Studia Scientiarum Mathematicarum Hungarica

AUXILIO
CONSILII INSTITUTI MATHEMATICI
ACADEMIAE SCIENTIARUM HUNGARICAE

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TOMUS VII
FASC. 1—2.
1972

AKADÉMIAI KIADÓ, BUDAPEST

Studia Scientiarum Mathematicarum Hungarica

A Magyar Tudományos Akadémia matematikai folyóirata

Szerkesztőség: 1061 Budapest V., Reáltanoda u. 13—15.

Technikai szerkesztő: Petruska Gy.

Kiadja az Akadémiai Kiadó, 1054 Budapest V., Alkotmány u. 21.

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Előfizetési ára belföldre 120,— Ft, külföldre 165,— Ft. Megrendelhető a belföld számára az Akadémiai Kiadónál, a külföld számára pedig a Kultúra Könyv és Hirlap Külkereskedelmi Vállalatnál (1011 Budapest II., Fő u. 32.).

Cserekapcsolatok felvétele ügyében kérjük a MTA Matematikai Kutató Intézete Könyvtárához (1061 Budapest V., Reáltanoda u. 13—15) fordulni.

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Studia Scientiarum Mathematicarum Hungarica is a journal of the Hungarian Academy of Sciences publishing original papers on mathematics, in English, German, French or Russian.

It is published semiannually, making up one volume per year.

Editorial Office: 1061 Budapest V., Reáltanoda u. 13—15, Hungary.
Technical Editor: Gy. Petruska

Subscription rate: $ 16.00 per volume. Orders may be placed with *Kultúra* Trading Co. for Books and Newspapers, Budapest 62, P. O. B. 149 or with its representatives abroad.

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A NEW PROOF OF S. A. TELYAKOVSKIĬ'S THEOREM ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY ALGEBRAIC POLYNOMIALS

by

R. B. SAXENA

1. The results of S. A. TELYAKOVSKIĬ * [12] and I. E. GOPENGAUZ [4] imply the validity of the following assertion: *for every function $f(x)$, continuous on $[-1, 1]$, and any natural number n, there is an algebraic polynomial $P_n(f, x)$ of degree $\leqq n$ such that for all $x \in [-1, 1]$*

$$(1) \qquad |f(x)-P_n(f, x)| \leqq A\omega\left(f; \frac{\sqrt{1-x^2}}{n}\right)$$

where A is an absolute constant and $\omega(f, \delta)$ is the modulus of continuity of $f(x)$. This sharpens the well-known theorem of A. F. TIMAN [11] which itself is a stronger form of the classical theorem of D. JACKSON [5].

In the present paper, the author gives a new proof of inequality (1) by constructing the interpolation polynomials $\Lambda_n(f, x)$ of degree $\leqq 4n+2$ which take the same values as $f(x)$ at the points $x_{kn} (k = 0, 1, 2, \dots, n+1)$. In this way, for fixed n, our polynomials $\Lambda_n(f, x)$ depend on a finite set of values of $f(x)$ only. We shall prove the following.

THEOREM. *For every function $f(x)$ continuous on $[-1, 1]$ and any natural number n there is a polynomial $\Lambda_n(f, x)$ of degree $\leqq 4n+2$ such that for all $x \in [-1, 1]$*

$$|f(x)-\Lambda_n(f, x)| \leqq 1285\omega\left(f; \frac{\sqrt{1-x^2}}{n}\right)$$

where $\omega(f, \delta)$ is the modulus of continuity of $f(x)$.

We observe that G. FREUD [2], M. SALLAY [10], the author [8] and P. VÉRTESI [13] have constructed the algebraic ** interpolation polynomials for which Jackson's inequality is satisfied. The interpolation polynomials satisfying Timan's inequality have been studied by G. FREUD & P. VÉRTESI [3], O. KIS & P. VÉRTESI [6] and the author [9].

2. We shall first describe the construction of the polynomials $\Lambda_n(f, x)$. Let

$$(2) \qquad x_{kn} = \cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n$$

be the zeros of

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta, \qquad n = 1, 2, \dots$$

* For its generalization to two variables, see MALAZEMOV [7].

** The trigonometric analogue is studied in [6].

the Čebyšev polynomial of second kind. The fundamental polynomial of Lagrange interpolation constructed on the nodes (2) is given by

$$l_{kn}(x) = \frac{(-1)^{k+n}(1-x_{kn}^2)}{n+1} \cdot \frac{U_n(x)}{x-x_{kn}}.$$

Further, let

$$v_{kn}(x) = 1 - \frac{3x_{kn}(x-x_{kn})}{1-x_{kn}^2}$$

and

$$\psi_n(t, u) = \frac{2}{n+1} \sum_{r=1}^{n-1} U_r'(t) U_r(u).$$

Set

$$\lambda_{kn} = \left(\frac{1-x^2}{1-x_{kn}}\right)^2 [v_{kn} l_{kn}^4(x) + 2(x-x_{kn}) l_{kn}^3(x) \cdot (1-x_{kn}^2)\psi_n(x_{kn}, x)], \tag{3}$$

the right hand side being a polynomial of degree $4n+1$, and

$$\Lambda_n(f, x) = \frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1) + \tag{4}$$

$$+ \sum_{k=1}^{n} \left[f(x_{kn}) - \left\{ \frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1) \right\} \right] \lambda_{kn}(x),$$

for an arbitrary function $f(x)$ defined for $-1 \leqq x \leqq 1$.

The interpolation process (4) was constructed by the author in [8]. It is easily seen that $\Lambda_n(f, x)$ are polynomials of degree $\leqq 4n+2$ and that

$$\Lambda_n(f, x_{kn}) = f(x_{kn}), \qquad k = 0, 1, 2, \ldots, n+1 \, (x_0 = 1, x_{n+1} = -1).$$

It was proved in [8] that *for every function* $f(x)$ *continuous on* $[-1, 1]$ *and any natural number* n *Jackson's inequality*

$$|f(x) - \Lambda_n(f, x)| \leqq 414\omega\left(f; \frac{1}{n}\right) \tag{5}$$

holds for all $x \in [-1, 1]$. We further observed in [9] that the above inequality can be sharpend and we replaced (5) by

$$|f(x) - \Lambda_n(f, x)| \leqq 384 \left[\omega\left(f; \frac{\sqrt{1-x^2}}{n}\right) + \omega\left(f; \frac{|x|}{n^2}\right) \right]. \tag{6}$$

The inequality (6) is due to A. F. TIMAN [11]. It was shown by V. K. DZYADYK [1] that, for functions of class Lip α ($\alpha < 1$), the result of A. F. TIMAN is the best possible as far as order is concerned.

3. In the sequel an essential role is played by the following

LEMMA. *For* $0 \leqq \theta \leqq \pi$, *we have*

$$\sum_{k=1}^{n} |\lambda_k(\cos\theta)| \leqq 130 \tag{7}$$

$$\sum_{k=1}^{n} \sin\frac{1}{2}|\theta-\theta_k|\,|\lambda_k(\cos\theta)| \leqq 74\sin\theta \tag{8}$$

$$\sum_{k=1}^{n} \sin^2\frac{1}{2}(\theta-\theta_k)\,|\lambda_k(\cos\theta)| \leqq 37\sin^2\theta. \tag{9}$$

PROOF. From equality (18) in [9], we have

$$\lambda_k(\cos\theta) = \frac{\sin^3(n+1)\theta\cos(n+1)\theta\sin^2\theta_k\sin\theta}{(n+1)^3(\cos\theta-\cos\theta_k)^3} + \frac{\sin^4(n+1)\theta\sin^4\theta_k}{(n+1)^4(\cos\theta-\cos\theta_k)^4} - \tag{10}$$

$$-\frac{2\sin^4(n+1)\theta\sin^2\theta_k\cos\theta_k}{(n+1)^4\cdot(\cos\theta-\cos\theta_k)^3} - \frac{\sin^4(n+1)\theta\sin^2\theta_k}{(n+1)^3(\cos\theta-\cos\theta_k)^2} +$$

$$+\frac{\sin^4(n+1)\theta\sin^2\theta_k}{4(n+1)^4(\cos\theta-\cos\theta_k)^2}\left[\operatorname{cosec}^2\frac{\theta-\theta_k}{2} + \operatorname{cosec}^2\frac{\theta+\theta_k}{2}\right].$$

Since

$$\sin\theta_k \leqq \sin\theta_k + \sin\theta = 2\sin\frac{\theta+\theta_k}{2}\cos\frac{\theta-\theta_k}{2} \leqq 2\sin\frac{\theta+\theta_k}{2}$$

and

$$\sin\theta \leqq \sin\theta + \sin\theta_k \leqq 2\sin\frac{\theta+\theta_k}{2},$$

therefore

$$|\lambda_k(\cos\theta)| \leqq \frac{|\sin^3(n+1)\theta|}{(n+1)^3\sin^3\frac{|\theta-\theta_k|}{2}} + \frac{\sin^4(n+1)\theta}{(n+1)^4\sin^4\frac{\theta-\theta_k}{2}} +$$

$$+\frac{\sin^4(n+1)\theta}{(n+1)^4\sin\frac{\theta+\theta_k}{2}\sin^3\frac{|\theta-\theta_k|}{2}} + \frac{\sin^4(n+1)\theta}{(n+1)^3\sin^2\frac{\theta-\theta_k}{2}} +$$

$$+\frac{\sin^4(n+1)\theta}{4(n+1)^4\sin^2\frac{\theta-\theta_k}{2}}\left[\operatorname{cosec}^2\frac{\theta-\theta_k}{2} + \operatorname{cosec}^2\frac{\theta+\theta_k}{2}\right].$$

But

$$\frac{1}{\sin\frac{\theta+\theta_k}{2}} \leqq \frac{1}{\sin\frac{|\theta-\theta_k|}{2}}, \qquad (0\leqq\theta\leqq\pi,\ 0\leqq\theta_k\leqq\pi),$$

therefore

$$(11)\qquad |\lambda_k(\cos\theta)| \leqq \frac{|\sin^3(n+1)\theta|}{(n+1)^3\sin^3\frac{|\theta-\theta_k|}{2}} + \frac{5}{2}\cdot\frac{\sin^4(n+1)\theta}{(n+1)^4\sin^4\frac{\theta-\theta_k}{2}} +$$

$$+\frac{\sin^4(n+1)\theta}{(n+1)^3\sin^2\frac{\theta-\theta_k}{2}} \leqq 2\left[\frac{|\sin(n+1)\theta|}{(n+1)\sin\frac{|\theta-\theta_k|}{2}}\right]^3 + \frac{5}{2}\left[\frac{\sin(n+1)\theta}{(n+1)\sin\frac{|\theta-\theta_k|}{2}}\right]^4.$$

From (11), because of the inequality

$$(12)\qquad |\sin(n+1)\theta| \leqq (n+1)\sin\theta,$$

we have

$$(13)\qquad \sin\frac{|\theta-\theta_k|}{2}\,|\lambda_k(\cos\theta)| \leqq$$

$$\leqq \sin\theta\left[2\left\{\frac{\sin(n+1)\theta}{(n+1)\sin\frac{\theta-\theta_k}{2}}\right\}^2 + \frac{5}{2}\left\{\frac{|\sin(n+1)\theta|}{(n+1)\sin\frac{|\theta-\theta_k|}{2}}\right\}^3\right].$$

Further, from (10), on account of (12), we have

$$|\lambda_k(\cos\theta)| \leqq \sin^2\theta\left\{\frac{\sin^2(n+1)\theta\sin^2\theta_k}{(n+1)^2|\cos\theta-\cos\theta_k|^3} + \frac{\sin^2(n+1)\theta\sin^4\theta_k}{(n+1)^2(\cos\theta-\cos\theta_k)^4} +\right.$$

$$+\frac{2\sin^2(n+1)\theta\sin^2\theta_k}{(n+1)^2|\cos\theta-\cos\theta_k|^3} + \frac{\sin^2(n+1)\theta\sin^2\theta_k}{(n+1)(\cos\theta-\cos\theta_k)^2} +$$

$$\left.+\frac{\sin^2(n+1)\theta\sin^2\theta_k}{4(n+1)^2(\cos\theta-\cos\theta_k)^2}\left[\operatorname{cosec}^2\frac{\theta-\theta_k}{2} + \operatorname{cosec}^2\frac{\theta+\theta_k}{2}\right]\right\} \leqq$$

$$\leqq \sin^2\theta\left[\frac{\sin^2(n+1)\theta}{2(n+1)^2\sin^4\frac{\theta-\theta_k}{2}} + \frac{\sin^2(n+1)\theta}{(n+1)^2\sin^4\frac{\theta-\theta_k}{2}} + \frac{\sin^2(n+1)\theta}{(n+1)^2\sin^4\frac{\theta-\theta_k}{2}} +\right.$$

$$\left.+\frac{\sin^2(n+1)\theta}{(n+1)\sin^2\frac{\theta-\theta_k}{2}} + \frac{\sin^2(n+1)\theta}{2(n+1)^2\sin^4\frac{\theta-\theta_k}{2}}\right]$$

and hence

$$(14)\qquad \sin^2\frac{\theta-\theta_k}{2}\,|\lambda_k(\cos\theta)| \leqq \left[3\left\{\frac{\sin(n+1)\theta}{(n+1)\sin\frac{\theta-\theta_k}{2}}\right\}^2 + \frac{\sin^2(n+1)\theta}{n+1}\right]\cdot\sin^2\theta.$$

Hence on using the inequality

$$\sum_{k=1}^{n}\left[\frac{\sin(n+1)\theta}{(n+1)\sin\frac{|\theta-\theta_k|}{2}}\right]^m \leqq 2^{m+1}+4, \qquad m \geqq 2$$

proved in [9] the inequalities (11), (13) and (14) at once give the lemma.

From this lemma, for $-1 \leqq x \leqq 1$, we get

$$\sum_{k=1}^{n} |\lambda_k(x)| \leqq 130 \tag{15}$$

$$\sum_{k=1}^{n} |x-x_k|\,|\lambda_k(x)| \leqq 222(1-x^2). \tag{16}$$

(15) is essentially the same as (7) for $x=\cos\theta$ while (16) may be proved as follows. Putting $x=\cos\theta$ and making use of the lemma, we have

$$\sum_{k=1}^{n} |x-x_k|\,|\lambda_k(x)| = \sum_{k=1}^{n} |\cos\theta_k-\cos\theta|\,|\lambda_k(\cos\theta)| =$$

$$= \sum_{k=1}^{n}\left|2\sin\frac{\theta-\theta_k}{2}\left\{\sin\theta\cos\frac{\theta-\theta_k}{2}-\cos\theta\sin\frac{\theta-\theta_k}{2}\right\}\right| \times |\lambda_k(\cos\theta)| \leqq$$

$$\leqq 2\sin\theta\sum_{k=1}^{n}\sin\frac{|\theta-\theta_k|}{2}\cdot|\lambda_k(\cos\theta)| + 2\sum_{k=1}^{n}\sin^2\frac{\theta-\theta_k}{2}\,|\lambda_k(\cos\theta)| \leqq$$

$$\leqq 148\sin^2\theta+74\sin^2\theta = 222\sin^2\theta = 222(1-x^2).$$

For the main proof of the theorem, besides (15) and (16), we shall need the inequality

$$\left|1-\sum_{k=1}^{n}\lambda_k(x)\right| \leqq 3, \qquad -1 \leqq x \leqq 1 \tag{17}$$

proved in [8].

4. Proof of the theorem. We shall use the following properties of modulus of continuity.

(i) $$\omega(\delta_1) \leqq \omega(\delta_2) \qquad 0<\delta_1<\delta_2$$

(ii) $$\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2)$$

(iii) $$\omega(n\delta) \leqq n\omega(\delta) \qquad n, \text{ a positive integer,}$$

(iv) $$\frac{\omega(\delta_1)}{\delta_1} \leqq \frac{2\omega(\delta_2)}{\delta_2}, \qquad \delta_1 \geqq \delta_2 > 0.$$

From these we at once have

$$\frac{1-x}{2}\omega(2) \leqq (1-x)\omega(1) \leqq 2\omega(1-x) \tag{18}$$

and

$$(19)\qquad \omega(1-x_k) \leqq \begin{cases} \omega(1-x), & x_k > x \\ 2\omega(1-x)+2\dfrac{x-x_k}{1-x}\,\omega(1-x), & x_k \leqq x < 1. \end{cases}$$

We now complete the proof of the theorem. First let $\sqrt{1-x^2} \geqq \dfrac{1}{n}$. On account of the inequality (6) we have:

$$(20)\qquad |f(x)-\Lambda_n(f,x)| \leqq 384\left[\omega\left(f;\frac{\sqrt{1-x^2}}{n}\right)+\omega\left(f;\frac{1}{n^2}\right)\right] \leqq 768\omega\left(f;\frac{\sqrt{1-x^2}}{n}\right).$$

Let now $\sqrt{1-x^2} < \dfrac{1}{n}$. For definiteness we shall assume that $x>0$. In this case, by virtue of the definition of the operator $\Lambda_n(f,x)$, we have

$$(21)\qquad |f(x)-\Lambda_n(f,x)| = \left| f(x)-f(1)+\frac{1-x}{2}\{f(1)-f(-1)\}\left(1-\sum_{k=1}^{n}\lambda_k(x)\right)+\right.$$

$$\left.+\sum_{k=1}^{n}\{f(1)-f(x_k)\}\lambda_k(x)\right| \leqq \omega(1-x)+\frac{1-x}{2}\,\omega(2)\left|1-\sum_{k=1}^{n}\lambda_k(x)\right|+$$

$$+\sum_{k=1}^{n}\omega(1-x_k)|\lambda_k(x)| \leqq \omega(1-x)+6\omega(1-x)+\sum_{k=1}^{n}\omega(1-x_k)\cdot|\lambda_k(x)|$$

because of (17) and (18). But owing to (15), (16) and (19)

$$\sum_{k=1}^{n}\omega(1-x_k)|\lambda_k(x)| = \sum_{\substack{k=1\\ x_k>x}}^{n}\omega(1-x_k)|\lambda_k(x)|+\sum_{\substack{k=1\\ x_k\leqq x}}^{n}\omega(1-x_k)|\lambda_k(x)| \leqq$$

$$\leqq \omega(1-x)\sum_{k=1}^{n}|\lambda_k(x)|+2\omega(1-x)\sum_{k=1}^{n}|\lambda_k(x)|+2\,\frac{\omega(1-x)}{1-x}\sum_{k=1}^{n}|x-x_k|\,|\lambda_k(x)| \leqq$$

$$\leqq 390\omega(1-x)+444(1+x)\omega(1-x) \leqq 1278\omega(1-x).$$

Hence

$$(22)\qquad |f(x)-\Lambda_n(f,x)| \leqq 1285\omega(1-x) \leqq 1285\omega(1-x^2).$$

Combining (20) and (22) we have the theorem.

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(Received October 20, 1970.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 11—20.

AN EXAMINATION OF NONNEGATIVITY AND QUASICONVEXITY CONDITIONS OF QUADRATIC FORMS ON THE NONNEGATIVE ORTHANT

by

G. KÉRI

Introduction

In this paper I shall be concerned with the following problems, emerged in the field of quadratic programming.

1. What quadratic forms $\mathbf{x}'C\mathbf{x}$ have nonnegative value for every vector $\mathbf{x}$ on the nonnegative orthant? Is it necessary for this property that the matrix [1] C be the sum of a matrix associated with a positive definite or positive semidefinite quadratic form and a matrix with only nonnegative elements?
The conjecture involved here is due to A. PRÉKOPA.

2. What quadratic forms $\mathbf{x}'C\mathbf{x}$ are quasiconvex on the nonnegative orthant?

An answer to the first part of question 1. will be given by Theorems 4. and 5. while the second part will be answered in this paper only in the case of not larger than 3×3 matrices. In this case the answer is "yes". The difficulty arises from the fact that, in general, one cannot easily decide it whether a given matrix of not too small size can be decomposed in the form of the sum of a matrix associated with a positive definite or semidefinite quadratic form and a matrix consisting entirely of nonnegative elements. Problem 2. will be answered in the second part of this article. However, the ideas being applied in the course of the proofs can be transferred without any difficulty to the more general case where the role of the nonnegative orthant is played by any convex polyhedron with the only restriction that it has a nonempty interior. Thus the second question will be discussed in this sense because it does not incorporate any surplus effort even if from the technical point of view.

The last thing I want to remark in advance is that from the answer to Problem 2 (Proposition 13. of Part II.) one can deduce the following corollaries: If $\mathbf{x}'C\mathbf{x}$ is equally quasiconvex and quasiconcave in a convex set K which has some interior points, and $C\neq 0$ then the rank of C is equal to one; if $\mathbf{x}'C\mathbf{x}$ is equally quasiconvex and nonnegative in a set K of the previous attributes then $\mathbf{x}'C\mathbf{x}$ is either positive definite or positive semidefinite. Let us add to the above assertions that the nonnegativity condition in convex cones, as Professor A. PRÉKOPA pointed at, is equivalent to the boundedness from below and the same can be said in the case of such convex sets which derive from a convex cone by removing a strictly bounded set. Consequently in the case of such sets the boundedness from below and the quasiconvexity of a quadratic form imply its semidefiniteness.

[1] In this paper I only deal with symmetric matrices but generally omitting the word „symmetric"

Part I.

1. Proposition: If $C = A+B$ where A is a matrix associated with a positive definite or positive semidefinite quadratic form and B is a matrix with only nonnegative elements then $\mathbf{x}'C\mathbf{x} \geqq 0$ for every $x \geqq 0$.

2. Definition: Let C be an $n\times n$ symmetric matrix. We say that a quadratic submatrix D of the matrix C is symmetrically selected if and only if D is made from C by crossing out some rows and columns chosen in a symmetrical way.

3. PROPOSITION. *If $\mathbf{x}'C\mathbf{x} \geqq 0$ for every $\mathbf{x} \geqq \mathbf{0}$ and D is a symmetrically selected submatrix of C then $\mathbf{y}'D\mathbf{y} \geqq 0$ for every $\mathbf{y} \geqq \mathbf{0}$.*

4. THEOREM. (Necessary condition for $\mathbf{x}'C\mathbf{x} \geqq 0$ in the nonnegative orthant): *If $\mathbf{x}'C\mathbf{x} \geqq 0$ for every $\mathbf{x} \geqq \mathbf{0}$ then any symmetrically selected submatrix D of the matrix C and any nonnegative eigenvector $\mathbf{s}$ of D satisfies $\mathbf{s}'D\mathbf{s} \geqq 0$* (that is every eigenvalue belonging to a nonnegative eigenvector of any symmetrically selected quadratic submatrix of C should be nonnegative).

5. THEOREM. (Sufficient condition for $\mathbf{x}'C\mathbf{x} \geqq 0$ in the nonnegative orthant): *If for any symmetrically selected submatrix D of the matrix C and for any (in all its components) positive eigenvector $\mathbf{s}$ of D, $\mathbf{s}'D\mathbf{s} \geqq 0$, then $\mathbf{x}'C\mathbf{x} \geqq 0$ for every $\mathbf{x} \geqq \mathbf{0}$.*

PROOF. Let us assume that we have an n-component vector $\mathbf{x}$ for which $\mathbf{x} \geqq \mathbf{0}$ and $\mathbf{x}'C\mathbf{x} < 0$. Let $\mathbf{x}_0$ be a vector that minimizes $\mathbf{x}'C\mathbf{x}$ in the closed bounded set

$$H = \{\mathbf{x}:\mathbf{x} \geqq \mathbf{0},\ \mathbf{x}'\mathbf{x} = 1\}$$

Then necessarily $\mathbf{x}_0'C\mathbf{x}_0 < 0$, and therefore $\mathbf{x}_0 \neq \mathbf{0}$. Let $\mathbf{y}_0$ be the vector consisting of the positive components of $\mathbf{x}_0$. Furthermore, let D be the corresponding symmetrically selected submatrix of C, that is if $\mathbf{x}_0$ has $x_{j_1}, x_{j_2}, \ldots, x_{j_{m-n}}$ as its zero components, then D should be the matrix made from C by omitting the j_1 st, j_2 nd, ..., j_{m-n} th rows and columns. Because of the extremity property of $\mathbf{x}_0$, also $\mathbf{y}_0$ minimizes the function $\mathbf{y}'D\mathbf{y}$ in the set $H^* = \{\mathbf{y}: \mathbf{y} \geqq 0,\ \mathbf{y}'\mathbf{y} = 1\}$ lying in the m-dimensional Euclidean space. Consequently, having written $\mathbf{y}_0'D\mathbf{y}_0 = \mathbf{x}_0'C\mathbf{x}_0 = \lambda$, we also have $\mathbf{y}'D\mathbf{y} = \lambda\mathbf{y}'\mathbf{y}$ for every $\mathbf{y} \geqq \mathbf{0}$.

Then $\mathbf{y}_0$ is asserted to be an eigenvector of D. To prove this, let $\mathbf{e}$ be an arbitrary unit-vector, orthogonal to $\mathbf{y}_0$. Because of $\mathbf{y}_0 \gg \mathbf{0}$ we get $\mathbf{y}_0 + u\mathbf{e} \geqq \mathbf{0}$ if u is contained in an appropriate neighbourhood of zero. Thus

$$(\mathbf{y}_0 + u\mathbf{e})'D(\mathbf{y}_0 + u\mathbf{e}) \geqq \lambda(\mathbf{y}_0 + u\mathbf{e})'(\mathbf{y}_0 + u\mathbf{e}).$$

Hence

$$(\mathbf{e}'D\mathbf{e} - \lambda)u^2 + 2(\mathbf{e}'D\mathbf{y}_0)u \geqq 0$$

is satisfied for those u (both positive and negative) which lie close enough to zero. This could not be true unless $\mathbf{e}'D\mathbf{y}_0 = 0$. Therefore $D\mathbf{y}_0$ is orthogonal to any vector that is orthogonal to $\mathbf{y}_0$, consequently $D\mathbf{y}_0$ is parallel to $\mathbf{y}_0$. In other words, the vector $\mathbf{y}_0 \gg 0$ is really an eigenvector of D. As the eigenvalue belonging to $\mathbf{y}_0$ is λ, and λ is negative, we are able to conclude that if $\mathbf{x} \geqq \mathbf{0}$ does not imply the inequality $\mathbf{x}'C\mathbf{x} \geqq 0$ then the premise of Theorem 5. is not true either.

Remark: Since the above sufficiency condition is seemingly (formally) less strict than the necessary one, both are equally necessary and sufficient conditions.

6. PROPOSITION. *If* $\mathbf{x}'C\mathbf{x} \geqq 0$ *for every* $\mathbf{x} \geqq \mathbf{0}$, *and*

$$C = \begin{bmatrix} c_{11} & c_{12} \cdots c_{1n} \\ c_{21} & c_{22} \cdots c_{2n} \\ \vdots & \\ c_{n1} & c_{n2} \cdots c_{nn} \end{bmatrix},$$

then $c_{ii}=0$ *involves that every element in the i-th row (and in the i-th column) of C is nonnegative.*

PROOF. Let us assume, on the contrary, that $c_{ii}=0$ and $c_{ij}<0$. Let $\mathbf{x} \geqq \mathbf{0}$ be a vector for which $x_j=1$ and $x_l=0$ for $l \neq i$ and $l \neq j$. Then

$$\mathbf{x}'C\mathbf{x} = 2c_{ij}x_i + c_{jj} < 0$$

provided x_i is sufficiently great.

7. PROPOSITION. *If* $\mathbf{x}'C\mathbf{x} \geqq 0$ *for every* $\mathbf{x} \geqq \mathbf{0}$ *and C has no positive elements outside of its main diagonal, then* $\mathbf{x}'C\mathbf{x}$ *is either positive definite or positive semidefinite.*

PROOF. Assume that $\mathbf{x}'C\mathbf{x}<0$ where $\mathbf{x}'=[x_1, x_2, \ldots, x_n]$. Let

$$\mathbf{y}'=[|x_1|, |x_2|, \ldots, |x_n|].$$

Clearly $\mathbf{y} \geqq \mathbf{0}$ and

$$\mathbf{y}'C\mathbf{y} \leqq \mathbf{x}'C\mathbf{x} < 0.$$

8. PROPOSITION. *Let C be a* 2×2 *symmetric matrix, written*

$$C = \begin{bmatrix} a & c \\ c & b \end{bmatrix}.$$

In this case $\mathbf{x}'C\mathbf{x} \geqq 0$ *for every* $\mathbf{x} \geqq \mathbf{0}$ *if and only if* $a \geqq 0$, $b \geqq 0$ *and* $c \geqq -\sqrt{ab}$.

PROOF. a) Necessity: If $a<0$ then for $\mathbf{x}'=[1, 0]$, $\mathbf{x}'C\mathbf{x}=a<0$. The same thing can be said with $\mathbf{x}'=[0, 1]$ for $b<0$. If $a \geqq 0$ and $b \geqq 0$ then, considering Proposition 7., it is necessary that either $c \geqq 0$ or $\mathbf{x}'C\mathbf{x}$ be positive definite or positive semidefinite, i.e. $-\sqrt{ab} \leqq c \leqq \sqrt{ab}$.

b) Sufficiency: Let $a \geqq 0$, $b \geqq 0$ and $c \geqq -\sqrt{ab}$. Then

$$C = \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} a & -\sqrt{ab} \\ -\sqrt{ab} & b \end{bmatrix} + \begin{bmatrix} 0 & c+\sqrt{ab} \\ c+\sqrt{ab} & 0 \end{bmatrix}. \tag{1}$$

In this manner C has been decomposed to a sum of a matrix belonging to a positive definite or semidefinite quadratic form and a matrix with only nonnegative elemenst. According to the 1. Proposition $\mathbf{x}'C\mathbf{x} \geqq 0$ for every $\mathbf{x} \geqq \mathbf{0}$.

9. THEOREM. *If C is a symmetric matrix with not more than* 3 *rows and* 3 *columns, and* $\mathbf{x}'C\mathbf{x} \geqq 0$ *for every* $\mathbf{x} \geqq \mathbf{0}$, *then*

$$C = A + B \tag{2}$$

where A is a matrix associated with a positive definite or positive semidefinite quadratic form, and B is a symmetric matrix with only nonnegative elements. Furthermore, a decomposition of type (2) *can always be carried out such that whenever an element in a position of C is nonnegative, then the element in the corresponding position of B is zero.*

PROOF. a) for $n=2$, the presentation (1) is adequate in all the cases where $c>0$. In the remaining cases, according to Proposition 7, $\mathbf{x}'C\mathbf{x}$ itself is a positive definite or semidefinite quadratic form, so $C = C+0$ can be considered to be a presentation of form (2).

b) Let $n=3$, and

$$C = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

Assume that $\mathbf{x}'C\mathbf{x} \geqq 0$ for every $\mathbf{x} \geqq \mathbf{0}$. By means of Propositions 3. and 8. we can imply that

$$\begin{aligned} a &\geqq 0, \quad d \geqq -\sqrt{ab}, \\ b &\geqq 0, \quad e \geqq -\sqrt{ac}, \qquad (3) \\ c &\geqq 0, \quad f \geqq -\sqrt{bc}. \end{aligned}$$

If C has a zero element in its main diagonal, say $c=0$ then, according to Proposition 6., $e \geqq 0$ and $f \geqq 0$, thus if for the matrix

$$C_1 = \begin{bmatrix} a & d \\ d & b \end{bmatrix}$$

we have $C_1 = A_1 + B_1$ as a decomposition of form (2), the appropriate decomposition of C is

$$C = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{C_1} & \begin{array}{c} e \\ f \end{array} \\ \hline e & f & c \end{array}\right] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_1} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|c} \multicolumn{2}{c|}{B_1} & \begin{array}{c} e \\ f \end{array} \\ \hline e & f & c \end{array}\right]. \qquad (4)$$

Now we are allowed to suppose that $a>0$, $b>0$ and $c>0$. If $d \leqq 0$, $e \leqq 0$ and $f \leqq 0$ then, because of Proposition 6., $C = C+0$ is a pertinent decomposition. If at least two numbers of d, e and f are nonnegative, say $e \geqq 0$ and $f \geqq 0$, then the decomposition (4) can be applied. We are only left to examine the case where two numbers of d, e, f are negative and the third one is positive. Let $d<0$, $e<0$, and $f>0$. Then the first two elements in the series of the descending principal minors of C are positive, the third element of this series is nonnegative. If the last, fourth element were positive or zero, then C would be positive semidefinite. Now, in the case of

$$f = \frac{de - \sqrt{(ab-d^2)(ac-e^2)}}{a} = f_0 \qquad (5)$$

the determinant of C equals zero, consequently C is positive semidefinite. For this reason, so long as $f \geqq f_0$, a pertinent decomposition of the matrix C is

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} = \begin{bmatrix} a & d & e \\ d & b & f_0 \\ e & f_0 & c \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f-f_0 \\ 0 & f-f_0 & 0 \end{bmatrix}. \tag{6}$$

Finally we can now demonstrate that in the case of $f<f_0$ there exists a vector $\mathbf{x} \geqq \mathbf{0}$ for which $\mathbf{x}'C\mathbf{x}<0$. To prove this, let us examine the polinomial

$$\det C = abc + 2def - af^2 - be^2 - cd^2$$

as a function of f. Clearly, the smaller root of this quadratic polinomial of f is f_0. Thus $\det C<0$ if $f<f_0$, consequently C has a negative eigenvalue λ.

From the (a priori redundant) system of equations defining the eigenvalues belonging to λ, leaving out the first equation, we get

$$\begin{bmatrix} d & b-\lambda & f \\ e & f & c-\lambda \end{bmatrix} \mathbf{x} = \mathbf{0}. \tag{7}$$

One solution of system (7) consists of the vector $\mathbf{x}_0$ whose components are

$$\begin{vmatrix} b-\lambda & f \\ f & c-\lambda \end{vmatrix}, \quad \begin{vmatrix} f & d \\ c-\lambda & e \end{vmatrix}, \quad \begin{vmatrix} d & b-\lambda \\ e & f \end{vmatrix}.$$

From the hypothetic inequalities

$$a>0, \quad d<0, \quad d^2 \leqq ab,$$
$$b>0, \quad e<0, \quad e^2 \leqq ac,$$
$$c>0, \quad 0<f<f_0,$$
$$\lambda<0,$$

and from $f_0 \leqq \dfrac{de}{a}$, which is obviously involved in (5), we get

$$\begin{vmatrix} b-\lambda & f \\ f & c-\lambda \end{vmatrix} = (b-\lambda)(c-\lambda)-f^2 > bc-f_0^2 \geqq bc-\frac{d^2e^2}{a^2} \geqq bc-\frac{ab\cdot ac}{a^2} = 0,$$

$$\begin{vmatrix} f & d \\ c-\lambda & e \end{vmatrix} = ef-d(c-\lambda) > ef_0-cd \geqq e\cdot\frac{de}{a}-cd = \frac{d}{a}\cdot e^2-cd \geqq \frac{d}{a}\cdot ac-cd = 0,$$

$$\begin{vmatrix} d & b-\lambda \\ e & f \end{vmatrix} = df-e(b-\lambda) > df_0-be \geqq d\cdot\frac{de}{a}-be = \frac{e}{a}\cdot d^2-be \geqq \frac{e}{a}\cdot ab-be = 0,$$

that is $\mathbf{x}_0 \gg 0$. Taking into account $\mathbf{x}_0'C\mathbf{x}_0 = \lambda\mathbf{x}_0'\mathbf{x}_0<0$, we have got the required contradiction.

Part II.

1. PROPOSITION. *If the quadratic form* $\mathbf{x}'C\mathbf{x}$ *is positive definite or positive semidefinite, then* $\mathbf{x}'C\mathbf{x}$ *is convex in the whole space.*

PROOF. $\mathbf{x}'C\mathbf{x}$ is convex in the whole space if and only if it is convex on any straight line of the space. $\mathbf{x}'C\mathbf{x}$ is convex on the line consisting of the points $x+u\mathbf{y}$ (where $\mathbf{x}$ and $\mathbf{y}$ are fixed vectors, $\mathbf{y}\neq 0$ and the parameter u ranges over the set of all real numbers) if

$$(\mathbf{x}+u\mathbf{y})'C(\mathbf{x}+u\mathbf{y})$$

is a convex function of u. At last,

$$(\mathbf{x}+u\mathbf{y})'C(\mathbf{x}+u\mathbf{y}) = (\mathbf{y}'C\mathbf{y})u^2+2(\mathbf{x}'C\mathbf{y})u+\mathbf{x}'C\mathbf{x}$$

is a convex function of u if $\mathbf{x}'C\mathbf{x}$ is positive definite or semidefinite.

2. PROPOSITION. *If* $\mathbf{x}'C\mathbf{x}$ *is neither positive definite nor positive semidefinite (or, in other words,* C *has one or more negative eigenvalues) then for any vector* $\mathbf{x}$ *there exists a line containing the point* $\mathbf{x}$ *such that* $\mathbf{x}'C\mathbf{x}$ *is strictly concave on it.*

PROOF. Let $\mathbf{y}$ be an eigenvector of C having negative eigenvalue. Then for arbitrary vector $\mathbf{x}$,

$$(\mathbf{x}+u\mathbf{y})'C(\mathbf{x}+u\mathbf{y})$$

is a strictly concave function of u, that is $\mathbf{x}'C\mathbf{x}$ is strictly concave on a line containing the point $\mathbf{x}$.

3. *Definition.* A function $f(\mathbf{x})$ is said to be quasiconvex in a convex set K if and only if

$$\mathbf{x}_1\in K,\quad \mathbf{x}_2\in K,\quad \mathbf{x} = \lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2 \qquad (0<\lambda<1)$$

imply that

$$f(\mathbf{x}) \leqq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}.$$

4. *Definition.* A given function $f(x)$ is said to be quasiconvex at a point $\mathbf{x}_0$ if and only if there exists a neighbourhood S of $\mathbf{x}_0$ in which $f(\mathbf{x})$ is defined, and $\mathbf{x}_1\in S$, $\mathbf{x}_2\in S$, $\mathbf{x}_0 = \lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2$ $(0<\lambda<1)$ imply

$$f(\mathbf{x}_0) \leqq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}.$$

5. PROPOSITION. *A quadratic form* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex in a convex open set* K *if and only if* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex at every point of* K.

6. PROPOSITION. *If the function* $f(\mathbf{x})$ *is continuous in a convex set* K, *and* $f(\mathbf{x})$ *is quasiconvex in the interior of* K, *which is assumed to be non-empty, then* $f(\mathbf{x})$ *is quasiconvex in* K.

7. COROLLARY. A quadratic form $\mathbf{x}'C\mathbf{x}$ is quasiconvex in a convex set K, which has a non-empty interior, if and only if $\mathbf{x}'C\mathbf{x}$ is quasiconvex at any interior point of K.

8. PROPOSITION. *A quadratic form* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex at a point* $\mathbf{x}_0$ *if and only if arbitrary line, containing the point* $\mathbf{x}_0$ *has another point* $\mathbf{x}\neq\mathbf{x}_0$ *on it, for which*

$$\mathbf{x}'C\mathbf{x} \geqq \mathbf{x}_0'C\mathbf{x}_0.$$

9. Proposition. *A quadratic form* $\mathbf{x}'C\mathbf{x}$ *is not quasiconvex at a point* $\mathbf{x}_0$ *if and only if there exists a vector* $\mathbf{y}$ *that satisfies*

$$\mathbf{x}_0'C\mathbf{y} = 0 \quad \textit{and} \quad \mathbf{y}'C\mathbf{y} < 0.$$

Proof. a) If there exists a vector $\mathbf{y}$ satisfying $\mathbf{x}_0'C\mathbf{y}=0$ and $\mathbf{y}'C\mathbf{y}<0$, then

$$(\mathbf{x}_0+u\mathbf{y})'C(\mathbf{x}_0+u\mathbf{y}) = \mathbf{x}_0'C\mathbf{x}_0+u^2(\mathbf{y}'C\mathbf{y}) < \mathbf{x}_0'C\mathbf{x}.$$

for every $u\neq 0$, thus using Proposition 8., we obtain that $\mathbf{x}'C\mathbf{x}$ cannot be quasiconvex at $\mathbf{x}_0$.

b) If $\mathbf{x}'C\mathbf{x}$ is not quasiconvex at $\mathbf{x}_0$, then by Proposition 8. there exists a vector $\mathbf{y}$ which satisfies

$$(\mathbf{x}_0+u\mathbf{y})'C(\mathbf{x}_0+u\mathbf{y}) < \mathbf{x}_0'C\mathbf{x}_0$$

for every $u\neq 0$, that is

$$(\mathbf{y}'C\mathbf{y})u^2+2(\mathbf{x}_0'C\mathbf{y})u < 0$$

for every nonzero u. This is impossible, unless $\mathbf{y}'C\mathbf{y}<0$ and $\mathbf{x}_0'C\mathbf{y}=0$.

10. Theorem. *If* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex at a point* $\mathbf{x}_0$, *then* C *has no or only one invariant direction with the corresponding eigenvalue being negative.*

Proof. Let $\mathbf{s}$, $\mathbf{t}$ be assumed to be orthogonal unit-eigenvectors having the eigenvalues $\lambda<0$ and $\mu<0$ respectively. Then for any nonzero vector $\mathbf{y} = \alpha\mathbf{s}+\beta\mathbf{t}$ in the plane spanned by $\mathbf{s}$ and $\mathbf{t}$,

$$\mathbf{y}'C\mathbf{y} = \alpha^2\lambda+\beta^2\mu < 0.$$

Let us attempt choosing the values of α and β such that $\mathbf{x}_0'C\mathbf{y}=0$. To do so, we have to solve the homogeneous linear equation of two unknowns

$$\mathbf{x}_0'C\mathbf{y} = \alpha\lambda\mathbf{x}_0'\mathbf{s}+\beta\mu\mathbf{x}_0'\mathbf{t} = 0$$

which clearly always has a nontrivial solution. Thus there exists a vector $\mathbf{y}$ having the property that $\mathbf{y}'C\mathbf{y}<0$ and $\mathbf{x}_0'C\mathbf{y}=0$. Consequently, taking into account Proposition 9, $\mathbf{x}'C\mathbf{x}$ cannot be quasiconvex at $\mathbf{x}_0$.

11. Theorem. *Let us assume that exactly one of the invariant directions of* C *has nonnegative eigenvalue. In this case*

a) *If* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex at a point* $\mathbf{x}$, *then* $\mathbf{x}'C\mathbf{x}\leqq 0$.

b) *If* $\mathbf{x}'C\mathbf{x}<0$, *then* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex at* $\mathbf{x}$.

Proof. Choosing a coordinate system of principal axes, our assertions are to be proved only for diagonalmatrices.

Let

$$\mathbf{x}'C\mathbf{x} = \sum_{i=1}^{n}\lambda_i x_i^2 \quad \text{where} \quad \lambda_1<0,\ \lambda_2\geqq 0,\ \lambda_3\geqq 0,\ \ldots,\ \lambda_n\geqq 0.$$

a) Suppose that $\sum_{i=1}^{n} \lambda_i x_i^2 > 0$. Then consider

$$\mathbf{y}' = \begin{cases} \left[\dfrac{-\sum_{i=2}^{n} \lambda_i x_i^2}{\lambda_1 x_1}, x_2, x_3, \dots, x_n \right] & \text{if} \quad x_1 \neq 0 \\ [1, 0, 0, \dots, 0] & \text{if} \quad x_1 = 0. \end{cases}$$

Obviously $\mathbf{x}'C\mathbf{y} = 0$ and

$$\mathbf{y}'C\mathbf{y} = \begin{cases} \dfrac{\left(-\lambda_1 x_1^2 + \sum_{i=1}^{n} \lambda_i x_i^2\right)\left(\sum_{i=1}^{n} \lambda_i x_i^2\right)}{\lambda_1 x_1^2} < 0 & (x_1 \neq 0) \\ \lambda_1 < 0 & (x_1 = 0) \end{cases}$$

thus according to Proposition 9, $\mathbf{x}'C\mathbf{x}$ cannot be quasiconvex at the point $\mathbf{x}$.

b) Let $\mathbf{x}$ be such vector that $\sum_{i=1}^{n} \lambda_i x_i^2 < 0$. In this case the proof has to be carried out for nonzero $\lambda_1, \lambda_2, \dots, \lambda_n$ only. In the opposite case we are allowed to examine the quadratic form of smaller size, obtained by leaving out the identically zero terms. Now let $\lambda_1 < 0$ and $\lambda_i > 0$ for $i = 2, 3, \dots, n$. Let us consider the set

$$H = \left\{\mathbf{x}: \sum_{i=1}^{n} \lambda_i x_i^2 < 0\right\},$$

H has no point in common with the hyperplane $x_1 = 0$. However, every hyperplane, parallel with the hyperplane $x_1 = 0$ intersects H in a bounded set. Consequently no straight line is contained in H. This fact involves that on any line the maximum of $\mathbf{x}'C\mathbf{x}$ is nonnegative, thus because of Proposition 8, $\mathbf{x}'C\mathbf{x}$ is quasiconvex at the points of H.

12. THEOREM. *Let* $\mathbf{x}'C\mathbf{x} = \sum_{i=1}^{n} \lambda_i x_i^2$ *where* $\lambda_1 < 0$, $\lambda > 0$ *for* $i = 2, 3, \dots, m$, $\lambda_j = 0$ *for* $j = m+1, m+2, \dots, n$ $(1 \leqq m \leqq n)$. *Let us consider the sets*

$$F_+ = \left\{\mathbf{x}: \sum_{i=1}^{n} \lambda_i x_i^2 \leqq 0,\ x_1 \geqq 0\right\},$$

$$F_- = \left\{\mathbf{x}: \sum_{i=1}^{n} \lambda_i x_i^2 \leqq 0,\ x_1 \leqq 0\right\}.$$

Then we state the following seven assertions.

a) F_+ *and* F_- *are convex closed cones.*

b) *If* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex at a point* $\mathbf{x}$, *then* $\mathbf{x} \in F_+ \cup F_-$.

c) *If* $\mathbf{x}$ *is an interior point of* F_+ *or* F_- *then* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex at* $\mathbf{x}$.

d) $\mathbf{x}'C\mathbf{x}$ *is quasiconvex in* F_+ *and in* F_-.

e) *With the assumption of* $\mathbf{x} \in F_+ \cup F_-$, $\mathbf{x}'C\mathbf{x}$ *is not quasiconvex at* $\mathbf{x}$ *if and only if* $\mathbf{x} \in F_+ \cap F_-$ *that is if* $x_1 = x_2 = \dots = x_n = 0$.

f) $\mathbf{x}'C\mathbf{x}$ *is quasiconvex on a convex set* K, *which has a non-empty interior if and only if* $K \subset F_+$ *or* $K \subset F_-$.

g) *If for the convex set* K, *which has a non-empty interior*, $K \subset F_+ \cup F_-$ *then* $K \subset F_+$ *or* $K \subset F_-$.

PROOF. a) It is obvious that F_+ and F_- are closed. The proof of that F_+ is a convex cone is as follows: Let $\mathbf{x} \in F_+$ and $\mathbf{y} \in F_+$, that is

$$x_1 \geqq 0, \quad \sum_{i=1}^{n} \lambda_i x_i^2 \leqq 0,$$

$$y_1 \geqq 0, \quad \sum_{i=1}^{n} \lambda_i y_i^2 \leqq 0.$$

Let furthermore $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$ where α and β are nonnegative real numbers. Then

$$z_1 = \alpha x_1 + \beta y_1 \geqq 0$$

and

$$\sum_{i=1}^{n} \lambda_i z_i^2 = \lambda_1(\alpha x_1 + \beta y_1)^2 + \sum_{i=2}^{n} \lambda_i(\alpha x_i + \beta y_i)^2 =$$

$$= \lambda_1(\alpha x_1 + \beta y_1)^2 + \alpha^2 \sum_{i=2}^{n} \lambda_i x_i^2 + \beta^2 \sum_{i=2}^{n} \lambda_i y_i^2 + 2\alpha\beta \sum_{i=2}^{n} (\sqrt{\lambda_i}\, x_i)(\sqrt{\lambda_i}\, y_i) \leqq$$

$$\leqq \lambda_1(\alpha x_1 + \beta y_1)^2 + \alpha^2 \sum_{i=2}^{n} \lambda_i x_i^2 + \beta^2 \sum_{i=2}^{n} \lambda_i y_i^2 + 2\alpha\beta \left(\sum_{i=2}^{n} \lambda_i x_i^2\right)^{1/2} \left(\sum_{i=2}^{n} \lambda_i y_i^2\right)^{1/2} \leqq$$

$$\leqq \lambda_1(\alpha x_1 + \beta y_1)^2 - \alpha^2 \lambda_1 x_1^2 - \beta^2 \lambda_1 y_1^2 + 2\alpha\beta(-\lambda_1 x_1^2)^{1/2}(-\lambda_1 y_1^2)^{1/2} = 0,$$

therefore $\mathbf{z} \in F_+$.

The assertions in b) and c) are nothing else but the two assertions of Theorem 11.

Assertion d) follows from c) and from Proposition 7.

e) As F_+ and F_- have no interior point in common, if $\mathbf{x}$ is an interior point of one of F_+ or F_-, the assertion e) is not more than simple repetition of the assertion c).

If $\mathbf{x} \in F_+ \cap F_-$ then taking $\mathbf{y}' = \mathbf{e}_1' = [1, 0, \ldots, 0]$, $\mathbf{x}'C\mathbf{y} = 0$ and $\mathbf{y}'C\mathbf{y} = \lambda_1 < 0$, therefore by Proposition 9, $\mathbf{x}'C\mathbf{x}$ is not quasiconvex at $\mathbf{x}$.

If $\mathbf{x}$ is a boundary point of F_+ and $\mathbf{x}'C\mathbf{x}$ is not quasiconvex at $\mathbf{x}$ then according to Proposition 9. there exists a vector $\mathbf{y}$ which satisfies $\mathbf{x}'C\mathbf{y} = 0$ and $\mathbf{y}'C\mathbf{y} < 0$. Then we can write

$$0 = \mathbf{x}'C\mathbf{y} = \lambda_1 x_1 y_1 + \sum_{i=2}^{n} \lambda_i x_i y_i \leqq \lambda_1 x_1 y_1 + \left(\sum_{i=2}^{n} \lambda_i x_i^2\right)^{1/2} \left(\sum_{i=2}^{n} \lambda_i y_i^2\right)^{1/2} \leqq$$

$$\leqq \lambda_1 x_1 y_1 + (-\lambda_1 x_1^2)^{1/2}(-\lambda_1 y_1^2)^{1/2} = 0,$$

therefore the equality sign is valid at every place, thus because of

$$\sum_{i=2}^{n} \lambda_i y_i^2 < -\lambda_1 y_1^2$$

we get

$$\sum_{i=2}^{n} \lambda_i x_i^2 = -\lambda_1 x_1^2 = 0,$$

that is $x_1 = x_2 = \cdots = x_m = 0$.

f) The sufficiency follows from d). To prove the necessity let us assume that $\mathbf{x}'C\mathbf{x}$ is quasiconvex in the convex set K, which has a non-empty interior. Because of the Proposition 7 and of b) all the interior points of K are contained in F_+ or in F_-, consequently $K \subset F_+ \cup F_-$. Because of e), K has no interior point in the set $F_+ \cap F_-$, thus $K \subset F_+$ or $K \subset F_-$.

g) If the convex set K has a non-empty interior, and $K \subset F_+ \cup F_-$, then the interior of K is contained in the union of the interior of F_+ and of that of F_-. Since the interior of K is connected, furthermore since F_+ and F_- have no interior point in common, the interior of K is contained in F_+ or F_-, and consequently so is K.

13. *Corollary*. A quadratic form $\mathbf{x}'C\mathbf{x}$ is quasiconvex in a convex set K, which is assumed to have a non-empty interior if and only if

a) either all the eigenvalues of C are nonnegative[2]

b) or exactly one of the invariant directions of C has nonnegative eigenvalue and $\mathbf{x}'C\mathbf{x} \leqq 0$ for every point $\mathbf{x}$ in K.

14. *Corollary*. If $\mathbf{x}'C\mathbf{x}$ is quasiconvex at $\mathbf{x}_0$ and $\mathbf{x}'C\mathbf{x} \geqq 0$ in a neighbourhood of $\mathbf{x}_0$ then $\mathbf{x}'C\mathbf{x}$ is positive definite or positive semidefinite.

15. THEOREM. *An infinite quadratic form* $\mathbf{x}'C\mathbf{x}$ *is quasiconvex in the positive orthant if and only if exactly one of the invariant directions of C has nonnegative eigenvalue and the elements of C are all nonpositive.*

The proof of the necessity of this theorem can be found in [5] (Theorems 1. and 4.), the sufficiency follows from Proposition 13, given here in Part II.

Acknowledgement: I am indebted and wish to express my thanks to Professor A. PRÉKOPA for encouragement, advices, and first of all for presenting the problems dealt with in this paper.

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(Received November 25, 1970.)

[2] That is $\mathbf{x}'C\mathbf{x}$ is positive definite or positive semidefinite.

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 21—24.

NOTES ON THE LAW OF THE ITERATED LOGARITHM

by

S. TAKAHASHI

§ 1. Introduction. The sequence of real measurable functions $\{\varphi_n(t)\}$ defined in the interval [0, 1], is called an equinormed strongly multiplicative system (ESMS) if

$$\begin{cases} \int_0^1 \varphi_n(t)\,dt = 0, \quad \int_0^1 \varphi_n^2(t)\,dt = 1 \qquad (n=1,2,\ldots) \\ \int_0^1 \prod_{m=1}^{k} \varphi_{n_m}^{\alpha_m}(t)\,dt = \prod_{m=1}^{k} \int_0^1 \varphi_{n_m}^{\alpha_m}(t)\,dt \quad (n_1<n_2<\cdots<n_k,\ k=1,2,\ldots), \end{cases}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are equal to 1 or 2.

F. MÓRICZ and P. RÉVÉSZ have considered the law of the iterated logarithm for a uniformly bounded ESMS (cf. [1] and [2]). In this note we discuss the law for a uniformly bounded multiplicative system $\{\varphi_n(t)\}$, that is, $\{\varphi_n(t)\}$ satisfies the conditions:

$$(1.1)\qquad \begin{cases} \|\varphi_n(t)\|_\infty \leqq K & (n=1,2,\ldots), \\ \int_0^1 \prod_{m=1}^{k} \varphi_{n_m}(t)\,dt = 0 & (n_1<n_2<\cdots<n_k;\ k=1,2,\ldots). \end{cases}$$

THEOREM 1. *If $\{\varphi_n(t)\}$ satisfies (1. 1), then we have*

$$\overline{\lim_{N\to\infty}}\, (2N \log\log N)^{-1/2} \sum_{n=1}^{N} \varphi_n(t) \leqq K \quad a.e. \,.$$

THEOREM 2. *Let $\{\varphi_n(t)\}$ satisfies (1. 1) and for any $\theta>1$,*

$$(1.2)\qquad \lim_{k\to\infty} \theta^{-k} \sum_{n=1}^{[\theta^k]} \varphi_n^2(t) = C^2 \quad a.e. \,, *$$

where C is a positive constant. Then we have

$$(1.3)\qquad \overline{\lim_{N\to\infty}}\, (2N \log\log N)^{-1/2} \sum_{n=1}^{N} \varphi_n(t) \leqq C \quad a.e. \,.$$

* For any real number a, $[a]$ denotes its integral part.

If $\{\varphi_n(t)\}$is ESMS, then $C=1$. Further considering the sequence $\{\sin 2\pi n_k t\}$, $n_{k+1}/n_k > 3$, the constant C in (1. 3) is the best possible one [3].

§ 2. Some Lemmas. In the following let $\{\varphi_n(t)\}$ satisfy the condition (1. 1) and let us put

$$S(t; N, M) = \sum_{n=N+1}^{N+M} \varphi_n(t), \quad N \geqq 0 \text{ and } M \geqq 1.$$

Since $|x| \leqq 1/2$ implies that $e^x \leqq (1+x) \exp\left(\frac{x^2}{2}+|x|^3\right)$, it is seen that if $|\lambda K|^3 M \leqq 1$, then for $M \geqq 8$,

$$\text{(2. 1)} \quad \exp\{\lambda S(t; N, M)\} \leqq \prod_{n=N+1}^{N+M} \left\{(1+\lambda\varphi_n(t)) \exp\left(\frac{\lambda^2}{2}\varphi_n^2(t)+|\lambda\varphi_n(t)|^3\right)\right\}$$

$$\leqq A \exp(\lambda^2 K^2 M/2) \prod_{n=N+1}^{N+M} (1+\lambda\varphi_n(t)),$$

and

$$\text{(2. 2)} \qquad \exp\left\{\lambda \sum_{n=1}^{M} \varphi_n(t) - \frac{\lambda^2}{2} \sum_{n=1}^{M} \varphi_n^2(t)\right\} \leqq 2A \prod_{n=1}^{M} (1+\lambda\varphi_n(t)),$$

where A is a constant independent of λ, M, N.

LEMMA 1. *If* $0 < y \leqq M^{1/6}/\sqrt{2}$, *then we have*

$$\left|\{t; |S(t; N, M)| \geqq yK\sqrt{2M}\}\right| \leqq Ae^{-y^2} \qquad (M \geqq 8,\ N \geqq 0).\ *$$

PROOF. Putting $\lambda = \sqrt{2}yK^{-1}M^{-1/2}$, then $|\lambda K|^3 M \leqq 1$. Hence, by (2. 1) and the multiplicative orthogonality of $\{\varphi_n(t)\}$,

$$\int_0^1 \exp\{\lambda|S(t; N, M)|\}dt \leqq \int_0^1 \exp\{\lambda S(t; N, M)\}\,dt + \int_0^1 \exp\{-\lambda S(t; N, M)\}\,dt$$

$$\leqq A \exp(\lambda^2 K^2 M/2)\left\{\int_0^1 \prod_{n=N+1}^{N+M} (1+\lambda\varphi_n(t))\,dt + \int_0^1 \prod_{n=N+1}^{N+M} (1-\lambda\varphi_n(t))\,dt\right\}$$

$$\leqq 2A \exp(\lambda^2 K^2 M/2).$$

Therefore, we obtain

$$\left|\{t; |S(t; N, M)| \geqq yK\sqrt{2M}\}\right| \leqq 2A \exp\left(-\lambda yK\sqrt{2M}+\lambda^2 K^2 M/2\right) = 2Ae^{-y^2}.$$

LEMMA 2. *There exists a constant B such that*

$$\left|\{t; \max_{1 \leqq m \leqq M} |S(t; N, m)| \geqq B\sqrt{M \log\log M}\}\right| \leqq 7Ae^{-2\log\log M},$$

for any $N \geqq 0$ *and* $M \geqq 8$.

---

* For any measurable set E, $|E|$ denotes its Lebesgue measure

PROOF. Let l be a positive integer such that $2^{l-1} < M \leqq 2^l$. Then we have

$$(2.3)\qquad \max_{1\leqq m\leqq M}|S(t;N,m)| \leqq \max_{1\leqq m\leqq 2^l}|S(t;N,m)|$$

$$\leqq \sum_{s=0}^{[l/2]} \max_{0\leqq p<2^s}|S(t;N+p2^{l-s},2^{l-s})| + \max_n \max_{0\leqq m\leqq 2^{[l/2]+1}}|S(t;n,m)|.$$

Since $\sqrt{4\log(2^s\log M)} < 2^{(l-s)/6}$ for $s \leqq [l/2]$, Lemma 1 implies that

$$\left|\left\{t; |S(t;N+p2^{l-s},2^{l-s})| \geqq K\sqrt{2^{l-s+2}\log(2^s\log M)}\right\}\right|$$
$$\leqq 2A\cdot 2^{-2s}e^{-2\log\log M}, \quad \text{for} \quad s\leqq[l/2] \quad \text{and} \quad l-s\geqq 4.$$

Putting $E = \bigcup_{s=0}^{[l/2]}\left\{t; \max_{0\leqq p\leqq 2^s}|S(t;N+p2^{l-s},2^{l-s})| \geqq K\sqrt{2^{l-s+2}\log(2^s\log M)}\right\}$, we have

$$|E| \leqq 2A\sum_{s=0}^{[l/2]}(2^s+1)2^{-2s}e^{-2\log\log M} \leqq 7Ae^{-2\log\log M}.$$

If $t\notin E$, then (2. 3) implies that

$$\max_{1\leqq m\leqq M}|S(t;N,m)| \leqq \sum_{s=0}^{[l/2]} K\sqrt{2^{l-s+2}\log(2^s\log M)} + 2K2^{l/2}$$

$$< B\sqrt{M\log\log M}, \text{ for some positive constant } B.$$

§ 3. Proof of the Theorems

(i) Let ε be any given positive number. Take a number θ such that $1<\theta<1+\varepsilon^2/2$ Then we have, by Lemma 2,

$$\left|\left\{t; \max_{1\leqq m\leqq\theta^{k+1}-\theta^k}|S(t;[\theta^k],m)| \geqq B\sqrt{([\theta^{k+1}]-[\theta^k])\log\log\theta^k}\right\}\right| = O(k^{-2}), \text{ as } k\to+\infty.$$

Since $[\theta^{k+1}]-[\theta^k] < 2[\theta^k](\theta-1) < \varepsilon^2[\theta^k]$ for $k>k_0$, we have

$$(3.1)\qquad \overline{\lim_{k\to\infty}} \max_{1\leqq m\leqq\theta^{k+1}-\theta^k}\frac{|S(t;[\theta^k],m)|}{\sqrt{[\theta^k]\log\log\theta^k}} \leqq \varepsilon B \quad \text{a.e.}\ .$$

(ii) From Lemma 1 we obtain

$$\left|\left(t; \sum_{n=1}^{[\theta^k]}\varphi_n(t) \geqq K\sqrt{2(1+\varepsilon)[\theta^k]\log\log\theta^k}\right)\right| = O(k^{-(1+\varepsilon)}), \quad \text{as} \quad k\to\infty.$$

Hence, we have

$$(3.2)\qquad \overline{\lim_{k\to\infty}}\frac{\sum_{n=1}^{[\theta^k]}\varphi_n(t)}{\sqrt{2[\theta^k]\log\log\theta^k}} \leqq K(1+\varepsilon), \quad \text{a.e.}\ .$$

Since ε is arbitrary, we can prove Theorem 1, by (3. 1) and (3. 2).

(iii) Considering (3. 1), for the proof of Theorem 2 it is sufficient to show that

$$\varlimsup_{k\to\infty} \frac{\sum_{n=1}^{[\theta^k]} \varphi_n(t)}{\sqrt{2[\theta^k]\log\log[\theta^k]}} \leqq C \quad \text{a.e.} \,. \tag{3.3}$$

From (2. 2) and the multiplicative orthogonality of $\{\varphi_n(t)\}$ it is seen that

$$\int_0^1 \exp\left\{\lambda \sum_{n=1}^{M} \varphi_n(t) - \frac{\lambda^2}{2} \sum_{n=1}^{M} \varphi_n^2(t)\right\} dt \leqq A.$$

Hence, by the Tchebyschev inequality, we have

$$\left|\left\{t;\ \sum_{n=1}^{M} \varphi_n(t) \geqq \frac{\lambda}{2} \sum_{n=1}^{M} \varphi_n^2(t) + y\right\}\right| \leqq Ae^{-\lambda y} \qquad (\lambda < \lambda^3 K^3 M < 1)$$

Putting $M_k = [\theta^k]$, $\lambda_k = \sqrt{2(\log\log M_k)/C^2 M_k}$ and $y_k = (1+\varepsilon)C\sqrt{M_k(\log\log M_k)/2}$, we have

$$\left|\left\{t;\ \sum_{n=1}^{M_k} \varphi_n(t) \geqq \frac{\lambda_k}{2} \sum_{n=1}^{M_k} \varphi_n^2(t) + y_k\right\}\right| = O(k^{-(1+\varepsilon)}), \quad \text{as} \quad k \to +\infty.$$

Hence for a.e. t, we can find a positive integer $k_0(t)$ such that

$$\sum_{n=1}^{M_k} \varphi_n(t) < \frac{\lambda_k}{2} \sum_{n=1}^{M_k} \varphi_n^2(t) + y_k, \quad \text{for} \quad k \geqq k_0(t).$$

On the other hand we have, by (1. 2)

$$\frac{\lambda_k}{2} \sum_{n=1}^{M_k} \varphi_n^2(t) + y_k \cong \left(1 + \frac{\varepsilon}{2}\right) C\sqrt{2[\theta^k]\log\log[\theta^k]}, \quad \text{a.e. } t.$$

Since ε is arbitrary, we can obtain (3. 3) from the above two relations.

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(Received December 15, 1970.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 25—26.

A NOTE TO A PAPER OF S. TAKAHASHI

by

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In this paper we show that Theorem 1. of [1] is the best possible in the following en se:

THEOREM. *There exists a multiplicative system* $\{\xi_i\}$ *for which*

$$|\xi_i| \leqq K \qquad (i=1, 2, \ldots; K>1)$$

$$\mathsf{D}^2(\xi_i) = 1 \qquad (i=1, 2, \ldots)$$

and

(1) $$\mathsf{P}\left\{\overline{\lim_{n\to\infty}} \frac{\xi_1+\xi_2+\cdots+\xi_n}{\sqrt{2n\log\log n}} = K\right\} > 0.$$

PROOF. Let $\{\Omega_1, \mathscr{S}_1, \mathsf{P}_1\}$ and $\{\Omega_2, \mathscr{S}_2, \mathsf{P}_2\}$ be probability spaces and define the probability space $\{\Omega, \mathscr{S}, \mathsf{P}\}$ as follows

$$\Omega = \Omega_1 + \Omega_2$$

$\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ i.e. $A \in \mathscr{S}$ if $A = A_1 + A_2$ where $A_1 \in \mathscr{S}_1$, $A_2 \in \mathscr{S}_2$, $\mathsf{P} = p\mathsf{P}_1 + q\mathsf{P}_2$ i.e. $\mathsf{P}(A) = p\mathsf{P}_1(A\Omega_1) + q\mathsf{P}_2(A\Omega_2)$
where

$$p = \frac{1}{2K^2}, \quad q = 1 - \frac{1}{2K^2}.$$

Further let $\zeta_1, \zeta_2, \ldots$ be a sequence of independent random variables defined on Ω_1 such that

$$\mathsf{P}\{\zeta_i = K\} = \mathsf{P}\{\zeta_i = -K\} = 1/2$$

and let $\eta_1, \eta_2, \ldots$ be a sequence of independent random variables defined on Ω_2 such that

$$\mathsf{P}\{\eta_i = L\} = \mathsf{P}\{\eta_i = -L\} = 1/2$$

where $L = \dfrac{K}{\sqrt{2K^2-1}}$.

Finally define the sequence $\{\xi_i\}$ on Ω such that

$$\xi_i(\omega) = \begin{cases} \zeta_i & \text{if } \omega \in \Omega_1 \\ \eta_i & \text{if } \omega \in \Omega_2 \end{cases}$$

Then clearly

(i) $\{\xi_i\}$ is a multiplicative system

(ii) $|\xi_i| \leqq K$ $(i=1, 2, \ldots)$

(iii)
$$D^2(\xi_i) = \frac{1}{2K^2} K^2 + \left(1 - \frac{1}{2K^2}\right) \frac{K^2}{2K^2-1} = 1 \qquad (i=1, 2, \ldots)$$

and by the law of the iterated logarithm for independent system we have

$$P\left\{\overline{\lim_{n\to\infty}} \frac{\xi_1+\xi_2+\cdots+\xi_n}{\sqrt{2n \log\log n}} = K|\Omega_1\right\} = 1$$

what proves (1).

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(Received May 20, 1971.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 27—42.

REMARKS ON A PAPER BY P. MEDGYESSY, A. RÉNYI, K. TETTAMANTI AND I. VINCZE

by

Á. PETHŐ

§ 0. Introduction and Summary

In the paper [1] referred to in the title a discrete diffusion process, where both time and position are discontinuous, was treated by the methods of Markov chains. The probability distribution of the displacement (position) of a particle (in the function of time) has been determined and proved to tend to the normal distribution for large times.

In § 1 of the present paper a non-Markovian treatment of a generalization of the discrete problem is given. The passage time of a particle (as the function of the position coordinate) is also defined and the discussion is focused on determing the displacement and passage time distributions and their asymptotic behaviour for large values of the time and position coordinate, respectively.

In § 2 the time-continuous analogue of the discrete process is discussed.

§ 1. Generalization of the discrete diffusion process

1. 1. *Preliminaries and statement of the problem.* Let us consider a sequence (cascade) of similar cells (stages), numbered by $m(=0, 1, \dots)$, containing two phases from which let one be fixed and the other flow from one cell into the other in the direction of increasing cell numbers. Let the volumes of the phases in the cells be independent of the cell number. Then, the motion of the mobile phase should be performed step by step, numbered by $n(=0, 1, \dots)$, in the following way: denoting the volume of this phase in one cell by V and by $r(=1, 2, \dots)$ a fixed positive integer, let a fraction of volume V/r be transferred during one step into the next cell. (If $r=1$, the simpler case treated in [1] is attained. If $r\to\infty$, the intermittent flow of the mobile phase becomes continuous, see § 2.) A solute with different solubilities in the phases is also present in the system, henceforth referred to as a *reactor*, supposing perfect mixing in both phases.

Let us now consider the progress of the nth step in the mth cell. Suppose that the time between any two consecutive steps is the same, say τ/r. Since during this time the fixed phase acts as a source (or sink), the amounts of the solute in the fixed and mobile phase depend, besides m and n, also on the time t, $0\leqq t\leqq\tau/r$. These amounts will be denoted by $X_m^{(n)}(t)$ and $Y_m^{(n)}(t)$; for $t=\tau/r$ (i.e. at the end of the step in question) we put

$$X_m^{(n)}(\tau/r) = X_m^{(n)} \quad\text{and}\quad Y_m^{(n)}(\tau/r) = Y_m^{(n)}.$$

Assuming the above (so-called *sorption*) process to be linear and reversible, for the rate of sorption we have

$$\frac{dX_m^{(n)}(t)}{dt} = k_1 Y_m^{(n)}(t) - k_2 X_m^{(n)}(t), \qquad 0 \leqq t \leqq \tau/r, \tag{1.1.1}$$

where k_1 and k_2 are positive constants. The limiting case, when $k_1 \to \infty$ and $k_2 \to \infty$ simultaneously but k_1/k_2 tends to a finite number, will be called the *equilibrium case*; then the quantities used later

$$p = \frac{k_1}{k_1 + k_2}, \quad q = 1 - p \tag{1.1.2}$$

have obviously their finite limits, too. Independently of the sorption process, the equation of conservation of matter must hold:

$$\frac{d}{dt}[X_m^{(n)}(t) + Y_m^{(n)}(t)] = 0, \qquad 0 \leqq t \leqq \tau/r. \tag{1.1.3}$$

The initial conditions to the system of differential equations (1. 1. 1) and (1. 1. 3) read, according to the intermittent flow in the reactor, as

$$X_m^{(n)}(0) = X_m^{(n-1)}, \quad Y_m^{(n)}(0) = \frac{r-1}{r} Y_m^{(n-1)} + \frac{1}{r} Y_{m-1}^{(n-1)}. \tag{1.1.4}$$

The solution of (1. 1. 1)—(1. 1. 3)—(1. 1. 4) is at $t = \tau/r$:

$$X_m^{(n)} = (1-Q) X_m^{(n-1)} + \frac{r-1}{r} P Y_m^{(n-1)} + \frac{1}{r} P Y_{m-1}^{(n-1)}, \tag{1.1.5}$$

$$Y_m^{(n)} = Q X_m^{(n-1)} + \frac{r-1}{r}(1-P) Y_m^{(n-1)} + \frac{1}{r}(1-P) Y_{m-1}^{(n-1)}, \tag{1.1.6}$$

where (see also (1. 1. 2))

$$P = p(1-\varepsilon), \quad Q = q(1-\varepsilon), \quad \varepsilon = e^{-(k_1+k_2)(\tau/r)}. \tag{1.1.7}$$

Consider the system before the 0th step to exist in the (-1)st one, and let, as the initial conditions,

$$X_m^{(-1)} = Y_m^{(-1)} = 0, \qquad m = 0, 1, \ldots \tag{1.1.8}$$

be defined. As for the boundary condition, we also define the (-1)st cell; the solute should be introduced in a single step into the reactor, assuming its total amount to be 1:

$$Y_{-1}^{(n)} = \begin{cases} r, & n = -1, \\ 0, & n = 0, 1, \ldots. \end{cases} \tag{1.1.9}$$

The system of difference equations (1. 1. 5)—(1. 1. 6) with the initial and boundary conditions (1. 1. 8)—(1. 1. 9) will constitute the basis of what follows.

1. 2. *Distribution of passage times and displacements, respectively.* Consider now a particle of the solute and the *time* (i.e. the step number) after which this particle has just left the mth cell. This time, denoted by $\vartheta_m(=0, 1, \ldots)$, will be called the *passage time*[1] of the particle with respect to the mth cell. Based on the physical nature of the problem, the probability distribution function of ϑ_m will be obviously

$$\mathsf{P}(\vartheta_m \leqq n) = \frac{\sum_{j=0}^{n} W_m^{(j)}}{\sum_{j=0}^{\infty} W_m^{(j)}}, \tag{1.2.1}$$

where the notation

$$W_m^{(j)} = \frac{1}{r} Y_m^{(j)}, \qquad j=0, 1, \ldots, n \tag{1.2.2}$$

has been introduced. Now we can prove the following

THEOREM 1. 1: *The denominator in* (1. 2. 1) *equals* 1 *and so the probability distribution of* ϑ_m *is* $W_m^{(n)}$.

PROOF. Let us determine the bivariate (double) generating functions of $X_m^{(n)}$ and $Y_m^{(n)}$. Generally we put

$$Z(z, w) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_m^{(n)} z^m w^n; \tag{1.2.3}$$

the bivariate generating function is related to the (simple) generating functions

$$Z^{(n)}(z) = \sum_{m=0}^{\infty} Z_m^{(n)} z^m, \quad Z_m(w) = \sum_{n=0}^{\infty} Z_m^{(n)} w^n \tag{1.2.4}$$

obviously as follows[2]:

$$Z(z, w) = \sum_{n=0}^{\infty} Z^{(n)}(z) w^n = \sum_{m=0}^{\infty} Z_m(w) z^m. \tag{1.2.5}$$

Multiplying (1. 1. 5)—(1. 1. 6) by $z^m w^n$ and summing up from $m=0$ to $m=\infty$ as well as from $n=0$ to $n=\infty$, taking (1. 1. 8)—(1. 1. 9) also into account, with the above notation we have

$$[1-(1-Q)w]X(z, w) - \frac{r-1+z}{r} PwY(z, w) = P,$$

$$QwX(z, w) + \left[-1 + \frac{r-1+z}{r}(1-P)w\right] Y(z, w) = P-1.$$

[1] Generally called first-passage time as well as waiting time in the theory of stochastic processes, also residence time in the theory of chemical reactors [2].

[2] Investigation of convergence concerning generating functions may be avoided while using generating functions in the sense of Mikusinski's operational calculus [3, 4].

The solution of this system of equations reads as

$$(1.2.6)\qquad X(z,w)=\frac{P}{1-\left[(1-Q)+(1-P)\dfrac{r-1+z}{r}\right]w+\varepsilon\dfrac{r-1+z}{r}w^2},$$

$$(1.2.7)\qquad Y(z,w)=\frac{1-P-\varepsilon w}{1-\left[(1-Q)+(1-P)\dfrac{r-1+z}{r}\right]w+\varepsilon\dfrac{r-1+z}{r}w^2}.$$

Putting now $w=1$ into (1. 2. 7) one obtains

$$W(z,1)=\frac{1}{r}Y(z,1)=\frac{1}{1-z},$$

that is

$$W_m(1)=\sum_{n=0}^{\infty}W_m^{(n)}=1.$$

Q.e.d.

We will now consider the displacement (position) of a particle of the solute at the time n and denote it by $\eta_n(=0,1,\dots)$. Based on the physical nature of the problem, the probability distribution function of η_n becomes obviously

$$(1.2.8)\qquad \mathsf{P}(\eta_n\leqq m)=\frac{\sum\limits_{k=0}^{m}T_k^{(n)}}{\sum\limits_{k=0}^{\infty}T_k^{(n)}},$$

where, for the total amount of the solute in the kth cell and nth step, the notation

$$(1.2.9)\qquad T_k^{(n)}=X_k^{(n)}+Y_k^{(n)},\qquad k=0,1,\dots,m$$

has been introduced.

THEOREM 1. 2: *The denominator in* (1. 2. 8) *equals* 1 *and so the probability distribution of* η_n *is* $T_m^{(n)}$.

PROOF. In view of (1. 2. 6)—(1. 2. 7)—(1. 2. 9) we have

$$(1.2.10)\qquad T(z,w)=\frac{1-\varepsilon w}{1-\left[(1-Q)+(1-P)\dfrac{r-1+z}{r}\right]w+\varepsilon\dfrac{r-1+z}{r}w^2}.$$

Putting $z=1$ into this equation one obtains

$$T(1,w)=\frac{1}{1-w},$$

that is

$$T^{(n)}(1)=\sum_{m=0}^{\infty}T_m^{(n)}=1.$$

Q.e.d.

Between the probability distributions of ϑ_m and η_n, i.e. between $W_m^{(n)}$ and $T_m^{(n)}$ the discrete analogue of the Fokker—Planck equation holds:

$$\sum_{j=0}^{n-1} W_m^{(j)} + \sum_{k=0}^{m} T_k^{(n)} = 1, \tag{1.2.11}$$

as it can be immediately justified in view of (1. 1. 5)—(1. 1. 6) and the notations (1. 2. 2)—(1. 2. 9).

