **22** (1989), 197—202. *MR* **90i**: 90109

[36] VARECZA, Á., Are two given elements neighbouring? *Discrete Math.* **42** (1982), 107—117. *MR* **84c**: 68050

[37] WEGENER, I., On separating systems whose elements are sets of at most $k$ elements, *Discrete Math.* **28** (1979), 219—222. *MR* **81b**: 05019

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# WAITING FOR THE COVERAGE OF THE STRASSEN'S SET

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**Abstract**

Let $\{W(t),\ t \geq 0\}$ be a Wiener process and for any $\varepsilon > 0$ define

$$\nu(t,\varepsilon) = \inf\{s :\ s \geq t,\ W(s) \geq ((1-\varepsilon)2s \log\log s)^{1/2}\}.$$

The limit properties of $\nu(t,\varepsilon)$ $(t \to \infty)$ and some other similar waiting times are investigated.

## 1. Erdős - Rényi strong law of large numbers

Let $\{W(t),\ t \geq 0\}$ be a Wiener process and let $a(t) \leq t$ $(t \geq 0)$ be a real valued function satisfying the conditions

(i) $a(t)$ is a positive nondecreasing function,

(ii) $t^{-1}a(t)$ is a nonincreasing function.

Further let

$$I(T, a(T)) = \sup_{0 \leq t \leq T - a(T)} (W(t + a(T)) - W(t)).$$

Then a special case of the Erdős - Rényi strong law of large numbers [5] runs as follows

THEOREM A. *For any $C > 0$ we have*

$$\lim_{T\to\infty} \frac{I(T, C\log T)}{\log T} = (2C)^{1/2} \quad a.s.$$

A generalization of this special case is the following

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THEOREM B. (Csörgő - Révész, [2].) *Assume that* $a(\cdot)$ *satisfies conditions* (i) *and* (ii). *Then*

$$\limsup_{T\to\infty} \gamma(T,a(T))I(T,a(T)) = 1 \quad a.s. \tag{1.1}$$

*where*

$$\gamma(T,a(T)) = \left(2a(T)\left(\log\frac{T}{a(T)} + \log\log T\right)\right)^{-1/2}.$$

*If* $a(T)$ *also satisfies the condition*

(iii) $\lim_{T\to\infty} \dfrac{\log(T(a(T))^{-1})}{\log\log T} = \infty$

*then*

$$\lim_{T\to\infty} \gamma(T,a(T))I(T,a(T)) = 1 \quad a.s. \tag{1.2}$$

Note that in case $a(T) = C\log T$ we obtain Theorem A as a special case and in case $a(T) = T$ we meet the classical law of iterated logarithm (LIL):

$$\limsup_{T\to\infty} b(T)W(T) = 1 \quad \text{a.s.} \tag{1.3}$$

where

$$b(T) = (2T\log\log T)^{-1/2}.$$

Theorem B gives the best possible result in the sense that if condition (iii) is not satisfied then (1.2) does not hold true. In fact we have

THEOREM C. (Book - Shore, [1].) *Assume that* $a(\cdot)$ *satisfies conditions* (i) *and* (ii). *Assume also that*

(iv) $\lim_{T\to\infty} \dfrac{\log(T(a(T))^{-1})}{\log\log T} = r \quad (0 \le r \le \infty).$

*Then*

$$\liminf_{T\to\infty} \gamma(T,a(T))I(T,a(T)) = \left(\frac{r}{r+1}\right)^{1/2} \quad a.s. \tag{1.4}$$

We also mention that a somewhat stronger version of the lower half of (1.3) is also valid:

THEOREM D. (Erdős [4], Feller [7], Kolmogorov - Petrowsky [8].) *For any* $p = 2,3,\ldots$ *there exists a random sequence* $0 < T_1 = T_1(\omega,p) < T_2 = T_2(\omega,p) < \ldots$ *with* $\lim_{n\to\infty} T_n = \infty$ *a.s. such that*

$$W(T_n) \ge \left(2T_n\left(\log_2 T_n + \frac{3}{2}\log_3 T_n + \log_4 T_n + \ldots + \log_p T_n\right)\right)^{1/2}. \tag{1.5}$$

*Here and in what follows*

$$\log_1 x = \log x \quad \textit{and} \quad \log_p x = \log(\log_{p-1} x).$$

Consider the process

$$\xi(t) = \sup\{s : 0 \le s \le t, W(s) \ge (b(s))^{-1}\}.$$

(1.5) with $p = 2$ implies

$$\lim_{t\to\infty} \xi(t) = \infty \quad \text{and} \quad \limsup_{t\to\infty} t^{-1}\xi(t) = 1 \quad \text{a.s.}$$

The liminf behaviour of $\xi(t)$ is described in the following

THEOREM E. (Erdős - Révész, [6].)

$$\liminf_{t\to\infty} \frac{(\log_2 t)^{1/2}}{(\log_3 t)\log t} \log \frac{\xi(t)}{t} = -C \quad a.s. \tag{1.6}$$

*where $C$ is a positive constant with*

$$2^{-2} \le C \le 2^{14}.$$

Let

$$\nu(t) = \inf\{s : s \ge t, W(s) \ge (b(s))^{-1}\}.$$

Then a trivial reformulation of Theorem E is the following

THEOREM F. *For any $\varepsilon > 0$ and $t$ big enough we have*

$$\nu(t) \le t^{1+\varepsilon(t)} \quad a.s.$$

*where*

$$\varepsilon(t) = (2^{14} + \varepsilon)(\log_3 t)(\log_2 t)^{-1/2}$$

*and there exists a random sequence $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \dots$ with $\lim_{n\to\infty} t_n = \infty$ a.s. such that*

$$\nu(t_n) \ge t_n^{1+\delta(t_n)}$$

*where*

$$\delta(t) = \left(\frac{1}{4} - \varepsilon\right)(\log_3 t)(\log_2 t)^{-1/2}.$$

It is natural to ask how Theorem F should be changed if the definition of $\nu(t)$ is changed as follows. For any $0 < \varepsilon < 1$ let

$$\nu(t,\varepsilon) = \inf\{s : s \ge t, W(s) \ge (1-\varepsilon)^{1/2}(b(s))^{-1}\}.$$

Then we have

THEOREM 1. *For any $t$ big enough and $0 < \mu < 1$ we have*

$$\nu(t,\varepsilon) \le f(t,\varepsilon) = t\exp\left((\log t)^{1-\mu\varepsilon}\right) \quad a.s. \tag{1.7}$$

*Further for any $1 < \lambda < \varepsilon^{-1}$ there exists a random sequence $0 < t_1 = t_1(\omega,\varepsilon,\lambda) < t_2 = t_2(\omega,\varepsilon,\lambda) < \dots$ with $\lim_{n\to\infty} t_n = \infty$ a.s. such that*

$$\nu(t_n,\varepsilon) \ge t_n \exp\left((\log t_n)^{1-\lambda\varepsilon}\right). \tag{1.8}$$

We are also interested to find the analogue of Theorem 1 when we replace the process $W(\cdot)$ by the process $I(\cdot,\cdot)$ in it. Let

$$\nu(t,a(t),\varepsilon) = \inf\{s:\ s \ge t,\ \gamma(s,a(s))I(s,a(s)) \ge (1-\varepsilon)^{1/2}\}.$$

Then by Theorem B we have

$$\liminf_{t\to\infty} t^{-1}\nu(t,a(t),\varepsilon) = 1 \quad \text{a.s.}$$

If $a(t)$ satisfies condition (iv) with

$$\frac{r}{r+1} > 1-\varepsilon$$

then by Theorem C we have: for any $t$ big enough

$$\nu(t,a(t),\varepsilon) = t \quad \text{a.s.}$$

Hence we are only interested in the lim sup behaviour of $\nu$ when

$$\frac{r}{r+1} < 1-\varepsilon.$$

In this case we have

THEOREM 2. *Assume that $a(t)$ satisfies conditions* (i), (ii), (iv) *with*

$$r > 0 \quad \text{and} \quad \frac{r}{r+1} < 1-\varepsilon.$$

*Then for any $\delta > 0$ and $t$ big enough we have*

$$\nu(t,a(t),\varepsilon) \le t\exp\left((\log t)^{(r+1)(1-\varepsilon)-r+\delta}\right). \tag{1.9}$$

## 2. Strassen's theorem

Define the Strassen's set $S \subset C(0,1)$ as follows: a function $f \in C(0,1)$ is an element of $S$ if and only if

(a) $f(0) = 0$,

(b) $f(\cdot)$ is absolutely continuous in $[0,1]$ and

$$\int_0^1 (f'(x))^2 dx \leq 1.$$

For any $\varepsilon > 0$ and for any set $G \subset C(0,1)$ let

$$G^\varepsilon = \{f(\cdot) \in C(0,1),\ \|f(\cdot) - G\| \leq \varepsilon\}$$

where

$$\|f(\cdot) - G\| = \inf_{g \in G} \sup_{0 \leq x \leq 1} |f(x) - g(x)|.$$

Consider the stochastic process

$$w_t(x) = b(t)W(xt) \quad (0 \leq x \leq 1,\ t > 0).$$

Then Strassen proved in [10]

THEOREM G. (1) *For any* $\varepsilon > 0$ *there exists a random variable* $t_0 = t_0(\varepsilon, \omega) > 0$ *such that*

$$w_t(x) \in S^\varepsilon \quad if \quad t \geq t_0 \tag{2.1}$$

(2) *For any* $s(\cdot) \in S$ *there exists a random sequence* $0 < t_1 = t_1(s,\omega) < t_2 = t_2(s,\omega) < \ldots$ *with* $\lim_{n\to\infty} t_n = \infty$ *a.s. such that*

$$\lim_{n\to\infty} \sup_{0 \leq x \leq 1} |w_{t_n}(x) - s(x)| = 0. \tag{2.2}$$

(2) of Theorem G implies that the net $\{w_t(x),\ 0 \leq x \leq 1\}$ $(t > 0)$ eventually "covers" the Strassen' set $S$. In fact we say that a set of functions $\mathcal{F} \subset C(0,1)$ $\varepsilon$-covers a set $G \subset C(0,1)$ if for any $g(\cdot) \in G$ there exists a function $f(x) = f(x;\ g(\cdot),\ \varepsilon) \in \mathcal{F}$ such that

$$\sup_{0 \leq x \leq 1} |f(x) - g(x)| \leq \varepsilon.$$

Then it is natural to ask what is the waiting time for the coverage of $S$. We prove

THEOREM 3. *Let*

$$\rho(t) = \inf\{u \geq t :\ \mathcal{F}_{t,u} = \{w_z(\cdot),\ t \leq z \leq u\}\ \varepsilon\text{-covers } S\}.$$

*Then for any $t$ big enough and $0 < \mu < 1$ we have*

$$\rho(t) \leq t \exp\left((\log t)^{1-\mu\varepsilon}\right) \quad a.s. \tag{2.3}$$

Note that Theorem 1 implies that (2.3) is the best possible result in the same sense as (1.7) gives the best possible rate in Theorem 1.

A common generalization of Theorems B and G is the following

THEOREM H. (Révész, [9].) *Let*

$$\Gamma(x,t) = \Gamma(x; t, a(T)) = \gamma(T, a(T))(W(t + xa(T)) - W(t)) \quad 0 \leq x \leq 1$$

*and define the random set $V_T = V(T, a(T)) \subset C(0,1)$ as follows*

$$V_T = V(T, a(T)) = \{\Gamma(x,t) :\ 0 \leq t \leq T - a(T)\}.$$

*Assume that $a(T)$ satisfies conditions* (i) *and* (ii). *Then*

(1) *for any $\varepsilon > 0$ there exists a random variable $T_0 = T_0(\varepsilon, \omega) > 0$ such that*

$$V_T \subset S^{\varepsilon} \quad if \quad T \geq T_0 \tag{2.4}$$

(2) *for any $s(\cdot) \in S$ and $\varepsilon > 0$ there exist random variables*

$$T = T(\varepsilon, s(\cdot); \omega) > 0 \quad and \quad 0 \leq t = t(\varepsilon, s(\cdot); \omega) \leq T - a(T)$$

*such that*

$$\sup_{0 \leq x \leq 1} |\Gamma(x,t) - s(x)| \leq \varepsilon. \tag{2.5}$$

*If $a(T)$ also satisfies condition* (iii) *then for any $\varepsilon > 0$ there exists a random variable $T_0 = T_0(\varepsilon, s(\cdot); \omega) > 0$ such that*

$$S \subset (V_T)^{\varepsilon} \quad for\ all \quad T \geq T_0. \tag{2.6}$$

As Theorem 3 is a sharpening of (2) of Theorem G we can give a sharpening of (2.5) and (2.6) as follows

THEOREM 4. *Let*

$$\rho(T, a(T), \varepsilon) = \inf\{U \geq T : \mathcal{F}(U, a(U)) = \bigcup_{T \leq Z \leq U} V(Z, a(Z))\ \varepsilon\text{-}covers\ S\}.$$

($\alpha$) *Assume that* $a(t)$ *satisfies conditions* (i), (ii), (iv) *with*

$$\frac{r}{r+1} > 1 - \varepsilon \quad (0 < \varepsilon < 1).$$

*Then for any* $T$ *big enough*

$$\rho(T, a(T), \varepsilon) = T \quad a.s. \tag{2.7}$$

($\beta$) *Assume that* $a(t)$ *satisfies conditions* (i), (ii), (iv) *with*

$$\frac{r}{r+1} < 1 - \varepsilon \quad (0 < \varepsilon < 1,\ r > 0).$$

*Then for any* $\delta > 0$ *and* $T$ *big enough we have*

$$\rho(T, a(T), \varepsilon) \leq T \exp\left((\log T)^{(r+1)(1-\varepsilon)-r+\delta}\right) \quad a.s. \tag{2.8}$$

## 3. Proof of Theorem 1

At first we prove (1.7). In order to do so introduce the following notations: for any $t > 0$ and $\varepsilon_1 > 0$ define the sequences

$$\begin{aligned} \tau_k &= \tau_k(t) = te^k \quad (k = 0, 1, 2, \ldots, [(\log t)^{1-\mu\varepsilon}] = K), \\ A_k &= A_k(t) = \{W(\tau_k) \geq (1 - \varepsilon_1)^{1/2}(b(\tau_k))^{-1}\}. \end{aligned}$$

Observe that

$$\begin{aligned} \mathbf{P}\{\overline{A}_0\overline{A}_1 \ldots \overline{A}_K\} &= \mathbf{P}\{\overline{A}_0\} \prod_{k=1}^{K} \mathbf{P}\{\overline{A}_k \mid \overline{A}_{k-1} \ldots \overline{A}_0\} \\ &= (1 - \mathbf{P}\{A_0\}) \prod_{k=1}^{K} (1 - \mathbf{P}\{A_k \mid \overline{A}_{k-1} \ldots \overline{A}_0\}) \end{aligned}$$

and

$$\begin{aligned} &\mathbf{P}\{A_k \mid \overline{A}_{k-1} \ldots \overline{A}_0\} \geq \mathbf{P}\{A_k, |W(\tau_{k-1})| \leq \tau_{k-1}^{1/2} \mid \overline{A}_{k-1} \ldots \overline{A}_0\} \\ &= \mathbf{P}\{A_k \mid |W(\tau_{k-1})| \leq \tau_{k-1}^{1/2}, \overline{A}_{k-1} \ldots \overline{A}_0\}\mathbf{P}\{|W(\tau_{k-1})| \leq \tau_{k-1}^{1/2} \mid \overline{A}_{k-1} \ldots \overline{A}_0\} \\ &\geq \mathbf{P}\{A_k \mid W(\tau_{k-1}) = -\tau_{k-1}^{1/2}\}\mathbf{P}\{|W(\tau_{k-1})| \leq \tau_{k-1}^{1/2} \mid \overline{A}_{k-1} \ldots \overline{A}_0\} \\ &\geq \frac{1}{\log\log \tau_k (\log \tau_k)^{1-\varepsilon_1}} \end{aligned}$$

for any $k$ big enough. Hence

$$\mathbf{P}\{\overline{A}_0\overline{A}_1\ldots\overline{A}_K\} \le \prod_{k=1}^{K}\left(1-\frac{1}{\log\log\tau_k(\log\tau_k)^{1-\varepsilon_1}}\right)$$
$$\le \exp\left(-\frac{(\log t)^{\varepsilon_1-\mu\varepsilon}}{(\log\log t)^2}\right).$$

Choosing $\varepsilon_1 > \mu\varepsilon$ and applying the above inequality with $t_\ell = \Theta^\ell$ ($\ell = 1,2,\ldots;\ \Theta > 1$) we obtain

$$\sup_{t_\ell\le s\le\tau_k(t_\ell)} b(s)W(s) \ge (1-\varepsilon_1)^{1/2} \quad \text{a.s.}$$

for all but finitely many $\ell$. That is (1.7) is proved for $t_\ell$. In order to prove it for any $t_\ell \le t \le t_{\ell+1}$ it is enough to prove that if $W(t_k) \ge (1-\varepsilon_1)^{1/2}(b(t_k))^{-1}$ then $W(t_{k+1}) \ge (1-\varepsilon)^{1/2}(b(t_{k+1}))^{-1}$. Clearly

$$\begin{aligned} W(t_{k+1}) &= W(t_{k+1}) - W(t_k) + W(t_k)\\ &\ge W(t_{k+1}) - W(t_k) + (1-\varepsilon_1)^{1/2}(b(t_k))^{-1}\\ &\ge (1-\varepsilon_1)^{1/2}(b(t_k))^{-1} - (1-\varepsilon)^{1/2}(b(t_{k+1}-t_k))^{-1}\\ &\ge (1-\varepsilon)^{1/2}(b(t_{k+1}))^{-1}\end{aligned}$$

if $\Theta - 1$ is small enough and $\varepsilon_1 < \varepsilon$. Consequently (1.7) is proved.

Now we turn to the proof of (1.8). Introduce the following notations: for any $t > 0$ let

$$\psi_k = \psi_k(t) = t\psi^k \quad \left(\psi > 1,\ k = 0,1,2,\ldots,\ \left[\frac{(\log t)^{1-\lambda\varepsilon}}{\log\psi}\right] = K = K(t)\right),$$
$$B_k = B_k(t) = \{W(\psi_k) \ge (1-\varepsilon_2)^{1/2}(b(\psi_k))^{-1}\}.$$

Observe that

$$\mathbf{P}\{\overline{B}_0\overline{B}_1\ldots\overline{B}_K\} = (1-\mathbf{P}\{B_0\})\prod_{k=1}^{K}(1-\mathbf{P}\{B_k \mid \overline{B}_{k-1}\ldots\overline{B}_0)$$

and

$$\mathbf{P}\{B_k \mid \overline{B}_{k-1}\ldots\overline{B}_0\} \le \mathbf{P}\{B_k\} \le (\log\psi_k)^{-1+\varepsilon_2} \le (\log t)^{-1+\varepsilon_2}.$$

Hence

$$\mathbf{P}\{\overline{B}_0\overline{B}_1\ldots\overline{B}_K\} \geq \prod_{k=0}^{K}(1-(\log\psi_k))^{-1+\varepsilon_2}$$
$$\geq (1-(\log t)^{-1+\varepsilon_2})^{(\log t)^{1-\lambda\varepsilon}/\log\psi} \geq 1-\frac{(\log t)^{\varepsilon_2-\lambda\varepsilon}}{\log\psi}$$

if $\varepsilon_2 < \lambda\varepsilon$ which implies that there exists a random sequence $0 < t_1 < t_2 < \ldots$ with $\lim_{n\to\infty} t_n = \infty$ a.s. such that

$$\max_{0\leq k\leq K(t_n)} b(\psi_k(t_n))W(\psi_k(t_n)) \leq (1-\varepsilon_2)^{1/2} \quad (n=1,2,\ldots).$$

Choosing $\varepsilon_2 > \varepsilon$ and $\psi$ close enough to 1 we obtain (1.8).

## 4. Proof of Theorem 2

Let

$$\begin{aligned}
W(i,j,T) &= W(e^jT+(i+1)a(e^{j+1}T)) - W(e^jT+ia(e^{j+1}T)),\\
&\quad (i=0,1,2,\ldots,[(e^{j+1}-e^j)T(a((e^{j+1}-e^j)T))^{-1}],\\
&\quad j=0,1,2,\ldots,(\log T)^{(r+1)(1-\varepsilon)-r+\delta}),\\
\gamma(j,T) &= \gamma(e^{j+1}T, a(e^{j+1}T)).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbf{P}\{\max_{0\leq i\leq(e-1)T(a((e-1)T))^{-1}} W(i,0,T) \geq (1-\varepsilon_1)^{1/2}(\gamma(0,T))^{-1}\}\\
&= 1-\prod_{i=0}^{(e-1)T(a((e-1)T))^{-1}}\{W(i,0,T) < (1-\varepsilon_1)^{1/2}(\gamma(0,T))^{-1}\}\\
&\geq 1-\prod_{i=0}^{(e-1)T(a(eT))^{-1}}\left(1-\frac{1}{(\log T)^{(r+1)(1-\varepsilon_3)}}\right)\\
&\geq 1-\exp\left(-\frac{1}{(\log T)^{(r+1)(1-\varepsilon_3)-r}}\right)
\end{aligned}$$

for any $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon < 1$ if $T$ is big enough. Similarly for any $j$ and $\delta < \varepsilon$ we have

$$\mathbf{P}\{\max_{0\leq i\leq(e^{j+1}-e^j)T(a((e^{j+1}-e^j)T))^{-1}} W(i,j,T) \geq (1-\varepsilon_1)^{1/2}(\gamma(j,T))^{-1}\}$$

$$\geq 1 - \exp\left(-\frac{1}{(\log T)^{(r+1)(1-\varepsilon_2)-r}}\right).$$

Consequently

$$\mathbf{P}\{\max_j \max_i \gamma(j,T)W(i,j,T) < (1-\varepsilon_1)^{1/2}\} \leq \exp(-(\log T)^{\delta-(r+1)(\varepsilon-\varepsilon_2)}).$$

Assuming that $\delta > (r+1)(\varepsilon - \varepsilon_2)$, $T_k = \Theta^k$, $\Theta > 1$ we obtain

$$\max_j \max_i \gamma(j,T_k)W(i,j,T_k) \geq (1-\varepsilon_1)^{1/2} \quad \text{a.s.}$$

for all but finitely many $k$. Choosing $\Theta$ close enough to 1 we easily obtain Theorem 2.

## 5. Proof of Theorem 3

Let

$$\begin{aligned} V(t) &= V(\alpha_1,\alpha_2,\ldots,\alpha_d;t) \\ &= \alpha_1 W(t) + \alpha_2(W(2t)-W(t)) + \ldots + \alpha_d(W(dt) - W(d-1)t)) \\ &\quad (\alpha_1^2+\alpha_2^2+\ldots+\alpha_d^2 = 1,\ t \geq 0), \\ \chi(t) &= \chi(\alpha_1,\alpha_2,\ldots,\alpha_d;t) = \inf\{s:\ s \geq t,\ V(s) \geq (1-\varepsilon)^{1/2}(b(s))^{-1}\}. \end{aligned}$$

Then

LEMMA 1. *For any $t$ big enough and $0<\mu<1$ we have*

$$\chi(t) \leq t\exp((\log t)^{1-\mu\varepsilon}) \quad a.s.$$

PROOF is essentially the same as that of (1.7). It will be omitted.
Let

$$\begin{aligned} \mathcal{C} &= \mathcal{C}(d) = \{(x_1,x_2,\ldots,x_d):\ x_1^2+x_2^2+\ldots+x_d^2 = 1\}, \\ \overline{V} &= \overline{V}(d,t) = (W(t), W(2t)-W(t),\ldots,W(dt)-W((d-1)t)), \\ \overline{\chi} &= \overline{\chi}(d,t) = \inf\{s:\ s \geq t,\ \mathcal{F}_s = \{\overline{V}(d,u);\ t \leq u \leq s\} \\ &\qquad \varepsilon\text{-covers the set } \mathcal{C}\}. \end{aligned}$$

Then

LEMMA 2. *For any $t$ big enough and $0<\mu<1$ we have*

$$\overline{\chi} \leq t\exp((\log t)^{1-\mu\varepsilon}) \quad a.s.$$

PROOF. We give the proof for $d = 2$. For larger $d$ the proof is similar and immediate. Lemma 1 implies that the set

$$\{\overline{V}(2,u),\ t \leq u \leq \overline{\chi}(2,t)\}$$

$\varepsilon$-covers the circle $\{(x_1, x_2) :\ x_1^2 + x_2^2 = 1\}$. In the same way one can prove that the set

$$\{\overline{V}(3,u), t \leq u \leq \overline{\chi}(3,t)\}$$

$\varepsilon$-covers the sphere $\{(x_1, x_2, x_3);\ x_1^2+x_2^2+x_3^2 = 1\}$. These two facts combined imply the statement for $d = 2$. Hence we have Lemma 2.

For any real valued function $f \in C(0,1)$ and positive integer $d$, let $f^{(d)}$ be the linear interpolation of $f$ over the points $i/d$ i.e.

$$f^{(d)}(x) = f\left(\frac{i}{d}\right) + d\left(f\left(\frac{i+1}{d}\right) - f\left(\frac{i}{d}\right)\right)\left(x - \frac{i}{d}\right)$$
$$\left(\frac{i}{d} \leq x \leq \frac{i+1}{d};\ i = 0,1,2,\ldots,d-1\right).$$

Then, as it is well-known, we have

$$\sup_{0\leq x\leq 1} |w_t(x) - w_t^{(d)}(x)| \leq \sup_{0\leq x\leq 1}\ \sup_{0\leq s\leq 1/d} |w_t(x+s) - w_t(x)| \leq d^{-1/2} \quad \text{a.s.}$$

if $t$ is big enough. Hence we have Theorem 3.

## 6. Proof of Theorem 4

Combining the proofs of Theorems 2, 3 and H we easily get the proof of Theorem 4.

## 7. A note

Assume (iv) with

$$\frac{r}{r+1} < 1 - \varepsilon.$$

Then Theorem C implies that $V_T$ will not $\varepsilon$-cover $S$ i.e. $\rho(T, a(T), \varepsilon) > T$ (cf. (2.7)) for infinitely many $T$. It is also easy to see that $V_T$ will not $\varepsilon$-cover $S$ for any $T$ big enough. However it is natural to think that $V_T$ $\varepsilon$-covers a big subset of $S$ for every $T$ big enough. In fact we have

THEOREM I. (Deheuvels - Révész, [3].) *Assume* (iv). *Then for any* $\varepsilon > 0$ *there exists a random variable* $T_0 = T_0(\varepsilon, \omega)$ *such that for any* $T \geq T_0$ *the set* $V_T$ *$\varepsilon$-covers* $(\frac{r}{r+1})^{1/2} S$ *where*

$$f \in \left(\frac{r}{r+1}\right)^{1/2} S \text{ if and only if } \left(\frac{r+1}{r}\right)^{1/2} f \in S.$$

*In other words* (cf. Theorem 4) *if*

$$\rho(T, a(T), \varepsilon, \lambda) = \inf\{U \geq T,\ \mathcal{F}(U, a(U))\varepsilon\text{-covers}\lambda S\}.$$

*Then for any $T$ big enough*

$$\rho\left(T, a(T), \varepsilon, \left(\frac{r}{r+1}\right)^{1/2}\right) = T \quad a.s. \tag{7.1}$$

(2.8) told us that

$$\begin{aligned} \rho(T, a(T), \varepsilon, 1) &= \rho(T, a(T), \varepsilon) \\ &\leq T \exp\left((\log T)^{(r+1)(1-\varepsilon)-r+\delta}\right). \end{aligned} \tag{7.2}$$

Comparing (7.1) and (7.2) it is natural to ask about the lim sup behaviour of $\rho(T, a(T), \varepsilon, \lambda)$ whenever $a(T)$ satisfies (iv) and

$$\left(\frac{r}{r+1}\right)^{1/2} < \lambda < 1.$$

## REFERENCES


[1] BOOK, S. A. and SHORE, T. R., On large intervals in the Csörgő - Révész theorem on increments of a Wiener process, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **46** (1978/79), 1—11. *MR* **80h**: 60052

[2] CSÖRGŐ, M. and RÉVÉSZ, P., How big are the increments of a Wiener process?, *Ann. Probab.* **7** (1979), 731—737. *MR* **80g**: 60025

[3] DEHEUVELS, P. and RÉVÉSZ, P., On the coverage of Strassen set by sequences of Wiener processes (submitted).

[4] ERDŐS, P., On the law of iterated logarithm, *Ann. of Math.* (2) **43** (1942), 419—436. *MR* **4**—16

[5] ERDŐS, P. and RÉNYI, A., On a new law of large numbers, *J. Analyse Math.* **23** (1970), 103—111. *MR* **42** # 6907

[6] ERDŐS, P. and RÉVÉSZ, P., A new law of the iterated logarithm, *Acta Math. Hungar.* **55** (1990), 125—131.

[7] FELLER, W., The general form of the so-called law of the iterated logarithm, *Trans. Amer. Math. Soc.* **54** (1943), 373—402. *MR* **5**—125

[8] PETROWSKI, I. G., Zur ersten Randwertaufgabe der Wärmeleitungsgleichung, *Compositio Math.* **1** (1935), 383—419. *Zbl.* **10**, 299

[9] RÉVÉSZ, P., A generalization of Strassen's functional law of iterated logarithm, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **50** (1979), 257—264. *MR* **82c**: 60057

[10] STRASSEN, V., An invariance principle for the law of iterated logarithm, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** (1964), 211—226. *MR* **30** # 5379

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# THE REGION OF CONVERGENCE OF THE LAPLACE TRANSFORM: ALMOST SURE ESTIMATION

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## 1. Introduction

Let $X$ be a non-negative random variable with distribution function $F$ on $[0,\infty)$ and with Laplace transform

$$L(s) := E(e^{-sX}) = \int_0^\infty e^{-sx} dF(x).$$

We assume that $L(s)$ converges for $s > -\sigma$, $0 < \sigma < \infty$. To any sample $X_1, X_2, \ldots, X_n$ of $X$, let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding set of order statistics where $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$.

There are different ways to obtain estimators for $\sigma$ or $\sigma^{-1}$. One of them is looking at the way $\sigma$ is calculated if $F$ is known. Theorem 2.4.d. of Widder [9] gives us that

$$\limsup_{x\to\infty} \frac{\log(1-F(x))}{x} = -\sigma.$$

If we replace in this formula the distribution $F$ by its empirical version $F_n$ and choose $x = X_{n-k:n}$, we get an estimator for $1/\sigma$, namely

$$\widehat{\left(\frac{1}{\sigma}\right)} = \frac{X_{n-k:n}}{\log \frac{n}{k}}.$$

We can look at this estimator for fixed $k$ or for $k_n \to \infty$ as $n \to \infty$. To prove any property of this estimator we will need at least that

(1) $$\lim_{x\to\infty} \frac{1}{x} \log[1-F(x)] = -\sigma.$$

This assumption tells us that we may find a slowly varying function $l(x)$ such that

(2) $$-\log[1-F(x)] = \sigma x l(x)$$

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and $l(x) \to 1$ as $x \to \infty$. If in addition to (1) $F$ is ultimately continuous, then we can use a result of J. Beirlant and M. Browniatovsky [2] to obtain that

$$e(x) := \int_x^\infty \frac{1-F(y)}{1-F(x)}dy \sim \frac{1}{\sigma l(x)}, \quad x \to \infty$$

and

$$\lim_{x\to\infty} e(x) = \frac{1}{\sigma}. \tag{3}$$

We get a Hill-type estimator for $1/\sigma$ by replacing $F$ by $F_n$ and $x$ by $X_{n-k:n}$. After some calculations this leads to

$$\widehat{e_{k,n}} = \hat{e}(X_{n-k:n}) = \frac{1}{k}\sum_{i=1}^{k} X_{n-i+1:n} - X_{n-k:n}.$$

To prove a.s. convergence for both estimators we will need results which found their inspiration in results by Rényi [8].

## 2. Almost sure convergence for $\widehat{\frac{1}{\sigma}}$ and $\widehat{e_{k,n}}$

Letting $n$ tend to $\infty$ while $k$ remains fixed, we get a.s. convergence of $\frac{X_{n-k:n}}{\log\frac{n}{k}}$ when (1) fulfilled.

THEOREM 1. *Assume that $F$ is a distribution function satisfying* (1) *for some* $0 < \sigma < \infty$. *If $k$ is fixed and $n \to \infty$, then*

$$\frac{X_{n-k:n}}{\log\frac{n}{k}} \to \frac{1}{\sigma} \quad a.s.$$

To prove this we make use of a result of Barndorff-Nielsen [1]. He mentioned in the preliminaries of his paper Lemma 1 and Lemma 2 of Rényi [8].

LEMMA 1. *If $A_n$ is the event $\{X_n \geq X_{n-1:n-1}\}$, then $A_1, A_2, \ldots, A_n, \ldots$ are independent and $P(A_k) = \frac{1}{k}$ for $k = 1, 2, \ldots$.*

LEMMA 2. *If $F$ is continuous and $r_k$ is the rank of $X_k$ in the set $X_1, X_2, \ldots, X_k$, then the random variables $r_1, r_2, \ldots, r_k, \ldots$ are independent and $P\{r_k = j\} = \frac{1}{k}$, $j = 1, 2, \ldots, k$.*

Letting $k_n$ tend to $\infty$ with $n$, but controlling it such that $\frac{k_n}{n} \to 0$, we also get a.s. convergence of $\widehat{\left(\frac{1}{\sigma}\right)}$ if (1) is fulfilled.

THEOREM 2. *Assume that $F$ is a distribution function satisfying* (1) *for some* $0 < \sigma < \infty$. *If* $k_n \to \infty$ *as* $n \to \infty$ *and* $k_n = o(n)$, *then*

$$\frac{X_{n-k_n:n}}{\log \frac{n}{k_n}} \to \frac{1}{\sigma} \quad a.s..$$

If we put stronger conditions on $F$, we can say even more about the relation between $X_{n-n_k:n}$ and $\frac{1}{\sigma}\log\frac{n}{k_n}$ according to the rate of growth of $k_n$.

THEOREM 3. *Assume that $F$ is ultimately continuous and satisfies*

$$\lim_{x\to\infty}[\sigma x + \log(1 - F(x))] = C \tag{4}$$

*with $C$ finite; then*

*(i) if* $k_n \sim \lambda \log\log n$ *as* $n \to \infty$, *where* $0 < \lambda < \infty$, *then*

$$\liminf_{n\to\infty}[X_{n-k_n:n} - \frac{1}{\sigma}\log\frac{n}{k_n}] = \frac{-1}{\sigma}\log(1+\alpha_\lambda^+) + \frac{C}{\sigma} \quad a.s.,$$
$$\limsup_{n\to\infty}[X_{n-k_n:n} - \frac{1}{\sigma}\log\frac{n}{k_n}] = \frac{-1}{\sigma}\log(1+\alpha_\lambda^-) + \frac{C}{\sigma} \quad a.s.,$$

*where* $-1 < \alpha_\lambda^- < 0 < \alpha_\lambda^+$ *are the roots of the equation*

$$\alpha_\lambda^\pm - \log(1+\alpha_\lambda^\pm) = \frac{1}{\lambda}; \tag{5}$$

*(ii) if* $k_n/\log\log n \to \infty$ *as* $n \to \infty$ *and* $k_n = o(n)$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty}[X_{n-k_n:n} - \frac{1}{\sigma}\log\frac{n}{k_n}] = \frac{C}{\sigma} \quad a.s.;$$

*(iii) if* $k_n/\log\log n \to 0$ *as* $n \to \infty$ *and* $k_n \geq 1$ *for all* $n \geq 1$, *then*

$$\liminf_{n\to\infty}[X_{n-k_n:n} - \frac{1}{\sigma}\log\frac{n}{k_n}] = -\infty \quad a.s..$$

To prove Theorem 3 we use Theorem 6 of Kiefer [7] which we also need in the proof of the next result on a.s. convergence for $\widehat{e_{k,n}}$.

THEOREM 4. *Assume that $F$ is a distribution function satisfying* (1) *for some* $0 < \sigma < \infty$ *and that $F$ is ultimately continuous; then* *(i) if* $k_n \sim \lambda \log\log n$ *as* $n \to \infty$, *where* $0 < \lambda < \infty$, *then*

$$\limsup_{n\to\infty} \pm(\widehat{e_{k,n}} - \frac{1}{\sigma}) = \pm\frac{1}{\sigma}\alpha_\lambda^\pm \quad a.s.$$

*where* $-1 < \alpha_\lambda^- < 0 < \alpha_\lambda^+$ *are the roots of the equation* (5);

*(ii) if* $k_n/\log\log n \to \infty$ *as* $n \to \infty$ *and* $k_n = o(n)$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} \widehat{e_{k,n}} = \frac{1}{\sigma} \quad a.s.;$$

*(iii) if* $k_n/\log\log n \to 0$ *as* $n \to \infty$ *and* $k_n \geq 1$ *for all* $n \geq 1$, *then*

$$\liminf_{n\to\infty} \widehat{e_{k,n}} = 0 \quad a.s.$$

*and*

$$\limsup_{n\to\infty} \widehat{e_{k,n}} = \infty \quad a.s..$$

Note that $\widehat{e_{k,n}}$ is a strongly consistent estimator of $\frac{1}{\sigma}$ only in case (ii).

The proof of this theorem follows along the same lines as that of the Theorem in [6]. For (i), P. Deheuvels, e.a., first prove this result for random variables with an exponential distribution with mean one. To obtain this, they choose two subsequences of $\{n\}$. These subsequences are suitable to enclose all $\widehat{e_{k,n}}$ with probability one. For the proof of the latter result, they need to introduce an exceedence number. To get limit results for this exceedence number, they use Lemma 1 [8] and a variant of Lemma 1; i.e. the fact that the events $\{X_n \geq X_{n-i:n-1}\}$ are for $n = i+1, i+2, \ldots$ independent and of probability $\frac{i}{n}$.

