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KANTENISOMORPHISMEN VON MENGENSYSTEMEN

von

W. DÖRFLER

In der Arbeit [3] (sh. auch [2]) von H. WHITNEY werden Kantenisomorphismen von endlichen Graphen betrachtet und es wird gezeigt, daß mit Ausnahme gewisser Graphen aus der Beziehung, kantenisomorph zu sein, die Isomorphie der Graphen folgt. Das entsprechende Resultat hat JUNG [1] für den Fall unendlicher Graphen bewiesen. In der vorliegenden Arbeit wird zunächst der Begriff der Kantenisomorphie für Mengensysteme erklärt und ein dem Satz von Whitney entsprechendes Resultat bewiesen.

Definition. Ein Mengensystem X besteht aus einer Menge $V=V(X)$ und einer Menge $E=E(X)$ von Teilmengen $a \subset V$ mit $|a| \geq 2$. Die Elemente von V heißen Knoten von X , die Elemente von E Kanten von X . Gibt es nur Kanten mit 2 Knoten, so erhält man einen Graphen.

Ein Knoten, der in keiner Kante von X liegt, heißt isoliert. Ist $|V(X)| < \infty$, so heißt X endlich. Wir beschränken uns auf endliche Mengensysteme, denn eine Übertragung der Ergebnisse auf den unendlichen Fall ist hier nicht möglich.

Ein Mengensystem X heißt zusammenhängend, wenn es zu je zwei Knoten $x, y \in V$ eine Folge a_1, a_2, \dots, a_k von Kanten in X gibt mit $x \in a_1, y \in a_k$ und $a_i \cap a_{i+1} \neq \emptyset$ für $i=1, 2, \dots, k-1$.

Definition. Eine bijektive Abbildung $\varphi: V(X) \rightarrow V(Y)$ heißt ein Isomorphismus vom Mengensystem X auf das Mengensystem Y , wenn $a \subset V(X)$ genau dann eine Kante in X ist, wenn φa eine Kante in Y ist. Gibt es einen Isomorphismus $\varphi: X \rightarrow Y$, so sind X und Y isomorph, $X \cong Y$.

Definition. Eine bijektive Abbildung $\psi: E(X) \rightarrow E(Y)$ heißt ein Kantenisomorphismus von X auf Y , wenn gilt

(I) für alle $a \in E(X)$ ist $|a| = |\psi a|$

(II) für alle $a, b \in E(X)$ ist $|a \cap b| = |\psi a \cap \psi b|$.

Es ist klar, daß für jeden Isomorphismus $\varphi: X \rightarrow Y$ die Abbildung $\psi: E(X) \rightarrow E(Y)$, definiert durch $\psi a := \varphi a$ für alle $a \in E(X)$, ein Kantenisomorphismus ist. Gilt diese Beziehung, so heißt ψ der von φ induzierte Kantenisomorphismus. Es gibt jedoch Kantenisomorphismen, die nicht von Isomorphismen induziert werden können.

Beispiel. $V(X) = \{1, 2, 3, \dots, 7\}$, $E(X) = \{\{1, 2, 3, 4\}, \{4, 5, 6\}, \{1, 2, 5, 6, 7\}\}$, $V(Y) = V(X) \cup \{8\}$, $E(Y) = \{\{1, 2, 3, 5\}, \{1, 4, 7\}, \{1, 5, 6, 7, 8\}\}$. Es sei $\psi: E(X) \rightarrow E(Y)$ gegeben durch: $\psi \{1, 2, 3, 4\} = \{1, 2, 3, 5\}$, $\psi \{4, 5, 6\} = \{1, 4, 7\}$, $\psi \{1, 2, 5, 6, 7\} = \{1, 5, 6, 7, 8\}$.

Es soll zunächst die Frage geklärt werden, wann ein induzierter Kantenisomorphismus durch genau einen Isomorphismus induziert wird. Dazu benötigen wir den folgenden Begriff.

Definition. Mit $S(X)$ bezeichnen wir folgende Relation auf der Knotenmenge eines Mengensystems X :

$(x, y) \in S$ genau dann, wenn (I)—(III) gelten:

(I) es gibt ein $a \in E(X)$ mit $x, y \in a$,

(II) aus $x \in b$, $b \in E(X)$ folgt $y \in b$,

(III) aus $y \in b$, $b \in E(X)$ folgt $x \in b$,

oder wenn $x=y$ ist.

Man sieht leicht, daß S eine Äquivalenzrelation auf $V(X)$ ist. Hat X keine isolierten Knoten, so gilt:

LEMMA. Die Äquivalenzklassen nach S sind Durchschnitte von Kanten von X und von Komplementen von Kanten von X .

BEWEIS. Es sei $x \in V(X)$ beliebig und a_1, \dots, a_n seien die Kanten von X mit $x \in a_i$, $i=1, \dots, n$ und $b_j, j=1, \dots, m$ die Kanten von X mit $x \notin b_j$. Wir zeigen, daß $D = \bigcap_{i=1}^n a_i \cap \bigcap_{j=1}^m \bar{b}_j$ die S -Klasse x/S von x ist. Dabei ist $\bar{b}_j = V(X) - b_j$. Zunächst zeigen wir, daß $D \subset x/S$ ist. Für $D = \{x\}$ ist nichts zu zeigen. Sei also $z \neq x$ in D . Dann ist $z \in a_i$, $x \in a_i$ und es gilt (I). Die Bedingung (II) ist nach Definition von D erfüllt. Ist $z \in b$, $b \in E(X)$, so muß nach Definition von D $b = a_i$ sein für ein i , also ist auch $x \in b$ und (III) erfüllt. Daher gilt $(x, z) \in S$. Für die Umkehrung nehmen wir an, daß für $z \in V(X)$ gilt $(z, x) \in S$. Da $x \in a_i$ ist für alle $i=1, \dots, n$, folgt, daß auch $z \in a_i$, $i=1, \dots, n$, gilt. Ebenso folgt aus $x \notin b_j$, daß auch $z \notin b_j$ ist, also ist $z \in \bar{b}_j$. Daher muß $z \in D$ sein.

SATZ 1. Ist X ein Mengensystem ohne isolierte Knoten und die Relation $S(X)$ die identische Relation, so wird jeder Kantenisomorphismus von X auf ein Mengensystem Y von höchstens einem Isomorphismus induziert.

BEWEIS. Es sei also $S(X) = \{(x, x) | x \in V(X)\}$ und φ_1, φ_2 seien zwei Isomorphismen von X auf Y , die den Kantenisomorphismus $\psi: E(X) \rightarrow E(Y)$ induzieren. Wir wählen einen beliebigen Knoten $x \in V(X)$ und es seien $a_i, i=1, \dots, n$, die Kanten von X mit $x \in a_i$ und b_j die Kanten mit $x \notin b_j, j=1, \dots, m$. Dann ist nach dem Lemma und nach der Voraussetzung des Satzes

$$x/S = \bigcap_{i=1}^n a_i \cap \bigcap_{j=1}^m \bar{b}_j.$$

Da φ_1, φ_2 beide ψ induzieren, ist für alle $a \in E(X)$ stets $\varphi_1 a = \psi a = \varphi_2 a$ und somit gilt

$$\begin{aligned} \varphi_1 x &= \varphi_1 \left(\bigcap_{i=1}^n a_i \cap \bigcap_{j=1}^m \bar{b}_j \right) = \bigcap_{i=1}^n \varphi_1 a_i \cap \bigcap_{j=1}^m \varphi_1 \bar{b}_j = \\ &= \bigcap_{i=1}^n \varphi_2 a_i \cap \bigcap_{j=1}^m \varphi_2 \bar{b}_j = \varphi_2 \left(\bigcap_{i=1}^n a_i \cap \bigcap_{j=1}^m \bar{b}_j \right) = \varphi_2 x. \end{aligned}$$

Also ist $\varphi_1 = \varphi_2$.

Bemerkung. Ist $S(X)$ nicht die identische Relation auf $V(X)$ und $\psi: E(X) \rightarrow E(Y)$ ein induzierter Kantenisomorphismus, so kann ψ sicher durch mindestens zwei Isomorphismen induziert werden. Das ergibt sich sofort daraus, daß für $x, y \in x/S$ die Abbildung φ mit $\varphi x = y$, $\varphi y = x$ und $\varphi z = z$, $z \neq x, y$ ein Automorphismus von X ist.

SATZ 2. Sind X und Y Mengensysteme ohne isolierte Knoten und gibt es einen Kantenisomorphismus von X auf Y , so sind X und Y entweder beide zusammenhängend oder beide nicht zusammenhängend.

BEWEIS. Es sei $\psi: E(X) \rightarrow E(Y)$ ein Kantenisomorphismus und X sei zusammenhängend. Wir wählen $x \neq y \in V(Y)$ beliebig und dazu $b, b' \in E(Y)$ mit $x \in b$, $y \in b'$. Es seien $a = \psi^{-1}b$, $a' = \psi^{-1}b'$ und $z_1 \in a$, $z_2 \in a'$. Dann gibt es eine Kantenfolge $a = a_1, a_2, \dots, a_n = a'$ mit $a_i \cap a_{i+1} \neq \emptyset$, $i = 1, \dots, n-1$. Dann bilden die Kanten $\psi a_i = b_i$ die gesuchte Kantenfolge in Y , die x mit y verbindet. Die Umkehrung wird ebenso bewiesen.

Definition. Ist X ein Mengensystem, so sei X/S das folgende Mengensystem

$$V(X/S) = V(X)/S(X) = \{z | z \text{ ist Äquivalenzklasse nach } S(X)\}$$

$$E(X/S) = \{a | a \subset V(X/S), \{x | x \in V(X) \wedge x \in z \wedge z \in a\} \in E(X)\}.$$

SATZ 3. Sind X und Y zwei Mengensysteme, so ist $X \cong Y$ genau dann, wenn es einen Isomorphismus $\psi: X/S \rightarrow Y/S$ gibt mit

$$|\psi z| = |z| \quad \text{für alle } z \in V(X/S).$$

BEWEIS. Ist $X \cong Y$ und $\varphi: X \rightarrow Y$ ein Isomorphismus, so ist die Abbildung ψ mit $\psi z := \varphi z$ für alle $z \in V(X/S)$ ein Isomorphismus von X/S auf Y/S und natürlich gilt für ψ , daß $|\psi z| = |z|$, $z \in V(X/S)$ ist.

Es sei nun umgekehrt ψ ein Isomorphismus von X/S auf Y/S mit der angegebenen Eigenschaft. Für jedes $z \in V(X/S)$ sei φ_z eine bijektive Abbildung von z auf ψz . Durch

$$\varphi x := \varphi_z x \quad \text{für } x \in z$$

wird dann ein Isomorphismus von X auf Y definiert.

Wir kommen jetzt im Satz 4 zur Formulierung einer hinreichenden Bedingung dafür, daß ein Kantenisomorphismus durch einen Isomorphismus induziert wird. Diese Bedingung ist nicht notwendig. Es lassen sich jedoch leicht Beispiele finden, daß es Kantenisomorphismen gibt, die nicht induziert werden können, wenn diese Bedingung nicht erfüllt ist.

Definition. Mit L bezeichnen wir die Klasse aller endlichen Mengensysteme X mit folgender Eigenschaft: Sind a_1, a_2, \dots, a_n $n \geq 3$ Kanten in X mit $\bigcap_{i=1}^n a_i \neq \emptyset$, so gibt es ein i , $1 \leq i \leq n$, und eine von den a_j verschiedene Kante d in X derart, daß $d \cap a_i = a_i - \bigcup_{j \neq i} a_j \neq \emptyset$ ist.

und auf Grund der Induktionsvoraussetzung

$$|\bar{N}(i_1, \dots, i_t; j_1, \dots, j_t)| = |N(i_1, \dots, i_t; j_1, \dots, j_t)| + (-1)^t s$$

wobei $t = k - 2$ die Anzahl der Stellen nach dem Strichpunkt ist, und das gilt für jede Auswahl verschiedener Kanten a_{i_1}, \dots, a_{i_t} und für $1 \leq t \leq k - 2$. Zum Beweis sei zuerst $t = 1$. Dann ist mit (A)

$$|a_{i_1} \cap a_{i_2} \cap \dots \cap a_{i_{k-1}}| = |N(i_1, i_2, \dots, i_{k-1}; i_k)| + \left| \prod_{i=1}^k a_i \right|$$

und

$$|\psi a_{i_1} \cap \psi a_{i_2} \cap \dots \cap \psi a_{i_{k-1}}| = |\bar{N}(i_1, i_2, \dots, i_{k-1}; i_k)| + \left| \prod_{i=1}^k \psi a_i \right|$$

und die behauptete Beziehung folgt, weil die linken Seiten gleich sind. Ist diese Beziehung schon gezeigt für $t \leq m - 1 \leq k - 2$, so erhält man aus (A) wieder unter Verwendung der obigen Induktionsannahme ($r = k - (m - 1)$)

$$\begin{aligned} & |N(i_1, i_2, \dots, i_r; j_1, j_2, \dots, j_m)| + \sum |N(i_1, \dots, i_r, i_{r+1}; j'_1, \dots, j'_{m-1})| + \\ & + \sum |N(i_1, \dots, i_r, i_{r+1}, i_{r+2}; j''_1, \dots, j''_{m-2})| + \dots + \sum |N(i_1, \dots, i_r, \dots, i_{k-1}; j)| + \\ & + \left| \prod_{i=1}^k a_i \right| = |\bar{N}(i_1, i_2, \dots, i_r; j_1, j_2, \dots, j_m)| + \\ & + \sum |N(i_1, \dots, i_r, i_{r+1}; j'_1, \dots, j'_{m-1})| + (-1)^{m-1} s \binom{m}{m-1} + \\ & + \sum |N(i_1, \dots, i_r, i_{r+1}, i_{r+2}; j''_1, \dots, j''_{m-2})| + (-1)^{m-2} s \binom{m}{m-2} + \\ & + \dots + \sum |N(i_1, \dots, i_r, \dots, i_{k-1}; j)| + (-1)^s \binom{m}{1} + \left| \prod_{i=1}^k a_i \right| + s \end{aligned}$$

woraus sofort die Behauptung folgt.

Es kann nun sein, daß hier schon ein Widerspruch auftritt, wenn sich aus der Gleichung für N und \bar{N} für ein t bereits negative Kardinalzahlen ergeben würden. Andernfalls erhält man wegen $|a_1| = |\psi a_1|$ und (A) mit den Gleichungen zwischen den N und \bar{N} :

$$\begin{aligned} & |\bar{N}(1; 2, 3, \dots, k)| - |N(1; 2, 3, \dots, k)| = \\ & = (-s) \left(1 - \binom{k-1}{1} + \binom{k-1}{2} - \binom{k-1}{3} \pm \dots + (-1)^{k-2} \binom{k-1}{k-2} \right) = s \cdot (-1)^{k-1} = \Delta. \end{aligned}$$

Für k ungerade und $s < 0$ bzw. für k gerade und $s > 0$ ist $\Delta < 0$. Das ist jedoch ein Widerspruch zu folgenden Beziehungen:

$$\begin{aligned} |\psi d \cap \psi a_i| &= |d \cap a_i| = 0 \quad \text{für } i = 2, \dots, k \quad \text{und} \\ |N(1; 2, 3, \dots, k)| &= |d \cap a_1| = |\psi d \cap \psi a_1|. \end{aligned}$$

Ist jedoch k gerade, $s < 0$ oder k ungerade, $s > 0$, so genügt es, die Rollen von X und

Y zu vertauschen, um den gleichen Widerspruch zu erhalten (im allgemeinen mit einem $b_i = \psi a_i$ mit $i \neq 1$), denn $\bigcap_{i=1}^k \psi a_i$ kann laut Voraussetzung nicht leer sein.

Als nächstes zeigen wir, daß für je n Kanten a_1, \dots, a_n in X und alle Auswahlen verschiedener Indizes i_1, \dots, i_r aus $\{1, \dots, n\}$ stets gilt

$$(*) \quad |N(i_1, \dots, i_r; j_1, \dots, j_t)| = |\bar{N}(i_1, \dots, i_r; j_1, \dots, j_t)|.$$

Denn wäre dies nicht der Fall, so gibt es ein kleinstes $t \geq 1$, für das $(*)$ nicht gilt für eine bestimmte Auswahl $\{i_1, \dots, i_r\}$, $r = n - t$. Wendet man jetzt die Beziehung (A) auf die dieser Auswahl entsprechenden a_{i_1}, \dots, a_{i_r} , an, so erhält man einen Widerspruch zu

$$\left| \bigcap_{j=1}^r a_{i_j} \right| = \left| \bigcap_{j=1}^r \psi a_{i_j} \right|.$$

Nach dem Lemma ist jedes Element aus X/S von der Form $N(i_1, \dots, i_k; j_1, \dots, j_t)$ für bestimmte $a_{i_1}, \dots, a_{i_k}; a_{j_1}, \dots, a_{j_t}$ und man sieht leicht, daß die Zuordnung

$$\tilde{\varphi} N(i_1, \dots, i_k; j_1, \dots, j_t) := \bar{N}(i_1, \dots, i_k; j_1, \dots, j_t)$$

ein Isomorphismus von X/S auf Y/S ist mit

$$|\tilde{\varphi} z| = |z| \quad \text{für alle } z \in V(X/S).$$

Konstruiert man wie in Satz 2 einen Isomorphismus von X auf Y , indem man von $\tilde{\varphi}$ ausgeht, so erhält man einen Isomorphismus, der den Kantenisomorphismus ψ induziert. Damit ist der Satz bewiesen, wenn man beachtet, daß X und Y beide keine isolierten Knoten besitzen.

Die im Satz 4 gemachte Voraussetzung über den Kantenisomorphismus ψ geht dort ein, wo für gerades k gezeigt wird, daß $\left| \bigcap_{i=1}^k a_i \right| = \left| \bigcap_{i=1}^k \psi a_i \right|$ ist. Es genügt daher diese Bedingung für gerades n zu fordern. In der Klasse L kann man Mengensysteme X und Y und einen Kantenisomorphismus $\psi: X \rightarrow Y$ angeben, der n Kanten $a_i \in E(X)$, n gerade, mit nichtleerem Durchschnitt auf Kanten $\psi a_i \in E(Y)$ abbildet, die einen leeren Durchschnitt haben. Man könnte versuchen, diesen Mangel zu beseitigen indem man in der Definition von L nicht mehr $\bigcap_{i=1}^n a_i \neq \emptyset$ voraussetzt. Das führt jedoch zu einer zu starken Einengung der Klasse L , die dann Typen von Mengensystemen nicht enthält, für die trivialerweise jeder Kantenisomorphismus induzierbar ist. Eine andere Lösung bringt die folgende Definition.

Definition. Die Klasse L_0 von Mengensystemen X ist diejenige Teilklasse von L , in der folgende Bedingung erfüllt ist:

Sind a_j , $j=1, 2, \dots, n$, $n \geq 4$ und gerade, Kanten aus $E(X)$ mit $\bigcap_{j=1}^n a_j \neq \emptyset$, so gibt es ein Paar $i_1 \neq i_2$ von Indizes aus $\{1, 2, \dots, n\}$, sodaß gilt

$$\left| \bigcap_{j=1}^n a_j \right| > \left| \bigcap_{j \neq i_1, i_2} a_j - \{a_{i_1} \cup a_{i_2}\} \right|.$$

SATZ 5. Sind X und Y zwei Mengensysteme ohne isolierte Knoten aus der Klasse L_0 und $\psi: E(X) \rightarrow E(Y)$ ein Kantenisomorphismus, so gibt es einen Isomorphismus $\varphi: X \rightarrow Y$, der ψ induziert.

BEWEIS. Um den Beweis von Satz 4 anwenden zu können, genügt es zu zeigen, daß ψ Kanten aus $E(X)$ mit nichtleerem Durchschnitt in Kanten aus $E(Y)$ mit nichtleerem Durchschnitt überführt. Für $n=2$ gilt dies nach Definition eines Kantenisomorphismus. Es sei also richtig für $n \leq k-1$ Kanten. Wir nehmen an, daß für gewisse k Kanten $a_i, i=1, \dots, k$, mit $\bigcap_{i=1}^k a_i \neq \emptyset$ der Durchschnitt $\bigcap_{i=1}^k \psi a_i = \emptyset$ ist. Ist

k ungerade, so ergeben die Überlegungen im Beweis von Satz 4 wegen $s = -\left| \bigcap_{i=1}^k a_i \right| < 0$ einen Widerspruch, weil X und Y auch in L liegen. Ist k gerade, so verwenden wir, daß X und Y in L_0 sind. Es gilt mit den Bezeichnungen und den Ergebnissen aus dem Beweis von Satz 4 für das nach Definition von L_0 existierende Paar $i_1 \neq i_2$ aus $\{1, \dots, k\}$ wegen $\left| \bigcap_{i=1}^k a_i \right| = -s$ die Beziehung

$$|\bar{N}(j_1, j_2, \dots, j_{k-2}; i_1, i_2)| = |N(j_1, j_2, \dots, j_{k-2}; i_1, i_2)| - \left| \bigcap_{i=1}^k a_i \right|$$

aus der folgt

$$|\bar{N}(j_1, j_2, \dots, j_{k-2}; i_1, i_2)| < 0.$$

Dabei ist $\{j_1, j_2, \dots, j_{k-2}\} = \{1, 2, \dots, k\} - \{i_1, i_2\}$. Wir sind also wieder zu einem Widerspruch gelangt. Die Relation zwischen \bar{N} und N kann man anwenden, weil aus der Induktionsannahme bereits folgt, daß für $n \leq k-1$ durch ψ die Kardinalzahlen der Durchschnitte von $n \leq k-1$ Kanten aus $E(X)$ nicht verändert werden.

Somit ist die Voraussetzung von Satz 4 erfüllt, und daraus ergibt sich der Beweis von Satz 5.

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SETS OF LATTICE POINTS THAT FORM NO SQUARES

by

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Consider the lattice points in a square of size $n \times n$, i.e. consider e.g. the points (i, j) in the Cartesian plane where $0 \leq i, j \leq n$. P. ERDŐS and R. GRAHAM [1] conjectured that given any positive constant c , any set of cn^2 elements from among the above lattice points contains four points that form a square the sides of which are parallel to the coordinate axes provided n is large enough, depending on c . We are unable to settle this problem here. Instead we consider the easier case of isosceles right-angled triangles whose sides are parallel to the axes; the consideration of this case was also suggested by ERDŐS and GRAHAM. First we prove the following

THEOREM 1. *There is a positive absolute constant c such that, given any positive integer n , there are $c(r_3(n))^2$ lattice points in a square of size $n \times n$ no three of which form an isosceles right-angled triangle whose sides are parallel to the axes.*

Here $r_3(n)$ denotes the largest integer m such that there are m integers less than n no three of which form an arithmetic progression. According to a result of F. BEHREND, we have $r_3(n) > n^{1-c/\sqrt{\log n}}$.

PROOF. Let a_1, \dots, a_k ($k = r_3([n/2])$, where $[\cdot]$ stands for the integral part of the bracketed number) be a sequence of integers less than n containing no arithmetic progression of three elements, and consider the set S of those lattice points in the square $0 \leq x, y \leq n$ that are in the intersection of lines of form $y = x + 2a_i$; and $y = -x + 2a_j$; ($1 \leq i, j \leq k$). As any two such line intersects each other in a lattice point, it is easy to see that S has at least $k^2/4$ elements. As $r_3([n/2])$ and $r_3(n)$ have the same orders of magnitude, it is clear that the cardinality of S is at least $c(r_3(n))^2$, where c is an easily computable absolute constant. We show that there are no three points belonging to S that form a right-angled isosceles triangle with sides parallel to the axes. Assume, on the contrary, that ABC is such a triangle, say $\overline{AC} = \overline{BC}$. We may assume, without loss of generality, that each of the points A, B , and C lies on different lines of form $y = x + 2a_i$; (the other possibility, when each of A, B , and C lies on different lines of form $y = -x + 2a_i$ can be dealt with similarly); say A lies on the line $y = x + 2a_r$, C on $y = x + 2a_s$, and B on $y = x + 2a_t$. Then a_r, a_s , and a_t form an arithmetic progression, which is a contradiction. The proof complete.

In the other direction, we prove the following:

THEOREM 2. *Let $c > 0$, and let S be a set of cn^2 of lattice points in a square of size $n \times n$. Then there are three points in S that form an isosceles right-angled triangle whose sides are parallel to the axes, provided n is large enough.*

PROOF. Assume that the assertion of the theorem fails, and let c_n be the largest constant such that there are $c_n n^2$ points that violate the assertion of the theorem, and put $c = \varliminf_{n \rightarrow \infty} c_n$. Then $c > 0$ holds according to our assumption. Let n be a large

integer, and let S be a set of $(c-\varepsilon)n^2$ points in a square of size $n \times n$ such that S does not contain an isosceles triangle whose sides are parallel to the axes. We are going to obtain a contradiction by showing that there is a square of size $k \times k$ that contains $(c+\varepsilon)k^2$ elements of S (i.e. there are a_1, a_2 , and d such that there are $(c+\varepsilon)k^2$ elements of S of form (a_1+id, a_2+jd) with $0 \leq i, j < k$).

First we show that, almost every line $x=i$ ($0 \leq i < n$) contains at most $(c+\sqrt{\varepsilon})n$ elements of S , i.e. the number of exceptional i 's is $o(n)$. Let us consider the lines $x=i$ for $t\sqrt{n} \leq i < (t+1)\sqrt{n}$, where $0 \leq t < \sqrt{n}$. It is enough to show that the number of exceptional i 's satisfying this condition is $o(\sqrt{n})$. In fact, assume, on the contrary, that there is a positive constant η independent of n that there are $\eta\sqrt{n}$ exceptional i 's with $t\sqrt{n} \leq i < (t+1)\sqrt{n}$ for some $0 \leq t < \sqrt{n}$. The set H of exceptional i 's satisfying this condition can contain an arithmetic progression $a+jd$ ($0 \leq j < k$) of length k , where k can be arbitrarily large, depending on n ; note that we use the following theorem here: an infinite sequence of integers of positive upper density contains an arbitrarily long arithmetic progression (see [2]). Now split the set of positive integers $\leq n$ as $T_0 \cup \bigcup_{0 < l < n/k} T_l$, T_l is an arithmetic progression of length k with difference

d and T_0 contains at most $\eta \cdot n$ elements (note that $d < \sqrt{n}$). Then there is an l so that $A_l = T_l \times \{a+jd: 0 \leq j < k\}$ contains at least $(c+2\varepsilon)k^2$ elements of S , since each $a+jd$ was exceptional. If k is large enough, then in view of the definition of c , the set $A_l \cap S$ contains an isosceles triangle with sides parallel to the axes, which contradicts the definition of S .

So we have shown that, loosely speaking only very few lines $x=i$ ($0 \leq i < n$) can contain "more than the average" number of elements of S ; therefore, only a few lines contain "less than the average" number of elements of S ; more precisely, an easy computation gives that there are less than $2\sqrt{\varepsilon}n$ lines that contain more than $(c+\varepsilon^{1/4})n$ elements of S .

Now consider the lines $y=x+h$, where $-n \leq h \leq n$. If ε is small enough, say $\varepsilon < c/100$ then there is such a line with $h=h_0$ containing $cn/4$ elements of S . Omit those points which lie on a vertical line containing less than $(c-\varepsilon^{1/4})n$ elements of S ; there still remain say $cn/6$ points. Let these elements of S have abscissas a_i ($0 \leq i < cn/6$). According to a theorem cited just before, there is an arithmetic progression $a+jd$ of length k each element of which is some a_i ; and we may also assume that $d < \sqrt{n}$. Now decompose the set of integers $\leq n$ as $T_0 \cup \bigcup_{0 < l < n/k} T_l$ as above (with the new parameters d and k).

The rectangle $[0, n] \times \{a+jd: 0 \leq j < k\}$ contains at least $(c-\sqrt{\varepsilon})nk$ points of S as each line $x=a+jd$ contains at least $(c-\sqrt{\varepsilon})n$; no square $T_l \times \{a+jd: 0 \leq j < k\}$ may contain more than $(c+\varepsilon)k^2$ points if k is large enough in view of the definition of c . So almost every T_l (i.e. except at most $\varepsilon^{1/4} \frac{n}{k}$) contains at least $(c-\varepsilon^{1/4})k^2$ elements of S . We may assume that all of them do, as those T_l 's which do not, may be added to T_0 , which will thus have cardinality $\leq 2\varepsilon^{1/4}n$. There is an l such that T_l contains at least $ck/8$ elements of a_i+h_0 . Note that no line $y=a_i+h_0$ can contain an element with abscissa a_j for any j (so, in particular, with abscissa belonging to the arithmetic progression $a+jd$) as then (a_i, a_i+h) , (a_i, a_j) , (a_j, a_j+h) would be an isosceles triangle. This means that at least $ck/8$ lines of the square

$$T_l \times \{a+jd: 0 \leq j < k\}$$

contains no element of S which is a contradiction, provided k is large enough as we saw that almost every line of a square must contain many elements of S . (We proved this for the original square n , provided S is large enough; but these conditions hold for the above smaller square too, as it contains at least $(c - \varepsilon^{1/4})k^2$ elements of S .) The proof is complete.

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EINE ANWENDUNG DES MAXIMUMPRINZIPES
VON L. S. PONTRJAGIN AUF EIN LAGRANGE —
PROBLEM DER KLASSISCHEN VARIATIONSRECHNUNG

von
H. BERNAU

Die Theorie der optimalen Prozesse wurde in den letzten Jahren sehr stark weiterentwickelt, und es ist bekannt, dass sie als eine allgemeine Erweiterung der Theorie der klassischen Variationsrechnung betrachtet werden kann. So kann man zu den Problemen der klassischen Variationsrechnung analoge Probleme in der Theorie der optimalen Prozesse betrachten. In diesen Problemklassen sind die entsprechenden Probleme der Variationsrechnung als Spezialfälle für das Bewegungssystem

$$\frac{dy}{dt} = u$$

enthalten, und die entsprechenden Bedingungen der Variationsrechnung können aus den allgemeineren Bedingungen der Theorie der optimalen Prozesse hergeleitet werden. In dieser Hinsicht sind besonders die Resultate von M. R. HESTENES zu erwähnen. Doch werden dabei für die Variationsprobleme im allgemeinen keine neuen Resultate erzielt. Weiterhin ist man bei dieser Betrachtungsweise gezwungen, auf Grund der Struktur der entsprechenden Steuerungsprobleme, als Menge der zulässigen Kurven, stets die Menge der stückweise glatten Kurven, zu betrachten. Im folgenden wird gezeigt, dass die Euler—Lagrange'schen Gleichungen und die Weierstrass-Bedingung für das Lagrange-Problem mit gewöhnlichen Gleichungen

$$\min \int_a^b F(x, y_i(x), y_i'(x)) dx$$

$$y_i(a) = A_i, \quad y_i(b) = B_i, \quad \text{für } i = 1, \dots, n,$$

$$\varphi_j(x, y_i) = 0, \quad \text{für } j = 1, \dots, m, \quad m < n,$$

direkt aus dem Maximumprinzip von L. S. PONTRJAGIN hergeleitet werden können, ohne eine Verallgemeinerung der dort betrachteten Steuerungsprobleme vorzunehmen. Dabei ist es möglich, diese notwendigen Bedingungen in der Klasse der glatten Kurven herzuleiten. Weiterhin können auf diese Weise die Lagrange'schen Multiplikatoren explizit bestimmt werden, und für die Weierstrass-Bedingung kann eine anschauliche geometrische Interpretation gegeben werden. Dazu wird zuerst ein Lemma bewiesen, welches eine Einschränkung auf die Klasse der glatten Kurven bei diesem Variationsproblem ermöglicht.

Es wird das folgende Variationsproblem betrachtet:

$$\min I(\gamma) = \min \int_a^b F(x, y_i(x), y_i'(x)) dx,$$

in Bezug auf alle glatten Kurven

$$\gamma = \{(x, y_i) \in E_{n+1} | y_i = y_i(x), x \in [a, b], i = 1, \dots, n\},$$

die die gegebenen Punkte

$$P_1 = (a, A_1, \dots, A_n), \quad P_2 = (b, B_1, \dots, B_n)$$

verbinden, d. h. die den Randbedingungen

$$y_i(a) = A_i, \quad y_i(b) = B_i, \quad \text{für } i = 1, \dots, n,$$

genügen, und für die zusätzlich

$$(1) \quad |y_i(x) - A_i| < \delta, \quad \text{für } x \in [a, b] \quad \text{und } i = 1, \dots, n,$$

gilt, wobei $\delta > 0$ eine vorgegebene Zahl ist. Über die Funktion $F(x, y_i, y_i')$ wird vorausgesetzt, dass sie stetige partielle Ableitungen bis zur zweiten Ordnung nach allen Variablen hat. Damit dieses Problem sinnvoll ist, muss man natürlich annehmen, dass

$$|B_i - A_i| < \delta, \quad \text{für } i = 1, \dots, n,$$

ist. Es gilt dann das folgende Lemma.

LEMMA. Wenn das oben definierte Variationsproblem eine Lösungskurve

$$\hat{\gamma} = \{(x, y_i) \in E_{n+1} | y_i = \hat{y}_i(x), x \in [a, b], i = 1, \dots, n\}$$

besitzt, so ist die glatte Kurve $\hat{\gamma}$ auch optimal in Bezug auf alle stückweise glatten Kurven, die die Punkte P_1 und P_2 verbinden und zusätzlich der Bedingung (1) genügen.

BEWEIS. Der Beweis erfolgt indirekt. Angenommen, es existiert eine stückweise glatte Kurve

$$\bar{\gamma} = \{(x, y_i) \in E_{n+1} | y_i = \bar{y}_i(x), x \in [a, b], i = 1, \dots, n\}$$

mit

$$(2) \quad \bar{y}_i(a) = A_i, \quad \bar{y}_i(b) = B_i,$$

$$|\bar{y}_i(x) - A_i| < \delta, \quad \text{für } x \in [a, b] \quad \text{und } i = 1, \dots, n$$

so dass

$$(3) \quad I(\bar{\gamma}) < I(\hat{\gamma})$$

ist. Es wird gezeigt, dass man die stückweise glatte Kurve $\bar{\gamma}$ in der Form durch eine glatte Kurve $\tilde{\gamma}$ approximieren kann, so dass diese Kurve ebenfalls für das Variationsproblem zulässig ist und $I(\tilde{\gamma}) < I(\hat{\gamma})$ im Widerspruch zur Optimalität der Kurve $\hat{\gamma}$ gilt. Seien

$$(4) \quad a < c_1 < c_2 \dots < c_k < b$$

die x -Werte, in denen wenigstens eine der Funktionen $\bar{y}'_i(x)$ eine Unstetigkeitsstelle 1. Art besitzt, d. h. in denen

$$\bar{y}'_i(c_j-0) \neq \bar{y}'_i(c_j+0)$$

für mindestens ein $i \in \{1, \dots, n\}$ ist. Die Approximation erfolgt komponentenweise für jede Funktion $\bar{y}_i(x)$ und soll am Beispiel von $\bar{y}_1(x)$ erläutert werden. Nach (4) kann man ein $\varepsilon_0 > 0$ so wählen, dass die Intervalle

$$[c_j, c_j + \varepsilon_0], \quad \text{für } j = 1, \dots, k$$

paarweise disjunkt sind, sowie

$$[c_j, c_j + \varepsilon_0] \subset [a, b], \quad \text{für } j = 1, \dots, k,$$

und dass weiterhin die Differenzenquotienten

$$\frac{\bar{y}_i(c_j + \varepsilon) - \bar{y}_i(c_j)}{\varepsilon}, \quad \text{für } j = 1, \dots, k, \quad i = 1, \dots, n,$$

für jedes $\varepsilon \in (0, \varepsilon_0]$ beschränkt sind, da die Grenzwerte $\bar{y}'_i(c_j+0)$ endlich sind. Weiterhin gilt, da die Intervalle $[c_j, c_j + \varepsilon_0]$ disjunkt sind, dass die Funktionen $\bar{y}_i(c_j + \varepsilon)$ und $\bar{y}'_i(c_j + \varepsilon)$ im abgeschlossenen Intervall $[0, \varepsilon_0]$ stetig und damit beschränkt sind, da die Funktionen $\bar{y}'_i(x)$ in den Punkten c_j von rechts stetig sind. Bildet man jetzt die Funktionen

$$\begin{aligned} z_j(\varepsilon, x) = & (\varepsilon \bar{y}'_1(c_j + \varepsilon) + \varepsilon \bar{y}'_1(c_j - 0) - 2\bar{y}_1(c_j + \varepsilon) + 2\bar{y}_1(c_j)) \left(\frac{x - c_j}{\varepsilon} \right)^3 \\ (5) \quad & + (-\varepsilon \bar{y}'_1(c_j + \varepsilon) - 2\varepsilon \bar{y}'_1(c_j - 0) + 3\bar{y}_1(c_j + \varepsilon) - 3\bar{y}_1(c_j)) \left(\frac{x - c_j}{\varepsilon} \right)^2 \\ & + \varepsilon \bar{y}'_1(c_j - 0) \left(\frac{x - c_j}{\varepsilon} \right) + \bar{y}_1(c_j) \end{aligned}$$

mit dem Definitionsbereich

$$D_j = \{(\varepsilon, x) \in E_2 \mid \varepsilon \in (0, \varepsilon_0], x \in [c_j, c_j + \varepsilon]\}, \quad \text{für } j = 1, \dots, k,$$

so sind diese Funktionen stetig und besitzen eine stetige partielle Ableitung nach x in D_j . Die Funktionen $z_j(\varepsilon, x)$ genügen dabei den folgenden Randbedingungen

$$(6) \quad \frac{\partial z_j(\varepsilon, c_j)}{\partial x} = \bar{y}'_1(c_j - 0), \quad z_j(\varepsilon, c_j) = \bar{y}_1(c_j),$$

$$\frac{\partial z_j(\varepsilon, c_j + \varepsilon)}{\partial x} = \bar{y}'_1(c_j + \varepsilon), \quad z_j(\varepsilon, c_j + \varepsilon) = \bar{y}_1(c_j + \varepsilon),$$

für $j = 1, \dots, k$ und jedes $\varepsilon \in (0, \varepsilon_0]$. Da $\left| \frac{x - c_j}{\varepsilon} \right| \leq 1$ ist, für $(\varepsilon, x) \in D_j$, folgt aus der Definition in (5), dass die Funktionen $z_j(\varepsilon, x)$ über D_j beschränkt sind,

$$(7) \quad |z_j(\varepsilon, x)| < K_1, \quad \text{für } (\varepsilon, x) \in D_j \quad \text{und } j = 1, \dots, k.$$

Analog erhält man für die partiellen Ableitungen, dass

$$(8) \quad \left| \frac{\partial z_j(\varepsilon, x)}{\partial x} \right| < K_2, \quad \text{für } (\varepsilon, x) \in D_j \quad \text{und } j = 1, \dots, k,$$

ist. Ohne Beschränkung der Allgemeinheit kann man annehmen, dass auch

$$(9) \quad \begin{aligned} |\bar{y}_1(x)| &< K_1, \\ |\bar{y}'_1(x)| &< K_2, \quad \text{für } x \in [c_j, c_j + \varepsilon_0] \quad \text{und } j = 1, \dots, k, \end{aligned}$$

gilt, wenn $\bar{y}'_1(c_j) = \bar{y}'_1(c_j + 0)$ gesetzt wird. Setzt man jetzt

$$\tilde{y}_1(\varepsilon, x) = \begin{cases} \bar{y}_1(x) & \text{für } x \in [a, b] \setminus \bigcup_{j=1}^k [c_j, c_j + \varepsilon] \\ z_j(\varepsilon, x) & \text{für } x \in [c_j, c_j + \varepsilon], \quad j = 1, \dots, k, \end{cases}$$

so ist $\tilde{y}_1(\varepsilon, x)$ nach (6) für jedes $\varepsilon \in (0, \varepsilon_0)$ eine Funktion von x , die im Intervall $[a, b]$ definiert ist und dort eine stetige Ableitung nach x besitzt. Weiterhin erfüllt sie nach der Wahl der Intervalle $[c_j, c_j + \varepsilon_0]$ die Randbedingungen

$$\tilde{y}_1(\varepsilon, a) = A_1, \quad \tilde{y}_1(\varepsilon, b) = B_1, \quad \text{für jedes } \varepsilon \in (0, \varepsilon_0).$$

Nach den Ungleichungen (7) und (8) gilt weiter

$$(10) \quad \begin{aligned} |\tilde{y}(\varepsilon, x)| &< K_1, \\ \left| \frac{\partial \tilde{y}_1(\varepsilon, x)}{\partial x} \right| &< K_2, \quad \text{wenn } x \in [c_j, c_j + \varepsilon], \quad j = 1, \dots, k, \end{aligned}$$

für jedes $\varepsilon \in (0, \varepsilon_0]$. Analog kann man Funktionen $\tilde{y}_i(\varepsilon, x)$ für $i = 2, \dots, n$ angeben. Betrachtet man nun die Kurven

$$\tilde{\gamma}(\varepsilon) = \{(x, y_i) \in E_{n+1} | y_i = \tilde{y}_i(\varepsilon, x), x \in [a, b], i = 1, \dots, n\},$$

so sind diese Kurven für jedes $\varepsilon \in (0, \varepsilon_0]$ glatte Kurven, die die Punkte P_1 und P_2 verbinden. Dabei gilt

$$(11) \quad I(\tilde{\gamma}(\varepsilon)) - I(\tilde{\gamma}) = \sum_{j=1}^k \int_{c_j}^{c_j + \varepsilon} \left[F\left(x, \tilde{y}_i(\varepsilon, x), \frac{\partial \tilde{y}_i(\varepsilon, x)}{\partial x}\right) - F(x, \bar{y}_i(x), \bar{y}'_i(x)) \right] dx,$$

für jedes $\varepsilon \in (0, \varepsilon_0]$.

Sei

$$R_j = \{(x, y_i, y'_i) \in E_{2n+1} | x \in [c_j, c_j + \varepsilon_0], |y_i| \leq K_1, |y'_i| \leq K_2, i = 1, \dots, n\},$$

für $j = 1, \dots, k$, sowie

$$R = \bigcup_{j=1}^k R_j.$$

Dann ist die Menge R kompakt und die Funktion $F(x, y_i, y'_i)$ über R folglich beschränkt, d. h.

$$|F(x, y_i, y'_i)| \leq M, \quad \text{für } (x, y_i, y'_i) \in R.$$

Dann erhält man nach (11)

$$(12) \quad I(\tilde{\gamma}(\varepsilon)) - I(\tilde{\gamma}) \leq \sum_{j=1}^k \int_{c_j}^{c_j+\varepsilon} 2M dx = 2\varepsilon kM,$$

für jedes $\varepsilon \in (0, \varepsilon_0]$. Nach (3) kann man nun ein $\varepsilon_1 \in (0, \varepsilon_0]$ so wählen, dass

$$\varepsilon_1 < \frac{I(\tilde{\gamma}^\circ) - I(\tilde{\gamma})}{2kM}$$

ist. Damit folgt aus (12)

$$(13) \quad I(\tilde{\gamma}(\varepsilon)) - I(\tilde{\gamma}) < I(\tilde{\gamma}^\circ) - I(\tilde{\gamma}) \text{ d. h.}$$

$$I(\tilde{\gamma}(\varepsilon)) < I(\tilde{\gamma}^\circ)$$

für jedes $\varepsilon \in (0, \varepsilon_1]$. Es verbleibt also noch zu zeigen, dass ein $\tilde{\varepsilon} \in (0, \varepsilon_1]$ existiert, so dass die Bedingung (1) erfüllt ist, d. h. dass

$$|\tilde{y}_i(\tilde{\varepsilon}, x) - A_i| < \delta, \quad \text{für } x \in [a, b] \text{ und } i = 1, \dots, n$$

gilt. Da die Funktionen $\tilde{y}_i(x)$ stetige Funktionen über $[a, b]$ sind, existiert

$$(14) \quad \alpha = \max_{x \in [a, b]} \left\{ \max_{i \in \{1, \dots, n\}} (|\tilde{y}_i(x) - A_i|) \right\},$$

wobei nach (2)

$$(15) \quad \alpha < \delta$$

gilt. Weiterhin gilt nach der Definition der Funktionen $\tilde{y}_i(\varepsilon, x)$ und der Relation (2)

$$(16) \quad |\tilde{y}_i(\varepsilon, x) - A_i| = |\tilde{y}_i(x) - A_i| < \delta, \quad \text{für } i = 1, \dots, n,$$

wenn $x \in [a, b] \setminus \bigcup_{j=1}^k [c_j, c_j + \varepsilon]$, für jedes $\varepsilon \in (0, \varepsilon_0]$. Falls $x \in [c_j, c_j + \varepsilon]$, so gilt nach (6) und (15)

$$|\tilde{y}_i(\varepsilon, x) - A_i| \leq |\tilde{y}_i(\varepsilon, c_j) - A_i| + |\tilde{y}_i(\varepsilon, x) - \tilde{y}_i(\varepsilon, c_j)| \leq \alpha + |\tilde{y}_i(\varepsilon, x) - \tilde{y}_i(\varepsilon, c_j)|.$$

Nach dem Mittelwertsatz der Differentialrechnung gilt

$$\tilde{y}_i(\varepsilon, x) - \tilde{y}_i(\varepsilon, c_j) = \frac{\partial \tilde{y}_i(\varepsilon, x_i)}{\partial x} (x - c_j), \quad \text{für } i = 1, \dots, n,$$

wobei $x_i \in [c_j, c_j + \varepsilon]$. Da $|x - c_j| \leq \varepsilon$, für $x \in [c_j, c_j + \varepsilon]$, erhält man somit nach (10)

$$(17) \quad |\tilde{y}_i(\varepsilon, x) - A_i| \leq \alpha + K_2 \varepsilon,$$

für $x \in [c_j, c_j + \varepsilon]$, $j = 1, \dots, k$, $i = 1, \dots, n$ und jedes $\varepsilon \in (0, \varepsilon_0]$. Nun kann man nach (15) ein $\varepsilon_2 \in (0, \varepsilon_0]$ so wählen, dass

$$\varepsilon_2 < \frac{\delta - \alpha}{K_2}$$

ist, dann folgt aus (16) und (17) für jedes $\varepsilon \leq \varepsilon_2$

$$(18) \quad |\tilde{y}_i(\varepsilon, x) - A_i| < \delta, \quad \text{für } x \in [a, b] \text{ und } i = 1, \dots, n.$$

Setzt man nun

$$\tilde{\varepsilon} = \min(\varepsilon_1, \varepsilon_2),$$

so ist die Kurve

$$\tilde{\gamma}(\tilde{\varepsilon}) = \{(x, y_i) \in E_{n+1} | y_i = \tilde{y}_i(\tilde{\varepsilon}, x), x \in [a, b], i = 1, \dots, n\}$$

eine glatte Kurve, die der Bedingung (1) genügt, und für die nach (13) $I(\tilde{\gamma}(\tilde{\varepsilon})) < I(\hat{\gamma})$ im Widerspruch zur Optimalität der Kurve $\hat{\gamma}$ gilt. Damit ist dieses Lemma bewiesen.

Die hier angegebene Approximation zur Glättung der Kurven ist nicht so elegant, wie sie zum Beispiel im Buch von H. SAGAN angegeben wird, doch ist es bei dieser Art der Approximation möglich, den Punkt c_j , in dem die Ableitung unstetig ist, als festen Randpunkt der entsprechenden Approximationsintervalle zu wählen. Dadurch wird die Untersuchung des Lagrange-Problems wesentlich vereinfacht.

Es kann nun zur Herleitung der notwendigen Bedingungen für das Lagrange-Problem mit gewöhnlichen Gleichungen übergegangen werden.

Das folgende Variationsproblem wird betrachtet:

$$(19) \quad \min I(\gamma) = \min \int_a^b F(x, y_i(x), y'_i(x)) dx$$

in Bezug auf alle glatten Kurven γ des E_{n+1} , für die

$$\varphi_j(x, y_1(x), \dots, y_n(x)) = 0, \quad \text{für } x \in [a, b], j = 1, \dots, m, m < n,$$

und

$$y_i(a) = A_i; \quad y_i(b) = B_i, \quad \text{für } i = 1, \dots, n.$$

Damit dieses Problem sinnvoll ist, muss man natürlich annehmen, dass $\varphi_j(a, A_1, \dots, A_n) = 0$ und $\varphi_j(b, B_1, \dots, B_n) = 0$, für $j = 1, \dots, m$, ist. Dabei sollen die Funktionen $F(x, y_i, y'_i)$ bzw. $\varphi_j(x, y_i)$ stetige partielle Ableitungen bis zur zweiten bzw. dritten Ordnung nach allen Variablen haben. Und weiterhin soll die Funktionalmatrix

$$\begin{pmatrix} \frac{\partial \varphi_j}{\partial y_i} \end{pmatrix}$$

den maximalen Rang m haben über der Menge

$$(20) \quad M = \{(x, y_i) \in E_{n+1} | \varphi_j(x, y_i) = 0, j = 1, \dots, m\}.$$

Sei jetzt

$$\hat{\gamma} = \{(x, y_i) \in E_{n+1} | y_i = \hat{y}_i(x), x \in [a, b], i = 1, \dots, n\}$$

eine optimale Kurve dieses Variationsproblems und \bar{x} ein beliebiger Punkt des Intervalles $[a, b]$. Dann gilt

$$\varphi_j(\bar{x}, \hat{y}_i(\bar{x})) = 0, \quad \text{für } j = 1, \dots, m.$$

Weiterhin kann man ohne Beschränkung der Allgemeinheit annehmen, dass die Determinante

$$\begin{vmatrix} \frac{\partial \varphi_1}{\partial y_1}, & \dots, & \frac{\partial \varphi_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial y_1}, & \dots, & \frac{\partial \varphi_m}{\partial y_m} \end{vmatrix}$$

im Punkt $(\bar{x}, \hat{y}_i(\bar{x}))$ verschieden von Null ist. Damit sind die Voraussetzungen des Hauptsatzes über implizite Funktionen erfüllt, und man kann das System

$$\varphi_j(x, y_i) = 0, \quad \text{für } j = 1, \dots, m,$$

in einer Umgebung des Punktes $(\bar{x}, \hat{y}_i(\bar{x}))$ nach den Variablen y_1, \dots, y_m auflösen, d. h. es existieren Funktionen

$$f_j(x, y_{m+1}, \dots, y_n), \quad \text{für } j = 1, \dots, m,$$

über dem Bereich

$$(21) \quad |x - \bar{x}| < \delta, \quad |y_k - \hat{y}_k(\bar{x})| < \delta, \quad \text{für } k = m+1, \dots, n,$$

mit den folgenden Eigenschaften:

$$(22) \quad y_j = f_j(x, y_{m+1}, \dots, y_n), \quad \text{für } j = 1, \dots, m \quad \text{und} \quad (x, y_i) \in M,$$

d. h. es gilt

$$(23) \quad \varphi_j(x, f_1(x, y_{m+1}, \dots, y_n), \dots, f_m(\dots, y_{m+1}, \dots, y_n)) = 0,$$

für $j=1, \dots, m$, über dem in (21) angegebenen Bereich. Weiterhin besitzen die Funktionen f_j über diesem Bereich stetige partielle Ableitungen bis zur 3. Ordnung nach allen Variablen. Dabei erhält man für die ersten partiellen Ableitungen

$$(24) \quad \frac{\partial f_j}{\partial x} = \sum_{r=1}^m \left[\frac{\partial \varphi_j}{\partial y_r} \right]^{-1} \cdot \frac{\partial \varphi_r}{\partial x},$$

$$\frac{\partial f_j}{\partial y_k} = \sum_{r=1}^m \left[\frac{\partial \varphi_j}{\partial y_r} \right]^{-1} \cdot \frac{\partial \varphi_r}{\partial y_k}, \quad \text{für } j = 1, \dots, m \quad \text{und} \quad k = m+1, \dots, n,$$

wobei $\left[\frac{\partial \varphi_j}{\partial y_r} \right]^{-1}$ die Elemente der zur regulären Matrix $\left(\frac{\partial \varphi_r}{\partial y_j} \right)$ inversen Matrix sind.

Die Funktionen $\hat{y}'_i(x)$ sind stetige Funktionen, d. h. man kann ein $\varepsilon > 0$ angeben, so dass

$$(25) \quad |\hat{y}'_i(x) - \hat{y}'_i(\bar{x})| < \frac{\delta}{2}, \quad \text{für } |x - \bar{x}| < \varepsilon \quad \text{und} \quad i = 1, \dots, n \text{ ist.}$$

Sei

$$(26) \quad \mu = \max \left\{ \max_{i \in \{1, \dots, n\}} (|\hat{y}'_i(\bar{x}) - \delta|), \max_{i \in \{1, \dots, n\}} (|\hat{y}'_i(\bar{x}) + \delta|) \right\},$$

sowie

$$(27) \quad \beta_1 = \min \left\{ \delta, \varepsilon, \frac{\delta}{\mu} \right\}.$$

Wählt man jetzt ein $\bar{x} \in [a, b]$ mit $\bar{x} \neq \bar{x}$, so dass

$$(28) \quad |\bar{x} - \bar{x}| = \beta < \beta_1$$

ist, so gilt nach den Definitionen von β_1 und μ

$$(29) \quad |\hat{y}'_i(x)| < \mu, \quad \text{für } x \in [\bar{x}, \bar{x}] \quad \text{und} \quad i = 1, \dots, n.$$

Daraus folgt durch eine einfache Abschätzung

$$(30) \quad |\mathring{y}_i(x) - \mathring{y}_i(\bar{x})| < \delta, \quad \text{für } x \in [\bar{x}, \bar{x}] \quad \text{und } i = 1, \dots, n.$$

Aus den Relationen (21) und (22) ergibt sich somit, da

$$(31) \quad \begin{aligned} &(x, \mathring{y}_1(x), \dots, \mathring{y}_n(x)) \in M, \quad \text{für } x \in [\bar{x}, \bar{x}], \\ &\mathring{y}_j(x) = f_j(x, \mathring{y}_{m+1}(x), \dots, \mathring{y}_n(x)), \quad \text{für } x \in [\bar{x}, \bar{x}] \quad \text{und } j = 1, \dots, m. \end{aligned}$$

Durch Differenzieren nach x erhält man aus (31)

$$(32) \quad \mathring{y}'_j(x) = \frac{\partial f_j}{\partial x} + \sum_{k=m+1}^n \frac{\partial f_j}{\partial y_k} \mathring{y}'_k(x), \quad \text{für } x \in [\bar{x}, \bar{x}] \quad \text{und } j = 1, \dots, m.$$

Damit erhält man durch Einsetzen aus (31) und (32)

$$(33) \quad \begin{aligned} &\int_{\bar{x}}^{\bar{x}} F(x, \mathring{y}_i(x), \mathring{y}'_i(x)) dx = \\ &= \int_{\bar{x}}^{\bar{x}} \bar{F}(x, \mathring{y}_{m+1}(x), \dots, \mathring{y}_n(x), \mathring{y}'_{m+1}(x), \dots, \mathring{y}'_n(x)) dx. \end{aligned}$$

Auf Grund der Voraussetzungen über die Funktion $F(x, y_i, y'_i)$ und der Eigenschaften der Funktionen $f_j(x, y_k)$ besitzt die Funktion $\bar{F}(x, y_{m+1}, \dots, y_n, y'_{m+1}, \dots, y'_n)$ stetige partielle Ableitungen bis zur zweiten Ordnung nach allen Variablen über dem Bereich

$$|x - \bar{x}| < \delta, \quad |y_k - \mathring{y}_k(\bar{x})| < \delta, \quad y'_k \in (-\infty, \infty), \quad \text{für } k = m+1, \dots, n.$$

Betrachtet man nun das Variationsproblem

$$(34) \quad \min I(\tau) = \min \int_{\bar{x}}^{\bar{x}} \bar{F}(x, y_k(x), y'_k(x)) dx,$$

in Bezug auf alle glatten Kurven

$$\tau = \{(x, y_k) \in E_{n-m+1} | y_k = y_k(x), x \in [\bar{x}, \bar{x}], k = m+1, \dots, n\},$$

die die Punkte

$$(35) \quad \bar{P} = (\bar{x}, \mathring{y}_k(\bar{x})), \quad \bar{\bar{P}} = (\bar{x}, \mathring{y}_k(\bar{x}))$$

verbinden, und zusätzlich der Bedingung

$$|y_k(x) - \mathring{y}_k(\bar{x})| < \delta, \quad \text{für } x \in [\bar{x}, \bar{x}] \quad \text{und } k = m+1, \dots, n,$$

genügen, so gilt der folgende Satz:

SATZ. Wenn die Kurve \mathring{y} zum Minimum des Funktionales (19) führt, so führt die Kurve

$$(36) \quad \hat{\tau} = \{(x, y_k) \in E_{n-m+1} | y_k = \mathring{y}_k(x), x \in [\bar{x}, \bar{x}], k = m+1, \dots, n\}$$

zum Minimum des Funktionales (34).

BEWEIS. Angenommen, es existiert eine glatte Kurve

$$\bar{\tau} = \{(x, y_k) \in E_{n-m+1} | y_k = \bar{y}_k(x), x \in [\bar{x}, \bar{\bar{x}}], k = m+1, \dots, n\},$$

mit

$$\bar{y}_k(\bar{x}) = \hat{y}_k(\bar{x}), \quad \bar{y}_k(\bar{\bar{x}}) = \hat{y}_k(\bar{\bar{x}})$$

$$|\bar{y}_k(x) - \hat{y}_k(\bar{x})| < \delta, \quad \text{für } x \in [\bar{x}, \bar{\bar{x}}] \quad \text{und } k = m+1, \dots, n,$$

so dass $I(\bar{\tau}) < I(\hat{\tau})$ ist.

Es sei jetzt $\varepsilon_0 > 0$ so gewählt, dass

$$\bar{x} + \varepsilon_0 \in [\bar{x}, \bar{\bar{x}}] \quad \text{und} \quad \bar{\bar{x}} - \varepsilon_0 \in [\bar{x}, \bar{\bar{x}}],$$

und dass die Intervalle $[\bar{x}, \bar{x} + \varepsilon_0]$ und $[\bar{\bar{x}} - \varepsilon_0, \bar{\bar{x}}]$ disjunkt sind. Dann kann man durch eine analoge Konstruktion, wie im Lemma, indem man einmal $\bar{\bar{x}}$ als rechten Randpunkt und einmal \bar{x} als linken Randpunkt der Approximationsintervalle wählt, Funktionen $\tilde{y}_k(\tilde{\varepsilon}, x)$ über dem Intervall $[\bar{x}, \bar{\bar{x}}]$ definieren, für $k = m+1, \dots, n$, mit den folgenden Eigenschaften:

$$\tilde{y}_k(\tilde{\varepsilon}, \bar{x}) = \bar{y}_k(\bar{x}) = \hat{y}_k(\bar{x}), \quad \tilde{y}_k(\tilde{\varepsilon}, \bar{\bar{x}}) = \bar{y}_k(\bar{\bar{x}}) = \hat{y}_k(\bar{\bar{x}}),$$

$$(37) \quad \frac{\partial \tilde{y}_k(\tilde{\varepsilon}, \bar{x})}{\partial x} = \hat{y}'_k(\bar{x}), \quad \frac{\partial \tilde{y}_k(\tilde{\varepsilon}, \bar{\bar{x}})}{\partial x} = \hat{y}'_k(\bar{\bar{x}}),$$

$$(38) \quad |\tilde{y}_k(\tilde{\varepsilon}, x) - \hat{y}_k(\bar{x})| < \delta, \quad \text{für } x \in [\bar{x}, \bar{\bar{x}}] \quad \text{und } k = m+1, \dots, n,$$

$$(39) \quad \int_{\bar{x}}^{\bar{\bar{x}}} \bar{F}(x, \tilde{y}_k(\tilde{\varepsilon}, x)) \frac{\partial \tilde{y}_k(\tilde{\varepsilon}, x)}{\partial x} dx < \int_{\bar{x}}^{\bar{\bar{x}}} \bar{F}(x, \hat{y}_k(x), \hat{y}'_k(x)) dx.$$

Nach der Wahl von $\bar{\bar{x}}$ gilt

$$(40) \quad |x - \bar{x}| < \delta, \quad \text{für } x \in [\bar{x}, \bar{\bar{x}}].$$

Damit folgt aus (38) und (40) nach den Relationen (26) und (27), wenn man

$$(41) \quad \tilde{y}_j(\tilde{\varepsilon}, x) = f_j(x, \tilde{y}_{m+1}(\tilde{\varepsilon}, x), \dots, \tilde{y}_n(\tilde{\varepsilon}, x)), \quad \text{für } x \in [\bar{x}, \bar{\bar{x}}], \quad j = 1, \dots, m,$$

setzt, so gilt

$$\varphi_j(x, \tilde{y}_1(\tilde{\varepsilon}, x), \dots, \tilde{y}_n(\tilde{\varepsilon}, x)) = 0, \quad \text{für } x \in [\bar{x}, \bar{\bar{x}}] \quad \text{und } j = 1, \dots, m.$$

D. h. die Kurve

$$\tilde{\gamma}(\tilde{\varepsilon}) = \{(x, y_i) \in E_{n+1} | y_i = \tilde{y}_i(\tilde{\varepsilon}, x), x \in [\bar{x}, \bar{\bar{x}}], i = 1, \dots, n\}$$

verläuft in der Menge M . Setzt man jetzt

$$(42) \quad \hat{y}_i(x) = \begin{cases} \tilde{y}_i(\tilde{\varepsilon}, x) & \text{für } x \in [\bar{x}, \bar{\bar{x}}] \\ \hat{y}_i(x) & \text{für } x \in [a, b] \setminus [\bar{x}, \bar{\bar{x}}], \end{cases} \quad \text{für } i = 1, \dots, n,$$

so ist die Kurve

$$\hat{\gamma} = \{(x, y_i) \in E_{n+1} | y_i = \hat{y}_i(x), x \in [a, b], i = 1, \dots, n\}$$

nach (37) eine glatte Kurve, die ganz in der Menge M verläuft und die Punkte P_1

und P_2 verbindet. Aus den Relationen (33), (39), und (42) folgt dann unmittelbar

$$I(\overset{\circ}{\gamma}) - I(\hat{\gamma}) > 0.$$

Dies ist aber ein Widerspruch zur Optimalität der Kurve $\hat{\gamma}$ und der Satz damit bewiesen.

Nach dem Lemma ist dann die Kurve $\overset{\circ}{\tau}$ aus (36) auch optimal in Bezug auf alle stückweise glatten Kurven des E_{n-m+1} , die die Punkte \bar{P} und $\bar{\bar{P}}$ verbinden und der Bedingung

$$(43) \quad |y_k(x) - \overset{\circ}{y}_k(\bar{x})| < \delta, \quad \text{für } x \in [\bar{x}, \bar{\bar{x}}] \quad \text{und } k = m+1, \dots, n,$$

genügen. Durch Anwendung des Maximumprinzipes werden jetzt notwendige Bedingungen für die Funktionen $\overset{\circ}{y}_k(x)$ hergeleitet. Dazu wird das folgende Steuerungsproblem formuliert. Man geht von dem Funktional (34) aus und setzt

$$(44) \quad y_{n+1} = t = x, \quad t_0 = \bar{x}, \quad t_1 = \bar{\bar{x}}.$$

Daraus folgt

$$I(\tau) = \int_{t_0}^{t_1} \bar{F}(y_{n+1}(t), y_{m+1}(t), \dots, y_n(t), y'_{m+1}(t), \dots, y'_n(t)) dt.$$

Als Steuervariablen wählt man y'_k , für $k = m+1, \dots, n$. Nur muss man den Steuerungsbereich jetzt einschränken, da die betrachteten Kurven der Bedingung (43) genügen müssen. Aus diesem Grund setzt man

$$y'_k(t) = u_k(t), \quad \text{mit } |u_k(t)| < \mu, \quad t \in [t_0, t_1], \quad k = m+1, \dots, n.$$

(Die Grösse μ wurde in (26) definiert.) Dadurch erhält man das folgende Steuerungsproblem:

$$(45) \quad \min \int_{t_0}^{t_1} \bar{F}(y_{n+1}(t), y_k(t), u_k(t)) dt,$$

$$\frac{dy_k}{dt} = u_k(t); \quad |u_k(t)| < \mu, \quad \text{für } t \in [t_0, t_1] \quad \text{und } k = m+1, \dots, n,$$

$$\frac{dy_{n+1}}{dt} = 1,$$

$$y_k(t_0) = \overset{\circ}{y}_k(\bar{x}), \quad y_k(t_1) = \overset{\circ}{y}_k(\bar{\bar{x}}), \quad k = m+1, \dots, n,$$

$$y_{n+1}(t_0) = \bar{x}, \quad y_{n+1}(t_1) = \bar{\bar{x}}.$$

Für jede zulässige Steuerung $u(t)$ ergibt sich dann für die dazugehörige Trajektorie $y(t) = \{y_{n+1}(t), y_{m+1}(t), \dots, y_n(t)\}$ $|y_k(t) - y_k(t_0)| < |t - t_0|\mu \leq |t_1 - t_0|\mu = \beta\mu < \beta_1\mu \leq \delta$, für $t \in [t_0, t_1]$ und $k = m+1, \dots, n$, d. h. die entsprechenden Trajektorien genügen der Bedingung (43). Folglich ist nach dem Satz und dem Lemma die Steuerung

$$(46) \quad \overset{\circ}{u}_k(t) = \overset{\circ}{y}'_k(t), \quad \text{für } t \in [t_0, t_1] = [\bar{x}, \bar{\bar{x}}] \quad \text{und } k = m+1, \dots, n,$$

optimal für das Steuerungsproblem (45). Die Hamilton-Funktion dieses Problemes hat die folgende Gestalt

$$H(\psi, y, u) = \psi_0 \bar{F}(y, u) + \sum_{k=m+1}^n \psi_k u_k + \psi_{n+1} \cdot 1.$$

Nach dem Maximumprinzip (siehe [1]) existiert dann eine Konstante ψ_0° und eine stetige, nicht identisch verschwindende Lösung

$$\psi(t) = \{\psi_{n+1}^{\circ}(t), \psi_{m+1}^{\circ}(t), \dots, \psi_n^{\circ}(t)\} \quad \text{für } t \in [\bar{x}, \bar{x}],$$

des Systemes

$$(47) \quad \frac{d\psi_k}{dt} = -\frac{\partial H(\psi, \dot{y}(t), \dot{u}(t))}{\partial y_k} = -\psi_0^{\circ} \bar{F}_{y_k}(\dot{y}(t), \dot{u}(t)), \quad k = m+1, \dots, n,$$

mit den folgenden Eigenschaften

$$(48) \quad H(\psi(t), \dot{y}(t), \dot{u}(t)) = \max_{|u_k| \leq \mu} H(\psi(t), \dot{y}(t), u) = 0, \quad \text{für } t \in [\bar{x}, \bar{x}], \quad \psi_0^{\circ} \equiv 0.$$

Aus der Relation (48) folgt, da $\dot{u}(t)$ für $t \in [\bar{x}, \bar{x}]$ ein innerer Punkt des Steuerungsbereiches ist,

$$\frac{\partial H(\psi(t), \dot{y}(t), \dot{u}(t))}{\partial u_k} = \psi_0^{\circ} \bar{F}_{u_k}(\dot{y}(t), \dot{u}(t)) + \dot{\psi}_k(t) = 0,$$

für $t \in [\bar{x}, \bar{x}]$ und $k = m+1, \dots, n$. Das heisst, es gilt

$$(49) \quad \dot{\psi}_k(t) = -\psi_0^{\circ} \bar{F}_{u_k}(\dot{y}(t), \dot{u}(t)), \quad \text{für } t \in [\bar{x}, \bar{x}] \quad \text{und } k = m+1, \dots, n.$$

Daraus folgt, dass $\psi_0^{\circ} \neq 0$ sein muss, und nach (44) erhält man die folgenden Gleichungen

$$\begin{aligned} & \bar{F}_{y_k}(x, \dot{y}_{m+1}(x), \dots, \dot{y}_n(x), \dot{y}'_{m+1}(x), \dots, y'_n(x)) \\ & - \frac{d}{dx} \bar{F}'_{y_k}(x, \dot{y}_{m+1}(x), \dots, \dot{y}_n(x), \dot{y}'_{m+1}(x), \dots, \dot{y}'_n(x)) = 0, \end{aligned}$$

für $x \in [\bar{x}, \bar{x}]$ und $k = m+1, \dots, n$. Berücksichtigt man nun den Zusammenhang zwischen den Funktionen $\bar{F}(x, y_k, y'_k)$ und $F(x, y_i, y'_i)$, erhält man aus den obigen Gleichungen sofort die Euler—Lagrange'schen Gleichungen

$$(50) \quad F_{y_i} - \frac{d}{dx} F_{y'_i} - \sum_{r=1}^m \dot{\lambda}_r(x) \frac{\partial \varphi_r}{\partial y_i} = 0, \quad \text{für } x \in [\bar{x}, \bar{x}], \quad i = 1, \dots, n.$$

Dabei haben die Lagrange'schen Multiplikatoren $\dot{\lambda}_r(x)$ die folgende Gestalt

$$(51) \quad \dot{\lambda}_r(x) = \sum_{j=1}^m \left(F_{y'_j} - \frac{d}{dx} F_{y_j} \right) \left[\frac{\partial \varphi_j}{\partial y_r} \right]^{-1}, \quad \text{für } x \in [\bar{x}, \bar{x}], \quad r = 1, \dots, m.$$

Da $\bar{x} \in [a, b]$ beliebig gewählt war, müssen die Gleichungen aus (50) im ganzen Intervall $[a, b]$ durch die Funktionen $\dot{y}_i(x)$ erfüllt sein. Es wird als nächstes die Weierstrass-Bedingung für dieses Variationsproblem untersucht. Bildet man die Lagrange-Funktion

$$L(x, y_i, y'_i) = F(x, y_i, y'_i) - \sum_{r=1}^m \dot{\lambda}_r(x) \varphi_r(x, y_i)$$

und betrachtet für diese Funktion die Weierstrass'sche E -Funktion, so erhält man

$$E(x, y_i, y'_i, K_i) = F(x, y_i, K_i) - F(x, y_i, y'_i) - \sum_{i=1}^n (K_i - y'_i) F_{y'_i}(x, y_i, y'_i).$$

Das heisst, die Funktionen φ_j treten in dieser Funktion nicht auf. Es ist deshalb zu erwarten, dass die Bedingung

$$(52) \quad E(x, \mathring{y}_i(x), \mathring{y}'_i(x), K_i) \cong 0$$

nicht für jeden beliebigen Vektor $\{K_i\} \in E_n$ erfüllt sein wird. Im folgenden wird gezeigt, wie die Menge der Vektoren $\{K_i\}$, für die die Bedingung (52) erfüllt ist, geometrisch interpretiert werden kann.

Nach der Relation (48) gilt

$$(53) \quad H(\mathring{\psi}(t), \mathring{y}(t), \mathring{u}(t)) \cong H(\mathring{\psi}(t), \mathring{y}(t), u),$$

für jeden Vektor $u = \{u_{m+1}, \dots, u_n\}$ mit

$$(54) \quad |u_k| < \mu, \quad \text{für } k = m+1, \dots, n.$$

Daraus folgt nach (49) da $\mathring{\psi}_0 < 0$ ist

$$(55) \quad \bar{F}(\mathring{y}(t), u) - \bar{F}(\mathring{y}(t), \mathring{u}(t)) - \sum_{k=m+1}^n \bar{F}_{u_k}(\mathring{y}(t), \mathring{u}(t))(u_k - \mathring{u}_k(t)) \cong 0,$$

für jeden Vektor u , der der Bedingung (54) genügt. Setzt man jetzt

$$u_j = \frac{\partial f_j}{\partial x} + \sum_{k=m+1}^n \frac{\partial f_j}{\partial y_k} u_k, \quad \text{für } j = 1, \dots, m,$$

und berücksichtigt, dass

- (i) $\mathring{u}_k(t) = \mathring{y}'_k(t), \quad \text{für } k = m+1, \dots, n,$
- (ii) $\mathring{y}'_j(t) = \frac{\partial f_j}{\partial x} + \sum_{k=m+1}^n \frac{\partial f_j}{\partial y_k} \mathring{y}'_k(t), \quad \text{für } j = 1, \dots, m,$
- (iii) $\mathring{y}_{n+1}(t) = t = x,$
- (iv) $\bar{F}_{u_k} = \bar{F}_{y'_k} = F_{y_k} + \sum_{j=1}^m F_{y_j} \frac{\partial f_j}{\partial y_k}, \quad \text{für } k = m+1, \dots, n,$

ist, erhält man aus der Ungleichung (55)

$$(56) \quad F(x, \mathring{y}_i(x), u_i) - F(x, \mathring{y}_i(x), \mathring{y}'_i(x)) - \sum_{k=m+1}^n (u_k - \mathring{y}'_k(x)) F_{y'_k} - \\ - \sum_{j=1}^m F_{y'_j} \left(\sum_{k=m+1}^n \left(u_k \frac{\partial f_j}{\partial y_k} - \mathring{y}'_k(x) \frac{\partial f_j}{\partial y_k} \right) \right) \cong 0,$$

für jeden Vektor $u = \{u_1, \dots, u_n\}$, mit

$$(57) \quad u_j = \frac{\partial f_j}{\partial x} + \sum_{k=m+1}^n \frac{\partial f_j}{\partial y_k} u_k, \quad |u_k| < \mu, \quad \text{für } k = m+1, \dots, n, \quad j = 1, \dots, m.$$

Damit folgt nach (ii) aus der Ungleichung (56)

$$F(x, \dot{y}_i(x), u_i) - F(x, \dot{y}_i(x), \dot{y}'_i(x)) - \sum_{i=1}^n (u_i - \dot{y}'_i(x)) F_{y'_i} \cong 0,$$

für jeden Vektor u , der die Gestalt (57) hat. Das heisst, die Weierstrass-Bedingung (52) ist erfüllt für jeden Vektor $\{K_i\} \in E_n$, mit

$$(58) \quad K_j = \frac{\partial f_j}{\partial x} + \sum_{k=m+1}^n \frac{\partial f_j}{\partial y_k} K_k, \quad \text{für } j = 1, \dots, m,$$

$$|K_k| < \mu, \quad \text{für } k = m+1, \dots, n.$$

Um eine geometrische Interpretation der Vektoren aus (58) zu erhalten, werden an Stelle der Vektoren $K = \{K_1, \dots, K_n\} \in E_n$ die Vektoren $\bar{K} \in E_{n+1}$ betrachtet, wobei

$$(59) \quad \bar{K} = \{1, K_1, \dots, K_n\}$$

ist, und die Komponenten K_i den Bedingungen in (58) genügen. Es wird sich zeigen, dass diese Vektoren \bar{K} aus (59) Tangentialvektoren an die Mannigfaltigkeit M im Punkt $(x, \dot{y}_i(x))$ sind, für $x \in [\bar{x}, \bar{x}]$. Nach den Relationen (21), (22) und (23) erhält man durch das System

$$\begin{aligned} x &= x, \\ y_j &= f_j(x, y_{m+1}, \dots, y_n), & \text{für } j &= 1, \dots, m, \\ y_k &= y_k, & \text{für } k &= m+1, \dots, n, \\ \text{für } |x - \bar{x}| < \delta, \quad |y_k - \dot{y}_k(\bar{x})| < \delta, & \text{für } k &= m+1, \dots, n, \end{aligned}$$

lokal eine parametrische Beschreibung der Mannigfaltigkeit M in einer Umgebung des Punktes $(\bar{x}, \dot{y}_i(\bar{x}))$. Nach der Relation (31) verläuft die optimale Kurve \hat{y} , für $x \in [\bar{x}, \bar{x}]$, ganz in dieser Umgebung. Das heisst, wenn $(x, \dot{y}_i(x))$, für $x \in [\bar{x}, \bar{x}]$, ein Punkt der Kurve \hat{y} ist, bilden die Vektoren

$$\begin{aligned} w_{n+1} &= \left\{ 1, \frac{\partial f_1}{\partial x}, \dots, \frac{\partial f_m}{\partial x}, 0, \dots, 0 \right\} \\ w_{m+1} &= \left\{ 0, \frac{\partial f_1}{\partial y_{m+1}}, \dots, \frac{\partial f_m}{\partial y_{m+1}}, 1, \dots, 0 \right\} \\ &\vdots \\ w_n &= \left\{ 0, \frac{\partial f_1}{\partial y_n}, \dots, \frac{\partial f_m}{\partial y_n}, 0, \dots, 1 \right\} \end{aligned}$$

eine Basis der Tangentialebene an die Mannigfaltigkeit M in diesem Punkt, d. h. jeder Tangentialvektor $\bar{K} = \{\bar{K}_{n+1}, \bar{K}_1, \dots, \bar{K}_n\}$ kann als Linearkombination dieser Vektoren dargestellt werden

$$(60) \quad \bar{K} = v_{n+1} w_{n+1} + \sum_{k=m+1}^n v_k w_k.$$

Aus der speziellen Gestalt der Basis ergibt sich, dass für jeden Tangentialvektor im Punkt $(x, \dot{y}_i(x))$ gilt

$$\bar{K}_j = \bar{K}_{n+1} \frac{\partial f_j}{\partial x} + \sum_{k=m+1}^n \bar{K}_k \frac{\partial f_j}{\partial y_k}, \quad \text{für } j = 1, \dots, m.$$

Somit ist nach (58) die Weierstrass-Bedingung (52) im Punkt $(x, \dot{y}_i(x))$ erfüllt für alle Vektoren $\{K_i\} \in E_n$, für die $\{1, K_i\}$ ein Tangentialvektor an die Mannigfaltigkeit M in diesem Punkt ist, und für deren Komponenten K_k , $k=m+1, \dots, n$, $|K_k| < \mu$ gilt. Die Beschränkung der Komponenten K_k ergibt sich aus der Tatsache, dass das System $\varphi_j(x, y_1, \dots, y_n) = 0$ nur lokal nach y_1, \dots, y_m aufgelöst werden konnte. Ist dies global möglich, so fallen diese Beschränkungen weg. Die besondere Bedeutung des Tangentialvektors w_{n+1} , die durch die Festlegung $v_{n+1} = 1$ in (60) zum Ausdruck kommt, ergibt sich aus der Tatsache, dass die Variable x auch gleichzeitig Kurvenparameter ist. Dadurch werden nur Tangentialvektoren zum Vergleich zugelassen, in deren Richtung sich der Durchlaufsinne der Kurve nicht ändert.

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DIGRAPHS WITH PERIODIC LINE DIGRAPHS

by

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Introduction

A digraph D is called periodic if $L^m(D) \cong L^{m+k}(D)$ for some integers m and k , $m \geq 0$, $k > 0$. The smallest such k is called the period of D . BEINEKE [1] first studied finite periodic digraphs. In [3], the author surveyed the known results and extended many of them—mostly to infinite digraphs. In this paper we complete the description of finite periodic digraphs and give a structural determination of their period.

Our terminology is standard, e.g., as found in [2], except we use vertex and edge in place of point and line, respectively. Also digraphs here can be infinite and have loops and multiple edges. We will use the notation (u, v) for edges in a digraph. If D is a digraph then $L(D)$ will denote the *line digraph* of D and is the digraph with $V(L(D)) = E(D)$ and with $(\alpha, \beta) \in E(L(D))$ if and only if the terminal vertex of α is the initial vertex of β . For $n \geq 1$ we let $L^n(D) = L(L^{n-1}(D))$ where $L^0(D)$ is D . If S is a subdigraph of D and $v \in V(S)$ we will use $A_v(S)$ to denote the subdigraph of D consisting of all vertices and edges that occur on dipaths from vertices of D to v that contain no vertices of S other than v . The subdigraph $C_v(S)$ is defined in a similar manner only using dipaths emanating from v . In this connection we note that *arborescences* are ditrees with all their edges oriented toward a root v of the ditree. A *counterarborescence* is a ditree with all of its edges oriented away from a root v of the ditree. We also note that we will frequently use $L(P)$ when P is a dipath, say $P = (v_1, v_2, \dots, v_n)$, to denote the line digraph of the subdigraph corresponding to P . Similarly for dicycles.

Periodic Digraphs

The following theorem was given, without proof, in [1]. We will include a proof here since it is contained in our development of the structural characteristics of periodic digraphs.

THEOREM 1. *Let D be a finite digraph. Then*

- (i) $|L^n(D)| \rightarrow 0$ as $n \rightarrow \infty$ if and only if D has no dicycles,
- (ii) $|L^n(D)| \rightarrow \infty$ as $n \rightarrow \infty$ if and only if D has two dicycles joined by a dipath (perhaps of length zero), and
- (iii) D is periodic if and only if D has at least one dicycle, but no two are joined by a dipath (perhaps of length zero).

PROOF. Since the three conditions on each side are mutually exclusive and those on the right side cover all possibilities we only need to establish the sufficiency of the conditions on the right.

If D has no dicycles, then D has a longest dipath P since D is finite. Such a dipath is a source to sink dipath. But P is a source to sink dipath in D of length p if and only if $L(P)$ is a source to sink dipath in $L(D)$ of length $p-1$. Thus $L^n(P)$ is a longest dipath in $L^n(D)$ and $|L^n(D)| \rightarrow 0$ as $n \rightarrow \infty$.

If D has two dicycles joined by a dipath of length $k \geq 0$, then $L(D)$ has two dicycles pairwise isomorphic with the two in D and joined by a dipath of length $k+1$. Thus $|L^n(D)| \rightarrow \infty$ as $n \rightarrow \infty$.

Finally, suppose that D has at least one dicycle, but no two are joined by a dipath. Let $R = (r_1, r_2, \dots, r_n, r_1)$ be a dicycle of D and let $U(R)$ be the unilateral component of D that contains R , i.e., $U(R) = R \cup \left(\bigcup_{i=1}^n A_i(R) \right) \cup \left(\bigcup_{i=1}^n C_i(R) \right)$ where $A_i(R)$ is $A_v(R)$ and $C_i(R)$ is $C_v(R)$ with $v = r_i$. While this need not be a disjoint union we do have $A_i \cap C_j \subseteq \{r_i\} \cap \{r_j\}$, for otherwise D would have two dicycles with a dipath of length zero joining them. If we let $e_i = (r_{i-1}, r_i)$ where subscripts are reduced module n , then $L(A_i(R) + e_{i+1}) = A_{i+1}(L(R))$ (the latter subscript refers to the vertex e_{i+1} of $L(R)$). Similarly $L(C_i(R) + e_i) = C_i(L(R))$. It follows that $U(L(R))$ is a subdigraph of $L(U(R))$ and hence that $U(L^m(R))$ is a subdigraph of $L^m(U(R))$ for $m \geq 1$ ($U(L^m(R))$ means the unilateral component in $L^m(D)$ that contains $L^m(R)$). If some $A_i(R)$ or $C_i(R)$ is not a ditree then there is some vertex of $U(R)$ that can reach to R , or be reached from R , by two different dipaths. If k is the distance between R and the closest such vertex to R then $U(L(R))$ has no such vertex at a distance less than $k+1$ from $L(R)$ and it may have no such vertex. In fact, since the maximum distance between vertices of $U(L(R))$ and $L(R)$ is never greater than the maximum distance between vertices of $U(R)$ and R , and since $U(R)$ is finite, it follows that for m sufficiently large $U(L^m(R))$ has no such vertex. In the same manner we see that other "parts of D " that "are attached to $U(R)$ " but that are edge disjoint from $U(R)$ "move away" from $L^m(R)$ as m increases and so for m sufficiently large they become disconnected from $U(L^m(R))$. We have purposefully used somewhat vague language here since to be more precise would likely be more confusing than helpful. However let us state the conclusions clearly: for m sufficiently large the weak component of $L^m(D)$ containing $L^m(R)$ is the weak component of $L^m(U(R))$ containing $L^m(R)$ and the $A_i(L^m(R))$ and $C_i(L^m(R))$ in $L^m(D)$ are respectively arborescences and counterarborescences rooted on $L^m(R)$. Thus, by the part of (i) we proved, we can assume, without loss of generality, that each weak component of D consists of a dicycle with arborescences and counterarborescences rooted at vertices of the dicycle but otherwise disjoint. We will call such a weakly connected digraph a *basic unicyclic digraph*. It remains to examine what happens to such a digraph under iteration of L .

Let $R = (r_1, r_2, \dots, r_n, r_1)$, let $A_i(R)$ and $C_i(R)$ be as above and again let $e_i = (r_{i-1}, r_i)$. Then, as before, $L(A_i(R) + e_{i+1}) = A_{i+1}(L(R))$ and $L(C_i(R) + e_i) = C_i(L(R))$. But now, since the A_i and C_i are ditrees, we can also conclude that $A_{i+1}(L(R)) \cong A_i(R)$ where e_{i+1} corresponds to r_i and that $C_i(L(R)) \cong C_i(R)$ where e_i corresponds to r_i . If we continue the subscript notation in the obvious way, then we have $A_{i+m}(L^m(R)) \cong A_i(R)$ and $C_i(L^m(R)) \cong C_i(R)$ where the i^{th} vertex of $L^m(R)$ corresponds to r_i . It follows that $U(L^{kn}(R)) \cong U(R)$ for all $k \geq 1$. But, by the conclusion above, the structure of $L^m(L^{kn}(D_R))$, where D_R is the weak component of D that contains R and where m is sufficiently large, depends only on $U(L^{kn}(R))$

and m . Hence, for k sufficiently large, we have that $L^{kn}(L^{kn}(U(R))) \cong L^{kn}(U(R))$. Thus, if we let m be a sufficiently large common multiple of the lengths of the dicycles in D (this is possible since D is finite), then $L^{2m}(D) \cong L^m(D)$, i.e., D is periodic.

That completes the proof of the theorem. However, for the sake of completeness, we describe the remaining structure of $L^m(D)$. From the above it is obvious that we can assume that $D=U(R)$, that R is the dicycle $(r_1, r_2, \dots, r_n, r_1)$, and that $A_i(R)$ and $C_i(R)$ are ditrees. Let $r_i^{(0)}=r_i$ and, for $m \geq 1$, let $r_i^{(m)}=(r_{i-1}^{(m-1)}, r_i^{(m-1)}) \in E(L^{m-1}(R))=V(L^m(R))$, $A_i(L^m(R))=A_v(L^m(R))$, and $C_i(L^m(R))=C_v(L^m(R))$ where $v=r_i^{(m)}$. Recall that under our assumptions $A_{i+m}(L^m(R)) \cong A_i(R)$ and $C_i(L^m(R)) \cong C_i(R)$. Two counterarborescences C_1 and C_2 with roots v_1 and v_2 respectively are said to be (p, q) -highed for $p, q > 0$, if the sets $P = \{v \in V(C_1) : d(v, v_1) < p\}$ and $Q = \{v \in V(C_2) : d(v, v_2) < q\}$ are disjoint while the subtree of C_1 induced on $V(C_1) - P$ is identical to the one induced on $V(C_2) - Q$; they are said to be *hinged* if they are (p, q) -hinged for some p and q . Note that being hinged is an equivalence relation on a given set of counterarborescences.

Let $S = \bigcup_{i=1}^n S_i$ where $S_i = V(A_i(L^m(R))) - V(L^m(R))$ and let $T = \{v : v \notin A_i(L^m(R)) \text{ for } 1 \leq i \leq n \text{ and } v \text{ is a source of } D\}$. Let $\mathcal{C} = \{C_v(A_i(L^m(R))) : 1 \leq i \leq n, v \in S_i\} \cup \{C_v(L^m(R)) : v \in T\}$. If $v \in S \cup T$, let C_v be the counterarborescence in \mathcal{C} that is rooted at v and let $C(v)$ be the graph union of the equivalence class of counterarborescences in \mathcal{C} that are hinged with C_v . In the following we will assume that, if $v \in S_i$, then the distance between v and $r_i^{(m)}$ in $A_i(L^m(R))$ is k .

- (1) If $u \in S \cup T$ and $C(u) \neq C(v)$, then $C(u)$ and $C(v)$ are vertex disjoint.
- (2) If $v \in S_i$, then $C(v)$ is incident to each vertex of $A_i(L^m(R))$ at distance k from $r_i^{(m)}$ and to no other vertex of $A_i(L^m(R))$ or of any $A_j(L^m(R))$ for $j \neq i$.
- (3) If $v \in S_i$, then $C(v)$ is incident to each vertex of $C_{i-k}(L^m(R))$ at distance k from $r_{i-k}^{(m)}$ and to no other vertex of $C_{i-k}(L^m(R))$ or of any $C_j(L^m(R))$ for $j \neq i-k$.
- (4) If $v \in T$ and $C(v) \neq C(u)$ for any $u \in S$, then either $C(v)$ is a weak component of $L^m(D)$ or for some g and h , $C(v)$ is incident to each vertex of $C_g(L^m(R))$ at distance h from $r_g^{(m)}$ and to no other vertex of $C_g(L^m(R))$ or of any $C_j(L^m(R))$ for $j \neq g$.
- (5) If $v \in S_i$, if $C(v)$ and $C_{i-k}(L^m(R))$ are not disjoint, and if u is a source of $C(v)$, then the distance from u to $C_{i-k}(L^m(R))$ is at most k and it is k if and only if $u \in A_i(L^m(R))$.
- (6) If u and w are sources of $C(v)$, $v \in S_i$, then $C_u(A_i(L^m(R)))$ and $C_w(A_i(L^m(R)))$ are (p, q) -hinged with $p, q \leq k$. Moreover, if $u, w \in A_i(L^m(R))$, then $p=q$ and p is the distance from u to the vertex of intersection of the dipaths from u and w to $r_i^{(m)}$.
- (7) Every edge of $L^m(D)$ not in $U(L^m(R))$ is in some $C(v)$, $v \in S \cup T$.

The above gives a detailed description of the structure of $L^m(D)$. The following sequence of digraphs, each one a *portion* of the line digraph of the preceding one will serve to illustrate these ideas. In the figures only the images of a given arborescence and counterarborescence, and the portion of the digraph that they generate, are drawn. They will also illustrate how the weak components of $L^m(D)$ not containing a dicycle arise.

Before proceeding to the actual determination of the period of a periodic digraph we give a result that describes a partitioning of periodic digraphs into classes whose elements eventually have the same line digraph iterates. Toward this end let D_i , $i=1, 2$, be two basic unicyclic digraphs with dicycles $R_i=(r_{i,1}, r_{i,2}, \dots, r_{i,n}, r_{i,1})$,

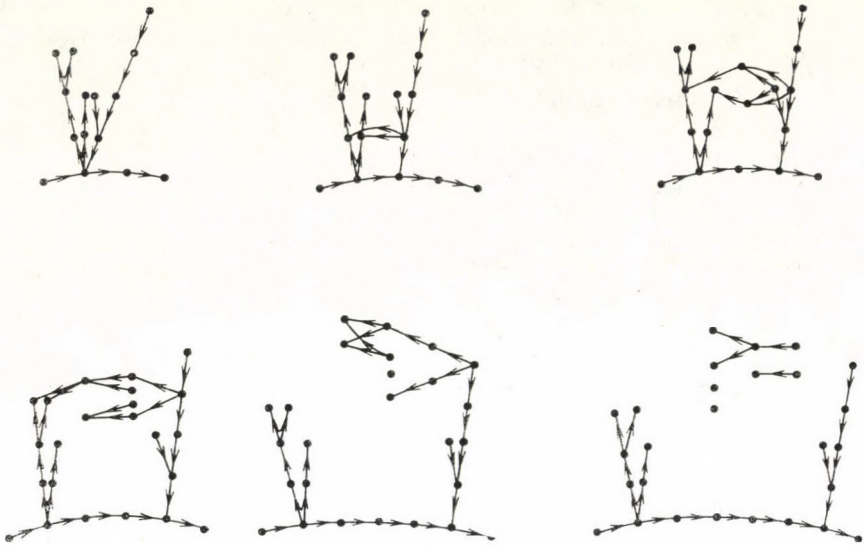


Fig. 1

with arborescences $A_{i,j}$ at $r_{i,j}$, and with counterarborescences $C_{i,j}$ at $r_{i,j}$, $1 \leq i \leq 2$, $1 \leq j \leq n$.

We say that D_1 and D_2 have the same arborescence sequence if there is an integer $p \geq 0$ such that $A_{1,j} \cong A_{2,j+p}$ for all j . Similarly D_1 and D_2 have the same counterarborescence sequence if there is an integer $q \geq 0$ such that $C_{1,j} \cong C_{2,j+q}$ for all j .

THEOREM 2. Let D_1 and D_2 be finite basic unicyclic digraphs. Then there are integers m_1 and m_2 such that $L^{m_1}(D_1) \cong L^{m_2}(D_2)$ if and only if D_1 and D_2 have the same arborescence sequence and the same counterarborescence sequence.

PROOF: If $\sigma: L^{m_1}(D_1) \cong L^{m_2}(D_2)$, then there is an integer q , $0 \leq q \leq n-1$, such that $\sigma(r_{1,j}^{(m_1)}) = r_{2,j+q}^{(m_2)}$ and thus such that $A_{1,j}(L^{m_1}(R_1)) \cong A_{2,j+q}(L^{m_2}(R_2))$ and $C_{1,j}(L^{m_1}(R_1)) \cong C_{2,j+q}(L^{m_2}(R_2))$. But $A_{i,j}(L^m(R)) \cong A_{i,j-m}(R)$ and $C_{i,j}(L^m(R)) \cong C_{i,j}(R)$ so $A_{1,j}(R_1) \cong A_{1,j+m_1}(L^{m_1}(R_1)) \cong A_{2,j+m_1+q}(L^{m_2}(R_2)) \cong A_{2,j+m_1+q-m_2}(R_2)$ and $C_{1,j}(R_1) \cong C_{1,j}(L^{m_1}(R_1)) \cong C_{2,j+q}(L^{m_2}(R_2)) \cong C_{2,j+q}(R_2)$. Take $p = m_1 + q - m_2$.

Conversely $U(L^k(D_1)) \cong D_2$ where $k = q - p$ if $p \leq q$ and $k = q - p + n$ if $p > q$. Thus $L^{m+k}(D_1) \cong L^m(D_2)$ for m sufficiently large.

We are now ready to determine the period of a finite periodic digraph.

THEOREM 3. Let D be a finite basic unicyclic digraph with dicycle $R = (r_1, r_2, \dots, r_n, r_1)$. Then D has period $t = (a, b)$ where a and b are the smallest positive integers such that $A_i(R) \cong A_{i+a}(R)$ and $C_i(R) \cong C_{i+b}(R)$ for all i , $1 \leq i \leq n$.

PROOF. Let $s = (a, b)$ where a and b are as in the theorem, and let t be the period of D . Thus there are integers x and y such that $ax + by = s$. Let $p \equiv ax \pmod{n}$ and $q \equiv ax - s \pmod{n}$ with $0 \leq p, q < n$. Thus $p - q \equiv s \pmod{n}$, so either $p - q = s$ or $p - q = s - n$. Assume that $p = q + s$. Then $A_{j-q}(L^s(R)) \cong A_{j-q-s}(R) \cong A_{j-p}(R) \cong A_j(R)$ since a divides p , and $C_{j-q}(L^s(R)) \cong C_{j-q}(R) \cong C_j$ since b divides q . Hence there

is an isomorphism $\sigma: D \cong U(L^s(D))$ where $\sigma(r_i) = r_i^{(s)}$. Thus, as before, $L^{m+s}(D) \cong L^m(D)$ for m sufficiently large. It follows that t divides s . But clearly a and b , and hence s , divide t so $t=s$. The situation is similar if $p=q+s-n$; again we find that $A_{j-q}(L^s(R)) \cong A_j(R)$ and $C_{j-q}(L^s(R)) \cong C_j(R)$.

We thus see that the correct version of Theorem 10 in [1] is given by the

Corollary: A finite unicyclic digraph has period one if and only if the lengths of the repetitive segments of the arborescence and of the counterarborescences are relatively prime.

It remains to find the period of a general periodic finite digraph. Let D be a digraph with dicycles R_i , $1 \leq i \leq r$, no two of which are joined by a dipath. Let D have period t and let $U(R_i)$ have period t_i . By the last theorem it is natural to conjecture that $t = \bar{i}$ where \bar{i} is the least common multiple of the t_i , $1 \leq i \leq r$. However, L might permute some of the components; for example if $L^{m+t}(D) \cong L^m(D)$ and $t > 1$, then the disjoint union of $L^m(D)$, $L^{m+1}(D)$, ..., and $L^{m+t-1}(D)$ is a digraph of period one.

We find t as follows. For each i , let $U(i)$ be the set of all $U(R_j)$ having the same line digraph orbits as $U(R_i)$ does, i.e., by Theorem 2, $U(i)$ is the set of all $U(R_j)$ having the same arborescence sequence and the same counterarborescence sequences as $U(R_i)$. Let t'_i be the smallest positive integer k such that the basic unicyclic parts of the sequence $U(R_i)$, $L^k(U(R_i))$, $L^{2k}(U(R_i))$, ..., $L^{t'_i-k}(U(R_i))$ (note that k must divide t_i) are paired with a sequence of distinct elements of $U(i)$ such that each pair are isomorphic. Note that if $U(R_j)$ is in this sequence then $U(R_j)$ will determine the same sequence (up to isomorphism). It is now clear that t is the least common multiple of the t'_i , $1 \leq i \leq r$.

In [3] the author left unsolved the characterization of three classes of infinite weakly connected digraphs D with $D \cong L(D)$. Two of these classes were distinguished by having a unique finite dicycle (an infinite dicycle is a two-way infinite dipath). By slight modifications of the above techniques one sees that there is a result like Theorem 3 for these two classes. However, the characterization of the periodic digraphs of the third type (an infinite number of infinite dicycles and no finite dicycles) is still out of sight.

Added in proof: L. Beineke and the author have since solved this case.

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**WEAK CONVERGENCE OF DISTRIBUTIONS FOR SUMS
OF INDEPENDENT HILBERT SPACE VALUED RANDOM VARIABLES**

by

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1°. One of the aims of this paper is to introduce a new numerical characteristic, V , of a probability distribution. V may, in a sense, be considered an analogue of the variance for the case when no moments are assumed to exist. The proof of theorem 1 (Section 5°) shows this concept to be a useful tool for the study of sums of Hilbert space valued independent random vectors. It may also be used to obtain new results for the distributions on the real line (theorem 2, Section 6°).

A similar concept was previously introduced in [3] and, in fact, the main results of the present paper, theorems 1, 2, might be obtained with the help of this rougher tool. The reason for a new definition is the fact (noticed by D. Szász) that the original version of V may lose some of its attractive properties (namely, 1) and 5) of p. 213 [3]) when the characteristic functional of the limiting distribution has too many zeroes (to avoid misunderstanding we remark this is not the case under study in [3]).

The author is deeply grateful to D. SZÁSZ for pointing out the shortcomings of the original definition of V . The author also acknowledges his gratitude to V. M. ZOLOTAREV for a helpful discussion of the results.

2°. Further we use the following notations

H is a separable Hilbert space; $\|u\|$, (u, v) are, respectively, the norm of an element and the scalar product; θ is the zero element of H .

$E(a)$ denotes the degenerate probability distribution (p.d.) concentrated at the element $a \in H$.

$L(\cdot, \cdot)$ is the Lévy—Prohorov's metric in the space of probability distributions in H defined by the relation (cf. [2]) $L(F, G) = \max \{\varepsilon_1, \varepsilon_2\}$,

$$\varepsilon_1 = \inf \{ \varepsilon : F(A) \leq G(A^\varepsilon) + \varepsilon \},$$

$$\varepsilon_2 = \inf \{ \varepsilon : G(A) \leq F(A^\varepsilon) + \varepsilon \},$$

where

$$A^\varepsilon = \{ u : \|u - v\| < \varepsilon, v \in A \}, \quad A \in \mathcal{A}$$

and \mathcal{A} is the class of all closed subsets of H . $I = \{y_n\}$ is a fixed countable (denumerable) set dense in H . $F * G$ stands for the convolution of p.d. F and G .

If F is a p.d., \bar{F} is the p.d. such that

$$\bar{F}(A) = F(-A);$$

$F^{(s)} = F * \bar{F}$ is the symmetrized p.d.

The characteristic functional (ch.f.) of a p.d. is defined by the formula

$$f(y) = \int_H \exp(i(y, u)) F(du).$$

The symbol $F = F[a, S, \mu]$ denotes the infinitely divisible (inf.div.) p.d., with the ch.f.

$$(1) \quad f(y) = \exp(i(a, y) - (Sy, y) + \int_H \mathbf{K}(y, u) \mu(du)),$$

$$\mathbf{K}(y, u) = \exp(i(y, u)) - 1 - \frac{i(y, u)}{1 + \|u\|^2},$$

where a is a fixed element of H , S — a fixed s -operator, μ is a spectral measure, that is, a measure on the Borel σ -algebra of H such that

$$\int_H \min \{\|u\|^2, 1\} \mu(du) < \infty.$$

Representation (1) is unique [1]. The measure μ is symmetric if for any Borel set A , $\mu(A) = \mu(-A)$.

A sequence of spectral measures $\{\mu_n\}$ converges weakly to the spectral measure μ if $\{\mu_n\}$ converges weakly to μ on the complement of each neighbourhood of the element θ with zero μ -measure of the boundary and

$$\sup_n \int_{\|u\| < 1} \|u\|^2 \mu_n(du) < \infty.$$

Let $F, F_j, j \geq 1$ be p.d.

Definition. The collection $\beta = (F_1, F_2, \dots)$ belongs to $U(F)$ if

$$1) \quad F = \prod_{j=1}^{\infty} * F_j,$$

2) $F_j = E(\theta)$ for $j \in I$, where $I = I(\beta)$ is a finite set.

Let F be a p.d. on the real line, f — its ch.f., $\Delta > 0$ — a fixed number such that $f(\Delta) \neq 0$. Following V. M. ZOLOTAREV [4] we call the number

$$C(\Delta, F) = \Delta^{-1} \operatorname{Im} \ln f(\Delta)$$

Δ -centre of the p.d. F .

It is worthwhile to point out some properties of Δ -centres:

1) If $F = F_1 * F_2$, then $C(\Delta, F) = C(\Delta, F_1) + C(\Delta, F_2)$,

2) if $L(F_n, F) \rightarrow 0$, then $C(\Delta, F_n) \rightarrow C(\Delta, F)$,

3) if $C(\Delta, F) = 0$, then $\int_{-\infty}^{\infty} \sin(\Delta u) F(du) = 0$.

3°. This section deals with a set of convenient numerical characteristics of a probability distribution.

Let us consider a convolution

$$F = \prod_j * F_j.$$

Using symbol \hat{f}_j to denote the ch.f. of inf.div. p.d. $\hat{F}_j[\theta, 0, F_j^{(s)}]$ corresponding to the component F_1 , we define $v(F_j, F)$ by the relation

$$V(F_j, F) = \sum_{n=1}^{\infty} 2^{-n} \frac{-\ln \hat{f}_j(y_n)}{1 - \sum_i \hat{f}_i(y_n)}.$$

For an arbitrary symmetric spectral measure μ we define the number

$$V(\mu) = \sum_{n=1}^{\infty} 2^{-n} \frac{-\ln \hat{f}(y_n)}{1 - \ln \hat{f}(y_n)},$$

where \hat{f} is the ch.f. of the inf.div. p.d. $\hat{F} = \hat{F}[\theta, 0, \mu]$. In particular, if the convolution F contains a single nontrivial component, $V(F, F) = V(F^{(s)})$. In order that the definition of the characteristic V is correct we must show that the series $\sum_i \ln \hat{f}_j(y)$ converges for any element $y \in H$. We need only consider the case when the convolution $F = \prod_j^* F_j$ consists of infinite number of components. For this purpose it is sufficient to prove that the product $\prod_{j \leq N} \hat{f}_j(y)$ converges to some ch.f. of inf.div. p.d. To prove this we note that the weak convergence of the convolution $F^{(s)} = \prod_j^* F_j^{(s)}$ theorem 2 [1] p. 457 and the generalized three series theorem (cf. [1], p. 455) imply the convergence for any $c > 0$ of the series

$$\sum_j \int_{\|u\| < c} \|u\|^2 F_j^{(s)}(du),$$

$$\sum_j F_j^{(s)}(\{u: \|u\| > c\}).$$

It follows from the definition of \hat{f}_j that the product $\prod_{j \leq N} \hat{f}_j$ converges to the ch.f. of inf.div. p.d. $G[\theta, 0, \sum_j F_j^{(s)}]$. We shall point out some properties of the numerical characteristics just introduced.

1) If the symmetric spectral measures μ_n converge weakly to the spectral measure μ , then $V(\mu_n) \rightarrow V(\mu)$.

2) If $L(\prod_j^* F_{nj}, \prod_j^* F_j) \rightarrow 0$, $L(F_{nj}, F_j) \rightarrow 0$ as $n \rightarrow \infty$, then

$$V(F_{ni}, \prod_j^* F_{nj}) \rightarrow V(F_i, \prod_j^* F_j), \quad i = 1, 2, \dots$$

as $n \rightarrow \infty$.

3) For each element $a \in H$ we have

$$V(F_1 * E(a), F_1 * F_2 * E(a)) = V(F_1, F_1 * F_2 * E(a)) = V(F_1, F_1 * F_2).$$

4) $\sum_i V(F_i, \prod_j^* F_j) = V(\sum_j F_j^{(s)})$.

5) If $V(F_1, F_2 * F_2) = 0$, then F_1 is a degenerate p.d. We shall only prove properties 1) and 5), the rest being fairly obvious. In order to prove 1) we note that from the weak convergence of spectral measures μ_n it follows ([1]) that $L(\hat{F}_n, \hat{F}) \rightarrow 0$, where

$\hat{F}_n = \hat{F}_n[\theta, 0, \mu_n]$, $\hat{F} = \hat{F}[\theta, 0, \mu]$. Let $\varepsilon > 0$ be an arbitrary positive number. We fix a number N such that $\sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon$. For each element $y \in H$, $\hat{f}_n(y) \rightarrow \hat{f}(y)$. Therefore, there exists N_1 such that for $n > N_1$

$$\sum_{m=1}^N 2^{-m} \left| \frac{\ln \hat{f}_n(y_m)}{1 - \ln \hat{f}_n(y_m)} - \frac{\ln \hat{f}(y_m)}{1 - \ln \hat{f}(y_m)} \right| < \varepsilon.$$

Hence, we have $|V(\mu_n) - V(\mu)| < 3\varepsilon$ for $n > N_1$. This proves 1) since $\varepsilon > 0$ is arbitrary.

In order to prove 5) we note that the equality $V(F_1, F_1 * F_2) = 0$ implies that $\hat{f}_1(y) = 1$ for $y \in I$. Therefore, $\hat{f}_1(y) = 1$ for every $y \in H$. The representation (1) being unique, it follows that $F_1^{(s)} = E(\theta)$.

4°. Let $\xi_{n1}, \xi_{n2}, \dots, \xi_{nm_n}$ be a double array of row-wise independent random variables with values in H , and let F be the p.d. with the ch.f. f . For each element $y \in H$ (ξ_{nj}, y), $1 \leq j \leq m_n$, $n = 1, 2, \dots$ are real random variables. If F_{nj} is the p.d. of random element ξ_{nj} , F_{nj}^y denotes the p.d. of the random variable (ξ_{nj}, y) . Let

$$F_n = \prod_{j=1}^{m_n} * F_{nj} = \prod_{j=1}^{\infty} * F_{nj} = \prod_j * F_j,$$

where $F_{nj} = E(\theta)$ for $j > m_n$, $n = 1, 2, \dots$

$$\Delta \equiv \Delta(y) = \sup \{v: |f(vy)| \geq \frac{1}{2}, 0 < v \leq 1\},$$

$$C(\Delta, F_{nj}^y) = \Delta^{-1} \operatorname{Im} \ln f_{nj}(\Delta y), \quad C(\Delta, F^y) = \Delta^{-1} \operatorname{Im} \ln f(\Delta y),$$

$$F_{nj}^y(u) = F_{nj}^y(u + c(\Delta, F_{nj}^y)),$$

$$\mathcal{F}_{nj}^y(u) = \int_{-\infty}^u \frac{z^2}{1+z^2} \hat{F}_{nj}^y(dz), \quad \bar{F}_n^y = \sum_j \mathcal{F}_{nj}^y$$

m_{nj}^y stands for a median of the random variable ξ_{nj}^y . Note that for every number $r > 0$ there exists a number $v = v(r) > 0$ such that $\Delta = \Delta(y) \geq v$ for $\|y\| \leq r$ (cf. [3], p. 219). We shall need some auxiliary statements.

LEMMA 1. Let $F_n = \prod_{j=1}^{m_n} * F_{nj}$ be a given sequence of p.d. such that

- 1) P.d. F_{nj} , $1 \leq j \leq m_n$ are uniformly asymptotically constants,
- 2) for any number $r > 0$ there exist $N_1 = N_1(r) > 0$ and $\alpha = \alpha(r) > 0$, such that

$$\inf_{n > N_1} \inf_{\|y\| \leq r} |f_n(y)| \geq \alpha,$$

then for any number $r > 0$ there exist $N = N(r) > 0$ and a number $c = c(r) > 0$ such that

$$\sup_{n > N} \sup_{\|y\| \leq r} \bar{F}_n^y(+\infty) \leq c.$$

PROOF. In accordance with lemma 7 [3] (see the proof) there exists a number $N_2 > 0$ such that for $n > N_2$ and any fixed element $y \in H$

$$\bar{F}_n^y(+\infty) \leq c_1(y) \sum_{j=1}^{m_n} Mg(\hat{\xi}_{nj}^y - m_{nj}^y),$$

$$g(x) = \frac{x^2}{1+x^2}, \quad c_1(y) = 2 + [A(y)]^{-2}.$$

We mentioned above that for every number $r > 0$ there existed a number $v = v(r) > 0$ such that $A(y) \geq v$. Denote $c_2 = c_2(r) = 2 + v^{-2}$. In accordance with estimate in [5] (cf. p.p. 319—320, (1) and (2) there exists a number N_3 such that for $n > N_3$

$$\sup_{\|y\| \leq r} \sum_{j=1}^{m_n} Mg(\hat{\xi}_{nj}^y - m_{nj}^y) \leq 2c(1) \sup_{\|y\| \leq r} \sum_{j=1}^{m_n} \int_0^1 [1 - |\hat{f}_{nj}(ty)|] dt \leq$$

$$\leq 4c(1) \sup_{\|y\| \leq r} \int_0^1 \left| \ln \left| \prod_{j=1}^{m_n} \hat{f}_{nj}(ty) \right| \right| dt \leq -4c(1) \ln \alpha \equiv c_3.$$

We may take $N = \max \{N_1, N_2, N_3\}$, $c = c_2 \cdot c_3$. The lemma is proved.

LEMMA 2. Let $L(F_n, F) \rightarrow 0$ as $n \rightarrow \infty$.

If there exists a sequence of collections $(G_{n1}, G_{n2}, \dots) \in U(F)$ such that

$$\limsup_{n \rightarrow \infty} \sup_j L(F_{nj}, G_{nj}) = 0$$

then for each $r > 0$

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} \sup_j L(\hat{F}_{nj}^y, \hat{G}_{nj}^y) = 0.$$

PROOF. Suppose that the conclusion of lemma 2 is wrong. In this case there exists a sequence of indices $\{n', j_{n'}\}$ and elements $y_{n'}$ (not necessarily different) such that

$$(2) \quad \lim_{n' \rightarrow \infty} L(\hat{F}_{n'j_{n'}}^{y_{n'}}, \hat{G}_{n'j_{n'}}^{y_{n'}}) = \delta > 0,$$

the sequences of p.d. $\{\hat{F}_{n'j_{n'}}^{y_{n'}}\}$ and $\{\hat{G}_{n'j_{n'}}^{y_{n'}}\}$ are shift-compact (ch. [5], p. 206). On the other hand (lemma 9, [3]):

$$\lim_{n \rightarrow \infty} \sup_j \sup_{\|y\| \leq r} |f_{nj}^y(y) - \hat{g}_{nj}(y)| = 0,$$

where functions \hat{g}_{nj} are defined by G_{nj} in the same way as \hat{f}_{nj} are defined by F_{nj} . The last relation and the shift-compactness of sequences $\{\hat{F}_{n'j_{n'}}^{y_{n'}}\}$ and $\{\hat{G}_{n'j_{n'}}^{y_{n'}}\}$ lead to a contradiction with (2). Lemma 2 is proved.

LEMMA 3. If $L(F_n, F) \rightarrow 0$ as $n \rightarrow \infty$ and there exists a sequence of collections $(G_{n1}, G_{n2}, \dots) \in U(F)$ such that

$$\lim_{n \rightarrow \infty} \sup_j L(F_{nj}, G_{nj}) = 0$$

then for each $r > 0$

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} \sup_j L(\mathcal{F}_{nj}^y, \mathcal{G}_{nj}^y) = 0,$$

where the p.d. \mathcal{G}_{nj} is defined by G_{nj} in the same way as \mathcal{F}_{nj}^y is defined by F_{nj} . If, in addition to that, for a sequence of sets $L_n \subset \{1, 2, \dots\}$ and arbitrary $r > 0$ the following conditions hold

- 1)
$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} L\left(\sum_{j \in L_n} \hat{F}_{nj}^y, \sum_{j \in L_n} \hat{G}_{nj}^y\right) = 0.$$
- 2) there exists a number $c = c(r) > 0$ such that

$$\sup_n \sup_{\|y\| \leq r} \sum_{j \in L_n} (\mathcal{F}_{nj}^y(+\infty) + \mathcal{G}_{nj}^y(+\infty)) \leq c$$

then

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} L\left(\sum_{j \in L_n} \mathcal{F}_{nj}^y, \sum_{j \in L_n} \mathcal{G}_{nj}^y\right) = 0.$$

The proof of this lemma is similar to the proof of lemma 2.

LEMMA 4. *If the sequence of the p.d. $F_n = \prod_j^* F_{nj}$ is shift-compact and*

$$\sup_{j \in L_n} V(F_{nj}, F_n) \rightarrow 0, \quad L_n \subset \{1, 2, \dots\}$$

as $n \rightarrow \infty$, then p.d. F_{nj} , $j \in L_n$ are uniformly asymptotically constants.

The proof of this lemma is in paper [3] (lemma 4). The only difference is that we use the characteristics defined in this paper instead of those of [3].

5°. In this section we prove a theorem which is a modification of theorem 1 in [3] where the limiting p.d. possesses a special property.

THEOREM 1. *Suppose that the ch.f. $f(y)$ of the p.d. F is not zero for every element $y \in H$. In order that*

$$\lim_{n \rightarrow \infty} L(F_n, F) = 0$$

the following conditions are necessary and sufficient:

- 1) there exists a sequence of collections $(G_{n1}, G_{n2}, \dots) \in U(F)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\alpha_n = \sup_j L(F_{nj}, G_{nj})$,

- 2) for all $r > 0$

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} \left| \sum_j [C(\Delta, F_{nj}^y) - C(\Delta, G_{nj}^y)] \right| = 0,$$

- 3) for all $r > 0$

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} L(\bar{F}_n^y, \bar{G}_n^y) = 0,$$

where \bar{G}_n^y is defined by G_{nj} , $n, j = 1, 2, \dots$ in the same way as \bar{F}_n^y is defined by F_{nj} , $n, j = 1, 2, \dots$

- 4) the sequence of measures $\hat{F}_n = \sum_j F_{nj}^{(s)}$ is bounded and weakly compact on the complement of every neighbourhood of the element θ ,

- 5) the sequence of s -operators $\{T_n(t)\}$

$$(T_n(t)y, y) = \int_{\|u\| \leq t} (u, y)^2 \hat{F}_n(du)$$

is compact for every number $t > 0$.

Remark. It is proved in [3] that conditions 1), 2), 4), 5) and two more conditions:

(a) for every element $y \in \{y: |f(y)| \neq 0\}$

$$\lim_{n \rightarrow \infty} \sum_j [\hat{f}_{nj}(y) - \hat{g}_{nj}(y)] = 0.$$

(b) there exists a number $l > 0$ such that

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq l} \left| \sum_j [\hat{f}_{nj}(y) - \hat{g}_{nj}(y)] \right| = 0$$

are necessary and sufficient for the weak convergence of p.d. F_n to the p.d. F . It is not easy to verify that if the ch.f. of the p.d. F is not zero for every element $y \in H$ then conditions (a) and (b) can be replaced by the only condition

3') for all $r > 0$

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} \left| \sum_j [\hat{f}_{nj}(y) - \hat{g}_{nj}(y)] \right| = 0.$$

The proof of this fact follows closely that of part (b) in theorem 1 [3].

PROOF of theorem. Necessity. By the previous reasoning we need only prove condition 3) assuming that 1), 2), 3'), 4), 5) are fulfilled. For this purpose we fix $r > 0$ and denote

$$\beta_n = \sup_j \sup_{\|y\| \leq r} |\hat{f}_{nj}(y) - \hat{g}_{nj}(y)|,$$

$$\delta_n = \sup_{\|y\| \leq r} \sup_j L(\hat{F}_{nj}^y, \hat{G}_{nj}^y),$$

$$L_n = \{j: V(F_{nj}, F_n) < \gamma_n\}, \quad \gamma_n = \max\{\sqrt{\beta_n}, \sqrt{\delta_n}\},$$

L'_n — the complement of L_n with respect to the set $\{1, 2, \dots\}$. The number of elements in L'_n is no greater than γ_n^{-1} since

$$\sum_{j=1}^{m_n} V(F_{nj}, F_n) = V(F_n, F_n) \leq 1.$$

In accordance with lemma 9 [3]

$$\lim_{n \rightarrow \infty} \beta_n = 0.$$

Therefore

$$(3) \quad \sup_{\|y\| \leq r} \sum_{j \in L_n} |\hat{f}_{nj}(y) - \hat{g}_{nj}(y)| \leq \sqrt{\beta_n} \rightarrow 0$$

as $n \rightarrow \infty$. 3) and (3) imply that

$$(4) \quad \sup_{\|y\| \leq r} \left| \sum_{j \in L_n} [\hat{f}_{nj}(y) - \hat{g}_{nj}(y)] \right| \rightarrow 0$$

as $n \rightarrow \infty$. From (4) it follows that for each fixed real number t

$$(5) \quad \sup_{\|y\| \leq r(t)} \left| \exp \left(\sum_{j \in L_n} \int_{-\infty}^{\infty} \mathbf{K}^y(t, u) \hat{F}_{nj}^y(du) \right) - \exp \left(\sum_{j \in L_n} \int_{-\infty}^{\infty} \mathbf{K}^y(t, u) \hat{G}_{nj}^y(du) \right) \right| \rightarrow 0$$

as $n \rightarrow \infty$, where $r(t) = r$, when $|t| \leq 1$ and $|t|^{-1}r$, when $|t| > 1$,

$$\mathbf{K}^y(t, u) = \exp(itu) - 1 - it \frac{\sin(\Delta(y)u)}{\Delta(y)}.$$

For every fixed real number t (cf. [3], (26) on p. 223)

$$(6) \quad \sup_{\|y\| \leq r(t)} \left| \sum_j [\ln \hat{f}_{nj}(ty) - \int_{-\infty}^{\infty} \mathbf{K}^y(t, u) \hat{F}_{nj}^y(du)] \right| \rightarrow 0$$

as $n \rightarrow \infty$. Relation (6) remains true if we replace functions \hat{f}_{nj} by functions \hat{g}_{nj} . From the relation $L(F_n, F) \rightarrow 0$ and condition 2) it follows that

$$\lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} |\hat{f}_n(y) - \hat{f}(y)| = 0.$$

Relation (3), the identity

$$(7) \quad \hat{f}_n - \hat{f} = \prod_{j \in L_n} \hat{f}_{nj} \left[\prod_{j \in L'_n} \hat{f}_{nj} - \prod_{j \in L'_n} \hat{g}_{nj} \right] + \prod_{j \in L'_n} \hat{g}_{nj} \left[\prod_{j \in L_n} \hat{f}_{nj} - \prod_{j \in L_n} \hat{g}_{nj} \right]$$

and the fact that

$$\inf_{\|y\| \leq r} \left| \prod_{j \in L'_n} \hat{g}_{nj}(y) \right| \geq \inf_{\|y\| \leq r} |\hat{f}(y)| > 0$$

imply that

$$(8) \quad \lim_{n \rightarrow \infty} \sup_{\|y\| \leq r} \left| \prod_{j \in L_n} \hat{f}_{nj}(y) - \prod_{j \in L_n} \hat{g}_{nj}(y) \right| = 0.$$

Suppose that condition 3) is wrong. In this case there exist a sequence of indexes $\{n'\}$ and elements $y_{n'} \in H$, $\|y_{n'}\| \leq r$ such that

$$(9) \quad \lim_{n' \rightarrow \infty} L(\bar{F}_{n'}^{y_{n'}}, \bar{G}_{n'}^{y_{n'}}) = \delta_1 > 0.$$

In accordance with (8) and lemma 1 functions

$$R_{n1}^y = \sum_{j \in L_n} \mathcal{F}_{nj}^y, \quad R_{n2}^y = \sum_{j \in L_n} \mathcal{G}_{nj}^y, \quad \|y\| \leq r, \quad n = 1, 2, \dots$$

are uniformly bounded. By the Helly's lemma ([5], p. 192) we can choose a sequence of indices $\{n''\} \subset \{n'\}$ such that functions $R_{nj}^{y_{n''}}$, $j=1, 2$ converge weakly to functions R_1 and R_2 , respectively. Let us prove that $R_1 = R_2$. For that purpose we suppose that numbers $\Delta(y_{n''})$ converge to some number $\delta > 0$. This is not a restriction since $1 \geq \Delta(y) \geq v(r) > 0$ for $\|y\| \leq r$ (see [3], p. 219). Therefore, if that is wrong we can take some subsequence of $\{n''\}$ which possesses the desired property. Note then that

$$(10) \quad \left| \frac{\sin \delta u}{\delta} - \frac{\sin(\Delta(y_{n''})u)}{\Delta(y_{n''})} \right| \leq \sum_{m=1}^{\infty} \frac{|\delta^{2m} - [\Delta(y_{n''})]^{2m}|}{(2m+1)!} |u|^{2m+1} \leq$$

$$\leq |\delta - \Delta(y_{n''})| \sum_{m=1}^{\infty} \frac{[\delta + \Delta(y_{n''})]^{2m-1}}{(2m)!} |u|^{2m+1},$$

since

$$|\delta^{2m} - [\Delta(y_{n''})]^{2m}| \leq |\delta - \Delta(y_{n''})| \cdot \sum_{j=0}^{2m-1} \delta^j [\Delta(y_{n''})]^{2m-1-j} \leq$$

$$\cong |\delta - \Delta(y_{n''})| \cdot \sum_{j=0}^{2m-1} [\delta + \Delta(y_{n''})]^{2m-1}.$$

On account of (10) the inf.div. p.d. defined by the ch.f.

$$\exp \left(\int_{-\infty}^{\infty} \mathbf{K}^{y_{n''}}(t, u) \frac{1+u^2}{u^2} R_{n''j}^{y_{n''}}(du) \right), \quad j = 1, 2; \quad n = 1, 2, \dots$$

converge to the inf.div. p.d. defined by the ch.f.

$$\exp \left(\int_{-\infty}^{\infty} \bar{\mathbf{K}}(t, u) \frac{1+u^2}{u^2} R_j(du) \right), \quad j = 1, 2,$$

$$\bar{\mathbf{K}}(t, u) = \exp(itu) - 1 - it \frac{\sin \delta u}{\delta}.$$

From (5), (6) and (8) it follows that $R_1 = R_2$. Therefore

$$(11) \quad \lim_{n'' \rightarrow \infty} L \left(\sum_{j \in L_{n''}} \mathcal{F}_{n''j}^{y_{n''}}, \sum_{j \in L_{n''}} \mathcal{G}_{n''j}^{y_{n''}} \right) = 0.$$

We have from obvious relation

$$(12) \quad \limsup_{n \rightarrow \infty} \sup_{\|y\| \leq r} L \left(\sum_{j \in L_n} \hat{F}_{nj}^y, \sum_{j \in L_n} \hat{G}_{nj}^y \right) \cong \limsup_{n \rightarrow \infty} \sup_{\|y\| \leq r} \sum_{j \in L_n} L(\hat{F}_{nj}^y, \hat{G}_{nj}^y) \cong \lim_{n \rightarrow \infty} \sqrt{\delta_n} = 0.$$

On account of lemma 3 (12) is true if we replace p.d. \hat{F}_{nj}^y and \hat{G}_{nj}^y by \mathcal{F}_{nj}^y and \mathcal{G}_{nj}^y , respectively. Using obvious inequality

$$L \left(\sum_j \mathcal{F}_{n''j}^{y_{n''}}, \sum_j \mathcal{G}_{n''j}^{y_{n''}} \right) \cong$$

$$= \max \left\{ L \left(\sum_{j \in L_{n''}} \mathcal{F}_{n''j}^{y_{n''}}, \sum_{j \in L_{n''}} \mathcal{G}_{n''j}^{y_{n''}} \right), L \left(\sum_{j \in L_{n''}} \mathcal{F}_{n''j}^{y_{n''}}, \sum_{j \in L_{n''}'} \mathcal{G}_{n''j}^{y_{n''}} \right) \right\}$$

we get a contradiction with (9). Necessity is proved.

Sufficiency. 3') follows from condition 3). Conditions 1), 2), 3'), 4), 5) are sufficient in order that p.d. F_n converge to the p.d. F (see the Remark in section 5°). The theorem is proved.

6°. Let $\xi_{n1}, \xi_{n2}, \dots, \xi_{nm_n}$ be a given double array of rowwise independent real random variables. We denote F_{nj} the p.d. of the random variable ξ_{nj} , $F_n = \prod_{j=1}^{m_n} F_{nj} = \prod_j^* F_{nj}$, where $F_{nj} = E(0)$ for $j > m_n$, $n = 1, 2, \dots$;

$$\hat{F}_{nj}(u) = F_{nj}(u + C(\Delta, F_{nj})), \quad \hat{F}(u) = F(u + C(\Delta, F)),$$

$C(\Delta, F_{nj})$, $C(\Delta, F)$ are Δ -centres of p.d. F_{nj} and F , where $\Delta > 0$ is to be specified later; $\hat{\xi}_{nj} = \xi_{nj} - C(\Delta, F_{nj})$, $n, j = 1, 2, \dots$; for every set $M_n \subset \{1, 2, \dots\}$ we define p.d.

$$R_1(M_n) = P \left(\min_{j \in M_n} \hat{\xi}_{nj} < u \right),$$

$$R_2(M_n) = P \left(\max_{j \in M_n} \hat{\xi}_{nj} < u \right).$$

Let $(G_{n1}, G_{n2}, \dots) \in U(F)$ be the collection from condition 1) of theorem 2. Consider a double array of rowwise independent random variables η_{nj} , $n, j=1, 2, \dots$ such that the random variable $\hat{\eta}_{nj}$ has the p.d. \hat{G}_{nj} . Let

$$R_3(M_n) = P(\min_{j \in M_n} \hat{\eta}_{nj} < u),$$

$$R_4(M_n) = P(\max_{j \in M_n} \hat{\eta}_{nj} < u).$$

THEOREM 2. Under the assumption that the ch.f. $f(t)$ of the p.d. F is not zero for all t , $-\infty < t < \infty$, in order that

$$\lim_{n \rightarrow \infty} L(F_n, F) = 0$$

the following conditions are necessary and sufficient:

1) there exist collections $(G_{n1}, G_{n2}, \dots) \in U(F)$ such that

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \alpha_n = \sup_j L(F_{nj}, G_{nj}),$$

2) there is a number $\Delta > 0$ such that

$$\lim_{n \rightarrow \infty} \sum_j [C(\Delta, F_{nj}) - C(\Delta, G_{nj})] = 0,$$

(3)

a) $\lim_{n \rightarrow \infty} L(R_1(L_n), R_3(L_n)) = 0,$

b) $\lim_{n \rightarrow \infty} L(R_2(L_n), R_4(L_n)) = 0,$

$$L_n = \{j: V(F_{nj}, F_n) + V(G_{nj}, F) < 2\sqrt{\alpha_n}\}.$$

PROOF. Necessity. On account of theorem 1 we must prove only condition 3). The number of elements in L'_n (L'_n is the complement of L_n with respect to set $\{1, 2, \dots\}$) is no greater than $\alpha_n^{-\frac{1}{2}}$. In accordance with lemma 2 [3]

$$L(\prod_{j \in L'_n}^* F_{nj}, \prod_{j \in L'_n}^* G_{nj}) \leq \sum_{j \in L'_n} L(F_{nj}, G_{nj}) \leq \sqrt{\alpha_n} \rightarrow 0$$

as $n \rightarrow \infty$. From the definition of Δ -centres (see section 2°) it follows

$$\lim_{n \rightarrow \infty} \lambda_n = 0,$$

$$\lambda_n = L(E(\sum_{j \in L'_n} C(\Delta, F_{nj})), E(\sum_{j \in L'_n} C(\Delta, G_{nj}))).$$

On account of lemma 2 [3] we have that

$$(13) \quad \lim_{n \rightarrow \infty} L(\prod_{j \in L'_n}^* \hat{F}_{nj}, \prod_{j \in L'_n}^* \hat{G}_{nj}) \leq \lim_{n \rightarrow \infty} (\sqrt{\alpha_n} + \lambda_n) = 0.$$

From condition 2) and the fact that $L(F_n, F) \rightarrow 0$ it follows that $L(\hat{F}_n, \hat{F}) \rightarrow 0$, as $n \rightarrow \infty$. In accordance with (7) and (13) we have that

$$(14) \quad \lim_{n \rightarrow \infty} L(\prod_{j \in L_n}^* \hat{F}_{nj}, \prod_{j \in L_n}^* \hat{G}_{nj}) = 0.$$

From the construction of sets L_n and lemma 4 it follows that p.d. $\hat{F}_{nj}, \hat{G}_{nj}, j \in L_n$ are uniformly asymptotically constant. Let us prove that they are uniformly asymptotically negligible. Indeed, let a_{nj} be numbers such that p.d. $\hat{F}_{nj} * E(a_{nj}), j \in L_n$, are uniformly infinitely small. Hence (see the properties of Δ -centres in section 2^o) uniformly over $1 \leq j \leq m_n$

$$a_{nj} = a_{nj} + C(\Delta, \hat{F}_{nj}) = C(\Delta, \hat{F}_{nj} * E(a_{nj})) \rightarrow 0$$

as $n \rightarrow \infty$. The same argument can be used to prove that p.d. $\hat{G}_{nj}, j \in L_n$ are uniformly asymptotically negligible. We shall prove, say 3), a) (3), b), can be proved in the same manner). Suppose that 3), a) is wrong. In this case there exists a sequence of indexes $\{n'\}$ such that

$$(15) \quad \lim_{n' \rightarrow \infty} L(R_1(L_{n'}), R_3(L_{n'})) = \delta > 0.$$

Since $\hat{F} = \prod_j^* \hat{G}_{nj}$, the sequence of p.d. $G_n = \prod_{j \in L_n}^* \hat{G}_{nj}$ is shift-compact. Therefore, there exist a sequence of indexes $\{n''\} \subset \{n'\}$ and numbers $a_{n''}$ such that p.d. $G_{n''} * E(a_{n''})$ converge weakly to some p.d. G . From (14) it follows that p.d. $\mathcal{F}_{n''} = \prod_{j \in L_{n''}}^* \hat{F}_{n''j} * E(a_{n''})$ also converge to the same p.d. G . We have mentioned above that p.d. \hat{G}_{nj} and $\hat{F}_{nj}, j \in L_n$, are uniformly asymptotically negligible. Therefore, the p.d. G is an inf.div. p.d. Applying the extremal criteria of [5] (p. 328) to p.d. $G_{n''}$ and $\mathcal{F}_{n''}$ we get a contradiction with (15). Necessity is proved.

Sufficiency. From the extremal criteria of [5] (p. 328), the shift-compactness of the sequence of p.d. G_n and condition 3) imply the validity of (14). Note that (13) is valid again. From (13) and (14) and lemma 2 [3] it follows that

$$(16) \quad \lim_{n \rightarrow \infty} L(\hat{F}_n, \hat{F}) = 0.$$

Condition 2) and (16) imply that

$$\lim_{n \rightarrow \infty} L(F_n, F) = 0.$$

Sufficiency is proved.

7^o. *Final remark.* In this section we introduce a different set of numerical characteristics of a p.d. The idea of this construction is due to V. M. ZOLOTAREV. For an arbitrary p.d. F with the ch.f. f let $\Gamma(F, r, r_1)$ be the set

$$\Gamma(F, r, r_1) = \{y: \|y\| \leq r_1, y \in Y, |f(y)| > r\}, \quad r_1 > 0, \quad r > 0.$$

Let

$$V(F, r, r_1) = \sum_{y_n \in \Gamma(F, r, r_1)} 2^{-n} \frac{-\ln |f(y_n)|}{1 - \ln |f(y_n)|}.$$

If the p.d. F is a convolution, $F = \prod_j^* F_j$

$$V(F_j; F, r, r_1) = \sum_{y_n \in \Gamma(F, r, r_1)} 2^{-n} \frac{\ln |f_j(y_n)|}{1 - \ln |f(y_n)|}.$$

It is not difficult to verify that these numerical characteristics possess properties 1)—5) of the section 3^o.

Remark. Numerical characteristics with similar properties may be introduced in a more general way. Let $F = \prod_j^* F_j$ be a given p.d. and μ be a given measure such that integrals

$$\int_H \frac{-\ln \hat{f}_j(y)}{1 - \sum_i \ln \hat{f}_i(y)} \mu(dy), \quad j = 1, 2, \dots$$

exist, where \hat{f}_j is the ch.f. of inf.div. p.d. determined by F_j as in the Section 3°. It is clear that the integrals above may be used to define analogues of $V(F_j, F)$ if the measure μ satisfies certain general conditions. The case under consideration in section 3° corresponds to the measure μ concentrated on the dense denumerable set $y = \{y_n\}$ with $\mu(\{y_n\}) = 2^{-n}$, $n = 1, 2, \dots$.

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ON THE EXISTENCE OF DENSITY ESTIMATORS

by
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Summary. Let \mathfrak{F} denote a class of density functions with respect to a σ -finite measure μ on a measurable space (X, \mathfrak{X}) , $P_f (f \in \mathfrak{F})$ the corresponding probability measures. The problem of estimating the density f based on n independent observations of a random variable, distributed according to P_f , is considered. — The following results are obtained:

1) For every $p > 1$

$$\delta_p(h) := \left[\sup_{f \in \mathfrak{F}} \int_{X^n \times X} |h(x^n, x) - f(x)|^p dP_f^n \otimes \lambda(x^n, x) \right]^{1/p}$$

is taken as a measure of deviation of an estimator (i.e. a measurable function defined on $X^n \times X$), λ being a σ -finite measure on (X, \mathfrak{X}) with $\sup_{f \in \mathfrak{F}} \int f^p d\lambda < \infty$. Then there is an estimator h_0 , minimizing $\delta_p(h)$ among all h 's with

$$(*) \quad \sup_{f \in \mathfrak{F}} \int |h|^p dP_f^n \otimes \lambda < \infty.$$

2) For $p > 1$, consider the class of all estimators h which are densities almost everywhere and satisfy (*). Among them there is an optimal one in the above sense.

3) Let \mathfrak{C} be a countably generated σ -algebra in \mathfrak{X}^n , sufficient for the class of product measures $\{P_f^n : f \in \mathfrak{F}\}$. For any h satisfying (*) the "conditional expectation" g of h , given $\mathfrak{C} \otimes \mathfrak{X}$, can be taken independent of f , satisfies (*) and $\delta_p(g) \leq \delta_p(h)$.

Results. Let (X, \mathfrak{X}) be a measurable space, μ and λ σ -finite measures defined on (X, \mathfrak{X}) . \mathfrak{F} denotes a given nonempty class of non-negative realvalued \mathfrak{X} -measurable functions f defined on X and satisfying $\int f d\mu = 1$. p is a fixed real number ≥ 1 . We further assume $\sup_{f \in \mathfrak{F}} \int f^p d\lambda$ to be finite for the p considered. $P_f := f\mu$ is the probability measure on (X, \mathfrak{X}) with density function f with respect to μ . For every $\mathfrak{X}^n \otimes \mathfrak{X}$ -measurable function h from $X^n \times X$ in $\overline{\mathbf{R}}$, the extended real line, we set

$$N_p(h; \mathfrak{F}, \lambda) := \left[\sup_{f \in \mathfrak{F}} \int |h|^p dP_f^n \otimes \lambda \right]^{1/p}.$$

For short, we sometimes write $\|h\|_{(p)}$, and, of course, $N_p(\cdot; \{f\}, \lambda)$ is the ordinary norm of the Lebesgue space $L_p(X^n \times X, \mathfrak{X}^n \otimes \mathfrak{X}, P_f^n \otimes \lambda) = L_p(P_f^n \otimes \lambda)$. $h_1 \sim h_2$ if $h_1 = h_2$ almost everywhere ($P_f^n \otimes \lambda$) for every $f \in \mathfrak{F}$ defines an equivalence relation in the set of all h 's as described above. We remark that this equivalence relation can be defined

by $h_1 = h_2$ a.e. ($\mu^n \otimes \lambda$), if the set of product measures $\{P_f^n: f \in \mathfrak{F}\}$ is equivalent to μ^n (cf. SCHMETTERER [5], WERTZ [8]). As usual we identify elements h with their equivalence classes, when no confusion arises. Let $A_p = A_p(\mathfrak{F}, \lambda)$ be the set of all h 's with $\|h\|_{(p)} < \infty$. It is known that A_p forms a Banach space under the norm $\|\cdot\|_{(p)}$, moreover A_p is a Riesz space under the pointwise order of functions (see WERTZ [8], PITCHER [4]).

The expression

$$\begin{aligned} \delta_p(h) &= \delta_p(h, \mathfrak{F}, \lambda) := \sup_{f \in \mathfrak{F}} N_p(h-f; \{f\}, \lambda) = \\ &= \left[\sup_{f \in \mathfrak{F}} \int_{X^n} \int_X |h(x^n, x) - f(x)|^p d\lambda(x) dP_f^n(x^n) \right]^{1/p} \end{aligned}$$

(where x^n stands for $(x_1, \dots, x_n) \in X^n$) serves as a measure of deviation of the estimator from the true density. In the case $p=1$, $\mu = \lambda =$ one-dimensional Lebesgue measure, δ_p has the following meaning: $\delta_p(h)$ is a uniform bound for the expected value of the area between the estimate $h(x^n, \cdot)$ (based on n observations $x^n = (x_1, \dots, x_n)$ of an X -valued random variable) and the density f underlying the distribution of this random variable, whatever $f \in \mathfrak{F}$ is true.

THEOREM 1. For each $p > 1$ there is an optimal density estimator h_0 in A_p , that is $\delta_p(h_0) = \min_{h \in A_p} \delta_p(h)$.

THEOREM 2. Among all h 's in A_p with $\int h(x^n, x) d\lambda(x) = 1$ for almost all x^n (i.e.: a.e. (P_f^n) for every $f \in \mathfrak{F}$) there is an optimal one.

THEOREM 3. Let \mathfrak{C} be a σ -algebra contained in \mathfrak{X}^n with a countable generator \mathfrak{G} and sufficient for $\{P_f^n: f \in \mathfrak{F}\}$. Let $p \geq 1$ be fixed. Then, for every $h \in A_p$ there exists a $\mathfrak{C} \otimes \mathfrak{X}$ -measurable function g in A_p with

$$(1) \quad \delta_p(g, \mathfrak{F}, \lambda) \leq \delta_p(h, \mathfrak{F}, \lambda).$$

Actually, g can be chosen such that:

- (i) $N_p(g-f; \{f\}, \lambda) \leq N_p(h-f; \{f\}, \lambda)$ for all $f \in \mathfrak{F}$;
- (ii) $g(\cdot, x)$ is a version of $E_f(h(\cdot, x) | \mathfrak{C})$, the conditional expectation of $h(\cdot, x)$ given \mathfrak{C} , for all x not belonging to a fixed set N of λ -measure zero.

In view of theorem 3 we can restrict our attention to functions in A_p , measurable with respect to $\mathfrak{C} \otimes \mathfrak{X}$.

If \mathfrak{C} contains all symmetric sets $A \in \mathfrak{X}^n$ (A is called symmetric, if $A = \{(x_{1/\pi}, \dots, x_{n/\pi}) : (x_1, \dots, x_n) \in A\}$ for every $\pi \in \mathfrak{S}_n$, the group of all permutations of $\{1, \dots, n\}$) only estimators symmetric in the observations are to be considered: let \mathfrak{A}_0 be the σ -algebra of all symmetric sets \mathfrak{X}^n ; let g^* be as described in theorem 3 and let g be defined by

$$(2) \quad g(x^n, x) := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} g^*(x_{1/\pi}, \dots, x_{n/\pi}, x).$$

g defines for every x a version of $E_f(g^*(\cdot, x) | \mathfrak{A}_0) = E_f(E_f(h(\cdot, x) | \mathfrak{C}) | \mathfrak{A}_0)$ and is clearly $\mathfrak{A}_0 \otimes \mathfrak{X}$ -measurable.

Hence we get as a

COROLLARY. If \mathfrak{C} contains all symmetric sets of \mathfrak{X}^n , for every $h \in A_p$ there is a $g \in A_p$ symmetric in the observations and satisfying (1) and i), ii) of theorem 3. g is given by formula (2).

In the case $p > 1$ the following obvious decision-theoretic interpretation of our results is possible: define by $L(f, g) := \int_X |g - f|^p d\lambda$ a loss function. Then there exists a minimax-estimator $h \in A_p$ for $f \in \mathfrak{F}$. If \mathfrak{C} is a countably generated σ -algebra, sufficient for $\{P_f^n\}$, then the $\mathfrak{C} \otimes \mathfrak{X}$ -measurable h 's in A_p form an essentially complete class of decision rules. Without restricting the generality we may assume that $\mathfrak{C} \subset \mathfrak{A}_0$, i.e. \mathfrak{C} consists only of symmetric sets, since in this special case $\mathfrak{C} \cap \mathfrak{A}_0$ is sufficient with \mathfrak{C} (for related problems see SCHMETTERER [7]).

Proofs. Theorem 1: Let $M_q = M_q(\mathfrak{F}, \lambda)$ be the set of all finite linear combinations $\sum_{i=1}^k \alpha_i g_i$ with k positive and integer, α_i real and $g_i \in L_q(X^n \times X, \mathfrak{X}^n \otimes \mathfrak{X}, P_{f_i}^n \otimes \lambda)$ with $f_i \in \mathfrak{F}$. q is determined by $(1/p) + (1/q) = 1$ and is possibly ∞ . Then $\langle h, \sum_i \alpha_i g_i \rangle := \sum_i \alpha_i \int h g_i dP_{f_i}^n \otimes \lambda$ defines a bilinear form on $A_p \times M_q$. By $\sigma(p, \mathfrak{F}, \lambda)$ we mean $\sigma(A_p(\mathfrak{F}, \lambda), M_q(\mathfrak{F}, \lambda))$, the weakest topology in $A_p(\mathfrak{F}, \lambda)$, for which all linear functionals $\langle \cdot, g \rangle$ ($g \in M_q$) are continuous (cf. BOURBAKI [1]). $\sigma(p, \mathfrak{F}, \lambda)$ is weaker than the weak topology of A_p (the term "weaker" being understood in the wide sense), since M_q is contained in A_p^* .

We recall that λ is assumed to be σ -finite; hence there is a probability measure ν , equivalent to λ and with $\varphi := d\nu/d\lambda > 0$ a.e. (λ); we may assume $\varphi(x) > 0$ for all $x \in X$. We have $d(P_f^n \otimes \nu)/d(P_f^n \otimes \lambda) = \varphi$, hence $h \in A_p(\mathfrak{F}, \lambda)$ iff $h\varphi^{-1/p} \in A_p(\mathfrak{F}, \nu)$. We set

$$\delta'_p = \delta'_p(h, \mathfrak{F}, \nu) := \sup_{f \in \mathfrak{F}} N_p(h - f\varphi^{-1/p}; \{f\}, \nu).$$

Now the measures $P_f^n \otimes \nu$ are dominated probability measures, hence the norm-closed balls of $A_p(\mathfrak{F}, \nu)$ are compact in the $\sigma(p, \mathfrak{F}, \nu)$ -topology (PITCHER [4]).

Define $\alpha := \inf_{h \in A_p(\mathfrak{F}, \nu)} \delta'_p(h, \mathfrak{F}, \nu)$. Since ω , the function vanishing identically, belongs to A_p , we have

$$0 \equiv \alpha \equiv \delta'_p(\omega, \mathfrak{F}, \nu) = \sup_{f \in \mathfrak{F}} \int_{X^n \times X} |f\varphi^{-1/p}|^p dP_f^n \otimes \nu = \sup_{f \in \mathfrak{F}} \int_{X^n \times X} f^p dP_f^n \otimes \lambda,$$

and this is finite by hypothesis. Hence there is a sequence in $A_p(\mathfrak{F}, \nu)$, say (h_k) , with $\alpha = \lim_{k \rightarrow \infty} \delta'_p(h_k, \mathfrak{F}, \nu)$. We easily conclude that (h_k) is contained in a norm-closed

ball, and we get the existence of a cluster point h' of $\{h_k\}$ in the $\sigma(p, \mathfrak{F}, \nu)$ -topology. Hence every neighbourhood of the form $\{h \in A_p(\mathfrak{F}, \nu) : \langle h - h', g \rangle \leq \varepsilon\}$ ($\varepsilon > 0$ and $g \in L_q(P_f^n \otimes \nu)$) contains a subsequence (h_{k_j}) of (h_k) . However, for every given g there is a subsequence (h_{k_j}) with $\lim_{j \rightarrow \infty} \langle h_{k_j} - h', g \rangle = 0$, and we conclude:

$$\begin{aligned} |\langle h' - f\varphi^{-1/p}, g \rangle| &= \lim_{j \rightarrow \infty} |\langle h_{k_j} - f\varphi^{-1/p}, g \rangle| \leq \\ &\equiv \left[\limsup_{j \rightarrow \infty} N_p(h_{k_j} - f\varphi^{-1/p}; \{f\}, \nu) \right] \cdot N_q(g; \{f\}, \nu) \equiv \alpha \cdot N_q(g; \{f\}, \nu). \end{aligned}$$

Considering $h' - f\varphi^{-1/p}$ as an element of $L_p(P_f^n \otimes \nu)$ we may interpret $h' - f\varphi^{-1/p}$ as an element of the bidual space $L_p^{**}(P_f^n \otimes \nu)$, and we get for every $f \in \mathfrak{F}$:

$$N_p(h' - f\varphi^{-1/p}; \{f\}, \nu) = \sup_{g \in L_q: N_q(g) \equiv 1} |\langle h' - f\varphi^{-1/p}, g \rangle| \leq \alpha.$$

We conclude $\delta'_p(h', \mathfrak{F}, \nu) = \alpha$. Now

$$\delta_p(h'\varphi^{1/p}) = \sup_{f \in \mathfrak{F}} \int_{X^n \times X} |h'\varphi^{1/p} - f|^p dP_f^n \otimes \lambda = \sup_{f \in \mathfrak{F}} \int_{X^n \times X} |h' - f\varphi^{-1/p}|^p dP_f^n \otimes \nu = \alpha.$$

Set $h_0 := h'\varphi^{1/p}$; we affirm h_0 is optimal in $A_p(\mathfrak{F}, \lambda)$. Suppose it is not; then there is an $h_1 \in A_p(\mathfrak{F}, \lambda)$ with $\delta_p(h_1) < \delta_p(h_0) = \alpha$, but $h_1\varphi^{-1/p} \in A_p(\mathfrak{F}, \nu)$ and $\delta'_p(h_1\varphi^{-1/p}) < \alpha$, which is a contradiction. Q.e.d.

Theorem 2: We set

$$D_\lambda := \{h \in A_p(\mathfrak{F}, \lambda): \int h(x^n, \cdot) d\lambda = 1 \text{ for almost all } x^n \in X^n\}$$

and

$$D_\nu := \{h \in A_p(\mathfrak{F}, \nu): \int h(x^n, \cdot) d\nu = 1 \text{ for almost all } x^n \in X^n\}.$$

Clearly D_ν is a convex set and is closed in the norm topology. By MAZUR's theorem (see HILLE—PHILLIPS [3]) D_ν is weakly closed and moreover closed in the $\sigma(p, \mathfrak{F}, \nu)$ -topology. Proceeding as in the proof of theorem 1 we can construct an $h' \in D_\nu$ minimizing $\delta'_p(h, \mathfrak{F}, \nu)$ with respect to $h \in D_\nu$. (We remark that $\delta'_p(h', \mathfrak{F}, \nu) \leq c$, anyhow, since we can take $h(x^n, x) := g(x)$, g being a density with respect to ν , and $\delta'_p(h', \mathfrak{F}, \nu) \leq \delta'_p(h, \mathfrak{F}, \nu) \leq c$ is valid). Then $h_0 := h'\varphi^{1/p}$ is in D_λ and minimizes $\delta_p(h, \mathfrak{F}, \lambda)$ with respect to $h \in D_\lambda$. Q.e.d.

Theorem 3: First we assume λ to be finite. We are going to prove that $\mathfrak{C} \otimes \mathfrak{X}$ is pairwise sufficient for the class $\mathfrak{P} := \{P_f^n \otimes \lambda: f \in \mathfrak{F}\}$, dominated by $\mu^n \otimes \lambda$. By a theorem of HALMOS and SAVAGE (see [2]) we have to show the $\mathfrak{C} \otimes \mathfrak{X}$ -measurability of $dP_f^n \otimes \lambda / d(P_f^n \otimes \lambda + P_g^n \otimes \lambda) =: \varphi_{f,g}$ for every pair $(f, g) \in \mathfrak{F} \times \mathfrak{F}$. Now $\varphi_{f,g}(x^n, x) = f_n(x^n) / [f_n(x^n) + g_n(x^n)]$ (f_n denoting the function $(x_1, \dots, x_n) \rightarrow \prod_{i=1}^n f(x_i)$), and this is \mathfrak{C} -measurable (considered as a function of x^n), since \mathfrak{C} is pairwise sufficient for $\{P_f^n: f \in \mathfrak{F}\}$. Therefore, $\mathfrak{C} \otimes \mathfrak{X}$ is pairwise sufficient for \mathfrak{P} , and since \mathfrak{P} is dominated, $\mathfrak{C} \otimes \mathfrak{X}$ is sufficient. Hence for every $h \in A_p$ there is a version $g \in A_p$ of the conditional expectation, independent of $f \in \mathfrak{F}$: $E_f(h | \mathfrak{C} \otimes \mathfrak{X})(x^n, x) = g(x^n, x)$ a.e. $(P_f^n \otimes \lambda) \forall f \in \mathfrak{F}$ (we speak of conditional expectation, although $P_f^n \otimes \lambda$ need not be a probability measure, however it is finite). We assert $g(\cdot, x)$ to be a version of $E_f(h(\cdot, x) | \mathfrak{C})$ for all x not belonging to a fixed null-set $(\lambda) N_1$. From the definition of conditional expectation we have:

$$\int_{C \times A} g dP_f^n \otimes \lambda = \int_{C \times A} h dP_f^n \otimes \lambda \quad \text{for all } C \in \mathfrak{C} \text{ and all } A \in \mathfrak{X}.$$

We conclude that for every $C \in \mathfrak{C}$ there is a set N_C of λ -measure zero with

$$\int_C g(\cdot, x) dP_f^n = \int_C h(\cdot, x) dP_f^n \quad \forall x \in \complement N_C.$$

In particular this equation holds for all $C \in \mathfrak{G}$, the countable generator of \mathfrak{C} ; without limitation of generality we may assume \mathfrak{G} to be a countable algebra. Setting $N_1 := \bigcup_{C \in \mathfrak{G}} N_C$ we get for each $C \in \mathfrak{G}$:

$$(3) \quad \int_C g(\cdot, x) dP_f^n = \int_C h(\cdot, x) dP_f^n \quad \forall x \in \mathfrak{L}N_1.$$

The class of all $C \in \mathfrak{C}$ fulfilling (3) is monotone and includes \mathfrak{G} , hence coincides with \mathfrak{C} , and we have shown (3) to hold for all C in \mathfrak{C} . By definition of conditional expectation, $g(\cdot, x)$ is a version of $E_f(h(\cdot, x)|\mathfrak{C})$ for all $x \in \mathfrak{L}N_1$.

Now we turn to the case of a σ -finite λ . There is an (almost) countable measurable decomposition $\{X_k\}$ of X with $\lambda(X_k) < \infty$. Set $\mathfrak{X}_k := X_k \cap \mathfrak{X}$ and let λ_k be the restriction of λ to \mathfrak{X}_k ; then $\lambda(A) = \sum_k \lambda_k(A \cap X_k)$ for every $A \in \mathfrak{X}$. By the foregoing we can construct a λ -null-set N_1 and $\mathfrak{C} \otimes \mathfrak{X}_k$ -measurable functions g_k such that $g_k(\cdot, x)$ is a version of $E_f(h(\cdot, x)c_{X_k}(x)|\mathfrak{C})$ for all $x \in \mathfrak{L}N_1$ (c denotes indicator functions). $g := \sum_k g_k$ is clearly $\mathfrak{C} \otimes \mathfrak{X}$ -measurable; integrating this formal series we get

$$\int_C g(\cdot, x) dP_f^n = \sum_k \int_C h(\cdot, x) c_{X_k}(x) dP_f^n = \int_C h(\cdot, x) dP_f^n$$

for all $C \in \mathfrak{C}$ and all $x \in \mathfrak{L}N_1$. Hence $g(\cdot, x)$ behaves like a conditional expectation a.e. (λ).

We note the inequality $|a+b|^p \geq |a|^p + pb|a|^{p-1} \operatorname{sgn} a$ for $p \geq 1$ and any real a, b (see SCHMETTERER [6]; the case $p=1$ is immediately verified), and set:

$$a = a(x^n, x) := g(x^n, x) - f(x), \quad b = b(x^n, x) := h(x^n, x) - g(x^n, x).$$

Since h belongs to A_p , $h(\cdot, x)$ is in $L_p(X^n, \mathfrak{X}^n, P_f^n)$ for all x 's not belonging to a λ -null-set N_2 . Because the functions $a(\cdot, x)$ and $\operatorname{sgn} a(\cdot, x)$ are \mathfrak{C} -measurable and the integrals occurring exist for every $x \in \mathfrak{L}(N_1 \cup N_2)$ we get by the rules for conditional expectations:

$$(4) \quad \begin{aligned} \int h(\cdot, x) |g(\cdot, x) - f(x)|^{p-1} \operatorname{sgn} a(\cdot, x) dP_f^n(x^n) &= \\ = \int g(\cdot, x) |g(\cdot, x) - f(x)|^{p-1} \operatorname{sgn} a(\cdot, x) dP_f^n(x^n). \end{aligned}$$

By integrating the above inequality with respect to P_f^n and λ , by taking equation (4) into consideration and by applying Fubini's theorem we get our assertion.