1. 3. *Determination of the mean and variance of the passage time distribution.* With the notation (1. 2. 4) one has, on the basis of Theorem 1. 1, for the mean and variance of the passage times,

$$\mathsf{M}(\vartheta_m) = \left.\frac{dW_m(w)}{dw}\right|_{w=1}, \tag{1.3.1}$$

$$\mathsf{D}^2(\vartheta_m) - \mathsf{M}(\vartheta_m) + \mathsf{M}^2(\vartheta_m) = \left.\frac{d^2W_m(w)}{dw^2}\right|_{w=1}. \tag{1.3.2}$$

By means of the bivariate generating function of ϑ_m (see (1. 2. 5)) the above equations can be transformed into

$$\sum_{m=0}^{\infty} \mathsf{M}(\vartheta_m) z^m = \left.\frac{\partial W(z, w)}{\partial w}\right|_{w=1},$$

$$\sum_{m=0}^{\infty} [\mathsf{D}^2(\vartheta_m) - \mathsf{M}(\vartheta_m) + \mathsf{M}^2(\vartheta_m)] z^m = \left.\frac{\partial^2 W(z, w)}{\partial w^2}\right|_{w=1}.$$

In view of (1. 2. 2) and (1. 2. 7) we have (see also (1. 1. 2))

$$\sum_{m=0}^{\infty} \mathsf{M}(\vartheta_m) z^m = \left.\frac{1}{r}\,\frac{p+qz}{q(1-z)^2}\right|_{z=(r-1+z)/r},$$

$$\sum_{m=0}^{\infty} [\mathsf{D}^2(\vartheta_m) - \mathsf{M}(\vartheta_m) + \mathsf{M}^2(\vartheta_m)] z^m = \left.\frac{2}{r}\,\frac{P(1-Q) + P(2Q-\varepsilon)z + Q^2z^2}{Q^2(1-z)^3}\right|_{z=(r-1+z)/r}.$$

From these equations, for example by an expansion into partial fractions using the rule

$$\frac{1}{(1-z)^k} = \sum_{m=0}^{\infty} \binom{m+k-1}{k-1} z^m, \qquad k = 1, 2, \ldots, \tag{1.3.3}$$

the mean and variance of ϑ_m can be obtained as

$$\mathsf{M}(\vartheta_m) = \frac{r}{q}\,m + \frac{r-1+p}{q} \tag{1.3.4}$$

and

$$\mathsf{D}^2(\vartheta_m) = \frac{r(r-1+p)}{q^2}\,(m+1)\,\frac{1+\dfrac{1-r+p}{r-1+p}\,\varepsilon}{1-\varepsilon}. \tag{1.3.5}$$

Note that in the case of equilibrium, i.e. by putting $\varepsilon=0$ according to (1. 1. 7), the simpler relation holds:

$$\mathsf{D}^2(\vartheta_m) = \frac{r(r-1+p)}{q^2}(m+1). \tag{1. 3. 6}$$

Asymptotic relations for large values of m can be obtained from (1. 3. 4) and (1. 3. 5):

$$\mathsf{M}(\vartheta_m) \sim \frac{r}{q} m, \quad \mathsf{D}^2(\vartheta_m) \sim \frac{r(r-1+p)}{q^2} m \frac{1+\dfrac{1-r+p}{r-1+p}\varepsilon}{1-\varepsilon}. \tag{1. 3. 7}$$

Note finally, that if substituting $r=1$ into the above equations, the corresponding special relations valid for the simpler process described in [1] may be attained.

1. 4. *Determination of the mean and variance of the displacement distribution.* With the notation (1. 2. 4) one has, on the basis of Theorem 1. 2, for the mean and variance of the displacements,

$$\mathsf{M}(\eta_n) = \left.\frac{dT^{(n)}}{dz}\right|_{z=1}, \tag{1. 4. 1}$$

$$\mathsf{D}^2(\eta_n) - \mathsf{M}(\eta_n) + \mathsf{M}^2(\eta_n) = \left.\frac{d^2 T^{(n)}(z)}{dz^2}\right|_{z=1}. \tag{1. 4. 2}$$

By means of the bivariate generating function of η_n (see (1. 2. 5)) the above equations may be transformed into

$$\sum_{n=0}^{\infty} \mathsf{M}(\eta_n) w^n = \left.\frac{\partial T(z, w)}{\partial z}\right|_{z=1},$$

$$\sum_{n=0}^{\infty} [\mathsf{D}^2(\eta_n) - \mathsf{M}(\eta_n) + \mathsf{M}^2(\eta_n)] w^n = \left.\frac{\partial^2 T(z, w)}{\partial z^2}\right|_{z=1}.$$

In view of (1. 2. 10) we have

$$\sum_{n=0}^{\infty} \mathsf{M}(\eta_n) w^n = \frac{1}{r} \frac{(1-P-\varepsilon w)w}{(1-\varepsilon w)(1-w)^2}, \tag{1. 4. 3}$$

$$\sum_{n=0}^{\infty} [\mathsf{D}^2(\eta_n) - \mathsf{M}(\eta_n) + \mathsf{M}^2(\eta_n)] w^n = \frac{2}{r^2} \frac{(1-P-\varepsilon w)^2 w^2}{(1-\varepsilon w)^2 (1-w)^3}. \tag{1. 4. 4}$$

From (1. 4. 3), for example by an expansion into partial fractions and using the rule

$$\frac{1}{(1-\varepsilon w)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \varepsilon^n w^n, \qquad k=1, 2, \ldots, \tag{1. 4. 5}$$

one gets

$$\mathsf{M}(\eta_n) = \frac{1}{r}\left[qn + p\,\frac{\varepsilon(1-\varepsilon^n)}{1-\varepsilon}\right]. \tag{1. 4. 6}$$

From (1. 4. 4), by an expansion into partial fractions, we have

$$\sum_{n=0}^{\infty} [\mathsf{D}^2(\eta_n) - \mathsf{M}(\eta_n) + \mathsf{M}^2(\eta_n)] w^n =$$

$$= \frac{2}{r^2}\left[\frac{A}{1-\varepsilon w} + \frac{B}{(1-\varepsilon w)^2} + \frac{C}{1-w} + \frac{D}{(1-w)^2} + \frac{E}{(1-w)^3}\right].$$

Now, instead of giving explicit expressions, we are rather interested in asymptotic relations for large values of n. By means of (1. 4. 5) and considering the fact that for $0<\varepsilon<1$ and $n\to\infty$

$$\binom{n+k-1}{k-1}\varepsilon^n \to 0, \qquad k=1, 2, \ldots,$$

we finally have

$$\mathsf{D}^2(\eta_n) - \mathsf{M}(\eta_n) + \mathsf{M}^2(\eta_n) \sim \frac{1}{r^2}[En^2 + (2D+3E)n].$$

In the expansion into partial fractions, therefore, only D and E are needed:

$$D = \frac{2q}{1-\varepsilon}(\varepsilon - q), \quad E = q^2,$$

hence, it follows that

$$(1.\,4.\,7) \qquad \mathsf{D}^2(\eta_n) - \mathsf{M}(\eta_n) + \mathsf{M}^2(\eta_n) \sim \frac{1}{r^2}\left[q^2 n^2 + q\,\frac{(1+3p)\varepsilon - q}{1-\varepsilon}\,n\right].$$

On account of the asymptotic relations, deduced from (1. 4. 6),

$$(1.\,4.\,8) \qquad \mathsf{M}(\eta_n) \sim \frac{q}{r}\,n,$$

$$\mathsf{M}^2(\eta_n) \sim \frac{1}{r^2}\left[q^2 n^2 + \frac{2pq\varepsilon}{1-\varepsilon}\,n\right],$$

from (1. 4. 7) one finally has

$$(1.\,4.\,9) \qquad \mathsf{D}^2(\eta_n) \sim \frac{q(r-1+p)}{r^2}\,n\,\frac{1+\dfrac{1-r+p}{r-1+p}\,\varepsilon}{1-\varepsilon}.$$

In the case of equilibrium ($\varepsilon=0$), owing to (1. 4. 3)—(1. 4. 4), the following holds:

$$\sum_{n=0}^{\infty} \mathsf{M}(\eta_n) w^n = \frac{1}{r}\,\frac{qw}{(1-w)^2},$$

$$\sum_{n=0}^{\infty} [\mathsf{D}^2(\eta_n) - \mathsf{M}(\eta_n) + \mathsf{M}^2(\eta_n)] w^n = \frac{2}{r^2}\,\frac{q^2 w^2}{(1-w)^3},$$

that is, in this case,

$$(1.\,4.\,10) \qquad \mathsf{M}(\eta_n) = \frac{q}{r}\,n, \quad \mathsf{D}^2(\eta_n) = \frac{q(r-1+p)}{r^2}\,n.$$

Note finally that substituting $r=1$ into the above equations, the corresponding special relations for the simpler process described in [1] can be obtained.

1. 5. *Determination of the passage time distribution.* According to (1. 2. 2) and (1. 2. 7) the generating function of ϑ_m becomes

$$(1.5.1)\quad W_m(w) = w^m \left\{ \frac{1-P-\varepsilon w}{r-[r(1+\varepsilon)-1+P]w+(r-1)\varepsilon w^2} \right\}^{m+1} \equiv w^m [G(w)]^{m+1}.$$

Let us define the new random variable

$$(1.5.2)\quad \tau_m = \vartheta_m - m,$$

whose generating function is clearly equal to $[G(w)]^{m+1}$, that is, the $(m+1)$th power of a generating function, namely of $G(w)$. Hence, τ_m is the sum of $m+1$ independent random variables of the same distribution and, therefore, according to the central limit theorem of Lindeberg—Lévy, after standardization τ_m will be asymptotically normally distributed. Obviously, it is not affected by the translation (1. 5. 2) of the random variable ϑ_m whether or not it tends to normality (if standardized). Thus we have the following

THEOREM 1. 3: *The limiting distribution of ϑ_m, after standardization, will be normal. That is (see* (1. 3. 7)),

$$(1.5.3)\quad \lim_{m\to\infty} \mathsf{P}\left[\left(\vartheta_m - \frac{r}{q} m\right) \Big/ \sqrt{\frac{r(r-1+p)}{q^2} m \frac{1+\dfrac{1-r+p}{r-1+p}\varepsilon}{1-\varepsilon}} < x \right] = \Phi(x)$$

where by $\Phi(x)$ the normal distribution function is denoted[3]:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

In the equilibrium case, i.e. if $\varepsilon=0$, (1. 5. 1) becomes

$$W_m(w) = w^m \left[\frac{q}{r-(r-1+p)w} \right]^{m+1}$$

and thus one has for the probability distribution of ϑ_m

$$(1.5.4)\quad W_m^{(n)} = \binom{n}{m} \frac{1}{r^{n+1}} q^{m+1} (r-1+p)^{n-m}, \qquad n=0, 1, \ldots.$$

The mean and variance of this distribution are given by (1. 3. 4) and (1. 3. 6). Note that in this special case the passage time ϑ_m is distributed according to the negative binomial distribution.

[3] This will be the same result, in the special case $r=1$, as obtained by Rényi [5] applying the theory of Markov chains.

1. 6. *Determination of the displacement distribution.* The distributions of the displacement and passage time of a particle are connected according to (1. 2. 11), which should now be written in the following form:

$$\mathsf{P}(\eta_n < m+1) = 1-\mathsf{P}(\vartheta_m < n). \tag{1. 6. 1}$$

Let us define

$$x = \left(m-\frac{q}{r}n\right)\Bigg/\sqrt{\frac{q(r-1+p)}{r^2}n\frac{1+\dfrac{1-r+p}{r-1+p}\varepsilon}{1-\varepsilon}},$$

then for large values of m [5] one has

$$n \sim \frac{r}{q}m - x\sqrt{\frac{r(r-1+p)}{q^2}m\frac{1+\dfrac{1-r+p}{r-1+p}\varepsilon}{1-\varepsilon}}.$$

With these expressions of x, for large values of m, (1. 6. 1) reads as

$$\mathsf{P}\left[\left(\eta_n-\frac{q}{r}n\right)\Bigg/\sqrt{\frac{q(r-1+p)}{r^2}n\frac{1+\dfrac{1-r+p}{r-1+p}\varepsilon}{1-\varepsilon}} < x\right] \sim$$

$$\sim 1-\mathsf{P}\left[\left(\vartheta_m-\frac{r}{q}m\right)\Bigg/\sqrt{\frac{r(r-1+p)}{q^2}m\frac{1+\dfrac{1-r+p}{r-1+p}\varepsilon}{1-\varepsilon}} < -x\right]$$

and so, in view of Theorem 1. 3 and (1. 4. 8)—(1. 4. 9), one obtains the following

THEOREM 1. 4: *The displacement η_n, if standardized, is asymptotically normally distributed:*

$$\lim_{n\to\infty}\mathsf{P}\left[\left(\eta_n-\frac{q}{r}n\right)\Bigg/\sqrt{\frac{q(r-1+p)}{r^2}n\frac{1+\dfrac{1-r+p}{r-1+p}\varepsilon}{1-\varepsilon}} < x\right] = 1-\Phi(-x) = \Phi(x),$$

where $\Phi(x)$ denotes the normal distribution function[4].

In the equilibrium case, i.e. if $\varepsilon=0$, (1. 2. 10) becomes (see also (1. 2. 7))

$$T(z, w) = \frac{1}{q}Y(z, w),$$

that is, in view of (1. 2. 2) and (1. 5. 4), the probability distribution of η_n will be

$$T_m^{(n)} = \binom{n}{m}\frac{1}{r^n}q^m(r-1+p)^{n-m}, \qquad m=0, 1, \ldots.$$

The mean and variance of this distribution are given by (1. 4. 10). Note that the displacement η_n is distributed according to the binomial distribution [1].

[4] This will be the same result, in the special case $r=1$, as obtained in [1].

§ 2. The time-continuous process

2. 1. *Statement of the problem.* Let us consider the limiting case $r \to \infty$ in the process described in 1. 1. Physically speaking, in this case the mobile phase flows continuously along the fixed phase. If defining the time coordinate as

$$t = \frac{n\tau}{r}, \tag{2. 1. 1}$$

the basic equations of the continuous process can readily be obtained in terms of t. To this end, let us write (1. 1. 5)—(1. 1. 6) as follows:

$$\tau \frac{X_m^{(n)} - X_m^{(n-1)}}{\tau/r} = -rQX_m^{(n-1)} + (r-1)PY_m^{(n-1)} + PY_{m-1}^{(n-1)},$$

$$\tau \frac{Y_m^{(n)} - Y_m^{(n-1)}}{\tau/r} = rQX_m^{(n-1)} - (r-1)PY_m^{(n-1)} - Y_m^{(n-1)} + (1-P)Y_{m-1}^{(n-1)}.$$

Considering that for $r \to \infty$ according to (1. 1. 2) and (1. 1. 7)

$$P \to 0, \quad Q \to 0, \quad rP \to k_1\tau, \quad rQ \to k_2\tau,$$

for $r \to \infty$ we have

$$\lim_{r\to\infty} \tau \frac{X_m^{(n)} - X_m^{(n-1)}}{\tau/r} = -k_2\tau X_m^{(n-1)} + k_1\tau Y_m^{(n-1)}, \tag{2. 1. 2}$$

$$\lim_{r\to\infty} \tau \frac{Y_m^{(n)} - Y_m^{(n-1)}}{\tau/r} = k_2\tau X_m^{(n-1)} - k_1\tau Y_m^{(n-1)} - Y_m^{(n-1)} + Y_{m-1}^{(n-1)}. \tag{2. 1. 3}$$

Now, in view of (2. 1. 1), writing $\Delta t = \tau/r$ and defining accordingly

$$X_m^{(n)} = X_m^{(t)}, \quad X_m^{(n-1)} = X_m^{(t-\Delta t)}, \qquad m = 0, 1, \dots, \tag{2. 1. 4}$$

$$Y_m^{(n)} = Y_m^{(t)}, \quad Y_m^{(n-1)} = Y_m^{(t-\Delta t)}, \qquad m = -1, 0, 1, \dots, \tag{2. 1. 5}$$

for $r \to \infty$ i.e. $\Delta t \to 0$, instead of (2. 1. 2)—(2. 1. 3) we have

$$\frac{dX_m^{(t)}}{dt} = -k_2 X_m^{(t)} + k_1 Y_m^{(t)}, \tag{2. 1. 6}$$

$$\frac{dY_m^{(t)}}{dt} = k_2 X_m^{(t)} - \left(k_1 + \frac{1}{\tau}\right) Y_m^{(t)} + \frac{1}{\tau} Y_{m-1}^{(t)}. \tag{2. 1. 7}$$

According to the definitions (2. 1. 4)—(2. 1. 5) the initial condition (1. 1. 8), in the limiting case $r \to \infty$ i.e. $\Delta t \to 0$, reads as

$$X_m^{(0)} = Y_m^{(0)} = 0, \tag{2. 1. 8}$$

and in this case the boundary condition (1. 1. 9) becomes, by introducing the "Dirac delta function" $\delta(t)$,

$$Y_{-1}^{(t)} = \tau\delta(t). \tag{2. 1. 9}$$

The system of difference — differential equations (2. 1. 6)—(2. 1. 7) with the supplementary conditions (2. 1. 8)—(2. 1. 9) will now constitute the basis of what follows. Note, however, that all the relations to come could also be derived directly from the corresponding ones in § 1 by passage to the limit $r \to \infty$. The way we will follow, i.e. going out from (2. 1. 6)—(2. 1. 7)—(2. 1. 8)—(2. 1. 9) without reference to §1, seems, however, more instructive in using generating functions and Laplace transforms in stochastic processes.

2. 2. *Distribution of passage times, and displacements, respectively.* Consider a particle of the solute and the time when this particle is just leaving the mth cell. This time, denoted by ϑ_m, will be called the passage time of the particle with respect to the mth cell. The probability distribution function of ϑ_m is, on the basis of the physical nature of the problem, obviously

$$\mathsf{P}(\vartheta_m < t) = \frac{\int_0^t W_m^{(t')}\,dt'}{\int_0^\infty W_m^{(t)}\,dt}.$$

(2. 2. 1)
$$W_m^{(t)} \equiv \frac{1}{\tau} Y_m^{(t)}.$$

THEOREM 2. 1: *The denominator* $\int_0^\infty W_m^{(t)}\,dt$ *equals* 1 *and so the density function of* ϑ_m *is* $W_m^{(t)}$.

PROOF. Let us define the generating function (regarding m) and Laplace transform (regarding t) of $X_m^{(t)}$ and $Y_m^{(t)}$. Generally we put

(2. 2. 2)
$$Z^{(t)}(z) = \sum_{m=0}^\infty Z_m^{(t)} z^m,$$
$$Z_m(s) = \mathscr{L} Z_m^{(t)} = \int_0^\infty e^{-st} Z_m^{(t)}\,dt,$$

and let us define $Z(z, s)$ as follows:

(2. 2. 3)
$$Z(z, s) = \mathscr{L} Z^{(t)}(z) = \sum_{m=0}^\infty Z_m(s) z^m.$$

Taking the Laplace transforms of (2. 1. 6) and (2. 1. 7), considering (2. 1. 8) as well, then multiplying the equations so obtained by z^m and summing up from $m=0$ to $m=\infty$ considering (2. 1. 9) too, after rearranging we finally have

(2. 2. 4)
$$X(z, s) = \frac{k_1}{s+k_2} Y(z, s),$$

(2. 2. 5)
$$Y(z, s) = \frac{\tau(s+k_2)}{\tau s^2 + [\tau(k_1+k_2)+1-z]s + k_2(1-z)}.$$

Putting now $s=0$ into (2. 2. 5) one has

$$W(z, 0) = \frac{1}{\tau} Y(z, 0) = \frac{1}{1-z},$$

that is

$$W_m(0) = \int_0^\infty W_m^{(t)}\, dt = 1.$$

Q.e.d.

We will now discuss the *displacement* (position) of a particle of the solute at time t; let us denote it by $\eta_t (=0, 1, \ldots)$. On the basis of the physical nature of the problem, the probability distribution function of η_t is obviously

$$\mathsf{P}(\eta_t \leqq m) = \frac{\sum_{k=0}^{m} T_k^{(t)}}{\sum_{k=0}^{\infty} T_k^{(t)}},$$

(2. 2. 6) $$T_k^{(t)} \equiv X_k^{(t)} + Y_k^{(t)}, \qquad k=0, 1, \ldots, m.$$

THEOREM 2. 2: *The denominator* $\sum_{k=0}^{m} T_k^{(t)}$ *equals* 1 *and so the probability distribution of* η_t *is* $T_m^{(t)}$.

PROOF. In view of (2. 2. 4) and (2. 2. 5) we have

(2. 2. 7) $$T(z, s) = X(z, s) + Y(z, s) = \frac{\tau(s+k_1+k_2)}{\tau s^2 + [\tau(k_1+k_2)+1-z]s + k_2(1-z)}.$$

Putting $z=1$ into this equation one gets

$$T(1, s) = \frac{1}{s}$$

that is

$$T^{(t)}(1) = \sum_{k=0}^{\infty} T_k^{(t)} = 1.$$

Q.e.d.

Between the density function $W_m^{(t)}$ and the probability distribution $T_m^{(t)}$ the Fokker—Planck equation holds:

(2. 2. 8) $$\int_0^t W_m^{(t')}\, dt' + \sum_{k=0}^{m} T_k^{(t)} = 1.$$

In order to prove this, take the sum of (2. 1. 6)—(2. 1. 7):

$$\frac{dT_m^{(t)}}{dt} = \frac{1}{\tau}(Y_{m-1}^{(t)} - Y_m^{(t)}).$$

Summing here from $k=0$ to m one has

$$\frac{d}{dt}\sum_{k=0}^{m} T_k^{(t)} = \frac{1}{\tau}(Y_{-1}^{(t)} - Y_m^{(t)}),$$

and the integration in view of (2. 1. 9), also considering (2. 2. 1), gives indeed (2. 2. 8).

2. 3. *Determination of the mean and variance of the passage time distribution.* On the basis of Theorem 2. 1, for the mean and variance of the passage times one has [6]

(2. 3. 1)
$$\mathsf{M}(\vartheta_m) = -\left.\frac{dW_m(s)}{ds}\right|_{s=0},$$

(2. 3. 2)
$$\mathsf{D}^2(\vartheta_m)+\mathsf{M}^2(\vartheta_m) = \left.\frac{d^2W_m(s)}{ds^2}\right|_{s=0}.$$

The generating functions of the means $\mathsf{M}(\vartheta_m)$ and variances $\mathsf{D}^2(\vartheta_m)$, where $m=0, 1, \dots$, can be written as

$$\sum_{m=0}^{\infty} \mathsf{M}(\vartheta_m) z^m = -\left.\frac{\partial W(z, s)}{\partial s}\right|_{s=0},$$

$$\sum_{m=0}^{\infty} [\mathsf{D}^2(\vartheta_m)+\mathsf{M}^2(\vartheta_m)] z^m = \left.\frac{\partial^2 W(z, s)}{\partial s^2}\right|_{s=0}.$$

In view of (2. 2. 5) we have

$$\sum_{m=0}^{\infty} \mathsf{M}(\vartheta_m) z^m = \frac{\tau}{q}\,\frac{1}{(1-z)^2},$$

$$\sum_{m=0}^{\infty} [\mathsf{D}^2(\vartheta_m)+\mathsf{M}^2(\vartheta_m)] z^m = \frac{2\tau^2}{q^2(1-z)^3}\left[1+\frac{p}{\tau(k_1+k_2)}(1-z)\right].$$

From these equations, applying the rule (1. 3. 3), we finally obtain

(2. 3. 3)
$$\mathsf{M}(\vartheta_m) = \frac{\tau}{q}(m+1),$$

(2. 3. 4)
$$\mathsf{D}^2(\vartheta_m) = \left(\frac{\tau}{q}\right)^2 (m+1)\left[1+\frac{2p}{\tau(k_1+k_2)}\right].$$

Note that in the case of equilibrium ($k_1 \to \infty$, $k_2 \to \infty$, but p and so q as well tend to finite limits) the simpler relation holds:

(2. 3. 5)
$$\mathsf{D}^2(\vartheta_m) = \left(\frac{\tau}{q}\right)^2 (m+1).$$

Note also the asymptotic relations, valid for large values of m:

(2. 3. 6)
$$\mathsf{M}(\vartheta_m) \sim \frac{\tau}{q} m,$$
$$\mathsf{D}^2(\vartheta_m) \sim \left(\frac{\tau}{q}\right)^2 m\left[1+\frac{2p}{\tau(k_1+k_2)}\right].$$

2. 4. *Determination of the mean and variance of the displacement distribution.* On the basis of Theorem 2. 2, the mean and variance of the displacement distribution may be determined by means of the generating function of $T_m^{(t)}$:

$$\mathsf{M}(\eta_t) = \left.\frac{dT^{(t)}(z)}{dz}\right|_{z=1}, \tag{2. 4. 1}$$

$$\mathsf{D}^2(\eta_t) - \mathsf{M}(\eta_t) + \mathsf{M}^2(\eta_t) = \left.\frac{d^2T^{(t)}(z)}{dz^2}\right|_{z=1}. \tag{2. 4. 2}$$

The Laplace transform of these equations can be written as

$$\mathscr{L}\mathsf{M}(\eta_t) = \left.\frac{\partial T(z,s)}{\partial z}\right|_{z=1},$$

$$\mathscr{L}[\mathsf{D}^2(\eta_t) - \mathsf{M}(\eta_t) + \mathsf{M}^2(\eta_t)] = \left.\frac{\partial^2 T(z,s)}{\partial z^2}\right|_{z=1}.$$

In view of (2.2.7) we have

$$\mathscr{L}\mathsf{M}(\eta_t) = \frac{s+k_2}{\tau s^2(s+k_1+k_2)} \tag{2. 4. 3}$$

and

$$\mathscr{L}[\mathsf{D}^2(\eta_t) - \mathsf{M}(\eta_t) + \mathsf{M}^2(\eta_t)] = \frac{2(s+k_2)^2}{\tau^2 s^3(s+k_1+k_2)^2}. \tag{2. 4. 4}$$

The inverse transform of (2. 4. 3) reads

$$\mathsf{M}(\eta_t) = \frac{q}{\tau}t + \frac{p}{\tau(k_1+k_2)}[1-e^{-(k_1+k_2)t}]. \tag{2. 4. 5}$$

Instead of giving the inverse transform of (2. 4. 4), we rather determine asymptotic relations for large values of t. From (2. 4. 5) one has

$$\mathsf{M}(\eta_t) \sim \frac{q}{\tau}t, \tag{2. 4. 6}$$

$$\mathsf{M}^2(\eta_t) \sim \frac{q}{\tau}t\left[\frac{q}{\tau}t + \frac{2p}{\tau(k_1+k_2)}\right],$$

further, from (2. 4. 4),

$$\mathsf{D}^2(\eta_t) - \mathsf{M}(\eta_t) + \mathsf{M}^2(\eta_t) \sim \frac{q}{\tau}t\left[\frac{q}{\tau}t + \frac{4p}{\tau(k_1+k_2)}\right].$$

Because of the above equations we finally have

$$\mathsf{D}^2(\eta_t) \sim \frac{q}{\tau}t\left[1 + \frac{2p}{\tau(k_1+k_2)}\right]. \tag{2. 4. 7}$$

Note that in the case of equilibrium ($k_1 \to \infty$, $k_2 \to \infty$ but p and so q also tend to finite limits) on account of (2. 4. 3) and (2. 4. 4) the following holds:

$$\mathscr{L}\mathsf{M}(\eta_t) = \frac{q}{\tau s^2}, \quad \mathscr{L}[\mathsf{D}^2(\eta_t) - \mathsf{M}(\eta_t) + \mathsf{M}^2(\eta_t)] = \frac{2q^2}{\tau^2 s^3},$$

that is

$$\mathsf{M}(\eta_t) = \frac{q}{\tau} t, \quad \mathsf{D}^2(\eta_t) = \frac{q}{\tau} t. \tag{2. 4. 8}$$

2. 5. *Determination of the passage time distribution.* By (2. 2. 5) the generating function of the Laplace transform $Y_m(s)$, $m=0, 1, \ldots$, was given. In view of (2. 2. 1) $W_m(s)$ is obtained as

$$W_m(s) = \left\{\frac{s+k_2}{\tau s^2 + [\tau(k_1+k_2)+1]s + k_2}\right\}^{m+1}. \tag{2. 5. 1}$$

Similarly to the considerations given in 1. 5 one can apply the central limit theorem of Lindeberg—Lévy obtaining the result that ϑ_m, if standardized, is asymptotically normally distributed. Taking into account (2. 3. 6), so we have the following

THEOREM 2. 3:

$$\lim_{m\to\infty} \mathsf{P}\left\{\left(\vartheta_m - \frac{\tau}{q} m\right) \Big/ \sqrt{\left(\frac{\tau}{q}\right)^2 m \left[1 + \frac{2p}{\tau(k_1+k_2)}\right]} < x\right\} = \Phi(x),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

In the equilibrium case (2. 5. 1) becomes

$$W_m(s) = \left(\frac{q/\tau}{s+q/\tau}\right)^{m+1}$$

and so the inverse transform reads

$$W_m^{(t)} = \frac{(q/\tau)^{m+1}}{m!} t^m e^{-(q/\tau)t}.$$

In other words, in the equilibrium case the passage time ϑ_m is distributed according to the gamma distribution. Its mean and variance are given by (2. 3. 3) and (2. 3. 5).

2. 6. *Determination of the displacement distribution.* The distributions of the displacement and passage time of a particle are connected according to (2. 2. 8), which should now be written in the following form:

$$\mathsf{P}(\eta_t < m+1) = 1 - \mathsf{P}(\vartheta_m < t). \tag{2. 6. 1}$$

Let us define

$$x = \left(m - \frac{q}{\tau} t\right) \Big/ \sqrt{\frac{q}{\tau} t \left[1 + \frac{2p}{\tau(k_1+k_2)}\right]},$$

then one has, for large values of m [5],

$$t \sim \frac{\tau}{q} m - x \sqrt{\left(\frac{\tau}{q}\right)^2 m \left[1 + \frac{2p}{\tau(k_1+k_2)}\right]}.$$

With these expressions of x, (2. 6. 1) reads for large values of m as

$$\mathsf{P}\left\{\left(\eta_t - \frac{q}{\tau} t\right) \Big/ \sqrt{\frac{q}{\tau} t \left[1 + \frac{2p}{\tau(k_1+k_2)}\right]} < x\right\} \sim$$

$$\sim 1 - \mathsf{P}\left\{\left(\vartheta_m - \frac{\tau}{q} m\right) \Big/ \sqrt{\left(\frac{\tau}{q}\right)^2 m \left[1 + \frac{2p}{\tau(k_1+k_2)}\right]} < -x\right\}$$

and thus, in view of Theorem 2. 3 and (2. 4. 6)—(2. 4. 7), one obtains the following

THEOREM 2. 4: *The displacement η_t, if standardized, is asymptotically normally distributed:*

$$\lim_{t\to\infty} \mathsf{P}\left\{\left(\eta_t - \frac{q}{\tau} t\right) \Big/ \sqrt{\frac{q}{\tau} t \left[1 + \frac{2p}{\tau(k_1+k_2)}\right]} < x\right\} = 1 - \Phi(-x) = \Phi(x),$$

where the normal distribution function is denoted by $\Phi(x)$.

In the equilibrium case (2. 2. 7) becomes

$$T(z, s) = \frac{\tau}{\tau s + q(1-z)},$$

that is

$$T_m^{(t)} = \frac{[(q/\tau)t]^m}{m!} e^{-(q/\tau)t}.$$

In other words, in the equilibrium case the displacement is distributed according to the Poisson distribution. Its mean and variance are given by (2. 4. 8).

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(Received April 16, 1968; revised July 18, 1969)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 43—46.

NONPARAMETRIC METHOD FOR DISCRIMINATING TWO INDEPENDENT GROUPS

by

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Let A and B be independent two-dimensional random samples. The task is to decide if they were drawn from different populations. The measurement level for both characteristics is at least *ordinal*. (About the ordinal scale and other kinds of measurement see e.g. SENDERS [5], chapter two.)

Let n and m be the sample sizes for A and B, and $n+m=N$. We give rank numbers to the united samples, r_i for one, and q_i for other characteristic ($i=1,2,\dots,N$). The r_i and q_i are, in the absence of ties, some permutations of the natural numbers up to N. (The presence of ties, however, does not affect the applicability of the method.) Graphically represented in the (r,q) coordinate system the sample elements lie at the rectangular lattice points determined by the natural numbers.

Our aim is to get new rank numbers, p_i say, in such a way that the two samples A and B be differentiated as much as possible. Applying the well-known MANN—WHITNEY test using rank sums as proposed by WILCOXON (see e.g. in HAYS [2], p. 634) this requirement is, as we can easily see, equivalent to that of demanding

$$D_p = \left| \frac{1}{n} \sum_A p_i - \frac{1}{m} \sum_B p_i \right| \tag{1}$$

to be maximal.

For getting the p_i ranks first we have to have one-dimensional sample elements. This is possible by weighted sum of the original ranks, r_i and q_i. (See e.g. RAO [4], p. 247.) The method for choosing the optimal weights is the following.

Starting from the (r,q) coordinate system, begin to rotate the axes in positive direction. The new axes, u and v, are determined by the angle α of rotation, so the new coordinate values of the sample points are functions of this angle. We need only one coordinate, u_i say, which is determined by the equation

$$u_i(\alpha) = r_i \cos\alpha + q_i \sin\alpha. \tag{2}$$

Since for all i

$$u_i(0) = r_i \tag{3}$$

and

$$u_i(\pi) = -r_i \tag{4}$$

we can confine ourselves to the domain $[0, \pi]$.

The problem is to determinate the "optimal" value of α, which makes the discrimination between A and B maximal. Unfortunately, this will not be the α maximizing D_u (a similar expression to (1) with u_i instead of p_i). The range of the u_i values changes as the axes are rotated; some standardization is therefore needed. For this purpose the *point-biserial correlation* (as defined e.g. in WALKER and LEV [7], p. 262) is the most convenient. One variable (the "serial") is the coordinate u the other (the "point") is the belonging to the samples A or B. Suited to our notations the point-biserial correlation is

$$r_{pb} = \frac{\bar{u}_A - \bar{u}_B}{\sqrt{\sum_1^N (u_i - \bar{u})^2}} \sqrt{\frac{nm}{N}}, \tag{5}$$

where $\bar{u}$ is the mean of all u_i values while $\bar{u}_A$ and $\bar{u}_B$ are the means of the u_i's belonging to the samples A and B, respectively.

To find the extreme value (maximum of minimum) of r_{pb} the derivative of (5) by α is needed. Before derivation, however, it is helpful to introduce some notations.

$$\sum_1^N r_i^2 - \frac{\left(\sum_1^N r_i\right)^2}{N} = \frac{N^3 - N}{12} = S \tag{6}$$

and similarly

$$\sum_1^N q_i^2 - \frac{\left(\sum_1^N q_i\right)^2}{N} = S \tag{7}$$

$$\sum_1^N r_i q_i - \frac{\left(\sum_1^N r_i\right)\left(\sum_1^N q_i\right)}{N} = \sum_1^N r_i q_i - \frac{N(N+1)^2}{4} = P \tag{8}$$

$$\frac{1}{n} \sum_A r_i - \frac{1}{m} \sum_B r_i = R \tag{9}$$

$$\frac{1}{n} \sum_A q_i - \frac{1}{m} \sum_B q_i = Q. \tag{10}$$

Using these notations

$$\bar{u}_A - \bar{u}_B = R \cos\alpha + Q \sin\alpha$$

$$\sum_1^N (u_i - \bar{u})^2 = \sum_1^N (r_i \cos\alpha + q_i \sin\alpha)^2 - \frac{\left(\sum_1^N (r_i \cos\alpha + q_i \sin\alpha)\right)^2}{N} = S + 2P \sin\alpha \cos\alpha.$$

So the derivative of r_{pb}:

$$\frac{dr_{pb}}{d\alpha} = \left(\frac{R\cos\alpha + Q\sin\alpha}{\sqrt{S + 2P\sin\alpha\cos\alpha}} \sqrt{\frac{nm}{N}}\right)' = \sqrt{\frac{nm}{N}} \times \tag{11}$$

$$\times \frac{(Q\cos\alpha - R\sin\alpha)\sqrt{S + 2P\sin\alpha\cos\alpha} - (R\cos\alpha + Q\sin\alpha)\dfrac{2P(\cos^2\alpha - \sin^2\alpha)}{2\sqrt{S + 2P\sin\alpha\cos\alpha}}}{S + 2P\sin\alpha\cos\alpha}.$$

Equated this expression to zero and multiplicated by the factors* $\sqrt{\frac{N}{nm}}$, $(S+2P\sin\alpha\cos\alpha)^{\frac{3}{2}}$ and $\cos^{-3}\alpha$ we get the equation

$$(Q-R\,\mathrm{tg}\,\alpha)\bigl(S(1+\mathrm{tg}^2\alpha)+2P\,\mathrm{tg}\,\alpha\bigr)=(R+Q\,\mathrm{tg}\,\alpha)P(1-\mathrm{tg}^2\alpha). \tag{12}$$

This is a cubic equation for $\mathrm{tg}\,\alpha$:

$$(PQ-SR)\,\mathrm{tg}^3\alpha+(SQ-PR)\,\mathrm{tg}^2\alpha+(PQ-SR)\,\mathrm{tg}\,\alpha+(SQ-PR)=0. \tag{13}$$

As we can directly see this equation has only one real root, namely

$$\mathrm{tg}\,\alpha=\frac{SQ-PR}{SR-PQ}, \tag{14}$$

that is, the solution wanted is

$$\alpha=\mathrm{arc\,tg}\frac{SQ-PR}{SR-PQ}. \tag{15}$$

As a consequence of (4), (5) has in $[0,\pi]$ positive and negative values as well, so the extreme value obtained by (15) is the *maximum* of $|r_{pb}|$.

In the presence of ties the denominator of (5) changes. Applying the usual corrections (see e.g. in SIEGEL [6], p. 206) in (6) and (7) we get

$$S_r=\frac{N^3-N-\sum_r(t_j^3-t_j)}{12} \tag{16}$$

and

$$S_q=\frac{N^3-N-\sum_q(t_j^3-t_j)}{12} \tag{17}$$

where t_j is the number of "equals" and the summations are extended for all the ties amongst the r_i and q_i ranks, respectively. The details omitted after similar computations as before we obtain

$$\alpha=\mathrm{arc\,tg}\frac{S_rQ-PR}{S_qR-PQ} \tag{18}$$

when tied observations occur.

We can give the solution (15) a more simple form by the aid of the Spearman rank correlation coefficient ϱ (as defined e.g. in GUILFORD [1], p. 287). By the usual formula (in the absence of ties)

$$\varrho=1-\frac{6\sum_1^N(r_i-q_i)^2}{N^3-N}. \tag{19}$$

* The second factor is equal to zero only if all u_i coincide, consequently any discrimination is meaningless. The discontinuity point of the third factor, $\alpha=\frac{\pi}{2}$, is a root of (11) if and only if it is a root of the equation (12) too.

Substituting (19) into (15), after some computations we get

$$\alpha = \text{arc tg}\,\frac{Q-\varrho R}{R-\varrho Q} \tag{20}$$

where R and Q are given, as before, by (9) and (10), respectively. The last expression (20) is a very suitable one to draw simple conclusions about the behaviour of the discriminating coordinate rotation. Unfortunately, when ties occur we cannot derive a similar simple relation.

As regards the maximum of $|r_{pb}|$ obtained at α determined by (15) or (18) it is easy to show that $r_{pb}=\pm 1$ only in the case if either sample has zero standard deviation (i.e. all elements of the sample are equal). Otherwise, even in the case of a "complete discrimination", $|r_{pb}|<1$. When the data are ranks (e.g. $\alpha=0$, see (3)) and the discrimination is complete (i.e. every element of B is smaller than any one of A) $|r_{pb}|$ is maximal when sample sizes are equal. In this case $|r_{pb}|$ tends from above very rapidly to $\sin\frac{\pi}{3}\left(=\frac{\sqrt{3}}{2}\right)$ as N tends to infinity:

$$r_{pb}^2 = \frac{3}{4\left(1-\frac{1}{N^2}\right)}.$$

We mention that the maximum of (1) is not restricted onto the point determined by (15). D_p is not a continuous function of α rather a stepwise one with at most nm different values: $D_p(0)=D_p(\pi)$ and between them there are nm jumping points when a u_i of A and a u_j of B coincide. This assertion implies that we may choose another α in the neighbourhood of that given by (15) to avoide incidental ties in p_i ranks.

The problem of the statistical comparison of the two groups is not solved, however, by the procedure described above. The choice of the *maximum* in every case hinders the difference in rank averages to follow the distribution simply derivable from that described by MANN and WHITNEY [3]. To determine the distribution of the maximum is still an open question. In any case, we can use the α determined by (15) or (18) (for the samples A and B) to give p_i ranks for a new, independently chosen pair of samples. In this latter case there are no objections to apply the MANN—WHITNEY test.

Finally, we note that the "rotated" ordinal variable, on which the p_i ranks refer, may be regarded as a *factor* in a psychological sense: this new, directly not (or not yet) measurable variate is just the characteristic feature in which the groups differ each another.

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ON THE ENUMERATION OF PSEUDO-SEARCH CODES

by

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1. Introduction. In his lecture notes [4], RÉNYI considered the problem of enumeration of search codes, motivated by the use of these codes in the theory of search. Besides other results, he obtained the number of regular search codes, which are special cases of search codes. In this paper, we modify RÉNYI's definition of search codes and enumerate these codes, from which RÉNYI's result follows.

2. Definitions.[1] For the sake of completeness, we list the necessary definitions. Except for trivial modifications, these definitions are due to RÉNYI.

Definition 2. 1. Any finite sequence of non-negative integers is a *codeword.*

We denote codewords by small Latin letters ($a, b, c, \ldots$). It is convenient to consider the empty sequence, e, as a codeword. We denote by Z the set of all codewords.

Definition 2. 2. A codeword b is called a *prefix* of a codeword c if there exists a codeword d such that $c=bd$, where bd is the concatenation of the codewords b and d.

Definition 2. 3. Any finite set C of different codewords is called a *code.*

The empty set is considered to be a code and is called the empty code. The code consisting of the empty codeword e is called the trivial code.

Definition 2. 4. If C is a code and a any codeword, then C_a is the set of all codewords $b\in Z$ such that $ab\in C$.

We denote by $N(C)$ the number of codewords in the code C.

Definition 2. 5. A code C is *branched* if one of the following occurs:

(1) C is the empty code.

(2) C is the trivial code.

(3) C does not contain e and there exists an integer $b(C)\geqq 1$, such that for k, the codeword consisting of the single letter k, $k=0, 1, 2, \ldots$, the code C_k is empty or non-empty according as $k\geqq b(C)$ or $k<b(C)$.

We call $b(C)$ the branching number of C. To complete the definition of branching number, it is convenient to put $b(C)=0$ if C is the empty or the trivial code.

Definition 2. 6. C is a *pseudo-search code* if C_a is branched for every $a\in Z$.

Definition 2. 7. For C a pseudo-search code, we call those $a\in Z$ for which $b(C_a)\geqq 1$ the branching points of C and $b(C_a)$ is the branching number of the branch point a.

Definition 2. 8. A pseudo-search code is called *regular* of degree $q \geqq 1$ if for each $a \in Z$ such that $b(C_a) \geqq 1$, $b(C_a) = q$.

It is convenient to say that the branching point a has $b(C_a)$ branches at the branching point a.

If a pseudo-search code does not contain e, then C is a search code according to RÉNYI if $b(C_a) \geqq 2$ for every branching point a of C. Also, he defines a regular search code as a regular pseudo-search code of degree $q \geqq 2$.

To clarify the above definitions consider the codes:

$$C^1 = (0, 10, 11, 2)$$

$$C^2 = (0, 11, 2)$$

$$C^3 = (0, 21).$$

C^1 is a pseudo-search code, C^2 is branched but is not a pseudo-search code (since e.g. C_1^2 is not branched) and C^3 is not branched ($C_1^3 = \emptyset$ but $C_2^3 = \{1\}$).

3. A 1:1 correspondence. Let C be a pseudo-search code and $B(C)$ the set of its branch points i.e. $B(C) = \{a : a \in Z \text{ and } b(C_a) \geqq 1\}$. We wish to label the branch points so that to each branch point we can assign a fixed branching number. It is convenient to order the codewords in $B(C)$ in lexicographic order and to label the codewords $1, 2, \ldots, k$ in that order. Let $S(q_1 \ldots q_k)$ be the set of all pseudo-search codes such that the ith branch point has q_i as its branching number $i = 1, 2, \ldots, k$. Let $R(q_1 \ldots q_k)$ (briefly R) be the number of pseudo-search codes in $S(q_1 \ldots q_k)$ (briefly S). It is easy to verify that

$$N(C) = \sum_{i=1}^{k} q_i - k + 1.$$

Let C be a code contained in S. We order the codewords in $C \cup B(C)$ in lexicographic order. Since the empty codeword e is always a branch point of any pseudo-search code, e is the first codeword to appear in the lexicographic order. To each code $C \in S$ we assign the vector $(c_1, c_2, \ldots, c_{n+k})$, $n = N(C) = \sum_{i=1}^{k} q_i - k + 1$, where $c_i = \pm 1$ according as the $(n+k-i+1)$st codeword in the lexicographic ordering of $C \cup B(C)$ belongs to C or $B(C)$. We note that c_1 is always $+1$, and c_{n+k}, which corresponds to e, is always -1.

Definition 3. 1. Let B, C be any 2 codes in S. B dominates C if and only if

$$\sum_{i=1}^{j} b_i \leqq \sum_{i=1}^{j} c_i \quad \text{for all} \quad j = 1, 2 \ldots n+k.$$

Let $C^0 \in S$ consist of the following codewords:

$$0, 1, 2, \ldots, q_1 - 2, \quad (q_1 - 1) \ldots (q_j - 1) r,$$

$$r = 0, \ldots, q_{j+1} - 2, \qquad j = 1, \ldots, k-1, \quad \text{and} \quad (q_1 - 1)(q_2 - 1) \ldots (q_k - 1).$$

An examination of the code C^0 shows that there are exactly k branch points, e,

$$(q_1-1),\ (q_1-1)(q_2-1),\ \dots,\ (q_1-1)(q_2-1)\dots(q_{j-1}-1),\ \dots,\ (q_1-1)(q_2-1)\dots$$
$$\dots(q_{k-1}-1).$$

The branch points as written down are in lexicographic order and hence the ith branch point has q_i as its branching number. Since the codewords in C^0 are already in lexicographic order, the vector assigned to C^0 is $(c_1^0, \dots, c_{n+1}^0)$ where

$$c_\alpha^0 = \begin{cases} -1 \text{ if } \alpha = n+k \text{ or } \alpha = n+k-\sum_{i=1}^{j} q_i \quad j=1,2\dots k-1. \\ +1 \text{ otherwise.} \end{cases}$$

LEMMA 3. 1. *C^0 dominates every $C \in S$.*

PROOF. We require to show that

$$\sum_{i=1}^{j} c_i^0 \leqq \sum_{i=1}^{j} c_i \quad \text{for every} \quad j=1,2\dots n+k.$$

The partial sums in a fixed sequence of $+1$'s and -1's will be as small as possible if all the -1's are as close to the beginning of the sequence as possible i.e. they appear as far from the end of the sequence as possible. For all $C \in S$, $c_{n+k}=-1$ and $c_1=1$. Since c_{n+k} represents the first branch point, the maximum number of $+1$'s allowed to precede it is q_1-1. Hence the farthest away the next -1 can appear is at c_{n+k-q_1}. Similarly, the maximum number of $+1$'s appearing before c_{n+k-q_1} is q_2-1 and thus the next -1 must appear at $c_{n+k-(q_1+q_2)}$. In general, after the jth -1 from the end appears, the maximum number of $+1$'s permitted to precede it is q_j-1 and $c_{n+k-\sum_{i=1}^{j} q_i}$ must be -1. When the last -1 from the end appears, there remain only q_k possible values which must all be $+1$. This completes the proof.

Let (m, n) be any point in the plane with non-negative, integral coordinates. It is well known [1], that any lattice path from $(0, 0)$ to (m, n) can be represented uniquely by a vector $(x_1, x_2, \dots, x_n)$, $0 \leqq x_1 \leqq x_2 \leqq \dots \leqq x_n \leqq m$, where x_i is the horizontal distance from the point $(m, n-i)$ to the path.

Definition 3. 2. A lattice path l corresponding to the vector $(x_1, x_2, \dots, x_n)$ dominates the lattice path l' corresponding to the vector $(x_1', x_2', \dots, x_n')$, if and only if $x_i' \leqq x_i$ for all i.

If we represent a unit horizontal step by $+1$ and a unit vertical step by -1, then the lattice path corresponding to $(x_1 \dots x_n)$ can be uniquely represented by a sequence of m $+1$'s and n -1's, where x_j is the total number of $+1$'s following the $(n-j+1)$st -1. For two lattice paths l and l', let the sequence of $+1$'s and -1's be $(l_1, l_2, \dots, l_{m+n})$ and $(l_1', l_2', \dots, l_{m+n}')$. A simple verification leads to the following result.

LEMMA 3. 2. *l dominates l' if and only if*

$$\sum_{i=1}^{j} l_i \leqq \sum_{i=1}^{j} l_i' \quad \text{for} \quad j=1,2\dots m+n.$$

Let $L(q_1, q_2, \ldots, q_k)$ (briefly L) denote the set of all lattice paths to the point $\left(a_k = \sum_{i=1}^{k} q_i - k+1, k\right)$ beginning with a unit horizontal step $(+1)$ and ending with a unit vertical step (-1), which are dominated by the lattice path l^0 corresponding to the vector $(0, a_1, a_2, \ldots, a_{k-1})$ where $a_j = \sum_{i=1}^{j} q_i - j$, $j = 1, 2, \ldots, k-1$.

THEOREM 3. 1. *There exists a* $1-1$ *correspondence between* S *and* L.

PROOF. To each code in S corresponds a lattice path through the representation by $+1$'s and -1's as already described. Furthermore, $C^0 \in S$ corresponds to the lattice path $l^0 \in L$, because the total number of $+1$'s following the jth -1 from the end in $(c_1^0, c_2^0, \ldots, c_{n+k}^0)$ is $\sum_{i=1}^{j} (q_i - 1) = \sum_{i=1}^{j} q_i - j = a_j$. From Lemmas 3. 1 and 3. 2, it follows that to each code in S corresponds a lattice path in L. Because two distinct codes in S have at least one different branch point, they correspond to two distinct lattice paths in L.

Next we give an algorithm, for a mapping from L to S. In the representation of $l \in L$ by $+1$'s and -1's, let n_r $(r = 1, 2, \ldots, k-1)$ be the number of $+1$'s between the rth and the $(r+1)$st -1's from the end. Without any ambiguity, we may use lattice path and its representation in $+1$'s and -1's interchangeably. Since l^0 dominates l, we note that:

$$(1) \qquad \sum_{i=1}^{j} n_i \leqq a_j = \sum_{i=1}^{j} q_i - j \qquad j = 1, 2 \ldots k-1.$$

The lattice path l ends with a -1. To this -1, we assign the codeword $b_1 = e$ and the set of codewords $B_1 = \{0, 1, \ldots, q_1 - 1\}$ written in lexicographic order. Now $n_1 \leqq a_1 = q_1 - 1$. To the next -1 from the end, we assign the codewords $b_2 = n_1$ and associate the set of codewords (in lexicographic order) $B_2 = \{b_2 0, b_2 1, \ldots, b_2(q_2 - 1)\}$ and at the same time delete from b_1 the first $n_1 + 1$ codewords. We then proceed to the mext -1 from the end to which we assign the codeword b_3 and the set of codewords $B_3 = \{b_3 0, b_3 1, \ldots, b_3(q_3 - 1)\}$. If $n_2 < q_2$, set $b_3 = b_2 n_2$ and delete the first $n_2 + 1$ codewords from B_2. On the other hand, if $n_2 \geqq q_2$, set $b_3 = n_1 + (n_2 - q_2) + 1$ and delete all the q_2 codewords from B_2 together with the first $n_2 - q_2 + 1$ codewords in lexicographic order that remained undeleted in B_1 viz. $\{n_1 + 1, \ldots, q_1 - 1\}$. Because of (1), the above construction is always possible.

To describe the procedure in general, suppose that to the rth -1 from the end, we have already assigned the codeword b_r and the associated set of codewords

$$B_r = \{b_r 0, b_r 1, \ldots, b_r(q_r - 1)\}$$

and have performed all the necessary deletions. For the ith -1 and the $(r+1)$st -1 from the end, $i < r$, we say that the ith -1 is incomplete relative to the $(r+1)$st -1 and denote such a -1 by $\varrho_{r+1}(i)$ if, and only if, the inequalities

$$\sum_{t=1}^{j} n_{i+t-1} + j < \sum_{t=1}^{j} q_{i+t-1}, \qquad j = 1, 2 \ldots r-i$$

are simultaneously satisfied. Let $i_{(j)}$ be the jth largest integer among the set of i's for which $\varrho_{r+1}(i)$ is defined. Consider the $(r+1)$st -1 from the end to which we assign codeword b_{r+1} and the set of codewords $B_{r+1} = \{b_{r+1}0, b_{r+1}1, \dots$ $\dots, b_{r+1}(q_{r+1}-1)\}$, where b_{r+1} is determined as described below. If $n_r < q_r$, delete the first n_r+1 codewords from B_r and put $b_{r+1} = b_r n_r$. If $n_r \geqq q_r$, we first delete all the q_r codewords in B_r and next delete all the undeleted codewords in $B_{i_{(1)}}$, then in $B_{i_{(2)}}$ and so forth until we have deleted exactly $n_r - q_r + 1$ codewords, the deletion always being done in lexicographic order. The last deleted codeword will occur in some $B_{i(j)}$ and let this codeword be $b_{i(j)}k$, $0 \leqq k \leqq q_{i(j)}-1$. In this case put $b_{r+1} = b_{i(j)}k$.

Proceed in this manner until we have assigned b_k and B_k to the last -1 from the end. Let $B = \{b_i\colon i = 2, 3, \dots, k\}$ and $C = \bigcup_{i=1}^{k} B_i - B$. We observe that C is a pseudo-search code and belongs to S because C_a is either the empty or trivial code for $a \notin B$ and C_{b_i} has the branching number q_i, $i = 1, 2, \dots, k$. Thus to each path in L, there corresponds a distinct code in S. The proof is therefore complete.

To illustrate this procedure consider the lattice path $l = (1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1)$ to the point $(7, 5)$ dominated by $(0, 3, 4, 5, 5)$.

Hence $q_1 = a_1 + 1 = 4$, $q_2 = a_2 - a_1 + 1 = 2$, $q_3 = a_3 - a_2 + 1 = 2$, $q_4 = a_4 - a_3 + 1 = 1$, $q_5 = a_5 - a_4 = 7 - 5 = 2$, and $n_1 = 1$, $n_2 = 0$, $n_3 = 2$, $n_4 = 0$.

Thus we can write:

$b_1 = e$, $B_1 = (0, 1, 2, 3)$. Since $n_1 = 1 < 4 = q_1$, set $b_2 = n_1 = 1$, $B_2 = (10, 11)$ and delete 0, 1 from B_1.

Since $n_2 = 0 < 2 = q_2$ set $b_3 = b_2 n_2 = 10$, $B_3 = (100, 101)$ and delete 10 from B_2. Now $n_3 = 2 = q_3$.

For $i = 1$, $n_1 + 1 = 2 < 4 = q_1$

$$n_1 + n_2 + 2 = 3 < 4 + 2 = q_1 + q_2.$$

Hence first -1 is incomplete relative to the 4th -1 from the end. Similarly for $i = 2$, $n_2 + 1 = 1 < 2 = q_2$.

Hence the 2nd -1 is incomplete relative to the 4th -1. Delete all the codewords in B_3. Delete $n_3 - q_3 + 1 = 2 - 2 + 1 = 1$ undeleted codeword from B_2 i.e. delete 11. Set $b_4 = 11$ and $B_4 = (110)$. Now $n_4 = 0$, $q_4 = 1$. Set $b_5 = b_4 n_4 = 110$ and $B_5 = (1100, 1101)$ and delete $n_4 + 1 = 1$ codeword from B_4 i.e. 110. Now

$$B = \{1, 10, 11, 110\}$$

$$\cup B_i = \{0, 1, 2, 3, 10, 11, 100, 101, 110, 1100, 1101\}.$$

Therefore $C = \{0, 2, 3, 100, 101, 1100, 1101\}$.

From C we obtain l as follows:

Write $C \cup \{b_i\colon i = 1, 2, \dots, k\}$ in lexicographic order (the starred numbers are branching points) and associate $+1$'s and -1's to codewords and branching points respectively.

*		*	*			*	*				
e	0	1	10	100	101	11	110	1100	1101	2	3
-1	1	-1	-1	1	1	-1	-1	1	1	1	1

Therefore $l = (1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1)$.

4. Enumeration. Because of the 1—1 correspondence, we are in a position to enumerate codes under various situations.

THEOREM 4. 1. *Let* $N(q_1, q_2, \ldots, q_k)$ *be the number of pseudo-search codes in* $S(q_1 \ldots q_k)$. *Then*

(2) $$N(q_1, q_2, \ldots q_k) = \det (a_{ij})_{(k-1)\times(k-1)}$$

where

$$a_{ij} = \begin{cases} 0 \text{ if } i > j+1 \\ \binom{a_{k-j}+1}{j-i+1} \text{ otherwise.} \end{cases}$$

PROOF: We have already shown that there is a 1—1 correspondence between the number of lattice paths from (0, 0) to the point (a_k, k) that are dominated by $(0, a_1, a_2, \ldots, a_{k-1})$ and the pseudo-search codes in $S(q_1, q_2, \ldots, q_k)$. The number of such paths has been determined by MOHANTY in [2] as in the theorem.

COROLLARY 1.

(3) $$N(q, q, \ldots q) = \frac{1}{k(q-1)+1}\binom{kq}{k}.$$

PROOF: By substituting $a_j = jq-j$ in the determinant of (2) and evaluating, the result is easily obtained. (See MOHANTY [2].)

The number $N(q, \ldots, q)$ represents the number of regular search codes as obtained by RÉNYI [5] using the generating function method. Here we remark that, if $q_i \geqq 2$ for each i, expression (2) give the number of search codes as defined by RÉNYI [5].

We know that to every lattice path from the origin to the point (a_k, k), dominated by $(0, a_1, \ldots, a_{k-1})$, there corresponds a pseudo-search code in $S(q_1 \ldots q_k)$ and conversely. From this fact, it follows that a particular path corresponds to several pseudo-search codes, one for each possible path dominating the given path. If this particular path can be characterized uniquely in terms of codes, we are then in a position to solve another enumeration problem. For this purpose denote by $S^*(r_1, r_2, \ldots, r_k)$ and $N^*(r_1, r_2, \ldots, r_k)$ the set and the number respectively of pseudo-search codes which, when arranged in lexicographic order including the branch points, have exactly r_i codewords between the ith and $(i+1)$st branching point, $i = 1, 2, \ldots, k-1$ and exactly r_k codewords following the last (kth) branching point.

THEOREM 4. 2.

(4) $$N^*(r_1, r_2, \ldots r_k) = \det (b_{ij})_{(k-1)\times(k-1)},$$

where

$$b_{ij} = \begin{cases} 0 \text{ if } i > j+1 \\ \binom{A_{k-j}+1}{j-i+1} \text{ otherwise.} \end{cases}$$

and

$$A_j = r_k + r_{k-1} + \cdots + r_{k-j+1} - 1, \qquad j = 1, 2, \ldots k-1.$$

PROOF: Because of our correspondence, we note that every code in S^* is represented by a given path, say l^* which has r_i horizontal unit steps between the

ith and $(i+1)$st vertical unit steps from the end, $i = 1, 2, \ldots, k-1$ and r_k horizontal unit steps before the first vertical step. However, any two distinct codes in S^* belong to two distinct $S(q_1 \ldots q_k)$'s. Thus $N^*(r_1, r_2, \ldots, r_k)$ is the number of paths from the origin $(a_k = \sum_1^k r_i, k)$ with the first step a unit horizontal step and the last step a unit vertical step which dominate l^*. Shifting the origin to $(a_k, k-1)$ and rotating the axis through 180°, we observe that the above number is equal to the number of paths from (0, 0) to $(a_{k-1}, k-1)$ that are dominated by the transformed l^*. The proof is complete by using the result in [2].

It is interesting to note that the number $N^{**}(r_1, r_2, \ldots, r_k)$, of search codes with the same property as that of $S^*(r_1, r_2, \ldots, r_k)$ cannot be obtained in an obvious manner. Nevertheless, by using the original technique for counting the number of paths dominated by a particular path as in [2], this number can be derived.

THEOREM 4. 3.

$$N^{**}(r_1, r_2, \ldots r_k) = \det(c_{ij})_{(k-1)\times(k-1)} \tag{5}$$

when

$$c_{ij} = \begin{cases} 0 & i > j+1 \\ \binom{A_{k-j}+j}{j-i+1} & i \leqq j+1,\ j = 1, 2, \ldots k-2. \\ \binom{A_{k-j}+j}{j-i+1} - \binom{k-2}{j-i+1} & \textit{for } i = 1, 2 \ldots k-1 \textit{ and } j = k-1. \end{cases}$$

PROOF: Because we are concerned with search codes, we can verify that N^{**} is equal to the number of paths from (0, 0) to $(a_k-1, k-1)$ which are dominated by the transformed l^* and must satisfy $(0 \leqq x_1 < x_2 < \cdots < x_{k-1})$ in their vector representation $(x_1, x_2, \ldots, x_{k-1})$. Thus

$$N^{**} = \sum_{x_1=0}^{A_1} \sum_{x_2=x_1+1}^{A_2} \cdots \sum_{x_{k-1}=x_{k-2}+1}^{A_{k-1}} 1. \tag{6}$$

For computational ease, we shall demonstrate the proof for $k=4$, the proof for any k being similar. Using (6),

$$N^{**}(r_1, r_2, r_3, r_4) = \sum_{x_1=0}^{A_1} \sum_{x_2=x_1+1}^{A_2} \sum_{x_3=x_2+1}^{A_3} 1$$

$$= \sum_{x_1=0}^{A_1} \sum_{x_2=x+1}^{A_2} \sum_{x_3=x_2+1}^{A_3} \begin{vmatrix} \binom{x_3}{0} & \binom{x_2+1}{1} & \binom{x_1+2}{2} \\ 0 & \binom{x_2+1}{0} & \binom{x_1+2}{1} \\ 0 & 0 & \binom{x_1+2}{0} \end{vmatrix}$$

$$= \sum_{x_1=0}^{A_1} \sum_{x_2=x_1+1}^{A_2} \begin{vmatrix} \binom{A_3+1}{1}-\binom{x_2+1}{1} & \binom{x_2+1}{1} & \binom{x_1+2}{2} \\ \binom{A_3+1}{0}-\binom{x_2+1}{0} & \binom{x_2+1}{0} & \binom{x_1+2}{1} \\ 0 & 0 & \binom{x_1+2}{0} \end{vmatrix}$$

$$= \sum_{x_1=0}^{A_1} \begin{vmatrix} \binom{A_3+1}{1} & \binom{A_2+2}{2}-\binom{x_1+2}{2} & \binom{x_1+2}{2} \\ \binom{A_3+1}{0} & \binom{A_2+2}{1}-\binom{x_1+2}{1} & \binom{x_1+2}{1} \\ 0 & \binom{A_2+2}{0}-\binom{x_1+2}{0} & \binom{x_1+2}{0} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{A_3+1}{1} & \binom{A_2+2}{2} & \binom{A_1+3}{3}-\binom{2}{3} \\ \binom{A_3+1}{0} & \binom{A_2+2}{1} & \binom{A_1+3}{2}-\binom{2}{2} \\ 0 & \binom{A_2+2}{0} & \binom{A_1+3}{1}-\binom{2}{1} \end{vmatrix}$$

which checks with (5).

Finally, we remark that there can be only one regular search code with given $r_1, r_2, \ldots, r_k$.

At the end of these enumeration it is worth while to mention that such combinatorial results in a more general setting were initiated by NARAYANA [3] in the context of an occupancy problem.

Acknowledgment. This work resulted from the inspiration provided by Prof. A. RÉNYI's talk on "Enumeration of Search Codes" at McMaster University.

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(Received March 22, 1970)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 55—60.

О ПОНЯТИИ ЦЕНТРА В СВЯЗНЫХ ГРАФАХ

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Введение

Хорошо известна следующая теорема К. Жордана [1]: Пусть F — дерево с n вершинами. Назовем вершину p дерева F центром дерева F, если каждая из веток, исходящих из p, содержит не больше чем $\frac{n}{2}$ ребер. Тогда

а) число центров F равно либо 1, либо 2;

б) в первом случае каждая из веток, исходящих из центра, содержит меньше чем $\frac{n}{2}$ ребер, а во втором случае центры смежны, и из обоих центров исходит по ветке, содержащей $\frac{n}{2}$ вершин каждая.

Жорданом были введены и другие понятия центра деревьев. В этой работе мы введем более общее понятие центра связных графов, обобщающее помимо четырех понятий центра деревьев, введенных Жорданом [1], также и понятие центра связных графов, введенное Гарари и Норманом [2].

В § 1. мы введем некоторые определения и обозначения. В § 2. докажем две теоремы, сопоставление которых и будет обобщением теоремы Жордана, сформулированной выше. В § 3., в качестве применения, докажем теорему, обобщающую один из результатов работы [3] М. Л. Цетлина.

§ 1.

Нам понадобятся следующие *определения и обозначения*[1]:

1. Пусть G — произвольный граф[2]. Соотношение „p есть вершина графа G“ будем обозначать через символ $p \in G$.

2. Для графа G и $p \in G$ разобьем множество вершин G, отличных от p, на классы, согласно следующему соотношению эквивалентности:

$p_1 \sim p_2$, если p_1 совпадает с p_2 или если существует путь, соединяющий p_1 с p_2 и не проходящий через p.

Эти классы эквивалентности определяют, если к каждому из них присоединить вершину p, подграфы графа G, содержащие единственную общую вершину p и не содержащие общих ребер. Эти подграфы будем называть вет-

[1] Определение понятий, которые здесь не поясняются, а также нужные сведения о них можно найти в [4], глава 14, § 1.

[2] Под графом условимся понимать *связный* неориентированный граф без петель и кратных рёбер.

ками графа G, исходящими из вершины p. Очевидно, что p не является точкой сочленения исходящих из него веток.

Пусть G — произвольный граф, и $p, p_1 \in G$. Через (p, p_1) будем обозначать ветку графа G, исходяшую из p и содержашую p_1. Далее, через $\mathcal{S}_G$ будем обозначать множество веток G.

3. Если для графа G и вершин $p_1, p_2 \in G$ найдется блок[3] B графа G, для которого $p_1, p_2 \in B$, то вершины p_1 и p_2 будем называть квазисмежными.

4. Если в графе S помечена одна вершина, не являющаяся точкой сочленения, то S будем называть корневым графом, а помеченную вершину — корнем S[4]. Корень графа S будем обозначать через $t(S)$. — Ветки, исходящие из вершины p некоторого графа G, будем считать корневыми графами с корнем p.

Графы без помеченных вершин будем иногда называть свободными графами.

5. Пусть S_1 и S_2 — корневые графы. S_1 будем называть веткой S_2 (и будем писать $S_1 \prec S_2$), если S_1 есть ветка свободного графа, полученного из S_2 удалением метки, которой было помечено $t(S_2)$, и если, кроме того, $t(S_2) \notin S_1$.

§ 2.

Пусть G — свободный граф. Предположим, что на $\mathcal{S}_G$ определена вещественная функция f, удовлетворяющая следующему условию:

$$(*) \qquad S_1 \prec S_2 \quad \text{влечет} \quad f(S_1) < f(S_2).$$

Определим функцию φ на множестве вершин графа G, полагая

$$\varphi(p) = \max\{f(S) : S \in \mathcal{S}_G, t(S) = p\} \qquad (p \in G),$$

и назовем вершину $p \in G$ центром графа G (относительно функции f), если

$$\varphi(p) = \min_{q \in G} \varphi(q).$$

В противном случае назовем p обыкновенной вершиной графа G. Назовем наконец вершину $p \in G$ локальным центром графа G (относительно f), если для всех $q \in G$, квазисмежных с p, имеет место неравенство

$$f((q, p)) \geqq f((p, q)).$$

Тогда верны следующие утверждения:

Теорема 1. *Все центры графа G, если их больше одного, расположены в одном и том же блоке графа G.*

Теорема 2. *Вершина $p \in G$ является центром графа G тогда и только тогда, если она — локальный центр.*

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[3] В [4] применяется термин „Glied“.

[4] Корневым *деревом* в литературе часто называют дерево, в котором помечена вершина, являющаяся не обязательно концевой.

Следствие. Граф G содержит по крайней мере один локальный центр, и все локальные центры, если их больше одного, содержатся в одном и том же блоке графа G.

Для доказательства теоремы 1 нам нужна следующая лемма:

Лемма 1. *Пусть $p \in G$. Если существуют вершины $q \in G$, отличные от p и такие, что $\varphi(q) \leqq \varphi(p)$, то все эти вершины содержатся в одной и той же из исходящих из p веток.*

В самом деле, покажем, что если вершины q_1 и q_2 расположены в различных из исходящих из p веток, то либо $\varphi(q_1) > \varphi(p)$, либо $\varphi(q_2) > \varphi(p)$. Действительно, пусть S — ветка, исходящая из p и такая, что $\varphi(p) = f(S)$; тогда по крайней мере одна из вершин q_1 и q_2 — скажем, q_2 — не содержится в S. Но тогда $S \prec (q_2, p)$, значит, в силу $(*)$, $\varphi(p) = f(S) < f((q_2, p)) \leqq \varphi(q_2)$.

Доказательство теоремы 1. Достаточно показать, что если p_1 и p_2 — центры G, то они квазисмежны. Предположим, что это неверно. Пусть W — некоторый путь[5] от p_1 к p_2. Поскольку W соединяет вершины, не квазисмежные одна другой, оно содержит по крайней мере одну проходную точку сочленения[6], скажем, вершину p. Но тогда $\varphi(p_1) \leqq \varphi(p)$, $\varphi(p_2) \leqq \varphi(p)$, и p_1 и p_2 расположены в различных из исходящих из p веток, что противоречит лемме 1.

Для доказательства теоремы 2 нам нужны следующие леммы:

Лемма 2. (Следствие леммы 1). *Пусть p_0 — центр, а p_1 — обыкновенная вершина графа G. Тогда φ строго убывает на множестве, состоящем из p_1 и из проходных точек сочленения любого пути от p_1 к p_0.*

Лемма 3. *Если p_0 — центр графа G и $p_0 \neq q \in G$, то для любой ветки $S \in \mathcal{S}_G$, исходящей из q и не содержащей вершину p_0, имеет место*

$$\varphi(q) = f((q, p_0)) > f(S).$$

В самом деле, предположим, что — в противоположность к утверждению — $\varphi(q) = f(S)$ для некоторой ветки S, исходяшей из q и не содержащей вершину p_0. Тогда $S \prec (p_0, q)$, значит, в силу $(*)$, $\varphi(q) = f(S) < f((p_0, q)) \leqq \varphi(p_0)$, что противоречит выбору p_0.

Доказательство теоремы 2. Пусть p_0 — центр, а p_1 — обыкновенная вершина G; покажем, что p_0 является локальным центром, в то время как p_1 не является локальным центром графа G. В силу леммы 3 имеем $f((q, p_0)) = \varphi(q) \geqq \varphi(p_0) \geqq f((p_0, q))$ для всех $q \in G$, отлинчых от p_0, и подавно для всех q, квазисмежных с p_0. Т. е. p_0 является локальным центром графа G.

С другой стороны, пусть $r = p_0$, если p_0 и p_1 квазисмежны, и пусть r — первая проходная точка сочленения любого пути от p_1 к p_0 в противном случае. В обоих случаях p_1 и r квазисмежны, и $\varphi(r) < \varphi(p_1)$. (Во втором случае это вытекает из леммы 2.) Но в силу леммы 3 $f((r, p_1)) \leqq \varphi(r) < \varphi(p_1) = f((p_1, p_0)) = f((p_1, r))$, значит, p_1 не является локальным центром графа G.

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[5] Под путем понимаем путь без кратных вершин.

[6] Пусть W путь в G. Вершину $p \in W$ называем проходной точкой сочленения относительно W, если p — промежуточная вершина W, и смежные с p вершины W не квазисмежны одна другой. (W может конечно не иметь проходных точек сочленения.)

Замечания. 1. Если f и $\tilde{f}$ — вещественные функции на $\mathscr{S}_G$, для которых соотношение $f(S_1) < f(S_2)$ влечет $\tilde{f}(S_1) < \tilde{f}(S_2)$, и наоборот ($S_1, S_2 \in \mathscr{S}_G$), то центры G относительно f совпадают с центрами G относительно $\tilde{f}$.

2. Все наши утверждения остаются в силе, если в качестве f допускать функции, принимающие значения в некотором упорядоченном множестве (т. е. функции, не обязательно вещественные). Как легко видеть, такая переформулировка на самом деле не прибавит общности нашим утверждениям, тем не менее, она может быть полезной.

Примеры. 1. Пусть S — корневой граф. Оьозначим через $\mu_b(S)$ число вершин S, через $\mu_\delta(S)$ — число блоков S, через $m(S)$ — расстояние от $t(S)$ до вершины, наиболее отдаленной от $t(S)$, а через $n(S)$ — длину самого длинного из путей, исходящих из $t(S)$.

Сужение на $\mathscr{S}_G$ любой из этих функций удовлетворяет условию $(*)$ при произвольном свободном графе G.

Частный случай теоремы 1, относящийся к функции m, был доказан Гарари и Норманом [2]. — Если G — *дерево*, то совпадают центры G, определяемые функциями μ_b и μ_δ, а также центры, определяемые функциями m и n. Понятия центра деревьев, соответствующие функциям μ_b и m, были введены Жорданом [1].

Обозначим далее для корневого графа S через $\nu_b(S)$ число вершин S, не являющихся точками сочленения, а через $\nu_\delta(S)$ — число блоков S, содержащих самое большее одну точку сочленения. Сужение на $\mathscr{S}_G$ любой из этих функций удовлетворяет условию $(*)$, если только G — такой свободный граф, каждая точка сочленения которого принадлежит по крайней мере трем блокам. Если при этом G — дерево, то центры G, определяемые функциями ν_b и ν_δ, совпадают.

2. Пусть S — корневое дерево. Заменим одним ребром те максимальные пути в S, промежуточные вершины которых имеют степень 2, в то время как концевые их вершины имеют степень, отличную от 2, относительно S. Обозначим полученное таким образом корневое дерево через $\bar{S}$. Каждому ребру $\bar{S}$ соответствует тогда некоторый путь в S, в том числе и ребру, исходящему из $t(\bar{S})$. Обозначим этот путь через $\bar{\bar{S}}$.

Пусть теперь F — свободное дерево. Положим при $S \in \mathscr{S}_F$

$$f_1(S) = \big(\mu_\delta(\bar{S}), \mu_\delta(\bar{\bar{S}})\big) \quad \text{и} \quad f_2(S) = \big(m(\bar{S}), m(\bar{\bar{S}})\big).$$

f_1 и f_2 суть отображения $\mathscr{S}_F$ во множество пар вещественных чисел. Если на этом множестве задан обычный порядок $(\bar{x}, \bar{\bar{x}}) < (\bar{y}, \bar{\bar{y}})$, если $\bar{x} < \bar{y}$ или если $\bar{x} = \bar{y}$ и $\bar{\bar{x}} < \bar{\bar{y}}$ ($\bar{x}, \bar{\bar{x}}, \bar{y}, \bar{\bar{y}}$ — вещественные числа), то f_1 и f_2 удовлетворяют условию $(*)$, и, следовательно, определяют центры в дереве F (см. замечание 2). Понятия центра деревьев, соответствующие функциям f_1 и f_2, были также введены Жорданом [1].

В дальнейшем будем рассматривать только деревья, поэтому повторим некоторые из предыдущих определений и утверждений для случая деревьев.

В дереве каждый блок состоит из единственного ребра, поэтому теорема 1 сводится к следующему:

Теорема 1'. *Если G — дерево, то число центров G равно либо 1, либо 2. В последнем случае центры смежны.*

Определение локального центра в случае деревьев эквивалентно следующему:

Вершина p дерева G является локальным центром G (относительно функции f), если для всех $q \in G$, смежных p, имеет место неравенство $f((q,p)) \geqq f((p,q))$.

Если G — дерево, то промежуточные вершины любого пути, содержащегося в G, суть проходные точки сочленения. Поэтому лемма 2 сводится к следующему:

Лемма 2'. *Если G — дерево, и $p_1 \in G$ — обыкновенная вершина, то φ строго убывает на пути от p_1 к p_0, где p_0 — ближайший от p_1 центр дерева G.*

§ 3.

В качестве применения теорем 1' и 2 докажем следующее утверждение:

Теорема 3. *Пусть g — сторого возрастающая и выпуклая снизу функция на множестве неотрицательных целых чисел, и пусть F — свободное дерево. Положим при $p \in F$*

$$h(p) = \sum_{q \in F} g(\varrho_{pq}),$$

где ϱ_{pq} обозначает расстояние вершин p и q. Тогда функция h обраща тся в минимум либо в единственной вершине, либо в двух смежных вершинах дерева F.

Доказательство. Положим при $S \in \mathcal{S}_F$

$$f(S) = \sum_{\substack{q \in S \\ q \neq t(S)}} [g(\varrho_{tq}) - g(\varrho_{tq}-1)],$$

где $t = t(S)$. Функция f удовлетворяет условию $(*)$. В самом деле, пусть $S_1, S_2 \in \mathcal{S}_F$, $S_1 \prec S_2$, $u = t(S_1)$ и $t = t(S_2)$. Тогда имеем

$$(3.1) \qquad f(S_2) - f(S_1) = \sum_{\substack{q \in S_2,\, q \neq t \\ q \notin S_1}} [g(\varrho_{tq}) - g(\varrho_{tq}-1)] + [g(\varrho_{tu}) - g(\varrho_{tu}-1)] +$$

$$+ \sum_{\substack{q \in S_1 \\ q \neq u}} \{g(\varrho_{tq}) - g(\varrho_{tq}-1) - [g(\varrho_{uq}) - g(\varrho_{uq}-1)]\}.$$

Из условий, наложенных на g, вытекает, что каждый член в правой части (3.1) неотрицательный, а член $g(\varrho_{tu}) - g(\varrho_{tu}-1)$ положительный, следовательно, $f(S_2) - f(S_1) > 0$.

Итак, теоремы 1' и 2 применимы к дереву F и к функции f.

С другой стороны, для любых смежных вершин $p_0, p_1 \in F$ вершины дерева F могут быть разделены на два непересекающихся класса: один класс состоит из вершин ветки (p_0, p_1), отличных от p_0, а другой — из вершин ветки (p_1, p_0), отличных от p_1. Поэтому имеем

$$(3.2)\quad h(p_0)-h(p_1)=\sum_{\substack{q\in(p_0,\,p_1)\\ q\neq p_0}}[g(\varrho_{p_0q})-g(\varrho_{p_1q})]+\sum_{\substack{q\in(p_1,\,p_0)\\ q\neq p_1}}[g(\varrho_{p_0q})-g(\varrho_{p_1q})]=$$

$$=\sum_{\substack{q\in(p_0,\,p_1)\\ q\neq p_0}}[g(\varrho_{p_0q})-g(\varrho_{p_0q}-1)]-\sum_{\substack{q\in(p_1,\,p_0)\\ q\neq p_1}}[g(\varrho_{p_1q})-g(\varrho_{p_1q}-1)]=$$

$$=f((p_0,p_1))-f((p_1,p_0)).$$

Но из (3.2) вытекает, что если для $p\in F$ $h(p)=\min_{q\in F}h(q)$, то p есть центр F относительно f. В самом деле, пусть p_1 — обыкновенная вершина дерева F, тогда, в силу теоремы 2, для некоторого $p_0\in F$, смежного с p_1, имеем $f((p_0,p_1))<$ $<f((p_1,p_0))$. Для этого p_0 в силу (3.2), имеем $h(p_0)<h(p_1)$.

Итак, поскольку в силу теоремы 1′ F содержит либо один центр, либо два смежных центра, теорема 3 доказана.