## 3. Proofs

PROOF OF THEOREM 1. Theorem 4.1. and 4.3. of Barndorff-Nielsen [1] tell us that a.s. convergence will hold if $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^k [1 - F((\frac{1}{\sigma} + \varepsilon)\log\frac{n}{k})]^{k+1} < \infty$$

and

$$\sum_{n=1}^{\infty} n^k [F((\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k})]^{n-k} [1 - F((\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k})]^{k+1} < \infty.$$

It follows from (1) that $\forall\gamma > 0$, $\exists x_0$ such that $\forall x \geq x_0$:

$$-\sigma - \gamma \leq \frac{\log[1 - F(x)]}{x} \leq -\sigma + \gamma. \tag{6}$$

For the first sum take $x = (\frac{1}{\sigma} + \varepsilon)\log\frac{n}{k}$ in (6), then we get for $n > M$,

$$\log\frac{n}{k} + \log[1 - F((\frac{1}{\sigma} + \varepsilon)\log\frac{n}{k})] \leq (\gamma(\frac{1}{\sigma} + \varepsilon) - \sigma\varepsilon)\log\frac{n}{k}.$$

Choose $\gamma < \frac{\varepsilon\sigma^2}{(1+\sigma\varepsilon)}$ such that $-\delta := \gamma(\frac{1}{\sigma} + \varepsilon) - \sigma\varepsilon < 0$ and

$$1 - F((\frac{1}{\sigma} + \varepsilon)\log\frac{n}{k}) \leq (\frac{n}{k})^{-\delta-1} \text{ for } n > M.$$

For the second sum note that

$$F((\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k}) \leq \exp\{-[1 - F((\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k})]\}.$$

Taking $x = (\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k}$ in (6), we get for $n > M$,

$$\log\frac{n}{k} + \log[1 - F((\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k})] \geq (-\gamma(\frac{1}{\sigma} - \varepsilon) + \sigma\varepsilon)\log\frac{n}{k}.$$

Choose $\gamma < \frac{\varepsilon\sigma^2}{(1-\sigma\varepsilon)}$ such that $\phi := -\gamma(\frac{1}{\sigma} - \varepsilon) + \sigma\varepsilon > 0$ and

$$1 - F((\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k}) \geq (\frac{n}{k})^{\phi-1} \text{ for } n > M.$$

We also have that $1 - F((\frac{1}{\sigma} - \varepsilon)\log\frac{n}{k}) \leq 1$. Now the boundedness of the second sum follows immediately. □

PROOF OF THEOREM 2. We can write condition (1) as in (2). We use Theorem 1.5.12 and Proposition 1.5.15 of [3] to obtain that inverse function satisfies

$$F^{\leftarrow}(1 - s) \sim \frac{-1}{\sigma}\log s l^{\#}(-\log s) \quad \text{for} \quad s \downarrow 0.$$

Furthermore, it follows from Theorem 1.5.13 [3] that if $l(x) \to 1$ as $x \to \infty$ so does $l^{\#}(x)$. This leads to the fact that there exists a slowly varying function $L(x)$ such that

$$F^{\leftarrow}(1 - s) = \frac{-1}{\sigma}\log s L(-\log s) \quad \text{and} \quad L(-\log s) \to 1 \quad \text{as} \quad s \downarrow 0.$$

We also know that $X_{n-k_n:n} \stackrel{D}{=} F^{\leftarrow}(1-(1-U_{n-k_n:n}))$, where $U_{n-k_n:n}$ is the $(n-k_n)$-th order statistic from the sample $U_1, U_2, \ldots, U_n$ of uniformly $(0,1)$ distributed random variables. Now we can write our estimator as

$$\frac{X_{n-k_n:n}}{\log \frac{n}{k_n}} = \frac{1}{\sigma} \frac{-\log(1-U_{n-k_n:n})}{\log \frac{n}{k_n}} L(-\log(1-U_{n-k_n:n})).$$

From Theorem 6 in [7] we derive that for $\frac{k_n}{n} \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{-\log(1-U_{n-k_n:n})}{\log \frac{n}{k_n}} = 1 \quad \text{a.s.}.$$

This fact also implies that $1 - U_{n-k_n:n} \to 0$ a.s. as $n \to \infty$, so that $L(-\log(1-U_{n-k_n:n})) \to 1$ a.s. as $n \to \infty$. □

PROOF OF THEOREM 3. We can write $X_{n-k_n:n} - \frac{1}{\sigma}\log\frac{n}{k_n}$ as

$$F^{\leftarrow}(1-(1-U_{n-k_n:n})) - F^{\leftarrow}(1-\frac{k_n}{n}) + F^{\leftarrow}(1-\frac{k_n}{n}) - \frac{1}{\sigma}\log\frac{n}{k_n}. \tag{7}$$

From (4) and the fact that $F$ is ultimately continuous it follows that

$$F^{\leftarrow}(1-\frac{k_n}{n}) - \frac{1}{\sigma}\log\frac{n}{k_n} \to \frac{C}{\sigma} \quad \text{as} \quad \frac{k_n}{n} \to 0. \tag{8}$$

(4) also implies by Fact 1.4 [4] that

$$F^{\leftarrow}(1-s) = \frac{-1}{\sigma}\log s + \log a(s) + \int_s^1 \frac{b(u)}{u} du, \tag{9}$$

for $s$ sufficiently small, where $a$ and $b$ are functions on $(0,1)$ which satisfy

$$\lim_{s\downarrow 0} a(s) = a_0 \quad \text{where} \quad 0 < a_0 < \infty,$$
$$\lim_{s\downarrow 0} b(s) = 0.$$

It follows immediately from (9) that

$$F^{\leftarrow}(1-\frac{1}{ux}) - F^{\leftarrow}(1-\frac{1}{x}) \to \frac{1}{\sigma}\log u \quad \text{as} \quad x \to \infty, \tag{10}$$

for all $u > 0$, uniformly in $u$. Using Theorem 6 in [7] (see also [5], [6]) and (7), (8) and (10) we obtain the three results. □

PROOF OF THEOREM 4. Using (9) we can write $\widehat{e_{k,n}}$ as

$$\begin{aligned}\widehat{e_{k,n}} = & -\frac{1}{\sigma}\frac{1}{k_n}\sum_{i=1}^{k_n}\log(1-U_{n-i+1:n}) + \frac{1}{\sigma}\log(1-U_{n-k_n:n}) \\ & + \frac{1}{k_n}\sum_{i=1}^{k_n}\log[a(1-U_{n-i+1:n})/a(1-U_{n-k_n:n})] \\ & - \frac{1}{k_n}\sum_{i=1}^{k_n}\int_{1-U_{n-k_n:n}}^{1-U_{n-i+1:n}}\frac{b(u)}{u}du.\end{aligned}$$

Note that in the first line of $\widehat{e_{k,n}}$ the sum is the Hill estimator. From here on the proof is exactly the same as that for the Hill estimator [6]. □

## REFERENCES


[1] BARNDORFF-NIELSEN, O., On the limit behaviour of extreme order statistics, *Ann. Math. Statist.* **34** (1963), 992—1002. *MR* **27** # 875

[2] BEIRLANT, J. and BROWNIATOVSKI, M., The mean residual life function and Chernoff-type large deviation, Tech. Report, March, 1991.

[3] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L., *Regular Variation*, Encyclopedia of mathematics and its applications, Vol. 27, Cambridge University Press, Cambridge - New York, 1987. *MR* **88i**: 26004

[4] CSÖRGŐ, S. and MASON, D. M., Central limit theorems for sums of extreme values, *Math. Proc. Cambridge Philos. Soc.* **98** (1985), 547—558. *MR* **87f**: 60028

[5] DEHEUVELS, P., Strong laws for the $k$-th order statistic when $k \le c\log_2 n$, *Probab. Theory Relat. Fields* **72** (1986), 133—154. *MR* **87j**: 60048

[6] DEHEUVELS, P., HAEUSLER, E. and MASON, D. M., Almost sure convergence of the Hill estimator, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), 371—381. *MR* **89i**: 62043

[7] KIEFER, J., Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$, *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Univ. California, Berkeley, Calif., 1970/71), Vol. I: Theory of statistics, Univ. California Press, Berkeley, Calif., 1972, 227—244. *MR* **53** # 6696

[8] RÉNYI, A., Théorie des éléments saillants d'une suite d'observations, *Colloquium on Combinatorial Methods in Probability Theory*, Aarhus, August 1-10, 1962, Matematisk Institut, Aarhus Universitet, 1962, 104—117. See *MR* **44** # 3376

[9] WIDDER, D., *The Laplace transform*, Princeton Mathematical Series, Vol. 6, Princeton University Press, Princeton, N. J., 1941. *MR* **3**—232

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# ON LÉVY-BAXTER THEOREM FOR GENERAL TWO-PARAMETER GAUSSIAN PROCESSES

CHEN XIONG and PAN XIA


**Abstract**

This note extends to a broad class of two-parameter Gaussian processes the theorem of G. Baxter: for a two-parameter Gaussian Process $X$, under some conditions,

$$\lim_{n\to\infty}\sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n}\Big[X(\frac{k_1}{2^n},\frac{k_2}{2^n})+X(\frac{k_1-1}{2^n},\frac{k_2-1}{2^n}) - X(\frac{k_1-1}{2^n},\frac{k_2}{2^n})-X(\frac{k_1}{2^n},\frac{k_2-1}{2^n})\Big]^2 = \int_0^1\int_0^1 f(s_1,s_2)ds_1ds_2 \quad \text{a.s.}$$


## 1. Introduction

As we know, P. Lévy [4] proved almost sure convergence to 1 of the quadratic variation $\sum_{k=1}^{2^n}[B(k2^{-n})-B((k-1)2^{-n})]^2$ of the Brownian motion on $[0,1]$. This result was extended to other processes with Gaussian increments defined on $[0,1]$ by G. Baxter [1].

In two-parameter case, there have been many results on quadratic variation of two-parameter martingales, which have played central role in this theory [2]. On the other hand, it seems that there is little discussion about the quadratic variation of other two-parameter processes. In this paper, for two-parameter Gaussian processes we shall give a result similar to that of Baxter. And as a corollary, we get the quadratic variation of Brownian Sheet and Two-Parameter Ornstein-Uhlenbeck process either.

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## 2. Results and proofs

LEMMA 2.1. *Let $R(s_1, s_2; t_1, t_2)$ be a real valued continuous function defined on $[0,1]^2 \times [0,1]^2$, having continuous and bounded third derivative on $[0,1]^2 \times [0,1]^2 - \Delta$, where $\Delta = \{(s,t;s,t),\ s,t \in [0,1]\}$. Then the following limits exist and are bounded, for any fixed $(s_1, s_2) \in [0,1]^2$*

$$
\begin{aligned}
D^{++}(s_1,s_2) &\triangleq \lim_{\substack{t_1 \downarrow s_1 \\ t_2 \downarrow s_2}} \frac{\partial^2 R}{\partial t_2 \partial t_1}(s_1,s_2;t_1,t_2) \\
D^{-+}(s_1,s_2) &\triangleq \lim_{\substack{t_1 \uparrow s_1 \\ t_2 \downarrow s_2}} \frac{\partial^2 R}{\partial t_2 \partial t_1}(s_1,s_2;t_1,t_2) \\
D^{+-}(s_1,s_2) &\triangleq \lim_{\substack{t_1 \downarrow s_1 \\ t_2 \uparrow s_2}} \frac{\partial^2 R}{\partial t_2 \partial t_1}(s_1,s_2;t_1,t_2) \\
D^{--}(s_1,s_2) &\triangleq \lim_{\substack{t_1 \uparrow s_1 \\ t_2 \uparrow s_2}} \frac{\partial^2 R}{\partial t_2 \partial t_1}(s_1,s_2;t_1,t_2)
\end{aligned}
\tag{2.1}
$$

*furthermore, we have*

$$
|D^{++}(s_1,s_2) - \frac{\partial^2 R}{\partial t_2 \partial t_1}(s_1,s_2;t_1,t_2)| \le c(|s_1 - t_1| + |s_2 - t_2|) \tag{2.2}
$$

*the similar results like (2.2) holds for $D^{\pm\pm}(s_1, s_2)$.*

PROOF. We only prove (2.2), the others are easy.

For any $(t_1', t_2') \in [0,1]^2$, by the mean value theorem,

$$
\begin{aligned}
&\left| \frac{\partial^2 R}{\partial t_2 \partial t_1}(s_1,s_2;t_1',t_2') - \frac{\partial^2 R}{\partial t_2 \partial t_1}(s_1,s_2;t_1,t_2) \right| \\
&\le \left| \frac{\partial^3 R}{\partial t_2 \partial t_1^2}(s_1,s_2;t_1^*,t_2^*) \right| |t_1' - t| + \left| \frac{\partial^3 R}{\partial t_2^2 \partial t_1}(s_1,s_2;t_1^*,t_2^*) \right| |t_2' - t| \\
&\le c(|t_1' - t_1| + |t_2' - t_2|),
\end{aligned}
$$

where $c$ is the bound for the third derivative. In the inequality, let $t_1' \downarrow t_1$, $t_2' \downarrow t_2$, we can prove (2.2).

THEOREM 2.2. *Suppose $\{X(t),\ t \in [0,1]^2\}$ is a mean zero Gaussian process, whose covariance function $R(s_1, s_2; t_1, t_2)$ satisfies the conditions stated in Lemma 2.1, then*

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k_1=1}^{2^n} \sum_{k_2=1}^{2^n} \Big[X(\frac{k_1}{2^n}, \frac{k_2}{2^n}) + X(\frac{k_1-1}{2^n}, \frac{k_2-1}{2^n}) \\
&\qquad - X(\frac{k_1-1}{2^n}, \frac{k_2}{2^n}) - X(\frac{k_1}{2^n}, \frac{k_2-1}{2^n})\Big]^2 \qquad (2.3)\\
&= \int_0^1 \int_0^1 f(s_1, s_2) ds_1 ds_2 \quad a.s.
\end{aligned}$$

*where,* $f(s_1, s_2) \hat{=} D^{++}(s_1, s_2) + D^{--}(s_1, s_2) - D^{+-}(s_1, s_2) - D^{-+}(s_1, s_2)$.

PROOF. Our method is similar to Baxter's, but more complicated.

(a) Denote $Y^{(n)}_{k_1 k_2} \hat{=} \Delta_{]\frac{k-1}{2^n}, \frac{k}{2^n}]} X$, where, $\frac{k}{2^n} \hat{=} (\frac{k_1}{2^n}, \frac{k_2}{2^n})$, $\frac{k-1}{2^n} \hat{=} (\frac{k_1-1}{2^n}, \frac{k_2-1}{2^n})$; $Z_n = \sum_{k_1=1}^{2^n} \sum_{k_2=1}^{2^n} \left[Y^{(n)}_{k_1 k_2}\right]^2$. It is easy to see that $\{Y^{(n)}_{k_1 k_2},\ k_i = 1, 2, \ldots, 2^n,\ i = 1, 2\}$ is a Gaussian system, which can be ordered as $\{Y^{(n)}_{k'},\ k' = 1, 2, \ldots, 2^{2n}\}$. Then

$$Z_n(\omega) = \sum_{k'=1}^{2^{2n}} \left[Y^{(n)}_{k'}\right]^2 = (Y_n, Y_n),$$

where $Y_n \hat{=} (Y^{(n)}_1, \ldots, Y^{(n)}_{2^{2n}})$ is a Gaussian random vector with mean zero and covariance matrix $V_n = (v^{(n)}_{j'k'},\ j', k' = 1, 2, \ldots, 2^{2n})$, $v^{(n)}_{j'k'} \hat{=} E(Y^{(n)}_{j'} Y^{(n)}_{k'})$.

By J. Yeh [3] Th.16.16, we have

$$\begin{aligned}
EZ_n &= E[(Y_n, Y_n)] = Tr(V_n) = \sum_{k'=1}^{2^{2n}} v^{(n)}_{k'k'} \qquad (2.4)\\
EZ_n^2 &= E[(Y_n, Y_n)^2] \\
&= 3 \sum_{j'=1}^{2^{2n}} \left(v^{(n)}_{j'j'}\right)^2 + 2 \sum_{j'=1}^{2^{2n}} \sum_{j'<k'} \left[v^{(n)}_{j'j'} v^{(n)}_{k'k'} + 2(v^{(n)}_{j'k'})^2\right]
\end{aligned}$$

So,

$$\sigma^2(Z_n) = EZ_n^2 - (EZ_n)^2 = 2 \sum_{j'=1}^{2^{2n}} \sum_{k'=1}^{2^{2n}} \left(v^{(n)}_{j'k'}\right)^2$$

$$= 2\sum_{j_1=1}^{2^n}\sum_{j_2=1}^{2^n}\sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n}(v_{jk})^2 = \lambda_n$$

where, $v_{jk}^{(n)} \triangleq EY_{j_1j_2}^{(n)}Y_{k_1k_2}^{(n)}$, $j=(j_1,j_2)$, $k=(k_1,k_2)$.

(b) By the Chebyshev inequality,

$$P(\omega : |Z_n - EZ_n| \geq \frac{n}{2^{n/2}}) \leq \frac{2^n}{n^2}\lambda_n \tag{2.5}$$

Since $\lim_{n\to\infty}\frac{n}{2^{n/2}} = 0$, if we can prove $\lambda_n = O(\frac{1}{2^n})$, then by the Borel-Cantelli lemma,

$$\lim_{n\to\infty}\{Z_n - EZ_n\} = 0 \quad \text{a.s.}$$

(c) Hence, we hope to estimate $\lambda_n$. Since,

$$\begin{aligned} v_{jk}^{(n)} &= EY_{j_1j_2}^{(n)}Y_{k_1k_2}^{(n)} \\ &= E\Delta_{]\frac{j-1}{2^n},\frac{j}{2^n}]}X \cdot \Delta_{]\frac{k-1}{2^n},\frac{k}{2^n}]}X \\ &= \overline{R}(k_1,k_2;j_1,j_2) + \overline{R}(k_1,k_2;j_1-1,j_2-1) \\ &\quad - \overline{R}(k_1,k_2;j_1-1,j_2) - \overline{R}(k_1,k_2;j_1,j_2-1) \\ &\quad + \overline{R}(k_1-1,k_2-1;j_1,j_2) + \overline{R}(k_1-1,k_2-1;j_1-1,j_2-1) \\ &\quad - \overline{R}(k_1-1,k_2-1;j_1-1,j_2) - \overline{R}(k_1-1,k_2-1;j_1,j_2-1) \\ &\quad - \overline{R}(k_1-1,k_2;j_1,j_2) - \overline{R}(k_1,k_2-1;j_1-1,j_2-1) \\ &\quad + \overline{R}(k_1-1,k_2;j_1-1,j_2) + \overline{R}(k_1-1,k_2;j_1,j_2-1) \\ &\quad - \overline{R}(k_1,k_2-1;j_1,j_2) - \overline{R}(k_1,k_2-1;j_1-1,j_2-1) \\ &\quad + \overline{R}(k_1,k_2-1;j_1-1,j_2) + \overline{R}(k_1,k_2-1;j_1,j_2-1) \\ &\triangleq I_1 + I_2 + I_1' + I_2' \end{aligned} \tag{2.6}$$

where, $\overline{R}(k_1,k_2;j_1,j_2) \triangleq R(\frac{k_1}{2^n},\frac{k_2}{2^n};\frac{j_1}{2^n},\frac{j_2}{2^n})$.

(d) By Taylor's expansion formula,

$$\begin{aligned} I_1 &= [\overline{R}(k_1,k_2;j_1,j_2) - \overline{R}(k_1,k_2;j_1-1,j_2)] \\ &\quad + [\overline{R}(k_1,k_2;j_1-1,j_2-1) - \overline{R}(k_1,k_2;j_1,j_2-1)] \\ &= \left[\frac{1}{2^n}\frac{\partial\overline{R}}{\partial t_1}(k_1,k_2;j_1-1,j_2) + \frac{1}{2^{2n}}\frac{\partial^2\overline{R}}{\partial t_1^2}(k_1,k_2;j_1-1,j_2)\right. \\ &\quad \left. + O(\frac{1}{2^{3n}})\right] \end{aligned}$$

$$- \left[ \frac{1}{2^n} \frac{\partial \overline{R}}{\partial t_1}(k_1, k_2; j_1 - 1, j_2 - 1) + \frac{1}{2^{2n}} \frac{\partial^2 \overline{R}}{\partial t_1^2}(k_1, k_2; j_1 - 1, j_2 - 1) + O(\frac{1}{2^{3n}}) \right]$$

$$= \frac{1}{2^n} \left[ \frac{\partial \overline{R}}{\partial t_1}(k_1, k_2; j_1 - 1, j_2) - \frac{\partial \overline{R}}{\partial t_1}(k_1, k_2; j_1 - 1, j_2 - 1) \right] + \frac{1}{2^{2n}} \left[ \frac{\partial^2 \overline{R}}{\partial t_1^2}(k_1, k_2; j_1 - 1, j_2) - \frac{\partial^2 \overline{R}}{\partial t_1^2}(k_1, k_2; j_1 - 1, j_2 - 1) \right] + O(\frac{1}{2^{3n}})$$

$$= \frac{1}{2^n} \left[ \frac{\partial^2 \overline{R}}{\partial t_2 \partial t_1}(k_1, k_2; j_1 - 1, j_2^*) \cdot \frac{1}{2^n} \right] + \frac{1}{2^{2n}} \frac{\partial^3 \overline{R}}{\partial t_2 \partial t_1^2}(k_1, k_2; j_1 - 1, j_2^{**}) \cdot \frac{1}{2^n} + O(\frac{1}{2^{3n}})$$

$$(2.7) \quad = \frac{1}{2^{2n}} \frac{\partial^2 \overline{R}}{\partial t_2 \partial t_1}(k_1, k_2; j_1 - 1, j_2^*) + O(\frac{1}{2^{3n}}),$$

where, $j_2^* \in (j_2 - 1, j_2)$, $j_2^{**} \in (j_2 - 1, j_2)$.

(e) By the same method,

$$I_1' = -\frac{1}{2^{2n}} \frac{\partial^2 \overline{R}}{\partial t_2 \partial t_1}(k_1 - 1, k_2; j_1 - 1, j_2^{*'}) + O(\frac{1}{2^{3n}})$$

where $j_2^{*'} \in (j_2 - 1, j_2)$. So,

$$I_1 + I_1' = \frac{1}{2^{2n}} \left[ \frac{\partial^2 \overline{R}}{\partial t_2 \partial t_1}(k_1, k_2; j_1 - 1, j_2^*) - \frac{\partial^2 \overline{R}}{\partial t_2 \partial t_1}(k_1 - 1, k_2; j_1 - 1, j_2^{*'}) \right] + O(\frac{1}{2^{3n}})$$

$$(2.8) \quad = \frac{1}{2^{2n}} \left[ \frac{\partial^3 \overline{R}}{\partial s_1 \partial t_2 \partial t_1} \cdot \frac{1}{2^n} + \frac{\partial^3 \overline{R}}{\partial t_2^2 \partial t_1} \cdot \frac{|j_2^* - j_2^{*'}|}{2^n} \right] + O(\frac{1}{2^{3n}}) = O(\frac{1}{2^{3n}}).$$

For the same reason, $I_2 + I_2' = O(\frac{1}{2^{3n}})$. Hence

$$(2.9) \qquad v_{jk}^{(n)} = O(\frac{1}{2^{3n}}), \quad (v_{jk}^{(n)})^2 = O(\frac{1}{2^{6n}}).$$

(f) We hope to estimate $v_{kk}^{(n)}$. Now,

$$\begin{aligned}
I_1 &= [\overline{R}(k_1,k_2;k_1,k_2) - \overline{R}(k_1,k_2;k_1-1,k_2)] \\
&\quad + [\overline{R}(k_1,k_2;k_1-1,k_2-1) - \overline{R}(k_1,k_2;k_1,k_2-1) \\
&= [\frac{\partial \overline{R}}{\partial t_1}(k_1,k_2;k_1-1,k_2)\cdot\frac{1}{2^n} + \frac{\partial^2 \overline{R}}{\partial t_1^2}(k_1,k_2;k_1-1,k_2)\cdot\frac{1}{2^{2n}} + O(\frac{1}{2^{3n}})] \\
&\quad - [\frac{\partial \overline{R}}{\partial t_1}(k_1,k_2;k_1-1,k_2-1)\cdot\frac{1}{2^n} + \frac{\partial^2 \overline{R}}{\partial t_1^2}(k_1,k_2;k_1-1,k_2-1)\frac{1}{2^{2n}} \\
&\quad + O(\frac{1}{2^{3n}})] \\
&= [\frac{\partial^2 \overline{R}}{\partial t_2 \partial t_1}(k_1,k_2;k_1-1,k_2)\cdot\frac{1}{2^{2n}} + O(\frac{1}{2^{3n}})] \\
&\quad + [\frac{\partial^3 \overline{R}}{\partial t_2 \partial t_1^2}(k_1,k_2;k_1-1,k_2)\cdot\frac{1}{2^{3n}}] + O(\frac{1}{2^{3n}}) \\
&= \frac{1}{2^{2n}}\frac{\partial^2 \overline{R}}{\partial t_2 \partial t_1}(k_1,k_2;k_1-1,k_2) + O(\frac{1}{2^{3n}}) \\
&= \frac{1}{2^{2n}}[D^{--}(\frac{k_1}{2^n},\frac{k_2}{2^n}) + O(\frac{1}{2^n})] + O(\frac{1}{2^{3n}}) \\
&= \frac{1}{2^{2n}}D^{--}(\frac{k_1}{2^n},\frac{k_2}{2^n}) + O(\frac{1}{2^{3n}}).
\end{aligned}$$

By the same approach as for $I_1$, we can evaluate $I_1'$, $I_2$, $I_2'$, hence,

$$\begin{aligned}
v_{kk}^{(n)} &= \frac{1}{2^{2n}}[D^{++}(\frac{k_1}{2^n},\frac{k_2}{2^n}) + D^{--}(\frac{k_1}{2^n},\frac{k_2}{2^n}) \\
&\quad - D^{+-}(\frac{k_1}{2^n},\frac{k_2}{2^n}) - D^{-+}(\frac{k_1}{2^n},\frac{k_2}{2^n})] + O(\frac{1}{2^{3n}}) \qquad (2.10)\\
&= \frac{1}{2^{2n}}f(\frac{k_1}{2^n},\frac{k_2}{2^n}) + O(\frac{1}{2^{3n}}).
\end{aligned}$$

So,

$$(v_{kk}^{(n)})^2 = \frac{1}{2^{4n}}f^2(\frac{k_1}{2^n},\frac{k_2}{2^n}) + O(\frac{1}{2^{5n}}) = O(\frac{1}{2^{4n}}).$$

Finally,

$$2^n\lambda_n = 2^{n+1}\{2^{2n}\cdot O(\frac{1}{2^{4n}}) + 2^{2n}(2^{2n}-1)O(\frac{1}{2^{6n}})\} = O(1).$$

Hence, $\lambda_n = O(\frac{1}{2^n})$, we have $Z_n - EZ_n \to 0$ $(n\to\infty)$ a.s. or, $\lim_{n\to\infty} Z_n = \lim_{n\to\infty} EZ_n$ a.s.

By (2.10)

$$\sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n} v_{kk}^{(n)} = \sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n} \frac{1}{2^{2n}} f(\frac{k_1}{2^n}, \frac{k_2}{2^n}) + O(\frac{1}{2^n}).$$

From (2.4),

$$EZ_n = \sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n} \frac{1}{2^{2n}} f(\frac{k_1}{2^n}, \frac{k_2}{2^n}) + O(\frac{1}{2^n}).$$

Let $n \to \infty$, $EZ_n \to \int_0^1\int_0^1 f(s_1, s_2)ds_1 ds_2$. So,

$$Z_n \to \int_0^1\int_0^1 f(s_1, s_2)ds_1 ds_2 \quad \text{a.s.} \quad \text{the end.}$$

NOTE. The condition $EX(t) = 0$ is not basic. One can replace it by some weaker conditions.

## 3. Corollaries

COROLLARY 3.1. *Let $\{W(t),\ t \in [0,1]^2\}$ be a Brownian Sheet, then*

$$\sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n} \Big[W(\frac{k_1}{2^n}, \frac{k_2}{2^n}) + W(\frac{k_1-1}{2^n}, \frac{k_2-1}{2^n}) - W(\frac{k_1-1}{2^n}, \frac{k_2}{2^n}) - W(\frac{k_1}{2^n}, \frac{k_2-1}{2^n})\Big]^2 \to 1 \quad a.s. \quad n \to \infty.$$

PROOF. Now

$$\begin{aligned} R(s_1, s_2; t_1, t_2) &\triangleq EW(s_1, s_2)W(t_1, t_2) \\ &= (s_1 \wedge t_1) \cdot (s_2 \wedge t_2), \quad (s_1, s_2) \neq (t_1, t_2). \end{aligned}$$

For any fixed $(s_1, s_2) \in [0,1]^2$

(i) When $t_1 \geq s_1,\ t_2 \geq s_2,\ R = s_1 s_2,\ \frac{\partial^2 R}{\partial t_2 \partial t_1} = 0,\ D^{++}(s_1, s_2) = 0$

(ii) When $t_1 \leq s_1,\ t_2 \geq s_2,\ R = t_1 s_2,\ \frac{\partial^2 R}{\partial t_2 \partial t_1} = 0,\ D^{-+}(s_1, s_2) = 0$

(iii) When $t_1 \leq s_1,\ t_2 \leq s_2,\ R = t_1 t_2,\ \frac{\partial^2 R}{\partial t_2 \partial t_1} = 1,\ D^{--}(s_1, s_2) = 1$

(iv) When $t_1 \geq s_1,\ t_2 \leq s_2,\ R = s_1 t_2,\ \frac{\partial^2 R}{\partial t_2 \partial t_1} = 0,\ D^{+-}(s_1, s_2) = 0$

So, $f(s_1, s_2) = 1$, by Th. 2.2:

$$\sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n}\Big[W(\frac{k_1}{2^n},\frac{k_2}{2^n}) + W(\frac{k_1-1}{2^n},\frac{k_2-1}{2^n}) - W(\frac{k_1-1}{2^n},\frac{k_2}{2^n}) - W(\frac{k_1}{2^n},\frac{k_2-1}{2^n})\Big]^2 \to 1 \quad \text{a.s.} \quad (n\to\infty).$$

COROLLARY 3.2. *Let $\{X(t),\ t\in[0,1]^2\}$ be two-parameter Ornstein-Uhlenbeck process,*

$$X(t) = \sigma e^{-\alpha_1 t_1 - \alpha_2 t_2}\int_0^{t_1}\int_0^{t_2} e^{\alpha_1 a_1 + \alpha_2 a_2} dW(a_1, a_2)$$

*where $t = (t_1, t_2)$, then*

$$\sum_{k_1=1}^{2^n}\sum_{k_2=1}^{2^n}\Big[\Delta_{]\frac{k-1}{2^n},\frac{k}{2^n}]}X\Big]^2 \to \sigma^2 \quad a.s. \quad (n\to\infty).$$

PROOF. Now,

$$R(s_1, s_2; t_1, t_2) = EX(s_1, s_2)X(t_1, t_2)$$
$$= \sigma^2 e^{-\alpha_1 s_1 - \alpha_2 s_2}\cdot e^{-\alpha_1 t_1 - \alpha_2 t_2}\cdot\frac{e^{2\alpha_1(s_1\wedge t_1)}-1}{2\alpha_1}\cdot\frac{e^{2\alpha_2(s_2\wedge t_2)}-1}{2\alpha_2}$$

(i)' When $t_1 \ge s_1,\ t_2 \ge s_2$,

$$R(s_1, s_2; t_1, t_2) = \sigma^2 e^{-\alpha_1(s_1+t_1)-\alpha_2(s_2+t_2)}\frac{e^{\alpha_1 s_1}-1}{2\alpha_1}\cdot\frac{e^{2\alpha_2 s_2}-1}{2\alpha_2}$$

$$\frac{\partial R}{\partial t_1} = \sigma^2(-\alpha_1)e^{-\alpha_1(s_1+t_1)-\alpha_2(s_2+t_2)}\cdot\frac{e^{2\alpha_1 s_1}-1}{2\alpha_1}\frac{e^{2\alpha_2 s_2}-1}{2\alpha_2}$$

$$D^{++}(s_1, s_2) = \lim_{\substack{t_1\downarrow s_1\\ t_2\downarrow s_2}}\frac{\partial^2 R}{\partial t_2\partial t_1} = \frac{\sigma^2}{4}(1-e^{-2\alpha_1 s_1})(1-e^{-2\alpha_2 s_2})$$

(ii)' When $t_1 \le s_1,\ t_2 \ge s_2$,

$$D^{-+}(s_1, s_2) = \frac{\sigma^2}{4}(1+e^{-2\alpha_2 s_2})(1-e^{-2\alpha_2 s_2}).$$

(iii)' When $t_1 \le s_1,\ t_2 \le s_2$,

$$D^{--}(s_1, s_2) = \frac{\sigma^2}{4}(1+e^{-2\alpha_1 s_1})(1+e^{-2\alpha_2 s_2}).$$

(iv)' When $t_1 \geq s_1$, $t_2 \leq s_2$,

$$D^{+-}(s_1, s_2) = \frac{\sigma^2}{4}(1 - e^{-\alpha_1 s_1})(1 + e^{-2\alpha_2 s_2}).$$

So,

$$f(s_1, s_2) = D^{++}(s_1, s_2) + D^{--}(s_1, s_2) - D^{+-}(s_1, s_2) - D^{-+}(s_1, s_2) = \sigma^2.$$

By Theorem 2.2,

$$\sum_{k_1=1}^{2^n} \sum_{k_2=1}^{2^n} \left[\Delta_{]( \frac{k_1-1}{2^n}, \frac{k_2-1}{2^n}),( \frac{k_1}{2^n}, \frac{k_2}{2^n})]} X\right]^2 \to \sigma^2 \quad \text{a.s.}$$

REMARK. After we finished this paper, Professor P. Révész called our attention to C. M. Deo's work related to the topic. Deo's result is more general. He considered general partitions of $[0,1]^q$, while we only discuss nested partition case. Anyway, for the nested partition case, our condition is weaker than that of Deo. We only require the covariance function has third continuous bounded derivative, but Deo's conclusion need fourth derivative and some other conditions. Since the methods and forms of our results are different from Deo's it seems that our paper has somewhat independent interest.

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## REFERENCES

[1] BAXTER, G., A strong limit theorem for Gaussian processes, *Proc. Amer. Math. Soc.* **7** (1956), 522—527. *MR* **19**—890

[2] IMKELLER, P., Two-parameter martingales and their quadratic variation, Lecture Notes in Math., Vol. 1308, Springer-Verlag, Berlin-New York, 1988. *MR* **89e**: 60098

[3] YEH, J., Stochastic processes and the Wiener integral, Pure and applied mathematics, Vol. 13, Marcel Dekker, New York, 1973. *MR* **57** # 14166

[4] LÉVY, P., Le mouvement brownien plan, *Amer. J. Math.* **62** (1940), 487—550. *MR* **2**—107

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# ON THE COINCIDENCE OF SOME NOTIONS OF QUASI-UNIFORM COMPLETENESS DEFINED BY FILTER PAIRS

J. DEÁK

**Abstract**

Four types of quasi-uniform completeness coincide in uniformly regular spaces satisfying a common generalization of symmetry and equinormality.

The aim of this note is to answer a question raised by Fletcher and Hunsaker [5]. First we have to recall some definitions; see [5] for the sources, and [6] for basic definitions; see also [2, 3].

A filter pair $(\mathfrak{f}, \mathfrak{g})$ in a quasi-uniform space $(X, \mathcal{U})$ is (i) *convergent* if there is an $x \in X$ such that $\mathfrak{f}\ \mathcal{U}^{-\mathrm{tp}}$-converges and $\mathfrak{g}\ \mathcal{U}^{\mathrm{tp}}$-converges to $x$; (ii) *Cauchy* if for any $U \in \mathcal{U}$ there are $F \in \mathfrak{f}$ and $G \in \mathfrak{g}$ with $F \times G \subset U$; (iii) *linked* if $F \cap G \neq \emptyset$ whenever $F \in \mathfrak{f}$ and $G \in \mathfrak{g}$.

The quasi-uniform space $(X, \mathcal{U})$ is (i) *pair complete* (equivalent to the double completeness of [1]) if each linked Cauchy filter pair is convergent; (ii) *D-complete* if the second element of each Cauchy filter pair is $\mathcal{U}^{\mathrm{tp}}$-convergent; (iii) *strongly D-complete* if the first element of each Cauchy filter pair has a $\mathcal{U}^{\mathrm{tp}}$-cluster point; (iv) *quiet* if for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $xUy$ whenever there is a Cauchy filter pair $(\mathfrak{f}, \mathfrak{g})$ with $Vx \in \mathfrak{g}$ and $V^{-1}y \in \mathfrak{f}$ (in the terminology of [3]: the system of all the Cauchy filter pairs is uniformly weakly concentrated); (v) *uniformly regular* if for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $\overline{Vx} \subset Ux$ for each $x \in X$ ($\overline{\phantom{x}}$ always denotes the $\mathcal{U}^{\mathrm{tp}}$-closure); (vi) *equinormal* provided that if $C$ is $\mathcal{U}^{\mathrm{tp}}$-closed, $H$ is $\mathcal{U}^{\mathrm{tp}}$-open, $C \subset H$ then there is a $U \in \mathcal{U}$ with $U[C] \subset H$; (vii) *locally symmetric* if for any $U \in \mathcal{U}$ and $x \in X$ there is a $V \in \mathcal{U}$ with $V^{-1}[Vx] \subset Ux$. Observe that, in contrast to [4, 5], $T_1$ is not assumed, not even in the definition of quietness.

According to [5] Corollary 3.2, the three notions of completeness defined above coincide in uniform spaces as well as in equinormal uniformly regular quasi-uniform spaces. Now [5] Questions 4.1 runs as follows:

*Does there exist a natural class of quasi-uniform spaces containing all uniform spaces and all equinormal quiet spaces in which the concepts of D-*

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*Key words and phrases.* Quasi-uniformity, pair complete, (strongly) $D$-complete, half-complete, uniformly regular, quiet, equinormal, locally symmetric.

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*completeness, strong D-completeness and pair completeness coincide? In particular, do the locally symmetric quiet spaces comprise such a class?*

We are going to give a counterexample for the second part of the question, and offer a solution to the first part, with quietness replaced by the more general notion of uniform regularity.

(By [4] Proposition 1.2, any quiet quasi-uniformity is uniformly regular: the entourage $V$ in the definition of quietness will also do in the definition of uniform regularity; indeed, if $y \in \overline{Vx}$ then let $\mathfrak{f}$ be generated by $\{\{y\}\}$, and $\mathfrak{g}$ by the trace on $Vx$ of the $\mathcal{U}^{\mathrm{tp}}$-neighbourhood filter of $y$.)

EXAMPLE. Let $X = \mathbf{R} \setminus \{0\}$, and define $\mathcal{U}$ by the quasi-metric

$$d(x,y) = \begin{cases} \min\{y-x, 1\} & \text{if } x < 0 < y, \\ 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Both $\mathcal{U}^{\mathrm{tp}}$ and $\mathcal{U}^{-\mathrm{tp}}$ are discrete, therefore $\mathcal{U}$ is locally symmetric (and so is $\mathcal{U}^{-1}$). $\mathcal{U}$ is quiet (even if $\mathrm{T}_1$ is required in the definition), pair complete, but not $D$-complete, because if $\mathfrak{f}$ is generated by $\{]-\varepsilon, 0[ : \varepsilon > 0\}$ and $\mathfrak{g}$ by $\{]0, \varepsilon[ : \varepsilon > 0\}$ then $(\mathfrak{f}, \mathfrak{g})$ is Cauchy, but $\mathfrak{g}$ is not $\mathcal{U}^{\mathrm{tp}}$-convergent.

DEFINITIONS. A quasi-uniform space $(X, \mathcal{U})$ is

a) *half-complete* (equivalent to completeness in the sense of [1] p. 228) if the second element of each linked Cauchy filter pair is $\mathcal{U}^{\mathrm{tp}}$-convergent;

b) *semi-symmetric* provided that if $C$ is $\mathcal{U}^{\mathrm{tp}}$-closed, $H$ is $\mathcal{U}^{\mathrm{tp}}$-open, $C \subset H$, and $U^{-1}[C] \subset H$ for some $U \in \mathcal{U}$ then there is a $V \in \mathcal{U}$ with $V[C] \subset H$.