Q.e.d.

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SOME ABSOLUTELY MONOTONIC FUNCTIONS

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Abstract. Recently the functions $(1-r)^{-2\alpha}(1-2xr+r^2)^{-\alpha}$ were shown to be absolutely monotonic, i.e. their power series have nonnegative coefficients, for $-1 \leq x \leq 1$, $\alpha > 0$. When $\alpha=2$, this result will be improved to give the absolute monotonicity of $(1-xr)(1-r)^{-3}(1-2xr+r^2)^{-2}$, $-1 \leq x \leq 1$.

1. Introduction. ASKEY and POLLARD [4] proved the following theorem.

THEOREM A. *The functions*

$$(1.1) \quad (1-r)^{-2\alpha}(1-2xr+r^2)^{-\alpha}$$

are absolutely monotonic for $-1 \leq x \leq 1$, $\alpha > 0$, i.e. their power series coefficients are nonnegative.

This result is best possible, in the sense that

$$(1-r)^{-\beta}(1-2xr+r^2)^{-\alpha}$$

is not absolutely monotonic when $x = -1$ for any $\beta < 2\alpha$. For the coefficient of r is $\beta - 2\alpha$. The proof of Theorem A used

$$\begin{aligned} g_\alpha(r) &= (1-r)^{-2\alpha}(1-2xr+r^2)^{-\alpha} = e^{zh(r)} \\ &= \sum_{n=0}^{\infty} \frac{\alpha^n [h(r)]^n}{n!} \end{aligned}$$

and the fact that $h(r)$ has nonnegative power series coefficients for $-1 \leq x \leq 1$. It was somewhat surprising that a sharp result could be obtained by this method, since $g_\alpha(r)$ was decomposed into the sum of absolutely monotonic functions. However, this is less surprising when we realize that this result is not really best possible, but only best possible of its type. Recall Fejér's classical result that

$$\frac{1-r^2}{(1-r)^2(1-2xr+r^2)} = \frac{1+r}{(1-r)(1-2xr+r^2)}$$

is absolutely monotonic for $-1 \leq x \leq 1$. This result implies Theorem A for $\alpha=1$. It is not clear if $(1+r)^\alpha(1-r)^{-\alpha}(1-2xr+r^2)^{-\alpha}$ is absolutely monotonic for $-1 \leq x \leq 1$, $\alpha \geq 1$, but it clearly is for $\alpha=2, 3, \dots$. When $\alpha=2$ there is a different extension of Theorem A.

THEOREM 1. $(1-xr)(1-r)^{-3}(1-2xr+r^2)^{-2}$ is absolutely monotonic for $-1 \leq x \leq 1$.

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Theorem 1 will be a consequence of other extensions of (1.1) when $\alpha=2$, but these extensions are only partial extensions, since they do not hold for $-1 \leq x \leq 1$, but only for part of this interval. The first result was proved by SCHWEITZER [5], after partial results had been obtained by FEJÉR and SZEGŐ.

THEOREM B. *The function $(1+r)(1-r)^{-2}(1-2xr+r^2)^{-2}$ is absolutely monotonic for $-\frac{1}{2} \leq x \leq 1$.*

Next came a strengthening of Theorem B for $0 \leq x \leq 1$, [2].

THEOREM C. *The function $(1-r)^{-2}(1-2xr+r^2)^{-2}$ is absolutely monotonic for $0 \leq x \leq 1$.*

Finally to complete the proof of Theorem 1 there must be a result for $-1 \leq x \leq -\frac{1}{2}$.

THEOREM 2. *The function $(1-xr)^2(1-r)^{-2}(1-2xr+r^2)^{-2}$ is absolutely monotonic for $-1 \leq x \leq 0$.*

Interesting results can probably be obtained for fractional powers of these functions. There is another problem which is equivalent to Theorem 1, and it almost surely has a fractional analogue.

THEOREM 3.

$$\sum_{k=0}^n \frac{(3)_{n-k}}{(n-k)!} \frac{(3)_k}{k!} \frac{\sin(k+1)\theta}{k+1} \geq 0, \quad 0 \leq \theta \leq \pi.$$

POCHHAMMER's notation for the shifted factorial is

$$(a)_n = a(a+1) \dots (a+n-1) = \Gamma(n+a)/\Gamma(a).$$

In [3] ASKEY and GASPER proved

THEOREM D.

$$\sum_{k=0}^n \frac{(\lambda+1)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{\sin(k+1)\theta}{k+1} \geq 0, \quad 0 \leq \theta \leq \pi, \quad -1 < \lambda \leq 1.$$

The following conjecture would connect Theorems 3 and D.

Conjecture 1.

$$\sum_{k=0}^n \frac{(\lambda+1)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{\sin(k+1)\theta}{k+1} \geq 0, \quad 0 \leq \theta \leq \pi, \quad \text{for } -1 < \lambda \leq 2.$$

The only open cases are $1 < \lambda < 2$. This inequality is false for each $\lambda > 2$. In fact much more is false.

THEOREM 4. *Let $\lambda > 2$. Then the inequality*

$$\sum_{k=0}^n \frac{(\mu)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{\sin(k+1)\theta}{k+1} \geq 0$$

fails for some θ , $\theta = \theta(n)$, $0 < \theta < \pi$, for infinitely many n .

Theorem 4 is a consequence of an even stronger result.

THEOREM 5. $\int_0^{\infty} e^{-xt} t^a \sin yt \, dt$ changes sign for $y \geq 0$ for x positive when $a > 1$.

In Section 2 we will give the reduction of Theorem 1 to the other results and Theorem 2 will be proved in Section 3. The remaining miscellaneous results will be given in Section 4.

2. Reduction of Theorem 1. To prove Theorem 1, we first consider $0 \leq x \leq 1$. Then

$$\frac{1-xr}{(1-r)^3(1-2xr+r^2)^2} = \frac{1-xr}{1-r} \cdot \frac{1}{(1-r)^2(1-2xr+r^2)^2}.$$

The second factor is absolutely monotonic by Theorem C and the first factor is absolutely monotonic, since it is

$$1 + \sum_{n=1}^{\infty} (1-x)r^n.$$

The product of absolutely monotonic functions is absolutely monotonic, so Theorem 1 holds for $0 \leq x \leq 1$.

When $-1 \leq x \leq 0$, write

$$\frac{1-xr}{(1-r)^3(1-2xr+r^2)^2} = \frac{1}{(1-xr)(1-r)} \cdot \frac{(1-xr)^2}{(1-r)^2(1-2xr+r^2)^2}.$$

The second factor is absolutely monotonic by Theorem 2 and the first factor is

$$\frac{1}{(1-xr)(1-r)} = \sum_{n=0}^{\infty} \left(\frac{1-x^{n+1}}{1-x} \right) r^n.$$

and so it is absolutely monotonic.

Theorem 2 is easy to prove except when x is close to zero, so it is of some interest to observe that Theorem 1 follows from Theorem B when $-\frac{1}{2} \leq x \leq 0$. For

$$\frac{1-xr}{(1-r)^3(1-2xr+r^2)^2} = \frac{(1-xr)}{1-r^2} \cdot \frac{1+r}{(1-r)^2(1-2xr+r^2)^2},$$

and the first factor is absolutely monotonic for $-1 \leq x \leq 0$ and the second is absolutely monotonic for $-\frac{1}{2} \leq x \leq 1$.

3. Proof of Theorem 2. A partial fraction decomposition gives

$$\frac{2 \sin \theta/2 (1-r \cos \theta)}{(1-r)(1-2r \cos \theta + r^2)} = \sum_{n=0}^{\infty} [\sin \theta/2 + \sin(n+1/2)\theta] r^n,$$

so Theorem 1 reduces to showing

$$(3.1) \quad \sum_{k=0}^n [\sin \theta/2 + \sin(n-k+1/2)\theta][\sin \theta/2 + \sin(k+1/2)\theta] \geq 0, \quad \pi/2 \leq \theta \leq \pi.$$

A summation shows that (3.1) is equivalent to

$$(3.2) \quad A = \frac{(n+1)}{2} \left[1 - \cos \theta - \cos (n+1)\theta + \frac{\sin (n+1)\theta}{(n+1) \sin \theta} \right] + 1 - \cos (n+1)\theta \geq 0.$$

When $\pi/2 \leq \theta \leq \pi$, $-\cos \theta \geq 0$, and $|\cos (n+1)\theta| \leq 1$, $|\sin (n+1)\theta / (n+1) \sin \theta| \leq 1$, so $A \geq 0$ holds when either $-\cos (n+1)\theta \geq 0$ or $\sin (n+1)\theta \geq 0$. This gives a simple proof for most θ , but a more detailed analysis is necessary to prove $A \geq 0$ for all θ , $\pi/2 \leq \theta \leq \pi$. First consider $\varphi = \pi - \theta$ with $0 \leq \varphi \leq \pi / (n+1)$. When n is even,

$$\sin (n+1)\theta = (-1)^n \sin (n+1)\varphi \geq 0, \quad 0 \leq \varphi \leq \pi / (n+1).$$

When n is odd,

$$-\cos \theta - \cos (n+1)\theta = \cos \varphi - \cos (n+1)\varphi$$

and this is nonnegative since $\cos u$ is a decreasing function for $0 \leq u \leq \pi$, and $0 \leq (n+1)\varphi \leq \pi$.

Next consider $\pi/2 \leq \theta \leq \pi - \pi / (n+1)$. Then

$$(3.3) \quad \left| \frac{\sin (n+1)\theta}{(n+1) \sin \theta} \right| \leq \frac{1}{3}, \quad \frac{\pi}{n+1} \leq \theta \leq \pi - \frac{\pi}{n+1}.$$

See [1]. If

$$(3.4) \quad 1 - \cos \theta - \cos (n+1)\theta - 1/3 \geq 0$$

then $A \geq 0$. But (3.4) holds if $x = \cos \theta \leq -1/3$. This would be enough to complete the proof of Theorem 1.

In the remaining interval $-1/3 < x \leq 0$,

$$2^{3/2}/3 \leq \sin \theta \leq 1$$

and

$$2A \geq (n+1)[1 - \cos \theta - \cos (n+1)\theta - 3 \cdot 2^{-3/2} / (n+1)] \geq 0$$

when $x = \cos \theta \leq -3 \cdot 2^{-3/2} / (n+1)$. Thus all of the real difficulty in proving Theorem 2 comes from a small interval near zero. Set $\theta = \pi/2 + \varphi$. From $-1/3 < x \leq 0$, $0 \leq \varphi \leq \pi/9$ follows. Then

$$0 \leq \varphi \leq \frac{\pi}{9} \frac{\sin \varphi}{\sin \pi/9} \leq \frac{\sin \varphi}{9(0.342)} \leq \frac{3\pi 2^{-3/2}}{(3.078)(n+1)} < \frac{\pi}{2(n+1)}.$$

Then when n is odd

$$-\cos (n+1)\theta = -\cos (n+1)(\pi/2 + \varphi) = -\cos (n+1)\pi/2 \cos (n+1)\varphi$$

and this is nonnegative when $n \equiv 1 \pmod{4}$. And

$$\sin (n+1)\theta = \cos (n+1)\pi/2 \sin (n+1)\varphi$$

which is nonnegative when $n \equiv 3 \pmod{4}$.

When n is even

$$\sin (n+1)\theta = \sin (n+1)\pi/2 \cos (n+1)\varphi$$

and this is nonnegative when $n \equiv 0 \pmod{4}$. Thus we need only consider $n = 4k + 2$.

In this case

$$2A = (4k+5)[1 - \sin(4k+3)\varphi] + (4k+3) \left[\sin \varphi - \frac{\cos(4k+3)\varphi}{(4k+3)\cos \varphi} \right]$$

and this is positive when $0 \leq \varphi \leq 3\pi / [(3.078)2^{3/2}(4k+3)]$. For

$$2A > (4k+5) \left[1 - \sin \frac{3\pi 2^{-3/2}}{3.078} \right] - 1 \cong 9(1 - .884) - 1 > 0$$

when $k \geq 1$. And when $k=0$ a simple calculation gives

$$2A = 4(1-x)^2 - x + 20x^3 > 4 - 20/27 > 0, \quad \text{since } -1/3 < x \leq 0.$$

I would like to thank JOHN STEINIG for pointing out an error in an earlier version of this proof. He supplied an alternate proof as well.

4. Miscellaneous results. Theorem 1 has an equivalent statement which suggests some very interesting problems.

LEMMA 1.

$$\frac{(1-r \cos \theta)}{(1-r)^3(1-2r \cos \theta + r^2)^2} = \sum_{n=0}^{\infty} r^n \sum_{k=0}^n \frac{(3)_{n-k}}{(n-k)!} \frac{(3)_k}{k!} \frac{\sin(k+1)\theta}{(k+1)\sin \theta}.$$

The easiest way to do this calculation is to multiply by $\sin \theta$, let $\sin(k+1)\theta = \operatorname{Im} e^{i(k+1)\theta}$ and sum the series on the right.

Theorem 1 and Lemma 1 immediately imply Theorem 2. Theorem D is a special case of the following problem for Jacobi polynomials.

Let $P_n^{(\alpha, \beta)}(x)$ be defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x)^{n+\alpha}(1+x)^{n+\beta}],$$

Problem 1. For which values of λ, α, β is

$$\sum_{k=0}^n \frac{(\lambda+1)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{P_k^{(\alpha, \beta)}(\cos \theta)}{P_k^{(\beta, \alpha)}(1)} \geq 0, \quad 0 \leq \theta \leq \pi, \quad n = 0, 1, \dots ?$$

In [3] this problem was solved for $-1-\beta < \alpha < \beta$, $-1 < \lambda \leq \alpha + \beta$ and when $|\beta| \leq \alpha \leq \beta + 1$ with $0 \leq \lambda \leq \alpha + \beta$, and it was conjectured to hold for $\beta \geq -\frac{1}{2}$ when $0 \leq \lambda \leq \alpha + \beta$. When $\alpha = \beta = \frac{1}{2}$,

$$\frac{P_n^{(1/2, 1/2)}(\cos \theta)}{P_n^{(1/2, 1/2)}(1)} = \frac{\sin(n+1)\theta}{(n+1)\sin \theta}$$

so the case $-1 < \lambda \leq 1 = \frac{1}{2} + \frac{1}{2}$ is Theorem D. But now we see that the condition $\lambda \leq \alpha + \beta$ is too restrictive in this case, and it probably is too restrictive for each $\beta > -\frac{1}{2}$. But until some more examples have been given I would hesitate to state a conjecture as to the largest possible value of λ .

To get some idea of how large λ could be we will now prove Theorems 4 and 5, which show that no $\lambda > 2$ will work when $\alpha = \beta = \frac{1}{2}$. We will first prove Theorem 5, which is the more basic result. Set

$$(4.1) \quad G(y) = \int_0^{\infty} e^{-xt} t^\alpha \sin yt \, dt; \quad x > 0, \quad y \geq 0.$$

This integral can be inverted to give

$$e^{-xt}t^a = \frac{2}{\pi} \int_0^{\infty} G(y) \sin yt \, dy$$

or

$$(4.2) \quad e^{-xt}t^{a-1} = \frac{2}{\pi} \int_0^{\infty} G(y) \frac{\sin yt}{yt} y \, dy.$$

If $G(y)$ were nonnegative then we would have

$$0 = \frac{2}{\pi} \int_0^{\infty} G(y)y \, dy$$

on letting $t \rightarrow 0$ in (4.2) and this is clearly impossible.

To see that Theorem 4 follows from Theorem 5 observe that

$$(4.3) \quad \int_0^r \left(1 - \frac{t}{r}\right)^{xr} t^a \sin yt \, dt \cong 0, \quad y \cong 0, \quad r \cong 0$$

for some $x > 0$ would imply $G(y) \cong 0$ for the same x since

$$\lim_{n \rightarrow \infty} t^a \left(1 - \frac{t}{r}\right)^{xr} = t^a e^{-xt}$$

dominatedly. But

$$(4.4) \quad \sum_{k=0}^n \frac{(\mu)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{\sin(k+1)\theta}{k+1} \cong 0$$

for $\theta = y/n_i$, $n = n_i$, $n_i < n_{i+1}$, $i = 1, 2, \dots$, would imply

$$(4.5) \quad \int_0^1 (1-v)^{\mu-1} v^{\lambda-1} \sin yv \, dv \cong 0$$

and a change of variables $v = t/r$ would give (4.3) for $a = \lambda - 1$, $x = \mu - 1$. And if (4.4) holds for some μ it also holds for $v > \mu$, so we may choose $x > 0$. But when $a = \lambda - 1 > 1$, $\lambda > 2$ and this proves Theorem 4.

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ON AN EXTREMUM PROBLEM OF INFORMATION THEORY

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The minimum amount of information needed to represent a message with average loss not exceeding a level d — called the *rate-distortion function* $R(d)$ — plays a fundamental role in source coding theory. The extremum problem of calculating $R(d)$ has been extensively treated for finite alphabets (for the results and the relevant literature we refer to GALLAGER's book [4] or BERGER's monograph [1]) but less attention was paid to the general case. Contrary to a not uncommon belief, the extension from finite to abstract alphabets is a non-trivial task and involves certain subtleties.

The aim of this paper is to derive generalizations of familiar finite alphabet results in a general setting and call attention to the limitations of their validity; such generality is called for when dealing with continuous time signals, in which case the "alphabets" are function spaces. A typical result is that the formula $R(d) = \max_{\alpha(x), s} \left[\int \log \alpha(x) P(dx) - sd \right]$, where the maximization refers to $\alpha(x) \geq 1$ and $s \geq 0$ satisfying $\int \alpha(x) e^{-s\varrho(x,y)} P(dx) \leq 1$ for all $y \in Y$, is generally valid, even if a joint distribution achieving $R(d)$ does not exist. Our secondary aim is to show that the method of Lagrange multipliers which tends to be a source of loose proofs in the literature of the subject can be completely dispensed with, gaining rather than losing in elegance of presentation.

1. Preliminaries

Let ξ and η be two RV's (random variables) with values in arbitrary measurable spaces (X, \mathcal{X}) and (Y, \mathcal{Y}) , respectively (the σ -algebras are assumed to contain the one point sets). If the joint distribution $P_{\xi\eta}$ of ξ and η , i.e., the measure defined on¹ $X \times Y$ by $P_{\xi\eta}(E) = \mathbf{P}\{(\xi, \eta) \in E\}$ is absolutely continuous with respect to the product of the marginal distributions P_ξ and P_η , with R-N derivative $a(x, y)$, the RV²

$$(1.1) \quad L_{\xi\eta} = \log a(\xi, \eta)$$

is called the *information density* and its expectation

$$(1.2) \quad I(\xi, \eta) = \mathbf{E}L_{\xi\eta}$$

¹ More exactly, the measure is defined on $(X \times Y, \mathcal{X} \times \mathcal{Y})$; but in the sequel we omit mentioning the σ -algebras.

² In this paper, we use natural logarithms.

is the *mutual information* of ξ and η . If $P_{\xi\eta} \not\ll P_\xi \times P_\eta$ then $I_{\xi\eta}$ is not defined and $I(\xi, \eta) = +\infty$.

Let $\varrho: X \times Y \rightarrow [0, \infty]$ be a given (measurable) function, called loss function; for convenience, we assume

$$(1.3) \quad \inf_y \varrho(x, y) = 0 \quad \text{for each } x \in X.$$

Suppose that a message considered as a random element of the alphabet X with given distribution P is to be represented by a random element of the alphabet Y with average loss or distortion not exceeding a certain level d . The minimum amount of information needed for such a representation, i.e.,

$$(1.4) \quad R_P(d) = \inf_{P_{\xi\eta} \leq d} I(\xi, \eta)$$

is called the *rate-distortion function*; the inf in (1.4) refers to such pairs of RV's (ξ, η) satisfying the given constraint for which the distribution of ξ equals the given P .

Since the mutual information of two RV's ξ and η depends on their joint distribution $P_{\xi\eta}$ only, for any PD (probability distribution = probability measure) \tilde{P} on the product space¹ $X \times Y$ we may define

$$(1.5) \quad I(\tilde{P}) = I(\xi, \eta), \quad (\xi, \eta): P_{\xi\eta} = \tilde{P}.$$

In the sequel the rate-distortion function (1.4) will be denoted by $R(d)$ (because P will be fixed throughout). We assume once for all that $R(d)$ is finite for some d and denote by d_0 the infimum of such d 's. The symbol \tilde{P} will be used for PD's on $X \times Y$ having X -marginal P , i.e.,

$$(1.6) \quad \tilde{P}(A \times Y) = P(A) \quad \text{for all } A \in \mathcal{X}.$$

Then (1.4) becomes

$$(1.7) \quad R(d) = \inf_{P(\tilde{P}) \leq d} I(\tilde{P}), \quad \varrho(\tilde{P}) = \int \varrho(x, y) \tilde{P}(dx, dy).$$

A useful equivalent definition of $I(\tilde{P})$ is

$$(1.8) \quad I(\tilde{P}) = I(\tilde{P} \| P \times Q_{\tilde{P}})$$

where $Q_{\tilde{P}}$ denotes the Y -marginal of \tilde{P} , i.e. $Q_{\tilde{P}}(B) = \tilde{P}(X \times B)$ for all $B \in \mathcal{Y}$. $I(\cdot, \cdot)$ denotes the KULLBACK—LEIBLER I-divergence, defined for any two PD's P_1 and P_2 on a measurable space (Z, \mathcal{Z}) by

$$(1.9) \quad I(P_1 \| P_2) = \begin{cases} \int \log \frac{P_1(dz)}{P_2(dz)} P_1(dz) & \text{if } P_1 \ll P_2 \\ +\infty & \text{else,} \end{cases}$$

or, equivalently — using a dominating measure λ — by

$$(1.9a) \quad I(P_1 \| P_2) = \int p_1(z) \log \frac{p_1(z)}{p_2(z)} \lambda(dz), \quad p_i(z) = \frac{P_i(dz)}{\lambda(dz)}, \quad i = 1, 2.$$

Now we formulate some simple lemmas which can be proved in our general case in the same way as for finite X and Y .

LEMMA 1.1. $R(d)$ is a non-decreasing convex function of d , and instead of (1.7) one may write

$$(1.10) \quad R(d) = \inf_{\varrho(\tilde{P})=d} I(\tilde{P}) \quad \text{if } d \leq d_1$$

where d_1 is the smallest non-negative number such that $R(d)$ is constant in (d_1, ∞) .

PROOF. Since $u \log \frac{u}{v}$ is a convex function of (u, v) , $I(P_1 \| P_2)$ is a convex function of (P_1, P_2) , see (1.9a). In view of (1.8), it follows that both $R(d)$ and the right hand side of (1.10), say $R'(d)$, are convex functions of d . But $R(d) = \inf_{d' \leq d} R'(d')$, completing the proof.

Let $F(s)$ denote the maximum of the vertical axis intercepts of the straight lines of slope $-s$ which have no point above the $R(d)$ curve, i.e., using (1.10) for $s > 0$

$$(1.11) \quad F(s) = \inf_{\tilde{P}} [I(\tilde{P}) + s\varrho(\tilde{P})], \quad s \geq 0.$$

Here and in the sequel we understand $s\varrho(\tilde{P}) = \infty$ if $\varrho(\tilde{P}) = \infty$ even for $s = 0$. This non-standard convention is necessary to cover the case when $R(d)$ equals a positive constant in (d_1, ∞) .

Let d_0 denote the infimum of those d 's for which $R(d) < \infty$. Since $R(d)$ is convex, to each $d > d_0$ there exists $s \geq 0$ (uniqueness is not claimed at this point) such that the straight line of slope $-s$ through $(d, R(d))$ is tangent to the $R(d)$ curve, i.e.

$$(1.12) \quad R(d') + sd' \geq R(d) + sd = F(s) \quad \text{for all } d'.$$

In this case we shall say that s is associated to d . Thus we have

LEMMA 1.2. For all $d > d_0$,

$$(1.13) \quad R(d) = \max_{s \geq 0} [F(s) - sd].$$

The maximum in (1.13) is attained iff s is associated to d . Further, if a \tilde{P}_0 achieves minimum in (1.11) then s is associated to $d = \varrho(\tilde{P}_0)$ and $R(d) = I(\tilde{P}_0)$.

Following a very useful idea of BLAHUT [2], we introduce the function

$$(1.14) \quad J(\tilde{P}, Q, s) = I(\tilde{P} \| P \times Q) + s\varrho(\tilde{P})$$

where Q is any PD on Y and $s \geq 0$.

If $J(\tilde{P}, Q, s)$ is finite then $\tilde{P} \ll P \times Q$ and the set $\{(x, y) : \varrho(x, y) < \infty\}$ must have positive $P \times Q$ -measure. Thus, using Fubini's theorem

$$(1.15) \quad \alpha_{Q,s}(x) = \left[\int e^{-s\varrho(x,y)} Q(dy) \right]^{-1}$$

is finite for P -a.e. $x \in X$, and there exists a PD $\tilde{P}_{Q,s}$ on $X \times Y$ with $P \times Q$ -density (R-N derivative) $\alpha_{Q,s}(x)e^{-s\varrho(x,y)}$; we write this symbolically as

$$(1.16) \quad \tilde{P}_{Q,s}(dx, dy) = \alpha_{Q,s}(x)e^{-s\varrho(x,y)} P(dx) Q(dy).$$

Both in (1.15) and (1.16) we understand $s\varrho(x, y) = \infty$ if $\varrho(x, y) = \infty$, even for $s = 0$.

Clearly, the X -marginal of $\tilde{P}_{Q,s}$ is P but its Y -marginal is in general different from Q .

LEMMA 1.3. For any \tilde{P} , Q and $s \geq 0$ with $J(\tilde{P}, Q, s) < \infty$

$$(1.17) \quad J(\tilde{P}, Q, s) = J(\tilde{P}, Q_{\tilde{P}}, s) + I(Q_{\tilde{P}} \| Q) = J(\tilde{P}_{Q,s}, Q, s) + I(\tilde{P} \| \tilde{P}_{Q,s}).$$

COROLLARY.

$$(1.18) \quad J(\tilde{P}, Q_{\tilde{P}}, s) = I(\tilde{P}) + sQ(\tilde{P})$$

and

$$(1.19) \quad J(\tilde{P}_{Q,s}, Q, s) = \int \log \alpha_{Q,s}(x) P(dx)$$

represent the minimum of $J(\tilde{P}, Q, s)$ with respect to Q or \tilde{P} , respectively. Moreover,

$$(1.20) \quad \inf_{\tilde{P}, Q} J(\tilde{P}, Q, s) = F(s) = \inf_Q \int \log \alpha_{Q,s}(x) P(dx).$$

PROOF. Simply by substitution, using the chain rule for R—N derivatives. Since I-divergence is non-negative, the corollary is immediate.

LEMMA 1.4. In order that a PD \tilde{P}_0 minimize $I(\tilde{P}) + sQ(\tilde{P})$, it is necessary and sufficient that its R—N derivative $a(x, y)$ with respect to $P \times Q_0$ (where $Q_0 = Q_{\tilde{P}_0}$) be of form

$$(1.21) \quad a(x, y) = \alpha(x) e^{-sQ(x, y)}$$

where $\alpha(x) \geq 1$ satisfies

$$(1.22) \quad \int \alpha(x) e^{-sQ(x, y)} P(dx) \leq 1 \quad \text{for all } y \in Y.$$

The same conditions together with $Q(\tilde{P}_0) = d$ (if $s > 0$) are necessary and sufficient in order that \tilde{P}_0 minimize $I(\tilde{P})$ subject to $Q(\tilde{P}) \leq d$ for $d > d_0$, where s is associated to d by (1.12). Moreover, for any $s \geq 0$ and $\alpha(x) > 0$ satisfying (1.22) we have

$$(1.23) \quad I(\tilde{P}) + sQ(\tilde{P}) \geq \int \log \alpha(x) P(dx) \quad \text{for all } \tilde{P},$$

with equality iff \tilde{P} can be represented as in (1.21), with the given $\alpha(x)$.

COROLLARY. If there exists a \tilde{P}_0 minimizing $I(\tilde{P}) + sQ(\tilde{P})$ then

$$(1.24) \quad F(s) = \max_{\alpha(x)} \int \log \alpha(x) P(dx)$$

and if there exists a \tilde{P}_0 minimizing $I(\tilde{P})$ subject to $Q(\tilde{P}) \leq d$ where $d > d_0$ then

$$(1.25) \quad R(d) = \max_{\alpha(x), s} \left[\int \log \alpha(x) P(dx) - sd \right],$$

where the maximization is under the constraints (1.22); it suffices to take only $\alpha(x) \geq 1$ into account.

Remarks. Of course, (1.21) and (1.22) imply that in the latter the equality

holds for Q_0 -a.e. y . Observe that (1.21) specifies the conditional distribution of ξ given $\eta=y$ for ξ and η having joint distribution \tilde{P}_0 . In fact, (1.21) means

$$(1.26) \quad P\{\xi \in A | \eta = y\} = \int_A \alpha(x) e^{-s\varrho(x,y)} P(dx) \quad (A \in \mathcal{X}).$$

The corollary generalizes a useful equivalent characterization of the rate-distortion function for finite alphabets. Here its value is limited by the restrictive hypothesis of existence of a minimizing \tilde{P}_0 . In § 2 we shall prove that this hypotheses can be dispensed with.

PROOF. Applying lemma 1.3 twice we obtain

$$(1.27) \quad J(\tilde{P}_0, Q_0, s) = J(\tilde{P}_1, Q_1, s) + I(\tilde{P}_0 \| \tilde{P}_1) + I(Q_1 \| Q_0)$$

where $\tilde{P}_1 = \tilde{P}_{Q_0, s}$ and $Q_1 = Q_{\tilde{P}_1}$. On account of (1.27) and (1.18), a necessary condition for \tilde{P}_0 to minimize $I(\tilde{P}) + s\varrho(\tilde{P})$ is $\tilde{P}_1 = \tilde{P}_0$, which means just (1.21) with $\alpha(x) = \alpha_{Q_0, s}(x)$, see (1.16). In order to show the necessity of (1.22), as well, consider $Q' = (1-\beta)Q_0 + \beta\delta_{y_0}$ where δ_{y_0} is the point mass at some fixed $y_0 \in Y$ and $0 \leq \beta < 1$ is arbitrary. Then, in view of (1.20),

$$(1.28) \quad F(s) = I(\tilde{P}_0) + s\varrho(\tilde{P}_0) = \int \log \alpha(x) P(dx) \leq \int \log \alpha'(x) P(dx)$$

where

$$(1.29) \quad \alpha'(x) = \alpha_{Q', s}(x) = \alpha(x)[1 - \beta + \beta e^{-s\varrho(x, y_0)} \alpha(x)]^{-1}.$$

(1.28) and (1.29) give

$$(1.30) \quad 0 \leq - \int \log [1 - \beta + \beta e^{-s\varrho(x, y_0)} \alpha(x)] P(dx).$$

Differentiating the right hand side with respect to β at $\beta=0$ (differentiation within the integral is permitted by monoton convergence of the difference quotients) we obtain

$$(1.31) \quad 0 \leq \int [1 - e^{-s\varrho(x, y_0)} \alpha(x)] P(dx).$$

Since $y_0 \in Y$ was arbitrary, (1.31) proves (1.22).

The sufficiency of (1.21), (1.22) for \tilde{P}_0 to minimize $I(\tilde{P}) + s\varrho(\tilde{P})$ will follow if we prove the last assertion. The corollary of lemma 1.3, the concavity of the log function, and (1.16) and (1.22) yield

$$(1.32) \quad \begin{aligned} I(\tilde{P}) + s\varrho(\tilde{P}) &\leq \int \log \alpha_{Q, s}(x) P(dx) = \\ &= \int \log \alpha(x) P(dx) - \int \log \frac{\alpha(x)}{\alpha_{Q, s}(x)} \tilde{P}_{Q, s}(dx, dy) \leq \\ &\leq \int \log \alpha(x) P(dx) - \log \int \alpha(x) e^{-s\varrho(x, y)} P(dx) Q(dy) \leq \\ &\leq \int \log \alpha(x) P(dx), \end{aligned}$$

where $Q = Q_{\tilde{P}}$. This proves (1.23) together with the condition of equality.

Combining the just proved necessary and sufficient condition for \tilde{P}_0 to minimize $I(\tilde{P}) + s\varrho(\tilde{P})$ with lemma 1.2, we obtain the assertion concerning the minimization of $I(\tilde{P})$ subject to $\varrho(\tilde{P}) \leq d$.

The first part of the corollary is immediate, and the second follows by lemma 1.2.

2. The results

We start with a simple lemma to be used in several approximation arguments.

LEMMA 2.1. Let ξ be an X -valued RV and η_i , $i=0, 1$ be Y -valued RV's with $I(\xi, \eta_i) < \infty$. Let \varkappa be a $\{0, 1\}$ -valued RV and set $\eta = \eta_\varkappa$. Then

$$(2.1) \quad |I(\xi, \eta) - E((1 - \varkappa)L_{\xi\eta_0} + \varkappa L_{\xi\eta_1})| \leq 2h(p)$$

where

$$(2.2) \quad p = P\{\varkappa = 0\}, \quad h(p) = -p \log p - (1 - p) \log(1 - p).$$

PROOF. Preferring not to rely on information-theoretic prerequisites, expand the R-N derivative of $P_{\xi\eta\varkappa}$ with respect to $P_\xi \times P_\eta \times P_\varkappa$ (which are PD's on the product space $X \times Y \times \{0, 1\}$) in two different ways by the chain rule, to obtain

$$(2.3) \quad a(x, y, k) P\{\varkappa = k | \eta = y\} = P\{\varkappa = k | \xi = x, \eta = y\} a(x, y).$$

Here $a(x, y, k)$ ($x \in X, y \in Y, k \in \{0, 1\}$) denotes the R-N derivative of $P_{\xi\eta\varkappa}$ with respect to $P_\xi \times P_{\eta\varkappa}$. Comparing (2.3) with the similar identity for η_i instead of η (this being indicated by indices i) and $k = i$, we obtain

$$(2.4) \quad a(x, y) P\{\varkappa = i | \xi = x, \eta = y\} P\{\varkappa = i | \eta_i = y\} = \\ = a_i(x, y) P\{\varkappa = i | \xi = x, \eta_i = y\} P\{\varkappa = i | \eta = y\},$$

since, by the definition of η , we have $a(x, y, i) = a_i(x, y, i)$, $i=0, 1$. Now observe that for any RV ζ (taking values in some measurable space (Z, \mathcal{Z}))

$$(2.5) \quad 0 \cong \int_{Z \times \{i\}} \log P\{\varkappa = i | \zeta = z\} P_{\zeta\varkappa}(dz, dk) = \\ = \int_Z P\{\varkappa = i | \zeta = z\} \log P\{\varkappa = i | \zeta = z\} P_\zeta(dz) \cong P\{\varkappa = i\} \log P\{\varkappa = i\}$$

(convexity of $f(u) = u \log u$). After taking logarithms in (2.4), integrate on $X \times Y \times \{i\}$ with respect to $P_{\xi\eta\varkappa}$; using (2.5) we arrive at

$$(2.6) \quad \left| \int_{X \times Y \times \{i\}} (\log a(x, y) - \log a_i(x, y)) P_{\xi\eta\varkappa}(dx, dy, dk) \right| \leq -2P\{\varkappa = i\} \log P\{\varkappa = i\}.$$

In view of (1.1), (1.2) and (2.2), this proves (2.1).

The following simple examples show some kinds of possible pathological behaviour of $R(d)$ in the general case.

Example 1. Let $X = Y = [0, 1]$ (with the Borel σ -algebra) and $\varrho(x, y) = 0$ for $x = y$ and $\varrho(x, y) = 1$ else. Then for any continuous distribution P we have $R(d) = +\infty$ for $0 \leq d < 1$ and $R(d) = 0$ for $d \geq 1$.

Example 2. Let $X = Y = \{1, 2, \dots\}$, let $P = \{p(x); x = 1, 2, \dots\}$ be a PD on X with infinite entropy, and let $\varrho(x, x) = 0$, $\varrho(x, y) = \frac{1}{p(x)}$ for $y \neq x$. Then $R(d)$ is identically infinite.

Example 3. Let $X = \{0, 1\}$, $Y = \{0, 1, 2, \dots\}$,

$$\varrho(0, 0) = \varrho(1, 1) = 0, \quad \varrho(0, 1) = \varrho(1, 0) = 1, \quad \varrho(0, k) = \varrho(1, k) = \frac{1}{k}$$

for $k \geq 2$; let $P = \{p, 1-p\}$ be arbitrary with $0 < p < 1$. Then $R(d) = 0$ for $d > 0$ and $R(0) = h(p) > 0$, showing that $R(d)$ may have a discontinuity at $d = 0$ even if it is finite everywhere.

As mentioned in § 1, we exclude the contingency that $R(d)$ is identically infinite, as in Example 2. I.e., we assume (without further reference) the existence of RV's ξ and η_0 with $P_\xi = P$ such that

$$(2.7) \quad I(\xi, \eta_0) < \infty, \quad E\varrho(\xi, \eta_0) < \infty.$$

The following theorem completes lemma 1.1 in characterising the $R(d)$ curve, under weak regularity hypotheses.

THEOREM 2.1. Suppose that to any $\varepsilon > 0$ there exist measurable subsets $A_1 \subset A_2 \subset \dots$ of X and finite subsets $B_1 \subset B_2 \subset \dots$ of Y such that $P(A_n) \rightarrow 1$ as $n \rightarrow \infty$ and

$$(2.8) \quad \min_{y \in B_n} \varrho(x, y) < \varepsilon \quad \text{for each } x \in A_n, \quad n = 1, 2, \dots$$

Then $R(d)$ is finite for all $d > 0$.

If, on the other hand, there exist A'_n 's as above and $y_n \in Y$ with

$$(2.9) \quad \max_{x \in A'_n} \varrho(x, y_n) < \infty, \quad n = 1, 2, \dots,$$

then $R(d) \rightarrow 0$ as $d \rightarrow \infty$.

Finally, setting

$$(2.10) \quad d^{**} = \lim_{K \rightarrow \infty} \left(\inf_y \int \varrho^{(K)}(x, y) P(dx) \right)$$

where

$$(2.11) \quad \varrho^{(K)}(x, y) = \min(\varrho(x, y), K),$$

$R(d)$ is positive for $d < d^{**}$ and if $d^{**} < \infty$ then $R(d)$ vanishes for $d > d^{**}$.

Remarks. The hypothesis of the first assertion is trivially fulfilled if Y is at most countable (recall the assumption (1.3)). It is also fulfilled if (X, \mathcal{X}) is polish (complete separable metric space with the Borel σ -algebra) and $\varrho(x, y)$ is continuous for each fixed y , or at least $\varrho^{(K)}(x, y)$ is upper semicontinuous for some fixed K ; in fact, in this case the A_i 's may be chosen as an increasing sequence of compact subsets of X , since on a polish space every PD is tight, see [7]. The hypothesis of the second assertion is trivially fulfilled if ϱ is finite valued. Since $R(d)$ is continuous in (d_0, ∞) (by convexity), the last assertion implies that $R(d)$ vanishes exactly for $d \geq d^{**}$ if $d^{**} > d_0$. (Recall that d_0 is the left endpoint of the interval where $R(d) < \infty$; by theorem 2.1, in most cases of interest $d_0 = 0$.) d^{**} may be strictly less than

$$(2.12) \quad d^* = \inf_y \int \varrho(x, y) P(dx)$$

for which $d > d^*$ trivially implies $R(d) = 0$.

PROOF. To prove the first and second assertions, consider RV's ξ and η_0 satisfying (2.7) and a set A_n of the postulated sequence. Let $\varkappa=1$ or 0 according as $\xi \in A_n$ or $\xi \notin A_n$ and let $\eta_1=f(\xi)$ in the first case, where $f: X \rightarrow B_n$ is a function with $\varrho(x, f(x)) < \varepsilon$ for $x \in A_n$, and $\eta_1=y_n$ in the second case. Lemma 2.1 guarantees for $\eta=\eta_\varkappa$ that $I(\xi, \eta)$ is finite in the first and arbitrarily small (for sufficiently large n) in the second case; on the other hand, $E\varrho(\xi, \eta) < 2\varepsilon$ if n is sufficiently large in the first case, and $E\varrho(\xi, \eta)$ is finite also in the second one. On account of (1.4), this proves the first two assertions.

The last assertion is easy for bounded ϱ , since then $d^{**}=d^*$, see (2.12), thus it suffices to show that $R(d)=0$ implies $d \cong d^*$. This follows from the fact that the variation distance of any two PD's can be bounded from above in terms of their I-divergence; to be exact, we have $|P_1 - P_2| \cong \sqrt{2I(P_1 \| P_2)}$, see e.g. [3]. This means, indeed, that $I(\tilde{P}_n) \rightarrow 0$ implies $|\tilde{P}_n - P \times Q_n| \rightarrow 0$ where $Q_n = Q_{\tilde{P}_n}$, see (1.8), thus — since ϱ is bounded — also

$$(2.13) \quad \int \varrho(x, y) \tilde{P}_n(dx, dy) - \int \varrho(x, y) P(dx) Q_n(dy) \rightarrow 0.$$

Here the second term is $\cong d^*$, thus if $\varrho(\tilde{P}_n) \cong d$, $n=1, 2, \dots$, then $d \cong d^*$.

The last assertion of theorem 2.1 follows from its just proved particular case if we show that to any given $d > 0$ and $\varepsilon > 0$ there exists such K that if the RV's ξ and η_1 (with $P_\xi = P$) satisfy $E\varrho^{(K)}(\xi, \eta_1) \cong d$, an η with

$$(2.14) \quad I(\xi, \eta) < I(\xi, \eta_1) + \varepsilon, \quad E\varrho(\xi, \eta) < E\varrho^{(K)}(\xi, \eta_1) + \varepsilon$$

can also be found. To this end, set $\varkappa=1$ if $\varrho(\xi, \eta_1) \cong K$ and $\varkappa=0$ else, and apply lemma 2.1 with η_0 satisfying (2.7). Since, by assumption, $\varrho^{(K)}(\xi, \eta_1) = K$ if $\varkappa=0$,

$$(2.15) \quad P\{\varkappa = 0\} \cong \frac{1}{K} E\varrho^{(K)}(\xi, \eta_1) \cong \frac{d}{K}.$$

Hence both $E(1-\varkappa)L_{\xi\eta_0}$ and $E(1-\varkappa)\varrho(\xi, \eta_0)$ are arbitrarily small if K is sufficiently large. On the other hand,

$$(2.16) \quad \begin{aligned} E(\varkappa L_{\xi\eta_1}) &= I(\xi, \eta_1) - \int_{\varrho(x, y) > K} \log a_1(x, y) P_{\xi\eta_1}(dx, dy) \cong \\ &\cong I(\xi, \eta_1) - P\{\varkappa = 0\} \log P\{\varkappa = 0\}, \end{aligned}$$

because, by convexity

$$(2.17) \quad \begin{aligned} \int_{\varrho(x, y) > K} \log a_1(x, y) P_{\xi\eta_1}(dx, dy) &= \int_{\varrho(x, y) > K} a_1(x, y) \log a_1(x, y) P_\xi(dx) P_{\eta_1}(dy) \cong \\ &\cong P\{\varrho(\xi, \eta_1) > K\} \log P\{\varrho(\xi, \eta_1) > K\}. \end{aligned}$$

Thus (2.14) follows from lemma 2.1, completing the proof of theorem 2.1.

For finite alphabets it is also known that $R(d) \rightarrow R(0)$ as $d \rightarrow 0$, and if ϱ is finite-valued then $R(d)$ is differentiable for each $d > 0$, $d \neq d^*$. The familiar proof of these facts (see e.g. [4], pp. 463—464) depends on the existence of a \tilde{P}_0 minimizing $I(\tilde{P}) + s\varrho(\tilde{P})$ for given $s \geq 0$. If such \tilde{P}_0 exists for each $s \geq 0$, the proof of differentiability easily extends to the general case while to the proof of $R(d) \rightarrow R(0)$ ($d \rightarrow 0$) some additional condition is needed, cf. example 3. In this direction we prove

THEOREM 2.2. *If X is polish and Y compact metric, with $\varrho(x, y)$ jointly continuous⁴, then $R(d)$ is finite for each $d \geq 0$, there exists for each $d > 0$ a \tilde{P}_0 minimizing $I(\tilde{P})$ subject to $\varrho(\tilde{P}) \leq d$, and*

$$(2.18) \quad R(d) \rightarrow R(0) \quad \text{as } d \rightarrow 0.$$

If we drop the assumption that X is polish and the continuity of $\varrho(x, y)$ is required in y only (for each fixed $x \in X$) there still exists to each $s \geq 0$ a \tilde{P}_0 minimizing $I(\tilde{P}) + s\varrho(\tilde{P})$.

Remarks. The first hypothesis is trivially fulfilled if X is at most countable and Y finite (using a trivial metric making all functions continuous). The proof of (2.18) given below reduces in that case to a very elementary one, simpler than that found in the literature for finite alphabets (loc. cit.).

We note that the second, weaker hypothesis of theorem 2.2 also implies, for each $d > d_0$, the existence of a \tilde{P}_0 minimizing $I(\tilde{P})$ subject to $\varrho(\tilde{P}) \leq d$. This is trivial if for s associated to d by (1.12), there is no other d having the same s , since then the \tilde{P}_0 minimizing $I(\tilde{P}) + s\varrho(\tilde{P})$ will certainly do. If d belongs to a linear segment of the $R(d)$ curve, however, the proof required more effort than the result deserves, thus we omit it. Observe that the weaker result also suffices to prove in the familiar way — for finite-valued ϱ — the differentiability of $R(d)$ for each $d > d_0$, $d \neq d^*$.

For the proof of theorem 2.3 below, it will be essential that the second hypothesis of theorem 2.2 is trivially fulfilled if Y is finite. Accordingly, we formulate for later reference

COROLLARY of theorem 2.2. *If Y is finite, the minimum of $I(\tilde{P}) + s\varrho(\tilde{P})$ is always achieved, for any $s \geq 0$.*

To the proof of theorem 2.2. we need

LEMMA 2.2. *For PD's on (the Borel σ -algebra \mathcal{Z} of) an arbitrary metric space Z , the I-divergence is lower semicontinuous with respect to weak convergence, i.e., $P_n \Rightarrow P$ and $Q_n \Rightarrow Q$ imply $\liminf_{n \rightarrow \infty} I(P_n \| Q_n) \geq I(P \| Q)$.*

PROOF. Let \mathcal{Z}_0 denote the class of Borel subsets of Z whose boundary has $(P+Q)$ -measure 0. Clearly, \mathcal{Z}_0 is an algebra generating the σ -algebra \mathcal{Z} . Thus, by a well known property of I-divergence (see [8], theorem 2.4.1) $I(P \| Q)$ equals the supremum of $\sum_{i=1}^k P(A_i) \log \frac{P(A_i)}{Q(A_i)}$ for all partitions (A_1, \dots, A_k) of Z consisting of sets $A_i \in \mathcal{Z}_0$. But $P_n(A) \rightarrow P(A)$, $Q_n(A) \rightarrow Q(A)$ if $A \in \mathcal{Z}_0$ (see [7], theorem 2.6.1), hence for such partitions

$$(2.19) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^k P_n(A_i) \log \frac{P_n(A_i)}{Q_n(A_i)} = \sum_{i=1}^k P(A_i) \log \frac{P(A_i)}{Q(A_i)}$$

provided that $P(A_i) + Q(A_i) > 0$, $i = 1, \dots, k$. The latter restriction is immaterial when taking the supremum of the right hand side of (2.19), and we obtain $\liminf_{n \rightarrow \infty} I(P_n \| Q_n) \geq I(P \| Q)$ as asserted.

⁴ Recall that ϱ is considered as a mapping $X \times Y \rightarrow [0, \infty]$, thus its continuity does not necessarily mean that it is finite valued.

PROOF of theorem 2.2. Under the first hypothesis, $R(d) < \infty$ for $d > 0$ follows from theorem 2.1. To prove the other assertions, consider a sequence of PD's on $X \times Y$ with

$$(2.20) \quad I(\tilde{P}_n) \rightarrow R(d), \quad \varrho(\tilde{P}_n) \equiv d$$

or

$$(2.20a) \quad I(\tilde{P}_n) \rightarrow \lim_{d \rightarrow 0} R(d), \quad \varrho(\tilde{P}_n) \rightarrow 0,$$

respectively. Since the common X -marginal P of the \tilde{P}_n 's is tight (as any PD on a polish space), in view of the compactness of Y the sequence \tilde{P}_n is uniformly tight. Thus, by Prohorov's theorem (see [7], theorem 6.7), there exists a weakly convergent subsequence, and without any loss of generality we may assume that \tilde{P}_n itself is weakly convergent, $\tilde{P}_n \Rightarrow \tilde{P}_0$, say. Then, with the notation (2.11), the integral of $\varrho^{(K)}(x, y)$ with respect to \tilde{P}_n converges to that with respect to \tilde{P}_0 for any K , implying

$$(2.21) \quad \varrho(\tilde{P}_0) \equiv \varliminf_{n \rightarrow \infty} \varrho(\tilde{P}_n).$$

By (1.8) and lemma 2.2 we also have

$$(2.22) \quad I(\tilde{P}_0) \equiv \varliminf_{n \rightarrow \infty} I(\tilde{P}_n).$$

(2.20), (2.21) and (2.22) give that \tilde{P}_0 minimizes $I(\tilde{P})$ subject to $\varrho(\tilde{P}) \equiv d$, while (2.20a), (2.21) and (2.22) yield

$$(2.23) \quad R(0) \equiv I(\tilde{P}_0) \equiv \lim_{d \rightarrow 0} R(d).$$

But (2.23) proves (2.18), since certainly $R(0) \equiv \lim_{d \rightarrow 0} R(d)$, by convexity.

Under the second hypothesis, consider a sequence of PD's on $X \times Y$ with

$$(2.24) \quad I(\tilde{P}_n) + s\varrho(\tilde{P}_n) \rightarrow F(s) = \inf [I(\tilde{P}) + s\varrho(\tilde{P})].$$

We may assume as above that the Y -marginals $Q_n = Q_{\tilde{P}_n}$ are weakly convergent, $Q_n \Rightarrow Q_0$, say.

Set $\alpha_n(x) = \alpha_{Q_n, s}(x)$, $\alpha_0(x) = \alpha_{Q_0, s}(x)$, $\tilde{P}'_n = \tilde{P}_{Q_n, s}$, $\tilde{P}_0 = \tilde{P}_{Q_0, s}$, see (1.15), (1.16). On account of the weak convergence $Q_n \Rightarrow Q_0$ and (1.15), the continuity of ϱ for fixed x implies $\alpha_n(x) \rightarrow \alpha_0(x)$ for each $x \in X$. Using the corollary of lemma 1.3 and Fatou's lemma (clearly $\log \alpha_n(x) \geq 0$) and (2.24), we obtain

$$(2.25) \quad \begin{aligned} F(s) &\equiv I(\tilde{P}_0) + s\varrho(\tilde{P}_0) \equiv J(\tilde{P}_0, Q_0, s) = \\ &= \int \log \alpha_0(x) P(dx) \equiv \varliminf_{n \rightarrow \infty} \int \log \alpha_n(x) P(dx) = \\ &= \varliminf_{n \rightarrow \infty} J(\tilde{P}'_n, Q_n, s) \equiv \lim_{n \rightarrow \infty} [I(\tilde{P}_n) + s\varrho(\tilde{P}_n)] = F(s), \end{aligned}$$

completing the proof of theorem 2.2.

In the above proof the compactness of Y has been used only to (trivially) ensure uniform tightness of the Y -marginals of the considered PD's \tilde{P}_n . If the latter is ensured by some other assumption, it suffices for Y to be polish. E.g., if $X = Y$ are Euclidean spaces, a sufficient condition is $\varrho(x, y) \rightarrow \infty$ as $\|x - y\| \rightarrow \infty$. Unfortunately, I did not find useful sufficient conditions for Banach spaces.

As a point of interest observe that the continuity assumptions in theorem 2.2 can be weakened to lower semicontinuity, requiring only small modification of the proof.

It is very easy to give examples where a minimizing PD does not exist. The following one is easily checked to be such, moreover, it also shows that $R(d)$ may have a discontinuity at $d=0$ without being constant for $d>0$ as in example 3.

Example 4. Let $X = \{0, 1, 2\}$, $Y = \{0, 1, 2, \dots\}$, $\varrho(0, 0) = \varrho(1, 1) = \varrho(2, 2) = 0$, $\varrho(1, 0) = \varrho(2, 0) = 1$, $\varrho(0, k) = 1$, $\varrho(i, k) = \frac{1}{k}$ for $k=1, 2, \dots$, $i=1, 2$, $i \neq k$. Then for any PD $P = \{p(0), p(1), p(2)\}$ on X with positive probabilities, we have

$$\lim_{d \rightarrow 0} R(d) = h(p(0)) < R(0) = H(P) = - \sum_{i=0}^2 p(i) \log p(i).$$

To the proof for theorem 2.3 we shall need

LEMMA 2.3. *Suppose that there exists a finite set $B \subset Y$ such that*

$$(2.26) \quad \int \varrho(x, B) P(dx) < \infty \quad \text{where} \quad \varrho(x, B) = \min_{y \in B} \varrho(x, y).$$

Then in the definition (1.7) of $R(d)$, it suffices to restrict attention to PD's \tilde{P} having Y -marginals concentrated on finite subsets of Y , if $d > d_0$.

PROOF. Modify the definition of $R(d)$ by restricting (1.4) to finite-valued RV's η , and denote the resulting function by $R'(d)$. The assertion is $R'(d) = R(d)$ (for $d > d_0$) and it suffices to prove $R'(d) \leq R(d)$.

Let ξ_1, \dots, ξ_L be independent X -valued RV's with the same distribution P and η_1, \dots, η_L be arbitrary RV's taking values in a finite subset of Y . Then we have — just as for the ordinary $R(d)$, see e.g. [4], theorem 9.2.1 and its extension on p. 470 —

$$(2.27) \quad \frac{1}{L} I((\xi_1, \dots, \xi_L), (\eta_1, \dots, \eta_L)) \leq R'(\bar{d}_L); \quad \bar{d}_L = E \frac{1}{L} \sum_{i=1}^L \varrho(\xi_i, \eta_i).$$

By the source-coding theorem ([4], theorem 9.6.2), for any $\delta > 0$ and sufficiently large L , there exists a mapping f of X^L into a finite subset of Y^L such that $(\eta_1, \dots, \eta_L) = f(\xi_1, \dots, \xi_L)$ satisfies $\bar{d}_L \leq d + \delta$ and the entropy of (η_1, \dots, η_L) is $\leq LR(d) + \delta L$. Since the latter entropy is an upper bound of the mutual information in (2.27), we obtain

$$(2.28) \quad R'(d + \delta) \leq R(d) + \delta.$$

Substituting d by $d - \delta$ and letting $\delta \rightarrow 0$, (2.28) gives $R'(d) \leq R(d)$.

Remark. It should be possible to prove lemma 2.3 without relying on the source-coding theorem. The reader is invited to find a direct proof. Observe, however, that condition (2.26) cannot be omitted as a simple modification of example 2 shows (let P have finite entropy).

THEOREM 2.3. Under the condition (2.26), for each $d > d_0$ holds

$$(2.29) \quad R(d) = \max_{\alpha(x), s} \left[\int \log \alpha(x) P(dx) - sd \right]$$

where the maximization refers to $\alpha(x) \geq 1$ and $s \geq 0$ under the constraints

$$(2.30) \quad \int \alpha(x) e^{-s\varrho(x, y)} P(dx) \leq 1 \quad \text{for each } y \in Y.$$

The maximizing s values are exactly those associated with d by (1.12).

PROOF. On account of lemmas 1.2 and 1.4, it suffices for $s \geq 0$ associated with d to exhibit an $\alpha(x) \geq 1$ satisfying the constraints (2.30) such that

$$(2.31) \quad F(s) = \int \log \alpha(x) P(dx).$$

To this end, consider a sequence \tilde{P}_n satisfying (2.24) for which the marginals $Q_n = Q_{\tilde{P}_n}$ are concentrated on finite subsets $Y_1 \subset Y_2 \subset \dots$ of Y . The existence of such \tilde{P}_n 's follows from lemmas 1.2 and 2.3; moreover, by the corollary of theorem 2.3 we may assume that \tilde{P}_n minimizes $I(\tilde{P}) + s\varrho(\tilde{P})$ within the class of \tilde{P} 's with $Q_{\tilde{P}}$ concentrated on Y_n . Then

$$(2.32) \quad I(\tilde{P}_n) + s\varrho(\tilde{P}_n) = \int \log \alpha_n(x) P(dx) = \max \int \log \alpha(x) P(dx)$$

where $\alpha_n(x) \geq 1$ satisfies (1.22) and the maximization is subject to that constraint — rewritten as (2.30) — with the modification that Y is replaced by Y_n in both cases.

Observe that for any (m, n) with $m < n$, say, $\frac{1}{2}(\alpha_m(x) + \alpha_n(x))$ satisfies the constraint for Y_m . Thus, using that for arbitrary u and v

$$(2.33) \quad 2 \log \frac{u+v}{2} - \log u - \log v = \log \frac{(u+v)^2}{4uv} = -\log \left(1 - \left(\frac{u-v}{u+v} \right)^2 \right) \geq \left(\frac{u-v}{u+v} \right)^2,$$

we obtain the inequality

$$(2.34) \quad I(\tilde{P}_m) + s\varrho(\tilde{P}_m) - I(\tilde{P}_n) - s\varrho(\tilde{P}_n) \geq 2 \int \log \frac{\alpha_m(x) + \alpha_n(x)}{2} P(dx) - \\ - \int \log \alpha_m(x) P(dx) - \int \log \alpha_n(x) P(dx) \geq \int \left(\frac{\alpha_m(x) - \alpha_n(x)}{\alpha_m(x) + \alpha_n(x)} \right)^2 P(dx).$$

Since clearly $\left(\frac{u^{(K)} - v^{(K)}}{u^{(K)} + v^{(K)}} \right)^2 \leq \left(\frac{u-v}{u+v} \right)^2$ where $u^{(K)} = \min(u, K)$, (2.31) and (2.34) imply that $\alpha_n^{(K)}(x)$ is a Cauchy-sequence in $L_2(P)$ -norm for each fixed K . Hence follows, in particular, that a subsequence $\alpha_{n_k}(x)$ of $\alpha_n(x)$ converges almost everywhere to some $\alpha(x) \geq 1$.

By the concavity of the log function, we have for any $A \in \mathcal{X}$

$$(2.35) \quad \frac{1}{P(A)} \int_A \log (\alpha_n(x) e^{-s\varrho(x, B)}) P(dx) \leq \log \frac{1}{P(A)} \int_A \alpha_n(x) e^{-s\varrho(x, B)} P(dx),$$

implying

$$(2.36) \quad \int_A \log \alpha_n(x) P(dx) \leq s \int_A \varrho(x, B) P(dx) + P(A) \log \frac{1}{P(A)} + \\ + P(A) \log \int_A \alpha_n(x) e^{-s\varrho(x, B)} P(dx).$$

In view of the assumption (2.26), this proves that $\log \alpha_n(x)$ is uniformly integrable, since the last integral is certainly \leq the number of elements of B , by (2.30) (assuming, as we may, that $B \subset Y_1$).

Thus, by (2.24) and (2.32), $\alpha_{n_k}(x) \rightarrow \alpha(x)$ implies (2.31).

Observe that if $\varrho(x, y)$ is bounded then so are the $\alpha_n(x)$'s (uniformly, see (1.15)), thus (2.34) implies their convergence, making the last arguments unnecessary.

We have still to show that $\alpha(x) = \lim \alpha_{n_k}(x)$ satisfies the constraints (2.30). This is clear for $y \in Y_\infty = \bigcup_{n=1}^{\infty} Y_n$ by Fatou's lemma. Consider any other $y_0 \in Y$ and repeat the above argument with $Y'_n = Y_n \cup \{y_0\}$. We arrive at a function $\alpha'(x) \equiv 1$ satisfying (2.31) as well as the constraints (2.30) for $y \in \bigcup_{n=1}^{\infty} Y'_n = Y_\infty \cup \{y_0\}$.

By the very construction, $F(s) = \inf [I(\tilde{P}) + sQ(\tilde{P})]$ gives the infimum also for those \tilde{P} 's for which $Q_{\tilde{P}}$ is concentrated on Y_∞ . Thus, in view of (1.23), for the function $\frac{1}{2}(\alpha(x) + \alpha'(x))$ — which also satisfies the constraints (2.30) for $y \in Y_\infty$ — we must have, using the concavity of the log function

$$(2.37) \quad F(s) \geq \int \log \frac{1}{2}(\alpha(x) + \alpha'(x)) P(dx) \geq \int \frac{1}{2}(\log \alpha(x) + \log \alpha'(x)) P(dx) = F(s).$$

This is possible only if $\alpha(x) = \alpha'(x)$ a.e., proving that $\alpha(x)$ satisfies the constraints (2.30) also for y_0 , hence for each $y \in Y$.

The proof of theorem 2.13 is complete.

Remarks. It is conceivable, in general, that there are several pairs $(\alpha(x), s)$ achieving maximum in (2.29), but then they must have different s values (else the strict concavity of the log function would lead to a contradiction). We do not go into the problem here, under what conditions (weaker than the existence of a \tilde{P}_0 achieving $R(d)$) can one assert uniqueness of s , i.e. differentiability of $R(d)$.

It is interesting to compare theorem 2.3 with an analogous result concerning channel capacity. A channel with input space X and output space Y is defined as a family $\{Q_x: x \in X\}$ of PD's on Y , and its capacity is

$$(2.38) \quad C = \sup_P I(\tilde{P})$$

where P ranges over the PD's on X and \tilde{P} is the PD on $X \times Y$ specified by $\tilde{P}(A \times B) = \int_A Q_x(B) P(dx)$ ($A \in \mathcal{X}, B \in \mathcal{Y}$). Then always holds

$$(2.39) \quad C = \min_Q \left[\sup_x I(Q_x \| Q) \right]$$

where the minimizing Q is the Y -marginal of \tilde{P}_0 corresponding to a P_0 achieving maximum in (2.38) if such P_0 exists, but (2.39) is valid also if the maximum in (2.38) is not achieved, cf. [5]. Thus, in case of channels, an „optimal” output distribution always exists even if a compatible input can not be found, while theorem 2.3 asserts, intuitively, the existence of an optimal backward conditional distribution — see (1.26) — although a compatible joint distribution need not exist.

Theorem 2.3 leaves the problem open whether a similar representation of $R(d)$ is possible also for $d = d_0$. In most cases of interest $d_0 = 0$ (see theorem 2.1) thus the

problem left is the representation of $R(0)$. Setting $q'(x, y) = 0$ if $q(x, y) = 0$ and $q'(x, y) = \infty$ else, we have $F'(s) = R(0)$ for any $s > 0$. If Y is compact metric and the set $\{y: q(x, y) = 0\}$ is closed for every $x \in X$ (in particular, if Y is finite) then the existence of a \tilde{P}_0 achieving $F'(s) = R(d)$ can be proved to the analogy of theorem 2.2. If such a \tilde{P}_0 exists, the corollary of lemma 1.4 gives

$$(2.42) \quad R(0) = \max_{\alpha(x)} \int \log \alpha(x) P(dx)$$

subject to

$$(2.43) \quad \int_{\{x: q(x, y) = 0\}} \alpha(x) P(dx) \leq 1 \quad \text{for each } y \in Y.$$

One would infer that the same holds even without assuming the existence of a minimizing \tilde{P}_0 . Unfortunately, this does not follow from theorem 2.3 since the modified loss function $q'(x, y)$ typically does not meet the condition (2.26).

If (2.42)—(2.43) holds, on the basis of basis of inequality (1.23) it is often possible to prove $\lim_{d \rightarrow 0} R(d) = R(0)$, e.g. if $\int \alpha(x) e^{-sq(x, y)} P(dx)$ converges uniformly to the right hand side of (2.43), as $s \rightarrow \infty$. This is the case, in particular, if Y is finite. Without going deeper into this problem, we conclude as an example by a simple proof of a generalization of a recent result of LINKOV, see [6], theorem 4.

THEOREM 2.4. *Let $P = \{p(x): x \in X\}$ be any PD on a finite or countable set X , let $Y \supset X$ be arbitrary, $q(x, x) = 0$ for each $x \in X$, and suppose that to any x_1 and x_2 in X there exists $\delta > 0$ such that*

$$(2.44) \quad \max(q(x_1, y), q(x_2, y)) > \delta \quad \text{for each } y \in Y.$$

Then $R(d)$ is finite⁵ for $d > 0$ and

$$(2.45) \quad \lim_{d \rightarrow 0} R(d) = R(0) = H(P) = - \sum_{x \in X} p(x) \log p(x).$$

LINKOV (loc.cit.) has proved (2.45) under the more restrictive conditions that $q(x, y)$ is a metric on Y and $H(P) < \infty$.

Remark. If each $p(x)$ is positive and $H(P) < \infty$, the sufficient condition of theorem 2.4 is easily seen to be necessary, as well.

PROOF. The finiteness of $R(d)$ for $d > 0$ follows from Theorem 2.1., even if $H(P) = \infty$. Since $I(\xi, \eta) \leq H(P)$ for any RV's ξ and η with $P_\xi = P$, on account of lemma 1.2 eq. (1.13) it suffices to prove

$$(2.46) \quad \lim_{s \rightarrow \infty} F(s) \leq H(P).$$

By lemma 1.4, inequality (1.23), we have

$$(2.47) \quad F(s) \leq \sum_{x \in X} p(x) \log \alpha(x)$$

⁵ Excluding, as we did throughout, the trivial case that $R(d)$ is identically infinite.

for any positive numbers $\alpha(x)$ satisfying (1.22), i.e.

$$(2.48) \quad \sum_{x \in X} p(x) \alpha(x) e^{-sq(x,y)} \leq 1 \quad \text{for all } y \in Y$$

(observe that here no reference is needed to the deeper result Theorem 2.3).

Given $\varepsilon > 0$, let A be a finite subset of X with

$$(2.49) \quad \sum_{x \in X \setminus A} p(x) < \frac{\varepsilon}{2},$$

and let $\alpha(x) = \frac{1-\varepsilon}{p(x)}$ for $x \in A$ and $\alpha(x) = 1$ for $x \in X \setminus A$. Then

$$(2.50) \quad \begin{aligned} \sum_{x \in X} p(x) \alpha(x) e^{-sq(x,y)} &\leq \frac{\varepsilon}{2} + (1-\varepsilon) \sum_{x \in A} e^{-sq(x,y)} \leq \\ &\leq \frac{\varepsilon}{2} + (1-\varepsilon)(1 + |A|e^{-s\delta}) \quad \text{for all } y \in Y, \end{aligned}$$

where $|A|$ denotes the size of A and δ is a positive number such that (2.44) holds for each pair (x_1, x_2) of elements of A . For sufficiently large s , the right hand side of (2.50) is ≤ 1 , and (2.47) gives

$$(2.51) \quad F(s) \geq - \sum_{x \in A} p(x) \log p(x) + \log(1-\varepsilon).$$

The right hand side of (2.51) tends to $H(P)$ as $\varepsilon \rightarrow 0$, proving (2.46) and thus (2.45), as well.

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PERFECT SINGLE-LEE-ERROR-CORRECTING CODE

by

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By a *code*, we shall mean throughout this paper a set of certain n -dimensional vectors the coordinates of which are nonnegative integers less than a given natural number q .

The elements of the code are called *codevectors* or *codewords*.

The *Lee-weight* of a vector $\mathbf{c}=(c_1 \dots c_n)$, $0 \leq c_i < q$, c_i integer, is defined to be the sum

$$W_L(\mathbf{c}) = \sum_{i=1}^n |d_{L,q}(c_i)|,$$

where $d_{L,q}(c_i) \equiv c_i \pmod q$ and $0 \leq |d_{L,q}(c_i)| \leq \frac{q}{2}$.

The *difference* \mathbf{c} of the codevectors \mathbf{c}^1 and \mathbf{c}^2 is defined by $\mathbf{c}=\mathbf{c}^1-\mathbf{c}^2$ where $c_i \equiv c_i^1 - c_i^2 \pmod q$ and $0 \leq c_i < q$.

The *Lee distance* $d_L(\mathbf{c}^1, \mathbf{c}^2)$ of the two vectors \mathbf{c}^1 and \mathbf{c}^2 is the Lee weight of the vector $\mathbf{c}=\mathbf{c}^1-\mathbf{c}^2$.

We denote the number of elements of a set A by $|A|$.

A code C is a perfect single-Lee-error-correcting code of block-length n over the alphabet of q letters if

$$|C| \cdot (2n + 1) = q^n$$

and for $\mathbf{c}^1 \neq \mathbf{c}^2$, $\mathbf{c}^1, \mathbf{c}^2 \in C$ $d_{L,q}(\mathbf{c}^1, \mathbf{c}^2) \geq 3$.

Let $L(n, q)$ denote such a code.

In [1], Theorem (3.21) E. R. BERLEKAMP deals with the problem of finding a necessary and sufficient condition for the existence of perfect single-Lee-error-correcting codes. In this paper we shall correct a small error in this result. We prove the following

THEOREM. *A perfect single-Lee-error-correcting code of block-length n over the alphabet of q letters exists for those and only those integers $q > 2$ and $n > 0$, for which*

$$(1) \quad n = \frac{\frac{q^r}{h} - 1}{2}$$

for appropriate integers $r > 0$ and $h > 0$.

PROOF. In the proof we shall use from BERLEKAMP's result the following

LEMMA. *If $q > 2$ and*

$$(2) \quad n = \frac{q^r - 1}{2}$$

for some integer $r > 0$, then there exists a perfect single-Lee-error-correcting code of block-length n over the alphabet of q letters.

Let us suppose that $n = \frac{q^r - 1}{2}$. Let $q' = \frac{q^r}{h}$. Then obviously we have $q' \geq 3$. Thus according to the Lemma, there exists a perfect single-Lee-error-correcting code $L(n, q')$ of block-length n over the alphabet of q' letters.

Let A be the set of all vectors \mathbf{a} for which $\mathbf{a} = \mathbf{c} + q'\mathbf{b}$, where $\mathbf{c} \in L(n, q')$ and \mathbf{b} is a vector for the coordinates of which the inequality $0 \leq b_i < \frac{h}{q^{r-1}}$ holds.

We claim that A is a perfect single-Lee-error-correcting code of block length n over the alphabet of q letters.

Obviously $\mathbf{c}^1 + q'\mathbf{b}^1 = \mathbf{c}^2 + q'\mathbf{b}^2$ can hold only if $\mathbf{c}^1 = \mathbf{c}^2$ and $\mathbf{b}^1 = \mathbf{b}^2$. Therefore

$$|A| = |L(n, q')| \cdot \left(\frac{h}{q^{r-1}} \right)^n.$$

Since $|L(n, q')| \cdot (2n+1) = (q')^n$, we have

$$|A| = \frac{(q')^n}{2n+1} \cdot \left(\frac{h}{q^{r-1}} \right)^n = \frac{\left(q' \cdot \frac{h}{q^{r-1}} \right)^n}{2n+1} = \frac{q^n}{2n+1}.$$

Now, we prove in an indirect way: if \mathbf{a}^1 and $\mathbf{a}^2 \in A$, $\mathbf{a}^1 \neq \mathbf{a}^2$, then $d_{L,q}(\mathbf{a}^1, \mathbf{a}^2) \geq 3$.

Obviously, if $\mathbf{a} = (a_1, \dots, a_n) \in A$ then $0 \leq a_i < q$. Therefore, if \mathbf{a}^1 and $\mathbf{a}^2 \in A$ and $d_{L,q}(a_i^1 - a_i^2) = 0$ then $a_i^1 = a_i^2$.

Assume now that for the vectors \mathbf{a}^1 and $\mathbf{a}^2 \in A$ $d_{L,q}(\mathbf{a}^1, \mathbf{a}^2) = 1$. Then there is a single index, say j , for which $|d_{L,q}(a_j^1 - a_j^2)| = 1$ and we have $d_{L,q}(a_i^1 - a_i^2) = 0$ for every $i = 1, 2, \dots, n$ $i \neq j$.

Using the relations

$$d_{L,q}(a_j^1 - a_j^2) \equiv a_j^1 - a_j^2 \pmod{q} \quad \text{and} \quad 0 \leq |a_j^1 - a_j^2| < q$$

we obtain $1 \equiv a_j^1 - a_j^2 \pmod{q}$ or $-1 \equiv a_j^1 - a_j^2 \pmod{q}$, that is

$$|c_j^1 - c_j^2 - q'(b_j^1 - b_j^2)| = 1 \quad \text{or} \quad |c_j^1 - c_j^2 - q'(b_j^1 - b_j^2)| = q - 1.$$

Because of the inequality $|c_j^1 - c_j^2| < q'$, the equality

$$|c_j^1 - c_j^2 - q'(b_j^2 - b_j^1)| = 1$$

can hold only if

$$|c_j^1 - c_j^2| = 1 \quad \text{and} \quad |b_j^2 - b_j^1| = 0$$

or

$$|c_j^1 - c_j^2| = q' - 1 \quad \text{and} \quad |b_j^2 - b_j^1| = 1.$$

On the other hand, from the equality

$$|c_j^1 - c_j^2 - q'(b_j^2 - b_j^1)| = q - 1$$

follows that

$$|c_j^1 - c_j^2| = q' - 1 \quad \text{and} \quad |b_j^2 - b_j^1| = \frac{h}{q^{r-1}} - 1.$$

Thus, we have in both cases

$$|c_j^1 - c_j^2| \equiv 1 \pmod{q'}$$

that is

$$|d_{L,q'}(c_j^1 - c_j^2)| = 1.$$

Further if $i \neq j$ then the equality $|c_i^1 - c_i^2| = 0$ follows from $d_{L,q}(a_i^1 - a_i^2) = 0$.

Thus we have

$$d_{L,q'}(\mathbf{c}^1, \mathbf{c}^2) = \sum_{i=1}^n |d_{L,q'}(c_i^1 - c_i^2)| = 1,$$

which contradicts the assumption that \mathbf{c}^1 and $\mathbf{c}^2 \in L(n, q')$.

Finally, assume that for the vectors \mathbf{a}^1 and $\mathbf{a}^2 \in A$, we have

$$d_{L,q}(\mathbf{a}^1, \mathbf{a}^2) = \sum_{i=1}^n |d_{L,q}(a_i^1 - a_i^2)| = 2.$$

This can be true only if

- a) There is a single index, say j , for which $d_{L,q}(a_j^1 - a_j^2) = 2$ and we have $d_{L,q}(a_i^1 - a_i^2) = 0$ for every $i = 1, 2, \dots, n \ i \neq j$.
- b) There are two indices, say k and l , for which

$$d_{L,q}(a_k^1 - a_k^2) = d_{L,q}(a_l^1 - a_l^2) = 1$$

and we have $d_{L,q}(a_i^1 - a_i^2) = 0$ for every $i = 1, 2, \dots \ i \neq k$ and $i \neq l$.

In both cases we have $d_{L,q}(\mathbf{c}^1, \mathbf{c}^2) = 2$ which can be proved in a similar way to the previous case. But this contradicts the assumption that \mathbf{c}^1 and $\mathbf{c}^2 \in L(n, q')$.

Now, we show that the relation $n = \frac{q^r - 1}{2h} - 1$ holds for every perfect single-Lee-error-correcting code for suitable positive integers r and h . The number of possible n -dimensional vectors is q^n . If the code is perfect, then $\frac{q^n}{2n+1} = K$ holds, where K is

integer. From this equality we get the relation $n = \frac{\frac{q^n}{K} - 1}{2}$.

The proof of the theorem is completed.

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A REMARK ON EXTREMAL SETS IN THE THEORY OF POLYNOMIAL INTERPOLATION

by
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1. Introduction and summary

Let the numbers x_1, x_2, \dots, x_n ($n \geq 2$) be prescribed on the interval $[-1, 1]$ with $-1 \leq x_1 < x_2 < \dots < x_n \leq 1$. Denoting the fundamental Lagrange interpolating polynomials by $l_i(x)$, we introduce the Lebesgue function $\lambda_n(x) = \sum_{i=1}^n |l_i(x)|$. The question arises as how to choose the nodes x_1, x_2, \dots, x_n in such a way that $A_n = \max_{-1 \leq x \leq 1} \lambda_n(x)$ will be as small as possible. (A set of nodes with this property is called an extremal set.) This is a famous unsolved problem in interpolation theory. Interesting results in connection with it can be found in BERNSTEIN [1], ERDŐS [4] and LUTTMANN—RIVLIN [5]; the paper by CHENEY—PRICE [3] deals with problems of the same kind in a more general setting.

A footnote in [1] is concerned with the particular case of three nodes. An extremal set consisting of the points $x_1 = -\frac{2}{3}\sqrt{2}$, $x_2 = 0$, $x_3 = \frac{2}{3}\sqrt{2}$ is exhibited with $A_3 = \frac{5}{4}$. (Actually, the nodes x_1, x_2, x_3 with $-x_1 = x_3 \geq \frac{2}{3}\sqrt{2}$, $x_2 = 0$ are also extremal sets, cf. CHENEY [2], p. 65.)

Assuming $-1 < x_1$ and $x_n < 1$, BERNSTEIN ([1], p. 1026—1027) conjectured that an extremal set of nodes is characterized by the property that the values of the $n+1$ local maxima of $\lambda_n(x)$ on the intervals $[-1, x_1], [x_1, x_2], \dots, [x_n, 1]$ are all equal. This conjecture can also be found in [4]. Attention should also be drawn to a paper by LUTTMANN and RIVLIN [5]. They show e.g. that one can always take the endpoints of $[-1, 1]$ as nodes in an extremal set.

The object of this note is to determine all extremal sets in case of three nodes. Furthermore, we propose a modification of Bernstein's conjecture.

2. Evaluation of the interpolation problem having three nodes

The problem we are concerned with may be formulated as follows. Let $-1 \leq x_1 < x_2 < x_3 \leq 1$ and consider the function $\lambda_3(x) = \sum_{i=1}^3 |l_i(x)|$. The nodes x_1, x_2, x_3 are to be determined in such a way that $\max_{-1 \leq x \leq 1} \lambda_3(x)$ has least value. Using the familiar

result $\sum_{i=1}^3 l_i(x) = 1$, it is easy to verify that we have

$$(1) \quad \max_{-1 \leq x \leq x_1} \lambda_3(x) = 1 + 2 \frac{(1+x_1)(1+x_3)}{(x_2-x_1)(x_3-x_2)}, \quad \max_{x_1 \leq x \leq x_2} \lambda_3(x) = 1 + \frac{(x_2-x_1)^2}{2(x_3-x_1)(x_3-x_2)},$$

$$\max_{x_2 \leq x \leq x_3} \lambda_3(x) = 1 + \frac{(x_3-x_2)^2}{2(x_2-x_1)(x_3-x_1)}, \quad \max_{x_3 \leq x \leq 1} \lambda_3(x) = 1 + 2 \frac{(1-x_1)(1-x_3)}{(x_2-x_1)(x_3-x_2)}.$$

ASSERTION 1. *In order that x_1, x_2, x_3 constitute an extremal set of nodes, it is necessary that*

$$(2) \quad x_2 = \frac{1}{2}(x_1 + x_3).$$

PROOF. Suppose $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are an extremal set with $\bar{x}_2 \neq \frac{1}{2}(\bar{x}_1 + \bar{x}_3)$. A careful examination of (1) then leads to a contradiction. We omit the calculational details.

We thus can remove one parameter in formulae (1) by assuming (2) and there remains the problem of choosing x_1 and x_3 in such a way that

$$(3) \quad A_3 - 1 = \max \left\{ \frac{8(1+x_1)(1+x_3)}{(x_3-x_1)^2}, \frac{1}{4}, \frac{8(1-x_1)(1-x_3)}{(x_3-x_1)^2} \right\}$$

is minimal. Obviously we have $A_3 \geq 5/4$. Taking into account Bernstein's example, it follows from (3) that a necessary and sufficient condition for $x_1, x_2 = \frac{1}{2}(x_1 + x_3), x_3$ to form an extremal set is that the inequalities

$$\frac{8(1+x_1)(1+x_3)}{(x_3-x_1)^2} \leq \frac{1}{4}, \quad \frac{8(1-x_1)(1-x_3)}{(x_3-x_1)^2} \leq \frac{1}{4}$$

are simultaneously satisfied. After elementary, but somewhat tedious calculations we arrive at the following result.

THEOREM 1. *In case of three nodes the extremal sets, as defined in section 1, can be given in terms of the position of the first node as follows:*

(i) if

$$-1 \leq x_1 \leq -\frac{2}{3}\sqrt{2} = -0.9428 \dots,$$

then

$$x_2 = \frac{1}{2}(x_1 + x_3), \quad (17 - 12\sqrt{2})x_1 + 12\sqrt{2} - 16 \leq x_3 \leq 1;$$

(ii) if

$$-\frac{2}{3}\sqrt{2} < x_1 \leq -24\sqrt{2} + 33 = -0.9411 \dots,$$

then

$$x_2 = \frac{1}{2}(x_1 + x_3), \quad (17 + 12\sqrt{2})x_1 + 12\sqrt{2} + 16 < x_3 \leq 1.$$

REMARK 1. It follows from theorem 1 that the nodes x_1 and x_3 need not be symmetric with respect to the origin. Moreover, as compared with CHENEY's example, the range of x_1 can slightly be enlarged; the range of x_3 is then diminished.

REMARK 2. If $-1 < x_1 < x_2 < x_3 < 1$, then there is only one extremal set with the property that the four local maxima of $\lambda_3(x)$ on $[-1, x_1]$, \dots , $[x_3, 1]$ are all equal. This is precisely BERNSTEIN's example. As is apparent from theorem 1 there are numerous examples of extremal sets for which the local maxima of $\lambda_3(x)$ at the points of $[-1, 1]$ are smaller than $5/4$.

3. A modification of Bernstein's conjecture

Recalling the result of LUTTMANN and RIVLIN [5] that in the general case of n nodes ($n > 2$) the extremal set is not unique, and taking into account remark 2 of section 2, we propose to modify the conjecture of Bernstein in the following way:

(i) there exists at least one set of nodes with the property that the $n+1$ local maxima of the Lebesgue function $\lambda_n(x)$ on the intervals $[-1, x_1]$, $[x_1, x_2]$, \dots , $[x_n, 1]$ are all equal; this set is extremal.

(ii) in order that x_1, x_2, \dots, x_n form an extremal set, it is necessary that the $n-1$ local maxima of $\lambda_n(x)$ on $[x_1, x_2]$, $[x_2, x_3]$, \dots , $[x_{n-1}, x_n]$ be all equal.

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A GENERAL METHOD FOR DENSITY ESTIMATION

by

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1. Introduction

Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of i.i.d.r.v.'s with common density function (d.f.) f and let $\Phi_k(x, y)$ ($k=1, 2, \dots$) be a sequence of Borel-measurable functions, (so-called kernel functions) defined on the Euclidean plane. Then the empirical d.f. can be defined as follows:

$$(1) \quad f_n(x) = \frac{1}{n} \sum_{k=1}^n \Phi_n(x, \xi_k).$$

First of all let us give some examples to clear the meaning of our definition.

Example 1. Let h be a d.f. and let $a_n \searrow 0$ be a sequence of positive numbers. Define $\Phi_n(x, y)$ as follows:

$$\Phi_n(x, y) = \frac{1}{a_n} h\left(\frac{x-y}{a_n}\right).$$

Then

$$f_n(x) = \frac{1}{na_n} \sum_{k=1}^n h\left(\frac{x-\xi_k}{a_n}\right)$$

i.e. we get the well-known Rosenblatt's definition ([3]).

Example 2. Let $\{\varphi_j\}$ be an orthonormal system and let $q_n \nearrow +\infty$ be a sequence of integers. Define

$$\Phi_n(x, y) = \sum_{j=1}^{q_n} \varphi_j(x) \varphi_j(y).$$

Then

$$f_n(x) = \sum_{j=1}^{q_n} \tilde{c}_j \varphi_j(x)$$

where

$$\tilde{c}_j = \frac{\varphi_j(\xi_1) + \varphi_j(\xi_2) + \dots + \varphi_j(\xi_n)}{n}$$

i.e. we get the expansion in orthogonal functions.

Example 3. Let

$$\dots < x_{-2}^{(n)} < x_{-1}^{(n)} < x_0^{(n)} < x_1^{(n)} < x_2^{(n)} < \dots \quad (n = 1, 2, \dots)$$

$$x_{i+1}^{(n)} - x_i^{(n)} = a_n \searrow 0$$

be a sequence of partitions of the real line and let

$$\varphi_j^{(n)}(z) = \begin{cases} 1 & \text{if } x_{j-1}^{(n)} \leq z < x_j^{(n)} \\ 0 & \text{otherwise} \end{cases}$$

Finally set

$$\Phi_n(x, y) = \frac{1}{a_n} \sum_{j=-\infty}^{\infty} \varphi_j^{(n)}(x) \varphi_j^{(n)}(y).$$

Then clearly

$$f_n(t) = \frac{v_j^{(n)}}{na_n} \quad \text{if } x_{j-1}^{(n)} \leq t < x_j^{(n)}$$

where $v_j^{(n)}$ is the number of elements of the sample $\xi_1, \xi_2, \dots, \xi_n$ lying in the interval $[x_{j-1}^{(n)}, x_j^{(n)})$ i.e. we get the so-called histogram definition (see e.g. [2]).

Example 4. Making use of the notations of Example 3 and setting

$$\tilde{x}_i^{(n)} = \frac{x_{i-1}^{(n)} + x_i^{(n)}}{2}$$

define

$$\tilde{\Phi}_n(x, y) = \Phi_n(\tilde{x}_i^{(n)}, y) + \frac{\Phi_n(\tilde{x}_{i+1}^{(n)}, y) - \Phi_n(\tilde{x}_i^{(n)}, y)}{a_n} (x - \tilde{x}_i^{(n)})$$

if $\tilde{x}_i^{(n)} \leq x < \tilde{x}_{i+1}^{(n)}$.

Then

$$\tilde{f}_n(x) = \frac{1}{n} \sum_{k=1}^n \tilde{\Phi}_n(x, \xi_k) = f_n(\tilde{x}_i^{(n)}) + \frac{f_n(\tilde{x}_{i+1}^{(n)}) - f_n(\tilde{x}_i^{(n)})}{a_n} (x - \tilde{x}_i^{(n)})$$

if $\tilde{x}_i^{(n)} \leq x < \tilde{x}_{i+1}^{(n)}$, i.e. we get a smoothed histogram (see [2], Definition 2.1.).

Remark 1. It is easy to modify our basic definition (1) as follows: Let $\Phi_{n,k}(x, y)$ ($k=1, 2, \dots, n=1, 2, \dots$) be a double array of Borel-measurable functions defined on the Euclidean plane and let

$$f_n^*(x) = \frac{1}{n} \sum_{k=1}^n \Phi_{n,k}(x, \xi_k).$$

Making use of this definition we can get the definition of WOLVERTON and WAGNER ([4], see also [5]) as a special case too. We wish to emphasize that our results can be reformulated to this more general case without any difficulty but we do not do this because of the sake of simplicity.

Remark 2. In [2] (see Definitions 1.2, 2.2, 3.2) the second author formulated some histogram type definitions based on a random partition. One of these definitions was investigated in [6], an other but similar definition was used in [7]. Unfortunately our present definition (1) does not contain these definitions.

In the present paper the asymptotic property of the probability

$$P\left(\sup_x |f_n(x) - f(x)| \geq \varepsilon\right) \quad (n \rightarrow \infty)$$

will be studied. In the paragraphs 2 and 3 we formulate some simple lemmas concerning the first and second moments of $f_n(x)$, in order to clear the meaning of our Theorem formulated later on. Finally a strong law will be formulated.

2. First moment

In this paragraph we formulate some well-known lemmas:

LEMMA 1. (See [8] or [9].) *If f is a d.f. and h is a bounded d.f. for which*

$$\lim_{|x| \rightarrow +\infty} |x|h(x) = 0$$

then

$$(2) \quad \frac{1}{a_n} \int_{-\infty}^{\infty} h\left(\frac{x_0 - y}{a_n}\right) f(y) dy \rightarrow f(x_0) \quad (n \rightarrow \infty)$$

at every continuity point x_0 of f where $a_n \searrow 0$ is a sequence of positive numbers. Further if f is satisfying the Lipschitz condition: $|f(x_1) - f(x_2)| < C|x_1 - x_2|$ then the convergence in (2) is uniform.

Since

$$E f_n(x) = \frac{1}{a_n} \int_{-\infty}^{\infty} h\left(\frac{x - y}{a_n}\right) f(y) dy$$

(provided that $\Phi_n(x, y) = \frac{1}{a_n} h\left(\frac{x - y}{a_n}\right)$; Example 1.) one can state that $f_n(x)$ is an asymptotically unbiased estimation of $f(x)$ if x is a continuity point of f .

LEMMA 2. *Suppose that*

$$\Phi_n(x, y) = \sum_{j=0}^{q_n} \varphi_j(x) \varphi_j(y) \quad (q_n \nearrow +\infty).$$

(Example 2) and

$$\varphi_j(x) = \cos jx \quad 0 < x \leq \pi.$$

Then

$$E f_n(x) = \sum_{j=0}^{q_n} c_j \cos jx$$

where

$$c_j = \int_0^{\pi} f(x) \cos jx dx$$

i. e.

$$E f_n(x) \rightarrow f(x) \quad (n \rightarrow \infty)$$

provided that f is a square integrable d.f. vanishing outside the interval $[0, \pi]$.

Moreover if f has a bounded derivative in $[0, \pi]$ then $E f_n \rightarrow f$ uniformly in $[\varepsilon, \pi - \varepsilon]$ for any $\varepsilon > 0$.

LEMMA 3. *Suppose that*

$$\Phi_n(x, y) = \sum_{j=1}^{q_n} h_j(x) h_j(y) \quad (q_n \nearrow +\infty)$$

and

$$h_j(x) = (2^j j! \pi^2)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_j(x)$$

where $H_j(x)$ is the j -th Hermite polynomial¹. Further suppose that f has a bounded derivative in $(-\infty, +\infty)$ then

$$E f_n(x) \rightarrow f(x) \quad (n \rightarrow \infty)$$

uniformly in any finite interval $[a, b]$.

LEMMA 4 (see e.g. [2]). If $\Phi_n(x, y)$ is the kernel function defined in Examples 3 or 4 then $E f_n(x) \rightarrow f(x)$ at any continuity point x of $f(x)$ and further if f has a bounded derivative in a finite or infinite interval (a, b) then $E f_n \rightarrow f$ uniformly in² $[a+\varepsilon, b-\varepsilon]$ for any $\varepsilon > 0$.

Finally we formulate a general lemma (which goes back to Lebesgue) what gives a condition implying the convergence $E f_n \rightarrow f$.

LEMMA 5 (see e.g. [10], Theorem 4.2.1). Suppose that $\Phi_n(x, y)$ is defined in a finite square $[a, b] \times [a, b]$ and

$$\lim_{n \rightarrow \infty} \int_{x-\delta}^{x+\delta} \Phi_n(x, y) dy = 1, \quad \lim_{n \rightarrow \infty} \int_{[a, b] - [x-\delta, x+\delta]} \Phi_n(x, y) dy = 0$$

for any $\delta > 0$, and

$$\sup_{y \in [a, b] - [x-\delta, x+\delta]} |\Phi_n(x, y)| \leq \Phi(\delta)$$

where $\Phi(\delta)$ is a number depending on δ and x but independent of n , further $\left| \int_a^b \Phi_n(x, y) f(y) dy \right| < \infty$ for any $f \in L^1_{[a, b]}$ and $\left| \int_a^b \Phi_n(x, y) dy \right| < K$, finally f is a function of finite variation in $[A, B] \subset [a, b]$, then

$$(3) \quad \int_a^b \Phi_n(x, y) f(y) dy \rightarrow f(x) \quad (x \in (A, B))$$

in any continuity point x of $f(x)$. Further if the conditions hold uniformly in $[A, B]$ and f is continuous then the convergence (3) holds uniformly too.

Remark 3. Since

$$(4) \quad E f_n(x) = \int_a^b \Phi_n(x, y) f(y) dy$$

this lemma shows that f_n is an asymptotically unbiased estimation of f if the conditions of our Lemma 5 hold and f is a d.f. vanishing outside a finite interval $[a, b]$.

¹ We remark that the $\{h_j(x)\}_{j=1}^{\infty}$ is a uniformly bounded system of orthonormal functions (see [12] p. 208).

² Further on in case of infinite interval (a, b) we denote by $[a+\delta, b-\delta]$ an arbitrary closed subinterval of (a, b) .

Remark 4. If $\Phi_n(x, y)$ is defined on the whole Euclidean plane (instead of a finite square) then a similar but somewhat more complicated lemma can be formulated.

3. Second moments

LEMMA 6. (See e.g. [8].) *Let*

$$\Phi_n(x, y) = \frac{1}{a_n} h\left(\frac{x-y}{a_n}\right)$$

and suppose that f and h are bounded d.f.-s then

$$D^2(\Phi_n(x, \xi_1)) \leq \frac{C}{a_n}$$

where C is a positive constant. Since

$$f_n(x) = \frac{1}{na_n} \sum_{k=1}^n h\left(\frac{x-\xi_k}{a_n}\right)$$

we get

$$D^2(f_n(x)) \leq \frac{C}{na_n}.$$

LEMMA 7. *If $\Phi_n(x, y)$ is the kernel function defined in Examples 3 or 4 and f is bounded then*

$$D^2(\Phi_n(x, \xi_1)) \leq \frac{C}{a_n}$$

further

$$D^2(f_n(x)) \leq \frac{C}{na_n}$$

where C is a positive constant.

This lemma can be proved in the same way like Lemma 5 of [2] was proved.

LEMMA 8. *Suppose that*

$$\Phi_n(x, y) = \sum_{j=1}^{q_n} \varphi_j(x) \varphi_j(y)$$

where $\{\varphi_j(x)\}$ is a uniformly bounded orthonormal system and $f(x)$ is a bounded d.f. $f(x) \leq K$, then

$$D^2(\Phi_n(x, \xi_1)) \leq Cq_n$$

i.e.

$$D^2(f_n(x)) \leq \frac{Cq_n}{n}.$$

PROOF. Our statements follow from the following inequality

$$\begin{aligned} D^2(\Phi(x, \xi_1)) &\leq \int_{-\infty}^{\infty} \left(\sum_{j=1}^{q_n} \varphi_j(x) \varphi_j(y) \right)^2 f(y) dy \leq \\ &\leq K \int_{-\infty}^{\infty} \left(\sum_{j=1}^{q_n} \varphi_j(x) \varphi_j(y) \right)^2 dy = K \int_{-\infty}^{\infty} \sum_{j=1}^{q_n} \varphi_j^2(x) \varphi_j^2(y) dy. \end{aligned}$$

4. The main theorem and its consequences

THEOREM 1. Let f be a d.f. having finite absolute moment of order γ , that is

$$(i) \quad \int_{-\infty}^{\infty} |x|^{\gamma} f(x) dx < C_0$$

with some $\gamma > 0$.

Further let

$$(ii) \quad |f(x_1) - f(x_2)| < K_0 |x_1 - x_2| \quad \text{if } x_1, x_2 \in [a, b]$$

where $[a, b]$ denotes a finite or infinite interval.

Moreover let $\Phi_n(x, y)$ be a sequence of Borel measurable functions. Suppose that there exist two positive numbers Ω and τ and an interval $[c, d] \supset [a, b]$ such that

(iii) for each n the interval $[c, d]$ can be divided into disjoint leftclosed intervals $I_s^{(n)}$ ($s = 1, 2, \dots$) for which

$$|I_s^{(n)}| \cong \frac{1}{n^{\Omega}}, \quad \bigcup_{s=1}^{\infty} I_s^{(n)} = [c, d],$$

$$|\Phi_n(x_1, y) - \Phi_n(x_2, y)| \cong n^{\tau} |x_1 - x_2|$$

provided that x_1 and x_2 belong to the same interval $I_s^{(n)}$. Suppose that

$$(iv) \quad \int_a^b \Phi_n(x, y) f(y) dy \rightarrow f(x) \quad (n \rightarrow \infty)$$

uniformly in $[a + \delta, b - \delta]$ for any $\delta > 0$.

Suppose that for each n

$$(v) \quad D^2(\Phi_n(x, \xi_1)) \cong h_n$$

and

$$(vi) \quad h_n \cong \frac{n}{\omega(n) \log n}$$

where

$$\omega(n) \nearrow +\infty \quad (n \rightarrow \infty)$$

$$(vii) \quad |\Phi_n(x, y)| \cong Ch_n$$

and C is a positive constant.

Suppose further that there exists a $\nu > 0$ and a sequence of positive numbers $\varepsilon_n \rightarrow 0$ such that

$$(viii) \quad |\Phi_n(x_n, y_n)| < \varepsilon_n \quad \text{whenever } |x_n - y_n| > n^{\nu}.$$

Then

$$(5) \quad P\left\{ \sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - f(x)| \cong \varepsilon \right\} \leq e^{-\frac{Kn}{h_n}}$$

if n is large enough, (where K is a positive constant depending on ε , δ , and $f(x)$).

Our Theorem 1 clearly implies

THEOREM 1.a. (Example 1.) *Let f be a bounded distribution function obeying conditions (i) and (ii) of Theorem 1. Further let h be a d.f. for which*

$$|h(x_1) - h(x_2)| \leq C|x_1 - x_2| \quad (-\infty < x_1 < x_2 < +\infty)$$

and

$$\lim_{|x| \rightarrow +\infty} |x|h(x) = 0$$

finally let $a_n \searrow 0$ be a sequence of real numbers for which

$$a_n \geq \frac{\omega(n) \log n}{n} \quad (\omega(n) \nearrow +\infty, n \rightarrow \infty)$$

then

$$P\left\{ \sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - f(x)| \geq \varepsilon \right\} \leq e^{-Kna_n}$$

where

$$f_n(x) = \frac{1}{na_n} \sum_{k=1}^n h\left(\frac{x - \xi_k}{a_n}\right).$$

THEOREM 1.b. *Let f be a bounded d.f. obeying conditions (i) and (ii) of Theorem 1 and let $\{\varphi_j(x)\}_{j=1}^{\infty}$ be a uniformly bounded orthonormal system defined on an interval $[c, d]$.*

Set

$$\Phi_n(x, y) = \sum_{j=1}^{q_n} \varphi_j(x) \varphi_j(y)$$

and suppose that $\Phi_n(x, y)$ satisfies condition (viii) of Theorem 1. Let

$$f_n(x) = \frac{1}{n} \sum_{k=1}^n \Phi_n(x, \xi_k)$$

where

$$q_n \leq \frac{n}{\omega(n) \log n} \quad (\omega(n) \rightarrow +\infty, n \rightarrow \infty).$$

Suppose that

(i) the Fourier series

$$\sum_{j=1}^n c_j \varphi_j, \quad c_j = \int_c^d \varphi_j f dx$$

of f converges to f uniformly in an interval $[a, b] \subset [c, d]$

(ii) for each n the interval $[c, d]$ can be divided into disjoint intervals $I_s^{(n)}$ ($s=1, 2, \dots$) such that

$$|I_s^{(n)}| \geq \frac{1}{n^\alpha} \quad \text{and} \quad |\varphi_j(x) - \varphi_j(y)| \leq j^\alpha |x - y| \quad (\alpha > 0)$$

provided that x and y are belonging to the same $I_s^{(n)}$ and $1 \leq j \leq q_n$. Then

$$P\left\{ \sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - f(x)| \geq \varepsilon \right\} \leq e^{-\frac{K\varepsilon n}{q_n}}$$

for any $\delta > 0$.

Theorem 1.b. easily implies the following Theorems 1.c. and 1.d.

THEOREM 1.c. (Example 2.). *Let f be a bounded d.f. obeying condition (i) of Theorem 1, having uniformly bounded derivative in $(0, \pi)$. Further let*

$$f_n(x) = \sum_{j=0}^{q_n} \tilde{c}_j \cos jx$$

where

$$\tilde{c}_j = \frac{1}{n} \sum_{k=1}^n \cos j\xi_k$$

and $q_n \nearrow +\infty$ is a sequence of integers for which

$$q_n \cong \frac{n}{\omega(n) \log n} \quad (\omega(n) \rightarrow \infty, n \rightarrow \infty).$$

Then

$$P\left\{ \sup_{\delta \cong x \cong \pi - \delta} |f_n(x) - f(x)| \cong \varepsilon \right\} \cong e^{-\frac{Kn}{q_n}}$$

for any $\delta > 0$.

THEOREM 1.d. (Example 2.) *Let f be a bounded d.f. obeying condition (i) of Theorem 1, and having uniformly bounded derivative in $(-\infty, +\infty)$, further let*

$$f_n(x) = \sum_{j=0}^{q_n} \tilde{c}_j h_j(x)$$

where $h_j(x)$ was defined in Lemma 3,

$$\tilde{c}_j = \frac{1}{n} \sum_{k=1}^n h_j(\xi_k)$$

and $q_n \nearrow +\infty$ is a sequence of integers for which $q_n \cong \frac{n}{\omega(n) \log n}$. Then

$$P\left\{ \sup_{a \cong x \cong b} |f_n(x) - f(x)| \cong \varepsilon \right\} \cong e^{-\frac{Kn}{q_n}}$$

in any finite interval $[a, b]$.

Remark. The validity of conditions (iii) and (viii) of Theorem 1 can be verified with the aid of the recursion

$$H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x)$$

and the Christoffel—Darboux form.

THEOREM 1.e. (Example 3.) *Let f be a bounded d.f. obeying condition (i) of Theorem 1. Let $f_n(x)$ and $\Phi_n(x, y)$ be defined as in Example 3. Suppose that $f(x)$ has a uniformly bounded derivative in a finite or infinite interval (a, b) . If*

$$a_n \cong \frac{\omega(n) \log n}{n} \quad (\omega(n) \nearrow +\infty, n \rightarrow \infty)$$

then

$$P\left\{ \sup_{a+\delta \cong x \cong b-\delta} |f_n(x) - f(x)| \cong \varepsilon \right\} \cong e^{-Kna_n}$$

5. The proof of Theorem 1

First of all we give some lemmas.

LEMMA 9. Under the conditions of Theorem 1 for any $x \in [a, b]$ we have

$$E(e^{\lambda_n(f_n(x) - Ef_n(x))}) \leq e^{\lambda_n^2 \frac{h_n}{n}}$$

provided that

$$0 < \lambda_n < \frac{n}{4Ch_n}$$

where C is the constant of condition (vii).

PROOF. Using the inequality $e^u \leq 1 + u + u^2$ ($|u| \leq \frac{1}{2}$) we have

$$E(e^{\lambda_n(f_n(x) - Ef_n(x))}) = \prod_{k=1}^n E(e^{\frac{\lambda_n}{n} [\Phi_n(x, \xi_k) - E\Phi_n(x, \xi_k)]}) \leq \prod_{k=1}^n \left(1 + \frac{\lambda_n^2}{n^2} h_n\right) \leq e^{\frac{\lambda_n^2 h_n}{n}}.$$

This Lemma and the Markov-inequality imply

LEMMA 10. Under the conditions of Theorem 1 for any $x \in [a, b]$ we have

$$P(|f_n(x) - Ef_n(x)| \geq \varepsilon) \leq e^{-\frac{K_0(\varepsilon)n}{h_n}}$$

where the constant $K_0(\varepsilon)$ does not depend on n and x .

PROOF of Theorem 1. Let for each positive integer n

$$-n^T = z_0^{(n)} < z_1^{(n)} < \dots < z_{l(n)}^{(n)} = n^T$$

(the number T will be determined later on) be a partition of the interval $[-n^T, n^T]$ having the following properties:

$$(a) \quad \frac{c_1}{n^{\Omega+2\tau}} \leq z_i^{(n)} - z_{i-1}^{(n)} \leq \frac{c_2}{n^{\Omega+2\tau}}$$

$$0 < c_1 < c_2 < +\infty, \quad i = 1, \dots, l(n)$$

(b) those endpoints of the intervals $I_s^{(n)}$ which belong to $[-n^T, n^T]$ are elements of the sequence $z_0^{(n)}, \dots, z_{l(n)}^{(n)}$.

By (a) and condition (iii)

$$|f_n(x) - f_n(y)| \leq C \frac{1}{n^{\Omega+\tau}} \quad \text{if } x, y \in [z_{i-1}^{(n)}, z_i^{(n)}]$$

and

$$|Ef_n(x) - Ef_n(y)| \leq C \frac{1}{n^{\Omega+\tau}} \quad \text{if } x, y \in [z_{i-1}^{(n)}, z_i^{(n)}].$$

Therefore we have

$$\begin{aligned} P\left(\sup_{-n^T \leq x < n^T} |f_n(x) - E f_n(x)| \geq \varepsilon\right) &\leq \sum_{i=1}^{l(n)} P\left(\sup_{z_i^{(n)} \leq x < z_{i+1}^{(n)}} |f_n(x) - E f_n(x)| \geq \varepsilon\right) \leq \\ &\leq l(n) \max_i \left\{ P\left(\sup_{z_i^{(n)} \leq x < z_{i+1}^{(n)}} |f_n(x) - E f_n(x)| \geq \varepsilon\right) \right\} \leq \\ &\leq l(n) \max_i \left\{ P(|f_n(z_i^{(n)}) - E f_n(z_i^{(n)})| \geq \varepsilon) \right\} \leq l(n) e^{-\frac{K_0(\varepsilon)n}{h_n}} \leq e^{-\frac{K_1(\varepsilon)n}{h_n}} \end{aligned}$$

if n is large enough.

On the other hand

$$\begin{aligned} P\left(\sup_{\substack{|x| \geq n^T \\ x \in [a+\delta, b-\delta]}} |f_n(x) - E f_n(x)| \geq \varepsilon\right) &\leq P\left(\sup_{\substack{|x| \geq n^T \\ x \in [a+\delta, b-\delta]}} |f_n(x)| \geq \frac{\varepsilon}{2}\right) + \\ &+ P\left(\sup_{\substack{|x| \geq n^T \\ x \in [a+\delta, b-\delta]}} |E f_n(x)| \geq \frac{\varepsilon}{2}\right) = P\left(\sup_{\substack{|x| \geq n^T \\ x \in [a+\delta, b-\delta]}} |f_n(x)| \geq \frac{\varepsilon}{2}\right) \end{aligned}$$

if n is large enough, because of (ii) and (iv). The above expression can be estimated as follows:

$$\begin{aligned} P\left(\sup_{|x| \geq n^T} |f_n(x)| \geq \frac{\varepsilon}{2}\right) &\leq P\left(\sup_{|x| \geq n^T} \frac{1}{n} \sum_{k: |\xi_k - x| \leq \delta} |\Phi_n(x, \xi_k)| \geq \frac{\varepsilon}{4}\right) + \\ &+ P\left(\sup_{|x| \geq n^T} \frac{1}{n} \sum_{\substack{k: |\xi_k - x| > \delta \\ \xi_k \in S\left(x, \frac{n^T}{2}\right)}} |\Phi_n(x, \xi_k)| \geq \frac{\varepsilon}{8}\right) + \\ &+ P\left(\sup_{|x| \geq n^T} \frac{1}{n} \sum_{\substack{k: |\xi_k - x| > \delta \\ \xi_k \notin S\left(x, \frac{n^T}{2}\right)}} |\Phi_n(x, \xi_k)| \geq \frac{\varepsilon}{8}\right) \leq 2P\left(\frac{1}{n} Ch_n \sum_{k: |\xi_k| \geq \frac{n^T}{2}} 1 \geq \frac{\varepsilon}{8}\right) + \\ &+ P\left(\sup_{|x| \geq n^T} \frac{1}{n} \sum_{\substack{k: |\xi_k - x| > \delta \\ \xi_k \notin S\left(x, \frac{n^T}{2}\right)}} |\Phi_n(x, \xi_k)| \geq \frac{\varepsilon}{8}\right) \end{aligned}$$

where $S\left(x, \frac{n^T}{2}\right)$ denotes the interval $\left[x - \frac{n^T}{2}, x + \frac{n^T}{2}\right]$.

Observe that

$$P\left(|\xi_k| > \frac{n^T}{2}\right) = P\left(|\xi_k|^\gamma > \frac{n^{T\gamma}}{2^\gamma}\right) \leq \frac{E(|\xi_k|^\gamma) 2^\gamma}{n^{T\gamma}} \leq \frac{c(\gamma)}{n^{T\gamma}},$$

where $c(\gamma)$ is a constant depending only on γ , and f . Consequently

$$P\left(\frac{1}{n} Ch_n \sum_{k: |\xi_k| > \frac{n^T}{2}} 1 \geq \frac{\varepsilon}{8}\right) \leq \left(\frac{nc(\gamma)}{n^{T\gamma}}\right)^{\frac{nc_1}{h_n}}.$$

Let us choose the number T as $T > \max\left(\frac{1}{\gamma}, \nu\right)$ and denote $T\gamma - 1 = \gamma^* > 0$. Then clearly

$$P\left(\frac{1}{n} Ch_n \sum_{k: |\xi_k| > \frac{n^T}{2}} 1 \cong \frac{\varepsilon}{8}\right) \leq e^{-\frac{\gamma^* K_2(\varepsilon)n}{h_n}}$$

Moreover

$$P\left(\sup_{|x| \cong n^T} \frac{1}{n} \sum_{k: \xi_k \in S\left(x, \frac{n^T}{2}\right)} |\Phi_n(x, \xi_k)| \cong \frac{\varepsilon}{8}\right) \leq P\left(\varepsilon_n \cong \frac{\varepsilon}{8}\right) = 0$$

if n is large enough by (viii). Thus if n is large enough,

$$P\left(\sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - f(x)| \cong \varepsilon\right) \leq P\left(\sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - E f_n(x)| \cong \frac{\varepsilon}{2}\right) + \\ + P\left(\sup_{a+\delta \leq x \leq b-\delta} |E f_n(x) - f(x)| \cong \frac{\varepsilon}{2}\right) = P\left(\sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - E f_n(x)| \cong \frac{\varepsilon}{2}\right) \leq e^{-\frac{K(\varepsilon)n}{h_n}}$$

which proves our theorem.

6. A strong law

Finally we formulate a strong law. The proof will be omitted, just we mention that the proof of this theorem is quite similar to the proof of Theorem 1 of [5].

THEOREM 2. *Suppose that we have a function $g(n)$ such that*

$$\sup_{a+\delta \leq x \leq b-\delta} |E f_n(x) - f(x)| \leq g(n)$$

and there exist a $\beta > 0$ with $g(n) > n^{-\beta}$. Moreover suppose that all conditions of Theorem 1 hold but instead of (viii) we have

$$(viii') \quad \frac{\min\left(\sqrt{\frac{n}{h_n}}, \frac{1}{g(n)}\right)}{\log n} |\Phi_n(x_n, y_n)| < \varepsilon_n$$

whenever

$$|x_n - y_n| > n^\nu.$$

Then

$$P\left(\lim_{n \rightarrow \infty} \frac{\min\left(\sqrt{\frac{n}{h_n}}, \frac{1}{g(n)}\right)}{\log n} \sup_{a+\delta \leq x < b-\delta} |f_n(x) - f(x)| = 0\right) = 1.$$

Remark that in case of the existence of higher derivatives of $f(x)$ G. HALÁSZ obtained a better rate of convergence of $f_n \rightarrow f$.

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ОБ ОДНОЙ НЕЛИНЕЙНОЙ ЗАДАЧЕ ОПТИМИЗАЦИИ

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Следующая задача* возникает в большом числе практических ситуаций:

Имеется N приборов, которые работают независимо друг от друга. Первый прибор включается в момент $T_1=0$, i -ый прибор в момент T_i . Мы обозначаем $N-1$ -мерный вектор (T_2, \dots, T_N) через T и всегда предполагаем, что $0 \leq T_2 \leq T_3 \leq \dots \leq T_N$. Время жизни системы, состоящей из N приборов, определяется следующим образом

$$\zeta_T = \inf \left\{ x : x > 0, x \notin \bigcup_{i=1}^N [T_i, T_i + \xi_i) \right\},$$

где ξ_i является временем жизни i -того прибора, $P(\xi_i < x) = F_i(x)$, $F_i(0) = 0$. Пусть \mathcal{F} множество всех функций распределения и

$$\mathcal{F}_i = \left\{ F : F \in \mathcal{F}, F(0) = 0, \int x dF(x) = m_i, \int (x - m_i)^2 dF(x) = \sigma_i^2 \right\},$$

где m_i и σ_i данные постоянные ($i=1, 2, \dots, N$). Пусть $w(x)$ — некоторая функция, определенная для $x \geq 0$. Задача состоит в том, чтобы определить значение

$$E = \sup_T \inf_{\substack{F_i \in \mathcal{F}_i \\ (1 \leq i \leq N)}} Mw(\zeta_T)$$

и вектор \hat{T} такой, что

$$E = \inf_{\substack{F_i \in \mathcal{F}_i \\ (1 \leq i \leq N)}} Mw(\zeta_{\hat{T}})$$

если такой \hat{T} существует.

В первом пункте настоящей работы мы дадим решение задачи в случае $N=2$, причем $w(x)$ имеет вид

$$(1) \quad w_1(x) = x \quad x \geq 0$$

или

$$(2) \quad w_2(x) = \begin{cases} 0 & 0 \leq x < R \\ 1 & x \geq R \end{cases}$$

где R положительное число. Во втором пункте мы укажем на проблемы, возникающие при решении задачи в случае общего N .

* Задача была поставлена на семинаре В. М. Золотарева и В. В. Токарева по теории надежности, работавшего в 1969 году в МИАН. Результаты работы докладывались на Международном совещании по теории надёжности в Тихане (1969. IX. 16—19).

Частный случай $m_1 = m_2$ задачи (1) рассматривался другим методом в лекциях Гермейера [2].

1. В этом пункте вместо $\zeta_T = \zeta_{(T_2)}$ мы будем писать ζ_T , где $T = T_2$, и для удобства мы принимаем условие $\xi_i > 0$, которое по существу не изменяет результаты. Если ввести обозначения

$$(3) \quad p = P(\xi_1 \leq T)$$

$$(4) \quad \mu^* = M(\xi_1 - m_1 | \xi_1 > T), \quad -\mu = M(\xi_1 - m_1 | \xi_1 \leq T)$$

$$(5) \quad s^{*2} = D(\xi_1 - m_1 | \xi_1 > T), \quad s^2 = D(\xi_1 - m_1 | \xi_1 \leq T)$$

то

$$(6) \quad p\mu = (1-p)\mu^*$$

и

$$(7) \quad ps^2 + (1-p)s^{*2} + p\mu^2 + (1-p)\mu^{*2} = \sigma_1^2.$$

Из известного свойства математического ожидания следует, что

$$(8) \quad Mw(\zeta_T) = M(w(\xi_1) | \xi_1 < T)p + M(w(\max(\xi_1, T + \xi_2)) | \xi_1 \geq T)(1-p),$$

где в случаях $p=0$ и $p=1$ одно из слагаемых исчезает. Очевидно, что

$$(9) \quad e(T) = \inf_{\substack{F_i \in \mathcal{F}_i \\ (i=1,2)}} Mw(\zeta_T) = \inf_{\substack{p, \mu, \mu^* \\ s, s^*}} \inf_{\substack{F_1 \in \mathcal{F}_1^* \\ F_2 \in \mathcal{F}_2}} Mw(\zeta_T),$$

где $\mathcal{F}_1^* = \mathcal{F}_1^*(p, \mu, \mu^*, s, s^*)$ обозначает подкласс тех функций распределения из \mathcal{F}_1 , вероятность и условные моменты (3), (4) и (5) которых равны $p, -\mu, \mu^*, s^2, s^{*2}$ и \inf берётся по системам параметров, удовлетворяющих уравнениям

(6), (7). Из (8) и (9) следует, что

$$(10) \quad e(T) = \inf_{\substack{p, \mu, \mu^* \\ s, s^*}} [p \inf_{F_1 \in \mathcal{F}_1^*} M(w(\xi_1) | \xi_1 \leq T) + (1-p) \inf_{\substack{F_1 \in \mathcal{F}_1^* \\ F_2 \in \mathcal{F}_2}} M(w(\max(\xi_1, T + \xi_2)) | \xi_1 > T)].$$

Значит, для вычисления $e(T)$ вообще надо решить следующие задачи:

α) Пусть $a > 0$,

$$\mathcal{H} = \left\{ G: G \in \mathcal{H}, G(0) = 0, G(a+0) = 1, \int x dG(x) = M, \int (x-M)^2 dG(x) = S^2 \right\}$$

и η случайная величина такая, что $P(\eta < x) = G(x)$. Ищется

$$\inf_{G \in \mathcal{H}} Mw(\eta).$$

γ_2) Пусть

$$(11) \quad \mathcal{G}_i = \left\{ G: G \in \mathcal{H}, G(0) = 0, \int x dG(x) = M_i, \int (x-M_i)^2 dG(x) = S_i^2 \right\}$$

и $P(\eta_i < x) = G_i(x)$ ($i = 1, 2$). Ищется

$$\inf_{\substack{G_i \in \mathcal{G}_i \\ (i=1,2)}} Mw(\max(\eta_1, \eta_2)).$$

При рассмотренных нами выборах функции $w(x)$ (см. (1) и (2)) задача α) тривиальна, решение задачи γ_2) дается следующими леммами:

Лемма 1. Если η_1 и η_2 независимы, то

$$\inf_{\substack{G_i \in \mathcal{G}_i \\ (i=1,2)}} M \max(\eta_1, \eta_2) = \max(M_1, M_2).$$

Лемма 2. Если η_1 и η_2 независимы, то

$$\inf_{\substack{G_i \in \mathcal{G}_i \\ (i=1,2)}} P(\max(\eta_1, \eta_2) \geq R) = 1 - k(R, M_1, S_1)k(R, M_2, S_2)$$

где

$$k(R, M, S) = \begin{cases} \frac{S^2}{S^2 + (M - R)^2} & R \leq M \\ 1 & R \geq M. \end{cases}$$

Доказательство леммы 1. Неравенство $M \max(\eta_1, \eta_2) \geq \max(M_1, M_2)$ очевидно. Определим G_i следующим образом

$$P(\eta_i = M_i - A_i) = p, \quad P(\eta_i = M_i + B_i) = 1 - p$$

где

$$A_i = S_i \sqrt{\frac{1-p}{p}}, \quad B_i = S_i \sqrt{\frac{p}{1-p}}$$

и $0 < p < 1$. Если p достаточно близко к 1, то $M_i - A_i \geq 0$ и так $G_i \in \mathcal{G}_i$. Теперь легко убедиться в том, что при $p \rightarrow 1$

$$M \max(\eta_1, \eta_2) = \max(M_1, M_2) + O(\sqrt{1-p}).$$

Доказательство леммы 2. Очевидно, что

$$\inf_{\substack{G_i \in \mathcal{G}_i \\ (i=1,2)}} P(\max(\eta_1, \eta_2) \geq R) = 1 - \sup_{G_1 \in \mathcal{G}_1} P(\eta_1 < R) \sup_{G_2 \in \mathcal{G}_2} P(\eta_2 < R)$$

и утверждение следует из теоремы Маркова-Крейна ([1], гл. III, теорема 2.1).
Следующие теоремы определяют функции

$$e_k(T) = \inf_{\substack{F_i \in \mathcal{F}_i \\ (i=1,2)}} M w_k(\zeta_T) \quad (k = 1, 2).$$

Пусть

$$T' = m_1 - \frac{m_2 + \sqrt{m_2^2 - 4\sigma_1^2}}{2}, \quad T'' = m_1 - \frac{m_2 - \sqrt{m_2^2 - 4\sigma_1^2}}{2}.$$

Теорема 1. Если $2\sigma_1 \geq m_2$, то для всех $T > 0$

$$(12) \quad e_1(T) = m_1.$$

Если $2\sigma_1 < m_2$, то при $T \notin (T', T'')$, $T > 0$ $e_1(T)$ определяется также с помощью (12), и при $T \in (T', T'')$, $T > 0$

$$e_1(T) = T + m_2 \frac{(m_1 - T)^2}{(m_1 - T)^2 + \sigma_1^2}.$$

Наконец,

$$(13) \quad e_1(0) = \max(m_1, m_2).$$

Теорема 2. Если $T \geq R$ или $0 < T < R - m_2$, то

$$e_2(T) = 1 - k(R, m_1, \sigma_1).$$

Если $0 < T < R < T + m_2$, то для $T \geq m_1$

$$e_2(T) = 0$$

и для $T < m_1$

$$e_2(T) = \min_{\mu^* \equiv R - m_1} \frac{m_1 - T}{m_1 - T + \mu^*} \times \\ \times \left[1 - \frac{\sigma_2^2}{\sigma_2^2 + (m_2 - R + T)^2} \cdot \frac{\sigma_1^2 - (m_1 - T)\mu^*}{\frac{m_1 - T}{m_1 - T + \mu^*} (\mu^* + m_1 - R)^2 + \sigma_1^2 - (m_1 - T)\mu^*} \right].$$

Наконец,

$$(14) \quad e_2(0) = 1 - k(R, m_1, \sigma_1)k(R, m_2, \sigma_2).$$

Доказательство теоремы 1. (13) вытекает из леммы 1. Предположим, что $T > 0$. Очевидно, что $e_1(T) \geq m_1$.

Случаи $T \geq m_1$, $T + m_2 \leq m_1$, $T < m_1 < T + m_2$ рассматриваются отдельно. Если $T \geq m_1$, тогда определим F_1 так же, как определена G_1 в доказательстве леммы 1, и пусть F_2 любое распределение, сосредоточенное на конечном интервале. Если p достаточно близко к 1, то $M\zeta_T = m_1$ и (12) выполняется. Если $T + m_2 \leq m_1$, тогда определим F_1 и F_2 таким же образом, как мы определили G_1 и G_2 в доказательстве леммы 1. Мы получим, что при $p \rightarrow 1$ $M\zeta_T = m_1 + O(\sqrt{1-p})$ и поэтому (12) выполняется.

Рассмотрим случай $T < m_1 < T + m_2$. Поставив в (10) $w_1(x)$ в место $w(x)$ и пользуясь леммой 1

$$e_1(T) = \inf_{\substack{p, \mu, \mu^* \\ s, s^*}} [p(m_1 - \mu) + (1-p) \max(m_1 + \mu^*, T + m_2)] = \\ = \inf_{\substack{p, \mu, \mu^* \\ s, s^*}} \{m_1 + \max[0, (T + m_2 - m_1 - \mu^*)(1-p)]\}.$$

Исходя из произвольных значений параметров, мы повторно будем изменять их значения таким образом, чтобы (6) и (7) остались в силе и выражение под знаком \inf не возрастало. Вначале фиксируем p , μ и μ^* и возьмем $s=0$. Из (6) и (7) получается, что

$$(15) \quad (1-p)s^{*2} + \frac{p}{1-p} \mu^2 = \sigma_1^2.$$

Теперь, если μ и $s (=0)$ фиксированы и p возрастает, то μ^* возрастает а s^* убывает. Выбирая $s^*=0$, из (6) и (15) мы получим, что p и μ^* , как функции от μ имеют вид

$$p(\mu) = \frac{\sigma_1^2}{\mu^2 + \sigma_1^2} \quad \text{и} \quad \mu^*(\mu) = \frac{\sigma_1^2}{\mu}.$$

Наконец, при фиксированных $s(=0)$, $s^*(=0)$ мы можем взять наименьшее значение μ , именно $m_1 - T$, значит

$$e_1(T) = m_1 + \max \left[0, T - m_1 + m_2 \frac{(m_1 - T)^2}{(m_1 - T)^2 + \sigma_1^2} \right].$$

Легко убедиться в том, что второй член под знаком максимума положителен тогда и только тогда, когда $m_2 > 2\sigma_1$ и $T \in (T', T'')$. Отсюда и вытекает утверждение теоремы.

Замечание. Элементарные вычисления показывают, что если $m_2 > 2\sigma_1 > 0$, то функция

$$h(T) = T + m_2 \frac{(m_1 - T)^2}{(m_1 - T)^2 + \sigma_1^2}$$

в интервале (T', T'') имеет единственный максимум в некоторой точке T''' . T''' вычисляется следующим образом: уравнение $z^4 - \alpha z + \alpha = 0$, где $\alpha = 4m_2^2/\sigma_1^2$ имеет два вещественных корня: z' и z'' ($1 < z' \leq z''$), и $T''' = m_1 - \sigma_1 \sqrt{z'' - 1}$.

Легко доказать следующее

Следствие теоремы 1. Если $\sigma_1 > 0$, то существует \hat{T} , для которого

$$E_1 = \max_{T \geq 0} \inf_{\substack{F_i \in \mathcal{F}_i \\ (i=1,2)}} M\zeta_T = \inf_{\substack{F_i \in \mathcal{F}_i \\ (i=1,2)}} M\zeta_{\hat{T}}$$

\hat{T} и E_1 определяются по следующей таблице:

		$2\sigma_1 < m_2$		$2\sigma_1 \geq m_2$
$m_1 \geq m_2$	\hat{T}	T'''		произвольно
	E_1	$h(T''')$		
$m_1 < m_2$		$T''' > 0$	$T''' \leq 0$	0
	\hat{T}	0 или T'''	0	
	E_1	$\max(m_2, h(T'''))$	m_2	m_2

Если $\sigma_1 = 0$, то

$$E_1 = \lim_{T \rightarrow m_1 - 0} \inf_{\substack{F_i \in \mathcal{F}_i \\ (i=1,2)}} M\zeta_T = m_1 + m_2.$$

Замечание. Легко видеть, что при фиксированных m_1 и m_2 и при $\sigma_1 \rightarrow 0$ мы имеем

$$\hat{T} = m_1 - \sqrt[3]{2m_2\sigma_1^{2/3}} + o(\sigma_1^{2/3})$$

и

$$E_1 = m_1 + m_2 - \frac{3}{2} \sqrt[3]{2m_2\sigma_1^{2/3}} + o(\sigma_1^{2/3}).$$

Доказательство теоремы 2. Основная мысль доказательства подобна тому, которой мы воспользовались в случае теоремы 1, поэтому здесь мы

будем только указывать на важные шаги доказательства. Рассмотрим только случай $0 < T < R < T + m_2$, $T < m_1$. Доказательство в остальных случаях получается с помощью элементарных конструкций. Положив в (10) функцию $w_2(x)$ и пользуясь леммой 2, мы получаем, что

$$(16) \quad e_2(T) = \inf_{\substack{p, \mu, \mu^* \\ s, s^*}} \{(1-p)[1 - k(R-T, m_1 + \mu^* - T, s^*)k(R-T, m_2, \sigma_2)]\}.$$

Принимая во внимание, что $R - T < m_2$, мы получим, что выражение под знаком \inf при $R \cong m_1 + \mu^*$ равняется

$$(17) \quad (1-p) \left(1 - \frac{\sigma_2^2}{\sigma_2^2 + (m_2 - R + T)^2} \right)$$

и при $R \cong m_1 + \mu^*$ оно равняется

$$(18) \quad (1-p) \left(1 - \frac{s^{*2}}{s^{*2} + (m_1 - \mu^* + R)^2} \cdot \frac{\sigma_2^2}{\sigma_2^2 + (m_2 - R + T)^2} \right).$$

Теперь снова изменяем значения параметров подобным образом, как в доказательстве теоремы 1.

При фиксированных p , μ , μ^* можно взять $s=0$. Потом при фиксированных $s(=0)$ и μ^* берётся наименьшее возможное значение μ , именно $\mu = m_1 - T$. Теперь фиксируем $s(=0)$ и $\mu(=m_1 - T)$ и пусть μ^* изменяется при условии $R \cong m_1 + \mu^*$. Тогда из (17) видно, что для μ^* можно взять его наибольшее значение, именно $\mu^* = R - m_1$. Но в этом случае правые стороны в (17) и (18) совпадают, и утверждение теоремы отсюда и из (16), (6) и (7) уже следует.

2. Здесь мы укажем на проблемы, возникающие при применении метода, использованного в первом пункте, для общего N .

Определим следующие интервалы:

$$I_1 = [0, T_2), \quad I_i = (T_i, T_{i+1}] \quad (2 \leq i \leq N-1), \quad I_N = (T_N, \infty).$$

Обозначим событие $\{T_i + \xi_i \in T_j\}$ через C_{ij} , и обозначим далее

$$(19) \quad P(C_{ij}) = p_{ij}, \quad M(T_i + \xi_i | C_{ij}) = \mu_{ij}, \quad D(T_i + \xi_i | C_{ij}) = \sigma_{ij}^2.$$

Из условий, наложенных на моменты случайных величин ξ_2, \dots, ξ_N следует, что

$$(20) \quad \sum_{j=1}^N p_{ij} \mu_{ij} = T_i + m_i$$

и

$$(21) \quad \sum_{j=1}^N p_{ij} [(\mu_{ij} - T_i - m_i)^2 + s_{ij}^2] = \sigma_i^2.$$

С другой стороны, применив известное свойство математического ожидания

$$(22) \quad e(T) = Mw(\zeta_T) = \sum_{j_1=1}^N \dots \sum_{j_i=i}^N \dots \sum_{j_N=N}^N \prod_{k=1}^N p_{k j_k} M(w(\zeta_T) | \bigcap_{l=1}^N C_{l j_l}).$$

Следующее равенство очевидно

$$(23) \quad e(T) = \inf_{\substack{p_{ij}, \mu_{ij}, s_{ij} \\ (1 \leq i \leq j \leq N)}} \inf_{\substack{F_i \in \mathcal{F}_i^* \\ (1 \leq i \leq N)}} Mw(\zeta_T),$$

где $\mathcal{F}_i^* = \mathcal{F}_i^*(p_{ij}, \mu_{ij}, s_{ij}; 1 \leq i \leq j \leq N)$

обозначает класс тех функций распределения из \mathcal{F}_i , для которых кроме первых двух моментов вероятности и условные моменты (19) также фиксированы, и первый \inf взят по системам параметров p_{ij}, μ_{ij}, s_{ij} ($1 \leq i \leq j \leq N$), удовлетворяющих уравнениям (20) и (21). (Если при этом хотя бы один из классов \mathcal{F}_i^* пустой, то \inf считается равной бесконечности.). Теперь из (22) и (23) мы

получаем, что

$$e(T) = \inf_{\substack{p_{ij}, \mu_{ij}, s_{ij} \\ (1 \leq i \leq j \leq N)}} \sum_{j_1=1}^N \dots \sum_{j_N=N}^N \left(\prod_{k=1}^N p_{k j_k} \right) \inf_{\substack{F_i \in \mathcal{F}_i^* \\ (1 \leq i \leq N)}} M(w(\zeta_T) | \bigcap_{l=1}^N C_{l j_l}).$$

Значит, чтобы определить $e(T)$, нам надо знать

$$(24) \quad \inf_{\substack{F_i \in \mathcal{F}_i^* \\ (1 \leq i \leq N)}} M(w(\zeta_T) | \bigcap_{l=1}^N C_{l j_l}) \quad (l \leq j_l \leq N)$$

при любых j_1, \dots, j_N ($l \leq j_l \leq N$). При определении (24) возникают *задача α*), *задача*

β_n) Пусть

$$\mathcal{H} = \left\{ G: G \in \mathcal{H}, G(0) = 0, G(a+0) = 1, \int x dG(x) = M_i, \int (x - M_i)^2 dG(x) = S_i^2 \right\}$$

и $P(\eta_i < x) = G_i(x)$ ($1 \leq i \leq N$). Ищется

$$\inf_{\substack{G_i \in \mathcal{H}_i \\ (1 \leq i \leq n)}} Mw(\max(\eta_1, \dots, \eta_n)),$$

где η_1, \dots, η_n независимые случайные величины, и *задача γ_n*), которая является естественным обобщением задачи γ_2). Заметим, что для решения задачи α) существует разработанная теория (см. [1]), однако решение задач β_n) и γ_n), вообще не известно. В своем докладе на совещании по теории надежности Б. В. Гнеденко указал на явление, что задачи такого типа в большом числе возникают в теории надежности и массового обслуживания (Тихань, 1969, IX. 16—19).

В заключение хочу выразить сердечную благодарность В. М. Золотареву за поставленную задачу и за ценные разговоры при ее решении.

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SOLID SETS OF CIRCLES

by

G. FEJES TÓTH

To the sixtieth birthday of my father

It is known [1] that in the Euclidean plane the face-incircles of a regular hexagonal tessellation form a densest packing, and the face-circumcircles of a regular hexagonal tessellation form a thinnest covering. Analogous results hold in spherical geometry for regular trihedral tessellations. But the extension of these results to the hyperbolic geometry involves difficulties because of the lack of a reasonable notion of density. In order to avoid these difficulties it is convenient to introduce the solidity of a circle-packing and circle-covering [2].

A *circle-packing* is said to be *solid* if no finite subset of the circles can be rearranged so as to form, with the rest of the circles, a packing not congruent to the original one. Analogously, a *circle-covering* is said to be *solid* if no finite subset of the circles can be rearranged so as to form, with the rest of the circles, a covering not congruent to the original one.

A theorem of Mrs. M. IMRE [3] implies that the face-incircles of any regular trihedral tessellation form a solid packing, and the face-circumcircles of any regular trihedral tessellation form a solid covering. It was conjectured by L. FEJES TÓTH [2] that this result continues to hold for all trihedral Archimedean tessellations. With the aid of a general theorem he was able to prove this conjecture for a lot of Archimedean tessellations having two kinds of faces. As an example we mention the "football tessellation", i.e. the spherical tessellation (5, 6, 6) and the hyperbolic tessellations (6, 6, 7) and (8, 8, 5). We shall prove the conjecture also for some trihedral Archimedean tessellations in which three types of polygons occur, e.g. for the hyperbolic tessellations (6, 8, 10), (8, 10, 12) and (10, 12, 14).

We shall obtain these results as special cases of a more general theorem. Let S be a set of convex domains, s a finite subset of S and A the area of the set of those points which are contained in at least one domain of s but not contained in any domain of $S - s$. Rearrange the domains of s in some way obtaining a set of domains \bar{s} . Let \bar{S} be the set of domains arising from S by replacing s by \bar{s} and \bar{A} the corresponding area. We say that S is a *solid set of domains* if for any s and \bar{s} not congruent to s we have $A > \bar{A}$.

Obviously, any solid circle-packing and any solid circle-covering is a solid set of circles. With the notion of the solidity of a set of domains the above mentioned theorem of Mrs. M. IMRE can be formulated as follows: A set of circles concentric with the faces of a regular trihedral tessellation such that the boundaries of adjacent circles intersect or touch each other on the common edge of the corresponding faces, is always solid.

It may be conjectured that this theorem still holds for all trihedral Archimedean tessellations. Our aim is to prove this conjecture in some cases.

We shall need a further definition in order to phrase our results. Let $1 \leq p, q \leq \infty$ be such that $1/p + 1/q = 1$. Consider two convex domains G and H , and let F be the intersection of them. We define the area deviation of H from G formed with the weights p and q , in short the *deviation* of H from G by

$$a(G, H) = p(G - F) + q(H - F).$$

Here and in what follows we denote a domain and its area with the same symbol. Denoting for certain weights p and q the minimal deviation of an n -gon from a circle of radius r with $d(n, r)$, we have the following

THEOREM. Consider a set S of circles concentric with the faces of the Archimedean tessellation (n_1, n_2, n_3) such that the boundaries of adjacent circles intersect or touch each other on the common edge of the corresponding faces. Let r_i be the radius of the circles concentric with the n_i -gonal faces of the tessellation. Let AB be an edge of the tessellation, A' and B' the intersection of the segment AB with the boundaries of the circles and M the midpoint of AB . We choose the weights p and $q = p/(p-1)$ defining the deviation $d(n, r)$ as follows:

$$p = \begin{cases} \frac{\sin MA}{\sin MA'} & \text{on the sphere,} \\ \frac{MA}{MA'} & \text{in the Euclidean plane,} \\ \frac{\text{sh } MA}{\text{sh } MA'} & \text{in the hyperbolic plane.} \end{cases}$$

If the inequality

$$d(n_i + 1, r_i) + d(n_j - 1, r_j) - d(n_i, r_i) - d(n_j, r_j) > 0$$

is satisfied for all $i, j = 1, 2, 3$ ($i \neq j$) then S is solid.

The proof of the Theorem rests on some lemmas.

LEMMA 1. The sequence $d(3, r), d(4, r), \dots$ is convex for all fixed values of p, q and r .

Since in the cases $p=1, q=\infty$ and $p=\infty, q=1$ Lemma 1 expresses wellknown facts [2], we can restrict ourselves to the case when $1 < p, q < \infty$.

Let C be a circle with centre O and radius r , σ a sector of C with angle $2\varphi < \pi$ and $\Delta = AOB$ a triangle such that A and B are on the half-lines bounding σ . From among these triangles we are looking for that one which has a minimal deviation $d = d(\varphi)$ from σ .

Obviously, we may suppose that the side AB intersects the boundary of C at two points A' and B' which may coincide. Small rotations of the side AB about the midpoint M of $A'B'$ show that $AA' = BB'$.

Consider a segment EF of length x . Let $t = t(x, y)$ be the area of the part of the parallel domain of the line EF at distance y lying between the lines perpendicular

to EF at E and F . Writing $v(y) = \frac{dt}{dx}$, we have

$$\frac{1}{2}v(y) = \begin{cases} \sin y & \text{on the sphere,} \\ y & \text{in the Euclidean plane,} \\ \text{sh } y & \text{in the hyperbolic plane.} \end{cases}$$

Small displacements of the line AB perpendicularly to OM show that for the triangle having minimal deviation from σ we have

$$pv(MA') = q(v(MA) - v(MA')),$$

thus

$$p = \frac{v(MA)}{v(MA')} = \begin{cases} \frac{\sin MA}{\sin MA'} & \text{on the sphere,} \\ \frac{MA}{MA'} & \text{in the Euclidean plane,} \\ \frac{\text{sh } MA}{\text{sh } MA'} & \text{in the hyperbolic plane.} \end{cases}$$

It is easily seen that the triangle $\Delta = AOB$ is unically determined by this relation.

We claim that $d(\varphi)$ is a convex function of φ . Consider the triangle $\Delta = AOB$ having a minimal deviation $d = d(\varphi)$ from σ . Let u be the segment cut off from the circle C by the line AB . Then

$$d(\varphi) = pu + q(\Delta - \sigma + u),$$

thus

$$\frac{d}{d\varphi}d(\varphi) = p \frac{du}{d\varphi} + q \left(\frac{d\Delta}{d\varphi} - \frac{d\sigma}{d\varphi} + \frac{du}{d\varphi} \right).$$

Let $g = g(\varphi, \varrho)$ be the area of a circular sector with angle 2φ cut out from a circle of radius ϱ . Writing $w(\varrho) = \frac{dg}{d\varphi}$ and $x = OM$, we have

$$\frac{d\Delta}{d\varphi} = w(OA) + \frac{dx}{d\varphi}v(MA) \quad \text{and} \quad \frac{du}{d\varphi} = -\frac{dx}{d\varphi}v(MA').$$

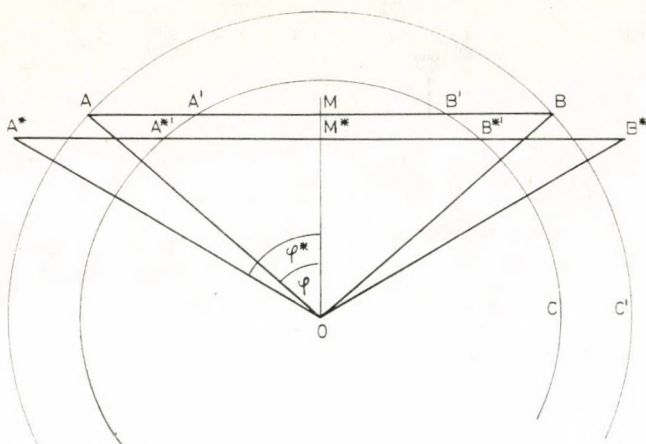
Thus, in view of

$$pv(MA') = q(v(MA) - v(MA')),$$

we have

$$\frac{d}{d\varphi}d(\varphi) = q(w(OA) - w(r)).$$

Since w is an increasing function of ϱ it suffices to prove that OA increases with φ . Consider a triangle $\Delta^* = A^*OB^*$ with $\sphericalangle A^*OB^* = 2\varphi^* > 2\varphi$ having a minimal deviation from the corresponding sector of C . Suppose that $OA^* < OA$. Let C' be the circle with centre O and radius OA and D one of the points of intersection of the line A^*B^* with the boundary of C' . Let the boundary of C intersect A^*B^* at A^{**} and B^{**} . Let M^* be the midpoint of A^*B^* (see the figure). Then we obtain by a simple computation that



$$\frac{v(MA)}{v(MA')} = \begin{cases} \sqrt{\frac{\cos^2 OM - \cos^2 OA}{\cos^2 OM - \cos^2 OA'}} \\ \sqrt{\frac{OA^2 - OM^2}{OA'^2 - OM^2}} \\ \sqrt{\frac{\text{ch}^2 OA - \text{ch}^2 OM}{\text{ch}^2 OA' - \text{ch}^2 OM}} \end{cases} \quad \text{and} \quad \frac{v(M^*D)}{v(M^*A'^*)} = \begin{cases} \sqrt{\frac{\cos^2 OM^* - \cos^2 OA}{\cos^2 OM^* - \cos^2 OA'}} \\ \sqrt{\frac{OA^2 - OM^{*2}}{OA'^2 - OM^{*2}}} \\ \sqrt{\frac{\text{ch}^2 OA - \text{ch}^2 OM^*}{\text{ch}^2 OA' - \text{ch}^2 OM^*}} \end{cases}$$

Hence, in view of $OM^* < OM$, we have

$$p = \frac{v(M^*A^*)}{v(M^*A'^*)} < \frac{v(M^*D)}{v(M^*A'^*)} < \frac{v(MA)}{v(MA')} = p,$$

which is a contradiction.

Let P be an n -gon having minimal deviation from the circle C . It is easily seen that P is convex. It follows that $O \in P$. Otherwise we can displace P in such a way as to increase the intersection F of P and C , decreasing thereby the deviation $a(C, P)$. Let $\Delta_1, \dots, \Delta_n$ be the triangles based on the sides of P and having O as a common apex. Let the angle of the triangle Δ_i at O be $2\varphi_i$. Let the deviation of Δ_i from the corresponding sector be d_i . Because of the convexity of $d(\varphi)$ we have

$$a(C, P) = d_1 + \dots + d_n \geq d(\varphi_1) + \dots + d(\varphi_n) \geq nd(\pi/n).$$

Thus P is a regular n -gon concentric with C , and we have

$$d(n, r) = nd(\pi/n).$$

It is known [2] that if $f(x)$ is a convex continuous function of $x > 0$ then so is the function $xf(1/x)$. Therefore the convexity of the function $d(\varphi)$ implies the convexity of $xd(\pi/x)$. This completes the proof of Lemma 1.

LEMMA 2. Let p_1, \dots, p_n be the number of sides of n non-overlapping convex

polygons joining along whole sides and having a connected pointset union U . Then

$$p_1 + \dots + p_n \leq 6(n+h-2) + f - b,$$

where h is the number of the connected components of which the complementary of U consists and f and b are the numbers of vertices lying on the boundary of U at which only two and more than two edges meet, respectively. Equality holds only if at each vertex at most three edges meet.

This lemma, which is Lemma 6 in [2] is a simple consequence of Euler's formula. We omit the proof here.

LEMMA 3. *The faces of a trihedral Archimedean tessellation form a solid set.*

Let T be a trihedral Archimedean tessellation. Consider a finite subset s of the faces, and rearrange them in some way, so as to obtain a new set \bar{s} of domains forming together with the rest of the faces of T a tessellation \bar{T} . We have to prove that \bar{T} is congruent with T .

First we show that the faces of \bar{T} join along whole sides. For, let A_1A_3 and A_2A_4 be sides of two faces of \bar{T} such that A_2 and A_3 are interior points of the segments A_1A_3 and A_2A_4 , respectively. It follows that there is a face of \bar{T} with a side A_3A_5 such that A_4 lies on A_3A_5 . Thus we obtain by induction a set of points \dots, A_{-1}, A_0 , such that A_4 lies on A_3A_5 . Thus we obtain by induction a set of points $\dots, A_{-1}, A_0, A_1, \dots$ such that $A_{i-1}A_{i+1}$ is a side of \bar{T} containing as an interior point A_i ($i=0, \pm 1, \pm 2, \dots$). In the Euclidean and hyperbolic plane this implies that the Archimedean tessellation T has faces joining along real partial segments of certain sides, which is impossible.

On the sphere this argument does not work. But the assumption made on the sides A_1A_3 and A_2A_4 shows that in A_2 some faces of \bar{T} must meet whose angles at A_2 add up to π . On the sphere there are seven trihedral Archimedean tessellations, and it can be checked that non of them satisfies this requirement.

In order to prove Lemma 3 we obviously may suppose that the pointset union U of the polygons of s is connected. Since the faces of \bar{T} join along whole sides we can apply Lemma 2 to the polygons of \bar{s} . Applying, on the other hand, Lemma 2 to the polygons of s evidently equality holds. Since the left hand sides are in both cases equal, we have equality also in the first case, showing that the tessellation \bar{T} is trihedral. But if two adjacent faces of \bar{T} are given, the construction of the rest of the faces is unicyclicly determined by the conditions that \bar{T} is trihedral and its faces join along whole sides.

Now we are ready to prove our theorem.

Consider the Archimedean tessellation (n_1, n_2, n_3) and the set S of circles described in the Theorem. Let the n_j -gonal faces of the tessellation be D_j^1, D_j^2, \dots . Let the circle of S concentric with D_j^i be C_j^i . Let s be a finite subset of S and \bar{s} a set of circles obtained by rearranging the circles of s in some way. Consider the set of circles $\bar{S} = \{\bar{C}_1^1, \bar{C}_1^2, \dots, \bar{C}_2^1, \bar{C}_2^2, \dots, \bar{C}_3^1, \bar{C}_3^2, \dots\}$ arising from S by replacing s by \bar{s} . We define the Dirichlet cell \bar{D}_j^i of \bar{C}_j^i as the set of points which have a smaller power with respect to \bar{C}_j^i than with respect to any other circle of \bar{S} . As the intersection of half-planes \bar{D}_j^i is a convex polygon. The Dirichlet cells of \bar{S} join along whole sides and fill the plane without overlapping and without interstices [2]. Since we have rearranged only a finite number of circles, there are positive integers m_1, m_2

and m_3 such that the pointset union U of the cells $\bar{D}_1^1, \dots, \bar{D}_1^{m_1}, \bar{D}_2^1, \dots, \bar{D}_2^{m_2}, \bar{D}_3^1, \dots, \bar{D}_3^{m_3}$ is connected and that for any $i > m_j$ D_j^i and \bar{D}_j^i are identical ($j=1, 2, 3$).

Let F be the part of U covered by the circles $C_1^1, \dots, C_1^{m_1}, C_2^1, \dots, C_2^{m_2}, C_3^1, \dots, C_3^{m_3}$ and \bar{F} the part of U covered by the circles $\bar{C}_1^1, \dots, \bar{C}_1^{m_1}, \bar{C}_2^1, \dots, \bar{C}_2^{m_2}, \bar{C}_3^1, \dots, \bar{C}_3^{m_3}$. It suffices to prove that if \bar{s} is not congruent to s then $F > \bar{F}$. Writing $F_j^i = D_j^i \cap C_j^i$ and $\bar{F}_j^i = \bar{D}_j^i \cap \bar{C}_j^i$ we have

$$F = \sum_{j=1}^3 \sum_{i=1}^{m_j} F_j^i$$

and

$$\bar{F} = \sum_{j=1}^3 \sum_{i=1}^{m_j} \bar{F}_j^i.$$

The proof of the inequality $F > \bar{F}$ rests on the fact, to be proved later, that if \bar{s} is not congruent to s then for the deviations formed with the specific weights defined in the Theorem we have

$$(*) \quad \sum_{j=1}^3 \sum_{i=1}^{m_j} a(\bar{C}_j^i, \bar{D}_j^i) > \sum_{j=1}^3 m_j d(n_j, r_j).$$

First we consider the case when S neither forms a packing nor a covering. Then the weights p and q defined in the Theorem are finite. We have, on the one hand,

$$\bar{F}_j^i = \bar{D}_j^i - (\bar{D}_j^i - \bar{F}_j^i),$$

on the other hand,

$$\bar{F}_j^i = \bar{C}_j^i - (\bar{C}_j^i - \bar{F}_j^i).$$

Hence

$$\bar{F} = \frac{\bar{F}}{p} + \frac{\bar{F}}{q} = \frac{1}{p} \sum_{j=1}^3 \sum_{i=1}^{m_j} \bar{D}_j^i + \frac{1}{q} \sum_{j=1}^3 \sum_{i=1}^{m_j} \bar{C}_j^i - \frac{1}{pq} \sum_{j=1}^3 \sum_{i=1}^{m_j} a(\bar{C}_j^i, \bar{D}_j^i).$$

But by the choice of p and q we have

$$d(n_j, r_j) = p(C_j^i - F_j^i) + q(D_j^i - F_j^i).$$

Thus in view of

$$\sum_{j=1}^3 \sum_{i=1}^{m_j} \bar{D}_j^i = \sum_{j=1}^3 \sum_{i=1}^{m_j} D_j^i$$

and

$$\sum_{j=1}^3 \sum_{i=1}^{m_j} \bar{C}_j^i = \sum_{j=1}^3 \sum_{i=1}^{m_j} C_j^i$$

the inequality $(*)$ implies that $F > \bar{F}$ whenever \bar{s} is not congruent to s .

Consider now the case when S is a packing. We assume that \bar{S} is also a packing, because otherwise the inequality $\bar{F} < F$ is trivial. Thus in view of the inequality $(*)$ we have

$$\bar{F} = \sum_{j=1}^3 \sum_{i=1}^{m_j} \bar{D}_j^i - \sum_{j=1}^3 \sum_{i=1}^{m_j} a(\bar{C}_j^i, \bar{D}_j^i) \leq \sum_{j=1}^3 \sum_{i=1}^{m_j} D_j^i - \sum_{j=1}^3 m_j d(n_j, r_j) = F$$

with equality only if \bar{S} is congruent to S . The case when S is a covering can be settled in a similar way.

We continue to prove the inequality $(*)$. Let n_j^i be the number of sides of the Dirichlet cell \bar{D}_j^i . Applying Lemma 2, on the one hand, to the polygons $D_1^1, \dots, D_1^{m_1}$,

$D_2^1, \dots, D_2^{m_2}, D_3^1, \dots, D_3^{m_3}$, on the other hand, to the polygons $\bar{D}_1^1, \dots, \bar{D}_1^{m_1}, \bar{D}_2^1, \dots, \bar{D}_2^{m_2}, \bar{D}_3^1, \dots, \bar{D}_3^{m_3}$, we find that

$$\sum_{j=1}^3 m_j n_j = 6(m_1 + m_2 + m_3 + h - 2) + f - b$$

and

$$\sum_{j=1}^3 \sum_{i=1}^{m_j} n_j^i \cong 6(m_1 + m_2 + m_3 + h - 2) + f - b,$$

whence

$$(n) \quad \sum_{j=1}^3 \sum_{i=1}^{m_j} n_j^i \cong \sum_{j=1}^3 m_j n_j.$$

With the above notations we obviously have

$$\sum_{j=1}^3 \sum_{i=1}^{m_j} a(\bar{C}_j^i, \bar{D}_j^i) \cong \sum_{j=1}^3 \sum_{i=1}^{m_j} d(n_j^i, r_j).$$

We want to find the minimum of the right hand side of this inequality under the condition (n). We may suppose that in (n) equality holds. Otherwise we replace some of the n_j^i 's by greater values, so as to obtain equality in (n), decreasing thereby the sum under consideration. Again, we may assume that

$$n_1^1 + \dots + n_1^{m_1} = m_1 n_1 \quad (j = 1, 2, 3).$$

For supposing e.g. that $n_1^1 + \dots + n_1^{m_1} < m_1 n_1$, we have either $n_2^1 + \dots + n_2^{m_2} > m_2 n_2$ or $n_3^1 + \dots + n_3^{m_3} > m_3 n_3$. Suppose that $n_2^1 + \dots + n_2^{m_2} > m_2 n_2$. Then for some k and l we have $n_1^k < n_1$ and $n_2^l > n_2$. Therefore, in view of the supposition

$$d(n_1 - 1, r_1) + d(n_2 + 1, r_2) - d(n_1, r_1) - d(n_2, r_2) > 0$$

and Lemma 1, we have

$$d(n_1^k, r_1) - d(n_1^k + 1, r_1) \cong d(n_1 - 1, r_1) - d(n_1, r_1) >$$

$$d(n_2, r_2) - d(n_2 + 1, r_2) \cong d(n_2^l - 1, r_2) - d(n_2^l, r_2),$$

i. e.

$$d(n_1^k, r_1) + d(n_2^l, r_2) > d(n_1^k + 1, r_1) + d(n_2^l - 1, r_2).$$

Thus replacing n_1^k by $n_1^k + 1$ and n_2^l by $n_2^l - 1$, and repeating this process until no n_1^k is less than n_1 , the sum $\sum_{j=1}^3 \sum_{i=1}^{m_j} d(n_j^i, r_j)$ decreases. The remaining cases can be settled

in the same way. Thus, by Lemma 1, $\sum_{i=1}^{m_j} d(n_j^i, r_j) \cong m_j d(n_j, r_j)$ ($j = 1, 2, 3$), showing that

$$\sum_{j=1}^3 \sum_{i=1}^{m_j} a(\bar{C}_j^i, \bar{D}_j^i) \cong \sum_{j=1}^3 m_j d(n_j, r_j).$$

The above proof shows that equality holds only if \bar{D}_j^i is congruent to D_j^i . Thus, in view of Lemma 3, equality holds only if \bar{s} is congruent to s .

This completes the proof of the Theorem.