Легко видеть, что при условиях теоремы 3 для $p\in F$ условие $h(p)=\min_{q\in F}h(q)$ эквивалентно условию „p есть центр дерева F относительно f", из чего, в силу лемм 2′ и 3 и равенства (3.2), вытекает

Лемма 2″. *При условиях теоремы* 3 *функция* h *строго убывает на пути от* p_1 *к* p_0, *где* p_1 *— вершина* F *такая, что* $h(p_1)>\min_{q\in F}h(q)$, *а* p_0 *— ближайшая от* p_1 *вершина* F, *для которой* $h(p_0)=\min_{q\in F}h(q)$.

Для случая $g(\varrho)\equiv\varrho$ теорема 3 была доказана Цетлиным [3]. Из проведенных выше рассуждений ясно, что центр, введенный Цетлиным, совпадает с центром, определяемым при помощи функции μ_b.

Замечание. Если дерево F не содержит вершин степени 2, и функция g обладает свойствами, описанными в теореме 3, то утверждение теоремы справедливо также для следующей функции вместо h:

$$\bar{h}(p)=\sum_{q\neq p}g(\varrho_{pq})\qquad(p\in F),$$

где сумма берется по *концевым* вершинам F, отличным от p.

Автор выражает глубокую благодарность профессорам Т. Галлаи, А. Рени и П. Рожа, а также рецензенту М. Шимоновичу за всестороннюю помощь.

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(Поступила: 1-ого августа 1970 г.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 61—66.

НЕКОТОРЫЕ ЗАДАЧИ ПЕРЕЧИСЛЕНИЯ НЕМАРКИРОВАННЫХ ДЕРЕВЬЕВ

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Введение

Цель этой работы — применить одно следствие теоремы Гарари и Нормана о характеристике неподобия для графов [1], а также результаты, полученные в [2], к некоторым задачам перечисления немаркированных деревьев. В § 1 мы докажем это следствие теоремы Гарари и Нормана, а в § 2 рассмотрим две конкретные задачи перечисления деревьев. Понятия, которые здесь не поясняются, см. в [1], [2] и [3]. — Под графом будем понимать неориентированный граф без петель и кратных ребер, но, в отличие от [2], не обязательно связный.

§ 1.

Нам нужны следующие определения и обозначения:

1. Если в дереве S помечена одна концевая вершина, то S будем называть корневым деревом,[1] а помеченную вершину — корнем S. Через $t(S)$ будем обозначать корень дерева S, а через $k(S)$ — смежную с $t(S)$ вершину. Деревья без помеченных вершин будем иногда называть свободными.

Пусть S_1 и S_2 — непересекающиеся корневые деревья. Если выбросить из них корни $t(S_1)$ и $t(S_2)$ вместе с ребрами, исходящими из них, а потом соединить вершины $k(S_1)$ и $k(S_2)$, то получим свободное дерево, которое будем обозначать через $\langle S_1, S_2\rangle$. Верщины $k(S_1)$ и $k(S_2)$ дерева $\langle S_1, S_2\rangle$ будем иногда обозначать через $t(S_2)$ и $t(S_1)$, соответственно.

2. Сочетание с повторениями, элементами которого являются корневые деревья, будем называть сочетанием веток. Сочетание корневых деревьев $S_1, S_2, \ldots, S_n$ будем обозначать через $\{S_1, S_2, \ldots, S_n\}$.

Если $\mathscr{A}$ — некоторое множество сочетаний веток, а F — свободное дерево, то через $c_{\mathscr{A}}(F)$ обозначаем подграф дерева F, содержащий те и только те вершины дерева F, исходящие из которых ветки образуют сочетание, принадлежащее $\mathscr{A}$, а также все ребра F, соединяющие вершины, принадлежащие $c_{\mathscr{A}}(F)$.

Имеет место следующее следствие теоремы Гарари и Нормана:

Теорема. *Пусть $\mathscr{F}$ — конечное множество свободных деревьев. Пусть $\mathscr{A}$ — множество сочетаний веток, удовлетворяющее следующим условиям:*

α) $c_{\mathscr{A}}(F)$ есть непустое дерево при любом $F\in\mathscr{F}$,

β) $c_{\mathscr{A}}(F)$ пусто для всех деревьев $F\notin\mathscr{F}$.

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[1] Корневым деревом в литературе часто называют дерево, в котором помечена вершина, являющаяся не обязатлельно концевой.

Обозначим через $\mathscr{A}'$ *множество неупорядоченных пар корневых деревьев* $\{S_1, S_2\}$ *таких, что*

(1. 1) $$t(S_i) \in c_{\mathscr{A}}(\langle S_1, S_2 \rangle)\ ^2 \qquad (i=1, 2)$$

и

$$S_1 \not\equiv S_2.\ ^3$$

Тогда имеет место равенство

(1. 2) $$\overline{\overline{\mathscr{F}}} = \overline{\overline{\mathscr{A}}} - \overline{\overline{\mathscr{A}'}}.\ ^4$$

Доказательство. Для дерева F подграф $c_{\mathscr{A}}(F)$ будем для краткости обозначать через F'. Далее, группу автоморфизмов графа G будем обозначать через $\mathscr{C}(G)$.

При $F \in \mathscr{F}$ F' есть непустое дерево, и ясно, что группа $\mathscr{C}(F)$ является подгруппой группы $\mathscr{C}(F')$. Обозначим при $F \in \mathscr{F}$ через $P(F; \mathscr{A})$ и $L(F; \mathscr{A})$, соответственно, числа неподобных относительно $\mathscr{C}(F)$ вершин и ребер дерева F', а через $R(F; \mathscr{A})$ — число ребер в F', содержащих подобные относительно $\mathscr{C}(F)$ вершины. (См. [1]; ясно, что $R(F; \mathscr{A})$ равно либо нулю, либо единице.) Применяя следствие 1 (Corollary 1) теоремы Гарари и Нормана к дереву F' и к группе $\mathscr{C}(F)$, получаем соотношение

(1. 3) $$1 = P(F; \mathscr{A}) - L(F; \mathscr{A}) + R(F; \mathscr{A}).$$

Просуммируя (1.3) по всем $F \in \mathscr{F}$, получаем

(1. 4) $$\overline{\overline{\mathscr{F}}} = \overline{\overline{\mathscr{A}}} - \overline{\overline{\mathscr{A}'_1}} + \overline{\overline{\mathscr{A}'_2}},$$

где $\mathscr{A}'_1$ есть множество *всех* неупорядоченных пар корневых деревьев $\{S_1, S_2\}$, удовлетворяющих условию (1.1), а $\mathscr{A}'_2$ — множество корневых деревьев S таких, что $t(S) \in c_{\mathscr{A}}(\langle S, S \rangle)$ (т. е. таких, что $\{S, S\} \in \mathscr{A}'_1$).

В самом деле, каждой паре (F, t), где $F \in \mathscr{F}$ и $t \in F'$, поставим в соответствие сочетание веток, исходящих из t. Ясно, что это есть отображение в $\mathscr{A}$. Более того, оно есть отображение *на* $\mathscr{A}$. Действительно, обозначим при $\{S_1, S_2, \ldots$ $\ldots, S_n\} \in \mathscr{A}$ через F свободное дерево, составленное из коревных деревьев S_i сращиванием их корней, а через t — общий корень веток S_i в F. Тогда $t \in c_{\mathscr{A}}(F)$, т. е. $F \in \mathscr{F}$, $t \in F'$, и паре (F, t) соответствует сочетание $\{S_1, S_2, \ldots, S_n\}$.

Далее, двум парам (F_1, t_1), (F_2, t_2) $(F_i \in \mathscr{F}, t_i \in F'_i, i=1, 2)$ соответствует одно и то же сочетание веток тогда и только тогда, если существует изоморфизм дерева F_1 на F_2, переводящий t_1 в t_2. Поэтому имеем $\sum_{F \in \mathscr{F}} P(F; \mathscr{A}) = \overline{\overline{\mathscr{A}}}$.

Поставим теперь в соответствие каждой паре (F, l), где $F \in \mathscr{F}$ и l есть ребро дерева F', неупорядоченную пару корневых деревьев $\{S_1, S_2\}$, где $F \equiv \langle S_1, S_2 \rangle$ и $t(S_1)$, $t(S_2)$ инцидентны ребру l. Поскольку l принадлежит $F' \equiv c_{\mathscr{A}}(\langle S_1, S_2 \rangle)$, имеем $t(S_i) \in c_{\mathscr{A}}(\langle S_1, S_2 \rangle)$ $(i=1, 2)$, т. е. $\{S_1, S_2\} \in \mathscr{A}'_1$. Этим отображением мы и исчерпываем $\mathscr{A}'_1$, так как обозначив при $\{S_1, S_2\} \in \mathscr{A}'_1$ через F дерево $\langle S_1, S_2 \rangle$,

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[2] $p \in S$ обозначает, что „p есть вершина графа S“.

[3] Пусть F_1 и F_2 — деревья, либо оба свободные, либо оба корневые; пишем $F_1 \equiv F_2$, если существует изоморфизм дерева F_1 на F_2, который в случае корневых деревьев вершину $t(F_1)$ переводит в $t(F_2)$.

[4] Пусть $\mathscr{H}$ — конечное множество; через $\overline{\overline{\mathscr{H}}}$ обозначим число элементов $\mathscr{H}$.

имеем $t(S_i)\in c_{\mathscr{A}}(F)=F'$ $(i=1, 2)$, следовательно, $F\in\mathscr{F}$, и ребро l, соединяющее $t(S_1)$ с $t(S_2)$ в F, принадпежит F'. А паре (F, l) соответствует как раз $\{S_1, S_2\}$.

Далее, двум парам (F_1, l_1), (F_2, l_2) $(F_i\in\mathscr{F}, l_i$ —ребро в F_i', $i=1, 2)$ соответствует одна и та же неупорядоченная пара корневых деревьев тогда и только тогда, если существует изоморфизм дерева F_1 на F_2, переводящий l_1 в l_2. Поэтому имеем $\sum_{F\in\mathscr{F}} L(F;\mathscr{A}) = \overline{\overline{\mathscr{A}_1'}}$.

Аналогично доказывается также и равенство $\sum_{F\in\mathscr{F}} R(F;\mathscr{A}) = \overline{\overline{\mathscr{A}_2'}}$.

Итак, верно (1. 4), из чего вытекает (1. 2), ведь $\overline{\overline{\mathscr{A}_1}} = \overline{\overline{\mathscr{A}_1'}}-\overline{\overline{\mathscr{A}_2'}}$.

Замечание. Пусть $\mathscr{F}$ — некоторое множество свободных деревьев; условиям α) и β) могут удовлетворять, вообще говоря, разные множества сочетаний веток. Выполнение условий α) и β) всегда можно обеспечить, выбрав $\mathscr{A}$ так, чтобы сочетание веток $\{S_1, S_2, \dots, S_k\}$ принадлежало $\mathscr{A}$ тогда и только тогда, если $F\in\mathscr{F}$, где F — свободное дерево, полученное сращиванием корней $t(S_i)$. Если $\mathscr{F}$ — множество свободных деревьев с n вершинами, то такой выбор $\mathscr{A}$ приводит к формуле Оттера [4]:

$$f_n = \overline{\overline{\mathscr{F}}}= \overline{\overline{\mathscr{A}}}-\overline{\overline{\mathscr{A}'}} = t_n-\left[\sum_{1\leq k<n/2} t_k t_{n-k}+\binom{t_{n/2}}{2}\right] = t_n-\frac{1}{2}\left[\sum_{k=1}^{n-1} t_k t_{n-k}-t_{n/2}\right],$$

где f_n — число свободных деревьев с n вершинами, а t_m — число корневых деревьев, содержащих m вершин вне корня. (Если n нечетно, то $t_{n/2}=0$. — Ясно что t_m равно числу сочетаний веток, содержащих всего $m-1$ вершин вне корней.)

§ 2.

В этом параграфе рассмотрим две задачи перечсиления девевьев.

1. Пусть $k\geqq 2$ и r — натуральные числа. Пусть $\mathscr{H}_k$ — множество свободных деревьев, все вершины которых имеют степень либо 1, либо k. Пусть $\tilde{\mathscr{H}}_k$ — множество всех деревьев, принадлежащих $\mathscr{H}_k$ и содержащих больше чем одно ребро. Пусть наконец $\mathscr{F}=\mathscr{F}(k, r)$ — множество всех тех деревьев из $\tilde{\mathscr{H}}_k$, диаметр которых не превышает $2r$. Хотим вычислить величину $\overline{\overline{\mathscr{F}}}(k, r)$.

Пусть S — корневое дерево; обозначим через $m(S)$ высоту[5] S, а через $|S|$ — свободное дерево, полученное из S удалением метки, которой было помечено $t(S)$.

Докажем соотношение

$$\overline{\overline{\mathscr{F}}}=\binom{\vartheta_{k,r}+k}{k}-\binom{\vartheta_{k,r}}{2}, \tag{2. 1}$$

где $\vartheta_{k,r}$ — число корневых деревьев S таких, что $|S|\in\tilde{\mathscr{H}}_k$ и $m(S)\leqq r$. Поскольку

[5] Под высотой корневого дерева S понимаем расстояние от $t(S)$ до вершины, наиболее отдаленной от $t(S)$.

величины $\vartheta_{k,s}$ определяются формулами

$$\vartheta_{k,1} = 0,$$

$$\vartheta_{k,s} = \binom{\vartheta_{k,s-1}+k-1}{k-1} \qquad (s=2, 3, \ldots),$$

формула (2. 1) в принципе применима к вычислению величин $\overline{\overline{\mathcal{F}(k, r)}}$.

Чтобы доказать (2. 1), обозначим через $\mathcal{A}$ множество сочетаний веток $\{S_1, S_2, \ldots, S_n\}$ таких, что

$$n = k, \quad |S_i| \in \mathcal{H}_k \quad \text{и} \quad m(S_i) \leqq r \qquad (i=1, 2, \ldots, k).$$

Множество $\mathcal{A}$ удовлетворяет условиям α) и β). В самом деле, очевидно, что $c_{\mathcal{A}}(F)$ пусто для всех деревьев $F \notin \tilde{\mathcal{H}}_k$. Зафиксируем теперь дерево $F \in \tilde{\mathcal{H}}_k$. Обозначим при $p \in F$ через $\varphi_F(p)$ высоту наивысшей из веток, исходящих из p, а через p_0 — центр или один из центров дерева F относительно функции m. (См. [2]; напомним, что p_0 есть такая вершина F, для которой $\varphi_F(p_0) = \min_{p \in F} \varphi_F(p)$.) Легко убедиться в том, что дерево F принадлежит $\mathcal{F}$ тогда и только тогда, если $\varphi_F(p_0) \leqq r$. Но поскольку p_0 не может быть концевой вершиной F, условия $\varphi_F(p_0) \leqq r$ и $p_0 \in c_{\mathcal{A}}(F)$ равносильны. Следовательно, равносильны также и условия $F \in \mathcal{F}$ и „$c_{\mathcal{A}}(F)$ не пусто“. Предположим наконец, что $F \in \mathcal{F}$. Применяя теорему 1′ и лемму 2′ [2] к дереву F и к функции φ_F, убеждаемся в том, что $c_{\mathcal{A}}(F)$ связно.

Итак, имеет место (1. 2). Подставляя в (1. 2) выражения

$$\overline{\overline{\mathcal{A}}} = \binom{\vartheta_{k,r}+k}{k}, \quad \overline{\overline{\mathcal{A}'}} = \binom{\vartheta_{k,r}}{2},$$

получаем (2. 1).

2. Пусть n и a — натуральные числа. Обозначим через $\mathcal{K}_n$ множество свободных деревьев с n вершинами. Положим при $F \in \mathcal{K}_n$ и $p \in F$

$$h_F(p) = \sum_{q \in F} \varrho_{pq},$$

где ϱ_{pq} обозначает расстояние вершин p и q. Пусть $\mathcal{F} = \mathcal{F}(n, a)$ — множество тех $F \in \mathcal{K}_n$, для которых $\min_{p \in F} h_F(p) \leqq a$.

Чтобы выразить $\overline{\overline{\mathcal{F}}}$ при помощи чисел элементов некоторых множеств, состоящих из корневых деревьев, введем следующие обозначения: Для корневого дерева S обозначим через $\mu(S)$ число вершин дерева S вне $t(S)$, а через $\sigma(S)$ — сумму $\sum_{q \in S} \varrho_{tq}$, где $t = t(S)$. Обозначим далее через $\tau_{m,s}$ число корневых деревьев S, удовлетворяющих условиям

$$\mu(S) = m \quad \text{и} \quad \sigma(S) = s \qquad (m, s = 1, 2, \ldots).$$

Докажем соотношение

(2. 2)

$$\overline{\overline{\mathcal{F}}} = \sum_{s \leqq a+n} \tau_{n,s} - \frac{1}{2}\Big[\sum_{n_1, n_2 \geqq 1,\, n_1+n_2=n} \; \sum_{s_1+s_2 \leqq a+\min\{n_1, n_2\}} \tau_{n_1,s_1} \cdot \tau_{n_2,s_2} - \sum_{2s \leqq a+n/2} \tau_{n/2,s}\Big],$$

где $\tau_{n/2,s}=0$, если n нечетно. Перед доказательством заметим, что из (2. 2) вытекает немного более простое выражение для $\overline{\overline{\mathscr{F}_1}}$, где $\mathscr{F}_1=\mathscr{F}_1(n, a)$ — множество тех $F\in\mathscr{K}_n$, для которых $\min\limits_{p\in F} h_F(p)=a$. Имеем

(2. 2')

$$\overline{\overline{\mathscr{F}_1}}=\tau_{n,a+n}-\frac{1}{2}\Big[\sum_{n_1, n_2\geqq 1,\ n_1+n_2=n,\ s_1+s_2=a+\min\{n_1, n_2\}} \tau_{n_1,s_1}\cdot\tau_{n_2,s_2}-\tau_{n/2,(n+2a)/4}\Big],$$

где $\tau_{n/2,(n+2a)/4}=0$, если $n+2a$ не делится на 4. Поскольку величины $\tau_{m,s}$ вычислимы при помощи перечисляющей теоремы Пойа [5], формулы (2. 2) и (2. 2') в принципе применимы к вычислению величин $\overline{\overline{\mathscr{F}(n, a)}}$ и $\overline{\overline{\mathscr{F}_1(n, a)}}$, соответственно.

Чтобы доказать (2. 2), обозначим через $\mathscr{A}$ множество сочетаний веток $\{S_1, S_2, \dots, S_l\}$ таких, что

$$\sum_{i=1}^{l}\mu(S_i)=n-1$$

и

$$\sum_{i=1}^{l}\sigma(S_i)\leqq a$$

(l — произвольное натуральное число). Множество $\mathscr{A}$ удовлетворяет условиям α) и β). В самом деле, очевидно, что для дерева F подграф $c_{\mathscr{A}}(F)$ не пуст тогда и только тогда, если $F\in\mathscr{F}$, далее, из деммы 2'' работы [2] следует, что $c_{\mathscr{A}}(F)$ связно при $F\in\mathscr{F}$. Итак, имеет место (1. 2).

Но из определения множества $\mathscr{A}$ ясно, что

$$\overline{\overline{\mathscr{A}}}=\sum_{s\leqq a+n}\tau_{n,s},$$

Чтобы выразить $\overline{\overline{\mathscr{A}'}}$ при помощи чисел $\tau_{m,s}$, заметим, что для корневых деревьев S_1, S_2 соотношение $\{S_1, S_2\}\in\mathscr{A}'$ имеет место тогда и только тогда, если

$$\mu(S_1)+\mu(S_2)=n$$

и

$$\sigma(S_1)+\sigma(S_2)\leqq a+\min\{\mu(S_1), \mu(S_2)\},$$

и потому $\overline{\overline{\mathscr{A}'}}$ равно второму члену в провой части формуы (2. 2). Итак, (2. 2) доказано.

Автор выражает глубокую благодарность профессорам Т. Галлаи, А. Рени и П. Рожа, а также рецензенту М. Шимоновичу за всестороннюю помощь.

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(Поступила: 1-ого августа 1970 г.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 67—81.

ON COLOUR-CRITICAL GRAPHS

by

M. SIMONOVITS

Notations. Since this paper deals with colouring problems, the graphs, considered by us are undirected graphs without loops and multiple edges. If G^n is a graph, n denotes the number of its vertices. $\chi(G^n)$ is the chromatic number of G^n. Let x be a vertex of G^n, st x denotes the star of x, i.e. the set of vertices, joined to x; $\sigma(x)$ is the valence of x. Let e be an edge (or x be a vertex) of G^n, then G^n-e (or G^n-x) denotes the graph obtained from G^n omitting e (or omitting x and the edges, incident with it).

Introduction

The concept of the critical graphs was introduced by G. DIRAC, [1], [2]. Let G be a k-chromatic graph and e be an edge of it. e is said to be critical, if $\chi(G-e) = k-1$. The graph G is critical, if each edge of it is critical. In our case G is called a critical k-chromatic or shortly a k-critical graph.

The following two problems of T. GALLAI are investigated in this paper:

(A) For given k, n and m how many independent vertices of valence $\geqq m$ can be contained by a k-critical graph of n vertices?

The maximum will be denoted by $i(k, n, m)$.

(B) $\sigma(G)$ denotes the minimum valence in the graph G. For given k and n how large can $\sigma(G^n)$ be if G^n is k-critical.

The first part of this paper contains the following results:

THEOREM 1. *Let* $4 \leqq k \leqq m+1 \leqq n$, *then*

(1) $$n-i(k,n,m) \geqq \frac{1}{2}\sqrt[k-1]{(k-2)!\,nm}\,.$$ [1]

Clearly $i(k, n) = i(k, n, k-1)$ is the maximum number of independent vertices a k-critical graph of n vertices can have. Theorem 1 implies

(2) $$n-i(k,n) \geqq \frac{1}{2}\sqrt[k-1]{(k-1)!\,n}\,.$$

This result is not too far from the best possible in the following sense:

[1] Clearly, this theorem gives an estimation for *every* k-critical graph, while theorems 2, 4, 5 are constructions „only".

THEOREM 2. *Let $k \geqq 4$. Then for infinitely many values of n*

$$n-i(k,n) = O\left(n^{\frac{1}{2\left[\frac{k-1}{3}\right]}}\right). \tag{3}$$

If $k=4$, one can improve Theorem 1 in the following way:

THEOREM 3. *Let $n \geqq m+1 \geqq 4$. There exists a constant $c_1>0$ such that*

$$n-i(4,n,m) \geqq c_1(nm)^{2/5}. \tag{4}$$

The technique applied to prove Theorem 3 gives also some sharpening in the case $k>4$, but the value of k is greater, the obtained result is relatively the worse and the proof becomes very complicated; therefore this case will not be investigated.

The second part of the paper contains a construction of a 4-critical graph W^n depending on some parameters. Choosing these parameters in two different suitable ways we obtain the following theorems:

THEOREM 4. *Let n be an even integer, large enough. Then*

$$n-i(4,n,m) \leqq 20\sqrt{nm}\,. \tag{5}$$

THEOREM 5. *Let n be an even integer, large enough. Then there exists a 4-critical graph W^n such that*

$$\sigma(W^n) \geqq \frac{\sqrt[3]{n}}{6}. \tag{6}$$

This result can be sharpened. Let ec (G) denote the edgeconnectivity of a graph G, i.e. the least integer ϑ such that omitting ϑ suitable edges of G we may obtain a disconnected graph from it. Clearly ec $(G) \leqq \sigma(G)$.

I. JACOBSEN asked, what can be said about the edge-connectivity of a 4-critical graph? The example, proving Theorem 5 also proves

THEOREM 6. *Let n be an even integer, large enough. There exists a* 4-*critical graph W^n such that*

$$\text{ec}\,(W^n) \geqq \frac{\sqrt[3]{n}}{6}. \tag{7}$$

Here I have to make some remarks on the history of these results. W. G. BROWN and J. W. MOON gave a construction [3] which proved for infinitely many n that

$$n-i(4,n) = O(\sqrt{n}).$$

Applying Lemma 1 to a 6-critical graph, constructed by G. DIRAC, which had $2n$ vertices and n^2+2n edges, I could prove only

$$n-i(6,n) = O(\sqrt{n}) \tag{3*}$$

(for infinitely many n) but this result is essentially weaker than the result of BROWN and MOON. Last September (1969) B. TOFT constructed a 4-critical graph $\Gamma(n)$ which had $4n$ vertices and $\approx n^2$ edges. Applying Lemma 1 to TOFT's graph I could already prove (3*). T. GALLAI pointed out that the set of many independent vertices in both proofs consists of vertices of valence 3. This note led B. TOFT and me to Problem (A). We solved this problem and besides Problem (B) as well independently from each other at the same time and in very similar ways: both our proofs apply my splitting method (i.e. Lemma 1) in a modified form to the original graph of B. TOFT. The most important differences between our constructions are that

(1) Instead of Lemma 1 B. TOFT applies two similar lemmas and applying them succesively to the graph $\Gamma(n)$ he builds up the desired graphs. In my proof no splitting method is used explicitely, I construct a graph W^n directly and prove, that it is 4-critical. (I used the splitting method only to find the graph W^n.) From this point of view B. TOFT's proof is algorithmic, while mine is more direct, constructive.

(2) My graph consists of 3 similar parts, each of which is very similar to the graph of B. TOFT corresponding to Problem (B). From this point of view my construction is a little more complicated. The reason, why I had to "stick" together three such blocks is, that I split the vertices of TOFT's original graph into vertices, having disjoint stars, while B. TOFT did not.

Both the methods have their advantages and thus B. TOFT and I decided to publish both of them and almost together: the next paper of this journal contains a short description of B. TOFT's results [4]. The reader can also find the description of B. TOFT's original graph in it. An other source to find $\Gamma(m)$ is [6]. Finally, one can obtain $\Gamma(n)$ from the block Q constructed in the second part of this paper taking $p=q=1$, $a=d=n$.

I recommend the reader to think over, that this graph is 4-critical, because this will help to understand many things, connected with my construction.

Finally I remark that in the proofs I shall never verify that a given colouring of a given graph is good or not. The reader can easily prove in this cases, that no two vertices of the same colour are joined.

Added in proof. (February, 1973.) Last Summer I gave a lecture on a „Working Sminar on Hypergraphs" (Columbus, Ohio, USA) the bases of which was the hypergraphtheoretical part of this paper. It turned out, that W. G. BROWN, P. ERDŐS and V. T. SÓS [AP 1] also proved Lemma 2 for the special case $s=2$. At the same time I succeded to prove that Lemma 2 is sharp for $s=3$ as well (AP 2]. Finally, what is perhaps the most important, L. LOVÁSZ [AP 3], improving the methods of this paper and the construction of [3] recently has proven that there exist two positive constants c_1 and c_2 such that

$$c_1 n^{1/k-2} \leqq n-i(k,n) \leqq c_2 n^{1/k-2}$$

(The corresponding 3 papers are sumbitted but not published yet.)

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§ 1. The upper bounds

First I introduce the concept of splitting a vertex x into the vertices $x_1, \dots, x_\nu$.

Definition. Let G and $\tilde{G}$ be two graphs and $x \in G$, $x_1, \dots, x_\nu \in \tilde{G}$ be vertices given in them such that

$$G - x = \tilde{G} - x_1 - x_2 - \dots - x_\nu$$

and $\operatorname{st} x = \bigcup_{i=1}^{\nu} \operatorname{st} x_i$ hold. We shall say that $\tilde{G}$ can be obtained from G by splitting x into $x_1, \dots, x_\nu$.

The following lemma is of great importance in our investigations:

LEMMA 1. *Let G be a k-critical graph and x be a vertex of it. There exists a k-critical graph $\tilde{G}$, which can be obtained from the graph G by splitting x into $\nu \geqq \frac{\sigma(x)}{k-1}$ new vertices $x_1, \dots, x_\nu$. Besides $\sigma(x_i) = k-1$, $i = 1, \dots, \nu$.*

PROOF. Let $s = \binom{\sigma(x)}{k-1}$ and let us consider s new vertices $x_1, \dots, x_s$. Let us join x_i to $k-1$ vertices of $\operatorname{st} x$ in the graph $G-x$ so that $\operatorname{st} x_i \neq \operatorname{st} x_j$ unless $i=j$. Thus we split x into $x_1, \dots, x_s$. This procedure does not increase the chromatic number, since $x_1, \dots, x_s$ are independent. Thus the obtained graph G^* is at most k-chromatic. We prove that $\chi(G^*)=k$.

If $\chi(G^*) \leqq k-1$ held, we could colour G^* by $1, 2, \dots, k-1$. Since each subset of $G-x$ consisting of $k-1$ elements is joined to an x_i, it must be coloured by at most $k-2$ colours. So $\operatorname{st} x$ is coloured by at most $k-2$ colours. Restricting this colouring of G^* to $G-x$ we can extend it to G giving to x the colour from $\{1, \dots, k-1\}$ which does not occur in $\operatorname{st} x$. This implies $\chi(G) \leqq k-1$ contradicting $\chi(G)=k$. Hence $\chi(G^*)=k$.

Each k-chromatic graph contains a k-critical subgraph and this subgraph contains all the critical edges and vertices of the graph (where a vertex is called critical if omitting it we obtain a $k-1$-chromatic graph).

Let $\tilde{G}$ be a k-critical subgraph of G^*.

a) *$\tilde{G}$ contains $G-x$.* In order to prove this it is enough to prove that each edge of $G-x$ is critical. If e is an edge of $G-x$, $\chi(G-e) = k-1$ since G is k-critical. G^*-e can be obtained from $G-e$ by splitting x into $x_1, \dots, x_s$, therefore $\chi(G^*-e) \leqq \chi(G-e) = k-1$, i.e. e is a critical edge in G^*. Thus e belongs to $\tilde{G}$.

b) *If y is a vertex of* $\operatorname{st} x$, *then there exists an x_i joined to y (in $\tilde{G}$).* Otherwise $\tilde{G}$ could be obtained from $G-(x, y)$ by splitting x into $x_1, \dots, x_s$ and then omitting some vertices and edges from the resulting graph. This would imply $\chi(G^*) \leqq k-1$. But this is a contradiction showing that $\operatorname{st} x = \bigcup_{x_i \in \tilde{G}} \operatorname{st} x_i$ where the stars $\operatorname{st} x_i$ are counted in $\tilde{G}$.

c) Now the proof is completed. Indeed, we know that $\tilde{G}$ is k-critical. a) and b) together state that $\tilde{G}$ can be obtained from G by splitting x into some x_i's. Since a k-critical graph does not contain vertices of valence $< k-1$, thus $\sigma(x_i) = k-1$ if $x_i \in \tilde{G}$. According to b) $\sigma(x) \leqq \sum_{x_i \in \tilde{G}} \sigma(x_i)$, i.e. the number of x_i's, belonging to $\tilde{G}$, is at least $\frac{\sigma(x)}{k-1}$. Q.e.d.

PROOF of Theorem 1. We have to prove that if G is a k-critical graph and $x_1, \ldots$ $\ldots, x_t$ are independent vertices of valence $\geqq m$ in it, then

$$t \leqq n - \frac{1}{2}\sqrt[k-1]{(k-2)!\,mn}\,.$$

Let us split the vertices $x_1, \ldots, x_t$ into the vertices $\{x_{i,j}\}_{\substack{i \leqq t \\ j \leqq v_i}}$ successively: the graph, obtained in the $(i-1)$th step contains x_i and x_i has the same valence in it as in the original graph. We split x_i into $x_{i,1}, \ldots, x_{i,v_i}$ so that the resulting graph is also k-critical and $v_i \geqq \dfrac{k-1}{\sigma(x_i)}$. Since $x_1, \ldots, x_t$ are independent, the vertices $x_{i+1}, \ldots, x_t$ remain untouched.

In the last step we obtain a k-critical graph G^N. Since G^N is k-critical, st $x_{i,j} \neq$ $\neq$ st $x_{k,l}$ unless $(i,j)=(k,l)$. Further, st $x_{i,j}$ is a subset of $G^n - x_1 - \cdots - x_t$, consisting of $k-1$ elements. Hence

$$\binom{n-t}{k-1} \geqq \sum v_i \geqq \frac{1}{k-1}\sum \sigma(x_i) \geqq \frac{1}{k-1}\,mt. \tag{8}$$

$$n-t \geqq \sqrt[k-1]{(k-2)!\,mt} \tag{9}$$

follows immediately from (8) and if $n \leqq 2^{k-1}t$, (8) implies (1). If $n > 2^{k-1}t$, then

$$n^{k-1}\left(1-\frac{1}{2^{k-1}}\right)^{k-1} > \left(\frac{n}{2}\right)^{k-1} > nm\,\frac{n^{k-3}}{2^{k-1}} > nm\,\frac{k^{k-3}}{2^{k-1}} > nm\,\frac{(k-2)!}{2^{k-1}}$$

which gives (1) also in this case:

$$n-t \geqq n\left(1-\frac{1}{2^{k-1}}\right) > \sqrt[k-1]{nm(k-2)!}\cdot\frac{1}{2} \quad \text{Q.e.d.}$$

Since (8) is a very rough estimation, one can try to improve it. As we mentioned already in the Introduction, this can be done, though for $k>4$ it is rather complicated. Thus we shall improve Theorem 1 only for $k=4$ and only later, since first we prove Theorem 2, showing, that Theorem 1 is sharp in a certain sense.

PROOF of Theorem 2. Let G^{n_l} be a k_l-critical graph of n_l vertices for $l=1, \ldots, T$ and $I_l = \{x_{i,l}\}_{i \leqq \xi_l}$ be a set of independent vertices of valence $k_l - 1$ in G^{n_l}. We construct a $\Sigma(k_l-1)+1$-critical graph G^N.

Let $\tilde{G}^N$ be the following graph: we join each vertex of $G^{n_l} - I_l$ to each vertex of $G^{n_m} - I_m$ for $1 \leqq l < m \leqq T$. Then we consider $\Pi\xi_l$ new vertices $P(i_1, \ldots, i_T)$ where $i_l = 1, 2, \ldots, \xi_l$. The vertices $P(i_1, \ldots, i_T)$ form a set of independent vertices of valence $\Sigma(k_l-1)$: $P(i_1, \ldots, i_T)$ is joined to a vertex $u \in G^{n_l} - I_l$ if and only if $x_{i_l, l}$ is joined to u in G^{n_l}:

$$\text{st}\,P(i_1, \ldots, i_T) = \bigcup_{l=1}^{T} \text{st}\,x_{i_l, l}.$$

1. $\chi(\tilde{G}_N) \geqq \Sigma(k_l-1)+1$. Indeed, $G^{n_l}-I_l$ is a k_l-1-chromatic graph and each vertex of $G^{n_l}-I_l$ is joined to each one of $G^{n_m}-I_m$ if $l \neq m$. Thus the subgraph G^*, spanned by the graphs $G^{n_l}-I_l$ must be coloured by at least $\Sigma(k_l-1)$ colours. If G^* is coloured by exactly $\Sigma(k_l-1)$ colours, then each $G^{n_l}-I_l$ is coloured by k_l-1 colours. Since $\chi(G^{n_l})=k_l$, there exists an $x_{\tau_l,l}$ such that st $x_{\tau_l,l}$ is coloured by at least k_l-1 colours. Hence st $P(\tau_1, \dots, \tau_T)$ is coloured by exactly $\Sigma(k_l-1)$ colours, i.e. $\chi(\tilde{G}^N) \geqq \geqq \Sigma(k_l-1)+1$.

2. Now we prove that each $P(i_1, \dots, i_T)$ is critical. Let us consider a $P(\tau_1, \dots, \tau_T)$. We have to prove that

$$\chi(\tilde{G}^N - P,(\tau_1 \dots, \tau_T)) = \sum (k_l-1).$$

Since $\chi(G^{n_l}-x_{\tau_l l}) = k_l-1$, $G^{n_l}-I_l$ can be coloured by k_l-1 colours so that each st $x_{i,l}$ but st $x_{\tau_l l}$, is coloured by $\leqq k_l-2$ colours. Let us fix such a colouring for each $G^{n_l}-I_l$. Now st $P(i_1, \dots, i_T)$ is coloured by at most $\Sigma(k_l-1)-1$ colours unless $i_1=\tau_1, \dots, i_T=\tau_T$. Thus the colouring of G^* by $\Sigma(k_l-1)$ colours can be extended onto $\tilde{G}^N-P(\tau_1, \dots, \tau_T)$, i.e. $\chi(\tilde{G}^N-P(\tau_1, \dots, \tau_T)) = \Sigma(k_l-1)$.

3. Let now G^M be a $\Sigma(k_l-1)+1$-critical subgraph of $\tilde{G}^N$. According to 2 G^M contains all the vertices $P(i_1, \dots, i_T)$. This proves

$$M - i\left(\sum (k_l-1)+1, M\right) \leqq \sum_{i=1}^{T} (n_l-\xi_l). \tag{10}$$

Let now G^n be a 4-critical graph with $n-O(\sqrt{n})$ independent vertices. According to [3] or according to Theorem 4 there exist such graphs for infinitely many n. Setting $G^{n_l}=G^n$ and $k_l=4$ we obtain a $3T+1$-critical graph G^M. Here

$$n_l-\xi_l = O(\sqrt{n_l}) = O(\sqrt{n})$$

and from this and (10) we get

$$M - i(3T+1, M) = O(\sqrt{n}).$$

Since $M \geqq \prod \xi_l \approx n^T$,

$$M - i(3T+1, M) = O(M^{\frac{1}{2T}}).$$

This proves Theorem 2 if $k = 3T+1$. In order to obtain Theorem 2 in the other cases we apply the trivial inequality

$$i(k+1, n+1) \geqq i(k, n) \tag{11}$$

Q.e.d.

Remarks. 1. If we apply our construction to an odd circuit, we get for infinitely many M

$$M - i(k, M) = O\left(M^{\frac{1}{\left[\frac{k-1}{2}\right]}}\right) \tag{12}$$

which is only slightly weaker, than Theorem 2; on the other hand we do not need Brown and Moon's result in this case.

2. *The graph $\tilde{G}^N$ in the proof of Theorem 2 is a critical graph, i.e. $\tilde{G}^N=G^M$.* This can be proved very easily.

Now we give the mentioned sharpening of Theorem 1. As we have mentioned in the introduction, we consider only the case $k=4$. In this case Theorem 1 gives

(13) $$n-i(4,n,m) \geqq \frac{1}{2}\sqrt[3]{2mn}$$

while Theorem 4 gives only

(14) $$n-i(4,n,m) = O(\sqrt{nm})$$

(where n is even and sufficiently large). Improving (8) we shall prove

THEOREM 3. *Let $4 \leqq m+1 \leqq n$, then there exists a constant $c>0$ such that*

(4) $$n-i(4,n,m) \geqq c(nm)^{2/5}.$$

PROOF. I shall not introduce the concept of triangle-graphs but refer to [5]. Some parts of the proof will be omitted.

Definition. $C_{3,s,t}$ denotes the following triangle-graph: the vertices of $C_{3,s,t}$ are $u_1, \dots, u_s, v_1, \dots, v_t$ and the triangles of it are the triplets $(u_i v_j v_{j+1})$, where $v_{t+1}=v_1$ and $i=1, \dots, s$; $j=1, \dots, t$.

LEMMA 2. *If G^m is a triangle-graph of m vertices which does not contain a $C_{3,s,t}$ for $t=3, 4, \dots$, then G^m has at most*

(15) $$\left(\frac{1}{3}+o(1)\right)m^{3-\frac{1}{s}}$$

triangles.

The proof of Lemma 2 will be given later. Now let us consider the vertices of $G^{n-t} = G^n - x_1 - \dots - x_t$ in the proof of Theorem 1 and the triplets st $x_{i,j}$; $i=1, \dots, t$, $j=1, \dots, v_i$. These vertices and triplets define a triangle-graph of Σv_i triangles. We prove that this triangle-graph does not contain a $C_{3,s,t}$, thus Lemma 2 gives $\Sigma v_i = O((n-t)^{5/2})$ instead of (8). From this we obtain Theorem 3 by the same way as we obtained Theorem 1 from (8). We call a triangle-graph "good" if for each triangle of it we can colour its vertices by 3 colours so that this triangle is coloured by 3 colours but all the others by at most two ones. The reader can easily prove that $C_{3,s,t}$ is not "good". Since a subgraph of a "good" graph is also "good", a "good" graph cannot contain $C_{3,s,t}$, thus a "good" graph of m vertices has at most $O(m^{5/2})$ triangles, (Lemma 2). Thus it is enough to prove that the triangle-graph, constructed on the vertices of $G^n - x_1 - \dots - x_t$ is "good". Since G^N is 4-critical, it has a 4-colouring, such that the colour of $x_{i,j}$ is not used in $G^N - x_{i,j}$. In this case st $x_{i,j}$ is coloured by 3 colours and st $x_{k,l}$ is coloured by at most two colours, if $(k,l) \neq (i,j)$. $G^n - x_1 - \dots - x_t$ is coloured by 3 colours. Thus the considered triangle-graph is "good". Q.e.d.

PROOF of Lemma 2. The Lemma is similar to Theorem 1 in a paper of P. ERDŐS [5] and the proof is also almost the same.

Let G^m be a triangle-graph, $1, \dots, m$ be its vertices and t denote the number of its triangles; let $A(i,j)$ denote the set of vertices k such that $\{i,j,k\}$ is a triangle of the graph. Clearly

(16) $$3t = \sum_{1 \leqq i < j \leqq m} |A(i,j)|$$

A set $u_1, \ldots, u_s$ and a pair (i, j) will be called a "flower" if $u_l \in A(i, j)$ $(l=1, \ldots, s)$. Since (i, j) is contained in $\binom{|A(i,j)|}{s}$ "flowers",

$$F = \sum_{1 \leqq i < j \leqq m} \binom{|A(i,j)|}{s} \tag{17}$$

is the number of "flowers" contained in G^m.

On the other hand, if $u_1, \ldots, u_s$ are given, at most n pairs (i, j) form a "flower" with $u_1, \ldots, u_s$, otherwise there would be a cycle $(v_1, \ldots, v_t, v_1)$ such that $u_1, \ldots, u_s$ formed a "flower" with each pair $(v_l v_{l+1})$ when $l=1, \ldots, t$, $v_{t+1}=v_1$. Thus the vertices $u_1, \ldots, u_s, v_1, \ldots, v_t$ would determine a $C_{3,s,t}$ in G^m. This is a contradiction and hence

$$n\binom{n}{s} \geqq F. \tag{18}$$

(16), (18) and the convexity of $\binom{x}{s}$ for $x \geqq s$ imply

$$t \leqq \left(\frac{2^{1/s}}{6} + o(1)\right) n^{3-\frac{1}{s}}$$ Q.e.d.

Remark. One can prove that Lemma 2 is sharp for $s=2$: if c is a positive, but sufficiently small constant and we select each triangle-graph having m vertices and $cm^{3-\frac{1}{2}}$ triangles with the same probability, then the selected graph will not contain any $C_{3,2,t}$ with probability, tending to 1 (when m tends to infinity). I do not know, whether Theorem 3 is sharp or not.

Now we construct a graph proving Theorems 4, 5, 6.

§ 2. The lower bounds for $i(k, n, m)$

We restrict our investigations to the case $k=4$, because

a) If G^n is a 4-critical graph and we join the vertices of a complete $(k-4)$-graph to each vertex of G^n, then we obtain a k-critical graph. Hence our constructions give also some lower bounds for the general case.

b) The problem (B) is not too interesting for $k \geqq 6$: Let $\gamma(n)$ denote a circuit of n vertices. If we join each vertex of a $\gamma(n)$ to each vertex of another $\gamma(n)$ and n is odd, then we obtain a 6-critical graph of $2n$ vertices with minimum valence $n+2$. (This construction is due to G. DIRAC.)

The desired results will be obtained by a construction: we construct a 4-critical graph depending on many different parameters and consisting of 3 or more similar blocks.

The block **Q**.

The vertices of the graph **Q** can be divided into four parts, which will be called the stories of the block. Let a, d, p, q be given odd integers.

The *second story* consists of apq independent vertices, denoted by $B(i, j, x)$, where $i=1, \ldots, p$; $j=1, \ldots, a$; $x=1, \ldots, q$.

The *third story* consists of dpq vertices, denoted by $C(k, l, y)$, where $k=1, \dots, q$; $l=1, \dots, d$; $y=1, \dots, p$. $B(i, j, k)$ is joined to $C(k, l, i)$ for every i, j, k and l. The set $\{B(i, j, x)\}_{j,x}$ will be called a class and denoted by C_i. Similarly, $\{C(k, l, y)\}_{l,y}$ is the class $\bar{C}_k$. The sets $\{B(i, j, x)\}_x$ will be called groups and denoted by $G_{i,j}$. Similarly, $\{C(k, l, y)\}_y$ is the group $\bar{G}_{k,l}$.

The *first story* consists of the vertices $A(i, j)$; $i=1, \dots, p$; $j=1, \dots, a$. These vertices form a circuit $\gamma(ap)$: $A(i, j)$ is joined to $A(i', j')$ if $i=i'$ and $|j-j'| = 1$ or if $i' = i+1$, $j=a$, $j'=1$ (or conversely) and $A(p, a)$ is joined to $A(1, 1)$. Fixing i we obtain the arcs α_i of the circuit $\gamma(ap)$. The vertices of the first story are also joined to some vertices of the second one: $A(i, j)$ is joined to $B(i, j, x)$ for $x=1, \dots, q$. The fourth story is similar to the first one: it consists of the vertices $D(k, l)$; $k=1, \dots, q$; $l=1, \dots, d$ which determine the circuit $\bar{\gamma}(dq)$ consisting of the arcs $\bar{\alpha}_k$. $D(k, l)$ and $C(k, l, y)$ are joined.

This is the block (graph) **Q**. First we investigates its 3-colourings. We need the following definition:

Definition. If γ is an odd circuit, a 3-colouring of it is called elementary if one of the three colours is used only once. This colour and the corresponding vertex are called the exceptional colour and vertex respectively. If γ is divided into arcs, a 3-colouring of it is called periodic if each arc is coloured by two colours.

LEMMA 3. *Let be given a 3-colouring of* **Q**. *Then either* $\gamma(ap)$ *or* $\bar{\gamma}(dq)$ *is coloured periodically.*

Remark. In the case $p=q=1$ $a=d=n$ we obtain B. TOFT's graph of $4n$ vertices and $\approx n^2$ edges. Since neither $\gamma(a)$ nor $\bar{\gamma}(d)$ has periodic colourings, $\chi(\mathbf{Q}) \geqq 4$. It is easy to prove that **Q** is a 4-critical graph.

PROOF of Lemma 3. If neither $\gamma(ap)$ nor $\bar{\gamma}(dq)$ is coloured periodically, then for an i and a k both $\gamma_i(a)$ and $\gamma_k(d)$ are coloured by 3 colours. Hence both the sets $\{B(i, j, k)\}_j$ and $\{C(k, l, i)\}_l$ are coloured by at least two colours. These sets span a complete bipartite graph, therefore the colours, used at the two sets are different, i.e. Q is coloured by at least 4 colours. This contradiction proves Lemma 3.

Lemma 3 concerns with the question, how Q can not be coloured by 3 colours. The next question is, how it can.

LEMMA 4. *For given i, j and k, if $q \geqq 3$, then* **Q** *can be coloured by* 1, 2 *and* 3 *so that the only vertex of* $\gamma(ap)$ *coloured by* 1 *is* $A(i, j)$ *and if* $D(k', l)$ *is coloured by* 2, *then* $k'=k$.

PROOF. Let us consider the following colouring of **Q**:

a) $\gamma(ap)$ is coloured by 1, 2 and 3 elementary: $A(i, j)$ is the only vertex coloured by 1.

b) $B(i, j, k)$ is coloured by 2, $B(i, j, x)$ by 3 $(x \neq k)$ and $B(i^*, j^*, x^*)$ by 1 if $(i^*, j^*) \neq (i, j)$.

c) $C(k^*, l^*, y^*)$ is coloured by 3 if $k^*=k$ and by 2 if $k^* \neq k$.

d) Finally we have to colour $\bar{\gamma}(dq)$. Let us colour $D(k, l^*)$ by 2 if l^* is odd and by 1 otherwise. The other vertices of $\bar{\gamma}(dq)$ span a path, which can be coloured by 1 and 3 since the vertices of the third story, joined to the vertices of this path are coloured by 2. The obtained 3-colouring proves Lemma 3.

Now we turn to the

Problem: How can $\mathbf{Q}-e$ be coloured by 3 colours if e is an edge of $\mathbf{Q}$?

LEMMA 5. *Let e and f be two edges of $\mathbf{Q}$:*

$$e=(B(i,j,x);\ C(k,l,y)),\quad (i=y, x=k)$$

and

$$f=(A(i,j):\ B(i,j,x)).$$

Then both $\mathbf{Q}-e$ and $\mathbf{Q}-f$ can be coloured by 3 colours so that $\gamma(ap)$ and $\bar{\gamma}(dq)$ are coloured elementary, $A(i,j)$ and $D(k,l)$ are the exceptional vertices.

If g is an edge of $\gamma(ap)$, then any given 3-colouring of $\bar{\gamma}(dq)$ can be extended onto $\mathbf{Q}-g$ so that the path $\gamma(ap)-g$ is coloured by two colours.

PROOF. A) Let us consider the following colouring of $\mathbf{Q}-e$: $\gamma(ap)$ and $\bar{\lambda}(dq)$ are coloured elementary, the exceptional vertices are $A(i,j)$ and $D(k,l)$ which are coloured by 1 and 2 respectively. $\mathbf{G}_{i,j}$ and $\overline{\mathbf{G}}_{k,l}$ are coloured by 3, all the other vertices of the second story are coloured by 1 and all the other vertices of the third story are coloured by 2. Trivially this is a good 3-colouring of $\mathbf{Q}-e$. If we colour $B(i,j,x)$ by 1 instead of 3, we obtain a good colouring of $\mathbf{Q}-f$. This proves the first part of our lemma.

B) Let be given a 3-colouring of $\bar{\gamma}(dq)$ by 1, 2 and 3. Now we colour the second story by 3, the first one by 1 and 2. Up to now no edge joins vertices of the same colour. This 3-colouring can be extended onto the third story, since each $C(k,l,y)$ is joined to some vertices of the second story, coloured by 3 and to one vertex of the fourth story. Thus $C(k,l,y)$ can be coloured either by 1 or by 2. Q.e.d.

We need also the following trivial lemma:

LEMMA 6. *Let $\gamma(ap)$ be coloured periodically. Then the vertices $A(i,1)$ and $A(i,a)$ are coloured by the same colour and the set $\{A(i,1)\}_i$ is coloured by exactly 3 colours.*

PROOF. Since a is odd and the arc $\{A(i,j)\}_j$ is coloured by 2 colours, $A(i,1)$ and $A(i,a)$ have the same colour. Let us consider an odd circuit $\gamma(p)$ and colour its ith vertex by the colour of $A(i,1)$. Since $A(i,a)$ has the same colour as $A(i,1)$ and is joined to $A(i+1,1)$, thus $A(i,1)$ and $A(i+1,1)$ have different colours. Thus we obtained a good 3-colouring of $\gamma(p)$ and therefore $\{A(i,1)\}_i$ is coloured by exactly 3 colours. Q.e.d.

Graphs with many independent vertices of high valence.

Let $\mathbf{Q}$ be a block, defined above, where

$$p=1,\quad d=m,\quad a=qm.$$

Let E be a new vertex and let us join each $D(k,1)$ to it. Thus we obtain a 4-chromatic graph $\tilde{S}^n$ which contains a 4-critical subgraph S^n.

1. $\chi(\tilde{S}^n)\geqq 4$. Indeed, if $\mathbf{Q}$ is coloured by 1, 2 and 3, $\bar{\gamma}(dq)$ must be coloured periodically, since $\gamma(a)$ has no periodic colouring. According to Lemma 6 $\{D(k,1)\}_k$ is coloured by 3 colours, hence E must have another, fourth colour.

2. Let $e=(B(1,j,x);\ C(x,l,1))$ and $l\neq 1$. Then $\chi(\tilde{S}^n-e)\leqq 3$. Indeed, according to Lemma 5 $\mathbf{Q}-e$ can be coloured by 1, 2 and 3 so that only $D(k,l)$ is coloured by 2 in $\bar{\gamma}(dq)$. Thus E can be coloured by 2. Similarly, one can prove that $\chi(\tilde{S}^n-f)\leqq 3$

and $\chi(\tilde{S}^n - g) \leqq 3$ if $f = (A(1,j); B(1,j,x))$ and g is an edge of $\gamma(a)$ or $\bar{\gamma}(dq)$. (The last two assertions will not really be needed.)

3. Since $\chi(\tilde{S}^n) \geqq 4$, $\tilde{S}^n$ contains a 4-critical subgraph S which contains all the edges of $\tilde{S}^n$ mentioned in 2. This and $\chi(\mathbf{Q})=3$ imply that S contains all the vertices of $\tilde{S}^n$. The vertices $B(1,j,x)$ are aq independent vertices of valence $\geqq m$. Thus $i(4,n,m) \geqq aq$. Here $a=qm$. Therefore

$$(19)\quad n - i(4,n,m) \leqq \big(a(q+1)+2mq+1\big) - dq = a + 2mq + 1 = 3mq + 1 \leqq 3\sqrt{mn}.$$

This proves Theorem 4 for every m for infinitely many n. Later we shall prove Theorem 4 for every even and sufficiently large value of n.

The 4-critical graph W^n.

First we define a 4-chromatic graph $\tilde{W}^n$ which has a lot of critical edges and then we select a 4-critical subgraph W^n of it which will be just the desired graph. (I.e. W^n will prove Theorems 5, 6.)

Let t be an odd integer and let us consider t blocks $\mathbf{Q}_1, \dots, \mathbf{Q}_t$ with the parameters

$$a_\tau, d_\tau, p_\tau, q_\tau, \qquad \tau = 1, \dots, t.$$

These blocks will be connected to each other in the following way:

$$E_\tau(i), \quad F_\tau(i), \quad G_\tau(k), \quad H_\tau(k)$$

are new vertices. $E_\tau(i)$ is joined to $A_\tau(i,1)$, $F_\tau(i)$ is joined to $A_\tau(i,a)$, $G_\tau(k)$ is joined to $D_\tau(k,l)$ and $H_\tau(k)$ is joined to $D_\tau(k,d)$. (Here e.g. $A_\tau(1,a)$ is the abbreviation of $A_\tau(i,a_\tau)$; generally the index showing, which block is meant will be omitted where it causes no confusion.) Let us join each $E_\tau(i)$ to each $F_{\tau+1}(j)$ and each $G_\tau(k)$ to each $H_{\tau+1}(l)$ for $\tau=1,\dots,t$, where $t+1 \equiv 1$. The obtained graph will be denoted by $\tilde{W}^n$.

Investigating the colouring properties of the graph $\tilde{W}^n$ first we shall colour the blocks $\mathbf{Q}_\tau$ by 3 colours and then extend this colouring onto the whole graph. The following assertion deals with the possibility of this extension:

(+) Let us suppose that some vertices of $\tilde{W}^n$ are coloured by 3 colours and no edge joins vertices of the same colour. Let us also suppose that for a fixed τ the vertices of $\gamma_\tau(ap)$ and $\gamma_{\tau+1}(ap)$ are coloured, the vertices of $\{E_\tau(i)\} \cup \{F_{\tau+1}(j)\}$ are not. We can extend this colouring onto $\{E_\tau(i)\} \cup \{F_{\tau+1}(j)\}$ if and only if either $\{A_\tau(i,1)\}_i$ or $\{A_{\tau+1}(j,a)\}_j$ is coloured by (at most) two colours.

Indeed, if both $\{A_\tau(i,1)\}$ and $\{A_{\tau+1}(j,a)\}$ are coloured by 3 colours, then both $\{E_\tau(i)\}$ and $\{F_{\tau+1}(j)\}$ must be coloured by at least 2 colours. Since each $E_\tau(i)$ is joined to each $F_{\tau+1}(j)$, the set $\{A_\tau(i,1)\} \cup \{A_{\tau+1}(j,a)\} \cup \{E_\tau(i)\} \cup \{F_{\tau+1}(j)\}$ must be coloured by at least 4 colours.

On the other hand, if e.g. $\{A_\tau(i,1)\}$ is coloured by 1 and 2, then we colour $E_\tau(i)$ by 3. Since each $F_{\tau+1}(j)$ is joined to exactly one vertex of $\gamma_{\tau+1}(ap)$, it can be coloured either by 1 or by 2. This completes the proof of (+).

2. Now we prove that $\chi(\tilde{W}^n) \geqq 4$. If we colour the blocks by 1, 2 and 3, then at least t circuits $\gamma_\tau(ap)$ and $\bar{\gamma}(dq)$ must be coloured periodically. Without the loss of generality we may assume that at least $\frac{t+1}{2}$ of the circuits $\gamma_\tau(ap)$ are coloured

periodically and therefore there exists a τ such that $\gamma_\tau(ap)$ and $\gamma_{\tau+1}(ap)$ are coloured periodically. According to Lemma 6 both $\{A_\tau(i, 1)\}$ and $\{A_{\tau+1}(j, a)\}$ are coloured by 3 colours and this and (+) give that $\chi(\tilde{W}^n) \geqq 4$.

3. Now we show that almost all the edges of $\tilde{W}^n$ are critical. Because of the symmetry of $\tilde{W}^n$ we may assume that the considered edges belong to the first block or are of the form $(G_t(k_0), H_1(k_1))$ or $(H_1(k_1), D_1(k_1, d))$. Let us colour the blocks $\mathbf{Q}_3, \mathbf{Q}_5, \dots, \mathbf{Q}_t$ by 1, 2 and 3 so that $\bar{\gamma}_\tau(dq)$ be coloured periodically and the colour 2 be used only on the arc $\bar{\alpha}_{\tau,k_0}$; $\gamma_\tau(ap)$ be coloured elementarily and only $A_\tau(1, 2)$ be coloured by 1. $\mathbf{Q}_1$ is coloured similarly: in $\gamma_1(ap)$ only $A_1(1, 2)$ is coloured by 1 and in $\bar{\gamma}_1(dq)$ only some vertices of $\bar{\alpha}_{1,k_1}$ have the colour 3. $\mathbf{Q}_2, \mathbf{Q}_4, \dots, \mathbf{Q}_{t-1}$ are coloured conversely: in $\bar{\gamma}_{2\nu}(dp)$ only $D_{2\nu}(k_2, 2)$ has the colour 1 and only on α_{τ, i_0} is used the colour 2. According to Lemma 4 these colourings do exist. The following scheme illustrates the situation:

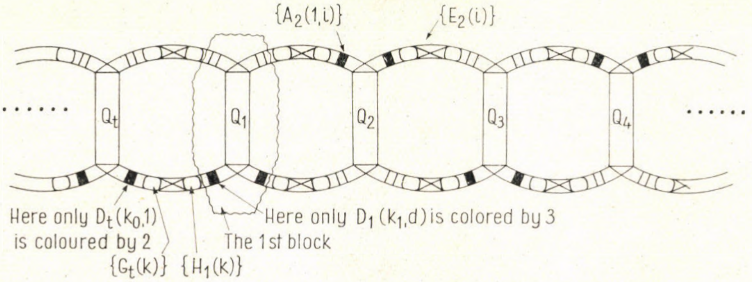

Fig. 1

Now we try to extend the given 3-colouring of the blocks onto the whole graph. (+) quarantees that the sets $\{E_\tau(i) \cup \{F_{\tau+1}(j)\}$ and $\{G_\tau(k)\} \cup H_{\tau+1}(l)\}$ can be coloured by 1, 2 and 3 except in the case of $\{G_t(k)\} \cup \{H_1(k)\}$. Now we colour $\{G_t(k)\}_{k \neq k_0}$ by 2 and $\{H_1(k)\}_{k \neq k_1}$ by 3 and colour $G_t(k_0)$ by 1. The only vertex of $\tilde{W}^n$ which is not coloured yet, is $H_1(k_1)$ and no edge joins vertices of the same colour. If we colour $H_1(k_1)$ by 1, then only $(G_t(k_0); H_1(k_1))$ has endpoints of the same colour; if we colour $H_1(k_1)$ by 3, then only $(H_1(k_1); D_1(k_1, d))$ has endpoints of the same colour. Thus both $(G_t(k_0); H_1(k_1))$ and $(H_1(k_1), D_1(k_1 d))$ are critical edges in $\tilde{W}^n$.

Because of the symmetry all the edges, not belonging to the blocks are critical. (Besides, $\chi(\tilde{W}^n - (G_0(k_0); H_1(k_1)) = 3$ implies $\chi(\tilde{W}^n) \leqq 4$. This and 2 give $\chi(\tilde{W}^n) = 4$.)

4. Now we turn to the following question: which edges of a block, say of $\mathbf{Q}_1$ are critical?

Let h be an edge of $\mathbf{Q}_1$ and the blocks $\mathbf{Q}_\tau$ be coloured as in 3 if $\tau \neq 1$. The block $\mathbf{Q}_1 - h$ is coloured by 1, 2 and 3 in the way, described in Lemma 5. Because of (+) all the vertices of $\tilde{W}^n$ but the vertices of $\{E_1(i)\} \cup \{F_2(j)\}$ and $\{G_t(k)\} \cup \{H_1(l)\}$ can be coloured by 1, 2 and 3 so that no edge joins vertices of the same colours. This colouring can be extended onto $\{E_1(i)\} \cup \{F_2(j)\}$ and $\{G_t(k)\} \cup \{H_1(l)\}$ if and only if both $\{D_1(k, d)\}_k$ and $\{A_1(i, 1)\}_i$ are coloured by two colours. According to Lemma 5 this can be achieved if

a) h is an edge belonging to $\gamma_1(ap)$ or to $\bar{\gamma}_1(dq)$.

b) $h = (A_1(i, j); B_1(i, j, x))$, $j \neq 1$.

c) $h=(B(i,j,k);\ C_1(k,l,i))$, $j\neq 1$ and $l\neq d_1$. Thus the edges described in a), b), c) are critical. Because of the symmetry the following edges are also citical:

b′) $h=(A_1(i,j);\ B_1(i,j,x))$, $j\neq d_1$.

c′) $h=(B_1(i,j,k);\ C_1(k,l,i))$, $j\neq a_1$ and $l\neq 1$. Thus all the edges of $\mathbf{Q}_1$ are critical except perhaps some edges of form $(B_1(i,j,k);\ C_1(k,l,i))$ where either $j=l=1$ or $j=a_1$, $l=d_1$.

5. Let W^n be a 4-critical subgraph of $\tilde{W}^n$. 3. and 4. give that W^n contains all the edges of $\tilde{W}^n$ except a few one of form $(B_\tau(i,j,k),\ C_\tau(k,l,i))$ where $j=l=1$ or $j=a_\tau$ $l=d_\tau$. Thus

$$\sigma(W^n)\geqq\sigma(\tilde{W}^n)-1=\min_\tau\min(p_\tau,q_\tau,a_\tau,d_\tau)=:m+1. \tag{20}$$

Here we applied that the valence of vertices $E_1(i)$, $A_1(i,j)$, $B_1(i,j,x)$ are at least p_2+1, q_2+2, d_1+1 respectively and the valence of the other vertices can be estimated from below similarly because of the symmetry.

If ec (G) denotes the edge-connectivity of the graph G, then trivially ec $(G)\leqq\sigma(G)$. In our case

$$\mathrm{ec}(W^n)=\sigma(W^n). \tag{21}$$

For the sake of simplicity we prove only

$$\mathrm{ec}(W^n)\geqq\sigma(W^n)-1=m. \tag{21*}$$

To prove this we need the following trivial notice:

(++) Let G be a graph and omit $m-1$ edges of it. Let G^* denote the obtained graph. If K is a connected component of G^* and A is a vertex of G, joined by m independent paths to m vertices of K in the graph G, then A also belongs to K.

Let us omit $m-1$ edges of W^n and denote the obtained graph by W^*. Let K_τ be the component of W^* containing $E_\tau(1)$. In order to prove (21*) we have to prove, that K_τ contains all the vertices of W^*.

The paths $E_\tau(1)-F_{\tau+1}(i)-E_\tau(i_0)$ are independent in W^n, thus $E_\tau(i_0)\in K_\tau$. The paths $F_{\tau+1}(i_0)-E_\tau(i)$ are also independent, hence $F_{\tau+1}(i_\tau)\in K_\tau$. Considering the paths

$$D_\tau(k,l)-C_\tau(k,l,i)-B_\tau(i,2,k)-A_\tau(i,2)-A_\tau(i,1)-E_\tau(i)$$

for $i=1,\ldots,m$ we obtain that $D_\tau(k,l)\in K_\tau$.

Because of the symmetry $D_\tau(k,l)$ belongs not only to the component of $E_\tau(1)$ but to the component of $F_\tau(1)$, i.e. to $K_{\tau-1}$ as well. Thus $K_\tau=K_{\tau-1}$, i.e. the components K_τ are identical with each other. This common component will be denoted by K hereafter. The paths $G_\tau(k_0)-H_{\tau+1}(k)-G_\tau(k)-D_\tau(k,1)$ for $k\leqq m$, $k\neq k_0$ and $G_\tau(k_0)-D_\tau(k_0,1)$ show that $G_\tau(k_0)\in K$. Similarly $H_\tau(k_0)\in K$, i.e. each vertex not belonging to the blocks belongs to K and the same holds for the vertices of the fourth stories. Because of the symmetry the vertices of the first stories also belong to K. The only assertion we have to prove is that $C_\tau(k_0,l_0,i_0)\in K$ too.

Since the paths $C_\tau(k_0,l_0,i_0)-B_\tau(i_0,j,k_0)-A_\tau(i_0,j)$ and $C_\tau(k_0,l_0,i_0)-$ $-D_\tau(k_0,l_0)$ are independent, $C_\tau(k_0,l_0,i_0)\in K$ and this completes the proof of (21*). (If $l_0=1$ or $l_0=d_\tau$, one of these paths is not contained by W^n!)

PROOF of Theorem 6. Let us consider W^* with the following parameters:

$$t=3, \quad p_\tau=q_\tau=a_\tau=d_\tau=v.$$

Trivially the number of vertices of W^n is

$$n = 6(v^3+v^2+2v) \tag{22}$$

while $\text{ec}\,(W^n)\geqq n$. This proves Theorem 6 for infinitely many integers.

If we wish to prove Theorem 6 for every sufficiently large even integer, we can do it in the following way:

First we suppose that n is not divisible by 4. Let us consider a W^n with 19 blocks and let

$$p_\tau = v+\varepsilon_\tau, \quad q_\tau = v-\varepsilon_\tau, \quad a_\tau = d_\tau = v$$

for $1\leqq\tau\leqq 4$, where t is odd, ε_τ is even,

$$p_\tau = v, \quad q_\tau = v+2, \quad a_\tau+d_\tau = 2v$$

for $5\leqq\tau\leqq 19$. Since W^n has

$$n = \sum p_\tau q_\tau(a_\tau+d_\tau)+(p_\tau a_\tau+q_\tau d_\tau)+2(p_\tau+q_\tau) \tag{23}$$

vertices in the general case, now it has

$$(38v^3+98v^2+76v+60)-2v\sum_{\tau=1}^{4}\varepsilon_\tau^2+2\sum_{\tau=5}^{19}d_\tau \tag{24}$$

vertices. Here the first and second terms are divisible by 4 while the third one is not. Therefore (24) is not divisible by 4.

Now we fix the least odd integer such that $38v^3\geqq n$. Clearly

$$98v^2 \leqq 38v^3+98v^2+76v-60-n = O(v^2).$$

Since every integer is the sum of four square numbers, we can achieve

$$n-20v \leqq 38v^3+98v^2+76v+60-2v\sum_{1}^{4}\varepsilon_\tau^2 \leqq n-12v.$$

Finally we select d_τ from $\left[\frac{12}{15}v, \frac{20}{15}v\right]$ so that (24) be equal to n. Since $\varepsilon_\tau=O(\sqrt{v})$ and $a_\tau, d_\tau\geqq\frac{10}{15}v$ thus Theorem 6 (and Theorem 5 too) is proved.

The case, when n is divisible by 4 can be treated similarly. The only change is that we take 5 blocks of the first type and 14 blocks of the second type. Thus W^n has

$$n = 38v^3+94v^2+76v+60-2v\sum_{\tau=1}^{5}\varepsilon_\tau^2+2\sum_{\tau=6}^{19}d_\tau$$

vertices, what is divisible by 4. This completes the proof.

A second proof of Theorem 4. We consider a W^n with 21 blocks. The first block has the parameters

$$q_1=q, \quad d_1=m, \quad a_1=qm \quad \text{and} \quad p_1=3.$$

Let $\tilde{\mathbf{Q}}_\tau$ denote the subgraph of W^n spanned by $\mathbf{Q}_\tau$ and the vertices $E_\tau(i)$, $F_\tau(j)$, $G_\tau(k)$, $H_\tau(l)$. The number of vertices of $\tilde{\mathbf{Q}}_1$ is

$$(25) \qquad 3q(qm+q)+3qm+qm+4(q+3) = 3q^2m+7qm+4q+12 =: f(q, m)$$

The proof of Theorem 6 shows that if n_1 is a sufficiently large even integer: $n_1 > n_0$, then the other parameters can be chosen so that the number of vertices of the blocks $\tilde{\mathbf{Q}}_2, \dots, \tilde{\mathbf{Q}}_{20}$ is exactly n_1. Let us choose q so that

$$f(q, m) \leqq n-n_0 \leqq f(q+1, m).$$

In this case

$$0 \leqq (n-n_0)-f(q, m) \leqq f(q+1, m)-f(q, m) = 6mq+10m+4.$$

If the parameters of the other blocks are chosen suitably, the graph W^n has exactly n vertices. Since the second story of $\mathbf{Q}_1$ contains $3q^2m$ independent vertices of valence $\geqq m$, we conclude

$$(26) \qquad n-i(4, n, m) = O(qm)+n_0 = O(\sqrt{nm})$$

(here we applied $qm = \sqrt{q^2m}\cdot\sqrt{m} = O\sqrt{nm}$.)

Final remarks. Theorem 4 holds not only for even integers, e.g. the first proof contains a construction having odd number of vertices. If we consider the same block $\mathbf{Q}$ as in the first proof of Theorem 4 and $E_1, E_2, \dots, E_q, F_1, \dots, F_s, G_1, \dots, G_t$ are new vertices, where $\{G_k\}_k$ span an odd circuit, F_l is joined to one of the vertices G_k and each F_l is joined to each E_i, finally E_i is joined to $D(i, 1)$, then we obtain a graph $\tilde{S}^n$ containing a 4-critical subgraph S^n, which proves Theorem 4 for every sufficiently large n.

It would be interesting to have some non-trivial upper bounds for (B). Finally I remark that the multiplicative constants can be improved in many statements of this paper. Since the order of magnitude of the upper and lower bounds are different, I was not interested in the constants.

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(Received July 30, 1970)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 83—89.

TWO THEOREMS ON CRITICAL 4-CHROMATIC GRAPHS

by

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Abstract

It is proved that for infinitely many integers n there exists a critical 4-chromatic graph Γ^n having n vertices and minimal-valency $>\binom{n}{2}^{\frac{1}{3}}$. We also obtain a lower bound for the possible number of independent vertices of given valency in critical 4-chromatic graphs.

O. Notation

We consider finite, undirected graphs without loops and multiple edges. We use the notation of [4]. $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices and the set of edges respectively of the graph Γ. If $A \subseteqq V(\Gamma)$ then $\Gamma(A)$ denotes the subgraph of Γ spanned by A.

1. The theorems

For an introduction to this note we refer to [3]. Our purpose is to prove:

THEOREM 1. *Let h be an integer $\geqq 3$. The following statement holds for infinitely many values of n:*

There exists a critical 4-chromatic graph Γ with n vertices containing a set I of independent vertices so that:

(1) *Each vertex of I has valency h in Γ.*

(2) $n-|I| \leqq 3(h-2)^{\frac{1}{2}} n^{\frac{1}{2}}$.

THEOREM 2. *Let h be an even integer $\geqq 4$. There exists a critical 4-chromatic graph having $2(h^2-4h+5)(h-1)$ vertices in which each vertex has valency $\geqq h$.*

The proofs of the theorems are made by giving examples. These examples are obtained from the examples of [4] using certain operations. The basic idea behind the operations is due to M. SIMONOVITS who obtained similar results in [3]. I express my thanks to M. SIMONOVITS for fruitful discussions.

I regard the results obtained in Theorems 1 and 2 as in a sense negative. They show that the property of being a critical 4-chromatic graph is less restrictive than one might have hoped for. Theorem 1 extends partly the result of [1]. Theorem 2 answers a question of T. GALLAI.

2. The examples of [4]

Let m be an odd integer $\geqq 3$. A graph Γ belongs to the class $H(m)$ if and only if it has the following structure:

α_1) $V(\Gamma) = A^1 \cup A^2 \cup A^3 \cup A^4$, where $A^i \cap A^j = \emptyset$ for $i \neq j$.

α_2) For $i=1, 2$ $\Gamma(A^i)$ is an odd circuit of length m with vertices $a_1^i, a_2^i, \ldots, a_m^i$ in cyclic order. Each vertex $a_j^i \in A^i$ ($1 \leqq i \leqq 2$ and $1 \leqq j \leqq m$) has valency 3 in Γ and is joined to a vertex a_j^{i+2} of A^{i+2}.

α_3) For $i=1, 2$ A^{i+2} is an independent set of m vertices in Γ. Each vertex of A^{i+2} is joined to precisely one vertex of A^i.

α_4) All $(A^3) \times (A^4)$-edges are in Γ.

The class $H(m)$ is in [4] denoted $H_{4,2}(m, m, m, m)$ and it has up to isomorphism only one member. It is easy to prove (see [4]):

(I) *If $\Gamma \in H(m)$ then Γ is critical 4-chromatic.*

3. The first operation

Let m be an odd integer $\geqq 3$. Let Γ be a graph satisfying $\beta_1), \ldots, \beta_4)$:
β_1) $V(\Gamma) = A \cup B \cup D \cup \{x\} \cup \{y\}$, where any two of the sets A, B, D, $\{x\}$ and $\{y\}$ are disjoint.
β_2) $\Gamma(A)$ is an odd circuit with m vertices $a_1, a_2, \ldots, a_m$ in cyclic order.
β_3) $|B|=|A|=m$ and each vertex of A is joined to precisely one vertex of B and each vertex of B is joined to precisely one vertex of A. The vertices of B are independent in Γ. Let b_i denote the vertex of B joined to a_i in Γ, $i=1, \ldots, m$.
β_4) The vertex x is joined to all vertices of $B \cup \{y\}$, but to no other vertices of Γ.
From the graph Γ we obtain a new graph Γ' as follows. Let $n_0, n_1, \ldots, n_s$ be a set of $s+1$ integers, where $2 \leqq s \leqq m-1$, satisfying:

$$(*) \qquad 1 = n_0 < n_1 < \cdots < n_{s-1} < n_s = m.$$

Consider the graph $\Gamma - x$. Introduce a set of s new, independent vertices $x_1, \ldots, x_s$. Join for each i, $1 \leqq i \leqq s$, x_i to y and to $b_{n_{i-1}}, b_{n_{i-1}+1}, \ldots, b_{n_i}$, but to no other vertices of $\Gamma - x$. The graph obtained is denoted Γ'. The operation leading from Γ to Γ' is a splitting-operation. In order to obtain Γ' the vertex $x \in V(\Gamma)$ is splitted into $x_1, \ldots, x_s$ (see Figure 1).

(II) *Let Γ in addition to $\beta_1), \ldots, \beta_4)$ satisfy:*

β_5) *Γ is critical 4-chromatic.*
β_6) *In any 3-colouring of $\Gamma - x$ the vertices of B get precisely two different colours.*
β_7) *Any colouring of the vertices of B with precisely two different colours can be extended to a 3-colouring of $\Gamma - x$.*
Then Γ' is critical 4-chromatic.

Remark to (II): If Γ satisfy $\beta_1), \dots, \beta_4)$, and if there is a vertex $d \in D$ joined to all vertices of B, then Γ satisfy $\beta_6)$. Let namely K be a 3-colouring of $\Gamma - x$. Because of d the vertices of B have $\leqq 2$ different colours in K. If they all have the same colour then we obtain a contradiction by $\beta_3)$, since this colour is used in the 3-chromatic graph $\Gamma(A)$.

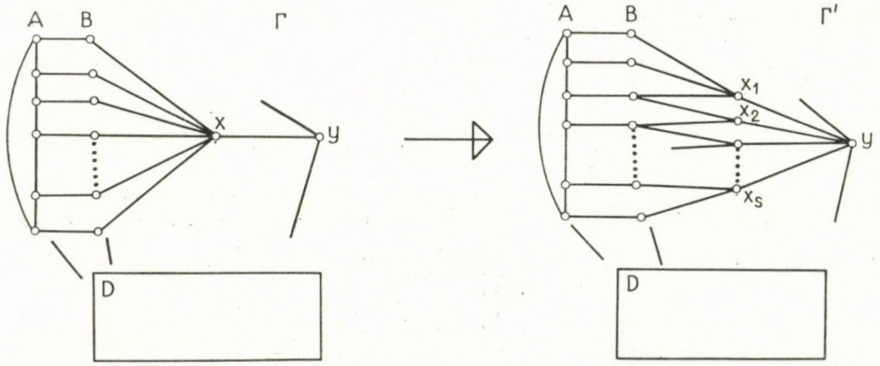

Fig. 1

Proof of (II). Let Γ satisfy $\beta_1), \dots, \beta_7)$. Suppose that Γ' has a 3-colouring K using the colours 1, 2 and 3. By $\beta_6)$ B has precisely two colours, say 1 and 2. One of the vertices $x_1, \dots, x_s$ is joined to two vertices of B having different colours, hence y has in K one of the colours 1 and 2. From the 3-colouring of $\Gamma' - x_1 - \cdots - x_s (= \Gamma - x)$ we obtain a 3-colouring of Γ by giving x the colour 3. By $\beta_5)$ this is a contradiction. A 4-colouring of Γ' can be obtained from a 4-colouring of Γ by giving the vertices $x_1, \dots, x_s$ of Γ' the colour of x. In order to prove that Γ' is critical 4-chromatic we have to prove for all $e \in E(\Gamma')$ that $\Gamma' - e$ is 3-colourable. Consider these five cases:

1) Let $e \in E(\Gamma')$ be an edge not incident with any of the vertices $x_1, \dots, x_s$. A 3-colouring of $\Gamma' - e$ can be obtained from a 3-colouring of $\Gamma - e$ by giving the vertices $x_1, \dots, x_s$ of $\Gamma' - e$ the colour of x in the 3-colouring of $\Gamma - e$.

2) Let $e = (x_i, b_j)$, where $b_j \in B$ is incident with precisely one of $x_1, \dots, x_s$ in Γ'. A 3-colouring of $\Gamma' - e$ is obtained from a 3-colouring of $\Gamma - (x, b_j)$ by giving $x_1, \dots, x_s$ of $\Gamma' - e$ the colour of x in the 3-colouring of $\Gamma - (x, b_j)$.