THEOREM. *The notions of D-completeness, strong D-completeness, pair completeness and half-completeness coincide in semi-symmetric uniformly regular spaces.*

REMARKS. a) It is evident that equinormal or symmetric spaces are semi-symmetric.

b) A semi-symmetric uniformly regular (or just regular) quasi-uniformity is locally symmetric: for $x \in X$ and $U \in \mathcal{U}$ with $Ux$ open, take a $V \in \mathcal{U}$ such that $Vx$ is closed and $V^2 \subset U$; let $C = X \setminus Ux$ and $H = X \setminus Vx$; now $V^{-1}[C] \subset H$, thus there is a $W \in \mathcal{U}$ with $W[C] \subset H$, i.e. $W^{-1}[Vx] \subset Ux$.

c) Parts of the proof below could be cited from [4, 5]; we prefer, however, to give a complete proof, mainly because $\mathrm{T}_1$ is assumed (although not used) throughout [4, 5].

PROOF. 1° Strong $D$-completeness implies $D$-completeness without any further assumption, since if $(\mathfrak{f}, \mathfrak{g})$ is Cauchy and $x$ is a $\mathcal{U}^{\mathrm{tp}}$-cluster point of $\mathfrak{f}$ then taking $F \in \mathfrak{f}$ and $G \in \mathfrak{g}$ with $F \times G \subset U$, we clearly have $G \subset U^2 x$.

2° A $D$-complete uniformly regular quasi-uniformity is pair complete: Assume that $(\mathfrak{f}, \mathfrak{g})$ is a Cauchy filter pair, and let the filter $\overline{\mathfrak{g}}$ be generated by $\{\overline{G}: G \in g\}$. $(\mathfrak{f}, \overline{\mathfrak{g}})$ is Cauchy, too, because if $\overline{Vx} \subset Ux$ $(x \in X)$ then $F \times G \subset V$, $F \in \mathfrak{f}$, $G \in \mathfrak{g}$ imply that $F \times \overline{G} \subset U$. Let $\mathfrak{g}$ $\mathcal{U}^{tp}$-converge to $x$; taking $F \in \mathfrak{f}$ and $G \in \mathfrak{g}$ with $F \times \overline{G} \subset U \in \mathcal{U}$, we have $F \subset U^{-1}x$ from $x \in \overline{G}$. This means that each Cauchy filter pair is convergent, whether linked or not.

3° It is evident that pair completeness implies half-completeness.

4° Assume now that $\mathcal{U}$ is half-complete, and let $(\mathfrak{f}, \mathfrak{g})$ be a Cauchy filter pair; we have to show that $\mathfrak{f}$ has a $\mathcal{U}^{tp}$-cluster point. We have already seen in 2° that $(\mathfrak{f}, \overline{\mathfrak{g}})$ is Cauchy; but so is $(\overline{\mathfrak{f}}, \overline{\mathfrak{g}})$, too, since if $F \in \mathfrak{f}$, $G \in \mathfrak{g}$ and $F \times \overline{G} \subset U$ then $\overline{F} \times \overline{G} \subset U^2$.

If $F \in \mathfrak{f}$, $G \in \mathfrak{g}$ and $\overline{F} \cap \overline{G} = \emptyset$ then we have $U[\overline{F}] \cap \overline{G} \neq \emptyset$ $(U \in \mathcal{U})$ from the Cauchy property, and hence $U^{-1}[\overline{F}] \cap \overline{G} \neq \emptyset$ $(U \in \mathcal{U})$ from the semi-symmetry; the last statement is evidently valid in the case $\overline{F} \cap \overline{G} \neq \emptyset$, too. Thus a filter $\mathfrak{h}$ is generated by

$$\{\overline{F} \cap U[\overline{G}]: F \in \mathfrak{f},\ G \in \mathfrak{g},\ U \in \mathcal{U}\}.$$

If $\overline{F} \times \overline{G} \subset U \in \mathcal{U}$, $F \in \mathfrak{f}$, $G \in \mathfrak{g}$ and $x, y \in \overline{F} \cap U[\overline{G}]$ then $xU^2y$; so $(\mathfrak{h}, \mathfrak{h})$ is linked and Cauchy, $\mathfrak{h}$ is $\mathcal{U}^{tp}$-convergent by the half-completeness, therefore $\overline{\mathfrak{f}} \subset \mathfrak{h}$ has a $\mathcal{U}^{tp}$-cluster point, and then so has $\mathfrak{f}$ itself.

## REFERENCES

[1] CSÁSZÁR, Á., *Foundations of general topology*, Pergamon Press, Oxford, 1963, *MR* **28** # 575

[2] DEÁK, J., Extensions of quasi-uniformities for prescribed bitopologies I, *Studia Sci. Math. Hungar.* **25** (1990), No. 1–2, 45–67. *MR* **92b**: 54058.

[3] DEÁK, J., Extensions of quasi-uniformities for prescribed bitopologies II, *Studia Sci. Math. Hungar.* **25** (1990), No. 1–2, 69–91. *MR* **92b**: 54058.

[4] FLETCHER, P. and HUNSAKER, W., Uniformly regular quasi-uniformities, *Topology Appl.* **37** (1990), 285–291. *MR* **92b**: 54062.

[5] FLETCHER, P. and HUNSAKER, W., Completeness using pairs of filters, *Topology Appl.* (to appear).

[6] FLETCHER, P. and LINDGREN, W. F., *Quasi-uniform spaces*, Lecture Notes in Pure Appl. Math. **77**, Marcel Dekker, New York, 1982. *MR* **84h: 54026**

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# ON POWERS OF $f$-DIVERGENCES DEFINING A DISTANCE

P. KAFKA, F. ÖSTERREICHER and I. VINCZE

**Summary**

This note deals with conditions on the value $\alpha \in (0,\infty)$ and the convex function $f$ so that the power $I_f^\alpha$ of the $f$-divergence $I_f$ defines a metric on the set of probability distributions on a measurable space. In Section 2, for a given $\alpha \in (0,1]$, a sufficient condition on $f$ is stated, whereas in Section 3, for a given $f$, two necessary conditions on $\alpha$ are given.

## 1. Preliminaries

Let $(\Omega, \mathcal{A})$ be a nondegenerate measurable space (i.e. $|\mathcal{A}| > 2$ and hence $|\Omega| > 1$) and let $\wp(\Omega, \mathcal{A})$ be the set of probability measures on $(\Omega, \mathcal{A})$. Furthermore let $\mathcal{F}$ be the set of convex functions $f\colon \mathbf{R}_+ \to \overline{\mathbf{R}}$ which are continuous at 0. And let the function $f^* \in \mathcal{F}$ be defined by

$$f^*(u) = u f\left(\frac{1}{u}\right) \quad \text{for} \quad u \in (0,\infty).$$

REMARK 1. Owing to the continuity of $f$ and $f^*$ at 0 and by setting $0 \cdot f\left(\frac{0}{0}\right) := 0 \quad \forall f \in \mathcal{F}$ it holds

$$x f^*\left(\frac{y}{x}\right) = y f\left(\frac{x}{y}\right) \qquad \forall x, y \in \mathbf{R}_+.$$

DEFINITION (cf. Csiszár [4] and Ali and Silvey [1]). Let $P, Q \in \wp(\Omega, \mathcal{A})$. Then

$$I_f(Q,P) = \int f\left(\frac{q}{p}\right) \cdot p\, d\mu$$

is called $f$-divergence of $P$ and $Q$. (As usual $q$ and $p$ denote the Radon–Nikodym-derivatives of $P$ and $Q$ with respect to a dominating $\sigma$-finite measure $\mu$.)

We are interested in conditions on $\alpha \in (0,\infty)$ and on $f \in \mathcal{F}$ so that

$$\varrho_\alpha(Q,P) = [I_f(Q,P)]^\alpha$$

defines a metric on $\wp(\Omega, \mathcal{A})$, i.e. that $\varrho_\alpha$ has the properties

(M1) $\quad \varrho_\alpha(Q,P) \geqq 0$ with equality iff $Q = P \quad \forall P, Q \in \wp(\Omega, \mathcal{A})$,

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(M2) $$\varrho_\alpha(Q,P) \equiv \varrho_\alpha(P,Q)$$

and

(M3,$\alpha$) $$\varrho_\alpha(Q,R) + \varrho_\alpha(R,P) \geqq \varrho_\alpha(Q,P) \qquad \forall P,Q,R \in \wp(\Omega,\mathcal{A}).$$

PROPOSITION 1. *The above definition is justified and*

$$I_f(Q,P) \geqq f(1) \qquad \forall P,Q \in \wp(\Omega,\mathcal{A}),$$

*where equality holds if $Q = P$. Moreover, $Q = P$ follows from equality iff $f$ is strict convex at* 1.

PROOF. This follows from the property of the (convex) function $f \in \mathcal{F}$

$$\exists c \in \mathbf{R}: \ f(u) \geqq f(1) + c(u-1) \qquad \forall u \in \mathbf{R}_+,$$

where strict convexity of $f$ at 1 is by definition equivalent with the fact that equality above implies $u = 1$. □

PROPOSITION 2. *Let $f \in \mathcal{F}$. Then $I_f(Q,P) \equiv I_f(P,Q)$ is equivalent with the condition $\exists c \in \mathbf{R}: \ f^*(u) \equiv f(u) + c(u-1)$.*

For the proof we refer to the Uniqueness and Symmetry Theorem 1.13 in [8]. Note, however, that owing to Remark 1 $I_f(Q,P) \equiv I_{f^*}(P,Q)$ holds, so that the converse direction is obvious.

REMARK 2. The properties (M1) and (M2) are equivalent with the following properties on the function $f \in \mathcal{F}$:

(f1) $f(1) = 0$ and $f$ is strict convex at 1 and

(f2) $\exists c \in \mathbf{R}: \ f^*(u) \equiv f(u) + c(u-1)$.

Unless otherwise stated we will assume in the sequel that $f \in \mathcal{F}$ has the properties (f1) and $f^* = f$. The latter since we can restrict ourselves to $c = 0$ in (f2) without loss of generality.

REMARK 3. Let $f \in \mathcal{F}$ satisfy $f(1) = 0$ and $f^* = f$. Then

$$f(u) \geqq 0 \qquad \forall u \in \mathbf{R}_+.$$

This follows from the subceeding consequence of Jensen's inequality

$$\frac{f(u) + f^*(u)}{u+1} = \frac{1}{u+1} f(u) + \frac{u}{u+1} f\left(\frac{1}{u}\right) \geqq f(1).$$

## 2. A sufficient condition concerning $f$ for some power $\alpha \in (0,1]$

PROPOSITION 3. *Let $\alpha \in (0,1]$ and let $f \in \mathcal{F}$ satisfy $f(u) \geqq 0 \ \forall u \in \mathbf{R}_+$. Then*

($*,\alpha$) $$\left(rf\left(\frac{q}{r}\right)\right)^\alpha + \left(pf\left(\frac{r}{p}\right)\right)^\alpha \geqq \left(pf\left(\frac{q}{p}\right)\right)^\alpha \qquad \forall p,q,r \in \mathbf{R}_+,$$

*is a sufficient condition for the validity of* (M3,$\alpha$).

PROOF. Let $P, Q, R \in \wp(\Omega, \mathcal{A})$, $\mu$ be a dominating $\sigma$-finite measure and $p, q, r$ be the corresponding Radon–Nikodym-derivatives of $P, Q, R$ with respect to $\mu$. Then the application of $(*,\alpha)$ and Minkowski's inequality yields

$$\begin{aligned}(I_f(Q,P))^\alpha &= \left(\int \left[\left(pf\left(\frac{q}{p}\right)\right)^\alpha\right]^{1/\alpha} d\mu\right)^\alpha \leqq \\ &\leqq \left(\int \left[\left(rf\left(\frac{q}{p}\right)\right)^\alpha + \left(pf\left(\frac{r}{p}\right)\right)^\alpha\right]^{1/\alpha} d\mu\right)^\alpha \leqq \\ &\leqq \left(\int \left[\left(rf\left(\frac{q}{r}\right)\right)^\alpha\right]^{1/\alpha} d\mu\right)^\alpha + \left(\int \left[\left(pf\left(\frac{r}{p}\right)\right)^\alpha\right]^{1/\alpha} d\mu\right)^\alpha = \\ &= (I_f(Q,R))^\alpha + (I_f(R,P))^\alpha \qquad \square\end{aligned}$$

Summarizing Propositions 1, 2 and 3 and Remark 3 we get Theorem 1.

THEOREM 1. *Let* $\alpha \in (0,1]$ *and let* $f \in \mathcal{F}$ *satisfy* (f1), $f^* = f$ *and* $(*,\alpha)$. *Then*

$$\varrho_\alpha(Q,P) = [I_f(Q,P)]^\alpha$$

*is a metric on* $\wp(\Omega, \mathcal{A})$. $\square$

Let $\alpha \in (0,1]$ and let $\mathcal{F}_2(\alpha)$ be the subset of functions $f \in \mathcal{F}$ satisfying $f(1) = 0$, $f^* = f$ and

$$(**,\alpha) \qquad f(u) = \frac{|u^\alpha - 1|^{1/\alpha}}{h(u)}$$

where $h: \mathbf{R}_+ \to \overline{\mathbf{R}}$ satisfies $h(0) < \infty$ and is decreasing on $[0,1]$ and continuous at 1. (In view of the class of Examples (E2), $s < 1/2$, it is not appropriate to assume in addition $h(1) \neq 0$).

REMARK 4. $f^* = f$ and $(**,\alpha)$ yield $h(u) \equiv h\left(\frac{1}{u}\right)$. Hence $h$ is increasing on $[1,\infty)$.

THEOREM 2. *Let* $f \in \mathcal{F}_2(\alpha)$ *for* $\alpha \in (0,1]$. *Then* $f$ *satisfies* (f1) *and* $(*,\alpha)$.

PROOF. *Proof of* (f1): In view of Remark 3 a function $f \in \mathcal{F}_2(\alpha)$ is nonnegative. This yields together with $(**,\alpha)$

$$0 < h(u) \leqq h(0) \qquad \forall u \in [0,1)$$

and, consequently, in view of the continuity of $h$ at 1 and Remark 4

$$f(u) = \frac{|u^\alpha - 1|^{1/\alpha}}{h(u)} \geqq \frac{|u^\alpha - 1|^{1/\alpha}}{h(0)} \qquad \forall u \in \mathbf{R}_+.$$

Because of $h(0)\in(0,\infty)$ this implies $f(u)>0$ for $u\neq 1$. □

*Proof of* $(*,\alpha)$: We can and do restrict ourselves to the case $q<p$ since $p<q$ can be led back to the above case by exchanging $p$ and $q$ and observing $f^*=f$ and since for $p=q$ $(*,\alpha)$ follows immediately from $f(1)=0$ or $0\cdot f\left(\frac{0}{0}\right)=0$, respectively.

*Proof of* $(*,\alpha)$ *for the cases* (i) $q<p\leqq r$ *and* (ii) $r\leqq q<p$: Since $f$ is decreasing on $[0,1]$ and $f$ is nonnegative,

(i) i.e. $\frac{q}{r}\leqq\frac{q}{p}<1$ implies $rf\left(\frac{q}{r}\right)\geqq rf\left(\frac{q}{p}\right)\geqq pf\left(\frac{q}{p}\right)$, and

(ii) i.e. $\frac{r}{p}\leqq\frac{q}{p}<1$ implies $pf\left(\frac{r}{p}\right)\geqq pf\left(\frac{q}{p}\right)$.

Consequently $(*,\alpha)$ holds in either case.

*Proof of* $(*,\alpha)$ *for the case* (iii) $q<r<p$: For $q=0<r<p$ $(*,\alpha)$ reduces to $f(u)\geqq f(0)|u^\alpha-1|^{1/\alpha}$ or, equivalently, to $h(0)\geqq h(u)$. Hence, let us assume further on $q>0$.

The form of $f$ implies

$$pf\left(\frac{q}{p}\right)=\frac{|p^\alpha-q^\alpha|^{1/\alpha}}{h\left(\frac{q}{p}\right)}.$$

Now let $\gamma=\dfrac{p^\alpha-r^\alpha}{p^\alpha-q^\alpha}$. Then $(*,\alpha)$ reduces to

$$(*',\alpha)\qquad (1-\gamma)\left(h\left(\frac{q}{r}\right)\right)^{-\alpha}+\gamma\left(h\left(\frac{r}{p}\right)\right)^{-\alpha}\geqq\left(h\left(\frac{q}{p}\right)\right)^{-\alpha}.$$

However, because of $\frac{q}{p}<\frac{q}{r}<1$ it holds $h\left(\frac{q}{p}\right)\geqq h\left(\frac{q}{r}\right)$ and because of $\frac{q}{p}<\frac{r}{p}<1$ $h\left(\frac{q}{p}\right)\geqq h\left(\frac{r}{p}\right)$. Hence

$$(1-\gamma)h\left(\frac{q}{r}\right)+\gamma h\left(\frac{r}{p}\right)\leqq h\left(\frac{q}{p}\right).$$

Together with the fact that the function $x\to x^{-\alpha}$ is decreasing and convex this and the application of Jensen's inequality yield

$$(1-\gamma)\left(h\left(\frac{q}{r}\right)\right)^{-\alpha}+\gamma\left(h\left(\frac{r}{p}\right)\right)^{-\alpha}\geqq$$

$$\geqq\left((1-\gamma)h\left(\frac{q}{r}\right)+\gamma h\left(\frac{r}{p}\right)\right)^{-\alpha}\geqq\left(h\left(\frac{q}{p}\right)\right)^{-\alpha}. \qquad \square$$

Let $\alpha \in (0,1]$ and let $\mathcal{F}_3(\alpha)$ be the subset of functions $f \in \mathcal{F}$ satisfying $f(1)=0$, $f^* = f$ and

$$(***,\alpha) \qquad f(u) = \frac{|u-1|^{1/\alpha}}{(u+1)^{1/\alpha - 1} h_0(u)}$$

where $h_0 \colon \mathbf{R}_+ \to \overline{\mathbf{R}}$ satisfies $h_0(0) < \infty$ and is decreasing on $[0,1]$ and continuous at 1.

REMARK 5. $f^* = f$ and $(***,\alpha)$ yield $h_0(u) \equiv h_0\left(\frac{1}{u}\right)$.

Hence $h_0$ is increasing on $[1,\infty)$.

PROPOSITION 4. *Let* $\alpha \in (0,1]$. *Then* $\mathcal{F}_3(\alpha)$ *is a subset of* $\mathcal{F}_2(\alpha)$.

PROOF. Let $f \in \mathcal{F}_3(\alpha)$. Then

$$f(u) = \frac{|u^\alpha - 1|^{1/\alpha}}{h_0(u) h_1(u)} \quad \text{with} \quad h_1(u) = \left(\frac{u^\alpha - 1}{u-1}\right)^{1/\alpha} (u+1)^{1/\alpha - 1}.$$

If $h_0$ and $h_1$ are nonnegative and decreasing on $[0,1]$ then so is $h = h_0 \cdot h_1$. However, $h_0$ has these properties by assumption and the nonnegativity of $h_0$ is obvious. To see that $h_0$ is decreasing on $[0,1]$ let $u \in (0,1)$. Then

$$h_1'(u) = \frac{1}{\alpha}\left(\frac{u^\alpha - 1}{u-1}\right)^{1/\alpha - 1} (u-1)^{-2}(1+u)^{1/\alpha - 1} k(u)$$

with $k(u) = 2 - \alpha + \alpha u - \alpha u^{\alpha - 1} - (2-\alpha)u^\alpha$. Since $k$ is concave and $k(1)=0$, $k'(1) = 0$ it holds $k(u) \leqq 0$ and hence, $h_1'(u) \leqq 0$.

From the following (classes of) examples

$$\text{(E1)} \qquad f_k(u) = \frac{1}{2}\left|u^{1/k} - 1\right|^k \qquad k \in [1,\infty)$$

$$\text{(E2)} \qquad f^s(u) = \frac{1}{2}\left[u + 1 - (u^s + u^{(1-s)})\right] \qquad s \in (0,1)$$

$$\text{(E3)} \qquad \Phi_k(u) = \frac{1}{2}\frac{|u-1|^k}{(u+1)^{k-1}} \qquad k \in [1,\infty)$$

$$\text{(E4)} \qquad f(u) = \sqrt{u^2+1} - (u+1)/\sqrt{2} = \frac{1}{2}\frac{(u-1)^2}{\sqrt{u^2+1} + (u+1)/\sqrt{2}}$$

obviously $f_k$ belongs to $\mathcal{F}_2(1/k)$ but not to $\mathcal{F}_3(1/k)$, $\Phi_k$ belongs to $\mathcal{F}_3(1/k)$ and for (E4) $f$ belongs to $\mathcal{F}_3(1/2)$. Finally, it can be shown that for the class (E2), for which $f^{1/2} = f_2$ from (E1), $f^s$ belongs to $\mathcal{F}_2(\min(s, 1-s))$.

For $f_k$ from the class of Examples (E1), introduced by Boekee in [2], Example 4.1.2, it is an immediate consequence of Minkowski's inequality that

$\varrho_{1/k}$ is a metric. According to [3], Bemerkung 3.1, the so-called Hellinger-distance, given by $f_2$, can even be related to the norm of a Hilbert space on the set of bounded signed measures on $(\Omega, \mathcal{A})$.

For $f^s$ from the class of Examples (E2) Csiszár and Fischer have shown in [3], §4 that $\varrho_{\min(s,1-s)}$ is a metric.

For $\Phi_2$ from the class of Examples (E3), introduced by Puri and Vincze in [10], a direct proof of (M3, 1/2) may be found in [7], chapter 4.2. For further information see also [11].

The $f$-divergence associated with Example (E3) has a geometric meaning because

$$\int \sqrt{q^2+p^2}\, d\mu + \sqrt{2}$$

is the perimeter of the risk set

$$R(P,Q) = \mathrm{co}\Big(\{(P(A), Q(A^c)) : A \in \mathcal{A} \quad \text{and} \quad P(A) + Q(A^c) \leqq 1\}\Big)$$

of the testing problem $(P,Q)$. It has been used in [9] to construct least favourable distributions. The conjecture that the corresponding $\varrho_{1/2}$ is a metric was made in [6], Example 8.

## 3. Two necessary conditions concerning the power $\alpha$

Once more, let $f \in \mathcal{F}$ satisfy $f(1) = 0$ and $f^* = f$. Now we discuss some necessary conditions concerning the power $\alpha \in (0, \infty)$ so that $\varrho_\alpha$ is a metric, or equivalently, so that (M3,$\alpha$) is satisfied.

The first part of the following remark deals with the existence of such a power $\alpha$.

REMARK 6. Owing to [5], Theorem 3 $f(0) < \infty$ is, in essence, a necessary condition for the existence of a power $\alpha \in (0, \infty)$ such that $\varrho_\alpha$ is a metric on $\wp(\Omega, \mathcal{A})$.

Furthermore, if $\varrho_\beta$ is a metric for $\beta = \alpha \in (0, \infty)$ then it is a metric for all $\beta \in (0, \alpha]$. This holds owing to the following general fact.

Let $\phi\colon \mathbf{R}_+ \to \mathbf{R}_+$ be increasing, concave, continuous at 0 and satisfy $\phi(0) = 0$. And let $\varrho$ be a metric. Then so is $\phi\varrho$.

Now let $\alpha_0$ be the maximal power for which $\varrho_{\alpha_0}$ is a metric on $\wp(\Omega, \mathcal{A})$.

Finally we mention two upper bounds for $\alpha_0$, reflecting the behaviour of $f$ at 1 and, respectively at 0.

PROPOSITION 5. *Let $f \in \mathcal{F}$ satisfy $f^* = f$, $f(0) \in (0, \infty)$ and*

$$(k_1) \qquad f(u) \sim c|u-1|^{k_1} \quad \text{for} \quad u \to 1, \quad k_1 \in (0,\infty), \quad c \in (0,\infty).$$

*Then $k_1 \geqq 1$ and $\alpha_0 \leqq 1/k_1$.*

PROOF. $k_1 \geqq 1$ is obvious by observing $f(u) \leqq f(0)|u-1|$.

Evaluating (M3,$\alpha$) for $\Omega = \{0,1\}$, $P = (p, 1-p)$, $Q = (1-p, p)$, $p \in [0,1]$ and $R = (1/2, 1/2)$ gives owing to $f^* = f$

$$2^{1-2\alpha}[f(2p) + f(2(1-p))]^\alpha \geqq \left[(1-p)f\left(\frac{p}{1-p}\right)\right]^\alpha$$

or, equivalently, $\alpha \leqq \left[2 - {}^2\log\left(\inf\{\psi(p) : p \in [0,1]\}\right)\right]^{-1}$ whereby

$$\psi(p) = \frac{f(2p) + f(2(1-p))}{(1-p)f\left(\frac{p}{1-p}\right)}.$$

Since $f^* = f$ the function $\psi \colon [0,1] \to \mathbf{R}_+$ is symmetric with respect to $p = 1/2$. Furthermore it can be shown to be convex. Hence

$$\inf\{\psi(p) : p \in [0,1]\} = \lim_{p \to \frac{1}{2}} \psi(p) = 2^{2-k_1},$$

the latter being a consequence of $(k_1)$. □

Let $f \in \mathcal{F}$ satisfy $f(0) + f^*(0) < \infty$ or, equivalently, $f(0), f^*(0) < \infty$ and let the function $g \colon \mathbf{R}_+ \to \overline{\mathbf{R}}$ be defined by

$$g(u) = f(0) + uf^*(0) - f(u).$$

REMARK 7. If, in addition, $f(1) = 0$ and $f^* = f$, then for $u \in [0,1]$ holds $f(u) \leqq f(0)(1-u)$ and hence $g(u) \geqq 2f(0)u$ (and hence $k_0 \in (0,1]$, $k_0$ from below) and $0 \leqq f(0)u - f(1-u)$.

PROPOSITION 6. *Let $f \in \mathcal{F}$ satisfy (f1), $f^* = f$, $f(0) < \infty$ and*

$(k_0)$ $\quad g(u) \sim cu^{k_0}$ *for* $u \downarrow 0$, $k_0 \in (0,1]$, $c \in (0,\infty)$.

*Then $\alpha_0 \leqq k_0$.*

PROOF. By considering $f(u)/2f(0)$ we can restrict ourselves to $f(0) = 1/2$. Then the evaluation of (M3,$\alpha$), $\alpha \in (0,1]$, for $\Omega = \{0,1\}$, $P = (0,1)$, $Q = (1,0)$ and $R = (r, 1-r)$, $r \in (0,1)$ gives owing to $f^* = f$

(1) $$[f(r) + (1-r)/2]^\alpha + [f(1-r) + r/2]^\alpha \geqq 1 \qquad \forall r \in [0,1].$$

The application of

$$f(r) + (1-r)/2 = 1 - [(r+1)/2 - f(r)] = 1 - g(r),$$
$$f(1-r) + r/2 = r - [r/2 - f(1-r)] \leqq r$$

and the inequality $(1+x)^\alpha \leqq 1 + \alpha x$ gives

$$[f(r) + (1-r)/2]^\alpha - 1 + [f(1-r) + r/2]^\alpha \leqq -\alpha g(r) + r^\alpha.$$

Hence

$$r^\alpha \geqq \alpha g(r) \qquad \forall r \in [0,1]$$

is a necessary condition for the validity of (1). Taking into consideration $(k_0)$ this implies $\alpha \leqq k_0$.

Since for the classes of Examples (E1) and (E3), $f_k \in \mathcal{F}_2(1/k)$ and $\Phi_k \in \in \mathcal{F}_3(1/k)$, Proposition 5 implies $\alpha_0 = 1/k$. For Example (E4), $f \in \mathcal{F}_3(1/2)$ and hence $\alpha_0 = 1/2$. For the class of Examples (E2), finally, Proposition 6 implies $\alpha_0 \leqq \min(s, 1-s)$. Together with $f^s \in \mathcal{F}_2(\min(s, 1-s))$ this yields $\alpha_0 = \min(s, 1-s)$.

## REFERENCES


[1] Ali, S. M. and Silvey, S. D., A general class of coefficients of divergence of one distribution from another, *J. Roy. Statist. Soc. Ser. B* **28** (1966), 131–142. *MR* **33** # 4963.

[2] Boekee, D. E., *A generalization of the Fisher information measure*, Delft University Press, Delft, 1977.

[3] Csiszár, I. und Fischer, J., Informationsentfernungen im Raum der Wahrscheinlichkeitsverteilungen, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 159–180. *MR* **32** # 9136.

[4] Csiszár, I., Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 85–108. *MR* **29** # 1671.

[5] Csiszár, I., On topological properties of $f$-divergences, *Studia Sci. Math. Hungar.* **2** (1967), 329–339. *MR* **36** # 2429.

[6] Feldman, D. and Österreicher, F. A note on $f$-divergences, *Studia Sci. Math. Hungar.* **24** (1989), 191–200. *MR* **91g**: 94006.

[7] Le Cam, L., *Asymptotic methods in statistical decision theory*, Springer-Verlag, New York–Berlin, 1986. *MR* **88a**: 62004.

[8] Liese, F. and Vajda, I., *Convex Statistical Distances*, Teubner-Texte zur Mathematik, Band 95, Teubner, Leipzig, 1987. *MR* **89g**: 62006.

[9] Österreicher, F., The construction of least favourable distributions is traceable to a minimal perimeter problem, *Studia Sci. Math. Hungar.* **17** (1982), 341–351. *MR* **85k**: 62009.

[10] Puri, M. L. and Vincze, I., Measure of information and contiguity, *Statist. Probab. Lett.* **9** (1990), 223–228. *MR* **91e**: 62011.

[11] Vincze, I., A measure of information and its role in uncertainty relations in mathematical statistics, *Exchangeability in Probability and Statistics* (Rome, 1981), North-Holland, Amsterdam, 1982, 331–335. *MR* **84c**: 62014.


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# MARTINGALES CONDITIONNELLES

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**Summary**

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space, $\mathcal{B}$ a sub-algebra of $\mathcal{A}$, and $\mathsf{E}$ be a Köthe space of $\mathcal{B}$-measurable real variables. We say that $X$, a real random variable on $(\Omega, \mathcal{A}, \mathrm{P})$, is $\mathsf{E}$-integrable if $E^{\mathcal{B}}(|X|) \in \mathsf{E}$. When $\mathsf{E}$ is weakly sequentially complete such a variable is a real analogue of $\mathsf{E}$-valued random with Pettis integrable absolute value. Then we study in a real setting generalization of Banach lattice-valued martingales, amarts and measures.

**Introduction.** Soit $(\Omega_1, \mathcal{F}_1, \mathrm{P}_1)$ un espace de probabilité et $\mathsf{E}$ un espace de Banach. Il est classique de considérer les variables aléatoires vectorielles $X$ de $\Omega_1$ dans $\mathsf{E}$ Pettis intégrables et de lier les propriétés géométriques de $\mathsf{E}$ à celles de ces variables. Lorsque $\mathsf{E}$ est de plus un bon espace réticulé, il est isomorphe à un espace de variables aléatoires réelles — v.a.r. — sur un autre espace de probabilité $(\Omega_2, \mathcal{F}_2, \mathrm{P}_2)$ ce qui permet de considérer $X$ comme une v.a.r. définie sur l'espace produit $(\Omega_1 \times \Omega_2,\ \mathcal{F}_1 \otimes \mathcal{F}_2,\ \mathrm{P}_1 \otimes \mathrm{P}_2)$ dont l'espérance conditionnelle par rapport à une sous-tribu appartient à $\mathsf{E}$. Cet aspect nous a conduit à l'étude des variables conditionnellement intégrables c'est-à-dire celles dont l'espérance conditionnelle appartient à un espace déterminé. D'autre part la notion d'amart a permis de développer simultanément la géométrie des Banach et la convergence des suites de variables aléatoires réelles ou vectorielles.

Une propriété fondamentale des amarts est la réticulation. Nous nous sommes attachés à étudier la réticulation des v.a. conditionnellement intégrables pour obtenir les meilleures convergences possibles.

La partie I donne les définitions de base et les premières propriétés de convergence des martingales et des amarts conditionnels.

La partie II s'intéresse à l'étude des mesures vectorielles et des densités conditionnelles.

La réticulation est abordée dans la partie III. On y trouve outre les propriétés de convergence, une version vectorielle du théorème d'Andersen et Jensen.

La partie IV traite de la théorie classique des bimesures qui trouvent ici un cadre naturel.

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Enfin nous étudions en V le cas des tribus indépendantes qui correspond au schéma traditionnel des v.a. vectorielles.

Nous avons adopté la présentation de [18] pour les espaces de Köthe et, celle de [19], pour les espaces de Riesz.

## I-1. Variables conditionnellement intégrables

Soit $(\Omega, \mathcal{F}, \mathrm{P})$ un espace de probabilité et $\mathcal{B}$ une sous-tribu de $\mathcal{F}$. Soit $\mathsf{E}$ un espace de Köthe de v.a.r. $\mathcal{B}$-mesurables c'est-à-dire un Banach tel que $L^\infty(\Omega, \mathcal{B}, \mathrm{P}) \subset \mathsf{E} \subset L^1(\Omega, \mathcal{B}, \mathrm{P})$ les injections étant continues. On note $\mathsf{E}^*$ le dual topologique de $\mathsf{E}$ et $\mathsf{E}' = \{Y \in L^1(\mathcal{B}); XY \in L^1(\mathcal{B})$, pour tout $X \in \mathsf{E}\}$ son dual de Köthe.

DEFINITION 1. Une v.a.r. définie sur $(\Omega, \mathcal{F}, \mathrm{P})$ est conditionnellement $\mathsf{E}$-intégrable ou, plus rapidement, $\mathsf{E}$-intégrable si $E^{\mathcal{B}}(|X|) \in \mathsf{E}$.

Faisons quelques remarques.

**a** — Reprenons les notations de l'introduction et posons $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{A} = \mathcal{F} \otimes \mathcal{B}$ et $\mathrm{P} = \mathrm{P}_1 \otimes \mathrm{P}_2$. Identifions $\mathcal{B}$ avec $\{\{\emptyset, \Omega_1\} \otimes \mathcal{B}\}$ et $\mathcal{F}$ avec $\{\mathcal{F} \otimes \{\emptyset, \Omega_2\}\}$. Une v.a. $X$ de $(\Omega_1, \mathcal{F}, \mathrm{P})$ dans $\mathsf{E} \subset \mathsf{L}^1(\Omega_2, \mathcal{B}, \mathrm{P})$ définit une v.a.r. sur $(\Omega, \mathcal{A}, \mathrm{P})$ que nous noterons de la même façon. Supposons que $|X|$ soit Pettis-intégrable [6] alors, pour $Y \in \mathsf{E}'_+$ $E(YE^{\mathcal{B}}(|X|)) = E(E^{\mathcal{B}}(Y|X|)) = E(Y|X|)$ donc $Y \cdot E^{\mathcal{B}}(|X|) \in \mathsf{L}^1$. Si $\mathsf{E}$ possède la propriété de Fatou [18] i.e. $(\mathsf{E}')' = \mathsf{E}$, on en déduit que $E^{\mathcal{B}}(|X|) \in \mathsf{E} : X$ est $\mathsf{E}$-intégrable.

Inversement, supposons $\mathsf{E}$ à norme continue pour l'ordre — n.c.o. — i.e. $\mathsf{E}' = \mathsf{E}^*$, et soit $X$ une v.a. $\mathsf{E}$-intégrable, alors pour tout $u \in \mathsf{E}^*_+$ $E(u|X|) = u(E(|X|))$ donc $|X|$ est Pettis-intégrable. Par suite, si $\mathsf{E}$ est faiblement séquentiellement complet — f.s.c. — nous pouvons identifier l'ensemble des v.a.r. $\mathcal{F} \otimes \mathcal{B}$-mesurables telles que $E^{\mathcal{B}}(|X|) \in \mathsf{E}$, et l'ensemble des v.a. $\mathcal{F}$-mesurables à valeurs dans $\mathsf{E}$ qui sont de valeur absolue Pettis-intégrables par la correspondance

$$X(\omega, \omega') = X(\omega)(\omega').$$

**b** — Soit $X$ une v.a.r. $\mathsf{E}$-intégrable, $\mathsf{E}$ n.c.o., l'application $A \to E^{\mathcal{B}}(1_A)$ définit une mesure vectorielle à valeurs dans $\mathsf{E}$ signée, à variation bornée dans $\mathsf{L}^1(\mathcal{B})$, mais qui, bien qu'elle soit absolument continue par rapport à P, n'a généralement pas de densité, — prendre $X = 1$ —. Le point de vue adopté ici est donc, d'une certaine façon, plus général que l'aspect vectoriel classique. Ils coïncident lorsque les tribus $\mathcal{F}$ et $\mathcal{B}$ sont indépendantes, voir §V.

**c** — Lorsque $\mathsf{E} = \mathsf{L}^1(\mathcal{B})$ les v.a.r. $\mathsf{E}$-intégrables qui sont aussi de valeur absolue Pettis-intégrable sont aussi Bochner-intégrables et l'identification du **a** — précédent est classique.

De manière évidente on voit que le théorème de Lebesgue est valable pour les v.a. $\mathsf{E}$-intégrables:

LEMME 2. *Soit* $(X_n)$ *une suite croissante de v.a.* $\mathsf{E}$*-intégrables, positives, telles que* $\operatorname{Sup} E^{\mathcal{B}}(X_n) \in \mathsf{E}$, *alors* $\operatorname{Sup} X_n$ *est* $\mathsf{E}$*-intégrable et* $\operatorname{Sup} E^{\mathcal{B}}(X_n) = E^{\mathcal{B}}(\operatorname{Sup} X_n)$.

LEMME 3. *Soit* $(X_n)$ *une suite de v.a.r.* $\mathsf{E}$*-intégrables, telles que* $\operatorname{Sup}|X_n|$ *soit* $\mathsf{E}$*-intégrable. Si* $X_n \to X$ *p.s. Alors* $X$ *est aussi* $\mathsf{E}$*-intégrable et, si* $\mathsf{E}$ *est n.c.o.,* $E^{\mathcal{B}}(X_n) \to E^{\mathcal{B}}(X)$ *p.s. et dans* $\mathsf{E}$.

Signalons en outre la possibilité d'étendre la définition 1 aux espaces intermédiaires entre $\mathsf{L}^1$ et $\mathsf{L}^0$, comme cela a été fait dans le cadre abstrait de [4]. En effet soit $\mathsf{E}$ un idéal complètement réticulé de $\mathsf{L}^0(\mathcal{B})$ tel que $\mathsf{L}^\infty(\mathcal{B})$ soit dense pour l'ordre dans $\mathsf{E}$. Une v.a.r. $X \in \mathsf{L}^0(\Omega, \mathcal{A}, \mathrm{P})$ est dite $\mathsf{E}$-intégrable si $\sup E^{\mathcal{B}}(|X| \wedge n) \in \mathsf{E}$. Le lemme 2 montre que cette définition coïncide avec la précédente, celle des espaces de Köthe. On retrouve ainsi l'intégrale pour l'ordre de [4].

## I–2 Martingales conditionnelles

On pourrait examiner, pour les v.a. conditionnellement intégrables les versions adaptées des théorèmes classiques comme la loi des grands nombres ou le central-limite. Nous nous limiterons aux martingales et aux notions adjacentes.

Soit $(\Omega, \mathcal{A}, \mathrm{P})$ un espace de probabilité, $\mathcal{B}$ une sous-tribu de $\mathcal{A}$ et $\mathsf{E}$ un espace de Köthe de v.a.r. $\mathcal{B}$-mesurables. Soit $(\mathcal{F}_n)$ une filtration croissante de sous-tribus de $\mathcal{A}$, telle que $\mathcal{F}_0 = (\emptyset, \Omega)$ et $\vee \mathcal{F}_n = \mathcal{F}$.