A set of computations, kindly performed by Mrs. ILDIKÓ ZSIGMOND, has shown that the inequalities of the Theorem are satisfied in many cases. Before presenting the results of the computations we mention that for integers n_1, n_2, n_3 greater than 2 a tessellation (n_1, n_2, n_3) exists if and only if either (i) $n_1 = n_2 = n_3$, or (ii) two of the n_i 's are equal and even, or (iii) non of the n_i 's is odd [4]. The tessellation is spherical, Euclidean or hyperbolic according as $1/n_1 + 1/n_2 + 1/n_3 \cong 1/2$, respectively.

Lemma 1 guarantees the fulfilment of the inequalities under consideration in the regular case (i). In the following table those further tessellations (n_1, n_2, n_3) with $n_1 \leq n_2 \leq n_3 \leq 20$ are listed which satisfy all inequalities of the Theorem for all admissible circle-systems.

(3, 4, 4)	(8, 10, 14)	(10, 10, 13)	(11, 18, 18)	(14, 14, 17)
(3, 4, 5)	(8, 10, 16)	(10, 10, 14)	(11, 20, 20)	(14, 14, 18)
(5, 6, 6)	(8, 10, 18)	(10, 10, 15)	(12, 12, 13)	(14, 14, 19)
(5, 8, 8)	(8, 12, 12)	(10, 10, 16)	(12, 12, 14)	(14, 14, 20)
(6, 6, 7)	(8, 12, 14)	(10, 10, 17)	(12, 12, 15)	(14, 16, 16)
(6, 6, 8)	(8, 12, 16)	(10, 10, 18)	(12, 12, 16)	(14, 16, 18)
(6, 6, 9)	(8, 12, 18)	(10, 10, 19)	(12, 12, 17)	(14, 16, 20)
(6, 6, 10)	(8, 12, 20)	(10, 10, 20)	(12, 12, 18)	(14, 18, 18)
(6, 8, 8)	(8, 14, 14)	(10, 12, 12)	(12, 12, 19)	(14, 18, 20)
(6, 8, 10)	(8, 14, 16)	(10, 12, 14)	(12, 12, 20)	(14, 20, 20)
(7, 8, 8)	(8, 14, 18)	(10, 12, 16)	(12, 14, 14)	(15, 16, 16)
(7, 10, 10)	(8, 14, 20)	(10, 12, 18)	(12, 14, 16)	(15, 18, 18)
(7, 12, 12)	(8, 16, 16)	(10, 12, 20)	(12, 14, 18)	(15, 20, 20)
(7, 14, 14)	(8, 16, 18)	(10, 14, 14)	(12, 14, 20)	(16, 16, 17)
(8, 8, 9)	(8, 16, 20)	(10, 14, 16)	(12, 16, 16)	(16, 16, 18)
(8, 8, 10)	(8, 18, 18)	(10, 14, 18)	(12, 16, 18)	(16, 16, 19)
(8, 8, 11)	(8, 18, 20)	(10, 14, 20)	(12, 16, 20)	(16, 16, 20)
(8, 8, 12)	(8, 20, 20)	(10, 16, 16)	(12, 18, 18)	(16, 18, 18)
(8, 8, 13)	(9, 10, 10)	(10, 16, 18)	(12, 18, 20)	(16, 18, 20)
(8, 8, 14)	(9, 12, 12)	(10, 16, 20)	(12, 20, 20)	(16, 20, 20)
(8, 8, 15)	(9, 14, 14)	(10, 18, 18)	(13, 14, 14)	(17, 18, 18)
(8, 8, 16)	(9, 16, 16)	(10, 18, 20)	(13, 16, 16)	(17, 20, 20)
(8, 8, 17)	(9, 18, 18)	(10, 20, 20)	(13, 18, 18)	(18, 18, 19)
(8, 8, 18)	(9, 20, 20)	(11, 12, 12)	(13, 20, 20)	(18, 18, 20)
(8, 10, 10)	(10, 10, 11)	(11, 14, 14)	(14, 14, 15)	(18, 20, 20)
(8, 10, 12)	(10, 10, 12)	(11, 16, 16)	(14, 14, 16)	(19, 20, 20)

The total number of the regular and Archimedean tessellations (n_1, n_2, n_3) with $2 \leq n_1 \leq n_2 \leq n_3 \leq 20$ amounts to 256. Thus our Theorem confirms for the majority of these tessellations, exactly for $130 + 19 = 149$ tessellations, the conjecture made in the introduction concerning the solidity of the corresponding circle-systems.

There are some tessellations such that all inequalities of the Theorem are satisfied for the face-incircles of the tessellation. But increasing the circles in the described way, at least one of the inequalities stops to hold up to a certain point. Among the 237 Archimedean tessellations (n_1, n_2, n_3) with $n_1 \leq n_2 \leq n_3 \leq 20$ there are 15 such tessellations, namely

(5, 10, 10)	(6, 10, 10)	(6, 12, 14)	(7, 16, 16)	(8, 8, 19)
(6, 6, 11)	(6, 10, 14)	(6, 14, 14)	(7, 18, 18)	(8, 8, 20)
(6, 8, 12)	(6, 12, 12)	(6, 16, 16)	(7, 20, 20)	(8, 10, 20)

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APPROXIMATION OF LACUNARY WALSH-SERIES WITH BROWNIAN MOTION

by

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1. § Introduction

It is well-known that lacunary subsequences of orthogonal systems behave similarly as independent random variables, e.g. they obey the central limit theorem, the law of the iterated logarithm and other related results. The proofs of such theorems are usually based on the method of SALEM and ZYGMUND which adapts the classical technique of characteristic functions and of exponential estimates for such dependent sequences. The crucial property of a lacunary subsequence $\{f_{n_k}\}$ of an orthogonal system $\{f_n\}$ which is used in the proofs is the fact that the sequence $\{f_{n_k}\}$ is not only orthogonal but “almost multiplicatively orthogonal” in the sense that the integrals $\int f_{n_{i_1}} f_{n_{i_2}} \dots f_{n_{i_m}} d\mu$ ($m \geq 3$) are small. The idea is simple, the calculations, however, are sometimes very involved, particularly in the case of the law of the iterated logarithm. The purpose of the present paper is to prove that by using a different method we can obtain much stronger results in an easier way, at least in the case of the Walsh-system. Namely, instead of imitating the classical methods for such sequences we shall construct a Brownian motion which approximates the partial sums of a lacunary Walsh-sequence very closely and then, using well-known limit theorems for the Brownian motion, we obtain the same theorems automatically for the Walsh system. In this way we can prove not only the central limit theorem and the law of the iterated logarithm but also the functional (Strassen-type) law of the iterated logarithm, the invariance principle, the strong form of the law of the iterated logarithm giving exact upper and lower class criteria, e.t.c. To obtain these results (especially the last one) by the classical methods seems to be very difficult.

Approximation with Brownian motion appeared first in the works of STRASSEN; he used it to obtain some attractive limit theorems for independent r.v.-s and martingales. Our present proof also uses martingale techniques. It seems likely that the results of this paper are valid (exactly in the same form) also for the trigonometric system; our present proof, however, makes essential use of the special properties of the Walsh-system. On the other hand, if the lacunarity condition is strong enough, the results of our paper can be extended for every orthogonal system. This will be proved in a subsequent paper.

2. § The approximation theorem

Definition. Let X_1, X_2, \dots and X'_1, X'_2, \dots be sequences of r.v.-s on the probability spaces (Ω, \mathcal{F}, P) and $(\Omega', \mathcal{F}', P')$, respectively. We say that the sequence X_1, X_2, \dots is equivalent to the sequence X'_1, X'_2, \dots if the finite dimensional distributions of the two sequences are identical.

THEOREM 1. Let $n_1 < n_2 < \dots$ be a sequence of positive integers such that for every $k \geq 1$

$$(2.1) \quad \frac{n_{k+1}}{n_k} \geq 1 + ck^{-\alpha} \quad c > 0, \quad 0 \leq \alpha < \frac{1}{2}.$$

Then there exists a probability space (Ω, \mathcal{F}, P) , a sequence X_1, X_2, \dots of r.v.-s and a Wiener-process $\zeta(t)$ such that the sequences X_1, X_2, \dots and w_{n_1}, w_{n_2}, \dots are equivalent¹ and

$$(2.2) \quad X_1 + \dots + X_n = \zeta(n) + O\left(n^{\frac{1}{2}-\mu}\right) \quad \text{a.s.}$$

where μ is a positive constant depending only on α, c .

3. § Some lemmas

LEMMA 1. Let Y_1, Y_2, \dots be a sequence of r.v.-s such that $E(Y_n | Y_1, \dots, Y_{n-1}) = 0$ a.s. ($n=1, 2, \dots$), $EY_1^2 > 0$, $b_n = \sum_{i=1}^n E(Y_i^2 | Y_1, \dots, Y_{i-1}) \rightarrow \infty$ a.s. and

$$(3.1) \quad \sum_{n=1}^{\infty} f(b_n)^{-2} E(Y_n^4 | Y_1, \dots, Y_{n-1}) < \infty \quad \text{a.s.}$$

where $f(x) > 0$ is a function on $(0, +\infty)$ such that $f(x) \uparrow \infty$, $x^{-1}f(x) \downarrow 0$ if $x \rightarrow \infty$. Then there exists a probability space (Ω, \mathcal{F}, P) , a sequence $\tilde{Y}_1, \tilde{Y}_2, \dots$ of r.v.-s and a Wiener-process $\zeta(t)$ (each is defined on (Ω, \mathcal{F}, P)) such that the sequences Y_1, Y_2, \dots and $\tilde{Y}_1, \tilde{Y}_2, \dots$ are equivalent and

$$\tilde{Y}_1 + \dots + \tilde{Y}_n = \zeta(b_n) + o\left((b_n f(b_n))^{\frac{1}{4}} \log b_n\right) \quad \text{a.s.}$$

as $n \rightarrow \infty$.

This lemma is an immediate consequence of Theorem 4.4 in [1].

LEMMA 2. Let $n_1 < n_2 < \dots$ be a sequence of positive integers satisfying (2.1) and put

$$A_N^{(m)} = \sum_{1 \leq i_1 < \dots < i_m \leq N} |E(w_{n_{i_1}} w_{n_{i_2}} \dots w_{n_{i_m}})|$$

Then we have

$$A_N^{(3)} \leq C_1 N^{1+\alpha}, \quad A_N^{(4)} \leq C_2 N^{2\alpha+1} \log N, \quad A_N^{(m)} \leq C_3 N^{2+\frac{1+\alpha}{3}m} \quad (m \geq 5)$$

for every $N \geq 1$, where C_1, C_2 depend on c, α ; C_3 depends on c, α, m .

PROOF. See the proof of Lemma 2 in [3].

¹ The underlying probability space in the case of the Walsh-system $\{w_n\}$ is $(\Omega_0, \mathcal{F}_0, P_0)$ where $\Omega_0 = [0, 1]$, \mathcal{F}_0 is the class of Borel-subsets of $[0, 1]$ and P_0 is the Lebesgue-measure on \mathcal{F}_0 .

LEMMA 3. Let $n_1 < n_2 < \dots$ satisfy (2.1). Then for every $K \geq 0$, $L \geq 1$ and for every $\varepsilon_v = \pm 1$ we have

$$\begin{aligned} E \left(\left(\sum_{v=K+1}^{K+L} \varepsilon_v w_{n_v} \right)^4 \right) &\leq C_4 L^2 + C_5 (K+L)^{2\alpha+1} \log^2 (K+L) \\ E \left(\left(\sum_{v=K+1}^{K+L} \varepsilon_v w_{n_v} \right)^{2p} \right) &\leq C_6 L^p + C_7 (K+L)^{2+\frac{1+\alpha}{3}2p} \quad (p > 2 \text{ integer}) \end{aligned}$$

where C_4, C_5 depend on c, α ; C_6, C_7 depend on c, α, p .

PROOF. From the polinomial theorem we obtain

$$(3.2) \quad E \left(\left(\sum_{v=K+1}^{K+L} \varepsilon_v w_{n_v} \right)^{2p} \right) \leq \sum \frac{(2p)!}{\beta_1! \dots \beta_j!} |E(w_{n_{v_1}}^{\beta_1} \dots w_{n_{v_j}}^{\beta_j})|$$

where β_1, \dots, β_j can be any positive integers whose sum is $2p$ and the indices v_1, \dots, v_j vary between $K+1$ and $K+L$. If β_1, \dots, β_j are all even then $j \leq p$ and since $\binom{L}{j} \leq L^j \leq L^p$ for $1 \leq j \leq p$, it follows that the sum of such terms in the right side of (3.2) is at most $C_p L^p$ where C_p depends only on p . On the other hand, using the fact that $w_{n_{v_s}}^{\beta_s} = 1$ or $w_{n_{v_s}}$ according as β_s is even or odd we obtain that the sum of the remaining terms in (3.2) is at most $C'_p (A_{K+L}^{(1)} + \dots + A_{K+L}^{(2p)})$. Applying Lemma 2, we obtain the desired result.

LEMMA 4. For any $p > q \geq 1$ we have $w_p w_q = w_r$ where $p - q \leq r \leq p + q$. Furthermore, if $p \in [2^k, 2^{k+1})$, $q \in [2^l, 2^{l+1})$ ($k > l$) then $w_p w_q = w_r$ where $r \in [2^k, 2^{k+1})$.

PROOF. Trivial.

LEMMA 5. Let $n_1 < n_2 < \dots$ satisfy (2.1) and put $Z_k = \sum_{i=Q+1}^{Q+k} w_{n_i}$. Then we have for every $Q \geq 0$ and every real t

$$(3.3) \quad P(\max_{1 \leq k \leq N} Z_k > t) \leq 2P(Z_N > t - 2\sqrt{N})$$

and

$$(3.4) \quad P(\max_{1 \leq k \leq N} |Z_k| > t) \leq 2P(|Z_N| > t - 2\sqrt{N})$$

provided that

$$(3.5) \quad N^{-1}(Q+N)^{2\alpha} \log^2(Q+N) \leq \delta_0$$

where δ_0 depends only on c, α .

This lemma is the analogue of a well-known maximal inequality of KOLMOGOROV (see [4] and also [7] where a similar inequality for Walsh-series with Hadamard gaps is derived). In the special case $Q=0$ condition (3.5) is satisfied for $N \geq N_0$.

Proof. Put

$$A = \left\{ \max_{1 \leq k \leq N} Z_k > t \right\}, \quad C = \{Z_N > t - 2\sqrt{N}\}, \quad A_1 = \{Z_1 > t\},$$

$$A_k = \{Z_1 \leq t, \dots, Z_{k-1} \leq t, Z_k > t\} \quad (2 \leq k \leq N)$$

$$B_k = \{Z_N - Z_k > -2\sqrt{N}\} \quad (1 \leq k \leq N).$$

It is obvious that the sets $A_k B_k$ are pairwise disjoint and $\bigcup_{k=1}^N A_k B_k \subset C$, hence

$$(3.6) \quad \sum_{k=1}^N P(A_k B_k) \leq P(C).$$

On the set \bar{B}_k we have $^2 (Z_N - Z_k)^2 \geq 4N$, thus the Markov-inequality gives

$$(3.7) \quad P(A_k \bar{B}_k) \leq \frac{1}{4N} \int_{A_k} (Z_N - Z_k)^2 dx = \frac{1}{4N} \left[(N-k)P(A_k) + 2 \sum_{\tilde{k} < i < j \leq \tilde{N}} \int_{A_k} w_{n_i} w_{n_j} dx \right]$$

where $\tilde{k} = Q + k$, $\tilde{N} = Q + N$. We shall prove that for $1 \leq k \leq N$

$$(3.8) \quad \left| \sum_{\tilde{k} < i < j \leq \tilde{N}} \int_{A_k} w_{n_i} w_{n_j} dx \right| \leq C_8 \tilde{N}^{2\alpha} [\log(c^{-1} \tilde{N}^\alpha) + 1] P(A_k)$$

where C_8 depends only on α, c . Using this relation, we obtain from (3.7) and (3.5) $P(A_k \bar{B}_k) \leq \frac{1}{4} P(A_k) + C_9 N^{-1} \tilde{N}^{2\alpha} \log \tilde{N} \cdot P(A_k) \leq \frac{1}{2} P(A_k)$, thus $P(A_k B_k) \geq \frac{1}{2} P(A_k)$ for $1 \leq k \leq N$. Combining this result with (3.6), one sees that $\frac{1}{2} P(A) = \frac{1}{2} \sum_{k=1}^N P(A_k) \leq P(C)$ and this is identical with (3.3). The same argument, applied to the sequence $\{-w_{n_i}\}$ instead of $\{w_{n_i}\}$, yields $P(\max_{1 \leq k \leq N} (-Z_k) > t) \leq 2P((-Z_N) > t - 2\sqrt{N})$ which, together with (3.3), easily implies (3.4).

It remains to prove (3.8) for every fixed k . To this aim, let us introduce the indices m, a, b for which $n_k \in [2^m, 2^{m+1})$, $n_i \in [2^a, 2^{a+1})$, $n_j \in [2^b, 2^{b+1})$, $m \leq a \leq b$. We state, following an observation due to A. FÖLDES, that the integral $\int_{A_k} w_{n_i} w_{n_j} dx$

can differ from zero only if $a = b < m + \log(c^{-1} \tilde{N}^\alpha) + 1$. This statement is sufficient for our purposes, since every interval $[2^l, 2^{l+1})$ contains at most $C_{10} \tilde{N}^\alpha$ members of the sequence $w_{n_1}, \dots, w_{n_{\tilde{N}}}$ (see Lemma 1 of [3]), hence the number of pairs (i, j) for which $\int_{A_k} w_{n_i} w_{n_j} dx \neq 0$ is not more than $(C_{10} \tilde{N}^\alpha)^2 [\log(c^{-1} \tilde{N}^\alpha) + 1]$. Estimating each of these integrals trivially by $\left| \int_{A_k} w_{n_i} w_{n_j} dx \right| \leq P(A_k)$, we obtain (3.8).

To prove the statement mentioned above, let us denote by $s(i, j)$ the index s for which $w_{n_i} w_{n_j} = w_s$ (see Lemma 4) and remark that if $s(i, j) \geq 2^{m+1}$, then $\int_{A_k} w_{n_i} w_{n_j} dx = E(\chi_{A_k} w_{s(i, j)}) = E(\chi_{A_k}) E(w_{s(i, j)}) = 0$ (χ_{A_k} is the indicator of A_k). In fact,

² \bar{B}_k denotes the complement of B_k .

A_k and χ_{A_k} are defined in terms of $w_{n_{Q+1}}, \dots, w_{n_k}$, i.e. in terms of the Rademacher-functions r_1, \dots, r_{m+1} ; on the other hand, if $s(i, j) \geq 2^{m+1}$ then the product form of $w_{s(i, j)}$ contains an r_v for which $v > m+1$ and thus the r.v.-s χ_{A_k} and $w_{s(i, j)}$ are independent. Now, if $a < b$, then by the second statement of Lemma 4 $s(i, j) \in [2^b, 2^{b+1})$, i.e. $s(i, j) \geq 2^{a+1} \geq 2^{m+1}$. Furthermore, if $a = b \geq m + \log_2(c^{-1}\tilde{N}^\alpha) + 1$, then $n_i \geq 2^{m + \log_2(c^{-1}\tilde{N}^\alpha) + 1} = 2^{m+1} c^{-1} \tilde{N}^\alpha$, i.e. by (2.1) and the first statement of Lemma 4 $s(i, j) \geq n_j - n_i \geq n_{i+1} - n_i \geq cn_i i^{-\alpha} \geq cn_i \tilde{N}^{-\alpha} \geq 2^{m+1}$. In each case we have $\int_{A_k} w_{n_i} w_{n_j} dx = 0$ and the proof of Lemma 5 is completed.

LEMMA 6. Let X_1, X_2, \dots be a sequence of r.v.-s on the probability space (Ω, \mathcal{F}, P) such that every X_i takes only the values ± 1 . Let $1 = a_1 < b_1 < a_2 < b_2 < \dots$ be a sequence of integers and put $T_k = \sum_{i=a_k}^{b_k} X_i$. Let $(\Omega', \mathcal{F}', P')$ be on other, atomless³ probability space and let T'_1, T'_2, \dots be a sequence of r.v.-s on $(\Omega', \mathcal{F}', P')$ which is equivalent to T_1, T_2, \dots . Then there exists a sequence X'_1, X'_2, \dots of r.v.-s on $(\Omega', \mathcal{F}', P')$ which is equivalent to X_1, X_2, \dots and $T'_k = \sum_{i=a_k}^{b_k} X'_i$.

The proof of this statement is simple routine and is omitted.

4. § Proof of Theorem 1

By assumption $\alpha < 1/2$, furthermore $(r+1)/r(1-\varepsilon) \rightarrow 1$ if $r \rightarrow \infty, \varepsilon \rightarrow 0$, thus we can choose an integer $r \geq 2$ and a rational number $\varepsilon > 0$ such that

$$(4.1) \quad \frac{1+\alpha}{3} \frac{r+1}{r(1-\varepsilon)} < \frac{1}{2}, \quad \varepsilon r > 8, \quad \frac{r+1}{r} (2\alpha+1) < 2, \quad r(1-2\alpha-2\varepsilon) > 2\alpha.$$

We can also achieve that $r(1-\varepsilon)$ is an integer. Let the intervals

$$A_1 = [p_1, q_1], \quad A'_1 = [p'_1, q'_1], \quad \dots, \quad A_k = [p_k, q_k], \quad A'_k = [p'_k, q'_k], \quad \dots$$

be defined by

$$p_1 = 1, \quad p'_1 = q_1 + 1, \quad \dots, \quad p_k = q'_{k-1} + 1, \quad p'_k = q_k + 1, \quad \dots$$

and

$$|A_k| = Ak^r \quad |A'_k| = Ak^{r(1-\varepsilon)} \quad (k = 1, 2, \dots)$$

where $|A|$ denotes, for any interval $A = [a, b]$, the number of integers contained in A , and A is a positive integer such that

$$(4.2) \quad A^{1-\alpha} \geq c^{-1} 3^\alpha 2^{r(1-\varepsilon)}.$$

Put

$$T_k = \sum_{i \in A_k} w_{n_i} \quad T'_k = \sum_{i \in A'_k} w_{n_i}.$$

³ A probability space $(\Omega', \mathcal{F}', P')$ is called atomless if for every $A \in \mathcal{F}'$ with $P'(A) > 0$ and for every $0 < p < P'(A)$ there exists a $B \in \mathcal{F}', B \subset A$ such that $P'(B) = p$.

We shall prove the following proposition, which implies the statement of Theorem 1 fairly easily:

PROPOSITION. *We have*

$$(4.3) \quad E(T_k | T_1, \dots, T_{k-1}) = 0 \quad (k \geq 1)$$

$$(4.4) \quad E(T_k^2 | T_1, \dots, T_{k-1}) = Ak^r \quad (k \geq k_0)$$

$$(4.5) \quad E(T_k^4 | T_1, \dots, T_{k-1}) \leq C_{11}(Ak^r)^2 \quad (k \geq k_0)$$

where C_{11} depends only on α, c .

To deduce the theorem from the proposition, let us first assume that (4.4) and (4.5) are also valid for every $k \geq 1$ and apply Lemma 1 for the sequence T_1, T_2, \dots with $f(x) = x^q$ where $0 < q < 1$ is a suitable constant. Put $B_k = \sum_{i=1}^k Ai^r$, then the general term of the series in (3.1) can be majorized by

$$B_n^{-2q} C_{11} (An^r)^2 \sim (C_{12} n^{r+1})^{-2q} C_{11} (An^r)^2 \sim C_{13} n^{2r - (r+1)2q}.$$

Here $2r - (r+1)2q \rightarrow -2$ if $q \rightarrow 1$ thus we can choose q in such a way that the series converges. For such a q the lemma is applicable and we obtain (using also Lemma 6) that there exists a probability space (Ω, \mathcal{F}, P) , a sequence X_1, X_2, \dots of r.v.-s and a Wiener-process $\zeta(t)$ such that the sequences X_1, X_2, \dots and w_{n_1}, w_{n_2}, \dots are equivalent and

$$(4.6) \quad V_1 + \dots + V_k = \zeta(B_k) + o\left(B_k^{\frac{1}{2} - \mu_0}\right) \quad \text{a.s.}$$

where $V_k = \sum_{i \in A_k} X_i$ and μ_0 is a positive constant. We can also assume, without loss of generality, that $\mu_0 < \frac{\epsilon}{8}$. If (4.4) and (4.5) are valid only for $k \geq k_0$, then instead

of (4.6) we obtain $V_1 + \dots + V_k = \zeta(B'_k) + o\left(B'_k^{\frac{1}{2} - \mu_0}\right)$ where $B'_k = \sum_{i=1}^k E(T_i^2 | T_1, \dots, T_{i-1})$. But $B'_k - B_k = O(1)$ and $B'_k \sim B_k$, hence the Borel-Cantelli lemma yields $\zeta(B'_k) - \zeta(B_k) = o(\log k) = o\left(B_k^{\frac{1}{2} - \mu_0}\right)$ and thus (4.6) is valid also in this case. Put now

$$S_n = \sum_{i=1}^n X_i, \quad V'_k = \sum_{i \in A'_k} X_i, \quad D_k = \sum_{i=1}^k Ai^r + \sum_{i=1}^k Ai^{r(1-\epsilon)} \sim C_{14} k^{r+1}.$$

Using the Markov-inequality, Lemma 3 and the fact that the sequences X_1, X_2, \dots and w_{n_1}, w_{n_2}, \dots are equivalent, we obtain

$$\begin{aligned} P\left(|V'_k| \geq |A'_k|^{\frac{1}{2} + \delta}\right) &= P\left(|T'_k| \geq |A'_k|^{\frac{1}{2} + \delta}\right) \leq \\ &\leq |A'_k|^{-2p} \left(\frac{1}{2} + \delta\right) E(|T'_k|^{2p}) \leq |A'_k|^{-2p} \left(\frac{1}{2} + \delta\right) \left(C_6 |A'_k|^p + C_7 D_k^{2 + \frac{1+\alpha}{3} 2p}\right) \end{aligned}$$

for every $\delta > 0$, $p > 2$. Here

$$D_k^{2 + \frac{1+\alpha}{3} 2p} \leq C_{15} k^{(r+1)\left(2 + \frac{1+\alpha}{3} 2p\right)} \leq C_{15} k^{2(r+1) + r(1-\varepsilon)p}$$

by the first relation of (4.1), hence, choosing p large enough, using

$$|A'_k| = Ak^{r(1-\varepsilon)}$$

and the Borel-Cantelli lemma we obtain

$$V'_k = O\left(|A'_k|^{\frac{1}{2} + \delta}\right) = O\left(k^{r(1-\varepsilon)\left(\frac{1}{2} + \delta\right)}\right) \quad \text{for any } \delta > 0.$$

This implies

$$V'_1 + \dots + V'_k = O\left(k^{r(1-\varepsilon)\left(\frac{1}{2} + \delta\right) + 1}\right)$$

and if we choose δ sufficiently small, we obtain

$$V'_1 + \dots + V'_k = O(k^{\gamma_0 r + 1})$$

i.e.

$$(4.7) \quad V_1 + \dots + V_k = \sum_{i=1}^{D_k} X_i + O(k^{\gamma_0 r + 1}) \quad \text{a.s.}$$

where $\gamma_0 = 1/2 - \varepsilon/4$. A similar argument yields for every $\delta > 0$

$$\begin{aligned} P\left(\left|\sum_{i=B_k+1}^{D_k} X_i\right| \geq (D_k - B_k)^{\frac{1}{2} + \delta}\right) &\leq (D_k - B_k)^{-2p\left(\frac{1}{2} + \delta\right)} E\left(\left(\sum_{i=B_k+1}^{D_k} W_{n_i}\right)^{2p}\right) \\ &\leq (D_k - B_k)^{-2p\left(\frac{1}{2} + \delta\right)} \left(C_6 (D_k - B_k)^p + C_7 D_k^{2 + \frac{1+\alpha}{3} 2p}\right). \end{aligned}$$

Using again

$$D_k^{2 + \frac{1+\alpha}{3} 2p} \leq C_{15} k^{2(r+1) + r(1-\varepsilon)p}, \quad D_k - B_k = \sum_{i=1}^k A_i r^{(1-\varepsilon)} \sim C_{16} k^{r(1-\varepsilon) + 1},$$

choosing p large enough and applying the Borel-Cantelli lemma we obtain

$$(4.8) \quad \begin{aligned} \sum_{i=1}^{D_k} X_i - \sum_{i=1}^{B_k} X_i &= \sum_{i=B_k+1}^{D_k} X_i = O\left((D_k - B_k)^{\frac{1}{2} + \delta}\right) \\ &= O\left(k^{[r(1-\varepsilon) + 1]\left[\frac{1}{2} + \delta\right]}\right) = O(k^{\gamma_0 r + 1}) \end{aligned}$$

if δ is also small enough. (4.6), (4.7) and (4.8) imply

$$(4.9) \quad S_{B_k} = \sum_{i=1}^{B_k} X_i = \zeta(B_k) + O(k^{\gamma_0 r + 1}) + o\left(B_k^{\frac{1}{2} - \mu_0}\right) = \zeta(B_k) + O\left(B_k^{\frac{1}{2} - \mu_0}\right) \quad \text{a.s.}$$

since $k^{\gamma_0 r + 1} \leq k^{\left(\frac{1}{2} - \frac{\varepsilon}{8}\right)r} = O\left(B_k^{\frac{1}{2} - \mu_0}\right)$ by $\mu_0 < \varepsilon/8$ and the second relation of (4.1). Relation (4.9) yields the statement of our theorem for the indices $n = B_k$.

To deduce (2.2) for every n , let us observe that $B_{k+1} \sim C_{17} k^{r+1}$, $B_{k+1} - B_k \sim Ak^r$ and hence $(B_{k+1} - B_k)^{-1} B_{k+1}^{2\alpha} \log B_{k+1} \sim C_{18} k^{r(2\alpha-1)+2\alpha} \log k$ which is $o(1)$ by the last relation of (4.1). Using this fact and the equivalence of the sequences X_k and w_{n_k} , we obtain from Lemma 5

$$\begin{aligned} & P \left(\max_{B_k+1 \leq n \leq B_{k+1}} |S_n - S_{B_k}| \geq (B_{k+1} - B_k)^{\frac{1}{2}+\delta} \right) \leq \\ & \leq 2P \left(|S_{B_{k+1}} - S_{B_k}| \geq (B_{k+1} - B_k)^{\frac{1}{2}+\delta} - 2(B_{k+1} - B_k)^{\frac{1}{2}} \right) \leq \\ & \leq 2P \left(\left| \sum_{i=B_k+1}^{B_{k+1}} X_i \right| \geq \frac{1}{2} (B_{k+1} - B_k)^{\frac{1}{2}+\delta} \right) \end{aligned}$$

for sufficiently large k . The latter probability can be estimated by the above argument (using $B_{k+1}^{2+\frac{1+\alpha}{3}2p} \leq C_{19} k^{2(r+1)+r(1-\varepsilon)p}$) and we obtain from the Borel-Cantelli lemma

$\max_{B_k+1 \leq n \leq B_{k+1}} |S_n - S_{B_k}| = O \left((B_{k+1} - B_k)^{\frac{1}{2}+\delta} \right)$ for every $\delta > 0$. By well-known properties of the Wiener-process

$$\begin{aligned} & P \left(\sup_{B_k \leq t \leq B_{k+1}} |\zeta(t) - \zeta(B_k)| \geq (B_{k+1} - B_k)^{\frac{1}{2}+\delta} \right) \leq \\ & \leq 2P \left(|\zeta(B_{k+1}) - \zeta(B_k)| \geq (B_{k+1} - B_k)^{\frac{1}{2}+\delta} \right) = 4[1 - \Phi((B_{k+1} - B_k)^\delta)] \leq \\ & \leq (2\pi)^{-\frac{1}{2}} (B_{k+1} - B_k)^{-\delta} \exp \left\{ -\frac{1}{2} (B_{k+1} - B_k)^{2\delta} \right\}, \end{aligned}$$

thus $B_{k+1} - B_k \sim Ak^r$ and the Borel-Cantelli lemma yield

$$\sup_{B_k \leq t \leq B_{k+1}} |\zeta(t) - \zeta(B_k)| = O \left((B_{k+1} - B_k)^{\frac{1}{2}+\delta} \right).$$

Here

$$(B_{k+1} - B_k)^{\frac{1}{2}+\delta} \leq C_{20} B_k^{\frac{r}{2}+\delta} = C_{20} B_k^{\frac{1}{2}-\eta}$$

where $\eta > 0$ if δ is sufficiently small. Consequently, for any $k \geq k_0$, $B_k + 1 \leq n \leq B_{k+1}$ we have

$$\begin{aligned} |S_n - \zeta(n)| & \leq |S_{B_k} - \zeta(B_k)| + \max_{B_k+1 \leq n \leq B_{k+1}} |S_n - S_{B_k}| + \\ & + \sup_{B_k \leq t \leq B_{k+1}} |\zeta(t) - \zeta(B_k)| = O \left(B_k^{\frac{1}{2}-\mu_0} \right) + O \left(B_k^{\frac{1}{2}-\eta} \right) = O \left(B_k^{\frac{1}{2}-\mu} \right) = O \left(n^{\frac{1}{2}-\mu} \right) \end{aligned}$$

where $\mu = \min(\mu_0, \eta)$. The last relation is identical with (2.2).

It remains to prove the Proposition. To this aim, let p and q denote the largest integer of the interval Δ_{k-1} and the smallest integer of the interval Δ_k , respectively; let further $w_{n_p} = r_s r_{s'} r_{s''} \dots$ ($s > s' > s'' \dots$). Then every Walsh-function appearing in T_1, T_2, \dots, T_{k-1} contains only the Rademacher-functions r_1, r_2, \dots, r_s and hence

every set of $\sigma(T_1, T_2, \dots, T_{k-1})^4$ can be obtained as a disjoint union of sets of the form $\{r_1 = \varepsilon_1, \dots, r_s = \varepsilon_s\} (\varepsilon_v = \pm 1)$. Our Proposition, therefore, will follow if we can show that

$$(4.10) \quad \int_E T_k dx = 0 \quad (k \geq 2)$$

and

$$(4.11) \quad P(E)^{-1} \int_E T_k^2 dx = Ak^r, \quad P(E)^{-1} \int_E T_k^4 dx \leq C_{11}(Ak^r)^2 \quad (k \geq k_0)$$

for every $E = \{r_1 = \varepsilon_1, \dots, r_s = \varepsilon_s\}$.

To prove (4.10) and (4.11), we first remark that

$$(4.12) \quad \frac{n_{q-1}}{n_p} \geq 2k^{\beta r} \quad k = 2, 3, \dots$$

where $\beta = 1 - \alpha - 2\varepsilon$. Indeed, $q = \sum_{i=1}^{k-1} Ai^r + \sum_{i=1}^{k-1} Ai^{r(1-\varepsilon)} + 1 \leq 3Ak^{r+1}$ and hence by (2.1), (4.2) and the second relation of (4.1) we have

$$\begin{aligned} n_{q-1}/n_p &\geq \prod_{i=p}^{q-2} (1 + ci^{-\alpha}) \geq \sum_{i=p}^{q-2} ci^{-\alpha} \geq cq^{-\alpha} |\Delta'_{k-1}| \geq \\ &\geq c(3Ak^{r+1})^{-\alpha} A(k/2)^{r(1-\varepsilon)} = [c3^{-\alpha} A^{1-\alpha} 2^{-r(1-\varepsilon)}] k^{r(1-\alpha-\varepsilon)-\alpha} \geq \\ &\geq 2k^{r(1-\alpha-2\varepsilon)} = 2k^{\beta r}. \end{aligned}$$

In particular, $n_q/n_p \geq 2$ which implies that in the product form of every term of the sum $T_k = \sum_{i \in \Delta_k} w_{n_i}$ there appears a Rademacher-function r_v with $v > s$. Let now

$E = \{r_1 = \varepsilon_1, \dots, r_s = \varepsilon_s\}$ and investigate a term $w_{n_i} = r_{v_1} \dots r_{v_l}$ of the sum $T_k = \sum_{i \in \Delta_k} w_{n_i}$

on the set E . From the decomposition $w_{n_i} = \prod_{v_j \leq s} r_{v_j} \cdot \prod_{v_j > s} r_{v_j}$ (where the second product is not empty by our remark above) it follows that $w_{n_i} = w_{t_i} w_{m_i}$ where w_{t_i} is constant on E ($+1$ or -1) and w_{m_i} does not contain r_1, \dots, r_s . (Of course, t_i can be 0 if the first product is empty, but $m_i > 0$ is always satisfied.) Therefore, on the set E we have $T_k = \sum_{i \in \Delta_k} w_{n_i} = \sum_{i \in \Delta_k} \varrho_i w_{m_i}$ where $\varrho_i = \pm 1$; observe now that $\sum_{i \in \Delta_k} \varrho_i w_{m_i}$ is independent of E and thus we have for every $j \geq 1, k \geq 2$

$$(4.13) \quad P(E)^{-1} \int_E T_k^j dx = P(E)^{-1} \int_E \left(\sum_{i \in \Delta_k} \varrho_i w_{m_i} \right)^j dx = \int_0^1 \left(\sum_{i \in \Delta_k} \varrho_i w_{m_i} \right)^j dx$$

The indices m_i are, in general, different from the n_i 's but we shall prove that they also satisfy a lacunarity condition

$$(4.14) \quad \frac{m_{t+1}}{m_t} \geq 1 + \frac{c}{2} \cdot t^{-\alpha} \quad (t, t+1 \in \Delta_k, \quad k \geq k_1)$$

furthermore

$$(4.15) \quad \frac{m_q}{n_{q-1}} \geq 1 + \frac{c}{2} \cdot (q-1)^{-\alpha} \quad (k \geq k_1)$$

⁴ $\sigma(T_1, T_2, \dots, T_{k-1})$ denotes the σ -algebra generated by T_1, T_2, \dots, T_{k-1} .

This implies, in particular, that all the m_i 's are different (if $k \geq k_1$) and hence, applying (4.13) with $j=2$, we obtain the first relation of (4.11). On the other hand, (4.14) and (4.15) show that the sequence $n_1, n_2, \dots, n_{q-1}, m_q, m_{q+1}, \dots, m_v$ (v is the largest integer of Δ_k) satisfies the lacunarity condition (2.1) with $c/2$ instead of c . Hence Lemma 3, the third relation of (4.1) and

$$v = \sum_{i=1}^k Ai^r + \sum_{i=1}^{k-1} Ai^{r(1-\varepsilon)} \leq 2Ak^{r+1}$$

imply

$$E\left(\left(\sum_{i \in \Delta_k} \varrho_i w_{m_i}\right)^4\right) \leq C_{21}(Ak^r)^2 + C_{22}(2Ak^{r+1})^{2\alpha+1} \log^2(2Ak^{r+1}) \leq C_{23}k^{2r} \leq C_{23}(Ak^r)^2.$$

By virtue of (4.13), this yields the second relation of (4.11). Finally, applying (4.13) with $j=1$ we obtain (4.10), and this is valid for every $k \geq 2$ since here we do not need (4.14) and (4.15), but only (4.12).

To obtain (4.14) and (4.15), let us observe that $w_{m_i} = w_{n_i}(w_{t_i})^{-1} = w_{n_i}w_{t_i}$ and hence, by Lemma 4, $n_i - t_i \leq m_i \leq n_i + t_i$. Now $w_{n_p} = r_s r_{s'} r_{s''} \dots$ implies $n_p \geq 2^{s-1}$, furthermore w_{t_i} can contain only r_1, \dots, r_s and thus $t_i \leq 2^s$. We have therefore $t_i \leq 2n_p$, i.e. $|n_i - m_i| \leq 2n_p$ for every $i \in \Delta_k$. Hence

$$\begin{aligned} m_{t+1} - m_t \left(1 + \frac{c}{2} t^{-\alpha}\right) &= (m_{t+1} - m_t) - \\ &- \frac{c}{2} m_t t^{-\alpha} \leq (n_{t+1} - n_t) - 4n_p - \frac{c}{2} n_t t^{-\alpha} - cn_p t^{-\alpha} \leq n_t c t^{-\alpha} - 4n_p - \\ &- \frac{c}{2} n_t t^{-\alpha} - cn_p t^{-\alpha} \leq \frac{c}{2} n_q t^{-\alpha} - 4n_p - cn_p t^{-\alpha} = \frac{1}{2} n_p t^{-\alpha} \left(c \frac{n_q}{n_p} - 8t^\alpha - 2c\right) \end{aligned}$$

for every $t, t+1 \in \Delta_k$. Here $c \cdot n_q/n_p \geq 2ck^{\beta r}$ by (4.12) and for any $t \in \Delta_k$ we have $t \leq v \leq 2Ak^{r+1}$ i.e. $8t^\alpha + 2c \leq 8(2Ak^{r+1})^\alpha + 2c \leq C_{24}k^{(r+1)\alpha} < 2ck^{\beta r}$ for $k \geq k_1$ by the last relation of (4.1). Thus $c \cdot n_q/n_p - 8t^\alpha - 2c > 0$ and (4.14) is proved. A similar argument yields

$$m_q - n_{q-1} \left(1 + \frac{c}{2} (q-1)^{-\alpha}\right) \geq n_p (q-1)^{-\alpha} \left[\frac{c}{2} \cdot \frac{n_{q-1}}{n_p} - 2(q-1)^\alpha\right] > 0$$

for sufficiently large k , hence (4.15) is also valid and the proof of Theorem 1 is completed.

5. § Some limit theorems for lacunary Walsh-series

In this paragraph we mention some consequences of Theorem 1. To obtain these results, we remark that (2.2) implies

$$(5.1) \quad \max_{1 \leq k \leq n} |S_k - \zeta(k)| = O\left(n^{-\frac{1}{2}-\mu}\right) \quad \text{a.s.}$$

where

$$S_k = X_1 + \dots + X_k.$$

Definition. Let Y_1, Y_2, \dots be a sequence of r.v.-s, $S_N = Y_1 + \dots + Y_N$ ($S_0 = 0$) and let $\chi_N(t)$ denote the random function defined in $[0, 1]$ which is linear in every interval $[(k-1)/N, k/N]$ ($1 \leq k \leq N$) and $\chi_N(k/N) = S_k$ ($0 \leq k \leq N$). Put $\varphi_N(t) = N^{-1/2} \chi_N(t)$, $\psi_N(t) = (2N \log \log N)^{-1/2} \chi_N(t)$. We say that the sequence Y_1, Y_2, \dots obeys the functional central limit theorem (invariance principle) if the sequence $\varphi_N(t)$ of processes converges weakly to the Wiener-process in the usual sense, i.e. the measures μ_N on the space $C[0, 1]$, corresponding to the processes $\varphi_N(t)$, converge weakly to the Wiener-measure. On the other hand, we say that the sequence Y_1, Y_2, \dots obeys the functional (Strassen-type) law of the iterated logarithm of the sequence $\psi_N(t)$ is equicontinuous with probability one and the set of its norm-limit points (in $C[0, 1]$ norm) coincides with the set

$$K = \left\{ x(t) : x(t) \text{ is absolutely continuous in } [0, 1], x(0) = 0 \text{ and } \int_0^1 \dot{x}(t)^2 dt \leq 1 \right\}.$$

THEOREM 2. Let $n_1 < n_2 < \dots$ be a sequence of positive integers satisfying (2.1). Then the sequence w_{n_k} ($k=1, 2, \dots$) obeys the functional central limit theorem.⁵

THEOREM 3. Let $n_1 < n_2 < \dots$ be a sequence of positive integers satisfying (2.1). Then the sequence w_{n_k} ($k=1, 2, \dots$) obeys the functional law of the iterated logarithm.

THEOREM 4. Let $n_1 < n_2 < \dots$ satisfy (2.1) and put $S_k = \sum_{i=1}^k w_{n_i}$. Then

$$\liminf_{N \rightarrow \infty} \frac{\max_{1 \leq k \leq N} |S_k|}{\sqrt{N/\log \log N}} = \frac{\pi}{\sqrt{8}}$$

for almost all $x \in [0, 1]$.

Definition. Let Y_1, Y_2, \dots be a sequence of r.v.-s, $S_n = Y_1 + \dots + Y_n$. We say that a function $\varphi(n)$ belongs to the upper class with respect to Y_1, Y_2, \dots if a.s. there exist only finitely many n such that $S_n > \sqrt{n} \varphi(n)$; furthermore, we say that $\varphi(n)$ belongs to the lower class if the inequality $S_n > \sqrt{n} \varphi(n)$ is satisfied for infinitely many n a.s.

THEOREM 5. Let $n_1 < n_2 < \dots$ satisfy (2.1) and let $\varphi(n) > 0$ be a monotone increasing function. Then $\varphi(n)$ belongs to the upper (lower) class with respect to w_{n_k} if and only if the sum $\sum_{n=1}^{\infty} \frac{1}{n} \varphi(n) e^{-\frac{1}{2} \varphi^2(n)}$ converges (diverges). In particular,

$$\varphi(n) = \left[2 \left(\log_2 n + \frac{3}{2} \log_3 n + \log_4 n + \dots + \log_{k-1} n + (1 + \varepsilon) \log_k n \right) \right]^{\frac{1}{2}}$$

belongs to the upper class if $\varepsilon > 0$ and to the lower class if $\varepsilon \leq 0$.

Theorem 2 follows immediately from (5.1), Theorems 3 and 4 follow from (5.1) and well-known theorems of STRASSEN and CHUNG (see [2], [5]). Finally, Theorem 5 follows from Theorem 1 by an argument similar to the one used in [1], pp. 337—338.

⁵ See footnote 1.

⁶ $\log_2 n = \log \log n$, $\log_k n = \log(\log_{k-1} n)$.

6. § Remarks

1. In the case $\alpha > 1/2$ Theorem 1 is no longer valid; in fact, for every $\alpha > 1/2$ there exists a sequence $n_1 < n_2 < \dots$ such that $n_{k+1}/n_k \cong 1 + k^{-\alpha}$ but the central limit theorem does not hold (see [3]).

2. A special case of Theorems 2 and 3 ($\alpha = 0$) is proved in our paper [6] by a different method.

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TWO LOWER BOUNDS FOR LYAPUNOV TYPE FUNCTIONALS

by
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We consider the differential equation

$$(1) \quad y'' + p(t)y = 0,$$

where $p(t) > 0$ for $t > 0$. Let $t_0 \geq 0, t_1, \dots, t_n$ be the consecutive zeros of a solution $y(t)$ of (1), and let t'_1, \dots, t'_n be the zeros of $y'(t)$ with $t_0 < t'_1 < t_1 < \dots < t'_n < t_n$. It is easy to see that $y'(t)$ has no more zeros on $[t_0, t_n]$. We say that the function $p(t)$ belongs to the set of functions $C_v[t_0; n]$ ($v > 0$) if $p(t)$ is nonnegative, non-decreasing and $[p(t)]^v$ is concave on $[t_0, t_n]$. Similarly, $p(t)$ belongs to $C_v[t_0; n - \frac{1}{2}]$ if $p(t)$ behaves in this way on $[t_0, t'_n]$.

Now we are interested in finding lower bounds for the Lyapunov type functional

$$(2) \quad I_n = I_n(p) = (t_n - t_0) \int_{t_0}^{t_n} p(t) dt$$

if $p \in C_v[t_0; n]$. The upper bound is already known (see [1]—[3]). Concerning the lower bound we mention an inequality due to A. S. GALBRAITH [4] which asserts

$$I'_n = (t'_n - t'_1) \int_{t'_1}^{t'_n} p(t) dt \geq (n - 1)^2 \pi^2$$

for nonnegative, nondecreasing and concave $p(t)$ on $[t'_1, t'_n]$. This yields as an immediate consequence the estimation $I_n > (n - 1)^2 \pi^2$ for $p \in C_v[t_0; n]$. A little later J. H. COHN [5] showed among others the relation

$$(3) \quad I_n(p) > \left[\left(n - \frac{1}{2} \right) \pi + \sqrt{\frac{2v + 1}{v + 1}} \right]^2 \quad \text{for } p \in C_v[t_0; n].$$

In [6] we have found the inequality

$$\int_{t_0}^{t_n} \sqrt{p(t)} dt \geq \left(n - \frac{1}{2} \right) \pi + j_{\frac{v}{1+2v}-1} \quad \text{for } p \in C_v[t_0; n],$$

where j_μ is the first positive root of the Bessel function $J_\mu(z)$ of first kind. Applying the Cauchy—Schwarz inequality we get

$$(4) \quad I_n \geq \left[\left(n - \frac{1}{2} \right) \pi + j_{\frac{v}{1+2v}-1} \right]^2 \quad \text{for } p \in C_v[t_0; n].$$

Since $j_{-\frac{2}{3}} = 1.243046\dots$, the lower bound in (4) is likely sharper for $v \geq 1$ than the one in (3).

From now on we restrict ourselves to the case $v=1$ and $t_0=0$. The latter means no restriction at all, because the functional $I_n(p)$ is invariant under the linear transformation $x=t-t_0$.

THEOREM. *With the above notations, the inequalities*

$$(5) \quad I_n(p) > \left[\left(n - \frac{1}{2} \right) \pi + \frac{3\sqrt{2}}{4} j_{-\frac{2}{3}} \right]^2$$

and

$$(6) \quad I_n(p) > \left(n - \frac{1}{2} \right)^2 \pi^2 + C \left(n - \frac{1}{2} \right) \pi + \frac{9}{8} j_{-\frac{2}{3}}^2$$

hold for $p \in C_1[0; n]$, where $C = 2.74964\dots$ is defined by

$$C = \min_{p(t) \in C_1 \left[0; \frac{1}{2} \right]} \left[T\sqrt{p(T)} + \frac{1}{\sqrt{p(T)}} \int_0^T p(t) dt \right] \quad (T = t'_1).$$

The constant C in (6) is the best one.

Remark. Since $C > 2 \cdot (3\sqrt{2}/4) j_{-\frac{2}{3}} = 2 \cdot 1.31844\dots$ the lower bound for $I_n(p)$ given in (6) is sharper than that in (5).

PROOF. Because the functions twice continuously differentiable and concave in $C_1[0; n]$ are everywhere dense in $C_1[0; n]$, it is sufficient to consider only these functions while we must take care of the inequality signs in (5) and (6).

Let us introduce the function $\varphi(t)$ defined by

$$(7) \quad \operatorname{tg} \varphi(t) = \frac{y(t)\sqrt{p(t)}}{y'(t)}, \quad \varphi(0) = 0.$$

The properties of $\varphi(t)$ are dealt in [7] and now the followings are required:

$$(8) \quad \varphi(t_i) = i\pi \quad i = 0, 1, \dots, n$$

$$\varphi(t'_i) = \left(i - \frac{1}{2} \right) \pi \quad i = 1, 2, \dots, n,$$

$$(9) \quad \varphi' = \sqrt{p} + \frac{p'}{4p} \sin 2\varphi$$

$$(10) \quad \varphi'(t) > 0 \quad (0 < t \leq t_n)$$

$$(11) \quad F(t) = \int_0^t \frac{p'(\tau)}{4p(\tau)} \sin 2\varphi(\tau) d\tau \geq 0 \quad (t \geq 0)$$

$$(12) \quad F(t'_i) \geq F(t'_{i+1}) \quad (i = 1, \dots, n-1)$$

$$(13) \quad 0 < \varphi(t) < t\sqrt{p(t)} \quad (t > 0).$$

Let $p_x(t)$ be defined by

$$(14) \quad p_x(t) = \begin{cases} p(t) & 0 \leq t \leq x \leq t_n \\ p(x) & t \geq x. \end{cases}$$

It is clear that $p_x \in C_1[0; n]$. The differential equation (1) with $p = p_x(t)$ has solution $y_x(t)$ with the initial conditions $y_x(0) = 0, y'_x(0) = y'(0)$, whose n -th zero is $t_{n,x}$. Determine the functional $I_n(p_x)$. Now $p_x(t)$ is twice continuously differentiable on the intervals $(0, x)$ and (x, ∞) . By the aid of (7) with $p = p_x(t)$ and $y = y_x(t)$ we can define the function $\varphi_x(t)$. Of course, $\varphi_x(t) = \varphi(t)$ for $0 \leq t \leq x$ and $\varphi_x(t)$ is continuous on $[0, \infty)$. By (9) $\varphi'_x(t) = \sqrt{p(x)}$ for $t > x$, hence integrating $\varphi'_x(t)$ from x to $t_{n,x}$ we have

$$n\pi - \varphi(x) = \sqrt{p(x)} (t_{n,x} - x),$$

i.e.

$$t_{n,x} = x + \frac{n\pi - \varphi(x)}{\sqrt{p(x)}}$$

and

$$(15) \quad I_n(p_x) = (n\pi - \varphi(x) + x\sqrt{p(x)}) \left(n\pi - \varphi(x) + \frac{1}{\sqrt{p(x)}} \int_0^x p(t) dt \right) = J(x) \quad (0 \leq x \leq t_n).$$

It is clear that $J(t_n) = I_n(p)$ and $J(0) = n^2\pi^2$. Next we shall show the relations

$$(16) \quad I_n(p) = J(t_n) \geq J(t'_n) \geq J(t'_{n-1}) \geq \dots \geq J(t'_1).$$

By the aid of (9) we can differentiate $J(x)$, thus

$$(17) \quad J'(x) = \frac{p'}{2p} \left[-\sin 2\varphi \left(n\pi - \varphi + \frac{1}{2} p^{-\frac{1}{2}}(x) \int_0^x p(t) dt + \frac{1}{2} x p^{\frac{1}{2}}(x) \right) + \right. \\ \left. + (n\pi - \varphi) \left(x p^{\frac{1}{2}}(x) - p^{-\frac{1}{2}}(x) \int_0^x p(t) dt \right) \right].$$

Here the second term in the brackets is nonnegative because $0 \leq \varphi \leq n\pi$ and $p(t)$ is nonnegative and nondecreasing. By (8) and (10) $(n - \frac{1}{2})\pi \leq \varphi(x) \leq n\pi$ on $[t'_n, t_n]$, and $\sin 2\varphi \leq 0$ there, and $J'(x) \geq 0$, i.e. $J(t_n) \geq J(t'_n)$.

Now we consider the interval $[0, t'_n]$, where $\varphi(x) \leq (n - \frac{1}{2})\pi$, hence $n\pi - \varphi \geq \pi/2$, thus

$$(18) \quad J'(x) \geq \frac{p'}{2p} \left[-n\pi \sin 2\varphi - \left(\varphi - \int_0^x \sqrt{p} dt \right) + \right. \\ \left. + \left(\int_0^x \sqrt{p} dt - \frac{1}{2} p^{-\frac{1}{2}}(x) \int_0^x p dt - \frac{1}{2} x p^{\frac{1}{2}}(x) \right) \right] + \\ + \frac{\pi}{2} \left(x p^{\frac{1}{2}} - p^{-\frac{1}{2}} \int_0^x p dt \right) = \frac{p'}{2p} \left[-n\pi \sin 2\varphi - |f_1 + f_2| + \frac{\pi}{2} \left(x p^{\frac{1}{2}} - p^{-\frac{1}{2}} \int_0^x p dt \right) \right].$$

From (9), (11) and (13) we find easily

$$\begin{aligned} 0 \leq f_1 &= \int_0^x \frac{p'}{4p} \sin 2\varphi dt \leq \int_0^x \frac{p'}{4p} 2t \sqrt{p} dt = x\sqrt{p} - \int_0^x \sqrt{p} dt = \\ &= \int_0^x [\sqrt{p(x)} - \sqrt{p(t)}] dt. \\ f_2 &= -\frac{1}{2\sqrt{p}} \int_0^x [p(x) - 2\sqrt{p(x)}\sqrt{p(t)} + p(t)] dt = \\ &= -\frac{1}{2\sqrt{p(x)}} \int_0^x (\sqrt{p(x)} - \sqrt{p(t)})^2 dt \leq -\frac{1}{2} \int_0^x (\sqrt{p(x)} - \sqrt{p(t)}) dt, \end{aligned}$$

hence

$$\begin{aligned} \frac{\pi}{2} \left(xp^{\frac{1}{2}} - p^{-\frac{1}{2}} \int_0^x p dt \right) - |f_1| - |f_2| &\geq \frac{\pi}{2} \int_0^x (\sqrt{p(x)} - \sqrt{p(t)}) dt - f_1 - |f_2| \geq \\ &\geq \left(\frac{\pi}{2} - \frac{3}{2} \right) \int_0^x (\sqrt{p(x)} - \sqrt{p(t)}) dt \geq 0, \end{aligned}$$

thus by (12) and (18) for $i=1, 2, \dots, n-1$

$$J(t'_{i+1}) - J(t'_i) \geq -n\pi \int_{t'_i}^{t'_{i+1}} \frac{p'}{2p} \sin 2\varphi dt = -2n\pi(F(t'_{i+1}) - F(t'_i)) \geq 0,$$

which proves (16).

Let $T=t'_1$ then by (15) and (16) we have

$$(19) \quad I_n(p) \geq \left(n - \frac{1}{2}\right)^2 \pi^2 + \psi(p) \left(n - \frac{1}{2}\right) \pi + T \int_0^T p dt,$$

where

$$(20) \quad \psi(p) = T\sqrt{p(T)} + \frac{1}{\sqrt{p(T)}} \int_0^T p dt.$$

To prove (5) and particularly (6) we shall make use of an inequality due to D. BANKS [8]: If $p(t)$ in the differential equation (1) is nonnegative and concave on $[a, b]$, where a and b are two consecutive zeros of a solution of (1) then

$$(b-a) \int_a^b p dt \geq \frac{9}{2} j_{-\frac{2}{3}}^2.$$

Let

$$p^*(t) = \begin{cases} p(t) & 0 \leq t \leq T \\ p(2T-t) & T \leq t \leq 2T. \end{cases}$$

The symmetry of $p^*(t)$ with respect to $t=T$ together with $y'(T)=0$ implies that the function

$$y^*(t) = \begin{cases} y(t) & 0 \leq t \leq T \\ y(2T-t) & T \leq t \leq 2T \end{cases}$$

is a solution of the differential equation $y'' + p^*(t)y = 0$. Whence the inequality of Banks states that

$$(21) \quad 2T \int_0^{2T} p^* dt = 4T \int_0^T p dt \cong \frac{9}{2} j^2_{-\frac{2}{3}}.$$

If we apply the inequality between the arithmetic and the geometric means we have in (20)

$$\psi(p) \cong 2 \left(T \int_0^T p dt \right)^{\frac{1}{2}},$$

thus by (21) and (19)

$$I_n(p) \cong \left[\left(n - \frac{1}{2} \right) \pi + \frac{3}{\sqrt{8}} j_{-\frac{2}{3}} \right]^2,$$

which is the same as (5) but the equality sign. The fulfilling of the equation here requires the equality between the arithmetic mean and the geometric mean both used above. This equality can occur only in case when $T\sqrt{p(T)} = 1/\sqrt{p(T)} \int_0^T p dt$, i.e. when $p(t) = \text{const}$. However, for $p(t) = \text{const}$ the value of $I_n(p)$ is $n^2\pi^2$ which is larger than $\left[\left(n - \frac{1}{2} \right) \pi + \frac{3}{\sqrt{8}} j_{-\frac{2}{3}} \right]^2$ thus (5) is proved completely.

Now we shall seek for a function $\tilde{p}(t) \in C_1[0; \frac{1}{2}]$ which minimizes the functional $\psi(p)$. It is clear that there is such a minimizing function. Let $\tilde{y}(t)$ be a solution of (1) with $p = \tilde{p}(t)$ which fulfils the conditions $\tilde{y}(0) = 0, \tilde{y}'(0) \neq 0$, and let T be the first positive zero of $\tilde{y}'(t) = 0$. The function $\tilde{p}(t)$ need not be twice continuously differentiable anymore but it is nonnegative, nondecreasing and concave, therefore it can be written in form

$$(22) \quad \tilde{p}(t) = \int_0^T \varrho_\xi(t) d\Gamma(\xi),$$

where the integral is taken in Lebesgue—Stieltjes sense and

$$(23) \quad \begin{aligned} \varrho_0(t) &= 1 & 0 \leq t \leq T \\ \varrho_x(t) &= \begin{cases} \frac{t}{x} & 0 \leq t \leq x \\ 1 & x \leq t \leq T, \end{cases} \end{aligned}$$

and $\Gamma(\xi)$ is a nondecreasing function bounded above with, say, $\Gamma(\xi) = 0$ for $\xi < 0$.

It is clear that if $\varepsilon > 0$ then $\tilde{p} + \varepsilon\varrho_x \in C_1[0; \frac{1}{2}]$ and by the definition of $\tilde{p}(t)$ we have

$$(24) \quad \Phi(x) = \lim_{\varepsilon \rightarrow +0} \frac{\psi(\tilde{p} + \varepsilon\varrho_x) - \psi(\tilde{p})}{\varepsilon} \cong 0.$$

We must evaluate this function. Let y_ε be the solution of the differential equation

$$(25) \quad y'' + (\tilde{p} + \varepsilon\varrho_x)y = 0$$

with the initial conditions $y_\varepsilon(0)=0$, $y'_\varepsilon(0)=\tilde{y}'(0)$. Let $T(\varepsilon)$ be the corresponding zero of $y'_\varepsilon(t)$. Now $y_\varepsilon(t)$ can be written in the form

$$y_\varepsilon(t) = \sum_{n=0}^{\infty} \varepsilon^n \eta_n(t),$$

where $\eta_0(t)=\tilde{y}(t)$ and η_n ($n=1, 2, \dots$) fulfils the differential equation

$$\eta_n'' + \tilde{p}\eta_n = -\varrho_x \eta_{n-1}$$

and the initial conditions are $\eta_n(0)=\eta_n'(0)=0$. Solving this differential equation via method of variation of coefficients we obtain

$$(26) \quad \eta_n = c_n \tilde{y} + d_n \tilde{\tilde{y}}, \quad \eta_n' = c_n \tilde{y}' + d_n \tilde{\tilde{y}}',$$

where

$$c_n(t) = \frac{1}{w} \int_0^t \varrho_x(\tau) \eta_{n-1}(\tau) \tilde{\tilde{y}}(\tau) d\tau$$

$$d_n(t) = -\frac{1}{w} \int_0^t \varrho_x(\tau) \eta_{n-1}(\tau) \tilde{y}(\tau) d\tau \quad (n = 1, 2, \dots)$$

and $\tilde{\tilde{y}}(t)$ is another solution of $y'' + \tilde{p}y=0$ such that \tilde{y} and $\tilde{\tilde{y}}$ are linearly independent, and $w=w(t)$ is the wellknown Wronskian $\tilde{y}\tilde{\tilde{y}}' - \tilde{y}'\tilde{\tilde{y}} = \text{const} \neq 0$. Thus by (26)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{y'_\varepsilon(T)}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \sum_{n=1}^{\infty} \varepsilon^{n-1} \eta_n'(T) = \eta_1'(T) = \tilde{\tilde{y}}'(T) d_1(T) = \\ &= -\frac{\tilde{\tilde{y}}'(T)}{w} \int_0^T \varrho_x(t) \tilde{y}^2(t) dt, \end{aligned}$$

and otherwise, by the mean value theorem

$$\lim_{\varepsilon \rightarrow 0} \frac{y'_\varepsilon(T)}{\varepsilon} = -\lim_{\varepsilon \rightarrow 0} \frac{y'_\varepsilon(T(\varepsilon)) - y'_\varepsilon(T)}{\varepsilon} = -\tilde{y}''(T) \lim_{\varepsilon \rightarrow 0} \frac{T(\varepsilon) - T}{\varepsilon} = \tilde{p}(T) \tilde{y}(T) \frac{dT(0)}{d\varepsilon}.$$

The Wronskian w taken at $t=T$ supplies $w=\tilde{y}(T)\tilde{\tilde{y}}'(T)$ and combining this with the last two limits, we have

$$(27) \quad \frac{dT(0)}{d\varepsilon} = -\frac{1}{\tilde{p}(T)\tilde{y}^2(T)} \int_0^T \varrho_x(t) \tilde{y}^2(t) dt.$$

Before going further we show that $\tilde{p}(t)$ is constant on an interval $[T-\delta, T]$ with $\delta > 0$. Let $\{p^{(m)}(t)\}_{m=1}^{\infty} \subset C_1[0; \frac{1}{2}]$ be a sequence of twice continuously differentiable functions satisfying

$$(28) \quad \lim_{m \rightarrow \infty} p^{(m)}(t) = \tilde{p}(t) \quad 0 \leq t \leq T.$$

We denote by $\varphi^{(m)}$ and $\tilde{\varphi}$ the corresponding functions, by $T^{(m)}$ the value of t at which $\varphi^{(m)}(t) = \frac{\pi}{2}$, and let $p_x^{(m)}$ be defined as p_x in (14). Now we can determine $\psi(p_x^{(m)})$ like $I_n(p_x)$ in (15),

(29)
$$\psi(p_x^{(m)}) = \pi - 2\varphi^{(m)}(x) + x\sqrt{p^{(m)}(x)} + \frac{1}{\sqrt{p^{(m)}(x)}} \int_0^x p^{(m)}(t) dt \quad \text{for } 0 \leq x \leq T,$$

hence

(30)
$$\frac{\partial \psi(p_x^{(m)})}{\partial x} = \frac{p'^{(m)}}{2p^{(m)}} \left[-\sin 2\varphi^{(m)} + x\sqrt{p^{(m)}} - \frac{1}{\sqrt{p^{(m)}}} \int_0^x p^{(m)} dt \right].$$

The sum in the brackets has the limit

$$g(x) = -\sin 2\tilde{\varphi}(x) + x\sqrt{\tilde{p}(x)} - \frac{1}{\sqrt{\tilde{p}(x)}} \int_0^x \tilde{p}(t) dt \quad 0 \leq x \leq T.$$

as $m \rightarrow \infty$. Since $\tilde{p}(t) \neq \text{const}$, $g(T) > 0$. Then there is a $\delta > 0$ satisfying $g(x) > \frac{1}{2}g(T)$ on $[T-\delta, T]$. Hence by (28) and (30) we have for $m > m_\delta$ and $T-\delta \leq x < T$

$$\psi(p^{(m)}) - \psi(p_x^{(m)}) \geq \int_x^{T^{(m)}} \frac{p'^{(m)}(t)}{2p^{(m)}(t)} \frac{1}{4} g(T) dt = \frac{g(T)}{8} \log \frac{p^{(m)}(T^{(m)})}{p^{(m)}(x)}.$$

Letting $m \rightarrow \infty$ it follows

$$0 \geq \psi(\tilde{p}) - \psi(\tilde{p}_x) \geq \frac{g(T)}{8} \log \frac{\tilde{p}(T)}{\tilde{p}(x)} \geq 0,$$

i.e.

(31)
$$\tilde{p}(x) = \tilde{p}(T) \quad \text{for } T-\delta \leq x < T, \quad \delta > 0.$$

By (27) $T(\varepsilon) < T$ when $\varepsilon > 0$ small enough. Then by (31)

$$\begin{aligned} \psi(\tilde{p} + \varepsilon \varrho_x) &= \psi(\tilde{p}) + \frac{\varepsilon}{2} \frac{T}{\sqrt{\tilde{p}(T)}} + 2\sqrt{\tilde{p}(T)}(T(\varepsilon) - T) - \\ &- \frac{\varepsilon}{2\tilde{p}^{\frac{3}{2}}(T)} \int_0^T \tilde{p} dt + \frac{\varepsilon}{\sqrt{\tilde{p}(T)}} \int_0^T \varrho_x dt + O(\varepsilon^2), \end{aligned}$$

hence by (24), (27)

(32)
$$\sqrt{\tilde{p}(T)} \Phi(x) = -\frac{2}{\tilde{y}^2(T)} \int_0^T \varrho_x \tilde{y}^2 dt + \int_0^T \varrho_x dt + \frac{T}{2} - \frac{1}{2\tilde{p}(T)} \int_0^T \tilde{p} dt.$$

The most important property of $\Phi(x)$ is that it is strictly convex on $[0, T]$. To prove this we differentiate twice the function $\Phi(x)$ given in (32) making use of (23), then

$$\sqrt{\tilde{p}(T)} \Phi''(x) = \frac{2}{x^3 \tilde{y}^2(T)} \int_0^x 2t[\tilde{y}^2(x) - \tilde{y}^2(t)] dt \quad \text{for } 0 < x < T$$

and since $\tilde{y}(0)=0$ and $\tilde{y}'(t)\neq 0$ on $[0, T)$, the function $\tilde{y}^2(t)$ is strictly increasing therefore $\Phi''(x)>0$ on $[0, T)$.

Let us extend the domain of definition of $\tilde{p}(t)$ from $[0, T]$ to $[0, \infty)$ by $\tilde{p}(t)=\tilde{p}(T)$ for $t\geq T$. Then a simple consideration shows that the relation (24) can be generalized as follows. If $\gamma(x)$ is a real function of bounded variation on $[0, T]$ and

$$\varrho(t) = \int_0^T \varrho_x(t) d\gamma(x),$$

then

$$(33) \quad \lim_{\varepsilon \rightarrow +0} \frac{\psi(\tilde{p} + \varepsilon\varrho) - \psi(\tilde{p})}{\varepsilon} = \int_0^T \Phi(x) d\gamma(x),$$

while we must assure only that the functions $\tilde{p} + \varepsilon\varrho$ belong to $C_1[0; \frac{1}{2}]$ at least for small positive ε 's and $\Phi(x)$ as same as in (32).

Concerning $\Phi(x)$ we shall prove that $\Phi(x)$ vanishes exactly at one point $x_0 \in [0, T]$. Since $\Phi(x) \geq 0$ and strictly convex it may have at most one zero. If $\Phi(x)$ would have no zero then it should be positive on $[0, T]$. Then choosing $\varrho(t) = -\tilde{p}(t)$ in (33) we would have by (22)

$$\lim_{\varepsilon \rightarrow 0} \frac{\psi((1-\varepsilon)\tilde{p}) - \psi(\tilde{p})}{\varepsilon} = - \int_0^T \Phi(\xi) d\Gamma(\xi) < 0,$$

thus for enough small $\varepsilon > 0$ it would be $\psi((1-\varepsilon)\tilde{p}) < \psi(\tilde{p})$ which would be a contradiction to the definition of \tilde{p} . Hence $\Phi(x)$ is strictly decreasing on $[0, x_0]$ and strictly increasing on $[x_0, T]$.

When $x_0 > 0$ then let $x_1 \in (0, x_0)$ be such that $\Gamma(x)$ is continuous at $x = x_1$ and we take

$$\varrho(t) = - \int_0^{x_1} \varrho_x(t) d\Gamma(x).$$

It is clear that $\tilde{p} + \varepsilon\varrho \in C_1[0; \frac{1}{2}]$ thus by (33) and by the above mentioned property of $\Phi(x)$

$$0 \leq \lim_{\varepsilon \rightarrow +0} \frac{\psi(\tilde{p} + \varepsilon\varrho) - \psi(\tilde{p})}{\varepsilon} = - \int_0^{x_1} \Phi(x) d\Gamma(x) \leq -\Phi(x_1)[\Gamma(x_1) - \Gamma(-0)],$$

i.e. $\Gamma(-0) = \Gamma(x_1)$. Since $\Gamma(x)$ is almost everywhere continuous we have $\Gamma(-0) = \Gamma(x_0-0)$ and similarly $\Gamma(x_0+0) = \Gamma(T+0)$, therefore by the notation $\gamma = \Gamma(x_0+0) - \Gamma(x_0-0)$ $\tilde{p}(t) = \gamma \varrho_{x_0}(t)$.

The transformation $t = \sqrt[3]{x_0/\gamma} s$ turns the differential equation of $\tilde{y}(t)$ into the differential equation of $Y(s) = \tilde{y}(t)$ of the form

$$(34) \quad Y'' + \hat{p}(s)Y = 0,$$

where

$$\hat{p}(s) = \begin{cases} s & 0 \leq s \leq \sigma \\ \sigma & s \geq \sigma. \end{cases}$$

By a simple calculation we could show that the functional $\psi(p)$ is invariant under this transformation. Now we have only one parameter, σ . If we take $p^{(m)}(t) = t$ in (29) we obtain for our case in (34)

$$\psi(\hat{p}) = \pi - 2\varphi(\sigma) + \frac{3}{2} \sigma^{\frac{3}{2}} = K(\sigma),$$

where $\varphi(\sigma)$ (according to (9)) is the solution of the differential equation

$$(34) \quad \varphi' = \sqrt{\sigma} + \frac{1}{4\sigma} \sin 2\varphi, \quad \varphi(0) = 0.$$

The function $K(\sigma)$ has only one minimum, i.e. $K'(\sigma)$ has only zero, if φ ranges over $[0, \pi/2]$. On $[0, \pi/4]$ we have $\sin 2\varphi \cong (4/\pi)\varphi$ and by (34) $\varphi' > \sqrt{\sigma}$, hence

$$\sin 2\varphi > \frac{8}{3\pi} \sigma^{\frac{3}{2}},$$

thus

$$K'(\sigma) = \frac{\sqrt{\sigma}}{4} - \frac{1}{2\sigma} \sin 2\varphi < \left(\frac{1}{4} - \frac{4}{3\pi} \right) \sqrt{\sigma} < 0 \quad \text{for } \sigma > 0, \quad \varphi(\sigma) < \frac{\pi}{4}.$$

On the other hand, the function $\sigma K'(\sigma)$ is strictly increasing if $\pi/4 \cong \varphi \cong \pi/2$ and if σ (and φ also) increases. At $\varphi = \pi/2$ $K'(\sigma) > 0$, therefore there is exactly one zero of $K'(\sigma)$, at which $K(\sigma)$ assumes its minimum. This value is denoted by C in Theorem. We remark here that there is an interval where the minimizing $\hat{p}(\sigma)$ is constant. Taking into consideration that in (21) the minimizing function for the functional $T \int_0^T p dt$ has no interval where it would be constant (see [8]), we see, that (19) and (21) proves (6) and thus our Theorem is proved completely.

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LOKALE WERTE UND GRENZWERTE VON PREDISTRIBUTIONEN*

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Zusammenfassung

In Abschnitt 1 wird die Konvergenz einer Folge von Predistributionen definiert und gezeigt, daß diese äquivalent zu der von E. BERZ in [1] gegebenen Definition der Konvergenz einer Folge von linearen Funktionalen über D , dem Raum aller reellen, beliebig oft differenzierbaren Funktionen über R mit kompakten Träger, ist. Weiterhin wird ein nichtstandarder Beweis des standarden Satzes über die Äquivalenz von starker und schwacher Konvergenz von standarden Distributionenfolgen gegeben.

Der Abschnitt 2 beschäftigt sich mit dem lokalen Wert einer Predistribution in einem endlichen Punkt. Die Definition des lokalen Wertes lehnt sich stark an den von S. ŁOJASIEWICZ in [2] geprägten Begriff eines lokalen Wertes einer standarden Distribution an. Die Äquivalenz der beiden Definitionen im standarden Fall wird in Satz 8 bewiesen.

Das bestimmte Integral einer Predistribution wird in Abschnitt 3 mit Hilfe der lokalen Werte definiert. Es wird ein hinreichendes Majorantenkriterium für die Existenz des bestimmten Integrals und ein Satz über die Vertauschbarkeit von Grenzwertbildung und bestimmter Integration einer Folge von Predistributionen bewiesen.

0. Es sei $^*\mathfrak{M}$ ein geeignetes nichtstandardes Modell höherer Stufe der Analysis, in welchem wir die folgenden Überlegungen ausführen können, und \mathfrak{M} ein standardes Modell der Analysis (vergl. [4]). Befinden wir uns im Modell $^*\mathfrak{M}$, bezeichnen wir, wenn es zu Verwechslungen führen kann, die entsprechenden Begriffe und Größen mit einem Stern. Größen, Begriffe, Mengen, Funktionen ... nennen wir standard, oder sagen „im standarden Sinne“, wenn sie im Modell \mathfrak{M} definiert sind und in $^*\mathfrak{M}$ beschrieben werden.

A. ROBINSON führte in [4], Abschnitt 5.3, Predistributionen folgendermaßen ein: Es sei Q die Menge aller inneren reellen Funktionen $f(x)$ einer reellen Variablen, die über jedem beschränkten Intervall im * Sinne von Lebesgue quadratisch integrierbar sind, und für die das folgende * Lebesgue-Integral $\int_{^*R} f(x)\varphi(x)dx$ für alle $\varphi(x)$ aus dem Raum D der reellen Schwartz'schen Grundfunktionen einer reellen Variablen in \mathfrak{M} endlich ist. In Q wird eine Äquivalenzrelation eingeführt: Zwei

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Funktionen $f_1(x)$ und $f_2(x)$ sind genau dann zueinander äquivalent, wenn für alle $\varphi(x)$ aus D gilt

$$\int_{*R} f_1(x) \varphi(x) dx =_1 \int_{*R} f_2(x) \varphi(x) dx.$$

Hierbei bedeutet das Zeichen $=_1$ „gleich bis auf eine infinitesimale Größe“, und $*R$ ist die nichtstandard Erweiterung der Menge aller reellen Zahlen in $*\mathfrak{M}$ [4]. Die so definierten Äquivalenzklassen werden Predistributionen genannt. Wie A. ROBINSON [4] zeigte, definiert jede Predistribution $\alpha = \{f_\alpha(x)\}$ ein lineares Funktional $T = T(\alpha)$ über D im Modell \mathfrak{M} :

$$(0.1) \quad (T, \varphi) = {}^0 \left[\int_{*R} f_\alpha(x) \varphi(x) dx \right].$$

Hierbei ist $f_\alpha(x)$ eine beliebige Funktion aus der Äquivalenzklasse α . Durch ${}^0[\]$ bezeichnen wir den Standardteil einer endlichen Zahl aus $*R$. Andererseits kann man jedes lineare Funktional T über D durch ein im allgemeinen nichtstandard Polynom $p(x)$ darstellen:

$$(0.2) \quad (T, \varphi) = \int_{*R} p(x) \varphi(x) dx.$$

Da $p(x)$ in einer Predistribution liegt, folgt aus (0.1) und (0.2), daß die Menge der Predistributionen und die Menge der linearen Funktionale über D algebraisch isomorph sind.

Wir sagen, daß eine Predistribution α eine standard Distribution ist, wenn das durch (0.1) definierte zugehörige lineare Funktional standard-stetig ist.

Im Raum der Predistributionen kann man einige Operationen (vergl. [4], [5], [6]) definieren, u. a. die Differentiation und die unbestimmte Integration: Die Ableitung einer Predistribution α ist die Predistribution γ , $\gamma = \alpha'$, die die Ableitungen der $*$ differenzierbaren Funktionen aus α enthält (jede Predistribution enthält mindestens ein Polynom, also insbesondere eine $*$ differenzierbare Funktion).

Jede Predistribution β , deren Ableitung α ist, wird unbestimmtes Integral von α genannt; $\beta = \alpha^{(-1)}$ enthält gewisse unbestimmte $*$ Integrale der Funktionen $f_\alpha(x)$ aus α .

1. Konvergenz einer Folge von Predistributionen

Definition 1. Eine Folge von Predistributionen $\{\alpha_n\}$ konvergiert genau dann zur Predistribution α , wenn für alle ω aus $*N - N$ $\alpha_\omega = \alpha$ ist. $*N$ ist die nichtstandard Erweiterung der Menge der natürlichen Zahlen N in $*\mathfrak{M}$. Eine andere Definition ist folgende (vergl. [1]):

Definition 2. Eine Folge von Predistributionen $\{\alpha_n\}$ konvergiert im Raum der Predistributionen, wenn die standard Zahlenfolge

$$\{c_{n,\varphi} = {}^0 \left[\int_{-\infty}^{+\infty} f_n(x) \varphi(x) dx \right]\}$$

für eine beliebige Funktion $\varphi(x)$ aus D im üblichen Sinn konvergiert. Hierbei ist $f_n(x)$ eine beliebige Funktion aus α_n .

SATZ 1. Die Definitionen 1 und 2 sind äquivalent.

BEWEIS. a) Die Folge $\{\alpha_n\}$ konvergiere nach Definition 1. Betrachten wir ein positives ε aus R . Es sei M_ε die innere Menge aller Zahlen n_0 aus *N , für die für alle $n > n_0$ Funktionen $f_n(x)$ aus α_n existieren, so daß

$$\left| {}^0 \left[\int_{-\infty}^{+\infty} f_n(x) \varphi(x) dx \right] - {}^0 \left[\int_{-\infty}^{+\infty} f(x) \varphi(x) dx \right] \right| < \varepsilon$$

ist für eine beliebige Funktion $\varphi(x)$ aus D . Hierbei ist $f(x)$ eine Funktion aus α . In M_ε liegen alle Zahlen n_0 aus ${}^*N - N$. In [4] wurde bewiesen, daß jede nichtleere innere Menge von Zahlen aus *N ein kleinstes Element besitzt. Dieses kleinste Element gehört zu N . Wir bezeichnen es mit n_ε . Damit haben wir folgende Aussage erhalten: Zu jedem $\varepsilon > 0$ gibt es ein $n_\varepsilon \in N$, so daß für jedes n mit $n > n_\varepsilon$ gilt

$$|c_{n,\varphi} - c_\varphi| < \varepsilon.$$

Hier wurde

$$c_\varphi = {}^0 \left[\int_{-\infty}^{+\infty} \varphi(x) f(x) dx \right]$$

gesetzt.

b) Die Folge $\{\alpha_n\}$ konvergiere nach Definition 2. Die Folge $\{c_{n,\varphi}\}$ habe den Grenzwert c_φ . Eine standard Zahlenfolge konvergiert aber genau dann, wenn

$$c_{\omega,\varphi} = {}_1 c_\varphi$$

für alle ω aus ${}^*N - N$ ist. Die Zahlen $c_{\omega,\varphi}$ haben die Darstellung

$$c_{\omega,\varphi} = {}^0 \left[\int_{-\infty}^{+\infty} f_\omega(x) \varphi(x) dx \right]$$

Damit sind die Funktionen $f_\omega(x)$ aus Q und liegen in ein- und derselben Predistribution, die wir α nennen.

Die Konvergenz ist mit der Differentiation und Integration in folgendem Sinn verträglich:

Konvergiert die Folge von Predistributionen $\{\alpha_n\}$ zur Predistribution α , so konvergiert die Folge der Ableitungen $\{\alpha'_n\}$ zur Predistribution α' und eine Folge von unbestimmten Integralen $\{\alpha_n^{(-1)}\}$ zu einem unbestimmten Integral $\alpha^{(-1)}$ von α .

In der Standard-Distributionentheorie gilt folgender Satz, der von mehreren Autoren, so z. B. von MIKUSIŃSKI [3] und BERZ [1] bewiesen wurde. Wir geben hier einen neuen nichtstandard Beweis.

SATZ 2. Eine Folge von standarden Distributionen $\{\alpha_n\}$ konvergiert genau dann im Raum der standarden Distributionen, wenn sich die α_n auf jedem beschränkten standard-Intervall I in folgender Weise charakterisieren lassen:

1° Es sei $\varphi(x)$ eine beliebige Funktion aus D , deren Träger in I liegt. Dann gilt für eine beliebige Funktion $f_n(x)$ aus α_n , daß

$${}^0 \left[\int_{-\infty}^{+\infty} f_n(x) \varphi(x) dx \right] = (-1)^k \int_{-\infty}^{+\infty} F_n(x) \varphi^{(k)}(x) dx$$

ist.

2° k ist eine Zahl aus N und hängt nur von I ab.

3° Die Funktionen $F_n(x)$ sind auf I standarde Funktionen, die dort im standarden Sinne stetig sind und gleichmäßig im standarden Sinn konvergieren.

BEWEIS. a) *Notwendigkeit.* Konvergiere die Folge von standarden Distributionen $\{\alpha_n\}$ gegen die standardde Distribution α . Betrachten wir ein abgeschlossenes standardes Intervall I und eine Funktion $\varphi(x)$ aus D , deren Träger in I liegt. Nach dem sogenannten Darstellungssatz für standardde Distributionen gilt für eine beliebige Funktion $f(x)$ aus α , daß

$${}^0 \left[\int_{*I} f(x) \varphi(x) dx \right] = (-1)^l \int_I F(x) \varphi^l(x) dx$$

ist, wobei l ein Zahl aus N und $F(x)$ eine standardde, im standarden Sinne stetige Funktion ist.

Betrachten wir die innere Menge M_1 aller Zahlen $n \in {}^*N$, für die es Funktionen $f_n(x)$ aus α_n gibt, so daß

$${}^0 \left[\int_{*I} f_n(x) \varphi(x) dx \right] = (-1)^l \int_I F_n(x) \varphi^l(x) dx$$

ist, wobei $F_n(x)$ im standarden Sinne stetige, standardde Funktionen sind. Zur Menge M_1 gehören alle Zahlen aus ${}^*N - N$. Jede nichtleere Menge von Zahlen aus *N besitzt ein kleinstes Element [4]. Dieses kleinste Element muß aus N sein. Wir bezeichnen es mit n_0 . Für die in den Predistributionen α_m , $m < n_0$, enthaltenen Funktionen $f_m(x)$ gilt

$${}^0 \left[\int_{*I} f_m(x) \varphi(x) dx \right] = (-1)^{l_m} \int_I F_m(x) \varphi^{(l_m)}(x) dx,$$

wobei l_m Zahlen aus N , und $F_m(x)$ im standarden Sinne stetige standardde Funktionen sind.

Es sei $k = \text{Max}_{i=1, \dots, m} (l_i, l)$. k ist eine Zahl aus N und für alle $n \in {}^*N$ gilt, daß

$$\int_{-\infty}^{+\infty} f_n(x) \varphi(x) dx = (-1)^k \int_I F_n(x) \varphi^{(k)}(x) dx$$

ist.

Weiterhin gilt folgender Satz [4]: Eine standardde Funktionenfolge $\{f_n(x)\}$ konvergiert genau dann gleichmäßig über einer Menge $M \subset R$, wenn für alle x aus *M gilt

$$f_\omega(x) = {}_1 f_{\omega'}(x) \quad \text{für alle } \omega, \omega' \in {}^*N - N.$$

Die Funktionenfolge $\{F_n(x)\}$ genügt diesem Satz.

b) *Hinlänglichkeit.* Betrachten wir eine beliebige Funktion $\varphi(x)$ aus D . Der Träger von $\varphi(x)$ befindet sich in einem abgeschlossenen standarden Intervall I . Für eine Funktion $f_n(x)$ aus α_n gilt, daß

$$c_{n,\varphi} = {}^0 \left[\int_I f_n(x) \varphi(x) dx \right] = (-1)^k \int_I F_n(x) \varphi^{(k)}(x) dx$$

ist. Die Folge $\{F_n(x)\}$ konvergiert gleichmäßig zu einer standarden im standarden Sinne stetigen Funktion $F(x)$, und daher konvergiert $c_{n,\varphi}$ zu $c_\varphi = (-1)^k \cdot \int_I F(x) \varphi^{(k)}(x) dx$ im standarden Sinne.

2. Der lokale Wert einer Predistribution in einem Punkt

Die in diesem Abschnitt geführten Beweise beruhen teilweise auf Beweisideen, die von A. ROBINSON in [4], Abschnitt 5.3, entwickelt werden. Es sei a ein Element aus *R . Die Menge $\mu(a) = \{x \in {}^*R : x = {}_1 a\}$ heißt *Monade* von a .

Definition 3. Eine Predistribution α hat im endlichen Punkt x_0 aus *R einen lokalen Wert, wenn es zu jedem λ aus $\mu(0)$ eine Funktion $f_\alpha^\lambda(x)$ aus α gibt, so daß für alle x aus einer standarden Umgebung von Null

$$f_\alpha^\lambda(\lambda x + x_0) = {}_1 C$$

ist. Hierbei ist C eine Konstante aus *R .

Definition 4. Eine Predistribution α hat in $+\infty$ einen lokalen Wert, wenn es zu jedem positiven unendlichen r aus *R eine Funktion $f_\alpha^r(x)$ aus α gibt, so daß für alle x aus einer standarden Umgebung von Null gilt

$$f_\alpha^r(x+r) = {}_1 C,$$

wobei C eine Konstante aus *R ist. Die Definitionen 3 und 4 sind unabhängig von der Auswahl der Funktionenmengen $\{f_\alpha^\lambda(x)\}_\lambda$ bzw. $\{f_\alpha^r(x)\}_r$. Das können wir wie folgt, einsehen: Nehmen wir an, daß die Funktionenmengen

$$W_{x_0, C_1} = \{f_\alpha^\lambda(x)\} \quad \text{und} \quad W_{x_0, C_2} = \{\tilde{f}_\alpha^\lambda(x)\}$$

aus α der Definition 3 genügen, wobei für die entsprechenden Konstanten C_1 und C_2 aus *R gelte, daß $C_1 \cong {}_1 C_2$ sei.

Die Funktionen $f_\gamma^\lambda(x) = f_\alpha^\lambda(x) - \tilde{f}_\alpha^\lambda(x)$ befinden sich in der Predistribution γ , die die Funktion $f_\gamma^v(x) \equiv 0$ enthält. Betrachten wir die innere Menge M aller Zahlen v aus *R , für die eine Funktion $f_\gamma^v(x)$ in γ existiert, so daß

$$f_\gamma^v(vx + x_0) > \frac{1}{2} (C_1 - C_2) = \frac{1}{2} C$$

für alle x aus einer standarden Umgebung von Null mit dem Radius ε ist. In M befinden sich alle infinitesimalen Zahlen. Da aber eine Monade keine innere Menge

ist [4], muß in M mindestens eine Zahl $a \neq 0$, a aus R , liegen. Es sei $\varphi(x)$ eine Funktion aus D mit

$$\text{supp } \varphi(x) \subset (-a\varepsilon + x_0, a\varepsilon + x_0),$$

$$\varphi(x) \geq 0, \quad \int_{-\infty}^{+\infty} \varphi(x) dx = 1.$$

Für diese Funktion $\varphi(x)$ gilt, daß

$$\int_{-\infty}^{+\infty} f_{\gamma}^a(x) \varphi(x) dx > \frac{1}{2} C$$

ist, was im Widerspruch zu der Annahme steht, daß γ die 0-Distribution ist. Nehmen wir weiterhin an, daß die Funktionenmengen

$$W_{\infty, C_1} = \{f_{\alpha}^r(x)\} \quad \text{und} \quad W_{\infty, C_2} = \{\tilde{f}_{\alpha}^r(x)\}$$

der Definition 4 genügen, wobei für die entsprechenden Konstanten C_1 und C_2 aus *R gelte, daß $C_1 - C_2 = C \neq 0$ sei. Die Funktionen $f_{\gamma}^r(x) = f_{\alpha}^r(x) - \tilde{f}_{\alpha}^r(x)$ befinden sich in der Predistribution γ , die die Funktion $f_{\gamma}(x) \equiv 0$ enthält. Betrachten wir die innere Menge M aller Zahlen ν aus *R , für die eine Funktion $f_{\gamma}^{\nu}(x)$ in γ existiert, so daß

$$f_{\gamma}^{\nu}(x + \nu) > \frac{1}{2} C$$

für alle x aus einer standarden Umgebung von Null mit dem Radius ε ist.

In M befinden sich alle positiven unendlich großen Zahlen aus *R . Diese bilden jedoch keine innere Menge [4]. Daher gibt es mindestens ein endliches a aus *R in M . Betrachten wir eine Funktion $\varphi(x)$ aus D mit

$$\text{supp } \varphi(x) \subset (-\varepsilon + a, a + \varepsilon), \quad \varphi(x) \geq 0, \quad \int_{-\infty}^{+\infty} \varphi(x) dx = 1.$$

Für diese Funktion $\varphi(x)$ gilt,

$$\int_{-\infty}^{+\infty} f_{\gamma}^a(x) \varphi(x) d\gamma > \frac{1}{2} C$$

was im Widerspruch zu der Annahme steht, daß γ die 0-Distribution ist. Ähnlich kann man zeigen, daß die Konstante C , die in der Definition 3 bzw. 4 auftritt, endlich ist:

Es sei x_0 eine endliche Zahl aus *R . Wir nehmen an, daß C eine positive unendlich große Zahl ist. Betrachten wir die innere Menge M aller Zahlen λ aus *R , für die eine Funktion $f_{\alpha}^{\lambda}(x)$ in α existiert, so daß

$$f_{\alpha}^{\lambda}(\lambda x + x_0) > \frac{1}{2} C$$

für alle x aus einer standarden Umgebung von Null, mit dem Radius ε ist. In M liegt mindestens eine Zahl $a \neq 0$, a aus R . Für eine Funktion $\varphi(x)$ aus D , deren Träger in $(-a\varepsilon + x_0, a\varepsilon + x_0)$ liegt, $\varphi(x) \geq 0$, $\int_{-\infty}^{+\infty} \varphi(x) dx = 1$, erhalten wir, daß

$$\int_{-\infty}^{+\infty} f_a^a(x) \varphi(x) dx > \frac{1}{2} C$$

ist. Da aber $f_a^a(x)$ aus Q ist, erhalten wir einen Widerspruch.

Kommen wir zur Definition 4. Wir nehmen an, daß C eine unendlich große positive Zahl sei. Es sei M die innere Menge aller positiven Zahlen r aus *R , für die eine Funktion $f_r^r(x)$ aus α existiert, so daß

$$f_r^r(x+r) > \frac{1}{2} C$$

für alle x aus einer standarden Umgebung von Null mit dem Radius ε ist. In M liegt eine endliche Zahl a . Betrachten wir eine Funktion $\varphi(x)$ aus D , mit $\varphi(x) \geq 0$, $\int_{-\infty}^{+\infty} \varphi(x) dx = 1$ und $\text{supp } \varphi(x) \subset (-\varepsilon + a, \varepsilon + a)$. Dann ist

$$\int_{-\infty}^{+\infty} f_a^a(x) \varphi(x) dx > \frac{1}{2} C,$$

was im Widerspruch dazu steht, daß $f_a^a(x)$ aus Q ist. Um den lokalen Wert einer Predistribution eindeutig festzulegen, führen wir folgende Definition ein:

Definition 5. Der lokale Wert einer Predistribution α im endlichen Punkt x_0 oder im Punkt ∞ ist, falls er existiert, der Standard-Teil der Konstanten C , die in der Definition 3 bzw. 4 auftritt. Wir schreiben:

$$\alpha|_{x_0} = {}^0[C], \quad \alpha|_{\infty} = {}^0[C].$$

Enthält eine Predistribution α eine, im endlichen Punkt x_0 im standarden Sinne stetige Funktion $f(x)$, so hat sie in x_0 den lokalen Wert ${}^0[f(x_0)]$. Für solche Funktionen gilt nämlich, [4], daß $f(x_0+h) = {}_1f(x_0)$ ist, für alle h aus der Monade von 0. A. ROBINSON definiert in [4]: Enthält eine Pre-Distribution eine im standarden Punkt x_0 standard-stetige Funktion $f(x)$, so hat sie in x_0 einen lokalen Wert.

Folgende Sätze sind leicht zu beweisen:

Haben die Predistributionen α und β im Punkt x_0 (x_0 ist endlich oder ∞) die lokalen Werte $\alpha|_{x_0}$ und $\beta|_{x_0}$, so hat die Predistribution $\gamma = \alpha \pm \beta$ den lokalen Wert $\alpha|_{x_0} \pm \beta|_{x_0}$. Hat die Predistribution α im endlichen Punkt x_0 den lokalen Wert $\alpha|_{x_0}$, dann hat die Predistribution

$$\beta = \omega(x) \cdot \alpha$$

in x_0 den lokalen Wert $\beta|_{x_0} = \omega(x_0)\alpha|_{x_0}$. Hierbei ist $\omega(x)$ eine Funktion aus C^∞ , der Menge aller standarden beliebig oft im standarden Sinne differenzierbaren Funktionen.

Wir untersuchen nun einige Eigenschaften der lokalen Werte, die ebenfalls in der Standardtheorie gelten und die für die Anwendung von Interesse sind.

SATZ 3. Hat die Predistribution α im endlichen Punkt x_0 einen lokalen Wert, so hat auch jedes unbestimmte Integral im Punkt x_0 einen lokalen Wert.

BEWEIS. Für jedes λ aus $\mu(0)$ befindet sich in α eine Funktion $f_\alpha^\lambda(x)$, so daß

$$f_\alpha^\lambda(\lambda x + x_0) = {}_1 C$$

für alle x aus einer standarden Umgebung von Null, $U_\varepsilon(0)$, ist. Es sei M die innere Menge aller Zahlen v aus *R , für die eine Funktion $f_\alpha^v(x)$ in α existiert, so daß

$$|f_\alpha^v(vx + x_0)| < \frac{3}{2} |C|$$

für alle x aus $U_\varepsilon(0)$ ist. In M befinden sich alle infinitesimalen Zahlen und damit mindestens eine Zahl $a \neq 0$, $a \in R$. Die zugehörige Funktion aus α bezeichnen wir mit $f_\alpha^a(x)$. Es sei $\alpha^{(-1)}$ ein unbestimmtes Integral von α . In $\alpha^{(-1)}$ liegt die Funktion

$$F_{\alpha^{(-1)}}^a(x) = \int_{x_0}^x f_\alpha^a(t) dt + K,$$

wobei K eine Konstante aus *R ist. Für diese Funktion gilt für jedes infinitesimale h , hier $h > 0$,

$$|F_{\alpha^{(-1)}}^a(x_0 + h) - F_{\alpha^{(-1)}}^a(x_0)| = \left| \int_{x_0}^{x_0+h} f_\alpha^a(t) dt \right| < \frac{3}{2} |C| h = {}_1 0$$

Damit ist $F_{\alpha^{(-1)}}^a(x)$ sogar standard-stetig im Punkt x_0 , und $\alpha^{(-1)}$ hat im Punkt x_0 den lokalen Wert ${}^0[F_{\alpha^{(-1)}}^a(x_0)]$. Jedes weitere unbestimmte Integral von α unterscheidet sich von $\alpha^{(-1)}$ nur durch eine Konstante aus R und hat damit im Punkt x_0 ebenfalls einen lokalen Wert.

Gleichzeitig haben wir folgendes Ergebnis erhalten:

Hat die Predistribution α im endlichen Punkt x_0 einen lokalen Wert, so enthält jedes unbestimmte Integral $\alpha^{(-1)}$ von α eine im Punkt x_0 standard-stetige Funktion.

SATZ 4. Enthält eine Predistribution α eine Funktion $f(x)$, die im endlichen Punkt x_0 standard differenzierbar ist und eine endliche Ableitung $f'(x_0)$ besitzt, dann hat die Predistribution α' im Punkt x_0 den lokalen Wert ${}^0[f'(x_0)]$.

BEWEIS. Die Funktion $f(x)$ ist nach Voraussetzung standard-differenzierbar im Punkt x_0 und daher dort standard-stetig. Betrachten wir das Polynom

$$p(x) = f(x_0) + f'(x_0)(x - x_0).$$

Es ist $p(x) - f(x) = {}_1 f(x_0) - f(x) = {}_1 0$ für alle x aus $\mu(x_0)$ und

$$p'(x) = p'(x_0) = f'(x_0).$$

Es sei $F(x)$ eine Stammfunktion von $f(x)$, die in Q liegt. Sie ist * stetig in $[-\omega, +\omega]$, $\omega \in {}^*N - N$, und daher gibt es ein Polynom $q(x)$, so daß

$$f(x) = {}_1 q(x)$$

für alle endlichen x ist. Das Polynom $q(x)$ ist im Punkt x_0 standard-stetig, und $q''(x)$ befindet sich in α' . Bezeichnen wir mit I_λ das Intervall $I_\lambda = [-\lambda\varepsilon + x_0, \lambda\varepsilon + x_0]$, wobei λ aus $\mu(0)$, $\lambda \neq 0$, und ε eine positive Zahl aus R ist. Wir zeigen, daß die Funktionen

$$g_\lambda(x) = \begin{cases} p'(x) & \text{für } x \in I_\lambda \\ q''(x) & \text{sonst} \end{cases}$$

in α' liegen.

Es sei $\varphi(x)$ eine beliebige Funktion aus D . Es gilt

$$\begin{aligned} \int_{-\infty}^{+\infty} g_\lambda(x) \varphi(x) dx &= \int_{-\infty}^{+\infty} g_\lambda^{(-2)}(x) \varphi''(x) dx = \\ &= \int_{I_\lambda} p^{(-1)}(x) \varphi''(x) dx + \int_{*R - I_\lambda} q(x) \varphi''(x) dx = \\ &= \int_{*R - I_\lambda} q(x) \varphi''(x) dx = \int_{*R} q(x) \varphi''(x) dx = \int_{*R} q''(x) \varphi(x) dx. \end{aligned}$$

Die Funktionen $g_\lambda(x)$ genügen der Definition 1.

SATZ 5. Hat die Predistribution α in zwei zueinander infinitesimal benachbarten Punkten x_1 und x_2 die lokalen Werte C_1 und C_2 , dann ist $C_1 = C_2$.

BEWEIS. Nehmen wir an, daß $C_2 - C_1 > 0$ sei. Es sei δ , $\delta > 0$, eine Zahl aus R . Wir betrachten die innere Menge M_δ aller Zahlen $\lambda \in *R$, für die Funktionen $f_{x_1}^\lambda(x)$, $f_{x_2}^\lambda(x)$ in α existieren, so daß

$$C_1 + \delta > f_{x_1}^\lambda(\lambda x + x_1) > C_1 - \delta$$

$$C_2 + \delta > f_{x_2}^\lambda(\lambda x + x_2) > C_2 - \delta$$

für alle x aus einer standarden Umgebung von Null, $U_\varepsilon(0)$, ist. In M_δ liegen alle infinitesimalen Zahlen und damit mindestens eine Zahl a_δ aus R , $a_\delta \neq 0$. Wir wählen nun δ so, daß $C_1 + \delta < C_2 - \delta$ ist. Es sei $\varphi(x)$ eine Funktion aus D , deren Träger in $(-a_\delta\varepsilon + x_1, x_1 + a_\delta\varepsilon) \cap (-a_\delta\varepsilon + x_2, x_2 + a_\delta\varepsilon)$ liegt, $\varphi(x) \geq 0$ und $\int_{-\infty}^{+\infty} \varphi(x) dx = 1$.

Eine solche Funktion existiert immer, da $x_1 =_1 x_2$ ist, und es ist

$$\begin{aligned} \int_{-\infty}^{+\infty} f_{x_2}^{a_\delta}(\lambda x) \varphi(x) dx &< C_1 + \delta \\ \int_{-\infty}^{+\infty} f_{x_2}^{a_\delta}(\lambda x) \varphi(x) dx &> C_2 - \delta. \end{aligned}$$

Da aber die Funktionen $f_{x_1}^{a_\delta}(x)$ und $f_{x_2}^{a_\delta}(x)$ aus der gleichen Predistribution α sind, erhalten wir einen Widerspruch.

SATZ 6. Es seien α und β zwei Predistributionen, die in jedem endlichen Punkt x_0 aus R die lokalen Werte $\alpha|_{x_0}$ bzw. $\beta|_{x_0}$ haben. Wenn $\alpha|_{x_0} = \beta|_{x_0}$ für jedes endliche x_0 aus R ist, dann ist $\alpha = \beta$.

BEWEIS. Die Predistribution $\gamma = \alpha - \beta$ hat in jedem endlichen Punkt x aus R den lokalen Wert 0. Wir müssen zeigen, daß γ die 0-Distribution ist, d. h., daß für jede Funktion $k(x)$ aus γ gilt,

$$\int_{-\infty}^{+\infty} k(x) \varphi(x) dx =_1 0 \quad \text{für alle } \varphi(x) \text{ aus } D.$$

Betrachten wir eine beliebige Funktion $\varphi(x)$ aus D . Ihr Träger befindet sich in einem endlichen Intervall $[a, b]$, (a, b sind aus R). Betrachten wir einen beliebigen Punkt x_0 aus R mit $a \leq x_0 \leq b$. Es sei λ eine Zahl aus $\mu(0)$. In γ befindet sich eine Funktion $f_{x_0}^\lambda(x)$ mit der Eigenschaft, daß

$$f_{x_0}^\lambda(\lambda x + x_0) =_1 0$$

ist.

Wir betrachten für ein $\delta > 0$, δ aus R , die innere Menge M_δ aller Punkte v aus $*R$, für die eine Funktion $f_{x_0}^v(x)$ in γ existiert, so daß

$$|f_{x_0}^v(vx + x_0)| < \delta$$

für alle x aus einer standarden Umgebung $U_{\varepsilon_{x_0}}(0)$ von Null mit dem Radius ε_{x_0} ist. In M_δ befinden sich alle infinitesimalen Zahlen und damit mindestens eine Zahl $a_{0,\delta}$ aus R , $a_{0,\delta} \neq 0$.

Wir erhalten so zu jedem standarden Punkt x_0 aus $[a, b]$ eine standardde Umgebung mit dem Radius $\varepsilon_{x_0} a_{0,\delta}$, $U_{\varepsilon_{x_0} a_{0,\delta}}(x_0)$, auf der eine Funktion aus γ (z. B. $f_{x_0}^{a_{0,\delta}}(x)$) dem Betrage nach kleiner als eine vorgegebene standardde Zahl $\delta > 0$ ist. Nach dem Heine—Borel'schen Überdeckungssatz bedecken schon endlich viele (im standarden Sinn) dieser Intervalle das Intervall $[a, b]$. Wir können also stets endlich viele Bedeckungsintervalle $U_{\varepsilon_{x_i} a_{i,\delta}}(x_i)$ $i=1, 2, \dots, m$ auswählen, die $[a, b]$ überdecken und deren Gesamtlänge nicht größer als $2(b-a)$ ist.

Jede Grundfunktion $\varphi(x)$ aus D kann in eine Summe von Grundfunktionen zerlegt werden

$$\varphi(x) = \sum_{i=1}^m \varphi_i(x),$$

wobei jede Funktion $\varphi_i(x)$ außerhalb des Intervalls $U_{i,\delta}(x_i) = U_{\varepsilon_{x_i} a_{i,\delta}}(x_i)$ verschwindet und

$$|\varphi_i(x)| \leq \text{Max}_x |\varphi(x)|$$

ist.

Sei $k(x)$ eine beliebige Funktion aus γ . Es ist

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} k(x) \varphi(x) dx \right| &= \left| \int_a^b k(x) \varphi(x) dx \right| = \left| \sum_{i=1}^m \int_{U_{\varepsilon_{x_i} a_{i,\delta}}(x_i)} k(x) \varphi_i(x) dx \right| = \\ &= \left| \sum_{i=1}^m \int_{U_{i,\delta}(x_i)} [k(x) - f_{x_i}^{a_{i,\delta}}(x) + f_{x_i}^{a_{i,\delta}}(x)] \varphi_i(x) dx \right| \leq \\ &\leq \left| \sum_{i=1}^m \int_{U_{i,\delta}(x_i)} [k(x) - f_{x_i}^{a_{i,\delta}}(x)] \varphi_i(x) dx \right| + \left| \sum_{i=1}^m \int_{U_{i,\delta}(x_i)} f_{x_i}^{a_{i,\delta}}(x) \varphi_i(x) dx \right| < \\ &< \eta + \delta \cdot 2(b-a) \text{Max}_x |\varphi(x)|. \end{aligned}$$

Dabei ist η infinitesimal, weil die Funktionen $k(x)$ und $f_{x_i}^{a_i, \delta}(x)$ in der gleichen Predistribution liegen. Die Zahl $\delta > 0$ war beliebig aus R gewählt. Daraus folgt

$$\int_{-\infty}^{+\infty} k(x) \varphi(x) dx = {}_1 0.$$

FOLGERUNGEN aus den Sätzen 4 und 5:

1. Eine Predistribution α hat genau dann in jedem endlichen Punkt einen lokalen Wert, wenn sie eine in jedem endlichen Punkt standard-stetige Funktion enthält.

2. Hat eine Predistribution α in jedem endlichen Punkt x_0 aus *R einen lokalen Wert und ist dieser in den Standardpunkten x_0 aus R bekannt, so ist die Predistribution α eindeutig bestimmt.

3. Hat eine Predistribution α in jedem endlichen Punkt x_0 aus *R einen lokalen Wert und sind diese bis auf eine Menge M von infinitesimalem * Lebesgue-Maße bekannt, so ist α eindeutig bestimmt. Es gibt nämlich zu jedem Punkt $x \in M$ einen Punkt $y \notin M$, der infinitesimal benachbart zu x ist.

Der folgende Satz ist ein hinreichendes und notwendiges Kriterium für die Existenz eines lokalen Wertes einer Predistribution in einem endlichen Punkt.

Wir bezeichnen mit D_ε die Menge aller Funktionen $\varphi(x)$ aus D mit $\varphi(x) \geq 0$, $\int_{-\infty}^{+\infty} \varphi(x) dx = 1$, $\text{supp } \varphi(x) \subset U_\varepsilon(0)$, wobei $U_\varepsilon(0) = \{x: x \in {}^*R, |x| < \varepsilon, \varepsilon \in R_+\}$ ist.

SATZ 7. Eine Predistribution α hat genau dann den lokalen Wert C im endlichen Punkt x_0 , wenn sie für jedes λ aus $\mu(0)$, $\lambda \neq 0$, eine Funktion $f_\alpha^\lambda(x)$ enthält, so daß
 1° $f_\alpha^\lambda(\lambda x + x_0)$ standard-stetig in jedem Punkt x aus $U_\varepsilon(0)$ ist,
 2° für eine beliebige Funktion $\varphi(x)$ aus D_ε gilt, daß

$$\int_{-\infty}^{+\infty} f_\alpha^\lambda(x) \frac{1}{\lambda} \varphi\left(\frac{x-x_0}{\lambda}\right) dx = {}_1 C$$

ist.

BEWEIS. a) *Notwendigkeit.* Wir überprüfen zunächst 1°. Die Predistribution α besitze in x_0 den lokalen Wert C . Dann existiert für jedes λ aus $\mu(0)$, $\lambda \neq 0$, eine Funktion $f_\alpha^\lambda(x)$ in α , so daß

$$f_\alpha^\lambda(\lambda x + x_0) = {}_1 C$$

für alle x aus $U_\varepsilon(0)$ ist. Da $U_\varepsilon(0)$ eine offene Menge ist, liegen mit dem Punkt x auch alle Punkte $x+h$, h aus $\mu(0)$, in $U_\varepsilon(0)$, und daher gilt

$$f_\alpha^\lambda(\lambda(x+h) + x_0) = {}_1 f_\alpha^\lambda(\lambda x + x_0).$$

Die Voraussetzung 2° ist ebenfalls erfüllt: Es sei λ eine Zahl aus $\mu(0)$, $\lambda \neq 0$, $\varphi(x)$ beliebig aus D_ε . Dann gilt für eine Funktion $f_\alpha^\lambda(x) \in \alpha$ nach dem verallgemeinerten

ersten Mittelwertsatz der Integralrechnung, daß

$$\int_{-\infty}^{+\infty} f_{\alpha}^{\lambda}(x) \frac{1}{\lambda} \varphi\left(\frac{x-x_0}{\lambda}\right) dx = \int_{-\infty}^{+\infty} f_{\alpha}^{\lambda}(\lambda x + x_0) \varphi(x) dx = f_{\alpha}^{\lambda}(\lambda \xi + x_0) = {}_1 C,$$

wobei ξ ein Wert aus $U_{\varepsilon}(0)$ ist.

b) *Hinlänglichlichkeit*. Es sei λ eine Zahl aus $\mu(0)$, $\lambda \neq 0$. Dann gilt für eine entsprechende Funktion $f_{\alpha}^{\lambda}(x)$ aus α , daß

$$(1) \quad \int_{U_{\varepsilon}(0)} [f_{\alpha}^{\lambda}(\lambda x + x_0) - C] \varphi(x) dx = {}_1 0$$

mit $\varphi(x)$ aus D_{ε} ist.

Wir nehmen an, daß in einem Punkt ξ aus $U_{\varepsilon}(0)$ $[f_{\alpha}^{\lambda}(\lambda \xi + x_0) - C] \neq {}_1 0$ sei. Infolge der Standard-Stetigkeit von $f_{\alpha}^{\lambda}(\lambda x + x_0)$ in jedem Punkt x aus $U_{\varepsilon}(0)$ gibt es eine Umgebung $U_{\delta}(\xi)$ mit dem Radius $\delta > 0$, δ aus R , so daß

$$[f_{\alpha}^{\lambda}(\lambda x + x_0) - C] > b$$

ist, für alle x aus $U_{\delta}(\xi)$, wobei b eine standard positive Zahl ist. Es sei $\varphi(x)$ eine beliebige Funktion aus D_{ε} , deren Träger in $U_{\delta}(\xi) \cap U_{\varepsilon}(0)$ liegt. Für diese Funktion gilt

$$\int_{U_{\varepsilon}(0)} [f_{\alpha}^{\lambda}(\lambda x + x_0) - C] \varphi(x) dx > b,$$

was im Widerspruch zu (1) steht.

Mit Hilfe von Satz 7 können wir beweisen, daß im standarden Fall die Definition von ŁOJASIEWICZ [2] und unsere Definition gleichwertig sind.

SATZ 8. *Eine standard Distribution α hat genau dann im Sinn von ŁOJASIEWICZ einen lokalen Wert im standarden Punkt x_0 , wenn sie nach Definition 1 einen lokalen Wert hat, und die lokalen Werte stimmen überein.*

BEWEIS. Wir setzen voraus, daß die standard Distribution α im Nullpunkt einen lokalen Wert im Sinn von ŁOJASIEWICZ hat. Es gibt dann, [2], eine Zahl k aus N und eine standard-stetige Funktion $F(x)$, so daß $\lim_{x \rightarrow 0} \frac{F(x)}{x^k} = \frac{c}{k!}$ und α die k -te Ableitung der Distribution β ist, die die Funktion $F(x)$ enthält.

In nichtstandarden Termini gilt für die Funktion $F(x)$, daß

$$\frac{F(\lambda)}{\lambda^k} = {}_1 \frac{c}{k!}$$

für alle infinitesimalen λ ist.

Insbesondere ist für jedes endliche x aus *R

$$\frac{F(\lambda x)}{(\lambda x)^k} = {}_1 \frac{c}{k!}$$

und damit

$$\frac{F(\lambda x)}{\lambda^k} = {}_1 \frac{cx^k}{k!}.$$

Wir betrachten ein festes λ . Es sei P_λ die Menge aller Polynome $p_\lambda(x)$, für die

$$|p_\lambda(x) - F(x)| < \lambda^{k+1}$$

für alle x aus dem Intervall $[-\omega, +\omega]$, $\omega \in {}^*N - N$, ist. Die Polynome $p_\lambda^{(k)}(x)$ befinden sich in α , und es ist

$$\frac{F(\lambda x)}{\lambda^k} =_1 \frac{p_\lambda(\lambda x)}{\lambda^k} =_1 \frac{cx^k}{k!}.$$

Aus der Menge P_λ wählen wir nun ein solches Polynom, daß außerdem

$$p_\lambda^{(k)}(\lambda x) = \left(\frac{p_\lambda(\lambda x)}{\lambda^k} \right)^{(k)} =_1 \left(\frac{cx^k}{k!} \right)^{(k)} = c$$

ist.

Umgekehrt: Die standard Distribution α besitze im Nullpunkt den lokalen Wert C im Sinne der Definition 1. Es sei δ eine positive Zahl aus R . Wir betrachten die innere Menge M_δ aller Zahlen aus *N , für die eine Funktion $f_\alpha^v(x)$ in α existiert, so daß

$$\left| \int_{-\infty}^{+\infty} f_\alpha^{1/v}(x) v\varphi(vx) dx - C \right| < \delta$$

für $\varphi(x)$ aus D_ε ist. Nach Satz 7 liegen in M_δ alle Zahlen aus ${}^*N - N$ (man setze nur $\frac{1}{v} = \lambda$). Das kleinste Element von M_δ ist eine Zahl n_0 aus N . Damit ist das standard Cauchy'sche Konvergenzkriterium erfüllt.

3. Das bestimmte Integral einer Predistribution

Wie in der standarden Theorie, [7], führen wir das bestimmte Integral einer Predistribution ein:

Es sei α eine Predistribution, $\alpha^{(-1)}$ ein unbestimmtes Integral von α . Hat $\alpha^{(-1)}$ in den endlichen Punkten a und b einen lokalen Wert, dann ist α über (a, b) integrierbar, und wir definieren

$$\int_a^b \alpha(dx) = \alpha^{(-1)}|_b - \alpha^{(-1)}|_a.$$

Da sich zwei unbestimmte Integrale von α nur durch eine Konstante unterscheiden, ist die obige Definition unabhängig von der Wahl des unbestimmten Integrals $\alpha^{(-1)}$. Enthält α eine über (a, b) standard Lebesgue-Integrierbare Funktion $f(x)$, so ist

$$\int_a^b \alpha dx = \int_a^b f(x) dx$$

Analog zur Definition 4 können der lokale Wert einer Predistribution α im Punkt $(-\infty)$ und entsprechend die uneigentlichen Integrale $\int_{-\infty}^{+\infty} \alpha dx$, $\int_a^{\infty} \alpha dx$, $\int_{-\infty}^a \alpha dx$

definiert werden. Für das Rechnen mit den bestimmten bzw. uneigentlichen Integralen von Predistributionen gelten analoge Rechenregeln wie im standarden Fall.

Um untersuchen zu können, wann das bestimmte Integral einer Predistribution existiert, definieren wir:

Definition 6. Die Predistribution α ist im endlichen Punkt x_0 beschränkt, wenn für jedes λ aus $\mu(0)$ eine Funktion $f_\alpha^\lambda(x)$ aus α existiert, so daß

$$|f_\alpha^\lambda(\lambda x + x_0)| < C$$

für alle x aus einer standarden Umgebung $U_\varepsilon(0)$ ist; C ist eine Konstante aus R . Diese Definition ist ein Spezialfall von folgender

Definition 7. Die Predistribution α hat im endlichen Punkt x_0 eine integrierbare Majorante, wenn für jedes λ aus $\mu(0)$ eine Funktion $f_\alpha^\lambda(x)$ aus α existiert, so daß

$$|f_\alpha^\lambda(\lambda x + x_0)| < g(\lambda x + x_0)$$

für alle x aus einer standarden Umgebung $U_\varepsilon(0)$ gilt. Dabei ist $g(x)$ eine in einer standarden Umgebung von x_0 , $U_\varepsilon(x_0)$, * Lebesgue-integrierbare Funktion, die ein im Punkt x_0 standard-stetiges unbestimmtes Integral $G(x)$ besitzt.

SATZ 9. Hat eine Predistribution α im endlichen Punkt x_0 eine integrierbare Majorante, dann hat jedes unbestimmte Integral von α in x_0 einen lokalen Wert.

BEWEIS. Wir betrachten die innere Menge M aller Zahlen v aus $*R$, für die eine Funktion $f_\alpha^v(x)$ aus α existiert, so daß

$$|f_\alpha^v(vx + x_0)| < g(vx + x_0)$$

für alle x aus $U_\varepsilon(0)$ ist. In M befinden sich alle infinitesimalen Zahlen und damit mindestens eine Zahl a aus R , $a \neq 0$. Es sei $\alpha^{(-1)}$ ein unbestimmtes Integral von α . In $\alpha^{(-1)}$ befindet sich eine Stammfunktion $F_\alpha^a(x)$ von $f_\alpha^a(x)$:

$$F_\alpha^a(x) = \int_{x_0}^x f_\alpha^a(t) dt + K$$

wobei K eine Konstante aus $*R$ ist.

Für ein h aus $\mu(0)$ ist

$$|F_\alpha^a(x_0 + h) - F_\alpha^a(x_0)| = \left| \int_{x_0}^{x_0+h} f_\alpha^a(t) dt \right| < \int_{x_0}^{x_0+h} g(t) dt = G(x_0 + h) - G(x_0) =_1 0.$$

Folglich ist $F_\alpha^a(x)$ im Punkt x_0 standard-stetig.

Die Definitionen 6 und 7 erinnern uns an die Voraussetzungen des Satzes von Lebesgue über die Vertauschbarkeit der Reihenfolge von Integration und Grenzwertbildung. Wir formulieren hier einen ähnlichen Satz für Predistributionen.

SATZ 10. Die Folge $\{\alpha_n\}$ von Predistributionen konvergiere gegen die Predistribution α . Wenn alle Predistributionen α_n , $n \in N$, gemeinsame, von n unabhängige, integrier-

bare Majoranten in den endlichen Punkten a und b haben, dann existiert

$$\int_a^b \alpha \, dx$$

und es ist

$$\lim_{n \rightarrow \infty} \int_a^b \alpha_n \, dx = \int_a^b \alpha \, dx$$

BEWEIS. Wir zeigen zunächst, daß $\int_a^b \alpha \, dx$ existiert. Die integrierbare Majorante im Punkt a sei $g(x)$, im Punkt b $\tilde{g}(x)$. Für jedes λ aus $\mu(0)$, n aus N , existieren Funktionen $f_{\alpha_n}^\lambda(x) \in \alpha_n$, $\tilde{f}_{\alpha_n}^\lambda(x) \in \alpha_n$, so daß

$$(2) \quad \begin{aligned} |f_{\alpha_n}^\lambda(\lambda x + a)| &< g(\lambda x + a) \\ |\tilde{f}_{\alpha_n}^\lambda(\lambda x + b)| &< \tilde{g}(\lambda x + b) \end{aligned}$$

für alle x aus einer standarden Umgebung $U_\varepsilon(0)$ ist. Halten wir λ fest und betrachten die innere Menge M_λ aller Zahlen $v \in {}^*N$, für die es keine Funktionen $f_{\alpha_v}^\lambda(x)$ und $\tilde{f}_{\alpha_v}^\lambda(x)$ in α_v gibt, die den Ungleichungen (2) genügen. Wenn M_λ leer ist, ist alles gezeigt. Nehmen wir an, daß M_λ nicht leer sei. Jede nichtleere innere Menge von natürlichen Zahlen enthält eine kleinste. Diese kann nicht aus N sein. Bezeichnen wir sie mit ω_λ . Für $\omega = \omega_\lambda - 1$ gibt es Funktionen $f_{\alpha_\omega}^\lambda(x)$ und $\tilde{f}_{\alpha_\omega}^\lambda(x)$ aus $\alpha_\omega = \alpha_{\omega_\lambda} = \alpha$, die den Ungleichungen (2) genügen. Somit haben wir einen Widerspruch erhalten, Nach

Satz 9 existiert $\int_a^b \alpha \, dx$.

Kommen wir zum zweiten Teil des Satzes. Eine Folge $\{\alpha_n^{(-1)}\}$ von unbestimmten Integralen konvergiert zu einem Integral $\alpha^{(-1)}$. Damit ist $\alpha_\omega^{(-1)} = \alpha^{(-1)}$ und

$$\alpha_\omega^{(-1)}|_b - \alpha_\omega^{(-1)}|_a = \alpha^{(-1)}|_b - \alpha^{(-1)}|_a$$

für alle ω aus ${}^*N - N$ auf Grund der Eindeutigkeit der lokalen Werte. Das ist aber gleichbedeutend mit der standarden Konvergenz der Zahlenfolge

$$\{\alpha_n^{(-1)}|_b - \alpha_n^{(-1)}|_a\}_{n \in N} \quad \text{zu} \quad \alpha^{(-1)}|_b - \alpha^{(-1)}|_a.$$

Bemerkung. Wie aus dem Beweis ersichtlich ist, gilt allgemeiner: Konvergiert eine Folge von über (a, b) integrierbaren Predistributionen $\{\alpha_n\}_{n \in N}$ zu einer über (a, b) integrierbaren Predistribution α , dann ist

$$\lim_{n \rightarrow \infty} \int_a^b \alpha_n \, dx = \int_a^b \alpha \, dx.$$

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ÜBER DIE REGULARISIERUNG UND ORDNUNG VON PREDISTRIBUTIONEN

von
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Zusammenfassung

Nicht jede Predistribution hat in einem beschränkten Gebiet eine eindeutig bestimmte Ordnung k , wobei k eine standard oder nichtstandard natürliche Zahl ist. Jede standard (übliche) Funktion, die eine nichtintegrierbare Singularität in einem standarden endlichen Punkt hat, besitzt eine Regularisierung im Bereich der Predistributionen. Es wird der Träger einer Predistribution definiert und die allgemeine Form der Regularisierung angegeben.

0. Es sei $^*\mathfrak{M}$ ein geeignetes nichtstandard Modell höherer Stufe der Analysis, in welchem wir die folgenden Überlegungen ausführen können, und \mathfrak{M} ein standardes Modell der Analysis (Vergl. [2]). Befinden wir uns im Modell $^*\mathfrak{M}$, bezeichnen wir, wenn es zu Verwechslungen führen kann, die entsprechenden Begriffe und Größen mit einem Stern. Größen, Begriffe, Mengen, Funktionen ... nennen wir standard, oder sagen „im standarden Sinne“, wenn sie im Modell \mathfrak{M} definiert sind und in $^*\mathfrak{M}$ beschrieben werden. Mit *N bezeichnen wir die nichtstandard Erweiterung der Menge der natürlichen Zahlen N in $^*\mathfrak{M}$.

A. ROBINSON führte in [2, Abschnitt 5.3] Predistributionen folgendermaßen ein: Es sei Q die Menge aller inneren reellen Funktionen $f(x)$ einer reellen Variablen, die über jedem beschränkten Intervall im * Sinne von Lebesgue quadratisch integrierbar sind, und für die das folgende * Lebesgue-Integral $\int_{^*R} f(x)\varphi(x)dx$ für alle $\varphi(x)$ aus dem Raum D in \mathfrak{M} der reellen Schwartz'schen Grundfunktionen einer reellen Variablen endlich ist. In Q wird eine Äquivalenzrelation eingeführt: Zwei Funktionen $f_1(x)$ und $f_2(x)$ sind genau dann zueinander äquivalent, wenn für alle $\varphi(x)$ aus D gilt

$$\int_{^*R} f_1(x)\varphi(x) dx =_1 \int_{^*R} f_2(x)\varphi(x) dx.$$

Hierbei bedeutet das Zeichen $=_1$ „gleich bis auf eine infinitesimale Größe“, und *R ist die nichtstandard Erweiterung der Menge aller reellen Zahlen in $^*\mathfrak{M}$ [2]. Die so definierten Äquivalenzklassen werden Predistributionen genannt. Wie A. ROBINSON [2] zeigte, definiert jede Predistribution $\alpha = \{f_\alpha(x)\}$ ein lineares Funktional $T = T(\alpha)$ über D im Modell \mathfrak{M} :

$$(0.1) \quad (T, \varphi) = {}^0 \left[\int_{^*R} f_\alpha(x)\varphi(x) dx \right].$$

Hierbei ist $f_\alpha(x)$ eine beliebige Funktion aus der Äquivalenzklasse α . Durch ${}^0[]$ bezeichnen wir den Standardteil einer endlichen Zahl aus *R . Andererseits kann man

jedes lineare Funktional T über D durch ein im allgemeinen nichtstandard Polynom $p(x)$ darstellen:

$$(0.2) \quad (T, \varphi) = \int_{*R} p(x) \varphi(x) dx.$$

Da $p(x)$ in einer Predistribution liegt, folgt aus (0.1) und (0.2), daß die Menge der Predistributionen und die Menge der linearen Funktionale über D algebraisch isomorph sind.

Wir sagen, daß eine Predistribution α eine standard Distribution ist, wenn das durch (0.1) definierte zugehörige lineare Funktional standard-stetig ist.

Im Raum der Predistributionen kann man einige Operationen (vergl. [2], [3], [4]) definieren, u. a. die Differentiation und die unbestimmte Integration: Die Ableitung einer Predistribution α ist die Predistribution γ , $\gamma = \alpha'$, die die Ableitungen der $*$ differenzierbaren Funktionen aus α enthält (jede Predistribution enthält mindestens ein Polynom, also insbesondere eine $*$ differenzierbare Funktion).

Jede Predistribution β , deren Ableitung α ist, wird unbestimmtes Integral von α genannt; β enthält gewisse unbestimmte $*$ Integrale der Funktionen $f_\alpha(x)$ aus α .

1. In der standarden Distributionentheorie gilt der Darstellungssatz: Jede in einem beschränkten Gebiet definierte Distribution ist die k -te Distributionenableitung einer stetigen Funktion. Ein analoger Satz würde in der nichtstandarden Theorie gelten, wenn wir unter Predistributionen $*$ lineare $*$ stetige Funktionale über $*D$ verstehen.

Da aber in unserem Falle Predistributionen lineare Funktionale über D sind, wo also Stetigkeit nicht unbedingt vorhanden ist, ist ein solcher Darstellungssatz im allgemeinen nicht zu erwarten.

Bei formaler Betrachtungsweise erhält man jedoch folgendes Ergebnis:

Jede Predistribution α enthält eine Funktion $f_\alpha(x)$, für die ein k -tes ($k \in *N$) unbestimmtes Integral in einem gegebenen abgeschlossenen beschränkten Gebiet standard-stetig ist.

Der Beweis dieses Satzes beruht darauf, daß jede Predistribution ein Polynom $p(x)$ enthält. Es kann ein k so konstruiert werden, daß ein k -tes unbestimmtes Integral von $p(x)$ in Q liegt und im standarden Sinne stetig ist.

Andererseits ist die k -te Ableitung, k aus $*N - N$ nicht eindeutig bestimmt, wie folgendes Beispiel zeigt: Die Null-Distribution enthält die Funktionen $f_1(x) \equiv 0$, $f_2(x) = \frac{x^k}{k!}$, k aus $*N - N$. Es ist $f_1^{(k)}(x) \equiv 0$, $f_2(x) \equiv 1$.

In diesem Fall geht die Eindeutigkeit der Definition der Ableitung aus folgendem Grund verloren:

SATZ 1. *Es sei k eine Zahl aus $*N - N$. Allein für die Funktion $\varphi(x) \equiv 0$ aus D gilt, daß $\varphi^{(k)}(x)$ auch aus D ist.*

BEWEIS. Wir nehmen an, daß $\varphi(x) \not\equiv 0$ sei. Wir bezeichnen mit a den kleinsten Punkt des Trägers von $\varphi(x)$, für den für alle k aus $*N$ gilt, daß $\varphi^{(k)}(a) = 0$ ist. Der Punkt a ist ein endlicher Punkt aus R . Die Funktion $\varphi(x)$ muß in mindestens einem Punkt x_0 , der von a einen Abstand kleiner als $\frac{1}{4}$ hat, nicht in eine Taylorreihe entwickelbar sein. Daher gibt es keine standard Umgebung des Nullpunktes $U_\varepsilon(0)$ mit dem Radius $\varepsilon > 0$, so daß das n -te Restglied

$$\varphi(x_0 + h) - \varphi(x_0) - \dots - \frac{h^n}{n!} \varphi^{(n)}(x_0) = R_n = \frac{h^{n+1}}{(n+1)!} \varphi^{(n+1)}(x_0 + \theta_n h), \quad \theta_n \in (0, 1),$$

für $n \rightarrow \infty$ für alle h aus $U_\varepsilon(0)$ gegen 0 im standarden Sinne konvergiert. Somit existiert eine standardde natürliche Zahl $k, k > 1$, für die

$$|\varphi^{(k)}(\xi_k)| > k$$

an einer geeigneten Stelle ξ_k aus $U_\varepsilon(x_0)$ ist. Nach dem Mittelwertsatz der Differentialrechnung gibt es einen Punkt ξ_{k+1} aus dem Intervall (ξ_k, a) , so daß

$$(1) \quad |\varphi^{(k+1)}(\xi_{k+1})| = \frac{|\varphi^{(k)}(\xi_k) - \varphi^{(k)}(a)|}{\xi_k - a} > k + 1$$

bei geeigneter Wahl von ε ist. (Man wähle ε z. B. so, daß $\xi_k - a < \frac{1}{2}$ ist). Setzen wir diesen Prozeß fort, indem wir ε festhalten und in Formel (1) k durch $k+1$ ersetzen, so erhalten wir eine standardde Zahlenfolge,

$$\{|\varphi^{(k)}(\xi_k)|\}_{k \in N},$$

die im standarden Sinne monoton gegen $+\infty$ strebt. In Modell $^*\mathfrak{M}$ gilt

$$|\varphi^{(\omega)}(\xi_\omega)| > \omega$$

für alle ω aus $^*N - N$.

Da die Funktion $\varphi^{(\omega)}(x)$ an einer Stelle ξ_ω aus *R dem Betrage nach unendlich groß ist, kann sie sich nicht in D befinden.

Diese Ausführungen zeigen, daß man die Ordnung einer beliebigen Predistribution in einem Gebiet nicht wie im standarden Fall definieren kann.

2. Regularisierung. Unter einer Regularisierung einer standarden Funktion $f(x)$, die im Punkt x_0 eine im standarden Sinne nichtintegrierbare Singularität hat, verstehen wir eine Predistribution $\alpha = \{f_\alpha(x)\}$ für die

$$\int_{^*R} f_\alpha(x) \varphi_0(x) dx =_1 \int_{^*R} f(x) \varphi_0(x) dx$$

ist, für jede Funktion $\varphi_0(x)$ aus D , die in einer Umgebung des Punktes x_0 identisch verschwindet. Im Gegensatz zu den standarden Distributionen gilt für Predistributionen folgender

SATZ 2. Jede standardde Funktion $f(x)$ mit einer nichtintegrierbaren Singularität in einem Punkt $x_0 \in R$ hat eine Regularisierung im Bereich der Predistributionen.

BEWEIS. Es sei D_0 die Menge aller Funktionen $\varphi_0(x)$ aus D , die in einer Umgebung von x_0 verschwinden. D_0 ist ein linearer Teilraum von D . Folglich kann das lineare Funktional T_0 definiert auf D_0 durch

$$(T_0, \varphi_0) = \int_{-\infty}^{+\infty} f(x) \varphi_0(x) dx$$

zu einem linearen Funktional T auf D fortgesetzt werden, so daß

$$(T, \varphi_0) = (T_0, \varphi_0)$$

für alle $\varphi_0(x)$ aus D_0 ist. Solch eine Fortsetzung ist auf Grund der Existenz einer linearen Basis eines linearen Raumes stets möglich. Die zu T gehörige Predistribution α ist eine Regularisierung der Funktion $f(x)$.

Im weiteren benötigen wir den Begriff „Träger einer Distribution“. Um ihn zu definieren, gehen wir von der standarden Definition des Trägers einer standarden Funktion $f(x)$ aus.

Definition 1. Der Punkt x_0 aus *R ist genau dann Nichtträgerpunkt einer Predistribution α , wenn es eine standarde Umgebung $U(x_0)$ von x_0 gibt, auf der eine Funktion $f_\alpha(x)$ aus α identisch verschwindet. Die Komplementärmenge aller Nichtträgerpunkte von α nennen wir den Träger der Predistribution α , und bezeichnen ihn mit $\text{supp } \alpha$.

E. BERZ gibt in [1] eine Definition des Trägers eines linearen Funktionals T über D :

Definition 2 (BERZ [1]). Es sei T ein lineares Funktional über dem Raum D . Der Punkt x_0 aus R ist genau dann Nichtträgerpunkt von T , wenn es eine Umgebung $U(x_0)$ von x_0 derart gibt, daß

$$(T, \varphi_u) = 0$$

für alle $\varphi_u(x)$ aus D mit $\text{supp } \varphi(x) \subset U(x_0)$ ist. Die Komplementärmenge aller Nichtträgerpunkte wird Träger von T genannt.

SATZ 3. Die Definitionen 1 und 2 sind äquivalent.

BEWEIS. a) Es sei der Punkt x_0 aus R Nichtträgerpunkt der Predistribution α im Sinne der Definition 1. Wir zeigen, daß x_0 Nichtträgerpunkt des linearen Funktionals

$$(T_\alpha, \varphi) = {}^0 \left[\int_{-\infty}^{+\infty} f_\alpha(x) \varphi(x) dx \right]$$

im Sinne der Definition 2 ist, wobei $f_\alpha(x)$ eine beliebige Funktion aus α ist.

Es sei $f_\alpha^0(x)$ die Funktion aus α , die auf einer Standard-Umgebung $U(x_0)$ identisch verschwindet. Betrachten wir eine beliebige Funktion $\varphi_u(x)$, deren Träger in $U(x_0)$ liegt. Es ist

$$(T_\alpha, \varphi_u) = {}^0 \left[\int_{-\infty}^{+\infty} f_\alpha^0(x) \varphi_u(x) dx \right] = 0$$

b) Es sei der Punkt x_0 aus R Nichtträgerpunkt des linearen Funktionals T über D im Sinne der Definition 2. Dem Funktional T entspricht eine Predistribution α_T . Wir zeigen, daß x_0 im Sinne der Definition 1 Nichtträgerpunkt von α_T ist. Es sei $p_\alpha(x)$ ein Polynom aus α_T . Wir behaupten, daß die Funktion

$$f_\alpha^0(x) = \begin{cases} p_\alpha(x) & \text{für alle } x \notin {}^*U(x_0) \\ 0 & \text{sonst} \end{cases}$$

ebenfalls in α_T liegt.

Dazu betrachten wir eine beliebige Funktion $\varphi(x)$ aus *D . Wir können $\varphi(x)$ auf $U(x_0)$ durch eine Folge von Funktionen $\varphi_n(x)$ aus D , deren Träger in $U(x_0)$ liegen, derart approximieren, daß die $\varphi_n(x)$ punktweise gegen $\varphi(x)$ auf $U(x_0)$ konvergieren und $|\varphi_n(x)| < C$ sind für alle n aus N , wobei C eine *K -Konstante aus R ist.

Die Funktionen $p_\alpha(x) \cdot \varphi_n(x)$ sind gleichmäßig durch eine Konstante aus *R über ${}^*U(x_0)$ beschränkt. Folglich gilt nach dem * Satz von Lebesgue, daß

$${}^*\lim_{n \rightarrow \infty} \int_{{}^*U(x_0)} p_\alpha(x) \varphi_n(x) dx = \int_{{}^*U(x_0)} p_\alpha(x) \varphi(x) dx$$

ist. Das bedeutet: Für ein $\lambda > 0$, λ aus $\mu(0)$, existiert ein ω_λ aus ${}^*N - N$ derart, daß für alle ω aus *N , $\omega > \omega_\lambda$,

$$\left| \int_{{}^*U(x_0)} p_\alpha(x) \varphi_\omega(x) dx - \int_{{}^*U(x_0)} p_\alpha(x) \varphi(x) dx \right| < \lambda$$

ist. Betrachten wir die standardde Zahlenfolge

$$\{c_n = {}^*0 \left[\int_{{}^*U(x_0)} p_\alpha(x) \varphi_n(x) dx \right]\}_{n \in N}.$$

Da für alle n aus N gilt, daß $c_n = 0$ ist, konvergiert diese Folge im standarden Sinne zum Wert 0. Das ist aber genau dann der Fall, wenn $c_\omega = {}^*0$ für alle ω aus ${}^*N - N$ ist. Daher gilt auch für alle $\omega > \omega_\lambda$, daß

$$\int_{{}^*U(x_0)} p_\alpha(x) \varphi_\omega(x) dx = {}^*1 \int_{{}^*U(x_0)} p_\alpha(x) \varphi(x) dx = {}^*1 \cdot 0$$

ist, und wir erhalten

$$\int_{-\infty}^{+\infty} f_\alpha^0(x) \varphi(x) dx = \int_{{}^*R - {}^*U(x_0)} p_\alpha(x) \varphi(x) dx = {}^*1 \int_{{}^*R} p_\alpha(x) \varphi(x) dx.$$

SATZ 4. Die Predistribution α , die eine Regularisierung einer standarden Funktion $f(x)$ ist, die nur im Punkt x_0 aus R eine nichtintegrierbare Singularität hat, enthält eine Funktion $f_\alpha(x)$ mit der Eigenschaft, daß

$$f_\alpha(x) = f(x)$$

für alle x aus R , $x \neq x_0$, ist.

BEWEIS. Es sei x_1 ein Punkt aus R , $x_1 \neq x_0$. Wir finden stets eine standardde Umgebung von x_0 , $U(x_0)$, die den Punkt x_1 nicht enthält. Wir betrachten ein Polynom $p_\alpha(x)$ aus α und zeigen, daß die Funktion

$$f_\alpha^0(x) = \begin{cases} p_\alpha(x) & \text{für alle } x \text{ aus } {}^*U(x_0) \\ f(x) & \text{sonst} \end{cases}$$

in α enthalten ist. Ähnlich wie im Teil b) des Beweises von Satz 3 stellen wir fest, daß

$$\int_{{}^*R - {}^*U(x_0)} p_\alpha(x) \varphi(x) dx = {}^*1 \int_{{}^*R - {}^*U(x_0)} f(x) dx$$

für eine beliebige Funktion $\varphi(x)$ aus D ist.

SATZ 5. Wenn eine standardde Funktion $f(x)$ in einem endlichen Punkt x_0 aus R eine nichtintegrierbare Singularität besitzt, und wenn die Ableitung $f'(x)$ im standarden Sinne in allen Punkten x aus R , $x \neq x_0$, existiert, dann ist die Ableitung einer Regularisierung von $f(x)$ eine Regularisierung von $f'(x)$.

BEWEIS. Es sei α eine Regularisierung von $f(x)$. Für jede Funktion $f_{\alpha'}(x)$ aus α' und jede Funktion $\varphi_0(x)$ aus D_0 gilt

$$\begin{aligned} \int_{-\infty}^{+\infty} f_{\alpha'}(x) \varphi_0(x) dx &= - \int_{-\infty}^{+\infty} f_{\alpha'}(x) \varphi_0'(x) dx = - \int_{-\infty}^{+\infty} f(x) \varphi_0'(x) dx = \\ &= \int_{-\infty}^{+\infty} f'(x) \varphi_0(x) dx. \end{aligned}$$

Um die allgemeine Form der Regularisierungen zu finden, führen wir den Begriff einer in einem Punkt x_0 aus R konzentrierten Predistribution ein.

Definition 3. Eine Predistribution α ist im Punkt x_0 aus R konzentriert, wenn $\text{supp } \alpha$ keinen weiteren standarden Punkt außer x_0 enthält.

SATZ 6. Es sei α eine Regularisierung der Funktion $f(x)$. Jede weitere Regularisierung β von $f(x)$ erhält man, indem man zu α eine im Punkt x_0 aus R konzentrierte Predistribution γ addiert.

BEWEIS. a) Die den Predistributionen $\beta = \alpha + \gamma$ und α entsprechenden linearen Funktionale T_{β} und T_{α} stimmen auf D_0 überein.

b) Seien α und β zwei Regularisierungen. Für alle Funktionen $\varphi_0(x)$ aus D_0 ist

$$\int_{-\infty}^{+\infty} [f_{\alpha}(x) - f_{\beta}(x)] \varphi_0(x) dx = 0,$$

wobei $f_{\alpha}(x)$ eine Funktion aus α und $f_{\beta}(x)$ eine Funktion aus β ist. Folglich ist jeder Punkt x aus R , $x \neq x_0$, Nichtträgerpunkt der Predistribution $\gamma = \alpha - \beta$.

Ohne Schwierigkeit kann man auch zu Regularisierungen von standarden Funktionen kommen, die abzählbar viele nichtintegrierbare Singularitäten haben, unter der Voraussetzung, daß diese sich im Endlichen nicht häufen. Eine solche Funktion $f(x)$ läßt sich nämlich immer als Summe von Funktionen darstellen, die nur in einem Punkt eine nichtintegrierbare Singularität haben.

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ÜBER KREISRINGE, DIE EINE EILINIE EINSCHLIESSEN

by

I. VINCZE

1. Es seien C eine geschlossene, konvexe ebene Kurve und G das von der Kurve C begrenzte, abgeschlossene Gebiet. Im folgenden beschäftigen wir uns mit Kreisringen, die die Kurve C enthalten und deren Mittelpunkte in G liegen. Es ist bekannt, dass es unter allen erwähnten Kreisringen genau einen einzigen mit minimaler Breite gibt. Führen wir die Bezeichnungen

$$R_p = \max_{Q \in C} \overline{PQ}, \quad r_p = \min_{Q \in C} \overline{PQ}$$

ein, so nimmt die Funktion $O_p = R_p - r_p$ ihren minimalen Wert in einem einzigen Punkte an (s. BONNESEN [1], BONNESEN—FENCHEL [2], KRITIKOS [3]). Der so eindeutig existierende Kreisring heißt der *Minimalkreisring* der Kurve C . Für den Mittelpunkt O des minimalkreisringes gilt die Relation

$$O_0 < O_p, \quad P \neq O, \quad P \in G$$

L. FEJES TÓTH hat die Frage aufgeworfen, ob die Funktion.

$$\tau_p = \frac{R_p}{r_p}, \quad P \in G$$

eine ähnliche Eigenschaft hat. Als Lösung dieses Problems beweisen wir im § 2 den folgenden

SATZ 1. Für die Funktion τ_p gilt die Ungleichheit

$$\tau_0 \leq \tau_p, \quad P \in G$$

wo O den Mittelpunkt des Minimalkreisringes bedeutet; das Gleichheitszeichen kann für $P \neq O$ nur im Falle bestehen, daß die Kurve C zwei Geradenstücke enthält, die sich in einem Punkt des äußeren Kreises des Minimalkreisringes treffen.

Ein ähnliches Resultat gilt für den sogenannten *kleinsten* Kreisring einer Eilinie dessen Flächeninhalt (d. h. $\pi(R_p^2 - r_p^2)$) minimal ist. (s. J. VON SZ. NAGY [4]).

Weiterhin wird im § 3. folgende Behauptung bewiesen:

SATZ 2. Die Niveaulinien der Funktion τ_p sind konvex: wenn für $P_1, P_2 \in G$ $\tau_{P_1} = \tau_{P_2} = \tau$ gilt, so besteht für den Mittelpunkt P_0 der Strecke P_1P_2 die Ungleichung

$$\tau_{P_0} \leq \tau.$$

Zur Formulierung einer Ungleichheit für τ_0 bedienen wir uns der folgenden weiteren Bezeichnungen:

Es bezeichne U den Mittelpunkt und R_U den Halbmesser des Umkreises der Kurve, welcher Kreis die Eigenschaft

$$R_U = \max_{Q \in C} \overline{UQ} \cong R_P = \max_{Q \in C} \overline{PQ}$$

für jedes $P \in G$ besitzt und mit dieser Eigenschaft eindeutig existiert. Ähnlicherweise sei I der Mittelpunkt und r_I der Halbmesser des (oder eines) Inkreises der Kurve C , der also der Relation

$$r_I = \min_{Q \in C} \overline{IQ} \cong r_P = \min_{Q \in C} \overline{PQ}$$

für jeden $P \in G$ genügt.

Die Abstände \overline{IO} und \overline{UO} sollen mit δ und Δ bezeichnet werden. In [5] wurden für die Halbmesser der Kreise des Minimalkreisringes die folgenden Relationen bewiesen:

$$(1.1) \quad R_U \cong R_O \cong R_U \left(1 + 2 \frac{\Delta}{R_U} - 3 \frac{\Delta^2}{R_U^2} \right)^{\frac{1}{2}} \cong \frac{2}{\sqrt{3}} R_U$$

$$(1.2) \quad r_I \cong r_O \cong \frac{R_U}{R_U + \delta} r_I > \frac{1}{2} r_I,$$

wo auch die Ungleichungen genau sind. Aus diesen Relationen folgt der

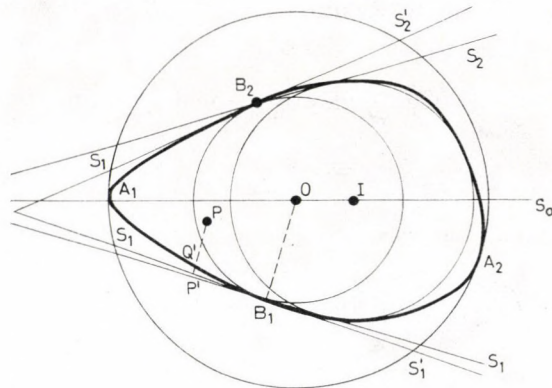


Fig. 1

SATZ 3. Für den minimalen Wert τ_0 der Funktion $\tau_P = \frac{R_P}{r_P}$ gilt die folgende Ungleichung:

$$1 \cong \frac{R_U}{r_I} \cong \tau_0 \cong \frac{R_U}{r_I} \left(1 + 2 \frac{\Delta}{R_U} - 3 \frac{\Delta^2}{R_U^2} \right)^{\frac{1}{2}} \left(1 + \frac{\delta}{R_U} \right) \cong \frac{4}{\sqrt{3}} \frac{R_U}{r_I}.$$

Hier kann das dritte Gleichheitszeichen — nach τ_0 — im Falle $\delta=0$ und/oder $\Delta=0$ gelten, und zwar für die extremalen Figuren, die zu den Relationen 1.1 und 1.2 gehören (siehe [5]).

2. In dem Beweis des Satzes 1 werden wir uns auf den folgenden Hilfssatz stützen:

LEMMA 1. *Es seien K eine Kreislinie mit dem Mittelpunkt M_0 und M ein beliebiges fester Punkt innerhalb K . H_0 und H seien diejenigen Punkte der Kreislinie, die mit M_0 und M auf einer Geraden — in der Reihenfolge H_0M_0MH — liegen. Wenn Q einen beliebigen Punkt von K und φ den absoluten Wert des Winkels $\sphericalangle QMH$ bezeichnen, so ist $r(\varphi) = \overline{QM}$ eine monoton zunehmende Funktion von φ im Intervalle $(0, \pi)$.*

Wir machen weiterhin von der bekannten geometrischen Charakterisierung des Minimalkreisringes Gebrauch (s. BONNESEN—FENCHEL [2]): Es gibt vier Punkte von C , die beim Durchlaufen von C in einer Richtung abwechselnd auf den äußeren und den inneren Kreis des Kreisringes fallen. Der Minimalkreisring ist durch diese Eigenschaft eindeutig bestimmt.

Es werden die folgenden weiteren Bezeichnungen angewendet: K_a und K_i bedeuten den äußeren bzw. inneren Kreis des Minimalkreisringes, die also die Halbmesser R_0 bzw. r_0 besitzen. Es seien $A_1(B_1)$ und $A_2(B_2)$ die gemeinsamen Punkte von $K_a(K_i)$ und C , die auf C in der Reihenfolge $B_1A_1B_2A_2$ liegen. K_i und C besitzen zwei gemeinsame Stützgeraden je im Punkte B_1 und B_2 , die mit s_1 und s_2 bezeichnet werden. Es bezeichne ω die Größe des (spitzen) Winkels, den s_1 und s_2 einschließen. Nun werden wir zwei Fälle betrachten:

a) $\omega = 0$. In diesem Falle ist K_i gleichzeitig (ein) Inkreis der Kurve C , d.h. es gilt $r_0 = r_i \cong r_p$ für jedes $P \in G$. Aus der Minimaleigenschaft des Punktes O , d.h. aus der Relation

$$R_0 - r_0 \cong R_p - r_p$$

folgt auch

$$\frac{R_0}{r_0} - 1 \cong \frac{R_p}{r_p} - 1, \quad \text{dh} \quad \tau_0 \cong \tau_p$$

b) Im Falle $\omega > 0$ gilt natürlich auch die Ungleichheit

$$\omega \cong \arcsin \frac{r_0}{R_0} < \frac{\pi}{2} \quad \text{dh} \quad (\sin \omega)^{-1} \cong \frac{R_0}{r_0} = \tau_0.$$

Die Geraden s_1 und s_2 besitzen eine Winkelhalbierende s_0 ; das Gebiet G wird nun durch die Strecken OB_1 und OB_2 und durch die Gerade s_0 in vier „Quadranten“ geteilt. Aus Symmetriegründen genügt es, sich mit zwei von diesen, die auf dieselbe Seite von s_0 fallen, zu beschäftigen. Die Gerade s_j ($j=0, 1, 2$) soll den Kreis in den Punkten S_j und S'_j schneiden, wo wir für die Bogen $\widehat{S_1S_2}$ und $\widehat{S'_1S'_2}$, die innerhalb des Winkels ω liegen, annehmen, daß der erste von diesen den Punkt A_1 von C und der zweite den Punkt A_2 von C enthält; ferner soll der erste der größere sein. Wir werden uns nun mit zwei Fällen beschäftigen, je nachdem der Punkt P von G 1) zum Viereck $B_1OS'_0S'_1$ oder 2) zu $S_1S_2OB_1$ gehört.

b.1 Die Projektion von P auf s_1 sei P' , und es bezeichne Q' denjenigen Punkt von C , in welchem die Strecke PP' die Kurve schneidet. Es besteht

$$\overline{PP'} > \overline{PQ'} > \min_{Q \in C} \overline{PQ} = r_p.$$

Nach unserem Lemma und nach der Definition von $A_1 \in C$ besteht weiterhin die Relation

$$\overline{PS_1} \cong \overline{PA_1} \cong \max_{Q \in C} \overline{PQ} = R_P.$$

Wegen der einfachen Tatsache

$$0 < \sphericalangle PS_1B_1 \cong \sphericalangle OS_1B_1 < \frac{\pi}{2}$$

können wir die folgende Ungleichung schreiben:

$$\tau_P = \frac{R_P}{r_P} \cong \frac{\overline{PS_1}}{\overline{PP'}} \cong \frac{\overline{OS_1}}{\overline{OB_1}} = \frac{R_O}{r_O} = \tau_O,$$

was in diesem Falle unsere Behauptung ergibt.

b.2 Wenn der Punkt P dem Viereck $S_1S_0OI_1$ angehört, so besteht für seine Projektion P' auf s_1 und für $Q' \in C \cap \overline{PP'}$ wieder

$$(2.1) \quad \overline{PP'} \cong \overline{PQ'} \cong \min_{Q \in C} \overline{PQ} = r_P.$$

Es besteht auch die andere Relation — $A_2 \in C$ —

$$(2.2) \quad \overline{PS'_1} \cong \overline{PA_2} \cong \max_{Q \in C} \overline{PQ} = R_P,$$

wo aber der erste Teil der Ungleichung nicht unmittelbar aus Lemma 1 folgt. Wenn wir nämlich mit M denjenigen Punkt von K_a bezeichnen, in welchem die durch die Punkte P und O gehende Gerade K_a schneidet, so kann M auf dem Bogen $S'_1S'_2$ liegen und A_2 auf den Teil $\widehat{MS_2}$ des Kreises K_a fallen; es liegt aber nahe, daß $\overline{PS'_2} < \overline{PS'_1}$ besteht und daraus folgt schon die Relation

$$\overline{PA_2} \cong \overline{PS_2} \cong \overline{PS'_1}, \quad \text{wo } A_2 \in \overline{AS'_2}.$$

Also gelten (2.1) und (2.2), und unsere Behauptung folgt ebenso wie in b.1.