3) Let $e = (x_i, b_j)$, where $j = n_i$ and $1 \leqq i \leqq s-1$. Colour $b_1, \dots, b_{j-1}$ with colour 1 and $b_j, \dots, b_m$ with the colour 2 and continue this 2-colouring of B to a 3-colouring K (with colours 1, 2, and 3) of $\Gamma - x (= \Gamma' - x_1 - \cdots - x_s)$. This is possible by $\beta_7)$. Since Γ is 4-chromatic y has in K the colour 3. Colour the vertices $x_1, \dots, x_i$ with colour 2 and $x_{i+1}, \dots, x_s$ with 1. The result is a 3-colouring of $\Gamma' - e$.

4) Let $e = (x_i, b_j)$, where $j = n_{i-1}$ and $2 \leqq i \leqq s$. This case is an analogue to 3).

5) Let $e = (x_i, y)$, where $1 \leqq i \leqq s$. There exists a 2-colouring of B, so that x_i as the only vertex of $x_1, \dots, x_s$ is joined (in Γ') to vertices of B having different colours. This 2-colouring of B can by $\beta_7)$ be extended to a 3-colouring of $\Gamma - x (= \Gamma' - x_1 - \cdots - x_s)$. Each of $x_1, \dots, x_s$ is in $\Gamma' - e$ joined to a set of vertices having $\leqq 2$ different colours, hence the 3-colouring of $\Gamma' - x_1 - \cdots - x_s$ can be extended to a 3-colouring of $\Gamma' - e$.

The cases 1), …, 5) exhaust all possible cases, hence (II) is proved.

4. The second operation

The second operation is similar to the first. Let again m be an odd integer $\geqq 3$ and let Γ be a graph satisfying $\gamma_1), \ldots, \gamma_6)$:

$\gamma_1)$ $V(\Gamma) = A \cup B \cup C \cup D \cup \{x\} \cup \{y\}$, where any two of the sets $A, B, C, D, \{x\}$ and $\{y\}$ are disjoint.

$\gamma_2)$ as $\beta_2)$

$\gamma_3)$ as $\beta_3)$

$\gamma_4)$ as $\beta_3)$ with B replaced by C and b_i replaced by c_i.

$\gamma_5)$ $B \cup C$ is an independent set of vertices in Γ.

$\gamma_6)$ as $\beta_4)$ with $B \cup \{y\}$ replaced by $B \cup C \cup \{y\}$.

Let $n_0, \ldots, n_s$ $(2 \leqq s \leqq m-1)$ denote $s+1$ integers satisfying $(*)$. As in § 3 we obtain from Γ a new graph Γ' by splitting x into a set of s new, independent vertices $x_1, \ldots, x_s$, where x_i in Γ' is joined to $y, b_{n_{i-1}}, b_{n_{i-1}+1}, \ldots, b_{n_i}, c_{n_{i-1}}, c_{n_{i-1}+1}, \ldots, c_{n_i}$, but to no other vertices of Γ'.

(III) *Let Γ in addition to $\gamma_1), \ldots, \gamma_6)$ satisfy:*

$\gamma_7)$ *as* $\beta_5)$.

$\gamma_8)$ *as* $\beta_6)$.

$\gamma_9)$ *as* $\beta_6)$ *with B replaced by C.*

$\gamma_{10})$ *Any colouring of the vertices of $B \cup C$ with precisely two different colours, where b_i and c_i get the same colour for each i, $1 \leqq i \leqq m$, can be extended to a 3-colouring of $\Gamma - x$.*

Then Γ' is 4-chromatic. Let Γ'' be a critical 4-chromatic graph contained in Γ'. Then:

$\delta_1)$ $V(\Gamma'') = V(\Gamma')$

$\delta_2)$ $\forall z \in A \cup D$: $\mathrm{val}(z, \Gamma'') = \mathrm{val}(z, \Gamma)$

$\delta_3)$ $\forall z \in B \cup C$: $\mathrm{val}(z, \Gamma'') \geqq \mathrm{val}(z, \Gamma)$

$\delta_4)$ *For* $i=1, \ldots, s$: $\mathrm{val}(x_i, \Gamma'') \geqq 2(n_i - n_{i-1}) + 1$

$\delta_5)$ $\mathrm{val}(y, \Gamma'') = \mathrm{val}(y, \Gamma) + s - 1$.

PROOF of (III). Let Γ satisfy $\gamma_1), \ldots, \gamma_{10})$. Suppose that Γ' has a 3-colouring K using the colours 1, 2 and 3. In K the vertices of B have by $\gamma_8)$ precisely two different colours, say 1 and 2. By $\gamma_9)$ also the vertices of C have precisely two colours. One of $x_1, \ldots, x_s$ is joined to two vertices of B having different colours, hence y has one of the colours 1 and 2. If the colours of C are 1 and 2, then from the 3-colouring of $\Gamma' - x_1 - \cdots - x_s (= \Gamma - x)$ we obtain a 3-colouring of Γ by giving x the colour 3. This is a contradiction by $\gamma_7)$, hence we may suppose that the colours of C are 2 and 3. As above it follows that y has one of the colours 2 and 3, hence y has colour 2 in K. None of $x_1, \ldots, x_s$ has therefore the colour 2 in K. For each i, $1 \leqq i \leqq m$, either b_i or c_i has the colour 2. If not, then b_i would have the colour 1 and c_i the colour 3, hence one of $x_1, \ldots, x_s$ would have the colour 2, which is not the case. Each vertex of the 3-chromatic graph $\Gamma'(A)$ is therefore joined to a vertex of $B \cup C$ of colour 2. This is a contradiction, since at least one vertex of A has colour 2. A 4-colouring of Γ' can be obtained from a 4-colouring of Γ by giving the vertices $x_1, \ldots, x_s$ in Γ' the colour of x. Hence Γ' is 4-chromatic.

By arguments similar to the arguments in 1), ..., 5) of § 3 (using $\gamma_{10})$ instead of $\beta_7)$) it follows that all edges of Γ' are critical, except possibly a subset of

$\bigcup_{i=1}^{s-1} \{(x_i, b_{n_i}), (x_i, c_{n_i}), (x_{i+1}, b_{n_i}), (x_{i+1}, c_{n_i})\}$, and that for each i, $1 \leqq i \leqq s-1$, the following four graphs are all 3-colourable:

$$\Gamma' - (x_i, b_{n_i}) - (x_i, c_{n_i}), \quad \Gamma' - (x_{i+1}, b_{n_i}) - (x_{i+1}, c_{n_i}),$$

$$\Gamma' - (x_i, b_{n_i}) - (x_{i+1}, b_{n_i}) \quad \text{and} \quad \Gamma' - (x_i, c_{n_i}) - (x_{i+1}, c_{n_i}).$$

(III) follows from this.

5. Proof of Theorem 1

Let h be an integer $\geqq 3$ and let s be an even integer $\geqq 2$. Let $m = (h-2)s+1$. m is odd and $\geqq 3$. Let $\Gamma = \Gamma^0 \in H(m)$ and let the notation be as in § 2. Split successively each of the vertices $a_1^4, a_2^4, \ldots, a_m^4$ into s new, independent vertices by the first operation (§ 3) using in the jth step $A = A^1$, $B = A^3$, $x = a_j^4$, $y = a_j^2$ and $n_i = i(h-2)+1$ for $i = 0, \ldots, s$. Let A_j^4 denote the set of s new vertices obtained by splitting a_j^4. $A_{j_1}^4 \cap A_{j_2}^4 = \emptyset$ if $j_1 \neq j_2$. Let Γ^j denote the graph obtained in the jth step, i.e. after splitting $a_1^4, \ldots, a_j^4$.

We shall prove that $\Gamma^0, \Gamma^1, \ldots, \Gamma^m$ are critical 4-chromatic. Γ^0 is critical 4-chromatic by (I). Suppose that Γ^j is critical 4-chromatic for a j-value satisfying $0 \leqq j \leqq m-1$. Using (II) we prove that Γ^{j+1} is critical 4-chromatic.

In any 3-colouring of $\Gamma^j - a_{j+1}^4$ the vertices of A^3 get precisely two different colours. If $j \leqq m-2$ this follows by the remark to (II), since a_m^4 is joined to all vertices of A^3. If $j = m-1$ then suppose that K is a 3-colouring of $\Gamma^{m-1} - a_m^4$ giving three different colours to A^3. In this case each of the two classes A_1^4 and A_2^4 have in $K \geqq 2$ different colours, and two of the colours in A_1^4 are also two of the colours in A_2^4, hence a_1^2 and a_2^2 have the same colour in K. But a_1^2 and a_2^2 are joined by an edge. Hence the vertices of A^3 get $\leqq 2$ colours in any 3-colouring of $\Gamma^{m-1} - a_m^4$. As in the remark to (II) it follows that they get $\geqq 2$ different colours.

Any 2-colouring of A^3 using precisely two colours 1 and 2 can be continued to a 3-colouring of $\Gamma^j - a_{j+1}^4$. The 2-colouring of A^3 can namely be continued to a 3-colouring of $\Gamma^j(A^1 \cup A^3)$ (see [2] Lemma 4. 1 or [4] Lemma 1). Each vertex of $A_1^4 \cup \cdots \cup A_j^4 \cup \{a_{j+2}^4, \ldots, a_m^4\} \cup \{a_{j+1}^2\}$ (where $A_1^4 \cup \cdots \cup A_j^4 = \emptyset$ if $j=0$ and $\{a_{j+2}^4, \ldots, a_m^4\} = \emptyset$ if $j = m-1$) is given the colour 3 and the remaining vertices of A^2 the colours 1 and 2. This can be done in such a way that the result is a 3-colouring of $\Gamma^j - a_{j+1}^4$ extending the given 2-colouring of A^3.

Using (II) we conclude that Γ^{j+1} is critical 4-chromatic. By induction Γ^m is critical 4-chromatic. Let $I = A_1^4 \cup A_2^4 \cup \cdots \cup A_m^4$. I is an independent set of vertices in Γ^m. Each vertex of I has valency h in Γ^m. Moreover:

$$|V(\Gamma^m)| = 3m + ms = m\left(3 + \frac{m-1}{h-2}\right) > \frac{m^2}{h-2}$$

$$|I| = ms.$$

It follows that:

$$|V(\Gamma^m)| - |I| = 3m < 3(h-2)^{\frac{1}{2}}(|V(\Gamma^m)|)^{\frac{1}{2}}.$$

For each even s we obtain a graph Γ^m. This proves Theorem 1.

6. Proof of Theorem 2

Let h be even and $\geqq 4$ and let $s = h-2$. Let Λ denote the critical 4-chromatic graph Γ^m obtained in § 5 in this case, where $m = (h-2)s+1 = h^2-4h+5$. A_j^4 denote the set of $h-2$ vertices obtained by the splitting of a_j^4. Let us denote these vertices $a_j^4(1), a_j^4(2), \ldots, a_j^4(h-2)$. All vertices of $A^2 \cup A_1^4 \cup A_2^4 \cup \cdots \cup A_m^4$ have valency h in Λ.

Let us consider A^3. The vertex $a_j^3 \in A^3$ is joined to a_j^1 and to vertices of $A_1^4 \cup \cdots \cup A_m^4$ as described below:

If $n_{i-1} < j < n_i$, where $1 \leqq i \leqq h-2$, or if $j = n_i$, where $i = h-2$, then a_j^3 is joined to precisely one vertex $a_q^4(i)$ of each class A_q^4, $q = 1, \ldots, m$.

If $j = n_0 = 1$ then a_j^3 is joined to precisely one vertex $a_q^4(1)$ of each class A_q^4.

If $j = n_i$, where $1 \leqq i \leqq h-3$, then a_j^3 is joined to precisely two vertices $a_q^4(i)$ and $a_q^4(i+1)$ of each class A_q^4.

Split in the graph Λ successively each of the vertices a_j^3, where $j \neq n_i$ $i = 1, \ldots, h-3$, into $h-2$ new, independent vertices by the first operation (§ 3) using $A = A^2$, $x = a_j^3$, $y = a_j^1$ and $n_i = i(h-2)+1$ for $i = 0, 1, \ldots, h-2$. (This "new" n_i is equal to the n_i already introduced in this §). Let A_j^3 denote the set of vertices obtained by the splitting of a_j^3. By an induction-argument as in § 5 using (II) it is seen that the graph obtained in each step is a critical 4-chromatic graph. In each case we can use the remark to (II) to see that β_6) holds, since one of $a_{n_1}^3, a_{n_2}^3, \ldots, a_{n_{h-3}}^3$ can be used as the vertex d. Let Λ' denote the critical 4-chromatic graph obtained after splitting the $m-(h-3)$ vertices.

(IV) *$\Lambda' - a_{n_1}^3 - a_{n_2}^3 - \cdots - a_{n_{h-3}}^3$ is 3-colourable and in any 3-colouring the set of vertices $T_i = \{a_1^4(i), a_2^4(i), \ldots, a_m^4(i)\}$ gets precisely two different colours for all i, $1 \leqq i \leqq h-2$.*

Proof of (IV). Since Λ' is critical 4-chromatic the first statement is clear. Let i satisfy $1 \leqq i \leqq h-2$ and let K be a 3-colouring of $\Lambda' - a_{n_1}^3 - a_{n_2}^3 - \cdots - a_{n_{h-3}}^3$. Because $\Lambda'(A^2)$ is 3-chromatic T_i has $\geqq 2$ different colours in K. If $h \geqq 6$ then the vertices $a_{n_i-1}^3$ and $a_{n_i-2}^3$ in Λ have been splitted into $A_{n_i-1}^3$ and $A_{n_i-2}^3$ in Λ', since then $n_i \geqq n_{i-1}+4$. If T_i in this case has 3 different colours in K then $A_{n_i-1}^3$ and $A_{n_i-2}^3$ have both $\geqq 2$ different colours and two of the colours in $A_{n_i-1}^3$ are the same as two of the colours in $A_{n_i-2}^3$. It follows that $a_{n_i-1}^1$ and $a_{n_i-2}^1$ have the same colour in K, which is a contradiction, since they are joined by an edge in Λ'. If $h=4$ and $i=1$ then the above argument is valid, since a_1^3 and a_2^3 of Λ have been splitted. If $h=4$ and $i=2$ then use a_4^3 and a_5^3 instead of a_1^3 and a_2^3. (IV) follows.

Split now successively each of the vertices $a_{n_1}^3, a_{n_2}^3, \ldots, a_{n_{h-3}}^3$ into $h-2$ new, independent vertices using the second operation (§ 4) using in the j'th step $A = A^2$, $x = a_{n_j}^3$, $y = a_{n_j}^1$ and $n_i = i(h-2)+1$ for $i = 0, 1, \ldots, h-2$ and remove after each splitting edges from the obtained graph in such a way that the remaining graph is critical 4-chromatic. This is possible by (III). By (IV) γ_8) and γ_9) are fulfilled in each step. Also γ_{10}) is fulfilled. This is seen by an argument similar to the argument that β_7) is fulfilled in each step in the proof of Theorem 1.

Let the obtained graph be denoted Λ^*. It is easy to see from the structure of Λ' and from (III) that each vertex of Λ^* has valency $\geqq h$. Moreover Λ^* is critical 4-chromatic and has $2m+2m(h-2) = 2(h^2-4h+5)(h-1)$ vertices. This proves Theorem 2.

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(Received July 30, 1970)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 91—109.

ON SOME ESTIMATION PROBLEMS

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1. Introduction

In this paper, for some specified forms of $F(x; \alpha)$ we deal with the estimation of the following random variables:

$$\xi = \begin{cases} \alpha; & X \leqq a \\ 0; & X < a \end{cases} \tag{1.1}$$

where X has a distribution function $F(x; \alpha)$ with parameter α and a is a given real number, and

$$\eta = \begin{cases} \alpha_1; & X_1 \leqq X_2 \\ \alpha_2; & X_2 < X_1 \end{cases} \tag{1.2}$$

where X_1 and X_2 are independent random variables having distribution functions $F(x; \alpha_1)$ and $F(x; \alpha_2)$ (α_1 and α_2 are parameters), respectively.

An estimation of the random variable ξ in (1. 1) was given by RUBINSTEIN [5], when X has a Poisson distribution. For cases when X has binomial and exponential (gamma) distributions, estimators have been derived by the author of the present paper in [1].

SARKADI [6] dealt with the estimation of the random variable η in (1. 2) in a general case and gave estimators of η when X_1 and X_2 have normal and Poisson distributions.

In the case of the normal distribution the estimation of the random variable ξ is in close connection with the estimation of η.

In Chapter 2 of our paper estimators of ξ are given in cases of negative binomial, normal and χ^2-distributions and some complementary remarks are discussed in cases of binomial and exponential (gamma) distributions.

In Chapter 3 we deal with the estimation of the random variable η in cases of binomial, negative binomial, normal, gamma and χ^2-distributions.

In Chapter 4 it is proved that estimators given in Chapter 2 characterize the negative binomial, normal and χ^2-distributions. Similar characterization theorems relating to Poisson, binomial and exponential (gamma) distributions were proved in paper [1].

2. Estimation of the random variable ξ in (1.1)

Let us consider the random variable ξ given in (1. 1) when X has a Poisson distribution with parameter $\lambda > 0$ i.e.

$$\xi_1 = \begin{cases} \lambda; & X \leqq a \\ 0; & X > a \end{cases} \tag{2.1}$$

where a is a given positive integer. An unbiased estimator of ξ_1 given by RUBINSTEIN [5] is as follows:

$$\hat{\xi}_1 = \begin{cases} X; & X \leqq a+1 \\ 0; & X > a+1 \end{cases} \tag{2.2}$$

In the following we deal with the cases when X has binomial, negative binomial, exponential (gamma), normal and χ^2-distributions, respectively.

2. 1. Binomial distribution

Let us suppose that X has a binomial distribution with parameters n and α $(0<\alpha<1)$ and a is a given positive integer for which the inequality $0<a\leqq n$ holds. The random variable ξ_2 to be estimated is as follows:

$$\xi_2 = \begin{cases} \alpha; & X \leqq a \\ 0; & X > a \end{cases} \tag{2.3}$$

In this case no unbiased estimator based on the original statistics X can be given for the random variable ξ_2 in (2. 3) [cf. KOLMOGOROV [2]]. Supposing that an auxiliary random variable $Y(1)$, being independent of X and having a binomial distribution with parameters 1 and α, is available, the following statistics:

$$\hat{\xi}_2(1) = \begin{cases} \dfrac{X+Y(1)}{n+1}; & X+Y(1) \leqq a+1 \\ 0; & X+Y(1) > a+1 \end{cases} \tag{2.4}$$

s an unbiased estimator of ξ_2 [see [1]].

Now let us suppose that, instead of $Y(1)$, the statistics $Y(s)$ being independent of X and having a binomial distribution with parameters s and α is known. $X+Y(s)$ is a sufficient statistics, thus the conditional expectation of the unbiased estimator $\hat{\xi}_2(1)$, relative to $X+Y(s)$ is another unbiased estimator. So we obtain the following statistics:

$$\hat{\xi}_2(s) = \mathsf{E}(\hat{\xi}_2(1)|X+Y(s)) \begin{cases} \sum_{r=1}^{s} f_r; & X+Y(s) \leqq a+1 \\ \sum_{r=j}^{s} f_r; & X+Y(s) = a+j \quad (j=2, 3, \dots, s), \\ 0; & X+Y(s) > a+s \end{cases} \tag{2.5}$$

where

$$f_r = \frac{\binom{n}{X+Y(s)-r}\binom{s-1}{r-1}}{\binom{n+s}{X+Y(s)}} \qquad \left(\binom{n}{k} = 0, \text{if } k<0\right)$$

is an unbiased estimator of the random variable ξ_2. If we consider the random variable

$$\xi_2^{(i)} = \begin{cases} \alpha^i; & X \leqq a \\ 0; & X > a \end{cases} \tag{2.6}$$

instead of ξ_2 then the following statistics is an unbiased estimator of $\xi_2^{(i)}$ for case $i \leqq s$:

$$\hat{\xi}_2^{(i)}(s) = \begin{cases} \sum_{r=i}^{s} f_r'\,; & X+Y(s) \leqq a+i \\ \sum_{r=j}^{s} f_r'\,; & X+Y(s) = a+j \quad (j=i+1, i+2, \dots, s) \\ 0; & X+Y(s) > a+s \end{cases} \tag{2.7}$$

where

$$f_r' = \frac{\binom{n}{X+Y(s)-r}\binom{s-i}{r-i}}{\binom{n+s}{X+Y(s)}} \qquad \left(\binom{n}{k} = 0, \text{if } k<0\right)$$

Note: The random variable $Y(1)$ or $Y(s)$ can be interpreted in a selection procedure described in paper [1]. The selection rule is given by the random variable ξ_2, $Y(1)$ or $Y(s)$ represents additional informations being available after the selection.

2. 2. Negative binomial distribution

Let us consider the random variable

$$\xi_3 = \begin{cases} \alpha; & X \leqq n+a \\ 0; & X > n+a \end{cases} \tag{2.8}$$

where X has a negative binomial distribution with parameters n and α $(0<\alpha<1)$, that is

$$\mathsf{P}(X = n+k) = \binom{n+k-1}{k}\alpha^k(1-\alpha)^n; \qquad k=0, 1, \dots$$

and a is a given positive integer.

The following statistics

$$\hat{\xi}_3 = \begin{cases} \dfrac{X-n}{X-1}; & X \leqq n+a+1 \\ 0; & X > n+a+1 \end{cases} \tag{2.9}$$

is an unbiased estimator of ξ_3.

The variance of $(\hat{\xi}_3 - \xi_3)$ is

$$S^2(\hat{\xi}_3) = \mathsf{D}^2(\hat{\xi}_3 - \xi_3) = \mathsf{E}[(\hat{\xi}_3 - \xi_3)^2] = \tag{2.10}$$
$$= -\alpha^2 \mathsf{P}(X \leqq n+a-1) + \alpha^2 \mathsf{P}(X = n+a) + \mathsf{E}(\hat{\xi}_3^2).$$

The statistics

$$\hat{S}^2(\hat{\xi}_3) = \begin{cases} \dfrac{(n-1)(X-n)}{(X-1)^2(X-2)}; & X \leqq n+a+1 \\ \dfrac{(X-n)(X-n-1)}{(X-1)(X-2)}; & X = n+a+2 \\ 0; & X > n+a+2 \end{cases} \tag{2.11}$$

is an unbiased estimator of $S^2(\hat{\xi}_3)$.

Our results concerning binomial and negative binomial distributions are summarized in the following theorem:

THEOREM 1. *The statistics* $\hat{\xi}_2(s)$ *in* (2. 5), $\hat{\xi}_2^{(i)}(s)$ *in* (2. 7) *and* $\hat{\xi}_3$ *in* (2. 9) *are unbiased estimators of random variables* ξ_2 *in* (2. 3), $\xi_2^{(i)}$ *in* (2. 6) *and* ξ_3 *in* (2. 8), *respectively, as well as the statistics* $\hat{S}^2(\hat{\xi}_3)$ *in* (2. 11) *is an unbiased estimator of* $S^2(\hat{\xi}_3)$ *in* (2. 10).

2. 3. Exponential (gamma) distribution

Let us suppose that the random variable T_r has a gamma distribution with parameters r and $\lambda>0$, and T is a given positive real number. The density function of T_r is as follows:

$$f(t)=\begin{cases}\dfrac{\lambda^r t^{r-1}}{(r-1)!}e^{-\lambda t}; & t>0\\ 0; & t<0\end{cases}$$

In this case we consider the random variable

$$\xi_4=\begin{cases}\lambda; & T_r\geqq T\\ 0; & T_r<T\end{cases} \tag{2.12}$$

Supposing that the statistics T_{r+1} having a gamma distribution with parameters $(r+1)$ and $\lambda>0$ is known, in paper [1] the following statistics is given as an unbiased estimator of ξ_4:

$$\hat{\xi}_4(1)=\begin{cases}\dfrac{r}{T_{r+1}}; & T_{r+1}\geqq T\\ 0; & T_{r+1}<T\end{cases} \tag{2.13}$$

First we prove the following lemma:

LEMMA 1. *For the random variable* ξ_4 *no unbiased estimator based on the original statistics* T_r *exists.*

PROOF. The proof of the lemma is based on the following theorem of PUTTER [see: [4]]:

Putter's theorem: Let us suppose that X is a continuous random variable with density function $f(x;\lambda)$ (λ is the parameter of the distribution). If $f(x;\lambda)=\lambda f(\lambda x)$, i.e. $f(x;\lambda)$ belongs to the scale parameter family of density functions, then no such $g(X;X_0)$ statistics exists which is unbiased estimator of the density function $f(x_0;\lambda)$ in a given point X_0.

In our case let us suppose that there exists a statistic $h(T_r)$ which is unbiased estimator of ξ_4, that is

$$\mathsf{E}(h(T_r))=\lambda\mathsf{P}(T_r\geqq T)=\frac{\lambda^r T^{r-1}}{(r-1)!}e^{-\lambda T}+\lambda\mathsf{P}(T_{r-1}\geqq T).$$

Taking into consideration that the expectation of the statistics

$$k(T_r) = \begin{cases} \dfrac{r-1}{T_r}; & T_r \geqq T \\ 0; & T_r < T \end{cases}$$

is equal to $\lambda \mathsf{P}(T_{r-1} \geqq T)$, thus we obtain that the statistics

$$g(T_r) = h(T_r) - k(T_r)$$

is an unbiased estimator of the density function

$$f(T; \lambda) = \frac{\lambda^r T^{r-1}}{(r-1)!} e^{-\lambda T} = \lambda f(\lambda T),$$

which contradicts Putter's theorem.

Let $T_{r+s} = T_r + T_s$ where T_s is independent of T_r and has a gamma distribution with parameters s and $\lambda > 0$. In a selection procedure T_s can be interpreted as resulting from additional informations observed after the selection. $\hat{\xi}_4(1)$ is an unbiased estimator of ξ_4. The statistics T_{r+s} has a gamma distribution with parameters $(r+s)$ and $\lambda > 0$, i.e. it is a sufficient statistics for λ therefore the conditional expectation of $\hat{\xi}_4(1)$, relative to T_{r+s} is an unbiased estimator, too. So we obtain that the following statistics

(2.14) $$\hat{\xi}_4(s) = \mathsf{E}(\hat{\xi}_4(1) \mid T_{r+s}) =$$

$$= \begin{cases} \dfrac{r+s-1}{T_{r+s}} \displaystyle\sum_{k=0}^{r-1} \binom{r+s-2}{k} \left(\frac{T}{T_{r+s}}\right)^k \left(1 - \frac{T}{T_{r+s}}\right)^{r+s-2-k}; & T_{r+s} \geqq T \\ 0; & T_{r+s} < T \end{cases}$$

is an unbiased estimator of ξ_4.

Similarly we obtain an unbiased estimator of the random variable

(2.15) $$\xi_4^{(i)} = \begin{cases} \lambda^i; & T_r \geqq T \\ 0; & T_r < T \end{cases}$$

The following statistics is an unbiased estimator of $\xi_4^{(i)}$ $(i \leqq s)$:

(2.16) $$\hat{\xi}_4^{(i)}(s) =$$

$$= \begin{cases} \dfrac{(r+s-1)(r+s-2)\dots(r+s-i)}{T_{r+s}^i} \displaystyle\sum_{k=0}^{r-1} \binom{r+s-i-1}{k} \left(\frac{T}{T_{r+s}}\right)^k \left(1 - \frac{T}{T_{r+s}}\right)^{r+s-i-1-k}; \\ \qquad T_{r+s} \geqq T \\ 0; \qquad T_{r+s} < T \end{cases}$$

Our results concerning the gamma distribution are summarized in the following theorem:

THEOREM 2. a) *For ξ_4 in (2. 12) no unbiased estimator based on the original statistics T_r exists.*

b) *The statistics $\hat{\xi}_4(s)$ in (2. 14) and $\hat{\xi}_4^{(i)}(s)$ in (2. 16) are unbiased estimators of the random variables ξ_4 in (2. 12) and $\xi_4^{(i)}$ in (2. 15), respectively.*

2. 4. Normal distribution

Suppose that X has a normal distribution with unknown expectation μ and known variance σ^2 and Z is a given real number. Let us consider the random variable

$$\xi_5 = \begin{cases} \mu; & X \leqq Z \\ 0; & X > Z \end{cases} \tag{2.17}$$

In paper [4] PUTTER proved that in general no unbiased estimator of ξ_5 based on the statistics X exists. The following biased estimator, however, has the property that its bias can be made arbitrary small with a suitable choice of a constant:

$$\hat{\xi}_5 = \begin{cases} X + \sigma\sqrt{c^2+1}\,\varphi\left(\dfrac{c(X-Z)}{\sigma}\right); & X \leqq Z \\ \sigma\sqrt{c^2+1}\,\varphi\left(\dfrac{c(X-Z)}{\sigma}\right); & X > Z \end{cases} \tag{2.18}$$

where $c>0$ is a constant, $\varphi(x)$ denotes the density function of the standardized normal distributions. The bias of the estimator $\hat{\xi}_5$ is

$$B = \mathsf{E}(\hat{\xi}_5 - \xi_5) = \sigma\left[\varphi\left(\frac{c(Z-\mu)}{\sigma\sqrt{c^2+1}}\right) - \varphi\left(\frac{Z-\mu}{\sigma}\right)\right]. \tag{2.19}$$

It can be seem from (2. 19) that B tends to zero, as $c \to \infty$. By a simple calculation we get the variance of $(\hat{\xi}_5 - \xi_5)$:

$$S^2(\hat{\xi}_5) = \mathsf{E}[(\hat{\xi}_5 - \xi_5)^2] + B^2 = \tag{2.20}$$

$$= \sigma^2 \Phi\left(\frac{Z-\mu}{\sigma}\right) - \sigma(Z-\mu)\varphi\left(\frac{Z-\mu}{\sigma}\right) + \frac{2\sigma^2 c^2}{c^2+1}(Z-\mu)\varphi\left(\frac{c(Z-\mu)}{\sigma\sqrt{c^2+1}}\right)\Phi\left(\frac{Z-\mu}{\sigma\sqrt{c^2+1}}\right) -$$

$$-\frac{2\sigma^2}{\sqrt{c^2+1}}\frac{1}{\sqrt{2\pi}}\varphi\left(\frac{\sqrt{c^2+1}\,(Z-\mu)}{\sigma}\right) + B^2 + \frac{(c^2+1)\sigma^2}{\sqrt{2c^2+1}}\frac{1}{\sqrt{2\pi}}\varphi\left(\frac{\sqrt{2}\,c(Z-\mu)}{\sqrt{\sigma^2(2c^2+1}}\right),$$

where $\Phi(X)$ denotes the standardized normal distribution function.

From (2. 20) it can be seen that the last member of $\mathsf{D}^2(\hat{\xi}_5 - \xi_5)$ in (2. 20) tends to infinity as $c \to \infty$. Therefore it is reasonable to choose the constant c in such a way that the bias and variance of the estimator $\hat{\xi}_5$ should be approximately of the same magnitude.

2. 4. 1. Normal distribution with parameters μ and $\alpha^2\mu$.

In the following we consider the case when X has a normal distribution with parameters μ and $\alpha^2\mu$ (μ and α are positive, α is known), that is, the expectation is proportional to the variance. In this case for the random variable

$$\xi_5 = \begin{cases} \mu; & X \leqq Z \\ 0; & X > Z \end{cases} \tag{2.21}$$

the statistics

$$\hat{\xi}_5^{(1)} = \begin{cases} X(1-e^{\frac{Z-X}{\alpha^2}}); & X \leqq Z \\ 0; & X > Z \end{cases} \tag{2.22}$$

and

$$\hat{\xi}_5^{(2)} = \begin{cases} X; & X \leqq Z \\ Xe^{\frac{Z-X}{\alpha^2}}; & X > Z \end{cases} \tag{2.23}$$

are unbiased estimators. The statistics X is not complete see [LEHMANN [3]], since $\mathsf{E}(\hat{\xi}_5^{(2)} - \hat{\xi}_5^{(1)}) = 0$ and $\hat{\xi}_5^{(2)} - \hat{\xi}_5^{(1)} = Xe^{\frac{Z-X}{\alpha^2}} \neq 0$. The variance of $(\hat{\xi}_5^{(1)} - \xi_5)$ is

$$S^2(\hat{\xi}_5^{(1)}) = \mathsf{E}[(\hat{\xi}_5^{(1)} - \xi_5)^2] = \tag{2.24}$$

$$= \alpha^2 \mu \Phi\left(\frac{Z-\mu}{\alpha\sqrt{\mu}}\right) - 2\alpha^2 \mu e^{\frac{2Z-\mu}{2\alpha^2}} \Phi\left(\frac{Z}{\alpha\sqrt{\mu}}\right) + (\alpha^2\mu + \mu^2)^2 e^{\frac{2Z}{\alpha^2}} \Phi\left(\frac{Z+\mu}{\alpha\sqrt{\mu}}\right).$$

An unbiased estimator of $S^2(\hat{\xi}_5^{(1)})$ is the following statistics:

$$\hat{S}^2(\hat{\xi}_5^{(1)}) = \tag{2.25}$$

$$= \begin{cases} \alpha^2 X(1-e^{\frac{Z-X}{\alpha^2}}) + X(e^{\frac{2(Z-X)}{\alpha^2}} - 2e^{\frac{Z-X}{\alpha^2}}) + \frac{X^3}{2\alpha^2} e^{\frac{Z-X}{\alpha^2}} + ZXe^{\frac{Z-X}{\alpha^2}} - \frac{Z^2 X}{2\alpha^2} e^{\frac{Z-X}{\alpha^2}}; & X \leqq Z \\ 0; & X > Z \end{cases}$$

The variance of $(\hat{\xi}_5^{(2)} - \xi_5)$ is

$$S^2(\hat{\xi}_5^{(2)}) = \mathsf{E}[(\hat{\xi}_5^{(2)} - \xi_5)^2] = \alpha^2 \mu \Phi\left(\frac{Z-\mu}{\alpha\sqrt{\mu}}\right) + (\alpha^2\mu + \mu^2) e^{\frac{2Z}{\alpha^2}} \Phi\left(-\frac{Z+\mu}{\alpha\sqrt{\mu}}\right). \tag{2.26}$$

An unbiased estimator of $S^2(\hat{\xi}_5^{(2)})$ is

$$\hat{S}^2(\hat{\xi}_5^{(2)}) = \tag{2.27}$$

$$= \begin{cases} \alpha^2 X; & X \leqq Z \\ \alpha^2 X e^{\frac{Z-X}{\alpha^2}} + X^2 e^{\frac{2(Z-X)}{\alpha^2}} + \frac{X^3}{2\alpha^2} e^{\frac{Z-X}{\alpha^2}} - ZXe^{\frac{Z-X}{\alpha^2}} - \frac{Z^2 X}{2\alpha^2} e^{\frac{Z-X}{\alpha^2}}; & X > Z \end{cases}$$

The statements discussed above are summarized in the following theorems:

THEOREM 3. *If X is a normally distributed random variable with parameters μ and σ^2 (σ^2 is known), then*

a) *No unbiased estimator of the random variable ξ_5 in* (2. 17) *exists.*

b) *The bias of the statistic $\hat{\xi}_5$ in* (2. 18) *tends to zero, but its variance tends to infinity as $c \to \infty$.*

THEOREM 3. 1. a) *In the case when $\sigma^2 = \alpha^2 \mu$ (μ and α are positive real numbers, α is known), the statistics $\hat{\xi}_5^{(1)}$ in* (2. 22) *and $\hat{\xi}_5^{(2)}$ in* (2. 23) *are unbiased estimators of ξ_5 in* (2. 21).

b) *The statistics X is not complete.*

c) *The statistics $\hat{S}^2(\hat{\xi}_5^{(1)})$ in* (2. 25) *and $\hat{S}^2(\hat{\xi}_5^{(2)})$ in* (2. 27) *are unbiased estimators of $S^2(\hat{\xi}_5^{(1)})$ in* (2. 24) *and $S^2(\hat{\xi}_5^{(2)})$ in* (2. 26), *respectively.*

2. 5. χ^2-distribution

Suppose again that X is a normally distributed random variable with parameters μ and σ^2 (σ^2 is unknown). Let $X_1, X_2, \dots, X_n$ be independent realizations of the random variable X and their mean $\bar{X} = \sum_{i=1}^{n} X_i/n$. In this case $Y/\sigma^2 = \sum_{i=1}^{n} (X_i-\bar{X})^2/\sigma^2$ has χ^2-distribution with parameter $m = n-1$ (degrees of freedom). Let us consider the random variable

$$\xi_6 = \begin{cases} \sigma^2; & Y \leqq Z \\ 0; & Y > Z \end{cases} \tag{2.28}$$

The following statistics is an unbiased estimator of ξ_6:

$$\hat{\xi}_6 = \begin{cases} \dfrac{Y}{m}; & Y \leqq Z \\ \dfrac{1}{m} \dfrac{Z^{\frac{m}{2}}}{Y^{\frac{m}{2}-1}}; & Y > Z \end{cases} \tag{2.29}$$

By a simple calculation we obtain the variance of $(\hat{\xi}_6 - \xi_6)$:

$$S^2(\hat{\xi}_6) = \mathsf{E}[(\hat{\xi}_6 - \xi_6)^2] = \tag{2.30}$$

$$= \frac{2\sigma^4}{m^2} \int_0^Z f_m(x)\,dx + \frac{2\sigma^4}{m^2}(m-2) Z f_m(Z) - \frac{2\sigma^2}{m^2} Z^2 f_m(Z) + \frac{Z^m}{m^2} \int_Z^\infty \frac{1}{x^{m-2}} f_m(x)\,dx$$

where

$$f_m(Z) = \begin{cases} \dfrac{Z^{\frac{m}{2}-1} e^{-\frac{Z}{2\sigma^2}}}{2^{\frac{m}{2}} \Gamma\left(\dfrac{m}{2}\right) \sigma^m}; & Z > 0 \\ 0; & Z \leqq 0 \end{cases}$$

The following statistics is an unbiased estimator of $S^2(\hat{\xi}_6)$:

$$\hat{S}^2(\hat{\xi}_6) = \begin{cases} \dfrac{Y^2}{m^2\left(\dfrac{m}{2}+1\right)}; & Y \leqq Z \\ \dfrac{1}{m^2} \dfrac{Z^m}{Y^{m-2}} + \dfrac{1}{2m} \dfrac{Z^{\frac{m}{2}}}{Y^{\frac{m}{2}-2}} - \dfrac{1}{2m} \dfrac{m+4}{m+2} \dfrac{Z^{\frac{m}{2}+1}}{Y^{\frac{m}{2}-1}}; & Y > Z \end{cases} \tag{2.31}$$

Our statements are summarized in the following theorem:

THEOREM 4. *The statistics $\hat{\xi}_6$ in* (2. 29) *and $\hat{S}^2(\hat{\xi}_6)$ in* (2. 31) *are unbiased estimators of ξ_6 in* (2. 28) *and $S^2(\hat{\xi}_6)$ in* (2. 30), *respectively.*

3. Estimation of the random variable η in (1.2)

The estimation of the random variable η in (1. 2) was discussed by SARKADI [6] in a general case and in cases when X_1 and X_2 have Poisson and normal distributions. In this paper estimators are given for cases when X_1 and X_2 have binomial, negative binomial, gamma and χ^2-distributions. Some complementary remarks are discussed for cases of Poisson and normal distributions.

3. 1. Poisson distribution

Let us suppose that X_1 and X_2 are independent random variables having Poisson distribution with parameters λ_1 and λ_2, respectively. The following random variable is to be estimated:

$$\eta_1 = \begin{cases} \lambda_1; & X_1 < X_2 \\ \dfrac{\lambda_1+\lambda_2}{2}; & X_1 = X_2 \\ \lambda_2; & X_2 < X_1 \end{cases} \tag{3. 1}$$

SARKADI [6] gave the following unbiased estimator of η_1:

$$\hat{\eta}_1 = t(X_1; X_2) = \begin{cases} X_1; & X_1 < X_2 - 1 \\ \frac{3}{2}X_2 - 1; & X_1 = X_2 - 1 \\ 2X_1; & X_1 = X_2 \\ t(X_2; X_1); & X_2 < X_1 \end{cases} \tag{3. 2}$$

The variance of the estimator $\hat{\eta}_1$ is

$$S^2(\hat{\eta}_1) = \lambda_1 \mathsf{P}(X_1 - X_2 < 0) + \lambda_2 \mathsf{P}(X_2 - X_1 < 0) + \frac{\lambda_1^2}{4}\mathsf{P}(X_2 - X_1 = 1) + \tag{3. 3}$$

$$+ \frac{\lambda_2^2}{4}\mathsf{P}(X_1 - X_2 = 1) + \frac{(\lambda_1+\lambda_2)^2 + (\lambda_1+\lambda_2)}{4}\mathsf{P}(X_1 = X_2).$$

An unbiased estimator of $S^2(\hat{\eta}_1)$ is the following statistic:

$$\hat{S}^2(\hat{\eta}_1) = h(X_1; X_2) = \begin{cases} X_1; & X_1 < X_2 - 2 \\ (X_2^2 + 3X_2)/4 - 2; & X_1 = X_2 - 2 \\ (X_2^2 + 4X_2)/4 - 1; & X_1 = X_2 - 1 \\ (X_1^2 + 4X_1)/2; & X_1 = X_2 \\ h(X_2; X_1); & X_2 < X_1 \end{cases} \tag{3. 4}$$

3. 2. Binomial distribution

Let us consider the following random variable:

$$\eta_2 = \begin{cases} \alpha_1; & X_1 < X_2 \\ \dfrac{\alpha_1+\alpha_2}{2}; & X_1 = X_2 \\ \alpha_2; & X_2 < X_1 \end{cases} \tag{3. 5}$$

where X_1 and X_2 are independent random variables having binomial distribution with parameters n and α_1 $(0<\alpha_1<1)$ and n and α_2 $(0<\alpha_2<1)$, respectively. No unbiased estimator based on the statistics of X_1 and X_2 exists [see [2]], but supposing the availability of the auxiliary random variables $Y_1(1)$ and $Y_2(1)$ which are independent of each other and of X_1, X_2, $Y_1(1)$, $Y_2(1)$ have binomial distribution with parameters 1 and α_i $(i=1, 2)$, the following unbiased estimator can be given:

$$\hat{\eta}_2 = s(X_1+Y_1(1); X_2+Y_2(1)) = \tag{3.6}$$

$$= \begin{cases} \dfrac{X_1+Y_1(1)}{n+1}; & X_1+Y_1(1)<X_2+Y_2(1)-1 \\ \dfrac{X_2+Y_2(1)-1}{n+1}+\dfrac{1}{2}\dfrac{X_2+Y_2(1)}{n+1}\left(1-\dfrac{X_2+Y_2(1)-1}{n+1}\right); & X_1+Y_1(1)=X_2+Y_2(1)-1 \\ \dfrac{X_1+Y_1(1)}{n+1}\left(2-\dfrac{X_1+Y_1(1)}{n+1}\right); & X_1+Y_1(1)=X_2+Y_2(1) \\ s(X_2+Y_2(1); X_1+Y_1(1)); & X_2+Y_2(1)<X_1+Y_1(1) \end{cases}$$

3. 3. Negative binomial distribution

Let us consider the following random variable

$$\eta_3 = \begin{cases} \alpha_1; & X_1<X_2 \\ \dfrac{\alpha_1+\alpha_2}{2}; & X_1=X_2 \\ \alpha_2; & X_2<X_1 \end{cases} \tag{3.7}$$

where X_1 and X_2 are independent random variables having negative binomial distribution with parameters n and α_i $(0<\alpha_i<1)$; $(i=1, 2)$ that is

$$\mathsf{P}(X_i=n+k) = \binom{n+k-1}{k}\alpha_i^k(1-\alpha_i)^k; \qquad k=0, 1 \ldots$$

Unbiased estimator of η_3 is given by the following statistics:

$$\hat{\eta}_3 = g(X_1, X_2) = \begin{cases} \dfrac{X_1-n}{X_1-1}; & X_1<X_2-1 \\ \dfrac{X_1-n}{X_1-1}+\dfrac{1}{2}\dfrac{X_2-n}{X_2-1}; & X_1=X_2-1 \\ \dfrac{2(X_1-n)}{X_1-1}; & X_1=X_2 \\ g(X_2; X_1); & X_2<X_1 \end{cases} \tag{3.8}$$

Our results concerning Poisson, binomial and negative binomial distributions are summarized in the following theorem:

THEOREM 5. *The statistics* $\hat{\eta}_1$ *in* (3. 2), $\hat{\eta}_2$ *in* (3. 6) *and* $\hat{\eta}_3$ *in* (3. 8) *are unbiased estimators of* η_1 *in* (3. 1), η_2 *in* (3. 5) *and* η_3 *in* (3. 7), *respectively, as well as the statistics* $\hat{S}^2(\hat{\eta}_1)$ *in* (3. 4) *is an unbiased estimator of* $S^2(\hat{\eta}_1)$ *in* (3. 3).

3. 4. *Exponential (gamma) distribution*

Let us consider the random variable

$$\eta_4 = \begin{cases} \lambda_1; & T_{1,r} \geqq T_{2,r} \\ \lambda_2; & T_{2,r} > T_{1,r}, \end{cases} \tag{3.9}$$

where $T_{1,r}$ and $T_{2,r}$ are independent random variables having gamma distribution with parameters r and $\lambda_i > 0$ $(i=1, 2)$. No unbiased estimator based on the statistics of $T_{1,r}$ and $T_{2,r}$ exists (see PUTTER [4] and Lemma 1. in Chapter 2 of this paper). An unbiased estimator based on $T_{1,r+1}$ and $T_{2,r+1}$ ($T_{i,r+1}$ has a gamma distribution with parameters $(r+1)$ and $\lambda_i > 0$; $i=1, 2$), however, can be given in the following form:

$$\hat{\eta}_4 = \begin{cases} \dfrac{r}{T_{1,r+1}} + r\dfrac{T_{2,r+1}^{r-1}}{T_{1,r+1}^{r}}; & T_{1,r+1} \geqq T_{2,r+1} \\[2ex] \dfrac{r}{T_{2,r+1}} + r\dfrac{T_{1,r+1}^{r-1}}{T_{2,r+1}^{r}}; & T_{2,r+1} > T_{1,r+1}. \end{cases} \tag{3.10}$$

THEOREM 6. *For the random variable* η_4 *in* (3. 9) *no unbiased estimator based on the original statistics* $T_{1,r}$ *and* $T_{2,r}$ *exists. The statistics* $\hat{\eta}_4$ *in* (3. 10) *based on statistics* $T_{1,r+1}$ *and* $T_{2,r+1}$, *however, is an unbiased estimator of* η_4.

3. 5. *Normal distribution*

Let us suppose that X_1 and X_2 are independent, normally distributed random variables with unknown expectations μ_i $(i=1, 2)$ and known variances σ_i^2 $(i=1, 2)$. Let us consider the random variable:

$$\eta_5 = \begin{cases} \mu_1; & X_1 \leqq X_2 \\ \mu_2; & X_2 < X_1 \end{cases} \tag{3.11}$$

If $X_2 = Z$ (constant) and $X_1 = X$, $\mu_1 = \mu$; $\sigma_1^2 = \sigma^2$ the expectation of η_5 differs from the expectation of ξ_5 given in (2. 17) in the term $Z\Phi\left(\frac{\mu - Z}{\sigma}\right)$. The estimator of $Z\Phi\left(\frac{\mu - Z}{\sigma}\right)$ is known, thus the estimation of the random variable ξ_5 can be obtained from the estimation of η_5.

SARKADI [6] gave the following estimator of η_5:

$$\hat{\eta}_5^{(1)} = X_1 + (X_2 - X_1)\Phi\left(\frac{c(X_1 - X_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) + c\sqrt{\sigma_1^2 + \sigma_2^2}\,\varphi\left(\frac{c(X_1 - X_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right), \tag{3.12}$$

where $\Phi(x)$ denotes the standardized normal distribution function, $\varphi(x)=\Phi'(x)$; $c>0$ is a constant.

The bias of $\hat{\eta}_5^{(1)}$ is

$$(3.13)\quad B_1 = \mathsf{E}(\hat{\eta}_5^{(1)}-\eta_5) = (\mu_2-\mu_1)\left[\Phi\left(\frac{c(\mu_1-\mu_2)}{\sqrt{(c^2+1)(\sigma_1^2+\sigma_2^2)}}\right)-\Phi\left(\frac{\mu_1-\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)\right].$$

From (3. 13) it can be seen that the bias of $\hat{\eta}_5^{(1)}$ can be made arbitrary small as $c\to\infty$. The variance of $(\hat{\eta}_5^{(1)}-\eta_5)$ cannot be given in an explicit form, but using the Cramer—Rao's lower bound it can be shown the variance of $(\hat{\eta}_5^{(1)}-\eta_5)$ tends to infinity as $c\to\infty$ (see [6]). Therefore it is reasonable to choose the constant c in such a way that the bias and the variance of $(\hat{\eta}_5^{(1)}-\eta_5)$ should be approximately of the same magnitude. Another estimator of η_5 given by SARKADI is

$$(3.14)\qquad \hat{\eta}_5^{(2)} = \min(X_1, X_2)+\sqrt{(c^2+1)(\sigma_1^2+\sigma_2^2)}\,\varphi\left(\frac{c(X_1-X_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right),$$

where $c>0$ is a constant.

The bias of $\hat{\eta}_5^{(2)}$ is

$$(3.15)\quad B_2 = \mathsf{E}(\hat{\eta}_5^{(2)}-\eta_5) = \sqrt{\sigma_1^2+\sigma_2^2}\left[\varphi\left(\frac{c(\mu_1-\mu_2)}{\sqrt{(c^2+1)(\sigma_1^2+\sigma_2^2)}}\right)-\varphi\left(\frac{\mu_1-\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)\right].$$

From (3. 15) it can be seen that the bias of $\hat{\eta}_5^{(2)}$ tends to zero as $c\to\infty$. The variance of $(\hat{\eta}_5^{(2)}-\eta_5)$ can be calculated. The calculation leads to a complicated formula which can be written in the following form:

(3. 16)

$$\mathsf{D}^2(\hat{\eta}_5^{(2)}-\eta_5) = A(\mu_1, \mu_2, \sigma_1, \sigma_2, c)+\frac{(c^2+1)(\sigma_1^2+\sigma_2^2)}{\sqrt{2c^2+1}}\frac{1}{\sqrt{2\pi}}\varphi\left(\frac{\sqrt{2}\,c(\mu_2-\mu_1)}{\sqrt{(2c^2+1)(\sigma_1^2+\sigma_2^2)}}\right).$$

where $A(\mu_1, \mu_2, \sigma_1, \sigma_2, c)$ tends to a constant and the second term on the left side of (3. 16) tends to infinity as $c\to\infty$. Therefore $\mathsf{D}^2(\hat{\eta}_5^{(2)}-\eta_5)$ tends to infinity as $c\to\infty$. Thus considerations about the choice of the constant c in the estimator $\hat{\eta}_5^{(1)}$ can be extended to this case.

In the above case no unbiased estimator of η_5 based on the statistics X_1 and X_2 exists (see PUTTER [4] and SARKADI [6]). An estimator of η_5 can be derived using the Bayes-method. Let us suppose having "a priori" information about the distributions of μ_1 and μ_2, that is the μ_i $(i=1, 2)$ are normally distributed, independent random variables with parameters m_i $(i=1, 2)$ and σ_i^2/k^2 $(i=1, 2)$. The "a posteriori" distribution of μ_i $(i=1, 2)$ is a normal one, too, with parameters $(X_i+k^2m_i)/(k^2+1)$ and $\sigma_i^2/(k^2+1)$(k is a constant), $(i=1, 2)$. In this case the estimator is as follows:

$$(3.17)\qquad \hat{\eta}_5^{(3)} = \begin{cases} \mathsf{E}(\mu_1|X_1) = \dfrac{X_1+k^2m_1}{k^2+1}; & X_1\leq X_2 \\[2ex] \mathsf{E}(\mu_2|X_2) = \dfrac{X_2+k^2m_2}{k^2+1}; & X_2<X_1 \end{cases}$$

The bias of $\hat{\eta}_5^{(3)}$ is

$$B_3 = \mathsf{E}(\hat{\eta}_5^{(3)} - \eta_5 | \mu_1, \mu_2) = \frac{k^2(m_1 - \mu_1)}{k^2+1} +$$

$$+ \frac{k^2[(m_2-m_1)-(\mu_2-\mu_1)]}{k^2+1} \Phi\left(\frac{\mu_1-\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right) - \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{k^2+1} \varphi\left(\frac{\mu_1-\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right).$$

Note: Another estimator of η_5 was given in paper [1] using a "pooling method".

3. 5. 1. Normal distribution with parameters μ_i and $\alpha^2\mu_i$

In the case, when X_i has a normal distribution with the expectation $\mu_i > 0$ and with the variance $\sigma_i^2 = \alpha^2 \mu_i$ $(i=1, 2)$, where $\alpha > 0$ is known, unbiased estimators can be given. In this case the unbiased estimators for η_5 are given by the following statistics:

$$\hat{\eta}_5^{(4)} = \min(X_1, X_2)\left(1 - e^{\frac{|X_2 - X_1|}{\alpha^2}}\right) \tag{3.18}$$

and

$$\hat{\eta}_5^{(5)} = \begin{cases} X_1 + X_2 e^{\frac{X_1 - X_2}{\alpha^2}}; & X_1 \leqq X_2 \\ X_2 + X_1 e^{\frac{X_2 - X_1}{\alpha^2}}; & X_2 < X_1 \end{cases} \tag{3.19}$$

The statistics $T = \{X_1, X_2\}$ is not complete (see LEHMANN [3]), since

$$\mathsf{E}[\hat{\eta}_5^{(5)} - \hat{\eta}_5^{(4)}] = 0,$$

but

$$\hat{\eta}_5^{(5)} - \hat{\eta}_5^{(4)} = X_1 e^{\frac{X_2 - X_1}{\alpha^2}} + X_2 e^{\frac{X_1 - X_2}{\alpha^2}} \neq 0.$$

Our results concerning normal distributions are summarized in the following theorems:

THEOREM 7. *If X_1 and X_2 are normally distributed random variables with parameters μ_i and σ_i^2 $(i=1, 2)$, then*

a) *No unbiased estimator of η_5 in* (3. 11) *based on the statistics of X_1 and X_2 exists.*

b) *The bias of estimators $\hat{\eta}_5^{(1)}$ in* (3. 12) *and $\hat{\eta}_5^{(2)}$ in* (3. 14) *tends to zero as $c \to \infty$ but the variances of these estimators tend to infinity as $c \to \infty$.*

THEOREM 7. 1. *In the case, when $\sigma_i^2 = \alpha^2 \mu_i$ $(i=1, 2)$ ($\alpha > 0$ is known), unbiased estimators of η_5 are given by the statistics $\hat{\eta}_5^{(4)}$ in* (3. 18) *and $\hat{\eta}_5^{(5)}$ in* (3. 19).

3. 6. χ^2-distribution

Let us assume that X_1 and X_2 are normally distributed, independent random variables with parameters μ_1, σ_1^2 and μ_2, σ_2^2, respectively. The independent realizations of X_i $(i=1, 2)$ are as follows: $X_{i1}, X_{i2}, \ldots, X_{in}$ $(i=1, 2)$. The sample mean is $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$ $(i=1, 2)$. Thus $Y_i/\sigma_i^2 = \sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2/\sigma_i^2$ $(i=1, 2)$ has χ^2-distribution with parameter $m = n-1$.

Let us consider the random variable:

$$\eta_6 = \begin{cases} \sigma_1^2; & Y_1 \leqq Y_2 \\ \sigma_2^2; & Y_2 < Y_1 \end{cases} \tag{3.20}$$

The following statistics is an unbiased estimator of η_6:

$$\hat{\eta}_6 = \begin{cases} \dfrac{Y_1}{m} + \dfrac{1}{m} \dfrac{Y_1^{\frac{m}{2}}}{Y_2^{\frac{m}{2}-1}}; & Y_1 \leqq Y_2 \\[2ex] \dfrac{Y_2}{m} + \dfrac{1}{m} \dfrac{Y_2^{\frac{m}{2}}}{Y_1^{\frac{m}{2}-1}}; & Y_2 < Y_1 \end{cases} \tag{3.21}$$

The variance of $(\hat{\eta}_6 - \eta_6)$ is

$$S^2(\hat{\eta}_6) = \frac{2\sigma_1^4}{m} + \frac{2(\sigma_2^4 - \sigma_1^4)}{m} \int_0^\infty f_{m,2}(z) \left[\int_z^\infty f_{m,1}(u)\, du \right] dz + \tag{3.22}$$

$$+ \frac{1}{m^2} \int_0^\infty z^m f_{m,1}(z) \left[\int_z^\infty u^{2-m} f_{m,2}(u)\, du \right] dz + \frac{1}{m^2} \int_0^\infty z^m f_{m,2}(z) \left[\int_z^\infty u^{2-m} f_{m,1}(u)\, du \right] dz +$$

$$+ \left(\frac{\sigma_2 \sigma_1}{\sigma_1^2 + \sigma_2^2} \right)^m \frac{\Gamma(m)}{\Gamma\left(\frac{m}{2}\right)} \frac{1}{m} \left[\frac{(\sigma_1^2 - \sigma_2^2)}{2} - \frac{1}{m} (\sigma_1^4 + \sigma_2^4) \right],$$

where

$$f_{m,i}(z) = \begin{cases} \dfrac{z^{\frac{m}{2}-1} e^{-\frac{z}{2\sigma_i^2}}}{2^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right) \sigma_i^m}; & z > 0 \\[2ex] 0; & z \leqq 0 \end{cases} \qquad (i = 1, 2).$$

Unbiased estimator of $S^2(\hat{\eta}_6)$ is

$$\hat{S}^2(\hat{\eta}_6) = h(Y_1; Y_2) = \tag{3.23}$$

$$= \begin{cases} \dfrac{Y_1^2}{m^2\left(\frac{m}{2}+1\right)} - \dfrac{1}{2m} \dfrac{m+4}{m+2} \dfrac{Y_1^{\frac{m}{2}+1}}{Y_2^{\frac{m}{2}-1}} + \dfrac{1}{2m} \dfrac{Y_1^{\frac{m}{2}}}{Y_2^{\frac{m}{2}-2}} + \dfrac{1}{m^2} \dfrac{Y_1^m}{Y_2^{m-2}}; & Y_1 \leqq Y_2 \\[2ex] h(Y_2; Y_1); & Y_2 < Y_1 \end{cases}$$

THEOREM 8. *The statistics $\hat{\eta}_6$ in* (3. 21) *and $\hat{S}^2(\hat{\eta}_6)$ in* (3. 23) *are unbiased estimators of η_6 in* (3. 20) *and $S^2(\hat{\eta}_6)$ in* (3. 22), *respectively.*

4. Characterization theorems

In this Chapter it will be shown that the negative binomial, normal and χ^2-distributions are characterized by unbiased estimators of a given form, i.e. by those given in Chapter 2. In paper [1] characterization theorems of binomial, Poisson and exponential distributions were proved.

THEOREM 9. *Let X be a discrete random variable taking on the values $n, (n+1), \dots, (n+k), \dots$ with probabilities $p_0(\alpha; n), p_1(\alpha, n), \dots, p_k(\alpha, n), \dots$ respectively, where α $(0<\alpha<1)$ and n (a positive integer) are the parameters of the distribution. If for every non-negative integer k $(k=0, 1, 2, \dots)$ the recursive formula*

$$p_{k+1}(\alpha, n) = \frac{n+k}{k+1}\alpha p_k(\alpha, n) \tag{4. 1}$$

holds, then

$$p_k(\alpha; n) = \binom{n+k-1}{k}\alpha^k(1-\alpha)^n; \qquad (k=0, 1, \dots), \tag{4. 2}$$

i.e. X has a negative binomial distribution with parameters α and n.

PROOF. Replacing in (4. 1) k successively by $0, 1, \dots, (k-1)$ and multiplying all the equations thus formed, we obtain

$$p_k = \binom{n+k-1}{k}\alpha^k p_0. \tag{4. 3}$$

Since $\sum_{k=0}^{\infty} p_k=1$ thus we get

$$p_0 = (1-\alpha)^n. \tag{4. 4}$$

It follows from (4. 3) and (4. 4) that

$$p_k(\alpha, n) = \binom{n+k-1}{k}\alpha^k(1-\alpha)^n.$$

As a corollary of Theorem 9. the following theorem holds:

THEOREM 10. *Let X be a discrete random variable taking on the values $n, n+1), \dots, (n+k), \dots$ with probabilities $p_0(\alpha, n), p_1(\alpha, n), \dots, p_k(\alpha, n), \dots$ respectively, where α $(0<\alpha<1)$ and n (positive integer) are the parameters of the distribution. If for every positive integer a the statistics*

$$\hat{f} = \begin{cases} \dfrac{X-n}{X-1}; & n \leqq X \leqq n+a+1 \\ 0; & X > n+a+1 \end{cases}$$

is an unbiased estimator of the random variable

$$f = \begin{cases} \alpha; & n \leqq X < n+a+1 \\ 0; & X \geqq n+a+1 \end{cases}$$

then X has a negative binomial distribution with parameters α and n.

PROOF. It follows from the assumptions that

$$\alpha \sum_{k=0}^{a} p_k = \sum_{k=0}^{a+1} \frac{k}{n+k-1} p_k. \tag{4. 5}$$

Replacing a in (4. 5) by $(a-1)$ and subtracting the equation obtained from the original one (4. 5) we obtain the recursive formula in (4. 1).

THEOREM 11. *Let X be a continuous random variable with density function $f(x: \mu; \sigma)$, where μ and $\sigma>0$ are the parameters of the distribution. If for every real number Z the relation*

$$\mathsf{E}(\hat{g}-g) = -\sigma f(Z; \mu; \sigma) \tag{4. 6}$$

holds, where

$$\hat{g} = \begin{cases} X; & X \leqq Z \\ 0; & X > Z \end{cases}$$

and

$$g = \begin{cases} \mu; & X \leqq Z \\ 0; & X > Z \end{cases}$$

then

$$f(x; \mu; \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(X-\mu)^2}{2\sigma^2}},$$

that is X normally distributed with parameters μ and σ.

PROOF. It follows from (4. 6) that

$$\mu \int_{-\infty}^{Z} f(x; \mu; \sigma)\, dx = \int_{-\infty}^{Z} x f(x; \mu; \sigma)\, dx + \sigma f(Z; \mu; \sigma). \tag{4. 7}$$

Differentiating both sides of the relation (4. 7) with respect to Z we obtain the following differential equation

$$\frac{f'(Z; \mu; \sigma)}{f(Z; \mu; \sigma)} = -\frac{Z-\mu}{\sigma}. \tag{4. 8}$$

The solution of the differential equation in (4. 8) is

$$f(Z; \mu; \sigma) = \frac{c}{\sigma} e^{-\frac{(Z-\mu)^2}{2\sigma^2}}.$$

Since $f(Z; \mu; \sigma)$ is a density function we determine the constant c and we obtain

$$f(Z; \mu; \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(Z-\mu)^2}{2\sigma^2}}.$$

THEOREM 12. *Let X be a continuous random variable with density function $f(x;\mu;\alpha)=f(x)$, where $\mu>0$ and $\alpha>0$ are the parameters of the distribution. If the statistics*

$$\hat{f}=\begin{cases} X(1-e^{\frac{Z-X}{\alpha^2}}); & X\leqq Z\\ 0; & X>Z\end{cases}$$

is an unbiased estimator for the random variable

$$f=\begin{cases}\mu; & X\leqq Z\\ 0; & X>Z\end{cases}$$

for every real number Z, then X is normally distributed with parameters μ and $\alpha\sqrt{\mu}$.

PROOF. It follows from the assumption that

$$\mu\int_{-\infty}^{Z} f(x)\,dx=\int_{-\infty}^{Z} xf(x)\,dx-\int_{-\infty}^{Z} xe^{\frac{Z-x}{\alpha^2}}f(x)\,dx. \tag{4.9}$$

Differentiating both sides of the relation (4. 9) with respect to Z we obtain the following differential equation:

$$\frac{F'(Z)}{F(Z)}=-\frac{1}{\alpha^2\mu}Z, \tag{4.10}$$

where

$$F(Z)=\int_{-\infty}^{Z} xe^{-\frac{x}{\alpha^2}}f(x)\,dx. \tag{4.11}$$

The solution of the differential equation (4. 10) is

$$F(Z)=Ce^{-\frac{1}{2\alpha^2\mu}Z^2} \tag{4.12}$$

From (4. 11) and (4. 12) we get

$$f(Z)=-\frac{C}{\alpha^2\mu}e^{-\frac{Z^2}{2\alpha^2\mu}+\frac{Z}{\alpha^2}}.$$

Taking into consideration that $f(Z)$ is a density function, we obtain

$$C=-\frac{\alpha\sqrt{\mu}}{\sqrt{2\pi}}e^{-\frac{\mu}{2\alpha^2}}.$$

Thus we have

$$f(Z;\mu;\alpha)=\frac{1}{\alpha\sqrt{\mu}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(Z-\mu)^2}{2\alpha^2\mu}}.$$

THEOREM 13. *Let X be a non-negative continuous random variable with density function $f(x;\sigma;m)=f(x)$, where $\sigma>0$ and m (m is a positive integer) are parameters. If the statistics*

$$\hat{f}=\begin{cases}\dfrac{X}{m}; & X\leqq Z\\[2mm] \dfrac{1}{m}\dfrac{Z^{\frac{m}{2}}}{X^{\frac{m}{2}-1}}; & X>Z\end{cases}$$

is an unbiased estimator of the random variable

$$f=\begin{cases}\sigma^2; & X\leqq Z\\ 0; & X>Z\end{cases}$$

for every real number $Z>0$, then $\dfrac{X}{\sigma^2}$ has a χ^2-distribution with parameter m.

PROOF. It follows from the assumption that

$$\sigma^2\int\limits_0^Z f(x)\,dx=\frac{1}{m}\int\limits_0^Z xf(x)\,dx+\frac{Z^{\frac{m}{2}}}{m}\int\limits_Z^\infty \frac{1}{x^{\frac{m}{2}-1}}f(x)\,dx. \tag{4.13}$$

Differentiating both sides of the equation (4.13) with respect to Z we obtain the following differential equation:

$$\frac{H'(Z)}{H(Z)}=-\frac{1}{2\sigma^2}, \tag{4.14}$$

where

$$H(Z)=\int\limits_Z^\infty \frac{1}{x^{\frac{m}{2}-1}}f(x)\,dx. \tag{4.15}$$

Solving the differential equation (4.14) we obtain

$$H(Z)=Ce^{-\frac{1}{2\sigma^2}Z}. \tag{4.16}$$

From (4.15) and (4.16) we obtain

$$f(Z)=\frac{C}{2\sigma^2}Z^{\frac{m}{2}-1}e^{-\frac{1}{2\sigma^2}Z}.$$

Taking into consideration that $f(Z)$ is a density function, we get

$$C=\frac{1}{(2\sigma^2)^{\frac{m}{2}-1}\,\Gamma\left(\frac{m}{2}\right)}.$$

Thus we get:

$$f(Z;\sigma;m)=\begin{cases}\dfrac{1}{2^{\frac{m}{2}}\,\sigma^m\,\Gamma\left(\frac{m}{2}\right)}\cdot Z^{\frac{m}{2}-1}\,e^{-\frac{Z}{2\sigma^2}}; & Z>0\\ 0; & Z\leqq 0.\end{cases}$$

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(Received September 22, 1970)

ON THE ORTHOGONALITY OF PROBABILITY MEASURES

by

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Summary

In this paper we prove that the measures according to the hypotheses in alternative hypothesis tests are orthogonal on the infinite dimensional product space provided the sequence of observations is a zero-one sequence under both hypotheses and inequality (8) holds. Especially two strictly stationary zero-one measures are orthogonal on the infinite product space provided they are different. This means that the probability of error (first or second kind) in testing alternate hypotheses under the above conditions tends to zero if the number of observations tends to infinity. The rate of convergence in a special case is also investigated.

Introduction

We list our notations and recall some results and concepts for later references.

— Accepting the Bayesian point of view we denote the prior probabilities by w_0 and w_1 respectively.

— Our decisions are based on the sequence of random variables $\xi_1, \xi_2, \dots$. We denote by $\mathfrak{M}_n^m$ the σ-algebra generated by $\{\xi_i,\ n\leqq i<m\}$. The probability measure under the hypothesis H_i will be denoted by $P^{(i)}$, $(i=0, 1)$, and let us denote the marginal distributions on the $\mathfrak{M}_n^m$ by $P^{(i)}_{(n,m)}$.

— The quantity

$$\Lambda = \Lambda(P^0, P^1) = \int \sqrt{\frac{dP^0}{dP}\cdot\frac{dP^1}{dP}}\,dP$$

is called the Hellinger integral of the measures P^0 and P^1 where P is an arbitrary measure, with respect to which P^0 and P^1 are absolutely continuous.

— The probability of a wrong decision is said to be the error of the decision and is denoted by $e(.)$. Define a decision Δ as follows: accept the hypothesis H_0 if $w_0\,\frac{dP^0}{dP} > w_1\cdot\frac{dP^1}{dP}$ and reject it otherwise. This so-called standard decision Δ is of minimal error, as it is known (e.g. RÉNYI [4]). The error of the standard decision Δ can be estimated by

$$\frac{e(\Delta)}{\sqrt{w_0 w_1}} \leqq \Lambda \leqq \frac{\sqrt{e(\Delta)}}{w_0 w_1} \tag{1}$$

(see NEMETZ [3]). This inequality means especially that having an infinite sequence of observations we can choose the true hypothesis with probability one if and only if the Hellinger-integral $\Lambda(P^0, P^1)$ is zero, i.e. the two measures are orthogonal on the infinite dimensional product-space. This is the situation e.g. when the sequence $\xi_1, \xi_2, \ldots$ consists of independent, identically distributed random variables. In this case the rate of convergence is exponential by (1). A. RÉNYI proved (see [5]) (under very weak assumption) that there exist numbers B and $0 \leqq \lambda < 1$ such that

$$e(\Delta_n) = \frac{B \cdot \lambda^n}{\sqrt{2\Pi n}} \big(1 + o(1)\big). \tag{2}$$

The observations in practice are not independent in general but there is a weak dependence among them. There is a lot of conditions concerning the weak dependence of the remote variables. (See ROSENBLATT [6], ARATÓ [1].) The weakest condition is the following: the sequence of random variables is called a zero-one sequence, if

$$\bigcap_{n=1}^{\infty} \mathfrak{M}_n^{\infty} = \mathfrak{N} \tag{3}$$

where $\mathfrak{N}$ consists of sets having probabilities 0 or 1 only. (3) is equivalent to

$$\sup_{B \in \mathfrak{M}_{N+n}^{\infty}} |\mathsf{P}(AB) - \mathsf{P}(A)\mathsf{P}(B)| = \gamma(A, N, n) \to 0 \tag{4}$$

if $n \to \infty$ for any fixed N and $A \in \mathfrak{M}_1^N$. (See NEMETZ, VARGA [2].)
Let us give one of the stronger conditions due to KOLMOGOROV

$$\sup_{C \in \mathfrak{M}_1^N \otimes \mathfrak{M}_{N+n}^{\infty}} |\mathsf{P}(C) - \mathsf{P}_{(1,N)} \otimes \mathsf{P}_{(N+n,0)}(C)| = \gamma^*(N, n) \to 0 \tag{5}$$

if $n \to \infty$ for every N. (Here $\otimes$ means the Cartesian product.) In this case the sequence is called strongly regular. If, in addition, there exists a function $\gamma^*(n) \to 0$ such that $\gamma^*(N, n) \leqq \gamma^*(n)$ independently of N, we call the sequence totally strongly regular.

The result.

Now we give a sharper form of a lemma due to ROZANOV (see [7], IV., lemma 11. 2).

Consider a sequence of integers $1 = M_1 < M_2 < \cdots < M_{2k}$ and let $N_i = M_{2i} - M_{2i-1}$ and $n_i = M_{2i+1} - M_{2i}$. For the sake of brevity we denote the σ-algebra $\mathfrak{M}_{M_{2i-1}}^{M_{2i}}$ by $\mathscr{F}_i$. Setting

$$A^{\varepsilon} = \begin{cases} A & \text{if } \varepsilon = 0 \\ \bar{A} & \text{if } \varepsilon = 1 \end{cases}$$

we seek an estimate for $\mathsf{P}(C)$, where $C = \bigcup_{(\varepsilon_1, \ldots, \varepsilon_k) \in E} A_1^{\varepsilon_1} A_2^{\varepsilon_2} \ldots A_k^{\varepsilon_k}$, $A_i \in \mathscr{F}_i$ in terms of $\mathsf{P}(A_i)$'s and $\gamma(A_i, M_{2i}, n_i)$'s. In order to obtain this we define the sets $B(\varepsilon_1, \ldots, \varepsilon_i)$, $i < k$ for every fixed $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ by

$$B(\varepsilon_1, \ldots, \varepsilon_i) = C \cap A_1^{\varepsilon_1} \cdot A_2^{\varepsilon_2} \cdot \cdots \cdot A_i^{\varepsilon_i}.$$

Here obviously $B(\varepsilon_1, \ldots, \varepsilon_i) \in \mathscr{F}_{i+1} \otimes \cdots \otimes \mathscr{F}_k \subset \mathfrak{M}_{M_{2i+1}}^{\infty}$ and

$$C = \bigcup A_1^{\varepsilon_1} \cdot \cdots \cdot A_i^{\varepsilon_i} B(\varepsilon_1, \ldots, \varepsilon_i)$$

where $(\varepsilon_1, \dots, \varepsilon_i)$ takes on all the possible 2^i values. It is easy to see that

$$|\mathsf{P}(C) - \sum_{(\varepsilon_1, \dots, \varepsilon_i)\in E} \mathsf{P}(A_1^{\varepsilon_1}) \cdot \dots \cdot \mathsf{P}(A_k^{\varepsilon_k})| \leqq$$

$$\leqq |\mathsf{P}\{A_1 B(0)\} - \mathsf{P}(A_1)\mathsf{P}\{B(0)\}| + |\mathsf{P}\{\bar{A}_1 B(1)\} - \mathsf{P}(\bar{A}_1)\mathsf{P}\{B(1)\}| +$$

$$+\sum_{i=2}^{k=1} \sum_{(\varepsilon_1, \dots, \varepsilon_{i-1})} \mathsf{P}(A_1^{\varepsilon_1}) \dots \mathsf{P}(A_{i-1}^{\varepsilon_{i-1}}) \Big\{ \sum_{\varepsilon_i=0,1} |\mathsf{P}\{A_i^{\varepsilon_i} B(\varepsilon_1, \dots, \varepsilon_i)\} - \mathsf{P}(A_i^{\varepsilon_i})\mathsf{P}\{B(\varepsilon_1, \dots, \varepsilon_i)\}|\Big\}$$

Taking into account that

$$B(\varepsilon_1, \dots, \varepsilon_{k-1}) = \begin{cases} \emptyset & \text{if } (\varepsilon_1, \dots, \varepsilon_{k-1}, \cdot) \notin E \\ A_k & \text{if } (\varepsilon_1, \dots, \varepsilon_{k-1}, 0) \in E, (\varepsilon_1, \dots, \varepsilon_{k-1}, 1) \notin E \\ \bar{A}_k & \text{if } (\varepsilon_1, \dots, \varepsilon_{k-1}, 0) \notin E, (\varepsilon_1, \dots, \varepsilon_{k-1}, 1) \in E \\ \Omega & \text{otherwise} \end{cases}$$

we have the following

LEMMA.

(7)
$$|\mathsf{P}\{\bigcup_{(\varepsilon_1, \dots, \varepsilon_k)\in E} A_1^{\varepsilon_1} \cdot \dots \cdot A_k^{\varepsilon_k}\} - \sum_{(\varepsilon_1, \dots, \varepsilon_k)\in E} \mathsf{P}\{A_1^{\varepsilon_1}\} \cdot \dots \cdot \mathsf{P}\{A_k^{\varepsilon_k}\}| \leqq 2 \sum_{i=1}^{k-1} \gamma(A_i, M_{2i}, n_i)$$

Here we used the identity $\gamma(A, N, n) = \gamma(\bar{A}, N, n)$. It is worthwhile to notice that

$$2(k-1)\varrho(n)$$

stands on the right-handside of this inequality if there exists a function $\varrho(n)$ such that $\gamma(A, N, n) \leqq \varrho(n)$ independently of N and A, and if $n_i \equiv n$.

Using this lemma we can prove the following theorem for testing alternate hypothesis:

THEOREM. *The two hypothetical measures are orthogonal on the infinite dimensional product space (or what is the same, the error of the standard decision tends to zero if the number of observation tends to infinity) provided the next assumptions hold:*
(i) *The sequence of observations is a zero-one sequence under both hypotheses.*
(ii) *The inequality*

(8)
$$A(P^{(0)}_{(N,\infty)}, P^{(1)}_{(N,\infty)}) \leqq \lambda_0 < 1$$

is valid for the Hellinger-integrals, where λ_0 does not depend on N.

PROOF. Let $\alpha(d)$ resp. $\beta(d)$ denote the error of the first (respectively second) kind of a decision d. If there exists a decision d with $\alpha + \beta \leqq \lambda < 1$ then there exists another decision $\tilde{d}$ being randomized in general such that

(9)
$$\alpha(\tilde{d}) = \beta(\tilde{d}) = \frac{\lambda}{1+\lambda} < \frac{1}{2}$$

as it is easy to see.

Fix the number $\lambda_0 < \lambda_1 < 1$. Then by (8) there is an integer M_2 such that

$$A(P^{(0)}_{(1,M_2)}; P^{(1)}_{(1,M_2)}) < \lambda_1$$

and choosing $w_0=w_1=1/2$ for the a priori probabilities we obtain from (1) that the error of the standard decision on the σ-algebra $\mathscr{F}_1$ (where $M_1=1$) is not greater than λ_1. Consequently we can define a decision d_1 on $\mathscr{F}_1$, for which

$$\alpha(d_1)=\beta(d_1)=\lambda=\frac{\lambda_1}{1+\lambda_1}<\frac{1}{2}$$

Let A_1 be the set where the hypothesis H_0 is accepted. Given an arbitrarily small $\varepsilon>0$ we can fix an integer k satisfying the inequality

$$\sum_{i=k+1}^{2k+1}\binom{2k+1}{i}\lambda^i(1-\lambda)^{2k+1-i}<\frac{\varepsilon}{2}$$

Let us define the number M_i and the sets A_i by induction on i: having given M_{2i} and A_i let M_{2i+1} be a fixed integer such that

$$\sup_{B\in\mathfrak{M}_{M_{2i+1}}^{\infty}}|\mathsf{P}^{(j)}(A_iB)-\mathsf{P}^{(j)}(A_i)\mathsf{P}^{(j)}(B)|<\frac{\varepsilon}{8k},\qquad j=0,1$$

$\left(\text{or what is the same } \gamma(A_i, M_{2i}, n_i)<\frac{\varepsilon}{8k}\right)$. Such a choice of M_{2i+1} is possible owing to our regularity condition. After this we choose M_{2i+2} with the property

$$\Delta(P^{(0)}_{M_{2i+1},M_{2i+2}};P^{(1)}_{M_{2i+1},M_{2i+2}})<\lambda_1$$

At last we define the decision d_{i+1} on $\mathscr{F}_{i+1}$ and the set A_{i+1} determined by it as we have done in case of d_1.