DÉFINITION 4. Une suite $(X_n)$ de v.a.r. adaptées à la filtration $(\mathcal{F}_n \vee \mathcal{B})$ est une $\mathsf{E}$-martingale conditionnelle, ou plus simplement, une $\mathsf{E}$-martingale sachant $\mathcal{B}$, si les $X_n$ sont $\mathsf{E}$-intégrables et si

$$E^{\mathcal{B}}(X_n \cdot 1_A) = E^{\mathcal{B}}(X_m \cdot 1_A) \quad \text{pour tout} \quad A \in \mathcal{F}_{n \wedge m}.$$

En notant $\mathcal{T}_1$ la famille des temps d'arrêts bornés liés à la filtration $(\mathcal{F}_n)$ et, $\mathcal{T}_2$ celle relative à la filtration $(\mathcal{F}_n \vee \mathcal{B})$ on obtient la caractérisation suivante:

LEMME 5. *Soit* $(X_n)$ *une suite de v.a. adaptées à* $(\mathcal{F}_n \vee \mathcal{B})$ $\mathsf{E}$*-intégrables où* $\mathsf{E}$ *est un espace de Köthe n.c.o. de v.a.r.* $\mathcal{B}$*-mesurables. Les assertions suivantes sont équivalentes:*

(i) $(X_n)$ *est une* $\mathsf{E}$*-martingale.*

(ii) $E^{\mathcal{B}}(X_\tau) = X_0$ *pour tout* $\tau \in \mathcal{T}_1$.

(iii) $E^{\mathcal{B}}(X_\tau) = X_0$ *pour tout* $\tau \in \mathcal{T}_2$.

(iv) $E^{\mathcal{B}}(X_n \cdot 1_A) = E(X_m \cdot 1_A)$ *pour tout* $A \in \mathcal{F}_{n \wedge m} \vee \mathcal{B}$.

*En particulier* $(X_n)$ *est une martingale pour la filtration* $(\mathcal{F}_n \vee \mathcal{B})$.

DÉMONSTRATION. Il suffit de montrer (i) $\Rightarrow$ (iv). Soit $\mu_n$ la mesure associée à $X$: $\mu_n(A) = E^{\mathcal{B}}(X_n \cdot 1_A)$ pour $A \in \mathcal{F}_n \vee \mathcal{B}$. Prenons alors $A \in \mathcal{F}_n$ et $B \in \mathcal{B}$: pour $m \geqq n$ on a:

$$\mu_n(A \cdot B) = 1_B \cdot E^{\mathcal{B}}(X_n \cdot 1_A) = 1_B \cdot E^{\mathcal{B}}(X_m \cdot 1_A) = \mu_m(A \cdot B).$$

Donc $\mu_n$ et $\mu_m$ coïncident sur l'algèbre engendrée par $\mathcal{F}$ et $\mathcal{B}$. Comme $\mathsf{E}$ est n.c.o., $\mu_n$ et $\mu_m$ sont $\sigma$-additives, elles coïncident sur $\mathcal{F} \vee \mathcal{B}$. □

Comme dans le cas réel nous obtenons une convergence p.s. ce qui constitue un résultat curieux.

PROPOSITION 6. *Soit* $(X_n)$ *une* $\mathsf{E}$*-martingale positive,* $\mathsf{E}$ *n.c.o., il existe une v.a.r.* $X$, $\mathsf{E}$*-intégrable, telle que* $X_n \to X$ *p.s.*

DÉMONSTRATION. $(X_n)$ est une martingale ordinaire, d'après le lemme 5, elle converge donc p.s. vers $X$. Le lemme de Fatou conditionnel montre que $E^{\mathcal{B}}(X) \leqq X_0$, par suite, $X$ est $\mathsf{E}$-intégrable. □

Il est naturel de considérer l'extension des notions liées aux martingales: Une suite $(X_n)$ de v.a.r. $\mathsf{E}$-intégrables adaptées à $(\mathcal{F}_n \vee \mathcal{B})$ est une $\mathsf{E}$-sous-martingale — resp. $\mathsf{E}$-sur-martingale — si:

(i') $n \leqq m$ et $A \in \mathcal{F}_n$ impliquent $E(X_n \cdot 1_A) \leqq E(X_m \cdot 1_A)$ — resp. $\geqq$ —.

Comme dans le lemme 5, cette condition est encore équivalente à l'une des trois suivantes, lorsque $\mathsf{E}$ est n.c.o.:

(ii') $E^{\mathcal{B}}(X_\tau) \nearrow$ lorsque $\tau \nearrow$ dans $\mathcal{T}_1$ — resp. décroit —

(iii') $E^{\mathcal{B}}(X_\tau) \nearrow$ lorsque $\tau \nearrow$ dans $\mathcal{T}_2$ — resp. décroit

(iv') $n \leqq m$ et $A \in \mathcal{F} \vee \mathcal{B} \Longrightarrow E^{\mathcal{B}}(X_n \cdot 1_a) \leqq E^{\mathcal{B}}(X_m \cdot 1_A)$ — resp. $\geqq$ —.

Les $\mathsf{E}$-sous- ou sur-martingales sont en particulier des sous- ou sur-martingales ordinaires. Il en résulte:

PROPOSITION 7. *Soit* $\mathsf{E}$ *un espace de Köthe f.s.c. et soit* $(X_n)$ *une* $\mathsf{E}$*-martingale telle que* $\operatorname{Sup} \|E^{\mathcal{B}}(|X_n|)\|_{\mathsf{E}} \leqq \infty$. *Alors* $X_n \to X$ *p.s. et* $X$ *est* $\mathsf{E}$*-intégrable.*

Soit $(X_n)$ une martingale à valeurs dans un Banach réticulé f.s.c. Si pour tout $n$, $|X_n|$ est Pettis-intégrable et de norme Pettis bornée, en représentant $\mathsf{E}$ comme un espace de Köthe, la Proposition 7 assure que $X_n$ converge p.s. vers $X$, avec $X$ Pettis-intégrable. Nous reviendrons sur cet aspect. Si, de plus, $\mathsf{E}$ vérifie la propriété de Radon–Nikodym et que les $X_n$ sont Bochner-intégrables et bornées pour cette norme, alors $X$ est la limite usuelle.

## I-3 Amarts conditionnels

Soit $\mathsf{E}$ un espace de Köthe f.s.c. et soit $(X_n)$ une suite $\mathsf{E}$-intégrable de v.a.r. adaptées à la filtration $(\mathcal{F}_n \vee \mathcal{B})$.

Définition 8. $(X_n)$ est un $\mathsf{E}$-amart fort — resp. faible — (amart conditionnel pour $\mathsf{E}$) si la famille $((E^{\mathcal{B}}(X_\tau))_{\tau\in\mathcal{T}_1}$ converge fortement — resp. faiblement — dans $\mathsf{E}$ selon le filtre $\mathcal{T}_1$.

Remarque. Contrairement à ce qui se passe dans la cas des martingales, les $\mathsf{E}$-amarts ne sont généralement pas des amarts réels, comme le montre l'exemple suivant. Choisissons $\mathcal{B} = \mathcal{A}$ et $\mathcal{F}_n = \mathcal{F} = \{\emptyset, \Omega\}$. La famille $\mathcal{T}_1$ se réduit aux temps constants. Dire que $(X_n)$ est un $\mathsf{E}$-amart fort signifie que la suite $(X_n)$ converge dans $\mathsf{E}$. Si maintenant $\mathsf{E} = \mathsf{L}^1$ et que $0 \leqq X_n \leqq 1$, $X_n$ convergeant vers 0 dans $\mathsf{L}^1$ mais pas p.s.; ce n'est pas un amart ordinaire pour la filtration constante.

La théorie des $\mathsf{E}$-amarts diffère donc de la théorie des martingales vectorielles. Nous allons montrer cependant qu'il est possible d'étendre des résultats de la théorie des amarts réels au cadre adopté ici.

Soit $(X_n)$ un $\mathsf{E}$-amart faible, pour $A \in \bigcup_{\sigma\in\mathcal{T}_1} \mathcal{F} \bigcup \mathcal{B}$, la famille $(E^{\mathcal{B}}(X_\tau \cdot 1_A))_{\tau\in\mathcal{T}_1}$ converge faiblement dans $\mathsf{E}$ vers une limite $m(A)$. C'est en effet évident si $B \in \mathcal{B}$ et si $A \in \mathcal{F}_{\sigma_0}$.

L'argument classique de la théorie des amarts réels, permet d'écrire, pour tout $Y \in \mathsf{E}'$ et $\sigma_0 \leqq \sigma \leqq \tau \in \mathcal{T}_1$,

$$E((X_\tau - X_\sigma)1_A Y) = E((X_\tau - X_{\tau'})Y) \to 0, \text{ en posant } \tau' = \sigma \cdot 1_A + \tau \cdot 1_{A^c}.$$

La limite, $m(A)$ est une fonction additive d'ensembles de l'algèbre engendrée par $\bigcup_\sigma \mathcal{F}_\sigma \bigcup \mathcal{B}$, à valeurs dans $\mathsf{E}$, et dont les restrictions à chacune des sous-tribus $\mathcal{F}_\sigma$ sont $\sigma$-additives.

On a de plus:

Proposition 13. *Soit $(X_n)$ un $\mathsf{E}$-amart faible (resp. fort), tel que* $\operatorname{Sup} \|E^{\mathcal{B}}(|X_n|)\|_{\mathsf{E}} < \infty$. *Alors $(X_n)$ s'écrit de manière unique sous la forme $X_n = M_n + Z_n$ dans laquelle $(M_n)$ est une $\mathsf{E}$-martingale et $(Z_n)$ un $\mathsf{E}$-amart faible (resp. fort), vérifiant $E^{\mathcal{B}}(Z_\sigma \cdot 1_A) \to 0$ faiblement (resp. fortement) dans $\mathsf{E}$.*

Démonstration. Soit $(X_n)$ un $\mathsf{E}$-amart faible et soit $n_0 \in \mathsf{N}$. Pour tout $A \in \mathcal{F}_{n_0}$, la suite $E^{\mathcal{B}}(X_n \cdot 1_A)$ converge faiblement dans $\mathsf{E}$. Il existe donc une v.a., $M_{n_0}$, $\mathcal{F}_{n_0} \vee \mathcal{B}$-mesurable et $\mathsf{E}$-intégrable telle que $E^{\mathcal{B}}(X_n \cdot 1_A) \to$ $\to E^{\mathcal{B}}(M_{n_0} \cdot 1_A)$ faiblement dans $\mathsf{E}$. Il est clair que la suite $(M_n)$ est une $\mathsf{E}$-martingale et, si l'on pose

$$Z_n = X_n - M_n,$$

la suite $(Z_n)$ est un $\mathsf{E}$-amart faible vérifiant

$$E^{\mathcal{B}}(Z_\tau \cdot 1_A) \to 0 \quad \text{faiblement dans} \quad \mathsf{E}, \quad \text{pour tout } A \in \mathcal{F}_\sigma.$$

Le cas des amarts forts est analogue. □

REMARQUES 1. Sous les hypothèses de la proposition précédente, la deuxième partie de la Proposition 11, montre que la $\mathsf{E}$-martingale $(M_n)$ vérifie $\sup_n \|E^{\mathcal{B}}(|M|)\|_{\mathsf{E}} < \infty$ et, par conséquent, converge p.s. vers une v.a. $X$, $\mathsf{E}$-intégrable (Proposition 7), qui apparait alors comme une limite de l'amart $(X_n)$ en un sens très élargi. En effet on ignore généralement si la suite $(Z_n)$ converge vers 0 p.s. (reprendre le contre-exemple du début pour lequel $M_n = 0$ et $Z_n = X_n$ ne converge pas p.s. vers 0).

2. Dans le cas réel, les potentiels $(Z_n)$ intervenant dans la décomposition de Riesz des amarts, convergent vers 0 dans $\mathsf{L}^1$. Nous ignorons si, dans le présent contexte, $(E^{\mathcal{B}}(|Z_\sigma|))_{\sigma\in\mathcal{T}_1}$ converge vers 0 en un sens raisonnable. Nous verrons plus loin que $E^{\mathcal{B}}(|Z_\sigma|) \to 0$ pour $\sigma(\mathsf{E},\mathsf{E}')$ sous des conditions plus fortes.

3. Si $\mathsf{E}$ possède la propriété de Radon–Nikodym, toutes les v.a. mises en jeu s'interprètent comme de vraies variables vectorielles (si les mesures associées sont à variation bornée) et on retrouve le théorème d'Edgar et Sucheston [11] classique. L'artifice utilisé ici permet d'étendre ce théorème à des espaces de Köthe qui, tel $\mathsf{L}^1$, ne possèdent pas la propriété de Radon–Nikodym.

## II-1 Mesures $\mathsf{E}$-conditionnelles

Afin d'étudier les propriétés de réticulation des $\mathsf{E}$-amarts, nous allons aborder maintenant l'aspect mesures $\mathsf{E}$-conditionnelles. Notons $\mathsf{P}_0(\mathsf{E})$ l'ensemble des v.a.r. $\mathsf{E}$-intégrables avec la norme $P_0(X) = \|E^{\mathcal{B}}(|X|)\|_{\mathsf{E}}$. Cet espace est complet seulement si $\mathsf{E} = \mathsf{L}^1(\mathcal{B})$. Par contre $\mathsf{P}_0(\mathsf{E})$ est complètement réticulé et l'injection $X \to E^{\mathcal{B}}(X \cdot 1)$ de $\mathsf{P}_0(\mathsf{E})$ dans l'ensemble $\mathcal{M}_0(\mathcal{A},\mathsf{E})$ des mesures signées de $\mathcal{A}$ dans $\mathsf{E}$ est un isomorphisme d'ordre:

$$\text{si} \quad \mu(A) = E^{\mathcal{B}}(X \cdot 1_A) \quad \text{alors} \quad \mu^+(A) = E^{\mathcal{B}}(X^+ \cdot 1_A).$$

Les propositions 6 et 7 permettent de situer la place de $\mathsf{P}_0(\mathsf{E})$ dans $\mathcal{M}_0(\mathcal{F},\mathsf{E})$. C'est l'objet du corollaire suivant qui est à rapprocher des résultats de [2] et [4].

Soit $\mathsf{E}$ un espace de Köthe faiblement séquentiellement complet de v.a.r. $\mathcal{B}$-mesurables (l'espace $\mathcal{M}_0(\mathcal{F},\mathsf{E})$ muni de la norme $\|\mu\| = \||\mu(\Omega)|\|_{\mathsf{E}}$ est aussi faiblement séquentiellement complet). Soit $\mathcal{F}$ une sous-tribu séparable de $\mathcal{A}$ et $(\mathcal{F}_n)$ une filtration atomique engendrant $\mathcal{F}$.

COROLLAIRE 14. *Soit $\mu \in \mathcal{M}_0(\mathcal{F},\mathsf{E})$ telle qu'il existe $Y$, $\mathsf{E}$-intégrable avec $|\mu(A)| \leqq E^{\mathcal{B}}(Y \cdot 1_A)$, pour tout $A \in \mathcal{F}$. Alors on peut écrire de manière* *unique* $\mu(A) = E^{\mathcal{B}}(X \cdot 1_A)$ *où* $X$ *est* $\mathsf{E}$*-intégrable et de plus* $|\mu|(A) = = E^{\mathcal{B}}(|X| \cdot 1_A)$.

DÉMONSTRATION. Supposons $\mu \geqq 0$, la restriction de $\mu$ à $\mathcal{F}_n$ s'écrit $\mu(A) = E^{\mathcal{B}}(X_n \cdot 1_A)$ pour $A \in \mathcal{F}_n$ et $X_n = \sum \frac{\mu(A_i)}{E^{\mathcal{B}}(1_{A_i})} 1_{A_i}$ la somme portant sur les $A_i$ atomes de $\mathcal{F}_n$. $(X_n)$ est alors une $\mathsf{E}$-martingale positive adaptée à la filtration $(\mathcal{F}_n \vee \mathcal{B})$ et sa limite p.s., $X$, vérifie:

$$E^{\mathcal{B}}(X_n \cdot 1_A) = E^{\mathcal{B}}(X \cdot 1_A) \quad \text{tout} \quad A \in \bigcup_n \mathcal{F}_n.$$

Donc

$$\mu(A) = E^{\mathcal{B}}(X \cdot 1_A) \quad \text{tout} \quad A \in \mathcal{F}.$$

Pour $\mu = \mu^+ - \mu^-$ on applique ce schéma à chaque composante. Supposons maintenant que $\mu(A) = E^{\mathcal{B}}(X \cdot 1_A) = E^{\mathcal{B}}(X' \cdot 1_A)$. On voit que l'égalité demeure pour $A \in \mathcal{F} \cup \mathcal{B}$ et donc pour $\mathcal{F} \vee \mathcal{B}$. La dernière partie n'offre guère de difficultés. □

De ce corollaire il résulte que $\mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$ se plonge canoniquement dans $\mathcal{M}_0(\mathcal{F}, \mathsf{E})$. Plus précisément, si $\mathrm{P}^{\mathcal{B}}$ est la mesure définie par $F \to E^{\mathcal{B}}(1_F)$, $F \in \mathcal{F}$ alors:

PROPOSITION 15. *L'espace* $\mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$ *est isomorphe à la bande engendrée par* $\mathrm{P}^{\mathcal{B}}$ *dans* $\mathcal{M}_0(\mathcal{F}, \mathsf{E})$.

DÉMONSTRATION. Soit $\mu$ une mesure vectorielle positive telle que

$$\mu \wedge n \cdot \mathrm{P}^{\mathcal{B}} \nearrow \mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E}).$$

Il existe $X_n \in \mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$ avec $\mu \wedge n \cdot \mathrm{P}^{\mathcal{B}}(A) = E^{\mathcal{B}}(X_n \cdot 1_A)$. La suite $(X_n)$ est croissante et sa limite p.s., $X$, vérifie

$$\mu(A) = E^{\mathcal{B}}(1_A). \quad \square$$

Nous aurons besoin en outre de propriétés de compacité faible, ce que nous examinons maintenant.

## II-2 Convergence faible dans $\mathsf{P}_0$ et $\mathcal{M}_0$

Dans toute cette partie nous nous donnons un espace probabilisé $(\Omega, \mathcal{A}, \mathrm{P})$, $\mathcal{A}$ séparable, $\mathcal{F}$ et $\mathcal{B}$ deux sous-tribus de $\mathcal{A}$, $\mathsf{E}$ un espace de Köthe faiblement séquentiellement complet de v.a. $\mathcal{B}$-mesurables. L'espace $\mathcal{M}_0 = \mathcal{M}_0(\mathcal{F}, \mathsf{E})$ n'est pas complet pour la converge simple faible ou forte induite par $\mathsf{E}$. On dispose cependant du résultat suivant:

PROPOSITION 16. *Soit* $(\mu_n)$ *une suite de mesures de* $\mathcal{M}_0$ *vérifiant* $\text{Sup} \, \||\mu_n|(\Omega)\|_{\mathsf{E}} < \infty$ *et, pour tout* $A \in \mathcal{F}$, *la suite* $(\mu_n(A))$ *converge faiblement dans* $\mathsf{E}$. *Il existe* $\mu \in \mathcal{M}_0$ *telle que*

$$\mu_n(A) \to \mu(A) \quad \textit{pour} \quad \sigma(\mathsf{E}, \mathsf{E}') \quad \textit{et} \quad A \in \mathcal{F}.$$

DÉMONSTRATION. La limite, $\mu(A)$, de la suite $(\mu_n(A))$, définit une mesure de $\mathcal{F}$ dans $\mathsf{E}$.

Pour montrer qu'elle est signée, il suffit d'établir que, pour toute famille finie, $(A_0, \dots, A_n)$ de $\mathcal{F}$ on a:

$$\left\| \bigvee_0^n \mu(A_i) \right\| \leqq C \text{ pour une certaine constante } C.$$

Prenons donc $A_0 = \emptyset$, $A_1, \dots, A_k$ une suite d'éléments de $\mathcal{F}$. Pour tout $n$:

$$0 \leqq \bigvee_0^k \mu_n(A_i) \leqq |\mu_n|(\Omega).$$

La suite $\bigvee_0^k \mu_n(A_i)$ est donc bornée dans $\mathsf{E}$.

Pour montrer qu'il existe une sous-suite convergeant faiblement dans $\mathsf{E}$, il suffit d'établir cette propriété dans $\mathsf{L}^1$, corollaire 3 de [7]. Comme chaque suite $(\mu_n(A_i))_n$ converge faiblement dans $\mathsf{E}$ donc dans $\mathsf{L}^1$, la suite $(\bigvee_i \mu_n(A_i))$ est faiblement compacte dans $\mathsf{L}^1$. Ainsi: $\|\vee \mu(A_i)\|_{\mathsf{E}} \leqq \text{Sup} \, \||\mu_n|(\Omega)\|_{\mathsf{E}}$ (Lemme 1 de [7]). □

REMARQUE. Cette proposition s'étend aux famille de mesure de $\mathcal{M}_0$ en utilisant le théorème d'Eberlein. Donnons, en application, l'analogue de cette propriété pour les v.a. de $\mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$.

PROPOSITION 17. *Soit* $(X_n)$ *une suite de v.a.* $\mathcal{F} \vee \mathcal{B}$*-mesurables, telles que* $\text{Sup}_n \|E^{\mathcal{B}}(|X_n|)\|_{\mathsf{E}} \leqq \infty$ *et pour tout* $A \in \mathcal{F}$, $(E^{\mathcal{B}}(X_n 1_A))$ *converge faiblement dans* $\mathsf{E}$. *Alors il existe une v.a.* $X$ *de* $\mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$ *telle que:* $E^{\mathcal{B}}(X_n 1_A) \to E^{\mathcal{B}}(X \cdot 1_A)$ *et de plus*

$$\|E^{\mathcal{B}}(X)\|_{\mathsf{E}} \leqq \text{Sup}_n \|E^{\mathcal{B}}(X_n)\|_{\mathsf{E}}.$$

DÉMONSTRATION. Posons $\mu_n(A) = E^{\mathcal{B}}(X_n 1_A)$ les hypothèses et la proposition précédente assurent qu'il existe $\mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$ telle que $\mu_n(A)$ converge faiblement vers $\mu(A)$. Pour tout $Y \in \mathsf{E}'_+$, la suite $(Y|X_n|)$ est équi-intégrable dans $\mathsf{L}^1$. Ainsi, pour une sous-suite convenable, $|X_{n_i}|$ et $|X_{n_i}| \cdot Y$ convergent faiblement dans $\mathsf{L}^1$, vers respectivement $Z$ et $Z \cdot Y$. Le Corollaire 3 de [7], assure que $Z$ est $\mathsf{E}$-intégrable et que

$$E^{\mathcal{B}}(X_{n_i}) \quad \text{converge, pour} \quad \sigma(\mathsf{E}, \mathsf{E}'), \quad \text{vers} \quad E^{\mathcal{B}}(Z).$$

En particulier, $\|E^{\mathcal{B}}(Z)\| \leqq \operatorname{Sup}_n \|E^{\mathcal{B}}(|X_n|)\|$. Enfin, en passant aux mesures associées, nous avons: $|\mu(A)| \leqq E^{\mathcal{B}}(Z \cdot 1_A)$. Le Corollaire 14 permet alors de conclure. □

Afin de généraliser la Proposition 13, introduisons les $\mathsf{E}$-amarts mesures. Soit $(\mu_n)$, $\mu_n \in \mathcal{M}(\mathcal{F}_n, \mathsf{E})$ une suite adaptée de mesures vectorielles [5, 6]. Nous dirons que $(\mu_n)$ est un faible — resp. fort — $\mathsf{E}$-amart mesure, si la famille $(\mu_\tau(\Omega))_{\tau \in T_1}$ converge faiblement — resp. fortement — dans $\mathsf{E}$. De même, c'est une $\mathsf{E}$-martingale mesure si $\mu_n(A) = \mu_m(A)$ pour tout $A \in \mathcal{F}_{n \wedge m}$.

PROPOSITION 18. *Soit $(\mu_n)$ un $\mathsf{E}$-amart mesure faible — resp. fort — tel que $\operatorname{Sup}_n \||\mu_n|(\Omega)\| < \infty$. Alors $(\mu_n)$ s'écrit de manière unique comme $\mu_n = m_n + \varrho_n$ où $(m_n)$ est une $\mathsf{E}$-martingale mesure et $(\varrho_n)$ un $\mathsf{E}$-amart mesure, vérifiant pour tout $A \in \bigcup_{\sigma \in T_1} \mathcal{F}_\sigma$,*

$$\varrho_\tau(A) \xrightarrow{\tau \in T_1} 0 \quad \textit{faiblement — resp. fortement —.}$$

## III-1 Famille de Riesz

Comme auparavant on se donne un espace de probabilité séparable $(\Omega, \mathcal{A}, \mathrm{P})$, $\mathcal{B}$ une sous-tribu et $\mathsf{E}$ un espace de Köthe f.s.c. de v.a $\mathcal{B}$-mesurables.

Soit $I$ un ensemble d'indices, filtrant croissant, et $(\mathcal{F}_\iota)_{\iota \in I}$ une filtration croissante de sous-tribus de $\mathcal{A}$. Pour chaque $i$, donnons nous une mesure signée $\mu_i \in \mathcal{M}_0(\mathcal{F}_i, \mathsf{E})$ — par exemple si $I = \mathsf{N}$, $(\mu_n)$ est la famille de mesures associées à un $\mathsf{E}$-amart $(X_n)$ —. Nous voulons déterminer, dans cette partie, des conditions sur la famille $(\mu_\iota)_{\iota \in I}$, assurant la convergence — au moins faible — de

$$(|\mu_\iota|(\Omega))_{\iota \in I}.$$

Il est évidemment nécessaire que $\operatorname{Sup}_\iota \||\mu_\iota|(\Omega)\|_{\mathsf{E}} < \infty$. Supposons, pour simplifier, que $I = \mathsf{N}$ et $\mathcal{F} = \mathcal{F}_n$. Soit $\mathsf{T}$ une topologie d'espace vectoriel sur $\mathcal{M}_0(\mathcal{F}, \mathsf{E})$ telle que $\mu_n \to 0$ pour $\mathsf{T}$, implique $|\mu_n|(\Omega) \to 0$ pour $\sigma(\mathsf{E}, \mathsf{E}')$: la topologie $\mathsf{T}$ est plus fine que celle induite par $\mathcal{M}_0(\mathcal{F}, \mathsf{L}^1(\mathcal{B}))$. D'autre part comme $\operatorname{Sup}_n \||\mu_n|(\Omega)\| < \infty$, la faible convergence de $(|\mu_n|(\Omega))$ dans $\mathsf{E}$ est équivalente à la faible convergence dans $\mathsf{L}^1(\mathcal{B})$.

Ainsi la plus faible définition acceptable est:

DÉFINITIONS 19. — Une famille adaptée de $\mathsf{E}$-mesures, $(\mu_\iota)_{\iota \in I}$, $\mu_i \in \mathcal{M}_0(\mathcal{F}_\iota, \mathsf{E})$, est une $\mathsf{E}$-famille de Riesz si

a) $\operatorname{Sup}_\iota \||\mu_\iota|(\Omega)\|_{\mathsf{E}} < \infty$

b) Pour tout $\varepsilon > 0$ il existe $i_\varepsilon \in I$ tel que $j > \iota > i_\varepsilon$ impliquent

$$E\Big[\operatorname*{Sup}_{A \in \mathcal{F}_\iota} |(\mu_j - \mu_\iota)(A)|\Big] \leqq \varepsilon.$$

— Une famille $(X_i)_{i\in I}$ de v.a. adaptées à la filtration $(\mathcal{F}_i \vee \mathcal{B})$ est une $\mathsf{E}$-famille de Riesz si, la famille de mesures associées par $\mu_i(A) = E^{\mathcal{B}}(X_i \cdot 1_A)$, $A \in \mathcal{F}_i$, est une $\mathsf{E}$-famille de Riesz de mesures.

EXEMPLES. Avec les notations de la partie précédente soit $(X_n)$ une suite de v.a. adaptées à $(\mathcal{F}_n \vee \mathcal{B})$ et telle que $\operatorname*{Sup}_{\tau\in\mathcal{T}_1} \|E^{\mathcal{B}}(|X_\tau|)\|_{\mathsf{E}} < \infty$.

* Si $(X_n)$ est une sous- ou sur- $\mathsf{E}$-martingale, alors $(X_\tau)_{\tau\in\mathcal{T}_1}$ est une $\mathsf{E}$-famille de Riesz.

* On dit que $(X_n)$ est un $\mathsf{E}$-o-amart si $(E^{\mathcal{B}}(X_\tau))_{\tau\in\mathcal{T}_1}$ est convergeant pour l'ordre dans $\mathsf{E}$, (voir [15] pour une version vectorielle), alors

$$\operatorname*{Sup}_{A\in\mathcal{F}_\sigma} |E^{\mathcal{B}}(X_\sigma - X_\tau)\cdot 1_A| \xrightarrow[\tau\geqq\sigma\in\mathcal{T}_1]{} 0$$

pour l'ordre de $\mathsf{E}$ et par suite $(X_\sigma)_{\sigma\in\mathcal{T}_1}$ est aussi une $\mathsf{E}$-famille de Riesz.

* On dit que $(X_n)$ est un uniforme $\mathsf{E}$-amart (voir [2] pour une version vectorielle) si

$$(*) \qquad \operatorname{var}_{\mathsf{E}}\left(\mu_\sigma - \mu_\tau|_{\mathcal{F}_\sigma}\right) \xrightarrow[\tau\geqq\sigma\in\mathcal{T}_1]{} 0;$$

en posant $\operatorname{var}_{\mathsf{E}}(\mu) = \operatorname{Sup}\sum_i \|\mu(A_i)\|_{\mathsf{E}}$; $(A_i)$ partition finie de $\mathcal{F}$ et où $\mu_\tau|\mathcal{F}_\sigma$ est la restriction de la mesure $\mu_\tau(A) = E^{\mathcal{B}}(X_\tau\cdot 1_A)$, à la tribu $\mathcal{F}_\sigma$.

De $(*)$ il résulte que $\operatorname{var}_{\mathsf{L}^1(\mathcal{B})}(\mu_\sigma - \mu_\tau|_{\mathcal{F}_\sigma}) \xrightarrow[\tau\geqq\sigma]{} 0$.

Maintenant il est aisé de voir que la norme $\operatorname{var}_{\mathsf{L}^1}$ est équivalente à celle de $\mathcal{M}_0(\mathcal{F}, \mathsf{L}^1(\mathcal{B}))$: $\|\mu\|_0 = E(|\mu|(\Omega))$ (voir [6] pour détails). Il s'ensuit que $(X_\tau)_{\tau\in\mathcal{T}_1}$ est une $\mathsf{E}$-famille de Riesz. Inversement si (et seulement si) $\mathsf{E} = \mathsf{L}^1(\mathcal{B})$, $(X_\tau)$ est une $\mathsf{E}$-famille de Riesz, alors $(X_n)$ est un uniforme $\mathsf{E}$-amart.

* Par contre, si $(X_n)$ est un faible $\mathsf{E}$-amart tel que

$$\operatorname*{Sup}_{\tau\in\mathcal{T}_1} \|E^{\mathcal{B}}(|X_\tau|)\|_{\mathsf{E}} < \infty.$$

Alors $(X_\tau)_{\tau\in\mathcal{T}_1}$ n'est pas en général une $\mathsf{E}$-famille de Riesz.

Donnons à présent le principal résultat de cette partie:

THÉORÈME 20. *Soit $(\mu_i)_{i\in I}$ une $\mathsf{E}$-famille de Riesz adaptée à la filtration $(\mathcal{F}_i)_{i\in I}$.*

1) *$\mu_i$ s'écrit de manière unique $\mu_i = m_i + \varrho_i$ où $(m_i)$ est une $\mathsf{E}$-martingale mesure et $|\varrho_i|(\Omega) \to 0$ faiblement dans $\mathsf{E}$.*

2) *$(\mu_i^+)$ est une $\mathsf{E}$-famille de Riesz et $\mu_i^+ = (m^+)|_{\mathcal{F}_i} + v_i$ où $m^+$ est la partie positive de $m$ dans l'espace complètement réticulé de toutes les fonctions additives, à valeurs dans $\mathsf{E}$, et définies sur $\bigcup_{i\in I}\mathcal{F}_i$ et, enfin, $|v_i|(\Omega) \to 0$ faiblement dans $\mathsf{E}$.*

*En particulier $m^+$ est limite de la* $\mathsf{E}$*-famille de Riesz* $(\mu_i^+)$.

DÉMONSTRATION. Fixons $i \in I$ et $A \in \mathcal{F}_i$; alors la famille $(\mu_j(A))_{j\in I}$ converge dans $\mathsf{L}^1(\mathcal{B})$ vers une limite, $m_i(A)$. La condition a) de la définition assure que $\mu_j(A) \xrightarrow[j\in I]{} m_i(A)$ faiblement dans $\mathsf{E}$.

Ceci montre, avec la Proposition 16, que $m_i \in \mathcal{M}_0(\mathcal{F}_i, \mathsf{E})$. Posons $\varrho_i = = \mu_i - m_i$. Comme $\operatorname{Sup}_i \| |\varrho_i|(\Omega)\|_{\mathsf{E}} < \infty$. Pour montrer que $|\varrho_i|(\Omega) \to 0$ faiblement dans $\mathsf{E}$, il suffit d'établir que $E(|\varrho_i|(\Omega)) \to 0$. Ce qui résulte de l'inégalité:

$$E(\operatorname*{Sup}_{A\in\mathcal{F}_i} |(\mu_j - m_i)(A)|) \leqq \varepsilon, \quad \text{pour } j \geqq i \text{ assez grands.}$$

Il est clair que $(m_i)$ est une $\mathsf{E}$-martingale mesure de la forme $m_i = m|_{\mathcal{F}_i}$, où $m$ est additive sur l'algèbre engendrée par les $\mathcal{F}_i$. Il est également clair que $(m_i^+)$ est une $\mathsf{E}$-sous-martingale mesure. Donc $(m_i^+(\Omega))_{i\in I}$ est une famille croissante convergente dans $\mathsf{L}^1(\mathcal{B})$.

De là, il résulte que $(m_i^+)$ et donc $(|m_i|)$ sont des $\mathsf{E}$-familles de Riesz. Montrons qu'il en est de même pour $(\mu_i^+)$. Comme $|m_i^+ - \mu_i^+| \leqq |\varrho_i|$ l'inégalité

$$|\mu_j^+|_{\mathcal{F}_i} - \mu_i^+| \leqq |\mu_j^+|_{\mathcal{F}_i} - m_j^+|_{\mathcal{F}_i}| + |m_j^+|_{\mathcal{F}_i} - m_i^+| + |m_j^+ - \mu_i^+|$$

implique que $E(|\mu_j^+|_{\mathcal{F}_i} - \mu_i^+|(\Omega)) \leqq \varepsilon$ pour $j \geqq i \geqq i_\varepsilon$. Et, par conséquent, $(\mu_i^+)$ est une $\mathsf{E}$-famille de Riesz. Finalement, montrons que la limite, dans $\mathsf{L}^1(\mathcal{B})$ et pour $A \in \bigcup \mathcal{F}_i$, de $(\mu_j^+(A))$ est $m^+(A)$.

Soit $A \in \mathcal{F}_{i_0}$; pour tout $\varepsilon > 0$, il existe $B_1, \dots, B_{N_\varepsilon} \in \bigcup_i \mathcal{F}_i$ et $f_\varepsilon \in \mathsf{L}^1$ tels que

$$B_i \subseteqq A, \quad E(f_\varepsilon) \leqq \varepsilon/2 \quad \text{et} \quad m^+(A) \leqq \bigvee_1^{N_\varepsilon} m(B_i) + f_\varepsilon.$$

Nous pouvons supposer, pour $j_0$ assez grand, que tous les $B_i$ appartiennent à $\mathcal{F}_{j_0}$. Donc pour $j$ grand, on peut écrire: $m(B_k) \leqq \mu_j(B_k) + g_k$ où $g_k \in \mathsf{L}^1_+$ est telle que $E(g_k) \leqq \frac{\varepsilon}{2N_\varepsilon}$. Il en résulte que: $m^+(A) \leqq \bigvee_1^{N_\varepsilon} \mu_j(B_i) + \sum g_k + f_\varepsilon \leqq$ $\leqq \mu_j^+(A) + h_\varepsilon$ où $E(h_\varepsilon) \leqq \varepsilon$. Or

$$|\mu_j^+(A) - m_j^+(A)| \leqq |\varrho_j|(\Omega) \to 0 \quad \text{dans} \quad \mathsf{L}^1(\mathcal{B}).$$

Ainsi nous avons montré que pour tout $\varepsilon > 0$ et tout $A \in \mathcal{F}_{i_0}$, il existe $j_0 \in I$ et $h \geqq 0$ avec $E(h) \leqq \varepsilon$, tels que, si $j \geqq j_0$:

$$m_j^+(A) \leqq m^+(A) \leqq m_j^+(A) + h + |\varrho_j|(\Omega).$$

C'est-à-dire: $\mu_j^+(A) \to m^+(A)$ fortement dans $\mathsf{L}^1(\mathcal{B})$. □

Nous allons donner maintenant une condition assurant une convergence forte de la partie potentielle, $\varrho_i = \mu_i - m_i$, $\||\varrho_i|(\Omega)\| \to 0$. Pour l'obtenir nous devons établir un lemme technique similaire à ceux de [7]. Rappelons qu'une suite $(X_n)$ de v.a. à valeurs dans un espace de Köthe $\mathsf{E}$ est $\mathsf{E}$-équi-intégrable si

$$\forall \varepsilon \exists \eta \quad \text{tel que:} \quad P(B) \leqq \eta \Rightarrow \|X_n \cdot 1_B\| < \infty.$$

Lemme 21. *Soit $(X_n)$ une suite bornée en norme d'un espace de Köthe f.s.c.. Alors elle converge fortement si, et seulement si, elle converge dans $\mathsf{L}^1$ et elle est $\mathsf{E}$-équi-intégrable.*

Démonstration. Supposons que $X_n \to X$ dans $\mathsf{L}^1$. Alors $X \in \mathsf{E}$ car $\operatorname{Sup} \|X_n\| < \infty$. Posons $Y_n = |X_n - X|$. La suite $(Y_n)$ converge vers 0 dans $\mathsf{L}^1$ et pour $\sigma(\mathsf{E}, \mathsf{E}')$. Le lemme de [16] montre que $\|Y_n \cdot 1_{(Y_n \leqq k)}\| \to 0$ pour tout $k$. Si $(X_n)$ est en outre $\mathsf{E}$-équi-intégrable il en est de même pour $(Y_n)$ et on a: $\|Y_n \cdot 1_{(Y_n > k)}\| \leqq \varepsilon$ pour $k$ suffisamment grand, ce qui achève la preuve. □

Remarquons qu'une famille est $\mathsf{E}$-équi-intégrable si, et seulement si, pour toute suite $(i_n)$ il existe une sous-suite extraite, $(i_n^*)$, telle que $(X_{i_n^*})$ soit $\mathsf{E}$-équi-intégrable.