3. Der Satz 2. ist eine einfache Folgerung aus den Tatsachen, daß R_P eine konvexe und r_P eine konkave Funktion des Punktes P sind; d. h. wenn P_0 den Mittelpunkt der Strecke P_1P_2 bezeichnet, $P_1, P_2 \in G$, dann gilt:

$$R_{P_0} \cong \frac{1}{2} (R_{P_1} + R_{P_2}),$$

$$r_{P_0} \cong \frac{1}{2} (r_{P_1} + r_{P_2}).$$

Wenn also

$$\frac{R_{P_1}}{r_{P_1}} = \frac{R_{P_2}}{r_{P_2}} = \tau$$

ausfällt, so ist

$$\tau_{P_0} = \frac{R_{P_0}}{r_{P_0}} \cong \frac{R_{P_1} + R_{P_2}}{r_{P_1} + r_{P_2}} = \tau = \frac{1}{2} (\tau_{P_1} + \tau_{P_2}),$$

w.z.b.w.

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ON PRODUCTS OF INTEGERS

by

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1. G. HALÁSZ raised the following conjecture (oral communication):

CONJECTURE (G. HALÁSZ). *Let ω be any real number and $k > k_0(\omega)$. Let $A = \{a_1, a_2, \dots, a_n\}$ be any finite, strictly increasing (i.e. $a_1 < a_2 < \dots < a_n$) sequence of positive integers such that it contains the first k positive integers, i.e.*

$$(1) \quad a_1 = 1, \quad a_2 = 2, \dots, a_k = k$$

(consequently, $n \geq k$). *Let us form all the products $a_i a_j$ ($i, j = 1, 2, \dots, n$). Then at least ωn of these products are different.*

(We remark that to get “many” different products, a condition of type (1) is necessary; otherwise e.g. the sequence $A = \{1, 2, 2^2, \dots, 2^{n-1}\}$ is a counterexample, namely for this sequence, the number of the different products is $2n - 1 = O(n)$.)

We will prove the conjecture of HALÁSZ in this paper in the following sharper form:

THEOREM. *Let ϱ be any number such that*

$$(2) \quad 0 < \varrho < \frac{1}{2}$$

and

$$(3) \quad f(\varrho) \stackrel{\text{def}}{=} -\varrho \log \varrho - (1 - \varrho) \log (1 - \varrho) - \left(1 - \frac{\varrho}{2}\right) \log 2 < 0$$

hold. *Let k be any positive integer, sufficiently large depending on ϱ :*

$$(4) \quad k > k_0(\varrho).$$

Let $A = \{a_1, a_2, \dots, a_n\}$ be any finite, strictly increasing sequence of positive integers such that it contains the first k positive integers, i.e. (1) holds. Then there exist at least

$$n \cdot \exp \left\{ \frac{\varrho \log 2}{4} \cdot \frac{\log k}{\log \log k} \right\} \text{ different products of form } a_i a_j \text{ (} i, j = 1, 2, \dots, n \text{).}$$

(It is easy to check that

$$\lim_{x \rightarrow 0^+} f(x) = -\log 2 < 0,$$

and therefore a number ϱ satisfying (2) and (3) exists. It is easy to show that e.g. $f(\frac{1}{4}) < 0$, i.e. $\varrho = \frac{1}{4}$ can be chosen. This means that for $k > k_0$, the number of the different products is at least

$$n \cdot \exp \left\{ \frac{\log 2}{16} \cdot \frac{\log k}{\log \log k} \right\} = n \cdot k^{\frac{\log 2}{16} \cdot \frac{1}{\log \log k}}$$

The sketch of the proof of the theorem is the following:

We are going to reduce our problem to a problem of purely combinatorial nature. The first step of the reduction is given in Section 2, where we reduce the problem practically to the case when all the elements of the given sequence are divisible only by "small" prime factors ("small" depending on k). More exactly, we show that it is sufficient to show that Lemma 1. holds. In Section 3., the case mentioned above, i.e. Lemma 1. is reduced to the even simpler case when the elements of our sequence (composed from small prime factors) are also squarefree. Furthermore, we show in this section that this simple case can be reduced to a combinatorial problem, i.e. to Lemma 2. This lemma is proved in Section 4.

2. PROOF of the theorem. Let us suppose that a sequence A satisfies (1) for some k . Let us denote the i^{th} prime number by p_i and define l by

$$(5) \quad \left(\prod_{i=1}^l p_i \right)^2 \leq k < \left(\prod_{i=1}^{l+1} p_i \right)^2$$

(1) and (5) imply that

$$(6) \quad p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_l^{\varepsilon_l} \in A$$

for any n -tuple $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l)$ such that

$$(7) \quad \varepsilon_1 = 0, 1 \text{ or } 2, \quad \varepsilon_2 = 0, 1 \text{ or } 2, \dots, \varepsilon_l = 0, 1 \text{ or } 2.$$

Furthermore, it is well known that for m tending to infinity,

$$(8) \quad \prod_{i=1}^m p_i = e^{(1+o(1))m \log m}.$$

(5) and (8) imply that for k tending to infinity,

$$e^{2(1+o(1))l \log l} = k,$$

or in equivalent form

$$2(1+o(1))l \log l = \log k.$$

Hence

$$(9) \quad l = \left(\frac{1}{2} + o(1) \right) \frac{\log k}{\log \log k}.$$

Let us write any $a \in A$ in the form

$$a = bc$$

where $p|b$ (p prime) implies that $p \leq p_l$ and $p|c$ (p prime) implies that $p > p_l$.

Let us classify the a_i 's according to the value of b . Let us denote the occurring values of b by b_1, b_2, \dots, b_s ($b_i \neq b_j$ for $i \neq j$) and the values of c_i for that $b_i c_i \in A$ holds (for fixed i , $1 \leq i \leq s$), by $c_1^{(i)}, c_2^{(i)}, \dots, c_{t_i}^{(i)}$ (where $c_1^{(i)} < c_2^{(i)} < \dots < c_{t_i}^{(i)}$). Let

$$C_i = \{c_1^{(i)}, c_2^{(i)}, \dots, c_{t_i}^{(i)}\} \quad (1 \leq i \leq s)$$

and

$$A(b_i) = \{b_i c_1^{(i)}, b_i c_2^{(i)}, \dots, b_i c_{t_i}^{(i)}\} = b_i \{c_1^{(i)}, c_2^{(i)}, \dots, c_{t_i}^{(i)}\} = b_i C_i \quad (1 \leq i \leq s).$$

Then A is, clearly, the disjoint union of the sequences $A(b_i)$:

$$(10) \quad A = \bigcup_{i=1}^s A(b_i)$$

and

$$(11) \quad A(b_i) \cap A(b_j) = \emptyset \quad \text{for } i \neq j.$$

We need the following lemma:

LEMMA 1. Let $q_1 (= q_1(q))$ be any number satisfying

$$(12) \quad (0 <) q < q_1 < \frac{1}{2}$$

and

$$(13) \quad f(q_1) < 0.$$

(Such number q_1 exists because of the definition of q and the continuity of $f(x)$ for $0 < x < 1$.) Let l be a sufficiently large integer, i.e. suppose that

$$(14) \quad l > l_1(q_1)$$

holds. Let b be a positive integer such that $p|b$ (p prime) implies that $p \leq p_l$ (also $b=1$ may occur) and let $C = \{c_1, c_2, \dots, c_t\}$ be a strictly increasing sequence of positive integers such that $p|c_u$ ($1 \leq u \leq t$) implies that $p > p_l$. Let

$$A(b) = \{bc_1, bc_2, \dots, bc_t\} = bC.$$

Then the number of the different products of form

$$(15) \quad (bc_u) \cdot (bc_v) = b^2(c_u c_v)$$

or

$$(16) \quad (bc_u) \cdot p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_l^{\varepsilon_l} = b(c_u p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_l^{\varepsilon_l})$$

(where $\varepsilon_j = 0, 1$ or 2 for every j , $1 \leq j \leq l$) is altogether at least

$$t \cdot \exp \left\{ \frac{q_1 \log 2}{2} l \right\}.$$

(The lemma says practically that our theorem is true when the elements of the given sequence are divisible by prime factors $\leq p_l$ only.)

Let us suppose now that Lemma 1. has been proved. Then we may complete the proof of the theorem in the following way:

Let us apply Lemma 1., replacing b by b_i , C by C_i and $A(b)$ by $A(b_i)$, for every i , $1 \leq i \leq s$. (We may apply the lemma because (14) holds by (4) and (9).) We obtain that the number of the products of form (15) or (16) is at least

$$(17) \quad \sum_{i=1}^s t_i \cdot \exp \left\{ \frac{q_1 \log 2}{2} l \right\} = \exp \left\{ \frac{q_1 \log 2}{2} l \right\} \cdot \sum_{i=1}^s t_i = \\ = n \cdot \exp \left\{ \frac{q_1 \log 2}{2} l \right\}$$

(by (10) and (11)). Furthermore, two products of form (15) are, clearly, different if we start out from different values of b ; in the same way, also products of form (16) corresponding to different values of b are different. Therefore, we may account any product at most twice (once as a product of form (15) and once as a product of form (16); in this case, one of the corresponding b 's must be the square of the other one). This means that the number of the different products of form (15) or (16) is altogether at least the half of the number in (17):

$$(18) \quad \frac{1}{2} n \cdot \exp \left\{ \frac{\varrho_1 \log 2}{2} l \right\} = \frac{1}{2} n \cdot \exp \left\{ \frac{\varrho_1 \log 2}{2} \left(\frac{1}{2} + o(1) \right) \frac{\log k}{\log \log k} \right\} >$$

$$> n \cdot \exp \left\{ \frac{\varrho \log 2}{4} \cdot \frac{\log k}{\log \log k} \right\},$$

by (4) and (9). On the other hand, by (6), (7) and the definition of the $A(b_i)$'s and C_i 's, all the products of form (15) or (16) are products of form $a_u a_v$, hence the number of the different products of form $a_u a_v$ is at least the number in (18), i.e. $n \cdot \exp \left\{ \frac{\varrho \log 2}{4} \cdot \frac{\log k}{\log \log k} \right\}$. This means that, indeed, Lemma 1. implies our theorem.

3. In this section, we will show that Lemma 1. can be reduced to a problem of purely combinatorial nature, formulated in Lemma 2.

PROOF of Lemma 1. Let us suppose that ϱ_1, l, b, c, A satisfy the assumptions of Lemma 1. Let us write every a_i ($1 \leq i \leq t$) in form

$$a_i = bc_i = bp_1^{\alpha_1^{(i)}} p_2^{\alpha_2^{(i)}} \dots p_l^{\alpha_l^{(i)}}$$

(where $\alpha_1^{(i)} \geq 0, \dots, \alpha_l^{(i)} \geq 0$ are integers), and let us divide every $\alpha_j^{(i)}$ by 2:

$$\alpha_j^{(i)} = 2q_j^{(i)} + r_j^{(i)} \quad (j = 1, 2, \dots, l)$$

where $q_j^{(i)} \geq 0$ is integer and $r_j^{(i)} = 0$ or 1. Then the a_i 's can be written in form

$$a_i = bc_i = b \left(p_1^{q_1^{(i)}} p_2^{q_2^{(i)}} \dots p_l^{q_l^{(i)}} \right)^2 p_1^{r_1^{(i)}} p_2^{r_2^{(i)}} \dots p_l^{r_l^{(i)}} \quad (1 \leq i \leq t).$$

Let us classify the a_i 's according to the l -tuples $(q_1^{(i)}, q_2^{(i)}, \dots, q_l^{(i)})$, determined by them. More exactly, a_i and a_j should belong to the same class if and only if

$$q_1^{(i)} = q_1^{(j)}, \quad q_2^{(i)} = q_2^{(j)}, \quad \dots, \quad q_l^{(i)} = q_l^{(j)}.$$

Then we obtain the sequence $A(b)$ as the disjoint union of subsequences of form

$$A(b, d_j) = \{bd_j^2 e_1^{(j)}, bd_j^2 e_2^{(j)}, \dots, bd_j^2 e_{t_j}^{(j)}\} =$$

$$= bd_j^2 \{e_1^{(j)}, e_2^{(j)}, \dots, e_{t_j}^{(j)}\} = bd_j E_j \quad (1 \leq j \leq u)$$

where the prime factors of b are greater than p_1 , the prime factors of d_j and the $e_i^{(j)}$'s are $\leq p_l$, the $e_i^{(j)}$'s are squarefree, and as we said,

$$(19) \quad A(b) = \bigcup_{j=1}^u A(b, d_j)$$

and

$$(20) \quad A(b, d_i) \cap A(b, d_j) = \emptyset \quad \text{for } i \neq j.$$

(19) and (20) imply that the number of the elements of the $A(b, d_j)$'s is altogether equal to the number of the elements of $A(b)$, i.e.

$$(21) \quad \sum_{j=1}^u t_j = t.$$

Let

$$(22) \quad \varphi(l) = \frac{1}{2} \cdot \exp \left\{ -\frac{q_1 \log 2}{2} l \right\} = 2^{-\frac{q_1}{2} l - 1}.$$

We have to distinguish two cases.

Case 1. Let us suppose that

$$(23) \quad \sum_{t_j > \varphi(l) 2^l} t_j < \frac{t}{2}.$$

Let us pick up one element of each subsequence $A(b, d_j)$, say

$$bd_j^2 e_1^{(j)} = bd_j^2 p_1^{\beta_1^{(j)}} p_2^{\beta_2^{(j)}} \dots p_l^{\beta_l^{(j)}},$$

and form all the products

$$(24) \quad (bd_j^2 e_1^{(j)}) \cdot p_1^{\gamma_1^{(j)}} p_2^{\gamma_2^{(j)}} \dots p_l^{\gamma_l^{(j)}}$$

where

$$\gamma_i^{(j)} = \begin{cases} 0 & \text{or } 1, & \text{if } \beta_i^{(j)} = 1, & \text{and} \\ 1 & \text{or } 2, & \text{if } \beta_i^{(j)} = 0. \end{cases}$$

Then, by the definition of the $\gamma_i^{(j)}$'s, we obtain products of form

$$(bd_j^2 p_1 p_2 \dots p_l) p_1^{\delta_1} p_2^{\delta_2} \dots p_l^{\delta_l},$$

where $\delta_i = 0$ or 1 for every i ($1 \leq i \leq l$). A number of this form uniquely determines the exponent of p_i in the canonical form of d_j for each i , and thus it determines also d_j itself. This implies that *all the products of form (24) are different*. The number of these products is, clearly, the number of the possible choices of j , multiplied by the number of the l -tuples $(\gamma_1^{(j)}, \gamma_2^{(j)}, \dots, \gamma_l^{(j)})$: $u \cdot 2^l$. On the other hand, all these products are products of the form (16). Hence the number of the different

products of form (16) is (by (21) and (23)) at least

$$\begin{aligned} u \cdot 2^l &= \left(\sum_{j=1}^u 1 \right) 2^l = \left(\sum_{t_j \leq \varphi(l)2^l} 1 + 1 \sum_{t_j > \varphi(l)2^l} 1 \right) 2^l \cong \\ &\cong \left(\sum_{t_j \leq \varphi(l)2^l} 1 \right) 2^l = \sum_{t_j \leq \varphi(l)2^l} 2^l \cong \sum_{t_j \leq \varphi(l)2^l} \frac{t_j}{\varphi(l)} = \\ &= \frac{1}{\varphi(l)} \cdot \sum_{t_j \leq \varphi(l)2^l} t_j \cong \frac{1}{\varphi(l)} \cdot \frac{t}{2} = 2 \exp \left\{ \frac{\varrho_1 \log 2}{2} l \right\} \cdot \frac{t}{2} = t \cdot \exp \left\{ \frac{\varrho_1 \log 2}{2} l \right\}, \end{aligned}$$

thus the assertion of the lemma holds in this case.

Case 2. Let us suppose that

$$(25) \quad \sum_{t_j > \varphi(l)2^l} t_j \cong \frac{t}{2}.$$

In Case 1., we constructed "many" products of form (16); in this case, we are going to prove, that the number of the products of form (15) is "large".

Let us take some subsequences $A(b, d_i)$, $A(b, d_j)$ and some elements $a_x, a_y \in A(b, d_i)$, $a_v, a_z \in A(b, d_j)$; let

$$(26) \quad \begin{aligned} a_x &= b d_i^2 p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_l^{\varepsilon_l}, & a_y &= b d_i^2 p_1^{\varphi_1} p_2^{\varphi_2} \dots p_l^{\varphi_l}, \\ a_v &= b d_j^2 p_1^{\mu_1} p_2^{\mu_2} \dots p_l^{\mu_l}, & a_z &= b d_j^2 p_1^{\nu_1} p_2^{\nu_2} \dots p_l^{\nu_l}, \end{aligned}$$

where

$$(27) \quad \varepsilon_h = 0 \quad \text{or} \quad 1, \quad \varphi_h = 0 \quad \text{or} \quad 1, \quad \mu_h = 0 \quad \text{or} \quad 1$$

and

$$v_h = 0 \quad \text{or} \quad 1 \quad \text{for} \quad h = 1, 2, \dots, l.$$

We assert that the products

$$(28) \quad a_x a_y = b^2 d_i^4 p_1^{\varepsilon_1 + \varphi_1} \dots p_l^{\varepsilon_l + \varphi_l}$$

and

$$(29) \quad a_v a_z = b^2 d_j^4 p_1^{\mu_1 + \nu_1} \dots p_l^{\mu_l + \nu_l}$$

are different for $i \neq j$. Namely, there exists at least one prime power p_h^{ξ} such that, say, p_h^{ξ}/d_i but $p_h^{\xi} \nmid d_j$ (or conversely), because $d_i \neq d_j$. The exponent of p_h is at least 4ξ in the canonical form of (28) but at most $(4\xi - 2)$ in the canonical form of (29), hence these products must be different, indeed.

Thus it is sufficient to show that the number of the different products $a_v a_z$ of form (29) (where $a_v \in A(b, d_j)$, $a_z \in A(b, d_j)$ for the same j) is "large" for each fixed j , for that

$$(30) \quad t_j > \varphi(l)2^l$$

holds.

The numbers a_v, a_z of form (26) are uniquely determined by the l -tuples $(\mu_1, \mu_2, \dots, \mu_l)$, resp. (v_1, v_2, \dots, v_l) (because of (27)). In the same way, the product (29) is, clearly, uniquely determined by the l -tuple $(\mu_1 + v_1, \dots, \mu_l + v_l)$; we may call this l -tuple as the sum of the l -tuples (μ_1, \dots, μ_l) and (v_1, \dots, v_l) :

$$(31) \quad (\mu_1, \dots, \mu_l) + (v_1, \dots, v_l) \stackrel{\text{def}}{=} (\mu_1 + v_1, \dots, \mu_l + v_l).$$

Applying this terminology, we have to prove the purely combinatorial fact that if we have "many" l -tuples (μ_1, \dots, μ_l) satisfying $\mu_h = 0$ or 1 for $h=1, 2, \dots, l$ (more exactly, we have more than $\varphi(l)2^l$ of these l -tuples by (30)), then there are "many" different sums of form (31), determined by them.

Indeed, we will prove the following lemma in the next section ($|S|$ denotes the number of the elements of the finite set S):

LEMMA 2. Let q_1 be any real number, satisfying (12) and (13), l any integer, sufficiently large depending on q_1 :

$$(32) \quad l > I_2(q_1).$$

Let S denote the set of the 2^l l -tuples $(\mu_1, \mu_2, \dots, \mu_l)$, satisfying $\mu_h = 0$ or 1 for $h=1, 2, \dots, l$. Let R be any subset of S for that

$$(33) \quad |R| > \varphi(l)2^l$$

(where $\varphi(l)$ is defined by (22)). Then the number of the different sums of form

$$(\mu_1 + v_1, \dots, \mu_l + v_l) = (\mu_1, \dots, \mu_l) + (v_1, \dots, v_l),$$

where $(\mu_1, \dots, \mu_l) \in R$ and $(v_1, \dots, v_l) \in R$, is greater than $[\varphi(l)]^{-1}|R|$.

Let us suppose that Lemma 2. has been proved. Then the proof of Lemma 1. in Case 2. can be completed in the following way:

Let us fix a j satisfying (30). Let us denote by R the set of the l -tuples $(\mu_1, \mu_2, \dots, \mu_l)$ for that $a_v = b d_j^2 p_1^{\mu_1} p_2^{\mu_2} \dots p_l^{\mu_l} \in A(b, d_j)$. Then we may apply Lemma 2. for this set R , because (32) holds for sufficiently large k by (9) and (33) holds by (30). We obtain that the number of the different l -tuples of form $(\mu_1 + v_1, \dots, \mu_l + v_l)$ (where $(\mu_1, \dots, \mu_l) \in R$ and $(v_1, \dots, v_l) \in R$), i.e. the number of the products $a_v a_z$ of form (29) (where $a_v \in A(b, d_j)$, $a_z \in A(b, d_j)$) is greater than $[\varphi(l)]^{-1}|R| = [\varphi(l)]^{-1}t_j$ for any fixed j . On the other hand, as we remarked, the products of form (29) were different for different j 's, thus the number of the different products of form (29) is altogether greater than

$$\sum_{t_j > \varphi(l)2^l} [\varphi(l)]^{-1}t_j = [\varphi(l)]^{-1} \sum_{t_j > \varphi(l)2^l} t_j \cong [\varphi(l)]^{-1} \frac{t}{2} = t \cdot \exp \left\{ \frac{q_1 \log 2}{2} l \right\},$$

which proves Lemma 1. also in Case 2.

4. To complete the proof of the theorem, we have to prove Lemma 2.

PROOF of Lemma 2. Let us fix an l -tuple $\mu = (\mu_1, \dots, \mu_l)$. Let us take all the l -tuples $v = (v_1, \dots, v_l)$ such that the number of the i 's satisfying $\mu_i = v_i$ is greater than $q_1 l + 5$:

$$(34) \quad \sum_{i: \mu_i = v_i} 1 > q_1 l + 5.$$

To give a lower estimation for the number of these l -tuples ν , we have to give an upper estimation for the number of the l -tuples ν satisfying the opposite inequality:

$$(35) \quad \sum_{i: \mu_i = \nu_i} 1 \leq \varrho_1 l + 5.$$

If (35) holds, then

$$\sum_{i: \mu_i = \nu_i} 1 = j$$

for some

$$0 \leq j \leq \varrho_1 l + 5.$$

If j is fixed, then the l -tuple ν is uniquely determined by the i 's for that $\mu_i = \nu_i$ holds. These i 's can be chosen in $\binom{l}{j}$ ways. Thus the number of the l -tuples ν satisfying (35) is

$$\sum_{j \leq \varrho_1 l + 5} \binom{l}{j}$$

(for fixed μ).

Applying the Stirling formula, this sum can be estimated in the following way (with respect to (12)):

$$\sum_{j \leq \varrho_1 l + 5} \binom{l}{j} < (\varrho_1 l + 6) \binom{l}{[\varrho_1 l + 5]} = \exp \{(-\varrho_1 \log \varrho_1 - (1 - \varrho_1) \log (1 - \varrho_1) + o(1))l\}.$$

We assert that this expression, i.e. the number of the l -tuples satisfying (35) is less than $\frac{|R|}{2}$, for sufficiently large l :

$$(36) \quad \exp \{(-\varrho_1 \log \varrho_1 - (1 - \varrho_1) \log (1 - \varrho_1) + o(1))l\} < \frac{|R|}{2} \quad \text{for } l > l_3.$$

By (33), it is sufficient to show that

$$\begin{aligned} & \exp \{(-\varrho_1 \log \varrho_1 - (1 - \varrho_1) \log (1 - \varrho_1) + o(1))l\} < \\ & < \frac{1}{2} \varphi(l) 2^l = \frac{1}{2} \cdot \frac{1}{2} \exp \left\{ -\frac{\varrho_1 \log 2}{2} l \right\} \cdot \exp \{(\log 2)l\} = \\ & = \exp \left\{ -\log 4 + \left(1 - \frac{\varrho_1}{2}\right) \log 2 \cdot l \right\} \end{aligned}$$

or

$$\exp \{[f(\varrho_1) + o(1)]l + \log 4\} < 1,$$

i.e.

$$[f(\varrho_1) + o(1)]l + \log 4 < 0.$$

But the limit of the left hand side is $-\infty$ by (13), thus this inequality holds for $l > l_3$, which proves (36).

Thus the number of the l -tuples satisfying (35) is less than $\frac{|R|}{2}$, i.e. the number of the l -tuples satisfying (34) is greater than $|R| - \frac{|R|}{2} = \frac{|R|}{2}$, for $l > l_3$. This means that, for any fixed μ , the l -tuple ν satisfying (34) can be chosen in more than $\frac{|R|}{2}$

ways. Furthermore, μ can be chosen in $|R|$ ways, hence the number of the ordered pairs (μ, ν) satisfying (34) is greater than

$$|R| \cdot \frac{|R|}{2} = \frac{|R|^2}{2}.$$

On the other hand, we assert that a sum

$$(37) \quad \mu + \nu = (\mu_1 + \nu_1, \dots, \mu_l + \nu_l),$$

determined by a pair μ, ν satisfying (34), can be obtained only from a "few" different pairs μ, ν . Is, namely, $\mu_i + \nu_i = 0$ or 2 for some i , then μ_i and ν_i are uniquely determined, and the number of these i 's is greater, than $q_1 l + 5$, by (34). The number of the remaining components of μ is less than $l - q_1 l - 5 = (1 - q_1)l - 5$, hence these components can be chosen in less than $2^{(1-e_1)l-5}$ different ways. Is μ given, then (37) uniquely determines ν . Thus if the right hand side of (37) is given, then the ordered pairs μ, ν satisfying (34) and (37) can be chosen in less than $2^{(1-e_1)l-5}$ ways.

Summarizing: the number of the ordered pairs μ, ν satisfying (34), is greater, than $\frac{|R|^2}{2}$, for $l > l_3$, and any sum of form (37) can be obtained from less than $2^{(1-e_1)l-5}$ such pairs. Thus the number of the different sums $\mu + \nu$ is greater than

$$\frac{|R|^2}{2 \cdot 2^{(1-e_1)l-5}} = \frac{|R|^2}{2^{(1-e_1)l-4}}.$$

To prove Lemma 2., we have to show that, for sufficiently large l , the number of the different sums $\mu + \nu$ is greater than $[\varphi(l)]^{-1}|R|$; thus we have to prove that

$$(38) \quad \frac{|R|^2}{2^{(1-e_1)l-4}} > [\varphi(l)]^{-1}|R|,$$

provided that (33) holds.

This inequality can be rewritten in the following way:

$$|R| > 2^{(1-e_1)l-4} [\varphi(l)]^{-1}.$$

By (33), it is sufficient to prove that

$$\varphi(l) 2^l > 2^{(1-e_1)l-4} [\varphi(l)]^{-1},$$

or, in equivalent form,

$$\varphi^2(l) > 2^{-e_1 l - 4}$$

$$\varphi(l) > 2^{-\frac{e_1}{2} l - 2}.$$

By (22), this inequality holds in trivial way. Thus (38) and, consequently, also Lemma 2 is proved.

5. Let k be any positive integer, $A = \{a_1, a_2, \dots, a_n\}$ be any finite, strictly increasing sequence of positive integers, satisfying (1). Let us denote the number of the different products $a_i a_j$ by $f(A, n, k)$ and let

$$F(n, k) = \min_A f(A, n, k),$$

where the minimum is taken for the sequences A satisfying the assumptions above.

Applying this notation, our theorem says that

$$(39) \quad \frac{F(n, k)}{n} > \exp \left\{ \frac{\varrho \log 2}{4} \cdot \frac{\log k}{\log \log k} \right\} \quad (\text{for } k > k_0).$$

Let $B_k = \{1, 2, \dots, k\}$. It is easy to check that

$$(40) \quad c \frac{k^2}{\log k} < f(B_k, k, k) < o(k^2).$$

Let us take, namely, the products $a_i a_j$ such that

$$a_i = tp, \quad a_j = q,$$

where p, q are any prime numbers greater than \sqrt{k} , t is any integer satisfying $1 \leq t \leq \frac{k}{p}$.

It is easy to see that the number of the different products $a_i a_j$ of this form is greater than $c \frac{k^2}{\log k}$, which gives that the first inequality of (40) holds. The second inequality of (40) can be deduced from the fact that "almost all" the products $a_i a_j$ ($a_i \in A, a_j \in A$) have "about" $2 \log \log k$ prime factors; on the other hand, "almost all" the integers exceeding k^2 have "about" $\log \log k^2 \sim \log \log k$ prime factors.

(40) implies that the exponential factor on the right hand side of (39) can not be replaced by ck , but I guess that it can be replaced by $k^{1-\varepsilon} \left(\frac{F(n, k)}{n} \right)$ is, probably, not much less than $\frac{f(B_k, k, k)}{k}$. The reason of that, that our estimation is much weaker, is that we have replaced assumption (1) by (6) and (7). If we start out from the assumption formed by (6) and (7), instead of (1), then our estimation is near the best possible.

Furthermore, I would like to remark that $\frac{F(n, k)}{n}$ is not much greater for fixed k and large n , than for $n=k$. This can be shown by the following construction: let $A = \{a_1, a_2, \dots, a_k\}$ be any strictly increasing sequence of positive integers and let $A^* = \{a_1^*, a_2^*, \dots, a_m^*\}$ be the sequence of the integers of form $p^i a_j$ where p is a fixed prime number greater than a_k , $i=1, 2, \dots, m$, $j=1, 2, \dots, k$, and m is any positive integer. Clearly,

$$\frac{f(A^*, n, k)}{n} < 2 \frac{f(A, k, k)}{k},$$

thus

$$\frac{F(n, k)}{n} < 2 \frac{F(k, k)}{k} \quad \text{for } k|n,$$

hence

$$\frac{F(n, k)}{n} < 4 \frac{F(k, k)}{k} \quad \text{for every } n.$$

Having the same assumptions, one may investigate also the minimal number of the different products of form $a_{i_1} a_{i_2} \dots a_{i_r}$ (r is fixed) or $\prod_{i=1}^n a_i^{\varepsilon_i}$ (where $\varepsilon_i=0$ or 1 for $i=1, 2, \dots, n$); in the second case, the minimal number of the different products should be between

$$n^2 \exp \left\{ c_1 \frac{k}{\log k} \right\} \quad \text{and} \quad n^2 \exp \left\{ c_2 \frac{k}{\log k} \right\},$$

where c_1 and c_2 are positive constants.

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**SPECTRUM, ALMOST-PERIODICITY AND EQUIDISTRIBUTION
MODULO 1**

by

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§. 1. Introduction

The object of this article is to relate the equidistribution (mod 1) of a sequence $A=(\lambda_n)$ to that of certain of its subsequences. Explicit applications will be discussed in the last paragraphs. Particular attention will be given to additive sequences.

§. 2. Almost-periodic functions and related sequences

Consider the complex vector space \mathfrak{Q} of arithmetic functions f such that

$$(1) \quad \|f\| = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} |f(k)| < \infty$$

and let \mathfrak{M} be the quotient space $\mathfrak{Q}/(0)$ where (0) is the set of null functions (i.e. functions f such $\|f\|=0$). The space \mathfrak{M} is known as the Marcinkiewicz space; it is a Banach space, the norm being induced by the semi-norm (1). From now on, we shall always identify any element $f \in \mathfrak{Q}$ with its class $f+(0) \in \mathfrak{M}$.

A trigonometric polynomial P is a finite linear combination of imaginary exponentials:

$$P(x) = \sum_v c_v e(-\alpha_v x)$$

where the summation extends over a finite range, where $e(\xi)$ denotes $\exp(2i\pi\xi)$ and where the c_v 's are complex numbers while the α_v 's are elements of the torus $\mathbf{T}=\mathbf{R}/\mathbf{Z}$. Let A be a nonempty subset of $\mathbf{T}=\mathbf{R}/\mathbf{Z}$. It is obvious that the family of trigonometric polynomials, the exponents α_v of which are to be taken in A , is a sub vector space of \mathfrak{M} . Its closure $\mathfrak{B}(A)$ is by definition *the vector space of almost-periodic functions with exponents in A* . $\mathfrak{B}(\mathbf{T})$ is the classical Besicovitch space of almost-periodic functions. (What we name "exponent" is usually known as elements of the "spectrum"; but as we shall later introduce a set which is convenient to be called "spectrum", and as we wish to avoid confusion, we prefer to describe A as the set of exponents.)

Let $M=(m_n)$ be a nondecreasing sequence of positive integers. For any integer $k \geq 1$, put

$$\chi_M(k) = \text{card } \{n | m_n = k\}$$

so that if M is strictly increasing, χ_M is the characteristic function of M . If M is not strictly increasing we still call χ_M the characteristic function of M . M is said to be

an A -sequence if the characteristic function χ_M belongs to $\mathfrak{B}(A)$ and if $\|\chi_M\| > 0$. Notice that if M is an A -sequence, it has positive density. Indeed

$$\lim_{n \rightarrow \infty} \frac{n}{m_n} = \lim_{n \rightarrow \infty} \frac{1}{m_n} \sum_{k \leq n} \chi_M(k) = \|\chi_M\| > 0.$$

Note also that $A_1 \subset A_2$ implies $\mathfrak{B}(A_1) \subset \mathfrak{B}(A_2)$ so that an A_1 -sequence is an A_2 -sequence. In particular, any A -sequence is a \mathbf{T} -sequence.

Explicit examples of A -sequences are easy to produce. We list below some important ones.

(i) Arithmetic progressions $(an+b)$ where $a \geq 1$ and $b \geq 0$ are integers, are (\mathbf{Q}/\mathbf{Z}) -sequences.

(ii) Let $\alpha > 0$ be a real number. The sequence $([an])$ is a \mathbf{T} -sequence.

(iii) The ordered set of squarefree integers is a (\mathbf{Q}/\mathbf{Z}) -sequence, and more generally, so are the sequences of r -free integers. For a proof, see MENDÈS FRANCE [4] or NOVOSELOV [5] from which it follows that any increasing integer sequence whose characteristic function is bounded and multiplicative, is a (\mathbf{Q}/\mathbf{Z}) -sequence.

The following notion will be used in the next paragraph. Let A be, as before, a subset of \mathbf{T} , possibly empty and recall that $\mathfrak{B}(A)$ is the set of almost-periodic functions with exponents in A . Consider the family of characteristic functions which belong to $\mathfrak{B}(A)$ and let $\mathfrak{B}'(A)$ denote the closure of the vector space spanned by these characteristic functions. Obviously $\mathfrak{B}'(A) \subset \mathfrak{B}(A)$. Let A^* be any subset of the torus \mathbf{T} such that $\mathfrak{B}(A^*) \subset \mathfrak{B}'(A)$. We then define the two subsets of \mathbf{T}

$$\hat{A} = \bigcup_{q=1}^{\infty} qA$$

and

$$\check{A} = \mathbf{T} \setminus \bigcap_{q=1}^{\infty} \frac{1}{q} (\mathbf{T} \setminus A)^*.$$

Notice that $\hat{\mathbf{T}} = \check{\mathbf{T}} = \mathbf{T}$ and $\hat{\emptyset} = \check{\emptyset} = \emptyset$.

§. 3. The notion of spectrum

In the sequel, we shall often identify a real number with its class modulo 1.

Let $A = (\lambda_n)$ be a given infinite sequence of real numbers. By definition, the spectrum of A is the set

$$\text{sp}(A) = \{\alpha \in \mathbf{T} | (\lambda_n - n\alpha) \text{ not equidistributed (mod 1)}\}.$$

(In a previous article [4], the spectrum had a slightly different definition.)

Metric properties of the spectrum will be discussed later. Observe however that the spectrum has Lebesgue measure 0 (see paragraph 4) so that in the following theorem the set S may be thought as having a 0 Lebesgue measure though in some cases it is convenient to choose S with positive measure.

THEOREM 1. *Let S be a proper subset of \mathbf{T} , possibly empty.*

(i) *If $\text{sp}(A) \subset S$ and if $M = (m_n)$ is any $(\mathbf{T} \setminus S)$ -sequence, then $A_M = (\lambda_{m_n})$ is equidistributed (mod 1).*

(ii) If all $(\mathbf{T} \setminus S)$ -sequences $M = (m_n)$ are such that $A_M = (\lambda_{m_n})$ is equidistributed (mod 1), then $\text{sp}(A) \subset S$.

PROOF. (i) Assume $\text{sp}(A) \subset S$ and let $M = (m_n)$ be any $(\mathbf{T} \setminus S)$ -sequence. We wish to prove that A_M is equidistributed (mod 1). According to Weyl's criterion, it is enough to show that

$$\sigma_n = \frac{1}{n} \sum_{k \leq n} e(q\lambda_{m_k}), \quad q \in \mathbf{Z} \setminus \{0\}$$

tends to zero when n goes to infinity. Let χ_M be the characteristic function of M . Then

$$\sigma_n = \frac{1}{n} \sum_{k \leq m_n} \chi_M(k) e(q\lambda_k).$$

Let P be any trigonometric polynomial in $\mathfrak{B}(\mathbf{T} \setminus S)$. Decompose σ_n into two sums

$$\sigma_n = \frac{1}{n} \sum_{k \leq m_n} (\chi_M(k) - P(k)) e(q\lambda_k) + \frac{1}{n} \sum_{k \leq m_n} P(k) e(q\lambda_k) = \sigma_n^1 + \sigma_n^2.$$

The sum σ_n^2 is a finite linear combination of expressions of the type

$$\sigma_n^3 = \frac{1}{n} \sum_{k \leq m_n} e(-\alpha k) e(q\lambda_k)$$

where $\alpha \in \mathbf{T} \setminus S$. As $M = (m_n)$ is a $(\mathbf{T} \setminus S)$ -sequence, the ratio $\frac{m_n}{n}$ tends to $\|\chi_M\|^{-1}$ so that

$$\sigma_n^3 \sim \|\chi_M\|^{-1} \frac{1}{m_n} \sum_{k \leq m_n} e\left(q\left(\lambda_k - \frac{\alpha}{q}k\right)\right).$$

The hypothesis $\text{sp}(A) \subset S$ implies that the sequence $A - \frac{\alpha}{q}\mathbf{N}$ is equidistributed (mod 1). Hence σ_n^3 tends to 0, and so does σ_n^2 . Thus

$$\limsup_{n \rightarrow \infty} |\sigma_n| = \limsup_{n \rightarrow \infty} |\sigma_n^1| \leq \|\chi_M\|^{-1} \|\chi_M - P\|.$$

Choosing then the polynomial P such that $\|\chi_M - P\| < \varepsilon \|\chi_M\|$, we conclude that σ_n tends to zero.

(ii) It remains now to prove the "converse". Assume that for any $(\mathbf{T} \setminus S)$ -sequence $M = (m_n)$

$$\sigma_n = \frac{1}{n} \sum_{k \leq n} e(q\lambda_{m_k}) = o(1), \quad q \in \mathbf{Z} \setminus \{0\}.$$

As before, we notice that m_n/n tends to a finite limit and therefore the hypothesis can be written

$$\frac{1}{m_n} \sum_{k \leq m_n} \chi_M(k) e(q\lambda_k) = o(1).$$

Using usual techniques, it is easily shown that this relation implies

$$(2) \quad \frac{1}{n} \sum_{k \leq n} \chi_M(k) e(q\lambda_k) = o(1).$$

Recall that the closure $\mathfrak{B}'(\mathbf{T} \setminus S)$ of the vector space spanned by the characteristic functions χ_M when M ranges over the family of all $(\mathbf{T} \setminus S)$ -sequences contains $\mathfrak{B}((\mathbf{T} \setminus S)^*)$. Hence, if $\alpha \in (\mathbf{T} \setminus S)^*$ and if $\varepsilon > 0$ is given, there exists a finite linear combination f of χ_M 's such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} |e(-\alpha k) - f(k)| < \varepsilon.$$

Consider the mean

$$\tau_n = \frac{1}{n} \sum_{k \leq n} e(-\alpha k) e(q\lambda_k) = \frac{1}{n} \sum_{k \leq n} (e(-\alpha k) - f(k)) e(q\lambda_k) + \frac{1}{n} \sum_{k \leq n} f(k) e(q\lambda_k).$$

According to (2), the last expression tends to 0 as n goes to infinity, hence

$$\limsup_{n \rightarrow \infty} |\tau_n| \leq \varepsilon.$$

We have thus proved that for all $\alpha \in (\mathbf{T} \setminus S)^*$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} e \left(q \left(\lambda_k - \frac{\alpha}{q} k \right) \right) = 0.$$

Consequently, if α belongs to the set $\bigcap_{q=1}^{\infty} \frac{1}{q} (\mathbf{T} \setminus S)^* = \mathbf{T} \setminus \check{S}$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} e(q(\lambda_k - \alpha k)) = 0.$$

The sequence $A - \alpha\mathbf{N}$ is thus equidistributed (mod 1) for all $\alpha \in \mathbf{T} \setminus \check{S}$. This proves that $\text{sp}(A) \subset \check{S}$ and completes the proof of the theorem.

We now give some consequences of the theorem:

COROLLARY 1. *If $\text{sp}(A)$ contains no rational numbers, then whatever the (\mathbf{Q}/\mathbf{Z}) -sequence $M = (m_n)$ be, the sequence $\Lambda_M = (\lambda_{m_n})$ is equidistributed (mod 1). In particular, for any integer $a \geq 1$ and any integer $b \geq 0$, the sequence $\Lambda_{a,b} = (\lambda_{an+b})$ is equidistributed (mod 1).*

Conversely, if all the sequences $\Lambda_{a,b}$ ($a \geq 1, b \geq 0$) are equidistributed (mod 1), then $\text{sp}(A)$ contains no rational numbers.

PROOF. Following BÉSINEAU, let $a \geq 1$ and $b \geq 0$ be two given integers and let $\chi_{a,b}$ denote the characteristic function of the arithmetical progression $(an+b)$. The two "reciprocal" identities (Fourier transforms on the additive group $\mathbf{Z}/(a\mathbf{Z})$)

$$\chi_{a,b}(k) = \frac{1}{a} \sum_{l=0}^{a-1} e \left(-\frac{bl}{a} \right) e \left(\frac{kl}{a} \right)$$

$$e \left(\frac{bk}{a} \right) = \sum_{l=0}^{a-1} e \left(\frac{bl}{a} \right) \chi_{a,l}(k)$$

valid for all integers $a \neq 0$, b and k imply the equality $\mathfrak{B}'(\mathbf{Q}/\mathbf{Z}) = \mathfrak{B}(\mathbf{Q}/\mathbf{Z})$ so that $(\mathbf{Q}/\mathbf{Z})^* = \mathbf{Q}/\mathbf{Z}$. Actually, the closure of the vector space spanned by the $\chi_{a,b}$'s is $\mathfrak{B}(\mathbf{Q}/\mathbf{Z})$. Setting $S = \mathbf{T} \setminus (\mathbf{Q}/\mathbf{Z})$ in theorem 1 and using the fact that $S = \hat{S} = \check{S}$, one then concludes that $A_{a,b}$ is equidistributed (mod 1) for all integers $a \equiv 1$, $b \equiv 0$ if and only if $\text{sp}(A) \subset S$, or equivalently, A_M is equidistributed (mod 1) for all (\mathbf{Q}/\mathbf{Z}) -sequences M if and only if $\text{sp}(A) \subset S$.

Remark. In the process of the proof, we have shown that if $A_{a,b}$ is equidistributed (mod 1) for all integers $a \equiv 1$, $b \equiv 0$ then A_M is equidistributed (mod 1) for all (\mathbf{Q}/\mathbf{Z}) -sequences M .

COROLLARY 2. *The two following statements are equivalent:*

- (i) $\text{sp}(A) = \emptyset$,
- (ii) for all \mathbf{T} -sequences $M = (m_n)$, the sequence $A_M = (\lambda_{m_n})$ is equidistributed (mod 1).

PROOF. Put $S = \emptyset$ in the theorem. (See [4].)

Actually, corollary 2 can be restated in a rather striking way. Define $B(A)$ as the set of real numbers x such that the sequence xA is equidistributed (mod 1). Let \mathfrak{C} be the class of \mathbf{T} -sequences.

COROLLARY 3. *For any sequence A*

$$\bigcap_{\alpha \in \mathbf{T}} B(A - \alpha \mathbf{N}) = \bigcap_{M \in \mathfrak{C}} B(A_M).$$

PROOF. The following implications are obvious:

$$\begin{aligned} x \in \bigcap_{\alpha \in \mathbf{T}} B(A - \alpha \mathbf{N}) &\Leftrightarrow xA - x\alpha \mathbf{N} \text{ is equidistributed (mod 1) for all } \alpha, \\ &\Leftrightarrow xA - \alpha \mathbf{N} \text{ is equidistributed (mod 1) for all } \alpha, \\ &\Leftrightarrow \text{sp}(xA) = \emptyset, \end{aligned}$$

and, according to corollary 2,

$$\begin{aligned} &\Leftrightarrow \forall M \in \mathfrak{C}, xA_M \text{ is equidistributed (mod 1),} \\ &\Leftrightarrow x \in \bigcap_{M \in \mathfrak{C}} B(A_M). \end{aligned}$$

§. 4. Empty spectra

It is easy to prove that all sequences A have 0 Lebesgue measure spectra. The result can be improved upon: a trivial consequence of a theorem due to H. WALLIN [7] shows indeed that *whatever the sequence A may be, $\text{sp}(A)$ has Hausdorff dimension 0*. One can furthermore establish that with respect to the Haar measure on the infinite torus $(\mathbf{R}/\mathbf{Z})^{\mathbf{N}}$, *almost all sequences A have empty spectra*, so that the situation described in corollary 2 is actually the most "common" one.

Explicit examples of empty spectrum sequences are simple to list. According to a well known theorem of VAN der CORPUT, any sequence $A = (\lambda_n)$ such that $(\lambda_{n+q} - \lambda_n)$

is equidistributed (mod 1) for all integers $q \geq 1$, has empty spectrum. As a consequence, if $g \geq 2$ is a given integer and if x is g -normal, then $\text{sp}((xg^n)) = \emptyset$. Also, if f is a real polynomial of degree $v \geq 2$

$$f(x) = a_v x^v + \dots + a_2 x^2 + a_1 x + a_0$$

with at least one irrational coefficient among $\{a_2, a_3, \dots, a_v\}$, then the sequence $f = (f(n))$ has an empty spectrum.

By the use of VINOGRADOV's results, G. RAUZY [6] has established that if f is a real entire function, other than a polynomial, and if for $|z|$ infinite

$$|f(z)| = O(\exp(\text{Log } |z|^\delta)), \quad 1 < \delta < \frac{5}{4},$$

then again $\text{sp}(f) = \emptyset$.

Finally if f is a real function with a continuous second derivative, slowly increasing in the following sense:

- (i) For all $\varepsilon > 0$, $f'(x) - f'(ex) = O(1)$
- (ii) $x^2 f''(x)$ tends to $\pm \infty$ when x approaches $+\infty$, then $\text{sp}(f) = \emptyset$.

The proof, which is quite straightforward, depends on VAN DER CORPUT's estimations relating

$$\sum_{a < n \leq b} e(f(n)) \quad \text{to} \quad \int_a^b e(f(u)) du$$

Functions f which verify the above conditions are for example $f(x) = (\text{Log } x)^\delta$ ($\delta > 1$) and $f(x) = x(\text{Log } x)^\delta$ ($\delta \in \mathbf{R} \setminus \{0\}$).

§. 5. Additive functions

A real arithmetic function f is said to be *additive* if for every couple of relatively prime positive integers m, n

$$f(mn) = f(m) + f(n).$$

THEOREM 2. *Let f be a real additive function. Then*

- (i) $\text{sp}(f) \cap (\mathbf{Q}/\mathbf{Z}) = \emptyset \Leftrightarrow (f(n))$ is equidistributed (mod 1).
- (ii) $\text{sp}(f) \cap (\mathbf{Q}/\mathbf{Z}) = \mathbf{Q}/\mathbf{Z} \Leftrightarrow (f(n))$ is not equidistributed (mod 1).

PROOF. (i) If $\text{sp}(f) \cap (\mathbf{Q}/\mathbf{Z}) = \emptyset$, then $0 \notin \text{sp}(f)$ and so $(f(n))$ is equidistributed (mod 1). Conversely, suppose that $(f(n))$ is equidistributed (mod 1). Let q and s be nonzero integers and let r be any integer. Define

$$\sigma_n \left(\frac{r}{s}, q \right) = \frac{1}{n} \sum_{k \leq n} e \left(q \left(f(k) - \frac{r}{s} k \right) \right).$$

It is obvious that

$$\sigma_n \left(\frac{r}{s}, q \right) = \sum_{h=0}^{s-1} \left(\frac{1}{n} \sum_{\substack{k \leq n \\ k \equiv h \pmod{s}}} e(qf(k)) \right) e \left(-\frac{qrh}{s} \right).$$

DELANGE [3] proved that the inner mean tends to zero. Hence $\sigma_n\left(\frac{r}{s}, q\right)$ tends to zero when n increases to infinity. WEYL's criterion then shows that $\left(f(n) - \frac{r}{s}n\right)$ is equidistributed (mod 1) so that finally $\frac{r}{s} \notin \text{sp}(f)$. Part (i) of the theorem is thus established.

(ii) It suffices to prove that if $(f(n))$ is not equidistributed (mod 1) then $\text{sp}(f) \cap \cap(\mathbf{Q}/\mathbf{Z}) = (\mathbf{Q}/\mathbf{Z})$. According to DELANGE [2], the nonequidistribution (mod 1) of $(f(n))$ is equivalent to the fact that

$$\sum_{p \text{ prime}} \frac{1}{p} \sin^2 m\pi(f(p) - t \text{Log } p) < \infty$$

for some nonzero integer m and for some real number t . The inequality

$$(3) \quad \sin^2 hx \leq h^2 \sin^2 x$$

valid for all integers h and all real numbers x , implies

$$\sum_{p \text{ prime}} \frac{1}{p} \sin^2 m\pi(hf(p) - ht \text{Log } p) < \infty$$

so that whatever the integer h is, the sequence $(hf(n))$ is not equidistributed (mod 1). Suppose now that the rational number $\frac{r}{s}$ does not belong to $\text{sp}(f) \cap (\mathbf{Q}/\mathbf{Z})$. Then $\left(f(n) - \frac{r}{s}n\right)$ is equidistributed (mod 1), and so is $(sf(n))$. This contradiction proves the theorem.

COROLLARY 4. *If f is a real additive function then either*

$$\text{sp}(f) = \emptyset \quad \text{or} \quad \text{sp}(f) = \mathbf{Q}/\mathbf{Z}.$$

PROOF. As a consequence of theorem 2, either $\text{sp}(f) \cap (\mathbf{Q}/\mathbf{Z}) = \emptyset$ or $\mathbf{Q}/\mathbf{Z} \subset \text{sp}(f)$. On the other hand DABOUSSI and DELANGE [1] proved that if F is multiplicative and if $|F(n)| \leq 1$, then for all irrational number α ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} F(k) e(-\alpha k) = 0.$$

Choosing $F(k) = \exp 2i\pi qf(k)$, it follows that $\text{sp}(f) \subset \mathbf{Q}/\mathbf{Z}$. Hence $\text{sp}(f)$ is either \emptyset or \mathbf{Q}/\mathbf{Z} .

COROLLARY 5. *Let f be a real additive function. If $f = (f(n))$ is equidistributed (mod 1), then whatever be the \mathbf{T} -sequence $M = (m_n)$, the sequence $(f(m_n))$ is equidistributed (mod 1).*

PROOF. The equidistribution (mod 1) of f implies $\text{sp}(f) = \emptyset$ if f is additive. The equidistribution (mod 1) of $(f(m_n))$ then follows from corollary 2.

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PERMUTATIONS WITH RESTRICTION OF MIDDLE STRENGTH

by

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Introduction

The problem of enumeration of restricted permutations of n elements has been examined by several authors. In this paper we are considered with the (probably most important) special case when the restriction is a bound on the distances between the elements and their images. If this bound is constant the restriction is *strong*; it is *weak* if the bound is $n - o(n)$ (see LEHMER [1]). In the present paper we deal with the case when the bound is between $\frac{n}{2}$ and n . We determine a generating function and by help of it we give several explicite formulas and estimations.

Let $p(n, k, l)$ be the number of permutations π of $1, 2, \dots, n$ satisfying $-(n-l) < i - \pi(i) < n - k$, where $k + l \leq n$. First we give a recursion for $p(n, k, l)$ by help of the almost trivial and well-known transformation of the problem to permanents of matrices. Let us consider the following matrix of order n

$$(1) \quad a_{ij} = \begin{cases} 1 & \text{if } -(n-l) < i-j < n-k \\ 0 & \text{otherwise.} \end{cases}$$

Then the permanent of this matrix is $p(n, k, l)$. We get the recursion by help of expansion by the first column. Expanding we get not-only matrices of (1) type but of the following type:

$$(2) \quad a_{ij} = \begin{cases} 0 & \text{if } i-j \leq -(n-l) \\ 0 & \text{if } i-j \geq n-k \\ 0 & \text{if } (i, j) \in \{(1, n-l), \dots, (t, n-l+t-1)\} \\ 1 & \text{otherwise} \end{cases}$$

Let $p(n, k, l, t)$ be the permanent of this matrix and $p(n, k, l, 0) = p(n, k, l)$. Then the first recursion for $p(n, k, l)$ is:

$$(3) \quad p(n, k, l) = \sum_{t=0}^{l-1} p(n-1, k-1, l-1, t) + (n-k-l)p(n-1, k-1, l)$$

By our assumption $k+l \leq n$ t runs till l . We express the permanent of a matrix of (2) type with the permanent of the matrices of (1) type. We prove the following equality by induction:

$$(4) \quad p(n, k, l, t) = \sum_{v=0}^t (-1)^v \binom{t}{v} p(n-v, k, l-v).$$

In case $t=0$ the equality holds. Since:

$$(5) \quad p(n, k, l, t) = p(n, k, l, t-1) - p(n-1, k, l-1, t-1)$$

(we have used the assumption $(k+l) \leq n$). Hence we have:

$$\begin{aligned} p(n, k, l, t) &= \sum_{v=0}^{t-1} (-1)^v \binom{t-1}{v} [p(n-v, k, l-v) - p(n-1-v, k, l-1-v)] = \\ &= \sum_{v=0}^t (-1)^v p(n-v, k, l-v) \left[\binom{t-1}{v} + \binom{t-1}{v+1} \right] = \\ &= \sum_{v=0}^t (-1)^v p(n-v, k, l-v) \binom{t}{v}. \end{aligned}$$

Putting (4) into (3) and changing the summation, we get the following recursion for $p(n, k, l)$:

THEOREM 1.

$$\begin{aligned} p(n, k, l) &= \sum_{v=0}^{l-1} (-1)^v \binom{l}{v+1} p(n-1-v, k-1, l-1-v) + \\ &\quad + (n-k-l)p(n-1, k-1, l) \end{aligned}$$

It will be convenient to set

$$r = n-k-l, \quad f(r, k, l) = p(n, k, l) = p(r+k+l, k, l).$$

Theorem 1 gives the recursion:

$$(6) \quad f(r, k, l) = \sum_{v=0}^{l-1} (-1)^v \binom{l}{v+1} f(r+1, k-1, l-1-v) + rf(r, k-1, l)$$

Let

$$g_{r,k}(z) = \sum_{l=0}^{\infty} f(r, k, l) \frac{z^l}{l!}.$$

From (6) we get:

$$g_{r,k}(z) = \sum_{l=0}^{\infty} \sum_{v=0}^{l-1} (-1)^v \binom{l}{v+1} f(r+1, k-1, l-1-v) \frac{z^l}{l!} + rg_{r,k-1}(z).$$

Hence

$$(7) \quad \text{Let} \quad g_{r,k}(z) = (1 - e^{-z})g_{r+1, k-1}(z) + rg_{r, k-1}(z).$$

Let

$$G(x, y, z) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} g_{r,k}(z) \frac{x^r y^k}{r! k!}$$

the exponential generating function for $f(r, k, l)$. From (7) we obtain a differential equation for $G(x, y, z)$:

$$G_y(x, y, z) = (1 + x - e^{-z})G_x(x, y, z)$$

with the boundary condition

$$G(x, 0, z) = \frac{1}{1 - xe^z}$$

(which easily follows from the explicit formula $f(r, 0, l) = r! r^l$). Solving the equation we get:

THEOREM 2.

$$(8) \quad G(x, y, z) = \frac{1}{e^y + e^z - (1+x)e^{y+z}}.$$

The generating function of $p(k+l, k, l) = f(0, k, l)$ is

$$G(0, y, z) = \frac{1}{e^y + e^z - e^{y+z}} = \frac{1}{1 - (1 - e^y)(1 - e^z)}.$$

This we transform into the following form

$$\begin{aligned} \frac{1}{1 - (1 - e^y)(1 - e^z)} &= \sum_{m=0}^{\infty} (1 - e^y)^m (1 - e^z)^m = \\ &= \sum_{m=0}^{\infty} \sum_{i=0}^m (-1)^i \binom{m}{i} e^{iy} \sum_{j=0}^m (-1)^j \binom{m}{j} e^{jz}. \end{aligned}$$

Hence we get an explicit formula for $f(0, k, l)$:

THEOREM 3.

$$f(0, k, l) = \sum_{m=0}^{\min(k,l)} S(k, m) S(l, m) (m!)^2.$$

($S(k, m)$ are the Stirling numbers of second kind)

It is interesting that using another transformation we can get an essentially different formula, which extends to the case $r \neq 0$ as well¹:

THEOREM 4.

$$f(r, k, l) = r! \sum_{\lambda=0}^{\min(k,l)} (-1)^{k+\lambda} \lambda! \binom{r+\lambda}{\lambda} (r+\lambda)! S(k+1, \lambda+1).$$

PROOF:

$$G(x, y, z) = \sum_{r=0}^{\infty} \frac{e^{(y+z)r}}{(e^y + e^z - e^{y+z})^{r+1}} x^r.$$

Now we transform the coefficient of x^r :

$$\frac{e^{(y+z)r}}{(e^y + e^z - e^{y+z})^{r+1}} = e^{-y} \sum_{\lambda=0}^{\infty} \binom{\lambda+r}{\lambda} e^{(\lambda+r)z} (1 - e^{-y})^\lambda.$$

¹ I am grateful to Prof. D. H. LEHMER for this observation.

Hence:

$$\frac{f(r, k, l)}{r!k!l!} = \sum_{\lambda=0}^{\infty} \binom{\lambda+r}{\lambda} \frac{(\lambda+r)^l}{\lambda} \sum_{i=0}^{\lambda} (-1)^{i+k} \binom{\lambda}{i} \frac{(i+1)^k}{k!}$$

$$f(r, k, l) = r! \sum_{\lambda=0}^{\infty} \binom{\lambda+r}{\lambda} (\lambda+r)^l (-1)^{k+\lambda} \frac{\lambda!}{(\lambda+1)!} \sum_{i=0}^{\lambda} (-1)^{\lambda+i} \binom{\lambda+1}{i+1} (i+1)^{k+1}$$

from this the identity is easily seen.

We obtain the estimation on $f(r, k, l)$ by help of the Cauchy-formula:

$$\frac{f(r, k, l)}{r!k!l!} = -\frac{1}{4\pi^2} \oint \oint \frac{e^{(y+z)r} dy dz}{(e^y + e^z - e^{y+z})^{r+1} y^{k+1} z^{l+1}}.$$

If we integrate on the circuits $|y|=|z|=a$ we get the estimation:

$$(9) \quad \frac{f(r, k, l)}{r!k!l!} \leq \frac{e^{2ar}}{(2e^a - e^{2a})^{r+1} a^{k+l}} = b(a).$$

Let

$$a = \left(1 - \frac{2(r+1)}{k+l}\right) \log 2$$

(this is near the minimum of $b(a)$).

Substituting a into $b(a)$ and keeping the essential factors only, we get:

THEOREM 5. When r is fixed and $k+l \rightarrow \infty$,

$$f(r, k, l) \leq (1+o(1)) \frac{r!k!l! \left(\frac{k+l}{2}\right)^{r+1}}{4(r+1)^{r+1} (\log 2)^{k+l+r+1}}.$$

Remark. In case $r=0, k=l$ this gives

$$f(0, k, k) \leq c(k!)^2 k \left(\frac{1}{\log^2 2}\right)^k,$$

whence

$$\frac{1}{k} \log \frac{f(0, k, k)}{(k!)^2} \leq \log \frac{1}{\log^2 2} + o(1) \approx \log 2, 1.$$

Trivial estimation gives

$$(k!)^2 < f(0, k, k) < (2k)!,$$

whence

$$0 < \frac{1}{k} \log \frac{f(0, k, k)}{(k!)^2} < \log 4.$$

It would be interesting to determine

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \frac{f(0, k, k)}{(k!)^2}.$$

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ON AN INTERPOLATORY ANALOGON OF THE DE LA VALLÉE-POUSSIN MEANS

by
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Recently, some attention has been paid to the discrete version of some well-known convolution operators. Thus R. BOJANIC and O. SHISHA [1] proved (among others) that interpolatory analogues of the Jackson integral and the Fejér—Korovkin operator approximate in the order $O\left(\omega\left(f, \frac{1}{n}\right)\right)$ where ω is the modulus of continuity of the function $f(x) \in C_{2\pi}$ to be approximated. A further example could be the so-called Jackson polynomial which is a discrete version¹ of the Fejér means, and that approximates $f(x)$ with an error $O\left(\omega\left(f, \frac{1}{n}\right) + \omega\left(\tilde{f}, \frac{1}{n}\right)\right)$ where $\tilde{f}(x)$ is the trigonometric conjugate to $f(x)$ (see [2]).

The aim of the present paper is to “discretize” the de la Vallée-Poussin means, and prove that the operator obtained in this way has the same advantages as the de la Vallée-Poussin means. Moreover, this operator will interpolate to the function in some equally distributed nodes, and in order to determine it, we need only these interpolated values. The latter can be important from computational point of view. The use of the de la Vallée-Poussin means from the point of view of interpolation theory, has already been pointed out by G. FREUD in [3].

Let

$$(1) \quad K_n(t) = \frac{\sin \frac{3nt}{2} \sin \frac{nt}{2}}{2n \sin^2 \frac{t}{2}} = \frac{1}{2} + \sum_{k=1}^n \cos kt + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} \cos kt$$

be the de la Vallée-Poussin kernel. Further let

$$t_k = t_{k,n} = \frac{2\pi k}{3n} \quad (k = 0, 1, \dots, 3n-1)$$

and consider the following operator:

$$(2) \quad V_n(f, x) = \frac{2}{3n} \sum_{k=0}^{3n-1} f(t_k) K_n(t_k - x) \quad (f(x) \in C_{2\pi}).^2$$

THEOREM. (i) $\deg V_n(f, x) \leq 2n-1$.

(ii) $V_n(f, t_k) = f(t_k) \quad (k=0, 1, \dots, 3n-1)$.

¹ We mention that the “discretization” of a convolution operator is not unique, and the order of convergence may depend on the manner we determine the discrete operator as a quadrature sum.

² The importance of this operator has been pointed out by R. DEVORE ([4], p. 17).

(iii) If $T_n(x)$ is an arbitrary trigonometric polynomial of degree at most n then $V_n(T_n, x) \equiv T_n(x)$.

(iv) $\left\| \sum_{k=0}^{3n-1} |K_n(t_k - x)| \right\| \leq \frac{9n}{2}$ where the norm is the usual maximum-norm.

(v) $\|f(x) - V_n(f, x)\| \leq 4E_n(f)$ for all $f(x) \in C_{2\pi}$ where $E_n(f)$ is the best approximation of $f(x)$ by trigonometric polynomials of degree $\leq n$.

PROOF. (i) is clear from the definition of $K_n(t)$. (ii) follows from

$$K_n(t_k) = \begin{cases} \frac{3n}{2} & \text{if } k = 0 \\ 0 & \text{if } k = 1, 2, \dots, 3n-1. \end{cases}$$

As for (iii), we need an other representation of $V_n(f, x)$. Define the step-function $\omega_n(t)$ as $\omega_n(t) = \frac{2\pi k}{3n}$ if $t_k \leq t < t_{k+1}$, $k=0, 1, \dots, 3n-1$; $t_{3n} = t_0 + 2\pi$. Then we have by (1) and (2)

$$\begin{aligned} V_n(f, x) &= \frac{1}{\pi} \int_0^{2\pi} f(t) K_n(t-x) d\omega_n(t) = \\ &= \frac{1}{2} A_{0,n}(f, x) + \sum_{k=1}^n A_{k,n}(f, x) + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} A_{k,n}(f, x) \end{aligned}$$

where

$$A_{k,n}(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos k(t-x) d\omega_n(t) \quad (k = 0, 1, \dots, 2n-1).$$

Now let $T_n(x)$ be an arbitrary trigonometric polynomial of degree at most n , and $n+1 \leq k \leq 2n-1$. Then, using the obvious relations

$$\int_0^{2\pi} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} jt \cdot \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} kt d\omega_n(t) = 0 \quad (0 \leq j \leq n < k \leq 2n-1)$$

we get

$$A_{k,n}(T_n, x) = \frac{1}{\pi} \int_0^{2\pi} T_n(t) \cos k(t-x) d\omega_n(t) \equiv 0 \quad (n+1 \leq k \leq 2n-1)$$

i.e.

$$V_n(T_n, x) = \frac{1}{2} A_{0,n}(T_n, x) + \sum_{k=1}^n A_{k,n}(T_n, x).$$

Hence $V_n(T_n, x)$ is a trigonometric polynomial of degree $\leq n$. But by (ii) we have $V_n(T_n, t_k) = T_n(t_k)$ ($k=0, 1, \dots, 3n-1$) and this implies (iii).

(iv) By the help of the Fejér-kernel

$$F_n(t) = \frac{1}{2(n+1)} \left(\frac{\sin \frac{n+1}{2} t}{\sin \frac{t}{2}} \right)^2 = \frac{1}{2} + \sum_{k=1}^n \frac{n-k+1}{n+1} \cos kt \geq 0$$

we may write $K_n(t) = 2F_{2n-1}(t) - F_{n-1}(t)$, and hence

$$(3) \quad \begin{aligned} \sum_{k=0}^{3n-1} |K_n(t_k - x)| &\leq \sum_{k=0}^{3n-1} [2F_{2n-1}(t_k - x) + F_{n-1}(t_k - x)] = \\ &= \frac{9n}{2} + \sum_{k=0}^{3n-1} \sum_{j=1}^{2n-1} \varrho_{j,n} \cos j(t_k - x) = \\ &= \frac{9n}{2} + \sum_{j=1}^{2n-1} \varrho_{j,n} \left(\cos jx \sum_{k=0}^{3n-1} \cos jt_k + \sin jx \sum_{k=0}^{3n-1} \sin jt_k \right) \end{aligned}$$

with

$$\varrho_{j,n} = \begin{cases} \frac{3n-2j}{n} & (1 \leq j \leq n-1) \\ \frac{2n-j}{n} & (n \leq j \leq 2n-1). \end{cases}$$

But

$$\sum_{k=0}^{3n-1} \cos jt_k + i \sum_{k=0}^{3n-1} \sin jt_k = \sum_{k=0}^{3n-1} e^{ijt_k} = \sum_{k=0}^{3n-1} e^{ikt_j} = \frac{1 - e^{i3nt_j}}{1 - e^{it_j}} = 0 \quad (1 \leq j \leq 2n-1)$$

and thus (3) yields (iv).

(v) Part (iii) and (iv) imply that if $T_n(x)$ denotes the best approximating trigonometric polynomial to $f(x)$ of degree $\leq n$ then

$$\begin{aligned} |f(x) - V_n(f, x)| &\leq |f(x) - T_n(x)| + |V_n(T_n - f, x)| \leq \\ &\leq \left[1 + \frac{2}{3n} \sum_{k=0}^{3n-1} |K_n(t_k - x)| \right] E_n(f) \leq 4E_n(f). \quad \text{Q.E.D.} \end{aligned}$$

REMARK. By a proper modification of the construction we could get an operator $V_n^{(\varepsilon)}(f, x)$ of degree $\leq n(1+\varepsilon)$ which interpolates at the nodes

$$t_k^{(\varepsilon)} = \frac{2\pi k}{[n(2+\varepsilon)]} \quad (k = 0, 1, \dots, [n(2+\varepsilon)] - 1)$$

but then in (v) we would have a coefficient $O\left(\log \frac{1}{\varepsilon}\right)$ ($\varepsilon \rightarrow 0$) instead of 4, for all fixed $\varepsilon > 0$.

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ON THE THINNEST NON-SEPARABLE LATTICE OF CONVEX PLATES

by

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Suggested by results of A. HEPPES [1] and J. HORVÁTH [2] (see also [3]) concerning transillumination of a lattice-packing of balls, G. FEJES TÓTH proposed the following problem: Find the thinnest n -dimensional lattice of closed n -dimensional balls such that any k -dimensional subspace ($0 \leq k \leq n-1$) has a point in common with at least one ball.

The case when $k=0$ is a well known problem: the problem of the thinnest lattice-covering of the space with balls [4]. Special attention is due also to the case when $k=n-1$. A family of n -dimensional bodies is said to be *separable* if there is an $(n-1)$ -dimensional plane not intersecting any of the bodies but containing on both sides a body. Thus the case of the above problem when $k=n-1$ can be formulated as follows: Find the thinnest non-separable lattice of n -dimensional balls.

Let $d_n(b)$ be the density of a thinnest non-separable lattice of an n -dimensional convex body b . Then we can ask: What is the infimum of $d_n(b)$ if b ranges over all convex bodies? A similar problem concerning connected lattices instead of non-separable ones was considered by GROEMER [5] (see also [6]).

In what follows we solve the above problems for $n=2$. We shall use the terms “disc” and “plate” for a closed 2-dimensional ball and a closed 2-dimensional convex body, respectively.

THEOREM 1. *The density of a non-separable lattice of discs is at least $\sqrt{3}\pi/8$. Equality holds only for a lattice generated by the excircles of a regular triangle (Fig. 1).*

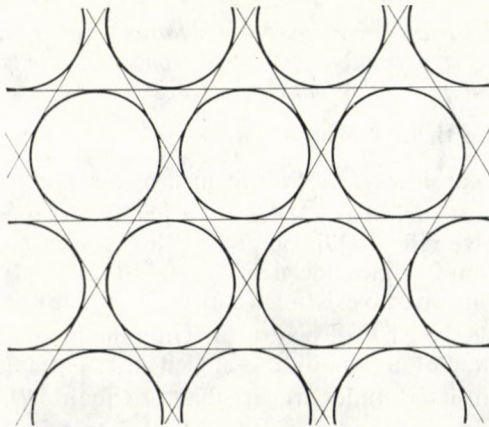


Fig. 1

PROOF. In a non-separable lattice of unit discs let A and B be two nearest centers. Consider a further center outside the line AB nearest to it. Among these centers there is one, say, C lying "above" the segment AB . Since in the triangle ABC AB is a shortest side, we have $\sphericalangle C \leq 90^\circ$. On the other hand, since C lies above the side AB , we have $\sphericalangle A \leq 90^\circ$ and $\sphericalangle B \leq 90^\circ$. Thus the triangle ABC is non-obtuse-angled. Obviously, it has the further property that any two of its side-vectors generate the lattice, in short, it is a lattice-triangle.

We claim that all altitudes of the triangle ABC are less than or equal to 2. For, assuming that for instance the altitude issuing from C is greater than 2, the disc with center C could be separated from the discs centered in A and B by a line parallel to AB . But this line would separate the whole lattice, in contradiction to the assumption that the lattice is non-separable.

We continue to show that among the non-obtuse-angled triangles ABC with altitudes ≤ 2 the equilateral triangle $A^*B^*C^*$ with altitude 2 has the greatest area. Let the feet of the altitudes issuing from A , B and C be A' , B' and C' . We may suppose that one of the altitudes, say, CC' is equal to 2. It suffices to show that $\overline{AB} \leq \overline{A^*B^*}$ with equality only if ABC is congruent to $A^*B^*C^*$.

Supposing that $\overline{AB} > \overline{A^*B^*}$ and, say, $\overline{AC} > \overline{CB}$, we move C in the direction BA until we have $\overline{AC} = \overline{CB}$. Meanwhile $\overline{AA'}$ obviously decreases. If in the new position $\sphericalangle C \geq 90^\circ$, then we have in the new position, and consequently in the original too, $\overline{AA'} > 2$, in contradiction to our assumption. Thus we assume that $\sphericalangle C < 90^\circ$, $\overline{AC} = \overline{CB}$ and $\overline{AB} > \overline{A^*B^*}$. Fixing the position of C , we displace A towards B and B towards A through the same distance until ABC becomes congruent to $A^*B^*C^*$. Observing that during this operation in the right-angled triangle $AA'C$ the hypotenuse AC decreases and $\sphericalangle A'AC$ increases, we see that $\overline{AA'}$ decreases, showing that the original altitude $\overline{AA'}$ was greater than 2.

Thus, denoting the area of ABC with Δ and that of $A^*B^*C^*$ with Δ^* , we have $\Delta \leq \Delta^* = 4/\sqrt{3}$, yielding for the density $d = \pi/2\Delta$ the desired inequality $d \geq \pi/\sqrt{3}/8$. Equality holds only in the case of a regular lattice-triangle with altitude 2. It is easy to check that the corresponding lattice is, in fact, non-separable.

THEOREM 2. *The density of a non-separable lattice of plates is at least $3/8$. Equality holds only if the plate is a triangle spanned by one vertex and the midpoints of the opposite sides of a basic parallelogram (Fig. 2).*

The proof rests on the following

LEMMA. A plate can always be transformed by an affinity in a new one having a diameter D and area A such that $\sqrt{3}D^2 \leq 4A$. Equality is claimed only for a triangle. As to the proof, we refer to [7], pp. 745—746.

PROOF of Theorem 2. Since the density of a lattice of plates, as well as the property of being non-separable is invariant under affine transformation, we may suppose that the lattice consists of plates satisfying the inequality $\sqrt{3}D^2 \leq 4A$.

Let Δ be the area of a non-obtuse-angled lattice-triangle. This triangle has the property that all of its altitudes are less than or equal to D . Therefore $\Delta \leq D^2/\sqrt{3}$, whence $\Delta \leq \frac{4}{3}A$. Since the density d of the lattice equals $A/2\Delta$, we have $d \geq 3/8$.

Equality holds only if both the lattice-triangle and the plate are regular triangles, the diameter (i.e. the side-length) of the latter being equal to the altitude of the former. A glance to Fig. 2 shows that this lattice is non-separable. This completes the proof of Theorem 2.

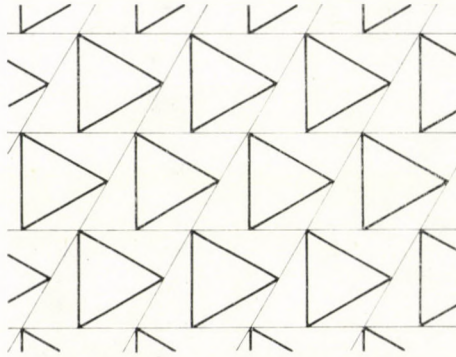


Fig. 2

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A REMARK ON LATTICE VARIETIES DEFINED BY PARTIAL LATTICES

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A lattice variety is a class of lattices, which is closed under the formation of direct products, subalgebras and homomorphic images.

It is well known that the two most important lattice varieties, the variety of modular lattices \mathcal{M} , and the variety of distributive lattices \mathcal{D} have every useful characterizations with sublattices. These lattices N_5 and M_5 are given in Figure 1.

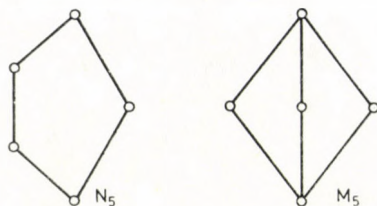


Fig. 1

A lattice L is distributive iff L has no sublattice isomorphic to M_5 or N_5 ; a lattice L is modular iff it has no sublattice isomorphic to N_5 ; a modular lattice is distributive iff it has no sublattice isomorphic to M_5 .

A similar characterization of lattice varieties is due from R. WILLE [4]. He defined the notation of primitive subset: a finite subset P of a lattice L is called *primitive* if $\bigwedge_{\substack{c, d \in P \\ c \not\leq d}} \Theta(c \vee d, d) > \omega$, i.e. if there exists a proper quotient a/b of L such

that a/b is weak projective into $c \vee d/d$ for all $c, d \in P, c \not\leq d$.

WILLE has proved the following

THEOREM (WILLE [4]). *Let \mathcal{P} be a set of finite posets, V a variety of lattices and $V_{\mathcal{P}}$ the class of all lattices in V having no primitive subset o-isomorphic¹ to a member of \mathcal{P} . Then $V_{\mathcal{P}}$ is a variety.*

N_5 and M_5 are subdirectly irreducible lattices, therefore if these lattices are sublattices of a lattice L then N_5, M_5 form primitive subsets of L . We have therefore: if L is a nonmodular lattice, then L contains a primitive subset which is o-isomorphic to N_5 . The converse statement is not true, the lattice K (Figure 2) is a (simple) modular lattice and the elements $0, a, b, c, 1$ form a primitive subset o-isomorphic to N_5 .

¹ Order-isomorphic.

It is easy to see that a modular lattice L is distributive iff L does not contain a primitive subset σ -isomorphic to M_5 . (The primitive subsets of a distributive lattice are namely the two element subchains.) For the variety \mathcal{D} we have also two possibilities, we can take M_5 as a sublattice or as a primitive subset.

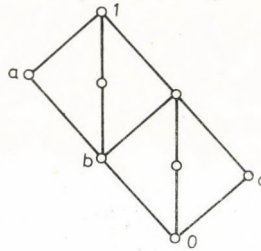


Fig. 2

We know a third characterization of lattice varieties, the characterization of n -distributive lattices.

A modular lattice satisfying the identity

$$x \vee \bigwedge_{i=0}^n y_i = \bigwedge_{j=0}^n \left(x \vee \bigwedge_{\substack{i=0 \\ i \neq j}}^n y_i \right)$$

is called a n -distributive lattice (A. HUHN [3]). Huhn has proved that the modular lattice L is n -distributive iff L does not contain a sublattice $B \cong 2^{n+1}$ and element x such that x is the relative complement of all atoms of B in interval $[\text{inf } B, \text{sup } B]$.

This characterization is different from the first two characterizations, the element x and the sublattice B can generate different sublattices, therefore this is not a "sublattice characterization". On the other hand the poset $B^* = \{x, B\}$ is primitive, but it is not possible to take B^* as a poset; the following counter-example is from A. HUHN (unpublished). Let L be the lattice represented by Fig. 3.

L is a 2-distributive lattice; the elements $u, a, b, c, d, e, f, 1$ form a poset which is σ -isomorphic to 2^3 , and x is a relative complement of a, b, c in interval $[u, 1]$.

All these characterizations are similar. In the first case we have (primitive) sublattices, in the second case (primitive) subsets and in the last case (primitive) partial lattices.

In this note we shall give a characterization of lattice varieties with partial lattices.

Let L be a lattice, and $\emptyset \neq P \subseteq L$. Suppose we have two partial binary operations \vee', \wedge' defined P such that if $a \vee' b = u, c \wedge' d = v$, then $a \vee b = u, c \wedge d = v$ in L . We say that $\mathcal{P} = \langle P, \vee', \wedge' \rangle$ is a weak sublattice of L . (See [1] p. 81.) This \mathcal{P} is weak partial lattice (for the definition we refer to [2]), and every weak partial lattice is a weak sublattice of a lattice L .

Let \mathcal{P} be a weak partial lattice and V a lattice variety. Take the class $V_{\mathcal{P}}$ of all lattices L in V having no weak sublattice \mathcal{P}' with the following two properties:

- (i) \mathcal{P}' is isomorphic to \mathcal{P} ;
- (ii) P' is a primitive subset of L .

$V_{\mathcal{P}}$ is obviously closed under the formation of subalgebras. We prove that $V_{\mathcal{P}}$ is closed under the formation of direct product. Let $L = \Pi(L_i; i \in I)$ the direct product of the lattices $L_i \in V_{\mathcal{P}}$. If L contains a weak sublattice \mathcal{P}' isomorphic to \mathcal{P} , and P' is a primitive subset of L , then there exists a proper quotient a/b of L such that a/b is weak projective into $c \vee d/d$ for all $c, d \in P'$, $c \not\equiv d$. Let φ_i be the projec-

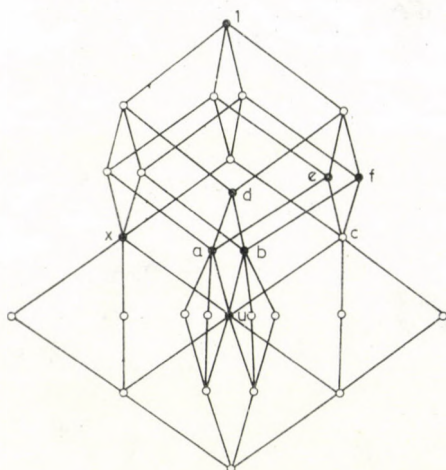


Fig. 3

tion from L onto L_i . We denote by Θ_i the congruence relation of L corresponding to φ_i . $L = \Pi(L_i; i \in I)$ implies $\bigwedge(\Theta_i; i \in I) = \omega$, hence there is an $i \in I$ such that $\varphi_i(a) \neq \varphi_i(b)$. The set $\varphi_i(P) = \{\varphi_i(p) | p \in P\} \subseteq L_i$ is therefore a weak subalgebra of L_i , which is isomorphic to P . By Lemma 5 of [4], $\varphi_i(P)$ is a primitive subset and so $\varphi_i(P)$ has the properties (i)—(ii). Thus $L_i \notin V_{\mathcal{P}}$, which is a contradiction.

$V_{\mathcal{P}}$ is not always closed under formation of homomorphic images. (It is easy to give two lattices L, L' such that L' is a homomorphic image of L , L' is subdirectly irreducible, and L does not contain a sublattice isomorphic to L' .)

We need the following

Definition. The partial lattice \mathcal{P}' has concerning V the property (H) if for any homomorphism

$$\varphi: L \rightarrow L' \quad (L, L' \in V)$$

if \mathcal{P}' is a weak sublattice of L' then there exists a weak sublattice \mathcal{P} of L isomorphic to \mathcal{P}' , and φ maps \mathcal{P} onto \mathcal{P}' .

We prove

THEOREM. Let \mathcal{P} be a finite weak partial lattice and V a lattice variety. If \mathcal{P} has the property (H) concerning V , then the class $V_{\mathcal{P}}$ of all lattices L in V having no weak sublattice \mathcal{P}' satisfying (i) and (ii) form a lattice variety.

PROOF. We must prove that $V_{\mathcal{P}}$ is closed under formation of homomorphism. Let $L \rightarrow L'$ be a homomorphism and $L, L' \in V$. Let \mathcal{P}' be a weak sublattice of L'

satisfying (i) and (ii), i.e. $L' \notin V_{\mathcal{P}}$. By the assumption that \mathcal{P} has the property (H) we get that L contains a weak sublattice $\bar{\mathcal{P}}$ isomorphic to \mathcal{P} . Applying Lemma 4 from [4], $\bar{\mathcal{P}}$ is a primitive subset of L . We have therefore $L \notin V_{\mathcal{P}}$, completing the proof of our theorem.

A sublattice is a weak sublattice; it is very easy to prove that M_5 and N_5 have the property (H) concerning the variety of all lattices resp. the variety \mathcal{M} . The characterization of \mathcal{M} and \mathcal{D} is also a special case of the theorem.

Every finite subset P is a weak sublattice if we define $a \vee b, a \wedge b$ only for comparable pairs. It is almost trivial that P has the property (H).

In the third case let $\varphi: L \rightarrow L'$ any homomorphism of the modular lattice L onto L' . Let us suppose that L' contains a sublattice $B \cong 2^{n+1}$ and an element x such that x is the relative complement of all atoms p_0, \dots, p_n of B in interval $[\inf B, \sup B]$. $\mathcal{P}' = \{x, B\}$ is a weak sublattice; $b_1 \vee b_2, b_1 \wedge b_2$ is defined for all pairs $b_1, b_2 \in B$ and $x \wedge b, x \vee b$ is defined only for the atoms of B . We set $y_i = \bigvee_{j \neq i} p_j$,

then $x \vee \bigwedge_{i=0}^n y_i = x, \bigwedge (x \vee \bigwedge_{i \neq j} y_i) = 1 = \sup B$ hence $x \vee \bigwedge y_i \neq \bigwedge (x \vee \bigwedge_{i \neq j} y_i)$. If $\bar{x}, \bar{y}_0, \bar{y}_n$ are arbitrary elements of L such that $\varphi(\bar{x}) = x, \varphi(\bar{y}_0) = y_1, \dots, \varphi(\bar{y}_n) = y_n$, then $\varphi(\bar{x} \vee \bigwedge \bar{y}_i) = x \vee \bigwedge y_i \neq \bigwedge (x \vee \bigwedge_{i \neq j} y_i) = \varphi(\bigwedge (\bar{x} \vee \bigwedge_{i \neq j} \bar{y}_i))$ implies $\bar{x} \vee \bigwedge \bar{y}_i \neq \bigwedge (\bar{x} \vee \bigwedge_{i \neq j} \bar{y}_i)$ in L .

A. HUHNS has proved [3] that there is a weak sublattice \mathcal{P} generated by \bar{x}, \bar{y}_i ($i=0, \dots, n$), such that \mathcal{P} is isomorphic to \mathcal{P}' and $\varphi(\mathcal{P}) = \mathcal{P}'$. \mathcal{P} has also the property (H).

Problem. Let \mathcal{P} be a weak sublattice of a free lattice in V , such that \mathcal{P} is a primitive subset. Has \mathcal{P} the property (H) concerning V ?

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EINE ABSCHÄTZUNG DER ENTROPIE VON FALTUNGSPOTENZEN UND IHRE ANWENDUNG AUF DEN NACHWEIS VON GLEICHVERTEILUNGSEIGENSCHAFTEN

von
F. LIESE

0. Einleitung

Ausgehend von einem allgemeinen Entropiebegriff, der in engem Zusammenhang zu der in [1] eingeführten f -Divergenz steht, wird die Entropie der durch eine bezüglich des Haarschen Maßes absolut stetigen Verteilung σ geglätteten Folge von Faltungspotenzen $\mu^n * \sigma$ auf einer lokalkompakten abelschen Gruppe G untersucht. Die Folge der Entropien $H_f(\mu^n * \sigma, \lambda)$ ist monoton wachsend, wenn das Entropiefunktional $H_f(\cdot, \lambda)$ mit Hilfe des Haarschen Maßes λ konstruiert wurde. $H_f(\mu^n * \sigma, \lambda)$ bleibt im Falle einer nicht kompakten Gruppe im allgemeinen nicht beschränkt. Im Abschnitt 2 wird die Wachstumsgeschwindigkeit von $H_f(\mu^n * \sigma, \lambda)$ für den Fall abgeschätzt, daß μ und σ einen kompakten Träger besitzen. Ohne die Kompaktheitsforderungen an den Träger werden Abschätzungen für den Spezialfall der additiven Gruppe der reellen Zahlen und $f = -x \log x$ (Shannonsche Entropie) angegeben. Mit Hilfe der Geschwindigkeitsabschätzungen läßt sich zeigen, daß

$$H_f(\mu^{n+1} * \sigma, \lambda) - H_f(\mu^n * \sigma, \lambda)$$

im Caesaro—Mittel gegen Null strebt. Es erhebt sich die Frage, welche Konsequenzen das Kleinwerden der obigen Entropiedifferenz für die Verteilungen $\mu^{n+1} * \sigma$ und $\mu^n * \sigma$ hat. Zur Behandlung diese Problems wird zunächst die obige Entropiedifferenz durch einen Entropieausdruck Δ_f von bestimmten, mit Hilfe von μ und σ gebildeten Verteilungen auf $G \times G$ abgeschätzt. Für den Spezialfall $f = -x \log x$ besteht zwischen Δ_f und dem Shannonschen Informationsgewinn gewisser zweidimensionaler Wahrscheinlichkeitsverteilungen ein enger Zusammenhang. Dieser Zusammenhang gestattet es, für Δ_f im Falle $f = -x \log x$ über die Konvergenz im Caesaro—Mittel hinausgehende Resultate abzuleiten. In Abschnitt 3 werden die Ergebnisse über das asymptotischen Verhalten von Δ_f verwendet, um auf informationstheoretischem Wege ein Ergebnis [6] über Gleichverteilungseigenschaften einer Folge von Faltungspotenzen abzuleiten.

1. Grundbegriffe

Es sei R eine Menge und \mathfrak{A} eine σ -Algebra von Teilmengen von R . \mathfrak{K} sei die Menge aller konkaven Funktionen $f: [0, \infty) \rightarrow (-\infty, +\infty)$, die in $[0, \infty)$ stetig sind und für die $f(0) = 0$ gilt.

μ, ν seien zwei Maße auf \mathfrak{A} . Alle in dieser Arbeit auftretenden Maße werden als positiv vorausgesetzt. μ möge eine Dichte p bezüglich ν besitzen, d. h. es möge eine bezüglich \mathfrak{A} meßbare Funktion $p: R \rightarrow [0, \infty)$ existieren mit

$$\mu(A) = \int_A p(x) \nu(dx) \quad \text{für alle } A \in \mathfrak{A}, \quad \text{mit } \nu(A) < \infty.$$

In diesem Falle sei

$$(1) \quad H_f(\mu, \nu) := \int_R f(p(x)) \nu(dx),$$

sofern das auftretende Integral existiert und endlich ist. Wir nennen $H_f(\mu, \nu)$ die *Entropie* von μ in bezug auf ν .

Für eine konvexe Funktion f wurde die *f-Divergenz* I_f in [1] definiert als

$$(2) \quad I_f(\mu, \nu) := \int_R f(p(x)) \lambda(dx).$$

Somit gilt

$$I_f(\mu, \nu) = -H_{-f}(\mu, \nu).$$

Es sei für $f \in \mathfrak{R}$

$$D_f(\nu) := \left\{ \mu : \mu \text{ besitzt eine Dichte } p \text{ bezüglich } \nu, \int_R |f(p(x))| \nu(dx) < \infty \right\}.$$

Die Existenz der Dichte p , die im weiteren auch durch $\frac{d\mu}{d\nu}$ bezeichnet wird, ist durch den Satz von Radon-Nikodym gesichert, falls eine der beiden folgenden Voraussetzungen erfüllt ist.

a) μ und ν sind σ -endlich und für jedes $A \in \mathfrak{A}$ mit $\nu(A) = 0$ gilt $\mu(A) = 0$

b) [5] R ist ein lokalkompakter Hausdorffscher Raum und \mathfrak{A} fällt mit dem System der Borelmengen \mathfrak{B}_R d. h. mit der kleinsten σ -Algebra die alle offenen Mengen enthält zusammen. Die Maße μ, ν sind Radonsche Maße auf \mathfrak{B}_R d. h. lokalendliche, nichtnegative und im Sinne von [5] reguläre Maße auf \mathfrak{B}_R und für die zugehörigen Systeme von lokalen Nullmengen \mathfrak{N}_μ und \mathfrak{N}_ν gilt $\mathfrak{N}_\mu \supseteq \mathfrak{N}_\nu$. Statt $\mathfrak{N}_\mu \supseteq \mathfrak{N}_\nu$ schreiben wir im weiteren auch $\mu \ll \nu$. Im Falle b) existiert nach [5] eine eindeutig bestimmte abgeschlossene Menge T_μ mit $\mu(R \setminus T_\mu) = 0$ und $\mu(T_\mu \cap E) \neq 0$ für alle offenen E mit $T_\mu \cap E \neq \emptyset$. T_μ nennen wir wie üblich den Träger von μ . Unter der Voraussetzung b) gilt $\mu \in D_f(\nu)$ sicher dann, wenn T_μ kompakt ist, $\mu \ll \nu$ gilt und $p = \frac{d\mu}{d\nu}$ beschränkt ist. Durch $\chi_A(x)$ bezeichnen wir die Indikatorfunktion der Menge A . Jedes $f \in \mathfrak{R}$ ist entweder monoton, oder es existiert ein $0 < y_0 < \infty$ derart, daß f in $0 \leq y \leq y_0$ monoton wachsend und in $y_0 < y < \infty$ monoton fallend ist.

In jedem Falle gilt deshalb mit einem geeigneten \tilde{y} , $0 \leq \tilde{y}$,

$$\begin{aligned} |f(\alpha y_1 + (1-\alpha)y_2)| &\leq |f(y_1)| + |f(y_2)| + \\ &+ |f(\tilde{y})| [\chi_{[0, \tilde{y}]}(y_1) \cdot \chi_{(\tilde{y}, \infty)}(y_2) + \chi_{[0, \tilde{y}]}(y_2) \cdot \chi_{(\tilde{y}, \infty)}(y_1)]. \end{aligned}$$

Hieraus erhalten wir, daß $D_f(\nu) := \{\mu : \mu \in D_f(\nu), \mu(R) < \infty\}$ konvex ist.

$H_f(\cdot, \nu)$ ist ein auf $D_f(\nu)$ definiertes konkaves Funktional.

$$(3) \quad \begin{aligned} H_f(\alpha\mu_1 + (1-\alpha)\mu_2, \nu) &= \int_G f(\alpha p_1(x) + (1-\alpha)p_2(x)) \nu(dx) \cong \\ &\cong \int_G [\alpha f(p_1(x)) + (1-\alpha)f(p_2(x))] \nu(dx) \cong \alpha H_f(\mu_1, \nu) + (1-\alpha)H_f(\mu_2, \nu) \end{aligned}$$

mit $\mu_i \in D_f(\nu)$, $\alpha\mu_1 + (1-\alpha)\mu_2 \in D_f(\nu)$, $0 \leq \alpha \leq 1$, $p_i =$ Dichte von μ_i bezüglich ν .

Ist die Funktion f streng konkav, d. h. enthält ihr Graph keine gerade Strecke, so steht in (3) das Gleichheitszeichen genau dann wenn

$$p_1 = p_2 \quad v\text{-fast sicher oder } \mu_1 = \mu_2$$

erfüllt ist. $H_f(\cdot, v)$ ist also für streng konkaves f selbst streng konkav.

Für zwei Radonsche Maße μ_1, μ_2 auf den lokalkompakten Hausdorffschen Räumen R_i bezeichnet $\mu_1 \times \mu_2$ das Produktmaß auf $[R_1 \times R_2, \mathfrak{B}_{R_1 \times R_2}]$. Für einen lokalkompakten Hausdorffschen Raum R sei $C_{00}(R)$ der Raum der stetigen reellen Funktionen φ , für die $\{x: \varphi(x) \neq 0\}$ relativ kompakt ist.

δ_t bezeichne im weiteren die Deltaverteilung im Punkt t .

Wir spezialisieren jetzt $[R, \mathfrak{B}_R]$.

G sei eine additive Hausdorffsche lokalkompakte abelsche topologische Gruppe und \mathfrak{B}_G die σ -Algebra ihrer Borelmengen. Durch

$$(4) \quad \int_G \int_G \varphi(x, y) (\mu_1 \oplus \mu_2)(dx \times dy) := \int_G \int_G \varphi(x, x+z) \mu_2(dz) \mu_1(dx)$$

für alle $\varphi \in C_{00}(G \times G)$ wird jedem Paar μ_1, μ_2 Radonscher Maße auf $[G, \mathfrak{B}_G]$ ein Maß $\mu_1 \oplus \mu_2$ auf $[G \times G, \mathfrak{B}_{G \times G}]$ zugeordnet. Insbesondere gilt für kompakte Mengen A, B

$$(\mu_1 \oplus \mu_2)(A \times B) = \int_A \mu_2(B-x) \mu_1(dx)$$

Ist ν ein weiteres Radonsches Maß auf $[G, \mathfrak{B}_G]$ mit $\mu_2 \ll \nu, p_2 := \frac{d\mu_2}{d\nu}$, so folgt aus (4)

$$\int_G \int_G \varphi(x, y) (\mu_1 \oplus \mu_2)(dx \times dy) = \int_G \int_G \varphi(x, y) p_2(y-x) (\mu_1 \oplus \nu)(dx \times dy)$$

Somit ist

$$\mu_1 \oplus \mu_2 \ll \mu_1 \oplus \nu$$

und

$$(5) \quad \frac{d(\mu_1 \oplus \mu_2)}{d(\mu_1 \oplus \nu)}(y, x) = p_2(y-x).$$

Sind die Maße μ_i endlich, dann bezeichne

$$(\mu_1 * \mu_2)(B) := (\mu_1 \oplus \mu_2)(G \times B), \quad B \in \mathfrak{B}_G$$

* ist die gewöhnliche Faltung von Maßen.

Durch μ^n bezeichnen wir die n -te Faltungspotenz des endlichen Maßes μ .

λ bezeichne im weiteren ein fest gewähltes Haarsches Maß, d. h. ein verschiebungsinvariantes Radonsches Maß mit $\lambda(U) > 0$ für alle offenen Mengen U , \mathcal{P} die Menge der Wahrscheinlichkeitsmaße auf $[G, \mathfrak{B}_G]$ und $\mathcal{P}_\lambda = \{\mu: \mu \in \mathcal{P}, \mu \ll \lambda\}$. Wie in [6] ordnen wir jedem $\mu \in \mathcal{P}$ das System \mathfrak{S} derjenigen abgeschlossenen Untergruppen \tilde{G} von G zu, für die ein $t_{\tilde{G}} \in \tilde{G}$ existiert mit

$$T_\mu \subseteq t_{\tilde{G}} + \tilde{G}$$

$$G_\mu := \bigcap_{\tilde{G} \in \mathfrak{S}} \tilde{G} \text{ heißt Gittergruppe von } \mu.$$

Offensichtlich ist G_μ die kleinste abgeschlossene Untergruppe von G , die die Menge $T_\mu - T_\mu = \{u - v : u, v \in T\}$ enthält.

Durch $\|\mu - \nu\|$, $\mu, \nu \in \mathcal{P}$, bezeichnen wir den Variationsabstand der Wahrscheinlichkeitsmaße μ und ν .

2. Abschätzung der Wachstumsgeschwindigkeit der Entropie

Es sei $\mu \in \mathcal{P}$, $\sigma \in \mathcal{P}_s$, dann gilt $\mu * \sigma \in \mathcal{P}_s$ und

$$\frac{d(\mu * \sigma)}{d\lambda} =: q = \int_G r(x-y) \mu(dy) \quad \text{mit} \quad r = \frac{d\sigma}{d\lambda}.$$

Gilt zusätzlich $\sigma \in D_f(\lambda)$, $\mu * \sigma \in D_f(\lambda)$ so erhält man aus der Jensenschen Ungleichung

$$\begin{aligned} (6) \quad H_f(\mu * \sigma, \lambda) &= \int_G f(q(x)) \lambda(dx) = \int_G f\left(\int_G r(x-y) \mu(dy)\right) \lambda(dx) \cong \\ &\cong \int_G \int_G f(r(x-y)) \mu(dy) \lambda(dx) = \int_G f(r(z)) \lambda(dz) = H_f(\sigma, \lambda) \end{aligned}$$

Die Ungleichung $H_f(\mu * \sigma, \lambda) \cong H_f(\sigma, \lambda)$ ergibt sich auch als Folgerung einer in [1] abgeleiteten Ungleichung für die f -Divergenz.

Ist die Gruppe G kompakt, und setzen wir in (6) $\mu = \lambda =$ normiertes Haarsches Maß, so folgt

$$H_f(\sigma, \lambda) \cong H_f(\lambda * \sigma, \lambda) = H_f(\lambda, \lambda) = f(1).$$

Somit gilt auch für $\mu^n * \sigma$

$$H_f(\mu^n * \sigma, \lambda) \cong f(1)$$

und nach (6) konvergiert deshalb $H_f(\mu^n * \sigma, \lambda)$ monoton wachsend gegen einen Wert, der kleiner oder gleich $f(1)$ ist. Ziel des ersten Teiles dieser Arbeit ist es, Aussagen über $H_f(\mu^n * \sigma, \lambda)$ ohne Voraussetzung der Kompaktheit an G zu machen.

Der Träger von $\nu \in \mathcal{P} \cap D_f(\lambda)$ möge in der kompakten Menge K liegen. Dann folgt aus der Jensenschen Ungleichung für $f \in \mathfrak{R}$

$$\begin{aligned} (7) \quad H_f(\nu, \lambda) &= \int_G f(q(x)) \lambda(dx) = \int_K f(p(x)) \lambda(dx) \cong \\ &\cong \lambda(K) f\left(\frac{1}{\lambda(K)}\right) = H_f(\varkappa[K], \lambda), \end{aligned}$$

mit $(\varkappa[K])(A) := \frac{\lambda(A \cap K)}{\lambda(K)}$. Wir nennen $\varkappa[K]$ die Gleichverteilung auf K .

Die Ungleichung (7) besagt, daß die Gleichverteilung auf K die Verteilung maximaler Entropie in der Klasse aller Verteilungen ist, deren Träger in K liegt.

□ Setzen wir voraus, daß die Träger von $\mu \in \mathcal{P}$ und $\sigma \in \mathcal{P}_s$ kompakt sind, dann ist auch $T_{\mu^k * \sigma}$ für jedes k kompakt und nach (7) gilt

$$H_f(\mu^k * \sigma, \lambda) \leq H_f(\chi[T_{\mu^k * \sigma}], \lambda).$$

Der folgende Satz zeigt, wie die Entropie von $\nu_k := \chi[T_{\mu^k * \sigma}]$ abgeschätzt werden kann.

SATZ 1. Es sei $f \in \mathfrak{R}$. $\mu \in \mathcal{P}$ und $\sigma \in D_f(\lambda)$ seien Wahrscheinlichkeitsmaße mit kompaktem Träger. Dann gilt $\mu^n * \sigma \in D_f(\lambda)$ und es existieren Konstanten c_1, c_2 mit

$$(8) \quad H_f(\mu^n * \sigma, \lambda) \leq H_f(\chi[T_{\mu^k * \sigma}], \lambda) \leq c_1 n^{c_2} f\left(\frac{1}{c_1 n^{c_2}}\right), \quad n = 1, 2, \dots$$

BEWEIS. 1. Wir setzen voraus, daß G die Darstellung $G = G_1 \times G_2$ mit zwei Hausdorffschen abelschen topologischen Gruppen G_1, G_2 gestattet. μ_1 bzw. μ_2 seien die Randverteilungen eines Wahrscheinlichkeitsmaßes μ auf $[G, \mathfrak{B}_G] = [G_1 \times G_2, \mathfrak{B}_{G_1 \times G_2}]$

$$\mu_1(B) := \mu(B \times G_2), \quad B \in \mathfrak{B}_{G_1}; \quad \mu_2(C) := \mu(G_1 \times C), \quad C \in \mathfrak{B}_{G_2}$$

Es sei $(x_0, y_0) \in T_\mu$. $U_{x_0}^1$ sei eine offene Umgebung von x_0 in G_1 und $U_{y_0}^2$ eine offene Umgebung von y_0 in G_2 . Dann folgt

$$0 < \mu(U_{x_0}^1 \times U_{y_0}^2) \leq \mu(U_{x_0}^1 \times G_2) = \mu_1(U_{x_0}^1)$$

$$0 < \mu(U_{x_0}^1 \times U_{y_0}^2) \leq \mu(G_1 \times U_{y_0}^2) = \mu_2(U_{y_0}^2)$$

und somit $(x_0, y_0) \in T_{\mu_1} \times T_{\mu_2}$. Damit ist

$$(9) \quad T_\mu \subseteq T_{\mu_1} \times T_{\mu_2}$$

bewiesen.

Da die Randverteilungen $(\mu * \nu)_i$ der Faltung $\mu * \nu$ mit der Faltung $\mu_i * \nu_i$ der Randverteilungen μ_i, ν_i übereinstimmen, folgt aus (9)

$$(10) \quad T_{\mu * \nu} \subseteq T_{\mu_1 * \nu_1} \times T_{\mu_2 * \nu_2}.$$

2. Wir zeigen jetzt, daß Konstanten c_1, c_2 mit

$$(11) \quad \lambda(T_{\mu^n * \sigma}) \leq c_1 n^{c_2}, \quad n = 1, 2, \dots$$

existieren, wenn μ und σ den Voraussetzungen in Satz 1 genügen. Ohne Beschränkung der Allgemeinheit können wir nach [9] annehmen, daß $G = R^m \times G_0$, $m \geq 0$, gilt. Hierbei ist G_0 eine lokalkompakte Hausdorffsche abelsche Gruppe, die eine kompakte Untergruppe B besitzt, für die G_0/B diskret ist. μ_1, μ_2 bzw. σ_1, σ_2 seien die Randverteilungen von μ bzw. σ auf R^m und G_0 . Die Anwendung von (10) liefert

$$(12) \quad T_{\mu^n * \sigma} \subseteq T_{\mu_1^n * \sigma_1} \times T_{\mu_2^n * \sigma_2}$$

Da das Haarsche Maß λ gleich dem Produkt aus Haarschen Maßen λ_1, λ_2 auf R^m und G_0 ist, brauchen wir somit die Behauptung (11) nur für die Spezialfälle R^m und G_0 nachzuweisen.

Da T_{μ_1} und T_{σ_1} kompakt sind, existieren endliche m -dimensionale Intervalle I_{μ_1}, I_{σ_1}

$$I_{\mu_1}: a_i \leq x_i \leq b_i; \quad I_{\sigma_1}: c_i \leq x_i \leq d_i, \quad i = 1, \dots, m$$

mit

$$T_{\mu_1} \subseteq I_{\mu_1}, \quad T_{\sigma_1} \subseteq I_{\sigma_1}.$$

Der Träger der Faltung von Wahrscheinlichkeitsmaßen ist gleich dem Abschluß der Summe der Träger (siehe [3]). Dies ergibt

$$T_{\mu_1^{*n} \sigma} \subseteq I^n$$

mit

$$I^n := \{x_1, \dots, x_m\}; \quad c_i + na_i \leq x_i \leq d_i + nb_i\}.$$

Da λ_1 Lebesgueschen Maß proportional ist, folgt mit einer Konstanten e_1

$$\lambda_1(I^n) \leq e_1 \left[\prod_{1 \leq i \leq m} (b_i - a_i + \frac{1}{n} (d_i - c_i)) \right] \cdot n^m.$$

Da T_{μ_2} und T_{σ_2} kompakt sind und G_0/B diskret ist, existieren

$$t_1, \dots, t_k \in G_0, \quad s_1, \dots, s_l \in G_0$$

mit

$$T_{\mu_2} \subseteq \bigcup_{i=1}^k (t_i + B)$$

$$T_{\sigma_2} \subseteq \bigcup_{j=1}^l (s_j + B).$$

Hieraus ergibt sich

$$T_{\mu_2^{*n} \sigma_2} \subseteq \bigcup_{j=1}^l (s_j + B) + \bigcup_{i=1}^k (t_i + B) + \dots + \bigcup_{i=1}^k (t_i + B).$$

Da alle möglichen Summen von n Elementen aus t_1, \dots, t_k und einem s_j höchstens

$$a_n = l \cdot \binom{n+k-1}{n}$$

verschiedene Elemente u_j ergeben, erhalten wir

$$\bigcup_{j=1}^l (s_j + B) + \bigcup_{i=1}^k (t_i + B) + \dots + \bigcup_{i=1}^k (t_i + B) \subseteq \bigcup_{j=1}^{a_n} (u_j + B).$$

Damit erhalten wir

$$\lambda_2(T_{\mu_2^{*n} \sigma_2}) \leq \lambda_2 \left(\bigcup_{j=1}^{a_n} (u_j + B) \right) \leq \sum_{j=1}^{a_n} \lambda_2(u_j + B) = a_n \lambda_2(B).$$

a_n läßt sich durch $e_2 n^{k-1}$ mit einer gewissen Konstanten e_2 abschätzen. Da B kompakt ist gilt $\lambda_2(B) < \infty$ und damit ist (11) auch für den zweiten Spezialfall nachgewiesen.

3. Es sei $x_1, x_2 \in (0, \infty)$, $x_1 < x_2$ und $\alpha = \frac{x_1}{x_2}$, dann folgt wegen $f \in \mathfrak{R}$

$$x_2 f\left(\frac{1}{x_2}\right) = x_2 f\left(\alpha \frac{1}{x_1} + (1-\alpha)0\right) \cong \alpha x_2 f\left(\frac{1}{x_1}\right) + f(0)(1-\alpha)x_2 = x_1 f\left(\frac{1}{x_1}\right).$$

$xf\left(\frac{1}{x}\right)$ ist somit monoton wachsend. Aus (7) und (11) erhalten wir (8).

Wir wollen jetzt die Voraussetzung, daß T_μ und T_σ kompakt sind fallen lassen, spezialisieren aber $G = R^1$ und $f = -x \log x$. Die zu $f = -x \log x$ und $\lambda =$ Lebesguesches Maß gehörige Entropie bezeichnen wir einfach durch $H(\cdot)$. Für alle $a > 0$, $\delta > 0$ bezeichnet $\pi_{a,\delta}$ das durch folgende Wahrscheinlichkeitsdichte $p_{a,\delta}$ definierte Wahrscheinlichkeitsmaß

$$(13) \quad p_{a,\delta} := c(a, \delta) e^{-a|x|^\delta}, \quad -\infty < x < +\infty,$$

$$[c(a, \delta)]^{-1} = \int_{-\infty}^{+\infty} e^{-a|x|^\delta} dx = 2\delta^{-1} a^{-\frac{1}{\delta}} \Gamma\left(\frac{1}{\delta}\right).$$

ν sei eine weitere bezüglich des Lebesgueschen Maßes absolut stetige Wahrscheinlichkeitsverteilung mit endlichem absoluten Moment m_ν der Ordnung δ , q sei die Dichte von ν . Nach [1] gilt für den Shannonschen Informationsgewinn

$$I(\nu, \pi_{a,\delta}) = \int_{-\infty}^{+\infty} q(x) \log \frac{q(x)}{p_{a,\delta}(x)} dx \cong 0$$

oder für die Shannonsche Entropie $H(\nu)$

$$H(\nu) \cong \int_{-\infty}^{+\infty} (-\log c(a, \delta) + a|x|^\delta) \nu(dx) = \log c(a, \delta) + am_\nu.$$

Es gelte zusätzlich

$$m_\nu \cong m_{\pi_{a,\delta}}$$

dann folgt

$$(14) \quad H(\nu) \cong -\log c(a, \delta) + am_{\pi_{a,\delta}} = H(\pi_{a,\delta})$$

Somit ist die durch die Dichte (13) definierte Wahrscheinlichkeitsverteilung die Verteilung maximaler Shannonscher Entropie in der Klasse aller Wahrscheinlichkeitsverteilungen, deren absolutes Moment der Ordnung δ kleiner oder gleich

$$(15) \quad m_{\pi_{a,\delta}} = c(a, \delta) \int_{-\infty}^{+\infty} |x|^\delta e^{-a|x|^\delta} dx = \frac{1}{a \cdot \delta}$$

ist. Siehe dazu auch [8].

Es gelten folgende elementare Ungleichungen :

$$\left| \sum_{i=1}^n y_i \right|^\delta \leq n^{\delta-1} \sum_{i=1}^n |y_i|^\delta \quad \text{für } \delta \geq 1$$

$$\left| \sum_{i=1}^n y_i \right|^\delta \leq \sum_{i=1}^n |y_i|^\delta, \quad 0 < \delta < 1, \quad y_i \in \mathbb{R}^1.$$

Hieraus folgt:

Sind μ und σ Wahrscheinlichkeitsverteilungen auf dem \mathbb{R}^1 mit endlichem absoluten Moment der Ordnung $\delta > 0$, so gilt

$$(16) \quad m_{\mu^n * \sigma} \leq b_n = \begin{cases} n^{\delta-1}(n \cdot m_\mu + m_\sigma) & \delta \geq 1 \\ nm_\mu + m_\sigma & 0 < \delta < 1. \end{cases}$$

Zu $b > 0$ sei

$$a := \frac{1}{\delta b}$$

dann gilt nach (15)

$$m_{\pi_{a,\delta}} = b.$$

Wir definieren jetzt a_n durch

$$a_n := \frac{1}{b_n \delta}$$

mit b_n aus (16), dann folgt aus (14)

$$(17) \quad H(\mu^n * \sigma) \leq H(\pi_{a_n, \delta}).$$

Analog zu Satz 1 gilt dann

SATZ 2. μ und σ seien Wahrscheinlichkeitsverteilungen auf den Borelmengen des \mathbb{R}^1 mit endlichem absoluten Moment der Ordnung $\delta > 0$. σ sei absolut stetig bezüglich des Lebesgueschen Maßes. $H(\sigma)$ möge existieren. Dann ist $H(\mu^n * \sigma)$ definiert und es gilt

$$(18) \quad H(\mu^n * \sigma) \leq \varphi(\delta) + \frac{1}{\delta} \log b_n$$

mit b_n aus (16) und

$$\varphi(\delta) = \log 2 + \log \Gamma\left(\frac{1}{\delta}\right) + \frac{1}{\delta} + \left(\frac{1}{\delta} - 1\right) \log \delta.$$

Der Beweis von Satz 2 folgt unmittelbar aus Ungleichung (17) und der Definition der b_n , wobei $\mu^n * \sigma \in D_{-x \log x}$ aus Ungleichung (14) und der Jensenschen Ungleichung folgt.

FOLGERUNG. Genügen μ, σ den Voraussetzungen von Satz 1 und gilt für f :

$$(19) \quad \lim_{x \rightarrow \infty} x^{1-\varepsilon} f\left(\frac{1}{x}\right) = 0 \quad \forall \varepsilon > 0$$

dann folgt

$$(20) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} [H_f(\mu^{k+1} * \sigma, \lambda) - H_f(\mu^k * \sigma, \lambda)] = 0.$$

Für $f = -x \log x$ gilt (20) auch unter den Voraussetzungen in Satz 2.

BEWEIS. Die Beziehung (20) ergibt sich unter den Voraussetzungen in Satz 1 durch Anwendung der Ungleichung (8)

$$\begin{aligned} 0 &\leq \frac{1}{n} \sum_{k=0}^{n-1} [H_f(\mu^{k+1} * \sigma, \lambda) - H_f(\mu^k * \sigma, \lambda)] = \\ &= \frac{1}{n} H_f(\mu^n * \sigma, \lambda) - \frac{1}{n} H_f(\sigma, \lambda) \leq -\frac{1}{n} H_f(\sigma, \lambda) + c_1 n^{c_2-1} f\left(\frac{1}{c_1 n^{c_2}}\right) \leq \\ &\leq -\frac{1}{n} H_f(\sigma, \lambda) + c_1^{c_2} x_n^{1-\varepsilon} f\left(\frac{1}{x_n}\right) \xrightarrow{n \rightarrow \infty} 0 \\ &x_n = c_1 n^{c_2}, \quad \varepsilon = \frac{1}{c_2}. \end{aligned}$$

Unter den Voraussetzungen von Satz 2 folgt (20) ebenso wegen

$$\frac{1}{n} \log b_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{mit } b_n \text{ aus (16).}$$

BEMERKUNG. Die Bedingung (19) bedeutet, daß $f(x)$ für $x \rightarrow 0$ hinreichend schnell gegen Null konvergiert. Der Beziehung (19) genügt zum Beispiel die Funktion

$$f(x) = -x \log x.$$

(19) wird natürlich erst recht von allen f , die eine endliche rechtsseitige Ableitung in $x=0$ besitzen, erfüllt. Die Aussage (20) ist in diesem Falle trivial, weil dann eine nur von f abhängige Konstante c_f mit

$$H_f(\sigma, \lambda) \leq c_f \quad \text{für alle } \sigma \in D_f(\lambda)$$

existiert und somit $H_f(\mu^n * \sigma, \lambda)$ als monoton wachsende und durch c_f beschränkte Folge konvergiert.

3. Eigenschaften der Informationsgröße Δ_f

$[R, \mathfrak{A}]$ sei ein meßbarer Raum, $\gamma_i, i=1, 2, 3$, Maße auf \mathfrak{A} mit $\gamma_i \in D_f(\gamma_3), i=1, 2, \frac{1}{2}(\gamma_1 + \gamma_2) \in D_f(\gamma_3)$. Sei

$$\Delta_f(\gamma_1, \gamma_2, \gamma_3) := H_f\left(\frac{1}{2}(\gamma_1 + \gamma_2), \gamma_3\right) - \frac{1}{2} H_f(\gamma_1, \gamma_3) - \frac{1}{2} H_f(\gamma_2, \gamma_3).$$

Da H_f ein konkaves Funktional ist, gilt

$$\Delta_f(\gamma_1, \gamma_2, \gamma_3) \geq 0.$$

Da H_f für streng konkaves f selbst streng konkav ist, steht in diesem Fall in der letzten Ungleichung das Gleichheitszeichen genau dann, wenn $\gamma_1 = \gamma_2$ gilt.

Δ_f läßt sich deshalb als Maß für die Unterschiedlichkeit von γ_1 und γ_2 deuten. Diese Deutung läßt sich im Spezialfall $f = -x \log x$ durch Ungleichungen präzisieren.

Bei allen auftretenden Informationsgrößen lassen wir im Spezialfall $f = -x \log x$ f als Index weg.

Es gilt

$$\begin{aligned} d(y_1, y_2) &:= -\frac{1}{2}(y_1 + y_2) \log \frac{1}{2}(y_1 + y_2) + \frac{1}{2}y_1 \log y_1 + \frac{1}{2}y_2 \log y_2 = \\ &= \frac{1}{2}y_1 \log \frac{y_1}{\frac{1}{2}(y_1 + y_2)} + \frac{1}{2}y_2 \log \frac{y_2}{\frac{1}{2}(y_1 + y_2)}. \end{aligned}$$

Ist p_i die Dichte von γ_i , $i=1, 2$, bezüglich γ_3 , so gilt

$$\begin{aligned} (21) \quad \Delta(\gamma_1, \gamma_2, \gamma_3) &= \int_{\mathcal{R}} d(p_1(x), p_2(x)) \gamma_3(dx) = \frac{1}{2} \int_{\mathcal{R}} p_1 \log \frac{p_1}{\frac{1}{2}(p_1 + p_2)} \gamma_3(dx) + \\ &+ \frac{1}{2} \int_{\mathcal{R}} p_2 \log \frac{p_2}{\frac{1}{2}(p_1 + p_2)} \gamma_3(dx) = I\left(\gamma_1, \frac{1}{2}(\gamma_1 + \gamma_2)\right) + I\left(\gamma_2, \frac{1}{2}(\gamma_1 + \gamma_2)\right) \end{aligned}$$

$\Delta(\gamma_1, \gamma_2, \gamma_3)$ hängt also nicht von γ_3 ab, und wir schreiben im weiteren hierfür nur $\Delta(\gamma_1, \gamma_2)$. $\Delta(\gamma_1, \gamma_2)$ ist für alle Wahrscheinlichkeitsmaße γ_1, γ_2 definiert, da man stets ein γ_3 finden kann, so daß $\gamma_i \in D_f(\gamma_3)$ gilt. Man kann etwa $\gamma_3 = \frac{1}{2}(\gamma_1 + \gamma_2)$ wählen.

Nach [2] gilt für den Shannonschen Informationsgewinn $I(q_1, q_2)$ zweier Wahrscheinlichkeitsmaße q_1, q_2 folgenden Ungleichung

$$\|q_1 - q_2\| \leq [2I(q_1, q_2)]^{\frac{1}{2}}$$

Mit der Identität (21) folgt für zwei Wahrscheinlichkeitsmaße γ_1, γ_2

$$\Delta(\gamma_1, \gamma_2) \leq \frac{1}{2} \left\| \gamma_1 - \frac{1}{2}(\gamma_1 + \gamma_2) \right\|^2 + \frac{1}{2} \left\| \gamma_2 - \frac{1}{2}(\gamma_1 + \gamma_2) \right\|^2 = \frac{1}{4} \|\gamma_1 - \gamma_2\|^2$$

oder

$$(22) \quad \|\gamma_1 - \gamma_2\| \leq 2[\Delta(\gamma_1, \gamma_2)]^{\frac{1}{2}}.$$

$\Delta(\gamma_1, \gamma_2)$ charakterisiert somit die Verschiedenheit von γ_1 und γ_2 in dem Sinne, daß aus der Kleinheit von $\Delta(\gamma_1, \gamma_2)$ die Kleinheit des Variationsabstandes von γ_1 und γ_2 folgt.

$[R, \mathfrak{A}]$, $[\bar{R}, \bar{\mathfrak{A}}]$ seien zwei meßbare Räume, q_1, q_2 zwei Wahrscheinlichkeitsmaße auf $[R, \mathfrak{A}]$ und P eine Übergangswahrscheinlichkeit von $[R, \mathfrak{A}]$ in $[\bar{R}, \bar{\mathfrak{A}}]$. Es sei

$$\bar{q}_i(A) := \int_{\mathcal{R}} P(x, A) q_i(dx), \quad A \in \bar{\mathfrak{A}}.$$

Dann gilt nach [1]

$$(23) \quad I(\bar{\varrho}_1, \bar{\varrho}_2) \cong I(\varrho_1, \varrho_2).$$

Wir setzen jetzt

$$[R, \mathfrak{A}] = [\bar{R}, \bar{\mathfrak{A}}] = [G \times G, \mathfrak{B}_{G \times G}]$$

und

$$P(x, y, C) := \int_G \chi_C(x, z+y) \mu(dz), \quad \mu \in \mathcal{P}$$

mit $C \in \mathfrak{B}_{G \times G}$ und $\chi_C =$ Indikatorfunktion der Menge C . Für $C = A \times B$, $A, B \in \mathfrak{B}_G$ gilt

$$P(x, y, A \times B) = \chi_A(x) \mu(B-y)$$

und somit

$$\begin{aligned} \int_G \int_G P(x, y, A \times B) (\mu_1 \oplus \mu_2)(dx \times dy) &= \int_G \int_G \chi_A(x) \mu(B-y) (\mu_1 \oplus \mu_2)(dx \times dy) = \\ &= \int_A (\mu * \mu_2)(B-x) \mu_1(dx) = (\mu_1 \oplus (\mu * \mu_2))(A \times B) \end{aligned}$$

oder

$$(24) \quad \overline{\mu_1 \oplus \mu_2} = \mu_1 \oplus (\mu * \mu_2).$$

Entsprechend gilt

$$(25) \quad \overline{\nu_1 \times \nu_2} = \nu_1 \times (\mu * \nu_2).$$

Die Ungleichung (23) liefert mit Hilfe der Identität (21) sowie (24) und (25) die folgende Aussage.

HILFSSATZ. Für beliebige $\mu_i \in \mathcal{P}$, $\nu_i \in \mathcal{P}$, $i=1, 2$; $\mu \in \mathcal{P}$ gilt

$$(26) \quad \Delta(\nu_1 \times \nu_2, \mu_1 \oplus \mu_2) \cong \Delta(\nu_1 \times (\mu * \nu_2), \mu_1 \oplus (\mu * \mu_2)).$$

Wir wollen jetzt einen Zusammenhang zwischen Δ_f und der Entropiedifferenz $H_f(\mu * \sigma, \lambda) - H_f(\sigma, \lambda)$ herstellen. Dazu dient der folgende Satz.

SATZ 3. Es sei $f \in \mathfrak{R}$, $\mu \in \mathcal{P}$, $\sigma_i \in \mathcal{P} \cap D_f(\lambda)$, $i=1, 2$, sowie $\mu * \sigma_2 \in D_f(\lambda)$. Dann gilt

$$(27) \quad H_f \left(\frac{1}{2} (\mu \times \sigma_1 + \mu \oplus \sigma_2), \mu \times \lambda \right) \cong H_f \left(\frac{1}{2} (\sigma_1 + \mu * \sigma_2), \lambda \right).$$

Für streng konkaves f steht in (27) genau dann das Gleichheitszeichen, wenn eine der beiden folgenden äquivalenten Bedingungen erfüllt ist

$$(28) \quad \sigma_2 = \delta_s * \sigma_2 \quad \forall s \in G_\mu$$

$$(29) \quad \mu \times (\mu * \sigma_2) = \mu \oplus \sigma_2.$$

BEWEIS. 1. Wir beweisen zunächst (27). p_i sei die Dichte von σ_i bezüglich λ . Dann gilt

$$\frac{d(\mu \times \sigma_1)}{d(\mu \times \lambda)}(x, y) = p_1(y)$$

und nach (5) sowie $\mu \oplus \lambda = \mu \times \lambda$

$$\left(\frac{d\mu \otimes \sigma_2}{d\mu \times \lambda} \right) (x, y) = p_2(y-x).$$

Hieraus folgt

$$(30) \quad H_f(\sigma_1, \lambda) = \int_G \int_G f(p_1(y)) \mu(dx) \lambda(dy) = H_f(\mu \times \sigma_1, \mu \times \lambda)$$

$$(31) \quad \begin{aligned} H_f(\sigma_2, \lambda) &= \int_G f(p_2(z)) \lambda(dz) = \\ &= \int_G \int_G f(p_2(y-x)) \mu(dx) \lambda(dy) = H_f(\mu \oplus \sigma_2, \mu \times \lambda) \end{aligned}$$

Somit gilt wegen $\sigma_i \in D_f(\lambda)$

$$\mu \times \sigma_1 \in D_f(\mu \times \lambda) \quad \text{und} \quad \mu \oplus \sigma_2 \in D_f(\mu \times \lambda).$$

Die Konvexität von $\hat{D}_f(\mu \times \lambda)$ ergibt $\frac{1}{2}(\mu \times \sigma_1 + \mu \oplus \sigma_2) \in D_f(\mu \times \lambda)$. q sei die Dichte von $\mu * \sigma_2$ bezüglich λ . Die Anwendung der Jensenschen Ungleichung liefert

$$\begin{aligned} H_f \left(\frac{1}{2}(\mu \times \sigma_1 + \mu \oplus \sigma_2), \mu \times \lambda \right) &= \int_G \int_G f \left(\frac{1}{2}(p_1(y) + p_2(y-x)) \right) \mu(dx) \lambda(dy) \leq \\ &\leq \int_G f \left(\int_G \frac{1}{2} [p_1(y) + p_2(y-x)] \mu(dx) \right) \lambda(dy) = \\ &= \int_G f \left(\frac{1}{2} [p_1(y) + q(y)] \right) \lambda(dy) = H_f \left(\frac{1}{2}(\sigma_1 + \mu * \sigma_2), \lambda \right). \end{aligned}$$

Ist f streng konkav, so steht in der letzten Ungleichung genau dann das Gleichheitszeichen, wenn für λ -fast alle y

$$\mu \{x : p_2(y-x) = q(y)\} = 1 \text{ gilt.}$$

Dies ist gleichbedeutend mit

$$p_2(y-x) = q(y) \quad \text{für} \quad \mu \times \lambda\text{-fast alle } x, y$$

oder nach (5) und $\mu \times \lambda = \mu \oplus \lambda$

$$\mu \oplus \sigma_2 = \mu \times (\mu * \sigma_2).$$

2. Wir weisen jetzt die Äquivalenz von (28) und (29) nach.

a) Es sei (28) erfüllt. Wegen $T_\mu - T_\mu \subseteq G_\mu$ gilt für alle u und v aus T_μ

$$\delta_u * \sigma_2 = \delta_v * \sigma_2$$

oder

$$\sigma_2(B-u) = \sigma_2(B-v) \quad \text{für alle } B \in \mathfrak{B}_G.$$

Hieraus folgt

$$\sigma_2(B-v) = \int_{T_\mu} \sigma_2(B-u) \mu(du) = \int_G \sigma_2(B-u) \mu(du) = (\mu * \sigma_2)(B) \quad \text{für alle } v \in T_\mu$$

und somit

$$\begin{aligned}\mu(A)(\mu * \sigma_2)(B) &= \int_A (\mu * \sigma_2)(B) \mu(dv) = \int_{T_\mu \cap A} (\mu * \sigma_2)(B) \mu(dv) = \\ &= \int_{T_\mu \cap A} \sigma_2(B-v) \mu(dv) = (\mu \oplus \sigma_2)((T_\mu \cap A) \times B) = (\mu \oplus \sigma_2)(A \times B).\end{aligned}$$

Aus der Beziehung

$$\mu(A)(\mu * \sigma_2)(B) = (\mu \oplus \sigma_2)(A \times B)$$

für alle $A, B \in \mathfrak{B}_G$ folgt (29).

b) Wegen $\sigma_2 \in \mathcal{P}_s$ ist die Abbildung $t \rightarrow \delta_t * \sigma_2$ von G in den metrischen Raum $[\mathcal{P}, \|\cdot\|]$ mit dem Variationsabstand als Metrik stetig. Bei festem $t_0 \in T_\mu$ existieren deshalb für $m=1, 2, \dots$ offene Umgebungen $U_{t_0, m}$ von t_0 mit

$$(32) \quad \|\delta_t * \sigma_2 - \delta_{t_0} * \sigma_2\| < \frac{1}{m} \quad \forall t \in U_{t_0, m}$$

$$(33) \quad \mu(U_{t_0, m}) > 0.$$

Sei

$$\mu_m := \frac{\mu(\cdot \cap U_{t_0, m})}{\mu(U_{t_0, m})}$$

Aus (29) folgt

$$\mu_m * \sigma_2 = \mu * \sigma_2$$

und nach (32)

$$\|\delta_{t_0} * \sigma_2 - \mu_m * \sigma_2\| < \frac{1}{m}.$$

Der Grenzübergang liefert

$$\delta_{t_0} * \sigma_2 = \mu * \sigma_2.$$

Da $t_0 \in T_\mu$ beliebig war, ergibt sich

$$\delta_u * \sigma_2 = \delta_v * \sigma_2 \quad \text{bzw.}$$

$$(34) \quad \delta_{u-v} * \sigma_2 = \sigma_2 \quad \forall u, v \in T_\mu.$$

$\{s: \delta_s * \sigma_2 = \sigma_2\}$ ist eine abgeschlossene Untergruppe G_0 von G . G_0 enthält infolge (34) $T_\mu - T_\mu$. Hieraus folgt $G_0 \supseteq G_\mu$, und dies beweist (28). Durch Benutzen von (30) und (31) ergibt sich aus Satz 3 die nachstehende Folgerung.

FOLGERUNG 1. *Unter den Voraussetzungen von Satz 3 gilt.*

$$(35) \quad \begin{aligned}\Delta_f(\mu \times \sigma_1, \mu \oplus \sigma_2, \mu \times \lambda) &\equiv \\ &\equiv \Delta_f(\sigma_1, \mu * \sigma_2, \lambda) + \frac{1}{2} [H_f(\mu * \sigma_2, \lambda) - H_f(\sigma_2, \lambda)]\end{aligned}$$

und für

$$\sigma_1 = \mu * \sigma_2$$

$$(36) \quad \Delta_f(\mu \times (\mu * \sigma_2), \mu \oplus \sigma_2, \mu \times \lambda) \equiv \frac{1}{2} [H_f(\mu * \sigma_2, \lambda) - H_f(\sigma_2, \lambda)].$$

Hierbei steht für streng konkaves f in (35) und (36) das Gleichheitszeichen genau dann, wenn (28) oder (29) erfüllt sind.

Wir spezialisieren jetzt $f = -x \log x$.

FOLGERUNG 2. Für beliebige $\mu, \sigma \in \mathcal{P}$ ist

$$\Delta(\mu \times (\mu^{n+1} * \sigma), \mu \oplus (\mu^n * \sigma))$$

monoton fallend n in und unter den Voraussetzungen von Satz 1 oder Satz 2 an μ und σ gilt.

$$(37) \quad \Delta(\mu \times (\mu^{n+1} * \sigma), \mu \oplus (\mu^n * \sigma)) \xrightarrow{n \rightarrow \infty} 0.$$

BEWEIS. Wir setzen in (26) $\nu_1 = \mu$, $\nu_2 = \mu^{n+1} * \sigma$, $\mu_1 = \mu$, $\mu_2 = \mu^n * \sigma$, dann folgt

$$\Delta(\mu \times (\mu^{n+2} * \sigma), \mu \oplus (\mu^{n+1} * \sigma)) \leq \Delta(\mu \times (\mu^{n+1} * \sigma), \mu \oplus (\mu^n * \sigma)).$$

Aus Ungleichung (36) in Folgerung 1 ergibt sich

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Delta(\mu \times (\mu^{n+1} * \sigma), \mu \oplus (\mu^n * \sigma)) &\leq \frac{1}{2} \liminf_{n \rightarrow \infty} [H(\mu^{n+1} * \sigma, \lambda) - H(\mu^n * \sigma, \lambda)] \leq \\ &\leq \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n [H(\mu^{k+1} * \sigma, \lambda) - H(\mu^k * \sigma, \lambda)]. \end{aligned}$$

Die rechte Seite der letzten Ungleichung ist nach (20) unter den Voraussetzungen in Satz 1 oder Satz 2 an μ und σ Null. Damit ist (37) gezeigt.

4. Eine Anwendung auf die Theorie der asymptotischen Gleichverteilung

Ist die lokalkompakte Gruppe G nicht kompakt, so existieren auf \mathfrak{B}_G keine verschiebungsinvarianten Wahrscheinlichkeitsmaße, d. h. es existiert keine Gleichverteilung.

Die zunehmende Gleichverteilung einer Folge von Wahrscheinlichkeitsverteilungen wurde in [6] durch die zunehmende Verschiebungsinvarianz erfaßt. Eine Folge $\sigma_n \in \mathcal{P}$ heißt nach [6] *asymptotisch gleichverteilt*, falls

$$\|\sigma_n - \delta_t * \sigma_n\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall t \in G$$

gilt. Nach [6] geht das Gewicht der bezüglich des Haarschen Maßes absolut stetigen Komponente einer asymptotisch gleichverteilten Folge von Verteilungen für $n \rightarrow \infty$ gegen 1. Gilt dies für die Folge $\mu_n \in \mathcal{P}$ nicht, so kann man Gleichverteilungseigenschaften nur von der geglätteten Folge $\sigma * \mu_n$, $\sigma \in \mathcal{P}_s$, erwarten. Für eine beliebige Folge $\mu_n \in \mathcal{P}$ sei

$$V((\mu_n)) := \{t : \|\sigma * \mu_n - \delta_t * \sigma * \mu_n\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall \sigma \in \mathcal{P}_s\}.$$

$V((\mu_n))$ ist eine abgeschlossene Untergruppe von G und heißt *Verschiebungsgruppe* der Folge $\mu_n \in \mathcal{P}$. Gilt $V((\mu_n)) = G$, so heißt $\mu_n \in \mathcal{P}$ *schwach asymptotisch gleichverteilt*. $V((\mu_n))$ charakterisiert die Gleichverteilungseigenschaft der μ_n durch die zunehmende Verschiebungsinvarianz der geglätteten Folgen $\sigma * \mu_n$, $\sigma \in \mathcal{P}_s$. In Anlehnung an [7] werden wir eine der Verschiebungsinvarianz äquivalente Aussage ableiten. Dieses Kriterium ergibt sich als Antwort auf folgende Frage:

Bleiben die Aussagen (28) und (29) in Satz 3 äquivalent, wenn die Gleichungen nicht exakt erfüllt sind, sondern mit einer Folge σ_n anstelle von σ_2 die Differenz der rechten und linken Seite von (28) und (29) im Sinne des Variationsabstandes für $n \rightarrow \infty$ gegen Null strebt?

SATZ 4. Es seien $\mu, \mu_n \in \mathcal{P}$, $\sigma \in \mathcal{P}_s$ sowie $\sigma_n := \sigma * \mu_n$. Dann sind folgende Aussagen äquivalent:

$$(38) \quad \|\sigma_n - \delta_s * \sigma_n\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall s \in G_\mu$$

$$(39) \quad \|\mu \times (\mu * \sigma_n) - \mu \oplus \sigma_n\| \xrightarrow{n \rightarrow \infty} 0$$

BEWEIS. 1. p_n bzw. q_n seien die Dichten von σ_n bzw. $\mu * \sigma_n$ bezüglich λ . Dann gilt mit (5) und $\mu \times \lambda = \mu \oplus \lambda$

$$(40) \quad \begin{aligned} \|\mu \times (\mu * \sigma_n) - \mu \oplus \sigma_n\| &= \int_G \int_G |q_n(y) - p_n(y-x)| \mu(dx) \lambda(dy) = \\ &= \int_G \int_G \left| \int_G p_n(y-z) \mu(dz) - p_n(y-x) \right| \mu(dx) \lambda(dy) \leq \\ &\leq \int_G \int_G \int_G |p_n(y-z) - p_n(y-x)| \mu(dz) \mu(dx) \lambda(dy) = \int_G \|\sigma_n - \delta_t * \sigma_n\| (\mu * \tilde{\mu})(dt) \end{aligned}$$

mit

$$\tilde{\mu}(A) := \mu(-A), A \in \mathfrak{B}_G.$$

G_μ ist die kleinste abgeschlossene Untergruppe von G , die $T_\mu - T_\mu$ enthält. Da $T_{\mu * \tilde{\mu}}$ der Abschluß von $T_\mu - T_\mu$ ist, gilt

$$(41) \quad T_{\tilde{\mu} * \mu} \subseteq G_\mu.$$

Ist jetzt (38) erfüllt, so folgt aus (40), (41) und dem Satz von LEBESGUE die Beziehung (39).

2. Es gilt

$$\begin{aligned} \|\mu \times (\mu * \sigma_n) - \mu \oplus \sigma_n\| &= \int_G \int_G |q_n(y) - p_n(y-x)| \mu(dx) \lambda(dy) = \\ &= \int_G \|\mu * \sigma_n - \delta_t * \sigma_n\| \mu(dt). \end{aligned}$$

Aus (39) folgt somit

$$(42) \quad \int_G \|\mu * \sigma_n - \delta_t * \sigma_n\| \mu(dt) \xrightarrow{n \rightarrow \infty} 0$$

$t_0 \in T_\mu$ sei fest gewählt und $K_{t_0} \subseteq T_\mu$ eine kompakte Umgebung von t_0 . Dann gilt

$$(43) \quad \mu(U_{t_0}) > 0$$

für jede Umgebung $U_{t_0} \subseteq K_{t_0}$ mit $U_{t_0} \in \mathfrak{B}_G$. Es sei

$$\varphi_n(t) := \|\mu * \sigma_n - \delta_t * \sigma_n\|$$

Die Folge φ_n besitzt folgende Eigenschaften

$$(44) \text{ (a)} \quad 0 \leq \varphi_n(t) \leq 2 \quad \forall t \in G$$

$$(b) \quad \begin{aligned} |\varphi_n(u) - \varphi_n(v)| &= \|\mu * \sigma_n - \delta_u * \sigma_n\| - \|\mu * \sigma_n - \delta_v * \sigma_n\| \leq \\ &\leq \|\delta_u * \sigma_n - \delta_v * \sigma_n\| = \|\delta_{u-v} * \sigma_n - \sigma_n\| \leq \|\delta_{u-v} * \sigma - \sigma\|. \end{aligned}$$

Wegen $\sigma \in \mathcal{P}_s$ ist $\|\delta_s * \sigma - \sigma\|$ eine stetige Funktion in s . Die Ungleichung (44) ergibt die gleichgradige Stetigkeit der φ_n in jedem Punkt von G .

Wir betrachten die Einschränkung der φ_n auf K_{t_0} . Nach dem Satz von ARZELA—ASCOLI existiert eine Teilfolge φ_{n_k} , die auf K_{t_0} gleichmäßig gegen eine stetige Funktion ψ konvergiert.

Ohne Beschränkung der Allgemeinheit kann man voraussetzen, daß die n_k so gewählt sind, daß

$$(45) \quad \lim_{k \rightarrow \infty} \varphi_{n_k}(t_0) = \overline{\lim}_{n \rightarrow \infty} \varphi_n(t_0)$$

gilt. Aus der Stetigkeit von ψ sowie (42) und (43) folgt $\psi(t_0) = 0$. Da t_0 beliebig war, ist

$$\|\mu * \sigma_n - \delta_t * \sigma_n\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall t \in T_\mu$$

gezeigt. Hieraus ergibt sich wegen

$$(46) \quad \begin{aligned} \|\delta_{u-v} * \sigma_n - \sigma_n\| &= \|\delta_u * \sigma_n - \delta_v * \sigma_n\| \leq \|\delta_u * \sigma_n - \mu * \sigma_n\| + \|\delta_v * \sigma_n - \mu * \sigma_n\| \\ \|\delta_{u-v} * \sigma_n - \sigma_n\| &\xrightarrow{n \rightarrow \infty} 0 \quad \forall u, v \in T_\mu. \\ V_\sigma &= \{s : \|\sigma_n - \delta_s * \sigma_n\| \xrightarrow{n \rightarrow \infty} 0\} \end{aligned}$$

ist wegen $\sigma \in \mathcal{P}_s$ eine abgeschlossene Untergruppe. V_σ enthält wegen (46) $T_\mu - T_\mu$ und somit gilt

$$V_\sigma \supseteq G_\mu$$

was (38) beweist.

Satz 4 zeigt, in welcher Weise informationstheoretische Beweise von Gleichverteilungseigenschaften möglich sind. Die Beziehung (39) in Satz 4 ist infolge Ungleichung (22) erfüllt, falls

$$\Delta(\mu \times (\mu * \sigma_n), \mu \oplus \sigma_n) \xrightarrow{n \rightarrow \infty} 0$$

gilt. $\Delta(\mu \times (\mu * \sigma_n), \mu \oplus \sigma_n)$ läßt sich nach (36) durch die Entropiedifferenz $\frac{1}{2}(H(\mu * \sigma_n, \lambda) - H(\sigma_n, \lambda))$ abschätzen. Damit erhalten wir Satz 5.

SATZ 5. Es seien μ, μ_n aus \mathcal{P} , $\sigma \in \mathcal{P}_s$ sowie $\sigma_n := \sigma * \mu_n$ und $\mu * \sigma_n$ aus $D(\lambda)$. Dann folgt aus

$$(48) \quad H(\mu * \sigma_n, \lambda) - H(\sigma_n, \lambda) \xrightarrow{n \rightarrow \infty} 0$$

$$(47) \quad \|\sigma_n - \delta_s * \sigma_n\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall s \in G_\mu.$$

Gilt jetzt $\mu_n = \mu^n$, dann ergibt sich

$$(49) \quad \Delta(\mu \times (\mu^{n+1} * \sigma), \mu \oplus (\mu^n * \sigma)) \xrightarrow{n \rightarrow \infty} 0, \text{ bereits dann}$$

wenn $H(\sigma * \mu^n, \lambda)$ für $n \rightarrow \infty$ nicht zu schnell wächst, wie aus dem Beweis der Folgerung 2 zu Satz 3 hervorgeht. Haben μ und σ kompakte Träger, liegt nach Satz 1 solch ein Wachstumsverhalten der Entropie vor und nach Folgerung 2 gilt (49). Die Anwendung von Satz 4 und Ungleichung (22) ergibt dann

$$\{s : \|\sigma * \mu^n - \delta_s * \sigma * \mu^n\| \xrightarrow{n \rightarrow \infty} 0\} \supseteq G_\mu.$$

Von der Voraussetzung der Kompaktheit der Träger kann man sich lösen und auf diesem Wege ergibt sich informationstheoretisch ein Resultat aus [6].

Satz 6. [6] Für alle $\mu \in \mathcal{P}$ gilt

$$V((\mu^n)) = G_\mu.$$

BEWEIS. 1. Zum Beweis benötigen wir eine Aussage, die eine einfache Modifizierung des Übertragungssatzes in [4] darstellt. Es seien ϱ, v, v_1 und v_2 aus \mathcal{P} , $0 < \alpha \leq 1$

$$\varkappa := \alpha v + (1 - \alpha)\varrho$$

sowie

$$\|v^n * v_1 - v^n * v_2\| \xrightarrow{n \rightarrow \infty} 0.$$

Dann gilt auch

$$(50) \quad \|\varkappa^n * v_1 - \varkappa^n * v_2\| \xrightarrow{n \rightarrow \infty} 0.$$

Beweis von (50):

$$\begin{aligned} \|\varkappa^n * v_1 - \varkappa^n * v_2\| &\leq \sum_{k=0}^n \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} \|v^k * \varrho^{n-k} * v_1 - v^k * \varrho^{n-k} * v_2\| \leq \\ &\leq \sum_{k=0}^n \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} \|v^k * v_1 - v^k * v_2\|. \end{aligned}$$

Zu $\varepsilon > 0$ existiert ein k_ε mit

$$(51) \quad \|v^k * v_1 - v^k * v_2\| < \frac{\varepsilon}{2} \quad \forall k \geq k_\varepsilon$$

sowie ein n_ε mit

$$(52) \quad \sum_{k=0}^{k_\varepsilon} \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} < \frac{\varepsilon}{2} \quad \forall n \geq n_\varepsilon.$$

Aus (51) und (52) ergibt sich $\|\varkappa^n * v_1 - \varkappa^n * v_2\| < \varepsilon \quad \forall n \geq n_\varepsilon$ und damit (50).

2. Für $\sigma \in \mathcal{P}_s$ existiert zu $\varepsilon > 0$ ein $\sigma_\varepsilon \in \mathcal{P}_s$ mit kompaktem Träger und beschränkter Dichte derart, daß

$$\|\sigma - \sigma_\varepsilon\| < \varepsilon$$

gilt. Dann folgt

$$\begin{aligned} \|\sigma * \mu^n - \delta_s * \sigma * \mu^n\| &\leq \|\sigma_\varepsilon * \mu^n - \delta_s * \sigma_\varepsilon * \mu^n\| + \|\sigma * \mu^n - \sigma_\varepsilon * \mu^n\| + \\ &+ \|\delta_s * \sigma * \mu^n - \delta_s * \sigma_\varepsilon * \mu^n\| \leq \|\sigma_\varepsilon * \mu^n - \delta_s * \sigma_\varepsilon * \mu^n\| + 2\varepsilon. \end{aligned}$$

Die Beziehung

$$(53) \quad \|\sigma * \mu^n - \delta_s * \sigma * \mu^n\| \xrightarrow{n \rightarrow \infty} 0 \quad \text{für alle } \sigma \in \mathcal{P}_s$$

ist also bereits richtig, falls (53) für alle $\sigma \in \mathcal{P}_s$ mit kompaktem Träger und beschränkter Dichte gilt.

3. Wir zeigen zunächst

$$G_\mu \subseteq V((\mu^n)).$$

Dazu reicht es

$$T_\mu - T_\mu \subseteq V((\mu^n))$$

nachzuweisen.

Seien $u, v \in T_\mu$ und $s = u - v$. $K_u \subseteq T_\mu$, $K_v \subseteq T_\mu$ seien kompakte Umgebungen von u und v sowie

$$K = K_u \cup K_v$$

$$\mu_K := \frac{\mu(\cdot \cap K)}{\mu(K)}.$$

Dann gilt

$$s \in G_{\mu_K}$$

und

$$(54) \quad \mu = \alpha \mu_K + (1 - \alpha) \varrho$$

mit

$$\alpha = \mu(K)$$

und

$$\varrho = \begin{cases} \frac{\mu(\cdot \cap (G \setminus K))}{\mu(G \setminus K)} & \mu(K) < 1 \\ \text{beliebig aus } \mathcal{P} & \mu(K) = 1 \end{cases}$$

$\sigma \in \mathcal{P}_s$ sei eine Verteilung mit kompaktem Träger und beschränkter Dichte. Dann gilt $\sigma * \mu_K^n \in D(\lambda)$ und aus Folgerung 2 zu Satz 3 ergibt sich

$$\Delta(\mu_K \times (\mu_K^{n+1} * \sigma), \mu_K \oplus (\mu_K^n * \sigma)) \xrightarrow{n \rightarrow \infty} 0.$$

Die Ungleichung (22) und Satz 4 ergeben wegen $s \in G_{\mu_K}$

$$\|\sigma * \mu_K^n - \delta_s * \sigma * \mu_K^n\| \xrightarrow{n \rightarrow \infty} 0.$$

Aus Teil 1 des Beweises folgt mit (54)

$$\|\sigma * \mu^n - \delta_s * \sigma * \mu^n\| \xrightarrow{n \rightarrow \infty} 0.$$

Da s beliebig war, ist

$$T_\mu - T_\mu \subseteq V((\mu^n))$$

nachgewiesen.

Der Beweis der Beziehung $V((\mu^n)) \subseteq G_\mu$ erfolgt wie in [6]. Sei $t \notin G_\mu$, dann existiert eine Umgebung U des Nullelements mit

$$t + U \cap G_\mu = \emptyset.$$

V sei eine offene Umgebung des Nullelements mit $V - V \subseteq U$, dann gilt

$$(t + V + G_\mu) \cap (G_\mu + V) = \emptyset.$$

Sei $\sigma \in \mathcal{P}_s$ mit $\sigma(V) = 1$. Dann gilt für $n = 1, 2, \dots$

$$\begin{aligned}(\sigma * \mu^n)(nt_{G_\mu} + G_\mu + V) &= 1 \\(\delta_t * \sigma * \mu^n)(t + nt_{G_\mu} + G_\mu + V) &= 1 \\(nt_{G_\mu} + G_\mu + V) \cap (t + nt_{G_\mu} + G_\mu + V) &= \emptyset.\end{aligned}$$

Hieraus ergibt sich

$$\|\sigma * \mu^n - \delta_t * \sigma * \mu^n\| = 2 \quad \forall n.$$

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ON A METRIZATION OF THE VAGUE CONVERGENCE

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1. Introduction

It is well-known that the weak topology in the space of probability distributions over a separable metric space S can be metrized. Such metrics have been defined by PROHOROV — the Prohorov-metric — and by FORTET and MOURIER — the dual-bounded-Lipschitz metric (cf. DUDLEY (1972)). In the case when S is a locally compact Abelian group the author (SZÁSZ (1971)) has defined a new metric equivalent to the weak topology. Modifying this definition we can metrize also the vague convergence. This metric can be used instead of the Prohorov metric when establishing vague convergence and not the weak one.

2. The metric $\varrho_{H,w}$

Let $\mathcal{M}(S)$ be the class of Borel measures defined in a metric space (S, d) with masses not exceeding 1 and denote by $\mathcal{P}(S)$ the subclass of probability measures. We recall that a sequence $\mu_1, \mu_2, \dots \in \mathcal{M}(S)$ tends *vaguely* to $\mu \in \mathcal{M}(S)$ (or briefly $\mu_n \Rightarrow \mu$) iff for every continuous real function $u(x)$ defined in S with a compact support

$$(1) \quad \lim_n \int u(x) \mu_n(dx) = \int u(x) \mu(dx)$$

(Integrals without specialization of their domain are to be taken over S .)

If $\mu_n, \mu \in \mathcal{P}(S)$ and (1) fulfils for every continuous bounded $u(x)$ then μ_n tends to μ *weakly* (or briefly $\mu_n \Rightarrow \mu$). In SZÁSZ (1971) it was supposed that

(2) (i) S is a locally compact second countable Abelian group:

(ii) $H \in \mathcal{P}(S)$ is absolutely continuous with a bounded and continuous density $h(x)$ and a non vanishing characteristic function;

and it was shown that $\mathcal{P}(S)$ supplied with the weak topology can be metrized by

$$(3) \quad \varrho_H(P, Q) = \text{Var} [(P - Q) * H]$$

(here Var denotes the total variation of the signed measure in brackets and $*$ denotes the convolution). ϱ_H can also be expressed in an analytic form

$$(4) \quad \varrho_H(P, Q) = \int \left| \int h(x-t) d(P-Q)(t) \right| dx$$

(dx denotes integral with respect to the Haar measure and $|A|$ the Haar measure of the set A .)

As the difference between weak and vague convergence is that the later one is insensitive to the behaviour of the measures of *infinity*, in order to metrize the vague convergence a natural way is to modify the form (4) of ϱ_H by introducing a weight function $w(x)$, i.e.

$$(5) \quad \varrho_{H,w}(P, Q) = \int \left| \int h(x-t) d(P-Q)(t) \right| w(x) dx.$$

About $w(x)$ we suppose that

- (6) $\alpha)$ $0 < w(x) \leq 1$;
 $\beta)$ for every $\varepsilon > 0$ there exists a compact $K = K(\varepsilon)$ such that $w(x) < \varepsilon$ if $x \notin K$;
 $\gamma)$ $w(x)$ is Borel measurable.

THEOREM 1. If S , H and w satisfy conditions (2) and (6) then $\varrho_{H,w}$ metrizes $\mathcal{M}(S)$.

PROOF. If $\varrho_{H,w}(P, Q) = 0$ then by the positivity of w

$$\int h(x-t) d(P-Q)(t) = 0$$

for a.e. x . Hence $H * P = H * Q$. The characteristic function of H does not vanish so from the equality of the characteristic functions of $H * P$ and $H * Q$ $P = Q$ follows (cf. PARTHASARATHY (1967), Chapter 4, § 3). Symmetry and triangle inequality are obvious.

3. Equivalence with the vague convergence

THEOREM 2. If the conditions (2), (6) and also the condition (iii) for every $\varepsilon > 0$ there exists a compact $K = K(\varepsilon)$ such that $h(x) < \varepsilon$ if $x \notin K$ are satisfied then the following two assertions are equivalent ($\mu_n, \mu \in \mathcal{M}(S)$)

$$(A) \quad \mu_n \Rightarrow \mu \quad (n \rightarrow \infty);$$

$$(B) \quad \lim_n \varrho_{H,w}(\mu_n, \mu) = 0.$$

PROOF. (A) \rightsquigarrow (B).

Let $\varepsilon > 0$ be given and choose compacts K and C such that $w(x) < \varepsilon$ if $x \notin K$ and $h(x) < \varepsilon$ if $x \notin C$. $K - C$ (algebraic difference!) is also compact so we can find a compact continuity set L of μ for which $L \supset K - C$. Then

$$\begin{aligned} \varrho_{H,w}(\mu_n, \mu) &= \int_K \left| \int_L h(x-t) d(\mu_n - \mu)(t) \right| w(x) dx + \\ &+ \int_K \left| \int_{L^c} \dots \right| + \int_{K^c} \left| \int \dots \right| = A_1 + A_2 + A_3. \end{aligned}$$

Put

$$u_n(x) = \int_L h(x-t) d(\mu_n - \mu)(t).$$

By the Helly—Bray lemma (see LOÈVE (1955), Chapter 4) $\lim_n u_n(x) = 0$. But $|u_n(x)| \leq 2 \sup h(x)$ so applying the Lebesgue convergence theorem for A_1 we obtain that $A_1 = o_x(1)$. A_2 can be estimated by noticing that if $x \in K$ and $t \in L^\varepsilon$ then $x - t \notin C$ and so $h(x - t) < \varepsilon$. Hence

$$A_2 \leq |K|2\varepsilon.$$

Finally it is obvious that

$$A_3 \leq \varepsilon \int \int h(x - t) d(\mu_n + \mu)(t) dx \leq 2\varepsilon.$$

(B) \rightsquigarrow (A).