We are in the position to define a suboptimal decision based on the observations $\xi_1,\xi_2,\dots,\xi_{M_{4k+2}}$. This decision d means that we accept the hypothesis H_0 if at least $(k+1)$ events from A_i''s, $i=1,\dots,2k$ occurred and reject H_0 otherwise.

In accordance with our lemma the errors of both kind must not exceed the quantity

$$\sum_{\sum\varepsilon_i=k+1}P^{(0)}(A_1^{\varepsilon_1})\cdot\dots\cdot P^{(0)}(A_k^{\varepsilon_k})+2\sum_{i=1}^{k}\gamma(A_i,M_{2i},n_i)\leqq$$

$$\leqq\sum_{i=k+1}^{2k+1}\binom{2k+1}{i}\lambda^i(1-\lambda)^{2k+1-i}+4k\frac{\varepsilon}{8k}<\varepsilon.$$

So we have proved our assertion.

Let us consider a sequence of observation which is strictly stationary under both hypotheses, and the two measures are different. For this case the assumption (8) is valid evidently so the next corollary is a special case of the above theorem but to emphasize it may be useful.

Corollary: Measures generated by stationary zero-one sequences are orthogonal on the infinite dimensional space, provided they are different.

In many cases it is the simple repetition of the previous idea which is needed only if the rate of convergence is investigated. Let a function $\varrho(n)$ mentioned in the lemma exist and let $\alpha(n)$ be the minimal error of first kind, assuming $\alpha(d)=\beta(d)$ where d depends on the first n observation only. (The rate of convergence of $\alpha(n)$ and that of $e(\Delta_n)$ is obviously equal.)

It is easy to see:

$$\alpha[(2k+1)N+2kn] \leqq \sum_{i=k+1}^{2k+1} \binom{2k+1}{i} \alpha^i(N)[1-\alpha(N)]^{2k+1-i} + 4k\varrho(n)$$

Here $\alpha(N) \to 0$ by our theorem, thus for every $0<p<1$ there exists a number N_p such that $N>N_p$ yields

$$\alpha[(2k+1)N+2kn] \leqq [\alpha(N)]^{pk} + 4k\varrho(n).$$

Taking $k=3$ we can immediately deduce from here for $\varrho(n) \leqq An^\alpha$, $\alpha<0$ that $\alpha(n) \leqq Bn^\alpha$ with a suitable B, too.

In this case our suboptimal decision is good enough because this is the best possible rate (in the sense that there is an example yielding exactly this rate of convergence). But this suboptimal decision can be and probable will be proved to be weak for the case of $\varrho(n) \approx \varrho_0^n$, $0<\varrho_0<1$. The rate of convergence in this situation is expected to be exponential as it is known for ergodic Markov chains. (See e.g. KORSCH [8].) But the best we can say is the next inequality for the Hellinger integral $\lambda(n)$ of the measures $P^{(0)}_{(1,n)}$ and $P^{(1)}_{(1,n)}$, assuming totally strongly regular sequence of observations:

$$\lambda[(k+1)N+kn] \leqq [\lambda(N)]^{k+1} + k\varrho_0^n, \qquad 0<\varrho_0<1,$$

and this inequality does not yield exponential rate of convergence.

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(Received November 16, 1970)

Studia Scientiarum Mathematicarum Hungarica 7. (1972) 117—130.

THE ALGEBRAIC DERIVATIVE AND INTEGRAL IN THE DISCRETE OPERATIONAL CALCULUS

by

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Introduction

MIKUSIŃSKI [1] has introduced the concept of the algebraic derivative in the operational calculus based on the functions of the real, continuous variable t varying on the half line $0 \leqq t < \infty$. GESZTELYI [2] has defined the inverse operation of the algebraic derivative, the so-called algebraic integral. GESZTELYI [2] has given a class of Mikusiński operators being algebraic integrable. Recently SCHATTE [3] generalized the results of papers [2], [4] and has given a class of operators being not algebraic integrable. The detailed discussion of the algebraic differential equations is given in WLOKA [5].

The algebraic derivative and integral has also a great practical importance in the solution of some problems of the classical analysis by the operational method. GESZTELYI [2] has solved differential equations with polynomial coefficients by application of the operational calculus. On the other hand FÉNYES [6], [7], [8] has solved linear and non-linear integral equations of the convolution type.

The mathematical foundation of the discrete analogue of the original Mikusiński's operational calculus and its application to the solution of ordinary and partial difference equations with constant coefficients, is to be found in papers ELIAS [9], FENYŐ [10], BERG [11], BUTZER—SCHULTE [12].

In this paper, we shall deal with the algebraic derivative and integral in the discrete operational calculus (see also MOORE [14]). We give the necessary and sufficient condition of the existence of the algebraic integral. The linear, first order algebraic differential equation will be discussed in detail. We give the existence criteria and the explicit form of the solutions of such equations.

As an application of the theory we shall be dealing with the operational solution of a discrete linear integral equation.

The operational notations and symbols of BUTZER—SCHULTE [12] will be generally used.

1. The brief summary of the elements of the discrete operational calculus given in [12]

The symbol $a = \{a_n\}$ denotes an arbitrary finite and real valued function defined in the discrete points $n = 0, 1, 2, \ldots$, of the positive half line. The symbol a_n denotes the values of the function $\{a_n\}$ for $n = 0, 1, 2, \ldots$.

The set of the above functions is denoted by E.

Two operations are defined in the set E:

Addition $a+b = \{a_n\}+\{b_n\} = \{a_n+b_n\},$

$$a, b \in E$$

Multiplication $ab = \{a_n\}\{b_n\} = \left\{\sum_{k=0}^{n} a_k b_{n-k}\right\}$

The set E is a commutative ring with respect to addition and multiplication defined above. It has no divisors of zero and can be extended to a quotient field M. The elements of M are called discrete convolution quotients or operators. They are of the form:

$$\frac{\{a_n\}}{\{b_n\}}, \quad \{a_n\}, \{b_n\} \in E, \quad \{b_n\} \neq 0$$

The field of the real numbers is denoted by K. E and K can be embedded isomorphic in the field M. The unit element of E is $\{\delta_{0,n}\}$ where $\delta_{0,n}$ denotes the Kronecker symbol. The unit element of E, M and K can be identified algebraically. They are denoted by 1. Similarly the zero element of E, M and K will be denoted by 0 since they can be identified too.

Every number is also a function in the discrete operational calculus, the value of the null-th component of which equals the value of the given number, the other components are zero.

In the sequel operators will be denoted also by the Latin letters $a, b, c, \ldots$. Sometimes we shall denote the numbers by Greek letters.

Special operators

The summing operator. The function $h=\{1\}$ having the value 1 for every $n=0, 1, \ldots$, defines the summing operator, since $h\{a_n\}=\{1\}\{a_n\}=\{\sum_{k=0}^{n} a_k\}$ for $\{a_n\} \in E$.

The difference operator. The operator

$$q = \frac{1}{h-1} \tag{1.1}$$

defines the difference operator. Obviously $q \notin E$. Its fundamental property is

$$q\{a_n\} = \{\Delta a_n\}+(1+q)a_0 \quad \text{for} \quad \{a_n\} \in E \tag{1.2}$$

where

$$\{\Delta a_n\} = \{a_{n+1}-a_n\}.$$

The translation operator. The translation operator v is defined by $v = \dfrac{1}{1+q}$. It holds that

$$\frac{1}{(1+q)^m} = \{\delta_{m,n}\} \qquad m=0, 1, 2, \ldots,$$

and

$$v^m\{a_n\} = \{b_n\}, \quad \text{where} \quad b_n = \begin{cases} a_{n-m}, & \text{if } n \geqq m \\ 0, & \text{if } 0 \leqq n < m \end{cases} \quad \text{for} \quad a \in E. \tag{1.3}$$

Moreover $v^i v^j = v^{i+j}$ (for every integers i, j). In the following the symbol $a^{(\nu)} = \{a_{\nu,n}\} \in E$, $\nu = 0, 1, \dots$, denotes a sequence of discrete functions. [12] defines a convergence in E as follows

$$\sum_{\nu=0}^{\infty} a^{(\nu)} = \sum_{\nu=0}^{\infty} \{a_{\nu,n}\} = \left\{ \sum_{\nu=0}^{\infty} a_{\nu,n} \right\}.$$

The operational form of an arbitrary $b = \{b_n\} \in E$ is

$$b = \{b_n\} = \sum_{\nu=0}^{\infty} \frac{b_\nu}{(1+q)^\nu}, \qquad b_n \in K \tag{1. 4}$$

in the sense of convergence defined above. (1. 4) is given in [12]. From [12] we can easily deduce that every operator $d \in M$ can be written as
(1. 5)

$$d = \sum_{\nu=\varkappa}^{-1} \frac{a_\nu}{(1+q)^\nu} + \sum_{\nu=0}^{\infty} \frac{a_\nu}{(1+q)^\nu} = \sum_{\nu=\varkappa}^{\infty} \frac{a_\nu}{(1+q)^\nu}, \qquad a_\nu \in K, \qquad \varkappa = \text{negative integer}.$$

There are only a finite number of terms not contained in E of (1. 5). (see also BERG [11]). An equivalent statement is the following. Every $b \neq 0$ operator can be written as

$$b = (1+q)^N \{\varrho_n\}, \quad \text{where} \quad \{\varrho_n\} \in E, \quad \varrho_0 \neq 0 \quad N \text{ is an integer.} \tag{1. 6}$$

If $N \leqq 0$, then $b \in E$.

2. The algebraic derivative and integral

We introduce in the ring E an operation by the definition

$$Da = D\{a_n\} = \{-na_n\}, \quad \text{for} \quad a \in E. \tag{2. 1}$$

It can be easily seen that

$$D\{a_n + b_n\} = D\{a_n\} + D\{b_n\}, \quad D[\{a_n\}\{b_n\}] = \{a_n\} D\{b_n\} + \{b_n\} D\{a_n\} \tag{2. 2}$$

The operation D will be termed the algebraic derivative (see also [14]).

The definition of the algebraic derivative can be extended to M by

$$D\frac{a}{b} = \frac{bDa - aDb}{b^2}, \qquad a, b \in E, \ b \neq 0 \tag{2. 3}$$

It is easy to verify that D retains properties (2. 2) in M. If $\alpha \in K$, then $D\alpha = 0$. Moreover, D is linear in M.

The algebraic derivative of the difference operator q is

$$Dq = 1 + q. \tag{2. 4}$$

Indeed, by (1. 1) and (1. 2) we can write

$$Dq = -\frac{D(h-1)}{(h-1)^2} = -\frac{Dh}{(h-1)^2} = \frac{\{n\}}{(h-1)^2} = q^2\{n\} = qh = 1+q$$

or

$$D(1+q) = 1+q \tag{2.5}$$

It is easy to verify by induction that

$$D(x^\nu)=\nu x^{\nu-1}Dx, \qquad \nu=\text{integer}, \quad x\in M, \quad [\text{see also [2]}]$$

and by choosing $x = 1+q$ we obtain

$$D[(1+q)^\nu] = \nu(1+q)^\nu \tag{2.6}$$

From [1. 5] and [2. 6] we see that

$$Dx = \sum_{\nu=\varkappa}^{\infty} \frac{-\nu a_\nu}{(1+q)^\nu} \quad \text{for} \quad x\in M \tag{2.7}$$

If $Dx=0$, for $x\in M$, then $x\in K$. Indeed, by (2. 7) we easily conclude that $a_\nu=0$, if $\nu\neq 0$ and a_0 is an arbitrary number. Consequently $x\in K$.

We shall now deal with power series whose terms are elements of the ring E i.e. with the series of the form

$$\sum_{\nu=0}^{\infty} \beta_\nu a^\nu \quad \text{where} \quad \beta_\nu\in K,\ a\in E$$

We prove the following

THEOREM 1. *Consider the power-series*

$$\sum_{\nu=0}^{\infty} \beta_\nu a^\nu \tag{2.8}$$

where $\beta_\nu\in K$, $a=\{a_n\}\in E$. *If* $a_0=0$ *then* (2. 8) *is convergent. If*

$$\sum_{\nu=0}^{\infty} \beta_\nu z^\nu$$

is an entire function of the complex variable z *then* (2. 8) *is convergent.*

PROOF. The first half of the Theorem is obvious. By introducing the notation

$$\{a_{\nu,n}\} = \{a_n\}^\nu = a^\nu \qquad a^0 = 1$$

we get

$$\sum_{\nu=0}^{\infty} \beta_\nu a^\nu = \sum_{\nu=0}^{\infty} \beta_\nu \{a_n\}^\nu = \left\{\sum_{\nu=0}^{\infty} \beta_\nu a_{\nu,n}\right\}. \tag{2.9}$$

By our assumption $a_0=0$. Consequently for every fixed n, the series (2. 9) has only a finite number of terms differing from zero:

$$a_{\nu,n} = 0 \quad \text{if} \quad \nu \geqq n+1. \tag{2.10}$$

The validity of the second half of the Theorem can be shown as follows. Let us fix N arbitrarily. Then

$$|a_n| < M, \quad \text{for} \quad n\leqq N,$$

and we obtain

$$|a_{1,n}| = |a_n| < M,$$

$$|a_{2,n}| = \left|\sum_{k=0}^{n} a_k a_{n-k}\right| \leqq \sum_{k=0}^{n} |a_k|\,|a_{n-k}| \leqq M^2(N+1)$$

$$|a_{3,n}| = \left|\sum_{k=0}^{n} a_{2,k} a_{n-k}\right| \leqq \sum_{k=0}^{n} |a_{2,k}|\,|a_{n-k}| \leqq M^3(N+1)^2$$

and

$$|a_{\nu,n}| \leqq M^\nu (N+1)^{\nu-1} \quad \text{for} \quad n \leqq N$$

Consequently

$$\left|\sum_{\nu=0}^{\infty} \beta_\nu a_{\nu,n}\right| \leqq |\beta_0| + \frac{1}{N+1} \sum_{\nu=1}^{\infty} |\beta_\nu| M^\nu (N+1)^\nu < \infty, \quad \text{if} \quad n \leqq N$$

since, by the assumption $\sum_{\nu=0}^{\infty} \beta_\nu z^\nu$ is an entire function. If $f(z)$ is analytic only in a circle of radius $\varrho < \infty$, then Theorem does not hold in general. For example

$$\frac{1}{1-z} = \sum_{\nu=0}^{\infty} z^\nu$$

is analytic in the unit circle nevertheless

$$\sum_{\nu=0}^{\infty} \{\delta_{0,n}\}^\nu = \sum_{\nu=0}^{\infty} 1 = \infty$$

Consider now the exponential function defined in the ring E. This has a role in the theory of algebraic differential equations of the first order.

$$e^a = e^{\{a_n\}} = \sum_{\nu=0}^{\infty} \frac{\{a_n\}^\nu}{\nu!} \tag{2.11}$$

Since e^z is entire function, the definition (2.11) is correct. It can easily be seen that

$$e^a e^b = e^{a+b}, \quad \text{for} \quad a, b \in E$$

Moreover, the exponential function

$$\{e_n\} = e^{\{a_n\}}, \quad \{a_n\} \in E \tag{2.12}$$

has the following properties:

THEOREM 2. $e_0 = e^{a_0}$, $e^{\{a_n\}} \neq 0$,

$$De^{\{a_n\}} = D\{a_n\} e^{\{a_n\}}$$

$$e^{\{a_n\}} \in K, \quad \textit{if and only if} \quad \{a_n\} \in K \tag{2.13}$$

PROOF.

$$\{e_n\} = e^a = 1 + \{a_n\} + \frac{\{a_n\}^2}{2} + \cdots = 1 + \{a_n\} + \frac{1}{2}\left\{\sum_{k=0}^{n} a_k a_{n-k}\right\} + \cdots,$$

hence

$$e_0 = 1 + a_0 + \frac{a_0^2}{2} + \cdots = e^{a_0}$$

$$e^{\{a_n\}} \neq 0, \quad \text{since} \quad e^{a_0} = e_0 \neq 0.$$

Moreover

$$De^a = D \sum_{\nu=0}^{\infty} \frac{\{a_n\}^\nu}{\nu!} = \sum_{\nu=0}^{\infty} D \frac{\{a_n\}^\nu}{\nu!}$$

holds, since at any fixed n the operation D is a simple multiplication by a constant. By the application of the formula

$$Dx^\nu = \nu x^{\nu-1} Dx$$

we obtain

$$(2.14) \qquad De^a = D\{e_n\} = \sum_{\nu=0}^{\infty} D \frac{\{a_n\}^\nu}{\nu!} = D\{a_n\} \sum_{\nu=1}^{\infty} \frac{\nu}{\nu!} \{a_n\}^{\nu-1} =$$

$$= D\{a_n\} \sum_{\nu=1}^{\infty} \frac{\{a_n\}^{\nu-1}}{(\nu-1)!} = D\{a_n\} e^{\{a_n\}} = \{e_n\} D\{a_n\} = e^a Da$$

The first half of the last statement of the Theorem is trivial. Moreover, if $e^{\{a_n\}} \in K$ holds for some $\{a_n\} \in E$, then by (2. 14) we have

$$De^a = e^a Da = 0$$

Since $e^a \neq 0$, we obtain finally $Da = 0$ and $a \in K$.

We introduce now the concept of the algebraic integral as the inverse of D. If for an arbitrary operator $x \in M$ there exists an operator $y \in M$ such that

$$Dy = x$$

then y is called the algebraic integral of the operator x and is denoted by

$$(2.15) \qquad y = \int x$$

It follows from (2. 15) that the algebraic integral is linear in M, i.e.

$$(2.16) \qquad \int (\alpha x + \beta z) = \alpha \int x + \beta \int z, \qquad x, z \in M, \ \alpha, \beta \in K$$

holds, provided that $x, z \in M$ are algebraic integrable. It can be easily shown that two integrals of an operator — if they exist — differ from each other in an arbitrary number.

We give now the simple necessary and sufficient condition of the algebraic integrability.

THEOREM 3. *Let $x \in M$ be an arbitrary operator given by*

$$(2.17) \qquad x = \sum_{\nu=\varkappa}^{\infty} \frac{a_\nu}{(1+q)^\nu}.$$

x is algebraic integrable, if and only if $a_0=0$. If so, the formula

(2.18)
$$\int x = \sum_{\nu=\varkappa}^{\infty} \frac{b_\nu}{(1+q)^\nu}$$

holds where

(2.19)
$$b_\nu = -\frac{a_\nu}{\nu}, \quad \text{for} \quad \nu \neq 0, \; b_0 = \text{arbitrary}.$$

PROOF. Let

$$\int x = y = \sum_{\nu=\varkappa}^{\infty} \frac{b_\nu}{(1+q)^\nu}$$

so we obtain

$$x = Dy = \sum_{\nu=\varkappa}^{\infty} \frac{-\nu b_\nu}{(1+q)^\nu} = \sum_{\nu=\varkappa}^{\infty} \frac{a_\nu}{(1+q)^\nu}$$

Comparing the coefficients we have

$$a_\nu = -\nu b_\nu, \quad b_\nu = -\frac{a_\nu}{\nu} \qquad \nu \neq 0$$

Choosing $\nu=0$ we get

$$a_0=0, \quad b_0=\text{arbitrary}.$$

Special cases. 1. Let $a_\nu=0$, for $\nu<0$. It is easy to see that a function $x=\{x_n\}\in E$ is algebraic integrable, if and only if, the initial value $x_0=0$. If so, the obtained algebraic integral is a function where the initial value is arbitrary. Positive and negative numbers are not integrable.

2. The difference operator is not integrable. Indeed,

$$q = q+1-1 \quad (a_0=-1).$$

3. The exponential function e^a is not integrable since the initial value $e_0=e^{a_0}\neq 0$.

3. The first order linear, algebraic differential equation

Consider now the linear, first order, algebraic differential equation

(3.1)
$$Dx - wx = y$$

where the operators $w, y \in M$ are given and we look for operators $x\in M$ which satisfy (3.1).

First we shall deal with the homogeneous equation, i.e. $y=0$. It is easy to show that, if

(3.2)
$$Dx - wx = 0$$

has a solution $x_0 \neq 0$, then the general solution of (3.2) is of the form

(3.3)
$$x = Cx_0 \quad C\in K.$$

The proof is quite analogous to that of MIKUSIŃSKI [13], where the statement is proved in the continuous case.

We look for a solution of (3. 2) as

(3. 4) $$x = (1+q)^m \{g_n\}, \qquad m = \text{integer}, \ \{g_n\} \in E, \ g_0 \neq 0$$

Then

$$Dx = m(1+q)^m \{g_n\} + (1+q)^m \{-ng_n\}$$

and

(3. 5) $$\frac{Dx}{x} = w = \frac{m\{g_n\} - \{ng_n\}}{\{g_n\}} = \frac{\{(m-n)g_n\}}{\{g_n\}}$$

Since $g_0 \neq 0$ by (1. 6) we can easily conclude that w is a function $w \in E$. If w is not a function, (3. 2) has only the trivial null solution.

If $w = \{f_n\}$ is a function, we obtain from (3. 5)

$$\{(m-n)g_n\} = \{f_n\}\{g_n\} = \left\{\sum_{k=0}^{n} f_k g_{n-k}\right\}$$

and for the initial value $n=0$, $mg_0 = f_0 g_0$ from which

$$f_0 = m$$

follows. If $w = \{f_n\}$ is a nonzero solution of (3. 2) then $w = \{f_n\}$ is a function having an integer initial value f_0.

The converse is also true. If $w = \{f_n\}$ is a function, having an integer initial value, then (3. 2) has nonzero solutions.

Namely, a nonzero solution is given by the formula

(3. 6) $$x = (1+q)^{f_0} \exp\left[\int \{F_n\}\right]$$

where

$$\{F_n\} = \{f_n\} - f_0$$

By substituting (3. 6) in (3. 2) where $w = \{f_n\}$ with (2. 6), (2. 13) we have

(3. 7) $$Dx - \{f_n\}x = f_0(1+q)^{f_0} \exp[\ldots] + (1+q)^{f_0}[\{f_n\} - f_0] \exp[\ldots] - $$
$$- \{f_n\}(1+q)^{f_0} \exp[\ldots] = 0$$

Thus (3. 7) is a nonzero solution of (3. 2).

Taking into account that the exponential function has nonzero initial value, $x \in E$ if and only if $f_0 \leqq 0$.

We have proved the following

THEOREM 4. *The algebraic differential equation*

(3. 8) $$Dx - wx = 0$$

has a nontrivial solution in M if and only if $w = \{f_n\} \in E$ where the initial value f_0 is an arbitrary integer. The general solution of (3. 8) *is*

(3. 9) $$x = C(1+q)^{f_0} \exp\left[\int \{F_n\}\right] \qquad C \in K$$

where

$$\{F_n\} = \{f_n\} - f_0.$$

For $C \neq 0$ the solution is a function if and only if $f_0 \leqq 0$.

It can easily be seen that the general solution of the inhomogeneous equation

$$Dx - wx = y \tag{3. 10}$$

is of the form

$$x = Cx_0 + x_i$$

where x_0 is a (nontrivial) solution of the homogeneous equation and x_i is a particular solution of the inhomogeneous equation, provided that they exist.

In the sequel it will be assumed that $x_0 \neq 0$ exists.

THEOREM 5. *Let $x_0 \neq 0$ exist. Then x_i exists, if and only if the algebraic integral*

$$\int \frac{y}{x_0}$$

exists. If so, x_i can be determined by the method of variation of parameters and the formula

$$x_i = x_0 \int \frac{y}{x_0} \tag{3. 11}$$

holds.

The proof is quite analogous to that of SCHATTE [3] where the statement is proved in the operational calculus based on continuous functions.

We write y as

$$y = (1+q)^N \{\varrho_n\} \qquad N = \text{integer}, \ \{\varrho_n\} \in E, \ \varrho_0 \neq 0 \tag{3. 12}$$

with (3. 7), (3. 11), (3. 12) we see that x_i exists, if and only if

$$\int \frac{y}{x_0} = \int (1+q)^{N-f_0} \{\varrho_n\} \exp\left[-\int \{F_n\}\right] \tag{3. 13}$$

exists.

Introducing the notation

$$\{H_n\} = \{\varrho_n\} \exp\left[-\int \{F_n\}\right] \tag{3. 14}$$

it can be seen that $H_0 \neq 0$. By theorem 3

if $N < f_0$, then (3. 13) exists,

if $N = f_0$, then (3. 13) does not exist,

if $N > f_0$, then (3. 13) exists, if and only if $H_{N-f_0} = 0$

Moreover, the formula

$$\int \frac{y}{x_0} = (1+q)^{N-f_0} \{G_n\} \tag{3. 15}$$

holds, where

$$G_n = \frac{H_n}{N - f_0 - n}, \quad \text{for} \quad n \neq N - f_0,$$

$$G_n = \text{arbitrary} \quad \text{for} \quad n = N - f_0$$

Namely, by algebraic differentiation we have

$$(1+q)^{N-f_0}\left\{\frac{-nH_n}{N-f_0-n}\right\}+(N-f_0)(1+q)^{N-f_0}\left\{\frac{H_n}{N-f_0-n}\right\}=$$

$$=(1+q)^{N-f_0}\left\{\frac{-nH_n+(N-f_0)H_n}{N-f_0-n}\right\}=(1+q)^{N-f_0}\{H_n\}=\frac{y}{x_0}$$

So a particular solution of (3. 10) is

$$x_i = x_0\int y/x_0 = (1+q)^N \exp\left[\int\{F_n\}\right]\{G_n\} \tag{3.16}$$

Since $G_0\neq 0$, it follows from (1. 6) that x_i is a discrete function $x_i\in E$, if and only if $N\leqq 0$. Finally, the general solution of (3. 10) is of the form

$$x = C(1+q)^{f_0}\exp\left[\int\{F_n\}\right]+(1+q)^N\{G_n\}\exp\left[\int\{F_n\}\right] \tag{3.17}$$

where the case $N=f_0$ is excluded, and for $N>f_0$

$$H_{N-f_0}=0$$

must hold. By Theorem 4 we see that (3. 17) is a function for every value of $C\in K$, if and only if, $N\leqq 0$ and $f_0\leqq 0$ hold. On the other hand, if $N\leqq 0$, $f_0>0$, (3. 17) is a function only for $C=0$. There exists only one solution in E.

For $N>0$, (3. 10) has only operational solutions. Since $y\notin E$, when $N>0$, there are no solutions in E. We have proved

THEOREM 6. *Consider the algebraic differential equation*

$$Dx-\{f_n\}x=(1+q)^N\{\varrho_n\} \qquad x\in M \tag{3.18}$$

where the functions f, $\varrho\in E$ *are given such that* $\varrho_0\neq 0$ *and* f_0 *are arbitrary integers. Let also* N *be an arbitrary integer. Moreover, let*

$$\{F_n\}=\{f_n\}-f_0$$

$$\{H_n\}=\{\varrho_n\}\exp\left[-\int\{F_n\}\right] \qquad n=0,1,\dots$$

and for $N\neq f_0$ *let* $\{G_n\}$ *be defined as*

$$G_n=\frac{H_n}{N-f_0-n} \quad \text{if} \quad n\neq N-f_0$$

$$G_n=\text{arbitrary} \quad \text{if} \quad n=N-f_0.$$

Then (3. 18) *is solvable in* M, *if and only if* $N<f_0$ *or*

$$N>f_0 \quad \textit{and} \quad H_{N-f_0}=0$$

If so, the general solution of (3. 18) *is given by the formula*

$$x=\left[C(1+q)^{f_0}+(1+q)^N\{G_n\}\right]\exp\left[\int\{F_n\}\right], \qquad C\in K.$$

This solution represents a function for every value of C, if and only if, $f_0 \leqq 0$, $N \leqq 0$ ($N \neq f_0$) *hold. For* $N \leqq 0$, $f_0 > 0$, *the solution belongs to E, only if* $C=0$. *In this case* (3. 18) *has only one solution in the ring E. If* $N > 0$ (3. 18) *has no solution in E.*

Remark. Practically the above solution formula seems to be complicated.

The application of computers seems to be useful in calculating the components of the solution for every value of n. Occasionally, it is more convenient to begin with the operational notations and express, y, f as a function of the operator q. In this manner in many cases the solutions appear in a very simple closed form. We shall illustrate this in the solution of discrete Volterra integral equations.

4. Discrete Volterra integral equations

We shall now solve discrete Volterra integral equations of the type

$$nf_n + \sum_{k=0}^{n} f_k g_{n-k} = h_n \tag{4. 1}$$

by the application of the operational calculus.

In (4. 1) g, h are given and f is unknown.

Operationally, (4. 1) can be written as

$$Df - fg = -h \tag{4. 2}$$

being an algebraic differential equation. We remark that (4. 2) can have also solutions in M. So (4. 1), (4. 2) are not equivalent. Every solution of (4. 1) is also a solution of (4. 2). The converse is not true in general. Equivalence holds if and only if, every solution of (4. 2) is contained in the ring E.

Consider here two examples.

In the continuous case FÉNYES [6] (p. 399) has solved the following examples.

$$tf(t) + \int_0^t f(\tau)e^{t-\tau}\,d\tau = 2te^t, \tag{4. 4}$$

$$tf(t) - \int_0^t f(\tau)e^{t-\tau}\,d\tau = 2te^t. \tag{4. 5}$$

The general operational solution of (4. 4) is

$$f = C(s-1) + \{e^t\}$$

where s is the differential operator.

(4. 5) has the general solution

$$f = Ce^t + 2e^t \log t$$

(4. 4) has only one, (4. 5) has infinitely many locally integrable solutions.

Let us now consider the discrete analogues of (4. 4), (4. 5)

$$nf_n + \sum_{k=0}^{n} f_k e^{n-k} = 2ne^n \tag{4.6}$$

$$nf_n - \sum_{k=0}^{n} f_k e^{n-k} = 2ne^n \tag{4.7}$$

The operational calculus is applicable, since g_0 is integer. Operationally, the algebraic differential equation

$$Df - f\{e^n\} = \{-2ne^n\}$$

corresponds to (4. 6)

Since $\{e^n\} = \dfrac{1+q}{1+q-e}$ (see [12] pg 17)

and

$$\{-ne^n\} = D\{e^n\} = \frac{(1+q)(1+q-e)-(1+q)^2}{(q+1-e)^2} = -\frac{e(1+q)}{(q+1-e)^2}$$

we obtain

$$Df - \frac{1+q}{1+q-e} f = -\frac{2e(1+q)}{(1+q-e)^2}. \tag{4.8}$$

Let us observe that a particular solution of the corresponding homogeneous equation is

$$x_0 = 1+q-e$$

so the general solution of the homogeneous equation is

$$f_h = C(q+1-e)$$

By (3. 16) a particular solution of (4. 8) is

$$x_i = x_0 \int \frac{y}{x_0} = -(q+1-e) \int \frac{2e(1+q)}{(1+q-e)^3}$$

The above algebraic integral can be easily determined by the aid of the formula $Dx^v = vx^{v-1} Dx$.

By an algebraic integration one has

$$x^v = \int vx^{v-1} Dx$$

and by choosing $x = q+1-e$, $v=-2$

$$\frac{1}{(1+q-e)^2} = -2 \int \frac{1+q}{(1+q-e)^3}$$

will be obtained.

Consequently

$$x_i = \frac{-2e(q+1-e)}{-2(1+q-e)^2} = \frac{e}{q+1-e} = \frac{e(1+q)}{(1+q)(1+q-e)} = \{e\delta_{1,n}\}\{e^n\} =$$

$$= \{\delta_{1,n}\}\{e^{n+1}\} = \{b_n\}, \quad \text{where} \quad b_n = \begin{cases} 0, & \text{if } n=0, \\ e^n, & \text{if } n>0. \end{cases}$$

or

$$x_i = \{e^n\} - 1$$

The general solution of (4. 8) is of the form

$$f = C(q+1-e) + \{e^n\} - 1 \tag{4. 9}$$

from which the only solution of (4. 6) in E is

$$f = \{f_n\} = \{e^n\} - 1. \tag{4. 10}$$

In the same manner,

$$Df + \frac{q+1}{q+1-e} f = \frac{-2e(q+1)}{(q+1-e)^2} \tag{4. 11}$$

corresponds to (4. 7)

The corresponding homogeneous equation has the solution

$$f_h = \frac{1}{q+1-e}.$$

However, (4. 11) has no solution! Indeed, the algebraic integral $\int \frac{y}{x_0}$ which occurs in (3. 16) does not exist in this case. Namely

$$\frac{y}{x_0} = \frac{-2e(1+q)}{q+1-e} = \{-2e^{n+1}\}$$

and by Theorem 3 we see that $\int \{-2e^{n+1}\}$ does not exist, since the initial value $\{-2e^{n+1}\}_{n=0} = -2e \neq 0$.

This is the case of $N = f_0$ of the Theorem 5 as one can easily see.

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(Received November 30, 1970)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 131—136.

ON THE TRANSFORMATION OF LACUNARY SERIES IN TWO VARIABLES

by

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0. Let Λ be a sequence of characters (or imaginary exponentials) defined in the plane, say $\exp 2\pi i(a_k x+b_k y)\equiv\chi_k(x, y)$, and suppose that the "norms" $\|\chi_k\| = =(a_k^2+b_k^2)^{\frac{1}{2}}$ are q-lacunary, that is, $\|\chi_{k+1}\|\geqq q\|\chi_k\|>0$ with $q>1$. By a method of ERDŐS [2; 1, p. 257] we can show that any series $\sum c_k\chi_k$, with complex coefficients c_k, converges or diverges almost everywhere in the plane, according to the convergence or divergence of $\sum |c_k|^2$. Thus for almost all y, the series $\sum c_k\chi(x, y)$ has the same property in regard to x. Suppose now that $F(x_1, x_2)$ is a measurable mapping of the plane and that $F^{-1}(E)$ has measure 0 whenever E has measure 0; then the same conclusions hold for series $\sum c_k\chi_k(F(x, y))$. Our main theorem shows that for certain mappings F, a stronger conclusion is possible than the one obtained by Fubini's theorem in the plane.

THEOREM 1. *Let $\Lambda=(\chi_k)_1^\infty$ be a sequence of characters, and $\chi_k(x, y)=\exp 2\pi i L_k(x, y)$ for linear forms L_k so that $\|L_{k+1}\|>q\|L_k\|$ $(q>1)$.*

Let μ be a probability measure on $(0, 1)$ *fulfilling a Lipschitz condition with exponent $\alpha>0$: $\mu(I)\ll(\text{diam } I)^\alpha$ for all intervals I.*

Finally, let $F(x, y)$ be a mapping of the plane whose co-ordinates are polynomials, such that for no linear form $L\neq 0$ does the real function $L(F(x, y))$ reduce to a sum of functions of one variable.

Then there is a null-set Z, depending only on Λ, F, μ such that for each $y\notin Z$

(1) *$\sum c_k\chi_k(F(x, y))$ converges for μ-almost all x, provided $\sum |c_k|^2<\infty$.*

(2) *The real part of $\sum c_k\chi_k(F(x, y))$ diverges for μ-almost all x, provided $\sum|c_k|^2 =+\infty$.*

Remark. For Lebesgue measure in the plane ERDŐS' method can be replaced by that of KOLMOGOROV (1924, for convergence) and ZYGMUND (1930, for divergence) [4, p. 203]. KOLMOGOROV's theorem is based on Fejér—Lebesgue summability, however, and no analogue of this is apparent for singular measures μ. In order to obtain functions of x, that are ordinary exponentials, we can choose $F(x, y)= =(yx, y^2x)$ for example.

The estimation necessary to Theorem 1 allows a variant theorem with a slight additional effort. Let us say that a sequence Λ is *admissible* for $\alpha>0$ and degree n if the theorem remains true for Λ, whenever μ admits the exponent α in a Lipschitz condition and F has degree at most n in x and y separately.

THEOREM 2. *To each $\alpha>0$ and $n=1, 2, 3, \dots$ there is an $r=r(\alpha, n)$ so that, for every sequence $X_k\geqq k^r$, one can find a sequence Λ admissible for α and n, while $\|\chi_k\|=X_k$.*

In a brief concluding paragraph, we obtain a fragmentary result concerning metrical properties of sets in relation to Theorem 2, and pose a problem.

The outline of the proofs is as follows. In the first paragraph we recall some standard estimations of exponential integrals, in particular of exponentials of polynomials. In the second we estimate certain double integrals, combining the estimates already obtained with particular estimates for the measure μ. Next we obtain an estimate of measures based on the work of ERDŐS and SIDON [2] and from this the proof of Theorem 1 is readily completed. The proof of Theorem 2 is brief.

1. It is convenient to write $e(u)\equiv e^{2\pi iu}$ and denote by $[a, b]$ the domain of functions occurring in the lemmas; except as stated, constants do not depend on a, b.

LEMMA 1. *If* $f'>0$ *on* $[a, b]$ *then*

$$\Big|\int_a^b e(f(x))\,dx\Big| \leqq \pi^{-1}|f'(a)|^{-1}+\pi^{-1}\int_a^b |f''|\,|f'|^{-2}\,dx.$$

LEMMA 2. *If* $f''\geqq d_2>0$ *on* $[a, b]$ *then* $\Big|\int_a^b e(f(x))\,dx\Big| \leqq A_2 d_1^{-\frac{1}{2}}$, (VAN DER CORPUT).

Lemma 1 is proved by a change of variable, and Lemma 2 then follows if a suitable interval — of length $<d_1^{-\frac{1}{2}}$ — is excluded from $[a, b]$; for details cf. [4, p. 197]. In the same way there follows now

LEMMA 2*n*. *If* $f^{(n)}\geqq d_n>0$ *for some* $n=2, 3, \ldots$, *then* $\Big|\int_a^b e(f(x))\,dx\Big|\leqq A_n d_n^{-1/n}$.

Let $p(x) = a_0+a_1x+\cdots+a_nx^n$ be a polynomial with with real coefficients, and $H\equiv H(p)\equiv \max(|a_1|, \ldots, |a_n|)$.

LEMMA 3*n*. $\Big|\int_a^b e(p(x))\,dx\Big|\leqq B_n H^{-1/n}$ *uuiformly for all intervals* $[a, b]\subseteqq[0, 1]$.

PROOF. Let k be the largest index so that $2^k n^{nk}|a_k|\geqq H$. When $k=n$ we can apply Lemma $2n$, since $p^{(n)}=n!a_n$. When $1\leqq k<n$ we find $|a_{k+1}|+\cdots+|a_n| < 2^{-k}n^{-nk}n^{-n}H < 2^{-k}n^{-n}|a_k|$. Thus, on $[0, 1]$,

$$|p^{(k)}-k!a_k| \leqq n^k|a_{k+1}|+\cdots < 2^{-k}n^{k-n}|a_k| \leqq 2^{-2}|a_k|.$$

Then $|p^{(k)}|>|a_k|>H$, and Lemma $2k$ applies, at least when $k\geqq 2$. With $k=1$, we observe that $\int_0^1 |f''|\cdot|f'|^{-2}\,dx < H^{-1}$ and the lemma is completely proved.

2. In this paragraph μ is a probability measure on $[0, 1]$ and α the exponent in Theorem 1.

LEMMA 4*n*. *Let f be n times differentiable on* $[0, 1]$ and $f^{(n)}\geqq d_n>0$. *Then (assuming* $0<\alpha<1$*)*

$$\int \inf(1, |f(x)|^{-1})\mu(dx) \ll d_n^{-\alpha/n} \qquad (n=1, 2, 3, \ldots).$$

PROOF. We can suppose $d_n>1$ and begin with $n=1$. The measure of the set $(|f|\leqq 1)$ is $\ll d_1^{-\alpha}$, while the contri bution from each set $(2^r\leqq|f|<2^{r+1})$ is $\ll(2^r d_1^{-1})^\alpha\cdot$

$\cdot 2^{-r} = d_1^{-\alpha} 2^{(\alpha-1)r}$. We sum this for $r \geqq 0$ to obtain Lemma 4. 1. For $n > 1$ we proceed by induction, excluding in each successive step a set $(|f^{(n)}| < d_{n+1}^{-e})$ of length $\ll d_{n+1}^{-1/n+1}$, namely $e = n/n+1$.

LEMMA 5*n*. *Let p be a polynomial of degree n. Then* $\int_0^1 \inf\left(1, \; |p(x)|^{-1}\right)\mu(dx) \ll H^{-\alpha/n}$.

This can be proved by the same argument as Lemma 3*n*, and has an immediate consequence.

Corollary. $\iint \inf\left(1, |p(x_1)-p(x_2)|^{-1}\right)\mu(dx_1)\mu(dx_2) \ll H^{-\alpha/n}$. This bound is satisfactory, but probably far from best-possible, for $n > 1$. For the partial integral $\int \inf\left(1, |p(x_1)-p(x_2)|^{-1}\right)\mu(dx_1)$ is somewhat smaller than $H^{-\alpha n}$ unless the zeroes of $p(x)-p(x_2)$ are near the zeroes of $p'(x)$, and this imposes an additional restraint on x_2.

When $q(x, y)$ is a polynomial in two variables, terms $Ax+By$ play a minor role, and we define, therefore, $H_0(q)$ as the maximum modulus of coefficients a_{jk} of mixed monomials $x^j y^k$ $(j \geqq 1, k \geqq 1)$.

THEOREM 3. *If* $q(x, y)$ *has degree at most n in x and y, then*

$$J = \int_0^1 \left|\int e\big(q(x, y)\mu(dx)\big)\right|^2 dy \ll H_0^{-b}, \quad b = \alpha n^{-2}.$$

PROOF. Inverting the order of integration we obtain

$$J = \iiint e\big(q(x_1, y) - q(x_2, y)\big)\, dy\, \mu(dx_1)\mu(dx_2)$$

The argument of the exponential is a polynomial in y, and the height H of $q(x_1, y) - q(x_2, y)$ (as a polynomial in y) is at least $p^*(x_1) - p^*(x_2)$; here p^* has degree at most n and $H(p^*) \geqq H_0(q)$. Applying Lemmas 3*n* and 5*n* we obtain

$$J \ll \iint \inf\left(1, |p^*(x_1) - p^*(x_2)|^{-\alpha/n}\right)\mu(dx_1)\mu(dx_2) \ll H_0^{-b},$$

the last by Hölder's inequality. This proves Theorem 3.

The hypothesis of Theorem 1 entails the existence of a constant $\delta > 0$ so that for each linear form $L(u, v) = au+bv$, $H_0\big(L(f(x, y))\big) \geqq \delta(a^2+b^2)^{\frac{1}{2}}$. This fact explains the application Theorem 3 to the proof of Theorem 1.

3. In this paragraph $(f_k)_1^\infty$ is a sequence of complexvalued functions on a space with measure $m \geqq 0$, having the following property P_4:

$$\int |\sum a_k f_k|^4\, dm \leqq C(\sum |a_k|^2)^2$$

for every sequence $(a_k)_1^\infty$ with $\sum |a_k|^2 < \infty$. A new sequence with exactly the same property can be formed by omitting some f_k, and inserting in its place $t_1 f_k, \ldots, t_r f_k$, subject to $|t_1|^2 + \cdots + |t_r|^2 = 1$. Also, the series $\sum a_k f_k$, when developed in the new sequence of functions, has the same l_2-norm (that is, of its sequence of coefficients).

For any numerical series $\sum b_k$, let $S^* = \max\left|\sum_1^r b_k\right|$. We seek to estimate the measure $m(S^*(x) > Y)$, where $S(x) = \sum_1^p a_k f_k(x)$, and p is a power of 2, $a = \max|a_k|$. To do so we divide S into two sums of $\frac{1}{2}p$ consecutive terms, then divide each of these, ... We obtain $\leqq 2^r$ sums of length $\leqq 2^{-r}p$, each of l_2-norm $\leqq a(2^{-r}p)^{\frac{1}{2}}$. Using the basic property of the sequence (f_k) we see that no sum of the r^{th} rank exceeds $(r+1)^{-2}Y$ in modulus, excepting a set of measure $< C2^r 4^{-r} p^2 a^4 (r+1)^8 Y^{-4}$. Summing for all $r \geqq 0$ we obtain a set of measure $< C_1 p^2 a^4 Y^{-4}$, outside of which $S^* < Y \sum_0^\infty (r+1)^{-2} = C_2 Y$.

Beginning now from a finite sequence $\sum b_k f_k$ we apply the splitting method described above to each f_k, obtaining eventually a sum $T = \sum c_k g_k$ in which every $c_k = 0$ or $|c_k| > \frac{1}{2}\max|c_j|$. By this method we obtain also $S^* \leqq T^*$, and applying the remarks before,

$$m\{S^* > Y\} \leqq m\{T^* > Y\} \ll \left(\sum |b_k|^2\right)^2 Y^{-4}.$$

From this we find that an infinite series $\sum_1^\infty a_k f_k$ converges almost everywhere when $\sum |a_k|^2 < +\infty$; this is essentially proved in [2].

To apply these observations in the proof of Theorem 1 we require a sufficient condition for property P_4, expressed in terms of the integrals $I(k, l; u, v) = \int f_k \bar{f}_l f_u \bar{f}_v$. Plainly

$$\int \left|\sum_1^N a_k f_k\right|^4 \leqq \sum \dots \sum |a_k a_l a_u a_v|\,|I(k, l; u, v)|.$$

Because each f_k shall have modulus 1, the terms in which $k = l$ and $u = v$ yield a sum $\left(\sum |a_k|^2\right)^2$. We shall also find that $\sum_R |I(k, l; u, v)| < +\infty$ where R is the set of quadruples $(k, l; u, v)$ in which the maximal term occurs just once. For the terms $k = l$ $u \neq v$, we shall obtain $\sum\sum \left|\int f_u \bar{f}_v\right| < +\infty$, and so finally

$$\int \left|\sum a_k f_k\right|^4 \ll \left(\sum |a_k|^2\right)^2.$$

(A more detailed treatment of the various cases is given in [3, p. 227].)

4. Assertion (1) of Theorem 1 is valid for the entire sequence Λ, provided Λ can be divided into a finite number of complementary subsequences $\Lambda = \bigcup \Lambda_r$ for which assertion (1) is known. Plainly, on the basis of the hypothesis, Λ can be divided into subsequences subject to $\|\chi_{k+1}\| > 5\|\chi_k\|$; we can complete the proof for these sequences. Thus, $f_k = \chi_k(F(x, y))$ and $m = \mu$. The integral

$$\int f_k \bar{f}_l f_u \bar{f}_v \, d\mu = \int e(L(F(x, y)))\, d\mu,$$

where L is a linear form depending on k, l, u, v. When $k > l, u, v$ we have $\|L\| \geqq \frac{2}{5}\|\chi_k\|$, and each index k occurs as the dominant term in at most k^3 coefficients. A similar estimation holds for integrals $\int f_u \bar{f}_v \, d\mu$. From Theorem 3 and the observation afterwards, we see that the coefficients $I(k, l; u, v)$ occurring in the verification of P_4 are in fact a summable sequence for almost all y.

5. The proof of assertion (2) is easier, except that the sequence Λ must not be divided. The series in question take the form $\sum a_k \cos(2\pi L_k F+\varphi_k)$, $a_k\geqq 0$, $\sum a_k^2=\infty$. To each set $E\subset(0,1)$ and $c<\frac{1}{2}$ there is an integer M so that

$$\int_E \left|\sum_M^N a_k \cos(2\pi L_k F+\varphi_k)\right|^2 \mu(dx) \geqq c\mu(E)\sum |a_k|^2;$$

this is to hold for almost all real numbers y. Because $\left\|\sum_M^N a_k \cos(2\pi L_k F+\varphi_k)\right\|_{L^4} \ll$ $\ll (\sum |a_k|^2)^2$, it is enough to prove the assertion for each set E in a suitable *sequence* of measurable sets. Then it is sufficient to prove that for each set E

$$\sum_{k=1}^{\infty}\left|\int_E \cos 4\pi L_k F(x,y)\mu(dx)\right|^2 < \infty$$

and a similar relation for $\sin 2\pi L_k F(x,y)$; further we require

$$\sum\sum_{l\neq k}\left|\int_E \cos(2\pi L_k F\pm L_l F)\mu(dx)\right| < \infty$$

and a relation involving sines. But $\|L_k \pm L_l\| \geqq (q-1)\|L_l\|$ when $k>l$, and from this point the proof follows the previous paragraph.

6. To prove Theorem 2 we use Theorem 3 and the following observation. If each combination $M = L_k+a_1L_l+a_2L_u+a_3L_v$ in which $k>\max(l,u,v)$, $a_i=0,+1,-1$ satisfies $\|M\|\gg k^{12b^{-1}}$, then the sequence $(\chi_k)_1^\infty$ is admissible for α and n. Let then (L_k) be a sequence of independent random variables, each L_k uniformly distributed on the circle with center 0 and radius X_k in the plane. The probability that the form M have length $<k^{12b^{-1}}$ is $\ll X_k^{-1}k^{12b^{-1}}$. Taking account of the number of forms involving k in the first position, we see that if $X_k>k^r$, with $r>4+12b^{-1}$, then almost all sequences $\chi_k=\exp 2\pi i L_k$ are admissible for α and n. In particular, choosing $F(x,y)=(xy, xy^2)$, we find $b=\alpha/4$ and our condition becomes $r>4+48\alpha^{-1}$; also, the functions of y constructed by this process are ordinary exponentials whose frequencies can have polynomial growth. We shall see that this is a best-possible result, in a certain sense.

7. Let $p>1$, $q>1$ and $d=p(q-1)q^{-1}$; let (χ_k) be a sequence of functions on $(-\infty,\infty)$ of modulus 1, and suppose that $\sup|\chi_k'|=o(k^d)$. Let B be a set of real numbers, contained in R intervals of length R^{-p}, for arbitrarily large integers R.

We shall outline a construction of a sequence of complex numbers so that $\sum |c_k|^q<\infty$, but $\sum c_k\chi_k$ diverges everywhere in B. In fact, to each integer R defined above, consider the set T of indices k satisfying $10|\chi_k'|<R^p$; then $R^p=o(|T|^d)$. Let T be divided into R adjacent subsets of nearly equal length, say $T=\bigcup_1^R T_j$, while $B=\bigcup B_j$ is the given covering of B. Then we can choose a_k $(k\in T)$ so that $|a_k|=1$ and $\operatorname{Re} a_k\chi_k(x)>\frac{1}{2}$ whenever $x\in B_j$ and $k\in T_j$. Thus the sum $S=\sum_T a_k\chi_k$ has a maximal function $|S^*(x)|>\frac{1}{4}R^{-1}|T|$ everywhere on B; now the l_q-norm of S, namely $|T|^{1/q}$, is $o(\inf_B S^*(x))$, because $|T|^{1/q}=o(R^{-1}|T|)$. (Recall that $R^p=o(|T|^d)$ and $dp^{-1}=(q-1)q^{-1}$.) Thus the series $\sum c_k\chi_k$ can be constructed.

To compare this with Theorem 2 we choose, of course, $q=2$ and obtain $d=2p$. Now the condition on B allows B to carry a measure μ with a Lipschitz condition in every exponent $\alpha<p^{-1}$. Thus the number r in Theorem 2 must increase to ∞ as α tends to 0.

It is well known that a closed set B carries a probability measure μ with a Lipschitz condition, if and only if B has positive Hausdorff dimension. Conversely, suppose $|\chi_k|=1$, $|\chi_k'|=O(k^r)$ and $\sum c_k\chi_k$ converges μ-almost everywhere, whenever $\sum |c_k|^2<\infty$. Must the support of μ have positive dimension?
Added in proof: The answer affirmative.

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(Received December 9, 1970)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 137—145.

A GEOMETRIC CHARACTERIZATION OF THE LINE GRAPH OF A SYMMETRIC BALANCED INCOMPLETE BLOCK DESIGN

by

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I. Introduction

A symmetric balanced incomplete block (SBIB) design $\pi(v, k, \lambda)$ is an arrangement of v objects, called points, into v sets, called blocks, such that each block consists of k distinct points, each point appears in k blocks, and each pair of distinct points appears in λ blocks. The parameters satisfy $\lambda < k$ and $\lambda(v-1) = k(k-1)$. The graph of $\pi(v, k, \lambda)$ is defined as the bipartite graph $H(\pi)$ whose vertices are the $2v$ points and blocks of π, with two vertices adjacent if and only if one is a block and the other is a point contained in the block. The line graph $G(\pi)$ of $\pi(v, k, \lambda)$ is the line graph of $H(\pi)$, i.e. the graph whose vertices are the edges of $H(\pi)$, with two vertices in $G(\pi)$ adjacent if and only if the corresponding edges in $H(\pi)$ have a common endpoint.

Let $\{G(\pi)\}$ denote the set consisting of the line graphs of all SBIB designs with parameters v, k, λ. In a recent paper, HOFFMAN and RAY—CHAUDHURI [4] proved that $G \in \{G(\pi)\}$ if and only if G is a regular connected graph on vk vertices and the distinct eigenvalues of the adjacency matrix of G are -2, $2k-2$, and $k-2\pm$ $\pm(k-\lambda)^{1/2}$, unless $v=4$, $k=3$, $\lambda=2$, when the sufficiency of these conditions fails due to the existence of a single exceptional graph. We give here a characterization of $\{G(\pi)\}$ in terms of several geometric properties of these graphs, with again one exceptional case (Figure 1) for $v=7$, $k=4$, $\lambda=2$. This result generalizes an earlier characterization [2] of the line graph of a finite projective plane ($\lambda=1$), but the conditions for $\lambda=1$ given here are slightly different from those of [2]. (See Remarks below.)

II. Definitions

By a *graph* G we mean a finite undirected graph without loops or multiple edges. $V(G)$, $E(G)$ denote, respectively, the *vertex set* and the *edge set* of G. We denote by $d(x, y)$ the *distance* between vertices x and y and define further

$$D_i(x) = \{y \in V(G): d(x, y) = i\}, \qquad i = 0, 1, \dots .$$

The *degree* of $x \in V(G)$ is written $\deg x(=|D_1(x)|)$. $\Delta(x, y) = |D_1(x) \cap D_1(y)|$ is the number of vertices adjacent to both x and y. G is *regular* if $\deg x$ is constant for all $x \in V(G)$, and *edge-regular* if, further, $\Delta(x, y)$ is constant for all $(x, y) \in E(G)$. A *clique* K is a set of vertices, any two of which are adjacent.

III. The theorem

THEOREM. *Let v, k, λ be positive integers with $\lambda<k$ and $\lambda(v-1)=k(k-1)$, and let $\{G(\pi)\}$ be the set of line graphs of all symmetric balanced incomplete block designs with parameters v, k, λ. If $G\in\{G(\pi)\}$, then G is connected and has the following properties:*

(P1) $|V(G)|=vk$,

(P2) $\deg x=2k-2$ *for all* $x\in V(G)$,

(P3) $\Delta(x,y)=k-2$ *if* $(x,y)\in E(G)$,

(P4) $\Delta(x,y)\leqq 2$ *if* $(x,y)\notin E(G)$,

(P5) $|D_3(x)\cap D_1(y)|=k-\lambda$ *if* $d(x,y)=2$, $\Delta(x,y)=1$,

(P6) $d(x,y)\leqq 3$ *for all* $x,y\in V(G)$.

Conversely, if G is a connected graph satisfying (P1)—(P6), *then* $G\in\{G(\pi)\}$ *or else* $v=7$, $k=4$, $\lambda=2$, *and G is the graph shown in Figure 1.*

Remarks. Properties (P1)—(P5) are the counterparts of the characterizing properties of the line graph of a finite projective plane ($\lambda=1$) given in [2], except that the analogue of (P4) in that paper is $\Delta(x,y)\leqq 1$ if $(x,y)\notin E(G)$. The increased upper bound for $\Delta(x,y)$ required when $\lambda>1$ necessitates the addition of (P6) here. That neither (P5) nor (P6) is redundant in all cases is demonstrated by the example given in [2] and the graph of Figure 2.

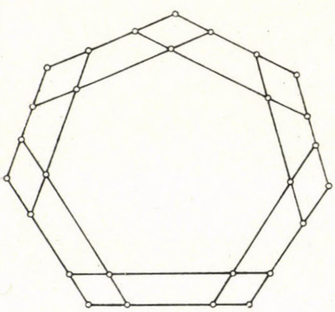

Figure 1. A graph $G\notin\{G(\pi)\}$ satisfying the properties (P1)—(P6) with $v=7$, $k=4$, $\lambda=2$

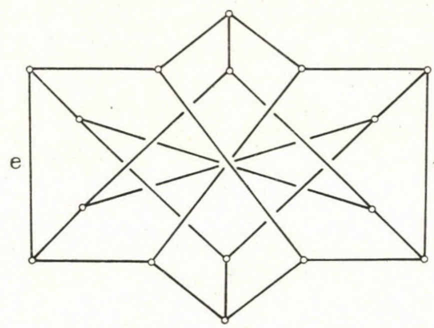

Figure 2. A graph $\bar{H}$ whose corresponding graph $G\notin\{G(\pi)\}$ and satisfies the properties (P1)—(P5) with $v=16$, $k=6$, $\lambda=2$

On comparing our characterization with that of HOFFMAN and RAY-CHAUDHURI [4], we meet with the intriguing problem of the relationship of the eigenvalues of the graph to its geometric properties. Apart from the fact that in the case of regular graphs the dominant eigenvalue is the degree of regularity, very little is known. An important tool is the polynomial of a graph [3], which for regular graphs gives an upper bound to the diameter. For example, the eigenvalues of $G(\pi)$ other than the degree are -2, $k-2\pm(k-\lambda)^{1/2}$, and the fact that there are just three immediately yields property (P6) of the theorem. Using certain "impossible subgraphs" of [4], the fact that -2 is the minimal eigenvalue is easily shown to imply (P4) except for one possible configuration. HOFFMAN and RAY-CHAUDHURI directly prove edge-regularity (P3) for $k \geqq 4$. The exact nature of the relationship between eigenvalues $k-2 \pm (k-\lambda)^{1/2}$ and condition (P5) is unknown.

IV. Proof of the theorem

The necessity of (P1)—(P6) when $G \in \{G(\pi)\}$ is easily verified and the proof is omitted here. We therefore assume that G is a connected graph and satisfies (P1)—(P6).

By (P3) it is evident that $|K| \leqq k$ for any clique K in G. Since we shall only consider cliques K in G such that $|K| = k$, let us agree to use the term "clique" in this restricted sense. We then have

LEMMA 1. *Each vertex of G is contained in exactly two cliques, and each edge o G is contained in exactly one clique.*

PROOF. If $k > 4$, the lemma follows from (P2)—(P4) by a theorem of BOSE and LASKAR [1] on edge-regular graphs. The cases $k = 2, 3, 4$ must be examined separately, and it is easily verified that (P2)—(P4) imply the lemma except for $k = 4$, when these three conditions alone do not rule out the possibility that a vertex $x \in V(G)$ exists such that the subgraph generated by $D_1(x)$ is a 6-cycle. If this is the case, connectivity implies that the subgraph generated by $D_1(y)$ is a 6-cycle for all $y \in V(G)$. We can then show with little difficulty that no such graph satisfies all the conditions (P1)—(P6). (We do encounter one graph that satisfies (P1)—(P6) if $\lambda = 4 (= k)$. This graph is the exceptional case in SHRIKHANDE's characterization [5] of the L_2 association scheme (square lattice graph). To avoid some special arguments in the proof of the theorem, we have assumed $\lambda < k$. However, the theorem remains valid when $\lambda = k = v$ provided we include this additional exception.) As the proof involves a case by case investigation and bears little connection to the main argument, it has been omitted.

COROLLARY 1. 1. *The number of cliques in G is* $2v$.

Let $C(G)$ denote the set of cliques in G, and $I(G)$ the set of unordered pairs (K, L) of distinct cliques such that $K \cap L \neq \emptyset$. Then by Lemma 1, $(K, L) \in I(G)$ if and only if $|K \cap L| = 1$. We denote the single vertex $x \in K \cap L$ by $x = \langle K, L \rangle$.

A *clique chain* $\mathscr{C} = (K_0, K_1, \ldots, K_m)$ is a sequence of distinct cliques such that $(K_i, K_{i+1}) \in I(G)$ and $\langle K_i, K_{i+1} \rangle \neq \langle K_j, K_{j+1} \rangle$ for $i \neq j$. The *length* of $\mathscr{C}$ is the number m of vertices $x_i = \langle K_i, K_{i+1} \rangle$. If $K_0 = K_m$ and no other clique appears twice, $\mathscr{C}$ is a *clique cycle*.

LEMMA 2. *G contains no clique cycle of length three.*

PROOF. Suppose $\mathscr{C} = (K_0, K_1, K_2, K_0)$ is a clique cycle in G. Let $x_i = \langle K_i, K_{i+1} \rangle$, (subscripts modulo 3). Then $x_0 \in D_1(x_1) \cap D_1(x_2)$ and $x_0 \notin K_2$, which implies

$$\Delta(x_1, x_2) \geqq k-1,$$

contradicting (P3).

For $x \in V(G)$, we define

$$D_2^{(j)}(x) = \{y : d(x,y) = 2,\ \Delta(x,y) = j\}$$

for $j=1,2$. Then by (P4), the sets $D_2^{(1)}(x)$, $D_2^{(2)}(x)$ partition $D_2(x)$.

LEMMA 3. *Let $x=\langle K_0, K_1\rangle$, $y=\langle L_0, L_1\rangle$ be two vertices of G. Then by properly labeling the four cliques K_i, L_j, $i,j=0,1$, we have*

(i) $y \in D_0(x)$ *iff* $K_0=L_0$, $K_1=L_1$.

(ii) $y \in D_1(x)$ *iff* $K_0=L_0$, $K_1 \neq L_1$.

(iii) $y \in D_2^{(2)}(x)$ *iff* $(K_0, L_0) \in I(G)$, $(K_1, L_1) \in I(G)$.

(iv) $y \in D_2^{(1)}(x)$ *iff* $(K_0, L_0) \in I(G)$, $(K_1, L_1) \notin I(G)$.

PROOF. Follows immediately from Lemmas 1 and 2.

LEMMA 4. *If $x=\langle K_0, K_1\rangle$, $y=\langle L_0, L_1\rangle$, where $(K_0, L_0) \in I(G)$, $(K_1, L_1) \notin I(G)$, then $y \in D_2^{(1)}(x)$ and there are exactly λ cliques, including L_0, which intersect L_1 and one of K_0, K_1.*

PROOF. Clearly $y \in D_2^{(1)}(x)$ by Lemma 3—(iv). By (P5) there are $k-\lambda$ vertices in $D_3(x) \cap D_1(y)$, and these must be in L_1. Hence the remaining λ vertices of L_1 are in $D_2^{(1)}(x)$. If $z=\langle L_1, K\rangle$ is one of these, then either $(K, K_0) \in I(G)$ or $(K, K_1) \in I(G)$.

LEMMA 5. *G contains no clique cycle of length five.*

PROOF. Suppose $\mathscr{C}=(K_0, K_1, K_2, K_3, K_4, K_0)$ is a clique cycle in G. Let $x_i = \langle K_i, K_{i+1}\rangle$ (subscripts modulo 5) and define α_i as the number of cliques intersecting both K_{i-1} and K_{i+1}, including K_i. Then $x_i \in D_2^{(1)}(x_{i+2})$, so by the preceeding lemma, $\alpha_{i-1}+\alpha_{i+1} = \lambda$. Since $(i+1)-(i-1) = 2$ is prime to 5, we infer that all $\alpha_i=\alpha$, say, and $\lambda=2\alpha$.

If $y \in D_2^{(1)}(x_0)$, $y=\langle K, L\rangle$, where K intersects neither K_0 nor K_1, and L intersects exactly one of K_0, K_1. Then $|K \cap D_2^{(1)}(x_0)| = \lambda = 2\alpha$. Since $2k$ cliques intersect K_0 or K_1 (including K_0, K_1) and G contains $2v$ cliques in all,

$$2k+(2\alpha)^{-1}|D_2^{(1)}(x_0)| \leqq 2v. \tag{1}$$

To show (1) is impossible, let L_0 be one of the $k-\alpha$ cliques intersecting K_0, but not K_2. Then if $y_0=\langle K_0, L_0\rangle$, we find $x_1 \in D_2^{(1)}(y_0)$, and so by Lemma 4, there are 2α cliques meeting K_2 which meet K_0 or L_0. But there are exactly $\alpha_1=\alpha$ such cliques meeting K_2 and K_0, so also there are α cliques meeting K_2 and L_0. Then by the same argument, there must be α cliques meeting L_0 and K_1. It follows that L_0 contains exactly $k-\alpha$ vertices of $D_2^{(1)}(x_0)$. We can repeat the argument for any of the $k-\alpha$ cliques L_1 meeting K_1, but not K_4, obtaining finally

$$|D_2^{(1)}(x_0)| \geqq 2(k-\alpha)^2,$$

which is easily seen to contradict (1).

COROLLARY 5.1. *If K_0, K_1 are two disjoint cliques in G, and L is a clique intersecting both K_0 and K_1, then the number of cliques intersecting both K_0 and K_1, including L, is either λ or k.*

PROOF. Let $x_i=\langle K_i, L\rangle$, $i=0, 1$. If there are fewer than k cliques meeting K_0 and K_1, then there exists $y_1\in K_1$ such that $x_0\in D_2^{(1)}(y_1)$. Now if $y_1=\langle K_1, L_1\rangle$, there are λ cliques meeting K_0 which meet one of K_1, L_1. But if such a clique were to meet K_0 and L_1, G would contain a clique cycle of length five.

Consider now a graph H, the *clique graph* of G, defined by $V(H)=C(G)$, $E(H)=$ $=I(G)$. The mapping $\Phi:(K, L)\to\langle K, L\rangle$ is a one-to-one function of $E(H)$ onto $V(G)$ such that Φ maps adjacent edges of H into adjacent vertices of G and Φ^{-1} maps adjacent vertices of G into adjacent edges of H. It follows that G is the *line graph* of H. Thus $G\in\{G(\pi)\}$ if and only if $H\in\{H(\pi)\}$. We shall restrict our attention henceforth to H. We first summarize some properties of H in

LEMMA 6. *H is connected and has the following properties:*

(Q1) $|V(H)|=2v$,
(Q2) $\deg K=k$ *for all* $K\in V(H)$,
(Q3) $\Delta(K, L)=\lambda$ *or* k *if* $d(K, L)=2$,
(Q4) *H contains no cycles of length three or five.*

PROOF. (Q1) is Corollary 1. 1, (Q2) follows from Lemma 1, (Q3) is Corollary 5. 1, and (Q4) follows from Lemmas 2 and 5, since a clique cycle in G corresponds to a cycle in H.

LEMMA 7. *If* $\Delta(K, L)<k$ *for all* $K, L\in V(H)$ *such that* $d(K, L)=2$, *then* $H\in\{H(\pi)\}$.

PROOF. In view of (Q3) we have $\Delta(K, L)=\lambda$ for all K, L such that $d(K, L)=2$. Let $K\in V(H)$ and define $V_1=\{K\}\cup D_2(K)$. Then (Q2)—(Q4) imply

$$|V_1| = 1+k(k-1)/\lambda = v.$$

If two vertices of V_1 are adjacent, then H contains a 5-cycle. Thus V_1 is an independent set, and so the vk edges incident with vertices of V_1 are all distinct. Since these are all the edges of H by (Q1) and (Q2), the complementary set $V_2=V(H)-V_1$ is also independent. Thus V is bipartite with vertex sets V_1, V_2.

Finally let n be the number of unordered pairs K_0, K_1 in V_1 such that $\Delta(K_0, K_1)=\lambda$. Then $n\lambda=vk(k-1)/2$, since each of the v vertices in V_2 is in $D_1(K_0)$ $\cap D_1(K_1)$ for exactly $k(k-1)/2$ such pairs K_0, K_1. Thus $n=v(v-1)/2$, i.e. $\Delta(K_0, K_1)=\lambda$ for all $K_0, K_1\in V_1$, $K_0\neq K_1$. If we identify the vertices of V_1 with points and those of V_2 with blocks, and define the point to be contained in a block if and only if the corresponding vertices are adjacent in H, then it is clear that H is the bipartite graph of an SBIB design $\pi(v, k, \lambda)$, i.e. $H\in\{H(\pi)\}$.[1]

We consider now the case where there exist two vertices, K, L in H such that $d(K, L)=2$, $\Delta(K, L)=k$. Let us define two vertices K_0, K_1 to be *equivalent* and write $K_0\equiv K_1$, if $K_0=K_1$ or $d(K_0, K_1)=2$, $\Delta(K_0, K_1)=k$. By (Q2) we then have $K_0\equiv K_1$ if and only if $D_1(K_0)=D_1(K_1)$. Let $\bar{K}$ denote the equivalence class containing $K\in V(H)$ ($\equiv$ is readily seen to be, in fact, an equivalence relation).

LEMMA 8. *Any two equivalence classes in* $V(H)$ *contain the same number* $t\geqq 2$ *of vertices.*

[1] It is interesting to note that we have not made use of (P6) to this point, assuming $k>4$.

PROOF. Let $K_0, K_1 \in V(H)$, $t_i = |\bar{K}_i|$, $i=0, 1$. Suppose first that $(K_0, K_1) \in E(H)$. If we can show that in this case $t_0 = t_1 = t$, then the lemma will follow by the connectedness of H. Consider then the number n of edges $(L_0, L_1) \in E(H)$ such that $(K_0, L_0) \in E(H)$, $(K_1, L_1) \in E(H)$, $L_0 \neq K_1$, $L_1 \neq K_0$. If $L_1 \in \bar{K}_0 - \{K_0\}$, L_0 can be chosen in $k-1$ ways, while if $L_1 \notin \bar{K}_0 - \{K_0\}$, L_0 can be choosen in $\lambda - 1$ ways. Hence $n = (t_0 - 1)(k-1) + (k - t_0)(\lambda - 1)$. Similarly if we first fix L_0 and choose L_1, we obtain $n = (t_1 - 1)(k-1) + (k - t_1)(\lambda - 1)$. Since $k - \lambda > 0$, this implies $t_0 = t_1$, and the proof is complete. (Note $t \geqq 2$ by hypothesis.)

Suppose $(K, L) \in E(H)$. Then $\bar{K} \neq \bar{L}$, and if $K_1 \in \bar{K}$, $L_1 \in \bar{L}$, then $(K, L_1) \in E(H)$ and therefore $(K_1, L_1) \in E(H)$. Hence the subgraph of H on the vertices of $\bar{K} \cup \bar{L}$ is the complete bipartite graph on $t+t$ vertices. The equivalence relation $\equiv$ defined on $V(H)$ induces a homomorphism $H \to \bar{H}$, where $\bar{H}$ is the graph defined by

$$V(\bar{H}) = \{\bar{K} : K \in V(H)\},$$

$$E(\bar{H}) = \{(\bar{K}, \bar{L}) : (K, L) \in E(H)\}.$$

LEMMA 9. *$\bar{H}$ is connected and has the following properties, where $\bar{v} = v/t$, $\bar{k} = k/t$, $\bar{\lambda} = \lambda/t$:*

$\overline{\text{(Q1)}}$ $|V(\bar{H})| = 2\bar{v}$,

$\overline{\text{(Q2)}}$ $\deg \bar{K} = \bar{k}$ *for all* $\bar{K} \in V(H)$,

$\overline{\text{(Q3)}}$ $\Delta(\bar{K}, \bar{L}) = \bar{\lambda}$ *if* $d(\bar{K}, \bar{L}) = 2$,

$\overline{\text{(Q4)}}$ *$\bar{H}$ contains no cycles of length three or five.*

PROOF. $\overline{\text{(Q1)}}$ and $\overline{\text{(Q2)}}$ are obvious. $\overline{\text{(Q3)}}$ follows from (Q3) of Lemma 6, since now $\Delta(\bar{K}, \bar{L}) = \bar{k}$ would imply $K \equiv L$, and hence $\bar{K} = \bar{L}$. We finally observe that if $\bar{H}$ were to contain a cycle of length three of five, then so would H, contradicting (Q4) of Lemma 6.

The equation relating $\bar{v}$, $\bar{k}$, $\bar{\lambda}$ reads

$$\bar{\lambda}(\bar{v}t - 1) = \bar{k}(\bar{k}t - 1). \tag{2}$$

(Note that $\bar{v}$ need not be an integer.)

Consider now a fixed edge $(\bar{K}_0, \bar{L}_0) \in E(\bar{H})$ and define

$$A_0 = \{\bar{K}_0\} = D_0(\bar{K}_0), \quad B_0 = \{\bar{L}_0\} = D_0(\bar{L}_0),$$

$$B_1 = D_1(\bar{K}_0) - D_0(\bar{L}_0), \quad A_1 = D_1(\bar{L}_0) - D_0(\bar{K}_0),$$

$$A_2 = D_2(\bar{K}_0) - D_1(\bar{L}_0), \quad B_2 = D_2(\bar{L}_0) - D_1(\bar{K}_0).$$

Then

$$D_0(\bar{K}_0) = A_0, \qquad D_0(\bar{L}_0) = B_0,$$

$$D_1(\bar{K}_0) = B_0 \cup B_1, \quad D_1(\bar{L}_0) = A_0 \cup A_1,$$

$$D_2(\bar{K}_0) = A_1 \cup A_2, \quad D_2(\bar{L}_0) = B_1 \cup B_2.$$

It follows from $\overline{(Q4)}$ that the six sets A_i, B_i $(i=0, 1, 2)$ are pairwise disjoint, and thus the two sets $D_i(\bar{K}_0)$, $D_i(\bar{L}_0)$ are disjoint for each $i=0, 1, 2$. Also by $\overline{(Q4)}$ we see that no two vertices of $D_i(\bar{K}_0)$ or of $D_i(\bar{L}_0)$ can be adjacent for $i=0, 1, 2$. Using $\overline{(Q2)}$ and $\overline{(Q3)}$ we can then easily determine the number of vertices in each set, obtaining

$$|A_0| = |B_0| = 1,$$

(3)
$$|B_1| = |A_1| = \bar{k}-1,$$

$$|A_2| = |B_2| = (\bar{k}-1)(\bar{k}-\bar{\lambda})/\bar{\lambda}.$$

Let C denote the set of vertices not in any of these sets. Then using $\overline{(Q1)}$, (2) and (3) we have

(4)
$$|C| = 2(\bar{k}-\bar{\lambda})(t-1)/\bar{\lambda}t.$$

Thus t divides $2(\bar{k}-\bar{\lambda})$, and so

(5)
$$2 \leqq t \leqq 2(\bar{k}-\bar{\lambda}).$$

LEMMA 10. *No two vertices of C are adjacent.*

PROOF. Clearly we have $d(\bar{K}_0, \bar{K}) \geqq 3$, $d(\bar{L}_0, \bar{K}) \geqq 3$ for all $\bar{K} \in C$. If $\bar{K}, \bar{L} \in C$ and $(\bar{K}, \bar{L}) \in E(\bar{H})$, then $(K, L) \in E(H)$. Let x be the vertex of G defined by $x=\langle K, L\rangle$, and let $x_0=\langle K_0, L_0\rangle$. Then by (P6), $d(x_0, x) \leqq 3$ in G. If $d(x_0, x)=3$, there exists a clique L_1 meeting one of K_0, L_0 and one of K, L, say K_0 and K. Then $d(K_0, K)=2$ in H, which implies either $\bar{K}_0=\bar{K}$ or $d(\bar{K}_0, \bar{K})=2$ in $\bar{H}$, a contradiction. If $d(x_0, x) \leqq 2$, the result is even simpler.

It follows from Lemma 10 that $D_1(\bar{K}) \subseteqq A_2 \cup B_2$ for all $\bar{K} \in C$. Hence $C \subseteqq D_3(\bar{K}_0) \cup D_3(\bar{L}_0)$.

LEMMA 11. *If $d(\bar{K}, \bar{L})=3$, then*

$$|D_2(\bar{K}) \cap D_1(\bar{L})| = |D_1(\bar{K}) \cap D_2(\bar{L})|.$$

PROOF. Let $r = |D_2(\bar{K}) \cap D_1(\bar{L})|$, $s = |D_1(\bar{K}) \cap D_2(\bar{L})|$. Consider the number n of edges joining $D_2(\bar{K}) \cap D_1(\bar{L})$ and $D_1(\bar{K}) \cap D_2(\bar{L})$. Using (Q3), we have $n=r\bar{\lambda}=s\bar{\lambda}$, i.e. $r=s$.

Let $\bar{K}$ be a fixed vertex of C and define

$$a = |D_1(\bar{K}) \cap A_2|,$$

$$b = |D_1(\bar{K}) \cap B_2|.$$

Then $a+b = \bar{k}$, since $A_2 \cap B_2 = \varnothing$. If we define

$$a_1 = |D_2(\bar{K}) \cap B_1|,$$

$$b_1 = |D_2(\bar{K}) \cap A_1|,$$

then it follows from Lemma 11 that $a_1=a$, since clearly $D_1(\bar{K}) \cap D_2(\bar{K}_0) = D_1(\bar{K}) \cap A_2$ and $D_2(\bar{K}) \cap D_1(\bar{K}_0) = D_2(\bar{K}) \cap B_1$. Similarly $b=b_1$. Since $a = |D_2(\bar{K}) \cap B_1| \leqq |B_1| = \bar{k}-1$, we have $b \geqq 1$. Similarly $a \geqq 1$. Hence $C = D_3(\bar{K}_0) \cap D_3(\bar{L}_0)$. Again $a \geqq 1$ implies $D_2(\bar{K}) \cap B_1 \neq \varnothing$, and hence there exists $\bar{L}_1 \in D_2(\bar{K}) \cap B_1$ with

$\Delta(\bar{K}, \bar{L}_1)=\bar{\lambda}$. Each of these vertices must be in $D_1(\bar{K})\cap A_2$, and therefore $a\geqq\bar{\lambda}$. Similarly $b\geqq\bar{\lambda}$ and we have

(6) $$\bar{\lambda} \leqq a, b \leqq \bar{k}-\bar{\lambda},$$

where $a+b = \bar{k}$.