On déduit du lemme précédent:

Corollaire 22. *Soit $(\mu_i)_{i \in I}$ une $\mathsf{E}$-famille de Riesz telle que la famille $(|\mu_i|(\Omega))$ soit $\mathsf{E}$-équi-intégrable. Dans la décomposition de Riesz, $\mu_i = m_i + \varrho_i$, les familles $(|m_i|(\Omega))$ et $(|\varrho_i|(\Omega))$ sont $\mathsf{E}$-équi-intégrables et $\||\varrho_i|(\Omega)\| \Rightarrow 0$.*

Démonstration. Il suffit de considérer des sous-suites, que nous noterons pour simplifier, encore $(\mu_n)$. Nous avons: $|\mu_n|_{\mathcal{F}_p}|(\Omega) \to |m_p|(\Omega)$ dans $\mathsf{L}^1$ et faiblement dans $\mathsf{E}$. Donc

$$\||m_p|(\Omega)\| \leqq \underline{\lim} \||\mu_n|(\Omega) \cdot 1_B\| \leqq \varepsilon \quad \text{si} \quad P(B) \leqq \eta.$$

Ce qui montre la $\mathsf{E}$-équi-intégrabilité des familles considérées. Enfin comme $|\varrho_n|(\Omega) \to 0$ dans $\mathsf{L}^1$, le lemme précédent montre que $\||\varrho_n|(\Omega)\| \to 0$. □

### III-2 Décomposition de Lebesgue

Les $\mathsf{E}$-famille de Riesz ont aussi une décomposition de Lebesgue. En effet si $(\mu_i)_{i \in I}$ est une telle famille, il est possible d'écrire de manière unique $\mu_i = \mu_i^1 + \mu_i^2$ où $(\mu_i^1)$ est une $\mathsf{E}$-famille de Riesz de mesures appartenant à la bande engendrée par $P$ dans $\mathcal{M}_0(\mathcal{F}_i, \mathsf{E})$, $i \in I$, i.e. $\mu_i^1 = \bigvee_n \mu_i \wedge nP$, pour tout $i$, et $(\mu_i^2)$ est aussi une $\mathsf{E}$-famille de Riesz, orthogonale pour l'ordre à $P$: $\mu_i^2 \wedge P = 0$.

La démonstration de cette décomposition est standard, par exemple [15].

APPLICATIONS. 1 — Les propositions précédentes permettent de retrouver tous les résultats classiques de la théorie des amarts pour l'ordre: réticulation et décomposition de Riesz.

2 — Si $(X_n)$ est un uniforme $\mathsf{E}$-amart, $(X_\tau)_{\tau\in\mathcal{T}_1}$, est, nous l'avons vu une $\mathsf{E}$-famille de Riesz. Il en est de même de $(X_\tau^+)$. La fonction additive d'ensembles limite de $(X_\tau^+)$, $m$, est la partie positive de la limite de $(X_\tau)$. En considérant les mesures associées on voit que: $|\mu_\tau^+ - m^+|_{\mathcal{F}_\tau}| \leqq |\mu_\tau - m|_{\mathcal{F}_\tau}|$ donc $\text{var}_{\mathsf{E}}(\mu_\tau^+ - m^+|\mathcal{F}_\tau) \xrightarrow[\tau\in\mathcal{T}_1]{} 0$. Il en résulte que l'espace des $\mathsf{E}$-amarts uniformes est réticulé lorsque $\mathsf{E}$ est un espace de Köthe f.s.c..

3 — Soit $(X_n)$ un $\mathsf{E}$-amart faible tel que $(X_\tau)_{\tau\in\mathcal{T}_1}$ soit une $\mathsf{E}$-famille de Riesz. Alors on peut écrire de manière unique $X_n = M_n + Z_n$ où $(M_n)$ est une $\mathsf{E}$-martingale convergeant p.s. et $(Z_n)$ est un $\mathsf{E}$-amart faible tel que $\|Z_\tau\|_1 \xrightarrow[\tau\in\mathcal{T}_1]{} 0$. En particulier $(Z_\tau)_{\tau\in\mathcal{T}_1}$ converge vers 0 en probabilité. Mais la suite $(Z_n)$ ne converge pas, en général, p.s..

4 — Soit $(M_n)$ une $\mathsf{E}$-martingale de limite, $m$, $(m(A) = E^{\mathcal{B}}(M_n \cdot 1_A)$, si $A \in \mathcal{F}_n)$. On vérifie facilement que les énoncés suivants sont équivalents:

(i) $M_n = E^{\mathcal{B}\vee\mathcal{F}_n}(M)$ pour une v.a. $M$ $\mathsf{E}$-intégrable.

(ii) $(M_n)$ est équi-intégrable.

(iii) $m \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$.

5 — Soit $(X_i)$ une $\mathsf{E}$-famille de Riesz, alors $(X_i)$ converge en probabilité. En effet par le Théorème 20, on se ramène au cas $0 \leqq X_i \leqq 1$, et on utilise le Corollaire 22. Remarquons enfin que dans le cas $\mathcal{B}$ triviale, on obtient la propriété classique de convergence p.s. des amarts $\mathsf{L}^1$ bornés [10].

### III-3 Le théorème d'Andersen et Jessen pour les familles de Riesz

Soit $(\Omega, (\mathcal{F}_n), \mathcal{F}, \mathrm{P})$ un espace probabilisé filtré, $\mathcal{F} = \vee\mathcal{F}_n$. Soit $\mu$ une mesure positive réelle et bornée sur $\mathcal{F}$, notons $\mu_n$ sa restriction à $\mathcal{F}_n$ et $\mu_n = X_n P + \nu_n$ sa décomposition de Lebesgue. Le théorème d'Andersen et Jessen [1] affirme que $X_n$ converge p.s. vers la dérivée, $X$, de $\mu$ sur $\mathcal{F}$ par rapport à $P$: $\mu = X \cdot P + \nu$. Ce théorème s'étend aux fonctions additives d'ensembles réelles et bornées voir [3] et [9].

Réciproquement, si une surmartingale positive $(X_n)$ converge p.s. vers $X$, la fonction additive d'ensemble limite, notée $\mu$: $\mu(A) = \lim E(X_n 1_A)$, à une décomposition de Lebesgue du type:

$$\mu = X \cdot \mathrm{P} + \nu.$$

Donnons tout d'abord quelques précisions sur les limites de Banach de v.a.r.. Pour cela notons $\mathsf{S}$ l'espace des suites $(X_n)$ de v.a.r. qui sont équi-intégrables dans $\mathsf{L}^1$.

PROPOSITION 23. *Il existe une application $L$ de $\mathbb{S}$ dans $\mathbb{L}^1$ telle que:*
*a)* $\underline{\lim} X_n \leq L((X_n)) \leq \overline{\lim} X_n$
*b)* $L((X_{n+1})) = L((X_n))$
*c)* $\|L((X_n))\|_1 \leqq \overline{\lim} \|X_n\|_1$.

DÉMONSTRATION. Pour $A \in \mathcal{A}$ la suite $(E(X_n 1_A))$ est bornée. Si $L_0$ est une limite de Banach sur $\ell^\infty$, la fonction d'ensemble $m(A) = L_0((E(X_n 1_A)))$ est $\sigma$-additive et absolument continue par rapport à P. Définissons $L((X_n))$ comme la dérivée de $m$. Il est aisé de vérifier les assertions de la proposition. □

Soit $(X_n)$ une suite de v.a. $\mathbb{E}$-intégrables adaptées à la filtration croissante $(\mathcal{F}_n \vee \mathcal{B})$, formée de sous-tribu de $\mathcal{A}$ et où $\mathbb{E}$ est un espace de Köthe f.s.c.

Supposons que $(X_\tau)_{\tau \in \mathcal{T}_1}$ soit une $\mathbb{E}$-famille de Riesz. Écrivons la décomposition de $m(A)$: $m(0) = E(X \cdot 1_0) + \nu(0)$ (limite faible dans $\mathbb{E}$ de $E(X_\tau 1_A)$, pour $A \in \bigcup_n \mathcal{F}_n$) dans la bande engendrée par $\mathrm{P}^{\mathcal{B}}$. La version vectorielle du théorème d'Andersen et Jessen devient:

THÉORÈME 24. *Sous les conditions précédentes on a:*
a) *La suite $(X_n)$ converge en probabilité vers $X$.*
b) $\nu(0) = \lim_n L\left(\left(E^{\mathcal{B}}\left(X_n \cdot 1_{0 \cap (|X_n| > k)}\right)\right)\right)$.
*De plus lorsque $\mathbb{E} = \mathbb{R}$ i.e. $\mathcal{B} = (\emptyset, \Omega)$ nous avons:*
c) *Il existe une v.a. $Y = X$ p.s. telle que*

$$m(0) = E^{\mathcal{B}}(Y \cdot 1_0) + \nu(0 \cap (|Y| = \infty));$$

*et $m \ll P$ si, et seulement si, $X_n \to X$ fortement dans $\mathbb{L}^1$.*

DÉMONSTRATION. Nous pouvons supposer $X_n \geqq 0$, avec le Corollaire 14 et le Théorème 20; l'assertion a) s'ensuit avec l'application 5 ci-dessus.

Montrons b). Si $\mu_n$ est la $\mathbb{E}$-mesure associée à $X_n$: $(\mu(0) = E^{\mathcal{B}}(X_n \cdot 1_0))$ on a:

$$(\mu_n - \mu_n \wedge kP)(0) = E^{\mathcal{B}}\left(X_n \cdot 1_{0 \cap (X_n > k)}\right) - k \cdot E^{\mathcal{B}}\left(1_{0 \cap (X_n > k)}\right).$$

Ce qui montre que $(\mu_n - \mu_n \wedge kP)(0)$ converge faiblement dans $\mathbb{L}^1$ vers $(m - m \wedge k \cdot P^{\mathcal{B}})(0)$ et la suite $\left(E^{\mathcal{B}}\left(1_{A \cap (X_n > k)}\right)\right)$ est trivialement équi-intégrable.

Prenons les limites de Banach, il vient:

$$(m - m \wedge k \cdot P^{\mathcal{B}})(0) = L\left(E^{\mathcal{B}}\left(X_n 1_{0 \cap (X_n > k)}\right)\right) = k \cdot L\left(E^{\mathcal{B}}\left(1_{0 \cap (X_n > k)}\right)\right)$$

d'où pour $k$ assez grand:

$$\overline{\lim_n} k \cdot \left\| E^{\mathcal{B}}\left(1_{A \cap (X_n > k)}\right) \right\|_1 \leqq k \cdot P(X > k) \leqq \varepsilon.$$

Par conséquent

$$\lim_k (m - m \wedge k \cdot P)(0) = E^{\mathcal{B}}(X \cdot 1_0) = \lim L\Big(E^{\mathcal{B}}\Big(X_n \cdot 1_{0 \cap (X_n > k)}\Big)\Big),$$

c'est l'assertion b).

Montrons c). Supposons $m$ $\sigma$-additive, $\nu$ et $P$ sont étrangères, et il existe $A \in \mathcal{F}$ tel que $\nu(A) = 0$ et $P(A) = 1$. On pose $Y = X \cdot 1_A + \infty \cdot 1_{A^c}$. Il est clair que $Y$ vérifie la première partie de l'assertion c). Si, enfin, $m \ll P$ alors $\nu = 0$ et $E(X_n) \to E(X)$ ce qui, avec la convergence en probabilité, équivaut à la convergence dans $\mathsf{L}^1$. La réciproque n'offre guère de difficultés. □

## IV Bi-mesures

Par analogie avec l'intégrabilité conditionnelle des v.a.r. ou la théorie réelle des v.a. vectorielles, il est naturel de considérer de la même façon, la théorie réelle des mesures vectorielles. Cette étude nous permettra notamment de caractériser simplement la bande engendrée par $P^{\mathcal{B}}$ dans $\mathcal{M}_0(\mathsf{E})$.

Soit donc $(\Omega, \mathcal{A}, \mathrm{P})$ un espace probabilisé, $\mathcal{F}$ et $\mathcal{B}$ deux sous-tribus de $\mathcal{A}$. Notons, en accord avec les écritures précédentes. $\mathcal{M}_0(\mathcal{B})$ l'ensemble des mesures réelles, signées, définies sur $\mathcal{B}$ et $\mathcal{M}(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ — resp. $\mathcal{M}_0(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ — l'ensembles des mesures vectorielles — resp. des mesures vectorielles signées — sur $\mathcal{F}$ à valeurs dans $\mathcal{M}_0(\mathcal{B})$.

Soit $\mathcal{C}$ l'algèbre engendrée par $\mathcal{F} \cup \mathcal{B}$ et $\mathsf{L}^\sigma(\mathcal{C})$ — resp. $\mathsf{L}^\sigma_0(\mathcal{C})$ — l'espace des fonctions additives d'ensembles, réelles, définies sur $\mathcal{C}$ — resp. additives signées, définies sur $\mathcal{C}$ et telles que leurs restrictions à chacune des sous-tribus $\mathcal{F}$ et $\mathcal{B}$ soient $\sigma$-additives. On aura reconnu la version ensembliste des bi-mesures, étudiées dans [22]. Introduisons enfin les notations suivantes:

$\mathcal{M}^*_0(\mathcal{F}, \mathcal{M}(\mathcal{B})) = (\mu \in \mathcal{M}_0(\mathcal{F}, \mathcal{M}_0(\mathcal{B})); \mu(F)(B) = 0$ dès que $F \cap B = \emptyset)$

$\mathcal{M}^*(\mathcal{F}, \mathcal{M}(\mathcal{B})) = (\mu \in \mathcal{M}(\mathcal{F}, \mathcal{M}_0(\mathcal{B})); \mu(F)(B) = 0$ dès que $F \cap B = \emptyset)$.

Il existe des isomorphismes canoniques, préservant l'ordre, entre $\mathcal{M}(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ et $\mathcal{M}(\mathcal{B}, \mathcal{M}_0(\mathcal{F}))$; $\mathcal{M}^*(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ et $\mathcal{M}^*(\mathcal{B}, \mathcal{M}_0(\mathcal{F}))$; et $\mathcal{M}^*_0(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ et $\mathcal{M}^*_0(\mathcal{B}, \mathcal{M}_0(\mathcal{F}))$. De plus on a:

THÉORÈME 25. *Il existe un isomorphisme $\varphi$ préservant l'ordre entre $\mathcal{M}^*(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ et $\mathsf{L}^\sigma(\mathcal{C})$. De plus, pour cet isomorphisme, les espaces $\mathcal{M}^*_0(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ et $\mathsf{L}^\sigma_0(\mathcal{C})$ se correspondent.*

DÉMONSTRATION. Il suffit de montrer que, pour $\mu \in \mathcal{M}^*(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$, la fonction additive d'ensembles $m(F \cap B) = \mu(F)(B)$ est bien définie et peut se prolonger, de manière unique, en un élément de $\mathsf{L}^\sigma(\mathcal{C})$. Supposons que

(1) $$F \cap B = \sum_{I \text{ fini}} F_i \cap B_i.$$

Posons $F_i^* = F \cap F_i$; $B_i^* = B \cap B_i$; $F_i = F_i^* + A_i$ et $B_i = B_i^* + C_i$. Des relations

$$F \cap B = \sum (F_i^* + A_i) \cap (B_i^* + C_i) = \sum F_i^* \cap B_i^*.$$

On déduit

$$A_i \cap B_i = \emptyset = F_i \cap C_i.$$

Donc

$$\mu(F)(B) = \mu(F^*)(B) + \mu(F^*)(B^*).$$

Par suite nous pouvons supposer, dans (1) que $F = \bigcup F_i$ et $B = \bigcup B_i$ et ainsi, que les $F_i$ sont disjoints. De sorte que: $F_i \cap B = F_i \cap B_i$. Il vient

$$\mu(F)(B) = \sum_i \mu(F_i)(B) = \sum_i \mu(F_i)(B_i).$$

Soit maintenant $(A_j)_{j \in J}$ les atomes de $\sigma(F_i, 1 \leqq i \leqq k)$ et $(C_\ell)_{\ell \in L}$ ceux de $\sigma(B_i, 1 \leqq i \leqq k)$. On a alors

$$F = \sum_{J_i} A_j \quad \text{et} \quad B = \sum_{L_i} C_\ell$$

et l'égalité: $F \cap B = \sum_{i,j} A_j \cap C_l = \sum_i (\sum_{J_i \times L_i} A_j \cap C_l)$ montre que $A_j \cap C_\ell = \emptyset$ si $j, \ell \notin \bigcup_i J_i \times L_i$. Par ailleurs

$$F \cap B = \sum_j A_j \cap B$$

et par conséquent

$$\mu(F)(B) = \sum_{j,\ell} \mu(A_j)(C_\ell) = \sum_i \sum_{J_i \times L_i} \mu(A_j)(C_\ell).$$

Il ne reste plus qu'à remarquer que $\mu(F_i)(B_i) = \sum_i \sum_{J_i \times L_i} \mu(A_j)(C_\ell)$ pour achever la première partie de la preuve.

Appelons $\varphi$ l'isomorphisme ainsi construit. Il préserve l'ordre. Remarquons enfin que $\mathsf{L}_0^\sigma$ est un espace de Banach réticulé pour la norme: $\|m\| = \operatorname*{Sup}_C |m(C)|$. Il en est de même pour $\mathcal{M}_0^\sigma(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ avec la norme $\|\mu\| = |\mu|(\Omega)(\Omega)$ qui est équivalente à la norme variation. Il en résulte que $(\varphi(\mu))^+ = \varphi(\mu^+)$ etc. ... □

Considérons maintenant, comme dans les paragraphes précédents, un espace de Köthe $\mathsf{E}$ de v.a.r. $\mathcal{B}$-mesurables, f.s.c.. Nous supposerons que les sous-tribus $\mathcal{F}$ et $\mathcal{B}$ sont complètes pour P. Toute mesure, $\mu$, sur $\mathcal{F}$ à valeurs $\mathsf{E}$, est alors nécessairement absolument continue par rapport à P.

Si $Y \in \mathsf{E}'$ la v.a.r. $Y \cdot \mu(F)$ appartient à $\mathsf{L}^1(\mathcal{B})$, pour tout $F \in \mathcal{F}$. Ce qui permet d'identifier $Y \cdot \mu$ comme mesure de $\mathcal{F}$ dans $\mathsf{L}^1(\mathcal{B})$. En effet si $F_n \searrow \emptyset$ $\|Y \cdot \mu(F_n)\| \leqq \|\mu(F_n)\|_{\mathsf{E}} \cdot \|Y\|_{\mathsf{E}'} \to 0$. Il est clair, d'autre part, que si $\mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$ et $Y \in \mathsf{E}'_+$, alors $|Y \cdot \mu| = Y \cdot |\mu|$. Nous pouvons aussi considérer $Y \cdot \mu$ comme un élément de $\mathcal{M}(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$ par la relation

$$Y \cdot \mu(F)(B) = E(Y \cdot \mu(F) \cdot 1_B).$$

Ainsi pour toute v.a.r. $Y \in \mathsf{E}'$, strictement positive p.s., nous avons construit une injection $\psi_Y$ de $\mathcal{M}(\mathcal{F}, \mathsf{E})$ dans $\mathcal{M}(\mathcal{F}, \mathsf{L}^1(\mathcal{B}))$ ou $\mathcal{M}(\mathcal{F}, \mathcal{M}_0(\mathcal{B}))$, définie par $\psi_Y(\mu) = Y \cdot \mu$. Cette injection préserve l'ordre et de plus

$$\psi_Y(\mathcal{M}_0(\mathcal{F}, \mathsf{E})) \subset \mathcal{M}_0(\mathcal{F}, \mathsf{L}^1(\mathcal{B}))$$

et

$$\psi_Y(\mathcal{M}_0^*(\mathcal{F}, \mathsf{E})) \subset \mathcal{M}_0^*(\mathcal{F}, \mathsf{L}^1(\mathcal{B})) \tag{2}$$

où $\mathcal{M}_0^*(\mathcal{F}, \mathsf{E})$ désigne l'espace des mesures $\mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$ telles que

$$\mu(F) \cdot 1_B = 0 \quad \text{dès que} \quad P(B \cap F) = 0.$$

L'inclusion (2) résulte du lemme plus précis suivant:

Lemme 26. *Soit* $\mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$. *Alors* $\mu \in \mathcal{M}_0^*(\mathcal{F}, \mathsf{E})$, *si et seulement si, pour tout* $Y \in \mathsf{E}'$, $Y \cdot \mu \in \mathcal{M}_0^*(\mathcal{F}, \mathsf{L}^1(\mathcal{B}))$.

Démonstration. Comme les espaces considérés sont réticulés, on peut supposer $Y$ et $\mu$ positives. Le lemme est alors évident en approchant $Y$ par des suites croissantes de v.a. étagées. □

Remarquons que les injections $\phi$ et $\psi_Y$ pour $Y \in \mathsf{E}'_+$ sont liées par la formule:

$$\phi \circ \psi_Y = \psi_Y \circ \phi \circ \psi_1.$$

On obtient enfin:

Théorème 27. *Soit* $\mu \in \mathcal{M}_0^*(\mathcal{F}, \mathsf{E})$; $\mu$ *peut s'écrire* $\mu(F) = E^{\mathcal{B}}(X \cdot 1_F)$, *si, et seulement si,* $\phi \circ \psi_Y(\mu)$ *est une mesure sur* $\mathcal{C}$ *absolument continue par rapport à* P.

Démonstration. La condition est évidemment nécessaire. Montrons qu'elle est suffisante. Soit $\mu \in \mathcal{M}_0^*(\mathcal{F}, \mathsf{E})$ telle que $\phi \circ \psi_1(\mu)$ soit une mesure absolument continue par rapport à P. On peut en outre supposer que $\mu$ est positive. Par hypothèse $\phi(\psi_1(\mu))(F \cap B) = E(\mu(F) \cdot 1_B) = E(X \cdot 1_{F \cap B})$. Pour toute v.a. étagée, $Z$, $\mathcal{B}$-mesurable, on a:

$$E(\mu(F) \cdot Z) = E(1_F \cdot Z \cdot X), \quad \text{pour tout} \quad F \in \mathcal{F}.$$

Cette égalité se prolonge, par approximation croissante, à toute v.a. $Y \in \mathsf{E}'_+$. Avec la propriété de Fatou sur $\mathsf{E}$, on en déduit que $X$ est $\mathsf{E}$-intégrable et que $\mu(F) = E^{\mathcal{B}}(X \cdot 1_F)$. □

Notons $\mathcal{M}_{P^{\mathcal{B}}}(\mathcal{F}, \mathsf{E})$ la bande engendrée par $P^{\mathcal{B}}$ dans $\mathcal{M}_0(\mathcal{F}, \mathsf{E})$, on déduit du résultat précédent:

COROLLAIRE 28. *Soit $\mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$; les énoncés suivants sont équivalents:*
(i) $\mu \in \mathcal{M}_{P^{\mathcal{B}}}(\mathcal{F}, \mathsf{E})$
(ii) $\mu \in \mathcal{M}_{P^{\mathcal{B}}}(\mathcal{F}, \mathsf{L}^1(\mathcal{B}))$
(iii) $E(\mu(F) \cdot 1_B)$ *définit une mesure sur* $\mathcal{C}$.

## V-1 Cas $\mathcal{F}$ et $\mathcal{B}$ indépendantes

Soit $(\Omega, \mathcal{A}, \mathrm{P})$ un espace de probabilité, $\mathcal{F}$ et $\mathcal{B}$ deux sous-tribus indépendantes, $\mathsf{E}$ un espace de Köthe f.s.c. de v.a.r. $\mathcal{B}$-mesurables. Nous avons clairement $\mathrm{P}^{\mathcal{B}} = \mathrm{P}$ et par suite $\mathcal{M}_{P^{\mathcal{B}}}(\mathcal{F}, \mathsf{E})$ est identique à la bande $\mathcal{M}_{\mathrm{P}}(\mathcal{F}, \mathsf{E})$. Notons $\overrightarrow{\mathsf{P}}_0(\mathcal{F}, \mathsf{E})$ l'espace des v.a. $\mathcal{F}$-mesurables à valeurs $\mathsf{E}$, $\overrightarrow{X}$, telles que $|\overrightarrow{X}|$ soit Pettis-intégrable. Sur $\overrightarrow{\mathsf{P}}_0(\mathcal{F}, \mathsf{E})$ nous considérons la norme $\|E(|\overrightarrow{X}|)\|_{\mathsf{E}}$. Soit $\Delta$ l'application de $\Omega$ dans $\Omega \times \Omega$ définie par $\Delta(\omega) = (\omega, \omega)$. Si $\overrightarrow{X} \in \overrightarrow{\mathsf{P}}_0(\mathcal{F}, \mathsf{E})$ il est clair que $X = \overrightarrow{X} \circ \Delta \in \mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$ espace des v.a. $\mathcal{F} \vee \mathcal{B}$-mesurables, $\mathsf{E}$-intégrables avec la norme $\|E^{\mathcal{B}}(|X|)\|_{\mathsf{E}}$. Plus précisement:

LEMME 29. *L'application $\overrightarrow{X} \to X = \overrightarrow{X} \circ \Delta$ est une isométrie pour l'ordre de $\overrightarrow{\mathsf{P}}_0(\mathcal{F}, \mathsf{E})$ sur $\mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$ et on a $E^{\mathcal{B}}(X \cdot 1_F) = E(\overrightarrow{X} \cdot 1_F)$ pour $F \in \mathcal{F}$.*

DÉMONSTRATION. L'unique chose à montrer est que $\overrightarrow{X} \to \overrightarrow{X} \circ \Delta$ est surjective. Soit $X \in \mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$. Définissons $\mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$ par $\mu(F) = E^{\mathcal{B}}(X \cdot 1_F)$; donc $\mu \in \mathcal{M}_{P^{\mathcal{B}}}(\mathcal{F}, \mathsf{E}) = \mathcal{M}_{\mathrm{P}}(\mathcal{F}, \mathsf{E})$. Comme $\mathsf{E}$ est f.s.c. il existe $\overrightarrow{X} \in \overrightarrow{\mathsf{P}}_0(\mathcal{F}, \mathsf{E})$ unique telle que

$$\mu(F) = E(\overrightarrow{X} \cdot 1_F).$$

Posons $\bar{X} = \overrightarrow{X} \circ \Delta$. Or $|\overrightarrow{X}|$ est Pettis-intégrable dans $\mathsf{E}$, donc Bochner-intégrable dans $\mathsf{L}^1(\mathcal{B})$. Ainsi

$$E(\overrightarrow{X} \cdot 1_F) = \int_F \overrightarrow{X}(\omega)(\omega') d\mathrm{P}(\omega) = E^{\mathcal{B}}(\bar{X} \cdot 1_F)$$

et les deux variables $X$ et $\tilde{X}$ définissent la même mesure $\mu$. On achève la preuve avec le Corollaire 14. □

REMARQUE. Soit $(\mathcal{F}_n)$ une filtration croissante engendrant $\mathcal{F}$, et soit $(\vec{X}_n)$ une martingale adaptée à cette filtration et à valeurs $\mathsf{E}$ telle que

$$\mathrm{Sup}\|E^{\mathcal{B}}(|\vec{X}_n|)\| < \infty.$$

Posons $X_n = \vec{X}_n \circ \Delta$, $(X_n)$ est clairement une $\mathsf{E}$-martingale telle que

$$\mathrm{Sup}\|E^{\mathcal{B}}(|X_n|)\| < \infty.$$

Ainsi elle converge p.s. vers une v.a. réelle $X$ appartenant à $\mathsf{P}_0(\mathcal{F} \vee \mathcal{B}, \mathsf{E})$.

Avec le Lemme 29, on lui associe une unique variable vectorielle

$$\vec{X} \in \vec{\mathsf{P}}_0(\mathcal{F}, \mathsf{E}) \quad \text{telle que} \quad \vec{X} \circ \Delta = X \quad \text{p.s.}.$$

Donc $\vec{X}_n \circ \Delta \to \vec{X} \circ \Delta$ p.s.. Comme précédemment, on peut aussi considérer $(\vec{X}_n(\omega)(\omega'))$ comme une martingale réelle, adaptée à $(\mathcal{F}_n \otimes \mathcal{B})$, dans l'espace de probabilité $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{B}, \mathrm{P} \otimes \mathrm{P})$ et, dans cet espace, $\vec{X}_n(\omega)(\omega') \to$ $\to \vec{X}(\omega)(\omega')$ p.s..

## V-2 Enveloppe de Snell

Soit $(X_n)$ une suite de v.a. $\mathsf{E}$-intégrables, adaptées à la filtration $(\mathcal{F}_n \vee \mathcal{B})$ et telle que $(X_\tau)_{\tau \in \mathcal{T}_1}$ soit une $\mathsf{E}$-famille de Riesz. Nous notons $\mu_n$ et $\vec{X}_n$ respectivement la $\mathsf{E}$-mesure et la v.a. à valeurs $\mathsf{E}$ associée à $X_n$. On a:

$$\mu_n(F) = E^{\mathcal{B}}(X_n \cdot 1_F) = E(\vec{X}_n \cdot 1_F) \quad \text{pour} \quad F \in \mathcal{F}_n.$$

THÉORÈME 30. *Si* $\bigvee_{\tau \in \mathcal{T}_1} E^{\mathcal{B}}(|X_\tau|) \in \mathsf{E}$ *alors* $(X_n)$ *est contrôlable au sens de Snell. C'est-à-dire qu'il existe deux suites* $(\nu_n)$ *et* $(\varrho_n)$ *respectivement sur et sous-martingales-mesures, adaptées à* $(\mathcal{F}_n)$ *à valeurs dans* $\mathsf{E}$ *et telles que:*

$$\nu_n \leqq \mu_n \leqq \varrho_n, \quad n \in \mathbb{N}; \quad \nu_\tau = \bigwedge_{\sigma \geqq \tau} (\mu_\sigma \mid \mathcal{F}_\tau)$$

*et* $\varrho_\tau = \bigvee_{\sigma \geqq \tau} (\mu_\sigma \mid \mathcal{F}_\tau)$, $\tau \in \mathcal{T}_1$; $\nu_n$ *et* $\varrho_n$ *appartiennent à* $\mathcal{M}_{\mathrm{P}}(\mathcal{F}_n, \mathsf{E})$. *De plus, si on écrit* $\nu_n = E(\vec{s}_n \cdot 1_F) = E^{\mathcal{B}}(s_n \cdot 1_F)$, *et*

$$\varrho_n = E(\vec{S}_n \cdot 1_F) = E^{\mathcal{B}}(S_n \cdot 1_F) \quad \textit{pour} \quad F \in \mathcal{F}_n.$$

*Alors* $\vec{s_n} \leqq \vec{X_n} \leqq \vec{S_n}$ *et en outre* $\vec{S_n} = \bigvee_{\tau \geqq n} E^{\mathcal{F}_n}(\vec{X_\tau})$; $\vec{s}_n = \bigwedge_{\tau \geqq n} E^{\mathcal{F}_n}(\vec{X_\tau})$. *Nous obtenons aussi les mêmes relations avec* $s$, $X$ *et* $S$: $s_n \leqq X_n \leqq S_n$ ... *etc.* ...

DÉMONSTRATION. $(\mu_\tau)_{\tau \in \mathcal{T}_1}$ est une famille de mesures pour l'ordre prenant ses valeurs dans la bande engendrée par $e_0 = \bigvee_{\tau \in \mathcal{T}_1} (|\mu_\tau|(\Omega) \vee 1)$ dans $\mathsf{E}$. Cet idéal est isomorphe à $\mathsf{L}^\infty$. Par suite pour chaque $\tau$, $\mu_\tau \in \mathcal{M}_{\mathrm{P}}(\mathcal{F}_\tau, \mathsf{L}^\infty)$. Appliquons le Théorème 5–3 de [17]: il existe deux familles $(\nu_\tau)$ et $(\varrho_\tau)$ respectivement sous et sur martingales mesures, contrôlées par $(\mu_\tau)$ et $|\nu_\tau|(\Omega) \vee |\varrho_\tau|(\Omega) \leqq e_0$. Il est aisé de vérifier que $\nu_\tau$ et $\varrho_\tau \in \mathcal{M}_{\mathrm{P}}(\mathcal{F}_\tau, \mathsf{L}^\infty)$. De là on en déduit leurs appartenance à $\mathcal{M}_{\mathrm{P}}(\mathcal{F}_\tau, \mathsf{E})$. La preuve s'achève alors de manière standard. □

Un cas particulièrement intéressant est celui où: $\varrho_n - \nu_n \to 0$. Dans ce cas $(\mu_\tau)$ est un $o$-amart et la réciproque est également vraie, (Corollaire 5–5 de [17]). On obtient une généralisation de la caractérisation des amarts réels de [14].

COROLLAIRE 31. *Sous les hypothèses du théorème précédent,* $(X_n)$ *est un* $\mathsf{E}$-*o-amart si et seulement si,* $E(S_n - s_n) \to 0$; *de plus* $X_n$ *converge p.s..*

## V-3 Hypomartingales

Dans cette partie nous considérons uniquement des variables à valeurs $\mathsf{E}$, Bochner-intégrables, et des $\mathsf{E}$-mesures à variation bornée. Rappelons que $\mu \in \mathcal{M}_0(\mathcal{F}, \mathsf{E})$ est à densité Bochner-intégrable, par rapport à P, si et seulement si, elle est à variation bornée et appartient à $\mathcal{M}_{\mathrm{P}}(\mathcal{F}, \mathsf{E})$ (comme d'habitude $\mathsf{E}$ est supposé f.s.c.) [6]. Pour simplifier les écritures, nous noterons $X$ (sans flêche) une v.a. à valeurs dans $\mathsf{E}$.

DÉFINITIONS 32. Une suite $(X_n)$ de v.a. positives, $\mathsf{E}$-Bochner-intégrables est dite

— équi-intégrable au sens de Bochner (ou e.i.) si la suite $(\|X_n(\Omega)\|_{\mathsf{E}})$ est équi-intégrable (dans $\mathsf{L}^1$);

— équi-intégrable pour l'ordre (ou o.e.i.) si

$$\lim_k [\operatorname{Sup}_n E(\|X_n - X_n \wedge k \cdot 1\|)] = 0 \qquad [15];$$

— faiblement équi-intégrable pour l'ordre (f.o.e.i.) si pour tout $u \in \mathsf{E}'$,

$$\lim_k [\operatorname{Sup}_n u(E(X_n - X_n \wedge k \cdot 1))] = 0.$$

REMARQUES 1. Ces notions s'étendent aux suites non nécessairement positives, en considérant les suites valeurs absolues.

2. L'espace $\mathsf{L}^1(\mathsf{E})$ des variables Bochner-intégrables est un nouvel espace de Köthe. Il est aisé de voir qu'une suite $(X_n)$, positive, de $\mathsf{L}^1(\mathsf{E})$ est o.e.i., si et seulement si, elle est $\mathsf{L}^1(\mathsf{E})$-équi-intégrable. De plus nous avons les implications suivantes: $(X_n)$ converge dans $\mathsf{L}^1(\mathsf{E}) \Rightarrow (X_n)$ est o.e.i. $\Rightarrow (X_n)$ est f.o.e.i. et si $E(X_n) \to 0$ alors $(X_n)$ est f.o.e.i..

Soit maintenant $(X_n)$ une suite à valeurs dans $\mathsf{E}$, adaptées à une filtration croissante $(\mathcal{F}_n)$ et Bochner-intégrables.

DÉFINITIONS 33. $(X_n)$ est une hypomartingale — resp. épimartingale — si on peut écrire $X_n = M_n - Z_n$ — resp. $X_n = M_n + Z_n$ — où $(M_n)$ est une martingale (à valeurs dans $\mathsf{E}$) et $(Z_n)$ un potentiel positif, faible.

Donnons quelques propriétés:

a) Si $\bigvee_{\tau\in\mathsf{T}} E(Z_\tau) \in \mathsf{E}$, où $\mathsf{T}$ est l'ensemble des temps d'arrêts finis, alors $(Z_n)$ est un potentiel fort: $\|Z_\tau\| \to 0$. En effet, comme $0 \leqq E(Z_\tau) \leqq \bigvee_{\tau\in\mathsf{T}} E(Z_\tau)$ et $E(Z_\tau) \to 0$, il suffit d'utiliser le lemme de [16].

b) $(X_n)$ est une hypomartingale — resp. épimartingale — si et seulement si, pour tout $A \in \bigcup_n \mathcal{F}_n$, la famille $(E(X_\tau \cdot 1_A))_{\tau\in\mathsf{T}}$ converge faiblement dans $\mathsf{E}$ en restant en dessous — resp. au dessus — de sa limite.

c) Soit $(X_n)$ une hypomartingale ou une épimartingale positive. Alors pour tout $x \in \mathsf{E}_+$, $(X_n \wedge x)$ est un amart convergeant fortement p.s. dans $\mathsf{L}^1(\mathsf{E})$.

En effet, pour une épimartingale: $X_n = M_n + Z_n$ on écrit

$$M_n \wedge x \leqq X_n \wedge x \leqq M_n \wedge x + Z_n \wedge x.$$

Or $(Z_n \wedge x)$ est un potentiel fort (assertion a) à valeurs dans le compact faible $[0, x]$, donc la Proposition 6 de [8] assure la convergence faible p.s. de ce potentiel, et le lemme de [16] montre la convergence forte p.s., et aussi celle dans $\mathsf{L}^1(\mathsf{E})$, vers 0.

De même, $(M_n \wedge x)$ est une surmartingale positive et on recommence le raisonnement, ce qui prouve l'assertion pour les épimartingales. Si, maintenant, $(X_n)$ est une hypomartingale positive, l'inégalité

$$M_n \wedge x - Z_n \leqq X_n \wedge x \leqq M_n \wedge x$$

montre, en utilisant la méthode précédente, que $(X_n \wedge x)$ est un amart fort à valeurs dans un compact faible, et on conclue de même.

LEMME 34. *Soit $(X_n)$ une hypomartingale positive à valeurs dans $\mathsf{E}$, $\mathsf{L}^1(\mathsf{E})$ bornée. Alors $(\|X_n\|)$ est une hypomartingale réelle, $\mathsf{L}^1$-bornée et* $\lim \|X_n(\omega)\| = \sup_{\|u\|_{\mathsf{E}'} \leq 1} \lim u(X_n(\omega))$ *P-p.s..*

DÉMONSTRATION. Il s'agit en fait d'une extension du Lemme V. 29 de [21] et d'une version vectorielle d'un résultat de [12] sur les sous-martingales.

Soit $u \in \mathsf{E}'_+$, il est clair que $(|u(X_n)|)$ est une hypomartingale réelle; par suite le lemme résulte de la propriété suivante.

Soit $((X_n^i)_n)$, $i \in I$, une famille dénombrable d'hypomartingales positives, réelles, et telle que $\operatorname{Sup}_n E(\operatorname{Sup}_I X_n^i) < \infty$. Alors $(\operatorname{Sup}_I X_n^i)_{n\in\mathsf{N}}$ est une hypomartingale $\mathsf{L}^1$ bornée, et

$$\lim_n \operatorname{Sup}_I X_n^i = \operatorname{Sup}_I \lim_n X_n^i \quad \text{p.s..}$$

On achève la preuve de manière classique. □

THÉORÈME 35. *Les assertions suivantes sont équivalentes:*

(i) $\mathsf{E}$ *a la propriété de Radon-Nikodym.*

(ii) *Toute hypomartingale positive,* $\mathsf{L}^1(\mathsf{E})$ *bornée, converge fortement P-p.s..*

(iii) *Pour toute hypomartingale* $(X_n)$*, positive,* $\mathsf{L}^1(\mathsf{E})$ *bornée, les propriétés suivantes sont équivalentes "*$(X_n)$ *est f.o.e.i."; "*$(X_n)$ *est o.e.i."; "*$(X_n)$ *est e.i.".*

(iv) *Même assertion que (iii) en remplaçant "hypomartingales" par "martingales".*

DÉMONSTRATION. (i) ⇒ (ii): Dans la décomposition $X_n = M_n - Z_n$ d'une hypomartingale positive, $\mathsf{L}^1(\mathsf{E})$ bornée, la partie martingale est aussi $\mathsf{L}^1(\mathsf{E})$ bornée et donc converge p.s.. De l'inégalité $0 \leqq Z_n \leqq M_n$ et, en utilisant un raisonnement similaire à la partie c) ci-dessus, on voit que $Z_n$ converge fortement p.s. vers 0.