Let μ_n and μ satisfy (B). If for some subsequence of integers

$$\mu_{n_k} \Rightarrow \nu \quad (k \rightarrow \infty, \nu \in \mathcal{M}(S))$$

then we have already proved that

$$\lim_k \varrho_{H,w}(\mu_{n_k}, \nu) = 0$$

hence by (B) and the triangle inequality $\nu = \mu$. But the Helly compactness theorem ensures the existence of at least one vaguely convergent subsequence. Hence the assertion.

4. Remarks

1. To define the metric $\varrho_{H,w}$ we need to give a measure H and a function w satisfying (ii), (iii) and α, β, γ resp. The most obvious choice for H is to choose it to be non singular normal. If S is a Euclidean space then this can be done while in the general case about densities of Gaussian distributions the author has no information. In the Euclidean case w can be chosen to be equal to the density of H .

2. Maybe the idea used here makes possible to define metrics equivalent to the vague or weak convergence of measures given in more general metric spaces than considered here. In connection with this it is worth to point out of that the reason why ϱ_H is a metric equivalent to the weak convergence is that the convolution operator

$$A_H : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$$

defined by

$$A_H P = H * P$$

determines an invertible transformation from $\mathcal{P}(S)$ endowed with the weak topology into $\mathcal{P}(S)$ endowed with the total variation topology and both A_H and A_H^{-1} are continuous.

3. The following simple property can be proved easily: if X and Y are random variables with values in S such that

$$P\{d(X, Y) > \varepsilon\} < \varepsilon$$

then

$$\varrho_{H,w}(\mathcal{L}(X), \mathcal{L}(Y)) \leq |K|(m_h(\varepsilon) + 2\varepsilon \sup_x h(x)) + 2\varepsilon$$

where

$$m_h(\varepsilon) = \sup_{d(x,y) < \varepsilon} |h(x) - h(y)|.$$

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* *Added in proof:* After this paper had been submitted the author found an interesting reference: D. A. SIBLEY: A metric for weak convergence of distribution functions. *Rocky Mountain J. Math.* **1** (1971) 427—430, where, in fact, vague convergence is metrized for distributions on the real line.

THE HERMITE—FEJÉR PROCESS ON THE TCHEBYCHEFF MATRIX OF SECOND KIND

by

R. B. SAXENA

1. Let $E_n = \{x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\}$ denote the n th row of a triangular matrix E and let

$$(1.1) \quad -1 \leq x_n < \dots < x_1 \leq 1.$$

For simplicity we shall write x_v for $x_v^{(n)}$.

The Hermite—Fejér interpolation polynomial $H_n(f, x)$ of degree $2n-1$ such that

$$H_n(f, x_v) = f(x_v), \quad H_n'(f, x_v) = 0$$

is given by

$$(1.2) \quad H_n(f, x) = \sum_{v=1}^n f(x_v) \left[1 - \frac{\omega''(x_v)}{\omega'(x_v)} (x - x_v) \right] l_v^2(x),$$

where

$$(1.3) \quad l_v(x) = \frac{\omega(x)}{\omega'(x_v)(x - x_v)}, \quad \prod_{v=1}^n (x - x_v) = \omega(x).$$

L. FEJÉR [4] proved that if the matrix of nodes (1.1) consists of the points

$$(1.4) \quad x_v^{(n)} = \cos \left(v - \frac{1}{2} \right) \frac{\pi}{n}, \quad v = 1, \dots, n; \quad n = 1, 2, \dots$$

which are the roots of $T_n(x)$ —the Tchebycheff polynomial of first kind, then for any function $f(x) \in C^*$ one has

$$(1.5) \quad H_n(f, x) \rightarrow f(x), \quad \text{for } n \rightarrow \infty$$

uniformly for $-1 \leq x \leq 1$.

D. L. BERMAN [1] and [2] has studied the Hermite—FEJÉR interpolation process for the system of nodes

$$(1.6) \quad x_0 = 1, \quad x_v = \cos \left(v - \frac{1}{2} \right) \frac{\pi}{n}, \quad v = 1, \dots, n, \quad x_{n+1} = -1,$$

which is obtained from (1.4) by adding the points $x = \pm 1$ as nodes. Unlike FEJÉR's result (1.5) he proved that the Hermite—Fejér process constructed for the nodes (1.6) diverges at $x=0$ for $f(x)=|x|$ and throughout $(-1, 1)$ for $f(x)=x^2$.

In view of these results one may expect that addition of the points $x = \pm 1$ as nodes may worsen the situation regarding the convergence of Hermite—Fejér process.

* C denotes the set of functions continuous in $[-1, 1]$.

As BERMAN [3] showed, this is not always so. He proved that the Hermite—Fejér interpolation process for $f \in C$ for the nodes

$$(1.7) \quad x_v^{(n)} = \cos \frac{v-1}{n-1} \pi, \quad v = 1, \dots, n; \quad n = 1, 2, \dots$$

converges at the point $x=0$, and posed the problem of convergence of this process at points different from $x=0$ in case of arbitrary $f \in C$. He [3] however partially answered this question by showing that for $f(x)=x$ and $f(x)=x^2$, $H_n(f, x)$ converges to $f(x)$ with an error of order $O\left(\frac{1}{n}\right)$ in $[-1, 1]$.

In this paper we shall answer his question for arbitrary $f \in C$ and $x \in [-1, 1]$. We shall prove the following

THEOREM. *The Hermite—Fejér interpolation process $H_n(f, x)$ constructed for the function $f \in C$ on the nodes (1.7) converges uniformly to $f(x)$ in $[-1, 1]$.*

In § 2 we obtain the explicit form of $H_n(f, x)$ and in § 3 we establish a trigonometric identity which, apart from its use here, is interesting in its own right. Further, in § 4 we obtain some auxiliary results and, finally, in § 5 we complete the proof of the theorem.

2. Explicit expression for $H_n(f, x)$

It is obvious that for the nodes (1.7)

$$(2.1) \quad \omega(x) = (1-x^2)U_{n-2}(x)$$

where $U_{n-2}(x)$ is the Tchebycheff polynomial of second kind given by

$$(2.2) \quad U_{n-2}(x) = \frac{\sin(n-1)\theta}{\sin \theta}, \quad \cos \theta = x.$$

The differential equation satisfied by $U_{n-2}(x)$ is

$$(2.3) \quad (1-x^2)U_{n-2}''(x) - 3xU_{n-2}'(x) + n(n-2)U_{n-2}(x) = 0.$$

We can easily verify that

$$(2.4) \quad \left. \begin{aligned} \omega'(x_v) &= (-1)^{v-1}(n-1) \\ \frac{\omega''(x_v)}{\omega'(x_v)} &= -\frac{x_v}{1-x_v^2} \end{aligned} \right\} \quad v = 2, \dots, n-1,$$

$$\omega'(1) = -2n(n-1) = (-1)^{n-1}\omega'(-1)$$

$$(2.5) \quad \frac{\omega''(1)}{\omega'(1)} = \frac{2n^2 - 4n + 3}{2} = -\frac{\omega''(-1)}{\omega'(-1)}.$$

Hence from (1.2), (1.3) using (2.4) and (2.5) we get

$$(2.6) \quad H_n(f, x) = f(1) \left[1 + \frac{2n^2 - 4n + 3}{3} (1-x) \right] \left[\frac{1+x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \\ + f(-1) \left[1 + \frac{2n^2 - 4n + 3}{3} (1+x) \right] \left[\frac{1-x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \\ + \sum_{v=2}^{n-1} f(x_v) \left[1 + \frac{x_v}{1-x_v^2} (x-x_v) \right] \left[\frac{(1-x^2) U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2.$$

3. A trigonometric identity

LEMMA 3.1. *The following identity holds:*

$$(3.1) \quad \sum_{k=2}^{n-1} \operatorname{cosec}^2 \frac{k-1}{n-1} \pi = \frac{n(n-2)}{3}.$$

PROOF. For $n=2m$ this identity has been established by D. L. BERMAN [3]. Therefore let $n=2m+1$. We shall introduce the Hermite-Fejér interpolation of degree $4m-1$ constructed on the nodes

$$(3.2) \quad x_k = \cos \frac{k-1}{2m} \pi, \quad k = 1, \dots, 2m+1; \quad k \neq m+1.$$

It is evident that, for the nodes (3.2), $\omega(x)$ in (1.2) is given by

$$(3.3) \quad \omega(x) = (1-x^2) U_{m-1}(2x^2-1)$$

where $U_{m-1}(t)$ is given by (2.2). From (2.3) we see that the differential equation satisfied by $U_{m-1}(2x^2-1)$ is

$$(3.4) \quad 4x^2(1-x^2)U''_{m-1}(x) - 3(2x^2-1)U'_{m-1}(2x^2-1) + (m^2-1)U_{m-1}(2x^2-1) = 0.$$

We can now easily verify that

$$(3.5) \quad \left. \begin{aligned} \omega'(x_k) &= (-1)^{k-1} \frac{m}{x_k} \\ \frac{\omega''(x_k)}{\omega'(x_k)} &= -\frac{2}{x_k} - \frac{x_k}{1-x_k^2} \end{aligned} \right\} \quad k = 2, \dots, 2m; \quad k \neq m+1,$$

$$(3.6) \quad \omega'(1) = -2m = -\omega'(-1)$$

$$\frac{\omega''(1)}{\omega'(1)} = \frac{8(m^2-1)}{3} + 1 = -\frac{\omega''(-1)}{\omega'(-1)}.$$

After simple calculations we have from (1.2) and (1.3)

$$(3.7) \quad H_{4m-1}(f, x) = f(1) \left[1 + (1-x) \left\{ \frac{8(m^2-1)}{3} + 1 \right\} \right] \left[\frac{1+x}{2} \cdot \frac{U_{m-1}(2x^2-1)}{m} \right]^2 + \\ + f(-1) \left[1 + (1+x) \left\{ \frac{8(m^2-1)}{3} + 1 \right\} \right] \left[\frac{1-x}{2} \cdot \frac{U_{m-1}(2x^2-1)}{m} \right]^2 + \\ + \sum_{k=2}^{2m} f(x_k) \left[1 + \left(\frac{2}{x_k} + \frac{x_k}{1-x_k^2} \right) (x-x_k) \right] \left[\frac{x_k(1-x^2)U_{m-1}(2x^2-1)}{m(x-x_k)} \right]^2.$$

Since $H_{4m-1}(1, x) = 1$, putting $f(x) = 1$ in (3.7) we get the identity

$$\left[1 + (1-x) \left\{ \frac{8(m^2-1)}{3} + 1 \right\} \right] \left[\frac{1+x}{2} \cdot \frac{U_{m-1}(2x^2-1)}{m} \right]^2 + \\ + \left[1 + (1+x) \left\{ \frac{8(m^2-1)}{3} + 1 \right\} \right] \left[\frac{1-x}{2} \cdot \frac{U_{m-1}(2x^2-1)}{m} \right]^2 + \\ + \sum_{\substack{k=2 \\ k \neq m+1}}^{2m} \left[1 + \left(\frac{2}{x_k} + \frac{x_k}{1-x_k^2} \right) (x-x_k) \right] \left[\frac{x_k(1-x^2)U_{m-1}(2x^2-1)}{m(x-x_k)} \right]^2.$$

Hence for $x=0$ we find that

$$\left[1 + \frac{4(m^2-1)}{3} \right] \frac{U_{m-1}^2(-1)}{m^2} - \sum_{\substack{k=2 \\ k \neq m+1}}^{2m} \left(1 + \frac{x_k^2}{1-x_k^2} \right) \frac{U_{m-1}^2(-1)}{m^2} = 1.$$

Since $U_{m-1}(-1) = (-1)^{m-1}m$, we have

$$\sum_{\substack{k=2 \\ k \neq m+1}}^{2m} \frac{1}{1-x_k^2} = \frac{4(m^2-1)}{3}.$$

ie.

$$\sum_{k=2}^{2m} \frac{1}{1-x_k^2} = \frac{4(m^2-1)}{3} + 1 = \frac{4m^2-1}{3} = \frac{(2m+1)(2m-1)}{3}$$

and (3.1) is proved.

4. Auxiliary results

LEMMA 4.1. For any $f \in C$ we have

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{1}{(n-1)^2} \sum_{k=2}^{n-1} f \left(\cos \frac{k-1}{n-1} \pi \right) \frac{1}{\sin^2 \frac{k-1}{n-1} \pi} = \frac{f(1)+f(-1)}{6},$$

$$(4.2) \quad \lim_{n \rightarrow \infty} \frac{1}{(n-1)^2} \sum_{k=2}^{n-1} f \left(\cos \frac{k-1}{n-1} \pi \right) \frac{\cos \frac{k-1}{n-1} \pi}{\sin^2 \frac{k-1}{n-1} \pi} = \frac{f(1)-f(-1)}{6}$$

The first part (4.1) of this lemma has been proved by BERMAN [3] in case when n is even. The proof makes use of the identity (3.1) which is now proved to be true for both even and odd n . (4.2) can be proved in the same way as BERMAN [3] proved (4.1).

LEMMA 4.2. For $0 \leq \theta \leq \pi$; $\theta_v = \frac{v-1}{n-1} \pi$

$$\sum_{v=2}^{n-1} \frac{\sin^2(n-1) \frac{\theta - \theta_v}{2}}{(n-1)^2 \sin^2 \frac{\theta - \theta_v}{2}} \leq 6.$$

PROOF. Let $\theta_i \leq \theta \leq \theta_{i+1}$. In the case $2 \leq v < i \leq n-2$; $3 \leq i \leq n-1$,

$$\theta_i - \theta_v \leq \theta - \theta_v \leq \pi,$$

therefore

$$\sin \frac{\theta - \theta_v}{2} \geq \sin \frac{\theta_i - \theta_v}{2} = \sin \frac{i-v}{n-1} \frac{\pi}{2} > \frac{i-v}{n-1}.$$

Hence

$$(4.3) \quad K_v \stackrel{\text{def}}{=} \frac{\sin^2(n-1) \frac{\theta - \theta_v}{2}}{(n-1)^2 \sin^2 \frac{\theta - \theta_v}{2}} \leq \frac{1}{(i-v)^2}.$$

In the case $i+1 \leq v \leq n-1$, $1 \leq i \leq n-3$

$$\theta_v - \theta_{i+1} \leq \theta_v - \theta \leq \pi,$$

we have

$$\sin \frac{\theta_v - \theta}{2} \geq \sin \frac{\theta_v - \theta_{i+1}}{2} = \sin \frac{v-i-1}{n-1} \cdot \frac{\pi}{2} > \frac{v-i-1}{n-1},$$

so that

$$(4.4) \quad K_v \leq \frac{1}{(v-i-1)^2}.$$

Also for $2 \leq i \leq n$,

$$(4.5) \quad K_i \leq 1,$$

and for $1 \leq i \leq n-1$,

$$(4.6) \quad K_{i+1} \leq 1.$$

Here we use that

$$|\sin(n-1)\theta| \leq (n-1)|\sin \theta|.$$

Thus from (4.3)–(4.6),

$$\begin{aligned} \sum_{v=2}^{n-1} K_v &= \sum_{v=2}^{i-1} K_v + K_i + K_{i+1} + \sum_{v=i+2}^{n-1} K_v = \sum_{v=2}^{i-1} \frac{1}{(i-v)^2} + 1 + 1 + \sum_{v=i+2}^{n-1} \frac{1}{(v-i-1)^2} \leq \\ &\leq 2 \left(1 + \sum_{j=1}^{\infty} \frac{1}{j^2} \right) \leq 6. \end{aligned}$$

LEMMA 4.3. For $-1 \leq x \leq 1$, we have

$$(4.7) \quad \sum_{v=2}^{n-1} \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 \leq 144,$$

$$(4.8) \quad \sum_{v=2}^{n-1} \left[\frac{x(1-x^2)U_{n-2}^2(x)}{(n-1)^2(x-x_v)} \right] \leq 72.$$

PROOF. Putting $x = \cos \theta$ and $x = \cos \theta_v$ in (4.7), we have

$$(4.9) \quad \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 = \left[\frac{\sin \theta \sin(n-1)\theta}{(n-1)(\cos \theta - \cos \theta_v)} \right]^2.$$

Since

$$\sin^2(n-1)\theta = [\sin(n-1)\theta - \sin(n-1)\theta_v]^2 =$$

$$= 4 \cos^2(n-1) \frac{\theta + \theta_v}{2} \sin^2(n-1) \frac{\theta - \theta_v}{2} \leq 4 \sin^2(n-1) \frac{\theta + \theta_v}{2},$$

and

$$\sin^2 \theta \leq (\sin \theta + \sin \theta_v)^2 = 4 \sin^2 \frac{\theta + \theta_v}{2} \cos^2 \frac{\theta - \theta_v}{2} \leq 4 \sin^2 \frac{\theta + \theta_v}{2}.$$

We have from (4.9)

$$\left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 \leq \left[\frac{2 \sin(n-1) \frac{\theta - \theta_v}{2}}{(n-1) \sin \frac{\theta - \theta_v}{2}} \right]^2 \leq 144$$

using lemma 4.2.

Similarly

$$\begin{aligned} \left| \frac{x(1-x^2)U_{n-2}^2(x)}{(n-1)^2(x-x_v)} \right| &= \left| \frac{\cos \theta \sin^2(n-1)\theta}{2(n-1)^2 \sin \frac{\theta + \theta_v}{2} \sin \frac{\theta - \theta_v}{2}} \right| \leq \\ &\leq 2 \left[\frac{\sin(n-1) \frac{\theta - \theta_v}{2}}{(n-1) \sin \frac{\theta - \theta_v}{2}} \right]^2 \end{aligned}$$

owing to

$$\frac{1}{\sin \frac{\theta + \theta_v}{2}} \leq \frac{1}{\sin \frac{|\theta - \theta_v|}{2}}; \quad 0 \leq \theta \leq \pi, \quad 0 \leq \theta_v \leq \pi,$$

and then using lemma 4.2 we at once get (4.5).

5. Proof of the theorem

We shall first simplify the expression for $H_n(f, x)$ in (2.6) in a form convenient to us in later use. Thus

$$(5.1) \quad H_n(f, x) = f(1) \left[\frac{1+x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + f(-1) \left[\frac{1-x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \\ + \sum_{v=2}^{n-1} f(x_v) \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 + \\ + \frac{2n^2-4n+3}{12(n-1)^2} [(1+x)f(1) + (1-x)f(-1)](1-x^2)U_{n-2}^2(x) + \\ + \frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \sum_{v=2}^{n-1} f(x_v) \frac{x_v(1-x^2)}{(x-x_v)(1-x_v^2)}.$$

Since

$$\frac{x_v(1-x^2)}{(x-x_v)(1-x_v^2)} = \frac{x_v}{x-x_v} - \frac{xx_v}{1-x_v^2} + 1 = \frac{x}{x-x_v} - \frac{xx_v}{1-x_v^2} - \frac{1}{1-x_v^2},$$

we get

$$\frac{2n^2-4n+3}{12(n-1)^2} [(1+x)f(1) + (1-x)f(-1)](1-x^2)U_{n-2}^2(x) + \\ + \frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \sum_{v=2}^{n-1} f(x_v) \frac{x_v(1-x^2)}{(x-x_v)(1-x_v^2)} = \\ = \frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \sum_{v=2}^{n-1} f(x_v) \frac{x_v}{x-x_v} + \\ + (1-x^2)U_{n-2}^2(x) \left[\frac{f(1)+f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} \frac{f(x_v)}{1-x_v^2} \right] + \\ + x(1-x^2)U_{n-2}^2(x) \left[\frac{f(1)-f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} f(x_v) \frac{x_v}{1-x_v^2} \right] + \\ + \frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \sum_{v=2}^{n-1} f(x_v) + \frac{(1-x^2)U_{n-2}^2(x)}{12(n-1)^2} [(1+x)f(1) + (1-x)f(-1)].$$

Hence

$$(5.2) \quad H_n(f, x) = f(1) \left[\frac{1+x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + f(-1) \left[\frac{1-x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \\ + \sum_{v=2}^{n-1} f(x_v) \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 + \sum_{v=2}^{n-1} f(x_v) \frac{x_v(1-x^2)U_{n-2}^2(x)}{(n-1)^2(x-x_v)} + \\ + (1-x^2)U_{n-2}^2(x) \left[\frac{f(1)+f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} \frac{f(x_v)}{1-x_v^2} \right] +$$

$$+ x(1-x^2)U_{n-2}^2(x) \left[\frac{f(1)-f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=1}^{n-1} f(x_v) \frac{x_v}{1-x_v^2} \right] + \\ + \frac{(1-x^2)U_{n-2}^2(x)}{12} \left[\frac{(1+x)f(1)+(1-x)f(-1)}{12} + \sum_{v=2}^{n-1} f(x_v) \right].$$

Putting $f(x)=1$ in (5.2) we have owing to the identity $H_n(1, x)=1$,

$$(5.3) \quad 1 = \left[\frac{1+x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \left[\frac{1-x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \sum_{v=2}^{n-1} \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 + \\ + \sum_{v=2}^{n-1} \frac{x_v(1-x^2)U_{n-2}^2(x)}{(n-1)^2(x-x_v)} + (1-x^2)U_{n-2}^2(x) \left[\frac{1}{3} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} \frac{1}{1-x_v^2} \right] - \\ - \frac{x(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \sum_{v=2}^{n-1} \frac{x_v}{1-x_v^2} + \frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \left[\frac{1}{6} + (n-1) \right].$$

Since

$$\sum_{v=2}^{n-1} \frac{x_v}{1-x_v^2} = 0,$$

we have on using the identity (3.1):

$$(5.4) \quad 1 = \left[\frac{1+x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \left[\frac{1-x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 + \sum_{v=2}^{n-1} \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 + \\ + \sum_{v=2}^{n-1} \left[\frac{x(1-x^2)U_{n-2}^2(x)}{(n-1)^2(x-x_v)} \right] + \frac{(1-x^2)U_{n-2}^2(x)}{2(n-1)^2}.$$

Multiplying (5.4) by $f(x)$ and subtracting from (2.8), we have

$$(5.5) \quad H_n(f, x) - f(x) = [f(1) - f(x)] \left[\frac{1+x}{2} \frac{U_{n-2}(x)}{n-1} \right]^2 + \\ + [f(-1) - f(x)] \left[\frac{1-x}{2} \frac{U_{n-2}(x)}{n-1} \right]^2 + \\ + \sum_{v=2}^{n-1} [f(x_v) - f(x)] \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 + \sum_{v=2}^{n-1} [f(x_v) - f(x)] \left[\frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2(x-x_v)} \right] + \\ + (1-x^2)U_{n-2}^2(x) \left[\frac{f(1)+f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} \frac{f(x_v)}{1-x_v^2} \right] + \\ + x(1-x^2)U_{n-2}^2(x) \left[\frac{f(1)-f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} f(x_v) \frac{x_v}{1-x_v^2} \right] + \\ + \frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \left[\frac{(1+x)f(1)+(1-x)f(-1)-6f(x)}{12} \right].$$

Now

$$(5.6) \quad [f(1) - f(x)] \left[\frac{1+x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 \rightarrow 0$$

uniformly for $-1 \leq x \leq 1$. This follows from the fact that the first factor vanishes at $x=1$ and is continuous for $-1 \leq x \leq 1$, while

$$\left[\frac{1+x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 = \left[\frac{\cos \frac{\theta}{2} \sin(n-1)\theta}{2(n-1) \sin \frac{\theta}{2}} \right]^2 \rightarrow 0$$

uniformly in each sub interval $-1 \leq x \leq 1 - \sigma$ ($0 < \sigma < 2$) and is bounded in the whole closed interval since

$$\left[\cos \frac{\theta}{2} \cdot \frac{\sin(n-1)\theta}{\sin \frac{\theta}{2}} \right]^2 \leq 4(n-1)^2.$$

Similarly

$$(5.7) \quad [f(-1) - f(x)] \left[\frac{1-x}{2} \cdot \frac{U_{n-2}(x)}{n-1} \right]^2 \rightarrow 0, \quad -1 \leq x \leq 1.$$

Since $f(x)$ is continuous in $[-1, 1]$, let

$$(5.8) \quad |f(x)| \leq M, \quad -1 \leq x \leq 1.$$

Given $\varepsilon > 0$ we can find a positive number δ such that

$$(5.9) \quad |x' - x''| \leq \delta \Rightarrow |f(x') - f(x'')| < \varepsilon,$$

the continuous function being also uniformly continuous for $-1 \leq x \leq 1$. Therefore

$$(5.10) \quad \left| \sum_{|x_v - x| \leq \delta} [f(x_v) - f(x)] \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 \right| \leq \\ \leq \varepsilon \sum_{|x_v - x| \leq \delta} \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 \leq 36\varepsilon$$

on using (4.7).

On the other hand

$$(5.11) \quad \left| \sum_{|x_v - x| > \delta} [f(x_v) - f(x)] \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 \right| \leq \\ \leq \frac{2M}{\delta^2} \sum_{|x_v - x| > \delta} \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right]^2 < \frac{2M}{(n-1)\delta^2} \leq \varepsilon$$

if n is large enough i.e. $n \geq N$, say. Here we used

$$(5.12) \quad \sqrt{1-x^2} |U_{n-2}(x)| \leq 1, \quad -1 \leq x \leq 1.$$

So by (5.10) and (5.11)

$$(5.13) \quad \left| \sum_{v=2}^{n-1} [f(x_v) - f(x)] \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_v)} \right] \right|^2 \leq 37\varepsilon$$

when $n \geq N$.

Similarly using (4.8) and arguing as above we can show that

$$(5.14) \quad \left| \sum_{v=2}^{n-1} [f(x_v) - f(x)] \left[\frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2(x-x_v)} \right] \right| \leq 73\varepsilon,$$

for $n \geq N$.

The terms

$$(5.15) \quad (1-x^2)U_{n-2}^2(x) \left[\frac{f(1)+f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} \frac{f(x_v)}{1-x_v^2} \right]$$

and

$$(5.16) \quad x(1-x^2)U_{n-2}^2(x) \left[\frac{f(1)-f(-1)}{6} - \frac{1}{(n-1)^2} \sum_{v=2}^{n-1} f(x_v) \frac{x_v}{1-x_v^2} \right]$$

tend to zero uniformly in $[-1, 1]$ on account of the lemma 4.1 and the inequality (5.12).

Finally the term

$$(5.17) \quad \left| \frac{(1-x^2)U_{n-2}^2(x)}{(n-1)^2} \left[\frac{(1+x)f(1) + (1-x)f(-1) - 6f(x)}{12} \right] \right| \leq$$

$$\leq \frac{10M}{12(n-1)^2} \leq \varepsilon, \quad n \geq N,$$

on account of (5.8) and (5.12). Thus (5.5), (5.6), (5.7), (5.13), (5.14), (5.15), (5.16) and (5.17) complete the proof of the theorem.

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О СВОЙСТВАХ МИНИМАЛЬНОЙ ПРЕДЕЛЬНОЙ ЛОГИКИ Q

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Введение

В этой работе мы изучаем свойства одной предельной логики Q [1], которая порождается одним элементом. Как будет показано, функции из Q и сам класс Q обладают определенными особенностями. Так, например, для задания любой функции из Q достаточно описать её на конечном отрезке натурального ряда, в логике Q существует лишь один предельный класс, любая функция из Q существенно зависит от всех переменных встречающихся в формуле, реализующей её и т. п.

Эта логика впервые встречалась в работах [1], [2] как выяснилось недавно [3], эта логика замечательная тем, что она является в некотором смысле простейшей среди предельных логик. Точнее, она может быть изоморфно вложена в любую предельную логику. Это обстоятельство придает ее исследованию особую значимость.

§ 1. Основные понятия

В этом разделе мы приведем определения основных понятий многозначной логики, которые будут необходимы нам в дальнейшем изложении, и введем ряд обозначений. Мы исходим из счетного алфавита переменных $W = \{w_1, w_2, w_3, \dots\}$. В дальнейшем во избежание употребления сложных индексов мы будем использовать метасимволы x, y, z с индексами или без них для обозначения букв этого алфавита. P_{\aleph_0} обозначает множество всех функций, переменные которых определены на множестве E_{\aleph_0} мощности \aleph_0 и сами функции принимают значения из этого же множества. В качестве E_{\aleph_0} возьмем множество всех целых неотрицательных чисел $\{0, 1, 2, \dots\}$. Множество P_{\aleph_0} называется счетнозначной логикой. Обычно, P_{\aleph_0} задается путем фиксации алфавита переменных $X = \{x_1, x_2, \dots\}$ и имен функций $f_v(x_{i_1}, \dots, x_{i_n})$, где v пробегает некоторое континуальное множество индексов (обозначение: $P_{\aleph_0}(X)$).

Функция $g(z_1, \dots, z_i, \dots, z_n)$ называется функцией k -значной логики P_k , $k \geq 2$, если ее аргументы определены на множестве $E_k = \{0, 1, \dots, k-1\}$ и любое значение $g(a_1, \dots, a_i, \dots, a_n)$ принадлежит множеству E_k при $a_i \in E_k$, $i = 1, 2, \dots, n$.

Понятия суперпозиции и замыкания множества определяются обычным образом [4], [5].

Пусть \mathfrak{M} и \mathfrak{N} -подмножества множества P_{\aleph_0} . Обозначим через

$$X = \{x_1, x_2, x_3, \dots\}$$

(соответственно $Y = \{y_1, y_2, y_3, \dots\}$) множество символов всех независимых переменных, встречающихся у функций $f \in \mathfrak{M}$ (соответственно $h \in \mathfrak{N}$).

Множество \mathfrak{M} функций f гомоморфно отображено на множество \mathfrak{N} функций h , если каждому символу x_i взаимнооднозначно соответствует символ y_i и каждой функции $f(x_1, \dots, x_m)$ отвечает одна и только одна функция $h(y_1, \dots, y_m)$, зависящая от соответствующих аргументов (следовательно, число аргументов у соответствующих функций одинаковое, области значений могут быть различными), и при этом выполнены следующие два условия:

- 1) если функции $f(x_1, \dots, x_m) \in \mathfrak{M}$ отвечает функция $h(y_1, \dots, y_m) \in \mathfrak{N}$ и $f(x_{i_1}, \dots, x_{i_m}) \in \mathfrak{M}$, то функции $f(x_{i_1}, \dots, x_{i_m})$ соответствует функция $h(y_{i_1}, \dots, y_{i_m})$;
- 2) если функциям

$$f(x_1, \dots, x_m), f_1(x_{11}, \dots, x_{1n}), f_2(x_{21}, \dots, x_{2n}), \dots, f_m(x_{m1}, \dots, x_{mn})$$

соответствуют функции

$$h(y_1, \dots, y_m), h_1(y_{11}, \dots, y_{1n}), h_2(y_{21}, \dots, y_{2n}), \dots, h_m(y_{m1}, \dots, y_{mn}),$$

где

$$f, f_i \in \mathfrak{M}(X); h, h_i \in \mathfrak{N}(Y); 1 \leq i \leq m;$$

и если $f(f_1, f_2, \dots, f_m) \in \mathfrak{M}$, то функции $f(f_1, f_2, \dots, f_m)$ соответствует принадлежащая системе \mathfrak{N} функция $h(h_1, h_2, \dots, h_m)$.

Подмножество функций $P \subset P_{\aleph_0}$ называется предельной логикой, если

- 1) P состоит из счетного числа функций;
- 2) P содержит гомоморфные прообразы всех k -значных логик P_k , $k \geq 2$, т.е. для всякого натурального k , $k \geq 2$ существует A_k из P , которое гомоморфно отображается на множество всех функций k -значной логики P_k .

Подмножество \mathfrak{N} из \mathfrak{M} , $\mathfrak{N} \subseteq \mathfrak{M} \subseteq P_{\aleph_0}$ называется полной системой в \mathfrak{M} , если $[\mathfrak{N}] \supseteq [\mathfrak{M}]$. Множество \mathfrak{N} называется предполным классом в \mathfrak{M} , если оно не является полной системой в \mathfrak{M} , но добавление к нему любой функции f из $\mathfrak{M} \setminus \mathfrak{N}$ дает полную систему в \mathfrak{M} .

§ 2. Единственность предполного класса в \mathcal{Q}

Разобьем множество E_{\aleph_0} следующим образом: $E_{\aleph_0} = \bigcup_{k=1}^{\infty} \mathfrak{E}_k$, где $\mathfrak{E}_1 = \{0\}$ и

$$\mathfrak{E}_k = \left\{ \frac{k(k-1)}{2}, \frac{k(k-1)}{2} + 1, \dots, \frac{k(k-1)}{2} + k - 1 \right\},$$

если $k \geq 2$. Вдальее, обозначим через s_k выражение $\frac{k(k-1)}{2}$.

Определим функцию $\varphi'_k(x_1, x_2)$, $k \geq 2$, областью определения которой является множество $\mathfrak{E}_k \times \mathfrak{E}_k$, а областью значений — множество \mathfrak{E}_k ;

$$\varphi'_k(x_1, x_2) = \begin{cases} s_k, & \text{если } \max(x_1, x_2) = s_{k+1} - 1; \\ \max(x_1, x_2) + 1 & \text{в остальных случаях.} \end{cases}$$

Множество $\underbrace{\mathfrak{E}_k \times \mathfrak{E}_k \times \dots \times \mathfrak{E}_k}_{m \text{ раз}}$ будем в дальнейшем называть k -ым ящиком размерности m .

Определим функцию $\varphi(x_1, x_2)$ следующим образом:

$$\varphi(x_1, x_2) = \begin{cases} \varphi'_k(x_1, x_2), & \text{если существует такое } k, k \geq 2, \\ & \text{что } (x_1, x_2) \in \mathfrak{E}_k \times \mathfrak{E}_k; \\ 0 & \text{в остальных случаях.} \end{cases}$$

Через Q обозначим класс $[\{\varphi(x_1, x_2)\}]$.

В дальнейшем если не оговорено противное, предполагается $f(x_1, \dots, x_m)$ содержится в Q .

Лемма 1. Q является предельной логикой.

Доказательство следует из определения функции $\varphi(x_1, x_2)$ и из [1], [2]. Исследуем предполные классы в предельной логике Q .

Каждая функция f является некоторой суперпозицией функции $\varphi(x_1, x_2)$. Индуктивно определяем понятие глубины суперпозиции:

- 1) переменная имеет глубину 0;
- 2) если f_1 и f_2 — имеют глубину a_1 и a_2 соответственно, то глубина $\varphi(f_1, f_2)$ равна $\max(a_1, a_2) + 1$.

Посредством $\Gamma[f(x_1, \dots, x_m)]$ будем обозначать глубину функции $f(x_1, \dots, x_m)$. $f^k(x_1, \dots, x_m)$ означает, что $f(x_1, \dots, x_m)$ имеет глубину k .

Лемма 2. Пусть функция $f(x_1, \dots, x_m)$ имеет глубину $l, l \geq 0$, тогда для любого $n, n \geq l$ имеет место

$$f(s_n, s_n, \dots, s_n) = s_n + l.$$

Доказательство. Докажем по индукции. Легко проверить, что утверждение леммы имеет место в случае $l=0$ и $l=1$. Пусть утверждение уже верно для $l < N$ и пусть $\Gamma[f] = N$.

Поскольку каждая функция порождается функцией $\varphi(x_1, x_2)$, то $f = \varphi(f_1, f_2)$, где $\Gamma[f_1] \leq N-1$ и $\Gamma[f_2] \leq N-1$, причем одна из $f_i, i=1, 2$, имеет глубину $N-1$. Пусть для определенности f_1^{N-1} . Тогда по предположению

$$f_1(s_n, s_n, \dots, s_n) = s_n + N - 1$$

и

$$f_2(s_n, s_n, \dots, s_n) \leq s_n + N - 1,$$

если $n \geq N-1$. Но тогда имеем при $n \geq N$, что

$$f(s_n, s_n, \dots, s_n) = \max(s_n + N - 1, f_2(s_n, s_n, \dots, s_n)) + 1 = s_n + N.$$

Теорема 1. Множество предполных классов в Q состоит из одного элемента $\Pi = Q \setminus \bigcup_{i, j (i \neq j)} \{\varphi(x_i, x_j)\}$.

Доказательство. Из леммы 2 вытекает, что если глубина f больше 1, то f не совпадает с $\varphi(x_1, x_2)$. Поэтому класс Π не полон в Q . Но очевидно, что он является предполным. Покажем, что в Q не могут существовать другие предполные классы. Действительно, поскольку замкнутый класс, содержащий $\varphi(x_1, x_2)$, совпадает с Q , то любой замкнутый класс, отличный от Q , содержится в Π .

Класс \mathcal{Q} , как это следует из определения Π , отличается от Π фактически единственной функцией вида $\varphi(x_i, x_j)$, $i \neq j$, поэтому каждый замкнутый класс в \mathcal{Q} отличный от \mathcal{Q} , содержится в Π и поэтому задача о полноте множества \mathcal{M} в \mathcal{Q} сводится к проверке включения $\mathcal{M} \subseteq \Pi$. Тем самым система \mathcal{M} является полной в \mathcal{Q} тогда и только тогда, когда при некоторых i и j , $i \neq j$, имеет место $\varphi(x_i, x_j) \in \mathcal{M}$. Теорема доказана.

§ 3. Однозначное восстановление всякой функции из \mathcal{Q} по поведению её в конечном числе некоторых точек

Покажем, что зная значения функции $f(x_1, \dots, x_m)$ в достаточно большой окрестности начала координат, можно восстановить значения функции на остальной части пространства. Кроме того оказывается, что определенность функции в наперед заданной достаточно большом ящике позволяет восстановить значения этой функции во всех точках, координат которых мажорируют координаты каких-либо точек этого ящика. Это восстановление возможно также и в некоторых точках, координаты которых сами мажорируются координатами некоторых точек из указанного ящика.

Лемма 3. Пусть $\Gamma[g(x)] = l, l \geq 1$. Тогда функция $g(x)$ в n -ом ящике $n \geq l$ на первых $n-l+1$ местах имеет следующий вид:

$$g(x) = \begin{cases} l+x, & \text{если } x = s_n+h, \text{ где } 0 \leq h \leq n-l-1; \\ s_n, & \text{если } x = s_n+n-l. \end{cases}$$

Доказательство леммы проведем индукцией по числу l . Легко проверить, что утверждение леммы имеет место в случае $l=1$ и $l=2$. Пусть утверждение верно для $l < N$ и пусть $\Gamma[g(x)] = N$.

Поскольку каждая функция порождается функцией $\varphi(x_1, x_2)$, то $g(x) = \varphi(g_1(x), g_2(x))$, где $\Gamma[g_i(x)] \leq N-1$, причем одна из функций g_i , $i=1, 2$ имеет глубину $N-1$. Пусть для определенности $\Gamma[g_1] = N-1$. Тогда по предположению

$$g_1(x) = \begin{cases} N-1+x, & \text{если } x = s_n+h, \text{ где } 0 \leq h \leq n-N; \\ s_n, & \text{если } x = s_n+n-N+1; \end{cases}$$

и

$$g_2(s_n+h) \leq g_1(s_n+h) \quad \text{при } 0 \leq h \leq n-N.$$

Функция $\varphi(x_1, x_2)$ на первых $n-N$ местах в n -ом ящике к максимальному значению добавляет 1, т.е.

$$g(s_n+h) = \max(s_n+h+N-1, g_2(s_n+h)) + 1 = s_n+h+N.$$

Лемма доказана.

Лемма 4. Пусть $\Gamma[g(x)] = l, l \geq 2$. Тогда функция $g(x)$ в n -ом ящике, $n \geq l$, в точке $s_n+n+1-l+i$, $i=0, 1, \dots, l-2$, может принимать только значения из множества: $\{d_{i,1}; d_{i,2}; \dots, d_{i,l}\}$, причем для любого i существуют функции

$g_{i,1}^l(x), \dots, g_{i,l}^l(x)$, такие, что $g_{i,j}^l(s_n + n + 1 - l + i) = d_{i,j}$; $j = 1, 2, \dots, l$; где

$$d_{ij} = \begin{cases} s_n + j - 1, & \text{если } j \leq i + 2 \text{ или } i + 2 = l; \\ s_{n+1} - l + j - 1 & \text{в остальных случаях т.е. если } j > i + 2 < l. \end{cases}$$

Доказательство проведем индукцией по числу l . Легко проверяется, что утверждение леммы верно в случае $l = 2$ и $l = 3$. Предположим, что утверждение верно в случае, когда глубина функций $l \leq N - 1$. Покажем, что для функций, имеющих глубину N , утверждение тоже верно.

Обозначим через B_0^N точку $s_n + n - N + 1$.

Функцию $g(x)$ можем получить только следующим способом:

$$g(x) = \varphi(g_1(x), g_2(x)).$$

Пусть для определенности $\Gamma[g_1(x)] = N - 1$ и $\Gamma[g_2(x)] \leq N - 1$.

По индуктивному предположению

$$g_1(B_0^{N-1} + i) = g_1(B_0^N + i) \in \{d_{i-1,1}; \dots; d_{i-1,i+1}; \dots, d_{i-1,N-1}\},$$

где $0 \leq i' \leq N - 3$ и $i' + 1 = i$,

а также следует

$$d_{i-1,1} \leq g_2(B_0^N + i) \leq d_{i-1,i+1} \quad \text{или} \quad s_n + n + 1 - N + i \leq g_2(B_0^N + i) \leq s_n + n - 1$$

причем в указанных случаях в силу того, что $0 \leq \Gamma[g_2(x)] \leq N - 1$ все значения параметров достижимы. Отсюда следует, что

$$g(B_0^N + i) \in \{d_{i-1,1} + 1 = d_{i,2}; d_{i-1,2} + 1 = d_{i,3}; \dots; d_{i-1,i+1} + 1 = d_{i,i+2};$$

$$d_{i-1,i+2} + 1 = d_{i,i+4}; d_{i-1,i+3} + 1 = d_{i,i+5}; \dots; d_{i-1,N+2} + 1 = d_{i,N};$$

$$d_{i-1,N-1} + 1 = d_{i,1}; \varphi(g_1(B_0^N + i), B_0^N + i) = d_{i,i+3}\}.$$

Сейчас вычислим значение функции $g(x)$ при $i = 0$, зависимости от того, что глубина функции $g_2(x)$ равна $N - 1$, $N - 2$ и $0 \leq \Gamma[g_2(x)] \leq N - 3$.

В случае, если $i = 0$, то $g^{N-1}(B_k^N) = s_n$ (лемма 3). Отсюда следует, что

$$g(B_0^N) = \varphi(g_1^{N-1}(B_0^N), g_2^{N-1}(B_0^N)) = s_n + 1,$$

$$g(B_0^N) = \varphi(g_1^{N-1}(B_0^N), g_2^{N-1}(B_0^N)) = s_n$$

и

$$g(B_0^N) = \varphi(g_1^{N-1}(B_0^N), g_2^t(B_0^N)) = s_n + n + 2 - N + t,$$

где $0 \leq t \leq N - 3$. Лемма доказана.

Лемма 5. Если функция $g^l(x)$ в точке $s_n + n + 1 - l + i + r \cdot n + s_r + r$ ($n + r$)-ого ящика принимает значение a_r , где $0 \leq i \leq l = 2, r \geq 0, l \geq 2$,

$$B_r^l = s_n + n + 1 - l + r \cdot n + s_r + r,$$

то либо $a_r = a_{r-1} + n + r - 1$ где $a_r \leq s_n + i + 1 + r \cdot n + s_r + r$, либо $a_r = a_{r-1} + n + r$, где $a_r \geq B_r^l + i + 1$.

Доказательство проведем индукцией по глубине функции. Легко проверить, что когда глубина функции 2 или 3, то утверждение леммы справедливо.

Пусть утверждение леммы верно, когда глубина функции $l < N$. Покажем, что лемма справедлива для функции, и имеющей глубину N .

Функцию $g(x)$ можем получить только следующим образом:

$$g(x) = \varphi(g_1(x), g_2(x)).$$

Пусть для определенности $\Gamma[g_1(x)] = N - 1$ и $0 \leq \Gamma[g_2(x)] \leq N - 1$. Пусть

$$g_1(B_r^N + i) = b_r \text{ и } g_2(B_r^N + i) = c_r,$$

$i = 1, 2, \dots, N - 2$. Рассмотрим следующие четыре случая а), б), в) и г).

а) Пусть $b_r = b_{r-1} + n + r - 1$ и $c_r = c_{r-1} + n + r - 1$.

Тогда по индуктивному предположению

$$b_r \leq d_{i-1, i+1} + r \cdot n + s_r + r \quad (i = 1, 2, \dots, N - 2)$$

и

$$c_r \leq d_{i-j, i-j+2} + r \cdot n + s_r + r \quad (i - j \geq 0, 1 \leq j \leq N - 2, \Gamma[g_2] = N - j).$$

Отсюда вытекает, что при $r \geq 0$ числа b_r и c_r меньше, чем $s_n + s_r + r \cdot n + n + r - 1$.

Заметим, что если $b_0 \geq c_0$ ($b_0 < c_0$), то для любого r справедливо

$$b_r \geq c_r \quad (b_r < c_r).$$

Действительно,

$$b_r = b_0 + r \cdot n + s_r \geq c_0 + r \cdot n + s_r = c_r.$$

Рассмотрим два подслучая.

1) Пусть $b_r \geq c_r$, тогда $a_r = b_r + 1$, то-есть $a_r = a_{r-1} + n + r - 1$ и потому $a_r \leq d_{i, i+2} + r \cdot n + s_r + r$.

2) Пусть $b_r < c_r$, тогда $a_r = c_r + 1$, то-есть для любого r , $r \geq 0$, справедливо $a_r = a_{r-1} + n + r - 1$ и потому $a_r \leq d_{i, i+2} + r \cdot n + s_r + r$.

б) Пусть $b_r = b_{r-1} + n + r$ и $c_r = c_{r-1} + n + r$. Прежде всего покажем, что если $b_0 \geq c_0$ ($b_0 < c_0$), то для любого r , $r \geq 0$, справедливо $b_r \geq c_r$ ($b_r < c_r$). Действительно, $b_r = b_0 + r \cdot n + s_r + r \geq c_0 + r \cdot n + s_r + r = c_r$. Рассмотрим три подслучая.

1) Пусть $b_r \geq c_r$ и $b_r < s_n + s_r + r \cdot n + n + r - 1$, тогда $a_r = b_r + 1$, то есть $a_r = a_{r-1} + n + r$ и потому $a_r \geq B_r^N + 1 + i$.

2) Пусть $b_r \geq c_r$ и $b_r = s_n + s_r + r \cdot n + n + r - 1$. Тогда $a_r = s_n + s_r + r \cdot n$, то-есть $a_r = a_{r-1} + n + r - 1$ и $a_r \leq d_{i, i+2} + r \cdot n + s_r + r$.

3) Пусть $b_r < c_r$. Аналогично доказывается, как 1) и 2).

в) Пусть $b_r = b_{r-1} + n + r - 1$ и $c_r = c_{r-1} + n + r$. Покажем, что в этом случае для любого r справедливо $c_r \geq b_r$. Действительно, по индуктивному предположению $b_r \leq s_n + i + r \cdot n + s_r + r$, $i = 1, 2, \dots, N - 2$ и $c_r \geq s_n + n - N + 1 + i + r \cdot n + s_r + r$ так как $N - 1 \geq \Gamma[g_2(x)] \geq 0$. Отсюда $c_r - b_r = n - N + 1 > 0$. Тогда $a_r = c_r + 1$ или $c_r = s_n + s_r + r \cdot n + n + r - 1$. Если $c_r < s_n + s_r + r \cdot n + n + r - 1$, то $a_r = a_{r-1} + n + r$ и $a_r \geq B_r^N + i + 1$. А если $c_r = s_n + s_r + r \cdot n + n + r - 1$, то $a_r = s_n + s_r + r \cdot n$, то-есть

$$a_r = a_{r-1} + n + r - 1 \text{ и } a_r \leq s_n + i + 1 + r \cdot n + s_r + r.$$

г) Пусть $b_r = b_{r-1} + n + r$ и $c_r = c_{r-1} + n + r - 1$. Подобно тому, как в пункте в), можно показать, что $b_r \geq c_r$. Отсюда получим, что $a_r = b_r + 1$ или $a_r = s_n + s_r + r \cdot n$. Дальше доказывается аналогично как в).

Пусть $i=0$. В этом случае $g_1(s_n+n-N+1+rn+s_r+r)=s_r+s_n+rn$ ($r \geq 0$).

Отсюда получим, что $c_r \geq b_r$, $r \geq 0$, то-есть $a_r = c_r + 1$ или $a_r = \frac{(n+r)(n+r-1)}{2}$.

Легко заметить, что если $c_r = c_{r-1} + n + r - 1$, то $\Gamma[g_2] = N - 1$. Отсюда следует, что $a_r = a_{r-1} + n + r - 1$ и $a_r \leq s_n + 1 + s_r + r + r \cdot n + i$. А если $\Gamma[g_2] < N - 1$, то

$$c_r = c_{r-1} + n + r.$$

Отсюда следует, что $a_r = a_{r-1} + n + r$ и $a_r \geq B_r^N + 1 + i$. Лемма доказана.

Теорема 2. Если функция $g(x)$ имеет глубину l , то зная значение функции в произвольном фиксированном n -ом ящике, где $n \geq l$, можно найти её значение в любом N -ом ящике, таком что $N \geq l$.

Доказательство. Из леммы 3 следует, что функция $g(x)$ в n -ом ящике на первых $n-l+1$ местах принимает следующий вид:

$$g(x) = \begin{cases} l+x, & \text{если } x = s_n+h, \text{ где } 0 \leq h \leq n-l-1; \\ s_n, & \text{если } x = s_n+n-l; \end{cases}$$

то-есть зная глубину функции $g(x)$ мы можем определить вид функции на первых $n-l+1$ местах.

Пусть $g(B_0^l+i) = a_0$ в n -ом ящике, где $0 \leq i \leq l-2$.

Из леммы 5 следует, что если $a_0 \leq s_n+i+1$, то $a_r = a_{r-1} + n + r - 1$ и

$$a_{-p} = a_{-p+1} - n + p,$$

где $n-p \geq l$. А если $a_0 \geq B_0^l+i+1$, то $a_r = a_{r-1} + n + r$ и $a_{-p} = a_{-p+1} - n + p - 1$, где $n-p \geq l$. Теорема доказана.

Перейдем теперь к изучению функций нескольких переменных.

Обозначим через $a_1 \setminus a_m$ выражение $n - a_m + a_1 + 1$.

Лемма 6. Для любых чисел $a_1, \dots, a_i, \dots, a_m$ таких что $a_1 \leq a_2 \leq \dots \leq a_m$ и $s_n + n \leq a_i \leq s_n + 2n$, и для любого k ($n \geq mk$), либо существует такое i ($2 \leq i \leq m$), что $a_i - a_{i-1} \geq k + 1$, либо справедливо $a_1 \setminus a_m \geq k + 1$ (где $i = 1, 2, \dots, m$).

Доказательство. Если числа i с указанным в лемме свойством не существуют, то-есть для любого i , $i = 2, 3, \dots, m$ справедливо $a_i - a_{i-1} \leq k$, то фиксируя значение a_1 для любого i имеем $a_i \leq a_1 + (i-1)k$. Отсюда следует, что $a_1 \setminus a_m = a_1 - a_m + n + 1 \geq a_1 - a_1 - (m-1)k + n + 1 = n - mk + k + 1 \geq k + 1$. Лемма доказана.

Определение. Будем говорить, что функция $g(x)$ моделирует функцию $f(x_1, \dots, x_m)$ в точке $\tilde{a} = (a_1, \dots, a_m)$, если существует значение a такое, что $g(a) = f(\tilde{a})$. Скажем, что множество функций G моделирует функцию $f(x_1, \dots, x_m)$ в точке \tilde{a} , если существует функция $g(x)$ из G , которая моделирует функцию $f(x_1, \dots, x_m)$ в точке \tilde{a} .

Лемма 7. Для любой функции $f(x_1, \dots, x_m)$ и любой точки $\tilde{a} = (a_1, \dots, a_m)$ n -го ящика размерности m , таких что $s_n \leq a_i \leq s_n + n - k$, $n \geq km$, $1 \leq i \leq m$, $\Gamma[f] = k$, существует функция $g(x)$, которая моделирует функцию $f(x_1, \dots, x_m)$ в точке \tilde{a} и справедливо $k \leq \Gamma[g(x)] \leq n$.

Доказательство. Обозначим через $\varphi^0(x) = x$, $\varphi^1(x) = \varphi(x, x)$, $\varphi^2(x) = \varphi^1(\varphi(x, x))$, ..., $\varphi^l(x) = \varphi^{l-1}(\varphi(x, x))$, Пусть $a_i = s_n + j_i$, $1 \leq i \leq m$, $0 \leq j_i \leq n + k$, тогда используя теоремы 2 мы имеем, что

$$f(a_1, \dots, a_i, \dots, a_m) = f(\varphi^{j_1}(s_n), \dots, \varphi^{j_i}(s_n), \dots, \varphi^{j_m}(s_n)) = g^h(s_n),$$

где

$$k \leq h \leq k + \max(j_1, \dots, j_i, \dots, j_m) \leq n.$$

Лемма доказана.

Лемма 8. Для любой функции $f(x_1, \dots, x_m)$ и любой точки

$$\tilde{b} = (a_1 + n, \dots, a_m + n),$$

$n+1$ -го ящика размерности m , таких что $s_n \leq a_i \leq s_n + n - k$, $1 \leq i \leq m$, $n \geq km$, $\Gamma[f] = k$ существует функция $g(x)$, которая моделирует функцию $f(x_1, \dots, x_m)$ в точках \tilde{a} и \tilde{b} и справедливо $k \leq \Gamma[g(x)] \leq n$.

Доказательство. Действительно,

$$\begin{aligned} f(b_1, \dots, b_i, \dots, b_m) &= f(a_1 + n, \dots, a_i + n, \dots, a_m + n) = \\ &= f(s_n + n + j_1, \dots, s_n + n + j_i, \dots, s_n + n + j_m) = \\ &= f(\varphi^{j_1}(s_n + n), \dots, \varphi^{j_i}(s_n + n), \dots, \varphi^{j_m}(s_n + n)) = g^h(s_n + n). \end{aligned}$$

Лемма доказана.

Следствие. Пусть $\tilde{b} = (b_1, \dots, b_m)$ точка $n-1$ ящика ($n-1 \geq mk$) такая что $b_i + n - 1 = a_i$, $i = 1, 2, \dots, m$. Тогда функция $g^h(x)$ моделирует функцию $f(x_1, \dots, x_m)$ в точке \tilde{b} . Следствие аналогично доказывается как лемма 8.

Пусть $\tilde{a} = (a_1, \dots, a_m)$ и $\tilde{a}' = (a'_1, \dots, a'_m)$ — точки n -го ящика размерности m . Рассмотрим следующее множество точек $D_{\tilde{a}}$, состоящее из таких точек $\tilde{a}' = (a'_1, \dots, a'_m)$ для каждой из которых существует натуральное число t ; $t = 0, 1, \dots, n-1$; такое, что для всех i , $1 \leq i \leq m$, выполняется соотношение

$$a'_i = \begin{cases} a_i + t, & \text{если } a_i + t \leq s_n + n - 1; \\ a_i + t - n, & \text{если } a_i + t > s_n + n - 1. \end{cases}$$

Будем говорить, что функция $g(x)$ моделирует функцию $f(x_1, \dots, x_m)$ в множестве точек $D_{\tilde{a}}$, если она моделирует функцию f в каждой точке этого множества, т.е. для любой точки \tilde{a}' из $D_{\tilde{a}}$ существует a' такая, что $g(a') = f(\tilde{a}')$.

Лемма 9. Если функция $g^h(x)$ моделирует функцию $f(x_1, \dots, x_m)$ в точке \tilde{a} , то $g^h(x)$ моделирует f в множестве точек $D_{\tilde{a}}$.

Доказательство. Действительно,

$$g^h(s_n + i) = f(\varphi^{j_1}(s_n + i), \dots, \varphi^{j_i}(s_n + i), \dots, \varphi^{j_m}(s_n + i)), \quad 0 \leq i \leq n-1.$$

Так как,

$$\varphi^{j_i}(s_n + i) = \begin{cases} s_n + j_i + i, & \text{если } j_i + i \leq n-1; \\ s_n + j_i + i, & \text{если } j_i + i > n-1; \end{cases}$$

т.е. $\varphi^{j_i}(s_n + i) = a'_i$. Отсюда вытекает, что

$$g^h(s_n + i) = f(a'_1, \dots, a'_i, \dots, a'_m).$$

Лемма доказана.

Пусть $\tilde{a} = (a_1, \dots, a_m)$ — точка n -го ящика размерности m . Обозначим через \tilde{a}^k , $k \geq 0$ следующую точку $n+k$ -ого ящика

$$(a_1 + kn + s_k, a_2 + kn + s_k, \dots, a_m + kn + s_k),$$

и обозначим через \tilde{a}^{-v} ; $v = 1, 2, \dots, u$; следующую точку $u-v$ -ого ящика

$$(a_1 - v \cdot n + s_v + v, a_2 - v \cdot n + s_v + v, \dots, a_m - v \cdot n + s_v + v),$$

где

$$n - u \geq mk \quad \text{и} \quad n - u - 1 < mk.$$

Следствие. Из лемм 7, 8, 9 вытекает, что если функция $g^h(x)$ моделирует функцию $f(x_1, \dots, x_m)$ в точке \tilde{a} , то она моделирует f в множестве $D_{\tilde{a}^k}$ и $D_{\tilde{a}^{-v}}$, где $k \geq 0$; $v = 1, 2, \dots, u$.

Пусть

$$G_{n+i}^f = \{g^h(x) : f(\varphi^{j_1}(x), \dots, \varphi^{j_i}(x), \dots, \varphi^{j_m}(x)), \text{ где } 0 \leq j_i \leq n - k + i\}, \text{ где } i \geq 0.$$

Лемма 10. Если $n \geq mk$, то множество функций G_n^f моделирует функцию $f(x_1, \dots, x_m)$ во всех точках $n+1$ -ого ящика размерности m .

Доказательство. Из леммы 8 следует, что G_n^f моделирует функцию $f(x_1, \dots, x_m)$ в точках $\tilde{b} = (b_1, \dots, b_t, \dots, b_m)$, $0 \leq t \leq m$, $s_n + n \leq b_t \leq s_n + n + n - k$. Покажем, что если по крайней мере, одна координата b'_t такова, что $s_n + 2n \geq b'_t \geq s_n + 2n - k + 1$, то существует функция $g^h(x)$ из G_n^f , которая моделирует f в некоторой точке \tilde{b} . Для определенности предположим, что $b'_1 \leq b'_2 \leq \dots \leq b'_m$. Тогда из леммы 6 следует, что существует i такое, что $b'_i - b'_{i-1} \geq k + 1$ ($b'_m \setminus b'_1 \geq k + 1$). Рассмотрим число d которое определяется с помощью следующего соотношения $b'_i + d = \frac{n(n+1)}{2} + n - k (= b_i)$. Легко видеть, что для любого t , $1 \leq t \leq m$, $b'_t + d \leq s_n + 2n - k (= b_t)$ или $b'_t + d \geq s_n + 2n$, но тогда вместо b_t возьмем $b'_t + d - n$. Лемма доказана.

Замечание. Аналогично доказывается утверждение о том, что функции G_n^f моделируют функцию $f(x_1, \dots, x_m)$ в $n-1$ -ом ящике размерности m , где $n-1 \geq m \cdot k$.

Теорема 3. Если функция $f(x_1, \dots, x_m)$ имеет глубину k , то зная значение функции в произвольном фиксированном ящике n , где $n \geq mk$, можно найти её значение в любом N -ом ящике, таком что $N \geq m \cdot k$.

Доказательство. Поведение функции $f(x_1, \dots, x_m)$ в $n+1$ -ом ($n+i+1$ -ом, $i \geq 1$) ящике в силу леммы 10 однозначно определяет множество функций $G_n^f (G_{n+i}^f)$. Далее из теоремы 2 следует, что определены значения функции $g^h(x)$ в $n+k$ -ом и $n-j$ -ом ($k \geq 0, n-j \geq mk$) ящике. Из лемм 7, 8 и 9 вытекает, что множество функций G_n^f, G_{n+1}^f, \dots моделируют функцию $f(x_1, \dots, x_m)$ в любой точке N -ого ящика. Теорема доказана.

§ 4. Построение функции, иммитирующей поведение заданной последовательности функций из Q

В этом параграфе мы установим существование для заданной последовательности функций из Q зависящих от одного и того же числа переменных, одной такой функции из Q , которая на заданной последовательности ящиков, номер которых попарно взаимно просты, поочередно принимает значения указанных функций.

Определение. Обозначим через $\bar{g}_{n_1, \dots, n_t}(x_1, \dots, x_m)$ функцию, определенную на n_1, \dots, n_t -ом ящике размерности m и совпадающую на этом множестве с функцией $g(x_1, \dots, x_m)$. Функцию $\bar{g}_{n_1, \dots, n_t}(x_1, \dots, x_m)$ назовем сужением функции $g(x_1, \dots, x_m)$ на n_1, \dots, n_t ящиках. Сужением множества функций $R, R \subseteq Q$ на n_1, \dots, n_t -ых ящиках назовем множество $\bar{R}_{n_1, \dots, n_t}$ сужений функций из R на n_1, \dots, n_t -ых ящиках. Очевидна следующая лемма.

Лемма 11. \bar{Q}_k изоморфно P_k .

Обозначим через

$$\varphi^0(x) = x, \quad \varphi^1(x_1, x_2) = \varphi(x_1, x_2), \quad \varphi^l(x) = \varphi(x, x),$$

$$\varphi^2(x_1, x_2) = \varphi^1(\varphi^1(x_1, x_2)), \dots, \varphi^l(x_1, x_2) = \varphi^1(\varphi^{l-1}(x_1, x_2)), \dots$$

Из определения функции $\varphi^1(x, x)$ следует, что

$$\overline{\varphi_k^1}(x) = (x - s_k + 1) \pmod{k} + s_k, \quad k \equiv 2.$$

Лемма 12.

$$\overline{\varphi_k^l}(x_1, x_2) = (\overline{\varphi_k}(x_1, x_2) - s_k + l - 1) \pmod{k} + s_k.$$

Доказательство. Доказываем индукцией по l . При $l=1$ и $l=2$ утверждение леммы очевидно. Пусть утверждение верно для $l < N$. Тогда

$$\varphi^N(x_1, x_2) = \varphi^1(\varphi^{N-1}(x_1, x_2)),$$

т.е.

$$\begin{aligned} \overline{\varphi_k^N}(x_1, x_2) &= \overline{\varphi_k^1}(\overline{\varphi_k^{N-1}}(x_1, x_2)) = \overline{\varphi_k^1}\{(\overline{\varphi_k}(x_1, x_2) - s_k + N - 2) \pmod{k} + s_k\} = \\ &= [(\overline{\varphi_k}(x_1, x_2) - s_k + N - 2) \pmod{k} + s_k - s_k + 1] \pmod{k} + s_k = \\ &= (\overline{\varphi_k}(x_1, x_2) - s_k + N - 1) \pmod{k} + s_k. \end{aligned}$$

Лемма доказана.

Следствие.

$$\overline{\varphi_k^l}(x_1, x_2) = (\overline{\max_k}(x_1, x_2) - s_k + l) \pmod{k} + s_k.$$

Пусть p_1, p_2, \dots, p_n попарно взаимно простые числа. Обозначим через

$$l_0 = p_1 \cdot p_2 \cdot \dots \cdot p_n \quad \text{и} \quad l_i = p_1 \cdot p_2 \cdot \dots \cdot p_{i-1} \cdot p_{i+1} \cdot \dots \cdot p_n, \quad 1 \leq i \leq n.$$

Лемма 13.

а) $\overline{\varphi_{p_j}^{l_0}}(x_1, x_2) = \overline{\max_{p_j}}(x_1, x_2)$, при $1 \leq j \leq n$.

б) $\overline{\varphi_{p_j}^{l_i}}(x_1, x_2) = \begin{cases} \overline{\varphi_{p_i}'}(x_1, x_2), & \text{если } j = i; \\ \overline{\max_{p_j}}(x_1, x_2), & \text{если } j \in \{1, 2, \dots, i-1, i+1, \dots, n\}; \end{cases}$

где $\overline{\varphi_{p_i}'}$ — некоторая функция Вебба в p_i -ом ящике [6].

Доказательство.

а) Это утверждение следует из того, что $l_0 \equiv 0 \pmod{p_j}$, $1 \leq j \leq n$.

б) Если $j = 1, 2, \dots, i-1, i+1, \dots, n$, то $l_i \equiv 0 \pmod{p_j}$.

Если $j = i$, то $(l_i, p_j) \equiv 1$ — взаимно простые числа, — поэтому $\overline{\varphi_{p_i}^{l_i}}(x_1, x_2)$ является функцией Вебба [6]. Лемма доказана.

Лемма 14. Какова бы ни была функция $g(x_1, \dots, x_m)$ существует функция $g'(x_1, \dots, x_m)$ такая что

$$\overline{g'_{p_j}}(x_1, \dots, x_m) = \begin{cases} \overline{\max_{p_j}}(x_1, \dots, x_m), & \text{если } j = 1, 2, \dots, i-1, i+1, \dots, n; \\ \overline{g_{p_i}}(x_1, \dots, x_m), & \text{если } j = i. \end{cases}$$

Доказательство. Пусть $\eta(x_1, \dots, x_m)$ — произвольная функция из $[\{\varphi^i(x_1, x_2)\}]$. Тогда, легко заметить, что $\overline{\eta_{p_j}}(x_1, \dots, x_m) = \overline{\max_{p_j}}(x_1, \dots, x_m)$, если $j = 1, 2, \dots, i-1, i+1, \dots, n$. В то же время по лемме 13 из $\varphi^i(x_1, x_2)$ можно получить любую функцию $g'(x_1, \dots, x_m)$ в p_i -ом ящике, т.е.

$$\overline{g'_{p_i}}(x_1, \dots, x_m) = g_{p_i}(x_1, \dots, x_m).$$

Лемма доказана.

Лемма 15. Каковы бы ни были функции $g^{(1)}(x), \dots, g^{(n)}(x)$ такие, что $g^{(i)}(x) \in \overline{Q}_{p_i}$, $1 \leq i \leq n$, найдется функция $g(x)$ такая, что для любого i :

$$\overline{g_{p_i}}(x) = g^{(i)}(x)$$

Доказательство. Из предыдущей леммы вытекает, что в Q существуют функции $g^{(1)'}(x), \dots, g^{(n)'}(x)$ такие, что в p_i -ом ящике совпадает с функцией $g^{(i)}(x)$, а в $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ -ом ящике совпадает с функцией $\mu(x) \equiv x$.

Легко видеть, что функция $g^{(1)'}(g^{(2)' }(\dots(g^{(n)' }(\dots))) = g(x)$ является искомой функцией. Лемма доказана.

Теорема 4. Каковы бы ни были $g^{(1)}(x_1, \dots, x_m), \dots, g^{(n)}(x_1, \dots, x_m)$ такие, что $g^{(i)}(x_1, \dots, x_m) \in \overline{Q}_{p_i}$, $1 \leq i, j \leq n$, $(p_i, p_j) \equiv 1$, найдется функция $g(x_1, \dots, x_m) \in Q$ такая, что для любого i : $\overline{g_{p_i}}(x_1, \dots, x_m) = g^{(i)}(x_1, \dots, x_m)$.

Доказательство. Возьмем функции $h^{(i)}(x), f^{(i)}(x_1, \dots, x_m)$, $1 \leq i \leq n$, определенные следующим образом:

$$f^{(i)}(x_1, \dots, x_m) = \begin{cases} g^{(i)}(x_1, \dots, x_m), & \text{если } (x_1, \dots, x_m) \in \mathfrak{E}_{p_i} \times \dots \times \mathfrak{E}_{p_i}; \\ \max_{p_j}(x_1, \dots, x_m), & \text{если } (x_1, \dots, x_m) \in \mathfrak{E}_{p_j} \times \dots \times \mathfrak{E}_{p_j}, \\ & \text{где } j \in \{1, 2, \dots, i-1, i+1, \dots, n\}. \end{cases}$$

$$h^{(i)}(x) = \begin{cases} x, & \text{если } x \in \mathfrak{E}_{p_i}; \\ s_{p_j}, & \text{если } x \in \mathfrak{E}_{p_j}, \text{ где } j \in \{1, 2, \dots, i-1, i+1, \dots, n\}. \end{cases}$$

Эти функции по лемме 14 и 15 принадлежат Q .

Строим функции $f^{(i)'}(x_1, \dots, x_m)$ следующим образом:

$$f^{(i)'}(x_1, \dots, x_m) = h^{(i)}(f^{(i)}(x_1, \dots, x_m)).$$

Из построения $f^{(i)'}(x_1, \dots, x_m)$ следует, что

$$f^{(i)'}(x_1, \dots, x_m) = \begin{cases} g^{(i)}(x_1, \dots, x_m), & \text{если } (x_1, \dots, x_m) \in \mathfrak{E}_{p_i} \times \dots \times \mathfrak{E}_{p_i}; \\ s_{p_j}, & \text{если } (x_1, \dots, x_m) \in \mathfrak{E}_{p_j} \times \dots \times \mathfrak{E}_{p_j}, \\ & \text{где } \{j = 1, 2, \dots, i-1, i+1, \dots, n\}. \end{cases}$$

Легко видеть, что функция

$$g(x_1, \dots, x_m) = \varphi^{l_0}(\varphi^{l_0}(\dots \varphi^{l_0}(\varphi^{l_0}(\varphi^{l_0}(f^{(1)'}, f^{(2)'}, f^{(3)'}, f^{(4)'}, \dots, f^{(n)'}))$$

является искомой функцией. Теорема доказана.

§ 5. Существенность каждого переменного в формуле, реализующей заданную функцию из Q

Здесь мы установим одно свойство функций из Q , характерное для логических сумм и произведений двухзначной логики. Именно, мы покажем, что любое переменное, встречающееся в формуле над системой $\{\varphi\}$ является существенным для функции реализуемой этой формулой.

Лемма 16. Пусть дана функция $f(x_1, \dots, x_m)$ и точки

$$\tilde{a} = (s_n + j_1, \dots, s_n + j_{i-1}, s_n + j_i, s_n + j_{i+1}, \dots, s_n + j_m)$$

и

$$\tilde{b} = (a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_m),$$

где

$$0 \leq j_l \leq n - k, \quad l \in \{1, 2, \dots, i-1, i+1, \dots, n\}, \quad 0 \leq j_i \leq n - k - 1, \quad \Gamma[f] = k$$

и пусть

$$g_1(x) = f(\varphi^{j_1}(x), \dots, \varphi^{j_{i-1}}(x), \varphi^{j_i}(x), \varphi^{j_{i+1}}(x), \dots, \varphi^{j_m}(x))$$

и

$$g_2(x) = f(\varphi^{j_1}(x), \dots, \varphi^{j_{i-1}}(x), \varphi^{j_{i+1}}(x), \varphi^{j_{i+1}}(x), \dots, \varphi^{j_m}(x)).$$

Тогда

$$\Gamma[g_1(x)] \leq \Gamma[g_2(x)] \leq \Gamma[g_1(x)] + 1.$$

Лемма с помощью индукции тривиально доказывается.

Определение. Скажем, что функция $g(x_1, \dots, x_i, \dots, x_m)$ существенно зависит от аргумента x_i , если найдутся два набора

$$\tilde{a} = (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_m) \quad \text{и} \quad \tilde{a}' = (a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_m)$$

такие, что $0 \neq g(\tilde{a}) \neq g(\tilde{a}') \neq 0$

Теорема 5. Каждая функция $g(x_1, \dots, x_i, \dots, x_m)$, $\Gamma[g] = k$, существенно зависит от всех своих переменных x_1, \dots, x_m .

Доказательство. Для простоты докажем, что $g(x_1, \dots, x_m)$ существенно зависит от x_1 . Рассмотрим ящик, номер которого $n \equiv mk$. Покажем, что $g(x_1, \dots, x_m)$ на наборах

$$\tilde{a}_0 = (s_n, s_n, \dots, s_n) \text{ и } \tilde{a}_{n-k} = (s_n + n - k, s_n, \dots, s_n)$$

принимает разные значения. Действительно

$$g(\tilde{a}_0) = s_n + k, \text{ а } g(\tilde{a}_{n-k}) = g(\varphi^{n-k}(s_n), s_n, \dots, s_n) = g^h(s_n) = b.$$

Так как $n \equiv h > k$, то $b > s_n + k$. Теорема доказана.

Замечание. Из доказательства теоремы 5 следует, что функция $g(x_1, \dots, x_m)$ существенно зависит от всех переменных в ящике, размер которого mk .

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ENUMERATING PARTITIONAL MATROIDS

by
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“He tried counting sheep which is sometimes a good way of getting to sleep...”

A. Milne: Winnie-the-Pooh

The concept of partitional matroids (the direct sum of uniform matroids) was first introduced by BERGE [1]. Certain properties of partitional matroids were treated in [4], emphasizing also their connection with transversal matroids and their applicability in the analysis of electric networks. If a set S has n labelled elements and a partition of S is considered then the number of different partitional matroids over this partition is at least $n-1$ and at most 2^n . The generating function for the number X_n of partitional matroids of S is determined and the result $n^{-1} \log X_n = \log n - \log \log n - 1 + o(1)$ is obtained, which implies that the average number of partitional matroids over a partition is $\exp(o(n))$.

Let S denote a finite set with n labelled elements. $\mathcal{S} = (S_1, S_2, \dots, S_p)$ is called a *partition* of S if $\bigcup_{i=1}^p S_i = S$ and if $S_i \cap S_j = \emptyset$ if and only if $i=j$. $\mathcal{A} = (a_1, a_2, \dots, a_p)$ is an ordered set of integers such that $0 \leq a_i \leq |S_i|$ ($i=1, 2, \dots, p$). The system $[\mathcal{S}, \mathcal{A}]$ corresponds to a matroid on S , namely a subset $X \subseteq S$ is independent if and only if $|X \cap S_i| \leq a_i$ for $i=1, 2, \dots, p$. The matroids of this type are called *partitional matroids*. In order to establish a one-one correspondence between the set of partitional matroids and the set of certain systems $[\mathcal{S}, \mathcal{A}]$ the convention of [4] is applied again:

All of the subsets S_i with $a_i=0$ or with $a_i=|S_i|$ are supposed to contain a single element only.

If A_n is a sequence of numbers then let the symbol $A_n \in \mathcal{A}$ abbreviate that

$$n^{-1} \log A_n = \log n - \log \log n - 1 + o(1).$$

If G_n denotes the number of partitions of the set S with n elements (the n^{th} Bell-number) then $G_n \in \mathcal{A}$, see e.g. [2]. Our statement that $X_n \in \mathcal{A}$ too, will directly be implied by the following theorem, since $\sum_{n=0}^{\infty} \frac{G_n t^n}{n!} = \exp(e^t - 1)$ and $\exp(e^t \cdot p(t)) = o(\exp(e^{(1+\varepsilon)t}))$ for any polynomial $p(t)$ and for any $\varepsilon > 0$.

THEOREM: $\sum_{n=0}^{\infty} \frac{X_n t^n}{n!} = \exp(e^t(t-1) + 2t + 1)$, if, by definition, $X_0 = 1$.

PROOF. Let $\mathcal{S} = (S_1, S_2, \dots)$ be an arbitrary partition of S and let t_i denote the number of the i -element subsets in \mathcal{S} ($i=1, 2, \dots$). The number of partitional matroids over \mathcal{S} is obviously $\prod_{i=1}^n \psi_i^{t_i}$ where $\psi_1=2$ and $\psi_i=i-1$ if $i>1$ (cf. the

above convention). The number of partitions of the labelled set, belonging to a certain system $T=(t_1, t_2, \dots)$ is obviously $\frac{n!}{\prod_{i=1}^n t_i!(i!)^{t_i}}$ where $\sum it_i=n$. Thus, there are

$$n! \prod_{i=1}^n \frac{\psi_i^{t_i}}{t_i!(i!)^{t_i}}$$

partitional matroids over all the partitions belonging to T . Hence

$$(1) \quad \frac{X_n t^n}{n!} = \sum \prod_{i=1}^n \frac{\left(\frac{\psi_i t^i}{i!}\right)^{t_i}}{t_i!}$$

where the summation is over all systems of integers t_i with $\sum_{i=1}^n it_i=n$. Finally

$$\sum_{n=0}^{\infty} \frac{X_n t^n}{n!} = \prod_{i=1}^{\infty} \left(\sum_{t_i=0}^{\infty} \frac{\left(\frac{\psi_i t^i}{i!}\right)^{t_i}}{t_i!} \right) = \prod_{i=1}^{\infty} \varphi_i(t),$$

which leads to the assertion since $\varphi_1(t)=e^{2t}$ and

$$\prod_{i=2}^{\infty} \varphi_i(t) = \exp \left(\sum_{i=2}^{\infty} \frac{t^i}{(i-1)!} - \sum_{i=2}^{\infty} \frac{t^i}{i!} \right) = \exp((t-1)e^t + 1).$$

REMARKS. 1. Putting $\psi_1=\psi_2=\dots=1$ the right hand side of (1) leads to the number G_n of the partitions of S (see RÉNYI [5]).

2. The partial ordering of the matroids of the set S (see [4]) is considered, i.e. $\mathcal{M}_1 \subseteq \mathcal{M}_2$ if any independent subset of the matroid \mathcal{M}_1 is independent in \mathcal{M}_2 too. Let Y_n denote the size of the longest antichain (i.e. the maximal number of pairwise incomparable elements) with respect to this partial ordering of the *partitional matroids* of S . We prove that $Y_n \in \mathcal{A}$. The upper bound is trivial, since $Y_n < X_n$. The lower estimation $n^{-2}G_n < Y_n$ will directly be implied by the pigeonhole-principle. Let $S(n, k)$ denote the Stirling numbers of the second kind (the number of partitions of n labelled elements into k subsets). The relation $\max_{1 \leq k \leq n} S(n, k) \in \mathcal{A}$ is well-known, see e.g. [3], and the existence of the function of dimension among the partitional matroids of the same rank, proved in [4], leads to the assertion.

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ON THE STRUCTURE OF DERIVATIVES

by

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1. Introduction

In certain problems it may have some importance to know criteria which assure that a Baire 1 function $f(x)$ cannot be a derivative even in the case if we know $f(x)$ only on a subset $H \subset [0, 1]$. If H is a scattered set then, by checking Darboux property one cannot obtain any information.

The idea of this paper originates from paper [1] where, for every $H \subset [0, 1]$ with $\lambda(H) > 0$, a Baire 1 function was constructed which could not be extended from H as a derivative. In this paper we are going to introduce the concepts of thin sets and irregular points and one of our main goals is to prove that the set of irregular points of a derivative is a thin set.

We recall here some notations and definitions.

— A set $H \subset [0, 1]$ is said to be D-open if every $x \in H$ is of inner density 1 in H . It is known that the D-open sets form a topology on $[0, 1]$, the so called D-topology. The topological space $([0, 1], D)$ has several remarkable properties. We mention here only the facts that the D-open sets are measurable and a real function $f(x)$ is continuous in the D-topology if and only if it is approximately continuous.

— The closure and interior of a set H will be denoted by $\text{cl } H$ and $\text{int } H$, respectively.

— For the sake of brevity the D-interior of a set H will be denoted by $\text{ind } H$.

— $\lambda(H)$ denotes the Lebesgue outer measure of H .

— A set H is said to be metrically dense in itself if, for any open interval I , $I \cap H \neq \emptyset$ implies $\lambda(I \cap H) > 0$.

— The family of derivatives defined in $[0, 1]$ is denoted by \mathcal{D} .

REMARK 1.1. If H is any set, then $H_1 = H \setminus \cup I$ is metrically dense in itself and $\lambda(H \setminus H_1) = 0$, if, in $\cup I$, I ranges over the open intervals with rational endpoints and $\lambda(I \cap H) = 0$. If further H is closed then H_1 is closed too.

2. Thin sets

DEFINITION 2.1. A set H is said to be thin if, for any closed set P , the set $H \cap \text{ind } P$ is nowhere dense in P . The system of thin sets is denoted by \mathcal{T} .

We remark that replacing the term “any closed set P ” by “any measurable set P ” we obtain an equivalent definition. In fact, suppose that M is a measurable set, $M \neq \emptyset$ and $\text{cl}(H \cap \text{ind } M) \supset M$. Then denoting $P = \text{cl } M$ we have $P \subset \text{cl}(H \cap \text{ind } M) \subset \text{cl}(H \cap \text{ind } P) \subset \text{cl } P = P$, i.e. $\text{cl}(H \cap \text{ind } P) = P$; in particular $\text{cl}(\text{ind } P) = P$ and thus P cannot possess isolated points. Thus we obtain that $H \cap \text{ind } P$ is everywhere dense in the nontrivial perfect set P .

LEMMA 2.2. *If $H \notin \mathcal{T}$ then there exists a non-empty closed set P metrically dense in itself such that*

$$(1) \quad \text{cl}(H \cap \text{ind } P) = P.$$

PROOF. If $H \notin \mathcal{T}$ then there exists a closed set Q and an open interval $I=(a, b)$ such that

$$\text{cl}(H \cap \text{ind } Q) \supset I \cap Q \neq \emptyset.$$

Let $P = \text{cl}(I \cap Q)$. Then $(\text{ind } Q) \cap I = \text{ind } P$, thus $\text{cl}(H \cap \text{ind } P) = \text{cl}(H \cap \text{ind } Q \cap I) \supset I \cap Q$, i.e. $\text{cl}(H \cap \text{ind } P) \supset P$ and hence $\text{cl}(H \cap \text{ind } P) = P$. In particular, $\text{cl}(\text{ind } P) = P$ implies that P is metrically dense in itself.

THEOREM 2.3.

- (i) *If $H_1 \subset H_2$ and $H_2 \in \mathcal{T}$, then $H_1 \in \mathcal{T}$.*
- (ii) *If $H_1, H_2 \in \mathcal{T}$ then $H_1 \cup H_2 \in \mathcal{T}$.*
- (iii) *If $H \in \mathcal{T}$ then $\lambda(H) = 0$.*

PROOF. Since $H_1 \subset H_2$ implies $(H_1 \cap \text{ind } P) \subset (H_2 \cap \text{ind } P)$ for any P , (i) is trivial. Also (ii) is obvious by

$$(H_1 \cap \text{ind } P) \cup (H_2 \cap \text{ind } P) = (H_1 \cup H_2) \cap \text{ind } P.$$

Suppose $\lambda(H) > 0$ and let H_1 denote the set given by Remark 1.1. Then $\lambda(H_1) > 0$ and $H_1 \cap I \neq \emptyset$ implies $\lambda(H_1 \cap I) > 0$. Let $P = \text{cl } H_1$. If for an open interval I the relation $I \cap H_1 \cap \text{ind } P = \emptyset$ holds then $I \cap H_1 \subset P \setminus \text{ind } P$ and by the density theorem $\lambda(I \cap H_1) = 0$, thus $I \cap H_1 = \emptyset$. Hence $P \cap I = \emptyset$ i.e. $H_1 \cap \text{ind } P$ is everywhere dense in P . Since $P \neq \emptyset$ we have $H_1 \notin \mathcal{T}$ and by (i), $H \notin \mathcal{T}$ as well.

Our next aim is to characterize the thin sets. A simple condition assuring a set to be thin is

$$(2) \quad H \cap \text{ind}(\text{cl } H) = \emptyset.$$

In fact, if $H \notin \mathcal{T}$ and P is the perfect set according to (1), then $P \subset \text{cl } H$, thus $\text{ind } P \subset \text{ind}(\text{cl } H)$ i.e. $H \cap \text{ind}(\text{cl } H) \neq \emptyset$. In particular, any subset of a closed set H with $\lambda(H) = 0$ is thin. We give below a transfinite classification of thin sets which decomposes any thin set into thin sets satisfying (2).

DEFINITION 2.4. Let H be any set. We define the set H_α for every countable ordinal number $\alpha < \omega_1$ as follows.

Let $H_0 = H$.

Suppose that H_β has been defined. Then $H_{\beta+1}$ is given by

$$(3) \quad H_{\beta+1} = H_\beta \cap (\text{ind}(\text{cl } H_\beta)).$$

Suppose that H_γ has been defined for $\gamma < \beta$ where β is a limit ordinal. Then we put

$$(4) \quad H_\beta = \bigcap_{\alpha < \beta} H_\alpha.$$

From (3) and (4) $H_{\beta_1} \supset H_{\beta_2}$ obviously follows for any $\beta_1 \leq \beta_2$. Hence $\text{cl } H_{\beta_1} \supset \text{cl } H_{\beta_2}$ and there exists a minimal ordinal $\alpha < \omega_1$ such that $\text{cl } H_\alpha = \text{cl } H_{\alpha+1} = \dots$. We shall refer to this α as the order of H and use the symbol $\alpha = o(H)$.

THEOREM 2.5. *Suppose $H \notin \mathcal{T}$ and let P denote the closed set defined in Lemma 2.2. Then, for every β*

$$(5) \quad H \cap \text{ind } P \subset H_\beta$$

holds. In particular, $P \subset \text{cl } H_\beta$.

PROOF. $H \cap \text{ind } P \subset H$ implies $\text{cl}(H \cap \text{ind } P) \subset \text{cl } H$ thus by (1) we have $P \subset \text{cl } H \subset \text{cl } H_0$. The proof proceeds by induction. Suppose that $H \cap \text{ind } P \subset H_\beta$ holds for some β . Referring to (1) this implies $P \subset \text{cl } H_\beta$. Hence $\text{ind } P \subset \text{ind}(\text{cl } H_\beta)$ follows, thus $H_\beta \cap \text{ind } P \subset H_\beta \cap \text{ind}(\text{cl } H_\beta) = H_{\beta+1}$. Recalling the induction hypothesis, $H \cap \text{ind } P \subset H_{\beta+1}$ and as above, this implies $P \subset \text{cl } H_{\beta+1}$. Let β be a limit ordinal and we assume $H \cap \text{ind } P \subset H_\gamma$ for every $\gamma < \beta$. Hence $H \cap \text{ind } P \subset \bigcap_{\gamma < \beta} H_\gamma = H_\beta$ and $P \subset \text{cl } H_\beta$.

THEOREM 2.6. *$H \in \mathcal{T}$ holds if and only if for any $\beta \cong o(H)$, $H_\beta = \text{cl } H_\beta = \emptyset$.*

PROOF. If $H \in \mathcal{T}$, then by Theorem 2.5 $\lambda(\text{cl } H_\beta) \cong \lambda(P) > 0$ holds for every β . Thus $\text{cl } H_\beta \neq \emptyset$ (and $H_\beta \neq \emptyset$) for every β .

On the other hand, let H be any set and suppose that $\text{cl } H_{o(H)} \neq \emptyset$. Since

$$(6) \quad \text{cl}(H_{o(H)} \cap \text{ind } \text{cl } H_{o(H)}) = \text{cl } H_{o(H)+1} = \text{cl } H_{o(H)} \neq \emptyset$$

the set $H_{o(H)} \cap \text{ind}(\text{cl } H_{o(H)})$ is everywhere dense in the nontrivial closed set $\text{cl } H_{o(H)}$, i.e. $H_{o(H)} \notin \mathcal{T}$. By Theorem 2.3 (i) and $H_{o(H)} \subset H$ we conclude $H \notin \mathcal{T}$.

THEOREM 2.7. *Let H be any set. Then*

$$(i) \quad H_{o(H)+1} = H_{o(H)+2} = \dots$$

$$(ii) \quad H = \bigcup_{\beta \cong o(H)} (H_\beta \setminus H_{\beta+1}) \cup H_{o(H)+1}$$

where $H_\beta \setminus H_{\beta+1} \in \mathcal{T}$ for $\beta \cong o(H)$ and $o(H_\beta \setminus H_{\beta+1}) \leq 1$ for $\beta \cong o(H)$. In particular, if $H \in \mathcal{T}$ then $H_{o(H)} = H_{o(H)+1} = \dots = \emptyset$ and

$$(iii) \quad H = \bigcup_{\beta < o(H)} (H_\beta \setminus H_{\beta+1})$$

is a disjoint countable decomposition of H into thin sets of order 1.

PROOF. Put for brevity $\alpha = o(H)$. Then $\text{cl } H_{\alpha+1} = \text{cl } H_\alpha$ thus (i) follows by $H_{\alpha+2} = H_{\alpha+1} \cap \text{ind } \text{cl } H_{\alpha+1} = (H_\alpha \cap \text{ind } \text{cl } H_\alpha) \cap \text{ind } \text{cl } H_\alpha = H_{\alpha+1}$. The inclusion “ \supset ” is trivial in (ii). We suppose that $x \in H = H_0$ and $x \notin H_{o(H)+1}$. Then we select a minimal ordinal γ such that $x \notin H_\gamma$. Then $\gamma > 0$ and by (4) it cannot be a limit ordinal. Thus $\gamma = \beta + 1$ and $x \in H_\beta \setminus H_{\beta+1}$. Hence

$$H = \bigcup_{\beta \cong o(H)} (H_\beta \setminus H_{\beta+1}) \cup H_{o(H)+1}$$

Let $D_\beta = H_\beta \setminus H_{\beta+1}$ ($\beta \cong o(H)$). Then $\text{cl } D_\beta \subset \text{cl } H_\beta$ thus $\text{ind } \text{cl } D_\beta \subset \text{ind } \text{cl } H_\beta$. Referring to (3) we obtain $D_\beta \subset \text{cl } H_\beta \setminus \text{ind } \text{cl } H_\beta$, hence $(\text{ind } \text{cl } D_\beta) \cap D_\beta \subset (\text{cl } H_\beta \setminus \text{ind } \text{cl } H_\beta) \cap$

$\bigcap(\text{ind cl } H_\beta) = \emptyset$. Thus $(D_\beta)_1 = \emptyset$, i.e. $D_\beta \in \mathcal{T}$ and

$$o(D_\beta) = \begin{cases} 1, & D_\beta \neq \emptyset \\ 0, & D_\beta = \emptyset, \end{cases}$$

making the proof complete.

COROLLARY 2.8. (i) For $H \in \mathcal{T}$, $o(H) = 0$ holds if and only if $H = \emptyset$.

(ii) If $H \neq \emptyset$, $\lambda(\text{cl } H) = 0$ then $H \in \mathcal{T}$ and $o(H) = 1$.

(iii) $H \cap \text{ind}(\text{cl } H) \in \mathcal{T}$ implies $H \in \mathcal{T}$.

(iv) The function $o(H)$ is monotone on \mathcal{T} , i.e. for $K \subset H \in \mathcal{T}$ the inequality $o(K) \leq o(H)$ holds.

PROOF. (i) and (ii) is trivial. Denoting $K = H \cap \text{ind cl } H$ we have $K = H_1$, thus the transfinite sequences $\{H_\beta\}$, $\{K_\beta\}$ consist of the same sets (apart from H_0). Hence theorem 2.6 implies (iii). Let $K \subset H \in \mathcal{T}$. Then $K_\beta \subset H_\beta$ for every β , i.e. by $K_{o(H)} \subset H_{o(H)} = \emptyset$ we obtain $K_{o(H)} = \emptyset$ therefore $o(K) \leq o(H)$.

We remark that this monotonicity property fails to hold in full generality: $o([0, 1]) = 0$ and $o(\text{Cantor set}) = 1$.

LEMMA 2.9. For every $\alpha < \omega_1$ there exists a system of open intervals $S_\alpha = \{I_\beta\}_{\beta < \alpha}$ such that

(i) $I_\beta \subset (0, 1]$ ($0 \leq \beta < \alpha$),

(ii) $I_\beta < I_\gamma$ if and only if $\beta > \gamma$. (By $I_\beta < I_\gamma$ we mean $x < y$ for every $x \in I_\beta$, $y \in I_\gamma$).

PROOF. We omit the straightforward proof.

COROLLARY 2.10. Let $\alpha < \omega_1$, $S_\alpha = \{I_\beta\} = \{(a_\beta, b_\beta)\}$ be the system given by the preceding lemma. Then we have $a_\beta < a_\gamma$ if and only if $\gamma < \beta$ and moreover

$$a_\beta < \inf \{a_\gamma : \gamma < \beta\}.$$

PROOF. The assertion is obvious by

$$a_\beta < b_\beta \leq \inf \{a_\gamma : \gamma < \beta\}.$$

THEOREM 2.11. Given $\alpha < \omega_1$, there exists $H \in \mathcal{T}$ such that $o(H) = \alpha$.

PROOF. Let α be given. We consider the set $\{a_\beta\}_{\beta < \alpha}$ as in the corollary above and define a transfinite sequence of perfect sets $\{P_\beta\}_{\beta < \alpha}$ by induction as follows.

(i) Let P_0 be a nowhere dense perfect set, $P_0 \subset [0, 1]$, $0, 1 \notin P_0$ and $\lambda(P_0) > a_0$. Such a set can be found by $a_0 < 1$.

(ii) If P_β has been defined such that $\lambda(P_\beta) > a_\beta$ then $\lambda(\text{ind } P_\beta) > a_\beta > a_{\beta+1}$, hence we can choose a perfect set $P_{\beta+1}$ satisfying $P_{\beta+1} \subset \text{ind } P_\beta$ and $\lambda(P_{\beta+1}) > a_{\beta+1}$.

(iii) If β is a limit ordinal and γ has been defined for every $\gamma < \beta$ such that $P_{\gamma_1} \subset \text{ind } P_{\gamma_2} \subset P_{\gamma_2}$ ($\gamma_2 < \gamma_1 < \beta$) and $\lambda(P_{\gamma_1}) > a_\gamma$, then we consider the closed set

$$Q = \bigcap_{\gamma < \beta} P_\gamma = \bigcap_{n=1}^{\infty} P_{\gamma_n}$$

where $\gamma_n < \beta$ is any monotone sequence tending to β . Since

$$\lambda(Q) = \lim_{n \rightarrow \infty} \lambda(P_{\gamma_n}) \geq \inf \{a_\gamma : \gamma < \beta\} > a_\beta,$$

we can find a perfect set P_β such that $P_\beta \subset \text{ind } Q$ and $\lambda(P_\beta) > a_\beta$ hold. It is obvious from the construction that the system $\{P_\beta\}_{\beta < \alpha}$ satisfies

$$(7) \quad P_{\beta_1} \subset \text{ind } P_{\beta_2} \quad (\beta_2 < \beta_1)$$

and

$$(8) \quad \lambda(P_\beta) > a_\beta > 0 \quad (\beta < \alpha).$$

Let K_β denote the set of the endpoints of P_β (by endpoints of a perfect set P we mean the endpoints of the open intervals contiguous to P), then we define H by

$$(9) \quad H = \bigcup_{\beta < \alpha} K_\beta.$$

We are going to prove that $H \in \mathcal{T}$ and $o(H) = \alpha$. By Theorem 2.6 it suffices to prove $o(H) = \alpha$ and $H_\alpha = \emptyset$. Let $\{H_\beta\}_{\beta < \alpha}$ denote the transfinite sequence according to $H = H_0$. We can easily verify that

$$(10) \quad H_\beta = \bigcup_{\gamma \cong \beta} K_\gamma \quad \text{for any } \beta < \alpha.$$

In fact, (7) implies $\bigcup_{\gamma \cong \beta} K_\gamma \subset P_\beta$ thus by $\text{cl } K_\beta = P_\beta$ we obtain $\text{cl} \left(\bigcup_{\gamma \cong \beta} K_\gamma \right) = P_\beta$. Relation (10) holds for $\beta = 0$ by the definition of H . Assuming (10) for a fixed β we have $H_{\beta+1} = H_\beta \cap \text{ind } \text{cl } H_\beta = \left(\bigcup_{\gamma \cong \beta} K_\gamma \right) \cap \text{ind } P_\beta = \bigcup_{\gamma \cong \beta+1} K_\gamma$. Considering a limit ordinal $\beta^* < \alpha$ and assuming (10) for $\beta < \beta^*$ we obtain

$$(11) \quad H_{\beta^*} = \bigcap_{\beta < \beta^*} H_\beta = \bigcap_{\beta < \beta^*} \left(\bigcup_{\gamma \cong \beta} K_\gamma \right).$$

Referring to (7)

$$K_{\gamma_1} \cap K_{\gamma_2} = \emptyset$$

holds whenever $\gamma_1 \neq \gamma_2$, thus (11) implies $H_{\beta^*} = \bigcup_{\gamma \cong \beta^*} K_\gamma$. Hence (10) has been verified and we have $\text{cl } H_\beta = P_\beta$ for every $\beta < \alpha$. By (7) $\text{cl } H_{\beta_1} \neq \text{cl } H_{\beta_2}$ if $\beta_1 < \beta_2 < \alpha$, i.e. $o(H) \cong \alpha$. Now we consider the set H_α . If $\alpha = \alpha^* + 1$ then $H_{\alpha^*} = \bigcup_{\gamma \cong \alpha^*} K_\gamma = K_{\alpha^*}$, thus

$$H_\alpha = H_{\alpha^*+1} = H_{\alpha^*} \cap \text{ind } \text{cl } H_{\alpha^*} = K_{\alpha^*} \cap \text{ind } \text{cl } K_{\alpha^*} = K_{\alpha^*} \cap \text{ind } P_{\alpha^*} = \emptyset.$$

If α is a limit ordinal then $H_\alpha = \bigcap \{H_\beta : \beta < \alpha\} = \bigcup_{\beta < \alpha} \left(\bigcup_{\gamma \cong \beta} K_\gamma \right)$ trivially implies $H_\alpha = \emptyset$.

In both cases $H_\alpha = \emptyset$ which implies $o(H) = \alpha$ and $H \in \mathcal{T}$.

The thin sets constructed by the above method will be called *canonical thin sets*. For later purposes we prove here some lemmas.

LEMMA 2.12. *Let H be a canonical thin set of order α . Let $Z = \bigcap_{\beta < \alpha} P_\beta$, $(a, b) \cap \bigcap Z = \emptyset$. If $[c, d] \subset (a, b)$ and $c, d \notin \bigcup_{\beta < \alpha} P_\beta = P_0$, then $H \cap [c, d]$ is a canonical thin set in $[c, d]$ and $o(H \cap [c, d]) < \alpha$.*

PROOF. Since $[c, d] \cap \bigcap_{\beta < \alpha} P_\beta = \emptyset$, we can find a minimal ordinal number $\gamma < \alpha$ such that $P_\beta \cap [c, d] = \emptyset$ for $\beta \cong \gamma$. If $\gamma > 0$ then we consider the sequence of non-

trivial perfect sets $Q_\beta = P_\beta \cap [c, d]$ ($\beta < \gamma$) and applying the construction for Q_β we obtain the canonical thin set $H \cap [c, d]$ having order γ .

LEMMA 2.13. Let $H \subset [0, 1]$ be arbitrary, G open. Then $(H \cap G)_\beta = H_\beta \cap G$ for every β .

PROOF. We prove by induction. Suppose $(H \cap G)_\beta = H_\beta \cap G$. Now $(H \cap G)_{\beta+1} = (H \cap G)_\beta \cap \text{ind}(\text{cl}(H \cap G)_\beta)$ and the induction hypothesis and the trivial equality

$$G \cap \text{ind cl}(H_\beta \cap G) = G \cap \text{ind cl } H_\beta$$

imply $(H \cap G)_{\beta+1} = H_{\beta+1} \cap G$. The transition step is obvious for limit numbers.

LEMMA 2.14. Let $\{(a_n, b_n)\}$ be a system of pairwise disjoint open intervals, let $H^n \subset (a_n, b_n)$ be a sequence of sets, $H = \bigcup_{n=1}^{\infty} H^n$. Then, for any β ,

$$(12) \quad H_\beta = \bigcup_{n=1}^{\infty} H_\beta^n.$$

In particular, if H^n are thin sets for $n=1, 2, \dots$ and $o(H^n) = \beta_n$, then H is thin and $o(H) = \sup \beta_n$.

PROOF. Taking $G = (a_n, b_n)$ we obtain (12) by Lemma 2.13. If H^n are thin sets then $(H^n)_\beta = \emptyset$ for $\beta \geq \beta_n$ thus $H_\beta = \emptyset$ if $\beta \geq \sup \beta_n$. Hence H is thin and $o(H) = \sup \beta_n$ is obvious.

3. Irregular points

DEFINITION 3.1. Let $f(x)$ be any measurable function in $[0, 1]$. The point x_0 is said to be a regular point of f if the following conditions hold.

- (i) There exists a perfect set P with $x_0 \in \text{ind } P$,
- (ii) $f|_P$ is bounded,

$$(iii) \quad f(x_0) = \lim_{\lambda(I) \rightarrow 0} \frac{1}{\lambda(I)} \int_{P \cap I} f(x) dx$$

where I denotes closed intervals such that $x_0 \in I$ and $\lambda(I) \rightarrow 0$. (As it is usual, we can suppose that either $I = [x_0, x_0 + h]$ or $I = [x_0 - h, x_0]$.) If $x_0 = 0, 1$ then $x_0 \in \text{ind } P$ is meant according to the one-sided density definitions and also the limit in (iii) should be taken from the right and left, respectively. The point x_0 is irregular if it is not regular. The set of irregular points of f will be denoted by T_f .

We remark that replacing the term "there exists a perfect set P " by "there exists a measurable set P " we obtain an equivalent definition.

It suffices to prove that if (i)–(iii) are fulfilled for a measurable set M then there exists a perfect set P with these listed properties. Let the perfect sets P_k, P'_k be chosen such that

$$P_k \subset M \cap \left[x_0 + \frac{1}{k+1}, x_0 + \frac{1}{k} \right] = M_k$$

and

$$\lambda(P_k) \cong \left(1 - \frac{1}{2^k}\right) \lambda(M_k),$$

and

$$P'_k \subset M \cap \left[x_0 - \frac{1}{k}, x_0 - \frac{1}{k+1}\right] = M'_k$$

and

$$\lambda(P'_k) > \left(1 - \frac{1}{2^k}\right) \lambda(M'_k),$$

respectively. Then we put

$$P = \bigcup_1^{\infty} P_k \cup \bigcup_1^{\infty} P'_k \cup \{x_0\}.$$

P is obviously perfect. Let now

$$I_h = [x_0, x_0 + h], \quad \frac{1}{k+1} \cong h < \frac{1}{k}.$$

Then

$$\begin{aligned} \frac{\lambda(P \cap I_h)}{\lambda(I_h)} &\cong \frac{\lambda\left(P \cap \left[x_0, x_0 + \frac{1}{k+1}\right]\right)}{\frac{1}{k}} = k \sum_{j=k+1}^{\infty} \lambda(P_j) \cong \\ &\cong k \sum_{j=k+1}^{\infty} \left(1 - \frac{1}{2^j}\right) \lambda(M_j) = k \sum_{j=k+1}^{\infty} \lambda(M_j) - k \sum_{j=k+1}^{\infty} \frac{\lambda(M_j)}{2^j} \cong \\ &\cong k \lambda\left(M \cap \left[x_0, x_0 + \frac{1}{k+1}\right]\right) - \frac{k}{2^k} = \frac{k}{k+1} \frac{\lambda\left(M \cap \left[x_0, x_0 + \frac{1}{k+1}\right]\right)}{\frac{1}{k+1}} - \frac{k}{2^k}. \end{aligned}$$

For $h \rightarrow +0$ we have $k \rightarrow +\infty$ and thus

$$\lim_{h \rightarrow +0} \frac{\lambda(P \cap I_h)}{h} = 1,$$

and similarly

$$\lim_{h \rightarrow +0} \frac{\lambda(P \cap I'_k)}{h} = 1 \quad \text{for } I'_k = [x_0 - h, x_0],$$

which proves $x_0 \in \text{ind } P$. If finally $f|_M$ is bounded then $f|_P$ is bounded too. Let I be a closed interval such that $x_0 \in I$ and $\lambda(I) = h$, then, for $h \rightarrow 0$,

$$\frac{1}{h} \int_{M \cap I} f(x) dx = \frac{1}{h} \int_{P \cap I} f(x) dx + \frac{1}{h} \int_{(M \setminus P) \cap I} f(x) dx$$

and

$$\frac{1}{h} \int_{(M \setminus P) \cap I} f(x) dx \rightarrow 0$$

because f is bounded on $M \setminus P$ and $M \setminus P$ is of density zero in x_0 . This implies property (iii).

THEOREM 3.2. *If $f(x)$ is bounded then $f \in \mathcal{D}$ holds if and only if $T_f = \emptyset$.*

PROOF. Trivial.

For the proof of our next result we recall two lemmas of [1].

LEMMA 3.3 (Lemma 4.17, [1]). *If $F(x)$ is differentiable in $[0, 1]$ and $H \subset [0, 1]$ is any perfect set, then there exists an open interval I and $K > 0$ such that $I \cap H \neq \emptyset$ and*

$$|F(x) - F(y)| \leq K|x - y| \quad (x, y \in H \cap I).$$

LEMMA 3.4 (Lemma 4.16, [1]). *If F is differentiable in $[0, 1]$ and for the perfect set $H \subset [0, 1]$ the estimation $|F(x) - F(y)| \leq K|x - y|$ ($x, y \in H$) holds, then there exists a differentiable function $G(x)$ such that $G(x) = F(x)$, $G'(x) = F'(x)$ hold for $x \in H$ and $|G(x) - G(y)| \leq (K+1)|x - y|$ for every $x, y \in [0, 1]$.*

THEOREM 3.5. *If $f \in \mathcal{D}$, then $T_f \in \mathcal{F}$.*

PROOF. Assuming $T_f \notin \mathcal{F}$ we consider the set P satisfying (1), i.e. we have
(13)
$$\text{cl}(T_f \cap \text{ind } P) = P.$$

Denoting by F a primitive of f we apply Lemma 3.3 for P . Then applying Lemma 3.4 for $P \cap I$ we obtain $G(x)$ satisfying $|G(x) - G(y)| \leq (K+1)|x - y|$, $x, y \in [0, 1]$. In particular, $g(x) = G'(x)$ is bounded, hence by Theorem 3.2 $T_g = \emptyset$. Thus every $x \in (\text{ind } P) \cap I$ is regular for g . By $g|_{P \cap I} = f|_{P \cap I}$, x is regular also for f , i.e. $T_f \cap P \cap I = \emptyset$, a contradiction with (13).

Now we apply the result for constructing Baire 1 functions which have no derivative extensions.

THEOREM 3.6. *Let P be a non-trivial perfect set metrically dense in itself. Then there exists a Baire 1 function $f(x)$ such that if $g(x) = f|_P(x)$ almost everywhere on P then g has no derivative extension onto $[0, 1]$.*

We remark that this result gives a negative answer to Problem 5.1.2, [1].

PROOF. Let x_1, x_2, \dots be a sequence everywhere dense in $\text{ind } P$. Then it is everywhere dense in P as well. We denote

$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq x_n \\ \frac{1}{2^n}, & x_n < x \leq 1, \end{cases}$$

and let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. The series is uniformly convergent, hence its sum is a Baire 1 function and $T_f = \{x_1, x_2, \dots\}$. Now, if $g = f|_P$ holds almost everywhere on P and g^* denotes any extension of g , then $T_{g^*} \supset \{x_1, \dots\}$. By the choice of the sequence, the set $\{x_1, \dots\}$ is not thin, thus by Theorem 2.3 (i) $T_{g^*} \notin \mathcal{F}$, hence by Theorem 3.5 g^* cannot be a derivative.

THEOREM 3.7. *If the derivative $f(x)$ has local extremum in x_0 then x_0 is regular for f .*

PROOF. We can suppose without loss of generality that $f(x_0)=0$ and $f(x)\geq 0$ (in a neighbourhood J of x_0). Then f is integrable (in this neighbourhood J) and let $F(x)=\int_{x_0}^x f(x)dx$ be a primitive of f . Let $\delta>0$ be arbitrarily given and consider the level set $H_\delta=\{x:f(x)\geq\delta\}$. Denoting $I_h=[x_0, x_0+h]$, we have

$$\begin{aligned} 0 = f(x_0) &= \lim_{h \rightarrow +0} \frac{F(x_0+h)}{h} = \lim_{h \rightarrow +0} \frac{1}{\lambda(I_h)} \int_{I_h} f(x) dx \equiv \\ &\equiv \overline{\lim}_{h \rightarrow +0} \frac{1}{h} \int_{H_\delta \cap I_h} f \equiv \delta \overline{\lim}_{h \rightarrow +0} \frac{\lambda(H_\delta \cap I_h)}{\lambda(I_h)}, \end{aligned}$$

and similar estimation holds for $h<0$. Hence we obtain that H_δ is of density zero in x_0 . Thus taking $M=J \cap \{x:f(x)<1\}$ we have $x_0 \in \text{ind } M$ and $0 \equiv f|_M < 1$. Now the inequality

$$\frac{1}{h} \int_{x_0}^{x_0+h} f \equiv \frac{1}{h} \int_{M \cap I_h} f$$

implies

$$\lim_{h \rightarrow +0} \frac{1}{h} \int_{M \cap I_h} f = 0 = f(x_0),$$

and similar relation holds for $h<0$. Hence x_0 is regular for f . (Actually we have proved the known result that the derivative $f(x)$ is approximately continuous in x_0 , if $f(x)$ has local extremum in x_0 . One can easily show that (in general) approximate continuity implies regularity.)

However trivial Theorem 3.2 is, in certain sense it is best possible as shown by the next theorem.

THEOREM 3.8. *Let α be any countable ordinal number. Then there exists a derivative $f \in \mathcal{D}$ such that $f \geq 0$ and $o(T_f) = \alpha$.*

More precisely, we prove that, if $H = \cup \{K_\beta : \beta < \alpha\}$ is a canonical thir. set of order α , then there exists a non-negative derivative f such that $o(T_f) = \alpha$,

$$K_\beta \subset (T_f)_\beta \setminus (T_f)_{\beta+1} \subset \text{cl } K_\beta \setminus \text{ind cl } K_\beta$$

for every β .

PROOF. The tedious proof is carried out by several steps. Though our construction is complicated to such an extent which may seem unnecessary, a deeper analysis shows that it cannot be simplified in any essential point. Actually, the possibility $T_f \neq \emptyset$ is not obvious either.

If f is any function defined in $[0, 1]$, we put

$$(14) \quad f(x; a, b) = f\left(\frac{x-a}{b-a}\right) \quad (a \leq x \leq b).$$

(i) In this section we are going to construct a derivative non-negative on $[0, 1]$ and having 0 as an irregular point.

For any integer $n \geq 4$ we define the continuous function $\varphi_n(x) \geq 0$ in $[0, 1]$ as follows. In $\left[0, \frac{1}{n}\right]$ $\varphi_n(x)$ is a trapezium-like function defined by the formulas

$$\varphi_n(x) = \begin{cases} n^3 x, & 0 \leq x \leq \frac{1}{n^2} \\ n, & \frac{1}{n^2} \leq x \leq \frac{2}{n^2} \\ -n^3 x + 3n, & \frac{2}{n^2} \leq x \leq \frac{3}{n^2} \\ 0, & \frac{3}{n^2} \leq x \leq \frac{1}{n} \end{cases}$$

then the definition is extended periodically by

$$\varphi_n(x) = \varphi_n\left(x - \frac{k}{n}\right) \quad \left(\frac{k}{n} \leq x \leq \frac{k+1}{n}\right), \quad k = 1, \dots, n-1.$$

The relations

$$(15) \quad \int_a^b \varphi_n(x) dx = 2(b-a), \quad a = \frac{k}{n}, \quad b = \frac{l}{n}, \quad 0 \leq k \leq l \leq n$$

and

$$(16) \quad \lambda(\{x : \varphi_n(x) = n\}) = \frac{1}{n}$$

are obvious. We denote by $p_1 < p_2 < \dots$ the sequence of prime numbers and by $p(n)$ the number of prime divisors of n , $p(1) = 1$. For $n \geq 1$, $0 \leq k \leq n$ we put $x_{n,k} = \frac{1}{2^n} + \frac{k}{n} \cdot \frac{1}{2^n}$. Obviously $x_{n+1, n+1} = x_{n,0}$. For any integer $t \geq 3$ we define $f_t(x)$ in $[0, 1]$ by

$$(17) \quad f_t(x) = \begin{cases} \varphi_{p(n)+t}(x; x_{n,k}, x_{n,k+1}), & x_{n,k} \leq x \leq x_{n,k+1}, \\ & 0 \leq k \leq n-1, \quad n = 1, 2, \dots; \\ 2, & x = 0. \end{cases}$$

According to the definition of φ , the interval $(x_{n,k}, x_{n,k+1})$ is divided into $p(n) + t$ equal subintervals by the numbers

$$(18) \quad x_{n,k} + l \frac{x_{n,k+1} - x_{n,k}}{p(n) + t}, \quad l = 0, 1, \dots, p(n) + t.$$

We shall refer to these points (including $x=0$) as the principal points of $[0, 1]$. We are to prove $f_t \in \mathcal{D}$ and $T_{f_t} = \{0\}$. Recalling (15), $\int_a^b f_t(x) dx = 2(b-a)$ if $a < b$ are arbitrary principal points, in particular $\int_0^h f_t(x) dx = 2h$ if h is a principal point.

For any $0 < h < 1$ we choose neighbouring principal points a, b such that $a \leq h < b$. Then

$$\frac{1}{b} \int_0^a f_t(x) dx \leq \frac{1}{h} \int_0^h f_t(x) dx \leq \frac{1}{a} \int_0^b f_t(x) dx$$

i.e.

$$2 \frac{a}{b} \leq \frac{1}{h} \int_0^h f_t(x) dx \leq 2 \frac{b}{a},$$

thus

$$\left| \frac{1}{h} \int_0^h f_t(x) dx - 2 \right| \leq 2 \left(\frac{b}{a} - 1 \right).$$

Referring to (18)

$$\begin{aligned} 2 \left(\frac{b}{a} - 1 \right) &= 2 \frac{b-a}{a} \leq 2 \frac{x_{n,k+1} - x_{n,k}}{p(n)+t} \frac{1}{x_{n,k}} \leq \frac{2}{t} \frac{x_{n,k+1} - x_{n,k}}{x_{n,k}} \leq \\ &\leq \frac{2}{t} \frac{\frac{1}{2^n n}}{\frac{1}{2^n} + \frac{k}{n} \frac{1}{2^n}} \leq \frac{2}{nt} \end{aligned}$$

where n and k are suitably chosen. Thus we obtain the estimation

$$(19) \quad \left| \frac{1}{h} \int_0^h f_t(x) dx - 2 \right| \leq \frac{2}{nt}, \quad \frac{1}{2^n} \leq h < \frac{1}{2^{n-1}}.$$

We observe that

$$\frac{1}{h} \int_a^{a+h} f_t(x; a, b) dx = \frac{b-a}{h} \int_0^{h/(b-a)} f_t(x) dx = \frac{1}{h'} \int_0^{h'} f_t(x) dx,$$

thus

$$(19a) \quad \left| \frac{1}{h} \int_a^{a+h} f_t(x; a, b) dx - 2 \right| \leq \frac{2}{nt}, \quad \frac{b-a}{2^n} \leq h < \frac{b-a}{2^{n-1}}.$$

Since $h \rightarrow 0$ implies $n \rightarrow \infty$ we obtain that $F_t(x) = \int_0^x f_t(x) dx$ is differentiable in 0 and $F_t'(0) = 2$. By the continuity of $f_t(x)$, $F_t'(x) = f_t(x)$ trivially holds for $x > 0$, i.e. $f_t \in \mathcal{D}$.

Let now N be an integer, $N \geq 1$ and consider the numbers $n = n_l = (p_1 p_2 \dots p_N)^l$, $l = 1, 2, \dots$. Then $p(n) = N$ and denoting

$$H_N = \{x : f_t(x) \geq N\}$$

we have

$$H_N \supset \bigcup_{k=0}^{n-1} \{x : \varphi_{p(n)+t}(x) = p(n)+t, x_{n,k} \leq x \leq x_{n,k+1}\},$$

for any fixed $n = n_t$. Recalling (16)

$$\lambda \left(H_N \cap \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \right) \cong \frac{\frac{1}{2^{n-1}} - \frac{1}{2^n}}{N+t} = \frac{1}{2^n} \frac{1}{N+1}.$$

Hence

$$2^{n-1} \lambda \left(H_N \cap \left[0, \frac{1}{2^{n-1}} \right] \right) \cong \frac{1}{2(N+t)},$$

thus

$$\varliminf_{h \rightarrow 0} \frac{1}{h} \lambda(H_N \cap [0, h]) \cong \frac{1}{2(N+t)} > 0$$

for $N=1, 2, \dots$. It yields that $f_t(x)$ cannot be bounded in any D -neighbourhood of 0. Since $f_t(x)$ is continuous for $x > 0$, we obtain $T_{f_t} = \{0\}$.

(ii) In this section we extend the previous construction in order to obtain a non-negative derivative exhibiting irregularity in every end-point of a non-trivial perfect set.

We put

$$g_t(x) = \begin{cases} f_t \left(x; 0, \frac{1}{2} \right), & 0 \leq x \leq \frac{1}{2} \\ g_t(1-x), & \frac{1}{2} < x \leq 1. \end{cases}$$

By $g_t \left(\frac{1}{2} \right) = 0$, $g_t(x)$ is continuous in $(0, 1)$ and according to (i), $T_{g_t} = \{0, 1\}$.

Let $P \subset [0, 1]$ be a nontrivial and nowhere dense perfect set, $0 \notin P$, $1 \notin P$. Let

$$(20) \quad [0, 1] \setminus P = [0, b_0] \cup (a_0, 1] \cup \bigcup_{k=1}^{\infty} (a_k, b_k)$$

be the decomposition of $[0, 1] \setminus P$ into intervals contiguous to P . Let

$$h(x) = \begin{cases} f_3(1-x; 0, b_0), & 0 \leq x < b_0 \\ f_3(x; a_0, 1), & a_0 < x \leq 1 \\ g_{t_k}(x; a_k, b_k), & a_k < x < b_k \\ 2, & x \in P \end{cases}$$

where t_k is chosen to satisfy $t_k(b_k - a_k) \cong 1$. Making use of (15),

$$\int_{\alpha}^{\beta} h(x) dx = 2(\beta - \alpha)$$

if α and β are either principal points in any of the intervals contiguous to P or belong to P . In particular, the integral $F(x) = \int_0^x h(y) dy$ is finite valued. $F'(x) = h(x)$ obviously holds for $x \notin P$ and by the construction of $f_t(x)$, $F'(b_k) = F'(a_k) = 2$, $k=0, 1, \dots$ if by $F'(b_k)$ and $F'(a_k)$ we mean left hand side and right hand side derivatives respectively.

Let $x_0 \in P$ and suppose that approaching x_0 say, from the right, x runs over an infinite number of intervals from the decomposition (20). If

$$a_k < x \leq \frac{1}{2}(a_k + b_k),$$

we write

$$\begin{aligned} \int_{x_0}^x h(y) dy &= \int_{x_0}^{a_k} h(y) dy + \int_{a_k}^x g_{t_k}(y; a_k, b_k) dy = \\ &= 2(a_k - x_0) + (2 + \varepsilon_k)(x - a_k) = 2(x - x_0) + \varepsilon_k(x - a_k) \end{aligned}$$

where $|\varepsilon_k| \leq 2/t_k$ holds by (19a).

For $\frac{1}{2}(a_k + b_k) < x < b_k$ we put

$$\begin{aligned} \int_{x_0}^x h(y) dy &= \int_{x_0}^{b_k} h(y) dy - \int_x^{b_k} g_{t_k}(y; a_k, b_k) dy = \\ &= 2(b_k - x_0) - (2 + \varepsilon'_k)(b_k - x) = 2(x - x_0) - \varepsilon'_k(b_k - x), \end{aligned}$$

where $|\varepsilon'_k| \leq 2/t_k$ holds again by (19a). Hence we have

$$\int_{x_0}^x h(y) dy = \begin{cases} 2(x - x_0) + \varepsilon_k(x - a_k), & a_k < x \leq \frac{1}{2}(a_k + b_k) \\ 2(x - x_0) - \varepsilon'_k(b_k - x), & \frac{1}{2}(a_k + b_k) < x < b_k. \end{cases}$$

Since

$$0 \leq \frac{x - a_k}{x - x_0} \leq 1$$

and

$$0 \leq \frac{b_k - x}{x - x_0} \leq \frac{b_k - \frac{1}{2}(a_k + b_k)}{\frac{1}{2}(a_k + b_k) - a_k} = 1$$

we obtain in either cases

$$\left| \frac{1}{x - x_0} \int_{x_0}^x h(y) dy - 2 \right| \leq \frac{2}{t_k},$$

which implies $F'(x_0) = 2$. We know $a_k \in T_h, b_k \in T_h$ by the construction given in (i) and we remark that, since h is constant on $P, T_h \cap \text{ind } P = \emptyset$ is trivial.

We observe that $h(0) = h(1) = 0$ and $h(x)$ is continuous in these points from the right and from the left, respectively.

(iii) Now we proceed by induction. Induction hypothesis: if $[a, b]$ is any interval and $H = \cup \{K_\beta : \beta < \alpha\}$ is a canonical thin set in $[a, b]$ such that $a, b \notin \text{cl } H$, then there exists a derivative $f \equiv 0$ on $[a, b]$ such that $o(T_f) = \alpha$,

$$(20a) \quad K_\beta \subset (T_f)_\beta \setminus (T_f)_{\beta+1} \subset \text{cl } K_\beta \setminus \text{ind cl } K_\beta \quad (\beta < \alpha),$$

$f(x)$ is continuous from one side in a and b and $f(a)=f(b)=0$. If $\alpha=1$, the required derivative is given by the preceding section (ii). Let $o(H)=\alpha>1$, $H=\cup\{K_\beta: \beta<\alpha\}$ where K_β are canonical thin sets of order 1, $\text{cl } K_\beta - \text{cl} (\cup_{\gamma \cong \beta} K_\gamma) \stackrel{\text{def}}{=} P_\beta$ and let $P = \bigcap_{\beta < \alpha} P_\beta$.

Now the induction hypothesis is assumed for every $\beta < \alpha$.

Observe that $P=P_\gamma$ if $\alpha=\gamma+1$, and one can say nothing more on P than that is a nowhere dense and non-empty closed set. Let $[0, 1] \setminus P = [0, B_0] \cup \cup_{k=1}^\infty (A_0, 1] \cup \cup_{k=1}^\infty (A_k, B_k)$ be the decomposition of $[0, 1] \setminus P$ into intervals contiguous

to P . If we consider the sequence $\{\dots a_{-1}, a_0, a_1, \dots\} = \left\{ \dots \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \dots \right\}$ i.e.

$a_n = \frac{1}{|n|+2}$, $n=0, -1, -2, \dots$ and $a_n = 1 - a_{-n}$, $n>0$, then for any integer n , $-\infty < n < +\infty$ we have $1 < \frac{a_{n+1}}{a_n} \leq \frac{3}{2}$ and $1 < \frac{1-a_n}{1-a_{n+1}} \leq \frac{3}{2}$. In every interval (A_k, B_k) we take a sequence $\{a_n^k\}_{n=-\infty}^{+\infty}$

$$a_n^k = \begin{cases} A_k + (B_k - A_k) \left(\frac{1}{|n|+2} + \varepsilon(n, k) \right), & n \leq 0 \\ B_k - (B_k - A_k) \left(\frac{1}{n+2} + \varepsilon(n, k) \right), & n > 0 \end{cases}$$

where $|\varepsilon(n, k)|$ are chosen so small that both

$$1 < \frac{a_{n+1}^k - A_k}{a_n^k - A_k} < 2, \quad 1 < \frac{B_k - a_n^k}{B_k - a_{n+1}^k} < 2 \quad (-\infty < n < +\infty)$$

and

$$a_n^k \notin P_0 = \text{cl } H \quad (-\infty < n < +\infty, k = 1, 2, \dots)$$

are satisfied. (This latter is possible since P_0 is nowhere dense. In $[0, B_0]$ and $(A_0, 1]$ we take one-sided sequences $\{a_n^k\}_{n=1}^\infty$ and $\{a_n^k\}_{n=-1}^\infty$ tending to B_0 and A_0 from the left and from the right, respectively.)

Recalling Lemma 2.12, $H_n^k = H \cap [a_n^k, a_{n+1}^k]$ is a canonical thin set the order α' of which is less than α :

$$H_n^k = \bigcup_{\beta < \alpha'} K_{n\beta}^k \quad \text{where} \quad K_{n\beta}^k = K_\beta \cap [a_n^k, a_{n+1}^k].$$

By the induction hypothesis there exists a derivative $f_n^k \cong 0$ on $[a_n^k, a_{n+1}^k]$ such that $o(T_{f_n^k}) = o(H_n^k)$,

$$K_{n\beta}^k \subset (T_{f_n^k})_\beta \setminus (T_{f_n^k})_{\beta+1} \subset \text{cl } K_{n\beta}^k \setminus \text{ind cl } K_{n\beta}^k, \quad \beta < \alpha'$$

f_n^k is continuous in a_n^k and a_{n+1}^k and vanishes in these points. We can assume

$$(21) \quad \int_{a_n^k}^{a_{n+1}^k} f \cong \frac{a_{n+1}^k - a_n^k}{|n|+k},$$

otherwise we replace f_n^k by cf_n^k where $c > 0$ is sufficiently small. We put

$$f(x) = \begin{cases} f_n^k(x), & x \in [a_n^k, a_{n+1}^k] \\ 0, & x \in P \end{cases}$$

and prove $f \in \mathcal{D}$. By (21) $F(x) = \int_0^x f$ is finite valued and $F'(x) = f(x)$ is obvious for $x \notin P$. Let $x_0 \in P$. If $x_0 = A_k$ and $h > 0$ then taking $a_n^k \leq A_k + h < a_{n+1}^k$ we have by (21)

$$(22) \quad \frac{1}{h} \int_{A_k}^{A_k+h} f \leq \frac{1}{a_n^k - A_k} \int_{A_k}^{a_{n+1}^k} f \leq \frac{a_{n+1}^k - A_k}{a_n^k - A_k} \cdot \frac{1}{|n| + k} < \frac{2}{|n| + k} \quad (n < 0)$$

and

$$(23) \quad \frac{1}{h} \int_{A_k}^{A_k+h} f \leq \frac{1}{h} \int_{A_k}^{B_k} f \leq \frac{B_k - A_k}{h} \cdot \frac{1}{k} \leq \frac{2}{k} \quad (n \geq 0).$$

If $x_0 \in P$ and for $h \rightarrow +0$ the variable $x_0 + h$ meets infinitely many intervals from $\{(A_k, B_k)\}$ then for $A_k \leq x_0 + h \leq B_k$

$$(24) \quad \frac{1}{h} \int_{x_0}^{x_0+h} f = \frac{1}{h} \int_{x_0}^{A_k} f + \frac{1}{h} \int_{A_k}^{x_0+h} f \leq \frac{A_k - x_0}{h} \frac{1}{m_k} + \frac{x_0 + h - A_k}{h} \cdot \frac{2}{k} \leq \frac{1}{m_k} + \frac{2}{k},$$

where the first integral is estimated by (21), the second one by (22) and (23), and m_k denotes the minimal index satisfying $(A_{m_k}, B_{m_k}) \subset (x_0, x_0 + h)$. In both cases

$\frac{1}{h} \int_{x_0}^{x_0+h} f \rightarrow 0$ and we have similar estimations for $h < 0$, hence $f \in \mathcal{D}$ is proved. Refer-

ring to Theorem 3.7 the points of P are regular for f , i.e. $T_f \cap P = \emptyset$.

By Lemma 2.14

$$(T_f)_\beta = \bigcup_k [T_f \cap (A_k, B_k)]_\beta \cup [T_f \cap [0, B_0]]_\beta \cup [T_f \cap (A_0, 1]]_\beta.$$

This and

$$K_\beta = \bigcup_k [K_\beta \cap (A_k, B_k)] \cup [K_\beta \cap [0, B_0]] \cup [K_\beta \cap (A_0, 1]] \quad (\beta + 1 < \alpha)$$

imply

$$(25) \quad K_\beta \subset (T_f)_\beta \setminus (T_f)_{\beta+1} \subset \text{cl } K_\beta \setminus \text{ind } \text{cl } K_\beta \quad ((\beta + 1) < \alpha)$$

and by the construction, f vanishes continuously in 0 and 1. Recalling Lemma 2.13, $(H \setminus P)_\beta = H_\beta \setminus P = K_\beta$ ($\beta + 1 < \alpha$) thus by (25) we have $o(T_f) = o(H \setminus P)$. In fact, if $\alpha = \gamma + 1$, then

$$(T_f)_\gamma \subset (H \setminus P)_\gamma = H_\gamma \setminus P = \emptyset,$$

and for $\beta < \gamma$ we have $\emptyset \neq K_\beta \subset (T_f)_\beta$. If α is a limit number then $o(T_f) = o(H \setminus P)$ is obvious, and for this case the proof is complete.

Let $\alpha = \gamma + 1$, in this case $H \cap P = H_\gamma = K_\gamma$ and we apply the basic construction (ii) for the perfect set $P_\gamma = \text{cl } K_\gamma$. Hence we obtain a derivative $g(x) \geq 0$ such that

$$(25a) \quad K_\gamma \subset T_g \subset P_\gamma \setminus \text{ind } P_\gamma = P \setminus \text{ind } P$$

and g vanishes continuously in 0 and 1. Finally we put $h(x)=g(x)+f(x)$. Since $T_g \cap T_f = \emptyset$ and $g \geq 0, f \geq 0$, we have $T_h = T_g \cup T_f$. We are going to prove

$$(26) \quad (T_h)_\beta = T_g \cup (T_f)_\beta \quad (\beta < \alpha)$$

by induction.

Let $\beta+1 < \alpha$. Then

$$(T_h)_{\beta+1} = (T_h)_\beta \cap \text{ind cl } (T_h)_\beta = (T_h)_\beta \cap \text{ind cl } [T_g \cup (T_f)_\beta] = (T_h)_\beta \cap \text{ind } (P_\gamma \cup P_\beta),$$

since by (25) and (25a) we have $\text{cl } (T_f)_\beta = P_\beta$ and $\text{cl } T_g = P_\gamma$. By $P_\gamma \subset P_\beta$ we obtain

$$(T_h)_{\beta+1} = [T_g \cup (T_f)_\beta] \cap \text{ind } P_\beta = (T_g \cap \text{ind } P_\beta) \cup ((T_f)_\beta \cap \text{ind } P_\beta) = T_g \cup (T_f)_{\beta+1},$$

referring the stronger inclusion $T_g \subset P_\gamma \subset \text{ind } P_\beta$.

The transition step is trivial for limit numbers, hence (26) is verified. Since $o(T_f) = o(H \setminus P) = o(H \setminus P_\gamma) = \gamma$, by $(T_f)_\gamma = \emptyset$ we obtain $(T_h)_\gamma = T_g$ and hence

$$(T_h)_{\gamma+1} = (T_g)_1 = \emptyset$$

i.e. $o(T_h) = \alpha$. Now $(T_h)_\beta \setminus (T_h)_{\beta+1} = (T_f)_\beta \setminus (T_f)_{\beta+1}$ if $\beta+1 < \alpha$ (i.e. $\beta < \gamma$), $(T_h)_\gamma \setminus (T_h)_{\gamma+1} = T_g$, and for $\beta > \gamma$ $(T_h)_\beta \setminus (T_h)_{\beta+1} = \emptyset$, thus by (25) and (25a) we obtain

$$K_\beta \subset (T_h)_\beta \setminus (T_h)_{\beta+1} \subset \text{cl } K_\beta \setminus \text{ind cl } K_\beta$$

for every β . $h(0)=h(1)=0$, and the continuity in these points is trivial by $h=f+g$, thus the theorem is proved.

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CONVERGENCE UNIFORME ET CHANGEMENT DE VARIABLE

par
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Dédié à L. ILJEFF à l'occasion de son 60-ième anniversaire

§ 1. Introduction

La note [12] de P. TURÁN a inspiré plusieurs recherches. Il a montré dans ce travail initiateur la proposition suivante: Soit ζ ($0 < |\zeta| < 1$) une constante complexe donnée à l'avance. Il existe alors une fonction

$$(1.1) \quad f_1(z) = \sum_{v=0}^{\infty} a_v z^v$$

analytique pour $|z| < 1$ et telle qu'en un point $z_1 = e^{it_1}$ la série $\sum a_v z_1^v$ converge, mais si l'on pose

$$(1.2) \quad f_1\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right) = f_2(z) = \sum_{n=0}^{\infty} b_n z^n \quad (b_n = b_n(\zeta)),$$

la série $\sum b_n z_2^n$ diverge, où $z_1 = (z_2 - \zeta)(1 - \bar{\zeta}z_2)^{-1}$, donc $z_2 = e^{it_2}$.

Plus tard, répondant aux questions posées également par TURÁN, J. CLUNIE [9] a prouvé que $f_1(z)$ tout en jouissant de la propriété mentionnée peut être simultanément continue sur la circonférence $|z| = 1$, tandis que K.-H. INDLEKOFER [11] a montré qu'il existe $f_1(z)$ telle que le module de continuité de $f_1(e^{it})$ remplit la condition

$$(1.3) \quad \omega(\delta; f_1(e^{it})) = O(|\log \delta|^{-1}) \quad (\delta \rightarrow +0).$$

Les résultats de CLUNIE et de INDLEKOFER, complétés avec une stipulation concernant la continuité des fonctions en question, portent donc sur le comportement local des séries considérées.

De notre part, en nous appuyant sur la note de CLUNIE, nous avons démontré [7] que malgré la convergence uniforme de $\sum a_v e^{ivt}$, la série $\sum b_n e^{int}$ peut être divergente sur un ensemble E dénombrable et partout dense dans $[0, 2\pi]$. Nous avons établi ensuite [8], utilisant le résultat de INDLEKOFER, que ces deux séries peuvent conserver leur caractère indiqué en dépit que (1.3) subsiste.

Dans la note présente nous étendons ces recherches sur certains changements de variable des séries de Fourier. Nous étions amené à envisager de problèmes de ce genre en étudiant les transformations conformes plus générales que les homographiques. Nous avons examiné des fonctions représentées par des séries de Faber dont la courbe de convergence est formée d'un seul arc analytique régulier fermé ([2], [3], [5]). Il s'est avéré que les cas considérés se ramènent à l'étude d'un certain type de changement de variable des séries de Fourier. On peut donc soulever de questions analogue sur les changements de variable des séries de Fourier uniformément convergentes sans se référer aux problèmes correspondants des séries de Faber.

Nous allons établir notamment la proposition ci-après.

THÉORÈME. — Soit $f(t)$ une fonction à variable et à valeur réelle, analytique sur l'axe des t et telle que $f'(t) \neq 0$ et $f(t) \equiv f(t+2\pi) \pmod{2\pi}$. Alors il existe une fonction

$$(1.4) \quad F(t) = \sum_{\nu=-\infty}^{\infty} A_{\nu} e^{i\nu t}$$

remplissant les conditions suivantes: (i) la série (1.4) converge uniformément mais non absolument; (ii) le module de continuité de $F(t)$ vérifie la relation

$$(1.5) \quad \omega(\delta; F) = O(|\log \delta|^{-1}) \quad (\delta \rightarrow +0);$$

(iii) et si l'on pose

$$(1.6) \quad F[f(t)] = G(t) \sim \sum_{n=-\infty}^{\infty} B_n e^{int}$$

la série (1.6) diverge sur un ensemble E dénombrable et partout dense dans l'intervalle $[0, 2\pi]$.

Si dans (1.5) on peut remplacer O par o , les séries (1.4) et (1.6) convergent uniformément.

La démonstration se fonde, outre les travaux cités de CLUNIE et de INDLEKOFER, sur un de nos résultats antérieurs [4]. CLUNIE, de sa part, a déterminé d'abord une fonction $f_2(z)$ (cf. (1.2)) satisfaisant aux exigences du problème, puis il en a déduit $f_1(z)$ (cf. (1.1)) au moyen de la transformation inverse en posant $f_1(z) = f_2[(z+\zeta)(1+\bar{\zeta}z)^{-1}]$. À cet effet il s'est servi des polynômes de Fejér définis par la formule ($n \geq 1$):

$$(1.7) \quad \varphi_n(z) = \frac{z}{n} + \frac{z^2}{n-1} + \dots + \frac{z^n}{1} - \dots - \frac{z^{n+1}}{1} - \dots - \frac{z^{2n}}{n} = \sum_{j=1}^n j^{-1} (z^{n-j+1} - z^{n+j}).$$

Nous allons suivre une voie pareille en utilisant une idée de FEJÉR esquissée dans [10]. Soit $\{t_n\}_{n=1}^{\infty} = E_0$ un ensemble dénombrable contenant une infinité de fois chaque élément de E . Alors $G(t)$ est définie de la manière ci-dessous:

$$(1.8) \quad G(t) = \sum_{n=1}^{\infty} 2^{-n} e^{ik_n(t-t_n)} \varphi_{2^{2n}}(e^{i(t-t_n)}) \sim \sum_{k=n^*}^{\infty} B_n e^{int}$$

où les exposants k_n sont des entiers positifs convenablement choisis. La série de fonctions (1.8) est absolument convergente, car il existe une constante absolue M telle que $|\varphi_j(e^{it})| \leq M$ uniformément en j et t . Si de plus les k_n vérifient l'inégalité

$$(1.9) \quad k_n + 2 \cdot 2^{2n} < k_{n+1} + 1,$$

alors en omettant dans (1.8) les parenthèses on obtient une série de Fourier formelle. Le choix particulier de la série de fonctions (1.8) s'adapte aux calculs de INDLEKOFER et est nécessaire pour établir (1.5). Nous allons voir dans le § 4 que la série de Fourier de $G(t)$ diverge sur E .

$f(t)$ étant monotone, par hypothèse, elle possède une fonction inverse, soit $f^{-1}(t) = g(t)$. Posons donc à l'instar des antécédents $F(t) = G[g(t)]$, c'est que

$$(1.10) \quad F(t) = \sum_{n=1}^{\infty} 2^{-n} e^{ik_n[g(t)-t_n]} \varphi_{2^{2n}}(e^{i[g(t)-t_n]}) \sim \sum_{\nu=-\infty}^{\infty} A_{\nu} e^{i\nu t}.$$

Il est à prouver ainsi que la série de Fourier de $F(t)$ converge uniformément et que son module de continuité est de l'ordre logarithmique. La formule (1.10) indique déjà la marche de la démonstration: nous pouvons encore disposer dans une large mesure librement des exposants k_n . Dans ce but il faut examiner d'abord le type de fonctions qui figure dans (1.10) et ensuite la série de fonctions (1.10) elle-même.

Conformément à ce programme le § 2 est consacré à certains lemmes, on s'occupe des polynômes de Fejér dans le § 3, enfin la preuve du Théorème est achevée dans le § 4. Au cours des démonstrations on désigne par c_j ($j=1, 2, \dots$) des constantes numériques positives.

§ 2. Quelques lemmes

Dans ce qui suit nous aurons besoin d'un de nos résultats précédents [4], dont un cas particulier a été déjà trouvé plus tôt ([1], § 6, pp. 307—312). Nous l'appelons ici Lemme 1. Nous le citons sans démonstration, mais nous faisons tout de même certaines observations qui permettent de voir l'origine de cette proposition.

REMARQUE. — $F(t)$ et $e^{if(t)}$ étant continues et 2π -périodiques, $G(t)$ l'est aussi. De plus, en raison de la continuité uniforme de la série de fonctions

$$G(t) = \sum_{\nu=-\infty}^{\infty} A_{\nu} e^{i\nu f(t)} \sim \sum_{n=n^*}^{\infty} B_n e^{int},$$

on peut écrire

$$B_n = \sum_{\nu=-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i[\nu f(t) - nt]} dt \right\} A_{\nu} = \sum_{\nu=-\infty}^{\infty} a_{n\nu} A_{\nu}$$

où l'on a évidemment

$$e^{i\nu f(t)} = \sum_{\nu=-\infty}^{\infty} a_{n\nu} e^{int}.$$

Il en résulte que

$$\sum_{n=l}^m B_n e^{int} = \sum_{\nu=-\infty}^{\infty} \left(\sum_{n=l}^m a_{n\nu} e^{int} \right) A_{\nu}.$$

Si, en particulier, $\sum |A_{\nu}| < \infty$ et $\left| \sum_{n=l}^m a_{n\nu} e^{int} \right|$ est uniformément bornée en ν, l, m et t , alors on peut démontrer que la série $\sum_{n=-\infty}^{\infty} B_n e^{int}$ converge uniformément [6]. Or, le Lemme 1 affirme précisément les suivants:

LEMME 1. — Soit $f(t)$ une fonction réelle, analytique et telle que $f(t) \equiv f(t+2\pi) \pmod{2\pi}$, soient de plus

$$e^{i\nu f(t)} = \sum_{n=-\infty}^{\infty} a_{n\nu} e^{int} \quad (\nu = 0, \pm 1, \pm 2, \dots)$$

et

$$S_{lm}^{\nu}(t; f) = S_{lm}^{\nu}(t) = \sum_{n=l}^m a_{n\nu} e^{int}.$$

Alors il existe une constante $K=K(f)$ telle que

$$|S_{lm}^v(t)| \leq K$$

uniformément en v, l, m et t .

Apercevons que cette fois-ci la monotonité de $f(t)$ n'est pas exigée. Cette dernière condition a été posée uniquement pour assurer l'existence de $g(t)=f^{-1}(t)$.

Les hypothèses adoptées sur $f(t)$ entraînent, d'autre part, que $[f(t+2\pi)-f(t)]/2\pi$ soit simultanément une fonction continue et un entier. C'est donc une constante à valeur entière. Il en découle que $f(t)=\varphi(t)+ht$ où $\varphi(t)$ est 2π -périodique et h est un entier. Lorsqu'on a aussi $f'(t)\neq 0$, on suppose, pour simplifier les calculs, que $h=1$ ou bien que $f(t)=\varphi(t)+t$. On en tire qu'alors $g(t)=\gamma(t)+t$ où $g(t)$ est encore réelle, analytique, $g'(t)\neq 0$ et $g(t+2\pi)-g(t)=2\pi$, c'est que $\gamma(t)$ est 2π -périodique. $f(t)$ et $g(t)$ sont donc exactement de la même nature, d'où la conclusion que voici.

COROLLAIRE. — Soit

$$(2.1) \quad e^{i\eta g(t)} = \sum_{v=-\infty}^{\infty} b_{vn} e^{ivt} \quad (n = 0, \pm 1, \pm 2, \dots).$$

Alors il existe une constante $K_0=K_0(g)$ telle que

$$(2.2) \quad |S_{\lambda\mu}^n(t; g)| = |S_{\lambda\mu}^n(t)| = \left| \sum_{v=\lambda}^{\mu} b_{vn} e^{ivt} \right| \leq K_0$$

uniformément en n, λ, μ et t .

En étudiant les fonctions analytiques et leurs transformations homographiques, le rôle de $f(t)$ resp. $g(t)$ a été attribué à

$$f_0(t) = -i \log \frac{e^{it} - \zeta}{1 - \bar{\zeta} e^{it}} = -i \log \frac{1 - \zeta e^{-it}}{1 - \bar{\zeta} e^{it}} + t \quad \text{resp.} \quad g_0(t) = -i \log \frac{1 + \zeta e^{-it}}{1 + \bar{\zeta} e^{it}} + t.$$

Le lemme suivant fournit une majoration des a_{nv} resp. b_{vn} . (Voir [4] Lemme 1, p. 191, et son cas particulier [1] Lemme 2, p. 296.)

LEMME 2. — $f(t)$ satisfait aux conditions du Lemme 1. Soient $t_1 < t_2 < \dots < t_k$ les racines de l'équation $f''(t)=0$ comprises dans l'intervalle $[-\pi, \pi)$ avec les multiplicités respectives $p_1-2, p_2-2, \dots, p_k-2$ et $p = \max p_j$ ($j=1, 2, \dots, k$). Alors

$$(2.3) \quad |a_{nv}| \leq c_1 |v|^{-1/p} \quad (v \neq 0)$$

uniformément en n , autrement dit $c_1 = c_1(f)$. Si $f'(t) \neq 0$, donc $g(t)$ existe, on a également

$$(2.4) \quad |b_{vn}| \leq c_2 |n|^{-1/p} \quad (n \neq 0)$$

uniformément en v , c'est que $c_2 = c_2(g)$.

DÉMONSTRATION. — (2.3) a été déjà établi dans les notes signalées. Il est à vérifier (2.4). Posons $u=f(t)$, $t=g(u)$. On a ainsi $g''(u) = -f''(t)[f'(t)]^{-3}$. À chaque t_j correspond donc un $u_j=f(t_j)$ racine de l'équation $g''(u)=0$. Les expressions des dérivées d'ordre plus élevé de $g(u)$ formées à l'aide de celles de $f(t)$ montrent que la multi-

plicité de u_j est aussi $p_j - 2$ ($j = 1, 2, \dots, k$). (2.4) n'est donc autre chose que (2.3) formulé pour les $b_{\nu n}$.

LEMME 3. — Soient $P(z)$ une fonction analytique et uniforme dans la couronne circulaire $R_1 < |z| < R_2$, $R_1 < 1 < R_2$; $M_0 = \max |P(e^{it})|$ et

$$(2.5) \quad e^{ing(t)} P(e^{ig(t)}) = \sum_{\nu=-\infty}^{\infty} \alpha_{\nu n} e^{i\nu t}.$$

Alors il existe un entier positif $n_0 = n_0(P, g)$ tel que pour tout entier $|n| \geq n_0$ les sommes partielles de la série (2.5) sont uniformément bornées, soit

$$(2.6) \quad |s_{\lambda\mu}^n(t)| = \left| \sum_{\nu=\lambda}^{\mu} \alpha_{\nu n} e^{i\nu t} \right| \leq 2(1 + K_0 M_0) = N_0,$$

c'est-à-dire que cette borne commune ne dépend que de M_0 et de K_0 définie sous (2.2).

DÉMONSTRATION. — Il est apparent qu'il suffit d'établir le lemme seulement pour

$$(2.7) \quad s_{0\nu}^n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{ing(u)}) \frac{e^{i(v+1)u} - e^{i(v+1)t}}{e^{iu} - e^{it}} du.$$

Lorsque $P(z) \equiv 1$, (2.7) fournit $S_{0\nu}^n(t)$ défini sous (2.2). Considérons maintenant l'expression

$$(2.8) \quad \begin{aligned} I_{n\nu} &= S_{0\nu}^n(t) - P(e^{ig(t)}) S_{0\nu}^n(t) = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P(e^{ig(u)}) - P(e^{ig(t)})}{e^{iu} - e^{it}} e^{ing(u)} [e^{i(v+1)u} - e^{i(v+1)t}] du = I'_{n\nu} - I''_{n\nu}. \end{aligned}$$

La fonction $[P(e^{ig(u)}) - P(e^{ig(t)})](e^{iu} - e^{it})^{-1} = Q_t(u)$ est analytique et uniforme dans une couronne circulaire contenant la circonférence unité. Par conséquent, la série de Fourier

$$(2.9) \quad Q_t(u) = \sum_{k=-\infty}^{\infty} \beta_k(t) e^{iku}$$

est absolument convergente et

$$(2.10) \quad \max_{-\pi \leq t \leq \pi} \sum_{k=-\infty}^{\infty} |\beta_k(t)| = H = H(P, g) < \infty.$$

Il résulte ainsi de (2.4) et de (2.8)–(2.10) que

$$\begin{aligned} |I'_{n\nu}| &\leq \sum_{k=-\infty}^{\infty} |\beta_k(t)| \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i[ng(u) + (v+k+1)u]} du \right| = \\ &= \sum_{k=-\infty}^{\infty} |\beta_k(t)| |b_{-(v+k+1), n}| \leq H c_2 |n|^{-1/p}. \end{aligned}$$

$|I''_{n\nu}|$ peut être majoré de la même façon et, par suite, $|I_{n\nu}| \leq 2H c_2 |n|^{-1/p}$. Cette dernière quantité est inférieure ou égale à 1 pour $|n| \geq n_0 > (2H c_2)^p$. On obtient finalement, vu (2.8),

$$|s_{0\nu}^n(t)| \leq 1 + K_0 M_0,$$

d'où il vient (2.6).

LEMME 4. — Si $\sum_{n=-\infty}^{\infty} |C_n| < \infty$, et si chaque terme de la suite de fonctions $\{P_n(z)\}_{n=-\infty}^{\infty}$ remplit les conditions:

(i) $P_n(z)$ est analytique et uniforme pour $R_1 < |z| < R_2$, $R_1 < 1 < R_2$;

(ii) $\max_t |P_n(e^{it})| \leq M_0$,

où M_0 est une constante indépendante de n , il existe une suite d'entiers $\{k_n\}_{n=-\infty}^{\infty}$ telle que la série de Fourier de la fonction

$$(2.11) \quad \Phi(t) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n g(t)} P_n(e^{ig(t)}) = \sum_{v=-\infty}^{\infty} D_v e^{ivt}$$

converge uniformément.

Nous ne détaillons pas ici la démonstration du Lemme 4 pour la raison suivante. Une série de Fourier analogue à (2.11) intervient aussi dans le travail de CLUNIE. Faute du cas particulier du Lemme 1 ([1], pp. 307—312), il a prouvé seulement la convergence locale de cette série, tandis que la preuve du Lemme 3 de la note [7], calquée sur celle de CLUNIE, a établi la convergence uniforme de la même série. Or, en possession du Lemme 3 de la note présente, la démonstration du Lemme 4 diffère guère de celle que nous avons exposée dans [7].

§ 3. Certaines propriétés des polynômes de Fejér

Les polynômes de Fejér ont été définis par la formule (1.7). Dans le lemme qui suit nous allons rappeler sans démonstration quelques-unes de leurs propriétés connues et simples.

LEMME 5. — (i) Il existe une constante absolue M telle que $|\varphi_n(z)| \leq M$ pour $|z| \leq 1$;
 (ii) $\varphi_n(1) = 0$, autrement dit la $2n$ -ième somme partielle de $\varphi_n(z)$ est égale à zéro pour $z = 1$;
 (iii) la n -ième somme partielle de $\varphi_n(z)$ excède $\log n$ pour $z = 1$;
 (iv) $|\varphi'_n(e^{it})| \leq 4Mn$.

La propriété (iv) est une conséquence de l'inégalité de Bernstein.

LEMME 6. — Si dans la formule (2.11) on remplace $P_n(e^{ig(t)})$ par $\varphi_n(e^{ig(t)})$ ($n > 0$), on peut prendre

$$(3.1) \quad k_n = k_1 n^{2p} \quad (n = 1, 2, \dots)$$

où $k_1 > 0$ est un entier convenablement choisi et p est défini dans le Lemme 2.

DÉMONSTRATION. — k_n joue maintenant le rôle de n_0 du Lemme 3. Nous allons évaluer $|I'_{nv}|$ (cf. (2.8)) en modifiant légèrement le raisonnement précédent. La fonction à intégrer dans l'expression de I'_{nv} peut s'écrire sous la forme

$$(3.2) \quad e^{ik_n g(u)} \frac{\varphi_n(e^{ig(u)}) - \varphi_n(e^{ig(t)})}{e^{ig(u)} - e^{ig(t)}} \cdot \frac{e^{ig(u)} - e^{ig(t)}}{e^{iu} - e^{it}} e^{i(v+1)u} \stackrel{\text{def}}{=} U_{in}(e^{ig(u)}) V_i(u),$$

avec

$$(3.3) \quad U_{tn}(e^{ig(u)}) = \sum_{k=0}^{2n-1} \gamma_{kn}(t) e^{i(k_n+k)g(u)},$$

$$V_t(u) = \sum_{j=-\infty}^{\infty} \delta_j(t) e^{iju}.$$

Il est simple de voir que

$$(3.4) \quad \max_t \sum_{k=0}^{2n-1} |\gamma_{kn}(t)| \leq 2n(\log n + 1),$$

$$\max_t \sum_{j=-\infty}^{\infty} |\delta_j(t)| = c_3 < \infty.$$

Par conséquent, en tenant compte de (2.1), (2.4) et de (3.2)–(3.4), on a

$$\begin{aligned} |I'_{nv}| &\leq \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} U_{tn}(e^{ig(u)}) V_t(u) du \right| \leq \\ &\leq \sum_{k=0}^{2n-1} |\gamma_{kn}(t)| \left[\sum_{j=-\infty}^{\infty} |\delta_j(t)| \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} e^{i[(k_n+k)g(u)+ju]} du \right| \right] \leq \\ &\leq \sum_{k=0}^{2n-1} |\gamma_{kn}(t)| \left[\sum_{j=-\infty}^{\infty} |\delta_j(t)| |b_{-j, (k_n+k)}| \right] \leq 2n(\log n + 1) c_2 c_3 k_n^{-1/p} = H_n k_n^{-1/p}. \end{aligned}$$

Il s'ensuit que $|I_{nv}| \leq |I'_{nv}| + |I''_{nv}| \leq 2H_n k_n^{-1/p}$, et $|I_{nv}| \leq 1$ pour $k_n \geq (2H_n)^p$, d'où il vient (3.1).

§ 4. Démonstration du Théorème

Nous allons montrer tout d'abord que la série de Fourier de $G(t)$ définie sous (1.8) diverge sur l'ensemble E . La série de fonctions donnant $G(t)$ a été formée au moyen d'un ensemble $E_0 = \{t_n\}_{n=1}^{\infty}$ contenant une infinité de fois chaque élément $\tau_\lambda \in E$. Il existe donc une suite infinie composée des éléments de E_0 qui sont égaux à τ_λ telle que

$$(4.1) \quad \tau_\lambda = t_{n_1} = t_{n_2} = \dots = t_{n_j} = \dots$$

où $n_j < n_{j+1}$, $j \rightarrow \infty$. De plus, dans la formule (1.8) n'intervient que des φ_j avec $j = 2^{2^n}$ ($n = 1, 2, \dots$) auxquels correspondent, d'après (3.1), des $k_j = k_1(2^{2^n})^{2^p}$. Mettons, pour simplifier l'écriture, $k_1(2^{2^n})^{2^p} = k_n$. Il est aisé de voir que ces k_n vérifient l'inégalité (1.9).