Consider $D_2(\bar{K})$. We wish to establish an upper bound on the number of vertices $\bar{L}\in D_2(\bar{K})$ which are not in C. Since $D_1(\bar{K}) = (D_1(\bar{K})\cap A_2)\cup(D_1(\bar{K})\cap B_2)$, such an $\bar{L}$ is adjacent to some vertex of $D_1(\bar{K})\cap A_2$ or to some vertex of $D_1(\bar{K})\cap B_2$. Assume without loss of generality the former and call the vertex $\bar{K}_2$. It then follows that $\bar{L}\in D_2(\bar{L}_0)$, and hence $\bar{L}\in D_1(\bar{K}_1)$ for some vertex $\bar{K}_1\in D_1(\bar{L}_0)$. If $\bar{K}_1\in \in D_2(\bar{K})\cap A_1$, then $\bar{H}$ contain a cycle of length five. Hence $\bar{K}_1\in A_0\cup(A_1-D_2(\bar{K}))$, a set of a vertices. Thus we have, since $\bar{K}_1\in D_2(\bar{K}_2)$, $|D_2(\bar{K}_2)\cap D_1(\bar{L}_0)|\leqq a$. By Lemma 11, $|D_1(\bar{K}_2)\cap D_2(\bar{L}_0)|\leqq a$. Thus at most a of the $\bar{k}$ vertices adjacent to $\bar{K}_2\in D_1(\bar{K})\cap \cap A_2$ are in $D_2(\bar{K})\cap D_2(\bar{L}_0)$. Since there are a vertices in $D_1(\bar{K})\cap A_2$, and since any vertex $\bar{L}_a\in D_2(\bar{K})\cap D_2(\bar{L}_0)$ adjacent to one vertex of $D_1(\bar{K})\cap A_2$ is adjacent to exactly $\bar{\lambda}$ of them (by $\overline{(Q3)}$, $\bar{L}_a\in D_2(\bar{K})\cap D_2(\bar{L}_0)$ cannot be adjacent to any vertex of $D_1(\bar{K})\cap B_2$), we have at most $a^2/\bar{\lambda}$ vertices $\bar{L}_a\in D_2(\bar{K})$ which are also in $D_1(\bar{K}_2)$ for some $\bar{K}_2\in D_1(\bar{K})\cap A_2$. A similar argument shows that there are at most $b^2/\bar{\lambda}$ vertices $\bar{K}_b\in D_2(\bar{K})$ which are also in $D_1(\bar{L}_2)$ for some $\bar{L}_2\in D_1(\bar{K})\cap B_2$. This now gives us

$$|D_2(\bar{K})-C| \leqq (a^2+b^2)/\bar{\lambda},$$

and thus together with (3)

(7) $$|C| \geqq 1+\bar{\lambda}^{-1}\cdot[\bar{k}(\bar{k}-1)-(a^2+b^2)].$$

Then by (4)

$$2(\bar{k}-\bar{\lambda})(1-1/t) \geqq \bar{\lambda}+\bar{k}(\bar{k}-1)-(a^2+b^2).$$

The inequality remains valid on replacing t by the upper bound $2(\bar{k}-\bar{\lambda})$ of (5), and we then have

$$a^2+b^2 \geqq \bar{k}^2-3\bar{k}+3\bar{\lambda}+1.$$

Substituting $\bar{k}-a$ for b and simplifying,

(8) $$a^2+\bar{k}a+1/2[3(\bar{k}-\bar{\lambda})-1] \geqq 0.$$

Assuming with no loss of generality that $a\leqq b$, it is easily verified that the only values of a, b, $\bar{k}$, $\bar{\lambda}$ satisfying both (6) and (8) are

(i) $\bar{k}\geqq 2,\ \bar{\lambda}=1,\ a=1,\ b=\bar{k}-1,$

(ii) $\bar{k}=4,\ \bar{\lambda}=1,\ a=b=2.$

If $\bar{k}=2$, $\bar{\lambda}=1$, $a=b=1$, then $t=2$ by (5), and from (3) and (4), we have exactly one vertex in each of the sets A_i, B_i $(i=0, 1, 2)$, and C. In this case $\bar{H}$ is a cycle of length seven, $v=7$, $k=4$, $\lambda=2$, and G is the graph of Figure 1 below. The lines in the figure represent cliques, two vertices being adjacent in G if and only if they are collinear.

Consider next the case $\bar{k}\geqq 3$, $\bar{\lambda}=1$, $a=1$, $b=\bar{k}-1$. Then if $\bar{K}_2$ is the vertex of A_2 adjacent to $\bar{K}\in C$, there are exactly $\bar{k}-1$ vertices in $D_1(\bar{K}_2)\cap C_2$ including $\bar{K}$, since $\bar{K}_2$ is already adjacent to a vertex $\bar{L}_1\in B_1 \subseteqq D_2(\bar{L}_0)$, and by the argument following (6) it is adjacent to at most $a=1$ vertices of $D_2(\bar{L}_0)$. Since $a=1$, $b=\bar{k}-1$ or $a=\bar{k}-1$, $b=1$ for every choice of $\bar{K}\in C$, it follows that each vertex $\bar{K}\in C$ is contained in at least one set C_0 of $\bar{k}-1$ vertices in C such that either $C_0 \subseteq D_1(\bar{K}_2)$ or $C_0 \subseteq D_1(\bar{L}_2)$ for some $\bar{K}_2 \in A_2$, $\bar{L}_2 \in B_2$. By (4) and (5) we have

$$\bar{k}-1 \leqq |C| \leqq 2\bar{k}-3, \tag{9}$$

so that if C_0, C_1 are two such sets, they must have a non-empty intersection. If $|C_0 \cap C_1| \geqq 2$, and $\bar{K}, \bar{L} \in C_0 \cap C_1$, then $\Delta(\bar{K}, \bar{L}) \geqq 2$, contradicting $(\overline{Q3})$, since $\bar{\lambda}=1$. Hence it follows from (9) that either there is only one such set $C_0 \subseteq C$, or else there are two sets $C_0, C_1 \subseteq C$ with $|C_0 \cap C_1| = 1$. In the former case, $|C| = \bar{k}-1$, while in the latter $|C| = 2\bar{k}-3$.

We can easily verify, using $(\overline{Q3})$, that the number of edges joining A_2 and C is equal to the number of edges joining B_2 and C. For $\bar{K}_i \in C$, set $a_i = |D_1(\bar{K}_i) \cap A_2|$ and $b_i = |D_1(\bar{K}_i) \cap B_2|$, and let $\bar{a}$ be the number of $\bar{K}_i$'s for which $a_i=1$, $b_i=\bar{k}-1$. Then since $\sum a_i = \sum b_i$, we obtain

$$\bar{a}+(|C|-\bar{a})(\bar{k}-1) = \bar{a}(\bar{k}-1)+(|C|-\bar{a}),$$

and since $\bar{k} \geqq 3$,

$$|C| = 2\bar{a}. \tag{10}$$

This rules our the possibility $|C| = 2\bar{k}-3$. As to the case $|C| = \bar{k}-1$, it follows from (10) that we must have two vertices $\bar{K}_1, \bar{K}_2 \in C$ with, say $a_1=1$ and $b_2=1$. Let $\bar{L}_1 \in A_2$ and $\bar{L}_2 \in B_2$ be the vertices adjacent to $\bar{K}_1$, $\bar{K}_2$, respectively, then by our earlier argument $\bar{L}_1, \bar{L}_2$ are both adjacent to all of C, which implies $\Delta(\bar{K}_1, \bar{K}_2) \geqq 2$, a contradiction.

The remaining case $\bar{k}=4$, $\bar{\lambda}=1$, $a=b=2$ can be shown to be impossible by a rather involved argument which demonstrates that $\bar{H}$ must contain a cycle of length five. We have omitted the proof here.

Concluding Remarks. The question whether the conditions (P1)—(P6) are redundant is difficult to answer in general. The example given in [2] with $v=7$, $k=3$, $\lambda=1$ shows that (P5) cannot be dropped without admitting additional exceptions, while the graph in Figure 2 below with $v=16$, $k=6$, $\lambda=2$ demonstrates that the same holds true for (P6). For ease of exposition we have drawn the graph $\bar{H}$ in Figure 2 with $t=2$, and thus $\bar{v}=8$, $\bar{k}=3$, $\bar{\lambda}=1$. The two vertices of G corresponding to the edges e and f in $\bar{H}$ are readily seen to be at distance 4 from each other.

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(Received December 10, 1970)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 147—153

FURTHER STATISTICAL PROPERTIES OF THE WALSH FUNCTIONS

by

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It is known that the trigonometric orthogonal system has subsequences which share important properties with sequences of independent random variables. In our present paper we study subsequences of the Walsh orthogonal system from this point of view. Let

$$R_k(x) = \text{sign}\sin 2^k \pi x \qquad 0 \leqq x \leqq 1$$

the kth Rademacher function, and for an integer $n = \sum \varepsilon_k 2^k$ ($\varepsilon_k = 0$, or $\varepsilon_k = 1$) let

$$w_n(x) = \prod_{\varepsilon_k = 1} R_k(x).$$

The Walsh functions $\{w_n(x)\}$ form a pairwise independent, but not mutually independent system. Since by [1] (see also [2])

$$\left| \sum_{k=1}^{n} w_k(x) \right| \leqq f(x)$$

where $f(x)$ is a measurable a.e. finite function, only lacunary subsequences of $\{w_n(x)\}$ will be interesting for us. By L_α we denote the set of sequences of integers satisfying $n_{k+1}/n_k \geqq 1 + k^{-\alpha}$.

THEOREM 1. *(Central limit theorem) If $\{n_k\} \in L_\alpha$ for $0 < \alpha < 1/2$ then*

$$\lim_{N \to \infty} \mathrm{P}\left(N^{-1/2} \sum_{k=1}^{N} w_{n_k}(x) < u \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{t^2}{2}}\, dt, \tag{1}$$

in turn, for every $1/2 < \alpha < 1$ there exists an $\{n_k\} \in L_\alpha$ for which this is no longer valid.

(1) generalizes results of G. MORGENTHALER [5] and P. RÉVÉSZ—M. WSCHEBOR [2], where $n_{k+1}/n_k \geqq q > 1$ was assumed. The idea of our proof is the same as in [2]. For the trigonometric case the analogous result was proved by P. ERDŐS [3].

THEOREM 2. *(Law of iterated logarithm) For $0 < \alpha < 1/2$ and $\{n_k\} \in L_\alpha$ we have for a.e. $x \in [0, 1]$*

$$\overline{\lim_{N \to \infty}}\, (2N \log\log N)^{-1/2} \sum_{k=1}^{N} w_{n_k}(x) \leqq 1.$$

The proof of Theorem 2 easily follows from our main lemma in § 2 by the argument discussed in [2]. Also the proof of (1) follows, after having the main lemma, the lines of the weaker result of [2] but we reproduce it in § 3 for the sake of completeness.

§ 1. Preliminary remarks

LEMMA 1. *Let* $0<\alpha<1$, $\{n_k\}\in L_\alpha$ *and* $n_r\in[2^\nu, 2^{\nu+1})$, *then at most* cr^α *numbers of* $\{n_k\}$ *belong to* $[2^\nu, 2^{\nu+1})$.

PROOF. Suppose that $n_{k+1}=x\in[2^\nu, 2^{\nu+1})$ but $n_k<2^\nu$, and $n_{l+1}<2x$ for an $l(>k)$. Then

$$2x > n_{l+1} \geqq n_{k+1} \prod_{j=k+1}^{l}\left(1+\frac{1}{j^\alpha}\right) = x\prod_{j=k+1}^{l}\left(1+\frac{1}{j^\alpha}\right)$$

Thus

$$\prod_{j=k+1}^{l}\left(1+\frac{1}{j^\alpha}\right) < 2$$

which easily implies; $l-k < ck^\alpha$.

REMARK 1. *Let* $0<\alpha<1$, $\{n_k\}\in L_\alpha$ *and* $l>k$. *Then the first* $[\log_2 k^\alpha]+2$ *binary digits of* n_k *are not all equal to the corresponding digits of* n_l.

PROOF. Let the first s binary digits of n_k and n_l be equal, then

$$2^{-s+1} \geqq \frac{n_l-n_k}{n_k} \geqq \frac{1}{k^\alpha}$$

consequently $s<[\log_2 k^\alpha]+2$.

§ 2. The main lemma

LEMMA 2. *Let* $0<\alpha<\frac{1}{2}$ *and* $\{n_k\}\in L_\alpha$ *then we have for an arbitrary complex* λ

$$\lim_{N\to\infty} \mathsf{E}\left(\prod_{k=1}^{N}(1+\lambda N^{-1/2}w_{n_k})\right) = 1.$$

This is an improvement of a lemma due to P. RÉVÉSZ—M. WSCHEBOR [2].

PROOF.

$$\prod_{k=1}^{N}(1+\lambda N^{-\frac{1}{2}}w_{n_k}) = 1+\lambda N^{-\frac{1}{2}}\sum_{k=1}^{N} w_{n_k}+\lambda^2 N^{-1}\sum_{1\leqq i<j\leqq N} w_{n_i}w_{n_j}+$$

$$+\lambda^3 N^{-\frac{3}{2}}\sum_{1\leqq i<j<l\leqq N} w_{n_i}w_{n_j}w_{n_l}+\cdots+\lambda^N N^{-\frac{N}{2}} w_{n_1}w_{n_2}\dots w_{n_N}.$$

Observe that the expectation of the second and third term is zero. So we have to prove that the limit of the expectation of the following part is zero as $N\to\infty$. Let us consider a term of that type:

$$\lambda^m N^{m/2}\sum_{1\leqq l_1<l_2<\cdots<l_m\leqq N} w_{n_{l_1}}w_{n_{l_2}}\dots w_{n_{l_m}} \tag{2}$$

In (2) the expectation of each term is 0 or 1, and it is 1 if and only if the exponent of each $R_k(x)$ in $w_{n_{l_1}}w_{n_{l_2}}\dots w_{n_{l_m}}$ is even. Thus we have to estimate for each fixed m the number of products having expectation 1.

Clearly, for choosing the largest index n_{l_m} we have at most N possibilities. Suppose that $n_{l_m} \in [2^\nu, 2^{\nu+1})$, it means $w_{n_{l_m}}$ contains $R_{\nu+1}$. The exponent of $R_{\nu+1}$ can be even in the product $w_{n_{l_1}} w_{n_{l_2}} \dots w_{n_{l_m}}$ only if we choose $n_{l_{m-1}}$ also from the interval $[2^\nu, 2^{\nu+1})$, which means by Lemma 1 at most cN^α possibilities. Put

$$w_{q^{(t)}} = w_{n_{l_m}} w_{n_{l_{m-1}}} \dots w_{n_{l_{m-t+1}}}.$$

By remark 1 $w_{q^{(2)}}$ must contain a Rademacher function whose index is larger than $(\nu+1)-([\alpha \log_2 N]+2) = \nu-[\alpha \log_2 N]-1$. To assure the even exponents of the R_k-s, we must choose $n_{l_{m-2}}$ in the interval $[2^{\nu-[\alpha \log_2 N]-1}, 2^{\nu+1})$, which means at most $([\alpha \log_2 N]+2)\cdot(cN^\alpha)$ possibilities.

Among the intervals determined by the $\{n_k\}$, $k=1, 2, \dots$ let us denote by $[n_{k(s)}, n_{k(s)+1})$ the unique one which contains the integer $s>0$.

Concerning the choice of the next term of the product we have the following alternatives:

1. $q^{(2)} = n_{k(q^{(2)})}$ and $n_{l_{m-2}} = n_{k(q^{(2)})}$
2. $q^{(2)} = n_{k(q^{(2)})}$ and $n_{l_{m-2}} \neq n_{k(q^{(2)})}$
3. $q^{(2)} \neq n_{k(q^{(2)})}$ and $n_{l_{m-2}}$ is either $n_{k(q^{(2)})}$ or $n_{k(q^{(2)})+1}$
4. $q^{(2)} \neq n_{k(q^{(2)})}$ and $n_{l_{m-2}}$ is neither $n_{k(q^{(2)})}$ nor $n_{k(q^{(2)})+1}$.

Let us investigate all these cases.

1. To choose $n_{l_{m-2}}$ we have only 1 possibility, thus for choosing the first three factors we have at most $N\cdot(cN^\alpha)\cdot 1$ possibilities. Moreover we have $w_{q^{(3)}}=(w_{q^{(2)}})^2=1$ so after these 3 steps reproduces itself the situation which we had before the first step.

2. and 4. Applying Remark 1 in case 2 for $q^{(2)}$ and $n_{l_{m-2}}$ and in case 4 for $n_{l_{m-2}}$ and $n_{k(q^{(2)})+1}$ we get, that in both cases there are at most $([\alpha \log_2 N]+2)(cN^\alpha)$ possibilities for choosing $n_{l_{m-3}}$. Among these possibilities for $n_{l_{m-3}}$ there are all the alternatives 1—4. which we had in the previous step for $n_{l_{m-2}}$. (Now instead of $q^{(2)}$ we shall consider $q^{(3)}$.)

3. To choose $n_{l_{m-2}}$ we have only 2 possibilities. We may suppose that, if $q^{(2)} \in [2^{\mu_1}, 2^{\mu_1+1})$ and $q^{(3)} \in [2^{\mu_2}, 2^{\mu_2+1})$ then $\mu_2 < \mu_1-[\alpha \log_2 N]-2$, which means, that the diadic interval containing $q^{(3)}$ lies at least $[\alpha \log_2 N]+2$ diadic intervals lower then the one containing $q^{(2)}$. (An interval is called diadic, if it is of the form $[2^p, 2^{p+1})$ $p \geqq 0$ integer.) If the above condition does not hold we can follow the same way as in case 2 and 4. In choosing $n_{l_{m-3}}$ we consider the following alternatives:

(i) $n_{l_{m-3}} \in [2^{\mu_2}, 2^{\mu_2+[\alpha \log_2 N]+2})$

(ii) $n_{l_{m-3}} \geqq 2^{\mu_2+[\alpha \log_2 N]+2}$

If neither (i) nor (ii) holds then R_{μ_2+1} would have an odd exponent in our product. In case (i) we have again at most $([\alpha \log_2 N]+2)(cN^\alpha)$ possibilities to choose $n_{l_{m-3}}$, and among them there are again all the types of possibilities which we have investigated in choosing $n_{l_{m-2}}$. In case (ii) we have only the trivial upper estimation N for the choice of $n_{l_{m-3}}$. But observe that in this case $n_{l_{m-3}}$ belongs to a higher diadic interval than $q^{(3)}$. Therefore $q^{(4)}$ belongs to the diadic interval which contains

$n_{l_{m-3}}$, thus we have to choose $n_{l_{m-4}}$ from the same diadic interval, which means again at most cN^α possibilities. (Any other choice would result an odd exponent of an R_k.)

Let us notice, that $w_{q^{(5)}}=w_{q^{(3)}}(w_{n_{l_{m-3}}} w_{n_{l_{m-4}}})$. By Remark 1 the index of the product $w_{n_{l_{m-3}}} w_{n_{l_{m-4}}}$ is larger than 2^{μ_2+1}, thus $q^{(5)}$ must belong to the diadic interval which contains the index of $w_{n_{l_{m-3}}} w_{n_{l_{m-4}}}$. But this diadic interval, as we have seen, lies at most $([\alpha \log_2 N]+2)$ diadic intervals lower, than the one which contains $n_{l_{m-4}}$. It means that to choose $n_{l_{m-5}}$ we have again at most $([\alpha \log_2 N]+2)(cN^\alpha)$ possibilities. We obtained, that to choose the consecutive 3 terms $n_{l_{m-2}}, n_{l_{m-3}}, n_{l_{m-4}}$ we have at most $2N(cN^\alpha)$ possibilities, and at the choice of the next term $n_{l_{m-5}}$ reproduces the same situation which we had for the choice of $n_{l_{m-2}}$.

Let us summarize our results:

1. After the first 3 steps we have again "almost" N possibilities, so we can consider the fourth step as a new beginning of our process, where the number of the factors of the product will change from m to $m-3$, and for these 3 steps we have at most $N\cdot(cN^\alpha)\cdot 1$ possibilities.

2., 3. (i), 4. In each step of these type we have at most $([\alpha \log_2 N]+2)\cdot(cN^\alpha)$ possibilities to choose the next term.

3. (ii) We have to choose a cycle containing 3 consecutive steps, where the corresponding possibilities can be estimated by $(2NcN^\alpha)$.

The only exception is the last term which can be choosen at most 1 way.

Being $\alpha<1/2$ we obtain;

$$\{([\alpha \log_2 N]+2)cN^\alpha\}^3 < 2NcN^\alpha \qquad \text{(if } N \text{ is large enough).}$$

As there are such steps where we have twice two alternatives to choose, we estimate the total number of choices by

$$4^m N^2 (2NcN^\alpha)^{\frac{m-4}{3}} \tag{3}$$

(In (3) we took into account that the $(m-1)$th step may be in the middle of a cycle.)

Estimation (3) is strong enough for our purpose if $m \geqq 5$. It is easy to see that for $m=3$ and $m=4$ the estimations $N\cdot(cN^\alpha)\cdot 1$ and $N\cdot(cN^\alpha)\cdot([\alpha \log_2 N]+2)N^\alpha\cdot 1$ are valid. Thus;

$$\lim_{N\to\infty}\left|\sum_{m=3}^{N} \lambda^m N^{-\frac{m}{2}} \mathsf{E}\Big(\sum_{1\leqq l_1<l_2<\ldots<l_m\leqq N} w_{n_{l_1}} w_{n_{l_2}} \ldots w_{n_{l_m}}\Big)\right| \leqq$$

$$\leqq \lim_{N\to\infty}\left\{N^{\alpha-\frac{1}{2}} c\big(|\lambda|^3(1+|\lambda|)\big)+\sum_{m=5}^{\infty}(4|\lambda|)^m \big(2cN^{1+\alpha-\frac{3}{2}}\big)^{\frac{m-4}{3}}\right\}=0 \quad \text{whenever } \alpha<\frac{1}{2}.$$

§ 3. Central limit theorem

PROOF of (1): Put $S_N = w_{n_1}+w_{n_2}+\cdots+w_{n_N}$ and $F_N(u)=\mathsf{P}(S_N N^{-1/2}<u)$. Denote $\varphi_N(\lambda)$ the characteristic function of $F_N(u)$. We have;

$$\varphi_N(\lambda)=\int_{-\infty}^{\infty} e^{i\lambda u}\,dF_N(u)=\mathsf{E}\big(e^{i\lambda\frac{S_N}{\sqrt{N}}}\big)=\mathsf{E}\Big(e^{i\lambda N^{-1/2}\sum_{k=1}^{N} w_{nk}}\Big)$$

Applying the identity:

$$\exp z = (1+z)\exp\left(\tfrac{1}{2}z^2+O(|z|^3)\right) \quad \text{if} \quad z\to 0$$

we get:

$$\exp\left(\frac{i\lambda}{\sqrt{N}}w_{n_k}\right) = \left(1+\frac{i\lambda}{\sqrt{N}}w_{n_k}\right)\exp\left(-\frac{1}{2}\frac{\lambda^2}{N}+P(\lambda,w_{n_k})\right) \text{ where } |P(\lambda,w_{n_k})| \leqq \frac{|\lambda|^3}{N^{3/2}}.$$

Therefore

$$\varphi_N(\lambda) = e^{-\frac{1}{2}\lambda^2}\,\mathsf{E}\left(\prod_{k=1}^{N}\left(1+\frac{i\lambda}{\sqrt{N}}w_{n_k}\right)\right)+O\left(\frac{|\lambda|^3}{N^{1/2}}e^{\frac{|\lambda|^3}{N^{1/2}}}\right).$$

Using the main lemma we have $\varphi_N(\lambda)\to e^{-\frac{1}{2}\lambda^2}$ if $N\to\infty$, which proves our statement.

§ 4. The lacunarity condition $n_{k+1} \geqq \left(1+\frac{1}{k^\alpha}\right)n_k, \alpha > \frac{1}{2}$ does not imply the central limit theorem

PROOF *of the counterexample to* (1): Let $I_\alpha(\beta)$ $(0<\alpha<1)$ be the set of integers v with the following properties:

a) If $2^{t-1}\leqq v<2^t$ then t satisfies

$$K = \beta+\left[\frac{\alpha}{1-\alpha}\log_2 t\right] > 0$$

where β is a fixed integer determined later.

b) All binary digits following the first K digits vanish.

Consider the sequence of the elements of the set of integers $\bigcup_{r=1}^{\infty} I_\alpha(\beta)\cap(0,\ 2^r)$ in increasing order and denote it by $\{n_k(\alpha)\}$. We show that for a convenient choice of β $\{n_k(\alpha)\}\in L_\alpha$. Let $N(r)$ be the number of members of the set $I_\alpha(\beta)\cap(0,2^r)$ By elementery computation we obtain, that

$$(4)\qquad (1-\alpha)2^{\beta-1}r^{\frac{1}{1-\alpha}}-A(\alpha,\beta) \leqq N(r) \leqq (1-\alpha)2^{\beta-1}(r+1)^{\frac{1}{1-\alpha}}-B(\alpha,\beta)$$

where $A(\alpha,\beta)$ and $B(\alpha,\beta)$ are functions of α and β, but they do not depend on r, so after fixing β, they can be considered as constants. Clearly,

$$n_{[N(r)](\alpha)} = 2^{r-1}+\left(2^{\beta+\left[\frac{\alpha}{1-\alpha}\log_2 r\right]-1}-1\right)2^{r-\left(\beta+\left[\frac{\alpha}{1-\alpha}\log_2 r\right]\right)},$$

$$n_{[N(r)+p+1](\alpha)} = 2^r+p\left(2^{r+1-\left(\beta+\left[\frac{\alpha}{1-\alpha}\log_2(r+1)\right]\right)}\right),$$

where

$$p = 0,1,2,\dots 2^{\beta+\left[\frac{\alpha}{1-\alpha}\log_2(r+1)\right]-1}-1.$$

Using estimation (4) it is easy to see, that with a convenient choice of β

$$\frac{n_{[N(r)+p+1](\alpha)}}{n_{[N(r)+p](\alpha)}} \geqq 1+\frac{1}{([N(r)+p](\alpha))^{\alpha}}$$

for $p=0,1,2,\dots 2^{\beta+\left[\frac{\alpha}{1-\alpha}\log_2(r+1)\right]-1}-1$, that is $\{n_k(\alpha)\}\in L_\alpha$ for every $0<\alpha<1$.

Making use of the Borel—Cantelli lemma, we shall show, that the partial sums

$$S_{n_{[N(r)](\alpha)}}(x)=\sum_{j=1}^{[N(r)](\alpha)} w_{n_j(\alpha)}(x)$$

are bounded for almost every $x\in[0,1]$, whenever $1/2<\alpha<1$. Clearly this contradicts to the central limit theorem.

We need the following remark. The obvious proof is omitted.

REMARK 2. *By $C(r,k)$ we denote the set of integers in $[2^{r-1}, 2^r)$ having vanishing last $r-k$ digits, then we have:*

a) $$T_r(x)=\sum_{j\in C(r,k)} w_j(x)=R_r(x)\prod_{j=r-k+1}^{r-1}(1+R_j(x)) \qquad x\in[0,1]$$

where $R_j(x)$ denotes the jth Rademacher function.

b) $$\prod_{j=r-k+1}^{r-1}(1+R_j(x))=\begin{cases}2^{k-1} & \text{if } x\in\left[\frac{t}{2^{r-k}},\frac{t}{2^{r-k}}+\frac{1}{2^{r-1}}\right) \\ 0 & \text{otherwise.}\end{cases} \quad t=0,1\dots 2^{r-k}-1.$$

c) $$\mathsf{P}\big(T_r(x)\neq 0\big)=\frac{1}{2^{k-1}}.$$

Let us denote by $A_r^{(\alpha)}$ $r=1,2,\dots$ the following events:

$$A_r^{(\alpha)}=\{x: S_{n_{[N(r)](\alpha)}}(x)-S_{n_{[N(r-1)](\alpha)}}(x)\neq 0\}$$

Applying Remark 2. we have;

$$\mathsf{P}(A_r^{(\alpha)})=\frac{1}{2^{\beta+\left[\frac{\alpha}{1-\alpha}\log_2 r\right]-1}}$$

Therefore:

$$\sum_{r=1}^{\infty}\mathsf{P}(A_r^{(\alpha)})\leqq\frac{1}{2^{\beta-2}}\sum_{r-1}^{\infty}\frac{1}{r^{\frac{\alpha}{1-\alpha}}}<\infty$$

whenever $\frac{\alpha}{1-\alpha}>1$, that means when $\alpha>\frac{1}{2}$. Thus from the Borel—Cantelli lemma follows that for almost every x at most finitely many

$$S_{n_{[N(r)](\alpha)}}(x) - S_{n_{[N(r-1)](\alpha)}}(x)$$

can be different from zero, and this implies our statement.

Added in proof: We remark that for $0 < \alpha < \frac{1}{2}$, $\{n_k\} \in L_\alpha$ also $\overline{\lim}_{N \to \infty} (2N \log\log N)^{-\frac{1}{2}} \sum_{k=1}^{N} w_{n_k}(x) = 1$ holds a. e. The proof is similar to the method of [7].

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(Received December 15, 1970)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 155—165.

JOINT DISTRIBUTIONS OF KOLMOGOROV—SMIRNOV STATISTICS AND RUNS

by

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1. Introduction

Investigation of the maximum difference and maximum absolute difference between the empirical distribution functions of two independent samples and of similar statistics, was begun by KOLMOGOROV [9]. He found the limiting distribution of the statistics $n^{1/2} \sup_x |F_n(x) - F(x)|$ suitable for testing the goodness-of-fit problem of whether $x_1, x_2, \ldots, x_n$ have a known continuous distribution function $F(x)$. The related statistics $n^{1/2} \sup_x [F_n(x) - F(x)]$ and the two-sample statistics were proposed by SMIRNOV [14].

The results for the one-sided statistics $D^+_{m,n} = \sup_x [F_m(x) - G_n(x)]$ and the two-sided statistics $D_{m,n} = \sup_x |F_m(x) - G_n x)|$ for the case where $m=n$ are due to GNEDENKO and KOROLYUK [6] and independently to DRION [5]. Proofs of the results were based on a random walk model.

As regards expressions for the distribution of the two-sided statistics $D_{m,n}$, there are also some attempts, e.g. by KOROLYUK [11] and BLACKMAN [2] for the integer multiple case and by DEPAIX [4] for the general case but the results are extremely complicated.

The asymptotic distributions of $D^+_{m,n}$ and $D_{m,n}$ were treated by SMIRNOV [14], [15], and by KOLMOGOROV [10] when the ratio of the sample sizes converges to a constant bounded away from zero and infinity.

Fine summaries of many results in this field are given by GNEDENKO [7], HÁJEK and SIDÁK [8], BARTON and MALLOWS [1] and DARLING [3].

The well-known run test was studied by WALD and WOLFOWITZ [16]. It is based on the number of maximal uninterrupted subsequences of elements of like kind in any ordered sequence of elements of two kinds. More general results on runs have been presented by MOOD [13].

Recurrence relations which help to compute the joint distribution of $D_{n,n}$ and runs $R_{n,n}$, were given by MOHANTY and PETROS [12]. In the present paper, exact expressions for the joint distributions of $\{D^+_{n,n}, R_{n,n}\}$ and $\{D_{n,n}, R_{n,n}\}$ under $H_0: F(x) = G(x)$ are presented. The conditional moments and the correlation coefficient for $\{D^+_{n,n}, R_{n,n}\}$ are then derived. The asymptotic distributions are also determined.

2. Joint distribution of $(D^+_{n,n}, R_{n,n})$ under H_0

Let us denote by $x_1, x_2, \dots, x_{m_1}$ and by $y_1, y_2, \dots, y_{n_1}$ samples drawn from populations with unknown continuous distribution functions $F(x)$ and $G(x)$, respectively. Let $F_{m_1}(x)$ and $G_{n_1}(x)$ be the corresponding empirical distribution functions. Denote by $z_1 < z_2 < \dots < z_{m_1+n_1}$ the ordered combined sample.

We further introduce the random variables

$$\theta_i = \begin{cases} +1 & \text{if } z_i = x_j \\ -1 & \text{if } z_i = y_k \end{cases}$$

and let the partial sum of the θ_i's be denoted by s_i, i.e.

$$s_i = \theta_1 + \theta_2 + \dots + \theta_i, \qquad i = 1, 2, \dots m_1 + n_1$$

$$s_0 = 0$$

Under the assumption $F(x) = G(x)$ each series $(\theta_1, \theta_2, \dots, \theta_{m_1+n_1})$ of the m_1 $(+1)$'s and the n_1 (-1)'s has the same probability $\binom{m_1+n_1}{n_1}$.

If the points (i, s_i) are represented in the plane and each of them is connected with the next one, then we obtain the usual illustrative figure of the paths, starting at the origin and reaching after (m_1+n_1) steps the point (m, n) where $m = m_1 + n_1$ and $n = m_1 - n_1$. We assume $m_1 \geqq n_1$.

Further, in the following a path as defined above will be said to have R runs if the total number of changes from positive direction to negative direction and vice versa is $R-1$. We shall distinguish this ordinary path from a composed path by defining the latter as one made up of runs where two consecutive runs are not necessarily of different kind.

For ease in writing we introduce the following symbols:

$E_{m,n}$: a path from $(0, 0)$ to (m, n)

$E^R_{m,n}$: an $E_{m,n}$ path with R runs

$E^{R+}_{m,n}$: an $E^R_{m,n}$ path starting with a positive step.

$E^{R-}_{m,n}$: an $E^R_{m,n}$ path starting with a negative step.

$E^{R,t}_{m,n}$: an $E^R_{m,n}$ path crossing the line $y = t$ at least once $(t > 0)$.

$E^{R+,t}_{m,n}$: an $E^{R+}_{m,n}$ path crossing the line $y = t$ at least once.

$E^{R-,t}_{m,n}$: an $E^{R-}_{m,n}$ path crossing the line $y = t$ at least once.

$E^{2r-,\cdot,l}_{m,n}(E^{(2r+1)-,\cdot,l}_{m,n})$: a composed path from $(0, 0)$ to (m, n) having r positive runs and $l+r$ $(l+r+1)$ negative runs where the first $l+1$ runs are negative and the remainder of the runs alternate.

$E^{2r-,t,l}_{m,n}(E^{(2r+1)-,t,l}_{m,n})$: An $E^{2r-,\cdot,l}_{m,n}(E^{(2r+1)-,\cdot,l}_{m,n})$ path crossing $y = t$.

$N(A)$: The number of all A paths. e.g. $N(E_{m,n}) = \binom{m}{\frac{m-n}{2}}$

THEOREM 1.

$$(2.1)\qquad N(E_{m,n}^{(2r+1)+,t}) = \binom{\frac{m-n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r} \qquad n \leqq t < \frac{m+n}{2}$$

PROOF. Let OPQ be an $E_{m,n}^{(2r+1)+,t}$ path (see Fig. 1) where P is the last point of intersection of the path with the line $y=t$. It can be shown that there exists a 1:1 correspondence between the $E_{m,n}^{(2r+1)+,t}$ paths and the $E_{m,2t-n}^{(2r+1)-}$ paths.

Reflect the section of the path from P to Q along the line $y=t$, i.e. change the direction of each step of the section from P to Q along $y=t$. Next reverse the order of the steps in the segment OP i.e. if $(\theta_1, \theta_2, \ldots, \theta_p)$ denotes the segment, we replace it by $(\theta_p, \theta_{p-1}, \ldots, \theta_2, \theta_1)$, (see Fig. 2.). The resulting path is an $E_{m,2t-n}^{(2r+1)-}$ path.

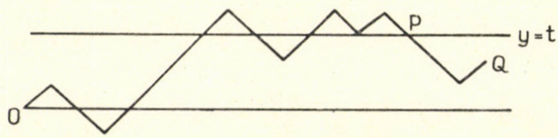

Fig. 1

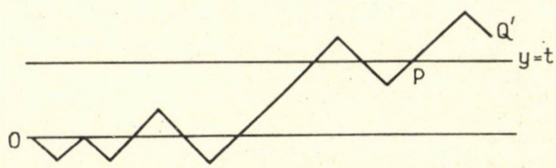

Fig. 2

By reverting this procedure it can easily be verified that the transformation is one to one.

The number of such paths is

$$\binom{\frac{m+n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r} \qquad \text{(see 16).}$$

The procedure applied above will henceforth be referred to as the "usual procedure".

THEOREM 2.

$$(2.2)\qquad N(E_{m,n}^{(2r+1)-,t,l}) = \binom{\frac{m-n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r+l} \qquad n \leqq t < \frac{m+n}{2}$$

PROOF. Let $OPQR$ denote an $E_{m,n}^{(2r+1)-,t,l}$ path with Q as its last point of intersection with $y=t$ and with P as the point where its first $(l+1)$ negative runs end.

Apply the „usual procedure" at the point Q to the path PQR. The resulting is an $E_{m,2t-n}^{2r-,\cdot,l+1}$ path (see Fig. 3. below).

The number of such composed paths is

$$\binom{\frac{m-n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r+l}$$

That the transformation is 1:1 can easily be verified by reversing the procedure about the point Q.

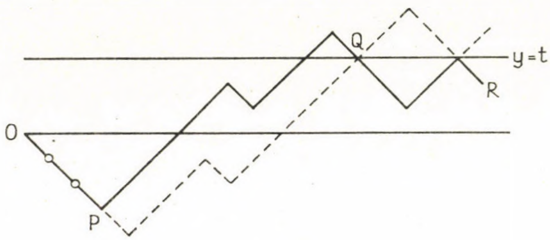

Fig. 3

Corollary: For an ordinary path $E_{m,n}^{(2r+1)-,t}$ the corresponding result got by putting $l=0$ in (2. 2) is

$$(2.\ 3)\qquad N(E_{m,n}^{(2r+1)-,t})=\binom{\frac{m-n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r}\qquad n\leqq t<\frac{m+n}{2}$$

THEOREM 3.

$$(2.\ 4)\qquad N(E_{m,n}^{2r+,t})=\binom{\frac{m-n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r-1}\qquad n\leqq t<\frac{m+n}{2}$$

PROOF. Let OPQ denote an $E_{m,n}^{2r+,t}$ path with P as its last point of intersection with $y=t$ (see Fig. 4).

By applying the same procedure as in the previous two theorems, it can easily be shown that the correspondence between the $E_{m,n}^{2r+,t}$ paths and the $E_{m,2t-n}^{2r-}$ paths in one-to-one. (See Fig. 5.)

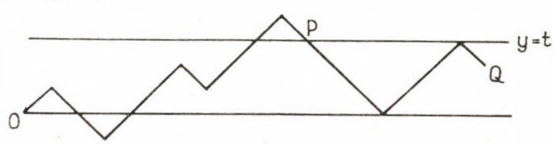

Fig. 4

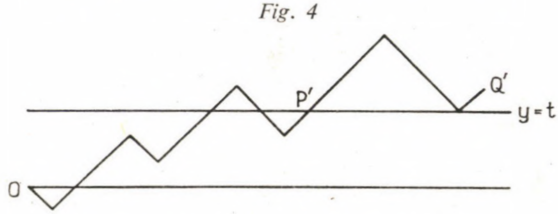

Fig. 5

Now

$$N(E^{2r-}_{m,2t-n}) = \binom{\frac{m-n}{2}+t-1}{r-1}\binom{\frac{m+n}{2}-t-1}{r-1}$$

and hence the theorem follows.

THEOREM 4.

$$(2.5)\qquad N(E^{2r-,t,l}_{m,n}) = \binom{\frac{m-n}{2}+t-1}{r-2}\binom{\frac{m+n}{2}-t-1}{r+l}\qquad n \leqq t < \frac{m+n}{2}.$$

PROOF. Let $OPQR$ denote an $E^{2r-,t,l}_{m,n}$ path with Q as its last point of intersection with $y=t$ and P the point where its first $(l+1)$ negative runs end.

By applying the „usual procedure" to the segment PQR at the point Q we get an $E^{(2r-1)-,\cdot,l+1}_{m,2t-n}$ path (see Fig. 6. below). The number of such composed paths is $\binom{\frac{m-n}{2}+t-1}{r-2}\binom{\frac{m+n}{2}-t-1}{r+l}$.

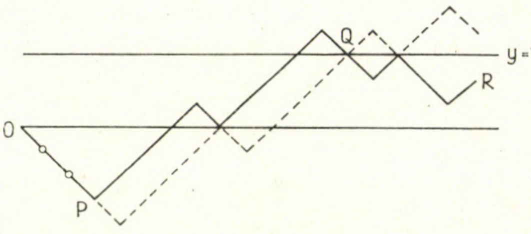

Fig. 6

By reversing the procedure about the point Q, it can easily be verified that the transformation is 1:1.

Corollary: For an ordinary path $E^{2r-,t\cdot}_{m,n}$ the corresponding result is a special case of (2. 5) got by putting $l=0$. Hence we have

$$(2.6)\qquad N(E^{2r-,t}_{m,n}) = \binom{\frac{m-n}{2}+t-1}{r-2}\binom{\frac{m+n}{2}-t-1}{r}\qquad n \leqq t < \frac{m+n}{2}.$$

Note: An alternative method of proof of the above theorems has been included in the author's dissertation to be submitted for the CSe. degree to the Hungarian Academy of Sciences.

From the preceeding result the joint distribution under $H_0: F(x)=G(x)$ of the pair of statistics $(D^+_{n,n}, R_{n,n})$ where

$$D^+_{n,n} = \max\,[F_n(x)-G_n(x)]$$

and $R_{n,n}$ is the number of runs in the combined sample of $2n$ elements, can be written down as follows.

THEOREM 5.

(2. 7)
$$\binom{2n}{n} \mathsf{P}(D^+_{n,n} \leqq t/n,\ R_{n,n}=2r) =$$

$$= \left[2\binom{n-1}{r-1}^2 - \binom{n+t-1}{r-2}\binom{n-t-1}{r} - \binom{n+t-1}{r-1}\binom{n-t-1}{r-1}\right] \qquad 0<t<n \quad 1\leqq r\leqq n$$

and

(2. 8)
$$\binom{2n}{n} \mathsf{P}(D^+_{n,n} \leqq t/n,\ R_{n,n}=2r+1) =$$

$$= 2\left[2\binom{n-1}{r-1}\binom{n-1}{r} - \binom{n-t-1}{r}\binom{n+t-1}{r-1}\right] \qquad 0<t<n \quad 1\leqq r\leqq n-1$$

PROOF. The derivation of the probability $\mathsf{P}(D^+_{n,n} \leqq t/n,\ R_{n,n}=s)$ involves the enumeration of paths from (0, 0) to $(2n, 0)$ not crossing the line $y=t$ and having s runs.

The theorem follows by making the substitutions $m=2n$, $n=0$ in the results already obtained.

3. The moments of $[D^+_{n,n},\ R_{n,n}]$

The conditional first and second moments of the random variable $R_{n,n}$ under the condition $D^+_{n,n} \leqq t/n$, the product moment and the correlation coefficient for $0<t<n$ are as follows:

(3. 1)
$$\mathsf{E}(R_{n,n} \mid D^+_{n,n} \leqq t/n) = \frac{\binom{2n}{n}(n+1) - \binom{2n-1}{n-t-1} 2(n-t)}{\binom{2n}{n} - \binom{2n}{n-t-1}}$$

(3. 2)
$$\mathsf{E}(R^2_{n,n} \mid D^+_{n,n} \leqq t|n) = \left[2\binom{2n-1}{n-1}(n+1)^2 + 2n\binom{2n-2}{n-2} - \right.$$
$$-4\binom{2n-4}{n-t-3}\{(2n-3)(2n-2)+3(n+t-1)(n+t-2)\} - 2(n-t+1)\binom{2n-1}{n-t-1} -$$
$$\left. -4\binom{2n-2}{n-t-2}(3n+2t-1)\right] \Big/ \left(\binom{2n}{n} - \binom{2n}{n-t-1}\right)$$

(3. 3)
$$\mathsf{E}(R_{n,n} \cdot D^+_{n,n}) =$$
$$= \left[\binom{2n}{n}\left(\frac{n-3}{2}\right) + 2^{2n-2}(2n+1) - \binom{2n-3}{n-3}(8n-6) - 3\binom{2n-2}{n-2}\right] \Big/ \binom{2n}{n}$$

The expectations and variances of $R_{n,n}$ and $D^+_{n,n}$ are:

$$\mathsf{E}(R_{n,n}) = n+1; \quad \sigma^2(R) = \frac{n(n-1)}{2n-1} \quad \text{(see 16)}$$

$$\mathsf{E}(D^+_{n,n}) = \frac{2^{2n-1}}{\binom{2n}{n}} - \frac{1}{2}; \quad \sigma^2(D^+_{n,n}) = n + \frac{1}{4} - \left[\frac{2^{n-1}}{\binom{2n}{n}}\right]^2.$$

Hence the correlation coefficient between $D^+_{n,n}$ and $R_{n,n}$ is

$$\varrho(R_{n,n}, D^+_{n,n}) = \frac{(n-1) - \dfrac{2^{2n-2}}{\binom{2n}{n}} - (4n-5)\binom{2n-2}{n-2} - 2\binom{2n-3}{n-3}}{\left[\dfrac{n(n-1)}{2n-1}\left\{n+\dfrac{1}{4}-\left(\dfrac{2^{2n-1}}{\binom{2n}{n}}\right)^2\right\}\right]^{1/2}}$$

4. The Asymptotic Distribution

For the joint limiting distribution we make use of the transformation

$$r = \frac{n}{2} + \frac{x\sqrt{n}}{2\sqrt{2}} \quad \text{and} \quad t = y\sqrt{2n}$$

which gives

$$\lim_{n\to\infty} \mathsf{P}\left(R_{n,n} \leqq n + \frac{x\sqrt{n}}{\sqrt{2}},\ D^+_{n,n} \leqq y\frac{\sqrt{2}}{\sqrt{n}}\right) = (1-e^{-2y^2}) \int_{-\infty}^{x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \tag{4. 1}$$

$$-\infty \leqq x \leqq \infty \quad 0 \leqq y \leqq \infty$$

This shows that $R_{n,n}$ and $D^+_{n,n}$ are independently distributed in the limit. Further as $n\to\infty$

$$\mathsf{E}(R_{n,n} \mid D^+_{n,n} \leqq t/n) \to n$$

$$\sigma^2(R_{n,n} \mid D^+_{n,n} \leqq t/n) \to n/2$$

$$\mathsf{E}(R_{n,n} \cdot D^+_{n,n}) \to n^{3/2}.$$

$$\varrho(R_{n,n}\, D^+_{n,n}) \to 0.$$

5. Joint distribution of $(D_{n,n}, R_{n,n})$ under H_0

Again as before we employ the random walk model. For the determination of the probability of the event $(D_{n,n} \leqq t/n,\ R_{n,n} = s)$ it is necessary to enumerate the number of paths from the point (0, 0) to $(2n, 0)$ not crossing the lines $y = \pm t$ and having s runs.

We further make use of the following notations in continuation of those used in (2). These notations hold both for $t>0$, $t'<n<t$ and for $t<0$, $t<n<t'$. Also by "Crossing" we mean a double intersection, one in either direction.

$E^{R,t(k+1),t'(k)}_{m,n}$: An $E^R_{m,n}$ path which at some stage crosses first $y=t$, then $y=t'$ and then goes on repeating these crossings k times, and finally again crosses $y=t$ for the $(k+1)$th time.

$E_{m,n}^{R,t(k),t'(k)}$: An $E_{m,n}^{R}$ path which at some stage crosses first $y=t$, then $y=t'$ and then goes on repeating these crossings k times, ending in a crossing of $y=t'$ for the kth time.

$E_{m,n}^{R+,t(k+1),t'(k)}$ $(E_{m,n}^{R-,t(k+1),t'(k)})$: An $E_{m,n}^{R,t(k+1),t'(k)}$ path starting with a positive (negative) step.

$E_{m,n}^{R+,t(k),t'(k)}$ $(E_{m,n}^{R-,t(k),t'(k)})$: An $E_{m,n}^{R,t(k),t'(k)}$ path starting with a positive (negative) step.

The corresponding composed paths will be defined as before.

In particular for $k=0$ we have

$$E_{m,n}^{R,t(1),t'(0)} = E_{m,n}^{R,t}$$

$$E_{m,n}^{R,t(0),t'(1)} = E_{m,n}^{R,t'}$$

In the following theorems $t>0$, $t'<n<t$.

THEOREM 6.

$$N(E_{m,n}^{2r-,t(k+1),t'(k)}) = N(E_{m,n}^{2r+,-t(k+1),-t'(k)}) = \tag{5.1}$$

$$= \binom{\frac{m-n}{2}+(k+1)t-kt'-1}{r-k-2}\binom{\frac{m+n}{2}-(k+1)t+kt'-1}{r+k}$$

PROOF. By applying the „usual procedure" at the appropriate points of intersection first to the path $E_{m,n}^{2r-,t(k+1),t'(k)}$ and the again to the path $E_{m,n}^{(2r-1)-,2t-2t'(k),t(k),1}$ we have the following recurrence relation:

$$N(E_{m,n}^{2r-,t(k+1),t'(k)}) = N(E_{m,n}^{(2r-1)-,2t-t'(k),t(k),1}) = N(E_{m,n+2t-2t'}^{(2r-2)-,3t-2t',2t-t',2})$$

so that

$$\sum_{i=0}^{k-1} N(E_{m,n_i}^{2r_i-,t_i(k_i+1),t_i'(k_i),2i}) = \sum_{i=0}^{k-1} N(E_{m,n_i+1}^{2(r_i-1)-,t_{i+1}(k_i),t_{i+1}'(k_i-1),2(i+1)})$$

where

$$r_i = r-i, \quad k_i = k-i, \quad n_i = n+2it-2it'$$

$$t_i = (2i+1)t-2it', \quad t_i' = 2it-(2i-1)t'$$

whence we obtain

$$N(E_{m,n}^{2r-,t(k+1),t'(k)}) = N(E_{m,n+2kt-2kt'}^{2(r-k)-,(2k+1)t-2kt'(1),2kt-(2k-1)t'(0),2k})$$

The required result now follows from theorem 4.

THEOREM 7.

$$N(E_{m,n}^{2r+,-t(k),-t'(k)}) = N(E_{m,n}^{2r-,t(k),t'(k)}) = \tag{5.2}$$

$$= \binom{\frac{m+n}{2}+kt-kt'-1}{r-k-1}\binom{\frac{m-n}{2}-kt+kt'-1}{r+k-1}$$

PROOF. As before the application of the „usual procedure" gives the following recurrence relation:

$$N(E_{m,n}^{2r-,t(k),t'(k)}) = N(E_{m,n}^{(2r-1)-,2t-t'(k),t(k-1),1}) = N(E_{m,n+2t-2t'}^{(2r-2)-,3t-2t'(k-1),2t-t'(k-1),2})$$

so that

$$\sum_{i=0}^{k=1} N(E_{m,n_i}^{2r_i-,t_i(k_i),t_i'(k_i),2i}) = \sum_{i=0}^{k-1} N(E_{m,n_i+1}^{2(r_i-1)-,t_{i+1}(k_i-1),t_{i+1}'(k_i-1),2(i+1)})$$

Therefore we obtain

$$N(E_{m,n}^{2r-,t(k),t'(k)}) = N(E_{m,n+2kt-2kt'}^{2(r-k)-,(2k+1)t-2kt'(0),2kt-(2k-1)t'(0),2k})$$

The required result now follows from Theorem 4.

Again by the application of the „usual procedure" and by using the preceeding two theorems, we obtain the following path enumerations in the form of a corollary.

Corollary 1.

(5.3)
$$N(E_{m,n}^{2r+,t(k+1),t'(k)}) = N(E_{m,-n}^{2r-,-t(k+1),-t'(k)}) =$$
$$= \binom{\frac{m-n}{2}+(k+1)t-kt'-1}{r-k-1}\binom{\frac{m+n}{2}-(k+1)t+kt'-1}{r+k-1}$$

(5.4)
$$(E_{m,n}^{2r+,t(k),t'(k)}) = N(E_{m,-n}^{2r-,-t(k),-t'(k)}) =$$
$$= \binom{\frac{m+n}{2}+kt-kt'-1}{r-k-1}\binom{\frac{m-n}{2}-kt+kt'-1}{r+k-1}$$

(5.5)
$$(E_{m,n}^{(2r+1)+,t(k+1),t'(k)}) = N(E_{m,-n}^{(2r+1)-,-t(k+1),-t'(k)}) =$$
$$= \binom{\frac{m-n}{2}+(k+1)t-kt'-1}{r-k-1}\binom{\frac{m+n}{2}-(k+1)t+kt'-1}{r+k}$$

(5.6)
$$(E_{m,-n}^{(2r+1)+,-t(k+1),-t'(k)}) = N(E_{m,n}^{(2r+1)-,t(k+1),t'(k)}) =$$
$$= \binom{\frac{m-n}{2}+(k+1)t-kt'-1}{r-k-1}\binom{\frac{m+n}{2}-(k+1)t+kt'-1}{r+k}$$

(5.7)
$$(E_{m,n}^{(2r+1)+,t(k),t'(k)}) = N(E_{m,-n}^{(2r+1)-,-t(k+1),-t'(k)}) =$$
$$= \binom{\frac{m+n}{2}+kt-kt'-1}{r-k}\binom{\frac{m-n}{2}-kt+kt'-1}{r+k-1}$$

(5.8)
$$(E_{m,-n}^{(2r+1)+,t(k),-t'(k)}) = N(E_{m,n}^{(2r+1)-,t(k),t'(k)}) =$$
$$= \binom{\frac{m+n}{2}+kt-kt'-1}{r-k-1}\binom{\frac{m-n}{2}-kt+kt'-1}{r+k}$$

THEOREM 8.

$$(5.9)\qquad \binom{2n}{n}\mathsf{P}(D_{n,n}\leqq t/n,\ R_{n,n}=2r) = 2\binom{n-1}{r-1}^2-$$

$$-2\left[\sum_{k=0}^{\infty}\binom{n+(2k+1)t-1}{r-k-1}\binom{n-(2k+1)t-1}{r+k-1}+\right.$$

$$\left.+\binom{n+(2k+1)t-1}{r-k-2}\binom{n-(2k+1)t-1}{r+k}\right]+2\left[2\sum_{k=1}^{\infty}\binom{n+2kt-1}{r-k-1}\binom{n-2kt-1}{r+k-1}\right]$$

PROOF. We know that if $E_{2n,0}^{R,\pm t}$ denotes an $E_{2n,0}^{R}$ path crossing either $y=t$ or $y=-t$ atleast once, then

$$N(E_{2n,0}^{R,\pm t}) = [N(E_{2n,0}^{R,t(1)})+N(E_{2n,0}^{R,-t(1)})]-[N(E_{2n,0}^{R,t(1),-t(1)})+N(E_{2n,0}^{R,-t(1),t(1)})]+$$

$$+[N(E_{2n,0}^{R,t(2),-t(1)})+N(E_{2n,0}^{R,-t(2),t(1)})]-\cdots+\cdots$$

Since

$$\binom{2n}{n}\mathsf{P}(D_{n,n}\leqq t/n,\ R_{n,n}=2r) = 2\binom{n-1}{r-1}^2-N(E_{2n,0}^{2r+,\pm t})-N(E_{2n,0}^{2r-,\pm t})$$

the required result follows by making use of 5. 1, 5. 2, 5. 3 and 5. 4 and by putting $m=2n$, $n=0$ and $t'=-t$ in them.

THEOREM 9.

$$(5.10)\qquad \binom{2n}{n}\mathsf{P}(D_{n,n}\leqq t/n,\ R_{n,n}=2r+1) =$$

$$=2\binom{n-1}{r-1}\binom{n-1}{r}-2\left[2\sum_{k=0}^{\infty}\binom{n+(2k+1)t-1}{r-k-1}\binom{n-(2k+1)t-1}{r+k}\right]+$$

$$+2\left[\sum_{k=1}^{\infty}\binom{n+2kt-1}{r-k}\binom{n-2kt-1}{r+k-1}+\binom{n+2kt-1}{r-k-1}\binom{n-2kt-1}{r+k}\right]$$

PROOF. As before since

$$\binom{2n}{n}\mathsf{P}(D_{n,n}\leqq t/n,\ R_{n,n}=2r+1) = 2\binom{n-1}{r-1}\binom{n-1}{r}-$$

$$-N(E_{2n,0}^{(2r+1)+,\pm t})-N(E_{2n,0}^{(2r+1)-,\pm t})$$

the required result follows from 5. 5, 5. 6, 5. 7 and 5. 8.

6. The Asymptotic distribution of $(D_{n,n}, R_{n,n})$

By making the transformation

$$r = \frac{n}{2} + \frac{x\sqrt{n}}{2\sqrt{2}}, \quad t = y\sqrt{2n}$$

in the joint distribution, we get

$$\lim_{n\to\infty} \mathsf{P}\left(D_{n,n} \leqq \frac{y\sqrt{2}}{\sqrt{n}},\ R_{n,n} \leqq n + \frac{x\sqrt{n}}{\sqrt{2}}\right) = \tag{6.1}$$

$$= \left[1 - 2\sum_{m=1}^{\infty} (-1)^{m+1} e^{-2m^2y^2}\right] \int_{-\infty}^{x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \qquad -\infty \leqq x \leqq \infty \quad 0 \leqq y \leqq \infty$$

which again shows that in the limit $R_{n,n}$ and $D_{n,n}$ are independently distributed.

7. Acknowledgement. My thanks are due to Prof. S. G. MOHANTY for initiating me into this piece of work.

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(Received December 15, 1970)

Studia Scientiarum Mathematicarum Hungarica 7. (1972) 167—179.

PSEUDO-QUADRATIC PROGRAMMING

by

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Abstract

This paper[1] considers the class of programs, so called "pseudo-quadratic" (p.q.-programs), of the type $\min_{x\in X} \psi(x)$ where $X\subset R^n$ is a convex polyhedron and the objective function has the form $\psi(x)=\varphi(c'x, x'Vx)$, x and c being n-vectors, V a given positive definite $n\times n$ matrix and $\varphi(u_1, u_2)$ a real-valued function defined on the set U, the image of X under the mapping $x\to(c'x, x'Vx)$. It is shown that under suitable assum ptions(convexity of ψ on X, differentiability of φ and strict monotonicity with respect to u_2 on U) this program can be reduced to a parametric quadratic program. Little extrawork besides the solution of the equation (6) involving the single unknown λ, is required in order to solve the original problem. Section 3 deals with some p.q.-programs arising in stochastic programming with random objective, for which the Basic Theorem is extended. The relationship with GEOFFRION's results on "Bi-criterion Programming" [7], as applied to p.q.-programming, is outlined (Section 4) and some numerical examples are given (Section 5).

1. Introduction

The present paper considers a class of programs which can be solved using any of the available techniques ([16], [12], etc.) for parametric quadratic programming.

We call "pseudo-quadratic" (p.q.-) the following program:

$$\min_{x\in X} \psi(x) = \varphi(c'x, x'Vx) \tag{1}$$

where c is a given n-vector, V a given $n\times n$ symetrix matrix and φ a real-valued function defined on the set $U=\{(u_1, u_2)|u_1=c'x, u_2=x'Vx, x\in X\}$. All vectors are considered column-vectors and the accent stands for transposition.

We consider X to be a convex polyhedron $X=\{x\in R^n|Ax\geqq b\}$ where b is a m-vector and A a $m\times n$ matrix. Although the Basic Theorem below holds if X is an arbitrary convex set defined by inequalities involving differentiable functions, for the sake of numerical computation we need however the linearity of constraints.

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[1] This is a revised version of a paper which appeared as a Report of the Center of Mathematical Statistics (April 1968) and was presented at the Third Colloquium on Probability Theory, Braşov, September 1968. Section 4 was added to the earlier version after GEOFFRION's paper [7] was brought to our knowledge.

Assumptions:

(i) φ has continuous partial derivatives on U (denoted φ_{u_1} and φ_{u_2} respectively) and $\varphi_{u_2}>0$ on U;

(ii) the matrix V is positive definite;

(iii) ψ is convex on X.

Essential for the results bellow are the assumptions (i) and (ii). The convexity requirement (iii) can be replaced by quasi-convexity of ψ on X (see Th. 2) if a weak condition (cf. [1]) is added to guarantee the sufficiency of Kuhn—Tucker conditions for (1). In the non-convex case the procedure below generates some stationary points of ψ on X among which there are all local-minima of ψ on X, and the optimal solution is easily found among them. If besides (i) φ is assumed convex on the convex hull of U, then ψ is strictly convex on X — as readily follows from (i) and (ii) — so that (1) has at most an optimal solution.

In the next section we shall prove that under assumptions (i)—(iii) above the p.q.-program (1) is equivalent with the quadratic program:

$$\min_{x\in X} \lambda c'x+x'Vx \tag{2}$$

for a suitable value of the real parameter λ. Notice first that, because of (ii), (2) has for each $\lambda\in R$ an unique optimal solution, which will be denoted by x_λ. Let us recall that the optimal solution function x_λ (and the corresponding Lagrange multipliers y_λ too) is, with respect to λ, continuous on R (cf. [2]) and piece-wise linear, i.e. there are critical values $-\infty=\lambda_0<\lambda_1<\dots<\lambda_s=+\infty$ so that:

$$x_\lambda = x_1^i+\lambda x_2^i \quad \text{for} \quad \lambda_i\leqq\lambda\leqq\lambda_{i+1} \qquad (i=0, 1, \dots, p-1). \tag{3}$$

We shall need below the following simple properties:

LEMMA 1. *If non-empty, the set* $\{\lambda|x_\lambda=x^0\}$ *is a closed interval.*

PROOF. The set is closed because of the continuity of x_λ. To prove it is convex suppose $\lambda_1\neq\lambda_2$ and $x_{\lambda_1}=x_{\lambda_2}=x^0$ and put $\lambda\equiv t\lambda_1+(1-t)\lambda_2$ where $0<t<1$. x^0, together with the multipliers y_{λ_i} satisfies Kuhn—Tucker conditions (10)—(11) for λ_i, $i=1, 2$. Multipling these inequalities by t, respectively $1-t$ and adding them we get that x^0, together with $y_\lambda\equiv ty_{\lambda_1}+(1-t)y_{\lambda_2}$, satisfies (10)—(11), so that x^0 is optimal in (2) for λ, i.e. $x_\lambda=x^0$. It then follows from (3) that there is an i such that $x_\lambda=x^0$ for each $\lambda\in[\lambda_i, \lambda_{i+1}]$.

LEMMA 2. *If the optimal solution of* (2) *is given by* (3) *then for each* $i=0, 1, \dots, p-1$ *we have:*

$$(\lambda c+2Vx_\lambda)'x_2^i = 0 \tag{4}$$

for each $\lambda\in(\lambda_i, \lambda_{i+1})$. *Moreover* $(x_1^i)'Vx_2^i=0$ *and*

$$c'x_2^i+2x_2^{i\prime}Vx_2^i = 0. \tag{5}$$

PROOF. The function $k(\mu) = \lambda c'x_\mu+x_\mu'Vx_\mu$, differentiable on $(\lambda_i, \lambda_{i+1})$ attains its minimum on this open interval at λ, so that

$$0=\frac{dk}{d\mu}(\lambda) = (\lambda c+2Vx_\lambda)'x_2^i$$

since $dx_\lambda/d\lambda=x_2^i$ on $(\lambda_i, \lambda_{i+1})$. The last assertions follow immediately from 4).

LEMMA 3. (i) ([16]) *The function* $\lambda \to c'x_\lambda$ *is nonincreasing on* R;

(ii) ([9]) *The function* $\lambda \to x'_\lambda V x_\lambda$ *is nonincreasing for* $\lambda \leqq 0$ *and nondecreasing for* $\lambda \geqq 0$;

(iii) ([10], [13]) *The function* $\lambda \to (x'_\lambda V x_\lambda)^{1/2}/\lambda$ *is nonincreasing for* $\lambda > 0$ *and also for* $\lambda < 0$.

2. Basic results

The theorems below are based on the fact that, under the assumptions (i)—(iii), both problems (1) and (2) are convex programs so that Kuhn—Tucker conditions [11] are necessary and sufficient for optimality and, moreover, these conditions coincide for the two problems if λ is a solution of the equation

$$\chi(\lambda) = \lambda \tag{6}$$

where

$$\chi(\lambda) \equiv \varphi_{u_1}(c'x_\lambda, x'_\lambda V x_\lambda)/\varphi_{u_2}(c'x_\lambda, x'_\lambda V x_\lambda)$$

or $\chi(\lambda) \equiv \varphi^\lambda_{u_1}/\varphi^\lambda_{u_2}$ denoting by $\varphi^\lambda_{u_i}$ $(i=1, 2)$ the partial derivatives of φ taken at the point x_λ. Notice that because of (i) and the continuity of the optimal solution function x_λ, χ is continuous on R.

THEOREM 1. *Under the assumptions* (i)—(ii), *if* x^* *is optimal in* (1) *then there exists a solution* λ^* *of the equation* (6) *such that* $x^* = x_{\lambda^*}$.

PROOF. Let

$$\lambda^* \equiv \varphi^*_{u_1}/\varphi^*_{u_2} \tag{7}$$

where $\varphi^*_{u_i}$ $(i=1, 2)$ denote the partial derivatives of φ taken at the point x^*. Since x^* is optimal in (1) there exist the multipliers $y^* \in R^m$ satisfying Kuhn—Tucker conditions:

$$\varphi^*_{u_1} c + 2\varphi^*_{u_2} V x^* - A'y^* = 0 \tag{8}$$

$$y^* \geqq 0, \quad Ax^* - b \geqq 0, \quad y^{*\prime}(Ax^* - b) = 0 \tag{9}$$

Then Kuhn—Tucker conditions for (2), i.e.

$$\lambda c + 2Vx - A'y = 0 \tag{10}$$

$$y \geqq 0, \quad Ax - b \geqq 0, \quad y'(Ax - b) = 0 \tag{11}$$

are satisfied, for $\lambda = \lambda^*$, by $x = x^*$ and $y = y^*/\varphi^*_{u_2}$ since $\varphi^*_{u_2} > 0$. Therefore x^* is optimal in (2) for $\lambda = \lambda^*$, i.e. $x^* = x_{\lambda^*}$. Then (7) is equivalent to $\chi(\lambda^*) = \lambda^*$ and the proof is complete.

COROLLARY 1. A necessary condition for (1) to have an optimal solution is that the equation (6) has at least a solution.

COROLLARY 2. If x^* is optimal in (1) then $x^* = x_{\lambda^*}$ where λ^* minimizes the function $\Phi(\lambda) \equiv \psi(x_\lambda)$ on R.

PROOF. According to Th. 1, $x^* \in X^* = \{x_\lambda | \lambda \in R\}$ and therefore (1) is equivalent to $\min_{x \in X^*} \psi(x) = \min_{\lambda \in R} \psi(x_\lambda) = \min_{\lambda \in R} \Phi(\lambda)$.

If the existence of an optimal solution in (1) is somehow guaranteed then the converse of this corollary is also true and, as in GEOFFRION's approach ([7], Th. 1), (1) is reduced to the one-dimensional problem of minimizing Φ on R:

THEOREM 2. *If assumptions* (i)—(ii) *are satisfied and X is compact then a necessary and sufficient condition for x_λ to be optimal in* (1) *is that λ minimizes the function Φ on R.*

PROOF. The necessity coincides with the corollary 2. Now let λ minimize Φ on R. Since X is compact (1) has at least an optimal solution x^*. Then (cf. corollary 2) there is λ^* minimizing Φ on R such that $x^* = x_{\lambda^*}$. It follows that $\Phi(\lambda) = \Phi(\lambda^*)$, i.e. $\psi(x^*) = \psi(x_\lambda)$ and therefore x_λ is optimal in (1).

THEOREM 3. *Under the assumption* (i)—(iii), *if λ is a solution of* (6) *then x_λ is optimal in* (1). *If moreover $\lambda c + 2Vx_\lambda \neq 0$ then the theorem still holds if the convexity assumption* (iii) *is replaced by quasi-convexity of ψ on X.*

PROOF. If x_λ and y_λ satisfy the conditions (10)—(11) then it follows, using (6), that $x^* = x_\lambda$ and $y^* = \varphi^\lambda_{u_2} y_\lambda$ satisfy Kuhn—Tucker conditions (8)—(9) and therefore x_λ is optimal in (1). To prove the last assertion notice that from (6) and (i) it follows that $\nabla\psi(x_\lambda) = (\lambda c + 2Vx_\lambda)\varphi^\lambda_{u_2} \neq 0$ and (cf. [1]) with, this condition, (8)—(9) are sufficient for optimality provided the objective is quasi-convex.

Notice that if the convexity assumption is dropped out then for a solution λ^* of (7), x_{λ^*} is generally a stationary point of ψ on X; it may be a local-optimal solution or even a saddle-point. Anyhow, if all the solutions of (6) have been found then the optimal solution in (1) can easily be selected among the resulting x_λ.

The following theorem is concerned with the set of solutions of the equation (6).

THEOREM 4. *Suppose the assumption* (i)—(ii) *hold and φ is convex on the convex hull of U. If the equation* (6) *has a solution then either the solution is unique or there is a whole interval of solutions $[\lambda_i, \lambda_{i+1}]$ where λ_i, λ_{i+1} are critical values in* (3).

PROOF. With the assumptions of the theorem ψ is strictly convex on X so that the optimal solution in (1) is unique provided it exists. Suppose λ_1 and λ_2 are different solutions of (6). Then (cf. Th. 2) both x_{λ_1} and x_{λ_2} are optimal in (1) and therefore $x_{\lambda_1} = x_{\lambda_2}$. It follows from lemma 1 that there is an interval $I = [\lambda_i, \lambda_{i+1}]$ in (3) such that, for each $\lambda \in I$, x_λ is optimal in (1). Then Theorem 1 implies that each $\lambda \in I$ is a solution of (6).

The theorems (1) and (2) are concentrated in the following Basic Theorem showing that (6) is a necessary and sufficient condition for x_λ to be optimal in (1).

BASIC THEOREM. *Under the assumptions* (i)—(iii) *the p.q.-program* (1) *has an optimal solution x^* if and only if there is a solution λ^* of* (6) *such that $x^* = x_{\lambda^*}$.*

This theorem can be naturally adapted for the general convex "bi-criterion" program (24). It is also easily seen to be true for the maximum problem if instead of (iii) and respectively (i), ψ is supposed concave on X and $\varphi_{u_2} < 0$ on U.

The procedure for solving (1), justified by the Basic Theorem, consists of solving the parametric quadratic program (2) till (6) is satisfied. Then x_λ is optimal in (1).

Provided φ is convex (6) has an unique solution so that in order to solve it is enough to compute the values of $\chi(\lambda)-\lambda$ only for critical values λ_i, to stop the solution of (2) when a change of sign occurs in the sequence $\chi(\lambda_i)-\lambda_i$ and to solve a single equation of the form (6). Even in the nonconvex case it is sometimes possible, as will be seen in the next section, to establish the unicity of the solution in (6) and therefore to extend the validity of the Basic Theorem.

3. Some stochastic models

The aim of this section is to apply or to extend the above procedure to some p.q.-programs arising as deterministic equivalents in stochastic programming with random objective. Suppose an activity vector x is to be selected from the convex polyhedron X to minimize the total cost $p'x$. If p is a random vector, with expectation c and covariance matrix V, then for each given $x \in X$ the total cost is a random variable with expected value $c'x$ and variance $x'Vx$. In different approaches for handling randomness in the objective various functionals of $c'x$ and $x'Vx$ have been considered as deterministic objective (see [14], [5], [9], [3]). According to our Basic Theorem each objective function satisfying assumptions (i), (ii) is equivalent with some linear combination of $c'x$ and $x'Vx$ and consequently the optimal solution is of the form x_λ with suitable $\lambda \in R$.

In this section we shall deal with three of these problems which have already been solved via reduction to (2). Although our requirement (iii) is sometimes not satisfied, our Basic Theorem can be established to hold for each of these particular p.q.-programs and the above procedure can be applied to solve any of them.

3. 1. In [9] KATAOKA has introduced the following "problem of the minimum level":

$$\min \{t \mid \mathsf{P}(p'x \leqq t) \geqq \beta,\ x \in X\}$$

where $\beta > 0.5$ is a given confidence level and P stands for probability. If p has a nondegenerate (then the covariant matrix V is positive definite) multi-normal distribution $N(c, V)$ the problem becomes:

$$\min_{x \in X} t = c'x + k(x'Vx)^{1/2} \tag{13}$$

where k is a positive constant. This "semi-quadratic" program was solved in [9] (see also [10] and [13]) and independently in [15] and [8].

Since our assumptions are satisfied if $0 \notin X$, from the Basic Theorem it follows: if $0 \notin X$ then x_λ is optimal in (13) if and only if λ is a (clearly pozitive) solution of the equation $\chi_1(\lambda) \equiv (x'_\lambda V x_\lambda)^{1/2}/\lambda = k/2$. This is KATAOKA's Theorem 3 of [9]. An additional condition can be introduced to guarantee the existence of the solution for the above equation. Suppose that $\min_{x \in X} c'x > -\infty$. Then (cf. [6]) there are $\bar{\lambda}$ and $\bar{x}$ such that $x_\lambda = \bar{x}$ for each $\lambda \geqq \bar{\lambda}$ and therefore $\chi_1(+\infty) = 0$. Furthermore the function χ_1 is nonincreasing for $\lambda > 0$ (cf. lemma 3 (iii)) and $\chi_1(+0) = +\infty$ because of the assumption $0 \notin X$. Consequently with the assumption that $0 \notin X$ and $c'x$ is bounded from below on X, the equation $\chi_1(\lambda) = k/2$ has a solution for any positive k.

We shall further consider the complementary case:

$$\min_{x \in X} c'x + (x'Vx)^{\alpha} \tag{14}$$

where $\alpha > 0.5$ is a given number. Notice that the corresponding equation (6) can be equivalently written as:

$$\chi_2(\lambda) \equiv (x_\lambda' V x_\lambda)^{1-\alpha}/\lambda = \alpha. \tag{15}$$

By specializing the Basic Theorem we obtain:

THEOREM 5. *If* $\alpha > 0.5$, $0 \notin X$ *and* V *is positive definite then the function* χ_2 *is strictly decreasing for* $\lambda > 0$, *the equation* (15) *has an unique positive solution* λ^* *and* x_{λ^*} *is optimal in* (14).

PROOF. The strict monotonicity of χ_2 will be proved below (lemma 5) and it assures the unicity of the solution in (15); the existence follows from the continuity of χ_2 and from the fact that, as readily seen from (3), $\chi_2(+0) = +\infty$, $\chi_2(+\infty) = 0$.

The following two lemma extend results from [13]:

LEMMA 4. *If* $\alpha \geqq 0.5$ *then the optimal solution of* (2), x_λ, *is also optimal in:*

$$\min_{x \in X} \lambda c'x + \frac{1}{\alpha}(x'Vx)^{\alpha}(x_\lambda' V x_\lambda)^{1-\alpha}. \tag{16}$$

PROOF. This is a convex program and Kuhn—Tucker conditions for x_λ coincide with (10)—(11).

LEMMA 5. *If* $\alpha > 0.5$ *the function* χ_2 *is continuous and strictly decreasing for* $\lambda > 0$ *and also for* $\lambda < 0$.

PROOF. Suppose $\lambda > \mu > 0$ and let x_λ, x_μ be the corresponding optimal solutions in (2). If $x_\lambda = x_\mu$ then $\chi_2(\lambda) < \chi_2(\mu)$ follows from $1/\lambda < 1/\mu$. If $x_\lambda \neq x_\mu$ then using lemma 4 we get:

$$\alpha\lambda c'x_\lambda + x_\lambda' V x_\lambda < \alpha\lambda c'x_\mu + (x_\mu' V x_\mu)^{\alpha}(x_\lambda' V x_\lambda)^{1-\alpha}$$

$$\alpha\mu c'x_\mu + x_\mu' V x_\mu < \alpha\mu c'x_\lambda + (x_\lambda' V x_\lambda)^{\alpha}(x_\mu' V x_\mu)^{1-\alpha}$$

the strict inequalities being due to the fact that (16) has an unique optimal solution because the objective is strictly convex. Dividing these inequalities by λ and μ respectively and adding then we obtain:

$$[(x_\lambda' V x_\lambda)^{\alpha} - (x_\mu' V x_\mu)^{\alpha}] \cdot [\chi_2(\lambda) - \chi_2(\mu)] < 0.$$

According to lemma 3 (ii) the expression within the first brackets is nonnegative so that $\chi_2(\lambda) < \chi_2(\mu)$. If $\mu < \lambda < 0$ then the sign of the last inequality should be inverted but the first brackets is now nonpositive.

3. 2. The "minimum risk solution with level t" of a stochastic program with random objective is the solution which minimize the probability that the total cost $p'x$ excedes the given level t, i.e. the optimal solution of the program:

$$\min_{x \in X} \mathsf{P}(p'x \geqq t)$$

This is a particular case of the P-model due to CHARNES and COOPER [5] (see also [3]). If the random vector p has a nondegenerate multinormal distribution $N(c, V)$, the problem becomes:

$$\min_{x \in X} \psi(x) = (c'x-t)/(x'Vx)^{1/2}. \tag{17}$$

This p.q.-program was solved independently by various authors [8], [10], [6], [4]. All approaches reduce (17) to a parametric quadratic program in a specific way. We shall express now our main result from [6] which shows that, although (17) does not satisfy the assumptions (i) and (iii), the conclusion of the Basic Theorem still holds.

We suppose the following (natural for the minimum risk model) assumptions to be satisfied:

$$t_* \equiv \min_{x \in X} c'x > -\infty \tag{18}$$

$$t > t_* \quad \text{and, if} \quad 0 \in X, \quad \text{then} \quad t < 0. \tag{19}$$

Notice the equation (6) corresponding to (17) can be equivalently written as:

$$\chi_3(\lambda) = t \tag{20}$$

where $\chi_3(\lambda) \equiv c'x_\lambda + 2x'Vx_\lambda/\lambda$. The following lemma shows that this equation is in fact of the extremely simple form $\alpha + \beta/\lambda = t$ so that its solution is $\lambda_t = \beta/(t-\alpha)$.

LEMMA 6. ([6]) *The function χ_3 is continuous for $\lambda \neq 0$ and piece-wise hyperbolic, i.e.*

$$\chi_3(\lambda) = \alpha_i + \beta_i/\lambda \quad \textit{for} \quad \lambda_i \leqq \lambda \leqq \lambda_{i+1} \qquad (i=0, 1, \dots, p-1)$$

where λ_i are the critical values appearing in (3). *Moreover $\beta_i > 0$ except may be for a single $\beta_{i_0} = 0$; then $\chi_3(\lambda) \equiv 0$ on $[\lambda_{i_0}, \lambda_{i_0+1}]$.*

LEMMA 7. ([6]) *Under the assumption* (18) $\chi_3(+\infty) = t_*$.

COROLLARY. Under the assumption (18) the equation (20) has a solution for each t satisfying (19). Moreover the solution is unique except perhaps for $t=0$.

PROOF. It suffices to combine the two lemma with the easily seen fact that $\chi_3(+0)=0$ if $0 \in X$ (since then $x_0=0$) and $\chi_3(+0) = +\infty$ if $0 \notin X$.

THEOREM 6. ([6]) *Under assumptions* (18)—(19) *the equation* (20) *has a (unique except perhaps for $t=0$) positive solution λ_t and x_{λ_t} is optimal in* (17).

PROOF. Since assumptions (i) and (iii) are not satisfied our Basic Theorem does not apply directly to the p.q.-program (17). Although a similar way as in Th. 2 could be followed (observe that $\varphi_{u_2}^{\lambda} > 0$ if λ is a solution of (20) and ψ is quasi-convex on the set $\{x | c'x - t < 0\}$ where because of the assumption (19) the minimum occurs) we shall however use the direct proof from [6]. Taking into account the above corollary, the only fact to be proved is that if λ is a solution of (20) then x_λ is optimal in (17). According to lemma 5(for $\alpha=0.5$) x_λ is optimal in (16) so that:

$$\lambda c'x_\lambda + 2x_\lambda' Vx_\lambda \leqq \lambda c'x + 2(x'Vx)^{1/2}(x_\lambda' Vx_\lambda)^{1/2}$$

for each $x \in X$. Dividing this inequality by $\lambda(x'Vx)^{1/2}$ and taking into account (20) and $2(x'_\lambda Vx_\lambda)^{1/2}/\lambda = -\psi(x_\lambda)$ as it results from (20), we obtain $\psi(x_\lambda) \leqq \psi(x)$ for each $x \in X$, so that x_λ is optimal in (17).