(i) ⇒ (iii) Soit $(X_n)$ une hypomartingale $\mathsf{L}^1(\mathsf{E})$ bornée, positive et f.o.e.i.. Il existe $M \in \mathsf{L}^1(\mathsf{E})$ telle que

$$X_n \wedge k \cdot e \to M \wedge k \cdot e \quad \text{p.s. et dans} \quad \mathsf{L}^1(\mathsf{E}),$$

$e$ étant un point quasi-intérieur de $\mathsf{E}$ (que nous pouvons supposer séparable).

En effet les arguments de c) assurent que

$$X_n \wedge k \cdot e \to M^k \quad \text{p.s. et dans} \quad \mathsf{L}^1(\mathsf{E}),$$

avec

$$M^{k+1} \wedge k \cdot e = M^k \quad \text{p.s. et} \quad \bigvee_k \|M^k\|_{\mathsf{L}^1(\mathsf{E})} < \infty.$$

Ce qui prouve l'existence de $M$ car $\mathsf{E}$ est f.s.c.. Enfin, si nous supposons que $(X_n)$ est e.i., alors la convergence p.s. de $X_n$ assure sa convergence dans $\mathsf{L}^1(\mathsf{E})$ donc que la suite est o.e.i..

(iii) ⇒ (iv) et (ii) ⇒ (i) sont triviales car toute martingale est la différence de deux hypomartingales positives.

(iv) ⇒ (i): Soit $(M_n)$ une martingale positive o.e.i., alors $M_n = E^{\mathcal{F}_n}(M)$.

En effet, $(M_n \wedge k \cdot e)$ converge p.s. et dans $\mathbb{L}^1(\mathbb{E})$ vers une v.a. $Y_k$. L'inégalité:

$$\|Y_p - Y_k\|_{\mathbb{L}^1(\mathbb{E})} \leqq \|Y_p - M_n \wedge p \cdot e\| + \|M_n \wedge p \cdot e - M_n \wedge k \cdot e\| + \|M_n \wedge k \cdot e - Y_k\|$$

montre qu'il existe $M \in \mathbb{L}^1(\mathbb{E})$ telle que $Y_k \to M$ dans $\mathbb{L}^1(\mathbb{E})$. Par ailleurs la suite $(Y_k)$ vérifie

$$Y_{k+1} \wedge k \cdot e = Y_k \quad \text{p.s., d'où} \quad Y_k = M \wedge k \cdot e.$$

Enfin, l'inégalité:

$$\|M_n - M\| \leqq \|M_n - M_n \wedge k \cdot e\| + \|M_n \wedge k \cdot e - M \wedge k \cdot e\| + \|M \wedge k \cdot e - M\|$$

montre que $M_n \to M$ dans $\mathbb{L}^1(\mathbb{E})$ donc $M_n = E^{\mathcal{F}_n}(M)$, ce qui achève notre preuve. □

## RÉFÉRENCES


[1] ANDERSEN, E. S. and JESSEN, B., Some limit theorems on set-functions, *Danske Vid. Selsk. Mat.-Fys. Medd.* **25**, no. 5, 1948. *MR* **10**-437.

[2] BELLOW, A., Les amarts uniformes, *C. R. Acad. Sci. Paris Sér. A-B* **284** (1977), 1295–1298. *MR* **56** # 3938.

[3] BOCHNER, S. and PHILLIPS, R. S., Additive set functions and vector lattices, *Ann. of Math.* (2) **42** (1941), 316–324. *MR* **2**-315.

[4] BRU, B., Espérance d'ordre, *C. R. Acad. Sci. Paris Sér. A-B* **290** (1980), 111–114. *MR* **80m**: 28012.

[5] BRU, B. and HEINICH, H., Sur l'espérance des variables aléatoires vectorielles, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **16** (1980), 177–196. *MR* **84g**: 60005.

[6] BRU, B. and HEINICH, H., Sur l'espérance des variables aléatoires à valeurs dans les espaces de Banach réticulés, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **16** (1980), 197–210. *MR* **84g**: 60006.

[7] BRU, B. and HEINICH, H., Meilleurs approximations et médianes conditionnelles, *Ann. Inst. H. Poincaré Probab. Statist.* **21** (1985), 197–224. *MR* **87g**: 60041.

[8] BRUNEL, A. and SUCHESTON, L., Sur les amarts à valeurs vectorielles, *C. R. Acad. Sci. Paris Sér. A-B* **283** (1976), 1037–1040. *MR* **55** # 11378.

[9] CHATTERJI, S. D., Differentiation along algebras, *Manuscripta Math.* **4** (1971), 213–224. *MR* **44** # 4173.

[10] EDGAR, G. A. and SUCHESTON, L., Amarts: a class of asymptotic martingales. A. Discrete parameter, *J. Multivariate Anal.* **6** (1976), 193–221. *MR* **54** # 1368.

[11] EDGAR, G. A. and SUCHESTON, L., The Riesz decomposition for vector-valued amarts, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **36** (1976), 85–92. *MR* **54** # 1369.

[12] GHOUSSOUB, N., Banach lattices valued amarts, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **13** (1977), 159–169. *MR* **57** # 10809.

[13] GHOUSSOUB, N., Order amarts: a class of asymptotic martingales, *J. Multivariate Anal.* **9** (1979), 165–172. *MR* **80i**: 60075.

[14] GHOUSSOUB, N. and SUCHESTON, L., A refinement of the Riesz decomposition for amarts and semiamarts, *J. Multivariate Anal.* **8** (1978), 146–150. *MR* **58** # 13334.

[15] HEINICH, H., Martingales asymptotiques pour l'ordre, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **14** (1978), 315–333. *MR* **80c**: 60067.

[16] HEINICH, H., Convergence des sous-martingales positives dans un Banach réticulé, *C. R. Acad. Sci. Paris Sér. A-B* **286** (1978), 279–280. *MR* **81e**: 60051.

[17] HEINICH, H., Mesures vectorielles dans les espaces réticulés, *Ann. Inst. H. Poincaré Sect. B (N.S.)* **19** (1983), 153–174. *MR* **85e**: 28012.

[18] LINDENSTRAUSS, J. and TZAFRIRI, L., *Classical Banach Spaces II. Functions spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 97, Springer-Verlag, Berlin–Heidelberg–New York, 1979. *MR* **81c**: 46001.

[19] LUXEMBURG, W. A. J. and ZAANEN, A. C., *Riesz spaces I*, North-Holland, Amsterdam–London, 1971. *MR* **58** # 23483.

[20] MAISONNEUVE, B., Surmartingales-mesures, *Séminaire de Probabilité XV* (Univ. Strasbourg, Strasbourg, 1979/1980), Lecture Notes in Math., no 850, Springer, Berlin, 1981, 347–350. *MR* **82h**: 60082.

[21] NEVEU, J., *Bases mathématiques du calcul des probabilités*, Masson, Paris, 1981.

[22] THOMAS, E., L'intégration par rapport à une mesure de Radon vectorielle, *Ann. Inst. Fourier* **20** (1970), 59–191. See *MR* **44** # 403.

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# ON A MULTIVARIATE EXTENSION OF PETROV'S LARGE DEVIATION THEOREM[1)]

E. DERSCH

**Abstract**

Let $\{X_n\}_{n\geqq 1}$ denote a sequence of independent $k$-dimensional random vectors, having the same strongly nonlattice respectively lattice distribution. Using moment generating function techniques and refinements in the central limit theorem a uniform asymptotic expansion of the probabilities of large deviations for the sum $S_n = X_1 + \ldots + X_n$ is proved. This result is a multivariate analogue of a well-known large deviation theorem on the real line due to Petrov (1965). As an application the limit distribution of $X_1$ under the condition $S_n \geqq na$ ($a > EX_1$ component-wise) is determined. The result is a multivariate analogue of a conditional limit theorem on the real line proved by Bártfai (1972).

## 1. Introduction

Petrov (1965) proved the following important large deviation theorem on the real line:

THEOREM A (Petrov). *Let $\{X_n\}_{n\geqq 1}$ be a sequence of independent identically distributed (iid.) real-valued random variables with moment generating function*

$$\varphi(t) = E\,\exp(tX_1) < \infty, \quad t \in [0, t_1).$$

*Then one has uniformly in $a \in [a_0, a_1] \subset A := \{\varphi'(t)/\varphi(t) : t \in (0, t_1)\}$*

$$P(S_n \geqq na) = \varrho^n(a)(2\pi n)^{-1/2} b_n(a)(1 + o(1)) \qquad (n \to \infty) \tag{1.1}$$

*with*

$b_n(a) = (h\sigma_h)^{-1}$ *for nonlattice distributions and*

$b_n(a) = \frac{d}{\sigma_h}\frac{\exp(-hd\Theta_n(a))}{(1-\exp(-hd))}$ *for lattice distributions with* span $d$.

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*Here $h = h(a)$ denotes the unique solution of the equation $\varphi'(t)/\varphi(t) = a$, $\varrho(a) := \inf_t \varphi(t)\exp(-ta) = \varphi(h)\exp(-ha)$ (Chernoff-function),*

$$\sigma_h^2 := \frac{d^2}{dt^2} \log \varphi(t)\Big|_{t=h}$$

*and*

$$\Theta_n(a) := \frac{x_0 - a}{d} n - \left[\frac{x_0 - a}{d} n\right] \quad \textit{if} \quad P(X_1 = x_0) > 0.$$

The goal of the present paper is to prove a multivariate analogue of Theorem A. As on the real line, we have to distinguish between the lattice and nonlattice case. A random vector $X$ is called a lattice random vector if there exists a lattice $L = A\mathbf{Z}^k$ with a nonsingular $k \times k$ matrix $A$, so that $P(X \in x_0 + L) = 1$ for some $x_0 \in \mathbf{R}^k$. $X$ is called strongly nonlattice if the modulus of its characteristic function equals one only at the origin. We will treat only these two cases, noting that for $k \geqq 2$ there exist nonlattice random vectors which are not strongly nonlattice.

Before stating our main result in Section 3 we collect some (well-known) facts about moment generating functions, conjugate distributions and the Chernoff-function. In Section 4 a refinement in the central limit theorem is discussed, which is used in the proof of the main result in Section 5. In Section 6 we apply our result to generalize Bártfai's (1972) conditional limit theorem.

## 2. Conjugate distributions and some related functions

Let $X$ be a $k$-dimensional random vector on a probability space $(\Omega, \mathfrak{A}, P)$ with moment generating function

$$\varphi(t) = E\exp(\langle t, X\rangle), \qquad t \in \mathbf{R}^k.$$

Define $D_\varphi := \{t \in \mathbf{R}^k : \varphi(t) < \infty\}$.

The family $\{X(t)\}_{t \in D_\varphi^0}$ of random vectors on $(\Omega, \mathfrak{A}, P)$ is said to be conjugate to $X$ if

$$(2.1) \qquad P(X(t) \in B) = \int_{\{X \in B\}} \frac{\exp(\langle t, X\rangle)}{\varphi(t)}\, dP \quad \text{for all} \quad B \in \mathfrak{B}^k.$$

$P_{X(t)}$ is called the conjugate distribution.

For fixed $t \in D_\varphi^0$ we have:

$$(2.2) \qquad \varphi_t(s) := E\exp(\langle s, X(t)\rangle) = \varphi(s + t)/\varphi(t) \quad \text{for} \quad s \in D_\varphi^0 - t.$$

$$E\,X(t) = \operatorname{grad}\varphi(t)/\varphi(t) = (\operatorname{grad}\log\varphi)(t) \tag{2.3}$$

$$\operatorname{Cov} X(t) = (\operatorname{Hess}\log\varphi)(t)$$

(Here Hess $f$ denotes the Hessian matrix of second partial derivatives of the function $f: \mathbf{R}^k \to \mathbf{R}$).

$$P(X \in B) = P_X(B) = \varphi(t) \int_B \exp(-\langle t, x\rangle) P_{X(t)}(dx), \quad B \in \mathfrak{B}^k. \tag{2.4}$$

The Chernoff-function $\varrho$ of the random vector $X$ is defined by

$$\varrho(a) := \inf_{t \in D_\varphi^0} \varphi(t) \exp(-\langle t, a\rangle). \tag{2.5}$$

If we define $A := \{(\operatorname{grad}\log\varphi)(t) : t \in D_\varphi^0\}$ and $X$ is nondegenerate we have for $a \in A$ the following assertions:

(2.6) The equation $\operatorname{grad}\varphi(t) = a\varphi(t)$ has a unique solution $t = t(a)$ in $A$. All partial derivatives of the function $t = t(a) : A \to D_\varphi^0$ exist.

$$\varrho(a) = \varphi(t(a)) \exp(-\langle t(a), a\rangle). \tag{2.7}$$

$$\operatorname{grad}\varrho(a) = -t(a)\varrho(a). \tag{2.8}$$

$$-\log\varrho(a) \quad \text{is strictly convex on} \quad A. \tag{2.9}$$

$$0 < \varrho(a) < 1 \quad \text{for all} \quad a \in A \quad \text{with} \quad t(a) \neq 0. \tag{2.10}$$

## 3. Main result

In the sequel we write for $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k) \in \mathbf{R}^k : a \leqq b$ $(a < b)$ iff $a_i \leqq b_i$ $(a_i < b_i)$ for all $i = 1, \ldots, k$. $\mathbf{R}_+^k := \{x \in \mathbf{R}^k : x > 0\}$.

The following multivariate extension of Petrov's Theorem A will be proved for strongly nonlattice as well as for lattice random vectors.

Note that, if $X$ is a nondegenerate lattice random vector, there exists a unique minimal lattice $L$ for $X$ in the sense (see e.g. Bhattacharya and Ranga Rao (1976), Lemma 21.4 and 21.5)

(i) $P(X \in x + L) = 1$ for all $x \in \mathbf{R}^k$ with $P(X = x) > 0$;

(ii) If $M$ is any closed subgroup of $\mathbf{R}^k$ such that $P(X \in y_0 + M) = 1$ for some $y_0 \in \mathbf{R}^k$, then $L \subset M$;

and

$$\det L := |\det A| \quad \text{if} \quad L = A\mathbf{Z}^k.$$

THEOREM 1. *Let $\{X_n : n \geqq 1\}$ be a sequence of iid. k-dimensional strongly nonlattice respectively nondegenerate lattice random vectors on some probability space $(\Omega, \mathfrak{A}, P)$ and let $D$ be an open, convex subset of $\mathbf{R}^k$ with*

(i) $$\varphi(t) := E\exp(\langle t, X_1\rangle) < \infty \quad \text{for} \quad t \in D$$

(ii) $$D_+ := D \cap \{x \in \mathbf{R}^k : x \geqq 0\} \neq \emptyset.$$

*Further let*

(iii) $$A := \{\operatorname{grad}\varphi(t)/\varphi(t) : t \in D_+\}^0$$
*and $I \subset A$ be a compact set.*

*Put $S_n = X_1 + \ldots + X_n$, $n = 1, 2, \ldots$. Then uniformly for $a \in I$ respectively for $a \in I \cap (x_0 + L/n)$ if $X_1$ is a lattice random vector with minimal lattice $L$ and $P(X_1 = x_0) > 0$:*

$$P(S_n \geqq na) = \frac{\varrho^n(a)}{(2\pi n)^{k/2}} \frac{1}{(\det V_h)^{1/2}} b(a)(1 + o(1)) \quad (n \to \infty), \tag{3.1}$$

*with*

$b(a) = (h_1 \cdot \ldots \cdot h_k)^{-1}$ *if $X_1$ is strongly nonlattice, and*

$b(a) = (\det L) \sum_{\substack{\alpha \in L \\ \alpha > 0}} \exp(-\langle h, \alpha\rangle)$ *if $X_1$ has a lattice distribution with minimal lattice $L$.*

*Here $h = h(a) = (h_1, \ldots, h_k)$ denotes the unique solution of $a = \operatorname{grad}\varphi(h)/\varphi(h)$,*

$$\varrho(a) = \inf_{t \in D} \varphi(t)\exp(-\langle t, a\rangle) = \varphi(h)\exp(-\langle h, a\rangle)$$

*and $V_h$ is the covariance matrix of a random vector $X_1(h)$ conjugate to $X_1$.*

REMARK 1. a) By definition of $A$ the solution $h = h(a)$ is strictly positive for $a \in I$, $h(a)$ is continuous in $a$ and $0 < \varrho(a) < 1$ for all $a \in I$. Condition (ii) is always satisfied if $0 \in D$.

b) If $X_1$ is strongly nonlattice the distributions of $X_1$ and $X_1(h)$ are nondegenerate. Hence $V_h$ is positive definite with $\det V_h > 0$.

## 4. Expansions in the central limit theorem

NOTATION. Let $f : \mathbf{R}^k \to \mathbf{R}$ be a bounded, measurable function. Then we set with $B(x : \varepsilon) := \{y \in \mathbf{R}^k : \|x - y\| < \varepsilon\}$, $x \in \mathbf{R}^k, \varepsilon > 0$, $\|.\| := \langle .,. \rangle^{1/2}$:

$$\omega_f(x : \varepsilon) := \sup\{|f(z) - f(y)| : z, y \in B(x : \varepsilon)\},$$

$$\overline{\omega}_f(\varepsilon : \mu) := \int \omega_f(x : \varepsilon)\mu(dx), \quad \mu \text{ a finite measure on } \mathbf{R}^k,$$

$$\omega_f^*(\varepsilon : \mu) := \sup_{y \in \mathbf{R}^k} \overline{\omega}_{f_y}(\varepsilon : \mu), \quad f_y(x) := f(x + y),$$

$$M_f(x:\varepsilon) := \sup\{f(y) \colon y \in B(x:\varepsilon)\},$$
$$m_f(x:\varepsilon) := \inf\{f(y) \colon y \in B(x:\varepsilon)\}.$$

The proof of Theorem 1 is mainly based upon the following lemmas (cf. also [4], Theorem 2.1, 2.2 and 2.3):

LEMMA 1a. *Let $\{X_n(h) \colon n \geqq 1\}_{h\in H}$ be a family of sequences of iid. $k$-dimensional random vectors on a probability space $(\Omega, \mathfrak{A}, P)$, satisfying the following conditions:*

(i) $E\,X_1(h) = 0$, $\operatorname{Cov} X_1(h) = V_h$, $E\|X_1(h)\|^3 \leqq p_3$.

(ii) *There exist constants $\gamma, \Gamma \in \mathbf{R}_+$, independent of $h$, so that the eigenvalues $\gamma_j(h)$ of $V_h$ satisfy*

$$0 < \gamma \geqq \gamma_j(h) \leqq \Gamma \qquad (j = 1, \dots, k).$$

(iii) *For $b_1, b_2 \in \mathbf{R}_+$ there exists a $\Theta(b_1, b_2) < 1$, so that*

$$\sup_{b_1 \leqq \|t\| \leqq b_2} |E\,\exp(i\langle t, X_1(h)\rangle)| \leqq \Theta(b_1, b_2) \quad \text{for all } h \in H. \tag{4.1}$$

*Then, for all bounded measurable functions $f \colon \mathbf{R}^k \to \mathbf{R}$,*

$$\left|\int f\, d(Q_{h,n} - \Phi_{0,V_h})\right| \leqq c_1\, n^{-1/2} M_f(\delta_n) + 2\omega_f^*(2\delta_n : \Phi_{0,V_h}) \tag{4.2}$$

*where $Q_{h,n}$ is the distribution of $n^{-1/2}(X_1(h) + \ldots + X_n(h))$, $\Phi_{0,V_h}$ is the $k$-dimensional normal distribution with mean 0 and covariance matrix $V_h$,*

*$c_1$ is a positive constant independent of $f$ and $h$,*

*$\delta_n = o(n^{-1/2})$ $(n \to \infty)$ independent of $f$ and $h \in H$,*

*and*

$$M_f(\delta_n) := \max\left\{\int |M_f(x:\delta_n)|\lambda^k(dx),\ \int |m_f(x:\delta_n)|\lambda^k(dx)\right\}$$

*provided these integrals exist ($\lambda^k$ denotes the Lebesgue–Borel measure on $\mathbf{R}^k$).*

In the lattice case we make use of

LEMMA 1b. *Under the assumptions* (i), (ii) *of Lemma* 1a, *however with $E\,X_1(h) = m_h$, let the following conditions be satisfied for $h \in H$:*

(iii') *$X_1(h)$ is a nondegenerate lattice random vector with minimal lattice $L$, fundamental domain $F^* = \{x \colon |\langle \zeta_j, x\rangle| < \pi$ for all $j = 1, \dots, k\}$ and $P(X_1(h) \in x_0 + L) = 1$ ($x_0 \in \mathbf{R}^k$, $(\zeta_1, \dots, \zeta_k)$ a basis of $L$).*

(iv) *For* $d \in \mathbf{R}_+$ *with* $E_d := \{x : \|x\| \leqq d\} \subset F^*$ *there exists a* $\delta_d < 1$*, such that*

$$\text{(4.3)} \qquad \sup\{|E\exp(i\langle t, X_1(h)\rangle)| : t \in F^* \setminus E_d\} =: \delta_d(h) \leqq \delta_d < 1.$$

*Then, for* $\xi_n \in nx_0 + L$ *and* $h \in H$*:*

$$\text{(4.4)} \qquad \sup_{\alpha \in L} |p_{h,n}(y_{\alpha,n}) - q_{h,n}(y_{\alpha,n})| \leqq \delta_2(n) = O(n^{-(k+1)/2}) \quad (n \to \infty)$$

*with* $\delta_2(n)$ *independent of* $\xi_n$ *and* $h$*,*

$$y_{\alpha,n} := n^{-1/2}(\alpha + \xi_n - nm_h)$$

$$p_{h,n}(y_{\alpha,n}) := P(X_1(h) + \ldots + X_n(h) = \alpha + \xi_n) = P\Big(n^{-1/2}\sum_{j=1}^{n}(X_j(h) - a) = y_{\alpha,n}\Big)$$

$$q_{h,n}(y_{\alpha,n}) := (\det L)(2\pi n)^{-k/2}(\det V_h)^{-1/2}\exp\left(-\langle y_{\alpha,n}, V_h^{-1} y_{\alpha,n}\rangle/2\right).$$

REMARK 2. a) For the properties of the fundamental domain $F^*$ we refer to Bhattacharya and Ranga Rao [2], p. 229. Note that

$$\overline{F^*} \cap \{t \in \mathbf{R}^k : |E\exp(i\langle t, X_1(h)\rangle)| = 1\} = \{0\},$$

so that $\delta_d(h) < 1$ for all $h \in H$.

b) Lemma 1b yields

$$\Big|\sum_{\alpha \in L} f(\alpha)(p_{h,n}(y_{\alpha,n}) - q_{h,n}(y_{\alpha,n}))\Big| = O(n^{-(k+1)/2}) \quad (n \to \infty)$$

uniformly in $n \in H$ for $f : \mathbf{R}^k \to \mathbf{R}$ with $\sum_{\alpha \in L} |f(\alpha)| < \infty$.

For the proof of Lemma 1 we need the following auxiliary facts (see e.g. Bhattacharya and Ranga Rao [2]):

LEMMA 2. *Let* $\mu, \nu$ *be finite measures on* $\mathbf{R}^k$*. Let* $\varepsilon$ *be a positive number and* $K_\varepsilon$ *be a probability measure on* $\mathbf{R}^k$ *satisfying*

$$\alpha \equiv K_\varepsilon(B(0 : \varepsilon)) > 1/2.$$

*Then for any bounded measurable function* $f : \mathbf{R}^k \to \mathbf{R}$,

$$\text{(4.5)} \qquad \Big|\int f\, d(\mu - \nu)\Big| \leqq (2\alpha - 1)^{-1}(\gamma^*(f : \varepsilon) + \omega_f^*(2\varepsilon : \nu))$$

*where*

$$\gamma_f^*(f : \varepsilon) := \sup_{y \in \mathbf{R}^k} \max\Big\{\int M_{f_y}(. : \varepsilon) d(\mu - \nu) * K_\varepsilon, -\int m_{f_y}(. : \varepsilon) d(\mu - \nu) * K_\varepsilon\Big\}.$$

PROOF. See Lemma 11.4 in [2].

The next lemma provides an expansion of the characteristic function of a normalized sum of iid. random vectors:

LEMMA 3. *Let $X$ be a $k$-dimensional random vector with mean zero, positive definite covariance matrix $V$, finite third absolute moment $p_3 = E\|X\|^3$ and characteristic function $\psi$. Then,*

$$|\psi^n(n^{-1/2}t) - \exp(-\langle t, Vt\rangle/2)| \leqq n^{-1/2} p_3 \left(\frac{\Gamma}{2\gamma}\right)^{3/2} \|t\|^3 \exp\left(-\frac{3\gamma}{8}\|t\|^2\right) \tag{4.6}$$

*for all $t \in \mathbf{R}^k$ satisfying*

$$\|t\| \leqq \frac{\gamma^{3/2}}{2p_3\Gamma^{1/2}} n^{1/2}. \tag{4.7}$$

*Here $\Gamma, \gamma$ denote the largest, smallest eigenvalue of $V$.*

PROOF. See Theorem 8.4 in [2].

PROOF OF LEMMA 1a. Let $K_\varepsilon$ be a probability measure on $\mathbf{R}^k$ with

$$K_\varepsilon(B(0:\varepsilon)) > 3/4 \quad \text{and} \quad \hat{K}_\varepsilon(t) = 0 \text{ for } \|t\| > c_2/\varepsilon \tag{4.8}$$

where $\hat{K}_\varepsilon$ denotes the characteristic function of $K_\varepsilon$. (This is possible due to Theorem 10.2 in [2].)

Let $g_{h,n,\varepsilon}$ denote the density of the finite signed measure $(Q_{h,n} - \Phi_{0,V_h})* *K_\varepsilon$ with respect to $\lambda^k$. By Lemma 2, for $\varepsilon > 0$,

$$\left|\int f\, d(Q_{h,n} - \Phi_{0,V_h})\right| \leqq 2\,\|g_{h,n,\varepsilon}\|_\infty\, M_f(\varepsilon) + 2\omega_f^*(2\varepsilon : \Phi_{0,V_h}) \tag{4.9}$$

with $\|g_{h,n,\varepsilon}\|_\infty = \sup_{y\in\mathbf{R}^k} |g_{h,n,\varepsilon}(y)|$.

We now set $\varepsilon = \eta n^{-1/2}$ and show that for all $\eta > 0$ there exists an $n(\eta) \in \mathbf{N}$ so that for $n \geqq n(\eta)$

$$\|g_{h,n,\varepsilon}\|_\infty \leqq c_1 n^{-1/2}. \tag{4.10}$$

This will complete the proof.

Using Fourier inversion we obtain

$$\begin{aligned} \|g_{h,n,\varepsilon}\|_\infty &\leqq (2\pi)^{-k} \int_{\mathbf{R}^k} \left|\hat{Q}_{h,n}(t) - \exp(-\langle t, V_h t\rangle/2)\right| |\hat{K}_\varepsilon(t)| \lambda^k(dt) \leqq \\ &\leqq (2\pi)^{-k} \int_{\{\|t\| \leqq c_2\eta^{-1}n^{+1/2}\}} \left|\psi_h^n(n^{-1/2}t) - \exp(-\langle t, V_h t\rangle/2)\right| \lambda^k(dt) \end{aligned} \tag{4.11}$$

where $\psi_h(t) := E\exp(i\langle t, X_1(h)\rangle)$.

Lemma 3 together with conditions (i) and (ii) of Lemma 1 yields for all $h \in H$

$$\int\limits_{\{\|t\| \leqq c_4 n^{+1/2}\}} \left| \psi_h^n(n^{-1/2}t) - \exp(-\langle t, V_h t\rangle/2)\right| \lambda^k(dt) \leqq c_5 n^{-1/2}. \tag{4.12}$$

Now by condition (iii) of Lemma 1a for small $\eta > 0$

$$\sup\left\{|\psi_h(n^{-1/2}t)| : c_4 n^{1/2} < \|t\| \leqq c_2\eta^{-1}n^{1/2}\right\} \leqq \Theta(c_4, c_2\eta^{-1}) < 1,$$

so that

$$\begin{aligned} \int\limits_{\{c_4 n^{1/2} < \|t\| \leqq c_2\eta^{-1}n^{1/2}\}} |\psi_h^n(n^{-1/2}t)|\lambda^k(dt) &\leqq c_6(\eta) n^{k/2}\Theta^n(c_4, c_2\eta^{-1}) \leqq \\ &\leqq n^{-1/2} \quad \text{for} \quad n \geqq n(\eta) \quad \text{(say)}. \end{aligned} \tag{4.13}$$

Thus (4.10) follows from

$$\begin{aligned} \int\limits_{\{\|t\| \geqq c_4 n^{1/2}\}} \exp(-\langle t, V_h t\rangle/2)\lambda^k(dt) &\leqq \\ \leqq \int\limits_{\{\|t\| \geqq c_4 n^{1/2}\}} \exp\left(-\frac{\gamma}{2}\|t\|^2\right)\lambda^k(dt) &\leqq c_7 n^{-1/2}. \end{aligned} \tag{4.14}$$

PROOF OF LEMMA 1b. Let $\psi_h$ denote the characteristic function of $X_1(h)$ and $\psi_{h,n}$ that of $n^{-1/2}(X_1(h) + \ldots + X_n(h) - nm_h)$. The inversion formula in the lattice case (see e.g. [2], p. 230) yields:

$$\begin{aligned} p_{h,n}(y_{\alpha,n}) &= (\det L)(2\pi)^{-k} \int\limits_{F^*} \psi_h^n(t)\exp(-i\langle t, \alpha + \xi_n\rangle)\lambda^k(dt) = \\ &= (\det L)(2\pi)^{-k} n^{-k/2} \int\limits_{n^{1/2}F^*} \psi_{h,n}(t)\exp(-i\langle t, y_{\alpha,n}\rangle)\lambda^k(dt). \end{aligned}$$

Also by Fourier inversion

$$q_{h,n}(y_{\alpha,n}) = (\det L)(2\pi)^{-k}n^{-k/2}\int\limits_{\mathbf{R}^k} \exp\left(-\frac{1}{2}\langle t, V_h t\rangle - i\langle t, y_{\alpha,n}\rangle\right)\lambda^k(dt).$$

Now choose $d > 0$ such that $E_d \subset F^*$ and $d \leqq \frac{\gamma^{3/2}}{2p_3\Gamma^{1/2}}$ (see Lemma 3). Hence

$$|p_{h,n}(y_{\alpha,n}) - q_{h,n}(y_{\alpha,n})| \leqq (\det L)(2\pi)^{-k}n^{-k/2}(I_1 + I_2 + I_3) \tag{4.15}$$

with

$$I_1 := \int\limits_{n^{1/2}E_d} \left|\psi_{h,n}(t) - \exp\Big(-\frac{1}{2}\langle t, V_h t\rangle\Big)\right| \lambda^k(dt) \leqq c_8 n^{-1/2},$$

$c_8$ independent of $h \in H$ and $\xi_n$ by Lemma 3 together with conditions (i) and (ii) of Lemma 1.

$$\begin{aligned} I_2 :=& \int\limits_{n^{1/2}(F^\bullet \setminus E_d)} |\psi_{h,n}(t)|\lambda^k(dt) \\ \leqq& n^{k/2}\delta_d^n \int\limits_{F^\bullet} 1\lambda^k(dt) \quad \text{(by condition (iv) of Lemma 1b)} \\ =& n^{k/2}\delta_d^n (2\pi)^k (\det L)^{-1} \quad \text{(see e.g. [2], p. 230)} \\ =& o(n^{-1/2}) \quad (n \to \infty) \quad \text{independent of } h \text{ and } \xi_n. \end{aligned}$$

$$\begin{aligned} I_3 :=& \int\limits_{\mathbb{R}^k \setminus n^{1/2}E_d} \exp\Big(-\frac{1}{2}\langle t, V_h t\rangle\Big)\lambda^k(dt) \\ \leqq& \int\limits_{\mathbb{R}^k \setminus n^{1/2}E_d} \exp\Big(-\frac{\gamma}{2}\|t\|^2\Big)\lambda^k(dt) = \\ =& o(n^{-1/2}) \quad (n \to \infty) \quad \text{independent of } h \text{ and } \xi_n. \end{aligned}$$

This completes the proof of Lemma 1.

## 5. Proof of Theorem 1

Let $a \in I$ respectively $a \in I \cap (x_0 + L/n)$ in the lattice case and $h = h(a)$ be the unique solution of the equation $\operatorname{grad}\varphi(h) = a\varphi(h)$. By the definition of $A$, $h \in \mathbb{R}_+^k$.

Further for each $n \in \mathbb{N}$ let $X_1(h), \ldots, X_n(h)$ denote iid. random vectors conjugate to $X_1, \ldots, X_n$, so that in distribution

$$S_n(h) \overset{D}{=} X_1(h) + \ldots + X_n(h) \tag{5.1}$$

where $S_n(h)$ is conjugate to $S_n = X_1 + \ldots + X_n$. For $B \in \mathfrak{B}^k$ we define with $m_h : EX_1(h) = \operatorname{grad}\varphi(h)/\varphi(h) = a$

$$P_{h,n}(B) := P(S_n(h) \in B) = P(X_1(h) + \ldots + X_n(h) \in B) \tag{5.2}$$

$$Q_{h,n}(B) = P(n^{-1/2}(X_1(h) + \ldots + X_n(h) - nm_h) \in B) = P_{h,n}(n^{1/2}B + nm_h).$$

Then, by the inversion formula (2.4) for conjugate distributions,

$$
\begin{aligned}
P(S_n \geqq na) &= \int\limits_{\{S_n \geqq na\}} dP = \\
&= \varphi^n(h) \int\limits_{\{x \geqq na\}} \exp(-\langle h, x\rangle) P_{h,n}(dx) = \varrho^n(a) I_n(h)
\end{aligned}
\tag{5.3}
$$

with

(5.4)

$$
I_n(h) = \begin{cases} \int\limits_{\{x \geqq 0\}} \exp(-n^{1/2}\langle h, x\rangle) Q_{h,n}(dx), & \text{if } X_1 \text{ is strongly nonlattice,} \\ \sum\limits_{\substack{\alpha \in L \\ \alpha \geqq 0}} \exp(-\langle h, \alpha\rangle) P(X_1(h) + \ldots + X_n(h) = na + \alpha), & \text{if } X_1 \text{ has a lattice distribution with } P(X_1 \in x_0 + L) = 1, \end{cases}
$$

and $\varrho(a) = \inf\limits_{t \in D} \varphi(t) \exp(-\langle t, a\rangle) = \varphi(h) \exp(-\langle h, a\rangle)$.

Note that in the lattice case $X_1(h)$ has the same minimal lattice as $X_1$. The above representation of $I_n(h)$ in this case is possible only when $na \in \in nx_0 + L$ and hence $P(S_n - na \in L) = 1$.

The following lemma shows that we can use Lemma 1 in the preceding section to get a precise estimate of $I_n(h(a))$ uniformly in $a \in I$ respectively in $a \in I \cap (x_0 + L/n)$.

LEMMA 4. *Let $X$ be a nondegenerate $k$-dimensional random vector on $(\Omega, \mathfrak{A}, P)$ with moment generating function $\varphi(t) = E \exp(\langle t, X\rangle) < \infty$ for $t \in D \subset \mathbf{R}^k$, $D$ open and convex. Further let $H \subset D$, $H$ compact and for $h \in H$ let $X(h)$ be a random vector conjugate to $X$ with characteristic function $\psi_h$. Then we have:*

(i) *If $X$ has a strongly nonlattice distribution, for every $b_1, b_2 \in \mathbf{R}_+$ there exists some $\Theta(b_1, b_2) < 1$, independent of $h \in H$, so that*

$$
\sup_{b_1 \leqq \|t\| \leqq b_2} |\psi_h(t)| \leqq \Theta(b_1, b_2) \quad \text{for all} \quad h \in H. \tag{5.5}
$$

(ii) *If $X$ has a lattice distribution with minimal lattice $L$ and fundamental domain $F^*$, for $d \in \mathbf{R}_+$ and $E_d = \{x : \|x\| \leqq d\} \subset F^*$ there exists some $\Theta(d) < 1$, independent of $h$ such that*

$$
\sup\{|\psi_h(t)| : t \in F^* \setminus E_d\} \leqq \Theta(d) \quad \text{for all } h \in H. \tag{5.6}
$$

(iii) *There exist positive constants* $p_s, \gamma, \Gamma \in \mathbf{R}_+$, *so that* $E\|X(h)\|^s \leqq p_s$ *for all* $h \in H$ *and* $s > 0$, *and the eigenvalues* $\gamma_j(h)$ *of the covariance matrix* $V_h$ *of* $X(h)$ *satisfy* $0 < \gamma \leqq \gamma_j(h) \leqq \Gamma$ *for* $j = 1, \dots, k$ *and all* $h \in H$.

PROOF. (i) Since $P_X$ is absolutely continuous with respect to $P_{X(h)}$, $|\psi_h(t_0)| = 1$ implies $|\varphi(it_0)| = 1$. Hence $X(h)$ is also strongly nonlattice and for $b_1, b_2 \in \mathbf{R}_+$ there exists a $\Theta(h, b_1, b_2) < 1$ so that

$$\sup_{b_1 \leqq \|t\| \leqq b_2} |\psi_h(t)| \leqq \Theta(h, b_1, b_2). \tag{5.7}$$

Now note that by the dominated convergence theorem

$$\begin{aligned}\lim_{\|\delta\| \to 0} \psi_{h+\delta}(t) &= \lim_{\|\delta\| \to 0} (\varphi(h+\delta))^{-1} \int \exp(i\langle t, x\rangle + \langle h+\delta, x\rangle) P_X(dx) = \\ &= \psi_h(t) \quad \text{uniformly in} \quad t \in \mathbf{R}^k.\end{aligned} \tag{5.8}$$

Consequently for all $h_0 \in H$ there exists a neighbourhood $U(h_0)$ of $h_0$ and a $\bar{\Theta}(h_0, b_1, b_2) < 1$ such that

$$\sup_{b_1 \leqq \|t\| \leqq b_2} |\psi_h(t)| \leqq \bar{\Theta}(h_0, b_1, b_2) \quad \text{for all} \quad h \in U(h_0) \tag{5.9}$$

and (5.5) follows from the assumption that $H$ is compact.

(ii) Since $P_X \approx P_{X(h)}$, $X$ and $X(h)$ have lattice distributions with same minimal lattice $L$ and therefore with same fundamental domain $F^*$. Let $d \in \mathbf{R}_+$ be such that $E_d = \{x : \|x\| \leqq d\} \subset F^*$. By definition of $F^*$ we have for $h \in H$: $\overline{F^*} \cap \{t \in \mathbf{R}^k : |\psi_h(t)| = 1\} = \{0\}$. Thus there exists a $\Theta(h, d) < 1$, so that

$$\sup\{|\psi_h(t)| : t \in F^* \setminus E_d\} \leqq \Theta(h, d). \tag{5.10}$$

Hence assertion (5.6) follows from (5.8) as in (i).