Posons ensuite

$$p_j = k_{n_j} + 2^{2^{n_j}}, \quad q_j = k_{n_j} + 2 \cdot 2^{2^{n_j}}$$

où les n_j sont ceux de (4.1). Notons enfin par $s_\mu(t)$ la μ -ième somme partielle symétrique de la série de Fourier de $G(t)$. On trouve, en vertu des propriétés (ii) et (iii) des polynômes de Fejér énumérées dans le Lemme 5, que

$$|s_{q_j}(e^{it_{n_j}}) - s_{p_j}(e^{it_{n_j}})| = |s_{q_j}(e^{i\tau_\lambda}) - s_{p_j}(e^{i\tau_\lambda})| > 2 \log 2, \quad j = 1, 2, \dots$$

La suite $\{s_\mu(e^{it\lambda})\}_{\mu=1}^\infty$ ne remplit pas ainsi le critère de convergence de Cauchy, elle est divergente.

La convergence uniforme de la série de Fourier de $F(t)$ (cf. (1.10)) est une conséquence immédiate des Lemmes 4 et 6. En effet, $\varphi_n(z)$ possède la propriété (i)

du Lemme 5 et $\sum_{n=1}^\infty 2^{-n} < \infty$.

D'autre part la série $\sum A_n e^{ivt}$ ne peut être absolument convergente, car, selon la Remarque du § 2, la série $\sum B_n e^{int}$ serait alors uniformément convergente et nous venons de voir qu'elle diverge sur l'ensemble E .

Il reste à établir encore que le module de continuité de $F(t)$ est de l'ordre logarithmique. Or, ce fait a été déjà établi par INDLEKOFER dans le cas des fonctions analytiques et des transformations homographiques. Les mêmes raisonnements s'appliquent dans notre cas sans aucun changement essentiel. On peut y apporter tout de même certaines modifications simples que voici.

LEMME 7. — *Les modules de continuité de $F(t)$ et de $G(t)$ sont de même ordre de grandeur:*

$$0 < c_4 < \omega(\delta; F)/\omega(\delta; G) < c_5 < \infty.$$

DÉMONSTRATION. — Soient $t=g(u)$ et $t+\delta=g(u+\theta)$. On a alors

$$F(t+\delta) - F(t) = G[g(u+\theta)] - G[g(u)]$$

et

$$\delta = g(u+\theta) - g(u) = \theta g'(u+\alpha\theta) \quad (0 \leq \alpha \leq 1).$$

$g'(u)$ étant bornée et différente de zéro, il vient que $\delta = O(\theta)$ et $\theta = O(\delta)$ ($\delta \rightarrow 0$), d'où le résultat.

Il suffit donc d'évaluer $\omega(\delta; G)$ à partir de la série de fonctions (1.8) plus simple que (1.10) donnant $F(t)$.

D'autre part, au cours de la démonstration de l'égalité $\omega(\delta; G) = O(|\log \delta|^{-1})$ il faut majorer la dérivée de la fonction $\psi_n(e^{it}) = e^{ik_n} \varphi_{2^{2n}}(e^{it})$. En appliquant (iv) du Lemme 5, on obtient

$$\left| \frac{d\psi_n(e^{it})}{dt} \right| = |k_n \varphi_{2^{2n}}(e^{it}) + e^{it} \varphi'_{2^{2n}}(e^{it})| \leq M(k_n + 2^{2n+2}) \leq c_6 k_n.$$

L'évaluation directe de $\omega(\delta; F)$ exige, par contre, le calcul de la dérivée de $e^{ik_n g(t)} \varphi_{2^{2n}}(e^{ig(t)})$.

La dernière assertion du théorème découle du critère de Dini—Lipschitz, selon lequel la relation $\omega(\delta; F) = o(|\log \delta|^{-1})$ entraîne la convergence uniforme de $\sum A_n e^{ivt}$. D'après le Lemme 7, on a également $\omega(\delta; G) = o(|\log \delta|^{-1})$ et $\sum B_n e^{int}$ converge aussi uniformément.

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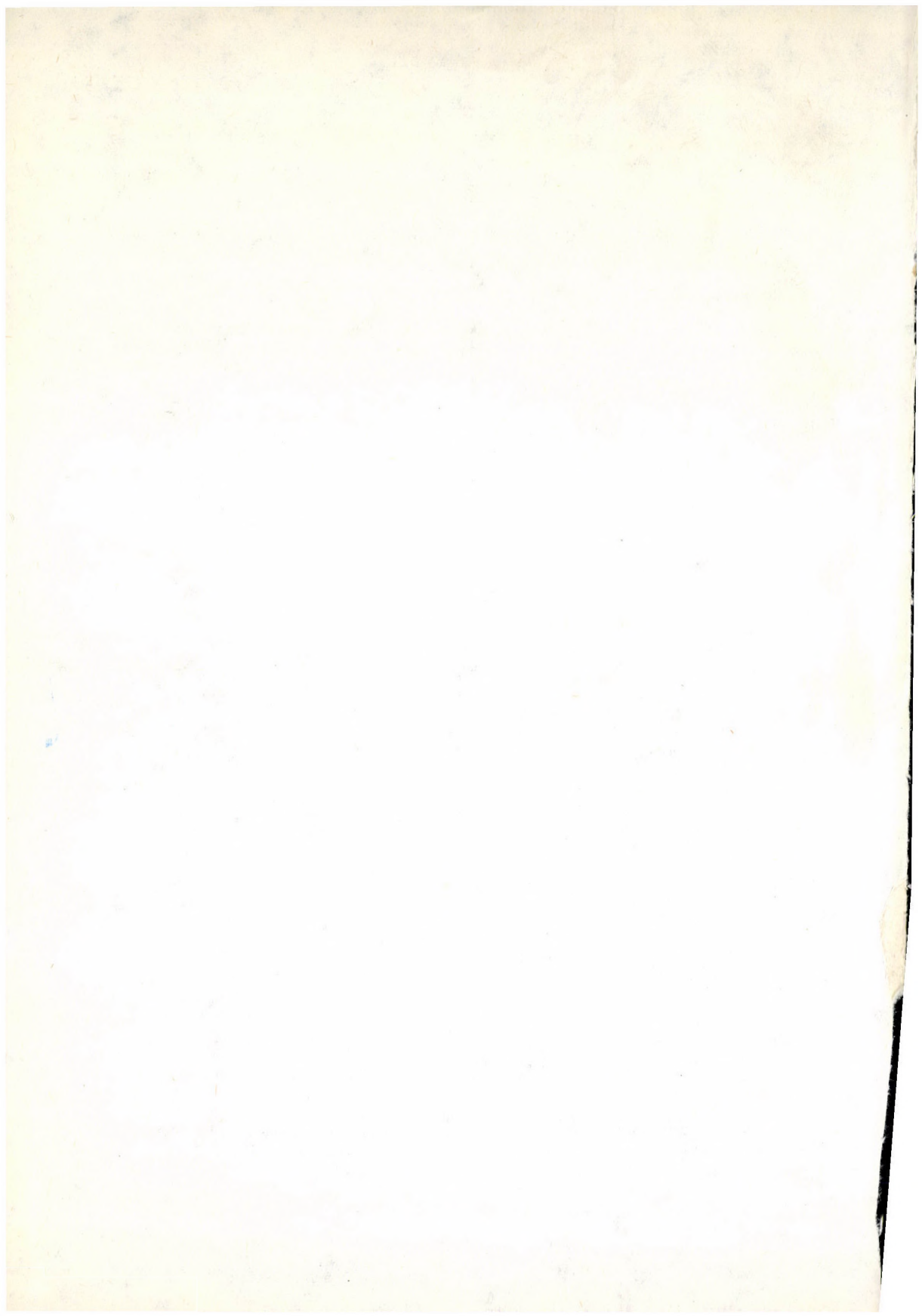
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ON THE SECOND MOMENTS OF RANDOM DETERMINANTS

by
W. SENDLER

§ 1. Let $A_n := (a_{ij})$, $i, j, = 1, 2, \dots, n$, be an $n \times n$ matrix, where the a_{ij} are real-valued, independent, identically distributed random variables. As well known, the precision of the computer-solutions of a large system of linear equations is closely connected with the distribution of the random determinant $D_n := \det(A_n)$. Various results on the moments of this distribution are known, but nearly all are based on the assumption, that a_{ij} is distributed symmetrically about 0, i.e. $E(a_{ij}) = 0$. Even in this case, without further restrictions on the distribution of a_{ij} it is only possible to give general formulas for the second and fourth moment of D_n (see [1], [2]).

Recently, results on higher moments were obtained (see [3]), and also results concerning the algebraic form of the moments ([3], p. 130, Theorem 3).

In this paper it is shown, that the second moment of D_n can be represented as a simple formula of the first and second moments of a_{ij} , assuming not more than the existence of those moments. In the terminology of [3] this means a formula for the Appel polynomial, connected with the second moment of D_n .

§ 2. By $p := (p(1), \dots, p(n))$ let us denote a permutation of the numbers $(1, 2, \dots, n)$. Then the determinant D_n reads:

$$D_n = \sum_{p \in \mathfrak{p}} a_{1p(1)} a_{2p(2)} \dots a_{np(n)} \varrho(p)$$

where $\varrho(p)$ is equal to $+1$ or -1 , if p is even or odd, respectively, and \mathfrak{p} is the set of all permutations.

We suppose the a_{ij} to be independent and identically distributed according to a certain random variable a ; the second moment of D_n reads

$$(1) \quad E(D_n^2) = \sum_{\substack{p \in \mathfrak{p} \\ q \in \mathfrak{p}}} E(a_{1p(1)} \dots a_{np(n)} a_{1q(1)} \dots a_{nq(n)}) \varrho(p) \varrho(q).$$

First of all we have to compute the number of independent factors in the products of the form

$$(*) \quad a_{1p(1)} \dots a_{np(n)} a_{1q(1)} \dots a_{nq(n)} \varrho(p) \varrho(q).$$

Obviously a certain entry a_{ij} can appear at most two times in $(*)$; this happens iff for some r , $1 \leq r \leq n$, we have $a_{rp(r)} \equiv a_{rq(r)}$, i.e. $p(r) = q(r)$.

For two permutations p, q let us call the numbers $p(s)$, $1 \leq s \leq n$, with $p(s) = q(s)$ fixpoints of p and q . Let m be the number of fixpoints of p and q , then the expecta-

tion of (*) is

$$E(a^2)^m [E(a)]^{2(n-m)} \varrho(p) \varrho(q).$$

To develop (1) into an overseable form, we have to know:

(a) the number of pairs (p, q) of permutations with exactly m fixpoints, $0 \leq m \leq n$;

(b) for how many pairs (p, q) of permutations with exactly m fixpoints is $\varrho(p) \varrho(q) = +1$ or -1 .

Let p_0 be a given permutation and denote by $f_{p_0; m}(n)$ the number of all permutations q having exactly m fixpoints with p_0 , then

$$f_{p_0; m}(n) = \frac{n!}{m!} \sum_{i=0}^{n-m} (-1)^i \frac{1}{i!}, \quad 0 \leq m \leq n.$$

Let us denote by $g_{p_0; m}(n)$ and $u_{p_0; m}(n)$ the number of even and odd permutations, respectively, having exactly m fixpoints with p_0 . We consider two cases:

(i) p_0 is even; let us choose m numbers of p_0 , say $p_0(j_1), \dots, p_0(j_m)$. We obtain a permutation q having exactly those m points as fixpoints with p_0 by fixing $p_0(j_1), \dots, p_0(j_m)$ and permuting the remaining $n-m$ numbers $p(j)$, $j \notin \{j_1, j_2, \dots, j_m\}$ in a fixpointfree manner; obviously q is even iff it results from p_0 by applying an even permutation to the remaining $n-m$ numbers $p_0(j)$. An analogous conclusion shows that q is odd iff the remaining $n-m$ numbers $p_0(j)$ are treated by an odd permutation.

Let us denote by $g_0(k)$ [$u_0(k)$] the number of permutations of k things resulting by application of an even [odd] permutation to the original order of this k things, having no fixpoints with this original order; then the foregoing consideration yields

$$(3) \quad g_{p_0; m}(n) = \binom{n}{m} g_0(n-m),$$

$$(3') \quad u_{p_0; m}(n) = \binom{n}{m} u_0(n-m)$$

(using the $\binom{n}{m}$ possibilities of choosing m special fixpoints).

(ii) p_0 is odd; using the notation of (i) we observe, that a permutation q , having exactly $p_0(j_1), \dots, p_0(j_m)$ as fixpoints with p_0 , is even iff it results from p_0 by applying an odd permutation to the remaining $n-m$ numbers $p_0(j)$, $j \notin \{j_1, j_2, \dots, j_m\}$, etc. Therefore we obtain

$$(4) \quad g_{p_0; m}(n) = \binom{n}{m} u_0(n-m),$$

$$(4') \quad u_{p_0; m}(n) = \binom{n}{m} g_0(n-m).$$

The numbers $g_0(k)$ [$u_0(k)$] can be obtained by taking the number of all permutations resulting by application of an even [odd] permutation on the k things, and then subtracting the numbers of such permutations, which result by application of an

even [odd] permutation on the k things, having exactly i fixpoints with the original order of the k things, $1 \leq i \leq k$. Without loss of generality we suppose the k things to be the numbers $1, \dots, k$ in their natural order. Hence, application of an even [odd] permutation to $(1, 2, \dots, k)$ produces an even [odd] permutation of $(1, 2, \dots, k)$ and we can use (3) and (3') to obtain the recurrence-relations

$$(5) \quad g_0(k) = \frac{k!}{2} - \sum_{i=1}^k \binom{k}{i} g_0(k-i),$$

$$(5') \quad u_0(k) = \frac{k!}{2} - \sum_{i=1}^k \binom{k}{i} u_0(k-i)$$

The computation of $g_0(k)$ and $u_0(k)$ can now be carried out easily by

$$g_0(k) + u_0(k) = f_{p_0;0}(k) = k! \sum_{i=0}^k (-1)^i \frac{1}{i!}$$

and the following result:

$$\text{LEMMA. } g_0(k) - u_0(k) = (k-1)(-1)^{k-1}.$$

PROOF. For $k=1$ the lemma is obvious; for arbitrary k , suppose the lemma to be correct for each j with $1 \leq j \leq k-1$; the desired result follows then by induction, using (5), (5') and the well-known relations

$$\binom{n}{m} = \binom{n}{n-m}, \quad \text{and} \quad \sum_{j=1}^k \binom{k}{j} j(-1)^j = 0.$$

Remark on the proof: let I_n be the matrix with elements $(I_n)_{ij} = 1 - \delta_{ij}$. Then it is known that $\det(I_n) = (n-1)(-1)^{n-1}$ ([4] p. 107, probl. 2.); this can be used as a proof of the lemma, if one considers the obvious fact, that $g_0(n) - u_0(n) = \det(I_n)$.

§ 3. The results of § 2 will now be used to compute $E(D_n^2)$. For this purpose, we use the notation

$$\mu_1 = E(a), \quad \mu_2 = (E(a^2))^{1/2}$$

If p, q have exactly m fixpoints, then the corresponding summand in (1) has the form

$$(6) \quad \mu_2^{2m} \mu_1^{2(n-m)} \varrho(p) \varrho(q).$$

Moreover, we have

$$\varrho(p) \varrho(q) = \begin{cases} +1, & \text{if } \varrho(p) = \varrho(q) = +1 \quad \text{or} \quad \varrho(p) = \varrho(q) = -1, \\ -1, & \text{if } \varrho(p) = -\varrho(q). \end{cases}$$

For each p out of the $\frac{n!}{2}$ even permutations, we have, using (3), $\binom{n}{m} g_0(n-m)$ even permutations q , having exactly m fixpoints with p ; for each p' of the $\frac{n!}{2}$ odd permutations, we have, using (4'), $\binom{n}{m} g_0(n-m)$ odd permutations, having exactly m fixpoints with p' . Hence, we obtain $\varrho(p) \cdot \varrho(q) = +1$ for $n! \binom{n}{m} g_0(n-m)$ pairs

(p, q) . An analogous calculation shows, that $q(p)q(q) = -1$ for $n! \binom{n}{m} u_0(n-m)$ pairs (p, q) . Adding all summands of (1) which have the weight $\mu_2^{2m} \mu_1^{2(n-m)}$, it turns out, that only knowledge of $g_0(n-m) - u_0(n-m)$ is needed; therefore the lemma can be used to determine the total contribution of the summands, having this weight, as

$$n! \binom{n}{m} (n-m-1) (-1)^{n-m-1} \mu_2^{2m} \mu_1^{2(n-m)}.$$

Hence, the second moment of D_n reads

$$(7) \quad E(D_n^2) = n! \sum_{m=1}^n \binom{n}{m} (n-m-1) (-1)^{n-m-1} \mu_2^{2m} \mu_1^{2(n-m)}.$$

Setting $\mu_1/\mu_2 = \alpha$ and $n-m=k$, (7) can be rewritten in the form

$$(7') \quad E(D_n^2) = n! \mu_2^{2n} \sum_{k=0}^n \binom{n}{k} (k-1) (-1)^{k-1} \alpha^{2k}.$$

Since

$$\sum_{k=0}^n \binom{n}{k} (k-1) (-1)^{k-1} \alpha^{2k} = n\alpha^2 (1-\alpha^2)^{n-1} + (1-\alpha^2)^n$$

and $\mu_2^2 = \mu_1^2 + \sigma^2$, we obtain after elementary computation:

$$(8) \quad E(D_n^2) = n! [\sigma^{2n} + n \cdot \mu_1^2 \cdot \sigma^{2(n-1)}].$$

For the case $\mu_1 = 0$, (8) reduces to the trivial and well known result $E(D_n^2) = n! \sigma^{2n}$ (cf. [1], [2]).

§ 4. Finally, we will use (8) to compare the asymptotic behaviour of the general case to the case where the matrix entries have mean value 0. For this purpose, let us denote from now the entries of A_n by $a_{ij}^{(n)}$ and suppose the $a_{ij}^{(n)}$ to be distributed like a certain random variable a , not depending on n . Instead of A_n we consider the "normalized" matrix B_n , defined by

$$B_n := \sigma^{-1} (n!)^{-1/(2n)} A_n, \quad \sigma^2 = \text{var}(a).$$

With the notation $\Delta_n := \det(B_n)$ we obtain from (8)

$$(9) \quad E(\Delta_n^2) = 1 + n! n \sigma^{-2} (n!)^{-1/n} \cdot \mu_1^2 \cdot (n!)^{-\frac{n-1}{n}} = 1 + n \cdot \frac{\mu_1^2}{\sigma^2}.$$

Let us denote by A'_n a random matrix $(a'_{ij}{}^{(n)})$ with the property $E(a'_{ij}{}^{(n)}) = 0$, $\text{var}(a'_{ij}{}^{(n)}) = \text{var}(a_{ij}^{(n)}) = \sigma^2$; denote by B'_n the "normalized" matrix $B'_n := \sigma^{-1} (n!)^{-1/(2n)}$;

then (9) yields

$$E(\Delta_n^2) = E(\Delta_n'^2) + n \cdot \frac{\mu_1^2}{\sigma^2}, \quad \text{where } \Delta_n' := \det(B_n').$$

This indicates, that for $n \rightarrow \infty$ the limiting distribution of Δ_n may have completely different behaviours for the cases $\mu_1=0$ and $\mu_1 \neq 0$, respectively.

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THE CENTRALITY OF VERTICES IN TREES

by

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To Professor László Kalmár on his seventieth birthday

Introduction

From an intuitive point of view, a centrality function in a tree T is a numerical function f defined on the set of vertices of T such that f takes its smallest value at some end vertices of T and it increases more and more as we move away the end vertices. It seems to be natural that f may be viewed as an appropriate centrality function only if f has the property named convexity in the sequel.

In §§ 1, 2, ..., 5 of this paper, five centrality functions f_1, f_2, \dots, f_5 will be studied (respectively). We shall examine their properties; among others, it will be proved that they are convex.

In Appendix I, three further functions f_6, f_7, f_8 (defined in a similar manner as the former ones, with minor modifications) are introduced and it is shown that they do not possess the convexity property.

The studying of f_1, \dots, f_5 is isolated from each other in the main parts of the article. Appendix II is devoted to the question what facts of independence or dependence are valid for these functions. We shall obtain that, among the ten two-element subsystems of the system $\{f_1, f_2, \dots, f_5\}$, independence (in certain sense) holds in eight cases and dependence is true for two pairs.

Chapter 5 of the classical work [2] contains a number of results concerning (essentially) the functions f_3 and f_4 . The most fundamental facts are also formulated on pp. 35-36 of [1]. The last sentences of our Theorems 3, 4 are included at these places (and in some publications mentioned), though the method of discussion of the subject seems to be different from ours.

Preliminaries

A vertex of degree 1 in a tree is called an *end vertex*. Any tree has at least two end vertices. A vertex is called a *simple vertex*, if its degree is 2; and a *star*, if its degree is at least 3. The tree consisting of an edge and two end vertices only is said to be *trivial*. A tree consisting of two end vertices, s (≥ 0) simple vertices and $s+1$ edges is said to be a *star-free graph*. We call a subtree V of a tree T a *star-free subgraph* of T if V is a star-free graph and all the simple vertices of V are of degree two also in T . A star-free subgraph V of T is called *maximal* if the degrees of its first and last vertex differ from 2 (in T).

The distance of the vertices A and B of a tree will be denoted by $\delta(A, B)$. For any vertex A , the set of vertices B for which $\delta(A, B)$ takes its maximum is denoted by $\varphi(A)$; this set consists only of end vertices.

Let A be a vertex of a tree and e be an edge incident to A . We introduce five notions depending on the pair A, e . We denote by $\psi(A, e)$ the set of end vertices B such that the (single) chain connecting A and B does not contain e . We denote by $\tau(A, e)$ the set of all vertices B such that the chain between A, B contains e ; let $\sigma(A, e)$ be the set of end vertices lying in $\tau(A, e)$. The cardinality $|\tau(A, e)|$ is denoted by $\nu(A, e)$. Finally, let $\mu(A, e)$ be the maximum of the distances $\delta(A, B)$ where B runs through the elements of $\sigma(A, e)$.

The function $\delta(e, A)$ denotes the distance $\delta(B, A)$ where B is that vertex incident to the edge e for which the chain between B, A passes through e . $\varphi(e)$ is defined analogously to $\varphi(A)$.

Let us consider a function f defined on the set of vertices of the tree T such that the values of f are non-negative integers. We shall use the notations $H(f), h(f)$ for the sets of vertices A of T where $f(A)$ takes its maximum and minimum, respectively. The function f is called *gapless* if, to any vertex A satisfying $f(A) > 0$, there exists a vertex B fulfilling $f(B) = f(A) - 1$. If f is gapless, then $f(A) = 0$ for some vertex A . The functions $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$, to be introduced, will be obviously gapless (therefore we shall not mention this property of them explicitly). f is called *jumpless* if

$$|f(A) - f(B)| \leq 1$$

is valid whenever A and B are adjacent vertices. f is called *convex* if

$$f(C) \cong \min(f(A), f(B))$$

whenever the chain connecting A and B contains C .

Let N be a finite set of integers. We introduce the function η , defined on N , in the following manner: for any $n (\in N)$, $\eta(n)$ is the number of elements m of N fulfilling $m > n$; similarly, we define $\eta^*(n)$ as the number of elements m satisfying $m < n$. η and η^* are clearly monotonic and one-to-one functions, their range is the set $\{0, 1, 2, \dots, |N| - 1\}$; η is decreasing and η^* is increasing.

The subsequent three propositions will be useful in proving the convexity of some functions.

PROPOSITION 1. *Suppose that the function f defined on the vertices of a tree is not convex. Then there exists a chain such that its vertices*

$$D_1, D_2, \dots, D_t, \quad (t \geq 3)$$

ordered as they follow in the chain, satisfy

$$f(D_1) > f(D_2) = f(D_3) = \dots = f(D_{t-1})$$

and

$$f(D_{t-1}) < f(D_t).$$

PROOF. Since f is not convex, there exists a chain with the vertices A_1, A_2, \dots, A_q such that $f(A_i) < f(A_1)$ and $f(A_i) < f(A_q)$ with a suitable i ($1 < i < q$). Choose the subscript i_0 such that $f(A_{i_0})$ should be minimal on the chain considered ($1 \leq i_0 \leq q$). If j is the largest subscript such that $1 \leq j < i_0$ and $f(A_j) > f(A_{i_0})$, moreover, k is the smallest subscript such that $i_0 < k \leq q$ and $f(A_k) > f(A_{i_0})$, then it is easy to see that the vertices A_j, A_{j+1}, \dots, A_k fulfil the assertion as D_1, D_2, \dots, D_t , respectively (where, of course, $t = k - j + 1$).

PROPOSITION 2. Suppose that the function f , defined on the vertices of a tree, satisfies the following assertion: whenever A, B and C are three different vertices such that B is adjacent both to A and to C , then $f(A) > f(B)$ implies $f(B) > f(C)$. In this case, f is convex.

PROOF. If f is not convex, then the vertices D_1, D_2, D_3 (occurring in Proposition 1) do not fulfil the assertion (whether $t=3$ or $t>3$).

PROPOSITION 3. Suppose that $f(A) \cong f(B)$ implies $f(B) > f(C)$ where A, B, C, f are as in Proposition 2. Then either $|H(f)|=1$ or $H(f)$ consists of two adjacent vertices.

PROOF. f is convex by Proposition 2, thus $H(f)$ is a connected subgraph of the tree. If $H(f)$ has at least three vertices, then there exist different vertices A, B, C , as above with $f(A)=f(B)=f(C)$, contradicting the assertion.

§ 1.

For any vertex A of a tree T , denote by $f_1^*(A)$ the maximum of the distances $\delta(A, B)$ where B runs through the end vertices of T . Define the function f_1 by $f_1(A) = \eta(f_1^*(A))$ where the range of f_1^* plays the role of N .*

PROPOSITION 4. Let e be an edge of T and A, B be the vertices incident to e . Then (exactly) one of the following three alternatives holds:

- (a) $f_1(A) = f_1(B) + 1$ and $\varphi(B) = \varphi(A) \cap \psi(A, e)$,
- (b) $f_1(A) + 1 = f_1(B)$ and $\varphi(A) = \varphi(B) \cap \psi(B, e)$,
- (c) $f_1(A) = f_1(B)$, $\varphi(A) \subseteq \psi(B, e)$ and $\varphi(B) \subseteq \psi(A, e)$.

PROOF. First we note the evident facts that $C \in \psi(A, e)$ implies

$$\delta(A, C) + 1 = \delta(e, C) = \delta(B, C)$$

and $C \in \psi(B, e)$ implies

$$\delta(A, C) = \delta(e, C) = \delta(B, C) + 1.$$

Case 1: $\varphi(e) \subseteq \psi(A, e)$. Then $D \in \psi(B, e)$ implies

$$\delta(e, D) \leq \max \delta(e, C) - 1$$

(where the maximum is taken for all vertices C in $\psi(A, e)$), hence

$$\varphi(B) = \varphi(e) = \varphi(A) \cap \psi(A, e)$$

follows by the first sentence of the proof; thus (a) is true.

Case 2: $\varphi(e) \subseteq \psi(B, e)$. Then we can verify (b) analogously to the preceding case.

* Evidently, $f_1(A) = d - f_1^*(A)$ where d is the maximum of f_1^* (i.e. the diameter of the tree).

Case 3: both $\varphi(e) \cap \psi(A, e)$ and $\varphi(e) \cap \psi(B, e)$ are non-empty. Then these sets equal $\varphi(B)$ and $\varphi(A)$, respectively; moreover,

$$f_1^*(A) = \max \delta(e, C) = f_1^*(B),$$

i.e. (c) is valid.

PROPOSITION 5. *Let A, B, C be three different vertices of T such that A and B are joined by the edge e_1 , moreover, B and C are joined by the edge e_2 . Then (precisely) one of the subsequent five assertions is valid:*

(i) $f_1(A) + 2 = f_1(B) + 1 = f_1(C)$,

(ii) $f_1(A) = f_1(B) + 1 = f_1(C) + 2$,

(iii) $f_1(A) + 1 = f_1(B) = f_1(C)$,

(iv) $f_1(A) + 1 = f_1(B) = f_1(C) + 1$,

(v) $f_1(A) = f_1(B) = f_1(C) + 1$.

PROOF. Suppose that none of (i)-(v) holds. Then the following situations can be imagined (without violating Proposition 4):

(vi) $f_1(A) = f_1(B) + 1 = f_1(C) + 1$,

(vii) $f_1(A) + 1 = f_1(B) + 1 = f_1(C)$,

(viii) $f_1(A) = f_1(B) + 1 = f_1(C)$,

(ix) $f_1(A) = f_1(B) = f_1(C)$.

Assume any of (vi), (vii), (viii), (ix). Let us apply the appropriate statements of Proposition 4 for the edges e_1 and e_2 . We get $\varphi(B) \subseteq \psi(A, e_1)$ and $\varphi(B) \subseteq \psi(C, e_2)$; these inclusions are obviously contradictory because $\psi(A, e_1)$ and $\psi(C, e_2)$ are disjoint and $\varphi(B)$ is not empty.

THEOREM 1. *Consider f_1 on an arbitrary tree T . f_1 is a jumpless and convex function. $h(f_1)$ is a (proper or nonproper) subset of the set of end vertices, furthermore, $|h(f_1)| \geq 2$. Either $|H(f_1)| = 1$ or $H(f_1)$ consists of two adjacent vertices.*

PROOF. f_1 is jumpless by Proposition 4. The convexity of f_1 follows from Propositions 2 and 5.

Any of the assertion (i)–(v) in Proposition 5 implies $f_1(B) \geq 1$. Thus $f_1(B)$ may be 0 only if B is an end vertex. We have evidently $f_1(A) = f_1(B) = 0$ for the first vertex A and the last one B of a longest chain of T , hence $|h(f_1)| \geq 2$.

The last statement of the theorem is implied by Propositions 3 and 5.

Remark. If A is an end vertex but no chain of maximal length of the tree terminates at A , then clearly $f_1(A) > 0$.

§ 2.

If A is an arbitrary vertex of a tree T , then we denote by $f_2^*(A)$ the sum of all distances $\delta(A, D)$ where D runs through the end vertices of T ; furthermore, we define f_2 by $f_2(A) = \eta(f_2^*(A))$ (with the range of f_2^* as N). f_2 is not necessarily jumpless (see Fig. 1).

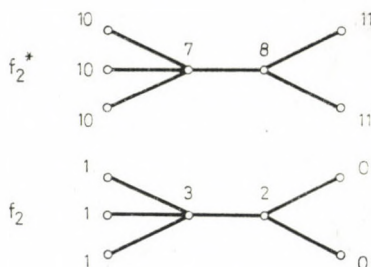


Fig. 1.

PROPOSITION. 6. Let e be an edge of T ; let A, B be the vertices incident to e . Then

$$f_2^*(A) + |\psi(A, e)| = f_2^*(B) + |\psi(B, e)|.$$

PROOF. If $D \in \psi(A, e)$, then

$$\delta(A, D) + 1 = \delta(B, D);$$

if $D \in \psi(B, e)$, then

$$\delta(A, D) = \delta(B, D) + 1$$

By adding these equalities we obtain

$$\sum \delta(A, D) + |\psi(A, e)| = \sum \delta(B, D) + |\psi(B, e)|$$

where the summations are taken for all end vertices D .

PROPOSITION 7. Let A, B, C be three different vertices such that B is adjacent both to A and to C . Then

$$f_2^*(B) \cong \frac{f_2^*(A) + f_2^*(C)}{2};$$

moreover, equality holds in this formula if and only if B is a simple vertex.

PROOF. Denote by e_1 the edge joining A and B , and by e_2 the edge between B and C . Proposition 6 and the obvious equalities

$$\psi(B, e_1) = (\psi(B, e_1) \cap \psi(B, e_2)) \cup \psi(C, e_2),$$

$$\psi(B, e_2) = (\psi(B, e_1) \cap \psi(B, e_2)) \cup \psi(A, e_1)$$

(where unions of disjoint sets stand on the right-hand sides) imply the validity of the following deduction:

$$\begin{aligned} f_2^*(A) + f_2^*(C) - 2f_2^*(B) &= (f_2^*(A) - f_2^*(B)) - (f_2^*(B) - f_2^*(C)) = \\ &= (|\psi(B, e_1) \cap \psi(B, e_2)| + |\psi(C, e_2)| - |\psi(A, e_1)|) - \\ &- (|\psi(C, e_2)| - |\psi(B, e_1) \cap \psi(B, e_2)| - |\psi(A, e_1)|) = 2|\psi(B, e_1) \cap \psi(B, e_2)| \geq 0. \end{aligned}$$

The set $\psi(B, e_1) \cap \psi(B, e_2)$ is empty exactly if B is a simple vertex.

PROPOSITION 8. *Let A, B, C be three vertices as in Proposition 7. If B is a simple vertex, then one of the following three alternatives holds:*

- (i) $f_2(A) = f_2(B) = f_2(C)$,
- (ii) $f_2(A) < f_2(B) < f_2(C)$,
- (iii) $f_2(A) > f_2(B) > f_2(C)$.

If B is a star, then either (ii) or (iii) or one of the subsequent alternatives (iv), (v), (vi) is true:

- (iv) $f_2(A) < f_2(B) = f_2(C)$,
- (v) $f_2(A) = f_2(B) > f_2(C)$,
- (vi) $f_2(A) < f_2(B)$ and $f_2(B) > f_2(C)$.

PROOF. If B is simple, then the corresponding statement follows from Proposition 7 and the definition of f_2 .

Suppose that B is a star. If $f_2(A) = f_2(B)$, then $f_2^*(A) = f_2^*(B)$, thus the formula stated in Proposition 7 takes the form

$$f_2^*(B) < \frac{f_2^*(B) + f_2^*(C)}{2},$$

and this implies $f_2(B) > f_2(C)$, therefore (v) is valid. If $f_2(A) < f_2(B)$, then one of (ii), (iv), (vi) is trivially fulfilled. If $f_2(A) > f_2(B)$, then $f_2^*(A) < f_2^*(B)$, consequently

$$f_2^*(C) - f_2^*(B) > f_2^*(B) - f_2^*(A) > 0$$

where the first inequality follows from Proposition 7. Hence $f_2(C) < f_2(B)$, i.e. (iii) holds.

THEOREM 2. *Consider f_2 on an arbitrary tree T that is not a star-free graph. The function f_2 is convex. $h(f_2)$ is a (proper or nonproper) subset of the set of end vertices. $H(f_2)$ consists either of a star only or of all the vertices of some maximal star-free subgraph V (having one or more edges) of T such that V contains no end vertex of T .*

PROOF. The convexity of f_2 is a consequence of Propositions 2 and 8.

The assertion of the theorem on $H(f_2)$ follows from the convexity of f_2 and the subsequent statements (A), (B), (C):

(A) If $B \in H(f_2)$ and B is a star, then, among the vertices adjacent to B , there is at most one vertex A fulfilling $A \in H(f_2)$.

(B) If the simple vertex B belongs to $H(f_2)$, then the vertices A, C , adjacent to B , also belong to $H(f_2)$.

(C) If A is an end vertex of T , then $f_2(A) < f_2(B)$ for the (single) vertex B adjacent to A .

First we are going to verify (A). Let A and C be different vertices adjacent to B . We have $f_2(A) \leq f_2(B)$, $f_2(C) \leq f_2(B)$ by the definition of $H(f_2)$. Since B is a star, the equality $f_2(A) = f_2(B) = f_2(C)$ is impossible (by Proposition 8).

If B is a simple vertex, then the conclusion of (B) holds because the alternative (i) in Proposition 8 must be valid for A, B, C .

Now we shall prove (C). Since $\psi(A, e) = \{A\}$ (where e is the edge joining A and B), we have (by Proposition 6)

$$f_2^*(A) + 1 = f_2^*(B) + |\psi(B, e)|.$$

Furthermore, $|\psi(B, e)| \geq 2$, because T is not a star-free graph. Thus $f_2^*(A) > f_2^*(B)$, hence $f_2(A) < f_2(B)$.

Finally, we prove the assertion of the theorem on $h(f_2)$. If B is a star, then Proposition 8 implies $B \notin h(f_2)$ immediately. Suppose that $B (\in h(f_2))$ is a simple vertex, and denote by V the maximal star-free subgraph containing B . The value

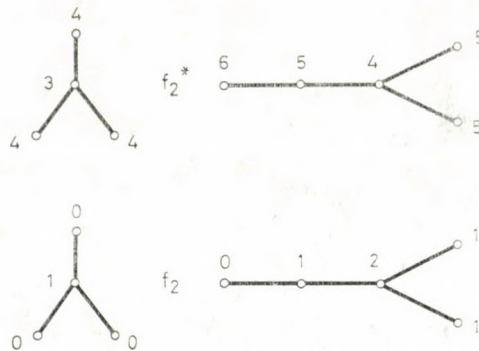


Fig. 2.

of f_2 is constant on V , because the above statement (B) holds also with $h(f_2)$ (instead of $H(f_2)$). Either the first or the last vertex (or both) of V is a star, thus the minimality of f_2 on V contradicts the assertions (iv), (v) in Proposition 8.

Remarks. f_2 is identically zero on every star-free graph. The trees on Fig. 2 show that $h(f_2)$ may consist of all end vertices and $|h(f_2)| = 1$ is also possible.

§ 3.

Let A be an arbitrary vertex of a tree T . Consider an edge e incident to A . Let $\lambda(A)$ be the sequence consisting of the values $\mu(A, e)$ (where e runs through all the vertices incident to A) such that the members of $\lambda(A)$ are ordered nonincreasingly.* We define $f_3(A)$ in the following manner:

if A is an end vertex, then $f_3(A)=0$,

if A is a simple vertex or a star, then let $f_3(A)$ be the second member** of the sequence $\lambda(A)$.

The tree on Fig. 3 shows that f_3 is not jumpless in general.

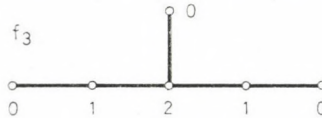


Fig. 3.

PROPOSITION 9. Let A, B be two adjacent vertices of T and e be the edge joining them. Suppose that neither A nor B is an end vertex. If $f_3(A) \cong f_3(B)$, then $\delta(A, F_0) \cong \delta(B, G_0)$ where F_0 is a vertex $F(\in \psi(A, e))$ for which $\delta(A, F)$ is maximal and G_0 is a vertex $G(\in \psi(B, e))$ for which $\delta(B, G)$ is maximal.***

PROOF. Suppose that the conclusion does not hold. i.e. $\delta(A, F_0) < \delta(B, G_0)$. Then we have

$$\delta(A, F_0) < \delta(B, G_0) + 1 = \delta(A, G_0),$$

hence $f_3(A) = \delta(A, F_0)$. Moreover, we can see that

$$\delta(B, F_0) = \delta(A, F_0) + 1 < \delta(B, G_0) + 1,$$

thus

$$f_3(A) + 1 = \delta(B, F_0) \cong f_3(B) \cong \delta(B, G_0)$$

holds (by the definition of f_3), therefore the assumption $f_3(A) \cong f_3(B)$ is false.

PROPOSITION 10. Let A, B and C be three different vertices such that B is adjacent both to A and to C . If $f_3(A) \cong f_3(B)$, then $f_3(B) > f_3(C)$.

PROOF. B cannot be an end vertex (hence $f_3(B) > 0$). If A is an end vertex, then the assumption is false. If C is an end vertex, then the conclusion is true. Therefore we may restrict ourselves to the case when none of A, B and C is an end vertex.

Suppose that the proposition is not valid, i.e. $f_3(A) \cong f_3(B)$ and $f_3(B) \cong f_3(C)$ for some triple of vertices. Denote by F_0 a vertex $F(\in \psi(A, e_1))$ for which $\delta(A, F)$ is maximal (where e_1 is the edge between A and B), by G_0 a vertex**** $G(\in \psi(B, e_1) \cap$

* Evidently, $\lambda(A)$ has as many members as the degree of A . The same number may occur more than once in $\lambda(A)$.

** We note that $f_1^*(A)$ and the function $f_6(A)$ (defined in Appendix I) are equal to the first and last member of $\lambda(A)$, respectively.

*** The maximality is understood in the sets $\psi(A, e)$ and $\psi(B, e)$, respectively.

**** $G_0 = B$ may occur (if B is a simple vertex).

$\cap \psi(B, e_2)$ for which $\delta(B, G)$ is maximal (where e_2 is the edge joining B and C), and by H_0 a vertex $H(\in \psi(C, e_2))$ for which $\delta(C, H)$ is maximal. Proposition 9 implies

$$\delta(A, F_0) \cong \max(\delta(B, G_0), \delta(B, H_0))$$

and

$$\delta(C, H_0) \cong \max(\delta(B, F_0), \delta(B, G_0)),$$

these formulae lead to the inequalities

$$\delta(A, F_0) \cong \delta(B, H_0) = \delta(C, H_0) + 1 > \delta(C, H_0)$$

and

$$\delta(C, H_0) \cong \delta(B, F_0) = \delta(A, F_0) + 1 > \delta(A, F_0),$$

resp., which contradict each other.

THEOREM 3. Consider f_3 on an arbitrary tree T . f_3 is a convex function. $h(f_3)$ consists of exactly the end vertices of T . Either $|H(f_3)|=1$ or $H(f_3)$ consists of two adjacent vertices.

PROOF. The statement on $h(f_3)$ is a trivial consequence of the definition of f_3 . The further two assertions follow from Propositions 2, 3 and 10.

PROPOSITION 11. Consider f_3 on an arbitrary tree T . Denote by T_q the graph consisting of the vertices A satisfying $f_3(A) \cong q$ and of the edges (of T) joining these vertices (where q can be 0, 1, 2, ...). Choose a number q such that T_q contains at least two vertices. Then T_q is a tree, furthermore, a vertex B is an end vertex of T_q if and only if $f_3(B)=q$ (where f_3 is meant in T).

PROOF. T_q is connected in consequence of the convexity of f_3 , therefore it is a tree (for it is a subgraph of a tree).

Suppose $f_3(B)=q$ for a vertex B of T_q . There exists a vertex A adjacent to B which is in T_q , i.e. which satisfies $f_3(A) \cong q$. Proposition 10 guarantees $f_3(C) < q$ for each vertex $C (\neq A)$ of T that is adjacent to B . Hence C does not belong to T_q , this means that B is an end vertex in T_q .

Assume $f_3(B)=p > q$. There exist two edges e_1, e_2 in T incident to B such that $\mu(B, e_1) \cong p$ and $\mu(B, e_2) = p$. Denote by A, C the vertices different from B that are incident to e_1, e_2 , respectively; moreover, by F_0 and G_0 such vertices $F(\in \psi(A, e_1))$, $G(\in \psi(C, e_2))$ for which $\delta(A, F)$ and $\delta(B, G)$ is maximal, respectively. We have

$$\delta(A, F_0) = \delta(B, F_0) - 1 \cong p - 1 \cong q,$$

$$\delta(A, G_0) = \delta(B, G_0) + 1 = p + 1 > q,$$

$$\delta(C, F_0) = \delta(B, F_0) + 1 \cong p + 1 > q,$$

$$\delta(C, G_0) = \delta(B, G_0) - 1 = p - 1 \cong q$$

in consequence of the choice of A, B, F_0, G_0 ; therefore both $f_3(A)$ and $f_3(C)$ are at least q , thus A and C belong to T_q and so B is not an end vertex in T_q .

An immediate consequence of Proposition 11 is the following statement:

PROPOSITION 12. Let T be an arbitrary tree. The subgraphs

$$T_0, T_1, T_2, \dots, T_q, \dots$$

(as long as they exist) can be produced by the following construction:

$$T_0 = T,$$

if T_q has already been constructed and contains at least three vertices, then delete the end vertices of the tree T_q and the edges incident to them; the remaining graph is T_{q+1} ,

if T_q has one vertex or two (adjacent) vertices, then the procedure is finished.

Some connections between the functions f_1 and f_3 will be explained in Appendix II of the paper.

§ 4.

Let A be an arbitrary vertex of a tree T . We denote by $f_4^*(A)$ the maximum of the values $v(A, e)$ where e runs through the edges incident to A . Define the function f_4 by $f_4(A) = \eta(f_4^*(A))$ where the range of f_4^* plays the role of N .

The tree on Fig. 4 shows that f_4 is not jumpless in general.

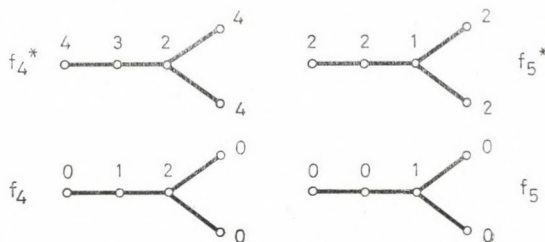


Fig. 4.

PROPOSITION 13. Let e be an edge of T ; let A, B be the vertices incident to e . If $f_4^*(A) \leq f_4^*(B)$, then $f_4^*(B) = v(B, e)$.

PROOF. Suppose $f_4^*(B) \neq v(B, e)$. Then, by the definition of f_4^* , there exists an edge e' (incident to B) such that

$$f_4^*(B) = v(B, e') > v(B, e).$$

The obvious inclusion $\tau(B, e') \subset \tau(A, e)$ implies

$$f_4^*(B) = v(B, e') < v(A, e) \leq f_4^*(A).$$

PROPOSITION 14. Let A, B, C be three different vertices such that B is adjacent both to A and to C . If $f_4(A) \cong f_4(B)$, then $f_4(B) > f_4(C)$.

PROOF. Denote by e_1 and e_2 the edges joining B with A and C , resp. For any $e' (\neq e_2)$, incident to C , the relation

$$v(C, e') < v(B, e_2) \leq f_4^*(B) = v(B, e_1) < v(C, e_2)$$

holds in consequence of the inclusions

$$\tau(C, e') \subset \tau(B, e_2), \quad \tau(B, e_1) \subset \tau(C, e_2)$$

and Proposition 13. Since e' is arbitrary, we have

$$f_4^*(C) = v(C, e_2) > f_4^*(B).$$

THEOREM 4. Consider f_4 on an arbitrary tree T . f_4 is a convex function. $h(f_4)$ consists of exactly the end vertices of T . Either $|H(f_4)|=1$ or $H(f_4)$ consists of two adjacent vertices.

PROOF. The first and third statements follow from Propositions 2, 3, and 14. The second one holds because of the evident formula $f_4^*(A) \equiv P-1$ (where P is the number of vertices of T) in which equality is true precisely when A is an end vertex.

§ 5.

Let A be an arbitrary vertex of a tree T . In analogy to the preceding §, we denote the maximum of $|\sigma(A, e)|$ by $f_5^*(A)$ and $\eta(f_5^*(A))$ by f_5 . Fig. 4 shows that f_5 is not jumpy in general.

PROPOSITION 15. If

$$A, B_1, B_2, \dots, B_t, C \quad (t \geq 1)$$

are the vertices of a maximal star-free subgraph (in natural ordering), then

$$f_5(B_1) = f_5(B_2) = \dots = f_5(B_t) = \min(f_5(A), f_5(C)).$$

PROOF. Denote the edges of the star-free subgraph by e_1, e_2, \dots, e_{t+1} (i.e. B_i is incident to e_i and e_{i+1} , $1 \leq i \leq t$). We clearly have

$$\sigma(A, e_1) = \sigma(B_1, e_2) = \sigma(B_2, e_3) = \dots = \sigma(B_t, e_{t+1})$$

and

$$\sigma(B_1, e_1) = \sigma(B_2, e_2) = \dots = \sigma(B_t, e_t) = \sigma(C, e_{t+1}).$$

Case 1: $|\sigma(B_1, e_1)| \leq |\sigma(A, e_1)|$. Then

$$f_5^*(A) = f_5^*(B_1) = f_5^*(B_2) = \dots = f_5^*(B_t) (= |\sigma(A, e_1)|) \geq f_5^*(C)$$

by the equalities above and the inequalities

$$|\sigma(A, e')| \leq |\sigma(B_1, e_1)|, \quad |\sigma(C, e'')| \leq |\sigma(A, e_1)|$$

where $e' (\neq e_1)$, $e'' (\neq e_{t+1})$ are arbitrary edges incident to A, C , resp.

Case 2: $|\sigma(A, e_1)| \leq |\sigma(B_1, e_1)|$. The treatment is symmetrical to Case 1.

THEOREM 5. Consider f_5 on an arbitrary tree T that is not a star-free graph. The function f_5 is convex. $h(f_5)$ consists of all vertices of degree ≤ 2 of the maximal star-free subgraphs terminating at end vertices. $H(f_5)$ consists either of a star only or of all the vertices of some maximal star-free subgraph V (having one or more edges) of T such that V contains no end vertex of T .

PROOF. Proposition 15 implies that it suffices to study f_5 only in trees without simple vertices (since every maximal star-free chain may be replaced by an edge, without altering f_5 essentially).

Propositions 13, 14, and Theorem 4 remain valid for f_5^* , $|\sigma(A, e)|$, f_5 (instead of f_4^* , $v(A, e)(=|\tau(A, e)|)$, f_4 , resp.) if T has no simple vertex. Concerning the proof of the modified statements, we note that the inclusion relations $\sigma(B, e') \subset \sigma(A, e)$ and $\sigma(B, e_1) \subset \sigma(C, e_2)$ are strict because B is a star, moreover, now the number of end vertices plays the role of P .

The assertions of Theorem 5 follow from Proposition 15 and the modified version of Theorem 4.

Remark. f_5 is identically zero on each star-free graph.

Some interrelations between f_2 and f_5 are studied at the end of Appendix II.

Appendix I.

In this section, we define three functions f_6, f_7, f_8 (somewhat related to the previous ones) on the vertex set of a tree T . We point out that any of these functions cannot be considered as an adequate characterization of the centrality of vertices, because it fails to be convex.

Let $f_6(A)$ be the minimum of $\delta(A, B)$ where (A is an arbitrary vertex and) B runs through the end vertices of T .

We denote by $f_7^*(A)$ and $f_8^*(A)$ the minimum of $v(A, e)$, $|\sigma(A, e)|$, respectively (where e runs through the edges incident to A), and we define the functions f_7^{**} , f_8^{**} as follows:

$$f_i^{**}(A) = \begin{cases} 0 & \text{if } A \text{ is an end vertex} \\ f_i^*(A) & \text{if } A \text{ is a simple vertex or a star} \end{cases}$$

(i can be 7 or 8)*. The obtained functions f_7^{**} , f_8^{**} are not necessarily gapless (see

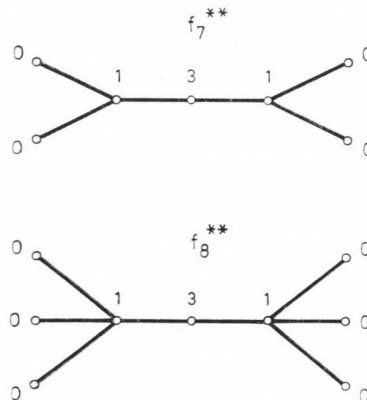


Fig. 5.

* The example of the star-free graphs shows why we modify f_i^* .

Fig. 5), therefore we introduce F_i by $f_i(A) = \eta^*(f_i^{**}(A))$ (i is again 7 or 8, and the range of f_i^{**} is chosen as N).

The functions f_6, f_7, f_8 are not jumpless in general (see Fig. 6). The next statement shows that they are far from being convex (in a certain sense). Simple

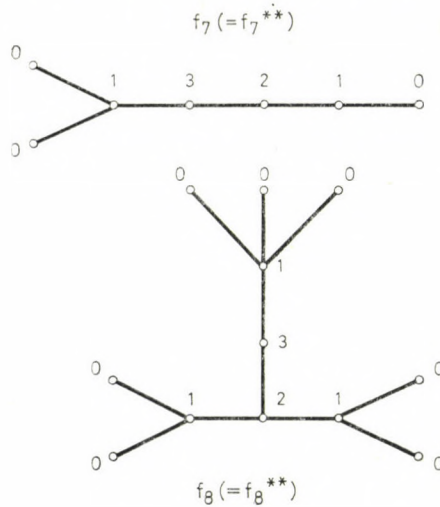


Fig. 6.

counterexamples disproving the convexity may be produced if the method (to be described in the proof) is applied to the star-free graph of length two such that W consists of the end vertices.

PROPOSITION 16. *Let i be one of the number 6, 7, 8; let T_1 be a tree and W be an arbitrary non-empty subset of the vertex set of T_1 . Then there exists a tree T such that T_1 is a subtree of T and $H(f_i)$ equals W .*

PROOF. In Part (A), we give the appropriate construction for f_6 and f_7 ; in part (B), for f_8 .

(A) To each vertex A of T_1 , we supplement the graph with an additional vertex A' and with an additional edge joining A and A' moreover, to each vertex $A (\in W)$ we introduce another vertex A'' and an edge between A' (introduced just now) and A'' . In the resulting tree T , it is easy to see that

$$f_i(A) = 2 \quad \text{if } A \in W,$$

$$f_i(A) = 1 \quad \text{if } A \text{ belongs to } T_1 \text{ and } A \notin W,$$

$$f_i(A') \leq 1 \quad \text{if } A' \text{ is a new vertex as above,}$$

$$f_i(A'') = 0 \quad \text{if } A'' \text{ is a new vertex as above}$$

(where i is 6 or 7).

(B) We introduce the new vertices A', A'' (and the edges between them) in full analogy to (A), furthermore, we supplement the graph with a vertex A''' and with an edge joining A' and A''' for each $A(\in W)$. The formulae in (A) hold with $i=8$ and we have $f_8(A''')=0$.

Appendix II.

In this section, we assert an independence statement (Proposition 17) and two dependence theorems as to how the functions f_1, f_2, f_3, f_4, f_5 are related to each other.

PROPOSITION 17. *Let i, j be two integers such that $1 \leq i < j \leq 5$. If the pair (i, j) differs from $(1, 3)$ and from $(2, 5)$, then there exists a tree T which satisfies $H(f_i) \cap H(f_j) = \emptyset$.*

PROOF. The tree on Fig. 7 ensures the statement for the following pairs (of

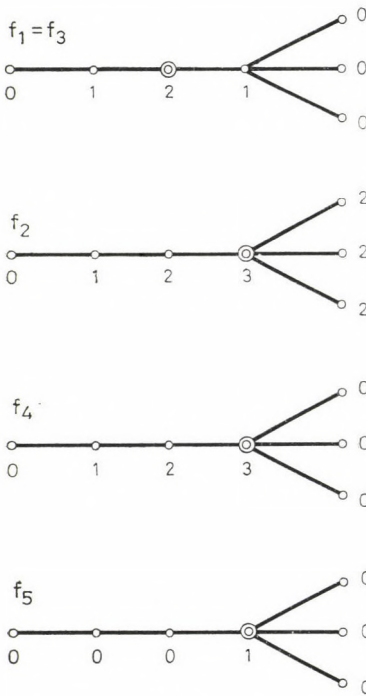


Fig. 7.

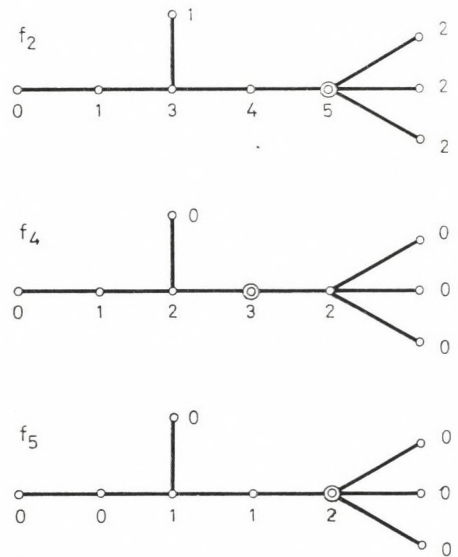


Fig. 8.

form (i, j)): $(1, 2), (1, 4), (1, 5), (2, 3), (3, 4), (3, 5)$. The tree on Fig. 8 guarantees the conclusion of the proposition for the pairs $(2, 4)$ and $(4, 5)$.

PROPOSITION 18. *Let T be a tree in which at least two vertices are of degree ≥ 2 , denote by T' the subtree which results from T by deleting the end vertices and the*

edges incident to them. The functions f_1, f_3 (defined in T), f'_1, f'_3 (defined in T') satisfy the equalities

$$f_1(A) = f'_1(A) + 1, \quad f_3(A) = f'_3(A) + 1$$

for each vertex A of T' .

PROOF. The diameters d and d' of T and T' (respectively) obviously fulfil $d = d' + 2$ and it is clear that

$$f_1^*(A) = f_1^{**}(A) + 1$$

for every vertex A of T . Hence

$$f_1(A) = d - f_1^*(A) = d' + 2 - (f_1^{**}(A) + 1) = f'_1(A) + 1$$

where A is an arbitrary vertex of T' .

Proposition 12 implies $T' = T_1, T'_1 = T_2, T'_2 = T_3, \dots$ (as long as the graphs in question exist), therefore we have

$$f_3(A) = f'_3(A) + 1$$

for the vertices of T' by the definition of the T_q 's.

PROPOSITION 19. Let B_1, B_2 be two (end) vertices of a tree such that $\delta(B_1, B_2)$ equals the diameter d of the tree. Denote the chain connecting B_1 and B_2 by c_1 . For any vertex A , let A' be the (single) vertex of c_1 for which the chain between A and A' does not contain any common vertex with c_1 but A' (possibly $A' = A$). Then

$$\delta(A, A') \leq \min(\delta(A', B_1), \delta(A', B_2)).$$

PROOF. If $\delta(A, A') > \delta(A', B_i)$ (where i is 1 or 2), then

$$\delta(B_{3-i}, A) = \delta(B_{3-i}, A') + \delta(A', A) = d - \delta(A', B_i) + \delta(A', A) > d,$$

and this contradicts the definition of d .

PROPOSITION 20. Let B_1, B_2 be as in Proposition 19. We have

$$f_1^*(A) = \max(\delta(A, B_1), \delta(A, B_2))$$

for each vertex A .

PROOF. Choose a vertex A , and let C be a vertex such that $\delta(A, C)$ is maximal (for all the vertices C for the tree).

Case 1: $A' \neq C'$. Let B_i be the vertex (among B_1, B_2) for which C' lies between B_i and A' . Proposition 19 guarantees the validity of the relations

$$\begin{aligned} f_1^*(A) = \delta(A, C) &= \delta(A, C') + \delta(C', C) \leq \delta(A, C') + \delta(C', B_i) = \delta(A, B_i) \leq \\ &\leq \max(\delta(A, B_1), \delta(A, B_2)), \end{aligned}$$

the converse inequality is trivial.

Case 2: $A' = C'$. Then

$$\begin{aligned} f_1^*(A) = \delta(A, C) &\leq \delta(A, A') + \delta(A', C) \leq \\ &\leq \delta(A, A') + \min(\delta(A', B_1), \delta(A', B_2)) \leq \max(\delta(A, B_1), \delta(A, B_2)). \end{aligned}$$

PROPOSITION 21. *The inequality $f_1(A) \leq f_1^*(A)$ is true for each vertex A of any tree. Equality in this formula holds if and only if $H(f_1) = \{A\}$.*

PROOF. We shall apply the notations used in Propositions 19, 20 and the assertion of Proposition 20 without explicit references.

Case 1: $A = A'$ and $\delta(A, B_1) = \delta(A, B_2)$. Then

$$f_1^*(A) = \delta(A, B_1) = \frac{d}{2} = f_1(A)$$

because $f_1(A) = d - f_1^*(A)$. For any vertex $C (\neq A)$, at least one of $C' \neq A'$ and $\delta(C, C') > 0$ is true, hence

$$f_1^*(C) = \delta(C, C') + \max(\delta(C', B_1), \delta(C', B_2)) > \frac{d}{2} = f_1^*(A),$$

therefore $f_1(C) < f_1(A)$, thus $H(f_1) = \{A\}$.

Case 2: either $A \neq A'$ or $\delta(A', B_1) \neq \delta(A', B_2)$. Then

$$f_1^*(A) = \delta(A, A') + \max(\delta(A', B_1), \delta(A', B_2)) > \frac{d}{2}$$

and

$$f_1(A) = d - f_1^*(A) < d - \frac{d}{2} = \frac{d}{2} < f_1^*(A).$$

Let $i (\in \{1, 2\})$ be a subscript determined by $\delta(A', B_i) \cong \delta(A', B_{3-i})$, and let D be the vertex which lies on the chain connecting A with B_i and is adjacent to A . We distinguish two subcases.

Case 2/a: $A \neq A'$. Then $D' = A'$ and

$$f_1^*(D) = \delta(D, A') + \delta(A', B_i) = \delta(A, A') - 1 + \delta(A', B_i) = f_1^*(A) - 1,$$

hence $f_1(D) > f_1(A)$ and $A \notin H(f_1)$.

Case 2/b: $A = A'$. Then $D' = D$ and

$$f_1^*(D) = \max(\delta(B_i, D), \delta(B_{3-i}, D)) = \max(\delta(B_i, A) - 1, \delta(B_{3-i}, D) + 1).$$

If

$$\delta(B_i, A) \cong \delta(B_{3-i}, A) + 2,$$

then we get $f_1^*(D) = f_1^*(A) - 1$ similarly to Case 2/a. We are left with the possibility

$$\delta(B_i, A) = \delta(B_{3-i}, A) + 1;$$

if this equality holds, then

$$f_1^*(D) = \delta(B_{3-i}, A) + 1 = \delta(B_i, A) = f_1^*(A) = \frac{d+1}{2}$$

(by $\delta(B_i, A) + \delta(B_{3-i}, A) = d$), thus

$$f_1(A) = f_1(D) = \frac{d-1}{2}$$

and

$$H(f_1) = \{A, D\}.$$

PROPOSITION 22. Let A be an arbitrary vertex of a tree. At least one of the formulae $A \notin H(f_1)$ and $f_1(A) = f_3(A)$ is true.

PROOF. Let B_1, B_2, c_1 be as in Proposition 19.

Case 1: $A \neq A'$. Let C be the vertex which is adjacent to A and lies on the chain between A, A' . The equality $f_1^*(C) = f_1^*(A) - 1$ follows from Proposition 20, thus A does not belong to $H(f_1)$.

Case 2: A is a vertex of c_1 and

$$|\delta(A, B_1) - \delta(A, B_2)| \cong 2.$$

Then $f_1^*(C) = f_1^*(A) - 1$, where C is one of the vertices adjacent to A of c_1 (as in Case 1).

Case 3: A is a vertex of c_1 and

$$|\delta(A, B_1) - \delta(A, B_2)| = 1.$$

Then we have the equalities

$$f_1^*(A) = \frac{d+1}{2}, \quad f_1(A) = \frac{d-1}{2},$$

$$\mu(A, e_1) = \frac{d+1}{2}, \quad \mu(A, e_2) = \frac{d-1}{2}, \quad \mu(A, e) \cong \frac{d-1}{2}$$

(the last of these follows from Proposition 19) where e_1 and e_2 are the edges of c_1 incident to A (the notation is so chosen that $\mu(A, e_1) \cong \mu(A, e_2)$), and e is an arbitrary further edge incident to A .

Case 4: $\delta(A, B_1) = \delta(A, B_2) = \frac{d}{2}$. Then each of $f_1(A), f_1^*(A), \mu(A, e_1), \mu(A, e_2)$ is equal to $\frac{d}{2}$ and $\mu(A, e) \cong \frac{d}{2}$, thus $f_3(A) = \frac{d}{2}$.

THEOREM 6. The formula $H(f_1) = H(f_3)$ holds for each tree. The relation $f_1(A) \cong f_3(A)$ is always valid; if $A \in H(f_1)$, then equality is true.

PROOF. First we verify $f_1(A) \cong f_3(A)$ by induction. If the tree T has at most two vertices of degree $\cong 2$, then the statement can easily be checked. In the general case, let T' be formed as in Proposition 18; if A belongs to T' , then

$$f_1(A) = f_1'(A) + 1 \cong f_3'(A) + 1 = f_3(A)$$

by Proposition 18; if A is an end vertex, then $f_1(A) \cong 0 = f_3(A)$.

The last assertion of the theorem is implied by Proposition 22, hence we also have $H(f_1) \subseteq H(f_3)$. The inclusion $H(f_1) \supseteq H(f_3)$ follows from the relation

$$f_3(B) \leq f_1(B) < f_1(A) = f_3(A)$$

where $A \in H(f_1)$ and $B \notin H(f_1)$.

PROPOSITION 23. *Let A, B be two adjacent vertices of a tree. If $f_2(A) \cong f_2(B)$, then $f_5(A) \cong f_5(B)$.*

PROOF. The assumption implies $f_2^*(A) \cong f_2^*(B)$, thus $|\psi(A, e)| \cong |\psi(B, e)|$ by Proposition 6; consequently,

$$f_5^*(A) \leq |\psi(A, e)| = |\sigma(B, e)| \leq f_5^*(B)$$

because of the definition of f_5^* and the obvious equality $\psi(A, e) = \sigma(B, e)$.

THEOREM 7. *The equality $H(f_2) = H(f_5)$ is valid for each tree T .*

PROOF. If T has no star, then the assertion is trivial (cf. the remarks at the ends of § 2 and § 5).

Suppose that at least one star occurs in T . The strict inclusions

$$\emptyset \subset H(f_2) \cap H(f_5) \subset H(f_2)$$

lead to a contradiction with Proposition 23, therefore we may have three possibilities:

- (i) $H(f_2) = H(f_5)$,
- (ii) $H(f_2) \cap H(f_5) = \emptyset$,
- (iii) $H(f_2) \subset H(f_5)$.

We are going to disprove (ii) and (iii).

If (ii) holds, then we have a chain consisting of some vertices

$$A_1, A_2, \dots, A_t \quad (t \geq 2)$$

such that $A_1 \in H(f_2)$, $A_t \in H(f_5)$ and $2 \leq i \leq t-1$ implies $A_i \notin H(f_2) \cup H(f_5)$. On the one hand, the inequalities

$$f_2(A_1) > f_2(A_2) > \dots > f_2(A_{t-1}) > f_2(A_t)$$

follow from the definition of $H(f_2)$ and from Proposition 8 (successively); on the other hand, the definition of $H(f_5)$ implies $f_5(A_{t-1}) < f_5(A_t)$. The two inequalities obtained for A_{t-1} and A_t contradict Proposition 23.

Suppose (iii). We distinguish two cases.

Case 1: $H(f_2) = \{A\}$ for some vertex A . A is a star by Theorem 2. Theorem 5 implies that there exists precisely one vertex $B (\in H(f_5))$ which is adjacent to A . We have $|\psi(A, e)| > |\psi(B, e)|$ by Proposition 6. Let e' be an edge, incident to A , such that $f_5^*(A) = |\sigma(A, e')|$. $|\sigma(A, e')| < |\psi(A, e)|$ evidently holds (whether $e' = e$

or $e' \neq e$). The relation

$$f_5^*(A) = |\sigma(A, e')| < |\psi(A, e)| = |\sigma(B, e)| = f_5^*(B)$$

implies $f_5(A) > f_5(B)$, this contradicts the definition of $H(f_5)$.

Case 2: $|H(f_2)| \geq 2$. By virtue of the assertions of Theorems 2 and 5 on the form of $H(f_2)$, $H(f_5)$ (resp.), a maximal star-free subgraph is properly included in another; this is clearly impossible.

PROPOSITION 24. *If the tree T has no simple vertex, then $f_2(A) = f_5(A)$ for each vertex A of T .*

PROOF. Proposition 23 holds now also in the modified form "if $f_2(A) > f_2(B)$, then $f_5(A) > f_5(B)$ " because the relation $|\psi(A, e)| > |\psi(B, e)|$ and the lack of simple vertices imply $f_5^*(A) < |\psi(A, e)|$ ($\cong f_5^*(B)$). The identity of f_2 and f_5 in T follows from this statement, the original version of Proposition 23, and the fact that f_2 and f_5 are gapless functions.

PROPOSITION 25. *The relation $f_2(A) \cong f_5(A)$ holds for any vertex A of a tree.*

PROOF. We use induction on the number of vertices. Denote by k the number of vertices of a tree T .

Case 1: T has no simple vertex. The assertion is given in Proposition 24.

Case 2: T has a simple vertex B . Delete B (and the edges incident to it), and draw a new edge between the vertices C and D adjacent to B . Denote the resulting tree (containing $k-1$ vertices) by T' . We have

$$f_5(A) = f_5'(A) \cong f_2'(A) \cong f_2(A)$$

for any vertex A ($\neq B$), and thus

$$f_5(B) = \min(f_5(C), f_5(D)) \cong \min(f_2(C), f_2(D)) \cong f_2(B)$$

(where f_2', f_5' are understood in T') by the induction hypothesis, the definitions of f_2 and f_5 , Proposition 15, and the convexity of f_2 .

The paper is finished with an open question (suggested by Dr. G. Pollák):

Problem. Does there exist a tree in which the sets $H(f_1)$, $H(f_2)$, $H(f_4)$ are pairwise disjoint?

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DENSITY OF COMPLEX φ -POLYNOMIALS

by

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Let φ be continuous on the complex plane. Let $V_n(\varphi)$ be the family of functions of the form

$$F(A, x) = \sum_{k=1}^n a_k \varphi(a_{n+k} x), \quad a_1, \dots, a_{2n} \text{ complex.}$$

The problem of approximation by $V_n(\varphi)$ is a special case of approximation by a curve of functions or equivalently approximation by γ -polynomials [1; 2; 3, 46ff]. A function in $V_n(\varphi)$ for some n is called a φ -polynomial. We are interested in determining sets of functions in which the set of φ -polynomials is dense, with respect to the uniform (Chebyshev) norm. The same problem for real functions is studied in [4].

Definition: A subset G of a set of functions H is fundamental in H if the set of (finite) linear combinations of functions in G is dense in H .

THEOREM. Let H be a set of functions defined on a bounded subset X of complex space. Let φ have a Taylor series $\sum_{k=0}^{\infty} c_k x^k$, convergent in a neighbourhood I of zero, such that the set of powers x^k with non-zero coefficient c_k is fundamental in H . If $\{\gamma_k\}$ is an infinite sequence of non-zero numbers converging to zero, then the set $\{\varphi(0), \varphi(\gamma_1 x), \varphi(\gamma_2 x), \dots\}$ is fundamental in H .

PROOF. It suffices to show that each power x^n with a non-zero coefficient c_n can be uniformly approximated by linear combinations of $\{\varphi(0), \varphi(\gamma_1 x), \varphi(\gamma_2 x), \dots\}$. It can be proven that for each $j > 0$ there is a sequence of linear combinations with the k -th linear combination equal to $x^n + O(k^{-j})$ on X . The proof is identical to the proof of the corresponding result in [4], where $X = [\alpha, \beta]$.

Applications

Let X be a subset of the complex plane. X will be called a *Runge region* if it is bounded, closed, and simply connected.

Let $C(X)$ be the space of continuous complex functions on X and $A(X)$ be the space of functions analytic on X . The subscript zero on C or A will denote restriction to functions vanishing at zero.

1. If all coefficients c_k of the Taylor series for φ are non-zero, the set of powers with non-zero coefficients is fundamental in $C(X)$, X any Jordan arc, and in $A(X)$, X a Runge region.

2. If $\varphi(0)=0$ and all but the zeroth coefficient of the Taylor series are non-zero, then the set of powers with non-zero coefficients is fundamental in $C_0(X)$, X any Jordan arc, and in $A_0(X)$, X any Runge region.

3. If φ is even and all even coefficients of the Taylor series are non-zero, the set of powers with non-zero coefficients is fundamental in $C[0, \beta]$.

4. If φ is odd and all odd coefficients of the Taylor series are non-zero, the set of powers with non-zero coefficients is fundamental in $C_0[0, \beta]$.

Generalizations

More generally we may be given φ continuous only on a neighbourhood J of zero and given a bounded region X . A (φ, J) -polynomial on X is any finite linear combination of terms of the form $a\varphi(bx)$ such that $bx \in J$ for all $x \in X$. Let $\{\gamma_k\} \rightarrow 0$. Then there exists K such that for $k \geq K$, $\gamma_k x \in J$ for all $x \in X$. It is trivial to extend the theorem to cover this case.

Let $\varphi(0)=0$ and all but the zeroth coefficient of the Taylor series for φ be non-zero. Let $\{\gamma_k\}$ be a non-zero sequence with limit 0. Let X be a Jordan arc. Let $f \in C(X)$ be given then $f - f(0) \in C_0(X)$ and can be uniformly approximated by linear combinations of $\{\varphi(\gamma_1 x), \varphi(\gamma_2 x), \dots\}$. Hence the set of linear combinations of $\{1, \varphi(\gamma_1 x), \varphi(\gamma_2 x), \dots\}$ is dense in $C(X)$. A similar result holds for X a Runge region and $f \in A(X)$. A similar result also holds for the case where $X = [0, \beta]$, φ is odd, and all odd coefficients of the Taylor series are non-zero.

Let φ be odd with the odd coefficients of its Taylor expansion non-zero. Let ψ be even with the even coefficients of its Taylor expansion non-zero. A linear combination of terms of the type $\varphi(ax)$ or $\psi(bx)$ is called a (φ, ψ) -polynomial. The set of (φ, ψ) polynomials is dense in $C(X)$, X a Jordan arc, and in $A(X)$, X a Runge region.

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**PERIODIC BOUNDARY VALUE PROBLEM FOR CERTAIN
NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATION**

by

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In this paper, we are concerned with the following non-linear second order differential equation

$$(1) \quad x'' + f(t, x, x') = 0$$

with periodic boundary conditions

$$(2) \quad \begin{aligned} x(0) &= x(\omega) \\ x'(0) &= x'(\omega) \end{aligned} \quad \omega \in [0, T].$$

It shall be shown that if certain conditions are imposed on the function f , then there exists at least one solution of (1) satisfying the boundary conditions (2).

If, in addition to our assumptions given below, the function f satisfies the following conditions

- (i) f is ω -periodic function in $t \in [0, T]$
- (ii) f satisfies the hypothesis of uniqueness with a given initial conditions, then the result of our paper gives an ω -periodic solution (see, [3]).

In the sequel it is assumed that:

(A₁) $f(t, x, x')$ is a real-valued, continuous function with domain $[0, T] \times \mathbb{R}^2, T > 0$,

(A₂) there exist a constant $K > 0$, and a function $H(t, r)$ with the following properties:

- 1° $H(t, r)$ is piecewise continuous in $t \in [0, T], r \geq 0$, continuous in $t \in [0, T]$, and nondecreasing (for fixed t) with respect to $r \geq 0$,
- 2° $|Kx - f(t, x, x')| \leq H(t, |x| + |x'|), t \in [0, T], (x, x') \in \mathbb{R}^2$,
- 3° $\pi M_c \leq 2Kc$ for some constant $c > 0$, where

$$M_c = \max_{t \in [0, T]} H(t, c).$$

THEOREM 1. *Under the above assumptions, there exists a positive real number $\omega_0 \left(0 < \omega_0 \leq \frac{\pi}{\sqrt{K}} \right)$ such that for each $\omega \in \left[\omega_0, \frac{\pi}{\sqrt{K}} \right]$ there exists at least one solution*

$x(t)$ of (1) which is continuous for $t \in \left[0, \frac{\pi}{\sqrt{K}}\right]$, satisfying the boundary conditions (2), and $|x(t)| \leq C$, $t \in \left[0, \frac{\pi}{\sqrt{K}}\right]$.

PROOF. It is obvious that the equation (1) together with the boundary conditions (2) is equivalent to

$$(3) \quad \begin{aligned} x'' + Kx &= Kx - f(t, x, x') \\ x(0) &= x(\omega), \quad x'(0) = x'(\omega). \end{aligned}$$

For $\omega \in \left(0, \frac{\pi}{\sqrt{K}}\right]$ let $G(t, s)$ be the Green function

$$G(t, s) = \begin{cases} \frac{1}{2\sqrt{K}} \cdot \frac{\cos \sqrt{K} \left(\frac{\omega}{2} + s - t\right)}{\sin \sqrt{K} \frac{\omega}{2}} & \text{for } 0 \leq s \leq t \leq \omega, \\ \frac{1}{2\sqrt{K}} \cdot \frac{\cos \sqrt{K} \left(\frac{\omega}{2} + t - s\right)}{\sin \sqrt{K} \frac{\omega}{2}} & \text{for } 0 \leq t \leq s \leq \omega. \end{cases}$$

Solution $x(t)$ of (3) is the same as that of the following nonlinear integral equation

$$(4) \quad x(t) = \int_0^{\omega} G(t, s) [Kx(s) - f(s, x(s), x'(s))] ds.$$

Using BIHARI'S Theorem 1, [1], we shall prove that (4) has at least one solution with the desired properties. First we assume $M_c > 0$. It is easy to see that $G(t, s)$ is continuous and non-negative in $0 \leq s \leq t \leq \omega$ and in $0 \leq t \leq s \leq \omega$. Since $\omega \in \left(0, \frac{\pi}{\sqrt{K}}\right]$, it follows that $\sqrt{K} \frac{\omega}{2} \in \left(0, \frac{\pi}{2}\right]$, and hence $\sin \sqrt{K} \frac{\omega}{2} \geq \frac{2}{\pi} \sqrt{K} \frac{\omega}{2}$ involving

$$G(t, s) \leq \frac{1}{2\sqrt{K}} \cdot \frac{1}{\frac{2}{\pi} \sqrt{K} \frac{\omega}{2}} = \frac{\pi}{2K\omega}.$$

Now, let

$$\omega_0 = \frac{\pi^2 M_c}{2K^{3/2}c}.$$

From 3°, it follows that

$$0 < \omega_0 \leq \frac{\pi}{\sqrt{K}},$$

and therefore the interval $\left[\omega_0, \frac{\pi}{\sqrt{K}}\right]$ actually exists. Moreover it is easily seen that

for any $\omega \in \left[\omega_0, \frac{\pi}{\sqrt{K}} \right]$ we have

$$G(t, s) \cong \frac{\pi}{2K\omega_0} = \frac{\sqrt{K} \cdot C}{\pi M_c}.$$

Thus $G(t, s)$ is continuous, bounded in $0 \leq s \leq t \leq \omega$ and in $0 \leq t \leq s \leq \omega$ for all $\omega \in \left[\omega_0, \frac{\pi}{\sqrt{K}} \right]$, and hence the first hypothesis of BIHARI's Theorem 1, [1] is satisfied.

From our assumptions it follows that the functions

$$F(t, x, x') = Kx - f(t, x, x') \quad t \in [0, T], \quad (x, x') \in R^2$$

$$H(t, |x| + |x'|)$$

satisfy the second hypothesis of BIHARI's Theorem 1, [1]. We also have

$$\frac{\sqrt{K} \cdot C}{\pi M_c} \int_0^\omega H(s, c) ds \cong \frac{\sqrt{K} \cdot C}{\pi M_c} \cdot \frac{\pi}{\sqrt{K}} M_c = C,$$

which shows that the third hypothesis of BIHARI's Theorem 1, [1] is satisfied.

Finally taking $z(t) \equiv 0$, we have shown that all the hypotheses of BIHARI's Theorem 1, [1] are satisfied, and hence conclusion of his theorem tells us that at least one solution $x(t)$ of (1) with the desired properties exists.

If $M_c = 0$, taking ω_0 to be any number in $\left(0, \frac{\pi}{\sqrt{K}} \right]$ the proof trivially follows.

Thus the proof of our Theorem 1 is completed.

COROLLARY 1. *Under the assumptions 1°, 2° and the assumption 4° $\pi M_c \cong Kc$ for some constant $c > 0$, where*

$$M_c = \text{Max}_{t \in [0, T]} H(t, c),$$

there exists a positive real number $\omega_0 \left(\frac{\pi}{\sqrt{K}} \cong \omega_0 \cong \frac{2\pi}{\sqrt{K}} \right)$ such that for each $\omega \in \left[\frac{\pi}{\sqrt{K}}, \omega_0 \right]$ there is a solution $x(t)$ of (1) satisfying the boundary conditions (2).

PROOF. Let $G(t, s)$ be as in Theorem 1. Since $|\sin u| \cong -\frac{2}{\pi}u + 2$ for $\frac{\pi}{2} \leq u \leq \pi$, we have for $\frac{\pi}{\sqrt{k}} \leq \omega \leq \frac{2\pi}{\sqrt{k}}$,

$$|G(t, s)| \cong \frac{1}{2\sqrt{K}} \frac{1}{-\frac{2}{\pi}\sqrt{K}\frac{\omega}{2} + 2} = \frac{\pi}{2\sqrt{K}(-\sqrt{K}\omega + 2\pi)} = \varphi(\omega).$$

Now, we let

$$\omega_0 = \frac{2\pi}{\sqrt{K}} - \frac{\pi^2 M_c}{K^{3/2} c}.$$

From 4°, it follows that

$$\frac{\pi}{\sqrt{K}} \cong \omega_0 \cong \frac{2\pi}{\sqrt{K}}.$$

Furthermore, for any $\omega \in \left[\frac{\pi}{\sqrt{K}}, \omega_0 \right]$ we have

$$\varphi(\omega) \cong \varphi(\omega_0) = \frac{\sqrt{K}c}{2\pi M_c}.$$

Now the remaining parts of the proof is similar to that of Theorem 1. Let us now consider a few applications of our Theorem 1.

(I) We consider the equation

$$(5) \quad x'' + a(t)x + b(t)f(x^2) = \mu p(t),$$

and assume

- (a) $a(t)$ is non-negative, continuous ω -periodic function,
- (b) $b(t)$ and $p(t)$ are continuous ω -periodic functions,
- (c) $f(x)$ is locally Lipschitzian, non-negative, non-decreasing for $x \geq 0$, and for some $c > 0$

$$\frac{\pi}{(\pi-2)c} (CA - Bf(c^2) - |\mu|E) \cong D,$$

where

$$A = \text{Min}_{t \in [0, \omega]} a(t), \quad D = \text{Max}_{t \in [0, \omega]} a(t), \quad E = \text{Max}_{t \in [0, \omega]} |p(t)|,$$

$$B = \text{Max}_{t \in [0, \omega]} |b(t)|.$$

Then under the above assumptions there exists constant $K > 0$ such that for each $\omega \in \left[0, \frac{\pi}{\sqrt{K}} \right]$ equation (5) has an ω -periodic solution.

PROOF. Let $K \cong D$, and let

$$F(t, x, x') = Kx - a(t)x - b(t)f(x^2) + \mu p(t),$$

$$H(t, |x| + |x'|) = (K - a(t))(|x| + |x'|) + |b(t)|f[(|x| + |x'|)^2] + |\mu||p(t)|.$$

It is obvious that F and H satisfy (A_1) , 1° and 2°. Now we should like to show for some $K \cong D$, there exists $c > 0$ such that 3° is also satisfied, that is, we need to show

$$\pi[(K - A)C + Bf(C^2) + |\mu|E] \cong 2KC$$

or

$$k \cong \frac{\pi}{(\pi-2)c} (Ac - Bf(c^2) - |\mu|E) = F_1.$$

Since by our assumption (c) for some $c > 0$ the right hand side of the last inequality is greater than or equal to D , we can be sure that for the given $c > 0$ (in (c)), and any $K \in [D, F]$, 3° is satisfied. Therefore from our Theorem 1 it follows that equation (5) has a solution satisfying the boundary conditions (2). Since F also satisfies (i) and (ii) the equation (5) has an ω -periodic solution. It should be mentioned that (5) had been discussed by BIHARI [2] regarding stability and boundedness of its solution for $\mu = 0$.

(II) Recently SEDZIWIY [4] has shown that the equation

$$x'' + f(x)x'^{2n} + g(x) = \mu p(t)$$

has at least one periodic solution with period ω provided $n \geq 1$ is an integer, all coefficients are continuous, $xg(x) > 0$ for $x \neq 0$, $\lim_{|x| \rightarrow \infty} \int_0^x g(u) du = \infty$, $\limsup_{x \rightarrow -\infty} \frac{g(x)}{x} = e$, where e is some constant, $0 < b \leq f(x) \leq d < \infty$ for all x , $p(t + \omega) = p(t)$ and $|\mu|$ is sufficiently small.

In this regard we consider the following equation

$$(6) \quad x'' + f(x)x'^n + ax = \mu p(t), \quad a > 0, \quad n \geq 2,$$

where f is locally Lipschitzian and all the coefficients are as above. We would like to show that (6) has an ω -periodic solution.

PROOF. Let $K \geq a$, and let

$$F(t, x, x') = Kx - ax - f(x)x'^n + \mu p(t)$$

$$H(t, |x| + |x'|) = (K - a)(|x| + |x'|) + d(|x| + |x'|)^n + |\mu|B$$

where

$$B = \text{Max}_{t \in [0, \omega]} |p(t)|.$$

Clearly F, H satisfy (A_1) , 1° and 2° . We would like to show that 3° is also satisfied.

Clearly $M_c = (K - a)c + dc^n + |\mu|B$, and we only need to show that for some $c > 0$

$$\pi[(K - a)c + dc^n + |\mu|B] \leq 2Kc.$$

Let $c = |\mu|^{1/n}$, then $dc^n + |\mu|B = O(|\mu|)$ and it suffices to require $\pi(K - a) \leq 2K$ or finally $a \leq K \leq \frac{\pi}{\pi - 2} a$. Thus all of the hypotheses of our Theorem 1 are satisfied.

Since F also satisfies (i) and (ii) the equation (6) has an ω -periodic solution.

Next let $0 < K \leq a$, and let

$$F(t, x, x') = Kx - ax - f(x)x'^n + \mu p(t),$$

$$H(t, |x| + |x'|) = (a - K)(|x| + |x'|) + d(|x| + |x'|)^n + |\mu|B.$$

With the similar calculation, it follows that for $c = |\mu|^{1/n}$ it suffices to require $\frac{\pi}{2 + \pi} a \leq K \leq a$. Thus the hypothesis 3° is satisfied. Hence for $\frac{2}{2 + \pi} a \leq K \leq a$ we also have periodic solution.

(III) We consider VAN DER POL's equation

$$(7) \quad x'' + \varepsilon x'(1 - x^2) + x = \mu p(t),$$

where $p(t)$ is continuous for $t \in [0, \omega]$ and is ω -periodic function. We shall show that there exists an ω -periodic solution, for sufficiently small $|\mu|$.

PROOF. Let

$$F(t, x, x') = Kx - \varepsilon x'(1 - x^2) - x + \mu p(t),$$

$$H(t, |x| + |x'|) = |K - 1|(|x| + |x'|) + \varepsilon(|x| + |x'|)[1 + (|x| + |x'|)^2] + |\mu|B,$$

where

$$B = \text{Max}_{t \in [0, \omega]} |p(t)|.$$

Clearly F and H satisfy (A_1) , 1° and 2° . Let c be any positive number, and let

$$\frac{\pi}{2 + \pi} \leq K \leq \frac{\pi}{\pi - 2}, \quad \varepsilon < \frac{2}{\pi(1 + c^2)},$$

and

$$|\mu| \leq \frac{c[2 - \varepsilon\pi(1 + c^2)]}{\pi B},$$

then the hypothesis 3° is satisfied. Since F also satisfies (i) and (ii), the equation (7) has an ω -periodic solution, for

$$\frac{\pi}{2 + \pi} \leq K \leq \frac{\pi}{\pi - 2}.$$

Remark. The result of this paper generalizes that of K. SCHMITT [5], since in [5] he assumes that the function $f(t, x, x')$ satisfies the strong condition $f(t, \alpha, 0) \leq 0 \leq f(t, \beta, 0)$ for constant $\alpha, \beta, \alpha \leq \beta$. This assumption strongly limits the class of differential equations for which K. SCHMITT's Theorem 1, [5], is applicable.

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**ON THE RAPIDITY OF CONVERGENCE OF α -QUASI
HERMITE—FEJÉR INTERPOLATION POLYNOMIAL**

by

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1. According to the classical result of L. FEJÉR [2] the Hermite—Fejér interpolation polynomials $H_n(f, x)$, given by

$$(1) \quad H_n(f, x) = \sum_{k=1}^n f(x_k) (1 - xx_k) \left(\frac{T_n(x)}{n(x - x_k)} \right)^2,$$

where $x_k = \cos \frac{2k-1}{2n} \pi$, $k=1, 2, \dots, n$, are roots of $T_n(x)$ — the n^{th} Tchebycheff polynomial of first kind, converge uniformly to $f(x)$ in $[-1, 1]$, when $f(x)$ is continuous in $[-1, 1]$. As to the rapidity of convergence E. MOLDOVAN [3] published the estimate

$$(2) \quad \max_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leq 2\pi\omega \left(\frac{\log n}{n} \right),$$

where $w(t)$ is the modulus of continuity of $f(x)$. This result was also established by O. SHISHA and B. MOND [6]. In [5] P. O. H. VÉRTESI has given an improvement of the estimation (2). He showed that

$$(3) \quad |f(x) - H_n(f, x)| = O(1) \left\{ \sum_{i=1}^n \frac{1}{i^2} \omega \left(\frac{i\sqrt{1-x^2}}{n} \right) + \sum_{i=1}^n \frac{1}{i^2} \omega \left(\frac{i^2}{n^2} \right) \right\}$$

Further, BOJANIČ [1] proved that

$$(4) \quad |f(x) - H_n(f, x)| \leq \frac{c}{n} \sum_{k=1}^n \omega \left(\frac{1}{k} \right).$$

In 1967 R. B. SAXENA [4] considered a Quasi Hermite-Fejér interpolation polynomial $S_{2n}(f, x)$ of degree $\leq 2n$ satisfying the following properties:

$$S_{2n}(f, -1) = f(-1), \quad S_{2n}(f, x_k) = f(x_k), \quad S'_{2n}(f, x_k) = 0, \quad k = 1, 2, \dots, n.$$

He proved that the sequence of polynomials $\{S_{2n}(f, x)\}$, constructed on the zeros of Jacobi polynomial $P_n^{(-\frac{1}{2}, \frac{1}{2})}(x)$, converges uniformly to $f(x)$ in $[-1, 1]$.

The aim of this paper is to obtain the rapidity of convergence of the sequence $\{\|f(x) - S_{2n}(f, x)\|\}$, where

$$\|f(x)\| = \max_{-1 \leq x \leq 1} |f(x)|.$$

2. A quasi Hermite—Fejér interpolation polynomial constructed on the zeros of $P_n^{(-\frac{1}{2}, \frac{1}{2})}(x)$ is given by [4]

$$(5) \quad S_{2n}(f, x) = f(-1) \frac{\omega_n^2(x)}{\omega_n^2(-1)} + \sum_{k=1}^n f(x_k) \frac{1+x}{1+x_k} \left(\frac{1-xx_k}{1-x_k^2} \right) l_k^2(x),$$

where

$$(6) \quad \omega_n(x) = \left[\frac{\cos(n+2)\theta}{\cos \frac{\theta}{2}} \right]_{x=\cos \theta}, \quad x_k = \cos \frac{2k-1}{2n+1} \pi,$$

and

$$(7) \quad l_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}, \quad k = 1, 2, \dots, n.$$

We shall prove the following theorems.

THEOREM 1. *If $f(x)$ is a continuous function in $[-1, 1]$, then for the sequence of polynomials $\{S_{2n}(f, x)\}$ in (5), we have*

$$(8) \quad |f(x) - S_{2n}(f, x)| \leq 88 \left\{ \sum_{i=1}^n \frac{1}{i^2} \omega \left(\frac{i\pi \sqrt{1-x^2}}{2n+1} \right) + \sum_{i=1}^n \frac{1}{i^2} \omega \left(\frac{i^2 \pi^2}{(2n+1)^2} \right) \right\}.$$

THEOREM 2. *For every $n \geq 2$ there exists a constant c such that*

$$(9) \quad \sup_{f \in C_M(\Omega)} \|f(x) - S_{2n}(f, x)\| \leq \frac{cM}{n} \sum_{k=1}^n \Omega \left(\frac{1}{k} \right)$$

holds in $[-1, 1]$.

Here Ω is an increasingly subadditive and continuous function on $\{x; x \geq 0\}$ with $\Omega(0) = 0$, and $C_M(\Omega)$ denotes the class of continuous functions defined on $[-1, 1]$, such that

$$f \in C_M(\Omega) \Leftrightarrow \omega(t) \leq M\Omega(t),$$

or, equivalently, $f \in C_M(\Omega)$ if and only if $|f(x_1) - f(x_2)| \leq M\Omega(|x_1 - x_2|)$ for all $x_1, x_2 \in [-1, 1]$.

It is obvious that

$$(10) \quad \Omega(\alpha t) \leq (\alpha + 1)\Omega(t) \quad \text{for } \alpha > 0, t \geq 0$$

and

$$(11) \quad \frac{\Omega(t_1)}{t_1} \leq \frac{2\Omega(t_2)}{t_2}; \quad t_1 > t_2 > 0.$$

3. To prove theorem 1, we need the following estimations.

Let

$$(12) \quad \frac{j-1}{n} \pi \leq \theta \leq \frac{j}{n} \pi, \quad j = 1, 2, \dots, n.$$

LEMMA 1.

$$(13) \quad |f(x_k) - f(x)| \leq \begin{cases} 2\omega \left(\frac{\pi \sin \theta}{2n+1} \right) + 2\omega \frac{\pi^2}{(2n+1)^2} & \text{for } k = j, \\ 4\omega \left(\frac{i\pi \sin \theta}{2n+1} \right) + 8\omega \left(\frac{i^2 \pi^2}{(2n+1)^2} \right) & \text{for } \begin{matrix} 1 \leq k = j-i < j, \\ j < k = j+i \leq n. \end{matrix} \end{cases}$$

PROOF. Let

$$(14) \quad \theta_k = \frac{2k-1}{2n+1} \pi, \quad k = 1, 2, \dots, n.$$

From Taylor's formula with Lagrange's remainder term, we have

$$x_k - x = \cos \theta_k - \cos \theta = (\theta - \theta_k) \sin \theta - \frac{1}{2} (\theta - \theta_k)^2 \xi,$$

where

$$\theta \leq \xi \leq \theta_k.$$

Therefore

$$(15) \quad |x_k - x| \leq |\theta_k - \theta| \sin \theta + \frac{1}{2} (\theta_k - \theta)^2.$$

From (12) and (14), we have

$$(16) \quad |\theta_j - \theta| \leq \frac{2\pi}{2n+1},$$

which with the help of (15) gives (13) for $k=j$.If $k < j$, $k = j-i$ and $i = 1, 2, \dots, j-1$, then by (14) and (16)

$$|\theta_k - \theta| \leq |\theta_k - \theta_j| + |\theta_j - \theta| \leq \frac{4i\pi}{2n+j}.$$

Using this and (15), we get (13) for $k=j-i$. Similarly we can prove (13) for $k=j+i$.

LEMMA 2. Let

$$x = \cos \theta, \quad 0 \leq \theta \leq \pi, \quad x_k = \cos \theta_k, \quad \theta_k = \frac{2k-1}{2n+1} \pi, \quad k = 1, 2, \dots, n$$

and

$$(17) \quad F_k \stackrel{\text{def}}{=} \frac{(1+x)}{(1+x_k)} - \frac{(1-xx_k)}{(1-x_k^2)} I_k^2(x),$$

then

$$(18) \quad 0 \leq F_k \leq \frac{8 \sin^2 \frac{2n+1}{2} \left(\frac{\theta - \theta_k}{2} \right)}{(2n+1)^2 \sin^2 \left(\frac{\theta - \theta_k}{2} \right)}.$$

One can easily prove this lemma, so we omit the proof. It is also easy to prove

$$(19) \quad \sin \left| \frac{\theta_k - \theta}{2} \right| \geq \frac{i}{2n+1} \quad \text{for } k \neq j.$$

4. PROOF of theorem 1.

From (5) and (17), we have the identity

$$(20) \quad \frac{\omega_n^2(x)}{\omega_n^2(-1)} + \sum_{k=1}^n F_k = 1.$$

Thus we can write

$$(21) \quad f(x) - S_{2n}(f, x) = (f(x) - f(-1)) \frac{\omega_n^2(x)}{\omega_n^2(-1)} + \sum_{k=1}^n (f(x) - f(x_k)) F_k,$$

which with the help of (13), (20), (18) and (19) gives

$$\begin{aligned} |f(x) - S_{2n}(f, x)| &\leq |f(x) - f(-1)| \frac{\omega_n^2(x)}{\omega_n^2(-1)} + |f(x) - f(x_j)| f_j + \\ &+ \sum_{k \neq j} |f(x) - f(x_k)| F_k \leq \omega(1+x) \frac{\omega_n^2(x)}{\omega_n^2(-1)} + 2 \left\{ \omega \left(\frac{\pi \sin \theta}{2n+1} \right) + \omega \left(\frac{\pi^2}{(2n+1)^2} \right) \right\} + \\ &+ 8 \sum_{k \neq j} \left\{ 4\omega \left(\frac{i\pi \sin \theta}{2n+1} \right) + 8\omega \left(\frac{i^2 \pi^2}{(2n+1)^2} \right) \right\} \left[\frac{\sin \frac{2n+1}{2} \left(\frac{\theta - \theta_k}{2} \right)}{(2n+1) \sin \frac{\theta - \theta_k}{2}} \right]^2 \leq \\ &\leq \{1 + (2n+1)^2(1+x)\} \omega \left(\frac{1}{(2n+1)^2} \right) \frac{\omega_n^2(x)}{\omega_n^2(-1)} + \\ &+ \frac{32}{(2n+1)^2} \sum_{i=1}^n \left\{ \omega \left(\frac{i\pi \sin \theta}{2n+1} \right) + 2\omega \left(\frac{i^2 \pi^2}{(2n+1)^2} \right) \right\} \left(\frac{2n+1}{i} \right)^2. \end{aligned}$$

Since

$$\frac{\omega_n^2(x)}{\omega_n^2(-1)} \leq 1, \quad 0 \leq (1+x)\omega_n^2(x) \leq 2 \quad \text{and} \quad \omega_n^2(-1) = (2n+1)^2,$$

therefore, we get (8).

Remark 1. From (8), we have

$$\begin{aligned} |f(x) - S_{2n}(f, x)| &= O(1) \omega \left(\frac{\log n}{n} \right) \sum_{i=1}^n \frac{1}{i^2} \left(\frac{i\sqrt{1-x^2}}{\log n} + 1 + \frac{i^2}{n \log n} + 1 \right) = \\ &= O(1) \omega \left(\frac{\log n}{n} \right). \end{aligned}$$

This estimation is of the Moldovan type.

Further, let $w(f) = t^\alpha$ ($0 < \alpha < 1$), then from (8), we have

$$|f(x) - S_{2n}(f, x)| = O(1) \left[\frac{\sqrt{(1-x^2)^\alpha}}{n^\alpha} \sum_{i=1}^n i^{\alpha-2} + \frac{1}{n^{2\alpha}} \sum_{i=1}^n i^{2\alpha-2} \right],$$

which gives

$$|f(x) - S_{2n}(f, x)| = O(1) \left\{ \frac{(1-x^2)^{\alpha/2}}{n^\alpha} + \frac{1}{n} \right\} \quad \text{for } 0 < \alpha < 1, \quad -1 \leq x \leq 1$$

and

$$|f(x) - S_{2n}(f, x)| = O(1) \left(\sqrt{1-x^2} \frac{\log n}{n} + \frac{1}{n} \right) \quad \text{for } \alpha = 1, \quad -1 \leq x \leq 1.$$

Remark 2. Let $g(x) = |x|$, $x = \cos \theta = 1$, $n = 1, 2, \dots$, then from (5) and (7), we have

$$g(1) - S_{2n}(g, 1) = \frac{2}{(2n+1)^2} \sum_{k=1}^n \frac{(1 - \cos \theta_k)^2}{(1 - \cos \theta_k)^2} = \frac{2n}{(2n+1)^2}.$$

Also, for $h(x) = |x|^\alpha$, $0 < \alpha < 1$; $x = \cos \theta = 1$, $n = 4, 5, 6, \dots$,

$$h(1) - S_{2n}(h, 1) \cong \frac{2}{(2n+1)^2} \sum_{k=1}^{\lfloor n/4 \rfloor} \frac{(1 - \cos^\alpha \theta_k)}{(1 - \cos \theta_k)} \quad \text{for } 0 < k \leq \left\lfloor \frac{n}{4} \right\rfloor$$

since by Lagrange Mean-value theorem for

$$x \cong \frac{1}{2} \quad \text{and} \quad 0 < k \leq \left\lfloor \frac{n}{4} \right\rfloor,$$

$$1 - |x_k|^\alpha \cong (1 - x_k).$$

5. PROOF of theorem 2.

To prove theorem 2, we shall use the following.*

For $m \geq 2$, we have

$$(22) \quad \sum_{r=1}^m \frac{1}{r^2} \Omega \left(\frac{(r+1)\pi}{m} \right) \leq \frac{8\pi}{m} \int_{\pi/m}^{\pi} \frac{\Omega(t)}{t^2} dt.$$

Now from (21), we have for any $f \in C_M(\Omega)$ and $x \in [-1, 1]$

$$(23) \quad |f(x) - S_{2n}(f, x)| \leq M\Omega(1+x) \frac{\omega_n^2(x)}{\omega_n^2(-1)} + M \sum_{k=1}^n \Omega(|x_n - x_k|) F_k = I_1 + I_2.$$

From (10), we have for $-1 \leq x \leq 1$

$$I_1 \leq M\Omega \left(\frac{\pi}{2n+1} \right) \frac{\omega_n^2(x)}{\omega_n^2(-1)} + \frac{2n+1}{\pi} M\Omega \left(\frac{\pi}{2n+1} \right) (1+x) \frac{\omega_n^2(x)}{\omega_n^2(-1)} < 2M\Omega \left(\frac{\pi}{2n+1} \right)$$

owing to the inequalities $\frac{\omega_n^2(x)}{\omega_n^2(-1)} \leq 1$, $(1+x)\omega_n^2(x) < 2$ and $\omega_n^2(-1) = (2n+1)^2$.

Using (17), (18) and $|\cos \theta - \cos \theta_k| \leq |\theta - \theta_k|$, we get

$$I_2 \leq \frac{8M}{(2n+1)^2} \sum_{k=1}^m \Omega(|\theta - \theta_k|) A_n(|\theta - \theta_k|),$$

* See [1].

where

$$A_n(|\theta - \theta_k|) = \left(\frac{\sin \frac{2n+1}{2} \left(\frac{\theta - \theta_k}{2} \right)}{\sin \frac{\theta - \theta_k}{2}} \right)^2.$$

Further, setting

$$E_r(n, \theta) = \left\{ k; \frac{r\pi}{2n+1} < |\theta - \theta_k| \leq \frac{(r+1)\pi}{2n+1} \right\}, \quad r = 0, 1, 2, \dots, 2n,$$

we get

$$I_2 \leq \frac{8M}{(2n+1)^2} \sum_{r=0}^{2n} \sum_{k \in E_r(n, \theta)} \Omega(|\theta - \theta_k|) A_n(|\theta - \theta_k|).$$

Since $\left| \sin n \frac{y}{2} \right| \leq n \left| \sin \frac{y}{2} \right|$ for $|y| \leq \pi$ and $|\theta - \theta_k| \leq \frac{\pi}{2n+1}$ holds for only one value of k , we have

$$\sum_{k \in E_0(n, \theta)} \Omega(|\theta - \theta_k|) A_n(|\theta - \theta_k|) \leq \left(\frac{2n+1}{2} \right)^2 \Omega \left(\frac{\pi}{2n+1} \right) \sum_{k \in E_0(n, \theta)} 1 \leq \left(\frac{2n+1}{2} \right)^2 \Omega \left(\frac{\pi}{2n+1} \right).$$

Also for $|y| \leq \pi$, $\left| \sin \frac{y}{2} \right| \geq \frac{|y|}{\pi}$ and for every $r=1, 2, \dots, 2n$,

$$\frac{r\pi}{2n+1} < |\theta - \theta_k| \leq \frac{(r+1)\pi}{2n+1}$$

holds for only one value of k , so we have

$$\begin{aligned} \sum_{k \in E_r(n, \theta)} \Omega(|\theta - \theta_k|) A_n(|\theta - \theta_k|) &\leq \pi^2 \sum_{k \in E_r(n, \theta)} \frac{\Omega(|\theta - \theta_k|)}{(\theta - \theta_k)^2} \leq \\ &\leq \left(\frac{2n+1}{r} \right)^2 \Omega \left(\frac{(r+1)\pi}{2n+1} \right) \sum_{k \in E_r(n, \theta)} 1 \leq \left(\frac{2n+1}{r} \right)^2 \Omega \left(\frac{(r+1)\pi}{2n+1} \right). \end{aligned}$$

Hence

$$I_2 \leq 2M\Omega \left(\frac{\pi}{2n+1} \right) + 8M \sum_{r=1}^{2n} \frac{1}{r^2} \Omega \left(\frac{(r+1)\pi}{2n+1} \right).$$

Since Ω is an increasing function, we have

$$\Omega \left(\frac{\pi}{2n+1} \right) \leq \frac{\pi}{2n} \int_{\pi/2n+1}^{\pi} \frac{\Omega(t)}{t^2} dt.$$

Using (22) for $m=2n+1$, we get

$$\sum_{r=1}^{2n} \frac{1}{r^2} \Omega \left(\frac{(r+1)\pi}{2n+1} \right) \leq \frac{8\pi}{(2n+1)} \int_{\pi/2n+1}^{\pi} \frac{\Omega(t)}{t^2} dt < \frac{4\pi}{n} \int_{\pi/2n+1}^{\pi} \frac{\Omega(t)}{t^2} dt.$$

Consequently, we have

$$|f(x) + S_{2n}(f, x)| \leq \frac{34\pi M}{n} \int_{\pi/2n+1}^{\pi} \frac{\Omega(t)}{t^2} dt.$$

At the last by monotonicity of Ω and by (10), we obtain

$$\int_{\pi/2n+1}^{\pi} \frac{\Omega(t)}{t^2} dt = \frac{1}{\pi} \int_1^{2n+1} \Omega\left(\frac{\pi}{t}\right) dt \leq 2 \int_1^{2n+1} \Omega\left(\frac{1}{t}\right) dt \leq \sum_{k=1}^n \Omega\left(\frac{1}{k}\right).$$

This completes the proof of the theorem.

Remark 3. Since the modulus of continuity $w(t)$ of any continuous function $f(x)$ on $[-1, 1]$ has the same properties as Ω and $f \in C_1(w)$, from (9) we have

$$\|f(x) - S_{2n}(f, x)\| \leq \frac{c}{n!} \sum_{k=1}^n \omega\left(\frac{1}{k}\right)$$

for any continuous function $f(x)$ on $[-1, 1]$. From this we get

$$\|f(x) - S_{2n}(f, x)\| = O(1) \omega\left(\frac{\log n}{n}\right).$$

Remark 4. One can get analogous results replacing the zeros of $P_n^{(-\frac{1}{2}, \frac{1}{2})}(x)$ by that of $P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)$.

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MODULES OVER qc-RINGS

by

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1. Introduction. A ring R is called a (right) qc-ring if each cyclic (right) R -module is quasi-injective. For various properties of these rings we refer to AHSAN [1] and KOEHLER [10]. The purpose of this note is to make a brief study of certain properties of modules over qc-rings. Among other results, we prove that a qc-ring R has the characterizing property that each submodule of a cyclic R -module is quasi-injective. Endomorphism rings of cyclic modules over qc-rings have also been studied.

2. Preliminaries. Throughout this note we assume that all rings have an identity and all modules are unitary right modules. If R is a ring and M is an R -module then $S(M)$ will denote the socle of M , i.e., the sum of all simple submodules of M . We shall write $A \subseteq_e B$ to denote that A is an essential submodule of B . A submodule A of a module M is called 'small' if for any submodule B of M , $A+B=M$ implies $B=M$. A ring R is said to be an 'FGS-ring' if each cyclic R -module has a finitely generated (or empty) socle. These rings have been studied by KURSHAN [11]. A ring all of whose (right) ideals are quasi-injective is called a (right) 'q-ring' (cf. JAIN, SINGH and MOHAMED [8]). Dualizing the notion of q-rings, KOEHLER studied rings all of whose cyclic modules are quasi-projective in [9]. Such rings have been called 'q*-rings'. By a (right) 'duo ring' we shall mean a ring all of whose (right) one-sided ideal is two-sided. Finally, a module M is called 'finite dimensional' (Goldie) if there do not exist infinitely many non-zero submodules whose sum is direct.

3. Main Results. We start with the following lemma.

LEMMA 3.1. *Let R be a local qc-ring. Then R is an FGS-ring.*

PROOF. Since R is a local qc-ring, R is a duo ring (see Proposition 1.2 and Corollary 3.4 of KOEHLER [10]). By Proposition 2.2 of KURSHAN [11], the lemma will follow if we show that each finitely generated R -module is finite dimensional. We do this by using the inductive argument. Let us suppose that M is an R -module generated by n elements. If $n=1$, then M is cyclic. Hence we can write $M \cong R/I$, where I is a right ideal of R . Since R is a duo ring, I is a two-sided ideal of R so that R/I becomes a factor ring of R . Hence by Lemma 2 of AHSAN [1], R/I is a qc-ring. Therefore R/I as an R -module is finite dimensional by AHSAN [1, Theorem 1]. Thus the lemma is proved for $n=1$. Let us now suppose that the result is true for $n=k$ and prove that the result is also true for $n=k+1$. Suppose $M = Rx_1 + \dots + Rx_{k+1}$ and let us write $A = Rx_1$ and $B = M/A$. By our assumption A and B are finite dimensional. Also, since $0 \rightarrow A \rightarrow M \rightarrow M/A \rightarrow 0$ is an exact sequence, Lemma 2.1 of KURSHAN [11] implies that M is also finite dimensional. The proof of the lemma is now complete.

COROLLARY. *If $T_0 \subseteq T_1 \subseteq \dots$; is any strictly increasing chain of submodules of a cyclic R -module C , then there exists an integer n such that $T_n \subseteq \sum T_i$.*

PROOF. Follows directly from the above lemma and Proposition 2.2 of KURSHAN [11].

LEMMA 3.2. *Let $(R_i)_{i \in I}$ be a family of rings and $R = \prod_{i \in I} R_i$ their direct product. If R is a qc-ring, then an R -module M will be finitely generated say by n elements (infinitely generated) if and only if it can be written $\prod_{i \in I} M_i$, where each M_i is an R_i -module generated by n elements (infinitely generated).*

PROOF. Since R is a qc-ring, the cardinality of I is finite ([1], Theorem 1), and now the result follows from Lemma 2.16 of BERGMAN [3].

LEMMA 3.3. *Let R be a qc-ring. Then $R = \bigoplus_{i=1}^n R_i$, where each R_i is either a semi-simple Artinian ring or a local qc- (duo) ring.*

PROOF. See Proposition 3.3 of KOEHLER [10].

The above lemmas together yield the following theorem.

THEOREM 3.4. *Let R be a qc-ring. Then R is an FGS-ring.*

Before we proceed further, we give an example to show that qc-rings need not always have non-zero socle.

Example 3.5. Let k be a field, x an indeterminate over k and

$$w = \{(x_i) | (x_i) \text{ is a well ordered sequence of positive reals}\}.$$

Let us form the ring

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^{\alpha_i} \mid a_i \in k \text{ and } (\alpha_i) \in w \right\}.$$

Let $R = T/xJ(T)$ where $J(T)$ denotes the Jacobson-radical of T . Then R is a commutative local hypercyclic ring with $S(R) \neq 0$ (see CALDWELL [4]). Since R is a commutative hypercyclic ring, R is qc (see AHSAN [2, Theorem 3.1]) and therefore $R/S(R)$ is a qc-ring which has zero socle.

COROLLARY. *Let R be a strongly qc-ring. Then each finitely generated R -module is a finite direct sum of quasi-injective modules.*

PROOF. Since R is a strongly qc-ring, R is uniserial (see AHSAN [1]). Hence R is a Köthe ring by FAITH [6] which means that each (right) R -module is a direct sum of cyclic submodules. Now each cyclic module over a qc-ring is quasi-injective and each finitely generated module over a qc-ring is finite dimensional by Theorem 3.4, hence the corollary follows.

We now study endomorphism rings of cyclic modules over qc-rings. We first prove the following lemma.

LEMMA 3.6. *Let R be a local qc-ring, M any cyclic R -module and $\Lambda = \text{Hom}_R(M, M)$. Then Λ is a qc-ring.*

PROOF. Since M is a cyclic R -module, $M \cong R/I$ where I is a right ideal of R . But R is a local qc- and hence a duo ring, therefore I is a duo-sided ideal of R . Hence $\Lambda = \text{Hom}_R(M, M) = \text{Hom}_R(R/I, R/I) = \text{Hom}_{R/I}(R/I, R/I) \cong R/I$. Now the fact that R is qc implies that R/I is a qc-ring by Lemma 2 of AHSAN [1]. Therefore Λ is a qc-ring.

The above lemma can be extended to obtain the following theorem.

THEOREM 3.7. *Let R be a qc-ring, M a cyclic R -module with $\Lambda = \text{Hom}_R(M, M)$. Then Λ is a qc-ring.*

PROOF. By Lemma 3.3, we can write $R = \bigoplus_{i=1}^n R_i$, where each R_i is either semi-simple Artinian or a local qc-ring. Therefore $M = \bigoplus_{i=1}^n M_i$, where M_i is a (cyclic) R_i -module. This implies that $\Lambda = \bigoplus_{i=1}^n \Lambda_i$, where $\Lambda_i = \text{Hom}_{R_i}(M_i, M_i)$. If R_i is semi-simple Artinian then it is easy to show that Λ_i is a qc-ring. Further if R_i is a local qc-ring then Λ_i is a qc-ring by the above lemma. Thus Λ is a finite direct sum of qc-rings. Therefore Λ is a qc-ring. Before we prove the next theorem, we set the following notation. If $\Lambda = \text{Hom}_R(M, M)$, where M is any R -module then we shall write $N(\Lambda) = \{\lambda \in \Lambda \mid \ker \lambda \subseteq M\}$.

THEOREM 3.8. *Let R be a qc-ring and M a finitely generated R -module. If $\Lambda = \text{Hom}_R(M, M)$ then $\Lambda/N(\Lambda)$ is embeddable in a semi-simple Artinian ring. Further, $\dim \Lambda/N(\Lambda) = \dim M$.*

PROOF. Since M is a finitely generated module over a qc-ring, M is finite dimensional by Theorem 3.4. The theorem now follows directly from Theorem 3 on page 310 of SHOCK [13].

Assuming that a finitely generated module over a qc-ring is also quasi-injective, we actually obtain the following

THEOREM 3.9. *Let R be a qc-ring and M a finitely generated R -module with $\Lambda = \text{Hom}_R(M, M)$. If M is quasi-injective then $\Lambda/J(\Lambda)$ is semi-simple Artinian where $J(\Lambda)$ is the Jacobson-radical of Λ .*

PROOF. Since M is finitely generated, M is finite dimensional by Theorem 3.4. Therefore Λ is finite dimensional. This means that $\Lambda/J(\Lambda)$ does not contain infinitely many orthonormal idempotents. Furthermore, since M is quasi-injective, $\Lambda/J(\Lambda)$ is regular in the sense of von Neumann by the Theorem on page 44 of FAITH [5]. Therefore $\Lambda/J(\Lambda)$ is semi-simple Artinian.

THEOREM 3.10. *Let R be a local strongly qc-ring and M an indecomposable quasi-injective R -module with $\Lambda = \text{Hom}_R(M, M)$. Then Λ is a local (duo) qc-ring.*

PROOF. Every quasi-injective module over a strongly qc-ring is quasi-projective. Therefore M is an indecomposable quasi-projective module over a local strongly qc-ring. Hence by Theorem on page 128 of FULLER [7], $M \cong R/L$ where L is a two sided ideal in R . Hence A is a local (duo) qc-ring.

We now state a result in the form of the following lemma which is due to MARES [12] and WARE [14]. These authors, in fact, proved the following result for projective modules but an examination of their proofs (MARES [12], Theorem 2.4 (i) and WARE [14], Proposition 1.1) shows that their arguments may be used to prove the result in the quasi-projective setting.

LEMMA 3.11. *Let R be any ring and M a quasi-projective R -module. If $A = \text{Hom}_R(M, M)$ then $J(A) = \{\lambda \in A \mid \text{Im } \lambda \text{ small in } M\}$; where $J(A)$ denotes the Jacobson-radical of A .*

THEOREM 3.12. *Let R be a qc-ring and M a cyclic R -module. If $A = \text{Hom}(M, M)$ then $A/J(A)$ is semi-simple Artinian and $J(A) = \{\lambda \in A \mid \ker \lambda \subseteq 'M\} = \{\lambda \in A \mid \text{Im } \lambda \text{ is small in } M\}$.*

PROOF. Since A is a qc-ring, $A/J(A)$ is semi-simple Artinian. The fact that $J(A) = \{\lambda \in A \mid \ker \lambda \subseteq 'M\}$ is well-known (see e.g. FAITH [5]). Further, since R is a qc-ring, it is a q^* -ring by KOEHLER [10, Corollary 3.4], M is also quasi-projective. Therefore by the above lemma, $J(A) = \{\lambda \in A \mid \text{Im } \lambda \text{ is small in } M\}$.

Finally, we prove a theorem which gives a module theoretic characterization of qc-rings. In order to do this, we first obtain the following lemma.

LEMMA 3.13. *Let R be a local qc-ring. Then each submodule of a cyclic R -module is quasi-injective.*

PROOF. Let M be any cyclic R -module. Then we can write $M \cong R/I$ where I is a (right) ideal of R . The lemma will follow if we show that each submodule of R/I is (R -) quasi-injective. Since R is a duo ring, I is a two sided ideal of R so that R/I is a qc-ring. This implies that R/I is a q -ring (Corollary 3.4, KOEHLER [10]). Now if A is a submodule of R/I then A is in fact a (right) ideal of R/I . Since R/I is a q -ring, A is R/I -quasi-injective. Then A is also (R -) quasi-injective by Lemma 1 of [1].

THEOREM 3.14. *Let R be any ring. Then the following statements are equivalent.*

- (1) R is a qc-ring.
- (2) Each submodule of a cyclic R -module is quasi-injective.

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RINGS ALL OF WHOSE IDEALS ARE FINITELY EMBEDDED

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1. Introduction P. VAMOS [8] introduced the notion of finitely embedded by dualizing the concept of finitely generated. An R -module M is said to be finitely embedded (f.e.) if $E(M) = E(P_1) \oplus \dots \oplus E(S_k)$ where $E(M)$ denotes the injective hull of M and each S_i is a simple R -module. It is well known that rings all of whose ideals are finitely generated are Noetherian Dually, VAMOS proved that a ring R is Artinian if and only if each factor ring of R is f.e. However, rings all of whose ideals are f.e. need not satisfy any chain condition. Rings with this property, in fact, properly contain the class of Artinian rings. The purpose of this paper is to prove that each ideal of a commutative local hypercyclic ring is f.e. Necessary and sufficient conditions on a finitely generated module over a commutative local hypercyclic ring to be f.e. will also be obtained.

2. Preliminaries We shall assume throughout this paper that all rings have an identity and all modules are unitary left modules. If R is a ring and M is an R -module then $E(M)$ will denote the injective hull of M and $S(M)$ its socle i.e. the sum of all simple submodules of M . The fact that A is an essential submodule of B will be denoted by $A \subseteq_e B$. A module M will be called 'finite dimensional' (Goldie) if it does not contain an infinite direct sum of non-zero submodules and the number of non-zero summands is called the 'dimension' of M denoted by $\dim M$.

A ring is said to be an 'FGS-ring' if each cyclic R -module has a finitely generated (or empty) socle. These rings have been studied by KURSHAN [8]. BEACHY [4] called a module M 'quasi-Artinian' if M contains an essential Artinian submodule. It can be shown that M is quasi-Artinian if and only if $S(M)$ is essential and finitely generated. Thus a ring R is 'quasi-Artinian' if and only if R has a finitely generated essential socle. In [5], CALDWELL called a ring R 'hypercyclic' if each R -module has a cyclic injective hull. CALDWELL proved that a perfect ring is hypercyclic if and only if it is uniserial. He has also shown, by means of an example, that even a commutative local hypercyclic ring need not satisfy any chain condition. A ring R is said to be a 'PF-ring' if and only if R is a semi-perfect (left) self-injective ring with essential socle. For various characterizations of these rings we refer to AZUMAYA [3] and UTUMI [9]. A ring R is called a (left) 'qc-ring' if each cyclic (left) R -module is quasi-injective.

3. Results We start with the following

THEOREM 3.1. *Let R be any ring. Then each ideal of R is f.e. if and only if R is quasi-Artinian.*

PROOF. If each ideal of R is f.e. then R_R is f.e. Therefore, by Lemma 1 of VAMOS [10], $S(R)$ is a finitely generated essential submodule of R_R . Hence R is a quasi-

Artinian ring. Conversely if R is a quasi-Artinian ring then R is f.e. Now for any (right) ideal I of R , the exactness of the sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ and the fact that R is f.e. imply that I is f.e. by Proposition 3* of VAMOS [10].

The following lemma is Theorem 3.1 of Ahsan [2].

LEMMA 3.2. *Let R be a commutative hypercyclic ring. Then R is qc.*

PROOF. In order to prove that R is a qc-ring, we need to show that each cyclic R -module is quasi-injective. Let A be a cyclic R -module with injective hull E . Since R is hypercyclic, E is cyclic. Let g be a generator of E , i.e. $E = gR$. Let $\theta \in A = \text{Hom}_R(E, E)$ and suppose $\theta(g) = gs$. Then if $a \in A$, we have $a = gr$ and $\theta(a) = \theta(g)r = gsr = grs = as$. Therefore $\theta(a) \in A$. Hence by JOHNSON and WONG [6, Theorem 1.1.], A is quasi-injective. Thus R is a qc-ring.

LEMMA 3.3. *Let R be a commutative qc-ring. Then R is an FGS-ring.*

PROOF. By proposition 2.2. of KURSHAN [8], the lemma will follow if we prove that each finitely generated R -module is finite dimensional.

Let us suppose that M is an R -module generated by n elements. We shall use the inductive argument to obtain the desired result. If $n=1$ then M is cyclic. This means that $M \cong R/I$ where I is an ideal of R . Since R is a qc-ring, so also is R/I (see Lemma 2 of [1]). Hence by [1], Theorem 1, R/I , as an R -module, is finite dimensional. This proves the result for $n=1$.

Let us now assume that the result is true for $n=k$ and prove that this is also true for $n=k+1$. Let $M = Rx_1 + \dots + Rx_{k+1}$.

Let us write $A = Rx_1$ and $B = M/A$. Then by our assumption, A and B are finite dimensional. Since $0 \rightarrow A \rightarrow M \rightarrow M/A \rightarrow 0$ is exact, Lemma 2.1 of KURSHAN [8] implies that M is also finite dimensional. This completes the inductive argument.

LEMMA 3.4. *If R is a commutative local hypercyclic ring, then $S(R) \subseteq R$.*

PROOF. See Theorem 2.10 of CALDWELL [5].

THEOREM 3.6. *Let R be a commutative local hypercyclic ring. Then each ideal of R is f.e.*

PROOF. Since R is a commutative local hypercyclic ring, R is qc by Lemma 3.2. Also, R is an FGS-ring by Lemma 3.3. Hence $S(R)$ is finitely generated (or empty). But Lemma 3.4. implies that $S(R) \subseteq R$. Hence R is f.e. by Lemma 1 of VAMOS [10]. Therefore each ideal of R is f.e.

We now prove the following theorem.

THEOREM 3.6. *Let R be a commutative hypercyclic ring. Then R is a PF-ring.*

PROOF. In order to prove that R is a PF-ring we show that R is a self-injective semi-perfect ring with an essential socle. Since R is a commutative hypercyclic ring, it follows from Lemma 3.2 that R is a qc-ring and so, in particular, R is a self-injective semi-perfect ring (see AHSAN [1]). Also, we can write $R = \bigoplus_{i=1}^n R_i$ where each R_i is a commutative local hypercyclic ring. Hence by Lemma 3.5, each R_i is a quasi-

Artinian ring. Since R is a finite direct sum of quasi-Artinian rings, R itself is a quasi-Artinian ring. Therefore $S(R)$ is finitely generated and essential in R . Hence R is a PF-ring.

VÁMOS [10], proved that a ring R is left Artinian if and only if each finitely generated R -module is f.e. In view of the fact that a commutative local hypercyclic ring need not be Artinian (see Example on page 42, CALDWELL [5]), we note that each finitely generated module over a commutative local hypercyclic ring need not be f.e. In form of the next theorem, we give a necessary and sufficient condition for a finitely generated module over a commutative local hypercyclic ring to be f.e. But first we state the following lemma which is due to CALDWELL [5].

LEMMA 3.7. *Let R be a commutative local hypercyclic ring. Then each submodule of a cyclic R -module is an essential submodule.*

THEOREM 3.8. *Let R be a commutative local hypercyclic ring and M a finitely generated R -module. Then M is f.e. if and only if the socle of each cyclic submodule of M is non-zero.*

PROOF. To prove the above theorem we shall again use the inductive argument.

Let M be an R -module generated by n elements. If $n=1$ then M is cyclic. Since R is a commutative hypercyclic ring, R is qc by Lemma 3.2. Therefore R is an FGS-ring by Lemma 3.3. Hence $S(M)$ is finitely generated (or empty). By our assumption $S(M) \neq 0$. Therefore $S(M) \subseteq M$ by Lemma 3.7. Thus M is f.e.

Let us now suppose that the result is true for $n=k$ and prove that the result is also true for $n=k+1$. Suppose $M = Rx_1 + \dots + Rx_{k+1}$ and let us write $A = Rx_1$ and $B = M/A$. Then by our assumption both A and B are f.e. Let us now consider the exact sequence

$$0 \rightarrow A \rightarrow M \rightarrow M/A \rightarrow 0.$$

Since A and M/A are f.e., M is f.e. by Proposition 3* of VÁMOS [10] and the inductive argument is complete.

Conversely if M is f.e. and A is any cyclic submodule of M then by Proposition 3* of VÁMOS [8], A is f.e. Therefore $S(A) \subseteq A$. This completes the proof of the theorem.

COROLLARY. *If M is a finitely generated R -module then M is f.e. if and only if the injective hull of each cyclic submodule of M is R itself.*

PROOF. This follows from the above theorem and theorem 2.12 of CALDWELL [5].

Finally, we note that a qc-ring need not be a quasi-Artinian ring, because such an assumption would imply that a qc-ring is Artinian which is not always the case (see example on p. 42 of [5]). A necessary and sufficient condition in order that each ideal of a qc-ring is f.e. may be formulated as follows:

THEOREM 3.10. *Let R be a qc-ring. Then the following statements are equivalent:*

- (1) R is a quasi-Artinian ring.
- (2) R is a PF-ring.

PROOF. Let us suppose that R is a *quasi-Artinian ring*. Since R is qc, R is self-injective and semi-perfect (see 1, theorem 1). Also because R is f.e., $S(R)$ is finitely generated and essential in R . This implies that R is a PF-ring.

Conversely if R is a PF-ring then R is always f.e. and hence a quasi-Artinian ring.

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THREE SPECIES OF CONNECTION IN COMPLEXES

by

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0. There are several natural generalizations of graph-theoretic connection to higher dimensions. One of these, (m, n) -connection, was used in [2] in characterizations of (m, n) -trees. Three other generalizations are studied in this paper and interesting relationships among these species are developed: although each generalizes a well-known graph-theoretic definition of connectivity, the three turn out to form a sequence of increasingly general concepts, at least in dimension two. One of the species of connection is equivalent to the condition that the k^{th} homology group of a complex be zero. The other two appear to be new and are more general than this.

1. Let K be a collection of subsets (called *simplexes*) of a set $V(K)$ of elements (called *vertices*). Then K is called a *complex* if

- (i) every subset of a simplex in K is a simplex in K .
- (ii) $\bigcup_{x \in K} x = V(K)$.

This definition is formally equivalent to the definition given in [5, p. 41] of an “abstract simplicial complex”. A simplex x of K is called an *n-simplex* if $|x| - 1 = n$. The collection of n -simplexes of K will be denoted by $K_{(n)}$. Two simplexes x and y are *incident* if $x \subset y$. The *n-degree* of an m -simplex x , $m < n$, is the cardinality of the set $\{y \in K_{(n)} : y \text{ is incident with } x\}$ and is denoted by $d^n(x)$. The complex K is said to have *dimension n* if it has at least one n -simplex but no $(n+1)$ -simplexes. Such a complex will be called here an *n-complex*. Obviously a graph is just an n -complex where $n \leq 1$.

Every complex by definition contains the empty set. A complex which contains only this set is called *null*. A complex K such that $V(K)$ consists of a single vertex is called *trivial*.

If K and L are complexes such that $L \subseteq K$, the L is called a *subcomplex* of K . For every vertex v in a complex K we define the subcomplex $\text{st}(v, K)$, called the *star of v in K*, as $\{x \in K : v \in x\}$ along with \emptyset the empty set. If S is a subset of a complex K , it is useful to define \bar{S} , the *closure of S*, as the smallest subcomplex of K containing S . This enables one to form a difference $K \dot{-} L$ of two complexes as $\bar{K} - \bar{L}$, always a subcomplex of K . This also enables one to define a boundary operator in set-theoretic terms: given an n -complex K , the $(n-1)$ -complex ∂K , called the *boundary of K*, is the closure of the set $\{x \in K_{(n-1)} : d^n(x) \equiv 1 \pmod{2}\}$. This definition is formally equivalent to the usual definition of the modulo 2 boundary of an n -chain [4, p. 235] and appears to be more useful in studying complexes from a purely combinatorial point of view. An n -complex is *pure* if every simplex is incident with some n -simplex of the complex. A pure n -complex Z having the property that $\partial Z = K_0^{-1}$ is called an *n-cycle*. The following concepts generalize those of “circuit” and “path” in

the theory of graphs: an n -circuit C is an n -complex satisfying two conditions:

- (i) C is an n -cycle
- (ii) C is minimal subject to (i).

Condition (ii) means that C contains no proper n -subcomplex satisfying condition (i). An n -path P is an n -complex which, for some fixed $(n-1)$ -circuit C satisfies the conditions

- (i) $\partial P = C$
- (ii) P is minimal subject to (i).

Thus, by condition (ii), P has no proper n -subcomplex P' such that $\partial P' = C$. Clearly a 0-circuit is a 0-complex having two vertices, while a 1-path is just the usual graph-theoretic concept [3, p. 13]. All graph-theoretic terms employed in this paper may be found in this last reference.

2. Three species of connection in dimension n are defined in this section.

The complex K will be called *hyperconnected in dimension n* if every $(n-1)$ -circuit in K is the boundary of some n -path in K . Clearly, to be hyperconnected in dimension 1 a 1-complex must be connected considered as a graph for in this case, it is required that every 0-circuit (pair of vertices) be the boundary of a 1-path (be connected by a path). It is easily proved that a complex K is hyperconnected in dimension n if and only if every $(n-1)$ -cycle in K is the boundary of some n -subcomplex of K . This condition, in turn, can readily be shown to be equivalent to requiring that the $(n-1)$ st mod 2 homology group of K vanish.

The next definition is inductive in nature and requires three preliminary ideas: let K be a complex. A subcomplex L of K is called a *cut* for K if there exist subcomplexes K_1 and K_2 of K such that

- (i) $K = K_1 \cup K_2$, $L = K_1 \cap K_2$ and
- (ii) every simplex of L is incident with a larger simplex in K_i , $i=1, 2$.

The union $K = K_1 \cup K_2$ is called an *L -decomposition* of K and K_1, K_2 are called *L -components* of the cut. Clearly a cut L for K may be specified by giving any such pair of L -components. The complex K will be called *section-connected in dimension 0* if $V(K) \neq \emptyset$. The subcomplex L of K is called *0-section for K* if L is a cut for K and if for some L -decomposition $K = K_1 \cup K_2$, the L -components K_i are section-connected in dimension 0, $i=1, 2$. The complex K is *section-connected in dimension n* if every $(n-1)$ -section of K is section connected in dimension $n-1$. It remains to complete this definition by defining an " n -section". First, an *n -component* of K is a subcomplex K' such that K' is section-connected in dimension n and such that K' is maximal subject to the first condition, i.e. K' is not a proper subcomplex of any subcomplex of K which is also section-connected in dimension n . Next, the subcomplex L of K is an *n -section** for K if K has an n -component K' such that

- (i) L is a cut for K' and
- (ii) for some L -decomposition $K' = K'_1 \cup K'_2$, the components K'_i are section-connected in dimension n , $i=1, 2$.

Before continuing to the third species of connection, the meaning of this last definition is briefly studied: when $n=0$ a complex being section-connected in dimension 0 means simply that it is not the null complex. A 0-section of a complex is simply a cut having at least one pair of non-null components. A complex is section-con-

* For some writers in Topology an " n -section" is what other writers mean by " n -skeleton".

nected in dimension 1 if every such cut is itself non-null. This is merely another way of saying that a complex K is connected if and only if there are not two non-null subcomplexes K_1 and K_2 such that $K=K_1 \cup K_2$ while $K_1 \cap K_2 = \emptyset$. Thus the definition of section-connectedness in dimension n not only generalizes what is meant by connectedness in a graph, but in a sense generalizes the standard definition for topological spaces [4, p. 14]. A 1-component of a complex is just a maximal connected subcomplex. A 1-section of a complex K is a cut L in some 1-component of the complex, a pair of whose components in some L -decomposition of K are both connected. Finally a complex K is section-connected in dimension 2 if all the 1-sections of K are section-connected in dimension 1, that is, connected in the ordinary sense.

A useful definition of connectedness in graphs employs partitions of its vertices into two non-empty sets. This definition has two rather natural generalizations: a complex K is *coherent in dimension n* if for every partition $V_1 + V_2 + \dots + V_{n+1}$ of $V(K)$ into $n+1$ non-empty sets, K has at least one n -simplex y which *spans* the partition, i.e. each set V_i contains at least one vertex in y . This condition imposes a heavy restriction on any complex satisfying it, for every n -subset of $V(K)$ must be present as an $(n-1)$ -simplex of K . Although this is true for any 1-complex, it is hardly true for complexes of dimension 2 or higher! A subsequent paper studies coherent complexes. The second generalization evades the restriction just encountered. A partition $\mathcal{V} = V_1 + V_2 + \dots + V_{k+1}$ of the vertices of a complex K into $k+1$ non-empty sets will be called a $(k+1)$ -*partition of $V(K)$* . Define $P_i(K, \mathcal{V})$ to be the number $|\{x \in K_{(k-1)} : x \text{ spans } V_1 + V_2 + \dots + \hat{V}_i + \dots + V_{k+1}\}|$. A $(k+1)$ -partition \mathcal{V} of $V(K)$ will be called *admissible** if there exists a $(k-1)$ -circuit C in K such that the parity of $P_i(C, \mathcal{V})$ is odd for each $i=1, 2, \dots, k+1$. At last, a complex K is *partition-connected in dimension n* if each admissible $(n+1)$ -partition of $V(K)$ is spanned by at least one n -simplex of K .

When $n=1$ an $(n+1)$ -partition $\mathcal{V} = V_1 + V_2$ is admissible since there exist vertices $v_1 \in V_1$ and $v_2 \in V_2$: the 0-complex consisting of these vertices alone is a 0-circuit such that $P_1(C, \mathcal{V})$ and $P_2(C, \mathcal{V})$ are both odd. Thus when $n=1$ the latter definition requires merely that every partition of $V(K)$ into two non-empty sets be spanned by at least one 1-simplex (edge) of K .

To summarize, three principal generalizations of connection in graphs to complexes have been given. Each particularizes to a unique definition of connection in graphs.

3. In the last definition met with in section 2 above, the condition that $P_i(C, \mathcal{V})$ be odd for each $i=1, 2, \dots, n+1$ is not so restrictive as it first may seem. The idea for the proof of the following results was suggested by Professor C. St. J. A. NASH-WILLIAMS.

THEOREM 1. *If \mathcal{V} is an $(n+1)$ -partition of $V(K)$ and C is an $(n-1)$ -circuit in K then the numbers $P_i(C, \mathcal{V})$, $i=1, 2, \dots, n+1$, are either all odd or all even.*

PROOF. Suppose that $P_i(C, \mathcal{V})$ is odd for some i with $1 \leq i \leq n+1$. Denote by X_0 the set of all $(n-1)$ -simplexes of C incident with exactly one $(n-2)$ -simplex

* This wording is motivated by the nearly equivalent concept of "admissible labelling" used in [6, p. 420].

spanning $V_1 + \dots + \hat{V}_i + \dots + \hat{V}_j + \dots + V_{n+1}$ where j is some integer different from i with $1 \leq j \leq n+1$. Denote by X_e the set of all $(n-1)$ -simplexes of C incident with none or two of such $(n-2)$ -simplexes. Since C is an $(n-1)$ -circuit, $d^{n-1}(z)$ is even for each $(n-2)$ -simplex z in C . Clearly $\sum d^{n-1}(z) = 2|x_e| + |X_0|$, the summation being taken over all $(n-2)$ -simplexes z of C which span $V_1 + \dots + \hat{V}_i + \dots + \hat{V}_j + \dots + V_{n+1}$. Hence $|X_0|$ is even. An $(n-1)$ -simplex lies in X_0 if and only if it spans either $V_1 + \dots + \hat{V}_i + \dots + V_j + \dots + V_{n+1}$ or $V_1 + \dots + V_i + \dots + \hat{V}_j + \dots + V_{n+1}$. Thus the numbers

spanning each n -partition have the same parity and $P(C, \mathcal{V}) \equiv P_j(C, \mathcal{V}) \pmod{2}$. This equality holds for each $j=1, 2, \dots, n+1$ and the lemma is proved.

The condition that $P_i(C, \mathcal{V})$ be odd for each $i=1, 2, \dots, n+1$ is known to be satisfied the moment it is known that at least one of these numbers is odd.

The diagram shown in Figure 1 below summarizes the results presented in this section. Each implication arrow between one kind of connection and another represents a theorem that the one kind of connection actually implies the other. The number appearing near the arrow indicates the dimension in

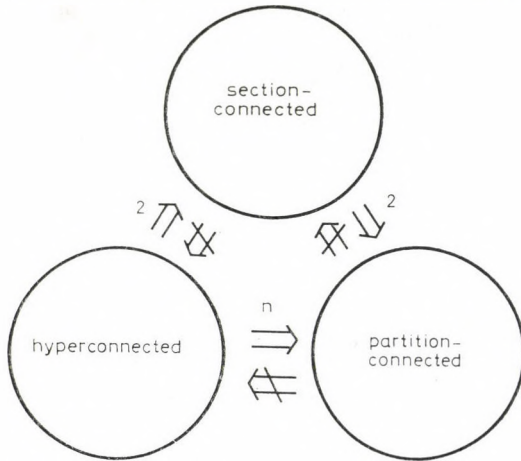


Figure 1. Structure of theory in Section 3

which the implication is proven. An implication arrow with a stroke through it indicates a counterexample to the corresponding implication.

THEOREM 2. *If a complex K is hyperconnected in dimension n , then K is partition-connected in dimension n .*

PROOF. Let K be hyperconnected in dimension n and let $\mathcal{V} = V_1 + V_2 + \dots + V_{n+1}$ be an admissible $(n+1)$ -partition of $V(K)$. Then there is an $(n-1)$ -circuit C in K such that $P_i(C, \mathcal{V})$ is odd for each $i=1, 2, \dots, n+1$.

Since K is hyperconnected in dimension n there is an n -path P in K such that $\partial P = C$. Denote by Y the set of n -simplexes of P which span \mathcal{V} , and by Y_i the set of n -simplexes of P which span $V_1 + \dots + \hat{V}_i + \dots + V_{n+1}$ but not \mathcal{V} , for some i such that $1 \leq i \leq n+1$. Let X_i denote the set of $(n-1)$ -simplexes of L which span $V_1 + \dots + \hat{V}_i + \dots + V_{n+1}$. Then

$$(i) \quad \sum_{x \in X_i} d_p^n(x) = 2|Y_i| + |Y|,$$

where d_p^n denotes the k -degree in P of the $(n-1)$ -simplex x . This equality follows from the fact that each n -simplex in Y_i has exactly two incident $(n-1)$ -simplexes in X_i while each n -simplex in Y has exactly one.

Let X_i be split into two subsets, X_i' consisting of the $(n-1)$ -simplexes x such that $d_p^n(x)$ is odd and X_i'' consisting of the $(n-1)$ -simplexes x such that $d_p^n(x)$ is even.

Then by (i)

$$(ii) \quad \sum_{x \in X_i} d_p^n(x) + \sum_{x \in X_i'} d_p^n(x) = 2|Y_i| + |Y|.$$

The $(n-1)$ -simplexes in X_i for which $d_p^n(x)$ is odd are precisely those lying in ∂P and spanning $V_1 + \dots + \hat{V}_i + \dots + V_{n+1}$. Thus $|X_i'| = P_i(C, \mathcal{V})$ and, since $P_i(C, \mathcal{V})$ is odd, so is the sum $\sum_{x \in X_i'} d_p^n(x)$. By equation (ii) above, this sum has the same parity as $|Y|$ whence $|Y|$ is odd and therefore Y contains at least one n -simplex, yielding the theorem.

Those familiar with Sperners Lemma will see in the above proof a close resemblance to the argument given in [6, p. 416]. (For an unrelated but interesting inductive argument, see COHEN [1].) The chief difference between Theorem 2 and Sperners Lemma is the greater generality of our result: Sperners Lemma requires an admissible labelling for the boundary of a triangulated (geometric) n -simplex, whereas our theorem requires an admissible labelling for the boundary of a triangulated pseudo- n -manifold.

The next result shows that in dimension 2 being section-connected is at least as general a property as being hyperconnected.

THEOREM 3. *If K is hyperconnected in dimension 2, then K is section-connected in dimension 2.*

PROOF. Let K be hyperconnected in dimension 2 and suppose that K is not section-connected in dimension 2. Then K has a 1-section L which is not connected. Let $K' = K'_1 \cup K'_2$ be an L -decomposition of a component K' of K such that K'_1 and K'_2 are both connected. Since L is not connected it has at least two components: denote one of these components by L_1 and let $L_2 = L - L_1$. Since K'_1 and K'_2 are both connected, any two vertices $u_i \in L_i, i = 1, 2$, are joined by $(0, 1)$ -paths P_1 and P_2 in K'_1 and K'_2 respectively.

Let Q_1 be a path of minimum length in K_1 joining a vertex of L_1 with a vertex of L_2 . Let the vertices so joined be $v_1 \in V(L_1)$ and $v_2 \in V(L_2)$. One observes that there exists a $(0, 1)$ -path Q_2 in K_2 which also joins v_1 to v_2 . In fact any such path has only the vertices v_1 and v_2 in common with Q_1 : for if Q_2 has an interior vertex v in common with Q_1 , then v is an interior vertex of Q_1 and hence lies in L . If $v \in V(L_1)$, then the portion of Q_1 from v to v_2 is shorter than Q_1 contradicting how Q_1 was chosen. Thus $v \notin V(L_1)$. Similarly $v \notin V(L_2)$. But then $v \notin V(L)$ and this last contradiction establishes the statement about Q_2 .

The union $Q_1 \cup Q_2$ is therefore a 1-circuit C . Since K is hyperco-

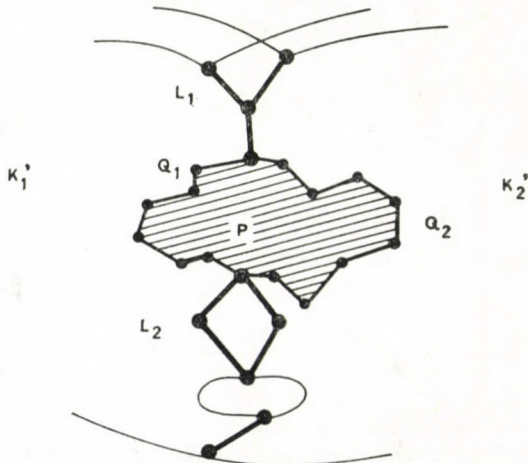


Figure 2. The 2-path P with boundary $C = Q_1 \cup Q_2$

nected in dimension 2 there is a 2-path P in K such that $\partial P = C$. Figure 2 below displays this situation somewhat schematically.

The complex L is represented by the heavy edges and large vertices. By the way Q_1 was chosen, none of its 1-simplexes lie in K'_2 . Thus any 2-simplex in P incident with a 1-simplex of Q_1 must lie in K'_1 . Therefore $Q_1 \subseteq Z$, where $Z = \partial(K'_1 \cap P)$. Since Z is a 1-cycle, there is a 1-path P_1 in Z having no 1-simplexes in common with Q_1 , but having the same end-vertices, namely v_1 and v_2 . Let x be an arbitrary 1-simplex of P_1 . Since x is incident with an odd number of 2-simplexes in $K_1 \cap P$, $x \in K'_1$. If $x \in K'_2$, then, of course, $x \in K'_1 \cap K'_2 = L$. If $x \notin K'_2$, then $\text{st}(x, P) = \text{st}(x, K'_1 \cap P)$, whence $x \in \partial P$. In this case $x \in Q_2 \subseteq K'_2$, contradicting the assumption that $x \notin K'_2$. We conclude that $P_1 \subseteq L$. But this is also a contradiction since v_1 and v_2 are both end-vertices of P_1 and lie in different components of L . Therefore the 2-path P cannot exist. This contradicts the fact that K is hyperconnected in dimension 2. Therefore K is section-connected in dimension 2.

The concept of an n -section as defined here does not seem to occur in the topological literature. In topological language Theorem 3, along with the next theorem, gives necessary conditions for an abstract simplicial 2-complex to be homologically trivial (all its homology groups vanish).

The final implication indicated in Figure 1 is now proved.

THEOREM 4. *If K is section-connected in dimension 2, then K is partition-connected in dimension 2.*

PROOF. Let K be a complex which is section-connected in dimension 2 and suppose that $\mathcal{V} = V_1 + V_2 + V_3$ is an admissible partition for $V(K)$ with C a 1-circuit in K such that $P_i(C, \mathcal{V})$ is odd, $i=1, 2, 3$. Denote by $M(\mathcal{V})$ the smallest number of 1-simplexes in K spanning any of the pairs $V_i + V_j$, $i \neq j$, $i, j=1, 2, 3$. Assume further that the theorem is not true and that \mathcal{V} is a 3-partition with the smallest value of $M(\mathcal{V})$ for which the theorem fails, i.e. no 2-simplex of K spans \mathcal{V} .

Suppose, without loss of generality, that the number of 1-simplexes in K spanning $V_1 + V_2$ is $M(\mathcal{V})$. Define K_1 to be the closure in K of the set of all simplexes spanning $V_1 + V_2$ as shown in Figure 3 below:

It will be shown first that K_1 is connected: If K_1 is not connected and if every component of K_1 contains an even number of 1-simplexes from C spanning $V_1 + V_2$,

then so would K_1 , implying that $P_3(C, \mathcal{V})$ is even. Therefore at least one component of K_1 , say K'_1 , has an odd number of 1-simplexes from C spanning $V_1 + V_2$. Alter the partition \mathcal{V} as follows: let $V'_1 = V_1 \cup V(K'_1)$, $V'_2 = V_2$ and let $V'_3 = V_3 \cup (V_1 \cap V(K_1 - K'_1))$. If the resulting partition \mathcal{V}' is spanned by a 2-simplex $y = \{v'_1, v'_2, v'_3\}$, then $v'_1 \in V(K_1)$, $v'_2 \in V_2$ and $v'_3 \in V(K_1 - K'_1)$. But in this case y spans $V_1 + V_2$ and has the 0-simplex $\{v'_1\}$ in common with K'_1 , whence $y \in K'_1$. This implies that $v'_3 \in V(K'_1)$, a contradiction with the fact that $v'_3 \in V(K_1 - K'_1)$. Therefore \mathcal{V}' is

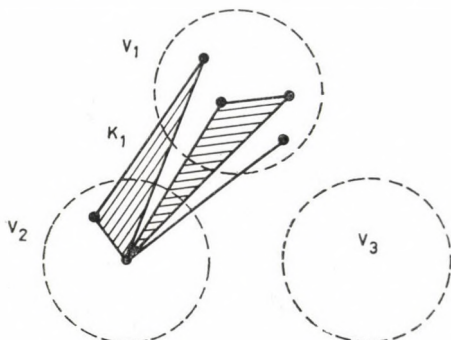


Figure 3. Constructing a 1-section

not spanned by any 2-simplex of K . The number $P_3(C, \mathcal{V}')$ is odd by the definition of K'_1 and by Theorem 5.1. the numbers $P_1(C, \mathcal{V}')$ and $P_2(C, \mathcal{V}')$ are also odd. Thus \mathcal{V}' is an admissible 3-partition of $V(K)$ such that $m(\mathcal{V}') < M(\mathcal{V})$, contradicting the original choice of \mathcal{V} . Therefore K_1 is connected.

Now let K' be the component of K containing K_1 . Clearly, at least one component K'_2 of $K' \setminus K_1$ has an odd number of 1-simplexes from C spanning $V_1 + V_3$. A simple parity argument shows that some portion of C connects a vertex in $V_3 \cap V(K'_2)$ with a vertex in $V_2 \cap V(K'_1)$ without passing through V_1 . In other words, $K'_2 \cap K_1$ has vertices in both V_1 and V_2 . Denote by K''_2 the union of all components of $K' \setminus K_1$ other than K'_2 (if any) and let $K'_1 = K_1 \cup K''_2$. Since K''_2 consists of components of $K' \setminus K_1$, each component has a vertex in common with K_1 . Therefore K'_1 is connected. Observe that $K' = K'_1 \cup K'_2$. It is verified that $L = K'_1 \cap K'_2$ is a cut for K' as follows: let x be a simplex of L . Then $x \notin K'_2$ since K'_2 has no simplexes in common with K'_1 . Therefore $x \in K'_1 \cap K'_2$ and x does not span $V_1 + V_2$ but is incident with such a simplex by the definition of K_1 . Since $x \in K' \setminus K_1$, it follows that x is incident with a larger simplex of $K' \setminus K_1$ and such a simplex must lie in K'_2 since K'_2 is connected. Therefore L satisfies conditions (i) and (ii) in the definition of a cut for K' .

Since K' , K'_1 and K'_2 are connected and since L is a cut for K' , L is a 1-section for K' and must be connected since K is section-connected. On the other hand, $L \supseteq K'_2 \cap K_1$ and L has vertices in both V_1 and V_2 . This implies that L is not connected since every simplex of L is contained either in V_1 or in V_2 . This contradiction shows that K cannot have been section-connected in dimension 2 to begin with, establishing the theorem.

There follow now the counterexamples indicated in Figure 1.

The complex KP shown in Figure 4 below represents a triangulation of the projective plane.

The two vertices labelled u in the figure above represent the same vertex in KP , as do the two vertices labelled v and the two vertices labelled w . Accordingly the two 1-simplexes $\{u, v\}$ in the above figure represent the same 1-simplex of KP and so forth. The complex KP supplies two counterexamples simultaneously.

THEOREM 5. *The complex KP is both partition-connected and section connected in dimension 2 but not hyperconnected in dimension 2.*

PROOF. KP is not hyperconnected in dimension 2 since the 1-circuit whose 1-simplexes are $\{u, v\}$, $\{v, w\}$, $\{w, u\}$ is not the boundary of any 2-subcomplex of KP .

To see that KP is section-connected in dimension 2 assume that some 1-section L for KP is not connected. Let $KP_1 \cup KP_2 = KP$ be an L -decomposition of KP where KP_1 and KP_2 are both connected. Any 1-simplex x in L is incident with a 2-simplex

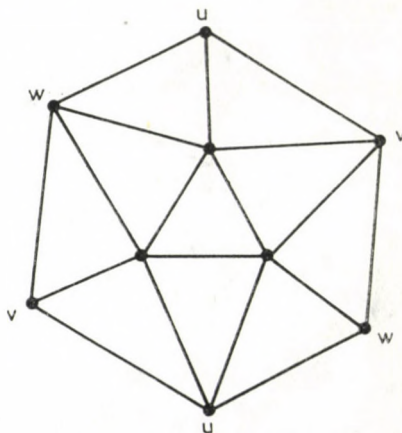


Figure 4. A triangulation of the projective plane

in each of KP_1 and KP_2 by condition (ii) in the definition for cut. Since every 1-simplex in KP is incident with exactly two 2-simplexes, $x \in \partial KP_1, \partial KP_2$. On the other hand if $x \in \partial KP_1$ then x is incident with exactly one 2-simplex in KP_1 and must also be incident with precisely one 2-simplex in KP_2 whence $x \in L$. We have shown that $L = \partial KP_1$. Similarly $L = \partial KP_2$. It follows that L is a 1-cycle. Since L is not connected L must consist of at least two disjoint 1-circuits C_1 and C_2 . Since KP has only 6 vertices, both 1-circuits must be triangles. Each 1-circuit is either a boundary $\partial \bar{y}$ where y is a 2-simplex of KP or else is a 1-circuit like C in the first paragraph of this proof. In fact both kinds of 1-circuits must be present, as shown in Figure 5 below.

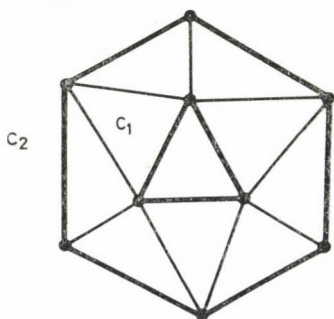


Figure 5. Possibilities for the 1-circuits in L

It is clear from the figure that the 1-simplexes of C_2 are all incident with 2-simplexes from exactly one of KP_1 or KP_2 . This contradiction shows that KP is section-connected in dimension 2. By Theorem 4 KP is also partition-connected in dimension 2.

It has been shown in dimension 2 at least that being hyperconnected is definitely less general a property than being either section-connected or partition-connected. To show that being partition-connected in dimension 2 is definitely more general than the other two properties an additional counterexample is required.

The 2-complex KD whose 1-skeleton is shown in Figure 6a below has been "pulled apart" and displayed in Figure 6b to show its 2-simplexes more clearly. These are listed to the right of the figure as a precaution against any misinterpretation.