Consequently the following algorithm [6] can be used to solve the "minimum risk problem" (17): solve (2) for positive λ, starting with $\lambda = 0$, and compute $\chi(\lambda_i)$ till we first obtain $\chi(\lambda_{i+1}) \leqq t$; if $\chi(\lambda) = \alpha + \beta/\lambda$ for $\lambda \in [\lambda_i, \lambda_{i+1}]$ then $x_{\beta/(t-\alpha)}$ is the minimum risk solution with level t.

3. 3. The minimum variability problem. For the ill-defined stochastic problem of minimizing the total profit $p'x$, where p is a random vector with expectation c and covariant matrix V, the coefficient of variability of $p'x$, i.e. the quotient between the standard deviation $(x'Vx)^{1/2}$ and the expected value $c'x$, is sometimes a suitable deterministic criterion. Notice it does not require the knowledge of the distribution of the random vector p but only estimations of c and V. For the p.q.-program:

$$\min_{x \in X} \psi(x) = (x'Vx)^{1/2}/c'x \tag{21}$$

we shall suppose that:

$$c'x > 0 \quad \text{if} \quad x \in X. \tag{22}$$

Then ψ is quasi-convex on X and Theorem 2 could be applied. A stronger form will however be proved directly. Notice that, with the assumption (22), (21) is a particular case (for $t = 0$) of the maximum problem (17).

THEOREM 7. *Under the assumption* (22) x_λ *is optimal in* (21) *if and only if* λ *is a negative solution of the equation:*

$$\lambda c'x_\lambda + 2x'_\lambda Vx_\lambda = 0. \tag{23}$$

PROOF. Since the assumption (ii) is satisfied, the necessity of the condition (23) follows from Th. 1. Its sufficiency will be proved directly. Let $\lambda < 0$ be any solution of (23). According to lemma 5 (for $\alpha = 0.5$), we have:

$$\lambda c'x + 2(x'Vx)^{1/2}(x'_\lambda Vx_\lambda)^{1/2} \geqq \lambda c'x_\lambda + 2x'_\lambda Vx_\lambda = 0$$

for each $x \in X$. Using (23) and (22) this inequality becomes $\psi(x_\lambda) \leqq \psi(x)$ for each $x \in X$, so that x_λ is optimal in (21).

4. Relationship with Geofrion's results

The idea of this paper originates in KATAOKA's paper [9] (see also the revisited form [10]) where first a p.q.-program was solved via reduction to the parametric quadratic program (2). Results on different but related models have also been obtained by MARKOWITZ [14], SINHA [15], LEMARIE [13], DRAGOMIRESCU [6] and especially by GEOFFRION [7], [8].

GEOFFRION's paper [7] is concerned with the more general "bi-criterion" program:

$$\min_{x \in X} \psi(x) = \varphi\big(u_1(f_1(x)), u_2(f_2(x))\big). \tag{24}$$

His main result is expressed in the following:

THEOREM ([7]). *Let X be a convex compact set in R^n; $f_1, f_2, u_1(f_1), u_2(f_2)$ convex functions on X, u_i being strictly increasing on the image of X under f_i $(i=1, 2)$, and φ nondecreasing and quasi-convex on the convex hull of the image of X under $(u_1(f_1), u_2(f_2))$, all functions being continuous. Assume that a parametric programming algorithm is available for solving*

$$\min_{x \in X} \gamma f_1(x) + (1-\gamma) f_2(x)$$

for each value of the parameter γ in the unit interval and that the resulting optimal solution function x_γ would be continuous on $[0, 1]$. *Then the function $\Phi(\gamma)=\psi(x_\gamma)$ in continous and unimodal on* $[0, 1]$ *and, if γ^* minimizes Φ on* $[0, 1]$ *then x_{γ^*} is optimal in* (24).

This theorem, specialized for the p.q.-program (1) by putting $f_1(x)=c'x, f_2(x)=$ $=x'Vx$, $u_1(f)=u_2(f)=f$ and introducing the new parameter $\lambda = \gamma/(1-\gamma)$, gives a result very similar to our Th. 2.

Corollary. If X is a bounded polyhedron, V a positive semidefinite matrix, φ is quasi-convex on the convex hull of U and nondecreasing in both variables and if x_λ denotes the optimal solution function for (2), then the function $\Phi(\lambda)=$ $=\varphi(c'x_\lambda, x'_\lambda V x_\lambda)$ is unimodal on R_+ and if λ^* minimizes Φ on R_+, then x_{λ^*} is optimal in (1).

This result should be compared with our Basic Theorem. Neither of the two results includes the other. Our result is essentially based on Kuhn—Tucker conditions, more precisely on differentiability of φ and monotonicity with respect to u_2, while GEOFFRION's one is based on compacity of X and monotonicity with respect to both variables u_1 and u_2. The consequence of the last assumption, natural if the program (1) is viewed as a "bi-criterion program", is that the optimal solution is "effective" (see [7], lemma 1) in the sense of MARKOWITZ [14] i.e., equivalently, of the form x_λ with $\lambda > 0$. The present paper does not envisage (1) as a bi-criterion program, we are not intersted in the effectiveness of the optimal solution (under our assumptions it may be of the form x_λ with $\lambda < 0$, i.e. not effective), but just in the reduction of (1) to (2).

From the practical point of view of computation the two methods are equivalent for most of the cases where both of them apply. GEOFFRION's approach requires the minimization of Φ while our involves the solution of the equation (6). Under assumption (i) Φ is differentiable for $\lambda \neq \lambda_i$ $(i=0, 1, \dots, p)$ and, using (3) and (4), we have:

$$\frac{d\Phi(\lambda)}{d\lambda} = \varphi^\lambda_{u_2}(\chi(\lambda)-\lambda)c'x^i_2$$

for $\lambda \in (\lambda_i, \lambda_{i+1})$, $i=0, 1, \dots, p$, where (cf. (5)) $c'x^i_2 \neq 0$ except for the case $x^i_2=0$, i.e. $x_\lambda = x^i_1$ for each $\lambda \in [\lambda_i, \lambda_{i+1}]$. It follows that, if the optimal solution is not among the critical points x_{λ_i} then its determination requires, in GEOFFRION's approach too, the solution of (6).

5. Numerical examples

In order to illustrate the proposed technique let us consider the following six p.q.-programs:

(A) $$\min_{x \in X} \psi_A(x) = c'x + (x'Vx)^2$$

(B) $$\min_{x \in X} \psi_B(x) = c'x + (x'Vx)^{1/2}$$

(C) $$\min_{x \in X} \psi_C(x) = (c'x+t)^2 + (x'Vx)^{1/2} \qquad (t \in R)$$

(D) $$\min_{x \in X} \psi_D(x) = (c'x-t)/(x'Vx)^{1/2} \qquad (t > -12)$$

(E) $$\min_{x \in X} \psi_E(x) = (c'x)^2 - (c'x)(x'Vx)$$

(F) $$\min_{x \in X} \psi_F(x) = (15+c'x)(x'Vx)$$

where:

$$X = \{(x_1, x_2) \mid x_1 + x_2 \leqq 2,\ 3x_1 + 2x_2 \geqq 3,\ x_1 \geqq 0,\ x_2 \geqq 0\}$$

$$c = \begin{pmatrix} -4 \\ -6 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Notice that $c'x < 0$ for $x \in X$, more precisely $-12 \leqq c'x \leqq -4$.
The optimal solution of the associated quadratic program

$$\min_{x \in X} \lambda(-4x_1 - 6x_2) + x_1^2 + 2x_1x_2 + 2x_2^2$$

is given in Table 1, together with the functions $c'x_\lambda$ and $x'_\lambda V x_\lambda$.

Table 1

λ	$\lambda \leqq 0,2$	$0,2 \leqq \lambda \leqq 0,6$	$0,6 \leqq \lambda \leqq 1$	$1 \leqq \lambda \leqq 2$	$\lambda \geqq 2$
x_1^λ	1	$1,2-\lambda$	λ	$2-\lambda$	0
x_2^λ	0	$-0,3+1,5\lambda$	λ	λ	2
$c'x_\lambda$	-4	$-5\lambda-3$	-10λ	$-2\lambda-8$	-12
$x'_\lambda V x_\lambda$	1	$2,5\lambda^2+0,9$	$5\lambda^2$	λ^2+4	8

Our Basic Theorem applies to the first four programs (for (D) cf. Th. 6). The last two programs violate the convexity assumption (iii) but, since (i) and (ii) are satisfied, according to Th. 1 the optimal solution (which exists because X is compact) must be in each case among the vectors x_{λ^*} with λ^* a solution of (6). The solution of the six considered programs is sumarized in Table 2 (for (A)—(D) the corresponding equation (6) is given in an equivalent form):

Table 2

Prog-ram	$\varphi(u_1, u_2)$	Equation for λ	λ^*	The optimal solution x_1^*	x_2^*
A	$u_1+u_2^2$	$\chi_A(\lambda)\equiv\lambda\cdot x_\lambda' V x_\lambda = 0{,}5$	0,23079	0,9692	0,0467
B	$u_1+u_2^{1/2}$	$\chi_B(\lambda)\equiv\frac{1}{\lambda}(x_\lambda' V x_\lambda)^{1/2} = 0{,}5$	5,6568	0	2
C	$(u_1+t)^2+u_2^{1/2}$	$\chi_C(\lambda)\equiv\frac{\lambda}{4(x_\lambda' Vx)^{1/2}}-c'x_\lambda=t$	1,429 (for $t=11$)	0,571	1,429
D	$(u_1-t)/u_2^{1/2}$	$\chi_D(\lambda)\equiv c'x_\lambda+\frac{2}{\lambda}x_\lambda' V x_\lambda = t$	1,333 (for $t=-2$)	0,667	1,333
E	$u_1^2-u_1u_2$	$\frac{x_\lambda' V x_\lambda+2(-c'x_\lambda)}{c'x_\lambda}=\lambda$	$-9/4$	1	0
F	$(15+u_1)u_2$	$\frac{x_\lambda' V x_\lambda}{15+c'x_\lambda}=\lambda$	$\lambda_1=1/11$ $\lambda_2=1$ $\lambda_3=4/3$ $\lambda_4=8/3$	1 1 2/3 0	0 1 4/3 2

For instance, in order to solve the program (A) we have to solve the equation:

$$\chi_A(\lambda)\equiv\lambda\cdot x_\lambda' V x_\lambda = 0{,}5.$$

Of course $\lambda_A>0$. The values of χ_A for critical values of λ are given in Table 3 below. Since 0,5 is between the critical values 0,2 and 1,08 it follows that the solution is $0{,}2<\lambda_A<0{,}6$; then $x_\lambda' V x_\lambda = 2{,}5\lambda^2+0{,}9$ (see Table 1) and therefore the above equation reduces to $25\lambda^3+9\lambda-5=0$. The unique real solution is $\lambda_A=0.23079$ so that the optimal solution in (A) is $x_1^A = 1{,}2-\lambda_A = 0{,}9692$ and $x_2^A=-0{,}3+1{,}5\,\lambda_A=$ $=0{,}0467$.

Table 3

λ	$+0$	0,2	0,6	1	2	$+\infty$
$c'x_\lambda$	-4	-4	-6	-10	-12	-12
$x'_\lambda Vx_\lambda$	1	1	1,8	5	8	8
$\chi_A(\lambda)$	0	0,2	1,08	5	16	$+\infty$
$\chi_B(\lambda)$	$+\infty$	5	2,236	2,236	1,414	0.
$\chi_C(\lambda)$	4	4,05	6,112	10,112	12,177	$+\infty$
$\chi_D(\lambda)$	$+\infty$	6	0	0	-4	-12

Similarly the functions χ_B, χ_C, χ_D are monotonic for $\lambda>0$. Their critical values are given in Table 3. For (B) we have to solve the equation $\chi_B(\lambda)=0.5$. Since 0.5 is between the critical values 1.414 and 0 it follows that $\lambda_B>2$ so that $\varkappa^B=(0, 2)$.

The optimal solution of (C) can be found for any given t. For instance for $t=11$ it follows from Table 3 that $1<\lambda_C<2$ and we have to solve the equation:

$$\frac{\lambda}{4\sqrt{\lambda^2+4}}+2\lambda+8 = 11.$$

It gives $\lambda_C=1,429$ and therefore $x_1^C = 2-\lambda_C = 0,571$, $x_2^C=\lambda_C=1,429$.

For the "minimum risk" program (D) let us take $t=-2$. Then $1<\lambda_D<2$ and the equation to be solved is:

$$\frac{8}{\lambda}-8 = t$$

It gives $\lambda_D=4/3$ and therefore $x_1^D=2/3$, $x_2^D=4/3$.

For the program (E) it is clear from Table 2 that any solution of the corresponding equation (6) must be negative. Since $x_\lambda=(1, 0)$ for any $\lambda<0$ it follows that (6) reduces to $\lambda_E=-9/4$. Since the solution is unique it follows from Th. 1 that $x_{\lambda_E}=(1, 0)$ is optimal.

The program (F) violates the convexity requirement too. The corresponding equation (6) has four solutions:

$$\lambda_1=1/11, \quad \lambda_2=1, \quad \lambda_3=4/3, \quad \lambda_4=8/3.$$

Since X is compact (F) has an optimal solution and, according to Th. 1, it must be among the points $x_{\lambda_1}=(1, 0)$, $x_{\lambda_2}=(1, 1)$, $x_{\lambda_3}=(2/3, 4/3)$, $x_{\lambda_4}=(0, 2)$. Computing $\psi_E(x_{\lambda_1})=11$, $\psi_E(x_{\lambda_2})=25$, $\psi_E(x_{\lambda_3})=25.037$, $\psi_E(x_{\lambda_4})=24$ it follows that the optimal solution is $x_{\lambda_1}=(1, 0)$; x_{λ_4} is a point of local-minimum while x_{λ_2} and x_{λ_3} are saddle-points.

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(Received July 1, 1969)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 181—188.

A LEXICOGRAPHIC COMPLEMENTARY PIVOT ALGORITHM FOR THE SOLUTION OF BIMATRIX GAMES

by

A. MAJTHAY

1. Introduction

The bimatrix game is a natural extension of the two person zero sum game. In [2] LEMKE and HOWSON established an algorithm for the solution of bimatrix games. They investigate the problem from a geometric point of view.

In this paper we consider the problem as a combinatorial one and get a lexicographic version of the Lemke—Howson algorithm. Lexicographic theory is applied in that form as it is presented in [7].

2. The bimatrix game

Denote $\mathbf{e}_i$ $i=1, \dots, m$ the ith unit vector in E^m, $\hat{\mathbf{e}}_j$ $j=1, \dots, n$ the jth unit vector in E^n. Let

$$\mathbf{g} = \mathbf{e}_1 + \dots + \mathbf{e}_m,$$
$$\hat{\mathbf{g}} = \hat{\mathbf{e}}_1 + \dots + \hat{\mathbf{e}}_n.$$

Let us denote by S the set of m-component stochastic vectors and by $\hat{S}$ the set of n-component stochastic vectors, i.e.

$$S = \{\mathbf{p} \,|\, \mathbf{p} \in E^m, \mathbf{p} \geqq \mathbf{0}, \mathbf{g}^T \mathbf{p} = 1\},$$
$$\hat{S} = \{\hat{\mathbf{p}} \,|\, \hat{\mathbf{p}} \in E^n, \hat{\mathbf{p}} \geqq \mathbf{0}, \hat{\mathbf{g}}^T \hat{\mathbf{p}} = 1\}.$$

Let $G=(g_{ij})$ an m by n and $\hat{G}=(\hat{g}_{ji})$ an n by m real matrix.

Definition. A Nash equilibrium point of the bimatrix game defined by G and $\hat{G}$ is a pair of vectors

$$\mathbf{p}_0 \in S, \quad \hat{\mathbf{p}}_0 \in \hat{S}$$

for which

(2. 1′) $$\mathbf{p}_0^T G \hat{\mathbf{p}}_0 \leqq \mathbf{p}^T G \hat{\mathbf{p}}_0 \quad \text{if} \quad \mathbf{p} \in S$$

and

(2. 1″) $$\hat{\mathbf{p}}_0^T \hat{G} \mathbf{p}_0 \leqq \hat{\mathbf{p}}^T \hat{G} \mathbf{p}_0 \quad \text{if} \quad \hat{\mathbf{p}} \in \hat{S}.$$

We can suppose here that

(2. 2) $$G > 0, \quad \hat{G} > 0,$$

as for any real number $\lambda \geqq 0$ the game defined by the matrices

$$G + \lambda \mathbf{g}\hat{\mathbf{g}}^T, \quad \hat{G} + \lambda \hat{\mathbf{g}}\mathbf{g}^T$$

has the same equilibrium points as the one defined by G and $\hat{G}$.

Let

$$\gamma = \mathbf{p}_0^T G \hat{\mathbf{p}}_0 \tag{2.3}$$
$$\hat{\gamma} = \hat{\mathbf{p}}_0^T \hat{G} \mathbf{p}_0$$

From (2. 2) we get $\gamma > 0$ and $\hat{\gamma} > 0$.

It is clear that (2. 1) holds if and only if the following system of inequalities holds

$$\gamma \mathbf{g} \leqq G\hat{\mathbf{p}}_0,$$
$$\hat{\gamma}\hat{\mathbf{g}} \leqq \hat{G}\mathbf{p}_0.$$

Let

$$G\hat{\mathbf{p}}_0 - \gamma\mathbf{g} = \gamma\mathbf{u},$$
$$\hat{G}\mathbf{p}_0 - \hat{\gamma}\hat{\mathbf{g}} = \hat{\gamma}\hat{\mathbf{u}}.$$

Let finally

$$\mathbf{z} = \frac{1}{\hat{\gamma}}\mathbf{p}_0, \tag{2.4}$$
$$\hat{\mathbf{z}} = \frac{1}{\gamma}\hat{\mathbf{p}}_0.$$

Now we are able to write (2. 1) and (2. 3) in another form more suitable for our purposes:

(2. 5a) $$\mathbf{u} - G\hat{\mathbf{z}} = -\mathbf{g},$$

(2. 5b) $$\mathbf{u} \geqq \mathbf{0}, \quad \hat{\mathbf{z}} \geqq \mathbf{0},$$

(2. 6a) $$\hat{\mathbf{u}} - \hat{G}\mathbf{z} = -\hat{\mathbf{g}},$$

(2. 6b) $$\hat{\mathbf{u}} \geqq \mathbf{0}, \quad \mathbf{z} \geqq \mathbf{0},$$

(2. 7) $$\mathbf{u}^T\mathbf{z} = 0, \quad \hat{\mathbf{u}}^T\hat{\mathbf{z}} = 0.$$

Suppose this system to be consistent and denote $\mathbf{u}$, $\hat{\mathbf{u}}$, $\mathbf{z}$, $\hat{\mathbf{z}}$ a solution. If we define

(2. 8) $$\gamma = \frac{1}{\sum_{j=1}^{n} \hat{z}_j}, \quad \hat{\gamma} = \frac{1}{\sum_{i=1}^{m} z_i}$$

we see that the pair of vectors

(2. 9) $$\mathbf{p}_0 = \hat{\gamma}\mathbf{z}, \quad \hat{\mathbf{p}}_0 = \gamma\hat{\mathbf{z}}$$

forms an equilibrium point. We have just proved the following

THEOREM 1. *The formulas* (2. 3), (2. 4), (2. 8) *and* (2. 9) *set up a one to one correspondence between the Nash equilibrium points of the bimatrix game defined by the positive matrices G and $\hat{G}$ and the solutions of the system* (2. 5a)—(2. 7).

3. L-feasible pairs of bases

The vectors $\mathbf{u}$, $\hat{\mathbf{z}}$ and $\hat{\mathbf{u}}$, $\mathbf{z}$ satisfying (2. 5a) and (2. 6a) are called solutions, the solutions satisfying (2. 5b) and (2. 6b) are called feasible solutions, finally the feasible solutions satisfying (2. 7) are called complementary solutions.

The components of $\mathbf{u}$, $\mathbf{z}$ and $\hat{\mathbf{u}}$, $\hat{\mathbf{z}}$ having the same subscript are called complementary variables, while the corresponding vectors $\mathbf{e}_i$, $-\hat{\mathbf{g}}_i$ $i=1, \dots, m$ and $\hat{\mathbf{e}}_j$, $-\mathbf{g}_j$ $j=1, \dots, n$ in the matrices

$$A = (I, -G)$$

$$\hat{A} = (\hat{I}, -\hat{G})$$

are called complementary vectors.

For the sake of simplicity we shall also denote

$$A = (\mathbf{a}_1, \dots, \mathbf{a}_{m+n}),$$

$$\hat{A} = (\hat{\mathbf{a}}_1, \dots, \hat{\mathbf{a}}_{n+m}),$$

i.e. we have

$$\mathbf{a}_p = \begin{cases} \mathbf{e}_p & \text{if } p = 1, \dots, m \\ -\mathbf{g}_{p-m} & \text{if } p = m+1, \dots, m+n, \end{cases}$$

$$\hat{\mathbf{a}}_p = \begin{cases} \hat{\mathbf{e}}_p & \text{if } p = 1, \dots, n \\ -\hat{\mathbf{g}}_{p-n} & \text{if } p = n+1, \dots, n+m. \end{cases}$$

Denoting

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \hat{\mathbf{z}} \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{u}} \\ \mathbf{z} \end{bmatrix},$$

$$\mathbf{a}_0 = -\mathbf{g}, \quad \hat{\mathbf{a}}_0 = -\hat{\mathbf{g}},$$

we can write the system (2. 5a)—(2. 6b) as

$$(3.1) \qquad A\mathbf{x} = \mathbf{a}_0, \quad \mathbf{x} \geqq \mathbf{0},$$

$$(3.2) \qquad \hat{A}\hat{\mathbf{x}} = \hat{\mathbf{a}}_0, \quad \hat{\mathbf{x}} \geqq \mathbf{0}.$$

The rank of the matrix A is m the one of $\hat{A}$ is n, so any basis of A has m vectors any basis of $\hat{A}$ has n vectors. Let us denote by B and $\hat{B}$ a basis of the columns of A and $\hat{A}$ respectively.

If

$$B = (\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_m}),$$

$$\hat{B} = (\hat{\mathbf{a}}_{\hat{i}_1}, \dots, \hat{\mathbf{a}}_{\hat{i}_n}),$$

then let

$$I = \{i_1, \dots, i_m\},$$

$$\hat{I} = \{\hat{i}_1, \dots, \hat{i}_n\}.$$

Any pair of bases B, $\hat{B}$ defines a unique pair of matrices

$$D = (\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_{m+n}) = \begin{bmatrix} \boldsymbol{\delta}_{i_1} \\ \vdots \\ \boldsymbol{\delta}_{i_m} \end{bmatrix} = (d_{i,p}),$$

$$\hat{D} = (\hat{\mathbf{d}}_0, \hat{\mathbf{d}}_1, \ldots, \hat{\mathbf{d}}_{n+m}) = \begin{bmatrix} \hat{\boldsymbol{\delta}}_{\hat{i}_1} \\ \vdots \\ \hat{\boldsymbol{\delta}}_{\hat{i}_n} \end{bmatrix} = (\hat{d}_{\hat{i},p}),$$

by the equations

$$BD = (\mathbf{a}_0, A),$$
$$\hat{B}\hat{D} = (\hat{\mathbf{a}}_0, \hat{A}).$$

In the sequel every symbol depending on a basis shall have the same superscript as the basis itself.

The basic solution of (3. 1) corresponding to B is defined by

$$x_p = \begin{cases} d_{p,0} & \text{if } p \in I \\ 0 & \text{if } p \notin I. \end{cases}$$

The basic solution of (3. 2) corresponding to $\hat{B}$ is defined analogously.

Definition. B and $\hat{B}$ form a complementary pair of bases if they do not contain complementary vectors.

Definition. The row vector $\boldsymbol{\delta}$ is called lexicographically positive

$$\boldsymbol{\delta} \succ \mathbf{0}$$

if its first nonvanishing component from the left is positive.
The vector $\boldsymbol{\delta}_1$ is lexicographically greater than the vector $\boldsymbol{\delta}_2$

$$\boldsymbol{\delta}_1 \succ \boldsymbol{\delta}_2$$

if their difference $\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2$ is lexicographically positive.

The relation $\succ$ is an order relation so if we have a finite number of vectors, there always exists a lexicographically greatest and least among them. We shall denote them by l-max and l-min respectively.

The matrices D and $\hat{D}$ are called lexicographically positive if every row of them is lexicographically positive. Clearly in such a case we have $\hat{\mathbf{d}}_0 \geqq \mathbf{0}$ and $\mathbf{d}_0 \geqq \mathbf{0}$, i.e. the basic solutions corresponding to B and $\hat{B}$ are feasible. That is the reason for the following

Definition. The basis B is called l-feasible if and only if the corresponding matrix D is lexicographically positive. The pair of bases B, $\hat{B}$ is l-feasible if both B and $\hat{B}$ are.

Let us suppose that we have an l-feasible basis B, and we choose one of the nonbasic vectors $\mathbf{a}_k$ $k \in \{1, \ldots, m+n\} - I$. We shall try to insert $\mathbf{a}_k$ into B and remove a vector $\mathbf{a}_j$ $j \in I$ from it, while keeping B l-feasible. If we are able to do so, we shall call the new l-feasible basis $B^{(1)}$ a neighbour of B.

THEOREM 2. *The l-feasible basis B has a neighbour which is formed by the insertion of $\mathbf{a}_k$ into B if and only if $\mathbf{d}_k$ has a positive component. If it has one then the neighbour is unique.*

PROOF. Insert $\mathbf{a}_k$ into B and delete a vector $\mathbf{a}_j$ from it. As the new matrix $B^{(1)}$ is a basis, we have $d_{j,k} \neq 0$. The transformation formulas between D and $D^{(1)}$ are as follows

$$(3.3) \qquad \boldsymbol{\delta}_k^{(1)} = \frac{1}{d_{j,k}} \boldsymbol{\delta}_j$$

$$(3.4) \qquad \boldsymbol{\delta}_i^{(1)} = \boldsymbol{\delta}_i - \frac{d_{i,k}}{d_{j,k}} \boldsymbol{\delta}_j \qquad i \in I^{(1)} - \{k\}$$

From the equation (3. 3) we see that necessarily $d_{j,k} > 0$. Let us suppose this is the case. If $d_{i,k} \leqq 0$ in (3. 4) then $\boldsymbol{\delta}_i^{(1)} \succ \mathbf{0}$. We only must investigate those subscripts i for which $d_{i,k} > 0$. There always exist such subscripts as for j we know that $d_{j,k} > 0$. For those subscripts $\boldsymbol{\delta}_i^{(1)} \succ \mathbf{0}$ if and only if

$$\frac{1}{d_{i,k}} \boldsymbol{\delta}_i \succ \frac{1}{d_{j,k}} \boldsymbol{\delta}_j$$

i.e. the subscript j is defined by the following equation

$$(3.5) \qquad \frac{1}{d_{j,k}} \boldsymbol{\delta}_j = l - \min_{\{i \in I | d_{i,k} > 0\}} \frac{1}{d_{i,k}} \boldsymbol{\delta}_i$$

The subscript is uniquely defined by (3. 5) as D does not have dependent rows. The theorem is proved.

Definition. The pair of l-feasible bases $B^{(1)}$, $\hat{B}^{(1)}$ is a neighbour of the pair of l-feasible bases B, $\hat{B}$ if either of the following conditions hold

$B^{(1)} = B$ and $\hat{B}^{(1)}$ is a neighbour of $\hat{B}$,

$B^{(1)}$ is a neighbour of B and $\hat{B}^{(1)} = \hat{B}$.

The following Theorem 3 is an immediate consequence of Theorem 2.

THEOREM 3. *The pair of l-feasible bases B, $\hat{B}$ has a neighbour which is formed by inserting $\mathbf{a}_k$ into B ($\hat{\mathbf{a}}_k$ into $\hat{B}$) if and only if $\mathbf{d}_k$ ($\hat{\mathbf{d}}_k$) has a positive component. In this case the neighbour is unique.*

4. The algorithm and its finiteness

Let $B^{(-2)} = (\mathbf{e}_1, \ldots, \mathbf{e}_m)$, $\hat{B}^{(-2)} = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_n)$, $D^{(-2)} = (\mathbf{a}_0, A)$, $\hat{D}^{(-2)} = (\hat{\mathbf{a}}_0, \hat{A})$. The pair of bases $B^{(-2)}$, $\hat{B}^{(-2)}$ is not feasible.

Define $B^{(-1)}$, $\hat{B}^{(-1)}$ as follows: let $k = m+1$ and insert $\mathbf{a}_k$ into $B^{(-2)}$. The subscript h of the vector to be deleted is defined by

$$(4.1) \qquad \frac{1}{d_{h,k}^{(-2)}} \boldsymbol{\delta}_h^{(-2)} = l - \max_{i \in I^{(-2)}} \frac{1}{d_{i,k}^{(-2)}} \boldsymbol{\delta}_i^{(-2)}.$$

As the rows of $D^{(-2)}$ are linearly independent h is unique. Let $\hat{B}^{(-1)} = \hat{B}^{(-2)}$.

In the next step we shall construct the pair $B^{(0)}$, $\hat{B}^{(0)}$. Let $B^{(0)}=B^{(-1)}$, $k=n+h$ and insert the vector $\hat{\mathbf{a}}_k$ into $\hat{B}^{(-1)}$. The subscript j of the vector to be deleted from $B^{(-1)}$ is defined by

$$(4.2)\qquad \frac{1}{\hat{d}_{j,k}^{(-1)}}\hat{\boldsymbol{\delta}}_j^{(-1)} = l-\max_{i\in\hat{I}^{(-1)}} \frac{1}{\hat{d}_{i,k}^{(-1)}}\hat{\boldsymbol{\delta}}_i^{(-1)},$$

j is unique as above.

It is easy to see that the pair $B^{(0)}$, $\hat{B}^{(0)}$ is l-feasible. In fact the equations (4. 1), (4. 2) have been constructed to satisfy this requirement.

The uniqueness already mentioned can be set up in the following form, too:

There is exactly one l-feasible basis containing $m-1$ $\mathbf{e}_i$ and $-\mathbf{g}_1$ and this is $B^{(0)}$.

There is exactly one l-feasible basis containing $n-1$ $\hat{\mathbf{e}}_i$ and $-\hat{\mathbf{g}}_h$ and this is $\hat{B}^{(0)}$.

As $\hat{\mathbf{d}}_j^{(0)}<\mathbf{0}$ the l-feasible pair of bases $B^{(0)}$, $\hat{B}^{(0)}$ does not have a neighbour constructed by the insertion of $\hat{\mathbf{a}}_j$.

Now we are ready to define the algorithm. Construct the sequence of basis pairs

$$(4.3)\qquad B^{(-2)},\hat{B}^{(-2)};\quad B^{(-1)},\hat{B}^{(-1)};\quad B^{(0)},\hat{B}^{(0)};\quad B^{(1)},\hat{B}^{(1)};\quad B^{(2)},\hat{B}^{(2)};\ldots$$

according to the following rules:

(a) if $\hat{B}^{(q)}=\hat{B}^{(q-1)}$ and $\mathbf{a}_j$ is the vector that left $B^{(q-1)}$, then let $B^{(q+1)}=B^{(q)}$, and denoting the complementary pair of $\mathbf{a}_j$ by $\hat{\mathbf{a}}_k$, we construct $\hat{B}^{(q+1)}$ by inserting $\hat{\mathbf{a}}_k$ into $\hat{B}^{(q)}$,

(b) if $B^{(q)}=B^{(q-1)}$ and $\hat{\mathbf{a}}_j$ is the vector that left $\hat{B}^{(q-1)}$, then let $\hat{B}^{(q+1)}=\hat{B}^{(q)}$ and denoting the complementary pair of $\hat{\mathbf{a}}_j$ by $\mathbf{a}_k$, we construct $B^{(q+1)}$ by inserting $\mathbf{a}_k$ into $B^{(q)}$.

Starting from $B^{(0)}$, $\hat{B}^{(0)}$ the vector $\hat{\mathbf{a}}_j$ is in (4. 2) defined, therefore the sequence is uniquely defined according to Theorem 2.

It is clear that every element in the sequence is an l-feasible pair of bases.

If either $\mathbf{a}_{m+1}=-\mathbf{g}_1$ or $\hat{\mathbf{a}}_1=\hat{\mathbf{e}}_1$ leaves the basis, the algorithm stops.

THEOREM 4. *The sequence* (4. 3) *is finite.*

PROOF. As the possible pairs of bases are finite, we only have to prove that no pair occurs twice in the sequence.

If we have a pair appearing twice, then we have a first one. The elements in the sequence are defined in such a manner that exactly one complementary pair $\mathbf{a}_{m+1}$, $\hat{\mathbf{a}}_1$ is always in basis, while exactly one complementary pair of vectors being out of basis. The neighbours of this element are constructed by inserting the vectors of the latter pair into the corresponding basis. Therefore Theorem 2 yields that every element in the sequence can have at most two neighbours there.

The first pair appearing twice cannot be $B^{(0)}$, $\hat{B}^{(0)}$, as we already know they do not have a neighbour constructed by inserting $\hat{\mathbf{a}}_j$.

The first element appearing twice has two different preceding element in the sequence as it is the first one, but this is impossible. The theorem is proved.

THEOREM 5. *The last element in the sequence* (4. 3) *defines a solution to the problem* (2. 4a)—(2. 7).

PROOF. We already know that there is a last pair of bases in the sequence. It is either a complementary pair of bases and then the theorem is proved, or it is the last pair because we cannot construct the next pair as the condition in Theorem 2 is not met. We shall prove the impossibility of this case by contradiction.

Denote B, $\hat{B}$ the last pair. As this pair is different from $B^{(0)}$, $\hat{B}^{(0)}$ we know that at least one of them is different. We also know that $-\mathbf{g}_1$ is in B while $\hat{\mathbf{e}}_1$ is in $\hat{B}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ $0 \leqq r < m$ denote the unit vectors contained in B, while $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_s$ $1 \leqq s < n$ those contained in $\hat{B}$. As the pairs only contain one complementary pair the other vectors in B form a subset of $\{-\mathbf{g}_{s+1}, \ldots, -\mathbf{g}_n\}$ and those in $\hat{B}$ form a subset of $\{-\hat{\mathbf{g}}_{r+1}, \ldots, -\hat{\mathbf{g}}_m\}$. There are $m+n+1$ vectors listed above therefore we know that exactly one is superfluous among them. We may suppose that it is either $-\mathbf{g}_{s+1}$ or $-\hat{\mathbf{g}}_{r+1}$. In the first case we have

$$B = (\mathbf{e}_1, \ldots, \mathbf{e}_r, -\mathbf{g}_1, -\mathbf{g}_{s+2}, \ldots, -\mathbf{g}_n),$$

(4. 4)

$$\hat{B} = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_s, -\hat{\mathbf{g}}_{r+1}, \ldots, -\hat{\mathbf{g}}_m),$$

while in the second one

$$B = (\mathbf{e}_1, \ldots, \mathbf{e}_r, -\mathbf{g}_1, -\mathbf{g}_{s+1}, \ldots, -\mathbf{g}_n)$$

$$\hat{B} = (\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_s, -\hat{\mathbf{g}}_{r+2}, \ldots, -\hat{\mathbf{g}}_m).$$

Both cases are possible as the systems

$$r+n-s = m, \qquad 0 \leqq r < m,$$

$$s+m-r = n, \qquad 0 < s < n,$$

and

$$r+n-s+1 = m, \qquad 0 \leqq r < m,$$

$$s+m-r-1 = n, \qquad 0 < s < n$$

are consistent.

The complementary pair not in the basis is $-\mathbf{g}_{s+1}$, $\hat{\mathbf{e}}_{s+1}$ in the first case and $\mathbf{e}_{r+1}$, $-\hat{\mathbf{g}}_{r+1}$ in the second one.

As we know one of them has merely nonpositive components with respect to the corresponding basis.

We only have to cheque all possible cases. In the first one we have either

$$\sum_{i=1}^{r} \mathbf{e}_i d_{i,m+s+1} - \mathbf{g}_1 d_{m+1,m+s+1} - \sum_{i=m+s+2}^{m+n} \mathbf{g}_{i-m} d_{i,m+s+1} = -\mathbf{g}_{s+1} \tag{4. 6}$$

or

$$\sum_{i=1}^{s} \hat{\mathbf{e}}_i \hat{d}_{i,s+1} - \sum_{i=n+r+1}^{n+m} \hat{\mathbf{g}}_{i-n} \hat{d}_{i,s+1} = \hat{\mathbf{e}}_{s+1}. \tag{4. 7}$$

Here we know that all $d_{i,k}$ are nonpositive. In (4. 6) every component of the right hand side vector is negative while the second and third term in the left hand side is nonnegative according to (2. 2). It follows that $r=m$ i.e. $B=B^{(-2)}$ which is not feasible. In (4. 7) we see that necessarily $s=n-1$ i.e. $B=B^{(0)}$ as we can see from (4. 5). Then $-\hat{\mathbf{g}}_h$ is contained in B as the pair contains only one complementary pair and that is $-\mathbf{g}_1$, $\mathbf{e}_1$, in other words $\hat{B}=\hat{B}^{(0)}$ which is impossible.

In the second case we have either

$$\sum_{i=1}^{r} \mathbf{e}_i d_{i,r+1} - \mathbf{g}_1 d_{m+1,r+1} - \sum_{i=m+s+1}^{m+n} \mathbf{g}_{i-m} d_{i,r+1} = \mathbf{e}_{r+1} \tag{4. 8}$$

or

$$\sum_{i=1}^{s} \hat{\mathbf{e}}_i \hat{d}_{i,m+r+1} - \sum_{i=n+r+2}^{n+m} \hat{\mathbf{g}}_{i-n} d_{i,n+r+1} = -\hat{\mathbf{g}}_{r+1}. \tag{4. 9}$$

The system (4. 8) can only be consistent if $r = m-1$ but this implies that either $\hat{B}=\hat{B}^{(0)}$ or $\hat{B}=\hat{B}^{(-1)}$ which is impossible.

Finally the system (4. 9) implies $s=n$ i.e. $\hat{B}=\hat{B}^{(-1)}$. The theorem is proved. An immediate consequence of this theorem is the following

THEOREM 6. *The bimatrix game has an equilibrium point.*

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(Received January 18, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 189—197.

A REMARK TO THE LAW OF THE ITERATED LOGARITHM

by

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§. 1.

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be independent, identically distributed random variables, $\mathsf{E}\xi_1=0$, $\mathsf{E}\xi_1^2=\sigma^2<+\infty$, $S_n=\xi_1+\cdots+\xi_n$. The HARTMAN—WINTNER version of the aw of the iterated logarithm (see [1]) asserts that

$$\overline{\lim_{n\to\infty}} \frac{|S_n|}{\sqrt{n \log\log n}} = \sigma\sqrt{2} \tag{1}$$

with probability one. Recently V. STRASSEN [2] proved that this theorem remains valid also in the case $\sigma=+\infty$, e.g. replacing the condition $\mathsf{E}\xi_1^2=\sigma^2<+\infty$ by $\mathsf{E}\xi_1^2=+\infty$ in the above theorem, instead of (1) we have

$$\overline{\lim_{n\to\infty}} \frac{|S_n|}{\sqrt{n \log\log n}} = +\infty \quad \text{a.s.} \tag{2}$$

STRASSEN's result — together with the HARTMAN—WINTNER theorem — seems to be complete, indeed, these two theorems together exactly characterize, how the variance of independent, identically distributed random variables affects the oscillations of their successive sums. Nevertheless the following question arises: Is STRASSEN's result best possible? Namely it is possible that $\mathsf{E}\xi_1=0$ $\mathsf{E}\xi_1^2=+\infty$ imply not only (2), but also

$$\overline{\lim_{n\to\infty}} \frac{|S_n|}{\sqrt{n \log\log n f(n)}} = +\infty \quad \text{a.s.,}$$

where $f(n)$ is an appropriate function such that $\lim_{n\to\infty} f(n)=+\infty$. If there exists such a function $f(n)$, this means that in the description of the fluctuations of successive sums of independent, identically distributed random variables with infinite variances the function $\sqrt{n \log\log n}$ does not play a characteristic role, and one ought to find a new theorem describing the "real" behaviour of the partial sums. But in the sequel we shall prove that no such function $f(n)$ exists, thus STRASSEN's result is best possible and it describes an essential property of random variables with infinite variances. More exactly we shall prove the next

THEOREM. *Let $f(n)$ be a function such that $\lim_{n\to\infty} f(n)=+\infty$; then there exists a sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ of independent, identically distributed random variables such that $\mathsf{E}\xi_1=0$ $\mathsf{E}\xi_1^2=+\infty$ and*

$$\lim_{n\to\infty} \frac{|S_n|}{\sqrt{n \log\log n f(n)}} = 0$$

with probability one.

Remark. Although by our theorem (2) is best possible, for certain sequences $\xi_1, \xi_2, \dots, \xi_n, \dots$ it can be sharpened; e.g. if we assume the stronger condition $E|\xi_1|^r = +\infty$, $1<r<2$ instead of $\mathsf{E}\xi_1^2 = +\infty$, then from a well-known theorem of MARCZINKIEWICZ (see [3]) follows

$$\varlimsup_{n\to\infty} \frac{|S_n|}{n^{1/r}} > 0 \quad \text{a.s.}$$

and — choosing a number c such that $\frac{1}{2} < c < \frac{1}{r}$ — this implies

$$\varlimsup_{n\to\infty} \frac{|S_n|}{n^c} = +\infty \quad \text{a.s.},$$

which is more than (2). Our theorem asserts only that $\mathsf{E}\xi_1^2 = +\infty$ itself does not imply more than (2).

The method of the construction — following STRASSEN's proof of (2) — is based on the SKOROKHOD representation theorem as well. However, that we shall need is not this representation theorem, but its — almost trivial — converse. We present it here, together with a result of FELLER, which will be also needed for our purposes.

LEMMA 1. *Let $\zeta(t)$ $(t \geqq 0)$ be a standard Brownian motion $(\zeta(0)=0)$, let $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ be independent, identically distributed nonnegative random variables,* $\mathsf{P}(\alpha_1 = k) = p_k$ $(k = 1, 2, \dots)$, $\sum_{k=1}^{\infty} p_k = 1$. *Assume that $\zeta(t)$ is independent of the α_k-s and define $\xi_1, \xi_2, \dots$ in the following manner:*

$$(3) \qquad \begin{aligned} \xi_1 &= \zeta(\alpha_1) \\ \xi_2 &= \zeta(\alpha_1+\alpha_2) - \zeta(\alpha_1) \\ &\vdots \\ \xi_n &= \zeta(\alpha_1+\dots+\alpha_n) - \zeta(\alpha_1+\dots+\alpha_{n-1}) \\ &\vdots \end{aligned}$$

Then $\xi_1, \xi_2, \dots$ are independent, identically distributed random variables.

As it can be seen, we assumed that $\zeta(t)$ is independent of the α_k-s, therefore Lemma 1 is not a precise converse of SKOROKHOD's representation theorem; however, its applicability depends on the same facts as SKOROKHOD's theorem.

The proof of Lemma 1 is immediate:

$$\mathsf{P}(\xi_k < x) = \sum_{l_1,\dots,l_k=1}^{\infty} \mathsf{P}(\xi_k < x \mid \alpha_1 = l_1, \dots \alpha_k = l_k)\mathsf{P}(\alpha_1 = l_1, \dots \alpha_k = l_k) =$$

$$= \sum_{l_1,\dots,l_k=1}^{\infty} \mathsf{P}(\zeta(\alpha_1+\dots+\alpha_k) - \zeta(\alpha_1+\dots+\alpha_{k-1}) < x \mid \alpha_1 = l_1, \dots \alpha_k = l_k) p_{l_1} \dots p_{l_k} =$$

$$= \sum_{l_1,\dots,l_k=1}^{\infty} \mathsf{P}(\zeta(l_1+\dots+l_k) - \zeta(l_1+\dots+l_{k-1}) < x \mid \alpha_1 = l_1, \dots \alpha_k = l_k) p_{l_1} \dots p_{l_k} =$$

$$= \sum_{l_1,\dots,l_k=1}^{\infty} \mathsf{P}(\zeta(l_1+\dots+l_k) - \zeta(l_1+\dots+l_{k-1}) < x) p_{l_1} \dots p_{l_k} =$$

$$= \sum_{l_1,\dots,l_k=1}^{\infty} \mathsf{P}(\zeta(l_k) < x) p_{l_1} \dots p_{l_k} = \sum_{l_k=1}^{\infty} \mathsf{P}(\zeta(l_k) < x) p_{l_k} = \sum_{r=1}^{\infty} \mathsf{P}(\zeta(r) < x) p_r$$

and this really does not depend on k. The independence of the ξ_k-s can be proved similarly.

LEMMA 2. (See FELLER [4].) *Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be independent, identically distributed random variables such that $\mathsf{E}|\alpha_1| = +\infty$; let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive numbers such that $\frac{a_n}{n}$ is nondecreasing and*

$$\sum_{n=1}^{\infty} \mathsf{P}(|\alpha_1| \geqq a_n) < +\infty$$

Then

$$\lim_{n\to\infty} \frac{\alpha_1 + \cdots + \alpha_n}{a_n} = 0$$

with probability one.

PROOF *of the theorem.* We prove the theorem in the following equivalent form: Let $g(n)$ be an arbitrary function such that $\lim\limits_{n\to\infty} g(n) = +\infty$, then there exists a sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ of independent, identically distributed random variables such that $\mathsf{E}\xi_1 = 0$ $\mathsf{E}\xi_1^2 = +\infty$ and

$$\lim_{n\to\infty} \frac{|S_n|}{\sqrt{ng(n) \log\log ng(n)}} = 0 \quad \text{a.s.} \tag{4}$$

where $S_n = \xi_1 + \cdots + \xi_n$. Without loss of generality we may assume that $g(n)$ is a nondecreasing, integer-valued function which satisfies the condition

$$g(n) \leqq n \qquad (n=1, 2, \ldots). \tag{5}$$

In fact, let us choose a nondecreasing, integer-valued function $g_1(n)$ such that $\lim\limits_{n\to\infty} g_1(n) = +\infty$, $g_1(n) \leqq g(n)$ and put

$$g_2(n) = \min\left(g_1(n), n\right)$$

Then the function $g_2(n)$ has the desired properties and since $g_2(n) \leqq g(n)$ it is sufficient to prove our theorem in the case when $g(n)$ is replaced by $g_2(n)$.

Let $d_1 > d_2 > \cdots > 0$ be a sequence of positive numbers such that

$$\sum_{i=1}^{\infty} d_i < +\infty \quad \sum_{i=1}^{\infty} d_i g(i) = +\infty .$$

(The construction of such a sequence is so simple, that we can omit it.) We define a sequence $c_1, c_2, \ldots$ such that

$$c_{1g(1)} = d_1, \quad c_{2g(2)} = d_2, \ \ldots \ c_{ng(n)} = d_n, \ \ldots$$

and

$$c_1 > c_2 > \cdots > c_n > \cdots > 0.$$

It is easy to verify that the sequence $c_1, c_2, \ldots, c_n, \ldots$ has the following properties:

1) $\sum_{i=1}^{\infty} c_i = +\infty$

2) $\sum_{i=1}^{\infty} c_{ig(i)} < +\infty$.

In fact, 2) is only a reformulation of $\sum_{i=1}^{\infty} d_i < +\infty$; on the other hand, by the monotonity of c_k and $g(k)$ we have

$$\sum_{i=g(1)}^{\infty} c_i = \sum_{k=1}^{\infty} \sum_{i=kg(k)}^{(k+1)g(k+1)-1} c_i \geqq \sum_{k=1}^{\infty} [(k+1)g(k+1)-kg(k)]c_{(k+1)g(k+1)} \geqq$$

$$\geqq \sum_{k=1}^{\infty} [(k+1)g(k+1)-kg(k+1)]c_{(k+1)g(k+1)} = \sum_{l=2}^{\infty} g(l)d_l = +\infty$$

We note that $\lim_{n\to\infty} c_n = 0$ (since by 2, we have $\lim_{k\to\infty} c_{kg(k)} = 0$ and $c_1 > c_2 > \cdots$); furthermore we may assume that $c_1 = 1$ (since in the opposite case we replace the sequence $c_1, c_2, \ldots$ by the sequence $\alpha c_1, \alpha c_2, \ldots$ where $\alpha = \frac{1}{c_1}$, this process does not change the validity of 1, 2). Let now

$$p_1 = c_1 - c_2, \quad p_2 = c_2 - c_3, \ldots, p_n = c_n - c_{n+1}, \ldots$$

then $p_1 > 0, p_2 > 0, \ldots$ and

$$\sum_{k=1}^{\infty} p_k = (c_1 - c_2) + (c_2 - c_3) + \cdots = \lim_{i\to\infty} (c_1 - c_i) = c_1 = 1$$

furthermore

$$\sum_{k=l}^{\infty} p_k = (c_l - c_{l+1}) + (c_{l+1} - c_{l+2}) + \cdots = c_l \qquad (l \geqq 1).$$

Now let $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be independent random variables with the common distribution defined by

$$\mathsf{P}(\alpha_i = k) = p_k \qquad (i, k = 1, 2, \ldots).$$

Then $\alpha_i > 0$ $(i = 1, 2, \ldots)$

$$\mathsf{E}\alpha_1 = \sum_{k=1}^{\infty} kp_k = \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} p_k = \sum_{l=1}^{\infty} c_l = +\infty$$

and

$$\sum_{n=1}^{\infty} \mathsf{P}(|\alpha_1| \geqq ng(n)) = \sum_{n=1}^{\infty} \sum_{r=ng(n)}^{\infty} p_r = \sum_{n=1}^{\infty} c_{ng(n)} < +\infty.$$

Applying Lemma 2 with $a_n = ng(n)$ we obtain

(6) $$\lim_{n\to\infty} \frac{\alpha_1 + \cdots + \alpha_n}{ng(n)} = 0 \quad \text{a.s.}$$

Now we are able to construct that sequence $\xi_1, \xi_2, \dots$ which proves our theorem. Consider a standard Brownian motion $\zeta(t)$, let further $\alpha_1, \alpha_2, \dots$ be the sequence defined above and let us suppose that $\zeta(t)$ is independent of the α_k-s (this latter condition can be guaranteed by the appropriate choice of our probability space). Let us define the sequence $\xi_1, \xi_2, \dots$ as in Lemma 1:

$$\begin{aligned}
\xi_1 &= \zeta(\alpha_1) \\
\xi_2 &= \zeta(\alpha_1+\alpha_2)-\zeta(\alpha_1) \\
&\vdots \\
\xi_n &= \zeta(\alpha_1+\dots+\alpha_n)-\zeta(\alpha_1+\dots+\alpha_{n-1}) \\
&\vdots
\end{aligned}$$

According to Lemma 1, $\xi_1, \xi_2, \dots$ are independent, identically distributed random variables. We shall prove, that $\mathsf{E}\xi_1=0$, $\mathsf{E}\xi_1^2=+\infty$ and the sequence $\xi_1, \xi_2, \dots$ satisfies (4). In fact, applying the Cauchy—Schwarz inequality and the well-known formula

$$\mathsf{E}\xi = \sum_k \mathsf{E}(\xi\,|\,A_k)\mathsf{P}(A_k) \quad \Big(\sum_k A_k = \Omega,\ A_iA_j = 0 \quad \text{if} \quad i\neq j\Big) \tag{7}$$

which is surely valid if $\xi\geqq 0$ or $\mathsf{E}|\xi|<+\infty$, we obtain

$$\begin{aligned}
\mathsf{E}|\xi_1| = \mathsf{E}|\zeta(\alpha_1)| &= \sum_{k=1}^{\infty} \mathsf{E}(|\xi(\alpha_1)|\,|\alpha_1=k)\mathsf{P}(\alpha_1=k) = \\
&= \sum_{k=1}^{\infty} \mathsf{E}(|\zeta(k)|\,|\alpha_1=k)\mathsf{P}(\alpha_1=k) = \sum_{k=1}^{\infty} \mathsf{E}|\zeta(k)|\,p_k \leqq \\
&\leqq \sum_{k=1}^{\infty} (\mathsf{E}|\zeta(k)|^2)^{1/2} p_k = \sum_{k=1}^{\infty} \sqrt{k}\,p_k = \mathsf{E}|\alpha_1|^{1/2}.
\end{aligned}$$

It follows from (5) and (6) that

$$\lim_{n\to\infty} \frac{\alpha_1+\dots+\alpha_n}{n^2} = 0 \quad \text{a.s.}$$

and an application of the earlier mentioned theorem of MARCZINKIEWICZ gives

$$\mathsf{E}|\alpha_1|^{1/2} < +\infty$$

thus

$$\mathsf{E}|\xi_1| \leqq \mathsf{E}|\alpha_1|^{1/2} < +\infty$$

therefore $\mathsf{E}\xi_1$ can be calculated by means of (7) and we obtain

$$\mathsf{E}\xi_1 = \mathsf{E}\zeta(\alpha_1) = \sum_{k=1}^{\infty} \mathsf{E}(\zeta(\alpha_1)\,|\,\alpha_1=k)\mathsf{P}(\alpha_1=k) = \sum_{k=1}^{\infty} \mathsf{E}\zeta(k)p_k = \sum_{k=1}^{\infty} 0\cdot p_k = 0.$$

Similarly

$$\begin{aligned}
\mathsf{E}\xi_1^2 = \mathsf{E}\zeta(\alpha_1)^2 &= \sum_{k=1}^{\infty} \mathsf{E}(\zeta(\alpha_1)^2\,|\,\alpha_1=k)\mathsf{P}(\alpha_1=k) = \\
&= \sum_{k=1}^{\infty} \mathsf{E}\zeta(k)^2 p_k = \sum_{k=1}^{\infty} kp_k = \mathsf{E}\alpha_1 = +\infty.
\end{aligned}$$

Now only (4) is to be proved. By the definition of the ξ_k-s, putting

$$\beta_n = \alpha_1 + \dots + \alpha_n$$

we have to prove

$$\frac{|\zeta(\beta_n)|}{\sqrt{ng(n) \log \log ng(n)}} \to 0 \quad \text{a.s.}$$

that is

$$\frac{|\zeta(\beta_n)|}{\sqrt{\beta_n \log \log \beta_n}} \sqrt{\frac{\beta_n}{ng(n)}} \sqrt{\frac{\log \log \beta_n}{\log \log ng(n)}} \to 0 \quad \text{a.s.} \tag{8}$$

Using the identity $\zeta(n) = (\zeta(1)-\zeta(0)) + (\zeta(2)-\zeta(1)) + \dots + (\zeta(n)-\zeta(n-1))$ and the fact that these summands are independent, normally distributed random variables with mean 0 and variance 1, the Hartman—Wintner theorem implies

$$\overline{\lim_{n\to\infty}} \frac{|\zeta(n)|}{\sqrt{n \log \log n}} = \sqrt{2} \quad \text{a.s.}$$

Since β_n is an integer-valued random variable and $\beta_n \geqq n$, from this we can conclude

$$\overline{\lim_{n\to\infty}} \frac{|\zeta(\beta_n)|}{\sqrt{\beta_n \log \log \beta_n}} \leqq \sqrt{2} \quad \text{a.s.} \tag{9}$$

By (6) we have

$$\frac{\beta_n}{ng(n)} \to 0 \quad \text{a.s.} \tag{10}$$

that is, with probability one for any n sufficiently large

$$e < n \leqq \beta_n \leqq ng(n)$$

therefore

$$0 < \log \log \beta_n \leqq \log \log ng(n)$$

which — using also (10) — gives

$$\sqrt{\frac{\beta_n}{ng(n)}} \sqrt{\frac{\log \log \beta_n}{\log \log ng(n)}} \to 0 \quad \text{a.s.} \tag{11}$$

Now (9) and (11) imply (8), and this completes the proof.

§. 2.

In this § we show that a slight modification makes our method applicable for the construction of other "pathological" sequences of independent — and also non-independent — random variables. An important class of non-independent random variables, where our method functions well is the class of multiplicatively orthogonal sequences introduced by G. ALEXITS. A sequence $\xi_1, \xi_2, \dots$ of random variables is called a strongly multiplicative system if $\mathsf{E}\xi_i = 0$ $(i=1, 2, \dots)$ and the products of the form $\xi_{i_1} \xi_{i_2} \dots \xi_{i_k} (i_1 < i_2 < \dots < i_k, k = 1, 2, \dots)$ constitute an orthogonal sequence. If these functions constitute not only an orthogonal but an orthonormed system, then the sequence $\xi_1, \xi_2, \dots$ is called an equinormed strongly multiplicative system (briefly ESMS). Such sequences are very interesting from probabilistic point of view, because they have similar properties as the independent random variables, e.g. if $\xi_1, \xi_2, \dots$ is a uniformly bounded ESMS, then the central limit theorem and the law of the iterated logarithm (this latter in the weaker form

$$\overline{\lim_{n\to\infty}} \frac{\xi_1 + \dots + \xi_n}{\sqrt{2n \log \log n}} \leq 1 \quad \text{a.s.})$$

holds for the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ (see [5], [6]). What makes our method applicable for the construction of multiplicative systems is the fact, that without the assumption of the mutual independence of the α_k-s in Lemma 1, the random variables defined by (3) are generally not independent, but the multiplicative orthogonality remains valid. More exactly we have the next

LEMMA 1*a*. *Let $\zeta(t)$ be a standard Brownian motion $(\zeta(0)=0)$, let $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ be nonnegative discret random variables such that*

$$\mathsf{E}\alpha_i^k < +\infty \qquad (i, k = 1, 2, \dots) \tag{12}$$

and suppose that $\zeta(t)$ is independent of the α_k-s. Then the sequence $\xi_1, \xi_2, \dots$ defined by (3) *is a strongly multiplicative system which satisfies the following relations:*

I. $$\mathsf{E}|\xi_i|^k < +\infty \qquad (i, k = 1, 2, \dots)$$

II. $$\mathsf{E}\xi_i^{2k} = \mathsf{E}\zeta(1)^{2k} \cdot \mathsf{E}\alpha_i^k \qquad (i, k = 1, 2, \dots)$$

in particular

$$\mathsf{E}\xi_i^2 = \mathsf{E}\alpha_i \qquad (i = 1, 2, \dots)$$

III. $$\mathsf{E}\xi_{i_1}^2 \xi_{i_2}^2 \dots \xi_{i_k}^2 = \mathsf{E}\alpha_{i_1}\alpha_{i_2} \dots \alpha_{i_k} \qquad (i_1 < i_2 < \dots < i_k,\ k = 1, 2, \dots).$$

Therefore if we assume

$$\mathsf{E}\alpha_{i_1}\alpha_{i_2} \dots \alpha_{i_k} = 1 \qquad (i_1 < i_2 < \dots < i_k,\ k = 1, 2, \dots) \tag{13}$$

then the sequence $\xi_1, \xi_2, \dots$ is an ESMS.

If $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ are identically distributed, then $\xi_1, \xi_2, \dots$ have the same property; if $\alpha_1, \alpha_2, \dots$ are independent, then $\xi_1, \xi_2, \dots$ are independent too.

We omit the proof, because it involves only simple computations similar to the proof of Lemma 1. We only stress an important feature of Lemma 1a: namely the fact, that in the construction of multiplicative systems we have a much greater freedom in the choice of $\alpha_1, \alpha_2, \dots$, than in the case of independent random variables. In fact, the independence of $\zeta(t)$ and the α_k-s can always be guaranteed by the appropriate choice of the probability space; (12) is a very weak restriction, and sometimes we can easily satisfy even condition (13) (see below).

To illustrate the usefulness of Lemma 1a, we present two simple applications of it.

I. There exists a (not uniformly bounded) ESMS $\xi_1, \xi_2, \dots$ such that

$$\overline{\lim_{n\to\infty}} \frac{\xi_1 + \dots + \xi_n}{\sqrt{2n \log \log n}}$$

is not a constant a.s.

II. Let $0 \leqq c \leqq 1$ and let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive integers such that $\lim_{n\to\infty} a_n = +\infty$ and $a_{n+1} - a_n = 0$ or 1 for every n. Then there exists a sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ of independent random variables such that

$$\mathsf{E}\xi_i = 0 \quad \mathsf{E}\xi_i^2 = 1 \quad \mathsf{E}|\xi_i|^k < +\infty \qquad (i, k = 1, 2, \ldots) \tag{14}$$

and with probability one

$$\overline{\lim_{n\to\infty}} \frac{\xi_1 + \cdots + \xi_n}{\sqrt{2a_n \log\log a_n}} = c. \tag{15}$$

This result — in the case $c=1$ — essentially means, that if $f(n)$ is an increasing function such that $f(n) \leqq \sqrt{2n \log\log n}$ and $f(n)$ tends to $+\infty$ "smoothly enough" (i.e. it has only little jumps), then there exists a sequence $\xi_1, \xi_2, \ldots$ of independent random variables such that (14) holds and

$$\overline{\lim_{n\to\infty}} \frac{\xi_1 + \cdots + \xi_n}{f(n)} = 1 \quad \text{a.s.}$$

The case $c<1$ is interesting only if $a_n = n$ $(n=1, 2, \ldots)$ when our example shows that in the independent case under conditions (14) the expression

$$\overline{\lim_{n\to\infty}} \frac{\xi_1 + \cdots + \xi_n}{\sqrt{2n \log\log n}}$$

can be arbitrary constant c such that $0 \leqq c \leqq 1$. It would be interesting to know, whether a similar result holds for every $c > 1$.

Since our method depends on (3), in order to construct the desired sequences, we have to define only the corresponding sequences $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ (and assure, that $\zeta(t)$ is independent of the α_k-s). Now in the first case let $\alpha_1, \alpha_2, \ldots$ be defined by

$$\alpha_n(\omega) = \begin{cases} \beta_n(\omega) & \text{if } \omega \in A \\ \gamma_n(\omega) & \text{if } \omega \in \bar{A} \end{cases} \qquad (n = 1, 2, \ldots)$$

where A is an arbitrary event with $0 < \mathsf{P}(A) < 1$; $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ resp. $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ are two sequences of independent random variables on the sets A resp. $\bar{A}$ (with respect to the conditional probability measures P_A resp. $\mathsf{P}_{\bar{A}}$) such that

$$\mathsf{P}_A(\beta_n = 0) = \mathsf{P}_A(\beta_n = 2) = \frac{1}{2}$$

$$(n = 1, 2, \ldots)$$

$$\mathsf{P}_{\bar{A}}(\gamma_n = 0) = 1 - \frac{1}{2n^2}, \quad \mathsf{P}_{\bar{A}}(\gamma_n = 2n^2) = \frac{1}{2n^2}.$$

In the second case let $\alpha_1, \alpha_2, \ldots$ be independent random variables such that

$$\mathsf{P}\big(\alpha_n = c^2(a_n - a_{n-1})\big) = 1 - \frac{1}{2n^2}$$

$$(n = 1, 2, \ldots, \; a_0 \overset{\text{def}}{=} 0)$$

$$\mathsf{P}\big(\alpha_n = 2n^2 - (2n^2 - 1)c^2(a_n - a_{n-1})\big) = \frac{1}{2n^2}.$$

It is not difficult to prove — using our Lemma 1a, the HARTMAN—WINTNER theorem, and other simple facts — that the corresponding sequences defined by (3) have the desired properties; in the case of the first sequence we can prove that

$$\overline{\lim_{n\to\infty}} \frac{\xi_1+\cdots+\xi_n}{\sqrt{2n\log\log n}} = \begin{cases} 1 & \text{a.s. on the set } A \\ 0 & \text{a.s. on the set } \bar{A} \end{cases}$$

The proof of these facts is left to the reader (we only mention, that we can easily verify (15) on the set of positive probability

$$\{\alpha_n = c^2(a_n - a_{n-1}) \text{ for every } n \geqq 1\}$$

thus the a.s. validity of (15) follows from the zero-one law).

It is necessary to remark that we can obtain these two counterexamples by a more elementary way, without using the concept of the Brownian motion; however, these examples are perhaps more instructive, since they illustrate the applicability of a general method.

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(Received February 2. 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 199—200.

COMPACTIFICATIONS AND A DUAL OF COMPACT SPACES

by

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We give a new proof of

THEOREM 1. *Let X be a completely regular space, $\mathscr{L}\subset C(X)$ (where $C(X)$ is the set of all bounded continuous real valued functions over X) a B-lattice, containing the constant functions and separating X. Then there is a compactification $\bar{X}$ of X, over which exactly the functions in $\mathscr{L}$ can be extended (from X).*

PROOF. We consider $E=\{f, f\in\mathscr{L}, 0\leqq\inf f\leqq\sup f\leqq 1\}$. For $f_1, f_2\in E$ denote $S(f_1,f_2)$ the closure of the range of the vector-valued function (f_1,f_2). We define the equivalence relation $f_1\sim f_2$ $(f_i\in E)$: $(x_1,x_2)\in S(f_1,f_2)$ implies $x_1=x_2=0$ or $x_1>0$, $x_2>0$. Denote $\tilde{f}$ the equivalence class containing f. Define $\tilde{f}_1<\tilde{f}_2$ whenever $f_1\in\tilde{f}_1$, $f_2\in\tilde{f}_2$, $(x_1,x_2)\in S(f_1,f_2)$, $x_2=0$ imply $x_1=0$. It follows then easily that $H=\{\tilde{f}, f\in E\}$ is a distributive lattice with maximal element $\tilde{1}$, and $\min(\tilde{f}_1,\tilde{f}_2)=\widetilde{\min(f_1,f_2)}$, $\max(\tilde{f}_1,\tilde{f}_2)=\widetilde{\max(f_1,f_2)}$.

Denote by $\bar{X}$ the set of the maximal ideals of H. $g\in\bar{X}$, $\tilde{f}_n\in g$, $f_n\to f$ uniformly imply $\tilde{f}\in g$. For, if $\tilde{f}\notin g$, then $[g,\tilde{f}]$, the ideal generated by g and $\tilde{f}$ should contain $\tilde{1}$, i.e. there exists an $\tilde{f}'\in g$, such that $\widetilde{\max(f',f)}=\tilde{1}$ i.e. $\inf[\max(f',f)]>0$, thus the last three statements hold for f_n, n large enough.

For $x\in X$ define $x^*=\{\tilde{f}, \tilde{f}\in H, f(x)=0\}$. $x_1\neq x_2$ implies $x_1^*\neq x_2^*$. $x^*\in\bar{X}$ since x^* is an ideal and for $f(x)>0$ $[x^*,\tilde{f}]\ni\tilde{1}$. $K_0=\{\{g, g\in\bar{X}, \tilde{f}\in g\}, \tilde{f}\in H\}$. K_0 is a lattice since $F_i=\{g,\tilde{f}_i\in g\}\in K_0$ $i=1,2$ implies $F_1\cap F_2=\{g, \widetilde{\max(f_1,f_2)}\in g\}\in K_0$, $F_1\cup F_2=\{g, \widetilde{\min(f_1,f_2)}\in g\}\in K_0$. K_0 is a closed base of a topology. Denote K the set of the closed sets of the topology generated by K_0.

$\bar{X}$ is a T_1 space. $\bar{X}$ is compact, since for $F_\gamma=\{g,\tilde{f}_\gamma\in g\}$ $\bigcap\limits_{\gamma} F_\gamma=\emptyset$ if and only if $[\tilde{f}_\gamma]\ni\tilde{1}$ thus these statements hold for a finite subset of γ-s. $X^*=\{x^*, x\in X\}$ is dense in $\bar{X}$, since $X^*\subset\{g,\tilde{f}\in g\}=F$ implies $f=0$, so $F=\bar{X}$. The correspondence $x\to x^*$ is a homeomorphism, since a closed base in X^* is $\{\{g,\tilde{f}\in g\}\cap X, f\in E\}=\{f^{-1}(0), f\in E\}$.

We show that $f\in\mathscr{L}$ can be extended to every $g_0\in\bar{X}$ (X identified with X^*). Let $0<\varepsilon<1$, we choose $c_1,\dots,c_n\in R$ so that $\bigcup\limits_{1}^{n}(c_i-\varepsilon, c_i+\varepsilon)$ contains the closure of the range of f. $f_c=\max(0,\varepsilon-|f-c|)$. $\bigcap\limits_{1}^{n}\{g,\tilde{f}_{c_i}\in g\}=\{g, \widetilde{\max f_{c_i}}\in g\}=\emptyset$. Thus for an i $g_0\in\{g,\tilde{f}_{c_i}\notin g\}=G$, G being open and $f(X\cap G)\subset(c-\varepsilon, c+\varepsilon)$.

$\bar{X}$ is completely regular. Let namely $F_0=\{g,\tilde{f}_0\in g\}$ and $\bar{f}_0$ the extension of f_0 to $\bar{X}$. We show $\bar{f}_0^{-1}(0)=F_0$. Let $\tilde{f}_0\in g_0$. Let $g_0\in\{g,\tilde{f}\notin g\}=G'$. If $\inf f_0(G'\cap X)>0$

then $\bar{f} < \bar{f}_0$ contrary to $\bar{f} \notin g_0$. $G' \ni g$ is arbitrary, so $\bar{f}_0(g_0)=0$. Let $\bar{f}_0 \notin g_0$. Then for some $\varepsilon > 0$ $\bar{f} \notin g_0$ where $f = \max(f_0 - \varepsilon, 0)$. $g_0 \in \{g, \bar{f} \notin g\} = G''$ and $\inf f_0(G'' \cap X) > 0$ thus $\bar{f}_0(g_0) > 0$.

$C(\bar{X})$ consists of the extensions of the functions in $\mathscr{L}$ because of the Stone—Weierstrass theorem.

A lattice H is a C-lattice if it is a distributive lattice with minimal and maximal elements (denoted by 0 and 1)

1. for every $u_1, u_2 \in H$, $u_1 \not\equiv u_2$ there exists a maximal ideal $g \subset H$, $u_2 \in g$, $u_1 \notin g$

2. if $u_1, u_2 \in H$, $u_1 \cup u_2 = 1$ then there exist $u_3, u_4, u_3 \cap u_4 = 0$, $u_3 \cup u_2 = u_4 \cup \cup u_1 = 1$

3. if $u \in H$ there exist $u_1, \dots, u_n, \dots \in H$ such that for a maximal ideal g $u \notin g \Leftrightarrow \exists n$, $u_n \in g$

4. if h is an intersection of maximal ideals of H, $u_n \in H$ and for a maximal ideal g $h \not\subset g \Leftrightarrow \exists n$, $u_n \in g$ then there exists a $u \in H$ that $h = [u]$.

THEOREM 2. *A dual of the category of compact T_2 spaces is the category of C-lattices.*

PROOF. For a compact T_2 space X H is the dual of the lattice of its closed G_δ sets. For a C-lattice H X is constructed as in [1], p. 173. X is compact T_1 this is proved similarly as in Theorem 1 for $\bar{X}$.

Condition 1 ensures that to different $u_1, u_2 \in H$ there correspond different closed sets of X. Condition 2 ensures that X is T_4. Conditions 3 and 4 ensure that the closed base of X obtained from H consists just of the closed G_δ sets.

To the continuous maps of the compact spaces there exist maps of the C-lattices corresponding in a natural way.

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(Received February 9, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 201—209.

ОБ ОЦЕНКАХ ПАРАМЕТРОВ ДИФФУЗИОННЫХ ПРОЦЕССОВ

А. А НОВИКОВ

0. Целью этой работы является изучение свойств оценок максимального правдоподобия (м. п.) параметров одного класса одномерных и двумерных процессов диффузионного типа. В частности, получены формулы для смещения и среднеквадратичного уклонения оценок м. п., применение которых иллюстрируется на примере процессов с линейным сносом. Кроме того, для оценки м. п. $\hat{\omega}_t(\xi)$ фазового параметра ω двумерного линейного процесса ξ_t (см. п. 6) доказывается следующий любопытный факт: случайная величина

$$(\hat{\omega}_t(\xi)-\omega)\Big(\int_0^t|\xi_s|^2\,ds\Big)^{1/2}$$

имеет в точности нормальное $N(0, 1)$ распределение. Этот факт был подмечен еще в работе [1], но его доказательство, по-видимому, приводится впервые.

1. Пусть на вероятностном пространстве $(\Omega, \mathscr{F}, \mathsf{P})$ задан одномерный случайный процесс ξ_t, $t\geqq 0$, удовлетворяющий стохастическому уравнению (определения и обозначения см. [6])

$$d\xi_t = f(\lambda, t, \xi)\,dt + dw_t, \tag{1}$$

где w_t — винеровский процесс с $w_0=\xi_0$, $\mathsf{M}w_t=\xi_0$, $\mathsf{D}w_t^2=t$; $f(\lambda, t, x)=f(\lambda, t, y)$, если $x_s=y_s$, $s\leqq t$, x и $y\in C$ — пространству непрерывных функций; λ — параметр, подлежащий оценке.

Обозначим μ_λ — меру процесса ξ_t при фиксированном значении параметра λ, μ_0 — винеровскую меру. Если

$$\mathsf{P}_\lambda\Big(\int_0^t f^2(\lambda, s, \xi)\,ds<\infty\Big) = 1, \tag{2}$$

то μ_λ абсолютно непрерывна относительно винеровской меры и

$$\frac{d\mu_\lambda}{d\mu_0}(\xi_0^t) = \exp\Big\{\int_0^t f(\lambda, s, \xi)\,d\xi_s - 1/2\int_0^t f^2(\lambda, s, \xi)\,ds\Big\}. \tag{3}$$

(Этот результат сообщен автору А. Н. Ширяевым.) Если, кроме того, выполняется условие

$$\mathsf{P}_0\Big(\int_0^t f^2(\lambda, s, w)\,ds<\infty\Big) = 1, \tag{4}$$

то, как следует из работы И. В. Гирсанова [5], меры μ_λ и μ_0 эквивалентны ($\mu_\lambda \sim \mu_0$).

2. Далее мы будем рассматривать только такие диффузионные процессы, у которых снос линейно зависит от оцениваемого параметра, т. е.

$$f(\lambda, t, \xi) = \lambda g(t, \xi)$$

и будем предполагать выполнение условий (2) и (4) для функции $g(t, \xi)$. Тогда из (3) следует, что

$$\frac{d\mu_\lambda}{d\mu_0}(\xi_0^t) = \exp\left\{\lambda \int_0^t g(s, \xi)\, d\xi_s - \lambda^2/2 \int_0^t g^2(s, \xi)\, ds\right\}.$$

Отсюда легко находим оценку м. п. параметра λ:

$$\hat{\lambda}_t(\xi) = \left(\int_0^t g^2(s, \xi)\, ds\right)^{-1} \int_0^t g(s, \xi)\, d\xi_s. \tag{5}$$

В работах [3], [6] изучались асимптотические свойства оценок м. п. параметров диффузионных процессов. В этом разделе мы выведем точные формулы для смещения и среднеквадратического уклонения оценки (5). При выводе этих формул будем предполагать, что параметр λ принимает значения из некоторого отрезка $[a, b]$ и для всех $\lambda \in [a, b]$

$$\mathsf{M}_\lambda \left(\int_0^t g^2(s, \xi)\, ds\right)^{-2} < \infty.\text{*} \tag{6}$$

Кроме того, предположим, что существует интегрируемая по мере μ_0 функция $H(t, w)$, независящая от λ, такая, что для всех $\lambda \in [a, b]$

$$\hat{\lambda}_t^2(w) \exp\left\{\lambda \int_0^t g(s, w)\, dw_s - \lambda^2/2 \int_0^t g^2(s, w)\, ds\right\} \leqq H(t, w). \tag{7}$$

Эти предположения носят технический характер и далее под условиями (6) и (7) можно подразумевать любые другие предположения, гарантирующие возможность дифференцирования под знаком математического ожидания.

Утверждение 1. *Если выполняются* (6) *и* (7), *то*

$$\mathsf{M}_\lambda(\hat{\lambda}_t(\xi) - \lambda) = \frac{\partial}{\partial \lambda} \mathsf{M}_\lambda \left(\int_0^t g^2(s, \xi)\, ds\right)^{-1}. \tag{8}$$

$$\mathsf{M}_\lambda(\hat{\lambda}_t(\xi) - \lambda)^2 = \frac{\partial^2}{\partial \lambda^2} \mathsf{M}_\lambda \left(\int_0^t g^2(s, \xi)\, ds\right)^{-2} + \mathsf{M}_\lambda \left(\int_0^t g^2(s, \xi)\, ds\right)^{-1}. \tag{9}$$

* Здесь и далее знак $\mathsf{M}_\lambda(\cdot)$ означает интегрирование по мере P при данном значении параметра λ, от которого зависит ξ_t.

Доказательство формулы (8) сводится к следующей цепочке выкладок, справедливых при перечисленных выше условиях

$$\mathsf{M}_\lambda(\hat\lambda_t(\xi)-\lambda) = \mathsf{M}_0(\hat\lambda_t(w)-\lambda)\exp\left\{\lambda\int_0^t g(s,w)\,dw_s-\lambda^2/2\int_0^t g^2(s,w)\,ds\right\} =$$

$$= \mathsf{M}_0\Big(\int_0^t g^2(s,w)\,ds\Big)^{-1}\Big(\int_0^t g(s,w)\,dw_s-\lambda\int_0^t g^2(s,w)\,ds\Big)\times$$

$$\times\exp\left\{\lambda\int_0^t g(s,w)\,dw_s-\lambda^2/2\int_0^t g^2(s,w)\,ds\right\} = \frac{\partial}{\partial\lambda}\mathsf{M}_\lambda\Big(\int_0^t g^2(s,\xi)\,ds\Big)^{-1}.$$

Аналогично выводится соотношение (9).

Если известно преобразование Лапласа

$$\mathsf{M}_\lambda\exp\left\{-\mu\int_0^t g^2(s,\xi)\,ds\right\}\equiv\psi(\mu;\lambda;t),\qquad \mu\geqq 0, \tag{10}$$

то отрицательные моменты, входящие в соотношения (8) и (9), вычисляются следующим образом:

$$\mathsf{M}_\lambda\Big(\int_0^t g^2(s,\xi)\,ds\Big)^{-k} = \int_0^\infty \mu^{k-1}\psi(\mu;\lambda;t)\,d\mu,\qquad k=1,2.$$

Отметим, что функцию $\psi(\mu;\lambda;t)$ можно найти иногда как решение некоторого уравнения в частных производных (см., например, [6, § 11]).

Замечание 1. Если известно преобразование Лапласа (10), то из него можно получить двумерное преобразование Лапласа

$$\mathsf{M}_\lambda\exp\left\{-\nu\int_0^t g(s,\xi)\,d\xi_s-\mu\int_0^t g^2(s,\xi)\,ds\right\}\equiv\varphi(\nu;\mu;\lambda;t).$$

В самом деле, в силу предположений (2) и (4) меры $\mu_\lambda\sim\mu_0$ и

$$\varphi(\nu;\mu;\lambda;t) = \mathsf{M}_0\exp\left\{(\lambda-\nu)\int_0^t g(s,w)\,dw_s-1/2\varkappa^2\int_0^t g^2(s,w)\,ds\right\} =$$

$$= \mathsf{M}_{\lambda-\nu}\exp\left\{-1/2(\varkappa^2-(\lambda-\nu)^2)\int_0^t g^2(s,\xi)\,ds\right\} = \psi\big(1/2\varkappa^2-1/2(\lambda-\nu)^2;\lambda-\nu;t\big),$$

где $\varkappa\equiv(\lambda^2+2\mu)^{1/2}$ и $\varkappa^2\geqq(\lambda-\nu)^2$, так как в (10) $\mu\geqq 0$. С помощью функции $\varphi(\nu;\mu;\lambda;t)$ можно тоже вычислить смещение и среднеквадратичное уклонение оценки (5), но этот способ менее удобен при численных вычислениях, чем описанный выше.