(iii) Let $\varphi, \varphi_h$ be the moment generating functions of $X, X(h)$. Then $\varphi_h(t) = \varphi(t+h)/\varphi(h)$ for $t \in D_h := D - h$. Since $h \in D^0 = D$ it follows $0 \in D_h^0 = D_h$. Hence all moments of $X(h)$ exist and $E\|X(h)\|^s \leqq p_s$ follows from

$$\begin{aligned}&\lim_{\|\delta\| \to 0} E\|X(h+\delta)\|^s = \\ &= \lim_{\|\delta\| \to 0} (\varphi(h+\delta))^{-1} \int \|x\|^s \exp(\langle h+\delta, x\rangle) P_X(dx) = E\|X(h)\|^s\end{aligned} \tag{5.11}$$

and the compactness of $H$.

Moreover $V_h$ is positive definite since $X(h)$ is nondegenerate. Hence the eigenvalues $\gamma_j(h)$ of $V_h$ are real and positive. Since $V_h = \operatorname{Cov} X(h) =$

$= (\text{Hess} \log \varphi)(h)$ the eigenvalues of $V_h$ are continuous functions of $h \in H$ and the assertion follows again from the assumption that $H$ is compact.

We now come back to the

*Estimation of $I_n(h)$ in the strongly nonlattice case*

Let

$$H := \{h(a) : a \in I\}. \tag{5.12}$$

Since $h(a) : A \to D_+$ is a continuous function and $I$ is a compact subset of $A$ it follows that $H$ is compact. Since $E\|X_1(h) - EX_1(h)\|^s \leqq 2^{s+1} E\|X_1(h)\|^s$ Lemma 4 allows for using Lemma 1a to estimate:

$$I_n(h) = \int\limits_{\{x \geqq 0\}} \exp\left(-n^{1/2}\langle h, x\rangle\right) \Phi_{0,V_h}(dx) + R_n(h) \tag{5.13}$$

with

$$|R_n(h)| \leqq c_1 n^{-1/2} M_{f_n}(\delta_n) + 2\omega^*_{f_n}(2\delta_n : \Phi_{0,V_h}) \tag{5.14}$$

where $\delta_n = o(n^{-1/2})$ $(n \to \infty)$ independent of $f_n$ and $h \in H$,

$$f_n(x) = \exp(-n^{+1/2}\langle h, x\rangle) I_{\{y \geqq 0\}}(x), \quad x \in \mathbf{R}^k$$

$V_h = \operatorname{Cov} X_1(h)$, $c_1 \in \mathbf{R}_+$ independent of $f_n$ and $h$. Now

$$\int\limits_{\{x \geqq 0\}} \exp(-n^{+1/2}\langle h, x\rangle)\Phi_{0,V_h}(dx) = (2\pi n)^{-k/2} (\det V_h)^{-1} \bar{I}_n(h)$$

with

$$\bar{I}_n(h) = \int\limits_{\{x \geqq 0\}} \exp\left(-\langle h, x\rangle - \frac{1}{2n}\langle x, V_h^{-1} x\rangle\right) \lambda^k(dx). \tag{5.15}$$

Thus the first part of Theorem 1 is proved if we can show that

$$\bar{I}_n(h) = (h_1 \cdot \ldots \cdot h_k)^{-1}(1 + o(1)) \quad (n \to \infty) \text{ uniformly in } h \in H \tag{5.16}$$

and

$$|R_n(h)| = o(n^{-k/2}) \quad (n \to \infty) \quad \text{uniformly in} \quad h \in H. \tag{5.17}$$

PROOF OF (5.16). Since $V_h$ is positive definite we have

$$\bar{I}_n(h) \leqq \int\limits_{\{x \geqq 0\}} \exp(-\langle h, x\rangle)\lambda^k(dx) = (h_1 \cdot \ldots \cdot h_k)^{-1}. \tag{5.18}$$

Now $\|V_h^{-1}\| \leqq \gamma^{-1}$ for all $h \in H$ (by Lemma 4) and hence $0 \leqq \langle x, V_h^{-1}x\rangle \leqq$ $\leqq \gamma^{-1}\|x\|^2$.

Since $H$ is a compact subset of $\mathbf{R}_+^k$ there exists a $h_* \in \mathbf{R}$, so that with $\underline{h} = (h_*, \dots, h_*) \in \mathbf{R}_+^k$: $0 < \underline{h} \leqq h$ for all $h \in H$. Thus, by the dominated convergence theorem

$$\left|(h_1 \cdot \ldots \cdot h_k)^{-1} - \tilde{I}_n(h)\right| =$$

$$= \int\limits_{\{x \geqq 0\}} \exp(-\langle h, x\rangle)\left(1 - \exp\left(-\frac{1}{2n}\langle x, V_h^{-1}x\rangle\right)\right)\lambda^k(dx) \leqq$$

$$\leqq \int\limits_{\{x \geqq 0\}} \exp(-\langle \underline{h}, x\rangle)\left(1 - \exp\left(-\frac{1}{2\gamma n}\|x\|^2\right)\right)\lambda^k(dx) =$$

$$= o(1) \quad (n \to \infty) \quad \text{uniformly in} \quad h \in H.$$

This proves (5.16).

PROOF OF (5.17). Recalling the definitions of $M_{f_n}$ and $\omega_{f_n}^*$ in Section 4 we see that

$$M_{f_n}(\delta_n) \leqq 2\int M_{f_n}(x : \delta_n)\lambda^k(dx), \tag{5.19}$$

since $f_n \geqq 0$, and

$$\omega_{f_n}^*(2\delta_n : \Phi_{0,V_h}) \leqq (2\pi\gamma)^{-k/2}\int \omega_{f_n}(x : 2\delta_n)\lambda^k(dx), \tag{5.20}$$

since $\det V_h \geqq \gamma^k$ by Lemma 4.

Let $\overline{\delta}_n = (\delta_n, \dots, \delta_n) \in \mathbf{R}_+^k$ and $\overline{h} = (h^*, \dots, h^*)$, $\underline{h} = (h_*, \dots, h_*) \in \mathbf{R}_+^k$ such that $0 < \underline{h} \leqq h \leqq \overline{h}$ for all $h \in H$. We write

$$\mathbf{R}^k = M_- \cup \bigcup_{l=0}^{k} M_l \tag{5.21}$$

with

$$M_l := \left\{x = (x_1, \dots, x_k) : x \geqq -\overline{\delta}_n,\ \#\{i \colon x_i < \delta_n\} = l\right\},\ l = 0, 1, \dots, k$$

$$M_- := \mathbf{R}^k \setminus \{x : x \geqq -\overline{\delta}_n\}$$

where $\#M$ denotes the number of elements of the set $M$. Now

$$\omega_{f_n}(x : \delta_n) = M_{f_n}(x : \delta_n) = 0 \quad \text{for} \quad x \in M_-, \tag{5.22}$$

$$\omega_{f_n}(x:\delta_n) \leqq M_{f_n}(x:\delta_n) \leqq 1 \quad \text{for } x \in M_k, \text{ since } 0 \leqq f_n \leqq 1, \tag{5.23}$$

$$M_{f_n}(x:\delta_n) \leqq \exp(-n^{1/2}\langle \underline{h}, x-\overline{\delta}_n\rangle) \quad \text{for } x \in M_0, \tag{5.24}$$

$$\begin{aligned} \omega_{f_n}(x:\delta_n) &\leqq \exp(-n^{1/2}\langle h, x-\overline{\delta}_n\rangle) - \exp(-n^{1/2}\langle h, x+\overline{\delta}_n\rangle) = \\ &= \exp(-n^{1/2}\langle h, x-\overline{\delta}_n\rangle)(1-\exp(-n^{1/2}\langle h, 2\overline{\delta}_n\rangle)) \leqq \\ &\leqq \exp(-n^{1/2}\langle \underline{h}, x-\overline{\delta}_n\rangle)4kn^{1/2}h^*\delta_n \end{aligned} \tag{5.25}$$

for $x \in M_0$, $n \geqq n_0$, which is possible since $\delta_n = o(n^{-1/2})$,

$$\omega_{f_n}(x:\delta_n) \leqq M_{f_n}(x:\delta_n) \leqq \exp(-n^{1/2}h_*((x_{l+1}-\delta_n)+\ldots+(x_k-\delta_n)) \tag{5.26}$$

for $x \in M_1$, $0 < l < k$ and, for example,

$$x_1, \ldots, x_l < \delta_n; \quad x_{l+1}, \ldots, x_k \geqq \delta_n.$$

Relations (5.21)–(5.26) yield

$$\begin{aligned} \int_{M_1} \omega_{f_n}(x:\delta_n)\lambda^k(dx) &\leqq \int_{M_1} M_{f_n}(x:\delta_n)\lambda^k(dx) \leqq \\ \leqq \delta_n^l (h_* n^{1/2})^{-(k-l)} \binom{k}{l} &= o(n^{-k/2}), \quad l = 1, \ldots, k \\ &= O(n^{-k/2}), \quad l = 0, \end{aligned} \tag{5.27}$$

$$\begin{aligned} \int_{M_0} \omega_{f_n}(x:\delta_n)\lambda^k(dx) &\leqq 4kh^* n^{1/2}\delta_n (h_* n^{1/2})^{-k} \\ &= o(n^{-k/2}), \quad \text{since } \delta_n = o(n^{-1/2}). \end{aligned} \tag{5.28}$$

(5.27) and (5.28) hold uniformly in $h \in H$. Thus assertion (5.17) follows with (5.19), (5.20) and (5.14).

This completes the proof of Theorem 1 in the strongly nonlattice case.

*Estimation of $I_n(h)$ in the strongly lattice case*

As before let $H = \{h(a) : a \in I\}$, i.e. $H$ compact. Now Lemma 4 again allows for using Lemma 1b to estimate $I_n(h)$ as follows: By Lemma 1b, with $\xi_n = na$ for all $a \in I \cap (x_0 + L/n) \subset I$ (note: $EX_1(h) = a$):

$$\begin{aligned} &P(X_1(h) + \ldots + X_n(h) = na + \alpha) = \\ &= (\det L)(2\pi n)^{-k/2}(\det V_h)^{-1/2} \exp\Big(-\frac{1}{2n}\langle \alpha, V_h^{-1}\alpha\rangle\Big) + R_n(h) \end{aligned} \tag{5.29}$$

where $|R_n(h)| \leqq r_n = o(n^{-k/2})$ $(n \to \infty)$, $r_n$ independent from $\alpha \in L$ and $h \in$ $\in H_n := \{h(a) : a \in I \cap (x_0 + L/n)\} \subset H$. Since $\left|\exp\left(-\frac{1}{2n}\langle \alpha, V_h^{-1}\alpha\rangle\right) - 1\right| \leqq$ $\leqq \frac{1}{n\gamma}\|\alpha\|^2$ for $n > n_\alpha$ (by Lemma 4) and $\sum_{\substack{\alpha \in L \\ \alpha \geqq 0}} \exp(-\langle h, \alpha\rangle)$ converges uniformly in $h \in H$ we get

(5.30)

$$\sum_{\substack{\alpha \in L \\ \alpha \geqq 0}} \exp(-\langle h, \alpha\rangle) \exp\left(-\frac{1}{2n}\langle \alpha, V_h^{-1}\alpha\rangle\right) = \left(\sum_{\substack{\alpha \in L \\ \alpha \geqq 0}} \exp(-\langle h, \alpha\rangle)\right)(1 + o(1))$$

uniformly in $h \in H$ if $n \to \infty$, and hence by (5.4), (5.29) for $h \in H_n \subset H$:

$$I_n(h) = (2\pi n)^{-k/2}(\det V_h)^{-1/2} b(a)(1 + \bar{R}_n(h(a))) \tag{5.31}$$

where $|\bar{R}_n(h)| \leqq \bar{r}_n = o(1)$ $(n \to \infty)$, $\bar{r}_n$ independent of $h \in H_n$ and thus independent of $a \in I \cap (x_0 + L/n)$, and $b(a) := (\det L) \sum_{\substack{\alpha \in L \\ \alpha \geqq 0}} \exp(-\langle h, \alpha\rangle)$.

This completes the proof of Theorem 1.

## 6. An application

Bártfai (1972) proved the following conditional limit theorem on the real line:

THEOREM B (Bártfai). *Let $\{X_n\}_{n \geqq 1}$ be a sequence of iid. real-valued random variables with moment generating function*

$$\varphi(t) = E \exp(tX_1) < \infty, \quad t \in [0, t_1).$$

*Then, for $x \in \mathbf{R}$ and*

$$a \in A := \{\varphi'(t)/\varphi(t) : t \in (0, t_1)\}$$

$$\lim_{n \to \infty} P(X_1 \leqq x \mid S_n \geqq na) = \frac{1}{\varphi(h)} \int_{(-\infty, x]} \exp(hz) P_{X_1}(dz) \tag{6.1}$$

*where $h = h(a)$ denotes the unique solution of the equation $\varphi'(h)/\varphi(h) = a$.*

Thus the limit distribution is equal to the distribution $P_{X(h)}$ of a random variable $X(h)$ conjugate to $X_1$. Note that $E X(h) = a > E X_1$ for $a \in A$.

The problem of determining the conditional limit above is interesting from the point of view of statistical mechanics and has many physical interpretations (see e.g. Vincze (1972)).

Recently Csiszár (1984) gave an elegant proof of a much more general result than Theorem B, considering random variables taking values in a convex topological vector space and making only use of the Sanov property (see Sanov (1957)) and generalized $I$-projections.

In the present section, as an application of Theorem 1, we prove a multivariate analogue of Theorem B in the case of strongly nonlattice or lattice random vectors and show that the convergence in (6.1) is uniform in $a \in I \subset \subset A$ for any compact $I$.

As before we write $a \leqq b$ iff $a_i \leqq b_i$ for all $i = 1, \ldots, k$.

THEOREM 2. *Under the assumptions of Theorem 1 one has, uniformly in $a \in I$ respectively in $a \in I \cap (x_0 + L/n) := I_n$ in the lattice case, for all $x \in \mathbf{R}^k$:*

$$\lim_{n\to\infty} P(X_1 \leqq x \mid S_n \geqq na) = \frac{1}{\varphi(h)} \int_{(-\infty,x]} \exp(\langle h, z\rangle) P_{X_1}(dz) \tag{6.2}$$

*where $h = h(a) \in \mathbf{R}^k_+$ denotes the unique solution of the equation* $\operatorname{grad}\varphi(h) = a\varphi(h)$.

PROOF OF THEOREM 2. The proof follows the lines of Bártfai (1972), using Theorem 1 instead of Petrov's large deviation theorem (Theorem A). By Theorem 1, for $a \in I$ (resp. $I_n$), $P(S_n \geqq na) > 0$ for $n \geqq n_0$. Hence, for $n \geqq n_0$ and $x \in \mathbf{R}^k$:

$$P(X_1 \leqq x \mid S_n \geqq na) = \frac{P(X_1 \leqq x, S_n \geqq na)}{P(S_n \geqq na)}. \tag{6.3}$$

By the definition of conditional probabilities we can write

$$\begin{aligned} P(X_1 \leqq x, S_n \geqq na) &= \int P(X_1 \leqq x, S_n \geqq na \mid X_1 = z) P_{X_1}(dz) = \\ &= \int_{(-\infty,x]} P(S_n \geqq na \mid X_1 = z) P_{X_1}(dz) = \\ &= \int_{(-\infty,x]} P(X_2 + \ldots + X_n \geqq na - z) P_{X_1}(dz) = \\ &= \int_{(-\infty,x]} P(S_{n-1} \geqq na - z) P_{X_1}(dz). \end{aligned}$$

Hence, for $n \geqq n_0$,

$$P(X_1 \leqq x \mid S_n \geqq na) = \int_{(-\infty,x]} \frac{P(S_{n-1} \geqq na - z)}{P(S_n \geqq na)} P_{X_1}(dz). \tag{6.4}$$

Let us first consider the strongly nonlattice case:

Dividing the numerator in (6.4) by $P(S_{n-1} \geqq na - y)$ ($y \in \mathbf{R}^k$ fixed) Theorem 1 yields (uniformly in $a \in I$):

$$\text{(6.5)} \quad \frac{P(S_{n-1} \geqq na - z)}{P(S_{n-1} \geqq na - y)} = \frac{\varrho^{n-1}\left(a - \frac{z-a}{n-1}\right)}{\varrho^{n-1}\left(a - \frac{y-a}{n-1}\right)} \frac{c\left(a - \frac{z-a}{n-1}\right)}{c\left(a - \frac{y-a}{n-1}\right)}(1 + o(1)) \quad (n \to \infty),$$

with $c(a) := (\det V_h)^{-1/2}(h_1 \cdot \ldots \cdot h_k)^{-1}$ and $h = h(a)$ defined by $\operatorname{grad} \varphi(h) = a\varphi(h)$. Since $h = h(a)$ is continuous on $A$ and $\det V_h$ is continuous on $H_0 = \{h(a) : a \in A\}$ we have

$$\text{(6.6)} \qquad \lim_{n\to\infty} c\left(a - \frac{z-a}{n-1}\right) \Big/ c\left(a - \frac{y-a}{n-1}\right) = 1$$

uniformly in $a \in I$ for all fixed $y, z \in \mathbf{R}^k$.

Since $\operatorname{grad} \log \varrho(a) = -h(a)$ we can write

$$\log \frac{\varrho^{n-1}\left(a - \frac{z-a}{n-1}\right)}{\varrho^{n-1}\left(a - \frac{y-a}{n-1}\right)} = (n-1)\left(\log \varrho\left(a - \frac{z-a}{n-1}\right) - \log \varrho\left(a - \frac{y-a}{n-1}\right)\right) =$$

$$\text{(6.7)} \qquad = \langle -h(a), y - z\rangle(1 + o(1)) \quad (n \to \infty)$$

uniformly for $a \in I$ and $y, z \in \mathbf{R}^k$ fixed. Thus, for $y, z \in \mathbf{R}^k$:

$$\text{(6.8)} \qquad \lim_{n\to\infty} \frac{P(S_{n-1} \geqq na - z)}{P(S_{n-1} \geqq na - y)} = \exp(-\langle h, y - z\rangle)$$

uniformly for $a \in I$.

Now let $y \geqq x$. Then for all $z \leqq x$ and large $n$

$$\frac{P(S_{n-1} \geqq na - z)}{P(S_{n-1} \geqq na - y)} \leqq 1,$$

and we can change the order of limit and integration to achieve

$$\text{(6.9)} \quad \lim_{n\to\infty} \int_{(-\infty,x]} \frac{P(S_{n-1} \geqq na - z)}{P(S_{n-1} \geqq na - y)} P_{X_1}(dz) = \int_{(-\infty,x]} \exp(-\langle h, y - z\rangle) P_{X_1}(dz)$$

uniformly in $a \in I$. Theorem 1 also yields, uniformly in $a \in I$,

$$\text{(6.10)} \quad \frac{P(S_n \geqq na)}{P(S_{n-1} \geqq na - y)} = \frac{\varrho^n(a)}{\varrho^{n-1}\left(a - \frac{y-a}{n-1}\right)} \frac{c(a)}{c\left(a - \frac{y-a}{n-1}\right)}(1 + o(1)) \ (n \to \infty).$$

By substituting $z = a$ in (6.7) and using the continuity of $c(a)$ it follows

$$(6.11)\quad \lim_{n\to\infty} \frac{P(S_n \geqq na)}{P(S_{n-1} \geqq na - y)} = \varrho(a)\exp(-\langle h, y-a\rangle) \quad \text{uniformly in } a \in I.$$

Combination of (6.9) and (6.11) yields

$$(6.12)\quad \begin{aligned} &\lim_{n\to\infty} \int\limits_{(-\infty,x]} \frac{P(S_{n-1} > na - z)}{P(S_n > na)} P_{X_1}(dz) = \\ &= \frac{1}{\varrho(a)\exp(\langle h, a\rangle)} \int\limits_{(-\infty,x]} \exp(\langle h, z\rangle) P_{X_1}(dz) \end{aligned}$$

uniformly in $a \in I$.

Taking into account that $\varrho(a) = \varphi(h)\exp(-\langle h, a\rangle)$ this proves Theorem 2 in the strongly nonlattice case in view of (6.4).

In the lattice case we have to show (6.12) uniformly in $a \in I_n := := I \cap (x_0 + L/n) \subset I$. Choose $y \in x_0 + L$ and also assume $z \in x_0 + L$. Then $a - \frac{z-a}{n-1}, a - \frac{y-a}{n-1} \in x_0 + \frac{L}{n-1}$, and we can use Theorem 1 in the lattice case to obtain (6.5) uniformly in $a \in I_n$ with

$$c(a) := (\det L)(\det V_h)^{-1/2} \sum_{\substack{\alpha\in L \\ \alpha \geqq 0}} \exp(-\langle h, \alpha\rangle).$$

Since $h = h(a)$ and $c(a)$ are continuous on $A$ and $I_n \subset I \subset A$ we get (6.8) uniformly in $a \in I_n$ for almost all $z$ with respect to $P_{X_1}$ and thus (6.9) uniformly in $a \in I_n$.

In the same manner we get (6.11) for $y \in x_0 + L$ uniformly in $a \in I_n$. Combination of (6.9) and (6.11) completes the proof of our theorem in the lattice case.

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## REFERENCES

[1] Bártfai, P., On a conditional limit theorem, *Progress in Statistics* (European Meeting of Statisticians, Budapest, 1972), Colloq. Math. Soc. J. Bolyai, Vol. 9, North-Holland, Amsterdam, 1974, 85–91. *MR* **51** # 1911.

[2] Bhattacharya, R. N. and Ranga Rao, R., *Normal approximation and asymptotic expansions,* Wiley, New York, 1976. *MR* **55** # 9219.

[3] Csiszár, I., Sanov property, generalized $I$-projection and a conditional limit theorem, *Ann. Probab.* **12** (1984), 768–793. *MR* **86h**: 60065

[4] Dersch, E., *Bedingte Grenzwertsätze bei Wahrscheinlichkeiten großer Abweichungen,* Diplomarbeit, Univ. Marburg, 1984.

[5] PETROV, V. V., On the probabilities of large deviations of sums of independent random variables, *Theory Prob. Appl.* **10** (1965), 287–298. See *MR* **32** # 3107.

[6] SANOV, I. N., On the probability of large deviations of random variables, English translation in *Select. Transl. Math. Statist. and Probability* **1** (1961), 213–244. *MR* **22** # 7166.

[7] VINCZE, I., On the maximum probability principle in statistical physics, *Progress in Statistics* (European Meeting of Statisticians, Budapest, 1972), Colloq. Math. Soc. J. Bolyai, Vol. 9, North-Holland, Amsterdam, 1974, 869–893. *MR* **51** # 12250.

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# ON $k$-INDEPENDENT SUBSETS OF A CLOSURE

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## 1. Introduction

A *closure operator* on a set $E$ is a function $\varphi\colon P(E)\to P(E)$ such that $X\subseteqq\varphi(X)=\varphi(\varphi(X))\subseteqq\varphi(Y)$ whenever $X\subseteqq Y\subseteqq E$. The *dimension* of $\varphi$ is

$$\dim(\varphi)=\min\{|A|:\varphi(A)=E\}.$$

A subset $X\subseteqq E$ is *independent* for $\varphi$ if $x\notin\varphi(X\setminus\{x\})$ for all $x\in X$; and we say that $X$ is *$\kappa$-independent*, where $\kappa$ is a cardinal, if

$$\varphi(Y)\cup X=Y \text{ for all } Y\in[X]^{\leq\kappa}=\{Y\subseteqq X:|Y|\preceq\kappa\}.$$

The closure $\varphi$ is *$\kappa$-generated* if, for every $X\subseteqq E$,

$$\varphi(X)=\cup\{\varphi(Y)\colon Y\in[X]^{\leq\kappa}\};$$

and $\varphi$ is said to be *algebraic* if it is $(<\aleph_0)$-generated, i.e.

$$\varphi(X)=\cup\{\varphi(Y)\colon Y\in[X]^{<\aleph_0}\}\qquad(X\subseteqq E).$$

Note that, if $\varphi$ is $\kappa$-generated, then $\kappa$-independence is the same as independence.

If $(E,\preceq)$ is a partially ordered set and we define $\varphi(X)=\{y\in E: y\preceq x$ for some $x\in X\}$ $(X\subseteqq E)$, then $\varphi$ is a 1-generated closure on $E$ and, in this case, the independent subsets are the antichains, and the dimension, $\dim(\varphi)=\mathrm{cf}(E,\preceq)$, is just the cofinality of the partial order. Milner and Pouzet [3] considered the question whether it is possible to extend to arbitrary closures, the following known result for partial orders (i.e. 1-generated closures) (see [1], [2], [4]):

THEOREM 1. *If $(P,\preceq)$ is a partially ordered set and the cofinality,* $\mathrm{cf}(P,\preceq)$, *is a singular cardinal $\lambda$, then $(P,\leqq)$ contains an antichain of size* $\mu=\min\{\mu'\colon\lambda^{\mu'}>\lambda\}$.

It is not known if $\mu$ can be replaced by $\mathrm{cf}(\lambda)$ in Theorem 1, although $\mu=\mathrm{cf}(\lambda)$ if we assume, for example, that $\lambda$ is a (singular) strong limit cardinal.

Milner and Pouzet [3] proved:

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THEOREM 2. *If $\varphi$ is a closure relation which has singular dimension, then there are arbitrarily large finite independent subsets for $\varphi$.*

This result is best possible since they also showed that there is an $\aleph_0$-generated closure $\varphi$, such that $\dim(\varphi)$ is a singular cardinal and there is no infinite independent subset for $\varphi$. (In fact, they noted that it is consistent, e.g. if $V = L$, that there is even an algebraic such $\varphi$.) On the other hand, Theorem 1 clearly implies that, if $\varphi$ is a 1-generated closure with dimension $\dim(\varphi) = \lambda > \mathrm{cf}(\lambda)$, then there are infinite independent sets, in fact of size $\mu = \min\{\mu' : \lambda^{\mu'} > \lambda\}$. They asked if the same is true if $\varphi$ is 2-generated.

We do not know the answer to this question for arbitrary singular $\lambda$, but there is a positive answer in the case when $\lambda$ is a singular strong limit cardinal. We prove the following theorem.

THEOREM 3. *If $\lambda$ is a singular strong limit cardinal, and if $\varphi$ is a closure relation on a set $E$ with dimension* $\dim(\varphi) = \lambda$, *then, for each $k < \omega$, there is a $k$-independent subset of $E$ having cardinality* $\mathrm{cf}(\lambda)$.

## 2. Proof of Theorem 3

Let $\lambda$ be a singular strong limit cardinal and let $\varphi$ be a closure relation on $E$ with dimension $\dim(\varphi) = \lambda$. Let $E' \in [E]^{\lambda}$ be such that $\varphi(E') = E$. For any set $X \subseteqq E$, we define

$$\delta(X) = \min\{|Y| : Y \subseteqq E', X \subseteqq \varphi(Y)\}.$$

Thus, in particular, $\delta(E') = \lambda$, and $\delta(X) < \lambda$ for $X \in [E']^{<\lambda}$.

Note first that, if $\lambda' < \lambda$ and $X \in [E']^{<\lambda}$, then there is a set $Y \in [E' \setminus \varphi(X)]^{<\lambda}$ such that $\delta(Y) > \lambda'$. To see this, write $E' \setminus \varphi(X) = \bigcup\{Y_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ where $|Y_\alpha| < \lambda$ $(\alpha < \mathrm{cf}(\lambda))$. If $\delta(Y_\alpha) \leqq \lambda'$ for each $\alpha < \mathrm{cf}(\lambda)$, then we obtain the contradiction

$$\delta(E') \leqq \delta(X) + \lambda' \cdot \mathrm{cf}(\lambda) < \lambda.$$

From the above observation, and the fact that $\lambda$ is a strong limit, it follows that there are sets $A_\alpha \in [E']^{<\lambda}$ $(\alpha < \mathrm{cf}(\lambda))$ such that

$$A_\alpha \subseteqq E' \setminus \varphi(\bar{A}_\alpha),$$

where $\bar{A}_\alpha = \bigcup_{\beta<\alpha} A_{\beta'}$ and

$$\delta(A_\alpha) > \lambda_\alpha = \left(2^{|\bar{A}_\alpha|} + \mathrm{cf}(\lambda)\right)^+.$$

(As usual, $\kappa^+$ denotes the cardinal successor of $\kappa$.)

Let $A = \cup\{A_\alpha \ \alpha < \mathrm{cf}(\lambda)\}$ and let

$$S = \{B \subseteqq A\colon \delta(B \cap A_\alpha) > \lambda_\alpha \quad \text{for all} \quad \alpha < \mathrm{cf}(\lambda)\}.$$

For $\alpha < \mathrm{cf}(\lambda)$ and $B \in S$ we write $B_\alpha = B \cap A_\alpha$ and $\tilde{B}_\alpha = B \cap \tilde{A}_\alpha$. Let $I = \{T \subseteqq A\colon |T \cap A_\alpha| \leqq 1 \text{ for all } \alpha < \mathrm{cf}(\lambda)\}$ be the set of partial transversals of the sets $A_\alpha (\alpha < \mathrm{cf}(\lambda)$. For $T \in I$ we define $L(T) = \cup\{A_\alpha\colon A_\alpha \cap T \neq \emptyset\}$.

For $V \subseteqq A$ and $k < \omega$, we say that a set $B \subseteqq A$ is $(V, k)$-*good* if

$$\varphi(V \cup T) \cap B \subseteqq L(T) \quad (\forall T \in [B]^{\leqq k} \cap I).$$

We will prove, by induction on $k < \omega$, that the statement $P(k)$: $\forall V \in [A]^{\leqq \mathrm{cf}(\lambda)}$ $\forall B \in S \exists B' \subseteqq B$ ($B' \in S$ and $B'$ is $(V, k)$-good) is true.

Note that the theorem follows immediately from the fact that $P(k)$ holds for $k < \omega$. For, given $k < \omega$, there is $B' \subseteqq A$ such that $B' \in S$ and $B'$ is $(\emptyset, k)$-good. Choose the set $H_k \subseteqq B'$ so that $|H_k \cap B'_\alpha| = 1$ for all $\alpha < \mathrm{cf}(\lambda)$. Then $|H_k| = \mathrm{cf}(\lambda)$. Also, if $X \in [H_k]^{\leqq k}$, then

$$\varphi(X) \cap H_k \subseteqq L(X) \cap H_k = X,$$

as required.

It remains to prove $P(k)$ $(k < \omega)$. Clearly, $P(0)$ holds; we simply put $B' = B \setminus \varphi(V)$. Suppose $P(k)$ holds for some $k < \omega$. We have to show that $P(k+1)$ also holds.

Let $V \in [A]^{\leqq \mathrm{cf}(\lambda)}$, $B \in S$. For $\alpha < \mathrm{cf}(\lambda)$ and $U \in [\tilde{B}_\alpha]^{<\omega}$ define a partition $\Delta_\alpha(U)$ of $B_\alpha$ so that $x \equiv y \pmod{\Delta_\alpha(U)}$ holds, if and only if,

$$\varphi(V \cup U \cup \{x\}) \cap \tilde{B}_\alpha = \varphi(V \cup U \cup \{y\}) \cap \tilde{B}_\alpha.$$

Let $\Delta_\alpha = \prod\limits_U \Delta_\alpha(U)$ be the common refinement of all the $\Delta_\alpha(U)$ $(U \in [\tilde{B}_\alpha]^{<\omega})$. Then $|\Delta_\alpha| < \lambda_\alpha$ $(\alpha < \mathrm{cf}(\lambda))$, and so there is a set $C_\alpha \subseteqq B_\alpha$ such that $\delta(C_\alpha) > \lambda_\alpha$ and $x \equiv y \pmod{\Delta_\alpha}$ for all $x, y \in C_\alpha$. Choose $c_\alpha \in C_\alpha$ $(\alpha < \mathrm{cf}(\lambda))$ and put $V' = \{c_\alpha\colon \alpha < \mathrm{cf}(\lambda)\}$.

Since $C = \cup\{C_\alpha\colon \alpha < \mathrm{cf}(\lambda)\} \in S$, it follows from the induction hypothesis that there is $C' \subseteqq C$ such that $C' \in S$ and $C'$ is $(V \cup V', k)$-good. Put $B'_\alpha = C'_\alpha \setminus \varphi(V \cup V' \cup \tilde{A}_\alpha)$ $(\alpha < \mathrm{cf}(\lambda))$. Then $B' = \cup\{B'_\alpha \ \alpha < \mathrm{cf}(\lambda)\} \subseteqq B$ and $B' \in S$. We claim that $B'$ is also $(V, k+1)$-good.

Consider any $T \in [B']^{<k+1} \cap I$. If $|T| \leqq k$, then, since $C'$ is $(V \cup V', k)$-good, we have

$$\varphi(V \cup T) \cap B' \subseteqq \varphi(V \cup V' \cup T) \cap C' \subseteqq L(T).$$

Suppose $|T| = k+1$. Let $\alpha = \max\{\beta\colon T \cap B'_\beta \neq \emptyset\}$ and let $T \cap B'_\alpha = \{t\}$, $T' = T \setminus \{t\}$. From the definition of the sets $B'_\gamma$ $(\gamma < \mathrm{cf}(\lambda))$, it is clear that

$$\varphi(V \cup T) \cap B' \subseteqq \tilde{B}'_\alpha \cup B'_\alpha.$$

Moreover, since $t \equiv c_\alpha \pmod{\Delta_\alpha(T')}$, it follows that

$$\varphi(V \cup T' \cup \{t\}) \cap \bar{B}'_\alpha = \varphi(V \cup T' \cup \{c_\alpha\}) \cap \bar{B}'_\alpha \subseteqq$$
$$\subseteqq \varphi(V \cup V' \cup T') \cap \bar{C}'_\alpha \subseteqq L(T') \subseteqq L(T),$$

since $C'$ is $(V \cup V', k)$-good. Since $B'_\alpha \subseteqq L(T)$ also, it follows that

$$\varphi(V \cup T) \cap B' \subseteqq L(T)$$

and so $B'$ is $(V, k+1)$-good, as claimed. □

## REFERENCES

[1] HAJNAL, A. and SAUER, N., Complete subgraphs of infinite multipartite graphs and antichains in partially ordered sets, *Discrete Math.* **59** (1986), 61–67. *MR* **87f**: 05008.

[2] MILNER, E.C. and POUZET, M., On the cofinality of partially ordered sets, *Ordered sets* (Banff, Alta., 1981), ed. by I. Rival, NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., **83**, Reidel, Dordrecht-Boston, 1982, 279–298. *MR* **83k**: 06003

[3] MILNER, E. C. and POUZET, M., On the independent subsets of a closure system with singular dimension, *Algebra Universalis* **21** (1985), 25–32. *MR* **87f**: 06005

[4] MILNER, E. C. and PRIKRY, K., The cofinality of a partially ordered set, *Proc. London Math. Soc.* (3) **46** (1983), 454–470. *MR* **84m**: 06004

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# REMARK ON SUBDIRECT REPRESENTATION THEOREM FOR ECE-VARIETIES OF PARTIAL ALGEBRAS

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One of the most important theorems in universal algebra is that of Birkhoff for varieties which reads that each variety is generated by its subdirectly irreducible members. In the case of partial algebras there are also theorems of this kind (see [1]). However, it seems to the author that for ECE-varieties of partial algebras it is impossible to prove an elegant, non-trivial and interesting analogon of Birkhoff Theorem. The best possible result for ECE-varieties in author's opinion is Theorem 2 below.

We shall freely use notation and terminology of [1].

Let us start with the following theorem proved by P. Burmeister and M. Siegmund-Schultze in [1].

THEOREM 1. *Each ECE-variety $K$ of partial algebras is the class of all $(H_c, S_c)$-subdirect products of $(H_c, S_c)$-subdirectly irreducible $K$-algebras.*

This Theorem looks very elegant. However, as the examples below show, the idea of Birkhoff Theorem to separate a proper subclass of relatively simple algebras of a variety, generating the whole variety is missed.

EXAMPLE 1. Let $K_1$ denote the class of partial algebras $\underline{A} = \langle A, \vee, \wedge, \neg, 0, 1, f, g\rangle$ of type $\langle 2,2,1,0,0,1,1\rangle$ axiomatized by the following set of ECE-identities

(1) $x \overset{e}{=} x \Rightarrow x \vee y \overset{e}{=} x \vee y, \quad x \overset{e}{=} x \Rightarrow x \wedge y \overset{e}{=} x \wedge y$;

(2) $x \overset{e}{=} x \Rightarrow \neg x \overset{e}{=} \neg x$;

(3) $x \overset{e}{=} x \Rightarrow 0 \overset{e}{=} 0 \quad x \overset{e}{=} x \Rightarrow 1 \overset{e}{=} 1$;

(4) $x \overset{e}{=} x \Rightarrow t_1 \overset{e}{=} t_2$, for every axiom $t_1 = t_2$ of Boolean algebras;

(5) $fx \overset{e}{=} fx \Rightarrow x \wedge y \overset{e}{=} x, \quad fx \overset{e}{=} fx \Rightarrow fx \wedge y \overset{e}{=} y$;

(6) $x \overset{e}{=} x \Rightarrow f0 \overset{e}{=} f0$;

(7) $gx \overset{e}{=} gx \Rightarrow x \wedge y \overset{e}{=} y, \quad gx \overset{e}{=} gx \Rightarrow gx \wedge y \overset{e}{=} gx$;

(8) $x \overset{e}{=} x \Rightarrow g1 \overset{e}{=} g1$.

The class $K_1$ is a non-trivial ECE-variety.

It is obvious that if $\underline{A} \in K_1$ then $\underline{A}^* = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and that any congruence relation on $\underline{A}$ is also a congruence relation on

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$\underline{A}^*$. As for any non-trivial congruence relation $R$ on the algebra $\underline{A}^*$ there is an element $a \in A$, $a \neq 1$ (in case $\underline{A}$ is non-trivial) such that $(a,1) \in R$ it follows that $\Delta$ — the identity relation — is the only closed congruence relation on $\underline{A}$ and consequently $\underline{A}$ itself is the only closed homomorphic image of $\underline{A}$. Thus we have the following

FACT. *Every element of the ECE-variety $K_1$ is $(H_c, S_c)$-subdirectly irreducible.*

COROLLARY. *A direct product of any family of members of $K_1$ is $(H_c, S_c)$-subdirectly irreducible.*

The above fact means that in this case the class of $(H_c, S_c)$-subdirectly irreducible partial algebras is too wide to be an analogon of a class of subdirectly irreducible algebras for a variety of total algebras. Let us note that to obtain a good subdirect representation theorem for the ECE-variety $K_1$ one should consider the class of all surjective full homomorphisms instead of closed ones, i.e. $(H_f, S_c)$-subdirectly irreducible algebras. Then it appears that there is exactly one non-trivial $(H_f, S_c)$-subdirectly irreducible partial algebra $\underline{Q}$ in $K_1$-two-element one and it generates $K_1$, i.e. $K_1 = \mathrm{Ps}^{(H_f,S_c)}\left(\mathrm{si}^{(H_f,S_c)}(\underline{Q})\right)$.