THEOREM 6. *The 2-complex KD is partition-connected in dimension 2 but not section-connected in dimension 2.*

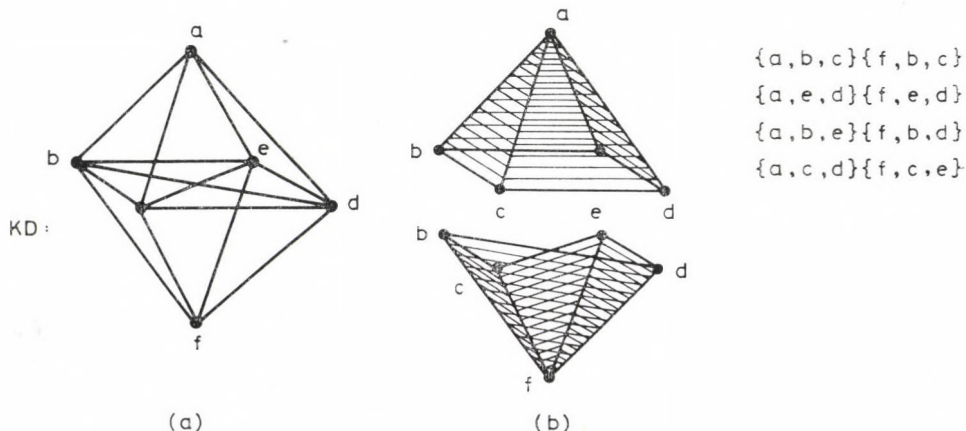


Figure 6. A complex which is partition-connected in dimension 2 but not section-connected in dimension 2

PROOF. The complex KD is not section connected in dimension 2: let $K_1 = \text{st}(a, KD)$, $K_2 = \text{st}(f, KD)$. Observe that $L = K_1 \cap K_2$ is a 1-section of KD and that $KD = K_1 \cup K_2$ is an L -decomposition of KD . Both K_1 and K_2 are connected but L is not connected. Thus KD is not section-connected in dimension 2.

We will prove that KD is partition-connected in dimension 2. Let $\mathcal{V} = V_1 + V_2 + V_3$ be an arbitrary admissible partition for KD . The argument is divided into two cases.

Case 1: The vertices a and f lie in the same set V_i : Without loss of generality a and f may be assumed to lie in V_1 . At least one of the vertices b, c, d or e must lie in V_2 and another in V_3 . Therefore at least one of the 6 1-simplexes containing these vertices spans $V_2 + V_3$. There is a 2-simplex incident with such a 1-simplex and containing either a or f . This 2-simplex spans $V_1 + V_2 + V_3$.

Case 2: The vertices a and f lie in different sets V_i : Without loss of generality a may be assumed to lie in V_1 and f in V_2 . Since $V_3 \neq \emptyset$, one of b, c, d, e must belong to V_3 and the symmetry of KD allows us to assume that $b \in V_3$. If $c \in V_1$ then $\{f, b, c\}$ spans \mathcal{V} and if $c \in V_2$ then $\{a, b, c\}$ spans \mathcal{V} : hence we may assume that $c \in V_3$. In this case $\{f, c, e\}$ spans \mathcal{V} if $e \in V_1$, $\{a, b, e\}$ spans \mathcal{V} if $e \in V_2$, $\{f, b, d\}$ spans \mathcal{V} if $d \in V_1$ and $\{a, c, d\}$ spans \mathcal{V} if $d \in V_2$: hence we may assume that $d, e \in V_3$.

Since no 1-simplex of KD spans $V_1 + V_2$, every 1-circuit C in KD has an even number of 1-simplexes spanning $V_1 + V_2$, whence the partition \mathcal{V} is not admissible. We have thus shown that all admissible partitions of $V(KD)$ are spanned by a 2-simplex of KD . Therefore KD is partition-connected in dimension 2.

4. A set of three generalizations of graph-theoretic connection has been given. These have been proved to form a succession of increasingly general concepts in dimension 2, at least. Although it is known (Theorem 2) that partition-connectedness is always at least as general a concept as hyperconnectedness in dimensions higher than 2, it is not known whether it is, in fact, strictly more general. It is also an open question whether section-connectedness is a concept of strictly intermediate generality in dimensions greater than 2.

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WAITING FOR CLUSTERS IN INHIBITED RENEWAL PROCESSES

by

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1. Introduction

We will in this paper work with the interaction of two random point processes such that points in a renewal point process, the *excitatory process*, are eliminated by points in an independent *inhibitory process*. Such a thinning situation was suggested by M. TEN HOOPEN and H. A. REUVER in [3] as a mathematical model for neuron firing. In [1] it is discussed how this thinning procedure applies to certain stochastic service systems. Various aspects of this kind of interaction of renewal processes is treated at length in [2]. Our interest in this paper is in the *response process* of those retained points of the excitatory process which complete a *cluster* of retained points, which are near each other in some sense. We will first consider clusters of k successive retained points with no eliminated points between them. Such clusters were considered in [2] for some simple elimination schemes. We will here extend and modify the treatment in [2]. We will also prove a limit theorem for $k \rightarrow \infty$. We will furthermore generalize to clusters of k retained points, such that at most r points have been eliminated between any two successive points. Finally, we will mention some applications.

2. Waiting for a Cluster

We assume that the excitatory process is a discrete time renewal process. Let r^g be the geometric transform (or probability generating function) of its inter-arrival distribution, $r^g(s) = \sum_{k=1}^{\infty} r_k s^k$. Furthermore we assume that the inhibitory process is a q -binomial process, independent of the excitatory process: that is, at each discrete time epoch a point of the inhibitory process is realized with probability $q = 1 - \varphi$, independently of what happens at other time epochs.

The elimination takes place according to the following scheme. Assume that the points of the excitatory process are realized at the times u_1, u_2, \dots . If one or more inhibitory points arrive at times $1, \dots, u_1$, the next j excitatory points starting with the one at u_1 will be eliminated with probability p_j , where $j \in \{0, 1, \dots\}$. Note that p_0 is the probability that the inhibitors have no elimination effect. For $n \geq 2$, assume that the point at u_n has not been eliminated by inhibitory points arriving at times $1, \dots, u_{n-1}$. If one or more inhibitory points arrive at times $u_{n-1} + 1, \dots, u_n$, the next j excitatory points starting with the one at u_n will be eliminated with probability p_j . A point of the inhibitory process has no elimination effect if a simultaneous or subsequent excitatory point already has been eliminated by previous inhibitors.

We are in this section interested in the response process consisting of those points of the excitatory process, which satisfy the following condition. Each such point is the last point in a sequence of k successive points of the excitatory process, none of which has been eliminated and of which only the last one is a point of the response process. Here k is a fixed positive integer.

Let p_k^g be the geometric transform of the inter-arrival distribution of this embedded renewal process. In order to derive an expression for $p_k^g(s)$ we will use the collective marks (or the additional event) method. Thus we add a marking process to the situation such that each time epoch is marked independently with probability $1-s$, and not marked with probability s . This implies that $r^g(s)$ is the probability that no marks are given during the excitatory interval.

We now define probabilities $d_0(s)$ and $d_1(s)$ as follows:

$d_0(s)$ = the probability that no marks are given during the first excitatory interval $[1, u_1]$ and that the point at the end of this interval is not eliminated,

$d_1(s)$ = the probability that the inhibitors arriving during the first excitatory interval eliminate one or more consecutive points of the excitatory process, and that up to and including the last elimination no marks have been given.

We now get by proper conditioning

$$(2.1) \quad d_0(s) = \sum_{k=1}^{\infty} r_k s^k (\varphi^k + (1 - \varphi^k) p_0) = r^g(\varphi s) + p_0(r^g(s) - r^g(\varphi s)).$$

Let furthermore $d_{1,j}(s)$ be the probability that no marks are given during the first excitatory interval and that the next j points are eliminated. (Time epochs after the first excitatory interval may or may not be marked.) Then $d_1(s) = \sum_{j=1}^{\infty} d_{1,j}(s) r^g(s)^{j-1}$.

We have for $j \in \{1, 2, \dots\}$

$$d_{1,j}(s) = \sum_{k=1}^{\infty} r_k s^k (1 - \varphi^k) p_j = p_j(r^g(s) - r^g(\varphi s)),$$

from which it follows with $p^g(s) = \sum_{j=0}^{\infty} p_j s^j$

$$(2.2) \quad d_1(s) = r^g(s)^{-1} (r^g(s) - r^g(\varphi s)) (p^g(r^g(s)) - p_0).$$

If we interpret $p_k^g(s)$ as the probability of no marks during the waiting time for the first point of the response process, we get by conditioning

$$p_k^g(s) = d_0(s)^k + \sum_{i=0}^{k-1} d_0(s)^i d_1(s) p_k^g(s),$$

which gives

$$p_k^g(s) = \frac{d_0(s)^k (1 - d_0(s))}{1 - d_0(s) - d_1(s) (1 - d_0(s)^k)}.$$

This result can be written as

$$(2.3) \quad 1 - p_k^g(s) = (1 - d(s)) \cdot \frac{1 - d_0(s)^k}{1 - d(s) + d_1(s)d_0(s)^k},$$

where $d(s) = d_0(s) + d_1(s)$. By dividing with $1 - s$ and letting $s \rightarrow 1$ we get an expression for the expectation $\mu_k = p_k^{g'}(1)$. With

$$\mu = r^{g'}(1), \quad a = 1 - r^g(\varphi), \quad b = r^g(\varphi)$$

and

$$\alpha = \sum_{k=1}^{\infty} k p_k$$

we get

$$(2.4) \quad \mu_k = \mu(b + a(\alpha + p_0)) \cdot \frac{1 - (b + p_0 a)^k}{a(1 - p_0)(b + p_0 a)^k}.$$

3. A limit theorem

In the previous section we have derived the probability distribution of the time intervals of the renewal response process. Let η_k be a random variable with this distribution.

THEOREM 1. *If $p_0 = 0$, then*

$$\lim_{k \rightarrow \infty} P(\eta_k / \mu_k \leq t) = 1 - e^{-t}, \quad t \geq 0.$$

PROOF. We observe that $\mu_k = \mu(b + \alpha s)(b^{-k} - 1)/a = O(b^{-k})$. We let s depend on k in such a way that $s = e^{-wb^k}$ for $w \geq 0$. We then get $\lim_{k \rightarrow \infty} d_0(s) = r^g(\varphi) = b$, $\lim_{k \rightarrow \infty} d_1(s) = 1 - b = a$. We can furthermore replace $d_0(s)^k = r^g(\varphi s)^k$ with b^k . This is justified as follows. Taylor expansions give

$$r^g(\varphi s) = b + (s - 1)O(s), \quad (s \rightarrow 1)$$

and

$$s - 1 = e^{-wb^k} - 1 = -wb^k O(k), \quad (k \rightarrow \infty).$$

Thus

$$(r^g(\varphi s)/b)^k = (1 - wb^{k-1} O(k))^k = \left(1 + O\left(\frac{1}{k}\right)\right)^k \rightarrow 1, \quad (k \rightarrow \infty).$$

We rewrite (2.3) as

$$p_k^g(s) = \frac{1 - d_0(s)}{\frac{1 - d(s)}{d_0(s)^k} + d_1(s)}.$$

Since

$$\lim_{k \rightarrow \infty} \frac{1 - d(s)}{d_0(s)^k} = \lim_{k \rightarrow \infty} \frac{1 - d(s)}{1 - s} \cdot \frac{1 - e^{-wb^k}}{b^k} = d'(1)w = \mu(b + \alpha a)w,$$

we find that

$$\lim_{k \rightarrow \infty} p_k^g(e^{-wb^k}) = \frac{1-b}{1-b+\mu(b+\alpha a)w} = \frac{1}{1+\mu \frac{b+\alpha a}{a} w}.$$

It follows that the Laplace—Stieltjes transform of η_k/μ_k for $k \rightarrow \infty$ tends to $\frac{1}{1+w}$ and thus the theorem is proved.

4. A generalization

We will in this section consider an elimination scheme such that $p^g(s)=s$, that is, one or more consecutive points of the inhibitory process eliminate only the next point of the excitatory process. We are interested in a response process, which consists of those points of the excitatory process, that satisfy the following condition. The point is the last point in a sequence of k successive retained points such that between any two consecutive points at most r points of the excitatory process have been eliminated and such that none is already a point of the response process. The case studied in previous sections corresponds to $r=0$.

Let $p_{k,r}^g$ be the geometric transform of the interarrival distribution of this response process. We will derive an expression for $p_{k,r}^g(s)$, interpreted as the probability for no marking during the response interval. For that purpose we define $h_0(s)$ to be the probability that counting from a retained point at most r points are eliminated before the next retained point and that no marks are given in the interval between the two successive retained points. Then

$$h_0(s) = \sum_{i=0}^r [r^g(s) - r^g(\varphi s)]^i r^g(\varphi s) = \frac{1 - [r^g(s) - r^g(\varphi s)]^{r+1}}{1 - r^g(s) + r^g(\varphi s)} r^g(\varphi s).$$

Let furthermore $h_1(s)$ be the probability that at least $(r+1)$ points are eliminated between two retained points and no marks given. Then

$$h_1(s) = [r^g(s) - r^g(\varphi s)]^{r+1}.$$

By proper conditioning we obtain

$$p_{k,r}^g(s) = r^g(\varphi s) h_0(s)^{k-1} + [r^g(s) - r^g(\varphi s)] p_{k,r}^g(s) + \sum_{i=0}^{k-2} r^g(\varphi s) h_1(s) h_0(s)^i,$$

which gives

$$(4.1) \quad p_{k,r}^g(s) = \frac{r^g(\varphi s) h_0(s)^{k-1} [1 - h_0(s)]}{1 - r^g(s) + r^g(\varphi s) [r^g(s) - r^g(\varphi s)]^{r+1} h_0(s)^{k-1}}.$$

From this follows

$$(4.2) \quad 1 - p_{k,r}^g(s) = \\ = \frac{1 - r^g(s)}{1 - r^g(s) + r^g(\varphi s) h_1(s) h_0(s)^{k-1}} \left[1 + r^g(\varphi s) h_0(s)^{k-1} \frac{[r^g(s) - r^g(\varphi s)]^{r+1} - 1}{1 - r^g(s) + r^g(\varphi s)} \right],$$

which easily gives the following formula for the corresponding expectation $\mu_{k,r}$.

$$(4.3) \quad \frac{\mu_{k,r}}{\mu} = \frac{1 - c^k}{b a^{r+1} c^{k-1}},$$

where

$$a = 1 - r^g(\varphi), \quad b = r^g(\varphi) \quad \text{and} \quad c = 1 - a^{r+1}.$$

It follows from (4.3) that

$$\lim_{r \rightarrow \infty} p_{k,r}^g(s) = \left[\frac{r^g(\varphi s)}{1 - r^g(s) + r^g(\varphi s)} \right]^k = p_r^g(s)^k,$$

where p_r^g is the geometric transform of the length of the time interval between successive retained points in this case. Let $\eta_{k,r}$ be a random variable with geometric transform $p_{k,r}^g$. We can then, with the aid of (4.1), prove the following theorem. The proof is similar to the proof of theorem 1.

THEOREM 2.

$$\lim_{k \rightarrow \infty} p(\eta_{k,r}/\mu_{k,r} \leq t) = 1 - e^{-t}, \quad t \geq 0.$$

5. Some concluding remarks

We have here considered the interaction between two discrete time point processes. It is easy, by a proper limit process to derive from our results formulas for the case when an inhibitory Poisson process with intensity λ interacts with a general renewal point process with inter-arrival distribution F .

Our results can be applied to the neuron firing process as follows. Stimuli arrive to a neuron and some of them are eliminated by an inhibitory process. The neuron will fire when it receives k stimuli, which are near each other, for instance between any two successive retained stimuli at most r stimuli have been deleted.

It is shown in [2] how results concerning the interaction of a renewal excitatory process and a binomial inhibitory process can be applied to a B/G/1 conveyor system, where customers arrive according to a binomial process and the service times have a general discrete probability distribution determined by the geometric transform r^g . Furthermore there is a finite waiting room and when it is empty the server will take a unit from a storage. Now assume that there is place for k units in this storage and that it is refilled in the following way. Each time the server has taken more than r units in a row from the conveyer a mechanism is released so that the storage is filled up to its full capacity. This implies that the only way the storage

can be emptied is that the server takes k units from the storage and that between any two of these units he has taken at most r units from the conveyor. But then $p_{k,r}^q$ in section 4 above is the geometric transform of the length of time from an epoch when the storage is full and the service of a unit is to begin until it is empty for the first time.

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BUSY CYCLE OF THE GI/M/1 QUEUE WITH FINITE WAITING ROOM

by

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1. Introduction

Consider the GI/M/1 queue with room for n customers ($n-1$ waiting places). Customers arriving when the system is full are lost. Let F be the inter-arrival time distribution function, β be the parameter of the service time distribution (mean service time $1/\beta$). Assume that

$$\mu = \int_0^{\infty} t dF(t) < \infty.$$

First it will be derived an expression for the Laplace—Stieltjes transform of the busy cycle, using the method of embedded Markov chain. This expression is employed to prove that the busy cycle, after norming to make the expectation unity, is, in the limit, exponentially distributed when β is small. Then we will use relations established between the determinants used here and determinants that are used in [3] in connection with the Laplace—Stieltjes transform of the busy period for an M/G/1 queue with finite waiting room and service time distribution F and intensity of arrival β . These relations are used to derive the Laplace—Stieltjes transform of the busy cycle for the GI/M/1 queue with infinite waiting room and to derive an expression for a generating function such that the transform for the finite waiting room case can be expressed in a simple way in terms of its coefficients. To derive this generating function we also use a result by COHEN [2], p. 826 (the expression for $b(p, \rho)$). For $s=0$ the generating function can also be obtained using a result by TOMKO [4], p 448, eq. (2). For this part most of the credit is of course due to the predecessors.

2. Basic formulas

Let us start with introducing the following quantities:

σ_i , ($i \geq 1$) arrival time of the i th customer;

$\gamma(t)$, ($t \geq 0$) number of customers present in the system at t ;

$$\gamma_i = \gamma(\sigma_i + 0), \quad (i \geq 1);$$

$\eta(t)$, ($t \geq 0$) waiting time, measured from t , for the first instant again a customer enters an empty system;

$$\eta_i = \eta(\sigma_i + 0), \quad (i \geq 1).$$

In what follows we shall be interested in describing the conditional probabilities

$$H_{n,k}(t) = P(\eta_i > t | \gamma_i = k).$$

The Markov chain $\{\gamma_i\}_1^\infty$ is finite and recurrent, so that the state 1 is visited infinitely many times. It follows that $1 - H_{n,k}$ is a proper d.f. Note that $1 - H_{n,1}$ is the d.f. of the busy cycle. — The reason for working with unity minus the d.f. and unity minus the Laplace—Stieltjes transform of the d.f. in the sequel is that we get formulas that are simpler and more easy to manipulate. By conditioning with respect to the time u when the next customer arrives and to the number r of served customers in $(0, u)$ we get

$$(1) \quad H_{n,k}(t) = \int_0^t \sum_{r=0}^{k-1} \frac{(\beta u)^r}{r!} e^{-\beta u} H_{n,k-r+1}(t-u) dF(u) + 1 - F(t),$$

$$k = 1, 2, \dots, n,$$

with an agreement $H_{n,n+1} = H_{n,n}$.

Now define

$$\hat{H}_{n,k}(s) = 1 - \int_0^\infty e^{-st} d(1 - H_{n,k}(t)).$$

Then (1) gives with $\hat{F}(s) = \int_0^\infty e^{-st} dF(t)$

$$(2) \quad \hat{H}_{n,k}(s) = \sum_{j=0}^{k+1} \left(\int_0^\infty e^{-(s+\beta)u} \frac{(\beta u)^{k+1-j}}{(k+1-j)!} dF(u) \right) \hat{H}_{n,j}(s) + 1 - \hat{F}(s),$$

$$k = 1, 2, \dots, n.$$

Put

$$\hat{H}_n(s) = \hat{H}_{n,1}(s).$$

Then $1 - \hat{H}_n$ is the Laplace—Stieltjes transform of the busy cycle.

We solve the linear system (2) by Cramer's rule.

Define the $n \times n$ determinant $R_n(s)$ of the matrix of coefficients by

$$R_n(s) = \begin{vmatrix} 1 & -a_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1-a_1 & -a_0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -a_2 & 1-a_1 & -a_0 & \dots & 0 & 0 & 0 \\ 0 & -a_3 & -a_2 & 1-a_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -a_{n-3} & -a_{n-4} & -a_{n-5} & \dots & 1-a_1 & -a_0 & 0 \\ 0 & -a_{n-2} & -a_{n-3} & -a_{n-4} & \dots & -a_2 & 1-a_1 & -a_0 \\ 0 & -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_3 & -a_2 & 1-a_1-a_0 \end{vmatrix}.$$

Replacing each element of the first column with 1 we define the determinant $T_n(s)$.

Here

$$a_r = a_r(s) = \int_0^\infty e^{-(s+\beta)u} \frac{(\beta u)^r}{r!} dF(u).$$

R_n is of course equal to its first principal subdeterminant.

We obtain

$$(3) \quad \hat{H}_n(s) = \frac{(1 - \hat{F}(s))T_n(s)}{R_n(s)}.$$

Dividing by s and letting $s \rightarrow 0$ we obtain for the expectation μ_n of the busy cycle

$$(4) \quad \mu_n = \frac{\mu T_n(0)}{R_n(0)}.$$

3. A limit theorem

We have

$$\lim_{\beta \rightarrow 0} a_r(s) = \begin{cases} \hat{F}(s), & r = 0, \\ 0, & r = 1, 2, \dots \end{cases}$$

Inserting these limits into R_n and T_n we obtain

$$(5) \quad \begin{cases} \lim_{\beta \rightarrow 0} R_n(s) = 1 - \hat{F}(s) \\ \lim_{\beta \rightarrow 0} T_n(s) = 1. \end{cases}$$

We thus have asymptotically

$$\mu_n \sim \frac{\mu}{R_n(0)} \quad (\beta \rightarrow 0).$$

Let η be the busy cycle length.

THEOREM.

$$\lim_{\beta \rightarrow 0} P\left(\frac{\eta}{E(\eta)} \leq t\right) = 1 - e^{-t}, \quad t \geq 0.$$

The theorem is equivalent to

$$\lim_{\beta \rightarrow 0} \hat{H}_n(sR_n(0)) = \frac{\mu s}{1 + \mu s}, \quad s > 0.$$

We have

$$\lim_{\beta \rightarrow 0} T_n(sR_n(0)) = 1,$$

because

$$\lim_{\beta \rightarrow 0} a_r(sR_n(0)) = \begin{cases} 1, & r = 0, \\ 0, & r = 1, 2, \dots \end{cases}$$

Thus

$$\lim_{\beta \rightarrow 0} \hat{H}_n(sR_n(0)) = \lim_{\beta \rightarrow 0} \frac{1 - \hat{F}(sR_n(0))}{sR_n(0)} \cdot \frac{sR_n(0)}{R_n(sR_n(0))} = \frac{\mu s}{\lim_{\beta \rightarrow 0} \left(\frac{R_n(sR_n(0))}{R_n(0)} \right)}.$$

We have

$$R_n(sR_n(0)) = R_n(0) + sR_n(0) R_n'(\theta sR_n(0)),$$

where $0 < \theta < 1$.

It now suffices to show that

$$\lim_{\beta \rightarrow 0} R_n'(\theta sR_n(0)) = \mu.$$

Since $R_n'(x)$ is a sum of products, where each factor is of the form

$$K \int_0^{\infty} e^{-(x+\beta)t} \beta^j t^r dF(t),$$

with $r=j$ or $r=j+1$, we have

$$\lim_{\beta \rightarrow 0} R_n'(x) = \lim_{\beta \rightarrow 0} R_n'(0), \quad \text{if } x \rightarrow 0 \quad \text{when } \beta \rightarrow 0.$$

If we define the function f_n by

$$f_n(\beta, s) = \begin{cases} \frac{R_n(s) - R_n(0)}{s}, & s > 0, \\ R_n'(0), & s = 0, \end{cases}$$

we may see that f_n is continuous for $(\beta, s) \cong (0, 0)$.

Then

$$\begin{aligned} \lim_{\beta \rightarrow 0} R_n'(\theta sR_n(0)) &= \lim_{\beta \rightarrow 0} R_n'(0) = \lim_{\beta \rightarrow 0} (\lim_{s \rightarrow 0} f_n(\beta, s)) = \\ &= \lim_{s \rightarrow 0} (\lim_{\beta \rightarrow 0} f_n(\beta, s)) = \lim_{s \rightarrow 0} \frac{1 - \hat{F}(s)}{s} = \mu. \end{aligned}$$

The theorem is proved.

We have studied the limiting behaviour of the busy cycle distribution when the service times get longer and longer. We may also allow the inter-arrival times to become shorter and shorter letting F depend on a parameter $\lambda > 0$ in such a way that $F(t) = G(\lambda t)$, where G is a d.f. for a non-negative random variable, and making $\lambda \rightarrow \infty$. We introduce the dependence on λ and β in our notation and write

$$\hat{H}_n(s) = \hat{H}_n(\lambda, \beta, s)$$

and

$$\mu_n = \mu_n(\lambda, \beta).$$

If we look at the a , we can see that

$$\hat{H}_n \left(\lambda, \beta, \frac{s}{\mu_n(\lambda, \beta)} \right) = \hat{H}_n \left(1, \frac{\beta}{\lambda}, \frac{s}{\mu_n(1, \beta/\lambda)} \right).$$

Thus the same exponential limiting distribution prevails if $\lambda \rightarrow \infty$, or β and λ change simultaneously in such a way that $\frac{\beta}{\lambda} \rightarrow 0$. We also note that the same limit, with the same norming factor, holds for the busy period; this follows from the fact that the difference (idle period) between the busy period and the busy cycle after norming converges in probability to 0.

— Our theorem is a counterpart to TOMKO's Theorem 2, p 451.

4. Connection with the M/G/1 queue; the infinite waiting room case

We define

$$R_1(s) = T_1(s) = 1.$$

We here study \hat{H}_n only for $n \geq 2$.

If we develop $T_n(s)$ along the first row we obtain

$$(6) \quad T_n(s) = R_n(s) + a_0(s)T_{n-1}(s), \quad n \geq 2.$$

Let $A_n(s)$ be the determinant obtained from $R_n(s)$ by replacing the element $1 - a_1(s) - a_0(s)$ in the right bottom corner with $1 - a_1(s)$. We set $A_1(s) = 1$. It follows from an elementary property of determinants that

$$(7) \quad T_n(s) = A_n(s).$$

If we now consider an M/G/1 queue with n waiting places, service time distribution F and intensity of arrival β , we have according to [3], eq. (5), for the Laplace—Stieltjes transform $\hat{F}_n(s)$ of the busy period

$$(8) \quad \hat{F}_n(s) = \frac{a_0(s)B_{n-1}(s)}{B_n(s)},$$

where $B_n(s) = B_n(s, 1)$ is a determinant defined in [3]. (Here $\lambda = \beta, p_1 = 1$.) A determinant $D_n(s) = D_n(s, 1)$ is defined by eq. (6) of [3]. We can see that

$$T_n(s) = (-1)^{n-1}D_{n-1}(s).$$

Defining $D_0(s) = B_0(s) = 1$ we have according to eq. (6) of [3]

$$(9) \quad B_n(s) = a_0(s)B_{n-1}(s) + (1 - \hat{F}(s))T_n(s), \quad n \geq 1.$$

For $s > 0$ we then have according to (6) and (9)

$$(10) \quad T_n(s) = (1 - \hat{F}(s))^{-1}[B_n(s) - a_0(s)B_{n-1}(s)],$$

$$(11) \quad R_n(s) = (1 - \hat{F}(s))^{-1}[B_n(s) - 2a_0(s)B_{n-1}(s) + a_0(s)^2B_{n-2}(s)].$$

Putting (10) and (11) into (3), dividing by $B_n(s)$ and using (8) we get

$$(12) \quad \hat{H}_n(s) = \frac{(1 - \hat{F}(s))(1 - \hat{F}_n(s))}{1 - 2\hat{F}_n(s) + \hat{F}_{n-1}(s)\hat{F}(s)}, \quad s > 0, \quad n \geq 2.$$

It is well known that the transform \hat{F}_n for $n \rightarrow \infty$ converges to the function $x(s)$ that is the smallest in absolute value solution of

$$x = \hat{F}(s + \beta - \beta x).$$

If $\mu\beta > 1$ the distribution defined by the Laplace—Stieltjes transform x is improper. We get immediately

$$\lim_{n \rightarrow \infty} \hat{H}_n(s) = \frac{(1 - \hat{F}(s))(1 - x(s))}{1 - 2x(s) + x(s)^2} = \frac{1 - \hat{F}(s)}{1 - x(s)}.$$

The Laplace—Stieltjes transform of the busy cycle of the GI/M/1 queue with infinite waiting room is thus

$$(13) \quad \frac{\hat{F}(s) - x(s)}{1 - x(s)}.$$

This result agrees with previous results, such as may for instance be found in [1], p 327, eq. (5. 198), by specializing ($r=1, n=1$).

5. An interesting generating function

Define

$$b_n(s) = \frac{B_{n-1}(s)}{a_0(s)^{n-1}}.$$

This definition is equivalent to the one given by COHEN [2], p 826, eq. (19) ($\beta(s) = \hat{F}(s), \alpha = 1/\beta$). Cohen gives the generating function

$$(14) \quad b(z, s) = \sum_{n=1}^{\infty} b_n(s) z^n = \frac{z}{1-z} \left\{ 1 - \frac{z(1 - \hat{F}(s))}{z - \hat{F}(s + \beta(1-z))} \right\}$$

for $|z|$ sufficiently small.

We further define

$$t_n(s) = \frac{T_n(s)}{a_0(s)^n}$$

and

$$t(z, s) = \sum_{n=1}^{\infty} t_n(s) z^n.$$

We get from (10) for $s > 0$

$$t_n(s) = (1 - \hat{F}(s))^{-1} [b_{n+1}(s) - b_n(s)]$$

and thus

$$\begin{aligned} t(z, s) &= (1 - \hat{F}(s))^{-1} \sum_{n=1}^{\infty} (b_{n+1}(s) z^n - b_n(s) z^n) = (1 - \hat{F}(s))^{-1} \left[\frac{1-z}{z} b(z, s) - 1 \right] = \\ &= \frac{z}{\hat{F}(s + \beta(1-z)) - z}. \end{aligned}$$

It is easy to check the validity of this formula also for $s=0$. We can then rewrite (3) and (4), using (6),

$$(15) \quad \hat{H}_n(s) = \frac{(1 - \hat{F}(s))t_n(s)}{t_n(s) - t_{n-1}(s)}$$

and

$$(16) \quad \mu_n = \frac{\mu t_n(0)}{t_n(0) - t_{n-1}(0)}, \quad n \geq 2,$$

where

$$(17) \quad \sum_{n=1}^{\infty} t_n(s) z^n = \frac{z}{\hat{F}(s + \beta(1-z)) - z}, \quad |z| < |x(s)|, \quad \text{Re}(s) \geq 0.$$

Following COHEN, we may evaluate $t_n(s)$ as a contour integral:

$$(18) \quad t_n(s) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{dz}{z^n (\hat{F}(s + \beta(1-z)) - z)},$$

where $0 < r < |x(s)|$.

6. Related variables; customer loss ratio

We consider in addition to the busy cycle η also the busy period ξ , the number φ of served customers during a busy period, and the number ζ of lost customers.

In a future paper we will treat the joint distribution of these four variables under more general conditions. The moments may be expressed as solutions of linear systems with matrix of coefficients as (2) for $s=0$. Anticipating the results for the special case studied here, we state that $E(\zeta)$ can be expressed as $C_n/R_n(0)$, where the determinant C_n is obtained from $R_n(0)$ by replacing in the first column the first element 1 with 0 and the last element 0 with $a_0(0) = \hat{F}(\beta)$. Thus

$$E(\zeta) = \frac{a_0(0)^n}{R_n(0)} = \frac{1}{t_n(0) - t_{n-1}(0)}.$$

Since η is a sum of $\varphi + \zeta$ inter-arrival times, each with mean μ , we have $E(\eta) = \mu(E(\varphi) + E(\zeta))$. With (16) of the present paper we thus obtain $E(\varphi)$. Also, since ξ is a sum of φ service times, $E(\xi) = \frac{1}{\beta} E(\varphi)$. We summarize, putting $t_n = t_n(0)$,

$$(19) \quad \begin{aligned} E(\eta) &= \mu t_n (t_n - t_{n-1})^{-1}, \\ E(\xi) &= \beta^{-1} (t_n - 1) (t_n - t_{n-1})^{-1}, \\ E(\varphi) &= (t_n - 1) (t_n - t_{n-1})^{-1}, \\ E(\zeta) &= (t_n - t_{n-1})^{-1}. \end{aligned}$$

The long run customer loss ratio, $E(\zeta)/(E(\varphi) + E(\zeta))$, is then

$$(20) \quad \frac{1}{t_n}.$$

We have

$$t_n = \frac{(-1)^{n-1} D_{n-1}(0)}{\hat{F}(\beta)^n}.$$

Comparing this formula with (9) of [3], or comparing (17) with TOMKO's [4] eq. (2), which is also cited in the beginning of [3], we see that μt_n is the expectation of the busy period of the above-mentioned M/G/1 queue with n waiting places.

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STABILITY OF D-PERIODIC SOLUTIONS OF THIRD-ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

by

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We shall consider in this paper the third-order differential equation

$$(1) \quad \ddot{u} + g(u, \dot{u}, \ddot{u}) = 0, \quad \left(\cdot = \frac{d}{dt} \right),$$

where $g: R^3 \rightarrow R$ and further assumptions on g will be made in section 1. First we shall deal with the existence of a D-periodic solution of (1), section 1. Sufficient conditions will be given for the stability of the D-periodic solution of (1), section 2. Also examples will be given to illustrate these conditions, section 3. The definition of a D-periodic function has been given in [1]. Certainly, a periodic function is a special case.

The results of this paper depend mainly on [8], [9] and [10].

1. Existence of a D-periodic solution of (1)

The basic assumption of this paper is the existence of a non-constant D-periodic solution $u_0(t)$ of (1). This solution can be written as

$$u_0(t) = \alpha_0(t) + a_0 t,$$

where

$$\alpha_0(t + \tau_0) \equiv \alpha_0(t) \quad \text{and} \quad \tau_0 > 0, \quad a_0 \in R.$$

We assume that the function g is analytic in the neighbourhood of $\{u_0(t), \dot{u}_0(t), \ddot{u}_0(t)\}$ ($0 \leq t \leq \tau_0$) and (for $a_0 \neq 0$) is periodic in u with period $a_0 \tau_0$.

In general, such a solution may not exist. However, there are many significant cases in which (1) has non-constant D-periodic solutions. For the existence of periodic solutions see e.g. [3], [4] pp. 353—364, [5] and [6]. The equation

$$(2) \quad \ddot{u} + a\ddot{u} + b\dot{u} + h(u) = 0,$$

where a, b are real constants, $h'(u) < ab$ for all u and $h \in C^1(-\infty, +\infty)$, in general, has no non-constant periodic solution (cf. [7]). But if it is assumed in addition that $a > 0, b > 0, h(u)$ is periodic with period 2π and $h(u) < 0$ (or $h(u) > 0$) for all u , then the equation (2) has at least one D-periodic solution (cf. [8]).

In section 3 we shall need the following theorem (cf. [8]) due to NAZAROV.

THEOREM A. *If the conditions stated above hold together with*

$$a^2 < 4b$$

and

$$\frac{d}{m+d} < \frac{a\sqrt{4b-a^2}}{4b}.$$

where

$$\min[-h(u)] = m,$$

$$\max[-h(u)] = m + 2d,$$

then the equation (2) has at least one D-periodic solution

$$u_0(t) = \alpha_0(t) + a_0 t,$$

where

$$\alpha_0(t + \tau_0) \equiv \alpha_0(t) \quad \text{and} \quad a_0 \tau_0 = 2\pi.$$

2. Stability of the D-periodic solution of (1)

Now we return to the general equation (1). The first variational equation of (1) corresponding to $u_0(t)$ is given by

$$(3) \quad \ddot{v} + g'_u(p(t))\ddot{v} + g'_u(p(t))\dot{v} + g'_u(p(t))v = 0$$

where, for sake of convenience, we introduced the notation

$$p(t) = (u_0(t), \dot{u}_0(t), \ddot{u}_0(t)).$$

Since (1) is an autonomous equation and $u_0(t)$ is a non-constant D-periodic solution of (1), it is well known that $v_1(t) = \dot{u}_0(t)$ is a periodic solution of (3). Without loss of generality, it will be assumed that

$$v_1(0) = 1, \quad \dot{v}_1(0) = 0, \quad \ddot{v}_1(0) = -d_0, \quad d_0 \geq 0.$$

$$v_3(0) = 0, \quad \dot{v}_3(0) = 0, \quad \ddot{v}_3(0) = 1.$$

Let $v_2(t), v_3(t)$ be two solutions of (3) such that

$$v_2(0) = 0, \quad \dot{v}_2(0) = 1, \quad \ddot{v}_2(0) = 0;$$

$$v_3(0) = 0, \quad \dot{v}_3(0) = 0, \quad \ddot{v}_3(0) = 1.$$

From FLOQUET'S theory the characteristic equation of (3) can be written as

$$\det \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -d_0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} v_1 & v_2 & v_3 \\ \dot{v}_1 & \dot{v}_2 & \dot{v}_3 \\ \ddot{v}_1 & \ddot{v}_2 & \ddot{v}_3 \end{bmatrix}_{\tau_0} - \varrho \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = 0,$$

and hence

$$(4) \quad \varrho^3 + R_1 \varrho^2 + R_2 \varrho + R_3 = 0,$$

where

$$R_1 = -[1 + \dot{v}_2(\tau_0) + d_0 v_3(\tau_0) + \ddot{v}_3(\tau_0)],$$

$$R_2 = \begin{vmatrix} 1 & v_2 \\ 0 & \dot{v}_2 \end{vmatrix}_{\tau_0} + \begin{vmatrix} 1 & v_3 \\ d_0 v_1 + \dot{v}_1 & d_0 v_3 + \ddot{v}_3 \end{vmatrix}_{\tau_0} + \begin{vmatrix} \dot{v}_2 & \dot{v}_3 \\ d_0 v_2 + \ddot{v}_2 & d_0 v_3 + \ddot{v}_3 \end{vmatrix}_{\tau_0},$$

$$R_3 = -\exp\left(-\int_0^{\tau_0} g'_u(p(t)) dt\right).$$

Since $\varrho_1=1$ is a solution of (4), it follows that the other two solutions satisfy the equation

$$(5) \quad \varrho^2 + (R_1 + 1)\varrho - R_3 = 0.$$

By solving (5), we find that

$$\varrho_{2,3} = \frac{-(R_1 + 1) \pm \sqrt{(R_1 + 1)^2 + 4R_3}}{2}.$$

Next we shall find sufficient conditions implying that the two characteristic multipliers ϱ_2 and ϱ_3 be in modulus less than one. There are two cases:

a) $(R_1 + 1)^2 + 4R_3 > 0$. The inequality $|-(R_1 + 1) \pm \sqrt{(R_1 + 1)^2 + 4R_3}| < 2$ is satisfied if $(|R_1 + 1| + \sqrt{(R_1 + 1)^2 + 4R_3}) < 2$, which can be rewritten in the form $2 - |R_1 + 1| > \sqrt{(R_1 + 1)^2 + 4R_3}$.

Taking the square of both sides we get

$$4 - 4|R_1 + 1| + (R_1 + 1)^2 > (R_1 + 1)^2 + 4R_3$$

i.e.

$$|R_1 + 1| < 1 - R_3.$$

b) $(R_1 + 1)^2 + 4R_3 \leq 0$. Then

$$\varrho_{2,3} = \frac{-(R_1 + 1) \pm i\sqrt{-(R_1 + 1)^2 - 4R_3}}{2}.$$

The inequality

$$|\varrho_{2,3}| = \sqrt{\frac{1}{4}\{(R_1 + 1)^2 - (R_1 + 1)^2 - 4R_3\}} = \sqrt{-R_3} < 1$$

is satisfied if $-R_3 < 1$. From these considerations it follows that $|\varrho_{2,3}| < 1$ if the following two conditions hold:

$$(6) \quad \varrho(\tau_0) = \int_0^{\tau_0} g'_u(p(t)) dt > 0$$

and

$$-1 - e^{-\delta(\tau_0)} < R_1 + 1 < 1 + e^{-\delta(\tau_0)}.$$

Applying the method used in [9], we now consider the equation

$$(7) \quad \ddot{v} + g'_u(p(t))\dot{v} - \lambda g'_u(p(t))v = 0,$$

where λ is real parameter.

Consider also the two linearly independent solutions $v_2(t, \lambda)$, $v_3(t, \lambda)$ of (7) with corresponding initial conditions

$$v_2(0, \lambda) = 0, \quad \dot{v}_2(0, \lambda) = 1, \quad \ddot{v}_2(0, \lambda) = 0;$$

$$v_3(0, \lambda) = 0, \quad \dot{v}_3(0, \lambda) = 0, \quad \ddot{v}_3(0, \lambda) = 1.$$

Since the coefficients of the linear equation (7) are analytic in t and λ , the solutions $v_j(t, \lambda)$ ($j=2, 3$) and their derivatives $\dot{v}_j(t, \lambda)$, $\ddot{v}_j(t, \lambda)$ are analytic. We expand $v_j(t, \lambda)$, $\dot{v}_j(t, \lambda)$ and $\ddot{v}_j(t, \lambda)$ by powers of λ :

$$(8) \quad v_j(t, \lambda) = \sum_{i=0}^{\infty} v_{ji}(t) \lambda^i,$$

$$\dot{v}_j(t, \lambda) = \sum_{i=0}^{\infty} \dot{v}_{ji}(t) \lambda^i,$$

$$\ddot{v}_j(t, \lambda) = \sum_{i=0}^{\infty} \ddot{v}_{ji}(t) \lambda^i \quad (j = 2, 3).$$

Clearly $v_j(t, -1) = v_j(t)$, $j=2, 3$, are the solutions of (3), which were considered.

In equation (7) we replace v , \dot{v} , \ddot{v} by the series (8) and compare coefficients of equal powers of λ on both sides, then we obtain an infinite system of linear differential equations for v_{ji} :

$$(9) \quad \ddot{v}_{ji} + g'_u(p(t))\dot{v}_{ji} = g'_u(p(t))\dot{v}_{j,i-1} + g'_u(p(t))v_{j,i-1}, \quad j = 2, 3, \quad i = 0, 1, 2, \dots,$$

where by definition

$$\dot{v}_{j,-1} \equiv v_{j,-1} \equiv 0.$$

Next we solve (9) with the initial conditions

$$v_{2,0}(0) = 0, \quad \dot{v}_{2,0}(0) = 1, \quad \ddot{v}_{2,0}(0) = 0;$$

$$v_{3,0}(0) = 0, \quad \dot{v}_{3,0}(0) = 0, \quad \ddot{v}_{3,0}(0) = 1;$$

$$v_{ji}(0) = \dot{v}_{ji}(0) = \ddot{v}_{ji}(0) = 0,$$

$$j = 2, 3; \quad i = 1, 2, \dots$$

Then we obtain

$$(10) \quad \begin{aligned} v_{2,0}(t) &= t, \\ v_{3,0}(t) &= \int_0^t ds \int_0^s e^{-\delta(s_1)} ds_1, \\ &\dots\dots\dots \end{aligned}$$

$$v_{ji}(t) = \int_0^t ds \int_0^s e^{-\delta(s_1)} ds_1 \int_0^{s_1} F_{ji}(s_2) e^{\delta(s_2)} ds_2,$$

where

$$F_{ji}(t) = g'_u(p(t)) \dot{v}_{ji-1} + g'_u(p(t)) v_{ji-1}, \quad j = 2, 3, \quad i = 1, 2, \dots,$$

$$\delta(t) = \int_0^t g'_u(p(s)) ds.$$

From G. A. Los's results [9] the solution $v_{ji}(t)$ can be written in the form

$$(11) \quad v_{ji}(t) = \int_0^t \ddot{\beta}^{-1}(s) F_{ji}(s) [s\dot{\beta}(s) - \beta(s) - t\dot{\beta}(s) + \beta(t)] ds, \quad j = 2, 3, \quad i = 1, 2, \dots,$$

where

$$\begin{aligned} \ddot{\beta}(t) &= e^{-\delta(t)}, \quad \ddot{\beta}^{-1}(t) = e^{\delta(t)}, \\ \dot{\beta}(t) &= \int_0^t e^{-\delta(s)} ds, \quad \beta(t) = \int_0^t ds \int_0^s e^{-\delta(s_1)} ds_1. \end{aligned}$$

From this follows

$$(12) \quad \begin{aligned} \dot{v}_{ji}(t) &= \int_0^t \ddot{\beta}^{-1}(s) F_{ji}(s) [\dot{\beta}(t) - \dot{\beta}(s)] ds, \\ \ddot{v}_{ji}(t) &= \int_0^t \ddot{\beta}^{-1}(s) F_{ji}(s) \ddot{\beta}(t) ds, \quad j = 2, 3, \quad i = 1, 2, \dots \end{aligned}$$

Here we need the following lemma due to Los (cf. [9], lemma 3.) to prove theorem 1.

LEMMA. *The functions $\beta(t)$ and $\dot{\beta}(t)$ satisfy the inequality*

$$(13) \quad t[\dot{\beta}(t) - \dot{\beta}(s)] \cong s\dot{\beta}(s) - \beta(s) - t\dot{\beta}(s) + \beta(t),$$

for $0 \leq s \leq t \leq \tau_0$.

It is clear that

$$(14) \quad t[\dot{\beta}(t) - \dot{\beta}(s)] \cong 0,$$

for $0 \leq s \leq t \leq \tau_0$.

We shall use the following notations

$$A = \int_0^{\tau_0} \ddot{\beta}^{-1}(s) [\dot{\beta}(\tau_0) - \beta(s)] [g'_u(p(s)) + |g'_u(p(s))|s] ds,$$

$$B = \int_0^{\tau_0} \ddot{\beta}^{-1}(s) \ddot{\beta}(\tau_0) [g'_u(p(s)) + |g'_u(p(s))|s] ds.$$

THEOREM 1. *Under the assumptions*

$$(15) \quad \delta(\tau_0) = \int_0^{\tau_0} g'_u(p(t)) dt > 0,$$

$$(16) \quad g'_u(p(t)) \geq 0, \quad 0 \leq t \leq \tau_0,$$

$$(17) \quad g'_u(p(t)) + g'_u(p(t))t \geq 0, \quad 0 \leq t \leq \tau_0,$$

$$(18) \quad 0 < A < 1,$$

$$(19) \quad -d_0 v_{3,0}(\tau_0) + d_0 v_{3,1}(\tau_0) + \ddot{v}_{3,1}(\tau_0) > \frac{A \dot{v}_{3,0}(\tau_0)}{1-A^2} (\tau_0 d_0 A + B),$$

$$(20) \quad -d_0 v_{3,0}(\tau_0) + \dot{v}_{2,1}(\tau_0) + \frac{\dot{v}_{3,0}(\tau_0)}{1-A^2} (\tau_0 d_0 A + B) < 2(1 + e^{-\delta(\tau_0)}),$$

the characteristic multipliers of (3) satisfy the conditions

$$\varrho_1 = 1, \quad |\varrho_2| < 1, \quad |\varrho_3| < 1$$

and thus the D-periodic solution $u_0(t)$ is asymptotically orbitally stable.

PROOF. First we can show with mathematical induction that $\dot{v}_{2k}(t)$ is non-negative, non-decreasing in the interval $0 \leq t \leq \tau_0$ and the inequality

$$0 \leq \dot{v}_{2k}(\tau_0) \leq A v_{2k-1}(\tau_0) \quad (k = 1, 2, \dots)$$

holds. From the formula (12) and the assumption (17) we have

$$\ddot{v}_{2,1}(t) = \int_0^t \ddot{\beta}^{-1}(s) \ddot{\beta}(t) [g'_u(p(s)) + g'_u(p(s))s] ds \geq 0, \\ 0 \leq t \leq \tau_0.$$

Hence the function $\dot{v}_{2,1}(t)$ is non-decreasing and from (10), (12) we see that

$$\dot{v}_{2,1}(t) \geq 0, \quad \dot{v}_{2,0}(t) \equiv 1.$$

Next from (12), (16) and (18) we can obtain

$$0 \leq \dot{v}_{2,1}(\tau_0) \leq A = A \dot{v}_{2,0}(\tau_0).$$

The inductive assumptions are as follows:

a) The function $\dot{v}_{2,k-1}(t)$ is non-negative and non-decreasing,

b) $0 \leq \dot{v}_{2k-1}(\tau_0) \leq A\dot{v}_{2k-2}(\tau_0)$, $k \geq 2$.

From the formula (12), $\ddot{v}_{2k}(t)$ has the form

$$\ddot{v}_{2k}(t) = \int_0^t \ddot{\beta}^{-1}(s) \ddot{\beta}(t) [g'_u(p(s)) \dot{v}_{2k-1}(s) + g'_u(p(s)) v_{2k-1}(s)] ds,$$

$$0 \leq t \leq \tau_0.$$

Since the function $\dot{v}_{2k-1}(s)$ is continuous, we can find a value $s^*(s)$, $0 \leq s^*(s) \leq s$, such that

$$v_{2k-1}(s) = \int_0^s \dot{v}_{2k-1}(s_1) ds_1 = \dot{v}_{2k-1}(s^*)s.$$

This leads to the equation

$$\ddot{v}_{2k}(t) = \int_0^t \ddot{\beta}^{-1}(s) \ddot{\beta}(t) [g'_u(p(s)) \dot{v}_{2k-1}(s) + g'_u(p(s)) s \dot{v}_{2k-1}(s^*)] ds.$$

By use of (16), (17) and inductive assumptions we obtain

$$\ddot{v}_{2k}(t) \geq \int_0^t \ddot{\beta}^{-1}(s) \ddot{\beta}(t) [g'_u(p(s)) + g'_u(p(s))s] \dot{v}_{2k-1}(s^*) ds \geq 0.$$

Therefore $\dot{v}_{2k}(t)$ is non-decreasing and from (12) we obtain

$$\dot{v}_{2k}(0) = 0, \quad \dot{v}_{2k}(t) \geq 0, \quad k \geq 2, \quad 0 \leq t \leq \tau_0.$$

Since

$$F_{2k}(s) = [g'_u(p(s)) \dot{v}_{2k-1}(s) + g'_u(p(s)) v_{2k-1}(s)] \geq$$

$$\geq [g'_u(p(s)) + g'_u(p(s))s] \dot{v}_{2k-1}(s^*) \geq 0;$$

applying (10)—(14) we get

$$(21) \quad t \dot{v}_{2k}(t) \geq v_{2k}(t) \geq 0, \quad 0 \leq t \leq \tau_0.$$

From (12), (16) and (21) we can show that

$$\dot{v}_{2k}(\tau_0) \geq \int_0^{\tau_0} \ddot{\beta}^{-1}(s) [\dot{\beta}(\tau_0) - \dot{\beta}(s)] [g'_u(p(s)) \dot{v}_{2k-1}(s) + |g'_u(p(s))| s \dot{v}_{2k-1}(s)] ds.$$

Since $\dot{v}_{2k-1}(s)$ is non-decreasing we finally obtain

$$\dot{v}_{2k}(\tau_0) \geq \int_0^{\tau_0} \ddot{\beta}^{-1}(s) [\dot{\beta}(\tau_0) - \dot{\beta}(s)] [g'_u(p(s)) +$$

$$+ |g'_u(p(s))|s] \dot{v}_{2k-1}(\tau_0) ds = A\dot{v}_{2k-1}(\tau_0).$$

The inductive conclusion has therefore been proved. From these results we see that

$$\lim_{k \rightarrow \infty} \dot{v}_{2k}(\tau_0) = 0,$$

and also $0 \leq \dot{v}_{2k}(\tau_0) \leq A^k \dot{v}_{2,0}(\tau_0)$, where from (18), $0 < A < 1$.

From these conclusions and by formula (8) the Leibniz series

$$\dot{v}_2(\tau_0) = \dot{v}_2(\tau_0, -1) = 1 - \dot{v}_{2,1}(\tau_0) + \dot{v}_{2,2}(\tau_0) - \dot{v}_{2,3}(\tau_0) + \dots$$

converges absolutely. This implies the inequality

$$(22) \quad 1 \geq \dot{v}_2(\tau_0) \geq 1 - \dot{v}_{2,1}(\tau_0) > 0.$$

We can show similarly for the functions $\dot{v}_{3i}(t)$ that

$$\ddot{v}_{3k}(t) \geq 0,$$

$$(23) \quad 0 \leq \dot{v}_{3k}(\tau_0) \leq A^k \dot{v}_{3,0}(\tau_0),$$

$$(24) \quad t \dot{v}_{3k}(t) \geq v_{3k}(t) \geq 0, \quad k = 0, 1, 2, \dots, 0 \leq t \leq \tau_0.$$

Now we shall estimate $\ddot{v}_{3i}(\tau_0)$. For $i=0$, we obtain

$$\ddot{v}_{3,0}(\tau_0) = e^{-\delta(\tau_0)}.$$

Next, from (12), (16) and (24) we can show that

$$\begin{aligned} \ddot{v}_{3,1}(\tau_0) &= \int_0^{\tau_0} \ddot{\beta}^{-1}(s) \ddot{\beta}(\tau_0) [g'_u(p(s)) \dot{v}_{3,0}(s) + g'_u(p(s)) v_{3,0}(s)] ds \leq \\ &\leq \int_0^{\tau_0} \ddot{\beta}^{-1}(s) \ddot{\beta}(\tau_0) [g'_u(p(s)) + |g'_u(p(s))| s] \dot{v}_{3,0}(s) ds. \end{aligned}$$

Since $\dot{v}_{3,0}(t)$ is non-decreasing, it follows that

$$\ddot{v}_{3,1}(\tau_0) \leq B \dot{v}_{3,0}(\tau_0).$$

Clearly,

$$\begin{aligned} \dot{v}_{3,0}(\tau_0) &> 0, \\ B &> 0. \end{aligned}$$

Similarly we can obtain the following inequalities

$$\ddot{v}_{3,2}(\tau_0) \leq \int_0^{\tau_0} \ddot{\beta}^{-1}(s) \ddot{\beta}(\tau_0) [g'_u(p(s)) + |g'_u(p(s))| s] \dot{v}_{3,1}(s) ds \leq B \dot{v}_{3,1}(\tau_0),$$

.....

$$\ddot{v}_{3k}(\tau_0) \leq B \dot{v}_{3k-1}(\tau_0), \quad k \geq 1.$$

By use of (23) we obtain

$$(25) \quad 0 \leq \ddot{v}_{3k}(\tau_0) \leq BA^{k-1}\dot{v}_{3,0}(\tau_0), \quad k \geq 1.$$

Finally, from (8) and (25) we find that

$$\begin{aligned} \ddot{v}_3(\tau_0) &= \ddot{v}_3(\tau_0, -1) = \ddot{v}_{3,0}(\tau_0) - \ddot{v}_{3,1}(\tau_0) + \ddot{v}_{3,2}(\tau_0) - \ddot{v}_{3,3}(\tau_0) + \dots \leq \\ &\leq \ddot{v}_{3,0}(\tau_0) - \ddot{v}_{3,1}(\tau_0) + \sum_{i=1}^{\infty} \ddot{v}_{3,2i}(\tau_0) \leq \ddot{v}_{3,0}(\tau_0) - \ddot{v}_{3,1}(\tau_0) + \sum_{i=0}^{\infty} A^{2i+1} B \dot{v}_{3,0}(\tau_0). \end{aligned}$$

From the condition $0 < A < 1$ the series $\sum_{i=0}^{\infty} A^{2i+1} B \dot{v}_{3,0}(\tau_0)$ converges; that is,

$$\sum_{i=0}^{\infty} A^{2i+1} B \dot{v}_{3,0}(\tau_0) = \frac{AB \dot{v}_{3,0}(\tau_0)}{1 - A^2}.$$

Then it follows that

$$(26) \quad \ddot{v}_3(\tau_0) \leq \ddot{v}_{3,0}(\tau_0) - \ddot{v}_{3,1}(\tau_0) + \frac{AB \dot{v}_{3,0}(\tau_0)}{1 - A^2}.$$

Similarly we can get a lower bound for $\ddot{v}_3(\tau_0)$:

$$\begin{aligned} \ddot{v}_3(\tau_0) &\geq \ddot{v}_{3,0}(\tau_0) - \sum_{i=0}^{\infty} \ddot{v}_{3,2i+1}(\tau_0) \geq \\ &\geq \ddot{v}_{3,0}(\tau_0) - \sum_{i=0}^{\infty} A^{2i} B \dot{v}_{3,0}(\tau_0) = \ddot{v}_{3,0}(\tau_0) - \frac{B \dot{v}_{3,0}(\tau_0)}{1 - A^2}, \end{aligned}$$

i.e.

$$(27) \quad \ddot{v}_3(\tau_0) \geq \ddot{v}_{3,0}(\tau_0) - \frac{B \dot{v}_{3,0}(\tau_0)}{1 - A^2}.$$

From the inequalities (23), (24) we have

$$\begin{aligned} v_3(\tau_0) &= v_3(\tau_0, -1) \leq v_{3,0}(\tau_0) - v_{3,1}(\tau_0) + \tau_0 \sum_{i=1}^{\infty} A^{2i} \dot{v}_{3,0}(\tau_0) = \\ &= v_{3,0}(\tau_0) - v_{3,1}(\tau_0) + \frac{\tau_0 A^2 \dot{v}_{3,0}(\tau_0)}{1 - A^2}; \end{aligned}$$

$$v_3(\tau_0) \geq v_{3,0}(\tau_0) - \tau_0 \sum_{i=0}^{\infty} A^{2i+1} \dot{v}_{3,0}(\tau_0) = v_{3,0}(\tau_0) - \frac{\tau_0 A \dot{v}_{3,0}(\tau_0)}{1 - A^2}.$$

Hence

$$(28) \quad v_{3,0}(\tau_0) - \frac{\tau_0 A \dot{v}_{3,0}(\tau_0)}{1 - A^2} \leq v_3(\tau_0) \leq v_{3,0}(\tau_0) - v_{3,1}(\tau_0) + \frac{\tau_0 A^2 \dot{v}_{3,0}(\tau_0)}{1 - A^2}.$$

From the estimates (22), (26), (27), (28) we finally obtain

$$\begin{aligned} & -d_0 v_{3,0}(\tau_0) + d_0 v_{3,1}(\tau_0) + \ddot{v}_{3,1}(\tau_0) - \frac{A \dot{v}_{3,0}(\tau_0)}{1-A^2} (\tau_0 d_0 A + B) - 1 - e^{-\delta(\tau_0)} \cong \\ & \cong -[\dot{v}_2(\tau_0) + d_0 v_3(\tau_0) + \ddot{v}_3(\tau_0)] \cong \\ & \cong -d_0 v_{3,0}(\tau_0) + \dot{v}_{2,1}(\tau_0) + \frac{\dot{v}_{3,0}(\tau_0)}{1-A^2} (\tau_0 d_0 A + B) - e^{-\delta(\tau_0)} - 1. \end{aligned}$$

Therefore, under the assumptions (15), (19) and (20) the inequality (6) is satisfied. Thus it follows that

$$|\varrho_{2,3}| < 1.$$

By the general ANDRONOV—VITT's theorem (cf. [10], pp. 323, theorem 2.2. for $a_0=0$ and [11], theorem 2.1. for $a_0 \neq 0$) the D-periodic solution $u_0(t)$ is asymptotically orbitally stable.

Remark 1. In the proof of theorem 1 the first estimates of $\dot{v}_2(\tau_0)$, $v_3(\tau_0)$ and $\ddot{v}_3(\tau_0)$ have been used. If we apply their second estimates, then we shall replace (19) and (20) by weaker (but more complicated) assumptions. In this case, instead of (22), (26), (27) and (28), we obtain

$$\begin{aligned} \dot{v}_{2,1}(\tau_0) - 1 & \cong -\dot{v}_2(\tau_0) \cong \dot{v}_{2,1}(\tau_0) - 1 - \dot{v}_{2,2}(\tau_0), \\ \ddot{v}_{3,1}(\tau_0) - \ddot{v}_{3,0}(\tau_0) - \ddot{v}_{3,2}(\tau_0) - \frac{A^3 B \dot{v}_{3,0}(\tau_0)}{1-A^2} & \cong -\ddot{v}_3(\tau_0) \cong \\ & \cong \ddot{v}_{3,1}(\tau_0) + \frac{A^2 B \dot{v}_{3,0}(\tau_0)}{1-A^2} - \ddot{v}_{3,0}(\tau_0) - \ddot{v}_{3,2}(\tau_0), \\ -d_0 v_{3,0}(\tau_0) + d_0 v_{3,1}(\tau_0) - d_0 v_{3,2}(\tau_0) - \frac{\tau_0 d_0 A^4 \dot{v}_{3,0}(\tau_0)}{1-A^2} & \cong \\ \cong -d_0 v_3(\tau_0) \cong -d_0 v_{3,0}(\tau_0) + d_0 v_{3,1}(\tau_0) + \frac{\tau_0 d_0 A^3 \dot{v}_{3,0}(\tau_0)}{1-A^2}. \end{aligned}$$

Hence we can replace (19) and (20) by the following inequalities

$$(19^*) \quad -d_0 v_{3,0}(\tau_0) + d_0 v_{3,1}(\tau_0) - d_0 v_{3,2}(\tau_0) + \dot{v}_{2,1}(\tau_0) + \ddot{v}_{3,1}(\tau_0) - \\ - \ddot{v}_{3,2}(\tau_0) > \frac{A^3 \dot{v}_{3,0}(\tau_0)}{1-A^2} (\tau_0 d_0 A + B),$$

$$(20^*) \quad -d_0 v_{3,0}(\tau_0) + d_0 v_{3,1}(\tau_0) + \dot{v}_{2,1}(\tau_0) - \dot{v}_{2,2}(\tau_0) + \ddot{v}_{3,1}(\tau_0) - \ddot{v}_{3,2}(\tau_0) + \\ + \frac{A^2 \dot{v}_{3,0}(\tau_0)}{1-A^2} (\tau_0 d_0 A + B) < 2(1 + e^{-\delta(\tau_0)}),$$

where $\dot{v}_{2,1}(\tau_0)$, $\dot{v}_{2,2}(\tau_0)$, $v_{3,0}(\tau_0)$, $v_{3,1}(\tau_0)$, $\dot{v}_{3,0}(\tau_0)$, $\ddot{v}_{3,1}(\tau_0)$ and $\ddot{v}_{3,2}(\tau_0)$ can be got from (10)—(12).

Remark 2. If equation (1) is considered as an unperturbed equation, then from the results of this paper we can still consider the existence and stability of a D-periodic solution of the three corresponding perturbed equations, where μ always denotes a small parameter:

a) $\ddot{u} + g(u, \dot{u}, \ddot{u}) = \mu\gamma(u, \dot{u}, \ddot{u}, \mu)$, where g and γ are analytic in all variables (cf. [10]).

b) $\ddot{u} + g(u, \dot{u}, \ddot{u}) = \mu\gamma(t, u, \dot{u}, \ddot{u}, \mu)$, where g and γ are analytic in all variables and γ is periodic in t (cf. [12]). The two above cases are considered with the condition $a_0 = 0$.

$$c) \ddot{u} + g(u, \dot{u}, \ddot{u}) = \mu\gamma\left(\frac{t}{\tau}, u, \dot{u}, \ddot{u}, \mu, \tau\right),$$

where g and γ are analytic in all variables, periodic in u with period $a_0\tau_0$ and γ is periodic in t with period τ , $|\tau - \tau_0| < \beta$ for some β , $0 < B < \tau_0$ (cf. [1], [2]).

3. Examples

Consider the equation

$$(29) \quad \ddot{u} + a\dot{u} + bu + \varepsilon h(u) - c = 0,$$

where $a > 0$, $b > 0$, $c > 0$, $|\varepsilon| \neq 0$ is sufficiently small, $h(u)$ is non-constant, analytic and periodic with period 2π :

$$h(u + 2\pi) \equiv h(u).$$

Now we denote

$$\begin{aligned} \max \varepsilon h(u) &= \varepsilon N, & \min \varepsilon h(u) &= \varepsilon M, \\ \max \varepsilon h'(u) &= \varepsilon P > 0, & \min \varepsilon h'(u) &= \varepsilon Q < 0, \end{aligned}$$

where for the case $\varepsilon > 0$:

$$\begin{aligned} N &= \max h(u), & M &= \min h(u), \\ P &= \max h'(u), & Q &= \min h'(u), \end{aligned}$$

for the case $\varepsilon < 0$:

$$\begin{aligned} N &= \min h(u), & M &= \max h(u), \\ P &= \min h'(u), & Q &= \max h'(u). \end{aligned}$$

With these notations we obtain

$$(30) \quad \begin{aligned} \min [-(\varepsilon h(u) - c)] &= c - \varepsilon N, \\ \max [-(\varepsilon h(u) - c)] &= c - \varepsilon M = c - \varepsilon N + 2 \frac{\varepsilon N - \varepsilon M}{2}. \end{aligned}$$

We assume that

$$(31) \quad a^2 < 4b.$$

Clearly, for sufficiently small $|\varepsilon|$ we have

$$\frac{\varepsilon N - \varepsilon M}{2c - \varepsilon N - \varepsilon M} < \frac{a\sqrt{4b - a^2}}{4b}.$$

By theorem A, (29) has a D-periodic solution $u_0(t)$, for which $u_0(t + \tau_0) - u_0(t) \equiv 2\pi$. By integration of (29) and letting $u = u_0(t)$, we obtain

$$\begin{aligned} [\underbrace{\ddot{u}_0(\tau_0) - \ddot{u}_0(0)}_{=0}] + a[\underbrace{\dot{u}_0(\tau_0) - \dot{u}_0(0)}_{=0}] + b[u_0(\tau_0) - u_0(0)] &= - \int_0^{\tau_0} (\varepsilon h(u_0(t)) - c) dt, \\ b[u_0(\tau_0) - u_0(0)] &= 2\pi b = - \int_0^{\tau_0} (\varepsilon h(u_0(t)) - c) dt. \end{aligned}$$

Then from (30) we see that

$$2\pi b \cong (c - \varepsilon M)\tau_0,$$

$$2\pi b \cong (c - \varepsilon N)\tau_0.$$

Hence

$$\frac{2\pi b}{c - \varepsilon M} \cong \tau_0 \cong \frac{2\pi b}{c - \varepsilon N}.$$

Obviously the function $\dot{u}_0(t) = \dot{\alpha}_0(t) + a_0$ is a periodic solution of the first variational equation

$$(32) \quad \ddot{v} + a\ddot{v} + b\dot{v} + \varepsilon h'_u(u_0(t))v = 0.$$

Since $\dot{u}_0(t)$ is periodic in t , analytic in ε and $a^2 < 4b$, we can show that

$$\dot{u}_0(t) = c_0 + \varepsilon r(t, \varepsilon), \quad v_1(t) = \frac{\dot{u}_0(t)}{\dot{u}_0(0)},$$

$$d_0 = -\ddot{v}_1(0) = \varepsilon \gamma(\varepsilon),$$

where $\gamma(\varepsilon)$ is an analytic function of ε .

Clearly, in this case the conditions (15), (16), (17) are satisfied for sufficiently small $|\varepsilon|$. Now we shall find new conditions, which are stronger than (18), (19) and (20). From (18) we see that

$$(33) \quad \begin{aligned} 0 < A &\cong \int_0^{\tau_0} e^{as} \left(\int_s^{\tau_0} e^{-as_1} ds_1 \right) (b + \varepsilon Ps) ds = \\ &= -\frac{b}{a^2} + \frac{b}{a^2} e^{-a\tau_0} + \frac{b}{a} \tau_0 + \varepsilon \Delta_1 < 1, \end{aligned}$$

where

$$\Delta_1 = P \int_0^{\tau_0} e^{as} \left(-\frac{1}{a} e^{-a\tau_0} + \frac{1}{a} e^{-as} \right) ds.$$

Next we can show that

$$B \equiv \int_0^{\tau_0} e^{as} e^{-a\tau_0} (b + \varepsilon Ps) ds = \frac{b}{a} - \frac{b}{a} e^{-a\tau_0} + \varepsilon A_2,$$

where

$$\begin{aligned} A_2 &= P e^{-a\tau_0} \int_0^{\tau_0} e^{as} s ds; \\ \dot{v}_{3,0}(t) &= \int_0^t e^{-as} ds = \frac{1}{a} - \frac{1}{a} e^{-at}; \\ \dot{v}_{3,0}(\tau_0) &= \frac{1}{a} - \frac{1}{a} e^{-a\tau_0}; \\ v_{3,0}(t) &= \int_0^t \left(\frac{1}{a} - \frac{1}{a} e^{-as} \right) ds = \frac{1}{a} t + \frac{1}{a^2} e^{-at} - \frac{1}{a^2}; \\ \ddot{v}_{3,1}(\tau_0) &\equiv \int_0^{\tau_0} e^{as} e^{-a\tau_0} (b \dot{v}_{3,0}(s) + \varepsilon Q v_{3,0}(s)) ds = \\ &= \frac{b}{a} e^{-a\tau_0} \left(\frac{1}{a} e^{a\tau_0} - \frac{1}{a} \right) - \frac{b}{a} e^{-a\tau_0} \tau_0 + \varepsilon A_3, \end{aligned}$$

where

$$\begin{aligned} A_3 &= Q e^{-a\tau_0} \int_0^{\tau_0} e^{as} \left(\frac{1}{a} s + \frac{1}{a^2} e^{-as} - \frac{1}{a^2} \right) ds; \\ \dot{v}_{2,1}(\tau_0) &\equiv A. \end{aligned}$$

From the above results and (19) we have

$$\begin{aligned} &\frac{b}{a^2} \left(1 - e^{-a\tau_0} - a e^{-a\tau_0} \tau_0 + \varepsilon \frac{a^2}{b} A_3 \right) + \varepsilon \gamma(\varepsilon) (v_{3,1}(\tau_0) - v_{3,0}(\tau_0)) > \\ &> \frac{1}{a} (1 - e^{-a\tau_0}) \frac{A}{1 - A^2} \left[\varepsilon \gamma(\varepsilon) \tau_0 A + \frac{b}{a} \left(1 - e^{-a\tau_0} + \varepsilon \frac{a}{b} A_2 \right) \right], \end{aligned}$$

then

$$(34) \quad e^{a\tau_0} - 1 - a\tau_0 + \varepsilon A_4(\varepsilon) > (e^{a\tau_0} - 2 - e^{-a\tau_0} + \varepsilon A_5(\varepsilon)) \frac{A}{1 - A^2},$$

where

$$\begin{aligned} A_4(\varepsilon) &= \frac{a^2}{b} e^{a\tau_0} A_3 + \frac{a^2}{b} e^{a\tau_0} \gamma(\varepsilon) [v_{3,1}(\tau_0) - v_{3,0}(\tau_0)], \\ A_5(\varepsilon) &= \frac{a}{b} (e^{a\tau_0} - 1) A_2 + \frac{a}{b} (e^{a\tau_0} - 1) \gamma(\varepsilon) \tau_0 A. \end{aligned}$$

Otherwise from above results and (20) we obtain

$$A + \frac{1}{a} (1 - e^{-a\tau_0}) \frac{1}{1 - A^2} \left[\varepsilon \gamma(\varepsilon) \tau_0 A + \frac{b}{a} \left(1 - e^{-a\tau_0} + \varepsilon \frac{a}{b} A_2 \right) \right] < 2(1 + e^{-a\tau_0}),$$

and hence

$$(35) \quad A + \frac{b(1 - e^{-a\tau_0})^2}{a^2(1 - A^2)} + \varepsilon \Delta_6(\varepsilon) < 2(1 + e^{-a\tau_0}),$$

where

$$\Delta_6(\varepsilon) = \frac{1}{a(1 - A^2)} (1 - e^{-a\tau_0}) [\gamma(\varepsilon) \tau_0 A + \Delta_2].$$

From the above results we can conclude that if the equation (29) satisfies the conditions (31), (33), (34) and (35) then there exists a D-periodic solution $u_0(t)$ and the characteristic multipliers of the first variational equation (32) satisfy the following conditions

$$\varrho_1 = 1, \quad |\varrho_2| < 1, \quad |\varrho_3| < 1.$$

Hence the solution $u_0(t)$ is asymptotically orbitally stable.

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**ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF THE
DIFFERENTIAL EQUATION**

$$y''(t) = t^\nu y(qt), \quad (0 < q < 1)$$

by

E. GYURKOVICS

The asymptotic behaviour and oscillatory character of the solution of the second order linear differential equations of the unstable type with deviating argument have been the subject of some investigations [3, 4]. KAMENSKY has proved in [3] that in the class of all solutions of the differential equation

$$(1) \quad y''(t) - p(t)y(t - \tau(t)) = 0, \quad p(t) \geq 0, \quad \tau(t) \geq 0,$$

with variable initial values of $y'(t_0)$ and with a fixed initial function there exist solutions $y_1(t), y_2(t)$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} y_1(t) &= \lim_{t \rightarrow \infty} y_1'(t) = -\infty, \\ \lim_{t \rightarrow \infty} y_2(t) &= \lim_{t \rightarrow \infty} y_2'(t) = +\infty. \end{aligned}$$

Furthermore the sets of initial values of $y'(t_0)$ for which the corresponding solutions tend to $-\infty$ or $+\infty$, respectively, are the intervals $(-\infty, A_1)$ and $(A_2, +\infty)$, where $A_1 \leq A_2$, and if $A_1 < A_2$ then for arbitrary $y'(t_0) = a_1 \in [A_1, A_2]$ the corresponding solution is unbounded and oscillatory. It has been proved in [4] that an equation more general than (1) has an oscillatory solution under certain growth conditions on the coefficients and the retarded arguments. To find conditions enough for $A_1 < A_2$ has been also arisen there. We prove that for the special differential equation (2) this inequality holds, i.e. this equation has infinitely many unbounded oscillatory solutions.

We consider the following second order linear differential equation of unstable type with deviating argument

$$(2) \quad y''(t) = t^\nu y(qt),$$

where $t \geq 0, 0 < q \leq 1$, and $\nu > -1$.

Let $u(t)$ and $v(t)$ be solutions of (2) with the initial conditions

$$(3) \quad u(0) = v'(0) = 1, \quad u'(0) = v(0) = 0.$$

Then the solutions $u(t)$ and $v(t)$ can be expressed by means of the series

$$(4) \quad \begin{aligned} u(t) &= \Gamma(1 - \mu) \sum_{l=0}^{\infty} \frac{q^{\frac{l(l-1)}{2\mu}}}{l! \Gamma(l + 1 - \mu)} \mu^{2l} t^{\frac{l}{\mu}}, \\ v(t) &= \Gamma(1 + \mu) \sum_{l=0}^{\infty} \frac{q^{l + \frac{l(l-1)}{2\mu}}}{l! \Gamma(l + 1 + \mu)} \mu^{2l} t^{1 + \frac{l}{\mu}}, \end{aligned}$$

where $0 < \mu = \frac{1}{2+v} < 1$ and $\Gamma(x)$ is the well-known gamma function. It is clear that $u(t) > 0$ and $v(t) \geq 0$ are linearly independent and

$$\lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} u'(t) = \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} v'(t) = +\infty.$$

The general solution $y(t)$ of (2) with the initial conditions

$$(5) \quad y(0) = a_0, \quad y'(0) = a_1$$

can be written in the form

$$(6) \quad y(t) = a_0 u(t) + a_1 v(t).$$

THEOREM. *If $0 < q < 1$ then we have $A_1 < A_2$ for the differential equation (2).*

PROOF. Without loss of generality we can assume that $a_0 = 1$. We denote the initial value $a_1 = a_1(\tau)$ for which the corresponding solution $y(t; a_1(\tau))$ of (2) is a solution of the boundary-value problem

$$y(0) = 1, \quad y(\tau) = 0.$$

Then it is clear by (6) that

$$(7) \quad a_1(\tau) = -\frac{u(\tau)}{v(\tau)}, \quad 0 < \tau < \infty,$$

and by KAMENSKY'S result we have that $a_1(\tau)$ is bounded, because the functions $y_1(t), y_2(t)$ are of the form

$$y_1(t) = u(t) + a_1^* v(t),$$

$$y_2(t) = u(t) + a_1^{**} v(t).$$

Therefore

$$a_1^* < -\frac{u(t)}{v(t)} < a_1^{**}, \quad \text{for } t > t_0.$$

To prove our theorem it is sufficient to show that $a_1(\tau)$ is not convergent, because this means that for arbitrary $a_1 \in [\liminf_{\tau \rightarrow \infty} a_1(\tau), \limsup_{\tau \rightarrow \infty} a_1(\tau)]$ the corresponding solution is oscillatory. We determine the asymptotic values of the functions $u(t)$ and $v(t)$ by the method used in [2] and we show that their quotient is not constant. Let

$$Q = q^{\frac{1}{2\mu}}, \quad z = \mu t^{\frac{1}{2\mu}} q^{-\frac{1}{4\mu}},$$

then by (4) we have

$$(8) \quad u(t) = \Gamma(1-\mu) \sum_{l=0}^{\infty} \frac{Q^{l^2} z^{2l}}{l! \Gamma(l+1-\mu)},$$

$$v(t) = \Gamma(1+\mu) t \sum_{l=0}^{\infty} \frac{Q^{l^2} (\sqrt{q}z)^2}{l! \Gamma(l+1+\mu)}.$$

Here we have $0 < Q < 1$, $0 \leq z < \infty$, since $\mu > 0$. Let $I_k = \{z: (k-1)Q^{-k+1} \leq z < kQ^{-k}\}$. If $z \in I_k$ then one of the largest terms in (8) is the k -th one. Hence it is useful to write (8) in the form

$$(9) \quad u(t) = \Gamma(1-\mu) \frac{Q^{k^2} z^{2k}}{k! \Gamma(k+1-\mu)} \sum_{r=-k}^{\infty} \alpha_{k,r} Q^{r^2} x^{2r}$$

$$v(t) = \Gamma(1+\mu) t \frac{Q^{k^2} (\sqrt{q}z)^{2k}}{k! \Gamma(k+1+\mu)} \sum_{r=-k}^{\infty} \beta_{k,r} Q^{r^2} (\sqrt{q}x)^{2r}$$

where

$$r = l - k, \quad \alpha_{k,r} = k^{2r} \frac{k!}{(k+r)!} \frac{\Gamma(k+1-\mu)}{\Gamma(k+r+1-\mu)},$$

$$x = \frac{z}{kQ^{-k}} \beta_{k,r} = k^{2r} \frac{k!}{(k+r)!} \frac{\Gamma(k+1+\mu)}{\Gamma(k+r+1+\mu)}.$$

Using the results of [2] we have

$$(10) \quad \alpha_{k,r} = 1 - \frac{r^2+r}{k} + \frac{\mu r}{k} + O\left(\frac{r^4}{k^2}\right),$$

$$|r| \leq \sqrt[4]{k}, \quad k \rightarrow \infty.$$

$$\beta_{k,r} = 1 - \frac{r^2+r}{k} - \frac{\mu r}{k} + O\left(\frac{r^4}{k^2}\right).$$

If we substitute (10) into (9), we get

$$(11a) \quad \sum_{r=-k}^{\infty} \alpha_{k,r} Q^{r^2} x^{2r} = \sum_{r=-\infty}^{\infty} Q^{r^2} x^{2r} - \frac{1}{k} \sum_{r=-\infty}^{\infty} (r^2+r) Q^{r^2} x^{2r} + \frac{\mu}{k} \sum_{r=-\infty}^{\infty} r Q^{r^2} x^{2r} + O\left(\frac{1}{k^2}\right),$$

$$(11b) \quad \sum_{r=-k}^{\infty} \beta_{k,r} Q^{r^2} (\sqrt{q}x)^{2r} = \sum_{r=-\infty}^{\infty} Q^{r^2} (\sqrt{q}x)^{2r} - \frac{1}{k} \sum_{r=-\infty}^{\infty} (r^2+r) Q^{r^2} (x\sqrt{q})^{2r} -$$

$$- \frac{\mu}{k} \sum_{r=-\infty}^{\infty} r Q^{r^2} (\sqrt{q}x)^{2r} + O\left(\frac{1}{k^2}\right).$$

The quotient of the factors in front of the sums in (9) is

$$(12) \quad \frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{(\sqrt{q})^{-2k}}{t} \frac{\Gamma(k+1+\mu)}{\Gamma(k+1-\mu)} = \frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{\mu^{2\mu}}{\sqrt{q}} x^{-2\mu} \left(1 + \frac{\mu}{k} + O\left(\frac{\mu^4}{k^2}\right)\right).$$

By (11a), (11b), (12) and (7), $a_1(t)$ can be written as

$$a_1(t) = -\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{\mu^{2\mu}}{\sqrt{q}} \left(1 + \frac{\mu}{k} + O\left(\frac{\mu^4}{k^2}\right)\right) \times$$

$$\frac{x^{-2\mu} \theta_3(x) - \frac{1}{k} \sum_{r=-\infty}^{\infty} (r^2+r(1-\mu)) Q^{r^2} x^{2r} + O\left(\frac{1}{k^2}\right)}{\theta_3(x) - \frac{1}{k} \sum_{r=-\infty}^{\infty} (r^2+r(1+\mu)) Q^{r^2} (\sqrt{q}x)^{2r} + O\left(\frac{1}{k^2}\right)},$$

where

$$\theta_3(x) = \sum_{k=-\infty}^{\infty} Q^{k^2} x^{2k} = q_0 \prod_{n=1}^{\infty} (1 + Q^{2n-1} x^2)(1 + Q^{2n-1} x^{-2}); \quad q_0 = \prod_{n=1}^{\infty} (1 - Q)^{2n}$$

is a well-known theta function [1].

Consequently

$$(13) \quad a_1(t) = -\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{\mu^{2\mu}}{\sqrt{q}} \frac{x^{-2\mu} \theta_3(x)}{\theta_3(\sqrt{qx})} \left(1 + O\left(\frac{1}{k}\right) \right),$$

where

$$\frac{k-1}{k} Q \equiv x < 1.$$

We shall be ready, if we show in (13) that the function

$$(14) \quad H(x) = \frac{x^{-2\mu} \theta_3(x)}{\theta_3(\sqrt{qx})}$$

is not constant on the interval $[Q, 1]$.

In order to prove this we suppose the contrary, i.e., $H(x) = \text{const.}$ on $[Q, 1]$. Then $H(x) = H(\sqrt{Q}x)$ on the interval $[\sqrt{Q}, 1]$. Since

$$H(\sqrt{Q}x) = \frac{x^{-2\mu} \theta_2(x)}{\theta_2(x\sqrt{q})},$$

where

$$\theta_2(x) = \sum_{k=-\infty}^{\infty} Q^{\left(k-\frac{1}{2}\right)^2} x^{2k-1} = q_0 Q^{\frac{1}{4}} \left(x + \frac{1}{x} \right) \prod_{n=1}^{\infty} (1 + Q^{2n} x^2)(1 + Q^{2n} x^{-2})$$

is also a theta function, it follows that

$$(15) \quad \theta_2^2(x) \theta_3^2(\sqrt{qx}) - \theta_2^2(\sqrt{qx}) \theta_3^2(x) = 0,$$

for

$$x \in [\sqrt{Q}, 1].$$

However, it is known ([1], p. 359) that

$$(16) \quad \theta_2^2(z) \theta_3^2(1) - \theta_2^2(z) \theta_3^2(1) = \theta_1^2(z) \theta_4^2(1)$$

for arbitrary $z \neq 0$, where

$$\theta_1(z) = \sum_{k=-\infty}^{\infty} (-1)^k Q^{\left(k-\frac{1}{2}\right)^2} z^{2k-1} = q_0 Q^{\frac{1}{4}} \left(x - \frac{1}{x} \right) \prod_{n=1}^{\infty} (1 - Q^{2n} x^2)(1 - Q^{2n} x^{-2})$$

and

$$\theta_4(z) = \sum_{k=-\infty}^{\infty} (-1)^k Q^{k^2} z^{2k} = q_0 \prod_{k=1}^{\infty} (1 - Q^{2n-1} x^2)(1 - Q^{2n-1} x^{-2})$$

are also theta functions. Let us substitute $x=1$ in (15) and $z=\sqrt{q}$ in (16), then we have

$$\theta_1^2(\sqrt{q}) \theta_4^2(1) = 0.$$

But this is impossible, because all of the zeros of $\theta_4(z)$ are $e^{\left(n+\frac{1}{2}\right)\log Q+m\pi i}$ ($m, n=0, \pm 1, \pm 2, \dots$), so $\theta_4^2(1) \neq 0$, and the zeros of $\theta_1(z)$ are $e^{n \log Q+mi}$, therefore $\theta_1^2(\sqrt{q})$ differs from zero for $\mu \neq n$ (notice that μ is not integer).

Remark 1. The range of $a_1(t)$ can be estimated as $t \rightarrow \infty$, i.e., there are $\gamma_1(Q)$, $\gamma_2(Q)$ such that

$$\gamma_1(Q) \equiv \liminf_{t \rightarrow \infty} a_1(t) < \overline{\lim}_{t \rightarrow \infty} a_1(t) \equiv \gamma_2(Q).$$

It is easy to see, that $\theta_3(x)$ is monotonically decreasing on the interval $[\sqrt{q}Q, 1]$, therefore

$$\frac{\theta_3(1)}{\theta_3(\sqrt{q}Q)} \equiv \frac{1}{x^{2\mu}} \frac{\theta_3(x)}{\theta_3(\sqrt{q}x)} \equiv \frac{1}{Q^2} \frac{\theta_3(Q)}{\theta_3(\sqrt{q})}.$$

On account of $\theta_3(Q) = \frac{1}{Q} \theta_3(1)$ the infinite product representation of $\theta_3(x)$ shows that

$$\frac{\theta_3(1)}{\theta_3(\sqrt{q}Q)} = qQ \prod_{k=1}^{\infty} \frac{(1+Q^{2k-1})^2}{(1+Q^{2k-1}q)(1+Q^{2k-1}q^{-1})}$$

and

$$\frac{1}{Q^2} \frac{\theta_3(Q)}{\theta_3(\sqrt{q})} = \frac{1}{qQ} \prod_{k=1}^{\infty} \frac{(1+Q^{2k-1})^2}{(1+Q^{2k-1}q)(1+q^{-1}Q^{2k-1})}.$$

Consequently

$$\gamma_1(Q) = -\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{\mu^{2\mu}}{q^2 Q} \prod_{k=1}^{\infty} \frac{(1+Q^{2k-1})^2}{(1+Q^{2k-1}q)(1+Q^{2k-1}q^{-1})}$$

and

$$\gamma_2(Q) = -\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \mu^{2\mu} Q \sqrt{q} \prod_{k=1}^{\infty} \frac{(1+Q^{2k-1})^2}{(1+Q^{2k-1}q)(1+Q^{2k-1}q^{-1})}.$$

We should like to remark that in the case of $\nu=0$ a correct estimation can be given. In this case $q=Q$ and

$$a_1(t) = -q^{-\frac{1}{4}} \frac{\theta_3(x)}{\theta_2(x)} \left(1 + O\left(\frac{1}{k}\right) \right).$$

Making use of the relation (see [1], p. 359)

$$\theta_3^2(x)\theta_3^2(1) - \theta_2^2(x)\theta_2^2(1) = \theta_4^2(x)\theta_4^2(1),$$

we have

$$\frac{\theta_3^2(x)}{\theta_2^2(x)} \equiv \frac{\theta_2^2(1)}{\theta_3^2(1)} \quad \text{or} \quad \frac{\theta_3(x)}{\theta_2(x)} \equiv \frac{\theta_2(1)}{\theta_3(1)},$$

where the equality holds when $\theta_4(x)=0$, i.e. at $x=\sqrt{q}$ on $[q, 1]$, hence

$$\limsup_{t \rightarrow \infty} a_1(t) = -q^{-\frac{1}{4}} \frac{\theta_2(1)}{\theta_3(1)} = -2 \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^2 = \tilde{\gamma}_2(q).$$

On the other hand a simple calculation shows

$$\frac{\theta_3(\sqrt{q}x)}{\theta_2(\sqrt{q}x)} = \frac{\theta_2(x)}{\theta_3(x)}$$

and

$$\liminf_{t \rightarrow \infty} a_1(t) = -q^{-\frac{1}{4}} \frac{\theta_3(1)}{\theta_2(1)} = -\frac{1}{2\sqrt{q}} \prod_{n=1}^{\infty} \left(\frac{1+q^{2n-1}}{1+q^{2n}} \right)^2 = \tilde{\gamma}_1(q).$$

Remark 2. It is clear that in the case of $\nu = -1$ (i.e. when $\mu = 1$) the solutions of (2), (3) cannot be written in the form of (4). ($u(t)$ in (4) does not exist if $\mu = 1$, since the function $\Gamma(s)$ has a pole at $s = 0$.) It is easy to see that $H(x)$ is continuous, and is a constant if $\mu = 1$.

Therefore

$$\lim_{\mu \rightarrow 1} [\max_{x \in [0, 1]} H(x) - \min_{x \in [0, 1]} H(x)] = 0.$$

In spite of this, we shall prove the following:

(A) The length of the interval $[A_1, A_2]$ tends to a positive constant as $\mu \rightarrow 1$.

Let $\mu = 1 - \varepsilon$. Then using the relation

$$\Gamma(\varepsilon)\Gamma(1-\varepsilon) = \frac{\pi}{\sin \pi\varepsilon}.$$

(see [5]), for the coefficient of $H(x)$ in (13) we have

$$(17) \quad \frac{1}{\sqrt{q}} (1-\varepsilon)^{2(1-\varepsilon)} \frac{\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)} = \frac{1}{\sqrt{q}} \frac{(1-\varepsilon)^{(1-2\varepsilon)}}{(\Gamma(1-\varepsilon))^2} \frac{\pi}{\sin \pi\varepsilon}.$$

On the other hand

$$x^{-2(1-\varepsilon)} = x^{-2}[1 + 2\varepsilon \log x + O(\varepsilon^2)]$$

and

$$\theta_3(x) = \sum_{k=-\infty}^{\infty} (\sqrt{q}^{k^2})^{\frac{1}{1-\varepsilon}} x^{2k} = \sum_{k=-\infty}^{\infty} \sqrt{q}^{k^2} x^{2k} + \varepsilon \log \sqrt{q} \sum_{k=-\infty}^{\infty} k^2 \sqrt{q}^{k^2} x^{2k} + O(\varepsilon^2).$$

Let us introduce the notations

$$\theta_3(x|\sqrt{q}) = \sum_{k=-\infty}^{\infty} \sqrt{q}^{k^2} x^{2k},$$

$$F_1(x) = \sum_{k=-\infty}^{\infty} k^2 \sqrt{q}^{k^2} x^{2k}, \quad F_2(x) = \sum_{k=-\infty}^{\infty} (k-1)^2 \sqrt{q}^{k^2} x^{2k},$$

$$\bar{\varepsilon} = \varepsilon \log \sqrt{q}.$$

Thus we have

$$H(x) = \frac{1 + 2\varepsilon \log x + O(\varepsilon^2)}{(\sqrt{q})^{-1}} \frac{\theta_3(x|\sqrt{q}) + \bar{\varepsilon}F_1(x) + O(\varepsilon^2)}{\theta_3(x|\sqrt{q}) + \bar{\varepsilon}F_2(x) + O(\varepsilon^2)}$$

and consequently

$$H(x) = \sqrt{q}(1 + \varepsilon \log x^2 + O(\varepsilon^2)) \left(1 + \bar{\varepsilon} \frac{F_1(x) - F_2(x)}{\theta_3(x|\sqrt{q}) + \bar{\varepsilon}F_2(x)} + O(\varepsilon^2) \right).$$

Therefore

$$(18) \quad \begin{aligned} & \max_{x \in [\sqrt{q}, 1]} H(x) - \min_{x \in [\sqrt{q}, 1]} H(x) = \\ & = \bar{\varepsilon} \left(1 + \frac{F_1(x_1) - F_2(x_1)}{\theta_3(x_1|\sqrt{q}) + \bar{\varepsilon}F_2(x)} - \frac{F_1(x_2) - F_2(x_2)}{\theta_3(x_2|\sqrt{q}) + \bar{\varepsilon}F_2(x_2)} + O(\bar{\varepsilon}) \right), \end{aligned}$$

for any $x_1, x_2 \in [\sqrt{q}, 1]$. By (17) and (18) (A) will be proved if we show that

$$\frac{F_1(x) - F_2(x)}{\theta_3(x|\sqrt{q})} = \text{const.}$$

does not hold identically on the interval $[\sqrt{q}, 1]$. Notice that

$$F_1(x) - F_2(x) = \sum_{k=-\infty}^{\infty} 2k \sqrt{q}^{-k^2} x^{2k} \theta_3(x|\sqrt{q}),$$

and

$$\frac{d}{dx} \theta_3(x|\sqrt{q}) = \sum_{k=-\infty}^{\infty} 2k x^{2k-1} \sqrt{q}^{-k^2}.$$

A simple computation shows that

$$\frac{x\theta_3'(x|\sqrt{q})}{\theta_3(x|\sqrt{q})} \neq \text{const.},$$

which completes the proof of (A).

Finally we give a numerical example. The calculation was made for the differential equation

$$y''(t) = y\left(\frac{1}{2}t\right)$$

by the computer CDC 3300.

Table 1

x	0,5	0,603553	0,707107
approx. values of $a_1(t)$ by (13)	-1,189204	-1,189207	-1,189210
values of t for x 's on the interval I_{10}	$0,102400 \times 10^5$	$0,123608 \times 10^5$	$0,144815 \times 10^5$
exact values of $a_1(t)$	-1,189188	-1,189206	-1,189223

$$\tilde{\gamma}_1\left(\frac{1}{2}\right) = -1,189210, \quad \tilde{\gamma}_2\left(\frac{1}{2}\right) = -1,189204.$$

Table 2
The exact values of $a_1(t)$ for small t 's

t_{\max}	$a_1(t)_{\max}$	t_{\min}	$a_1(t)_{\min}$
2,3	-1,176889	4,7	-1,192039
8,7	-1,188145	15	-1,189724
25	-1,188911	40,8	-1,189310
65,4	-1,189076	100	-1,189301

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ON THE UPDATED MAXIMUM LIKELIHOOD ESTIMATORS

by

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1. Introduction

For the problem of estimation of the parameters of time series the method of BOX and JENKINS [1] is a popular one. It is based on the model

$$\xi_n = \sum_{j=1}^p \varphi_j \xi_{n-j} - \sum_{j=1}^q \psi_j \varepsilon_{n-j} + \varepsilon_n,$$

where ξ_n is the n -th element of the investigated process, the random variables ε_n are supposed to be independent or at least orthogonal, p, q are known and $\varphi_1, \dots, \varphi_p, \psi_1, \dots, \psi_q$ are unknown parameters. BOX and JENKINS developed a modified version of the maximum likelihood method for estimating the unknown parameters of the process. Their estimators are "off-line" in the sense that a set of observations is made and estimates of the system parameters are computed on the basis of the entire set of observations. There is a general aim in the engineering literature producing on-line estimation methods. The advantages of the on-line methods are the following:

- i) they need small memory-capacity;
- ii) they need short computer time;
- iii) in case of on-line control of the process they are able to follow slow changes of the parameters.

The disadvantage of the on-line methods is usually their inefficiency. On-line methods in time series analysis were developed by SAKRISON [8], who used a different model, and solved roughly speaking the case $q=0$, and by MEHRA [7] who investigated the KALMAN—BUCY model and concluded by updating the roots of the YULE—WALKER equations, which are very inefficient estimators.

In a problem connected with the basis weight control in a paper making process the method of stochastic approximation was used too. We started with the simplest iteration

$$\varphi_k^{(n)} = \varphi_k^{(n-1)} + c_n \varepsilon_n \xi_{n-k} \quad (k = 1, 2, \dots, p)$$

$$\psi_k^{(n)} = \psi_k^{(n-1)} - c_n \varepsilon_n \varepsilon_{n-k} \quad (k = 1, 2, \dots, q)$$

for $n=1, 2, \dots, N$ where N is the sample size, and with $c_n=0.2/n$.

The first results were very encouraging. For parameters p, q small enough the iteration was convergent, stable and served as nearly as good an estimator as the modified maximum likelihood method of BOX and JENKINS. For large p and q the iteration was convergent but the estimators created by it were very poor. These preliminary results have shown that the application of the stochastic approximation for the estimation of the parameters of the BOX—JENKINS model needs some theo-

retical investigation. This investigation has begun in this paper and will be continued hopefully in subsequent papers.

In the present paper updating of maximum likelihood estimators is investigated in case of independent sample elements. It will be proved that the maximum likelihood estimator and its updated version are asymptotically equal. This statement is subject to generalization for time series. For the sake of proving our statement a theorem on stochastic approximation was developed which has interest in stochastic approximation, too.

2. The updated maximum likelihood estimator

Let ξ, ξ_1, ξ_2, \dots be independent, identically distributed random variables with density function $f(x, \vartheta)$ where ϑ is a one-dimensional real parameter. The loglikelihood function of the first N sample elements is

$$L(x_1, x_2, \dots, x_N; \mu) = \sum_{i=1}^N \log f(x_i, \mu).$$

If there are no constraints on μ the maximum likelihood estimator is the root of the equation

$$\sum_{i=1}^N K(x_i, \mu) = 0,$$

where

$$K(x, \mu) = \frac{\partial}{\partial \mu} \log f(x, \mu) = \frac{f_\mu(x, \mu)}{f(x, \mu)}.$$

The asymptotic variance of the ML estimator is $1/NV(\vartheta)$, where

$$V(\mu) = D_\vartheta^2 K(\xi, \mu) = -E_\vartheta \left(\frac{\partial}{\partial \mu} K(\xi, \mu) \right).$$

The regression function

$$R_K(\mu) = E_\vartheta K(\xi, \mu)$$

has the root $\mu = \vartheta$, that is why the stochastic approximation type iteration

$$\vartheta_n = \vartheta_{n-1} + c_n K(\xi_n, \vartheta_{n-1}) \quad (n = 1, 2, \dots, N)$$

may be suggested. The only question remained how to choose the coefficients c_n .

It is well-known in stochastic approximation (cf. WASAN [9]) that in the case $c_n = \frac{c}{n}$ under some particular conditions the asymptotic variance of ϑ_n is v/n , where

$$v = \frac{c^2}{2\alpha c - 1} \sigma^2,$$

with

$$\alpha = -R'_K(\vartheta) = V(\vartheta)$$

and

$$\sigma^2 = D_\vartheta^2 K(\xi, \vartheta) = V(\vartheta).$$

This v has its minimum when $c=1/V(\vartheta)$. In this case $v=1/V(\vartheta)$, i.e. the asymptotic variance of our estimator would be equal to that of the ML estimators. The only problem is that ϑ is the parameter that we are looking for, so we do not know this optimal c .

The situation sketched above motivates the choice of the function

$$H(x, \mu) = \frac{K(x, \mu)}{V(\mu)}.$$

In this case the regression function

$$R_H(\mu) = E_{\vartheta} H(\xi, \mu)$$

has the same root $\mu = \vartheta$ as $R_k(\mu)$ had, but now

$$R'_H(\vartheta) = -1.$$

So the optimal coefficients in this case are $c_n=1/n$. The updated form of the ML estimators is given by the iteration

$$\vartheta_n = \vartheta_{n-1} + \frac{1}{n} H(\xi_n, \vartheta_{n-1}).$$

This estimator was proposed by M. B. NEVELSON and R. Z. HASMINSKII [5]. They proved that ϑ_n is asymptotically as good as the ML estimator.

The investigation of the asymptotic property of the ML estimators is based on the fact that they are near to the average

$$\tilde{\vartheta}_N = \frac{1}{N} \sum_{i=1}^N H(\xi_i, \vartheta)$$

in the sense that $\sqrt{N}(\tilde{\vartheta}_N - \hat{\vartheta}_N)$ tends to 0 stochastically (cf. CRAMER [2]) where $\hat{\vartheta}_N$ is the ML estimator. We shall prove the similar result for the updated ML estimators. Hence we prove that ϑ_N is not only as good as the ML estimator but they are asymptotically equal.

3. A theorem on the stochastic approximation

Let ξ, ξ_1, ξ_2, \dots be independent, identically distributed random variables and let $H(x, \mu)$ be a function of two variables.

Condition 1. For arbitrary $\delta > 0$ there exist $A > 0$ and $\varepsilon > 0$ such that

$$ET(H(\xi, \mu), a) \leq -\varepsilon, \quad \text{for all } a \geq A, \quad \mu \geq \delta,$$

and

$$ET(H(\xi, \mu), a) \geq \varepsilon, \quad \text{for all } a \geq A, \quad \mu \leq -\delta,$$

where

$$T(x, y) = \begin{cases} x & \text{if } |x| \leq y \\ y & \text{if } x > y \\ -y & \text{if } x < -y, \end{cases}$$

i.e. $T(x, y)$ gives an x truncated at y .

Condition 2. There exist $\Delta > 0$ and $c_1 > 0$ such that

$$EH^2(\xi, \mu) \leq c_1 \quad \text{for all } |\mu| \leq \Delta.$$

Condition 3. The function

$$R(\mu) = EH(\xi, \mu)$$

is differentiable for $\mu=0$ with $R(0)=0$, $R'(0)=-1$.

Condition 4. The function $h(\mu)=H(x, \mu)$ is twice differentiable for all real x with respect to the variable μ , and there exist $\Delta_1 > 0$ and $c_2 > 0$, $c_3 > 0$ such that

$$E(H''(\xi, \mu))^2 \leq c_2 \quad \text{for all } |\mu| \leq \Delta_1,$$

and

$$EH^4(\xi, \mu) \leq c_3 \quad \text{for all } |\mu| \leq \Delta_1,$$

where $H''(x, \mu)=h''(\mu)$ is the second partial derivative of the function $H(x, \mu)$ with respect to the variable μ .

THEOREM. *Conditions 1—4 imply that*

$$\sqrt{N}(\vartheta_N - \tilde{\vartheta}_N)$$

tends stochastically to 0, where ϑ_N is generated by the iteration

$$\vartheta_n = \vartheta_{n-1} + T \left(\frac{1}{n} H(\xi_n, \vartheta_{n-1}), \Delta \right) \quad (n = 1, 2, \dots)$$

with arbitrary ϑ_0 ,

$$\tilde{\vartheta}_N = \frac{1}{N} \sum_{i=1}^N H(\xi_i, 0),$$

and $T(x, y)$ and Δ are defined in conditions 1 and 2.

Remark. The idea to truncate the steps in the original Robbins—Monro iteration is due to KOMLÓS and RÉVÉSZ [6].

4. Lemmas used in the proof of the theorem

LEMMA 1. *Condition 1 implies that*

$$\lim_{N \rightarrow \infty} P(\min(\vartheta_k, \vartheta_{k+1}, \dots, \vartheta_N) \geq \delta) = 0$$

and

$$\lim_{n \rightarrow \infty} P(\max(\vartheta_k, \vartheta_{k+1}, \dots, \vartheta_N) \leq -\delta) = 0$$

for all $\delta > 0$ and $k=0, 1, 2, \dots$.

PROOF. We now prove the first assertion. Define $\tilde{\varepsilon}_n$ by

$$\tilde{\varepsilon}_n = \begin{cases} 1 & \text{if } \vartheta_n \geq \delta \\ 0 & \text{if } \vartheta_n < \delta, \end{cases}$$

and ε_N by

$$\varepsilon_N = \prod_{n=k}^{N-1} \tilde{\varepsilon}_i.$$

Then the events $\{\min(\vartheta_k, \vartheta_{k+1}, \dots, \vartheta_N) \geq \delta\}$ and $\{\varepsilon_{N+1} = 1\}$ are equal. The random variable $\varepsilon_n \vartheta_n$ is for $\varepsilon_n = 0$ equal to 0 (of course) and for $\varepsilon_n = 1$ it is at least $\delta - \Delta$ because $\varepsilon_n = 1$ implies that $\vartheta_{n-1} \geq \delta$ and $\vartheta_n - \vartheta_{n-1} \leq -\Delta$. Hence

$$E\varepsilon_n \vartheta_n \geq C$$

with $C = \min(0, \delta - \Delta)$. Moreover

$$\varepsilon_n \vartheta_n = \begin{cases} \varepsilon_{n-1} \vartheta_{n-1} + T\left(\frac{1}{n} H(\xi_n, \vartheta_{n-1}), \Delta\right) & \text{if } \varepsilon_n = 1, \\ 0 & \text{if } \varepsilon_n = 0 \end{cases}$$

and

$$E\left(T\left(\frac{1}{n} H(\xi_n, \vartheta_{n-1}), \Delta\right) \middle| \varepsilon_n = 1\right) \leq -\frac{\varepsilon}{n}$$

if $n\Delta > A$. Hence

$$E\varepsilon_n \vartheta_n \leq E\varepsilon_{n-1} \vartheta_{n-1} - \frac{\varepsilon}{n} P(\varepsilon_n = 1)$$

if

$$n > n_0 = \max\left(k, \left\lceil \frac{A}{\Delta} \right\rceil\right),$$

and

$$E\varepsilon_n \vartheta_n \leq \varepsilon_{n_0} \vartheta_{n_0} - \varepsilon \sum_{j=n_0+1}^n \frac{1}{j} P(\varepsilon_j = 1) \leq E\varepsilon_{n_0} \vartheta_{n_0} - \varepsilon P(\varepsilon_n = 1) \sum_{j=n_0+1}^n \frac{1}{j},$$

because $P(\varepsilon_j = 1)$ is monotone decreasing in j . So we have proved that

$$P(\varepsilon_n = 1) \leq \frac{E\varepsilon_{n_0} \vartheta_{n_0} - C}{\varepsilon \sum_{j=n_0+1}^n \frac{1}{j}},$$

which tends to 0.

LEMMA 2. Condition 2 implies that for all $\alpha < \frac{1}{2}$, and $\delta > 0$

$$\lim_{N \rightarrow \infty} P(\sup_{n \geq N} n^\alpha |\vartheta_n - \vartheta_{n-1}| \varepsilon_{n-1} \leq \delta) = 1,$$

where

$$\varepsilon_j = \begin{cases} 1 & \text{if } |\vartheta_j| \leq \Delta \\ 0 & \text{if } |\vartheta_j| > \Delta. \end{cases}$$

PROOF. Condition 2 implies that

$$E(\vartheta_n - \vartheta_{n-1})^2 \varepsilon_n^2 \leq \frac{1}{n^2} \sup_{|\mu| \leq \Delta} EH^2(\xi, \mu) \leq \frac{C_1}{n^2}.$$

Hence by the Markov inequality

$$\begin{aligned} \mathbf{P}(\sup_{n \geq N} n^\alpha |\vartheta_n - \vartheta_{n-1}| \varepsilon_n > \delta) &\leq \sum_{n=N}^{\infty} \mathbf{P}(n^\alpha |\vartheta_n - \vartheta_{n-1}| \varepsilon_n > \delta) \leq \\ &\leq \frac{C_1}{\delta^2} \sum_{n=N}^{\infty} n^{-2(1-\alpha)} \leq \frac{C_1}{\delta^2} N^{2\alpha-1}, \end{aligned}$$

which tends to 0 as $N \rightarrow \infty$ for $\alpha < \frac{1}{2}$.

LEMMA 3. Define the random variables ε_n by

$$\varepsilon_n = \begin{cases} 1 & \text{if } |\vartheta_n| \leq \delta \\ 0 & \text{if } |\vartheta_n| > \delta, \end{cases}$$

$\tilde{\varepsilon}_n$ by

$$\tilde{\varepsilon}_n = \begin{cases} 1 & \text{if } n^\alpha |\vartheta_n - \vartheta_{n-1}| \leq \delta \\ 0 & \text{if } n^\alpha |\vartheta_n - \vartheta_{n-1}| > \delta, \end{cases}$$

and η_n by

$$\eta_N = \prod_{n=k+1}^N \varepsilon_n \tilde{\varepsilon}_n.$$

The conditions 2—3 imply that for every $\alpha < \frac{1}{2}$ there is a $\delta > 0$ such that

$$\mathbf{E}(\vartheta_N^2 | \vartheta_k, \eta_N = 1) \leq \left(\frac{k+1}{N+1} \right)^{2\alpha} \vartheta_k^2 + \frac{1}{N} C_4,$$

if $|\vartheta_k| \leq \delta$, $N > k$, for arbitrary k .

PROOF. Denote by $G_n(x, \mu)$ the sum

$$G_n(x, \mu) = T(H(x, \mu), n\Delta) + \mu.$$

Then

$$\vartheta_n = \left(1 - \frac{1}{n} \right) \vartheta_{n-1} + \frac{1}{n} G_n(\xi_n, \vartheta_{n-1}).$$

It is easy to see that Conditions 2—3 imply that

$$|\mathbf{E}(G_n(\xi_n, \vartheta_{n-1}) | \vartheta_{n-1}, \tilde{\varepsilon}_n = 1)| \leq c |\vartheta_{n-1}|$$

$$\mathbf{E}(G_n^2(\xi_n, \vartheta_{n-1}) | \vartheta_{n-1}, \tilde{\varepsilon}_n = 1) \leq C_4.$$

if $|\vartheta_{n-1}| < \delta$ with constant C_4 and arbitrarily small c if δ is small enough. Hence

$$\begin{aligned} \mathbf{E}(\vartheta_n^2 | \vartheta_{n-1}, \tilde{\varepsilon}_n = 1) &= \left(1 - \frac{1}{n} \right)^2 \vartheta_{n-1}^2 + \\ &+ \frac{2}{n} \left(1 - \frac{1}{n} \right) \vartheta_{n-1} \mathbf{E}(G_n(\xi_n, \vartheta_{n-1}) | \vartheta_{n-1}, \tilde{\varepsilon}_n = 1) + \\ &+ \frac{1}{n^2} \mathbf{E}(G_n^2(\xi_n, \vartheta_{n-1}) | \vartheta_{n-1}, \tilde{\varepsilon}_n = 1) \leq \left(1 - \frac{\alpha}{n} \right)^2 \vartheta_{n-1}^2 + \frac{C_5}{n^2}, \end{aligned}$$

if $|\vartheta_{n-1}| < \delta$, and $\alpha = 1 - c$. Moreover

$$E(\vartheta_n^2 | \vartheta_{n-1}, \tilde{\varepsilon}_n = 1, \varepsilon_n = 1) \leq E(\vartheta_n^2 | \vartheta_{n-1}, \tilde{\varepsilon}_n = 1),$$

leading to the conclusion of the statement of the Lemma by elementary calculations.

Remark. It is easy to see from the proof of the lemma 3 that its statement remains valid with the modification

$$\eta_N = \prod_{n=k+1}^N \varepsilon_n.$$

We refer to this statement as Lemma 3*.

LEMMA 4. *Conditions 1—3 imply that*

$$\lim_{N \rightarrow \infty} P(\min_{k \leq j \leq N} |j^\alpha \vartheta_j| \geq \delta) = 0$$

for arbitrary

$$\alpha < \frac{1}{2}, \quad \delta > 0, \quad k \geq 0.$$

PROOF. It is easy to prove the above statement for $\alpha = 0$ with the method used for proving Lemma 1 and Lemma 3. Using this fact we can conclude that it is sufficient to prove that there is a $p > 0$ (which does not depend on k) such that

$$\lim_{N \rightarrow \infty} P(\min_{k \leq j \leq N} |j^\alpha \vartheta_j| \geq \delta) \geq p$$

for arbitrary $\alpha < \frac{1}{2}$, $\delta > 0$, $k \geq 0$. That is what we shall prove.

Denote $\tilde{\varepsilon}_n, \eta_N$ for the same variables as in the proof of Lemma 3, but with

$$\varepsilon_n = \begin{cases} 1 & \text{if } 0 \leq \vartheta_n \leq \delta \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq \vartheta_k \leq \delta$

$$E(\vartheta_N^2 | \eta_{N-1} = 1) \leq \left(\frac{k+1}{N+1} \right)^{2\alpha} \delta^2 + \frac{C_4}{N}$$

as Lemma 3 stated it. Hence

$$E(\vartheta_N^2 | \eta_{N-1} = 1) \leq \frac{C_6}{N}$$

if N is large enough, so the conditional probability

$$P(N^\alpha |\vartheta_N| < \delta | \eta_{N-1} = 1)$$

is as near to 1 as we want it if N is large enough. Define τ_N by

$$\tau_N = \vartheta_k + \sum_{n=k+1}^N \eta_{n-1} \tilde{\varepsilon}_n (\vartheta_n - \vartheta_{n-1}),$$

Then $E(\tau_N | \vartheta_k) \leq \vartheta_k$, hence $P(\tau_N < 0 | \eta_{N-1} = 0)$ is bounded from below if $P(\eta_{N-1} = 0)$ is large enough. If $\tilde{\varepsilon}_N = \eta_N = 1$ and $\vartheta_N < 0$, then $\vartheta_N > -\delta n^{-\alpha}$ so we can prove our Lemma by gathering the conditional probabilities.

LEMMA 5. Define ε_n by

$$\varepsilon_n = \begin{cases} 1 & \text{if } |\vartheta_n| \leq \delta \\ 0 & \text{if } |\vartheta_n| > \delta \end{cases}$$

and η_N by

$$\eta_N = \prod_{n=k+1}^N \varepsilon_n.$$

Conditions 1–4 imply that if $k^\alpha |\vartheta_k| \leq \delta$ for some $\alpha \leq \frac{1}{2}$ and with δ small enough, then

$$E(\vartheta_N^2 | \eta_N = 1) \leq \frac{C_7}{N^{2\alpha}}$$

$$E(\vartheta_N^4 | \eta_N = 1) \leq \frac{C_8}{N^{4\alpha}}$$

for all $N > k$.

PROOF. The first inequality follows from Lemma 3*. The proof of the second inequality is similar to the proof of Lemma 3*.

5. The proof of the theorem

Define ε_n and η_n again as in Lemma 5 by

$$\varepsilon_n = \begin{cases} 1 & \text{if } |\vartheta_n| \leq \delta \\ 0 & \text{if } |\vartheta_n| > \delta \end{cases}$$

$$\eta_N = \prod_{n=k+1}^N \varepsilon_n,$$

and denote the event $\{\eta_N = 1\}$ by B_{kN} . Define $\tilde{\varepsilon}_n$ and $\tilde{\eta}_k$ by

$$\tilde{\varepsilon}_n = \begin{cases} 1 & \text{if } n^\alpha |\vartheta_n| \leq \delta \\ 0 & \text{if } n^\alpha |\vartheta_n| < \delta \end{cases}$$

$$\tilde{\eta}_k = \prod_{n=1}^k \tilde{\varepsilon}_n$$

and denote the event $\{\tilde{\eta}_{k-1} = 1, \tilde{\eta}_k = 0\}$ by A_k . The statement of Lemma 4 implies that

$$P\left(\sum_{k=1}^{\infty} A_k\right) = 1.$$

We prove that $P(B_{kN})$ is as near to 1 as we want it for suitable α and δ , and that the conditional moment $E((\vartheta_N - \tilde{\vartheta}_N)^2 | A_k B_{kN})$ is $o\left(\frac{1}{N}\right)$. These assertions imply the statement of our theorem.

Denote by $G(x, \mu)$ the function

$$G(x, \mu) = H(x, \mu) + \mu - H(x, 0).$$

Condition 4 implies that there is a μ_x such that

$$G(x, \mu) = (H'(x, 0) + 1)\mu + \frac{1}{2}H''(x, \mu_x)\mu^2$$

and for $|\mu| < \Delta_1$

$$E[H''(\xi, \mu_\xi)]^2 \leq C_2.$$

If $\delta < \frac{1}{2}\Delta$, and the event $A_k B_{kN}$ occurred, there is no need for truncation between k and N in the iteration, hence in this case

$$\vartheta_n = \left(1 - \frac{1}{n}\right)\vartheta_{n-1} + \frac{1}{n}(H(\xi_n, 0) + G(\xi_n, \vartheta_{n-1})).$$

Denote by τ_n the difference $\vartheta_n - \tilde{\vartheta}_n$, where

$$\tilde{\vartheta}_n = \frac{1}{n} \sum_{i=1}^n H(\xi_i, 0),$$

then

$$\tilde{\vartheta}_n = \left(1 - \frac{1}{n}\right)\tilde{\vartheta}_{n-1} + \frac{1}{n}H(\xi_n, 0),$$

so

$$\tau_n = \left(1 - \frac{1}{n}\right)\tau_{n-1} + \frac{1}{n}G(\xi_n, \vartheta_{n-1})$$

as long as $k < n \leq N$ and the event $A_k B_{kN}$ occurred. Nevertheless we can define the sequence τ_n by the above iteration starting from some value

$$\tau_k = \vartheta_k - \tilde{\vartheta}_k$$

without any reference to the occurrence of the event $A_k B_{kN}$. In this case the above calculation shows that the iterated τ_N equals $\vartheta_N - \tilde{\vartheta}_N$ if $A_k B_{kN}$ has occurred. That is why estimating the moments of the τ_n produced by the above unrestricted iteration we get estimates of the conditional moments of the difference $(\vartheta_N - \tilde{\vartheta}_N)$.

Denote by \mathcal{A}_n the σ -algebra generated by the random variables $\{\xi_1, \xi_2, \dots, \xi_n\}$. Let us note that it has no importance whether ϑ_0 is a random variable or it is a constant. If ϑ_0 is a random variable, however, then let \mathcal{A}_n be the σ -algebra generated by $\{\vartheta_0, \xi_1, \xi_2, \dots, \xi_n\}$.

$$\begin{aligned} E(\tau_n^2 | \mathcal{A}_{n-1}) &= \left(1 - \frac{1}{n}\right)^2 \tau_{n-1}^2 + \frac{2}{n} \left(1 - \frac{1}{n}\right) \tau_{n-1} \vartheta_{n-1}^2 E(G_1(\xi_n, \vartheta_{n-1}) | \mathcal{A}_{n-1}) + \\ &+ \frac{1}{n^2} E(G^2(\xi_n, \vartheta_{n-1}) | \mathcal{A}_{n-1}), \end{aligned}$$

where

$$G_1(x, \mu) = \mu^{-2} [G(x, \mu) - (H'(x, 0) + 1)\mu].$$

Condition 4 implies that for $|\vartheta_{n-1}| < \delta < A_1$

$$E(G_1^2(\xi_n, \vartheta_{n-1}) | \mathcal{A}_{n-1}) \leq C_2$$

and

$$E(G^2(\xi_n, \vartheta_{n-1}) | \mathcal{A}_{n-1}) \leq C_9 \vartheta_{n-1}^2.$$

Using the Schwarz-inequality twice and by Lemma 5 we can conclude that in case the event $A_k B_{kN}$ has occurred, then

$$E(\tau_n^2 | \mathcal{A}_{n-1}) \leq \left(1 - \frac{1}{n}\right)^2 \tau_{n-1}^2 + C_{10} n^{-1-3\alpha}.$$

Denote $\frac{1}{P(A_k)} \int_{A_k B_{kN}} \tau_n^2 dP$ by δ_n then for $k < n$

$$\delta_n \leq \left(1 - \frac{1}{n}\right)^2 \delta_{n-1} + C_{10} n^{-1-3\alpha},$$

hence

$$\delta_n \leq \left(\frac{k}{n}\right)^2 \delta_k + C_{10} n^{-2} \sum_{j=k+1}^n n^{-3\alpha+1} \leq \left(\frac{k}{n}\right)^2 \delta_k + C_{11} n^{-3\alpha}.$$

Here $\delta_k < C_{12} k^{-2\alpha}$, which is unfortunately large. That is why we have to change our initial τ_k to 0, which would imply that

$$\vartheta_n - \tilde{\vartheta}_n = \tau_n - \frac{1}{n} \sum_{i=1}^k H(\xi_i, 0),$$

if $A_k B_{kN}$ had occurred. In this case

$$\delta_n \leq C_{11} n^{-3\alpha}$$

and the conditional probability $P(\bar{B}_{kN} | A_k)$ is estimable by

$$C_{12} \sum_{n=k+1}^N \delta_n \leq C_{13} k^{1-3\alpha}$$

if $\alpha > \frac{1}{3}$. Hence $P(\bar{B}_{kN} | A_k)$ is as small as we want it and we can finish our proof by elementary calculation.

6. The multi-dimensional case

In the one-dimensional case there is no difference between the iterations on the functions K and H from the point of view of computer time. In the multidimensional case, however, the function H would be equal to

$$V^{-1}(\mu)K(x, \mu)$$

where $K(x, \mu)$ is a vector formed from the partial derivatives of the likelihood function L with respect to the coordinates of the parameter μ , and $V(\mu)$ is the covariance matrix of the vector $K(\xi, \mu)$. The inversion of the matrix $V(\mu)$ in each step of the iteration would be a great disadvantage in the multidimensional version of our iteration. That is why our theorem was not generalized to the multidimensional case.

There are methods of non-linear optimization without inversion, developed first by FLETCHER and POWELL [4] (see also in [3]). Their iteration includes a separate iteration for $V^{-1}(\vartheta_n)$, and that is what we should like to apply in the multidimensional case. Hopefully the asymptotic equality of the ML estimator and its updated version remains valid for this case too.

7. The updated Box—Jenkins estimator

Let us return to the ARMA (p, q) model

$$\zeta_n = \sum_{j=1}^p \varphi_j \zeta_{n-j} - \sum_{j=1}^q \psi_j \varepsilon_{n-j} + \varepsilon_n$$

investigated in the introduction. The basic idea of the BOX—JENKINS method is to replace the clumsy exact likelihood function by its approximation based on the error term

$$\varepsilon_n = \zeta_n - \sum_{j=1}^p \varphi_j \zeta_{n-j} + \sum_{j=1}^q \psi_j \varepsilon_{n-j}.$$

These ε_n are calculated step-by-step from some initial values $\varepsilon_1, \dots, \varepsilon_q$ and from the sample ζ_1, \dots, ζ_N . Our function K is now

$$K = -\frac{\partial}{\partial \vartheta} \varepsilon^2 = -2\varepsilon \frac{\partial}{\partial \vartheta} \varepsilon,$$

where ϑ is the vector formed from the parameters $(\varphi_1, \dots, \varphi_p, \psi_1, \dots, \psi_q)$. So for the sake of updating the BOX—JENKINS estimators we have to calculate by iteration the derivatives $\frac{\partial}{\partial \vartheta} \varepsilon$ too.

Let us denote these derivatives by γ , a vector of dimension $(p+q)$. The iteration for γ is:

$$\gamma_j^{(n)} = -\zeta_{n-j} + \sum_{i=1}^q \psi_i^{(n-1)} \gamma_j^{(n-i)} \quad (j = 1, \dots, p),$$

$$\gamma_j^{(n)} = \varepsilon_{n-j} + \sum_{i=1}^q \psi_i^{(n-1)} \gamma_j^{(n-i)} \quad (j = p+1, \dots, p+q).$$

(Let us remark that the lack of this iteration very likely caused the bias of the first computer-method mentioned in the introduction for large p, q .)

Denote the covariance matrix of the stationary solution of this iteration for fixed $\mathfrak{g}^{(n)} = \mathfrak{g}$ by $V(\mathfrak{g})$, then the updated Box—Jenkins estimator is the following:

$$\eta^{(n)} = \mathfrak{g}^{(n-1)} - \frac{2}{n} \varepsilon_n V^{-1}(\mathfrak{g}^{(n-1)}) \gamma^{(n)},$$

where ε_n is a scalar, $\gamma^{(n)}$ is vector of dimension $r = p + q$ and $V^{-1}(\mu)$ is matrix of dimension $r \times r$. The sequence ε_n is generated by

$$\varepsilon_n = \xi_n - \sum_{j=1}^p \varphi_j^{(n-1)} \xi_{n-j} + \sum_{j=1}^q \psi_j^{(n-1)} \varepsilon_{n-j},$$

and $\gamma^{(n)}$ is generated by the above written iteration.

In the special case $p=0$, $q=1$ the iteration has the form

$$\varepsilon_n = \xi_n + \mathfrak{g}^{(n-1)} \varepsilon_{n-1},$$

$$\gamma^{(n)} = \varepsilon_{n-1} + \mathfrak{g}^{(n-1)} \gamma^{(n-1)},$$

$$\mathfrak{g}^{(n)} = \mathfrak{g}^{(n-1)} - \frac{1}{n} \varepsilon_n [1 - (\gamma^{(n-1)})^2] \gamma^{(n)},$$

where all the quantities are scalar. Perhaps some truncation on $\mathfrak{g}^{(n)}$ would be suitable, e.g. to cut it down to an absolute value of 1. In the general case the easiest way is to check the error terms ε_n and in case of extremely large absolute values to cut down $\mathfrak{g}^{(n)}$ simply to 0.

The disadvantage of the present form of the iteration is the inversion of the matrix V in each step. We intend to substitute a FLETCHER—POWELL type iteration for it and in the final form the iteration will be running on four lines:

- (i) the iteration for the error term,
- (ii) the iteration for the derivatives,
- (iii) the iteration for the inverse matrix,
- (iv) the iteration for the parameter that we are looking for.

A further slight disadvantage in our method is its dependence on the initial values $\gamma^{(0)}$, $\mathfrak{g}^{(0)}$, $\varepsilon_1, \dots, \varepsilon_q$; that is why a certain feedback is proposed for off-line situations. After the first run we can make the next runs using the final parameters of the previous run as starting values and run over the same data field as many times as we want, counting the seps in each run again from 1 or counting them continuously from 1 to kN where N is the sample size and is the number of runs. In the latter case, if the iteration is convergent, its limit equals the Box—Jenkins estimators, so our method provides a method of numerical evaluation. Our impression is that for small N a relatively large k is needed and for sample sizes large enough k is small, perhaps 2 or 3. Having only asymptotic results on all types of estimators the question is still open whether it is worthwhile for small N to locate the exact minimum of the likelihood function. The finite sample properties of our iteration with small k might be equally good (or bad) as the finite sample properties of the ML estimators.

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PARAMETRIC APPROXIMATION OF CONTINUOUS FUNCTIONS WITH BOUNDED VARIATION

by

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The problem of parametric approximation of functions was stated by BL. SENDOV [1].

Let H_n be the set of all algebraic polynomials of degree n and let us denote

$$\hat{H}_m = \{p(x) | p(x) \in H_m, p'(x) \geq 0, x \in [a, b]; p(a) = a, p(b) = b\}.$$

For every function continuous in the interval $[a, b]$ the number

$$E_{n,m}(f) = \inf_{p \in \hat{H}_m} \inf_{Q \in H_n} \max_{x \in [a,b]} |f(p(x)) - Q(x)|$$

is called the best uniform parametric approximation of order (m, n) for the function f .

The uniform parametric approximation has the following properties: for some classes of functions with singularities the order of $E_{n,n}(f)$ is better than the order of usual uniform polynomial approximation and the order of $E_{n,n}(f)$ in these cases is about the same as the order of the rational approximation.

For example:

I. Let us consider the function $|x|$, $x \in [-1, 1]$.

1. For the best uniform (Chebishev) approximation by means of algebraic polynomials of degree n we have [3]

$$E_n(|x|) = O\left(\frac{1}{n}\right).$$

2. For the best uniform rational approximation by means of rational functions of degree n D. NEWMAN [4] has obtained that

$$E_n^R(|x|) = O(e^{-\sqrt{n}}).$$

3. For the best uniform parametric approximation BL. SENDOV [2] obtains that

$$E_{n,n}(|x|) \sim 4(1 + \sqrt{2})^{-2n}.$$

II. If the function f is piecewise analytic in $[a, b]$ then:

1. For the best uniform approximation by means of algebraic polynomials we have

$$E_n(f) = O\left(\frac{1}{n}\right).$$

2. For the best uniform rational approximation P. SZÜSZ and P. TURÁN [5] have obtained that

$$E_n^R(f) = O(e^{-c\sqrt{n}}).$$

3. For the best uniform parametric approximation J. SZABADOS [6] has obtained that

$$E_{n, c_1 \sqrt{n}}(f) \leq k(f) e^{-c\sqrt{n}}.$$

In this paper we obtain that this analogy between rational and parametric approximation exists also in the case when f is a continuous function with bounded variation in $[0, 1]$.

The following results are known:

III. If f is a continuous function with bounded variation and $f \in \text{Lip } \alpha$, ($0 < \alpha < 1$), then:

1. For the best uniform polynomial approximation we have

$$E_n(f) = O(n^{-\alpha}).$$

2. For the best uniform rational approximation G. FREUD [7] obtains that

$$E_n^R(f) = O\left(\frac{\log^2 n}{n}\right).$$

IV. If f is a continuous function with bounded variation in $[a, b]$ and

$$\omega(f; \delta) \leq k \log^{-\alpha} \frac{1}{\delta}, \quad (\alpha > 0)$$

where $\omega(f; \delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|$ is the modulus of continuity of f then:

1. For the best uniform polynomial approximation we have

$$E_n(f) = O(\log^{-\alpha} n).$$

2. For the best uniform rational approximation G. FREUD [7] obtains that

$$E_n^R(f) = O(n^{-\frac{\alpha}{2+\alpha}}).$$

In Theorem 1 of the present paper we prove that if f is a continuous function with bounded variation in $[0, 1]$ then there exists an absolute constant c_0^* such that for every $k > 0$ and every natural number n such that $kn \geq 1$ we have

$$E_{n, n}(f) \leq c_0 \left[\omega\left(f; \frac{\log kn}{kn}\right) + (v+1) \frac{\log kn}{n} \right]$$

From here we obtain an analogy between the parametric and rational approximation also in the cases III and IV.

* Here and in what follows c_0, c_1, \dots denote positive constants.

1. In Theorem 2 we prove that if f is a continuous function with bounded variation in $[0, 1]$ and

$$\omega(f; \delta) \leq c_2 \delta^\alpha, \quad (\alpha < 1),$$

then

$$E_{n,n}(f) = O\left(\frac{\log n}{n}\right).$$

2. In Theorem 3 we prove that if f is a continuous function with bounded variation v in $[0, 1]$ and

$$\omega(f; \delta) \leq k \log^{-\alpha} \frac{1}{\delta}, \quad (\alpha > 0),$$

then

$$E_{n,n}(f) \leq c_3(v, \alpha) n^{-\frac{\alpha}{1+\alpha}}.$$

LEMMA. For every continuous function f with bounded variation v in the interval $[0, 1]$ and for every $\varepsilon > 0$ there exists a monotone continuous function

$$\varphi_\varepsilon(t), \quad t \in [0, 1], \quad \varphi_\varepsilon(0) = 0, \quad \varphi_\varepsilon(1) = 1$$

such that

$$\omega(f(\varphi_\varepsilon(t)); \delta) \leq v\delta + \varepsilon.$$

PROOF. Let $\varepsilon > 0$ be arbitrary. Let us consider the function $g_\varepsilon(x) = \sqrt[\varepsilon]{f(x) + \frac{\varepsilon}{2}x}$.

The function g_ε is strictly monotone and continuous in $[0, 1]$ and $g_\varepsilon \in \left[0, v + \frac{\varepsilon}{2}\right]$.

Consequently there exists a monotone and continuous function $\varphi_\varepsilon(x) = g_\varepsilon^{-1}\left(vx + \frac{\varepsilon}{2}x\right)$ in $[0, 1]$. For the function φ_ε we have $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(1) = 1$ and for every t and $u \in [0, 1]$

$$\begin{aligned} |f(\varphi_\varepsilon(t)) - f(\varphi_\varepsilon(u))| &\leq \left| f\left(g_\varepsilon^{-1}\left(vt + \frac{\varepsilon}{2}t\right)\right) - f\left(g_\varepsilon^{-1}\left(vu + \frac{\varepsilon}{2}u\right)\right) \right| \leq \\ &\leq \frac{g_\varepsilon^{-1}\left(vt + \frac{\varepsilon}{2}t\right)}{g_\varepsilon^{-1}\left(vu + \frac{\varepsilon}{2}u\right)} \left| \sqrt[\varepsilon]{f(x) + \frac{\varepsilon}{2}x} - \sqrt[\varepsilon]{f(x) + \frac{\varepsilon}{2}x} \right| \leq \\ &\leq \left| g_\varepsilon\left(g_\varepsilon^{-1}\left(vt + \frac{\varepsilon}{2}t\right)\right) - \frac{\varepsilon}{2}g_\varepsilon^{-1}\left(vt + \frac{\varepsilon}{2}t\right) - g_\varepsilon\left(g_\varepsilon^{-1}\left(vu + \frac{\varepsilon}{2}u\right)\right) + \frac{\varepsilon}{2}g_\varepsilon^{-1}\left(vu + \frac{\varepsilon}{2}u\right) \right| \leq \\ &\leq \left| vt + \frac{\varepsilon}{2}t - \frac{\varepsilon}{2}u - vu + \frac{\varepsilon}{2}\left[g_\varepsilon^{-1}\left(vu + \frac{\varepsilon}{2}u\right) - g_\varepsilon^{-1}\left(vt + \frac{\varepsilon}{2}t\right)\right] \right| \leq v|t - u| + \varepsilon. \end{aligned}$$

This proves the lemma.

Remark 1. It is easy to see that if f is monotone then for every $\varepsilon > 0$ there exists a monotone continuous function $\varphi_\varepsilon(t)$, $t \in [0, 1]$, $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(1) = 1$ such that

$$\left| f(\varphi_\varepsilon(t)) - \left[\left(v + \frac{\varepsilon}{2}\right)t + f(0) \right] \right| \leq \varepsilon.$$

In the proof of the next theorem we need the following result from [8]:

THEOREM A. For every continuous monotone function $f(t)$, $t \in [0, 1]$, $|f(t)| \leq M$ and for every k and every natural number n such that $kn \geq l$, there exists a monotone algebraic polynomial $p_n(t) \in H_n$, $t \in [0, 1]$ such that

$$\varphi(k; f, p_n) \leq c_4 \frac{\log kn}{kn}$$

where

$$\begin{aligned} \varphi(k; f, p_n) = \max_{x \in [0, 1]} \max_{y \in [0, 1]} \{ \min_{z \in [0, 1]} \max_{y \in [0, 1]} [k^{-1}|x-y|, |f(x) - p_n(y)|], \\ \max_{z \in [0, 1]} \max_{x \in [0, 1]} [k^{-1}|x-y|, |f(y) - p_n(x)|] \} \end{aligned}$$

is the Hausdorff's distance with a parameter k between f and p_n in the interval $[0, 1]$.

THEOREM 1. Let f be a continuous function with bounded variation v in the interval $[0, 1]$. Then there exists an absolute constant c_0 such that for every k and every natural number n where $kn \geq l$ we have

$$E_{n,n}(f) \leq c \left[\omega \left(f; \frac{\log kn}{kn} \right) + (v+1) \frac{\log kn}{n} \right].$$

PROOF. Since f is a continuous function with bounded variation v from Lemma 1 it follows that for every $\varepsilon > 0$ there exists a monotone function φ_ε , $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(1) = 1$ such that $\omega(f(\varphi_\varepsilon(t)); \delta) \leq v\delta + \varepsilon$. Let us set $\delta = \varepsilon = \frac{1}{n}$. From JACKSON's theorem (see [3]) it follows that there exists an algebraic polynomial $q_n \in H_n$ such that $E_n = \omega(f(\varphi_{1/n}(t)); q_n(t)) \leq 12 \frac{v+1}{n}$. Let us consider the function $\varphi_{1/n}$. From Theorem A it follows that for $\varphi_{1/n}$ in $[0, 1]$ and for every k and natural number n where $kn \geq l$ there exists a monotone algebraic polynomial $p_n \in H_n$ such that

$$\tau(k; \varphi_{1/n}(t), p_n(t)) \leq c_4 \frac{\log kn}{kn}.$$

From the definition of Hausdorff's distance with a parameter k it follows that for every $t \in [0, 1]$ there exists $u \in [0, 1]$ such that

$$(1) \quad |t-u| \leq c_4 \frac{\log kn}{n}, \quad |p_n(t) - \varphi_{1/n}(u)| \leq c_4 \frac{\log kn}{kn}.$$

Let us set

$$\tilde{p}_n(t) = \frac{p_n(t) - p_n(0)}{p_n(1) - p_n(0)}.$$

\tilde{p}_n is a monotone polynomial and $\tilde{p}_n(0) = 0$, $\tilde{p}_n(1) = 1$. Let us estimate $|p_n(0)|$ and $|1 - (p_n(1) - p_n(0))|$:

Since p_n and $\varphi_{1/n}$ are monotone and $\varphi_{1/n}(0) = 0$, $\varphi_{1/n}(1) = 1$, from Theorem A (from (1)) it follows:

1. There exists a point $u \in [0, 1]$ such that

$$|u| \leq c_4 \frac{\log kn}{n}, \quad |p_n(u) - \varphi_{1/n}(0)| \leq c_4 \frac{\log kn}{kn}.$$

2. There exists a point $t \in [0, 1]$ such that

$$|t| \leq c_4 \frac{\log kn}{n}, \quad |\varphi_{1/n}(t) - p_n(0)| \leq c_4 \frac{\log kn}{kn}.$$

Then we have

$$p_n(0) = p_n(u) - \varphi_{1/n}(0) \leq p_n(u) - \varphi_{1/n}(0) \leq c \frac{\log kn}{kn},$$

$$-p_n(0) \leq \varphi_{1/n}(0) - p_n(0) \leq \varphi_{1/n}(t) - p_n(0) \leq c \frac{\log kn}{kn},$$

and consequently

$$|p_n(0)| \leq c \frac{\log kn}{kn}.$$

We have also

$$|1 - (p_n(1) - p_n(0))| = |\varphi_{1/n}(1) - p_n(1) + p_n(0)| \leq |\varphi_{1/n}(1) - p_n(1)| + |p_n(0)|.$$

From Theorem A it follows:

1. There exists a point $u \in [0, 1]$ such that

$$|1 - u| \leq c \frac{\log kn}{n}, \quad |\varphi_{1/n}(1) - p_n(u)| \leq c \frac{\log kn}{kn}.$$

2. There exists a point $t \in [0, 1]$ such that

$$|1 - t| \leq c \frac{\log kn}{n}, \quad |p_n(1) - \varphi_{1/n}(t)| \leq c \frac{\log kn}{kn}.$$

From here we obtain

$$\varphi_{1/n}(1) - p_n(1) \leq \varphi_{1/n}(1) - p_n(u) \leq c \frac{\log kn}{kn},$$

$$p_n(1) - \varphi_{1/n}(1) \leq p_n(1) - \varphi_{1/n}(t) \leq c \frac{\log kn}{kn}.$$

Consequently

$$|\varphi_{1/n}(1) - p_n(1)| \leq c \frac{\log kn}{kn}$$

and

$$|1 - (p_n(1) - p_n(0))| \leq c \frac{\log kn}{kn}.$$

Let $t \in [0, 1]$ be arbitrary. Let u be such that t and u satisfy (1). Then

$$\begin{aligned} & |f(\tilde{p}_n(t)) - q_n(t)| \leq |f(\tilde{p}_n(t)) - f(\varphi_{1/n}(u))| + \\ & + |f(\varphi_{1/n}(u)) - f(\varphi_{1/n}(t))| + |f(\varphi_{1/n}(t)) - q_n(t)| \leq \\ & \leq |f(\tilde{p}_n(t)) - f(\varphi_{1/n}(u))| + v|u - t| \cdot \frac{1}{n} + 12 \cdot \frac{v+1}{n}. \end{aligned}$$

On the other hand

$$\begin{aligned} |\tilde{p}_n(t) - \varphi_{1/n}(u)| &= \left| \frac{p_n(t) - p_n(0)}{p_n(1) - p_n(0)} - \varphi_{1/n}(u) \right| = \\ &= \left| \frac{p_n(t) - (p_n(1) - p_n(0))\varphi_{1/n}(u) - p_n(0)}{p_n(1) - p_n(0)} \right|, \\ 1 - c_4 \frac{\log kn}{kn} &\leq p_n(1) - p_n(0) \leq 1 + c_4 \frac{\log kn}{kn}. \end{aligned}$$

Consequently

$$p_n(1) - p_n(0) \leq 1 + \alpha_n, \quad |\alpha_n| \leq c_4 \frac{\log kn}{kn}.$$

Then

$$\begin{aligned} (3) \quad |\tilde{p}_n(t) - \varphi_{1/n}(u)| &\leq c_5 |p_n(t) - \varphi_{1/n}(u) - \alpha_n \varphi_{1/n}(u) - p_n(0)| \leq \\ &\leq c_5 [|p_n(t) - \varphi_{1/n}(u)| + |p_n(0)| + |\alpha_n \varphi_{1/n}(u)|] \leq c_6 \frac{\log kn}{kn} \end{aligned}$$

because

$$|\varphi_{1/n}(u)| \leq 1, \quad |\alpha_n \varphi_{1/n}(u)| \leq |\alpha_n| \leq c_4 \frac{\log kn}{kn}.$$

Then

$$|f(\tilde{p}_n(t)) - q_n(t)| \leq c_0 \left[\omega \left(f; \frac{\log kn}{kn} \right) + (v+1) \frac{\log kn}{n} \right].$$

The theorem is proved.

Remark 2. If f is a monotone function, from Remark 1 and the proof of the Theorem 1 it follows that

$$E_{1,n}(f) \leq c_0 \left[\omega \left(f; \frac{\log kn}{kn} \right) + (v+1) \frac{\log kn}{n} \right].$$

THEOREM 2. Let f be continuous function with bounded variation in the interval $[0, 1]$ and $\omega(f; \delta) \leq c_2 \delta^\alpha$, $\alpha < 1$. Then

$$(4) \quad E_{n,n}(f) = O \left(\frac{\log n}{n} \right) \quad \text{in } [0, 1]$$

PROOF. Let us set in Theorem 1 $k = n^{\frac{1-\alpha}{\alpha}}$. Then

$$E_{n,n}(f) \leq c_7 \left[\left(\frac{\log kn}{kn} \right)^\alpha + (v+1) \frac{\log kn}{n} \right] = O \left(\frac{\log n}{n} \right).$$

The theorem is proved.

Remark. It is easy to show that the order of $E_{n,n}(f)$ in (4) cannot be better than $\frac{1}{2(n+1)}$. Really, let us consider for every n the function

$$\psi_n(x) = \begin{cases} (-1)^i \frac{1}{2(n+1)}, & \text{if } x = \frac{i}{n+1}, \quad i = 0, \dots, n+1, \\ \text{linear in the interval } \left[\frac{i}{n+1}, \frac{i+1}{n+1} \right], & i = 0, \dots, n. \end{cases}$$

It is evident that $E_{n,n}(\psi_n) = \frac{1}{2(n+1)}$ and $v(\psi_n) = 1$. We don't know if the order $\frac{\log n}{n}$ is the best possible.

Remark 3. If f is a monotone function and $\omega(f; \delta) \leq c\delta^\alpha$ then from Remark 2 it follows that

$$E_{1,n}(f) = O\left(\frac{\log n}{n}\right).$$

THEOREM 3. Let f be a continuous function with bounded variation v in the interval $[0, 1]$ and $\omega(f; \delta) \leq k_1 \log^{-\alpha} \frac{1}{\delta}$ in $[0, 1]$, ($\alpha > 0$). Then

$$E_{n,n}(f) \leq c_0(v, \alpha) n^{-\frac{\alpha}{1+\alpha}}$$

PROOF. Let us set in Theorem 1 $k = e^n^{-\frac{1}{1+\alpha}}$. Then

$$\begin{aligned} E_{n,n}(f) &\leq c_1 \left[\log^{-\alpha} \frac{kn}{\log kn} + (v+1) \frac{\log kn}{kn} \right] \leq \\ &\leq c_3 \left\{ c_9(\alpha) n^{-\frac{\alpha}{1+\alpha}} + (v+1) \left[n^{-\frac{\alpha}{1+\alpha}} + \frac{\log n}{n} \right] \right\} \leq c_3(v, \alpha) n^{-\frac{\alpha}{1+\alpha}}. \end{aligned}$$

The theorem is proved.

Remark 4. If f is a monotone function and $\omega(f; \delta) \leq k_1 \log^{-\alpha} \frac{1}{\delta}$ then from Remark 2 it follows that

$$E_{1,n}(f) \leq c_3(v, \alpha) n^{-\frac{\alpha}{1+\alpha}}.$$

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THE BAHADUR EFFICIENCY OF THE REIMANN—VINCZE STATISTICS

by

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Summary

In two sample testing problems it is known that REIMANN—VINCZE (R—V) statistics [5] have null distributional advantages over KOLMOGOROV—SMIRNOV (K—S) statistics when sample sizes are nearly equal. In this note we show that their relative Bahadur efficiency is unity in this case. This lends force to the argument that R—V statistics are preferable to K—S statistics when sample sizes are close.

1. Introduction

Suppose we have two independent samples $X_1, \dots, X_m; Y_1, \dots, Y_n$, from continuous distributions F, G respectively, and wish to test $H_0: F=G$ against one of the alternative hypotheses $H_1: F \neq G$ or $H_2: F > G$. Writing $F_m[G_n]$ for the first [second] empirical cumulative distribution function, it is well known that the Kolmogorov—Smirnov statistics

$$(1.1) \quad \begin{aligned} K_N &= (mn/N)^{1/2} \sup_x |F_m(x) - G_n(x)|, \\ K_N^+ &= (mn/N)^{1/2} \sup_x \{F_m(x) - G_n(x)\}, \\ N &= m + n \end{aligned}$$

have certain desirable properties, including, when $m=n$, the existence of simple closed expressions for the probabilities $P_2 = P(K_N > x)$, $P_1 = P(K_N^+ > x)$ under H_0 (see e.g. HODGES [3]).

HODGES also demonstrates in [3] that in case $m \neq n$, the problem of computing these probabilities is of considerably greater difficulty. He notes (p. 477) that the case where m and n are nearly equal is of practical importance (e.g. equal sample sizes are intended, but some observations lost) and develops a theory in which an asymptotic expression ([3], eq. 4.10) is derived for P_1 in case $m-n$ is bounded as $m, n \rightarrow \infty$. He concludes that the error of the SMIRNOV approximation to P_1 (i.e. e^{-2x^2}) may be quite large even when n is large, particularly when $m-n=1$. P_2 presents even greater problems.

To overcome the above difficulties REIMANN and VINCZE [5] proposed the statistics B_N^+ and B_N , where

$$B_N^+ = N^{-1/2} \sup_x \{mF_m(x) - nG_n(x)\}$$

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and

$$B_N = N^{-1/2} \left\{ \sup_x \left| mF_m(x) - nG_n(x) + \frac{n-m}{2} \right| - \frac{n-m}{2} \right\}.$$

Note that if

$$B_N^- = N^{-1/2} \sup_x \{ nG_n(x) - mF_m(x) \},$$

then

$$(1.2) \quad B_N = \max \left\{ B_N^+, B_N^- - \frac{n-m}{\sqrt{N}} \right\},$$

and that when $m=n$, $B_N^+ = K_N^+$ and $B_N = K_N$.

In [5], REIMANN and VINCZE show that tests based on $B_N[B_N^+]$ are consistent against $H_1[H_2]$ provided $n \geq m$ (which involves no loss of generality), $N \rightarrow \infty$ and $N^{-1/2}(n-m) \rightarrow 2c \geq 0$. The case $c=0$ clearly covers the situation $m-n$ bounded dealt with by HODGES [3]. They also show that the small sample null distributions of these "R—V statistics" are much simpler than those of K—S statistics. They derive expressions for the asymptotic null distributions of their statistics when the above limiting procedure obtains. Stirling's formula may be used to show that these limits are approached in an orderly way (QUINE and ROBINSON [4]) — a decided advantage over K—S tests in this situation. While having the above advantages, the R—V tests continue to share other properties of K—S tests. For example, the statistics are all easily computed, and the tests based on B_N^+ and K_N^+ are unbiased against H_2 (because both are monotone rank tests).

The purpose of our note is to show that when $N^{-1/2}(n-m) \rightarrow 2c \geq 0$, the R—V test is as efficient, in the Bahadur sense, as the K—S test. Thus one may prefer to use the R—V test when n and m are not too far apart.

2. Bahadur efficiency

For an elaborate discussion of Bahadur efficiency we refer to [2]. From there we recall the following

LEMMA 1 (Theorem 7.2 of [2]). Suppose T_N is a test statistic to test the hypothesis $\theta \in \Theta_0$ vs $\theta \in \Theta_1 = \Theta - \Theta_0$; $\Theta \subset R$. Suppose

$$(i) \quad N^{-1/2} T_N \rightarrow b(\theta) \text{ a.s. when } \theta \in \Theta_1$$

and

$$(ii) \quad \lim_{N \rightarrow \infty} N^{-1} \log P_\theta(T_N \geq N^{1/2} t) = f(t) \forall \theta \in \Theta_0$$

and for each t in an open interval containing the range of $b(\theta)$ and on which f is continuous. Then the exact slope of the test $\{T_N\}$ is

$$C(\theta) = 2f(b(\theta)).$$

For our purposes we will be taking $\Theta = \{(F, G); F, G \text{ continuous cdf's on } R\}$, $\Theta_0 = \{(F, G) \in \Theta; F \equiv G\}$, $\Theta_1 = \{(F, G) \in \Theta; 0 < \sup_x |F(x) - G(x)| < 1\}$. We shall also need $\Theta_1^+ = \{(F, G) \in \Theta; 0 < \sup_x (F(x) - G(x)) < 1\}$.

Finally we would like to recall from [2] that the Bahadur efficiency of test $\{T_{N^1}\}$ relative to $\{T_{N^2}\}$ is defined as the ratio of their respective slopes, $C_1(\theta)/C_2(\theta)$, $\theta \in \Theta_1$.

We now proceed to derive the exact slope of R—V statistics.
Let

$$(2.1) \quad g(t) = (1+t) \log(1+t) + (1-t) \log(1-t), \quad 0 < t < 1.$$

PROPOSITION 1. If $n > m$; $(n-m)N^{-1/2} \rightarrow 2c$, $c \geq 0$ as $N = m+n \rightarrow \infty$, then the exact slope of B_N -test is

$$(2.2) \quad C_{RV}(\theta) = g(d(\theta)); \quad d(\theta) = \sup_x |F(x) - G(x)|$$

and that of B_N^+ -test

$$(2.3) \quad C_{RV}^+(\theta) = g(d^+(\theta)); \quad d^+(\theta) = \sup_x (F(x) - G(x)),$$

with

$$\theta = (F, G) \in \Theta_1.$$

PROOF. From [5] we recall that

$$(2.4) \quad P_0(B_N^- \cong kN^{-1/2}) = \binom{N}{m-k} / \binom{N}{n}, \quad k = 0, 1, \dots, m,$$

where P_0 stands for probability being computed under the null hypothesis $H_0: F=G$.

Now let $\{t_N\}$ be a sequence of numbers such that $t_N \rightarrow t$ and $\{Nt_N\}$ are integers. Then we have

$$P_0(B_N^- \cong N^{1/2}t_N) = \frac{n! m!}{(m - Nt_N)! (n + Nt_N)!},$$

since $N = m+n$

$$\sim \frac{n^{n+1/2} m^{m+1/2}}{(m - Nt)^{m - Nt + 1/2} (n + Nt)^{n + Nt + 1/2}}$$

using Stirling's formula and $t_N \rightarrow t$

$$(2.5) \quad \sim \frac{1}{2} \frac{\{(1-2t)^{1/2-t} (1+2t)^{1/2+t}\}^{-N}}{\left[\left(\frac{1}{2}-t\right)\left(\frac{1}{2}+t\right)\right]^{1/2}}$$

using $n/N, m/N \rightarrow 1/2$.

In the above \sim means that the ratio of two sides tends to 1 as $N \rightarrow \infty$ in the above fashion. Hence from (2.5) above we have for any sequence $t_N \rightarrow t$,

$$(2.6) \quad -N^{-1} \log P_0(B_N^- \cong N^{1/2}t_N) \rightarrow f(t) = \frac{1}{2} g(2t).$$

Considerations of symmetry and the fact that $n \sim m$ imply for any sequence $\{t_N\} \rightarrow t$,

$$(2.7) \quad -N^{-1} \log P_0(B_N^+ \cong N^{1/2}t_N) \rightarrow f(t).$$

Applying (2.6) with $t_N = t - (n-m)N^{-1}$, recalling that $(n-m)N^{-1} \rightarrow 0$, we have

$$(2.8) \quad -N^{-1} \log P_0(B_N^- - (n-m)N^{-1/2} \cong tN^{1/2}) \rightarrow f(t).$$

Using (2.7) with $t_N \equiv t$, (2.8) and (1.2) one sees that

$$(2.9) \quad -N^{-1} \log P_0(B_N \cong tN^{1/2}) \rightarrow f(t)$$

where f is defined by (2.6) and (2.1).

Note that f is continuous on $(-1/2, 1/2)$. Next note that in view of $n/N \rightarrow 1/2 \leftarrow m/N$, we have by the Glivenko—Cantelli lemma that

$$(2.10) \quad N^{-1/2} B_N^+ \rightarrow \frac{1}{2} d^+(\theta) = \frac{1}{2} \sup_x (F(x) - G(x)), \quad \text{a.s.} \quad \forall \theta \in \Theta_1,$$

$$N^{-1/2} B_N^- \rightarrow \frac{1}{2} d^-(\theta) = \frac{1}{2} \sup_x (G(x) - F(x)) \quad \text{a.s.} \quad \forall \theta \in \Theta_1.$$

Using (2.10), (1.2) and the fact that $(n-m)N^{-1} \rightarrow 0$ we have

$$N^{-1/2} B_N \rightarrow \frac{1}{2} d(\theta) = \frac{1}{2} \sup_x |F(x) - G(x)| \quad \text{a.s.}, \quad \theta = (F, G) \in \Theta_1.$$

Thus Lemma 1 above is applicable to the statistic $T_N = B_N$ with $b(\theta) = 1/2 d(\theta)$ and f as in (2.6). Hence the exact slope of B_N is

$$C_{RV}(\theta) = 2f(b(\theta)) = 2 \cdot \frac{1}{2} g(2b(\theta)) = g(d(\theta)).$$

Note that $0 < b(\theta) < 1/2$ for $\theta \in \Theta_1$. Hence the interval on which f is continuous contains the range $\{b(\theta); \theta \in \Theta_1\} = (0, 1/2)$. Similarly the slope of B_N^+ is seen to be one given by $C_{RV}^+(\theta); \theta \in \Theta_1^+$. We terminate the proof.

Using Stirling's formula, it can be shown that in case $n=m$,

$$(2.11) \quad -\lim N^{-1} \log P_0(K_N \cong tN^{1/2}) = f(t),$$

for all $t \in (0, 1/2)$. Furthermore, Theorem 3 of ABRAHAMSON [1] shows that in the general case of possibly unequal sample sizes the above limit depends on $v = \lim (m/N)$, but not upon the manner in which v is approached. It follows that (2.11) obtains whenever $v=1/2$, i.e. whenever $m \sim n$.

Again using Glivenko—Cantelli it further follows that

$$N^{-1/2} K_N \rightarrow \frac{1}{2} d(\theta) \quad \text{a.s.} \quad \forall \theta \in \Theta_1,$$

and hence in view of Lemma 1 above and (2.11) the exact Bahadur slope of the K_N -test when $(n-m)N^{-1/2} \rightarrow 2c$, $c \geq 0$ as $N \rightarrow \infty$ is

$$C_{KS}(\theta) = g(d(\theta)), \quad \theta = (F, G) \in \Theta_1.$$

We thus have

COROLLARY 1. *The Bahadur efficiency of two sided R—V test relative to two sided K—S test is 1 provided $(n-m)N^{-1/2} \rightarrow 2c \geq 0$ as $N \rightarrow \infty$.*

ABRAHAMSON's work [1] does not deal with the exact slope of the one sided K—S statistic K_N^+ . It is of some independent interest to note that some information may be gained in this regard from the present results.

PROPOSITION 2. *If $(n-m)N^{-1/2} \rightarrow 2c \geq 0$ as $N \rightarrow \infty$, then the exact Bahadur slope of K_N^+ is given by*

$$(2.12) \quad C_{KS}^+(\theta) = C_{RV}^+(\theta) \forall \theta \in \Theta_1.$$

PROOF. First note that

$$N^{-1/2}K_N^+ \rightarrow \frac{1}{2}d^+(\theta) \text{ a.s. } \forall \theta \in \Theta_1^+.$$

Using the identity

$$mF_m(x) - nG_n(x) = \frac{2mn}{N}(F_m(x) - G_n(x)) + (m-n)H_N(x),$$

where $H_N(\cdot)$ is the pooled empirical cumulative, we have

$$\frac{1}{2} \frac{N}{(mn)^{1/2}} \left(B_N^+ + \frac{n-m}{N^{1/2}} \right) \cong K_N^+ \cong K_N.$$

Using (2.7) with $t_N = t - (n-m)N^{-1}$ and (2.11) above concludes the proof.

Propositions 1 and 2 imply the following

COROLLARY 2. *The Bahadur efficiency of the one sided R—V test B_N^+ relative to the K—S test K_N^+ is 1 as long as $(n-m)N^{-1/2} \rightarrow 2c \geq 0$.*

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**ON JACKSON AND BERNSTEIN TYPE APPROXIMATION
THEOREMS IN THE CASE OF APPROXIMATION
BY ALGEBRAIC POLYNOMIALS IN \mathcal{L}_p -SPACES**

by

NGUYEN XUAN KY

1. Introduction

Let $\mathcal{L}_p[-1, 1]$ ($1 \leq p \leq \infty$) be the Banach-space of integrable functions defined in $[-1, 1]$ with norm

$$\|f(x)\|_p = \left\{ \int_{-1}^1 |f(x)|^p dx \right\}^{1/p} \quad (1 \leq p < \infty)$$

and for $p = \infty$ the space of bounded measurable functions with norm

$$\|f(x)\|_\infty = \text{vrai} \cdot \sup |f(x)|,$$

respectively.

Let \mathcal{P}_n be the set of algebraic polynomials of degree not greater than n ($n=0, 1, \dots$). For $f(x) \in \mathcal{L}_p[-1, 1]$, let

$$(1) \quad E_n^{(p)}(f) = \inf_{q \in \mathcal{P}_n} \|f - q\|_p, \quad n = 0, 1, \dots$$

In the present paper we define one modulus $\omega(\mathcal{L}_p; f; \delta)$ for each $f \in \mathcal{L}_p[-1, 1]$ and we give the analogue of the Jackson—Bernstein approximation theorem in the case of trigonometric approximation for our case, that is, for $E_n^{(p)}(f)$ and $\omega(\mathcal{L}_p; f; \delta)$.

In our paper c_1, c_2, \dots denote absolute constants, and $c_1\langle p \rangle, c_2\langle p \rangle, \dots$ denote constants, which depend only on p . For each $f(x)$ defined in $[-1, 1]$, we denote

$$(2) \quad f^*(\theta) = f(\cos \theta), \quad 0 \leq \theta \leq \pi.$$

For

$$f(x) \in \mathcal{L}_p[-1, 1] \quad (1 \leq p < \infty),$$

we define the continuity modulus of $f(x)$ as follows:

$$(3) \quad \omega(\mathcal{L}_p; f; \delta) = \sup_{0 \leq t \leq \delta} \left\{ \int_0^{5\pi/8} |f^*(\theta+t) - f^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} + \\ + \sup_{0 \leq t \leq \delta} \left\{ \int_{3\pi/8}^\pi |f^*(\theta-t) - f^*(\theta)|^p \sin \theta d\theta \right\}^{1/p}, \quad 0 \leq \delta \leq \frac{\pi}{3}.$$

Since

$$\sin \theta \leq c_1 \sin(\theta + t), \\ 0 \leq \theta \leq \frac{5\pi}{8}, \quad 0 \leq t \leq \frac{\pi}{3}$$

and

$$(4) \quad \sin \theta \leq c_1 \sin(\theta - t),$$

$$\frac{3\pi}{8} \leq \theta \leq \pi, \quad 0 \leq t \leq \frac{\pi}{3},$$

we have

$$\left\{ \int_0^{3\pi/8} |f^*(\theta + t) - f^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \leq (1 + c_1) \|f\|_p$$

and

$$\left\{ \int_{3\pi/8}^{\pi} |f^*(\theta - t) - f^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \leq (1 + c_1) \|f\|_p.$$

Thus the modulus $\omega(\mathcal{L}_p; f; \delta)$ exists for $0 \leq \delta \leq \pi/3$.

It is obvious that

$$\omega(\mathcal{L}_p; f; \delta) \rightarrow 0 \quad (\delta \rightarrow 0).$$

We are going to prove the following two theorems:

THEOREM 1. Let $1 \leq p < \infty$. For each $f \in \mathcal{L}_p[-1, 1]$ we have

$$(5) \quad E_n^{(p)}(f) \leq c_1 \langle p \rangle \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right), \quad n = 1, 2, \dots$$

THEOREM 2. Let $1 \leq p < \infty$. For each $f \in \mathcal{L}_p[-1, 1]$, we have

$$(6) \quad \omega(\mathcal{L}_p; f; \delta) \leq c_2 \langle p \rangle \delta \sum_{0 \leq n \leq \delta^{-1}} E_n^{(p)}(f).$$

Remark. From Theorem 1 and Theorem 2, it follows that:

(i) *Bernstein-type theorem:* For each $f \in \mathcal{L}_p$ ($1 \leq p < \infty$) we have

$$(7) \quad E_n^{(p)}(f) = O \left(\frac{1}{n^\alpha} \right) \Leftrightarrow \omega(\mathcal{L}_p; f; \delta) = O(\delta^\alpha), \quad 0 < \alpha < 1.$$

(ii) In the case $\alpha = 1$ we obtain only:

$$(8) \quad E_n^{(p)}(f) = O \left(\frac{1}{n} \right) \Rightarrow \omega(\mathcal{L}_p; f; \delta) = O \left(\delta \ln \frac{1}{\delta} \right).$$

2. Some linear polynomial operators

We denote by $P_n(x)$ ($n=0, 1, \dots$) the n -th Legendre orthogonal polynomial. For $f \in \mathcal{L}_p[-1, 1]$ ($1 \leq p \leq \infty$), let $S(f; x)$ be the generalized Fourier series of f with respect to the system $\{P_n(x)\}$, that is

$$(9) \quad S(f; x) = \sum_{k=0}^{\infty} c_k(f) P_k(x),$$

where

$$(10) \quad c_k(f) = \int_{-1}^1 f(t)P_k(t) dt \quad k = 0, 1, \dots$$

Let $S_n(f; x)$ be the n -th partial-sum of series (9). We shall denote by $\sigma_n^{[2]}(f; x)$ the n -th $(C, 2)$ -mean of series (9), that is we have

$$(11) \quad \sigma_n^{[2]}(f; x) = \int_{-1}^1 K_n^{[2]}(t, x)f(t) dt.$$

The kernel $K_n^{[2]}(t, x)$ is defined by

$$(12) \quad K_n^{[2]}(t, x) = \sum_{k=0}^n \frac{\binom{n-k+2}{2}}{\binom{n+2}{2}} P_k(t)P_k(x).$$

It was proved by L. FEJÉR [1] that

$$(13) \quad K_n^{[2]}(t, x) \geq 0 \quad (-1 \leq t, x \leq 1), \quad n = 0, 1, \dots$$

Thus, by the identical

$$\int_{-1}^1 K_n^{[2]}(t, x) dt = 1$$

we obtain

$$\|\sigma_n^{[2]}(f; x)\|_\infty \leq \|f(x)\|_\infty, \quad f \in \mathcal{L}_\infty[-1, 1] \quad n = 0, 1, \dots$$

and

$$\|\sigma_n^{[2]}(f; x)\|_1 \leq \|f(x)\|_1, \quad f \in \mathcal{L}_1[-1, 1] \quad n = 0, 1, \dots$$

Thus, from the Riesz—Thorin interpolation theorem, it follows that

$$(14) \quad \|\sigma_n^{[2]}(f; x)\|_p \leq \|f(x)\|_p, \quad f \in \mathcal{L}_p[-1, 1] \quad (1 \leq p \leq \infty), \quad n = 0, 1, \dots$$

We introduce the following operators:

$$(15) \quad V_{3n}(f; x) = \begin{cases} \frac{\sum_{j=0}^2 (-1)^{2-j} \binom{2}{j} \binom{(j+1)n+2}{2} \sigma_{(j+1)n}^{[2]}(f; x)}{2n^2}, & n = 1, 2, \dots, \\ 3\sigma_1^{[2]}(f; x) - 2\sigma_0^{[2]}(f; x) = S_1(f; x), & n = 0. \end{cases}$$

$V_{3n}(f; x)$ is a polynomial of degree $3n$ at most (for $n=1, 2, \dots$), degree 1 (for $n=0$) and we have by (14), (15) (see G. FREUD and J. SZABADOS [2])

$$(16) \quad (i) \quad V_{3n}(\Pi_n; x) \equiv \Pi_n(x), \quad \Pi_n \in \mathcal{P}_n,$$

$$(17) \quad (ii) \quad V_0(\Pi_1; x) \equiv \Pi_1(x), \quad \Pi_1 \in \mathcal{P}_1,$$

$$(18) \quad (iii) \quad \|V_{3n}(f; x)\|_p \leq c_3 \langle p \rangle \|f(x)\|_p, \quad f \in \mathcal{L}_p[-1, 1] (1 \leq p \leq \infty), n = 0, 1, \dots$$

Let now $f \in \mathcal{L}_p[-1, 1]$ ($1 \leq p \leq \infty$) and let $p_n(x)$ be an algebraic polynomial of best approximation of order n to f in $\mathcal{L}_p[-1, 1]$. We have by (16), (17) and (18)

$$(19) \quad \begin{aligned} \|f - V_{3n}(f)\|_p &\leq \|f - p_n\|_p + \|V_{3n}(f - p_n)\|_p \leq \\ &\leq E_n^{(p)}(f) + c_3 \langle p \rangle E_n^{(p)}(f) = c_4 \langle p \rangle E_n^{(p)}(f), \quad n = 0, 1, \dots \end{aligned}$$

3. Estimation for the best approximation of differentiable functions in

From the Jackson-theorem it follows that, if $f(\theta)$ is continuous, 2π -periodic and $f'(\theta)$ is essentially bounded in $[0, 2\pi]$, then we have the following estimation for the best approximation $E_n^*(f)$ of f by trigonometric polynomial in $C_{2\pi}$ -space:

$$(20) \quad E_n^*(f) \leq \frac{12}{n} \operatorname{urai} \cdot \sup_{0 \leq \theta \leq 2\pi} |f'(\theta)|.$$

It is a consequence of (20) that, if $f(x)$ is absolutely continuous in $[-1, 1]$ and $\sqrt{1-x^2} f'(x) \in \mathcal{L}_\infty[-1, 1]$ then

$$(21) \quad E_n^{(\infty)}(f) \leq \frac{12}{n} \|\sqrt{1-x^2} f'(x)\|_\infty.$$

Now, we shall prove inequality (21) in every $\mathcal{L}_p[-1, 1]$ -space.

THEOREM 3. *If $f(x)$ is absolutely continuous in $[-1, 1]$ and $\sqrt{1-x^2} f'(x) \in \mathcal{L}_p[-1, 1]$ ($1 \leq p \leq \infty$), then*

$$(22) \quad E_n^{(p)}(f) \leq \frac{c_5 \langle p \rangle}{n} \|\sqrt{1-x^2} f'(x)\|_p, \quad n = 1, 2, \dots$$

PROOF. We denote by S_p the set of functions satisfying the conditions in Theorem 3.

a) We begin by proving (22) for the case $p=1$. Let $\tilde{P}_n(x)$ be the n -th orthonormal Legendre polynomial. Let

$$(23) \quad \lambda_n(\xi) = \left[\sum_{k=0}^{n-1} \tilde{P}_k^2(\xi) \right]^{-1}, \quad n = 1, 2, \dots$$

the n -th Christoffel function with respect to the system $\{\tilde{P}_n(x)\}$ at point ξ . It has been proved by G. P. NÉVAI [8] that

$$(24) \quad \lambda_n(\xi) \sim \begin{cases} n^{-2} \left(1 - \frac{c_2}{n^2} \leq \xi \leq 1 \right), \\ \frac{1}{n} \sqrt{1-\xi^2} \left(-1 + \frac{c_3}{n^2} \leq \xi \leq 1 - \frac{c_2}{n^2} \right), \\ n^{-2} \left(-1 \leq \xi \leq -1 + \frac{c_3}{n^2} \right). \end{cases}$$

Let $\Gamma_\xi(t)$ be the following function:

$$(25) \quad \Gamma_\xi(t) = \begin{cases} 0, & t < \xi, \\ 1, & t \geq \xi. \end{cases}$$

It is known that

$$(26) \quad E_n^{(1)}(\Gamma_\xi) \leq \lambda_n(\xi), \quad (-1 \leq \xi \leq 1), \quad n = 1, 2, \dots$$

(see G. FREUD [5]).

(i) If $1 - \frac{c_2}{n^2} \leq \xi \leq 1$, then

$$1 - \xi \leq \frac{c_4}{n} \sqrt{1 - \xi^2}$$

thus, from (25), taking the polynomial $q_0(x) \equiv 0$ as an approximant, we have

$$(27) \quad E_n^{(1)}(\Gamma_\xi) \leq \int_{\xi}^1 dx \leq \frac{c_4}{n} \sqrt{1 - \xi^2}.$$

(ii) If $-1 \leq \xi \leq -1 + \frac{c_3}{n^2}$, then

$$1 + \xi \leq \frac{c_5}{n} \sqrt{1 - \xi^2}.$$

Thus, again from (25), by taking the polynomial $q_1(x) \equiv 1$ as an approximant we have

$$(28) \quad E_n^{(1)}(\Gamma_\xi) \leq \int_{-1}^{\xi} dx \leq \frac{c_5}{n} \sqrt{1 - \xi^2}.$$

Now, from (24), (26), (27) and (28) we obtain

$$(29) \quad E_n^{(1)}(\Gamma_\xi) \leq \frac{c_6}{n} \sqrt{1 - \xi^2}, \quad (-1 \leq \xi \leq 1), \quad n = 1, 2, \dots$$

From (29) by the method of G. FREUD in his paper [3], we obtain that, if $f \in S_1$ then we have (22). Thus (22) is proved in the case $p=1$.

b) We define the following operators for

$$f_* \in S_p^* = \{f_*(x) = \sqrt{1-x^2} f'(x) : f \in S_p\} \quad (1 \leq p \leq \infty)$$

as follows:

$$(30) \quad T_n(f_*; x) = f(x) - V_{3n}(f; x), \quad n = 0, 1, \dots$$

We have by (17), (21) and a)

$$(31) \quad \begin{cases} \|T_n(f_*; x)\|_p \leq c_6 \langle p \rangle \frac{1}{n} \|f_*(x)\|_p \\ \|T_0(f_*; x)\|_p \leq c_6 \langle p \rangle \|f_*(x)\|_p, \quad p = 1, \infty. \end{cases}$$

That is, T_n is (1, 1) and (∞, ∞) type operator. From the Riesz—Thorin interpolation theorem it follows that T_n is also (p, p) type, more precisely

$$(32) \quad \|T_n(f_*; x)\|_p \leq c_6 \langle p \rangle \frac{1}{n} \|f_*(x)\|_p,$$

$$\|T_0(f_*; x)\|_p \leq c_6 \langle p \rangle \|f_*(x)\|_p, \quad 1 \leq p \leq \infty.$$

Now, we have by (15), (30) and (32)

$$E_n^{(p)}(f) \leq \frac{c_7 \langle p \rangle}{n} \|\sqrt{1-x^2} f'(x)\|_p, \quad f \in S_p (\leq p \leq \infty), \quad n = 1, 2, \dots$$

This completes our proof.

4. Proof of Theorem 1

We begin with the following inequality which holds for $n=1, 2, \dots$

$$(34) \quad \left\{ \int_{3\pi/8}^{5\pi/8} \left| 2n \int_{n^{-1/2}}^{n^{-1}} [f^*(\theta+t) - f^*(\theta-t)] dt \right|^p d\theta \right\}^{1/p} \leq$$

$$\leq 2n \int_{n^{-1/2}}^{n^{-1}} \left\{ \int_{3\pi/8}^{5\pi/8} |f^*(\theta+t) - f^*(\theta-t)|^p d\theta \right\}^{1/p} dt \leq$$

$$\leq \sin^{-1/p} \frac{3\pi}{8} \cdot 2n \int_{n^{-1/2}}^{n^{-1}} \left\{ \int_{3\pi/8}^{5\pi/8} |f^*(\theta+t) - f^*(\theta-t)|^p \sin \theta d\theta \right\}^{1/p} dt \leq$$

$$\leq \sin^{-1/p} \frac{3\pi}{8} \sup_{n^{-1/2} \leq t \leq n^{-1}} \left\{ \int_{3\pi/8}^{5\pi/8} |f^*(\theta+t) - f^*(\theta-t)|^p \sin \theta d\theta \right\}^{1/p} \leq$$

$$\leq \sin^{-1/p} \frac{3\pi}{8} \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right).$$

From (34) it follows that there exists a θ_n with

$$\frac{3\pi}{8} \leq \theta_n \leq \frac{5\pi}{8},$$

so that

$$(35) \quad \left| 2n \int_{n^{-1/2}}^{n^{-1}} [f^*(\theta_n+t) - f^*(\theta_n-t)] dt \right| \leq$$

$$\leq \left(\frac{4}{\pi} \right)^{1/p} \sin^{-1/p} \frac{3\pi}{8} \cdot \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right), \quad n = 1, 2, \dots$$

We introduce the following function

$$(36) \quad \varphi_n^*(\theta) = \begin{cases} 2n \int_{-1/2}^{n^{-1}} f^*(\theta+t) dt, & 0 \leq \theta \leq \theta_n, \\ 2n \int_{n^{-1/2}}^{n^{-1}} f^*(\theta-t) dt + 2n \int_{n^{-1/2}}^{n^{-1}} [f^*(\theta_n+t) - f^*(\theta_n-t)] dt, & \theta_n < \theta \leq \pi. \end{cases}$$

The function $\varphi_n^*(\theta)$ is absolutely continuous in $[0, \pi]$. Thus the function $\varphi_n(x) = \varphi_n^*(\arccos x)$ is absolutely continuous in $[-1, 1]$. We have by (36)

$$(37) \quad \varphi_n^*(\theta) - f^*(\theta) = \begin{cases} 2n \int_{n^{-1/2}}^{n^{-1}} [f^*(\theta+t) - f^*(\theta)] dt, & 0 \leq \theta \leq \theta_n, \\ 2n \int_{n^{-1/2}}^{n^{-1}} [f^*(\theta-t) - f^*(\theta)] dt + 2n \int_{n^{-1/2}}^{n^{-1}} [f^*(\theta_n+t) - f^*(\theta_n-t)] dt, & \theta_n < \theta \leq \pi. \end{cases}$$

Thus, from (35) we obtain

$$(38) \quad \begin{aligned} \|\varphi_n(x) - f(x)\|_p &= \left\{ \int_0^\pi |\varphi_n^*(\theta) - f^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} \leq \\ &\leq 2n \int_{n^{-1/2}}^{n^{-1}} \left\{ \int_0^{\theta_n} |f^*(\theta+t) - f^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} dt + \\ &+ 2n \int_{n^{-1/2}}^{n^{-1}} \left\{ \int_{\theta_n}^\pi |f^*(\theta-t) - f^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} dt + \\ &+ \left(\frac{4}{\pi} \right)^{1/p} \sin^{-1/p} \frac{3\pi}{8} \cdot \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right) \left\{ \int_{\theta_n}^\pi \sin \theta d\theta \right\}^{1/p} \leq \\ &\leq \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right) + \\ &+ \left(\frac{4}{\pi} \right)^{1/p} \sin^{-1/p} \frac{3\pi}{8} \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right) \cdot \left(1 + \cos \frac{3\pi}{8} \right)^{1/p} = \\ &= \left[1 + \left(\frac{4}{\pi} \right)^{1/p} \sin^{-1/p} \frac{3\pi}{8} \cdot \left(1 + \cos \frac{3\pi}{8} \right)^{1/p} \right] \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right). \end{aligned}$$

Again, from (36) we obtain

$$(39) \quad \varphi_n^{*'}(\theta) = \begin{cases} 2n \left[f^*(\theta + n^{-1}) - f^*\left(\theta + \frac{n^{-1}}{2}\right) \right], & 0 \leq \theta \leq \theta_n, \\ 2n \left[f^*(\theta - n^{-1}) - f^*\left(\theta - \frac{n^{-1}}{2}\right) \right], & \theta_n < \theta \leq \pi. \end{cases}$$

Thus, we have

$$(40) \quad \begin{aligned} \|\sqrt{1-x^2} \varphi_n'(x)\|_p &= \left\{ \int_0^\pi |\varphi_n^{*'}(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \leq \\ &\leq 2n \left\{ \int_0^{\theta_n} \left| f^*(\theta + n^{-1}) - f^*\left(\theta + \frac{n^{-1}}{2}\right) \right|^p \sin \theta \, d\theta \right\}^{1/p} + \\ &+ 2n \left\{ \int_{\theta_n}^\pi \left| f^*(\theta - n^{-1}) - f^*\left(\theta - \frac{n^{-1}}{2}\right) \right|^p \sin \theta \, d\theta \right\}^{1/p} \leq 4n\omega \left(\mathcal{L}_p; f; \frac{1}{n} \right). \end{aligned}$$

From (36), and (40) it follows that $\varphi_n(x) \in S_p$, thus by Theorem 3, (38), (40) we obtain

$$(41) \quad \begin{aligned} E_n^{(p)}(f) &\leq \|\varphi_n - f\|_p + E_n^{(p)}(\varphi_n) \leq \\ &\leq \|\varphi_n - f\|_p + \frac{c_5 \langle p \rangle}{n} \|\sqrt{1-x^2} \varphi_n'(x)\|_p \leq \\ &\leq \left[1 + \left(\frac{4}{\pi} \right)^{1/p} \sin^{-1/p} \frac{3\pi}{8} \cdot \left(1 + \cos \frac{3\pi}{8} \right)^{1/p} + 4c_5 \langle p \rangle \right] \omega \left(\mathcal{L}_p; f; \frac{1}{n} \right), \end{aligned}$$

q.e.d.

5. Markov-type inequality

It was proved by S. N. BERNSTEIN as a refinement of Markov's inequality that for every $\Pi_n \in \mathcal{P}_n$, the following inequality is true:

$$(42) \quad |\Pi_n'(x)| \leq n \frac{\max_{x \in [-1, 1]} |\Pi_n(x)|}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

Now, we shall prove the inequality (42) in every $\mathcal{L}_p[-1, 1]$ ($1 \leq p < \infty$) and we deduce some of its consequences.

THEOREM 4.

(i) Let $1 \leq p \leq \infty$. For every $\Pi_n \in \mathcal{P}_n$ we have

$$(43) \quad \|\sqrt{1-x^2} \Pi_n'(x)\|_p \leq c_8 \langle p \rangle n \|\Pi_n(x)\|_p.$$

(ii) Let $1 \leq p < \infty$. For every $\Pi_n \in \mathcal{P}_n$ we have

$$(44) \quad \left\{ \int_0^{5\pi/8} |\Pi_n^*(\theta+h) - \Pi_n^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \leq \\ \leq c_9 \langle p \rangle h n \left\{ \int_0^\pi |\Pi_n^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p}, \quad 0 \leq h \leq \frac{\pi}{3}$$

and

$$(45) \quad \left\{ \int_{3\pi/8}^\pi |\Pi_n^*(\theta-h) - \Pi_n^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \leq \\ \leq c_{10} \langle p \rangle h n \left\{ \int_0^\pi |\Pi_n^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p}, \quad 0 \leq h \leq \frac{\pi}{3}.$$

PROOF. a) In the case $p=1$, (43) was proved by STEIN E. M. [9].

b) We define linear operators τ_n for $n=1, 2, \dots$ as follows:

$$(46) \quad \tau_n(f; x) = \sqrt{1-x^2} V'_{3n}(f; x), \quad f \in \mathcal{L}_p[-1, 1] \quad (1 \leq p \leq \infty).$$

We have by (15), (42) and (16)

$$(47) \quad \left\| \sqrt{1-x^2} V'_{3n}(f; x) \right\|_\infty \leq 3n \|V_{3n}(f; x)\|_\infty \leq \\ \leq 3nc_3 \langle \infty \rangle \|f(x)\|_\infty, \quad f \in \mathcal{L}_\infty[-1, 1].$$

We have analogously

$$(48) \quad \left\| \sqrt{1-x^2} V'_{3n}(f; x) \right\|_1 \leq 3c'_8 \langle 1 \rangle n \|V_{3n}(f; x)\|_1 \leq \\ \leq 3C_8 \langle 1 \rangle C_3 \langle 1 \rangle n \|f(x)\|_1, \quad f \in \mathcal{L}_1[-1, 1].$$

Thus, τ_n is a $(1, 1)$ and (∞, ∞) type operator. From the Riesz—Thorin interpolation theorem it follows that

$$(49) \quad \left\| \sqrt{1-x^2} V'_{3n}(f; x) \right\|_p \leq c_{11} \langle p \rangle n \|f(x)\|_p, \quad f \in \mathcal{L}_p[-1, 1], \quad (1 \leq p \leq \infty).$$

From (15), inserting a polynomial $\Pi_n \in \mathcal{P}_n$ for f in (49), we obtain (43). (44) and (45) are consequences of (43) and the following inequalities:

$$(50) \quad (i) \quad \left\{ \int_0^{5\pi/8} \left| \frac{\Pi_n^*(\theta+h) - \Pi_n^*(\theta)}{h} \right|^p \sin \theta \, d\theta \right\}^{1/p} = \\ = \left\{ \int_0^{5\pi/8} \left| \frac{1}{h} \int_0^h \Pi_n^{*'}(\theta+t) \, dt \right|^p \sin \theta \, d\theta \right\}^{1/p} \leq \\ \leq c_1 \frac{1}{h} \int_0^h \left\{ \int_0^{5\pi/8} |\Pi_n^{*'}(\theta+t)|^p \sin(\theta+t) \, d\theta \right\}^{1/p} dt \leq \\ \leq c_1 \frac{1}{h} \int_0^h \left\{ \int_0^\pi |\Pi_n^{*'}(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} dt = c_1 \left\{ \int_0^\pi |\Pi_n^{*'}(\theta)|^p \sin \theta \, d\theta \right\}^{1/p}.$$

$$(51) \quad \begin{aligned} & \text{(ii)} \\ & \left\{ \int_{3\pi/8}^{\pi} \left| \frac{\Pi_n^*(\theta-h) - \Pi_n^*(\theta)}{h} \right|^p \sin \theta \, d\theta \right\}^{1/p} \cong \\ & \cong c_1 \left\{ \int_0^{\pi} |\Pi_n^{*'}(\theta)|^p \sin \theta \, d\theta \right\}^{1/p}. \end{aligned}$$

Thus, we completely proved the Theorem 4.

6. Proof of Theorem 2

We apply the following inequality:

$$(52) \quad \sum_{i=0}^k 2^i E_{2^i}^{(p)}(f) \cong c_7 \sum_{0 \leq n < 2^k} E_n^{(p)}(f), \quad k = 0, 1, \dots$$

The inequality (52) is a special case of a more general inequality (see G. G. LORENTZ [6]).

Let $p_n(x)$ be an algebraic polynomial of best approximation of degree n to $f(x)$ in $\mathcal{L}_p[-1, 1]$. We have for $k=0, 1, \dots$

$$(53) \quad \begin{aligned} & \left\{ \int_0^{5\pi/8} |f^*(\theta+h) - f^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \cong \\ & \cong c_5 E_{2^k}^{(p)}(f) + \left\{ \int_0^{5\pi/8} |p_{2^k}^*(\theta+h) - p_{2^k}^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p}. \end{aligned}$$

From (50), we obtain

$$(54) \quad \left\{ \int_0^{5\pi/8} |p_{2^k}^*(\theta+h) - p_{2^k}^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \cong c_1 h \left\{ \int_0^{\pi} |p_{2^k}^{*'}(\theta)|^p \sin \theta \, d\theta \right\}^{1/p}.$$

We have

$$(55) \quad p_{2^k}^{*'}(\theta) = [p_1^*(\theta) - p_0^*(\theta)]' + \sum_{i=1}^k [p_{2^i}^*(\theta) - p_{2^i-1}^*(\theta)]'.$$

We notice that

$$(56) \quad \begin{aligned} & \left\{ \int_0^{\pi} |p_1^*(\theta) - p_0^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \cong 2E_0^{(p)}(f), \\ & \left\{ \int_0^{\pi} |p_{2^i}^*(\theta) - p_{2^i-1}^*(\theta)|^p \sin \theta \, d\theta \right\}^{1/p} \cong 2E_{2^i-1}^{(p)}. \end{aligned}$$

Now, we have by (55), (44) and (56) and (51)

$$\begin{aligned}
 (57) \quad & \left\{ \int_0^\pi |p_{2^k}^{*k}(\theta)|^p \sin \theta d\theta \right\}^{1/p} \cong \left\{ \int_0^\pi |[p_1^*(\theta) - p_0^*(\theta)]'|^p \sin \theta d\theta \right\}^{1/p} + \\
 & + \sum_{i=1}^k \left\{ \int_0^\pi |[p_{2^i}^*(\theta) - p_{2^{i-1}}^*(\theta)]'|^p \sin \theta d\theta \right\}^{1/p} \cong \left\{ \int_0^\pi |p_1^*(\theta) - p_0^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} + \\
 & + \sum_{i=1}^k 2^i \left\{ \int_0^\pi |p_{2^i}^*(\theta) - p_{2^{i-1}}^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} \cong 2E_0^{(p)}(f) + 4 \sum_{i=0}^{k-1} 2^i E_{2^i}^{(p)}(f) \cong \\
 & \cong c_{12} \langle p \rangle \sum_{0 \leq \mu < 2^k - 1} E_n^{(p)}(f).
 \end{aligned}$$

But

$$(58) \quad \sum_{0 \leq n \leq 2^k} E_n^{(p)}(f) \cong 2^k E_{2^k}^{(p)}(f).$$

Thus from (53), (54), (57) and (58) we obtain

$$\begin{aligned}
 (59) \quad & \left\{ \int_0^{5\pi/8} |f^*(\theta+h) - f^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} \cong \\
 & \cong c_{13} \langle p \rangle (h + 2^{-k}) \sum_{0 \leq n \leq 2^k} E_n^{(p)}(f).
 \end{aligned}$$

Similarly as before, we have by application of (45)

$$\begin{aligned}
 (60) \quad & \left\{ \int_{3\pi/8}^\pi |f^*(\theta-h) - f^*(\theta)|^p \sin \theta d\theta \right\}^{1/p} \cong \\
 & \cong c_{14} \langle p \rangle (h + 2^{-k}) \sum_{0 \leq n \leq 2^k} E_n^{(p)}(f).
 \end{aligned}$$

It is now clear that if we select $k=0, 1, \dots$ in (59) and (60) in such a way that $2^k \leq h^{-1} < 2^{k+1}$ we shall have (6).

This completes our proof.

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THE NUMBER OF SQUARES IN AN ARITHMETIC PROGRESSION

by

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P. ERDŐS conjectured that an arithmetic progression of k elements contains at most $o(k)$ squares (see [1]). We are going to give a short proof of this assertion here. To this end, assume that we are given an arithmetic progression

$$c + id \quad (0 \leq i < k)$$

of k elements, where k is sufficiently large, and consider the sequence

$$c + i_t d \quad (i_0 < i_1 \dots, 0 \leq t < m)$$

of squares in this arithmetic progression, where m denotes the number of squares in the former sequence. According to a well-known observation of EULER, there are no four squares that form an arithmetic progression; hence the sequence of squares contains no arithmetic progression of four elements. Clearly, the same is true for the sequence i_t ($0 \leq t < m$). This means that $m = o(k)$ does indeed hold, as a consequence of the theorem that an infinite sequence of positive upper density contains an arithmetic progression of four elements (see [2]; in fact, recently I succeeded in proving that such a sequence contains arbitrarily long finite arithmetic progressions see [3]). This completes the proof.

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ÜBER RINGE MIT MINIMALBEDINGUNG FÜR HAUPTRECHTSIDEALE, II

von

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§ 1. Einleitung

Unter einem Ring verstehen wir in dieser Arbeit stets einen *assoziativen* Ring. Ein Ring heißt artinsch (noethersch), wenn er der Minimalbedingung (Maximalbedingung) für Rechtsideale genügt. Genügt ein Ring der Minimalbedingung für Hauptrechtsideale (für die in Hauptrechtsidealen liegenden Rechtsideale), so heißt er ein MHR-Ring (MH_1R -Ring) (siehe F. SZÁSZ [8], [9]). Wir nennen einen Ring MH_1^*R -Ring, wenn er die Minimalbedingung für die in Hauptrechtsidealen liegenden Unterringe erfüllt.

In [1] haben wir MHR-Ringe mit der Eigenschaft (*) betrachtet. Ein Ring R besitze die Eigenschaft (*), wenn gilt: Ist $x \in R$ mit $x^2=0$, so gilt $xe=exe$ für jedes $e \in R$ mit $e=e^2$. In dieser Arbeit werden wir weitere Ergebnisse über MHR-Ringe (bzw. über MH_1R -, MH_1^*R -Ringe) mit der Eigenschaft (*) angeben. Anschließend zeigen wir, daß ein MHR-Ring, dessen additive Gruppe keine Untergruppe vom Typ $Z(p^\infty)$ enthält, ist genau dann artinsch, wenn er noethersch ist.

Wir stellen zunächst die in der Arbeit benutzten Bezeichnungen zusammen. \oplus und \sum^\oplus bezeichnen die gruppentheoretischen, \boxplus und \sum^\boxplus die ringtheoretischen diskreten direkten Summen. Für einen Ring R sei $(R, +)$ die additive Gruppe von R ; als Radikal von R betrachten wir nur das JACOBSONSche und bezeichnen es mit $J(R)$. Der Ring heißt vollständig primär, wenn R ein Einselement enthält und der Faktorring $R/J(R)$ ein Schiefkörper ist. Für eine Primzahl p sei $Z(p^i)$ die zyklische Gruppe der Ordnung p^i , falls i eine natürliche Zahl ist, für $i=\infty$ die PRÜFERSche p -Gruppe (Gruppe vom Typ $Z(p^\infty)$).

§ 2. MHR-Ringe mit der Eigenschaft (*)

Es sei M eine Untermenge eines Ringes R . Mit $(M:R)_1$ bezeichnen wir die Menge aller solchen Elemente x aus R , die die Bedingung $xR \subseteq M$ erfüllen. Ist insbesondere M ein Rechtsideal von R , so ist es auch $M \subseteq (M:R)_1$.

SATZ 1. Für jeden MHR-Ring mit der Eigenschaft (*) sind die folgenden Aussagen äquivalent:

(i) R besitzt die direkte Zerlegung:

$$R = \sum_{v \in \Gamma}^\boxplus R^{(v)},$$

wobei die $R^{(v)}$ ($v \in \Gamma$) vollständig primäre MHR-Ringe sind;

(ii) für jedes Element $x \in R$ gibt es ein $e_x \in R$ mit $xe_x = x$;

(iii) für jedes Nilrechtsideal N von R gilt $(N:R)_1 \cong N$.

BEWEIS. (i) \Rightarrow (ii) ist trivial.

(ii) \Rightarrow (iii): R sei ein MHR-Ring mit der Eigenschaft (*). N sei ein Rechtsideal von R . Ist $x \in R$ mit $x \notin N$, so gilt wegen (ii) $xR \not\subseteq N$. Es gilt also (iii).

(iii) \Rightarrow (i): Nach [1] (Satz 1) gilt

$$(1) \quad R = \sum_{v \in \Gamma}^{\oplus} e_v R e_v \oplus U,$$

wobei $e_v R e_v$ ($e_v^2 = e_v$) einander annullierende vollständig primäre Ringe sind und U eine Untergruppe von $(J(R), +)$ mit $e_v U e_v = (0)$ für jedes $v \in \Gamma$ ist. Es sei e ein Idempotent von R . Wegen (*) gilt nach [1] $xe = exe$ für jedes $x \in R$. Daher ergibt sich für jedes $v \in \Gamma$

$$(2) \quad U(e_v R e_v) = (U e_v) e_v R e_v = (e_v U e_v) e_v R e_v = (0).$$

Es seien $a, b \in U$. Wegen (1) gilt etwa

$$ab = e_1 r_1 e_1 + \dots + e_t r_t e_t + c$$

($e_i r_i e_i \in e_i R e_i, c \in U$). Wäre etwa $e_1 r_1 e_1 \neq 0$, so gälte

$$0 = a b e_1 = a (b e_1) = a (e_1 b e_1) = e_1 r_1 e_1 \neq 0.$$

Daher ist U ein Unterring und wegen (2) sogar ein Rechtsideal von R . Wir nehmen an, es sei $U \neq (0)$. Dann enthält U ein minimales Rechtsideal U_1 von R mit $U_1^2 = (0)$, das von einem Element $x \in U$ erzeugt ist: $U_1 = (x)_r$. Wäre $U_1 U \neq (0)$, so gälte $U_1 U = U_1$. Es gäbe ein $y \in U$ mit $x = xy$. Da y ein nilpotentes Element ist, gäbe es eine natürliche Zahl n ($n > 1$) mit $xy^n = 0, xy^{n-1} \neq 0$. Hieraus folgte

$$0 \neq xy^{n-1} = (xy)y^{n-1} = xy^n = 0.$$

Es gilt also $U_1 U = (0)$, und folglich ist $U_1 R = (0)$. Das ist ein Widerspruch zu (iii). Daher muß $U = (0)$ gelten. Aus (1) folgt dann die Behauptung (i).

Damit ist der Beweis des Satzes erbracht.

Ein Ring R heißt torsionsfrei, falls $(R, +)$ eine torsionsfreie Gruppe ist.

SATZ 2. Für jeden torsionsfreien MHR-Ring R sind äquivalent:

(i) R ist die ringtheoretische diskrete direkte Summe von vollständig primären MHR-Ringen;

(ii) R besitzt die Eigenschaft (*).

Zum Beweis benötigen wir den folgenden auch an sich interessanten Hilfssatz.

HILFSSATZ 3. Jedes Rechtsideal eines torsionsfreien MHR-Ringes ist teilbar.

BEWEIS. Eine additive abelsche Gruppe G heißt teilbar, falls $nG = G$ für jedes natürliche n gilt. Ein Rechtsideal A eines Ringes R heißt teilbar, wenn $(A, +)$ eine teilbare Gruppe ist. Es sei A ein Rechtsideal eines torsionsfreien MHR-Ringes R . Wir nehmen an, A sei nicht teilbar. Die Menge \mathfrak{M} aller teilbaren Rechtsideale A_i von R mit $(0) \cong A_i \subset A$ ist offenbar nicht leer. Nach dem ZORN'schen Lemma (vgl. [5], Satz 1.5) enthält \mathfrak{M} ein maximales Element M . Es gilt

$$(1) \quad (A, +) = (M, +) \oplus (B, +).$$

Da $M \subset A$ gilt, gibt es ein $a \in A$ mit $a \notin M$. Unter den Hauptidealidealen $(na)_r$ von R gibt es ein minimales, etwa $(ma)_r = m(a)_r$, wobei m, n natürliche Zahlen sind. Da die Faktorgruppe $(A, +)/(M, +)$ wegen (1) torsionsfrei ist, gilt $m(a)_r \not\subseteq M$. Andererseits ist $m(a)_r$ teilbar. Daher ist auch $M' = M + m(a)_r$ teilbar. Offensichtlich gehört M' zu \mathfrak{M} mit $M' \supset M$. Das ist ein Widerspruch zur Maximalität von M in \mathfrak{M} . Daher ist A teilbar.

BEWEIS des Satzes 2. (i) \Rightarrow (ii) ist trivial.

(ii) \Rightarrow (i): Es sei N ein Rechtsideal eines torsionsfreien MHR-Ringes R . Dann ist $(R, +)/(N, +)$ nach Hilfsatz 3 torsionsfrei. Es sei $x \in R$ mit $xR \subseteq N$. Wir nehmen an, es sei $x \notin N$. Wir betrachten die absteigende Kette von Hauptidealidealen $(2^i x)_r$ von R ($i=0, 1, 2, \dots$):

$$(2) \quad (x)_r \supseteq (2x)_r \supseteq \dots \supseteq (2^i x)_r \supseteq \dots$$

Wäre etwa $(2^i x)_r = (2^{i+1} x)_r$, so gäbe es eine ganze rationale Zahl m und ein Element $r \in R$ mit $2^i x = m2^{i+1} x + 2^{i+1} xr$. Daher gälte $2^i(1-2m)x = 2^{i+1} xr \in N$. Hieraus folgte $2^i(1-2m)x = 0$. Das bedeutete $1-2m=0$, es wäre also $m=1/2$. Dieser Widerspruch zeigt, daß (2) eine unendliche streng absteigende Kette sein muß. Das ist aber unmöglich, denn R ist ein MHR-Ring. Daher ist $x \in N$, d.h. R erfüllt die Bedingung (iii) des Satzes 1. Besitzt R die Eigenschaft (*), so erfüllt R nach Satz 1 die Bedingung (i) des Satzes 2.

Aus Satz 2 folgt unmittelbar die

FOLGERUNG 4. Jeder kommutative torsionsfreie MHR-Ring ist die ringtheoretische diskrete direkte Summe von vollständig primären MHR-Ringen.

Betrachten wir MH_1R -Ringe, so haben wir die

FOLGERUNG 5. Es sei R ein MH_1R -Ring mit der Eigenschaft (*). Dann sind die folgenden Aussagen äquivalent:

(i) R ist die ringtheoretische diskrete direkte Summe von vollständig primären artinschen Ringen;

(ii) für jedes $x \in R$ gibt es ein $e_x \in R$ mit $xe_x = x$;

(iii) für jedes Nilrechtsideal N von R gilt $(N:R)_1 \subseteq N$.

Der Beweis folgt aus Satz 1 und aus der Tatsache, daß jeder MH_1R -Ring mit Einselement ein artinscher Ring ist.

FOLGERUNG 6. Für jeden torsionsfreien MH_1R -Ring R sind die folgenden Aussagen äquivalent:

(i) R ist die ringtheoretische diskrete direkte Summe von vollständig primären artinschen Ringen;

(ii) R erfüllt die Eigenschaft (*).

FOLGERUNG 7. Jeder kommutative torsionsfreie MH_1R -Ring ist die ringtheoretische diskrete direkte Summe von vollständig primären artinschen Ringen.

Für MH_1^*R -Ringe gilt:

FOLGERUNG 8. Es sei R ein MH_1^*R -Ring mit der Eigenschaft (*). Dann sind die folgenden Aussagen äquivalent:

- (i) R ist die ringtheoretische diskrete direkte Summe unendlicher Körper mit Minimalbedingung für Unterkörper und endlicher vollständig primärer Ringe
 (ii) für jedes $x \in R$ gibt es ein $e_x \in R$ mit $xe_x = x$;
 (iii) für jedes Nilrechtsideal N von R gilt $(N:R)_1 \subseteq N$.

Für den Beweis benutze man die Folgerung 5 und die folgende Tatsache: Jeder vollständig primäre MH_1^+R -Ring genügt der Minimalbedingung für Unter-
 ringe. Demzufolge ist er entweder endlich oder ein unendlicher Körper mit Minimal-
 bedingung für Unterkörper. Die Struktur der unendlichen Körper mit Minimal-
 bedingung für Unterkörper ist wohlbekannt (siehe [7]).

§ 3. Zusammenhang zwischen artinschen Ringen und noetherschen Ringen

Nach einem Satz aus [4] ist ein artinscher Ring genau dann noethersch, wenn
 seine additive Gruppe keine Untergruppe vom Typ $Z(p^\infty)$ enthält. In [6] hat A. KER-
 TÉSZ die notwendigen und hinreichenden Bedingungen dafür angegeben, daß ein
 noetherscher Ring artinsch ist. Bezüglich der Untersuchung dieser Art siehe auch
 [2] und [3]. Im folgenden zeigen wir, daß die Begriffe „artinsch“ und „noethersch“
 für Ringe in der Klasse der MHR-Ringe, deren additive Gruppe keine Untergruppe
 vom Typ $Z(p^\infty)$ enthält, gleichbedeutend sind.

SATZ 9. Enthält die additive Gruppe $(R, +)$ eines MHR-Ringes R keine Unter-
 gruppe vom Typ $Z(p^\infty)$, so sind äquivalent:

- (i) R ist artinsch;
 (ii) R ist noethersch.

BEWEIS. Nach [4] gilt (i) \Rightarrow (ii).

(ii) \Rightarrow (i): R sei ein noetherscher MHR-Ring und $J(R)$ sei das Radikal von R .
 Nach [8] ist $J(R)$ ein nil-Ring. Da R noethersch ist, ist $J(R)$ dann nilpotent. Es sei
 k der Nilpotenzgrad von $J(R)$. Wir beweisen die Behauptung durch vollständige
 Induktion nach k .

Für $k=1$ ist $J(R)=(0)$, d.h. R ist ein halbeinfacher MHR-Ring, besitzt also
 die Struktur vom Satz 2 aus [8]. Da R auch noethersch ist, muß R dann artinsch
 sein, (siehe [9], Satz 6.1).

Es sei $k > 1$, und es sei vorausgesetzt, daß die Behauptung für alle Ringe wie
 im Satz 9, deren Radikal den kleineren Nilpotenzgrad als k hat, richtig sei. Der
 Faktorring $R/J(R)^{k-1}$ ist ein noetherscher MHR-Ring. Daher besitzt $(R/J(R)^{k-1}, +)$
 keine Untergruppe vom Typ $Z(p^\infty)$. Nach Induktionsannahme ist dieser Faktor-
 ring artinsch. Wir betrachten $J(R)^{k-1}$ als R -Rechtsmodul und machen es zu einem
 \bar{R} -Rechtsmodul (mit $\bar{R} \stackrel{\text{def}}{=} R/J(R)$) durch folgende Definition:

$$x\bar{r} \stackrel{\text{def}}{=} xr \quad (x \in J(R)^{k-1}, r \in R, \bar{r} \in \bar{R}).$$

(Diese Definition ist zulässig, da aus $\bar{r}_1 = \bar{r}_2 \in \bar{R}$, d.h. aus $r_1 - r_2 \in J(R)$ für jedes
 $x \in J(R)^{k-1}$

$$xr_1 - xr_2 = x(r_1 - r_2) \in J(R)^k = (0),$$

d.h. $x\bar{r}_1 = x\bar{r}_2$ folgt.) Daher stimmen die R -Untermoduln und die \bar{R} -Untermoduln von $J(R)^{k-1}$ überein. Insbesondere ist $J(R)^{k-1}$ als \bar{R} -Rechtsmodul noethersch. Da \bar{R} ein halbeinfacher artinscher Ring ist, gilt nach [5] (Satz 9.3) eine direkte Zerlegung:

$$(1) \quad J(R)^{k-1} = H \oplus M_1 \oplus \dots \oplus M_t,$$

wobei H ein trivialer \bar{R} -Rechtsmodul (d.h. $H\bar{R} = (0)$) und M_i ($i=1, 2, \dots, t$) einfacher \bar{R} -Rechtsmodul ist. Da jede Untergruppe von $(H, +)$ ein Rechtsideal von R ist, genügt $(H, +)$ der Maximalbedingung für Untergruppen und der Minimalbedingung für zyklische Untergruppen, d.h. $(H, +)$ ist eine endlich erzeugbare Torsionsgruppe. Daher ist $(H, +)$ endlich. Folglich ist $J(R)^{k-1}$ ein artinscher R -Rechtsmodul. Nach [5] (Satz 6.1) ist R artinsch.

Der Satz ist damit bewiesen.

FOLGERUNG 10. Ein noetherscher Ring ist genau dann artinsch, wenn er ein MHR-Ring ist.

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A THEOREM ON SPERNER-SYSTEMS SATISFYING AN ADDITIONAL CONDITION

by

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1. Introduction

Let X be a finite set. $|X|=n$ denotes that X consists of n elements. A family \mathcal{A} of subsets of X is said to have the Sperner-property if there are no two different sets belonging to \mathcal{A} such that one of them contains the other. SPERNER's theorem

asserts that if \mathcal{A} has the Sperner-property then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$ and if n is even then equality holds only if \mathcal{A} consists exactly of the $n/2$ -element subsets of X .

The following theorem is a generalization of Sperner's theorem:

Let \mathcal{A} be a family of subsets of X , $|X|=n$, \mathcal{A} has the Sperner-property and there exist k, l such that $l=0$ or 1 and for any two sets A_1, A_2 in \mathcal{A} $|A_1 \cup A_2| \leq 2k + l \leq n$ then $|\mathcal{A}| \leq \binom{n}{k}$ and for $l=0$ equality holds only if \mathcal{A} consists exactly of the k -element subsets of X .

This theorem is proved in E. C. MILNER [2] though it is stated there for intersections instead of unions.

In this paper we shall prove the following:

THEOREM. *Let h, k, r be given, $r < k$. Let \mathcal{A} be a family of subsets of X , $|X|=n$. Let \mathcal{A} have the Sperner-property and satisfy that for any k elements A_1, \dots, A_k of \mathcal{A} $|\bigcup_{i=1}^k A_i| \leq hk + r$; then for $n > n(h, k)$*

$$|\mathcal{A}| \leq \sum_{i=0}^s \binom{n-r}{h-i} \binom{r}{\lfloor \frac{r}{2} \rfloor + i} \quad \text{holds, where } s = \min \left(h, r - \left\lfloor \frac{r}{2} \right\rfloor \right)$$

and for $r \neq 1$ equality holds if and only if there is a subset C of X , $|C|=r$ and \mathcal{A} consists exactly of those subsets of X which satisfy

$$(1) \quad |A| = h + \left\lfloor \frac{r}{2} \right\rfloor, \quad |A \cap (X \setminus C)| \leq h.$$

Remark: It can be easily verified that it is not sufficient to suppose that $hk + r \leq n$ for $k \geq 3$. No attempt has been made to obtain explicit bounds on n .

The theorem for the special cases $r=0$ or 1 was independently proved by E. BOROS, too.

2. The proof of the theorem. First we prove that the system of sets defined by (1) satisfies the assumptions of the theorem.

As it consists of sets of the same cardinality it has the Sperner-property. If A_1, \dots, A_k belong to this system then (1) implies:

$$\left| \left(\bigcup_{i=1}^k A_i \right) \cap (X \setminus C) \right| \leq hk,$$

whence

$$\left| \bigcup_{i=1}^k A_i \right| = \left| \bigcup_{i=1}^k A_i \cap (X \setminus C) \right| + \left| \bigcup_{i=1}^k A_i \cap C \right| \leq hk + r.$$

Obviously this system has the cardinality of the theorem.

We prove the theorem by induction on k . For $k=2$ the assertions follow from the theorem cited before. Now let $k>2$. We may suppose that we have a system \mathcal{A} of maximal cardinality.

I. Let us suppose first that $r \neq k-1$.

If for any choice of A_1, \dots, A_{k-1} belonging to \mathcal{A} $\left| \bigcup_{i=1}^{k-1} A_i \right| \leq h(k-1) + r$ then the assertion follows from the induction hypothesis. Hence we may assume that there exist B_1, \dots, B_{k-1} in \mathcal{A} such that $h(k-1) + r < \left| Y = \bigcup_{i=1}^{k-1} B_i \right| \leq hk + r$.

Let B_k be an arbitrary set in \mathcal{A} ; then $\left| \bigcup_{i=1}^k B_i \right| = |Y| + |B_k \cap (X \setminus Y)|$, whence it follows that this latter is less than h . As there are at most $2^{|Y|}$ subsets of X having the same intersection with $X \cap Y$ so we arrive to the estimate:

$$|\mathcal{A}| \leq 2^{hk+r} \left[\sum_{i=0}^{k-1} \binom{n}{i} \right],$$

which leads to a contradiction for $n > n(h, k)$.

II. $r = k-1 \geq 2$. We may assume, by I, the theorem to be proved for smaller values of r ; as a consequence we may suppose that there exist sets A_1, \dots, A_k belonging to \mathcal{A} such that $\left| \bigcup_{i=1}^k A_i \right| = hk + k - 1$. Let $Y = \bigcup_{i=1}^k A_i$, $A'_i = Y \setminus \bigcup_{j \neq i} A_j$. For $i \neq j$ $A'_i \cap A'_j = \emptyset$, whence it follows that for some j $|A'_j| \leq h$. By symmetry reasons we may suppose $|A'_k| \leq h$.

Let A be an arbitrary element of \mathcal{A} ; then

$$(2) \quad \left| \bigcup_{i=1}^{k-1} A_i \cup A \right| \leq hk + k - 1 \quad \text{implies} \quad |A \cap (X \setminus Y)| \leq h.$$

We shall make use of the following construction: Let be given $Z \subset X$ such that

$$Y \subseteq Z, \quad |Z| \leq k(h+1)^2.$$

We pick out one by one sets C_i belonging to \mathcal{A} and satisfying the following conditions:

- a) $|C_i \cap (X \setminus Z)| = h$,
 b) $C_i \cap C_j \subseteq Y$ for $i \neq j$,
 c) $C_i \cap (Y \setminus (\bigcup_{j < i} C_j)) \neq \emptyset$.

After less than k steps the process stops. Otherwise we would have $\left| \bigcup_{i=1}^k C_i \right| > hk + k - 1$, which would be a contradiction.

Nevertheless we keep on picking out C_i 's satisfying a and b, up to the k -th step.

First we prove that this is possible. Let us suppose that on the contrary after having chosen C_1, \dots, C_i we are unable to find C_{i+1} i.e. for any $A \in \mathcal{A}$

$$(3) \quad |A \cap (X \setminus (Z \cup C_1 \cup \dots \cup C_i))| \leq h - 1$$

(as otherwise condition b, would follow by (2)). As above (3) leads us to the conclusion:

$$|\mathcal{A}| \leq 2^{(|Z|+hi)} \sum_{j=0}^{h-1} \binom{n}{j} \leq 2^{k(h+1)^2+hk} \binom{h-1}{j},$$

which is a contradiction for $n > n(h, k)$.

Let us suppose that $\left| \bigcup_{i=1}^k C_i \cap Y \right| < k - 1$. We divide the elements of \mathcal{A} into two disjoint classes:

$$\mathcal{A}_1 = \{A \in \mathcal{A} \mid |A \cap (X \setminus (Z \cup C_1 \cup \dots \cup C_k))| < h\},$$

$$\mathcal{A}_2 = \left\{ A \in \mathcal{A} \mid \left| A \cap \left(X \setminus \left(\bigcup_{i=1}^k C_i \cup Z \right) \right) \right| = h \right\}; \quad \text{by (2) } \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2.$$

As we have proved $|\mathcal{A}_1| \leq c(h, k)n^{h-1}$, $c(h, k)$ does not depend on n .

Let $A \in \mathcal{A}_2$; then by (2) $A \cap \left(Z \cup \left(\bigcup_{i=1}^k C_i \right) \right) \subseteq Y$. As A cannot satisfy condition c,

so we have $A \cap Y \subseteq \left(\bigcup_{i=1}^k C_i \cap Y \right) = D$, $|D| < k - 1$.

Let E be any h -element subset of $X \setminus Y$ and let $\mathcal{E} = \{A \in \mathcal{A}_2 \mid (A \cap X \setminus Y) = E\}$. By the Sperner-property of the sets belonging to \mathcal{A} the sets $A \cap Y$, $A \in \mathcal{E}$, form a family of subsets of D which has the Sperner-property, whence by Sperner's theorem

$$|\mathcal{E}| \leq \binom{|D|}{\lfloor \frac{|D|}{2} \rfloor} < \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor} \quad \text{as } k \geq 3.$$

So we have:

$$|\mathcal{A}_2| \cong \binom{D}{\lfloor \frac{|D|}{2} \rfloor} \binom{n - (hk + k - 1)}{h},$$

$$|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| < \binom{n - k + 1}{h} \binom{k - 1}{\lfloor \frac{k - 1}{2} \rfloor} \quad \text{for } n > n(h, k),$$

which contradicts the maximality of \mathcal{A} .

As $hk + k - 1 \cong \left| \bigcup_{i=1}^k C_i \right| = hk + |D|$, we have $|D| = k - 1$. Now we shall apply our construction inductively.

First let $Y = Z$ and let us denote C_i by B_i^0 . If we have already constructed the systems $\{B_i^0\}, \dots, \{B_i^s\}, i = 1, \dots, k, s < h$, then let

$$Z = Y \cup \left(\bigcup_{t \leq s} \bigcup_{i=1}^k B_i^t \right), \quad |Z| = |Y| + (s + 1)kh \cong kh^2 + kh + k - 1 < k(h + 1)^2,$$

so we may apply our construction to pick out $B_i^{s+1} = C_i$. Let $D_j = Y \cap \left(\bigcup_{i=1}^k B_i^j \right)$.

We wish to prove that for any $0 < j \leq h, D_0 = D_j$. Let us suppose that there would exist an index j such that $D_j \neq D_0$. As $|D_j| = k - 1 = |D_0|$ so there is an index s and a set B_s^j such that $B_s^j \cap Y \not\subseteq D_0$. The definition of B_s^j implies that

$$\left| B_s^j \cap \left(X \setminus \left(Y \cup \left(\bigcup_{i=1}^k B_i^0 \right) \right) \right) \right| \cong h, \quad \text{so we have } \left| \bigcup_{i=1}^{k-1} B_i^0 \cup B_s^j \right| \cong hk + k$$

(as B_k^0 did not satisfy condition c) and as a consequence

$$\left| Y \cap \left(\bigcup_{i=1}^{k-1} B_i^0 \cup B_s^j \right) \right| \cong k,$$

which contradicts the assumptions of the theorem. Let $D = D_0 = D_1 = \dots = D_h$.

Now we are able to prove that for any $A \in \mathcal{A} |A \cap (X \setminus D)| \cong h$. From the definition of B_i^s we know that for $j \neq l$

$$\left(\bigcup_{i=1}^k B_i^j - D \right) \cap \left(\bigcup_{i=1}^k B_i^l - D \right) = \emptyset.$$

If for an $A \in \mathcal{A}, |A \cap X \setminus D| \cong h + 1$, then either there would exist an index $0 \leq j \leq h$ such that

$$A \cap \left(\bigcup_{i=1}^k B_i^j \setminus D \right) = \emptyset \quad \text{yielding} \quad \bigcup_{i=1}^{k-1} B_i^j \cup A \cong h(k - 1) + k - 1 + h + 1 > hk + k - 1$$

or we would have $|A \cap X \setminus Y| \cong h + 1$ contradicting (2).

Up till now from the assumptions that \mathcal{A} is maximal and n is large enough we have constructed a set D , $|D|=k-1$ such that for any $A \in \mathcal{A}$, $|A \cap (X \setminus D)| \leq h$. We shall examine this situation more precisely.

LEMMA. Let $X=Y \cup Z$, $Y \cap Z = \emptyset$. Let \mathcal{A} be a family of subsets of X having the Sperner-property and satisfying the additional condition that for any $A \in \mathcal{A}$

$$(4) \quad |A \cap Z| \leq h.$$

If $|Y| \geq 2$, $|Z| \geq (h(2+|Y|/2))$, then the cardinality of \mathcal{A} is maximal if and only if \mathcal{A} consists exactly of the $(h+|Y|/2)$ -element subsets of X satisfying the condition (4).

Now the theorem follows obviously from the lemma.

We shall use the following estimate which is due to Sperner (see SPERNER [1]): Let \mathcal{R} be a family of k -element subsets of a finite set F , $|F|=f$ and let us define

$$\mathcal{R}^* = \{S | S \subseteq F, \exists R \in \mathcal{R} | R \subset S, |S| = k+1\},$$

$$\mathcal{R}_* = \{S | S \subseteq F, \exists R \in \mathcal{R} | R \supset S, |S| = k-1\}.$$

Then

$$(5) \quad |\mathcal{R}^*| \leq \frac{f-k}{k+1} |\mathcal{R}|,$$

$$(6) \quad |\mathcal{R}_*| \leq \frac{k}{f-k+1} |\mathcal{R}|.$$

Let us suppose that the assertion of the lemma is not valid i.e. there is a system \mathcal{A} of maximal cardinality which contains subsets of X which do not have the desired cardinality.

a) There is a set belonging to \mathcal{A} which has cardinality smaller than $h+[|Y|/2]$, let $|Y|=y$, $|Z|=z$. Let \mathcal{B} consist exactly of the sets of minimal cardinality, t belonging to \mathcal{A} . We classify the elements of \mathcal{B} according to the cardinality of $|A \cap Z|$ i.e. $\mathcal{B}_i = \{B \in \mathcal{B} | |B \cap Z| = i\}$, obviously for $i > h$. $\mathcal{B}_i = \emptyset$.

If $\mathcal{B}_h = \emptyset$, then \mathcal{B}^* satisfies (4), and as $|X| > 2t+1$ so $|\mathcal{B}^*| > |\mathcal{B}|$. Obviously $(\mathcal{A} \setminus \mathcal{B}) \cup \mathcal{B}^*$ satisfies the assumptions of the lemma which contradicts the maximality of \mathcal{A} .

Let E be a subset of Y , and F be a subset of Z . We define:

$${}^E\mathcal{B}_{h-1} = \{B \in \mathcal{B}_{h-1} | B \cap Y = E, \quad {}^F\mathcal{B}_h = \{B \in \mathcal{B}_h | B \cap Z = F\},$$

$$\overline{{}^E\mathcal{B}_{h-1}} = \{A \subseteq X | \exists B \in {}^E\mathcal{B}_{h-1}, A \cap Y = E, |A| = |B| + 1, A \supset B\},$$

$$\overline{{}^F\mathcal{B}_h} = \{A \subseteq X | \exists B \in {}^F\mathcal{B}_h, A \cap Z = F, A \supset B, |A| = |B| + 1\};$$

by (5) we have

$$|\overline{{}^F\mathcal{B}_h}| \leq |{}^F\mathcal{B}_h| \frac{y-(t-h)}{t-h+1}, \quad |\overline{{}^E\mathcal{B}_{h-1}}| \leq |{}^E\mathcal{B}_{h-1}| \frac{z-(h-1)}{h}.$$

Defining further

$$\underline{\mathcal{B}}_h = \bigcup_{|F|=h, F \subset Z} ({}^F\mathcal{B}_h) \quad \text{and} \quad \overline{\mathcal{B}}_{h-1} = \bigcup_{E \subset Y, |E|=t-h+1} (\overline{{}^E\mathcal{B}_{h-1}})$$

and using the indirect assumption $t < h + [y/2]$ we get

$$|\underline{\mathcal{B}}_h| \cong \frac{y+2}{y} |\mathcal{B}_h| \cong |\mathcal{B}_h| + \frac{2}{y} |\mathcal{B}_h|,$$

$$|\overline{\mathcal{B}}_{h-1}| \cong |\mathcal{B}_{h-1}| + \frac{z-2h+1}{h} |\mathcal{B}_{h-1}|.$$

The systems

$$\mathcal{A}_1 = (\mathcal{A} \setminus (\mathcal{B}_h \cup \mathcal{B}_{h-1})) \cup \underline{\mathcal{B}}_h$$

and

$$\mathcal{A}_2 = (\mathcal{A} \setminus (\mathcal{B}_h \cup \mathcal{B}_{h-1})) \cup \overline{\mathcal{B}}_{h-1}$$

satisfy the assumptions of the lemma as it can be easily verified, so by the maximality of \mathcal{A} we get:

$$\frac{2}{y} |\mathcal{B}_h| \cong |\mathcal{B}_{h-1}|, \quad \frac{z-2h+1}{h} |\mathcal{B}_{h-1}| \cong |\mathcal{B}_h|.$$

As we have proved $|\mathcal{B}_h| \neq 0$, $|\mathcal{B}_{h-1}| \neq 0$ either; as a consequence we can multiply the two above inequalities and divide both sides by $|\mathcal{B}_h| |\mathcal{B}_{h-1}|$. In such a way we get:

$$\frac{2(z-2h+1)}{yh} \cong 1 \quad \text{i.e.} \quad z+1 \cong h \left(\frac{y}{2} + 2 \right),$$

which contradicts our assumptions.

b) \mathcal{A} contains a set of cardinality greater than $h + [y/2]$. Let \mathcal{F} consist of all sets belonging to \mathcal{A} and having maximal cardinality m . We divide the elements of \mathcal{F} into disjoint subfamilies \mathcal{F}_i exactly as we did with the elements of \mathcal{B} . We define as above the families ${}^F\mathcal{F}_i$, $i=0, \dots, h$; $F \subseteq Z$, and let us define:

$${}^F\mathcal{F}'_i = \{A \subset X | \exists B \in {}^F\mathcal{F}_i, A \subset B, |A| = m-1, A \cap Z = F\}, \quad \mathcal{F}'_i = \bigcup_{F \subset Z} ({}^F\mathcal{F}'_i);$$

it is a disjoint union (the same was true for the ${}^F\mathcal{B}_h$ -s too) hence using (6) we arrive to the estimate:

$$|\mathcal{F}'_i| \cong |\mathcal{F}_i| \frac{m-i}{y+1-(m-i)} \cong \frac{h + \left[\frac{y}{2} \right] + 1 - i}{i - h + y - \left[\frac{y}{2} \right]} \cong \frac{(h-i) + \frac{y+1}{2}}{(i-h) + \frac{y+1}{2}} \cong 1 \quad \text{as} \quad h \cong i$$

and equality is possible only if $h=i$. The system

$$\mathcal{A}' = (\mathcal{A} \setminus \mathcal{F}) \cup \left(\bigcup_{i=0}^h \mathcal{F}'_i \right)$$

satisfies the hypothesis of the lemma and its cardinality is greater than that of \mathcal{A} unless $\mathcal{F}_i = \emptyset$ for $i < h$ in which case we could take $\mathcal{A}^* = \mathcal{A}' \cup \{Y\}$ to obtain the desired contradiction and we are done.

Remark: As it was pointed out to me by G. KATONA, the fact that the given system in the lemma is maximal can be easily deduced from [3] but we had to prove the uniqueness, too.

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A GENERALIZATION OF MORAN'S QUASI-POISSON PROCESS*

by

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Summary

For each integer $k \geq 2$ an example is given of a point process $Q(k)$ on the real line \mathbf{R} which is not a Poisson process, but which mimics a Poisson process in the following sense. The joint distributions of the counts of $Q(k)$ in any $k-1$ contiguous intervals of \mathbf{R} are the same as those of a Poisson process of unit rate. When $k=2$ an example given by MORAN is recovered.

1. Introduction

Let $k \geq 2$ be an integer. A point process Q on the real line \mathbf{R} will be said to satisfy the condition $C(k)$ if the respective counts (numbers of events) N_i ($i=1, 2, \dots, k-1$) of Q in any $k-1$ contiguous intervals $[a_i, a_{i+1})$ of \mathbf{R} are independent random variables and each N_i has a Poisson distribution with mean $a_{i+1}-a_i$ (we write $N_i \sim P(a_{i+1}-a_i)$). RÉNYI (1967) showed that only the Poisson process itself satisfies $C(k)$ simultaneously for all k .

GOLDMAN (1967, quoting an example of L. SHEPP) exhibited a process which was not a Poisson process, but which still satisfied $C(2)$. The key property of the Poisson process used in SHEPP's construction is that, conditionally on the number of events occurring in a Borel subset A of \mathbf{R} , their positions are independently and uniformly distributed over A . MORAN (1967) constructed a different non-Poisson process satisfying $C(2)$, by imbedding in \mathbf{R} an interval sequence which mimics a sequence of independent exponential variables.

SZÁSZ (1970) generalised SHEPP's example to obtain for each fixed k a process which was not Poisson but which satisfied $C(k)$ ** . In this note a complementary generalisation of MORAN's process is given. The construction is carried out for renewal processes as this leads to little extra complication.

2. Construction of the process

THEOREM. *Let $k \geq 2$ be an integer and let F be a distribution function with $F(0)=0$ and finite mean which is not concentrated on fewer than $2k$ points of \mathbf{R} . Then there exists a k -tuple of random variables (X_1, X_2, \dots, X_k) such that*

* *Key words.* Point process; Poisson process; Quasi-Poisson process; Renewal process.

** In fact, SZÁSZ constructed for every fixed k a process which was not Poisson and satisfied $C(k)$ without requiring in it the contiguity of the intervals. However, our construction also yields a process of this type if we make it for $2k-2$ instead of k .

- (i) each X_i has the distribution $F(\cdot)$,
 (ii) the $(k-1)$ -tuples $(X_1, X_2, \dots, X_{k-1})$ and (X_2, X_3, \dots, X_k) are each $(k-1)$ -tuples of independent random variables,
 (iii) for each i ($1 \leq i \leq k-1$) the $(k-1)$ -tuples $(X_1, X_2, \dots, X_i + X_{i+1}, X_{i+2}, \dots, X_k)$ have the joint distributions that they would have if (X_1, X_2, \dots, X_k) were mutually independent, but
 (iv) (X_1, X_2, \dots, X_k) are not mutually independent.

PROOF. It is sufficient to construct a signed measure $G(\cdot)$ on \mathbf{R}_+^k such that $F_k = F \times F \times \dots \times F - G$ is a probability measure on \mathbf{R}_+^k and such that the following are satisfied for any Borel subsets S_1, S_2, \dots, S_{2k} of \mathbf{R} and $S \subset \mathbf{R}^2$ of the form $\{x_i + x_{i+1} \leq y\}$:

- a) $G(\mathbf{R}, S_2, \dots, S_k) = G(S_1, S_2, \dots, \mathbf{R}) = 0$;
 b) $G(S_1, S_2, \dots, S_{i-2}, S, S_{i+1}, \dots, S_k) = 0$.

There exist $\delta > 0$ and Borel subsets A_1, A_2, \dots, A_{2k} of \mathbf{R} such that $F(A_i) > \delta$. Hence there exists also for each i ($1 \leq i \leq 2k$) a positive measure m_i on the Borel subsets of A_i such that $m_i(A_i) = \delta$ ($1 \leq i \leq 2k$) and such that $F - m_i$ is a positive measure on A_i . Let $B_j = A_{2j-1} \cup A_{2j}$ ($j=1, \dots, k$) and for each permutation $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(k))$ of $(1, 2, \dots, k)$ put $C_\alpha = \{x = (x_1, x_2, \dots, x_k) : x_i \in B_{\alpha(i)}\}$. If $i = (i(1), i(2), \dots, i(k))$ is a k -tuple of distinct integers with $1 \leq i(j) \leq 2k$ for all j let $\tau(i) = +1$ or -1 according as $\sum i(j)$ is even or odd. Let $R(i)$ be the rectangle $R(i) = A_{i(1)} \times \dots \times A_{i(k)}$. Then the $R(i)$ are mutually disjoint and, for some unique $\alpha = \alpha(i)$, $R(i) \subset C_\alpha$. We take $G \equiv \text{sgn} \{\alpha(i)\} \tau(i) m_{i(1)} \times \dots \times m_{i(k)}$ on each $R(i)$, and elsewhere $G \equiv 0$.

To see that G has the required properties, note that

- (a) $G(A_{2j-1}, S_2, \dots, S_k) + G(A_{2j}, S_2, \dots, S_k) = 0$ and
 (b) if x' is obtained from $x = (x_1, x_2, \dots, x_k)$ by interchanging x_i and x_{i+1} for some i , then $G(dx) = -G(dx')$.

Remark. If $k=2$, $F(x) = 1 - e^{-x}$, $A_i = (i-1, i]$ ($i=1, 2, 3, 4$) and each m_i is a multiple of Lebesgue measure, then MORAN's joint density is recovered.

The interval sequence $\{\dots X_{-1}, X_0, X_1, \dots\}$ of $Q(k)$ will be given by a sequence of independent k -tuples

$$(1) \quad \{(X_{mk+1}, X_{mk+2}, \dots, X_{(m+1)k}) : m = \dots -1, 0, 1, \dots\}.$$

Let

$$Y_m = X_{mk+1} + X_{mk+2} + \dots + X_{(m+1)k} \quad (m = \dots -1, 0, 1, \dots).$$

Then the Y_m are mutually independent and identically distributed and have finite mean. A stationary (equilibrium) renewal process may be constructed from the Y_m in the usual way (Cox, 1962, p. 28). The occurrences of this process will be termed E -points. The process $Q(k)$ is obtained by interpolating $k-1$ occurrences (called I -points) between each pair of neighboring E -points according to the joint distribution F_k , conditioned on $\sum_{i=1}^k X_i = Y_m$, the distance between these E -points.

3. Verification that $Q(k)$ satisfies $C(k)$,

From now on it will be assumed that $F(x) = 1 - e^{-x}$, so that we are dealing with the property $C(k)$ as formulated, although it will be seen that the analogous result will hold for any renewal process whose failure-time distribution satisfies the hypothesis of Theorem 1. The verification that $Q(k)$ satisfies $C(k)$ seems to require a rather intricate inclusion-exclusion argument of the form given below. Occurrences of $Q(k)$ will be called Q -points.

THEOREM 2. *Suppose that $k \geq 3$ and that the real numbers a_1, a_2, \dots, a_k satisfy $a_1 < a_2 < \dots < a_k$. Then if*

$$N_i = N[a_i, a_{i+1}) \quad (i = 1, 2, \dots, k-1),$$

where N is the counting measure of $Q(k)$, the N_i are independently distributed, $N_i \sim P(a_{i+1} - a_i)$.

PROOF. This is in two stages. We first consider the probability

$$\pi(n) = \Pr \{N_E = 0, N_1 = n_1, N_2 = n_2, \dots, N_{k-1} = n_{k-1}\},$$

where N_E is the number of E -events in $[a_1, a_k)$ and $n = (n_1, n_2, \dots, n_k)$ is a k -tuple of non-negative integers. Clearly, $\pi(n) = 0$ if $\sum n_i > k - 1$. It will be shown by a detailed enumeration of particular cases that $\pi(n)$ is invariant, i.e. it is unchanged by setting $G \equiv 0$. In each case $\pi(n)$ is evaluated by conditioning on either the position $a_1 - u$ of the last E -point before a_1 or on the position $a_k + v$ of the first E -point after a_k . To show that $\pi(n)$ is invariant it is sufficient to show that one of these conditional probabilities, denoted by $\pi_b(n, u)$ and $\pi_f(n, v)$ is invariant. It is convenient to include in both conditioning events the serial number m of the last E -event before a_1 , and to denote the interval X_{mk+i} by Z_i ($i = 1, 2, \dots, k$).

The various (non-exclusive) cases listed below include all possible values of n with $\sum n_i \leq k - 1$:

- (i) $\sum n_i \leq k - 2$;
- (ii) for some i , $n_i \geq 3$;
- (iiia) $n_1 = 2$; (iiib) $n_{k-1} = 2$;
- (iva) for some $s \geq 1$, $\sum_{i=1}^s n_i \geq s + 1$,
- (ivb) for some $s \geq 1$ $\sum_{i=k-s}^{k-1} n_i \geq s + 1$;
- (v) $n_i = 1$ for all i .

Case (i). If $\sum n_i \leq k - 2$ then either the first Q -point after a_k or the last Q -point before a_1 (or both) must be I -points. Then respectively $\pi_b(n, u)$, which depends only on the joint distribution of $(Z_1, Z_2, \dots, Z_{k-1})$, and $\pi_f(n, v)$, which depends only on the joint distribution of (Z_2, Z_3, \dots, Z_k) are invariant.

Case (ii). If $n_i \geq 3$ then, for some j , $\pi_f(n, v)$ depends only on the joint distribution of $(Z_1, Z_2, \dots, Z_{j-1}, Z_j + Z_{j+1}, Z_{j+2}, \dots, Z_k)$ and so is invariant.

Case (iiia). Consider $\pi(n, v)$ for fixed n_2, \dots, n_{k-1} with $n_2 + n_3 + \dots + n_{k-1} \leq k - 3$ and $n_1 = l$ allowed to vary. Then $\sum_i \pi_f(n, v)$ and $\pi_f(n, v)$ with $l=0$ and 1 are all invariant by (i) above. But the probabilities $\pi_f(n, v)$ for $l \geq 3$ (if they are non-zero) are invariant by (i). The result follows by subtraction. Case (iiib) is dealt with similarly.

Case (iv). Here we use induction on s , noting that for $s=1$ (iva) reduces to (iiia) or to (ii). Suppose that, if $1 \leq r \leq s$,

$$(1) \quad \sum_{i=1}^r n_i \geq r + 1 \Rightarrow \pi_f(n, v) \text{ is invariant.}$$

It will be shown that (1) holds also if $r=s+1$. If $n_{s+1} \geq 3$ then (ii) applies. If $n_{s+1}=2$ and $n_s=0$, or if $n_{s+1} \leq 1$ the induction hypothesis applies. If $n_{s+1}=2$ and $n_s \geq 1$ then $n_s + n_{s+1} \geq 3$ and (ii) applies if the s -th and $(s+1)$ -th intervals are pooled. The result now follows by subtraction, since it holds for all other n with the same values of $(n_1, n_2, \dots, n_{s-1}, n_s + n_{s+1}, n_{s+2}, \dots, n_{k-1})$. Hence the induction goes through and $\pi_f(n, v)$ is invariant in (iva). A similar argument applies to $\pi_b(n, u)$ in (ivb).

Case (v). If $\sum n_i = k - 1 \geq 2$ and (v) does not hold then one of (iva) and (ivb) must hold. The result for (v) now follows by subtraction, since $\Pr \{N_E = 0, \sum N_i = k - 1\}$ is invariant by (iiia).

This completes the first stage of the proof. The second stage is to evaluate probabilities of the form of $\pi(n)$ when $N_E \neq 0$. These may be determined by conditioning on the locations of all E -points within $[a_1, a_k)$, and using the independence of interpolations within different intervals of the E -process to factorise the relevant conditional probabilities. Each factor then corresponds to a set of at most $(k-1)$ contiguous intervals $\{[b_1, b_2), [b_2, b_3), \dots, [b_{l-1}, b_l)\}$, where either b_1 or b_k or possibly both are E -points. The joint distributions of counts over each such set may be evaluated by considering the same sequence of particular cases as above. Here, however, there is no need to condition further, i.e. on the positions of E -points outside $[a_1, a_k)$. The details, which are similar to those given in the first stage of the proof, are omitted. This completes the proof of Theorem 2.

Remark. For MORAN's process ($k=2$) the proof given requires a slight modification. Specifically,

$$\Pr \{N_E = 0, N_1 = 1\} = \Pr \{N_E = 0\} - \Pr \{N_E = 0, N_1 = 0\},$$

and the right-side is invariant.

4. Discussion

The interval sequence $\{X_i: i = \dots, -1, 0, 1, \dots\}$ of $Q(k+1)$ has a property that might be termed k -th order *quasi-stationarity*. For any integers $i(j)$ with $0 = i(0) < i(1) < \dots < i(k)$ and any n , the random variables

$$Y_j = X_{n+i(j-1)} + X_{n+i(j-1)+1} + \dots + X_{n+i(j)} \quad (1 \leq j \leq k)$$

have a joint distribution independent of n . This result holds also if the distribution of the k -tuples (1) depends on m (so long as it satisfies Theorem 1). Since

$$2 \sum_{i=1}^l \text{cov}(X_1, X_{i+1}) = \text{var} \left(\sum_{i=1}^{l+1} X_i \right) - \text{var}(X_{l+1}) - \text{var} \left(\sum_{i=1}^l X_i \right),$$

it follows that even first order quasi-stationarity implies weak stationarity.

It is interesting to note that MORAN's process and its generalisations have the same second order properties of both counts and intervals (COX and LEWIS, 1966, Chapter 4, p. 66) as the Poisson process.

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THE NEUTRIX DISTRIBUTION PRODUCT $x_+^{-r, \delta(r-1)}(x)$

by

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1. Introduction

A neutrix N , (see J. G. VAN DER CORPUT [1]) is a commutative additive group of functions $v(\xi)$ defined on a domain N' with values in an additive group N'' , where further if for some v in N , $v(\xi) = \gamma$ for all ξ in N' , then $\gamma = 0$. The functions in N are called negligible functions.

Now let N' be a set contained in a topological space with a limit point b which does not belong to N' . If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi) - \beta$ is negligible in N , then β is called the neutrix limit or N -limit of f as ξ tends to b and we write

$$N\text{-}\lim_{\xi \rightarrow b} f(\xi) = \beta.$$

This limit is unique if it exists.

The product of two distributions f and g was defined in [2] as the limit of the sequence $\{f_n g_n\}$, provided this sequence is regular, where

$$f_n = f * \delta_n, \quad g_n = g * \delta_n, \quad \delta_n(x) = n\varrho(nx)$$

for $n = 1, 2, \dots$ and ϱ is a fixed infinitely differentiable function having the following properties:

$$(1) \quad \varrho(x) = 0 \quad \text{for} \quad |x| \geq 1,$$

$$(2) \quad \varrho(x) \geq 0,$$

$$(3) \quad \varrho(x) = \varrho(-x),$$

$$(4) \quad \int_{-1}^1 \varrho(x) dx = 1.$$

The following definition of the product of two distributions was given in [4] so that further products of distributions can be defined.

DEFINITION. Let f and g be distributions and let

$$f_n = f * \delta_n, \quad g_n = g * \delta_n.$$

We will say that the product fg of f and g exists and is equal to the distribution h if

$$N\text{-}\lim_{n \rightarrow \infty} (f_n g_n, \varphi) = (h, \varphi)$$

for all test functions φ in the space K of infinitely differentiable functions with compact support, where N is the neutrix having domain $N' = \{1, 2, \dots\}$ and range N'' the real numbers. The negligible functions are linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n; \lambda > 0, \quad r = 1, 2, \dots$$

and all functions $f(n)$ for which $\lim_{n \rightarrow \infty} f(n) = 0$.

It is obvious that if the product $f\delta$ exists by the former definition it will exist by this new definition and will define the same distribution.

2.

We will now consider the neutrix product $x_+^{-r} \delta^{(r-1)}(x)$ and prove that

$$x_+^{-r} \delta^{(r-1)}(x) = \frac{(-1)^r (r-1)!}{4(2r-1)!} \delta^{(2r-1)}(x)$$

for $r=1, 2, \dots$.

By definition we have

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} \frac{d^r}{dx^r} \ln x_+$$

and so

$$(x_+^{-r})_n = \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^x \ln(x-t) \delta_n^{(r)}(t) dt$$

for $|x| \leq 1/n$. The support of $(x_+^{-r})_n \delta_n^{(r-1)}(x)$ is obviously contained in the interval $(-1/n, 1/n)$ and

$$\begin{aligned} \int_{-1/n}^{1/n} (x_+^{-r})_n \delta_n^{(r-1)}(x) x^m dx &= \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^{1/n} \delta_n^{(r-n)}(x) x^m \int_{-1/n}^x \ln(x-t) \delta_n^{(r)}(t) dt dx \\ &= \frac{(-1)^{r-1}}{(r-1)!} n^{2r+m-1} \int_{-1}^x \varrho^{(r-1)}(x) x^m \int_{-1}^x \ln\left(\frac{x-t}{n}\right) \varrho^{(r)}(t) dt dx. \end{aligned}$$

It follows that

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} (x_+^{-r})_n \delta_n^{(r-1)}(x) x^m dx = 0$$

for $m=0, 1, 2, \dots, 2r-2$.

Now it was proved in [3] that

$$\begin{aligned} \int_{-1/n}^{1/n} (x^{-r})_n \delta_n^{(r-1)}(x) x^{2r-1} dx &= -\frac{1}{2} (-1)^r (r-1)! \\ &= \int_{-1/n}^{1/n} [(x_+^{-r})_n + (-1)^r (x_-^{-r})_n] \delta_n^{(r-1)}(x) x^{2r-1} dx \end{aligned}$$

from which it follows that

$$\int_{-1/n}^{1/n} (x_+^{-r})_n \delta_n^{(r-1)}(x) x^{2r-1} dx = -\frac{1}{4} (-1)^r (r-1)!.$$

It is easily proved that

$$\int_{-1/n}^{1/n} |(x_+^{-r})_n \delta_n^{(r-1)}(x) x^{2r}| dx = O(1/n)$$

so that

$$\lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} |(x_+^{-r})_n \delta_n^{(r-1)}(x) x^{2r}| dx = 0.$$

Now let Φ be an arbitrary test function. Then

$$\Phi(x) = \sum_{m=0}^{2r-1} \frac{x^m}{m!} \Phi^{(m)}(0) + \frac{x^{2r}}{(2r)!} \Phi^{(2r)}(\xi x)$$

where $0 \leq \xi \leq 1$. It follows from what we have just proved that

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} & \left| \left((x_+^{-r})_n \delta_n^{(r-1)}(x), \Phi \right) - \frac{(-1)^r (r-1)!}{4(2r-1)!} (\delta^{(2r-1)}, \Phi) \right| \\ &= N\text{-}\lim_{n \rightarrow \infty} \frac{1}{(2r)!} \left| \int_{-1/n}^{1/n} (x_+^{-r})_n \delta_n^{(r-1)}(x) x^{2r} \Phi^{(2r)}(\xi x) dx \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2r)!} \left| \int_{-1/n}^{1/n} (x_+^{-r})_n \delta_n^{(r-1)}(x) x^{2r} \Phi^{(2r)}(\xi x) dx \right| \leq \\ &\leq \lim_{n \rightarrow \infty} \frac{K}{(2r)!} \int_{-1/n}^{1/n} |(x_+^{-r})_n \delta_n^{(r-1)}(x) x^{2r}| dx = 0, \end{aligned}$$

where $K = \sup |\Phi^{(2r)}(\xi x)|$. Our result now follows.

It follows similarly that

$$x_-^{-r} \delta^{(r-1)}(x) = \frac{(r-1)!}{4(2r-1)!} \delta^{(2r-1)}(x)$$

for $r=1, 2, \dots$

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DENSITY ESTIMATION FOR DEPENDENT SAMPLE

by

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1. Introduction

Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a strictly stationary sequence of random variables defined on a probability space (Ω, \mathcal{B}, P) . For $a \leq b$ define \mathcal{M}_a^b as the σ -field generated by the random variables ξ_a, \dots, ξ_b and define \mathcal{M}_a^∞ as the σ -field generated by ξ_a, ξ_{a+1}, \dots .

We shall say that the sequence $\{\xi_n\}$ is φ -mixing if, for each k ($1 \leq k < \infty$) and for each n ($n \geq 1$) $E_1 \in \mathcal{M}_1^k$ and $E_2 \in \mathcal{M}_{k+1}^\infty$ together imply

$$(I) \quad |P(E_1 E_2) - P(E_1)P(E_2)| \leq \varphi(n)P(E_1)$$

where $\varphi(n)$ $n=1, 2, \dots$ is a nonnegative function of integers such that

$$\lim_{n \rightarrow \infty} \varphi(n) = 0.$$

This definition is equivalent with the following one:

$$|P(E | \mathcal{M}_1^k) - P(E)| \leq \varphi(n),$$

with probability 1 if $E \in \mathcal{M}_{k+n}^\infty$. (See [1].)

Let $\Phi_k(x, y)$ ($k=1, 2, \dots$) be a sequence of Borel-measurable functions, defined on the Euclidean plane. We shall consider the following empirical density function:

$$f_n(x) = \frac{1}{n} \sum_{k=1}^n \Phi_n(x, \xi_k)$$

where $\{\xi_k\}$ is a φ -mixing sequence of identically distributed random variables having common density function $f(x)$. This definition is a common generalization of the hystogram, the well-known Rosenblatt's density estimation and the expansion in orthogonal functions. The same definition of empirical density function in case of independent identically distributed random variables was given in [2]. In the present paper the asymptotic property of the probability

$$P(\sup_x |f_n(x) - f(x)| > \varepsilon), \quad n \rightarrow \infty,$$

will be studied.

2. Some lemmas

At first we mention the following well-known lemma (see [1]).

LEMMA 1. Let $\{\xi_k\}$ be a φ -mixing stationary sequence and let the random variables ξ, η be measurable with respect to \mathcal{M}_1^k and \mathcal{M}_{k+n}^∞ respectively. If

$$E(|\xi|^p) < \infty, \quad E(|\eta|^q) < \infty$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \leq 2(\varphi(n))^{1/p} \mathbb{E}^{1/p}(|\xi|^p) \mathbb{E}^{1/q}(|\eta|^q).$$

The following lemma is probably well-known, but we couldn't find any reference.

LEMMA 2. Let $\{\xi_k\}$ be a φ -mixing stationary sequence and let ξ be a bounded random variable $|\xi| < M$, and measurable with respect to \mathcal{M}_{k+n}^∞ . Then with probability 1

$$|\mathbb{E}(\xi|\mathcal{M}_1^k) - \mathbb{E}(\xi)| \leq 2\varphi(n)M.$$

PROOF. Denoting by ξ^+ , ξ^- the positive and negative parts of ξ , we have

$$|\mathbb{E}(\xi|\mathcal{M}_1^k) - \mathbb{E}(\xi)| \leq |\mathbb{E}(\xi^+|\mathcal{M}_1^k) - \mathbb{E}(\xi^+)| + |\mathbb{E}(\xi^-|\mathcal{M}_1^k) - \mathbb{E}(\xi^-)|$$

with probability 1. Therefore it is enough to prove that if $\xi \geq 0$ then

$$|\mathbb{E}(\xi|\mathcal{M}_1^k) - \mathbb{E}(\xi)| \leq \varphi(n)M.$$

Define the A_{lm} -sets as follows:

$$A_{lm} = \left\{ \omega; \xi(\omega) > \frac{lM}{m} \right\}, \quad l = 0, 1, \dots, m-1, \\ m = 1, 2, \dots$$

Let

$$\xi_m(\omega) = \frac{M}{m} \sum_{l=0}^{m-1} \chi_{A_{lm}}(\omega)$$

where $\chi_{A_{lm}}$ denotes the indicator function of the A_{lm} set. Then we have for each ω

$$|\xi_m(\omega) - \xi(\omega)| \leq \frac{M}{m} \quad m = 1, 2, \dots$$

Moreover

$$|\mathbb{E}(\xi - \xi_m|\mathcal{M}_1^k) - \mathbb{E}(\xi - \xi_m)| \leq \frac{2M}{m}.$$

Concerning the ξ_m variables we have with probability 1:

$$\begin{aligned} & |\mathbb{E}(\xi_m|\mathcal{M}_1^k) - \mathbb{E}(\xi_m)| = \\ & = \left| \mathbb{E} \left(\frac{M}{m} \sum_{l=0}^{m-1} \chi_{A_{lm}}(\omega) | \mathcal{M}_1^k \right) - \mathbb{E} \left(\frac{M}{m} \sum_{l=0}^{m-1} \chi_{A_{lm}}(\omega) \right) \right| \leq \\ & \leq \frac{M}{m} \sum_{l=0}^{m-1} |\mathbb{P}(A_{lm} | \mathcal{M}_1^k) - \mathbb{P}(A_{lm})| \leq M\varphi(n) \end{aligned}$$

which proves our lemma.

LEMMA 3. Let $\{\xi_k\}$ be a φ -mixing sequence of identically distributed random variables. Suppose that for each $n=1, 2, \dots$ the first and second moment of $\Phi_n(x, \xi_1)$ exist, for each $x \in (a_0, b_0)$ where (a_0, b_0) is a finite or infinite interval. Let us denote $D^2 \Phi_n(x, \xi_1) = h_n(x)$ and suppose that $h_n(x) \leq h_n$ for $x \in (a_0, b_0)$. Denote by

$$U_k^{k+\alpha}(n, x) = \frac{1}{n} \sum_{j=k+1}^{k+\alpha} [\Phi_n(x, \xi_j) - E\Phi_n(x, \xi_j)]$$

where α is arbitrary integer $1 \leq \alpha \leq \alpha_0 \leq n$.

Suppose moreover that

$$\sum_{i=1}^{\infty} \varphi^{1/2}(i) < c.$$

Then we have

$$E(U_k^{k+\alpha}(n, x))^2 \leq \frac{(1+4c)\alpha_0 h_n}{n^2}$$

for each $x \in (a_0, b_0)$.

PROOF. Using Lemma 1

$$\begin{aligned} E(U_k^{k+\alpha}(n, x))^2 &\leq \frac{1}{n^2} \sum_{j=k+1}^{k+\alpha} h_n(x) + \\ &\frac{1}{n^2} 4 \sum_{k+1 \leq j < l \leq k+\alpha} \varphi(l-j)^{1/2} h_n(x) \leq \frac{h_n(x)\alpha}{n^2} \left(1 + 4 \sum_{j=1}^{\alpha} \varphi^{1/2}(j) \right) \leq \\ &\leq \frac{(1+4c)\alpha h_n(x)}{n^2} \leq \frac{(1+4c)\alpha_0 h_n}{n^2}. \end{aligned}$$

for each $x \in (a_0, b_0)$.

3. The main lemma

In this section the following probability will be estimated:

$$P(|f_n(x) - Ef_n(x)| > \varepsilon).$$

Clearly it is enough to consider $P(f_n(x) - Ef_n(x) > \varepsilon)$ and with the same method can be estimated $P(Ef_n(x) - f_n(x) > \varepsilon)$ too.

The basic idea of our method is the following: we divide the n summands $\frac{1}{n} [\Phi_n(x, \xi_j) - E\Phi_n(x, \xi_j)]$, $j=1, 2, \dots, n$ of $f_n(x) - Ef_n(x)$ into equal groups with length α . Then we consider the sum of each second groups and separately the sum of the remainder groups. We may suppose therefore that

$$n = 2\alpha\mu \quad \text{where} \quad \alpha = \alpha(n), \quad \mu = \mu(n).$$

If this supposition doesn't hold it results that our last two groups are shorter, than α or empty, but this fact doesn't alter our proof.

Let us introduce the following notations:

$$\begin{aligned} V(n, x, 1) &= U_1^\alpha(n, x), \\ R(n, x, 1) &= U_{\alpha+1}^{2\alpha}(n, x), \\ V(n, x, 2) &= U_{2\alpha+1}^{3\alpha}(n, x), \\ &\vdots \\ V(n, x, \mu) &= U_{2(\mu-1)\alpha+1}^{(2\mu-1)\alpha}(n, x), \\ R(n, x, \mu) &= U_{(2\mu-1)\alpha+1}^{2\mu\alpha}(n, x), \\ S(n, x, 1) &= \sum_{j=1}^{\mu} V(n, x, j), \\ S(n, x, 2) &= \sum_{j=1}^{\mu} R(n, x, j). \end{aligned}$$

Then

$$\begin{aligned} P(f_n(x) - E f_n(x) > \varepsilon) &= P(S(n, x, 1) + S(n, x, 2) > \varepsilon) \cong \\ &\cong P\left(S(n, x, 1) > \frac{\varepsilon}{2}\right) + P\left(S(n, x, 2) > \frac{\varepsilon}{2}\right). \end{aligned}$$

Thus we shall consider only the probability

$$P(S(n, x, 1) > \varepsilon).$$

LEMMA 4. *Suppose that the conditions of Lemma 3 hold, and suppose that*

$$(3.1) \quad |\Phi_n(x, y)| \leq ch_n \text{ for each pair } (x, y).$$

Then we have

$$(3.2) \quad E(e^{\lambda_n S(n, x, 1)}) \leq e^{\frac{\lambda_n^2 ch_n}{n}} (1 + c\varphi(\alpha))^\mu$$

whenever

$$(3.3) \quad 0 < \lambda_n < \frac{cn}{\alpha h_n},$$

where with the same letter c we denote here and in what follows different nonnegative constants.

PROOF.

$$\begin{aligned} E(e^{\lambda_n S(n, x, 1)}) &= E\left(e^{\lambda_n \sum_{j=1}^{\mu} V(n, x, j)}\right) = \\ &= E\left(e^{\lambda_n \sum_{j=1}^{\mu-1} V(n, x, j)} E(e^{\lambda_n V(n, x, \mu)} | V(n, x, 1), \dots, V(n, x, \mu-1))\right) = \\ &= E\left(e^{\lambda_n \sum_{j=1}^{\mu-1} V(n, x, j)} [E(e^{\lambda_n V(n, x, \mu)} | V(n, x, 1), \dots, V(n, x, \mu-1)) - E(e^{\lambda_n V(n, x, \mu)})]\right) + \\ &\quad + E\left(e^{\lambda_n \sum_{j=1}^{\mu-1} V(n, x, j)}\right) E(e^{\lambda_n V(n, x, \mu)}). \end{aligned}$$

By (3.1)

$$(3.4) \quad |V(n, x, \mu)| \leq \frac{\alpha ch_n}{n}.$$

Using Lemma 2, we have with probability 1:

$$|E(e^{\lambda_n V(n, x, \mu)} | V(n, x, 1), \dots, V(n, x, \mu - 1)) - E(e^{\lambda_n V(n, x, \mu)})| \leq 2e^{\frac{\lambda_n \alpha ch_n}{n}} \varphi(\alpha).$$

Thus

$$E(e^{\lambda_n S(n, x, 1)}) \leq E\left(e^{\lambda_n \sum_{j=1}^{\mu-1} V(n, x, j)}\right) (E(e^{\lambda_n V(n, x, \mu)}) + 2\varphi(\alpha) e^{\frac{\lambda_n \alpha ch_n}{n}}).$$

Using the inequality $e^u \leq 1 + u + u^2 \left(|u| < \frac{1}{2} \right)$ and Lemma 3 we have

$$\begin{aligned} E(e^{\lambda_n V(n, x, \mu)}) + 2\varphi(\alpha) e^{\frac{\lambda_n \alpha ch_n}{n}} &\leq E\left(1 + \lambda_n V(n, x, \mu) + \lambda_n^2 V^2(n, x, \mu)\right) + 2\varphi(\alpha) e^{\frac{\lambda_n \alpha ch_n}{n}} \leq \\ &\leq e^{\lambda_n^2 E V^2(n, x, \mu)} + 2\varphi(\alpha) e^{\frac{\lambda_n \alpha ch_n}{n}} \leq e^{\frac{\lambda_n^2 \alpha ch_n}{n^2}} + 2\varphi(\alpha) e^{\frac{\lambda_n \alpha ch_n}{n}} \end{aligned}$$

whenever

$$|\lambda_n V(n, x, \mu)| < \frac{1}{2}$$

that means whenever (3.3) holds. Consequently we have

$$E(e^{\lambda_n V(n, x, \mu)}) + 2e^{\frac{\lambda_n \alpha ch_n}{n}} \varphi(\alpha) \leq e^{\frac{\lambda_n^2 \alpha ch_n}{n^2}} \left(1 + 2e^{\frac{\lambda_n \alpha ch_n}{n} - \frac{\lambda_n^2 \alpha ch_n}{n^2}} \varphi(\alpha)\right).$$

By condition (3.3)

$$(3.5) \quad \frac{\lambda_n \alpha ch_n}{n} - \frac{\lambda_n^2 \alpha ch_n}{n^2} < c.$$

Thus from (3.5) in case of (3.3) we get

$$E(e^{\lambda_n S(n, x, 1)}) \leq E\left(e^{\lambda_n \sum_{j=1}^{\mu-1} V(n, x, j)}\right) e^{\frac{\lambda_n^2 \alpha ch_n}{n}} (1 + c\varphi(\alpha)).$$

Repeating this argument we get

$$E(e^{\lambda_n S(n, x, 1)}) \leq e^{\frac{\lambda_n^2 \alpha ch_n \mu}{n^2}} (1 + c\varphi(\alpha))^\mu$$

which proves our lemma.

LEMMA 5. *Suppose that all conditions of Lemma 4 hold. Suppose moreover, that*

$$(3.6) \quad \overline{\lim} (1 + c\varphi(\alpha))^\mu < c$$

Then we have

$$P(|f'_n(x) - Ef'_n(x)| > \varepsilon) \leq e^{-\frac{c\varepsilon}{\alpha h_n}}$$

for each $x \in (a_0, b_0)$.

PROOF. By condition (3.6)

$$E(e^{\lambda_n S(n, x, 1)}) \leq ce^{\frac{\lambda_n^2 ch_n}{n}}.$$

Consequently by the Markov inequality

$$P(S(n, x, 1) > \varepsilon) \leq c \exp\left(\frac{\lambda_n^2 ch_n}{n} - \lambda_n \varepsilon\right).$$

Let

$$\lambda_n = \frac{cn\varepsilon}{2\alpha h_n},$$

then using once more (3.6)

$$P(S(n, x, 1) > \varepsilon) \leq ce^{-\frac{cn}{\alpha h_n}}.$$

Clearly a similar estimation can be given for $S(n, x, 2)$. Thus

$$P(f_n - E(f_n) > \varepsilon) \leq e^{-\frac{cn}{\alpha h_n}}.$$

Similarly we get

$$P(E(f_n) - f_n > \varepsilon) \leq e^{-\frac{cn}{\alpha h_n}}$$

and our lemma is proved.

4. The main theorem

THEOREM 1. Let $\{\xi_k\}$ be a φ -mixing sequence of identically distributed random variables having common density function $f(x)$ for which

$$(4.1) \quad |f(x_2) - f(x_1)| \leq K|x_1 - x_2|$$

and

$$\int_{-\infty}^{\infty} |x|^\gamma f(x) dx < c_0 \quad \text{for some } \gamma > 0.$$

Suppose that

$$(4.2) \quad \sum_{i=1}^{\infty} \varphi^{1/2}(i) < c.$$

Further let $\Phi_n(x, y)$ $n=1, 2, \dots$ be a sequence of Borel-measurable functions. Suppose that there exist finite or infinite intervals $[a, b] \subset [c, d]$ and positive numbers Ω and τ such that

(4.3) for each n the interval $[c, d]$ can be divided into disjoint intervals $I_s^{(n)}$ ($s=1, 2, \dots$) for which

$$|I_s^{(n)}| > \frac{1}{n^\Omega} \quad \text{and} \quad \bigcup_{s=1}^n I_s^{(n)} = [c, d],$$

$$|\Phi_n(x_1, y) - \Phi_n(x_2, y)| \leq n^\tau |x_1 - x_2|$$

provided that x_1 and x_2 belong to the same interval $I_s^{(n)}$, and that

$$(4.4) \quad \int_a^b \Phi_n(x, y) f(y) dy \rightarrow f(x), \quad n \rightarrow \infty$$

uniformly in $[a+\delta, b-\delta]$ for any $\delta > 0$. Set $h_n(x) = D^2 \Phi_n(x, \xi_1)$ and let h_n be a sequence of real numbers for which $h_n(x) \leq h_n$, $x \in [a, b]$. Suppose further that

$$(4.5) \quad |\Phi_n(x, y)| < ch_n$$

and suppose that there exists a $\nu > 0$, and for each $\varepsilon > 0$, there is an $n_0(\varepsilon)$ such that:

$$(4.6) \quad |\Phi_n(x_n, y_n)| < \varepsilon$$

whenever

$$|x_n - y_n| > n^\nu$$

and

$$n > n_0(\varepsilon).$$

For any choice of the sequence $\alpha = \alpha(n)$, ($n = 2\alpha(n)\mu(n)$) for which

$$(4.7) \quad \overline{\lim} (1 + c\varphi(\alpha))^\mu < c$$

and

$$(4.8) \quad \alpha h_n \leq \frac{n}{\omega(n) \log n} \quad \text{for some } \omega(n) \nearrow \infty$$

hold, we have

$$P\left(\sup_{x \in [a+\delta, b-\delta]} |f_n(x) - f(x)| > \varepsilon\right) \leq e^{-\frac{cn}{\alpha h_n}}.$$

Our proof will be similar to that of [2].

PROOF. Let for each positive integer n

$$-n^T = z_0^{(n)} < z_1^{(n)} < \dots < z_{l(n)}^{(n)} = n^T$$

(the number T will be determined later on) be a partition of the interval $[-n^T, n^T]$ having the following properties:

a)

$$\frac{c_1}{n^{\Omega+2\tau}} \leq z_i^{(n)} - z_{i-1}^{(n)} \leq \frac{c_2}{n^{\Omega+2\tau}}$$

$$0 \leq c_1 < c_2 < +\infty, \quad i = 1, 2, \dots, l(n);$$

b) those endpoints of the intervals $I_s^{(n)}$ which belong to $[-n^T, n^T]$ are elements of the sequence $z_0^{(n)}, \dots, z_{l(n)}^{(n)}$.

By a) and (4.3)

$$|f_n(x) - f_n(y)| \leq c \frac{1}{n^{\Omega+\tau}} \quad \text{if } x, y \in [z_{i-1}^{(n)}, z_i^{(n)})$$

and

$$|E f_n(x) - E f_n(y)| \leq c \frac{1}{n^{\Omega+\tau}} \quad \text{if } x, y \in [z_{i-1}^{(n)}, z_i^{(n)}].$$

Therefore we have

$$\begin{aligned} P\left(\sup_{-n^T \leq x < n^T} |f_n(x) - E f_n(x)| > \varepsilon\right) &\leq \sum_{i=1}^{l(n)} P\left(\sup_{z_{i-1}^{(n)} \leq x < z_i^{(n)}} |f_n(x) - E f_n(x)| > \varepsilon\right) \leq \\ &\leq l(n) \max_i \left\{ P\left(\sup_{z_{i-1}^{(n)} \leq x < z_i^{(n)}} |f_n(x) - E f_n(x)| > \varepsilon\right) \right\} \leq \\ &\leq l(n) \max_i \left\{ P(|f_n(z_{i-1}^{(n)}) - E f_n(z_{i-1}^{(n)})| > \varepsilon) \right\} \leq l(n) e^{-\frac{c n}{a h_n}}, \end{aligned}$$

by Lemma 5, if n is large enough. On the other hand

$$\begin{aligned} P\left(\sup_{\substack{|x| \leq n^T \\ x \in [a+\delta, b-\delta]}} |f_n(x) - E f_n(x)| > \varepsilon\right) &\leq P\left(\sup_{\substack{|x| \leq n^T \\ x \in [a+\delta, b-\delta]}} |f_n(x)| > \frac{\varepsilon}{2}\right) + \\ &+ P\left(\sup_{\substack{|x| \leq n^T \\ x \in [a+\delta, b-\delta]}} |E f_n(x)| > \frac{\varepsilon}{2}\right) = P\left(\sup_{\substack{|x| \leq n^T \\ x \in [a+\delta, b-\delta]}} |f_n(x)| > \frac{\varepsilon}{2}\right) \end{aligned}$$

whenever n is large enough because of (4.1) and (4.4).

Now we have

$$\begin{aligned} (4.9) \quad P\left(\sup_{|x| \leq n^T} |f_n(x)| > \frac{\varepsilon}{2}\right) &\leq P\left(\sup_{|x| \leq n^T} \frac{1}{n} \sum_{k; |\xi_k - x| \leq \delta} \Phi_n(x, \xi_k) > \frac{\varepsilon}{4}\right) + \\ &+ P\left(\sup_{|x| \leq n^T} \frac{1}{n} \sum_{\substack{k; |\xi_k - x| > \delta \\ \xi_k \in S\left(x, \frac{n^T}{2}\right)}} |\Phi_n(x, \xi_k)| > \frac{\varepsilon}{8}\right) + \\ &+ P\left(\sup_{|x| \leq n^T} \frac{1}{n} \sum_{\substack{k; |\xi_k - x| > \delta \\ \xi_k \notin S\left(x, \frac{n^T}{2}\right)}} |\Phi_n(x, \xi_k)| > \frac{\varepsilon}{8}\right) \leq \\ &\leq 2P\left(\frac{1}{n} \sum_{\substack{k; |\xi_k| \leq \frac{n^T}{2}}} 1 > \frac{\varepsilon}{8}\right) + \\ &+ P\left(\sup_{|x| \leq n^T} \frac{1}{n} \sum_{\substack{k; |\xi_k - x| > \delta \\ \xi_k \in S\left(x, \frac{n^T}{2}\right)}} |\Phi_n(x, \xi_k)| > \frac{\varepsilon}{8}\right) \end{aligned}$$

where $S\left(x, \frac{n^T}{2}\right)$ denotes the interval

$$\left[x - \frac{n^T}{2}, x + \frac{n^T}{2}\right].$$

Let us denote $\chi_n(u)$ the following indicator function:

$$\chi_n(u) = \begin{cases} 1 & \text{if } |u| \geq \frac{n^T}{2}, \\ 0 & \text{if } |u| < \frac{n^T}{2}. \end{cases}$$

Then we have to estimate the following probability

$$(4.10) \quad P\left(\sum_{k=1}^n \chi_n(\xi_k) > \frac{c^* n \varepsilon}{h_n}\right)$$

Observe that $\{\chi_n(\xi_k)\}_{k=1}^n$ is also a φ -mixing sequence with the same φ as the sequence $\{\xi_k\}$. Therefore (4.10) will be estimated by the method of Lemma 5.

$$\begin{aligned} P\left(\sum_{k=1}^n \chi_n(\xi_k) > \frac{c^* n \varepsilon}{h_n}\right) &\leq P\left(\sum_{k=1}^n (\chi_n(\xi_k) - E\chi_n(\xi_k)) > \frac{c^* n \varepsilon}{2h_n}\right) + \\ &+ P\left(\sum_{k=1}^n E\chi_n(\xi_k) > \frac{c^* n \varepsilon}{2h_n}\right) \\ E(\chi_n^2(\xi_k)) &= E\chi_n(\xi_k) = P\left(|\xi_k| > \frac{n^T}{2}\right) = \\ &= P\left(|\xi_k|^\gamma > \frac{n^{T\gamma}}{2^\gamma}\right) \leq \frac{E(|\xi_k|^\gamma) 2^\gamma}{n^{T\gamma}} < \frac{c}{n^{T\gamma}}, \end{aligned}$$

where $c=c(\gamma)$ is a constant depending only on γ . Let us choose T to be so large that

$T\gamma - 1 > 0$, moreover by (4.8) $\frac{c^* \varepsilon n}{2h_n}$ does not go to 0, thus

$$P\left(\sum_{k=1}^n E\chi_n(\xi_k) > \frac{c^* n \varepsilon}{2h_n}\right) = 0$$

if n is large enough.

Further, if $0 < \lambda_n < \frac{c}{\alpha}$,

$$\begin{aligned} P\left(\sum_{k=1}^n (\chi_n(\xi_k) - E\chi_n(\xi_k)) > \frac{c^* n \varepsilon}{2h_n}\right) &\leq \frac{E\left(e^{\lambda_n \sum_{k=1}^n (\chi_n(\xi_k) - E\chi_n(\xi_k))}\right)}{e^{\frac{\lambda_n c^* n \varepsilon}{2h_n}}} \leq \\ &\leq c \left(1 + \lambda_n^2 \frac{\alpha c}{n^{T\gamma}}\right)^\mu e^{-\frac{\lambda_n c^* n \varepsilon}{2h_n}} \leq \\ &\leq \exp\left(\frac{\lambda_n^2 \alpha c \mu}{n^{T\gamma}} - \frac{\lambda_n c^* n \varepsilon}{2h_n}\right) = \exp\left(\frac{\lambda_n n}{h_n} \left(\frac{\lambda_n c}{n^{T\gamma}} - c_1 \varepsilon\right)\right) = \exp\left(-\frac{cn}{\alpha h_n}\right) \end{aligned}$$

with the choice

$$\lambda_n = \frac{\tilde{c}}{\alpha}.$$

To finish our proof only remained the estimation of the probability

$$P \left(\sup_{|x| > n^T} \frac{1}{n} \sum_{\substack{k; |\xi_k - x| > \delta \\ \xi_k \in S \left(x, \frac{n^T}{2} \right)}} |\Phi_n(x, \xi_k)| > \frac{\varepsilon}{8} \right).$$

This probability is 0 if n is large enough and T is chosen so that

$$T > \max \left(v, \frac{1}{\gamma} \right), \text{ by (4.6),}$$

which finishes the proof of the theorem.

5. A simple application

One of the most important but not trivial examples of φ -mixing sequence is the Markov process. It is well-known [5], that if

- (i) $\{\zeta_n\}$ is a Markov process with infinite state space (in our case the real line),
- (ii) $\{\zeta_n\}$ satisfies Doeblin's condition,
- (iii) there is only a single ergodic set, and this set contains no cyclically moving subsets,

then there exist positive constants a and $\varrho < 1$ such that

$$\varphi(n) = a\varrho^n.$$

It is easy to see that in this case the mixing conditions of our theorem hold if we choose $\alpha(n) = c_0 \log n$ (where c_0 is a constant depending only on ϱ), then (4.7) holds, thus for a Markov process we get almost the same convergence rate as for independent sequences.

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**ON THE RATE OF CONVERGENCE OF KESTEN'S
"ACCELERATED STOCHASTIC APPROXIMATION"**

by

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Introduction

Let $M(x)$ ($-\infty < x < +\infty$) be an unknown monotonically increasing function with $M(\theta) = 0$ and $M(x) \neq 0$ if $x \neq \theta$. Suppose that we can measure the value of $M(x)$ only with some random error Y_x i.e. the value $M(x) + Y_x$ for any x can be obtained by an experiment. Our aim is to find the root θ .

ROBBINS and MONRO ([1]) constructed the following sequence: let X_1 be an arbitrary real number and define the sequence $\{X_n\}$ by the recursion

$$(1) \quad X_{n+1} = X_n - \frac{1}{an} Z_n \quad (n = 1, 2, \dots; a > 0)$$

where $Z_n = M(X_n) + Y_{X_n}$. The process $\{X_n\}$ will be called Robbins—Monro Process (RMP).

BLUM ([2]) under some simple conditions proved that $P(X_n \rightarrow \theta) = 1$.

To characterize the rate of convergence of $\{X_n\}$ is clearly an interesting question. A result in these topics is the following: suppose that $M(x)$ is differentiable at the root θ and $0 < M'(\theta) = \alpha < \infty$, then (under some further conditions see e.g. [3]) we have

$$E(X_n - \theta)^2 = \begin{cases} O\left(\frac{1}{n}\right) & \text{if } \alpha > \frac{a}{2}, \\ O\left(\frac{\log}{n}\right) & \text{if } \alpha = \frac{a}{2}, \\ O(n^{-2\alpha}) & \text{if } \alpha < \frac{a}{2}. \end{cases}$$

Another characterization of the rate of convergence is the central limit theorem (see e.g. [10] or [11]) stating that $[a(2\alpha - a)]^{1/2}(X_n - \theta)$ is asymptotically normally distributed with parameters 0, 1 if $\alpha > \frac{a}{2}$ and some further conditions hold.

A law of the iterated logarithm i.e. a characterization of the rate of strong convergence of $\{X_n\}$ was proved in [4]. Both of these results (see also [5]) show that the rate of convergence is getting worse and worse if α is getting less and less. To avoid this difficulty VENTER ([6]) proposed a modification of the RMP. He and FABIAN ([7]) proved that this modified process is tending to θ as $1/\sqrt{n}$ provided that $0 < \alpha < \infty$. (Here we do not want to specify the exact meaning of this statement.) The rate of convergence in the case $\alpha = 0$ seems to be completely unknown.

Another modification of the RMP was proposed by KESTEN ([8]). His idea is the following: if the results Z_n of our measurements are positive (or negative) in a long run then we can conjecture that our estimations are far away from the root θ (near to the root a frequent change of the signs of Z_n is expected) and therefore we have to modify the values of X_n in a stronger way.

KESTEN called his method "an accelerated stochastic approximation" but it is not really proved that his method is really accelerated. This method is also mentioned in the most important monographies without any further explanation see e.g. [12] and [13].

In this paper we intend to investigate the KESTEN's process. In order to compare more precisely the rate of convergence of RMP and that of KESTEN's process, in paragraph 1 a law of iterated logarithm will be proved for RMP (Prof. G. KERSTING was kind enough to inform me that he proved similar theorems). A law of iterated logarithm and a central limit theorem for KESTEN's process are given in paragraph 2.

1. A law of the iterated logarithm for RMP

In this paragraph a law of the iterated logarithm will be proved for the original RMP in the case when the errors are i.i.d.r.v.'s. Namely we prove our

THEOREM 1. *Let Y_1, Y_2, \dots be a sequence of i.i.d.r.v.'s with $EY_i=0, EY_i^2=1$ ($i=1, 2, \dots$), further let $M(x)$ ($-\infty < x < +\infty$) be a measurable function for which*

- a) *there exist positive constants c and d such that $|M(x)| \leq c + d|x|$,*
- b) *$M(x) < 0$ if $x < 0$ and $M(x) > 0$ if $x > 0$ (i.e. we assume $\theta=0$),*
- c) *$\inf_{\delta_1 \leq |x| \leq \delta_2} |M(x)| > 0$ for every pair of positive numbers δ_1, δ_2 ,*
- d) *$M(x)$ is twice differentiable at 0,*
- e) *$M'(0) = \alpha \cong a/2$.*

Then

$$P \left(\overline{\lim}_{n \rightarrow \infty} \left(\frac{a(2\alpha - a)n}{2 \log \log n} \right)^{1/2} |X_n| = 1 \right) = 1 \quad \text{if} \quad M'(0) = \alpha > \frac{a}{2}$$

and

$$P \left(\overline{\lim}_{n \rightarrow \infty} \left(\frac{n}{\log n \log \log n} \right)^{1/2} |X_n| = 1 \right) = 1 \quad \text{if} \quad M'(0) = \alpha = \frac{a}{2}.$$

Before the proof¹ of this theorem we give three lemmas. The first and the third will be used in the proof of this theorem and later on, the second one will be used in the proof of the third.

¹ For the sake of simplicity we give the proof only in case $a=1$ but the general case can be treated quite similarly.

LEMMA A ([9]). Let Y_1, Y_2, \dots be a sequence of i.i.d.r.v.'s with $EY_i=0, EY_i^2=1$ ($i=1, 2, \dots$). Then

$$P\left(\overline{\lim}_{n \rightarrow \infty} \frac{n^\epsilon}{\sqrt{n}} \left(\frac{1-2\epsilon}{2 \log \log n}\right)^{1/2} \sum_{j=1}^n \frac{1}{j^\epsilon} Y_j = 1\right) = 1 \quad \text{if} \quad -\infty < \epsilon < \frac{1}{2},$$

$$P\left(\overline{\lim}_{n \rightarrow \infty} \frac{1}{(2 \log n \log \log \log n)^{1/2}} \sum_{j=1}^n \frac{1}{\sqrt{j}} Y_j = 1\right) = 1 \quad \text{if} \quad \epsilon = \frac{1}{2},$$

and

$$P\left(\overline{\lim}_{n \rightarrow \infty} \frac{n^\epsilon}{\sqrt{n}} \left(\frac{2\epsilon-1}{2 \log \log n}\right)^{1/2} \sum_{j=n}^{\infty} \frac{1}{j^\epsilon} Y_j = 1\right) = 1 \quad \text{if} \quad \frac{1}{2} < \epsilon < +\infty.$$

LEMMA B ([10]). Let $\{b_n^2\}$ be a sequence of positive numbers for which

$$(2) \quad b_{n+1}^2 \leq \left(1 - \frac{\rho}{n}\right) b_n^2 + \frac{A}{n^2} \quad (n = 1, 2, \dots)$$

where $0 < \rho < 1$ and $A > 0$. Then $b_n^2 \leq Bn^{-\rho}$ where B is a constant depending on A, ρ and b_1 .

LEMMA¹ 1. Let $X_n = X_n(\omega)$ ($\omega \in \Omega$) be a RMP obeying the conditions of Theorem 1. Then for any $\epsilon > 0, \delta > 0$ there exists a measurable set $F \subset \Omega$ such that

$$P(F) > 1 - \delta$$

and

$$(3) \quad \int_F X_n^2 dP \leq \frac{1}{n^{1-\epsilon}}$$

if n is big enough.

PROOF. By BLUM's theorem ([2]) our conditions imply that $X_n \rightarrow 0$ with probability 1. For any $\epsilon' > 0$ one can choose an $\eta = \eta(\epsilon') > 0$ such that $M(x) = \alpha x + \epsilon(x)x$ where $|\epsilon(x)| \leq \epsilon'$ whenever $|x| < \eta$ (where $M'(0) = \alpha$). Define $F = \bigcap_{n \geq n_0} \{\omega : |X_n| < \eta\}$. If $n_0 = n_0(\delta)$ is large enough then $P(F) \geq 1 - \delta$. Let $F_0 = \{\omega : |X_{n_0}| < \eta\}$ and $F_k = \bigcap_{n=n_0}^k \{\omega : |X_n| < \eta\}$ ($k \geq n_0$). (1) implies

$$\int_{F_n} X_{n+1}^2 \leq \int_{F_n} X_n^2 - \frac{2}{n} (\alpha - \epsilon') \int_{F_n} X_n^2 + \frac{2}{n^2} \int_{F_n} X_n^2 + \frac{1}{n^2}$$

for $n \geq n_0$. Taking into account that $\int_{F_{n+1}} X_{n+1}^2 \leq \int_{F_n} X_{n+1}^2$ and $2(\alpha - \epsilon') - \frac{2}{n} \geq 1 - \epsilon$ if n is large enough, we get

$$b_{n+1}^2 \leq b_n^2 \left(1 - \frac{1-\epsilon}{n}\right) + \frac{1}{n^2}.$$

Hence by lemma B we have $b_n^2 = O\left(\frac{1}{n^{1-\epsilon}}\right)$ what implies (3).

¹ This lemma and its proof is closely related to Lemma 3 of [5].

PROOF OF THEOREM 1. By the iteration of (1) one can obtain

$$X_{n+1} = \prod_{k=1}^n \left(1 - \frac{\alpha}{k}\right) X_1 - \sum_{k=1}^n \frac{1}{k} (U(X_k) + Y_k) \prod_{j=k+1}^n \left(1 - \frac{\alpha}{j}\right)$$

where $U(x) = M(x) - \alpha x$, and applying the relations

$$1 - \frac{\alpha}{i} = e^{-\frac{\alpha}{i} + o\left(\frac{1}{i^2}\right)} \sum_{i=n}^m \frac{1}{i} = \log \frac{m}{n} + O\left(\frac{1}{n}\right),$$

$$\sum_{i=n}^{\infty} \frac{1}{i^2} = O\left(\frac{1}{n}\right)$$

we get

$$(4) \quad n^\alpha X_{n+1} = O(1) X_1 - \sum_{k=1}^n \frac{1}{k^{1-\alpha}} \left(1 + O\left(\frac{1}{k}\right)\right) (U(X_k) + Y_k).$$

By Lemma A we have

$$\overline{\lim}_{n \rightarrow \infty} \frac{\sqrt{n}}{n^\alpha} \left(\frac{2\alpha - 1}{2 \log \log n}\right)^{1/2} \sum_{k=1}^n \frac{1}{k^{1-\alpha}} \left(1 + O\left(\frac{1}{k}\right)\right) Y_k = 1$$

with probability 1 if $\alpha > 1/2$ and

$$(5) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{(2 \log n \log \log n)^{1/2}} \sum_{k=1}^n \frac{1}{k^{1-\alpha}} \left(1 + O\left(\frac{1}{k}\right)\right) Y_k = 1$$

with probability 1 if $\alpha = 1/2$.

Lemma 1 easily implies:

$$(6) \quad \begin{aligned} 0 &\leq \overline{\lim}_{n \rightarrow \infty} \frac{\sqrt{n}}{n^\alpha} \sum_{k=1}^n \frac{1}{k^{1-\alpha}} \left(1 + O\left(\frac{1}{k}\right)\right) |U(X_k)| \leq \\ &\leq \overline{\lim}_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sqrt{k}}{k^\alpha} \frac{1}{k^{1-\alpha}} \left(1 + O\left(\frac{1}{k}\right)\right) |U(X_k)| = 0 \end{aligned}$$

with probability 1. (4), (5) and (6) together prove our Theorem 1.

Remark 1. Our Theorem 1 in some sense characterizes the rate of convergence of X_n to 0 in the case when $M'(0) = \alpha \geq \frac{a}{2}$. The case $0 < \alpha < \frac{a}{2}$ was investigated in [5]. The following result was proved there²:

THEOREM A. *Under the conditions of Theorem 1 there exists a random variable Z such that $P(n^\alpha X_n \rightarrow Z) = 1$.*

² In [5] only the case $a = 1$ was treated but the method can be applied without any change in the general case.

Roughly speaking the statements of Theorems 1 and A can be formulated as follows: under the conditions of Theorem 1 X_n tends to 0 like

$$\left(\frac{2 \log \log n}{a(2\alpha - a)n}\right)^{1/2} \quad \text{if } \alpha > \frac{a}{2}.$$

$$\left(\frac{2 \log n \log \log \log n}{n}\right)^{1/2} \quad \text{if } \alpha = \frac{a}{2}$$

$$O\left(\frac{1}{n^\alpha}\right) \quad \text{if } \alpha < \frac{a}{2}.$$

2. On the Kesten's process

At first we give the precise definition of the Kesten's process in the case when the errors are i.i.d.r.v.'s.

Let $M(x)$ be a measurable function and let Y_1, Y_2, \dots be a sequence of i.i.d.r.v.'s with $EY_i = 0, EY_i^2 = 1$ ($i = 1, 2, \dots$). Define the sequences $\{K_n\}$ and $\{\mu_n\}$ as follows: let K_1 be an arbitrary real number and let $\mu_1 = 1$, and put

$$K_{n+1} = K_n - \frac{1}{\mu_n}(M(K_n) + Y_n)$$

and

$$\mu_{n+1} = \begin{cases} \mu_n & \text{if } (M(K_n) + Y_n)(M(K_{n-1}) + Y_{n-1}) \geq 0, \\ \mu_n + 1 & \text{otherwise.} \end{cases}$$

The sequence $\{K_n\}$ will be called Kesten's process (K.P.).

Now we can formulate our

THEOREM 2. Let $M(x)$ be a measurable function satisfying conditions a)—d) of Theorem 1 and suppose $M'(0) = \alpha > 0$. Further let Y_1, Y_2, \dots be a sequence of discrete i.i.d.r.v.'s with

$$EY_i = 0, \quad EY_i^2 = 1 \quad (i = 1, 2, \dots), \quad P(Y_1 = x_j) = p_j$$

$$\left(j = 1, 2, \dots, r; \sum_{j=1}^r p_j = 1 \right) \quad \text{and suppose } x_j \neq 0 \quad (j = 1, 2, \dots, r).$$

Then

$$\left. \begin{aligned} &P\left(\overline{\lim}_{n \rightarrow \infty} Q \sqrt{\frac{n}{2 \log \log n}} |K_n| = 1\right) = 1 \\ &\lim_{n \rightarrow \infty} P(Q \sqrt{n} K_n < y) = \Phi(y) \end{aligned} \right\} \quad \text{if } \alpha > pq$$

$$\left. \begin{aligned} &P\left(\overline{\lim}_{n \rightarrow \infty} \left(\frac{n}{2 \log n \log \log \log n}\right)^{1/2} |K_n| = 1\right) = 1 \\ &\lim_{n \rightarrow \infty} P\left(\left(\frac{n}{\log n}\right)^{1/2} K_n < y\right) = \Phi(y) \end{aligned} \right\} \quad \text{if } \alpha = pq$$

and there exists a r.v. Z such that

$$P(n^\beta K_n \rightarrow Z) = 1$$

if $\alpha < pq$ where $p = P(Y_j > 0)$, $q = P(Y_j < 0)$,

$$Q^2 = 2pq(2\alpha - 2pq), \quad \beta = \frac{\alpha}{2pq}.$$

Remark 2. The condition saying Y_i is discrete, not equal to 0 is certainly superfluous but to prove the theorem without this condition seems to be much harder.

Remark 3. It is clear enough that $\mu_n \approx 2pqn$ (see Lemma 5) what suggests that the behaviour of K_n is the same as that of X_n , in case $a = 2pq$. In fact the statement of Theorem 2 says that this conjecture is correct.

Remark 4. Theorem 2 also says that the rate of convergence of the VENTER'S Process is better than that of the KESTEN'S if $\alpha > 0$. It would be interesting to investigate the rate of convergence of the KESTEN'S process when $\alpha = 0$.

Remark 5. It is very likely that the KESTEN'S Process converges better than the VENTER'S in the time when the place of measurement is far from the root (i.e. when X_n resp. K_n is far from θ). Hence it is natural to propose the using of the KESTEN'S method in the first few steps and to turn to VENTER'S later on. Clearly one has to decide the time of turning. The author intends to deal with this question elsewhere.

Before the proof some notations will be introduced:

1. Let $S_0 = 0$, and let S_k ($k = 1, 2, \dots$) be the largest integer for which $\mu_{s_k} = k$.
2. $v_k = S_k - S_{k-1}$ ($k = 1, 2, \dots$).
3. $\min_{1 \leq i \leq r} |x_i| = \varrho$, $\max_{1 \leq i \leq r} |x_i| = K$.
4. Let δ_0 be a positive number for which $|M(x)| \leq \varrho/2$ if $|x| \leq \delta_0$.
5. For any $\varepsilon > 0$ and $\delta > 0$ define the integer $\tilde{n} = \tilde{n}(\varepsilon, \delta)$ and the measurable set $F = F(\varepsilon, \delta) \subset \Omega$ (the underlying probability space) such that $P(F) \geq 1 - \varepsilon$ and $|K_n(\omega)| \leq \delta$ if $n \geq \tilde{n}$ and $\omega \in F$. Since by Kesten's theorem ([8]) $P(K_n \rightarrow 0) = 1$, for any $\varepsilon > 0$ and $\delta > 0$ there exist such F and \tilde{n} . Especially let $F_0 = F_0(\varepsilon) = F(\varepsilon, \delta_0)$ and $n_0 = n_0(\varepsilon) = \tilde{n}(\varepsilon, \delta_0)$.

6. Let \varkappa_0 be the smallest integer for which $S_{\varkappa_0} \geq n_0$.

7. Define the sequence $\{\tilde{v}_k\}$ as follows: let \tilde{v}_1 be the largest integer for which:

$$\text{sign } Y_{s_{\varkappa_0}} = \text{sign } Y_{s_{\varkappa_0+1}} = \dots = \text{sign } Y_{s_{\varkappa_0+\tilde{v}_1-1}},$$

similarly \tilde{v}_2 is the largest integer for which

$$\text{sign } Y_{s_{\varkappa_0+\tilde{v}_1}} = \text{sign } Y_{s_{\varkappa_0+\tilde{v}_1+1}} = \dots = \text{sign } Y_{s_{\varkappa_0+\tilde{v}_1+\tilde{v}_2-1}}$$

and in general \tilde{v}_k is the largest integer for which

$$\text{sign } Y_{s_{\varkappa_0+\tilde{v}_1+\dots+\tilde{v}_{k-1}}} = \dots = \text{sign } Y_{s_{\varkappa_0+\tilde{v}_1+\dots+\tilde{v}_k-1}}.$$

8.

$$\tilde{S}_k = \begin{cases} s_k & \text{if } k \leq \varkappa_3 \\ s_{\varkappa_0} + \tilde{v}_1 + \tilde{v}_2 + \dots + \tilde{v}_{k-\varkappa_0} & \text{if } k > \varkappa_0. \end{cases}$$

9. $\tilde{\mu}_k = l$ if $\tilde{S}_{l-1} < k \leq \tilde{S}_l$.

10. Define the process $\{\tilde{K}_n\}$ by $\tilde{K}_1 = K_1$ and

$$\tilde{K}_{n+1} = \tilde{K}_n - \frac{1}{\tilde{\mu}_n} (M(\tilde{K}_n) + Y_n).$$

Since $\text{sign}(M(K_k) + Y_k) = \text{sign } Y_k$ if $|K_k| \leq \delta_0$ we have:

LEMMA 2. $K_n(\omega) = \tilde{K}_n(\omega)$ if $n \leq \tilde{S}_{x_0} = S_{x_0}$ or $\omega \in F_0$.

In order to simplify our notations from now on we assume $Y_{S_{x_0}+1} > 0$; the case $Y_{S_{x_0}+1} < 0$ can be treated quite similarly. Now we formulate our

LEMMA 3. $\tilde{v}_1, \tilde{v}_3, \tilde{v}_5, \dots$ are i.i.d.r.v.'s with $P(\tilde{v}_1 = j) = qp^{j-1}$ ($j = 1, 2, \dots$) and $\tilde{v}_2, \tilde{v}_4, \dots$ are i.i.d.r.v.'s with $P(\tilde{v}_2 = j) = pq^{j-1}$ ($j = 1, 2, \dots$). Hence $E\tilde{v}_1 = 1/q$, $D^2\tilde{v}_1 = p/q^2$, $E\tilde{v}_2 = 1/p$, $D^2\tilde{v}_2 = q/p^2$.

PROOF is trivial.

Our next lemma follows from the well-known forms of the law of iterated logarithm.

LEMMA 4. We have

$$P\left(\overline{\lim}_{n \rightarrow \infty} \frac{\tilde{v}_1 + \tilde{v}_2 + \dots + \tilde{v}_n - \frac{n}{2pq}}{(n \log \log n)^{1/2}} = \frac{p^3 + q^3}{(pq)^2}\right) = 1.$$

This lemma easily implies

LEMMA 5.

$$P\left(\overline{\lim}_{n \rightarrow \infty} \left| \frac{\tilde{\mu}_n - 2pqn}{\sqrt{n \log \log n}} \right| = 2\sqrt{2} \frac{p^3 + q^3}{\sqrt{pq}}\right) = 1.$$

PROOF OF THEOREM 2. By Lemma 5 we have

$$\tilde{K}_{n+1} = \tilde{K}_n - \frac{1}{2pqn} (1 + V_n)(M(\tilde{K}_n) + y_n)$$

where $\{V_n\}$ is a sequence of r.v.'s for which

$$\overline{\lim}_{n \rightarrow \infty} \sqrt{\frac{n}{\log \log n}} |V_n| < \infty$$

with probability 1. Now our theorem 2 follows from the following simple remark: the central limit theorem and the law of iterated logarithm remain true if we modify the definition of the Robbins—Monro process as follows: let \tilde{X}_1 be a bounded r.v. and let

$$(1^*) \quad \tilde{X}_{n+1} = \tilde{X}_n - \frac{1}{an} (1 + o_n(1))(M(\tilde{X}_n) + Y_n)$$

where $o_n(1)$ is a sequence of r.v.'s independent of Y_n ($n = 1, 2, \dots$) and tending to 0 as $n \rightarrow \infty$. The proof of this remark is so easy, copying the proof of Theorem 1, that we can omit it.

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BOOK REVIEW

N. BOURBAKI: *Éléments de mathématique, XXXVII. Groupes et algèbres de Lie.*

CHAPITRE II: ALGÈBRES DE LIE LIBRES. CHAPITRE III: GROUPES DE LIE. —
ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES NO. 1349. HERMANN, PARIS, 1972.

Après les chapitres 1, 4, 5 et 6, ont paru enfin également les chapitres 2 et 3 des «Groupes et algèbres de Lie» de N. Bourbaki. Comme il est tout à fait superflu de rappeler tant l'importance des groupes et algèbres de Lie que le niveau des traités de Bourbaki, il est aussi inutile de dire à quel point cet ouvrage a été attendu.

Pour présenter la matière de ces deux chapitres, on s'est servi de tout l'arsenal des fascicules précédents des *Éléments de mathématique*. Ainsi, des connaissances considérables sont nécessaires pour la compréhension — ce qui n'est pas tellement surprenant, d'ailleurs.

Quant au contenu de l'ouvrage, en voici les titres des paragraphes. Les précisions mises entre parenthèses sont destinées à donner une idée plus proche de la matière traitée.

Chapitre II: § 1. Bigèbre enveloppante d'une algèbre de Lie (compléments de la théorie des bigèbres (algèbres de Hopf) avec la spécialisation indiquée.) — § 2. Algèbres de Lie libres (leur construction, polynômes de Lie, graduations, dérivations, bases de Hall). — § 3. Algèbre enveloppante de l'algèbre de Lie libre. — § 4. Filtrations centrales (filtrations et fonctions d'ordre). — § 5. Algèbres de Magnus (groupes de Magnus et groupes libres). — § 6. La série de Hausdorff (groupe de Hausdorff, séries formelles de Lie, la série de Hausdorff). — § 7. Convergence de la série de Hausdorff (cas réel ou complexe). — § 8. Convergence de la série de Hausdorff (cas ultramétrique). — Appendice: Fonction de Möbius.

Chapitre III: § 1. Groupes de Lie (définition, propriétés élémentaires, relation avec des variétés analytiques). § 2. Groupe des vecteurs tangents à un groupe de Lie. — § 3. Passage d'un groupe de Lie à son algèbre de Lie (algèbre des distributions ponctuelles sur un groupe de Lie, algèbre de Lie d'un groupe de Lie, représentations, formules de Maurer-Cartan, formes différentielles). — § 4. Passage des algèbres de Lie aux groupes de Lie (passage, applications exponentielles, passage des lois d'opérations infinitésimales aux lois d'opérations). — § 5. Calculs formels dans les groupes de Lie. — § 6. Groupes de Lie réels ou complexes (avec la complexification d'un groupe de Lie réel de dimension finie). — § 7. Groupes de Lie sur un corps ultramétrique. — § 8. Groupes de Lie sur R ou Q_p (les §§ 6—8 contiennent une spécialisation aux cas indiqués). — § 9. Commutateurs, centralisateurs, normalisateurs dans un groupe de Lie (définitions, groupes de Lie nilpotents et résolubles, radical et semi-simplicité). — § 10. Le groupe des automorphismes d'un groupe de Lie. — Appendice: Opérations sur les représentations linéaires.

Continuant les traditions de Bourbaki, le présent ouvrage contient une Note historique éclairante et une riche collection d'exercices, où, parmi les exemples et les applications des résultats obtenus dans le texte, on trouve des théorèmes bien connus ainsi que des résultats inédits.

L. Márki

S. FENYŐ—T. FREY: *Modern mathematical methods in technology.*

Volume 1. (North-Holland Series in applied mathematics and mechanics. Volume 9.) NORTH-HOLLAND PUBLISHING COMPANY, AMSTERDAM—LONDON, 1969. XII + 407 p. 82 figures, Dfl. 70.00.

This book is a somewhat modified translation (by W. U. Sirk) of the work: "Moderne mathematische Methoden in der Technik" (Birkhäuser-Verlag, Basel—Stuttgart, 1967) of the authors. Its objective is to acquaint the reader with mathematical disciplines important for recent applicati-

ons in natural science and engineering, which are not yet taught in universities. — Chapter I (Extension of the classical concept of an integral; pp. 1—71) is focused, after some preliminary considerations on point sets and metric spaces, on Lebesgue and Stieltjes integrals. — Chapter II (the operational calculus; pp. 72—148) presents the relevant foundation of J. Mikusiński, which is shown to be more powerful than the operational calculus of Heaviside or even the Laplace transform technique. The treatment begins with certain algebraical concepts and goes further by yielding the operational calculus of number sequences and, then, that of functions. Solution of differential and Volterra integral equations, applications to certain problems in electrical engineering and the theory of beams are also dealt with by the presented methods. — Chapter 3 (Fundamentals of distribution theory; pp. 149—282) is devoted to the theory of generalized functions created by L. Schwarz (Fourier analysis and function regularization inclusive). As applications, ordinary linear differential equations (Green's function, Cauchy's problem etc.) are treated by the help of the theory. — Chapter 4 (Analysis of nonlinear differential equations. The theory of nonlinear vibrations; pp. 283—403) is concerned with certain nonlinear problems in case of ordinary differential equations. Existence and uniqueness problems, stability questions, the properties of solutions, Lyapunov stability of linear and almost linear equations, the analysis of the structure of the integral curve, autonomous systems, singular points, limit cycles, nonlinear vibration problems are the main topics in that chapter. Numerous problems and applications in electronics and automation are also given.

The subjects are treated rigorously but with such a lucid and balanced style that technologists also would comprehend them easily. The independence of the chapters was also aimed at. Elementary algebra, analysis and geometry are the only requirements from the reader. It must be emphasized that also mathematicians interested in the latest fields of application will profit much from the study of the examples and problems in the book. — The Bibliography (a selection) contains 11 well-chosen items. A subject index closes the book.

P. Medgyessy

V. P. MASLOV: *Théorie des perturbations et méthodes asymptotiques.*

Suivi de deux notes complémentaires de V. I. Arnold et V. C. Bouslaev. Traduit par J. Lascoux et R. Senor. DUNOD ET GAUTHIER—VILLARS, PARIS, 1972. XVI+384 p. 11 figures. 170 F.

This book consists of two main parts. Part I (Theory of perturbations) generalizes the problem of solution of the classical problem of perturbation theory — that of the finding of eigenvalues and eigenvectors of the matrix $A(\varepsilon) = A + \varepsilon B$ if the corresponding characteristics of the matrix A are known, B is given and ε is sufficiently small. This generalization consists in taking an unbounded operator in an infinite dimensional space as A ; then the notions connected with the matrix case are to be defined and solved in a totally new way. In Ch. 1 (Problem of regularization of the perturbation theory) and Ch. 2 (Asymptotic behaviour of the eigenfunctions and perturbation theory for equations with operator coefficients) cases where the spectrum of A is discrete or possesses at least one isolated point are considered. Relevant problems of perturbation theory are treated by means of the regularization method due to A. N. Tihonov according to which an unbounded operator can be substituted by a sequence of certain bounded operators, whose subsequent application yields already good approximations of the solution belonging to the original operator occurring in the perturbed equation. Ch. 3. (Strong convergence of operator equations) and Ch. 5 (Weak convergence of operators), provides a further generalization of the main problems, namely the construction of the approximation of the inverses of members of certain family $\{T_\varepsilon\}$ of operators by help of the limit T of T_ε and by T^{-1} . Here the existence and the approximation by means of T_ε^{-1} of T^{-1} are also considered. Ch. 4 (Perturbations of one-parameter semigroups) considers one-parameter semigroups of operators and the relation between the convergence of these semigroups and the convergence of the corresponding generators. — Part II. (Global theory of characteristics and asymptotic methods in the theory of differential equations of operator coefficients) deals with equations of quantum mechanics and optics considering them as particular cases of equations with operator coefficients in certain function spaces. By this aspect the asymptotic formulas of different fields of physics can be shown to be interconnected. After Ch. 1 (Formulation of the problem) discussing the mentioned fundamental views, Ch. 2 (The canonical operator) aims at obtaining a class of asymptotic formulas being uniform in the coordinates of location. To this end a „canonical” operator transforming the space of functions defined on curves of the phase space (p, q) into the space of functions on the line (q) is constructed and studied. Ch. 3 (Asymptotics of the solutions of partial differential equations) and Ch. 4 (Equations with operator coefficients) consider the solution of the equations of the mentioned type as well as the behaviour of the

eigenvalues of selfadjoint differential operators. Ch. 5 (Local characteristic representation for equations of wave equation type) deals with the quasiclassical local asymptotics of the equations of quantum mechanics, which is analogous to that of a system of hyperbolic equations with discontinuous or oscillating initial data and was investigated only formally in the literature. Ch. 6 (Local asymptotics of operator equations with partial derivatives) presents the method of stationary phase, used in local asymptotics, according to the author's method. Ch. 7 (Global asymptotics of the solutions of the abstract equations) comprises the proofs of the theorems presented in chapters 2, 3 and 4. Ch. 8 (Quasi—classical formulas for the solutions of the equations with $[n/2]+4$ times differentiable coefficients of quantum mechanics) yields an introduction of asymptotic formulas that differs essentially from that given in Ch. 7, partly by the aid of the theory of Morse. Ch. 9 (Regularization of perturbation theory for calculating correction terms and the quasi-classical formula of Bohr) gives the higher terms in the considered asymptotic expansions by means of the methods of perturbation theory, combined with tools of the method of regularization. Results of Part I are utilized at several places of Part II.

An Appendix (Discontinuities in the asymptotics of the solutions of equations of tunnel effect type) shows new applications of the method used in the study of the discontinuities of the solutions of equations of wave equation type. A Supplement (B. Doubnov: On the global existence of the solutions of Hamilton's equations) and two Complements (V. I. Arnol'd: A characteristic class occurring in the quantification conditions; V. C. Bouslaev: Generating integral and Maslov's canonical operator for the W. K. B. method) present a new approach and supplementary material to certain problems of the main text.

The book (published in Russian in 1965) is of high mathematical level, is very comprehensive and presumes knowledge of several fields of mathematics. Although the problem solutions belong generally to the field of quantum mechanics, clarifying several questions not dealt with in full rigour up to now and yielding a very valuable help to research work on theoretical physics, they are worth being studied also by the mature mathematician.

There stands a main bibliography after the Supplement comprising 90 items. The two complements possess separate bibliographies of 5 and 11 items respectively. Unfortunately there is no index and several author names (Russian and non-Russian) have been printed with striking errors.

P. Medgyessy

H. G. GARNIR—M. DE WILDE—J. SCHMETS: Analyse fonctionnelle, Tome II, Mesure et intégration dans l'espace euclidien E_n .

LEHRBÜCHER UND MONOGRAPHIEN AUS DEM GEBIETE DER EXAKTEN WISSENSCHAFTEN, MATHEMATISCHE REIHE, BAND 37, BIRKHÄUSER VERLAG, BASEL—STUTTGART, 1972.

Comme le titre l'indique, ce livre est une introduction à la théorie de la mesure et de l'intégration dans un ouvert de l'espace euclidien E_n . Il comprend les chapitres suivants:

- I. Mesure.
- II. Intégration.
- III. Fonctions et ensembles boréliens.
- IV. Produit de mesures.
- V. Relations entre mesures.
- VI. Intégration de fonctions à valeurs dans un espace linéaire à semi-normes.
- VII. Mesures à valeurs dans un espace linéaire à semi-normes.

Les cinq premiers chapitres sont indépendants du Tome I de l'ouvrage. Ils présentent un exposé classique de la théorie de la mesure et de l'intégration. Cette exposition a deux particularités qui rendent le livre différent des autres ouvrages traitant le même sujet: l'une, c'est la restriction du domaine indiquée dans le titre; l'autre, c'est qu'on considère des mesures complexes dès le début. Bien sûr, le traitement est plus simple dans un espace euclidien. Et, comme les auteurs l'écrivent dans l'introduction, «le degré de généralité ainsi atteint nous paraît suffire à la plupart des utilisateurs, et constitue une introduction utile pour ceux qui veulent aborder la théorie de la mesure dans ses développements les plus abstraits.» C'est surtout en lisant le chapitre V qu'on réalise que les auteurs envisageaient aussi des applications: on y trouve, par exemple, le principe du bang-bang et le théorème de Lyapounov, repris à la théorie du contrôle, et une étude des ensembles de mesures dominées par une mesure, ce qui est très important en statistique.

Les chapitres VI et VII présentent une extension de la théorie classique de la mesure. On s'appuie également ici sur des résultats développés dans le premier tome de l'ouvrage.

L'exposition est claire et précise. Le livre contient de nombreux exercices. La solution peut en être facilitée par les suggestions qui suivent les cas les plus difficiles.

L. Márki

N. BOURBAKI, Topologie Générale, I.

(Chapitres 1 à 4.) XVI + 128 + 44 + 90 + 98 = 376 PAGES, HERMANN, PARIS, 1971.

Il n'est guère nécessaire de présenter au lecteur avisé le traité fondamental de Bourbaki. Bien connu est également le fait, que même parmi les livres constituant ce fameux traité, une place de choix est due à la «Topologie Générale». La parution, en 1940, de la première édition des deux premiers chapitres de ce livre marque en effet la naissance de la topologie générale telle comme on l'entend aujourd'hui.

Pendant les décennies écoulées depuis cette date, la présentation Bourbakiennne de la topologie a toujours conservé son importance capitale, et cela grâce à des rééditions mises au point selon les dernières exigences de l'évolution de la science. — A la hauteur des exigences des années soixante-dix, du point de vue formel l'édition ci-considérée se distingue de ces prédécesseurs par le fait, que le matériel réparti auparavant sur deux fascicules brochés est réuni ici dans un tome solide (bien relié et très bien imprimé), qui contient ainsi les chapitres suivants: I. Structures Topologiques. — II. Structures Uniformes. — III. Groupes Topologiques. — IV. Nombres Réels.

Agréable aux yeux et aisé à manier, ce beau volume contribuera sans doute à élargir encore l'influence du personnage collectif célèbre qui est Monsieur Nicolas Bourbaki.

S. Gacsályi

ROLF KLÖTZER: Mehrdimensionale Variationsrechnung.

BIRKHÄUSER VERLAG, BASEL—STUTTGART, 1970. 299 p.

Variational calculus in several dimension is a rapidly developing area of mathematical analysis that provides a lot of difficult problems and has several applications in different fields of mathematics as well as in physics and engineering practice. The present book seems to be an effective tool for a reader who has some general mathematical knowledge to reach the up-to-date level in a fairly extended domain.

Chapter 1. contains the exact formulation of the Bolza problem treated in the book. A functional of the form

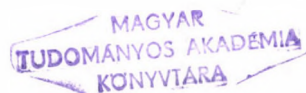
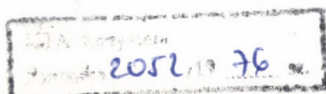
$$J(x) = \int_G f(t, x, x_t) dt + \int_{\partial G} g(t, x) d\sigma$$

has to be minimized where $x(t) = (x^1, x^2, \dots, x^m) \in R^m$. $G \subset R^m$ is a domain with piecewise smooth boundary ∂G and $x(t)$ satisfies the boundary conditions $h_\nu(t, x, x_t) = 0$ ($\nu = 1, 2, \dots, r < m$) and $R_\alpha(x)|_{\partial G} = 0$ ($\alpha = 1, 2, \dots, \rho$).

It is shown that most variational problems (isoperimetric problems, etc.) can be reduced to an appropriate Bolza problem. Moreover, assuming regularity (or quasiregularity) each Bolza-problem can be transformed into the usual canonical form by a Legendre transformation. The chapter concludes with a survey on parametric problems.

Chapter 2. presents an account of functional analysis that will later be needed. The definition and elementary properties of a linear normed space (especially Hilbert space) are given, mostly without proofs. The usual simple completion process and the theory of Sobolev-spaces are developed in more details, including Sobolev's embedding theorem and the Gårding inequality. Some simple properties of quadratic forms are listed.

Chapter 3 starts with the general definition of the variation as a Gateaux-derivative. The usual controlability condition implies the applicability of the variational principle. The open mapping theorem is used to prove the necessary integral-condition of the Lagrange-type. Assuming appropriate smoothness of the Lagrange multipliers, the fundamental lemma of the variational calculus yields the Euler-Lagrange differential equations. Haar's lemma and its converse are proved. Among their application is the "integrated" Euler-equation. In the case of boundary conditions the usual transversality conditions are developed. Regarding the second variation, a generalization of the Legendre—Hadamard-condition is given. This also implies the necessary condition of Weierstrass (for strong minimum).



In Chapter 4 a general sufficient condition for minimum of a function, defined on subset of a Hilbert space, is given. This is then used to generalize the sufficient condition of Van Hove for weak relative minimum. Two "structural" assumptions together with the strong Legendre condition imply weak relative minimality. A systematic treatment of the second variation $\delta^2 J^*(x_0, \xi)$ is given. The chapter concludes with Hölder's theory, i.e. the theory of sufficient conditions for minima in terms of eigenvalues of appropriate eigenvalue problems. Especially the Jacobi conditions are proved.

Sufficient conditions for a strong relative minimum are presented in the fifth chapter. After a short description of the field-theory, especially that of Caratheodory, the field theory of de Donder—Weyl is treated in great details. This is the most valuable part of the book, involving the authors own results. Assuming positive regularity (again) the construction of the field is done for the cases $m=1$ and $m>1$ separately. The possibility of an appropriate generalization of the field theory is also sketched.

In chapter 6 a result of the previous chapter is applied for a case when the integrand depends on a real parameter. This reduces to an eigenvalue problem which is exhaustively studied. The second part of the chapter deals with field-theoretic methods providing lower bounds for the functional $I(x)$.

Though the book is essentially selfcontained, some familiarity with classical variational calculus and elementary analysis may be useful for reading. Several examples make understanding easier and provide a large scale of applications.

A. Szép

I. J. MADDOX: Elements of functional analysis.

CAMBRIDGE UNIVERSITY PRESS, 1970, X+208 pp.

This is a really introductory, though non-trivial course on abstract methods of analysis for students with some knowledge of classical real and complex variable theory.

In Chapters 1—3 sets, metric spaces, topological spaces, linear spaces and linear metric spaces are discussed in detail. In Chapter 4 the Banach—Steinhaus, the open mapping and the Hahn—Banach theorem are proved in the context of normed linear spaces. Chapter 5, dealing with Banach algebras contains the Gelfand—Mazur theorem and a weakened form of the Gelfand representation theorem. The short Chapter 6, on Hilbert spaces, treats orthonormal systems, the projection theorem, and the Riesz—Fréchet theorem. Chapter 7 is concerned with matrix transformations and summation methods, an area of special interest to the author.

The book is written in a highly instructive manner. The style is clear and vivid. Concepts are soundly motivated, theorems are accompanied by numerous comments, examples, and exercises. (Since Lebesgue theory is not assumed, most applications involve sequence, rather than function, spaces.) Also the display of basic terms before defining them seems to be a good idea. Short historical remarks as well as outlook to more advanced topics help better orientation.

To sum up, this is an excellent first reading for all who wish to get thoroughly acquainted with basic notions, results and applications of topology and functional analysis.

J. Bognár

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