EXAMPLE 2. Let us consider ECE-variety $K_2$ of partial algebras consisting of algebras $\underline{A} = \langle A, \vee, \wedge, \neg, 0, 1, f, g, h, k\rangle$ of type $\langle 2,2,1,0,0,1,1,1,1\rangle$ such that $\underline{A}' = \langle A, \vee, \wedge, \neg, 0, 1, f, g\rangle \in K_1$, satisfying the following ECE-identity

(9) $hx \stackrel{e}{=} hx \;\;\&\;\; kx \stackrel{e}{=} kx \Rightarrow hx \stackrel{e}{=} kx.$

It is easy to see that each member of $K_2$ is $(H_c, S_c)$-subdirectly irreducible. Thus as in Example 1 we can try to replace the class $H_c$ by a class of full surjective homomorphisms.

Now let $\underline{B} = \langle\{0,1,a,b\}, \vee, \wedge, \neg, 0, 1, f, g, h, k\rangle \in K_2$ be a four-element partial algebra, where $\operatorname{dom} h = \{a\}$, $ha = b$, $\operatorname{dom} k = \{1\}$, $k1 = 1$. It is easily verified that the following partial algebras $\underline{B}_1$, $\underline{B}_2$ are full homomorphic images of $\underline{B}$ by full homomorphisms $F_1$, $F_2$, respectively,

$$\underline{B}_1 = \langle\{0,1\}, \vee, \wedge, \neg, 0, 1, f, g, h, k\rangle \text{ where } \operatorname{dom} h = \operatorname{dom} k = \{1\},\;\; h1 = 0,\; k1 = 1$$

$$\underline{B}_2 = \langle\{0,1\}, \vee, \wedge, \neg, 0, 1, f, g, h, k\rangle \text{ where } \operatorname{dom} h = \{0\},\;\; h0 = 1,\; \operatorname{dom} k = \{1\},\; k1 = 1.$$

Moreover, ECE-identity (9) fails to hold in $\underline{B}_1$ and it does hold in $\underline{B}_2$, i.e. $\underline{B}_1 \notin K_2$, $\underline{B}_2 \in K_2$. Note that $F_2$ is not closed and $\underline{B}$ is isomorphic to $\underline{B}_1 \times \underline{B}_2$.

Let us observe that such a situation could not happen in the case of a variety $V$ of total algebras, for the product of a family of algebras belongs to $V$ if and only if all members of this family belong to $V$.

(Taking $\underline{B}'$ an inner extension of $\underline{B}$, where $\operatorname{dom} k = \{1\}$, $\operatorname{dom} h = \{a, b\}$,

$hb = a$ one can easily check that $\underline{B}'$ is a product of two (isomorphic) partial algebras neither of which belongs to $K_2$).

The above examples show that it will be very difficult or even impossible to find a uniform elegant subdirect representation theorem for ECE-varieties. There are two reasons of such a situation. If we have given ECE-variety $K$ there can be non-closed homomorphisms preserving the axioms of $K$. The second reason is that it may happen that the axiomatization of an ECE-variety can be replaced by another one consisting of e.g. E-identities. Here the situation is similar to that concerning quasivarieties and varieties of total algebras. To answer the question whether a given quasivariety of total algebras is a variety, i.e. if a set of quasiidentities axiomatizing $V$ can be replaced by a set of identities is usually very difficult. The same is probably for different kinds of varieties of partial algebras.

REMARK. In the case of the ECE-variety $K_2$ it follows from the subdirect representation theorem for E-varieties that it cannot be axiomatized by any set of E-identities.

In author's opinion the best possible but neither elegant nor useful is the following

THEOREM 2. *Let $K$ be any ECE-variety of partial algebras, $H_K$ a class of all surjective full homomorphisms such that $H_K(K) \subseteq K$. Then*

$$K = \mathrm{Ps}^{(H_K,S_c)}\left(\mathrm{si}^{(H_K,S_c)}(K)\right).$$

Here it should be noted the following

THEOREM 3. *Under the assumptions of Theorem 2 if $H_K = H_c$ then $K$ is a class of total algebras.*

PROOF. Let us note that in any ECE-variety of partial algebras there is at least one total algebra — one element algebra $\underline{T}$. Now, let $\underline{A}$ be any member of $K$. Then $\underline{T} \times \underline{A}$ belongs to $K$ and projection $p\colon \underline{T} \times \underline{A} \to \underline{T}$ belongs to $H_K$. By assumption $p$ is a closed homomorphism which is equivalent to the totalness of $\underline{A}$.

We end this paper with the following

QUESTION. Is it true that if $H_K = H_f$ then $K$ is an E-variety?

The answer is probably negative.

## REFERENCE

[1] BURMEISTER, P., A model theoretic oriented approach to partial algebras, Part I, Mathematical Research, **32**, Akademie-Verlag, Berlin, 1986. *MR* **87m**: 08007a

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# RANDOM MEASURES WITH VALUES IN TOPOLOGICAL ABELIAN GROUPS

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**Abstract**

Let $X$ be a random measure defined on a $\sigma$-ring $\mathbf{R}$ of subsets of a set $\mathfrak{X}$ with values in a topological Abelian group $(G,\tau)$. The purpose of this paper is to describe the distribution of $X(E)$, $E\in\mathbf{R}$.

The main theorem presented in this note can be considered as a generalization of the results given in [1], [3].

## 1. Introduction

Let $(\Omega,\mathcal{A},P)$ be a probability space, $(\mathfrak{X},\mathbf{R})$ a measurable space (thus $\mathbf{R}$ is a $\sigma$-ring of subsets of $\mathfrak{X}$). A classical random measure $X$ is a real-valued function. $X(E,\omega)$ defined for $E\in\mathbf{R}$ and $\omega\in\Omega$ with the properties that for a fixed $\omega$ it is a measure on $(\mathfrak{X},\mathbf{R})$, and for a fixed $E$, it is a random variable. We consider group-valued random measures that is, $X(E,\omega)\in G$ for some abstract Abelian group $G$. Measures $X(\cdot,\omega)$, $\omega\in\Omega$ are $\sigma$-additive thus we need to consider infinite sums of elements of $G$: to this end we assume that $G$ is a topological group, endowed with a Hausdorff topology $\tau$.

Following Kingman [6], we declare $X$ to be completely random if, for any finite collection $E_1,E_2,\dots,E_n$ of disjoint members of $\mathbf{R}$, the random elements $X(E_1),X(E_2),\dots,X(E_n)$ are mutually independent.

In addition we assume that the completely random measure $X$ is such that:

(A) There exists an open neighbourhood $\mathbf{U}$ of $\theta$ ($\theta$ is the null-element of the Abelian group $G$), such that $\sum u_i=\theta$, where $u_i\in\mathbf{U}$ and $u_i$ are *possible values of the measure* $\mathbf{X}(E,\omega)$, implies that $u_i=\theta$.

We shall say, that a set $E\in\mathbf{R}$ is *small* (with respect to $X$ and $\mathbf{U}$) if, for each point $\omega\in\Omega$, $X(E,\omega)\in\mathbf{U}$, and if $X(E)=\theta$ a.s.

For $E\in\mathbf{R}$, let $\{\mathbf{P}_n(E),n\geqq 1\}$ be a sequence of finite partitions of $E$, i.e., $E=\sum_{E_j^{(n)}\in\mathbf{P}_n(E)} E_j^{(n)}$, where $E_j^{(n)}\in\mathbf{R}$ and $E_i^{(n)}\cap E_j^{(n)}=\emptyset$, for $i\neq j$.

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The sequence $\{\mathbf{P}_n(E), n \geqq 1\}$ will be called a fundamental sequence of partitions of $E$, if for all $n \geqq 1$, the partition $\mathbf{P}_{n+1}(E)$ refines $\mathbf{P}_n(E)$ and sets of the form $\bigcap_{n\geqq 1} E_{j_n}^{(n)}$ are small.

It is easy to see, if $E_1 \cap E_2 = \emptyset$ and $\mathbf{P}_n(E_1)$, $\mathbf{P}_n(E_2)$ are fundamental sequence of partitions of $E_1$, $E_2$, respectively, then $\mathbf{P}_n(E_1) \cup \mathbf{P}_n(E_2)$ constitute the fundamental sequence of partitions of $E_1 \cup E_2$.

The purpose of the present paper is to describe the distribution of random elements $X(E)$, $E \in \mathbf{R}$.

The main theorem of the paper can be considered as a generalization of the result obtained by Brown and Kupka [1]. In particular cases we obtain the Poisson measure (see [4]) and the bivariate Poisson measure (see [10], [5]). The method of the proof used in the present paper is not quite elementary, since it uses both operator theoretic methods, with relevant concepts described by Le Cam [7] and Ramsey's theorem [9]. We also use the technique given by Góźdź and Polak [3].

In the next section a simple combinatorial principle will be used known as Ramsey's theorem [9].

THEOREM. *Let each pair $(i, j)$ of disjoint positive integers be coloured either "red" or "blue" in any arbitrary fashion. Then there exists a strictly increasing sequence $\{n_i, i \geqq 1\}$ of positive integers such that all of pairs from this sequence have the same colour.*

## 2. The parameter of a random measure $X$

In what follows we assume that the following assumptions hold:

(a) for every $E \in \mathbf{R}$, there exists a fundamental sequence of partitions $\{\mathbf{P}_n(E), n \geqq 1\}$ of $E$,

(b) there exists a neighbourhood $\mathbf{V}_\theta$ of the null element $\theta \in G$ such that

$$\lim_{n\to\infty} P[X(E_n) \in \mathbf{V}_\theta \setminus \{\theta\}] = 0$$

provided $\{E_n, n \geqq 1\}$ is a decreasing sequence of elements $E_n \in \mathbf{P}_n(E)$.

THEOREM 2.1. *If $X$ is a random measure satisfying the assumption* (a), *then for every $\omega \in \Omega$, there exists an integer $n_\omega$ such that, for every integer $n \geqq n_\omega$, and for every $E_j^{(n)} \in \mathbf{P}_n(E)$, we have $X(E_j^n, \omega) \in \mathbf{U}$.*

PROOF. In order to prove Theorem 2.1, we suppose to the contrary that there exists a subsequence $\mathbf{P}_{n_i}(E)$ of $\mathbf{P}_n(E)$ containing sets $E_{n_i} \in \mathbf{P}_{n_i}(E)$ for which $X(E_{n_i}) \in \mathbf{U}^c$. By Ramsey's theorem there exists a strictly increasing sequence $\{m_i, i \geqq 1\}$ of positive integers such that either $E_{n_{m_i}} \cap E_{n_{m_j}} \neq 0$

or $E_{n_{m_i}} \subset E_{n_{m_j}}$ for $m_i < m_j$. Suppose that $E_{n_{m_i}} \cap E_{n_{m_j}} \neq 0$. Then this contradicts to the fact that

$$\sum_{i \geqq 1} X(E_{n_{m_i}}, \omega) = X\Big(\sum_{i \geqq 1} E_{n_{m_i}}, \omega\Big) \in G$$

(the necessary condition of convergence of the series $\sum\limits_{i \geqq 1} X(E_{n_{m_i}}, \omega)$ in topology $\tau$ is not satisfied for the neighbourhood $\mathbf{U}$).

Now, suppose that $E_{n_{m_i}} \subset E_{n_{m_j}}$ for $m_i < m_j$. Then the sets $E_{n_{m_i}}$ form decreasing subsequence of $\{E_{n_i}\}$ whose intersection $\bigcap\limits_{i \geqq 1} E_{n_{m_i}}$ is small by (a). It follows that

$$\tau - \lim X(E_{m_{n_i}}, \omega) = X\Big(\bigcap_{i \geqq 1} E_{n_{m_i}}\Big) \in \mathbf{U},$$

which contradicts that the set $\mathbf{U}^c$ is closed. Therefore Theorem 2.1 is established.

COROLLARY. *For each point* $\omega \in \Omega$ *and for every* $E \in \mathbf{R}$, *we have* $X(E) \in \in (\mathbf{U})$, *where* $(\mathbf{U})$ *is a semigroup generated by* $\mathbf{U}$.

THEOREM 2.2. *If* $X$ *is a completely random measure satisfying the assumptions* (a), *then the set function*

$$\mu(E) = -\ln P[X(E) = \theta], \quad E \in \mathbf{R}$$

*is well defined and it is a finitely additive measure on* $\mathbf{R}$.

PROOF. At first we shall show that the set function $\mu$ is finite for every set $E \in \mathbf{R}$.

Suppose that $P[X(E) = \theta] = 0$ for at least one set $E \in \mathbf{R}$. Then $X(E) \neq \theta$ a.s. Let $\{\mathbf{P}_n(E), n \geqq 1\}$ be a fundamental sequence of partitions of $E \in \mathbf{R}$. The inequality

$$0 = P[X(E) = \theta] = P\left[\sum_{E_j^{(1)} \in \mathbf{P}_1(E)} X(E_j^{(1)}) = \theta\right] \geqq$$
$$\geqq \prod_{E_j^{(1)} \in \mathbf{P}_1(E)} P[X(E_j^{(1)}) = \theta]$$

shows that $X(E_j^{(1)}) \neq \theta$ a.s. for at least one set $E_{j_1}^{(1)} \in \mathbf{P}_1(E)$. We repeat this argument with $E_{j_1}^{(1)}$ in place of $E$ and continue by induction to produce a decreasing sequence of sets $E_{j_n}^{(n)} \in \mathbf{P}_n(E)$, $n \geqq 1$ such that $X(E_{j_n}^{(n)}) \neq \theta$ a.s.

By (a) the set $\bigcap_{n\geq 1} E_{j_n}^{(n)}$ is small and therefore $X\left(\bigcap_{n\geq 1} E_{j_n}^{(n)}\right) = \theta$ a.s. Since for each point $\omega \in \Omega$

$$\tau - \lim X(E_{j_n}^{(n)}) = X\left(\bigcap_{n\geq 1} E_{j_n}^{(n)}\right),$$

it is easy to see that, for every neighbourhood $\mathbf{V}_\theta$ of $\theta$,

$$P\left[X(E_{j_n}^{(n)}) \in \mathbf{V}_\theta\right] \to 1 \text{ as } n \to \infty.$$

But, from assumption (b) it follows that there exists a neighbourhood $\mathbf{V}_\theta$ of $\theta$ such that

$$P\left[X(E_{j_n}^{(n)}) \in \mathbf{V}_\theta\right] = P\left[X(E_{j_n}^{(n)}) \in \mathbf{V}_\theta \setminus \{\theta\}\right] \to 0, \text{ as } n \to \infty.$$

This is a contradiction.

Now we will show the finite additivity of $\mu$.

Taking into account Theorem 2.1 it can be proved that

$$\prod_{E_j^{(n)} \in \mathbf{P}_n(E)} P\left[X(E_j^{(n)}) \in \mathbf{U}\right] \to 1, \quad n \to \infty. \tag{2.1}$$

Indeed, for every $\omega \in \Omega$, we have

$$I\left\{\bigcap_{E_j^{(n)} \in \mathbf{P}_n(E)} \left[X(E_j^{(n)} \in \mathbf{U}\right]\right\}(\omega) \to 1$$

where $I[A]$ denotes the indicator of an event $A$.

In view of the last relation and Lebesgue's theorem we may write

$$\prod_{E_j^{(n)} \in \mathbf{P}_n(E)} P\left[X(E_j^{(n)}) \in \mathbf{U}\right] = P\left[\bigcap_{E_j^{(n)} \in \mathbf{P}_n(E)} \left[X(E_j^{(n)}) \in \mathbf{U}\right]\right] =$$
$$= \int_\Omega \left(I\left\{\bigcap_{E_j^{(n)} \in \mathbf{P}_n(E)} \left[X(E_j^{(n)}) \in \mathbf{U}\right]\right\}\right) dP \to 1, \text{ as } n \to \infty.$$

Hence the proof of (2.1) is complete.

For disjoint sets $E_1, E_2 \in \mathbf{R}$. Let $\mathbf{P}_n(E_1)$ and $\mathbf{P}_n(E_2)$ be fundamental sequences of partitions of $E_1$ and $E_2$, respectively.

It is known that $\mathbf{F}_n(E_1+E_2)=\mathbf{P}_n(E_1)\cup\mathbf{P}_n(E_2)$ is a fundamental sequence of partitions of $E_1+E_2$.

Let us put

$$A_n=\bigcap_{F_j^{(n)}\in\mathbf{F}_n(E_1+E_2)}\left[X(F_j^{(n)})\in\mathbf{U}\right].$$

Since $P(A_n)\to 1$, as $n\to\infty$, therefore

$$P\left[X(E_1+E_2)=\theta, A_n\right]\to P\left[X(E_1+E_2)=\theta\right],\ \text{as } n\to\infty.$$

Now, for every $\varepsilon>0$ and $n$ sufficiently large, we have

$$P\left[X(E_1+E_2)=\theta\right]-\varepsilon\leqq P\left[X(E_1+E_2)=\theta, A_n\right]=$$

$$=P\left\{\sum_{F_j^{(n)}\in\mathbf{F}_n(E_1+E_2)}X(F_j^{(n)})=\theta,\ \bigcap_{F_j^{(n)}\in F_n(E_1+E_2)}\left[X(F_j^{(n)})\in\mathbf{U}\right]\right\}=$$

$$=P\left[\bigcap_{F_j^{(n)}\in\mathbf{F}_n(E_1+E_2)}\left[X(F_j^{(n)})=\theta\right],\ \bigcap_{F_j^{(n)}\in F_n(E_1+E_2)}\left[X(F_j^{(n)})\in\mathbf{U}\right]\right].$$

The last equality follows from (A).

Furthermore

$$\bigcap_{F_j^{(n)}\in F_n(E_1+E_2)}\left[X(F_j^{(n)})=\theta\right]\subseteqq A_n,$$

thus

$$P\left[X(E_1+E_2)=\theta\right]-\varepsilon\leqq P\left[X(E_1)=\theta\right]P\left[X(E_2)=\theta\right].$$

The inverse inequality is obvious, so the proof of finite additivity of $\mu$ is complete.

Notice that, if a set $E\in\mathbf{R}$ is small, then $\mu(E)=0$.

THEOREM 2.3. *If $X$ is a completely random measure satisfying the assumptions* (a), (b), *then for every number $\varepsilon>0$, there exists an integer $n_\varepsilon$ such that, for every integer $n\geqq n_\varepsilon$, and every set $E_j^{(n)}\in\mathbf{P}_n(E)$, we have $\mu(E_j^{(n)})<\varepsilon$.*

The proof of this theorem is essentially the same as the proof of Theorem 2.1.

PROOF. We suppose to the contrary that, there exists $\varepsilon>0$ and subsequence $\mathbf{P}_{n_i}(E)$ of $\mathbf{P}_n(E)$ containing sets $E_{n_i}\in\mathbf{P}_{n_i}(E)$ for which $\mu(E_{n_i})\geqq\varepsilon$. From Ramsey's theorem there exists a strictly increasing sequence $\{m_i, i\geqq\geqq 1\}$ of positive integers such that either $E_{n_{m_i}}\cap E_{n_{m_j}}=\emptyset$ or $E_{n_{m_i}}\subset E_{n_{m_j}}$

for $m_i < m_j$. Suppose that $E_{n_{m_i}} \cap E_{n_{m_j}} = \emptyset$ then this contradicts the fact that the measure $\mu$ is finite.

Now, suppose that $E_{n_{m_i}} \subset E_{n_{m_j}}$ for $m_i < m_j$. Then the sets $E_{n_{m_i}}$ form a decreasing subsequence of the sets $E_{n_i}$ whose intersection $\bigcap_{i \geq 1} E_{n_{m_i}}$ is small by (a). So we have

$$P\Big[X\Big(\bigcap_{i \geq i} E_{n_{m_i}}\Big) = \theta\Big] = 1.$$

By the countable additivity of $X$, we may write

$$P\left[\tau - \lim X(E_{n_{m_i}}) = \theta\right] = 1,$$

and in consequence for every neighbourhood $\mathbf{V}_\theta$ of $\theta$, we have

(2.2) $$\lim_{i \to \infty} P\left[X(E_{n_{m_i}}) \notin \mathbf{V}_\theta\right] = 0.$$

It is evident that

$$P\left[X(E_{n_{m_i}}) = \theta\right] = 1 - P\left[X(E_{n_{m_i}}) \in \mathbf{V}_\theta \setminus \{\theta\}\right] - P\left[X(E_{n_{m_i}}) \notin \mathbf{V}_\theta\right],$$

where $\mathbf{V}_\theta$ is certain neighbourhood of $\theta \in G$.

From the assumption (b) and taking into account (2.2) we obtain

$$\lim_{i \to \infty} P\left[X(E_{n_{m_i}}) = \theta\right] = 1.$$

Finally

$$\lim_{i \to \infty} \mu(E_{n_{m_i}}) = -\ln \lim_{i \to \infty} P\left[X(E_{n_{m_i}}) = \theta\right] = 0.$$

The last equalities contradict the fact that $\mu(E_{n_{m_i}}) \neq 0$.

## 3. Construction of the probability measure $\mathbf{G_E}$

Now we assume that $\{\theta\} \in \mathbf{G}$, i.e., the set that contains only the null-element of the group $G$ is an element of the $\sigma$-field $\mathbf{G}$ generated by $\tau$.

Let us put $\delta(E) = P[X(E) \neq \theta]$ and $\mathcal{L}X(E)$ denote the distribution of the random element $X(E, \cdot)$. For each $E \in \mathbf{R}$, let us put

(3.1) $$\mathcal{L}X(E) = I + \delta(E)(M_E - I),$$

where $I$ is the probability measure whose mass is entirely concentrated at the point $\theta \in G$ and $M_E$ is the probability measure such that

(3.2) $$M_E(\theta) = 0.$$

For each $E \in \mathbf{R}$, we define

$$\lambda(\mathbf{P}_n(E)) = \sum_{E_j^{(n)} \in \mathbf{P}_n(E)} \delta(E_j^{(n)}). \tag{3.3}$$

Taking into account Theorem 2.3 and the fact that $\delta(E) \leqq \mu(E)$, it is easy to see that for every $E \in \mathbf{R}$

$$\lim_{n\to\infty} \lambda(\mathbf{P}_n(E)) = \mu(E). \tag{3.4}$$

Now let us define

$$M(\mathbf{P}_n(E)) = \frac{1}{\lambda(\mathbf{P}_n(E))} \sum_{E_j^{(n)} \in \mathbf{P}_n(E)} \delta(E_j^{(n)}) M_{E_j^{(n)}}. \tag{3.5}$$

Let $v$ be a finite signed measure on $\mathbf{G}$. For any finite signed measure $v$ on $\mathbf{G}$ let $||v||$ be the norm defined by

$$||v|| = |v|(G),$$

where $|v|(G)$ is the total variation of $v$.

Let $\mathbf{M}$ be the system of all finite signed measure on $\mathbf{G}$. It is not difficult to show that $\mathbf{M}$, with the norm defined as above is a Banach space.

It may also be shown that

$$\sup_{A\in\mathbf{G}} |v(A)| \leqq ||v|| \leqq 2 \sup_{A\in\mathbf{G}} |v(A)| \tag{3.6}$$

(see Gihman, I. I., and Skorohod, A. V. [2] for details).

THEOREM 3.1. *If $X$ is a completely random measure satisfying the assumptions* (a), (b), *then for every $E \in \mathbf{R}$*

$$G_E = \lim_{n\to\infty} M(\mathbf{P}_n(E)) \tag{3.7}$$

*is a well defined probability measure on* $\mathbf{G}$ *such that* $G_E(\mathbf{U} \setminus \{\theta\}) = 1$.

Before the proof of Theorem 3.1, we will prove the following Lemmas:

LEMMA 3.1. *If $X$ is a completely random measure satisfying the assumptions* (a), (b), *then for every disjoint set $E_1, E_2 \in \mathbf{R}$, we have*

$$||\delta(E_1 \dotplus E_2)M_{E_1+E_2} - \delta(E_1)M_{E_1} - \delta(E_2)M_{E_2}|| \leqq 2\mu(E_1)\mu(E_2), \tag{3.8}$$

*where $\delta$, $M_E$ and $\mu$ are defined as above.*

PROOF. Let us put

$$v = \delta(E_1 + E_2)M_{E_1+E_2} - \delta(E_1)M_{E_1} - \delta(E_2)M_{E_2}.$$

Taking into account (3.1) and (3.2) it is easy to see that for each $\theta \notin A$, $v(A) = v(A \setminus \{\theta\})$. Now, we may write

$$v(A) = \begin{cases} P[X(E_1+E_2) \in A] - P[X(E_1) \in A] - P[X(E_2) \in A] & \text{if } A \not\ni \theta, \\ 0 & \text{if } A = \{\theta\}. \end{cases}$$

Hence for each $A \not\ni \theta$, we have

$$\begin{aligned} v(A) \geqq P[X(E_1) \in A, X(E_2) = \theta] + P[X(E_1) = \theta, X(E_2) \in A] + \\ - P[X(E_1) \in A] - P[X(E_2) \in A] = P[X(E_1) \in A] + \\ - P[X(E_1) \in A, X(E_2) \neq \theta] + P[X(E_2) \in A] + \\ - P[X(E_2) \in A, X(E_2) \neq \theta] \geqq \\ -2P[X(E_1) \neq \theta] P[X(E_2) \neq \theta] \geqq -2\mu(E_1)\mu(E_2). \end{aligned}$$

For every $A \not\ni \theta$, we put

$$\kappa(A) = P[X(E_1+E_2) \in A] - P[X(E_1) \in A, X(E_2) = \theta] + \\ - P[X(E_1) = \theta, X(E_2) \in A].$$

It may be seen that $\kappa(A) \geqq 0$ and also $v(A) \leqq \kappa(A)$. Hence

$$|v(A)| \leqq \max[2\mu(E_1)\mu(E_2), \kappa(A)].$$

Therefore

$$\sup_{\theta \notin A \subset G} |v(A)| \leqq \max[2\mu(E_1)\mu(E_2), \kappa(G \setminus \{\theta\})].$$

It is obvious that if $X$ is a completely random measure, then

$$\begin{aligned} \kappa(G \setminus \{\theta\}) = P[X(E_1+E_2) \neq \theta] - P[X(E_1) \neq \theta] + \\ + P[X(E_1) \neq \theta, X(E_2) \neq \theta] - P[X(E_2) \neq \theta] + \\ + P[X(E_1) \neq \theta, X(E_2) \neq \theta] \leqq \mu(E_1)\mu(E_2), \end{aligned}$$

which proves that

$$\sup_{\theta \notin A \subset G} |v(A)| \leqq 2\mu(E_1)\mu(E_2).$$

Taking into account (3.6) and the last inequality, we obtain (3.8).

LEMMA 3.2. *If $X$ is a completely random measure satisfying the assumptions* (a), (b), *then for any disjoint sets $E_1, E_2, \dots, E_n \in \mathbf{R}$*

$$\left\| \delta\Big(\sum_{j=1}^n E_j\Big) M_{\sum_{j=1}^n E_j} - \sum_{j=1}^n \delta(E_j) M_{E_j} \right\| \leqq 2\mu^2\Big(\sum_{j=1}^n E_j\Big). \tag{3.9}$$

PROOF. The proof of this Lemma will be settled by induction argument. Let $E_1, E_2 \in \mathbf{R}$ and $E_1 \cap E_2 = \theta$. Then using Lemma 3.1, we have

$$\|\delta(E_1+E_2)M_{E_1+E_2} - \delta(E_1)M_{E_1} - \delta(E_2)M_{E_2}\| \leqq 2\mu^2(E_1+E_2).$$

From (3.9) and (3.8), we obtain

$$\left\|\delta\Big(\sum_{j=1}^{n} E_j\Big)M_{\sum\limits_{j=1}^{n} E_j} - \sum_{j=1}^{n}\delta(E_j)M_{E_j}\right\| \leqq$$

$$\leqq \left\|\delta\Big(\sum_{j=1}^{n} E_j\Big)M_{\sum\limits_{j=1}^{n-1} E_j} - \sum_{j=1}^{n-1}\delta(E_j)M_{E_j}\right\| +$$

$$+\left\|\delta\Big(\sum_{j=1}^{n} E_j\Big)M_{\sum\limits_{j=1}^{n} E_j} - \delta\Big(\sum_{j=1}^{n-1} E_j\Big)M_{\sum\limits_{j=1}^{n-1} E_j} - \delta(E_n)M_{E_n}\right\| \leqq$$

$$\leqq 2\mu\Big(\sum_{j=1}^{n-1} E_j\Big) + 2\mu\Big(\sum_{j=1}^{n-1} E_j\Big)\mu(E_n) \leqq 2\mu^2\Big(\sum_{j=1}^{n} E_j\Big).$$

The last inequality ends the proof of Lemma 3.2.

PROOF of Theorem 3.1. Taking into account that $\mathbf{M}$ is a Banach space it is enough to prove that $\{M(\mathbf{P}_n(E)), n \geqq 1\}$ is a Cauchy sequence. Suppose that $E_j^{(n)}$, $j = 1, 2, \ldots, n$ are elements of the partition $\mathbf{P}_n(E)$ of the set $E$, and let $E_{j_k}^{(m)}$, $k = 1, 2, \ldots, n_j$, $j = 1, 2, \ldots, n$ be elements of the partition $\mathbf{P}_m(E)$, $m \geqq n$ of the same set $E$.

Assume (without loss of generality) that $E_j^{(n)} = \sum\limits_{k=1}^{n_j} E_{j_k}^{(m)}$, $j = 1, 2, \ldots, n$. From Lemma 3.2 we obtain

$$\left\|\sum_{j=1}^{n}\sum_{k=1}^{n_j}\left(E_{j_k}^{(m)}\right) M_{E_{j_k}^{(m)}} - \sum_{j=1}^{n}(E_j^{(n)})M_{E_j^{(n)}}\right\|$$

$$\cdot\sum_{j=1}^{n}\left\|\sum_{k=1}^{n_j}\left(E_{j_k}^{(m)}\right) M_{E_{j_k}^{(m)}} - \Big(\sum_{k=1}^{n_j} E_{j_k}^{(m)}\Big) M_{\sum\limits_{k=1}^{n_j} E_{j_k}^{(m)}}\right\| \leqq$$

$$\leqq 2\sum_{j=1}^{n}\mu^2\Big(\sum_{k=1}^{n_j} E_{jk}^{(n_2)}\Big) = 2\sum_{j=1}^{n}\mu^2(E_j^{(n)}) \leqq 2\mu(E)\max_{1\leqq j\leqq n}\mu(E_j^{n}).$$

Theorem 2.3 ends the proof of the existence of $\lim_{n\to\infty} M(\mathbf{P}_n(E))$.

Now, we will show that $G_E$ does not depend on the choice of the fundamental sequence of partitions $\{\mathbf{P}_n(E), n \geqq 1\}$. We suppose that $\{\mathbf{Q}_n(E), n \geqq 1\}$ is another fundamental sequence of partitions of the set $E$.

Let $\mathbf{Z}_n(E)$ be the sequence of partitions of the set $E$ consisting of all sets of the form $E_j^{(n)} \cap F_j^{(n)}$, where $E_j^{(n)} \in \mathbf{P}_n(E)$, $F_j^{(n)} \in \mathbf{Q}_n(E)$.

It is easy to see that for all $n$, the partition $\mathbf{Z}_n(E)$ refines the partitions $\mathbf{P}_n(E)$ and $\mathbf{Q}_n(E)$. Hence, and from the first part of the proof, we have

$$\|M(\mathbf{P}_n(E)) - M(\mathbf{Q}_n(E))\| \leqq \|M(\mathbf{P}_n(E)) - M(\mathbf{Z}_n(E))\| +$$
$$+\|M(\mathbf{Z}_n(E)) - M(\mathbf{Q}_n(E))\| \leqq 2\mu(E)\left[\max\mu(E_j^{(n)}) + \max\mu(F_j^{(n)})\right].$$

Therefore $G_E$ does not depend on the choice of the fundamental sequence of partitions $\mathbf{P}_n(E)$.

In fact $G_E$ is countably additive, which follows from the fact that $M(\mathbf{P}(E))$ is uniformly countably additive [8].

In order to finish the proof of this Theorem, we will show that $G_E(\mathbf{U}\setminus\{\theta\}) = 1$.

Taking into account (3.1) and (3.5) it is sufficient to prove that

$$\lim_{n\to\infty} \sum_{E_j^{(n)}\in\mathbf{P}_n(E)} P\left[X(E_j^{(n)}) \notin \mathbf{U}\right] = 0. \tag{3.10}$$

To prove (3.10), we consider the following inequality

$$x \leqq -\ln(1-x).$$

Now putting $x = P\left[X(E_j^{(n)}) \notin \mathbf{U}\right]$, we obtain

$$0 \leqq \sum_{E_j^{(n)}\in\mathbf{P}_n(E)} P\left[X(E_j^{(n)}) \in \mathbf{U}\right] \leqq -\ln \prod_{E_j^{(n)}\in\mathbf{P}_n(E)} P\left[X(E_j^{(n)}) \in \mathbf{U}\right].$$

Finally (2.1) proves (3.10). The proof of Theorem 3.1 is now complete.

## 4. The main result

We can now state the main theorem of this paper.

THEOREM 4.1. *If $X$ is a completely random measure satisfying the assumptions* (a), (b), *then*

$$\mathcal{L}X(E) = \exp \mu(E)(G_E - I),$$

*where* $\mu$ *and* $G_E$ *are defined as above.*

First we will prove the following

LEMMA 4.1. *If* $X$ *is a completely random measure satisfying the assumptions* (a), (b), *then*

$$\|\mathcal{L}X(E) - \exp \lambda(\mathbf{P}_n(E))\,(M(\mathbf{P}_n(E)) - I)\| \leqq 2 \sum_{E_j^{(n)} \in \mathbf{P}_n(E)} \mu^2(E_j^{(n)}),$$

*where* $\lambda$ *and* $M$ *are defined by* (3.3) *and* (3.5), *respectively.*

PROOF. The method used in the proof of this Lemma is essentially the same as the one given by Le Cam [7].

For the sake of simplicity we assume that the fundamental sequence of partitions $\mathbf{P}_n(E)$ consists of $n$ sets. Let us put

$$F_{E^{(n)}} = \exp \delta(E_j^{(n)}) \left( M_{E_j^{(n)}} - I \right), \quad j = 1, 2, \dots, n$$

and let

$$R_{E_1^{(n)}} = \prod_{j=2}^{n} {}^{*}\mathcal{L}X(E_j^{(n)}), \quad R_{E_n^{(n)}} = \prod_{j=1}^{n-1} {}^{*}F_{E_j^{(n)}}.$$

For $1 < k < n$, we define

$$R_{E_k^{(n)}} = \Big( \prod_{1 \leqq j \leqq k-1} {}^{*}F_{E_j^{(n)}} \Big) * \Big( \prod_{j \geqq k+1}^{n} {}^{*}\mathcal{L}X(E_j^{(n)}) \Big).$$

Then, for $k = 1, 2, \dots, n$, we have

$$R_{E_k^{(n)}} * F_{E_k^{(n)}} = R_{E_{k+1}^{(n)}} * \mathcal{L}X(E_{k+1}^{(n)}).$$

Now, it is easy to see that

$$\prod_{j=1}^{n} {}^{*}\mathcal{L}X(E_j^{(n)}) - \prod_{j=1}^{n} {}^{*}F_{E_j^{(n)}} = \sum_{j=1}^{n} R_{E_j^{(n)}} * \left[ \mathcal{L}X(E_j^{(n)}) - F_{E_j^{(n)}} \right].$$

Since $R_{E_j^{(n)}}$ is a probability measure, this implies

$$\left\| \prod_{j=1}^{n} {}^{*}\mathcal{L}X(E_j^{(n)}) - \prod_{j=1}^{n} {}^{*}F_{E_j^{(n)}} \right\| \leqq \sum_{j=1}^{n} \left\| \mathcal{L}X(E_j^{(n)}) - F_{E_j^{(n)}} \right\|.$$

The difference $F_{E_j^{(n)}} - \mathcal{L}X(E_j^{(n)})$ can be written as follows

$$F_{E_j^{(n)}} - \mathcal{L}X(E_j^{(n)}) = \left[e^{-\delta(E_j^{(n)})} - 1 + \delta(E_j^{(n)})\right] I +$$

$$+\delta(E_j^{(n)}) \left(e^{-\delta(E_j^{(n)})} - 1\right) M_{E_j^{(n)}} + e^{-\delta(E_j^{(n)})} \sum_{k \geqq 2} \delta^k(E_j^{(n)}) \left(M_{E_j^{(n)}}\right)^{*k} / k!.$$

Hence

$$\left\| F_{E_j^{(n)}} - \mathcal{L}X(E_j^{(n)}) \right\| \leqq 2\delta(E_j^{(n)}) \left(1 - e^{-\delta(E_j^{(n)})}\right) \leqq 2\mu^2(E_j^{(n)}).$$

This proves the desired result.

PROOF of Theorem 4.1. At first we note that from Lemma 4.1, we have

$$\|\mathcal{L}X(E) - \exp\mu(E)(G_E - I)\| \leqq 2 \sum_{E_j^{(n)} \in \mathbf{P}_n(E)} \mu^2(E_j^{(n)}) +$$

$$+\| \exp\lambda(\mathbf{P}_n(E))[M(\mathbf{P}_n(E)) - I] - \exp\mu(E)(G_E - I)\|.$$

Taking into account (3.4) and Theorem 3.1 it is easy to see that the right-hand side of the last inequality tends to 0.

The proof of Theorem 4.1 is complete.

REMARK. If the topology $\tau$ is discrete, then the assumption (a) implies (b) and all the results are valid under (a).

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## REFERENCES

[1] BROWN, T. C. and KUPKA, J., Ramsey's theorem and Poisson random measures, *Ann. Probab.* **11** (1983), 904–908. *MR* **85c:** 60074

[2] GIHMAN, I. I. and SKOROHOD, A. V., *The Theory of Stochastic Processes* I, Die Grundlehren der mathematischen Wissenschaften, Band 210, Springer, New York, 1974. *MR* **49** # 11603

[3] GÓŹDŹ, CZ. and POLAK, M., An approximation theorem for the Poisson processes defined on an abstract space, *Studia Sci. Math. Hungar.* **22** (1987), 169–174. *MR* **89b:** 60128

[4] GYŐRFI, L., Poisson processes defined on an abstract space, *Studia Sci. Math. Hungar.* **7** (1972), 243–248. *MR* **48** #7365

[5] KAWAMURA, K., The structure of bivariate Poisson distribution, *Kōdai Math. Sem. Rep.* **25** (1973), 246–256. *MR* **48** #5143

[6] KINGMAN, J. F. C., Completely random measures, *Pacific J. Math.* **21** (1967), 59–78. *MR* **35** #1079

[7] LE CAM, L., An approximation theorem for the Poisson binomial distribution, *Pacific J. Math.* **10** (1960), 1181–1197. *MR* **25** #5567

[8] NIKODYM, O., Contribution à la théorie des fonctionnelles linéaires en connexion avec la théorie de la mesure des ensembles abstraits, *Mathematica (Cluj)* **5** (1931), 130–141. *Zbl* **2**, 252

[9] ODELL, E., Applications of Ramsey theorems to Banach space theory, *Notes in Banach spaces*, Univ. Texas Press, Austin, TX, 1980, 379–404. *MR* **83g:** 46018

[10] POLAK, M., Bivariate Poisson processes defined on an abstract space, *Studia Sci. Math. Hungar.* **17** (1982), 409–416. *MR* **85j:** 60089

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