3. В этом разделе мы проиллюстрируем полученные выше результаты на примере процесса ξ_t, имеющего линейный снос $g(t,\xi)=-\xi_t$, т. е. удовлетворяющего уравнению

$$d\xi_t = -\lambda\xi_t\,dt+dw_t,\quad \xi_0=\text{const}. \tag{11}$$

Для этого процесса условия (2) и (4) заведомо выполняются и из (3) вытекает, что

$$\frac{d\mu_\lambda}{d\mu_0}(\xi_0^t) = \exp\Big\{-\lambda\int_0^t \xi_s\,d\xi_s - \lambda^2/2\int_0^t \xi_s^2\,ds\Big\}.$$

Оценка м. п. параметра λ в этом случае имеет вид

(12) $$\hat\lambda_t(\xi) = -\Big(\int_0^t \xi_s^2\,ds\Big)^{-1}\int_0^t \xi_s\,d\xi_s.$$

Отметим также, что для рассматриваемого процесса

(13) $$\int_0^t \xi_s\,d\xi_s = 1/2(\xi_t^2-\xi_0^2-t).$$

(Это соотношение легко проверяется по формуле Ито.)

Как было показано в п. 2, для вычисления смещения и среднеквадратичного уклонения оценки (12) надо знать преобразование Лапласа

(14) $$\mathsf{M}_\lambda \exp\Big\{-\mu\int_0^t \xi_s^2\,ds\Big\} \equiv \psi_1(\mu;\lambda;t), \qquad \mu\geqq 0.$$

Ниже будут описаны два метода вычисления функции $\psi_1(\mu;\lambda;t)$ (без решения уравнения в частных производных), а сейчас мы сформулируем соответствующий результат.

Утверждение 2. *Для процесса ξ_t, удовлетворяющего уравнению* (11), *при всех $\lambda\in(-\infty,\infty)$*

(15) $$\psi_1(\mu;\lambda;t) = (\operatorname{ch}\varkappa t+\lambda/\varkappa\operatorname{sh}\varkappa t)^{-1/2}\exp\{\lambda t/2-\mu\xi_0^2(\varkappa\operatorname{cth}\varkappa t+\lambda)^{-1}\},$$

где $\varkappa = (\lambda^2+2\mu)^{1/2}$.

Для доказательства этого утверждения нам понадобится следующая лемма, вытекающая из результатов работы Камерона и Мартина [4] (в принятых нами обозначениях).

Лемма 1. (Камерон—Мартин). *Пусть неслучайная функция $p(s)$ определена на* $[0, T]$ *и неотрицательна. Тогда при* $\mu\geqq 0$

$$\mathsf{M}_0\exp\Big\{-\mu\int_0^T p(s)w_s^2\,ds\Big\} = \big(y(0)\big)^{1/2}\exp\{w_0^2y'(0)/2y(0)\},$$

где $y(0)$ и $y'(0)$ находятся из уравнения

(16) $$y''(s)-2\mu p(s)y(s)=0,\quad y(T)=1,\quad y'(T)=0.$$

Для того, чтобы воспользоваться этим результатом, заметим, что процесс ξ_t, удовлетворяющий уравнению (11), представляется в виде

(17) $$\xi_t = \exp(-\lambda t)\Big(\int_0^t \exp(\lambda s)\,dw_s+\xi_0\Big).$$

Если сделать замену времени (см., например, [9, гл. VI, Лемма 1.1])

$$x(t) = \int_0^t \exp(2\lambda s)\, ds, \tag{18}$$

то стохастический интеграл $\int_0^t \exp(\lambda s)\, dw_s$ преобразуется в некоторый винеровский процесс $\tilde{w}_s$ $(0 \leqq s \leqq x(t),\ \tilde{w}_0 = 0)$. Делая подстановки (17) и (18) в (14), получим

$$\psi_1(\mu;\lambda;t) = \tilde{\mathsf{M}}_0 \exp\Big\{-\mu \int_0^{x(t)} (2\lambda s + 1)^{-2} (\tilde{w}_s + \xi_0)^2\, ds\Big\}.$$

Теперь осталось решить уравнение (16) с $p(s) = (2\lambda s + 1)^{-2}$ и $T = x(t)$. Наконец сделав замену аргумента в (16), обратную к (18), получим, что $y(0)$ и $y'(0)$ находятся из уравнения

$$y''(s) - 2\lambda y'(s) - 2\mu y(s) = 0; \quad y(t) = 1, \quad y'(t) = 0.$$

Решив это уравнение, с помощью леммы 1 получим (15).

Замечание 2. Формула (15) справедлива при всех λ. В частности, при $\lambda = 0$ и $\xi_0 = 0$ из нее следует результат, полученный в работе [4]:

$$\mathsf{M}_0 \exp\Big\{-\mu \int_0^t w_s^2\, ds\Big\} = (\operatorname{ch} \sqrt{2\mu}\, t)^{-1/2}.$$

Если же в (11) и, соответственно, в (14) считать $\lambda > 0$ и ξ_0 нормально распределенными $N(0, 1/2\lambda)$ (это соответствует стационарному гауссовскому марковскому процессу с корреляционной функцией, равной $1/2\lambda \exp\{-\lambda|\tau|\}$), то после соответствующего усреднения (15) получим формулу, согласующуюся с результатом М. Арато [2, формула 11].

Замечание 3. Способ, которым получено (15), применим для вычисления функции $\psi(\mu; \lambda; t)$ и тогда, когда снос процесса ξ_t имеет вид $g(t, \xi) = q(t)\xi_t$, где $q(t)$ — неслучайная функция. При этом решение уравнения (1) имеет представление, аналогичное (17), и можно воспользоваться леммой 1.

Замечание 4. Формулу (15) можно получить также следующим простым способом, в котором существенно используется формула (13). В силу абсолютной непрерывности мер μ_λ и μ_0

$$\psi_1(\mu; \lambda; t) = \mathsf{M}_0 \exp\Big\{-\lambda \int_0^t w_s\, dw_s - \varkappa^2/2 \int_0^t w_s^2\, ds\Big\} = \tag{19}$$

$$= \mathsf{M}_\varkappa \exp\Big\{-(\lambda + \varkappa) \int_0^t \xi_s\, d\xi_s\Big\}, \qquad \varkappa = (\lambda^2 + 2\mu)^{1/2}.$$

Теперь заметим, что процесс ξ_t гауссовский, причем

$$\mathsf{M}_\varkappa \xi_t = \exp(-\varkappa t)\xi_0; \quad \mathsf{D}_\varkappa \xi_t = \exp(-2\varkappa t) \int_0^t \exp(2\varkappa s)\, ds.$$

Легко видеть, что в силу соотношения (13) правая часть (19) сводится к классическому интегралу, после вычисления которого получим (15).

Используя результат утверждения 1, можно проверить выполнение условий (6), (7) и на основании утверждения 2 найти асимптотические разложения смещения и среднеквадратичного уклонения оценки (12). Например, справедливо следующее следствие из утверждений 1 и 2, которое выводится стандартными методами асимптотического анализа (см. [10]).

Утверждение 3. *Если процесс* ξ_t *удовлетворяет уравнению* (11) *с* $\xi_0=0$, *то при* $t = const > 0$

$$\mathsf{M}_\lambda(\hat{\lambda}_t(\xi)-\lambda) = \begin{cases} 2/t\bigl(1-3/(4\lambda t)+O(1/\lambda^2)\bigr), & \lambda\to\infty,\\ 1{,}78/t\bigl(1+2{,}34\lambda t+O(\lambda^2)\bigr), & \lambda\to 0,\\ (4\pi|\lambda|^3 t)^{1/2}\exp(-|\lambda|t)\bigl(1+O(1/\lambda)\bigr), & \lambda\to-\infty,\end{cases}$$

$$\mathsf{M}_\lambda(\hat{\lambda}_t(\xi)-\lambda)^2 = \begin{cases} 2\lambda/t\bigl(1+13(2\lambda t)+O(1/\lambda^2)\bigr), & \lambda\to\infty,\\ 13{,}3/t^2\bigl(1+0{,}156\lambda t+O(\lambda^2)\bigr), & \lambda\to 0,\\ (4\pi|\lambda|^5 t)^{1/2}\exp(-|\lambda|t)\bigl(1+O(1/\lambda)\bigr), & \lambda\to-\infty.^{*)}\end{cases}$$

4. Результаты первого и второго разделов настоящей работы легко распространить на случай двумерного процесса ξ_t, компоненты которого ζ_t и η_t удовлетворяют стохастическому уравнению

$$d\zeta_t = \bigl(\lambda u(t,\xi)+\omega v(t,\xi)\bigr)\,dt+d\varrho_t,$$
$$d\eta_t = \bigl(\omega u(t,\xi)-\lambda v(t,\xi)\bigr)\,dt+d\chi_t, \tag{20}$$

где ϱ_t и χ_t — независимые винеровские процессы с начальными условиями $\varrho_0=\zeta_0$, $\chi_0=\eta_0$; λ и ω — параметры, подлежащие оценке. (В комплексных обозначениях система (20) примет вид уравнения (1).) Оценки м. п. параметров λ и ω в этом случае имеют вид

$$\hat{\lambda}_t(\xi) = \Bigl(\int_0^t \bigl(u^2(s,\xi)+v^2(s,\xi)\bigr)\,ds\Bigr)^{-1}\Bigl(\int_0^t u(s,\xi)\,d\zeta_s-\int_0^t v(s,\xi)\,d\eta_s\Bigr),$$
$$\hat{\omega}_t(\xi) = \Bigl(\int_0^t \bigl(u^2(s,\xi)+v^2(s,\xi)\bigr)\,ds\Bigr)^{-1}\Bigl(\int_0^t u(s,\xi)\,d\eta_s+\int_0^t v(s,\xi)\,d\zeta_s\Bigr), \tag{21}$$

а их смещения и среднеквадратичные уклонения вычисляются по формулам (8) и (9), где только надо положить $g^2(t,\xi)=u^2(t,\xi)+v^2(t,\xi)$ и брать производные по соответствующему параметру.

5. Рассмотрим теперь систему (20) в том случае, когда $u(t,\xi)=-\zeta_t$, $v(t,\xi)=\eta_t$, т. е. линейное стохастическое уравнение

$$d\zeta_t = (-\lambda\zeta_t+\omega\eta_t)\,dt+d\varrho_t,$$
$$d\eta_t = (-\omega\zeta_t-\lambda\eta_t)\,dt+d\chi_t. \tag{22}$$

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* Значения констант, данных с тремя знаками, являются приближенными. Подробно метод вычисления этих асимптотических разложений описан в работе: Новиков, А.А.: *Стохастические интегралы и последовательное оценивание, Диссертация*, МИАН, Москва, 1972, (замечание при корректуре).

Найдем преобразование Лапласа

$$\mathsf{M}_{\lambda,\omega} \exp\left\{-\mu \int_0^t |\xi_s|^2 \, ds\right\} \equiv \psi_2(\mu; \lambda; t),$$

где $|\xi_s|^2 = \zeta_t^2 + \eta_t^2$, $\mu \geqq 0$, а ξ_0 будем предполагать не случайным.

Утверждение 4. *Для процесса ξ_t, удовлетворяющего уравнению* (22), *при всех $\lambda \in (-\infty, \infty)$ и $\omega \in (-\infty, \infty)$*

$$\psi_2(\mu; \lambda; t) = (\operatorname{ch} \varkappa t + \lambda/\varkappa \operatorname{sh} \varkappa t)^{-1} \exp\{\lambda t - \mu |\xi_0|^2 (\varkappa \operatorname{cth} \varkappa t + \lambda)^{-1}\}.$$

Для доказательства этого утверждения нам понадобится следующая лемма о преобразовании стохастических интегралов. (В самом общем виде для многомерных стохастических интегралов эта лемма сформулирована, например, в [8].)

Лемма 2. *Пусть ϱ_t и χ_t — независимые винеровские процессы и пусть случайные функции* $x(t, \omega)$ и $y(t, \omega)$, *принадлежащие области определения стохастических интегралов, таковы, что п. в.*

$$x^2(t, \omega) + y^2(t, \omega) = 1$$

Тогда процессы β_t и γ_t, определяемые соотношениями

$$d\beta_t = x(t, \omega)\, d\varrho_t + y(t, \omega)\, d\chi_t,$$

$$d\gamma_t = y(t, \omega)\, d\varrho_t - x(t, \omega)\, d\chi_t,$$

являются независимыми винеровскими.

Используя эту лемму, покажем, что распределение процесса $|\xi_t^2|$ не зависит от параметра ω. В самом деле, по формуле Ито

$$d|\xi_t|^2 = 2dt + 2\zeta_t \, d\zeta_t + 2\eta_t \, d\eta_t = 2(1 - \lambda |\xi_t|^2)\, dt + 2|\xi_t| \, d\beta_t, \tag{23}$$

где процесс β_t, определяемый соотношением

$$|\xi_t| d\beta_t = \zeta_t \, d\varrho_t + \eta_t \, d\chi_t. \tag{24}$$

является согласно лемме 2 некоторым новым винеровским процессом.

Теперь заметим, что решение уравнения (23) единственно (в сильном смысле), поскольку коэффициенты сноса и диффузии этого уравнения удовлетворяют условиям теоремы 1 из недавней работы Ямаду и Ватанабе [11]. (Функцию $\varrho(u)$, фигурирующую в указанной теореме, в рассматриваемом случае надо взять равной $u^{1/2}$.) Очевидно, из единственности решения уравнения (23) вытекает, что распределение $|\xi_t|^2$, а следовательно и $\int_0^t |\xi_s|^2 ds$, не зависит от параметра ω. Искомую функцию $\psi_2(\mu; \lambda; t)$ теперь можно найти, если в (20) положить $\omega = 0$. При этом компоненты процесса ξ_t будут независимыми. Воспользовавшись соотношением (15), получим результат утверждения 4.

6. Используя формулы (8) и (9) с учетом замечаний, сделанных в п. 4, теперь можно найти смещения и среднеквадратичные уклонения оценок $\hat{\lambda}_t(\xi)$ и $\hat{\omega}_t(\xi)$. В частности, получим

$$\mathsf{M}_{\lambda,\omega}\hat{\omega}_t(\xi) = \omega; \quad \mathsf{D}_{\lambda,\omega}\hat{\omega}_t(\xi) = \int_0^\infty \psi_2(\mu; \lambda; t)\, d\mu.$$

Кроме того, справедлив следующий факт.

Утверждение 5. (Арато—Колмогоров—Синай.) *Если процесс ξ_t удовлетворяет уравнению* (20), *то случайная величина*

$$(\hat{\omega}_t(\xi)-\omega)\Big(\int_0^t |\xi_s|^2\, ds\Big)^{1/2}$$

имеет нормальное $N(0, 1)$ распределение.

Доказательство. Из (22) и (21) после некоторых простых преобразований получим

$$(\hat{\omega}_t(\xi)-\omega)\Big(\int_0^t |\xi_s|^2\, ds\Big)^{1/2} = \Big(\int_0^t |\xi_s|^2\, ds\Big)^{-1/2}\int_0^t |\xi_s|\, d\gamma_s,$$

где процесс γ_t, определяемый соотношением

$$|\xi_t|\, d\gamma_t = \eta_t\, d\varrho_t - \xi_t\, d\chi_t,$$

является согласно лемме 2 винеровским и, что очень существенно, независящим от β_t из (24). Так как процесс $|\xi_t|^2$ удовлетворяет уравнению (23), то он является функционалом от β_t и, следовательно, процессы $|\xi_t|$ и β_t тоже независимы. Обозначим σ-алгебру событий, порожденную процессом $|\xi_t|$, через $\sigma(|\xi|_0^t)$. Тогда в силу известных свойств стохастических интегралов и условных математических ожиданий

$$\mathsf{M}_{\lambda,\omega}\exp\Big\{i\mu(\hat{\omega}_t(\xi)-\omega)\Big(\int_0^t |\xi_s|^2\, ds\Big)^{1/2}\Big\} =$$

$$= \mathsf{M}_{\lambda,\omega}\Big[\mathsf{M}_{\lambda,\omega}\exp\Big\{i\mu\Big(\int_0^t |\xi_s|^2\, ds\Big)^{-1/2}\int_0^t |\xi_s|\, d\gamma_s\Big\}\Big|\,\sigma(|\xi|_0^t)\Big] = \exp\{-\mu^2/2\}.$$

Это и требовалось показать.

7. В заключение несколько слов о возможных обобщениях.

На протяжение всей работы рассматривались процессы, имеющие единичный коэффициент диффузии. Однако, полученные формулы для смещения и среднеквадратичного уклонения рассматриваемых оценок м. п. легко распространяются и на случай произвольного коэффициента диффузии.

Из примеров, разобранных в работе, видно, что оценки м. п., вообще говоря, умеют смещение (случай, разобранный в п. 6, является исключительным), а их среднеквадратичное уклонение существенно зависит от оцениваемого параметра. Возникает вопрос, а существуют ли вообще несмещенные оценки параметров рассматриваемых процессов? Оказывается, что вопрос имеет положительное решение в классе последовательных оценок. О применении последовательного метода оценивания к диффузионным процессам, автор намерен сообщить в последующих работах.

Автор выражает глубокую благодарность А. Н. Ширяеву и Р. Ш. Липцеру за многочисленные полезные замечания, а также Т. С. Баршт за помощь в оформлении работы.

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(Поступила 16-ого февраля, 1971 г.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 211—224.

THE PROBLEM OF THE SECOND SEATING AND GENERALIZATIONS[1]

by

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1. Introduction. At the first session of a conference n people sit at a round table. At the second session they can be seated in

$$(1)\qquad \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-1-k)!+(-1)^n, \qquad n\geqq 3,$$

ways subject to the

(a) one-sided condition: no one (immediately) follows (in a clockwise direction) a person he followed at the first session. (Cf. [7], [8, p. 105], [5, p. 121].)

In section 2 we find that at the second session they can be seated in

$$(2)\qquad (n-1)!+2(-1)^n+\sum_{k=1}^{n-1}\sum_{i=1}^{k}(-1)^k 2^i\frac{n}{n-k}\binom{n-k}{i}\binom{k-1}{i-1}(n-1-k)!, \qquad n\geqq 3,$$

ways, subject to the more natural

(b) two sided condition: no one sits next to a person he sat next to at the first session.

These numbers are counts of certain classes of restricted permutations. In section 3 we describe generalizations and their second-seating interpretations. In section 4 we obtain explicit expressions for the generalizations; these complete the corresponding results we obtained [2] for straight-line permutations. Section 5 contains recurrence formulas for some simple cases, and section 6 deals with the situation where the first seating is at several tables while the second seating is at a single table.

2. Circular second-seatings. A permutation is a straight-line arrangement of $1, 2, \ldots, n$, while a circular permutation is an arrangement in a circle. Thus there are $n!$ permutations and only $(n-1)!$ circular permutations of $1, 2, \ldots, n$. A seating of n people in a straight line (i.e., row) is described by a permutation; a seating of n people at a round table is described by a circular permutation. A circular permutation is said to contain the 2-sequence (i.e., sequence of length 2) ij if, looking in the clockwise direction, j immediately follows i. Two 2-sequences are *consistent* if some permutation contains both of them; a collection of 2-sequences is consistent if each two of them are consistent. It is easy to see that the number of circular permutations containing a particular choice of k consistent 2-sequences is equal to $(n-k-1)!$ if $k<n$ and equal to 1 if $k=n$ (cf. [4]).

[1] This work was done, in part, when the authors were at the Summer Research Institute of the Canadian Mathematical Congress, Winnipeg, 1969.

The circular one-sided second-seating problem is that of finding the number of circular permutations containing none of the 2-sequences listed in the left column of Table I. Since any k of these 2-sequences are consistent, the Principle of Inclusion and Exclusion yields (1). Multiplying (1) by n gives the number of permutations π of the n residue classes mod n such that $\pi(i+1)-\pi(i) \not\equiv 1 \pmod n$, $i = 0, 1, \ldots, n-1$, [7].

1 2	2 1
2 3	3 2
⋮	⋮
$n-1$ n	n $n-1$
n 1	1 n

Table I

In the circular two-sided second-seating problem, the circular permutations contain none of the 2-sequences in Table I (both columns). Two 2-sequences in Table I are consistent provided they come from different rows, and if from adjacent rows, then from the same column, the first and last row being considered adjacent. Thus, letting $h(n:k)$, $1 \leqq k \leqq n$, $n \geqq 3$, denote the number of ways of choosing k consistent 2-sequences from Table I and using the Principle of Inclusion and Exclusion, the solution to the circular second-seating problem is

$$(3) \qquad (n-1)!+\sum_{k=1}^{n-1}(-1)^k h(n:k)(n-1-k)!+(-1)^n h(n:n), \qquad n \geqq 3.$$

Since $h(n:n)=2$, $n \geqq 3$, and

$$(4) \qquad h(n:k) = \sum_{i=1}^{k} 2^i \langle\langle n:k \mid i \rangle\rangle, \qquad 1 \leqq k < n,\ n \geqq 3,$$

where

$$(5) \qquad \langle\langle n:k \mid i \rangle\rangle = \frac{n}{n-k}\binom{n-k}{i}\binom{k-1}{i-1}, \qquad 1 \leqq k < n,\ 1 \leqq i,\ n \geqq 3,$$

is the number of ways of choosing, from n objects arranged in a circle, k of them so that the k-choice has exactly i parts ([1, p. 270] [2, p. 1252, expression (25)]), (3) (4) and (5) yield (2). Multiplying (2) by n gives the number of the permutations π of the n residue classes mod n such that $\pi(i+1)-\pi(i) \not\equiv 1$ or $n-1 \pmod n$, $i = 0, 1, \ldots, n-1$.

3. Generalizations and interpretations. The sequences in Table I are of length 2. There is an obvious generalization, for circular permutations, to sequences of length $w \geqq 2$. Similar problems for permutations are discussed in [2], and elsewhere for some special cases. We describe these generalizations and interpretations.

A permutation is said to contain the w-sequence (i.e., sequence of length w)

$$(6) \qquad ijk \ldots st$$

if, looking from left to right, j immediately follows i, k immediately follows j, …, t immediately follows s. A circular permutation is said to contain the w-sequence (6) if, looking in the clockwise direction, j immediately follows i, k immediately follows j, …, t immediately follows s. Two w-sequences are consistent if some permutation (circular permutation) contains both, and a collection of w-sequences is consistent if each two of them are consistent.

Now consider the w-sequences in the following tables:

$$\underbrace{\underbrace{\begin{matrix} 1 & 2 & \dots & w \\ 2 & 3 & \dots & w+1 \\ & & \vdots & \\ n-w+1 & n-w+2 & \dots & n \end{matrix}}_{\delta} \qquad \begin{matrix} w & w-1 & \dots & 1 \\ w+1 & w & \dots & 2 \\ & & \vdots & \\ n & n-1 & \dots & n-w+1 \end{matrix}}_{\Delta}$$

$$\underbrace{\underbrace{\begin{matrix} 1 & 2 \dots w-1 & w \\ 2 & 3 \dots \ w & w+1 \\ & \vdots & \\ n-w+1 & \dots n-2 & n \\ n-w+2 & \dots \ n & 1 \\ & \vdots & \\ n & 1 \quad w-2 & w-1 \end{matrix}}_{\lambda} \qquad \begin{matrix} w & w-1 \dots 2 & 1 \\ w+1 & w \ \dots 3 & 2 \\ & \vdots & \\ n & n-1 \ \dots & n-w+1 \\ 1 & n \ \dots & n-w+2 \\ & \vdots & \\ w-1 & w-2 \quad 1 & n \end{matrix}}_{\Lambda}$$

Let

$$P_\delta(n, w, r), \quad P_\Delta(n, w, r), \quad P_\lambda(n, w, r), \quad P_\Lambda(n, w, r) \tag{7)}$$

denote the number of permutations of degree n containing precisely r of the n-sequences in the δ, Δ, λ, Λ lists respectively. Let

$$Q_\delta(n, w, r), \quad Q_\Delta(n, w, r), \quad Q_\lambda(n, w, r), \quad Q_\Lambda(n, w, r) \tag{8}$$

denote the number of circular permutations of degree n containing precisely r of the w-sequences in the δ, Δ, λ, Λ lists respectively. Of course $Q_\lambda(n, 2, 0)$ and $Q_\Lambda(n, 2, 0)$ and given by (1) and (2) respectively.

The numbers (7) were given chess-board interpretations in [2]. They, and (8), have second-seating interpretations. For example, the following statement is valid when the three blanks are filled in, respectively, with the entries in any row of the Table II: If n people sit in a ____ at the first seating, then ____ is the number of second-seatings of the n people in a ____ such that (at each second seating) there are precisely r occurrences of: two people who sat next to each other at the first seating do so at the second seating.

First Seating	Number	Second Seating
row	$P_\Delta(n, 2, r)$	row
circle	$P_\Lambda(n, 2, r)$	r ow
row	$Q_\Delta(n, 2, r)$	circle
circle	$Q_\Lambda(n, 2, r)$	circle

Table II

4. Explicit expressions. Explicit expressions for the numbers (7) and (8) involve counts of certain k-choices which we now describe. Henceforth we take

$$\binom{n}{k} = \begin{cases} n!/k!(n-k)!, & 0 \leqq k \leqq n, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

Let

$$x_1 < x_2 < \cdots < x_k, \qquad 1 \leqq k \leqq n, \tag{10}$$

be a k-choice from $\{1, 2, \ldots, n\}$. Call (10) a $(n{:}k|a, b, c, \ldots)$-choice if

$$a = \sum_{x_{i+1}-x_i>1} 1, \quad b = \sum_{x_{i+1}-x_i>2} 1, \ \ldots, \tag{11}$$

where the sums are taken with $i = 1, 2, \ldots, k-1$. Define, for $1 \leqq i, j \leqq n$,

$$\overline{ij} = \begin{cases} j-i, & i<j, \\ j-i+n, & j \leqq i. \end{cases} \tag{12}$$

Call (10) a $\langle n{:}k|a, b, c, \ldots\rangle$-choice if (when $1 \leqq k < n$)

$$a = \sum_{\overline{x_i, x_{i+1}}>1} 1, \quad b = \sum_{\overline{x_i, x_{i+1}}>2} 1, \ \ldots, \tag{13}$$

where the sums are taken with $i = 1, 2, \ldots, k$ (and $x_{k+1} = x_1$). Of course $a \geqq b \geqq c \geqq \cdots \cdots \geqq 0$.

In [2] we proved that the number of $(n{:}k|a, b, c, \ldots, p, q)$-choices is

$$((n{:}k|a, b, c, \ldots, p, q)) = \binom{k-1}{a}\binom{a}{b}\binom{b}{c}\cdots\binom{p}{q}\binom{n-k-a-b-\cdots-p+1}{q+1}, \quad 1 \leqq k \leqq n, \tag{14}$$

while the number of $\langle n{:}k|a, b, c, \ldots, p, q\rangle$-choices is

$$\langle\langle n{:}k|a, b, c, \ldots, p, q\rangle\rangle = \frac{n}{a}\binom{k-1}{a-1}\binom{a}{b}\binom{b}{c}\cdots\binom{p}{q}\binom{n-k-a-b-\cdots-p-1}{q-1}, \quad 1 \leqq k < n, \ 0 < q. \tag{15}$$

For $w \geqq 2$, expressions involving the numbers (14) and (15) are given for

(16)

$P_\delta(n, w, r)$ in [2; the number $R_r(n, w)$ of Theorem 2(iii)];

$P_\Delta(n, w, r)$ in [2; the number $N_r(n, w)$ of Theorem 2(i)];

$P_\lambda(n, w, r)$ in [2; the number $Q_r(n, w)$ of Theorem 4(iii)];

$P_\Lambda(n, w, r)$ in [2; the number $M_r(n, w)$ of Theorem 4(i)].

We now turn our attention to the numbers $Q_\Lambda(n, w, 0)$, $2 \leqq w < n$. Observe that in a circular permutation of $1, 2, \ldots, n$ the joint occurrence of $k(\geqq 1)$ of the w-sequences in the Λ list is possible if and only if

(i) no two come in the same row, and

(ii) two which come froms rows s and v,

with $\overline{s, v} \leqq w-1$ or $\overline{v, s} \leqq w-1$ must come from the same column.

Remark. If a circular permutation contains two w-sequences (from Λ) which come from rows s and v with $\overline{s, v} \leqq w-1$ then by (ii) the two w-sequences come from the same column and the circular permutation also contains the $w-2$ w-sequences in that column and in rows $s+1, s+2, \ldots, v-1$ (numbers here modulo n).

Let us focus our attention on a particular choice of k consistent w-sequences from Λ. We prove

LEMMA 1. *If the $k \geqq 1$ rows, from which the particular k consistent w-sequences in Λ come, form a $\langle n; k | a_1, a_2, \ldots, a_{w-1}\rangle$-choice of the rows, then the number of circular permutations containing these k w-sequences is*

$$\begin{cases} (n-k-a_1-\cdots-a_{w-2}-1)! & \text{if } a_{w-1} \geqq 1, \\ 1 & \text{if } a_{w-1} = 0. \end{cases} \tag{17}$$

PROOF. If $a_{w-1} > 0$, any permutation containing the k chosen w-sequences contains (by condition (ii) and the Remark) precisely

$$k + a_1 - a_2 + 2(a_2 - a_3) + \cdots + (w-2)(a_{w-2} - a_{w-1})$$
$$= k + a_1 + a_2 + \cdots + a_{w-2} - (w-2)a_{w-1}$$

w-sequences (from Λ) and their rows form a

$$\langle n{:}k + a_1 + \cdots + a_{w-2} - (w-2)a_{w-1} | b_1, \ldots, b_{w-1}\rangle\text{-choice},$$
$$b_1 = b_2 = \cdots = b_{w-1} = a_{w-1},$$

of the n rows. This choice has a_{w-1} parts, and two w-sequences in rows belonging to different parts involve no common integers. Let $\alpha_1, \alpha_2, \ldots, \alpha_{a_{w-1}}$ be the lengths of the a_{w-1} parts. They involve, respectively

$$\alpha_1 + w - 1, \alpha_2 + w - 1, \ldots, \alpha_{a_{w-1}} + w - 1$$

different integers, and hence the k ("original") chosen w-sequences involve

$$\sum_{i=1}^{a_{w-1}} (\alpha_i + w - 1) = k + a_1 + \cdots + a_{w-2} - (w-2)a_{w-1} + (w-1)a_{w-1}$$
$$= k + a_1 + \cdots + a_{w-2} + a_{w-1}.$$

different integers. So there are $n-k-a_1-a_2-\cdots-a_{w-2}-a_{w-1}$ integers not involved in the k chosen w-sequences. Treating these integers, and the a_{w-1} parts, as $n-k-a_1-a_2-\cdots-a_{w-2}$ distinct entities, we have $(n-k-a_1-a_2-\cdots-a_{w-2}-1)!$ circular permutations.

If $a_{w-1}=0$, by condition (ii) the k w-sequences come from the same column and, by the Remark, any permutation containing them contains as well all the w-sequences in that column; there is just one such permutation, either

$$\begin{array}{ccc} & n & \\ n-1 & & 1 \\ & & 2 \\ \cdot & & \cdot \end{array}$$

containing all the w-sequences in the left column of Λ, or

$$\begin{array}{ccc} & n & \\ 1 & & n-1 \\ 2 & & \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

containing all the w-sequences in the right column of Λ.

This completes the proof of Lemma 1, and we have as well

LEMMA 2. *Let the non-negative integers $a_1 \geqq a_2 \geqq \dots \geqq a_{w-1}$ be given. Then the number of ways of choosing k consistent (with respect to circular permutations) w-sequences from the Λ list subject to the condition that the rows from which they come are a $\langle n:k|a_1, a_2, \dots, a_{w-1}\rangle$-choice (of the rows) is*

$$(18)\qquad \begin{cases} 2^{a_{w-1}}\langle\langle n:k\,|\,a_1, a_2, \dots, a_{w-2}, a_{w-1}\rangle\rangle & \text{if } a_{w-1} \geqq 1, \\ 2\langle\langle n:k\,|\,a_1, a_2, \dots, a_{w-2}, 0\rangle\rangle & \text{if } a_{w-1} = 0. \end{cases}$$

Lemmas 1 and 2, and the Principle of Inclusion and Exclusion, now yield

$$(19)\qquad Q_\Lambda(n, w, 0) = (n-1)!$$
$$+ \sum_{k=1} (-1)^k \sum_{k \geqq a_1 \geqq \dots \geqq a_{w-1} \geqq 1} \langle\langle n:k\,|\,a_1, \dots, a_{w-1}\rangle\rangle \cdot 2^{a_{w-1}}(n-k-a_1-\dots-a_{w-2}-1)!$$
$$+ \sum_{k=1} (-1)^k \sum_{k \geqq a_1 \geqq \dots \geqq a_{w-2} \geqq 0} \langle\langle n:k\,|\,a_1, \dots, a_{w-2}, 0\rangle\rangle \cdot 2 \cdot 1$$

and, if $0 < r \leqq n$,

$$(20)\qquad Q_\Lambda(n, w, r) =$$
$$= \sum_{k=0} (-1)^k \binom{r+k}{r} \sum_{r+k \geqq a_1 \geqq \dots \geqq a_{w-1} \geqq 1} \langle\langle n:r+k\,|\,a_1, \dots, a_{w-1}\rangle\rangle 2^{a_{w-1}} \times$$
$$\times (n-r-k-a_1-\dots-a_{w-2}-1)! + \sum_{k=0} (-1)^k \binom{r+k}{r} \times$$
$$\times \sum_{r+k \geqq a_1 \geqq \dots \geqq a_{w-2} \geqq 0} \langle\langle n:r+k\,|\,a_1, \dots, a_{w-2}, 0\rangle\rangle 2.$$

We will simplify (19) and (20) after proving that if r and m are non-negative integers, then

$$(21)\qquad \sum_{k=0} (-1)^k \binom{m}{k}\binom{k}{r} = \begin{cases} 0, & \text{if } r \neq m, \\ (-1)^m, & \text{if } r = m. \end{cases}$$

Proof of (21). Let the left hand side be denoted by $f(m, r)$. It is obvious that $f(m, m)=(-1)^m$ and, if $0\leqq m<r$ then $f(m, r)=0$. If $m>0$, then $f(m, 0) = = \sum_{k=0}(-1)^k\binom{m}{k} = 0$. If $0<r<m$,

$$f(m, r) = \sum_{k=r}(-1)^k\binom{m-1}{k-1}\frac{m}{k}\binom{k-1}{r-1}\frac{k}{r} = -\frac{m}{r}f(m-1, r-1)$$

and a repetition of this yields

$$f(m, r) = (-1)^r\binom{m}{r}f(m-r, 0) = 0,$$

completing the proof.

Since $\sum_{k\geqq a_1\geqq\ldots\geqq a_{w-2}\geqq 0}\langle\langle n:k\,|\,a_1, \ldots, a_{w-2}, 0\rangle\rangle$ is the number of k-choices (10) satisfying

$$x_{i+1}-x_i \leqq w-1, \qquad i=1, 2, \ldots, k-1,\ n+x_1-x_k \leqq w-1,$$

[it is equal to

$$\frac{n}{k}\sum_{j=0}(-1)^j\binom{k}{j}\binom{n-j(w-1)-1}{k-1}, \qquad 1\leqq k\leqq n,\ 2\leqq w,$$

3, p. 675, formula (1)], [6, formula (29)].

Hence, the last term of (19) is

$$(22)\qquad 2\sum_{k=1}(-1)^k\frac{n}{k}\sum_{j=0}(-1)^j\binom{k}{j}\binom{n-j(w-1)-1}{k-1}$$

$$= 2\sum_{k=1}(-1)^k\frac{n}{k}\left\{\binom{n-1}{k-1}+\sum_{j=1}(-1)^j\binom{k}{j}\binom{n-j(w-1)-1}{k-1}\right\}$$

$$= 2\sum_{k=1}(-1)^k\binom{n}{k}-2n\sum_{j=1}(-1)^j\frac{1}{j}\sum_{k=1}(-1)^{k-1}\binom{k-1}{j-1}\binom{n-j(w-1)-1}{k-1}$$

$$= 2(-1)-2n\sum_{j=1}(-1)^j\frac{1}{j}\sum_{k=0}(-1)^k\binom{k}{j-1}\binom{n-j(w-1)-1}{k}.$$

Now (21) with $m = n-j(w-1)-1$ and $r = j-1$ shows that

$$(23)\qquad \sum_{k=0}(-1)^k\binom{k}{j-1}\binom{n-j(w-1)-1}{k} = \begin{cases}0, & \text{if } n-jw\neq 0,\\ (-1)^{j-1}, & \text{if } n-jw=0.\end{cases}$$

Hence, if w does not divide n every term in the sum on j in (22) is equal to zero; while if w divides n the only non-zero term occurs when $j=n/w$. Thus, we finally have that the last term of (19) is equal to

$$2(w-1) \quad \text{if} \quad w\,|\,n,$$

$$-2 \quad \text{if} \quad w\nmid n.$$

Expression (20) simplifies when $r \geqq n-w+2$. In this case, necessarily $a_{w-1}=0$, so

$$Q(n, w, r) = \sum_{k=0} (-1)^k \binom{r+k}{r} \sum_{r+k \geqq a_1 \geqq \ldots \geqq a_{w-2} \geqq 0} \langle\langle n: r+k \,|\, a_1, \ldots, a_{w-2}, 0\rangle\rangle 2 =$$

$$= 2 \sum_{k=0} (-1)^k \binom{r+k}{r} \frac{n}{r+k} \sum_{j=0} (-1)^j \binom{r+k}{j} \binom{n-j(w-1)-1}{r+k-1}$$

(and, because $r \geqq n-w+1$)

$$= 2 \sum_{k=0} (-1)^k \binom{r+k}{r} \frac{n}{r+k} \binom{n-1}{r+k-1} = 2 \sum_{k=0} (-1)^k \binom{r+k}{r} \binom{n}{r+k} = \begin{cases} 0, & \text{if } r<n, \\ 2, & \text{if } r=n. \end{cases}$$

Of course

(24)
$$Q_\lambda(n, w, 0) = (n-1)!$$
$$+ \sum_{k=1} (-1)^k \sum_{k \geqq a_1 \geqq \ldots \geqq a_{w-1} \geqq 1} \langle\langle n: k \,|\, a_1, \ldots, a_{w-1}\rangle\rangle (n-k-a_1-\cdots-a_{w-2}-1)!$$
$$+ \begin{cases} w-1, & \text{if } w \mid n, \\ -1, & \text{if } w \nmid n, \end{cases} \qquad 2 \leqq w < n,$$

while

(25)
$$Q_\lambda(n, w, r) = \sum_{k=0} (-1)^k \times$$
$$\times \binom{r+k}{r} \sum_{r+k \geqq a_1 \geqq \ldots \geqq a_{w-1} \geqq 1} \langle\langle n: r+k \,|\, a_1, \ldots, a_{w-1}\rangle\rangle (n-r-a_1-\cdots-a_{w-2}-1)!$$
$$+ \sum_{k=0} (-1)^k \binom{r+k}{r} \sum_{r+k \geqq a_1 \geqq \ldots \geqq a_{w-2} \geqq 0} \langle\langle n: r+k \,|\, a_1, \ldots, a_{w-2}, 0\rangle\rangle,$$
$$2 \leqq w < n, \quad 1 \leqq r \leqq n-w+1,$$

and

(26)
$$Q_\lambda(n, w, r) = \begin{cases} 0, & \text{if } n-w+2 \leqq r < n, \\ 1, & \text{if } r=n. \end{cases}$$

Similar arguments show that

(27)
$$Q_\Delta(n, w, r) = \sum_{k=0}^{n-w+1-r} (-1)^k \binom{r+k}{r} \sum_{r+k-1 \geqq a_1 \geqq \ldots \geqq a_{w-1} \geqq 0} 2^{a_{w-1}+1} \times$$
$$\times ((n-w+1: r+k \,|\, a_1, \ldots, a_{w-1})) (n-r-k-a_1-\cdots-a_{w-2}-w+1)!,$$
$$2 \leqq w < n, \quad 1 \leqq r < n,$$

(28)
$$Q_\Delta(n, w, 0) = (n-1)! - \sum_{k=1}^{n-w+1} (-1)^k \sum_{k-1 \geqq a_1 \geqq \ldots \geqq a_{w-1} \geqq 0} 2^{a_{w-1}+1} \times$$
$$\times ((n-w+1: k \,|\, a_1, \ldots, a_{w-1})) (n-k-a_1-\cdots-a_{w-2}-w+1)!, \qquad 2 \leqq w < n,$$

$$Q_\delta(n, w, r) = \sum_{k=0}^{n-w+1-r} (-1)^k \binom{r+k}{r} \sum_{r+k-1 \geqq a_1 \geqq \ldots \geqq a_{w-1} \geqq 0} \times \tag{29}$$

$$\times ((n-w-1:r+k|a_1, \ldots, a_{w-1}))(n-r-k-a_1-\cdots-a_{w-2}-w+1)!,$$

$$2 \leqq w < n, \ 1 \leqq r < n,$$

$$Q_\delta(n, w, 0) = (n-1)! + \sum_{k=1}^{n-w+1} (-1)^k \times \tag{30}$$

$$\times \sum_{k-1 \geqq a_1 \geqq \ldots \geqq a_{w-1} \geqq 0} ((n-w+1:k|a_1, \ldots, a_{w-1}))(n-k-a_1-\cdots-a_{w-2}-w+1)!.$$

5. The Case $w=2$. When $w=2$ the numbers are:

$$P_\Delta(n, 2, 0) = n! + \sum_{k=1}^{n-1} (-1)^k \sum_{i=0}^{k-1} ((n-1:k|i))2^{i+1}(n-k)!, \qquad n \geqq 2, \tag{31}$$

[2, Theorem 2 (ii) with $w=2$];

$$P_A(n, 2, 0) = n! + \sum_{k=1}^{n-1} (-1)^k \sum_{i=1}^{k} \langle\langle n:k|i \rangle\rangle 2^i (n-k)!, \qquad n \geqq 3, \tag{32}$$

[2, Theorem 4 (ii) with $w=2$];

$$Q_\Delta(n, 2, 0) = (n-1)! + \sum_{k=1}^{n-1} (-1)^k \sum_{i=0}^{k-1} ((n-1:k|i))2^{i+1}(n-k-1)!, \qquad n \geqq 3, \tag{33}$$

from (28) with $w=2$; and

(34)

$$Q_A(n, 2, 0) = (n-1)! + \sum_{k=1}^{n-1} (-1)^k \sum_{i=1}^{k} \langle\langle n:k|i \rangle\rangle 2^i (n-k-1)! + (-1)^n 2, \qquad n \geqq 3,$$

from (19) with $w=2$. A simple manipulation shows that $nQ_\Delta(n, 2, 0) = P_A(n, 2, 0)$.

The symbolic method is an effective way of computing these numbers. With

$$((n:k|i)) = \begin{cases} 1, & k=0, \ i=-1, \\ \binom{k-1}{i}\binom{n-k+1}{i+1}, & \text{otherwise,} \end{cases} \tag{35}$$

and

$$\langle\langle n:k|i \rangle\rangle = \begin{cases} 1, & k=n \text{ and } i=1, \text{ or } k=i=0, \\ 0, & 1 \leqq k \leqq n \text{ and } i=0, \\ \frac{n}{i}\binom{k-1}{i-1}\binom{n-k-1}{i-1}, & \text{otherwise,} \end{cases} \tag{36}$$

we can write

$$P_\Delta(n, 2, 0) = \sum_{k=0}^{n-1} (-1)^k \sum_{i=0}^{k} ((n-1:k|i-1))2^i(n-k)! = \sum_{k=0}^{n-1} (-1)^k g(n-1:k)(n-k)!,$$

$$n \geqq 2,$$

$$P_A(n, 2, 0) = \sum_{k=0}^{n-1} (-1)^k \sum_{i=0}^{k} \langle\langle n:k|i \rangle\rangle 2^i (n-k)! = \sum_{k=0}^{n-1} (-1)^k h(n:k)(n-k)!, \qquad n \geqq 3,$$

$$Q_\Delta(n,2,0)=\sum_{k=0}^{n-1}(-1)^k\sum_{i=0}^{k}((n-1:k\,|\,i-1))2^i(n-k-1)!$$

$$=\sum_{k=0}^{n-1}(-1)^k g(n-1:k)\,(n-k-1)!,\qquad n\geqq 3,$$

(37) $$Q_\Delta(n,2,0)=\sum_{k=0}^{n-1}(-1)^k\sum_{i=0}^{k}\langle\langle n:k\,|\,i\rangle\rangle 2^i(n-k-1)!+2(-1)^n$$

$$=\sum_{k=0}(-1)^k h(n:k)\,(n-k-1)!+2(-1)^n,$$

where, for $n\geqq 1$,

(38) $$g(n:k)=\sum_{i=0}^{k}((n:k\,|\,i-1))2^i;\quad h(n:k)=\sum_{i=0}^{k}\langle\langle n:k\,|\,i\rangle\rangle 2^i.$$

If we write, in symbolic form, $E^m\varphi_0=\varphi_m=m!$ then

$$P_\Delta(n,w,0)=EG_n(E)\varphi_0,\qquad n\geqq 2;$$

$$G_n(E)=\sum_{k=0}^{n-1}(-1)^k g(n-1:k)E^{n-k-1},\qquad n\geqq 2;$$

$$P_\Delta(n,2,0)=EH_n(E)\varphi_0,\qquad n\geqq 3;$$

(39) $$H_n(E)=\sum_{k=0}^{n-1}(-1)^k h(n:k)E^{n-k-1},\qquad n\geqq 3;$$

$$Q_\Delta(n,2,0)=G_n(E)\varphi_0,\qquad n\geqq 2;$$

$$Q_\Delta(n,2,0)=\big(H_n(E)+2(-1)^n\big)\varphi_0,\qquad n\geqq 3.$$

Recurrences for the $G_n(E)$, $H_n(E)$ are obtained by noting that for $n\leqq 3$, $1\leqq k\leqq n-1$, $-1\leqq i\leqq k-1$,

$$((n:k|i))=((n-1:k|i))+((n-1:k-1|i))-((n-2:k-1|i))+((n-2:k-1|i-1))$$

[2, p. 1247], and for $n\geqq 4$, $1\leqq k\leqq n-2$, $0\leqq i\leqq k$,

$$\langle\langle n:k|i\rangle\rangle=\langle\langle n-1:k|i\rangle\rangle+\langle\langle n-1:k-1|i\rangle\rangle-\langle\langle n-2:k-1|i\rangle\rangle+\langle\langle n-2:k-1|i-1\rangle\rangle$$

(proved by directly substituting (36)), so

(40) $g(n:k)=g(n-1:k)+g(n-1:k-1)+g(n-2:k-1)$, for $n\geqq 3$, $1\leqq k\leqq n-1$,

(41) $h(n:k)=h(n-1:k)+h(n-1:k-1)+h(n-2:k-1)$, for $n\geqq 4$, $1\leqq k\leqq n-2$,

and hence

(42) $$G_n(E)=(E-1)G_{n-1}(E)-EG_{n-2}(E),\qquad n\geqq 4,$$

$$G_2(E)=E-2,\quad G_3(E)=E^2-4E+2;$$

(43) $$H_n(E)=(E-1)H_{n-1}(E)-EH_{n-2}(E)+2(-1)^{n-1},\qquad n\geqq 5,$$

$$H_3(E)=E^2-6E+6,\quad H_4(E)=E^3-8E^2+16E-8.$$

Tables III and IV carry the computation up to $n=8$. The recurrences (40) and (41) permit the numbers $g(n:k)$ and $h(n:k)$ to be arranged in a triangular form (Fig. 1).

n	$g(n:k)$							
1	1	2						
2	1	4	2					
3	1	6	8	2				
4	1	8	18	12	2			
5	1	10	32	38	16	2		
6	1	12	50	88	66	20	2	
7	1	14	72	170	192	102	24	2

n	$h(n:k)$								
1	1	2							
2	1	4	2						
3	1	6	6	2					
4	1	8	16	8	2				
5	1	10	30	30	10	2			
6	1	12	48	76	48	12	2		
7	1	14	70	154	154	70	14	2	
8	1	16	96	272	384	272	96	16	2

Fig. 1

The triangle is generated from the initial values (computed from (38), using (35) and (36))

$$g(n:0)=1, \quad g(n:n)=2, \quad n \geqq 1, \quad g(2:1)=4;$$

$$h(n:0)=1, \quad h(n:n)=2, \quad n \geqq 1, \quad h(n:n-1) = 2n \quad n \geqq 2.$$

Any other entry is the sum of the three closest numbers above it (see Figure 2).

$$h(n-2:k-1)$$

$$h(n-1:k-1) \qquad\qquad h(n-1:k)$$

$$h(n:k)$$

Fig. 2

6. First seating at several tables. Suppose that at the first seating $n = n_1 + \cdots + n_m$ ($m \geqq 2$, $n_i \geqq 1$, $i=1, 2, \ldots, m$) people sit at m tables $T_1, T_2, \ldots, T_m$, n_i of them at table T_i. Then the number of ways they can be seated at a single table T

n	$G_n(E)$	$H_n(E)$
2	$E-2$	
3	E^2-4E+2	E^2-6E+6
4	E^3-6E^2+8E-2	$E^3-8E^2+16E-8$
5	$E^4-8E^3+18E^2-12E+2$	$E^4-10E^3+30E^2-30E+10$
5	$E^5-10E^4+32E^3-38E^2+16E-2$	$E^5-12E^4+48E^3-76E^2+48E-12$
7	$E^6-12E^5+50E^4-88E^3+66E^2-$ $-20E+2$	$E^6-14E^5+70E^4-154E^3+154E^2-$ $-70E+14$
8	$E^7-14E^6+72E^5-170E^4+192E^3-$ $-102E^2+24E-2$	$E^7-16E^6+96E^5-272E^4+384E^3-$ $-272E^2+96E-16$

Table III

n	$P_\Delta(n, 2, 0)$	$P_\Lambda(n, 2, 0)$	$Q_\Delta(n, 2, 0)$	$Q_\Lambda(n, 2, 0)$
2	0			
3	0	0	0	0
4	2	0	0	0
5	14	10	2	2
6	90	60	10	6
7	646	462	66	46
8	5242	3920	490	354

Table IV

subject to the condition that no one sits next to a person he sat next to at the first seating is

$$\sum_{k=0} (-1)^k \sum_{\substack{a_1+\dots+a_m=k \\ 0\leqq a_i\leqq n_i-1}} \prod_{i=1}^{m} f(n_i:a_i)\times\begin{cases}(n-k)! & \text{if } T \text{ is straight,}\\ (n-k-1)! & \text{if } T \text{ is round,}\end{cases}$$

where $f(1:0)=1$ and

$$f(n_i:a_i)=\begin{cases} g(n_i-1:a_i) & \text{if } T_i \text{ is straight and } n_i\geqq 2,\\ h(n_i:a_i) & \text{if } T_i \text{ is round and } n_i\geqq 3,\\ 1 & \text{if } T_i \text{ is round, } n_i=2,\ a_i=0,\\ 2 & \text{if } T_i \text{ is round, } n_i=2,\ a_i=1,\end{cases}$$

or, in symbolic form,

$$\sum_{k=0} (-1)^k \sum_{\substack{a_1+\dots+a_m=k \\ 0\leqq a_i\leqq n_i-1}} \left\{\prod_{i=1}^{m} (-1)^{a_i} f(n_i:a_i)\, E^{n_i-a_i-1} E^{m-\varepsilon}\right\}\varphi_0,$$

$$\text{where } \varepsilon=\begin{cases} 0 & \text{if } T \text{ is straight,}\\ -1 & \text{if } T \text{ is round,}\end{cases}$$

or

$$\left\{E^{m-\varepsilon}\prod_{i=1}^{m}\sum_{a_j=0}^{n_i-1}(-1)^{a_j}f(n_i:a_j)E^{n_i-a_j-1}\right\}\varphi_0 = E^{m-\varepsilon}F_{n_1}(E)F_{n_2}(E)\dots F_{n_m}(E)\varphi_0,$$

where

$$F_{n_i}(E) = \begin{cases} 1 & \text{if } n_i=1, \\ G_{n_i}(E) & \text{if } T_i \text{ is straight, } n_i \geqq 2, \\ H_{n_i}(E) & \text{if } T_i \text{ is round, } n_i \geqq 3, \\ E-2 & \text{if } T_i \text{ is round, } n_i=2. \end{cases}$$

Example 1 $n_1=2$, $n_2=4$,

First Seating				Second Seating		Number of Second Seatings
T_1		T_2		T		
round	straight	round	straight	round	straight	
	2		4		6	136
	2	4			6	80
2			4		6	136
2		4			6	80
	2		4	6		14
	2	4		6		8
2			4	6		14
2		4		6		8

Example 2. Four separate decks (the four suits) of 13 cards each, with the cards in natural order (i.e., A, 2, 3, …, 10, J, Q, K) are put together and shuffled. What is the probability that no two cards which were adjacent in the original (small) decks are adjacent in the shuffled (large) deck?

Answer:
$$\frac{1}{52!}E^4\{G_{13}(E)\}^4\varphi_0.$$

If in addition the A and K of each small deck are considered adjacent, (but the top and bottom card of the shuffied deck are not considered adjacent) the answer is

$$\frac{1}{52!}E^4\{H_{13}(E)\}^4\varphi_0.$$

If the A and K of each small deck are considered adjacent, and the top and bottom card of the shuffled deck are also considered to be adjacent, the probability is

$$\frac{1}{51!}E^3\{H_{13}(E)\}^4\varphi_0.$$

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(Received February 16, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 225—232.

ON 4-VALENT GRAPHS IMBEDDED IN ORIENTABLE 2-MANIFOLDS

by

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1. Introduction. For a 3-connected regular 4-valent graph (without loops and multiple edges) imbedded in a closed connected orientable 2-manifold P let $p_k(M)$ denote the number of k-gonal faces (cells, countries) of the map M defined by the imbedded graph. From Euler's formula $\left(\sum_{k\geqq 3} p_k(M)+v(M) = h(M)+2(1-g),\right.$ $-v(M)$ or $h(M)$ denotes the number of vertices or edges of the map M respectively, g is the genus of P) follows

$$\sum_{k\geqq 3}(4-k)p_k(M) = 8(1-g) \tag{1}$$

Clearly, (1) does not impose restrictions on the number $p_4(M)$. Thus the following question can be asked: Given a sequence $p=(p_3, p_4, \dots, p_s)$ of non-negative integers satisfying

$$\sum_{k\geqq 3}(4-k)p_k = 8(1-g) \tag{2}$$

does there exist a map M with a 4-valent graph on the orientable surface P_g of genus g such that $p_k(M)=p_k$ for all $k\neq 4$? (If so then the sequence p is called *4-realizable on P_g.*)

For $g=0$ the solution is given in GRÜNBAUM [3, 5], for $g=1$ (as well as $g=0$) in BARNETTE—JUCOVIČ—TRENKLER [1] (cf. ZAKS [9]). The aim of this paper is to present the solution for all finite $g\geqq 3$, and to give a partial answer for $g=2$.

Our main result is contained in

THEOREM 1. *Every sequence $p=(p_3, p_4, \dots, p_s)$ of non-negative integers satisfying* (2) *with $g\geqq 3$ is 4-realizable on the closed connected orientable 2-manifold of genus g.*

2. PROOF **of Theorem 1** will be carried out by constructing, for every sequence p satisfying the assumption, the required map on the appropriate manifold. (We recall that there is, to every $g\geqq 0$, exactly one topological type of an orientable 2-manifold of genus g. For our purposes ist interpretation by a sphere with g "handles" will be convenient.) The construction is decomposed into several stages with two different procedures of construction. Luckily there is no need to consider many cases.

First let us decompose the sequence p into two sequences $r=(r_3, r_5, \dots, r_s)$

and $p'=(p'_3, p'_5, \dots, p'_s)$ of non-negative integers (p_4, r_4, p'_4 need not to be considered) such that

(i) $p_i = p'_i + r_i$ for all i, and

(ii) $\sum_{k \geqq 3} (4-k)r_k = 8(1-g)$, and

(iii) $p'_3 \neq 1$ holds, and

(iv) there exists no sequence $r'=(r'_3, \dots, r'_s)$ with $r'_i \leqq r_i$ for all i, different from r, satisfying conditions (i), (ii), (iii).

Notice that a) for the sequence p' holds $\sum_{k \geqq 3} (4-k)p'_k = 0$, b) from (ii) follows $\sum_{i \geqq 5} r_i$ to be even for i odd, if $r_3=0$.

The stages of our construction, in rough outline, are:

I. Constructing on the surface of the sphere a map R with r_i i-gonal faces and $2g$ "other" faces to be used as openings for forming the handles. Only the vertices of these last polygons are trivalent, all other vertices are 4-valent.

II. Constructing, on a tube, a map with p'_i i-gonal faces for all i. On the openings of the tube there are equal numbers of vertices and they are all 3-valent. All the other vertices are 4-valent.

III. Joining the tube from II. with the sphere in I. and adding $(g-1)$ handles decomposed into quadrangles.

I. The map R will be obtained from that on Fig. 1 consisting of one $(8g-4)$-gon $A_1 A_2 \dots A_{4g-3} A_{4g-2} B_{4g-2} \dots B_2 B_1 = \mathsf{A}$, two triangles $C_1 D_1 E_1, C_{2g} D_{2g} E_{2g}$ (the outer handle-openings), $2g-2$ hexagons $C_i D_i E_i Z_i V_i U_i$, $i = 2, \dots, 2g-1$ (the inner handle openings) and $16g-8$ quadrangles. (We designate by 0_i that handle-opening whose vertices have indices i.) In the sequel the face-aggregate obtained by dissecting the polygon A by new edges will be called *submap A*; the remaining part of the map in Fig. 1 as well as its transforms will be called *submap B*.

The vertices C_i, D_i, E_i, U_i, V_i, Z_i are trivalent, all other vertices are 4-valent. The r_k k-gons, $k \geqq 5$, are cut off from the polygon A, the triangles are inserted in the submap B.

Consider first the case when

α) $r_3 = 0$.

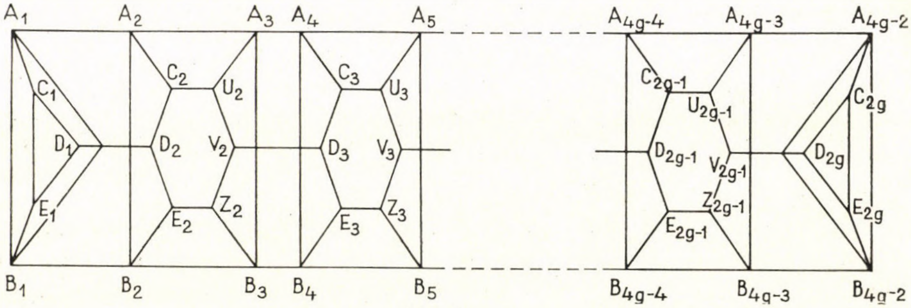

Fig. 1

If $k=2m$ choose a point Q_1 between A_{m-1} and A_m, and a point Q_2 between B_{m-1} and B_m. Joining Q_1 and Q_2 by an edge a k-gon is cut off from the polygon A. The vertices Q_1, Q_2 are still trivalent, however only vertices on the "handle openings" can remain so. If the line Q_1Q_2 does not meet such an opening we draw it and create quadrangles and 4-valent vertices in submap B only. In the opposite case the point Q_1 is joined with a point Q_3 between the points C_j and U_j, and the point Q_2 likewise with the point Q_4 lying on the edge E_jZ_j, $j=\frac{m+1}{2}$. (See Fig. 2). Analogously any further polygon with an even number of edges is cut off from the polygon A.

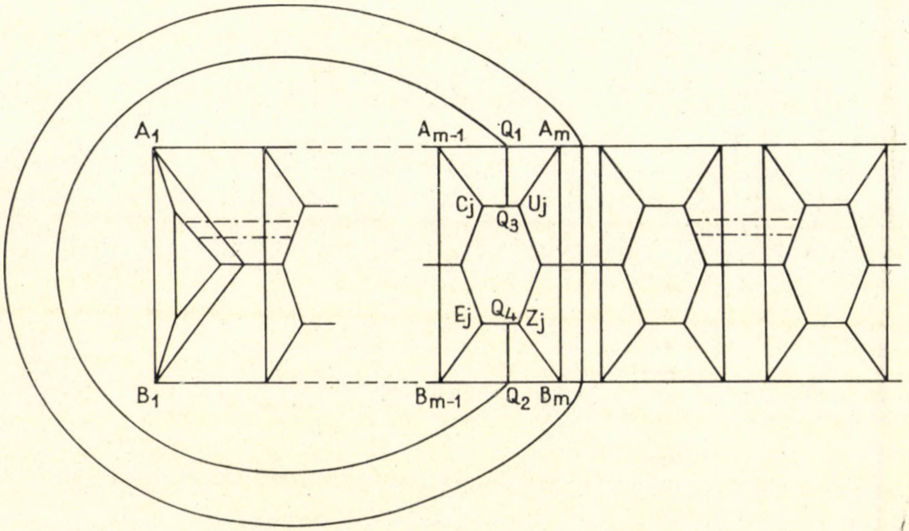

Fig. 2

Faces with odd numbers of edges are cut off in pairs. So if a k-gon, $k=2m+1$, and a t-gon, $t=2n+1$, are needed first a $2(m+n-1)$-gon is cut off, by an edge PQ, the point P lying between the points B_{m+n-2} and B_{m+n-1}, the point Q lying between the points A_{m+n-2} and A_{m+n-1}. (For simplicity's sake let the k-gon be the first we are constructing in submap A. If this is not so, then the indices of the points A_i, B_i are increased.)

Now join a point P_1 between the points A_m and A_{m-1} with a point Q_2 between the points B_m and P or B_m and B_{m+1} (if $t>5$), and a point P_2 between A_m and Q or A_m and A_{m+1} (if $t>5$) with a point Q_1 between B_m and B_{m-1}. A k-gon and a t-gon and two quadrangles $P_1A_mP_2X$, $Q_1B_mQ_2X$ are created in submap A. (X is the intersection point of the lines P_2Q_1 and P_1Q_2, see Fig. 3.) Increasing the degree of the vertices P_1, P_2, Q_1, Q_2 is performed as before.

However, in forming any pair of odd-gonal faces and sometimes in forming an even-gonal face the number of vertices on a handle-opening is increased by two. Therefore the numbers of vertices on the inner handle openings must be balanced up in any step when this number is increased at one of them.

Suppose the number of vertices of an inner handle-opening O_i was increased by two. Pairs of openings on the right hand side from O_i are joined by two paths each going from one opening to the other (dot-and-dashed lines in Fig. 2) increasing the number of their vertices by two and forming new quadrangles and 4-valent vertices only. The same is done with the openings on the left hand side from O_i.

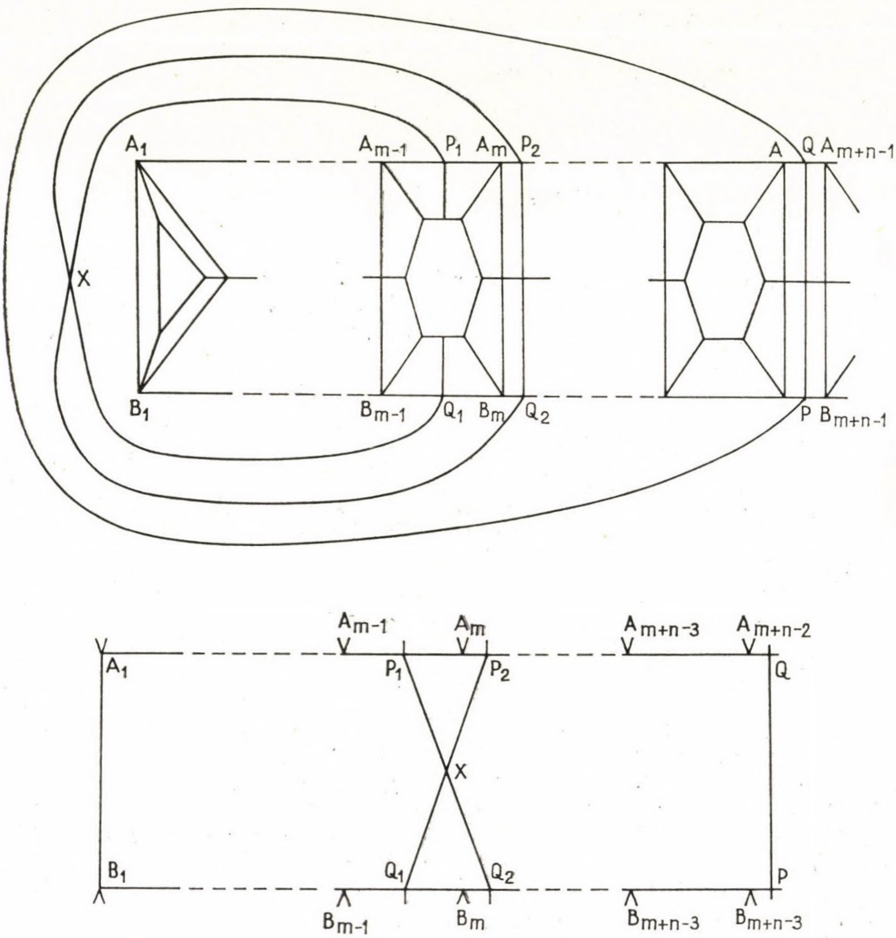

Fig. 3

In this way all the inner openings receive the same number of vertices while this need not be the case with the outer openings. However, in any case the outer openings have an odd number of vertices.

It remains, after dissecting the polygon A into r_i i-gons for all i and balancing the numbers of vertices on the inner openings to balance up the number of vertices of the outer openings when $a > b$, a or b being the numbers of vertices of these openings, respectively. Let e.g. b be the number of vertices of the opening O_1. We choose between the vertices D_1 and $E_1 (a-b)$ points $H_1, \ldots, H_{a-b}$, and between the points D_2 and E_2 points $G_1, \ldots, G_{a-b}$ and join these points by paths forming new quadrangles and 4-valent vertices only (see Fig. 4 where the points arised in balancing the numbers of vertices of the inner openings are not depicted.) As $g \geqq 3$ there are at least three inner openings different from O_2. Therefore we choose $\frac{1}{2}(a-b) = r$ points $L_1, \ldots, L_r$ situated between Z_3 and V_3. The points $N_1, \ldots, N_r$, $P_1, \ldots, P_r$ and $S_1, \ldots, S_r$ are likewise chosen to lie between E_4 and D_4, between Z_4 and V_4,

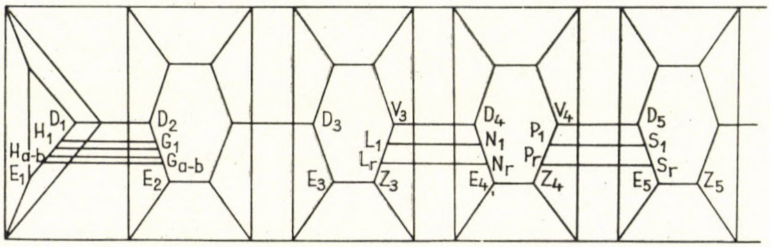

Fig. 4

and between E_5 and D_5, respectively. Then we join L_i with N_i and P_i with S_i, $i=1, \ldots, r$, increasing the number of 3-valent vertices of the openings O_3 and O_5 or O_4 by r or $(a-b)$, respectively, and forming new quadrangles and 4-valent vertices. At the end of this operation the pairs of openings O_1 and O_{2g}, O_2 and O_4, O_3 and O_5 have the same number of vertices. Of course all other openings $O_6, \ldots, O_{2g-1}$ have the same number of vertices too. The III. stage of the construction, the "handle forming" can follow.

β) $r_3 > 0$.

1. First let us settle the case of a sequence $p=(p_3, \ldots, p_s)$ which can be decomposed into the sequence r^1 and ${}^1p'$ satisfying conditions (i), (ii) and (iv) with $r_3^1=0$, but contradicting condition (iii), i.e. $r_3=1$. In such a decomposition we should have ${}^1p_3'={}^1p_5'=1$, ${}^1p_i'=0$ for $i>5$ and the second stage of the construction could not be performed as described belove. But the realization of the sequence r^1 in submap A can be (and has been) carried out. This is done wihthout balancing up the numbers of vertices of the openings Q_1 and O_{2g}. A "wedge" (triangle) is put next to the opening O_1 (as in Fig. 5) giving a triangle and a pentagon in submap B of the constructed map. Now balancing the number of vertices of the outer openings can follow.

2. Suppose that p is not decomposable into sequences r^1 and ${}^1p'$ satisfying conditions (i), (ii) and (iv) and contradicting (iii). Then $r_3 < i-4$ for every $i>5$ for which $r_i \neq 0$. Take such an i and analogously as in α) realize in submap A the sequence $r'=(r_3', \ldots, r_s')$ with $r_3'=0$, $r_i' = r_i-1$, $r_{i-r_3}' = r_{i-r_3}+1$, $r_j'=r_j$ for all $j \neq i$, $i-r_3$. One $(i-r_3)$-gon G is from the polygon A cut off as the first. The numbers of vertices of the inner handle-openings are balanced up. Then r_3 "wedges" are put next to the opening O_1 so that their vertices not lying on the opening O_1 belong to one edge of G, (see Fig. 6 where $r_3=3$). The $(i-r_3)$-gon G becomes an i-gon. Then balancing up the number of vertices of the outer openings is performed.

3. Let the sequence p be decomposable into r^1 and ${}^1p'$ satisfying (i), (ii) and (iv) and contradicting (iii), and let $r_3>1$, i.e. $r_3^1 \geqq 1$, ${}^1p_3' = {}^1p_5' = 1$, ${}^1p_i' = 0$ for $i \geqq 6$. The procedure in 1. is combined with that in 2. First the sequence r^1 is realized as in 2. Take an $i>5$ with $r_i \neq 0$ and cut off from the polygon A an $(r_i - r_3^1)$-gon, construct all remaining j-gons, $j \geqq 5$, in submap A, balance up the numbers of vertices on the inner openings, put r_3^1 "wedges" next to the openings O_1 as in Fig. 6 etc. Further a pentagon and a triangle is added in submap B as in 1. (Fig. 5).

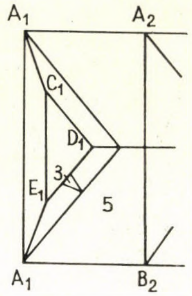

Fig. 5

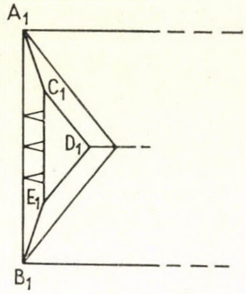

Fig. 6

II. The realization of the sequence p' on a tube is the same as in [1]. We bring this construction here for completeness' sake.

Let W be an w-gon, $w = 4 + \sum_{k \geqq 5} (k-4)p_k'$, with vertices $a_1, \ldots, a_w$ and right angles in the vertices a_1 and a_d, $d = \left[\frac{w+1}{2}\right]$. (See Fig. 7.)

The intersection points of the lines $a_1 a_2$ and $a_d a_{d-1}$, or $a_w a_1$ and $a_d a_{d+1}$ are the points b or c, respectively. The lines through the points a_i parallel to the line $a_1 c$ intersect the segment $a_1 b$ or $a_d c$ in the points u_i. Analogously we obtain the points v_i on the segments $a_1 c$, ba_d. Essential is the fact that if $\sum_{k \geqq 5} kp_k$ is even then the segments $a_1 b$ and ca_d or ca_1 and $a_d b$ contain the same number $(d-1)$ of vertices u_i or v_i. If $\sum_{k \geqq 5} kp_k$ is odd then the segments $a_1 b$ and ba_d contain $(d-1)$ points u_i

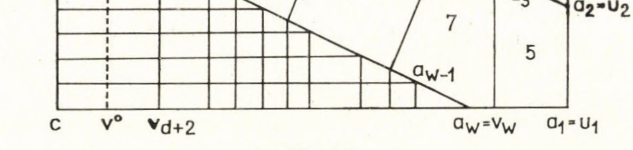

Fig. 7

or v_i, but in each of the segments $a_d c$, ca_1 there are only $(d-2)$ vertices v_i or u_i. In this case we add a new edge $u^{\cdot}q^{\cdot}$ and a series of edges on the arc $q^{\cdot}v^{\cdot}$, where the point $u^{\cdot}$ or $q^{\cdot}$ or $v^{\cdot}$ lies between the points u_{d+1} and u_{d+2} or u_{d+2} and a_{d+2} or c and v_{d+2}, respectively (dashed lines on Fig. 7). Thus all sides of the rectangle $a_1 b a_d c$ have the same number of points u_i or v_i.

Next we cut off from the polygon W the required k-gons. For some k, let $p'_k \neq 0$. We choose a point q_1 between a_n and a_{n+1}, $n<d$, and a point q_2 between a_m and a_{m+1}, $m>d$, such that $q_1 a_{n+1} a_{n+2} \dots a_d a_{d+1} \dots a_m q_2$ is a k-gon. The point q_1 is joined with a point between v_n and v_{n+1} and the point q_2 with a point between v_m and v_{m+1} by an arc.

Analogously all other required i-gons, $i \geqq 5$, are cut off from the polygon W. Along all sides of the rectangle $a_1 b a_d c$ series of quadrangles can be added. Then the trivalent vertices on one pair of opposite sides of the rectangle are unified to get a tube (or a handle).

The IIIrd stage of the construction is now simple. Pairs of the handle openings on the map R have the same number of trivalent vertices. So the handle constructed in stage II. and $(g-1)$ handles decomposed into quadrangles can be put on the map R. (The vertices on the openings of the handles are trivalent!) If the number of vertices on the opening of the tube is greater than on each handle-opening on R, the number of vertices on two handle-openings on R is increased as described in stage I. — A 2-manifold of genus g decomposed by a map realizing the sequence p is constructed.

3. THEOREM 2: *Every sequence $p=(p_3, p_4, \dots, p_s)$ of non-negative integers satisfying*

$$\sum_{k \geqq 3} (4-k) p_k = -8$$

for which neither $p_3=p_{13}=1, p_i=0$ for $i \neq 3, 4, 13$, nor $p_5=p_{11}=1, p_i=0$ for $i \neq 4, 5, 11$ holds, is 4-realizable on the orientable surface of genus 2.

The proof of Theorem 2 follows analogously as the proof of Theorem 1. Again the sequence p is decomposed into the sequences r and p'. However, balancing the number of vertices on the handle openings cannot be performed here as before. The cases in which this balancing should be necessary must be investigated separately. Because the number of these sequences r is relatively great we forsake to state them here. Unfortunately, we were not able either to realize the two exceptional sequences mentioned or to prove their non-realizability.

4. Remark. a) In our theorems we were interested in 4-realizing the sequence p with *any* number of quadrangles. Of course one could ask: What numbers of quadrangles can appear in these realizations? No definite answer has been produced at the time of writing.

b) The theorems 1 and 2 are analogoues of EBERHARD's theorem [2] in which first the question above was treated for 3-valent maps on the sphere. After the appearance of [3] much work was done with ramifications and analogues of EBERHARD's theorem. (See [1, 4, 5, 6, 7, 8, 9].)

Added in proof: In the meantime the sequences whose realizability is not decided by Theorem 2 turned out to be 4-realizable. (See JUCOVIČ—TRENKLER: On the structure of cell-decompositions of orientable 2-manifolds. *Mathematika* (to appear).

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(Received February 24, 1971. Revised September 23, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 233.

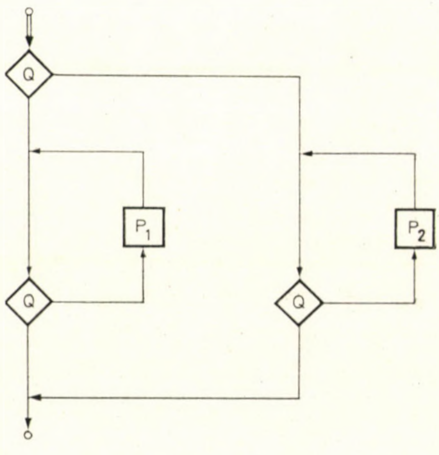
ЗАМЕТКА В СВЯЗИ С СТАТЬЮ Л. ДЬЮРИШ: О РЕГУЛЯРНЫХ ПРОГРАММАХ (6 (1971), 307—316)

Л. ДЬЮРИШ

Следующие исправления окажутся нужными в связи с рисунками, относящимися к соотношениям I. (21), I. (22). I. (23) в конце 2. §. I.:

1. Иллюстрация под правой стороной I. (23) является ненужной; а к четырем предыдущим рисункам относится, что все нужно положить с одним местом вперед (последний еще обратить тоже). То есть рисунок под левой стороной I. (21) положится под правую сторону I. (21), что там находится „идет" под I. (22) и т. д..

2. Следующая фигура нужна под левую сторону I. (21):



Рисс. 1

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 235—237.

BOOK REVIEW

N. BOURBAKI: Élements de mathématique, XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines.

— ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES NO. 1337. HERMANN, PARIS, 1971.

Bien que le présent ouvrage porte le titre «Groupes et algèbres de Lie», ces structures ne sont mentionnées que dans l'Introduction et dans la Note historique. Le sujet proprement dit traite des systèmes de racines, groupes de Coxeter et systèmes de Tits. Pour en savoir l'origine, citons un extrait de la Note historique: „Les groupes étudiés dans ces chapitres sont apparus à propos de questions variées de Géométrie, d'Analyse et de Théorie des groupes de Lie, tantôt sous forme de groupes de permutations, tantôt sous forme de groupes de déplacements en géométrie euclidienne ou hyperbolique, et ces divers points de vue n'ont été coordonnés qu'à date récente". L'introduction propose des exemples repris à la théorie des groupes et algèbres de Lie qui conduisent à la considération des structures traitées dans le livre, mais le traité lui-même est indépendant de la théorie des groupes de Lie.

Quant au contenu de l'ouvrage, en voici les titres des paragraphes. Les précisions mises entre parenthèses sont destinées à donner une idée plus proche de la matière traitée.

Chapitre IV: § 1. Groupes de Coxeter (Caractérisation par la «condition d'échange», matrices et graphes de Coxeter). — § 2. Systèmes de Tits (Relations avec les systèmes de Coxeter, théorème de simplicité de Tits). — Annexe: Graphes (quelques notions élémentaires).

Chapitre V: § 1. Hyperplans, chambres et facettes. — § 2. Réflexions. — § 3. Groupes de déplacements engendrés par des réflexions (Relation des chambres et des réflexions avec les systèmes de Coxeter, thévrèmes de finitude, décompositions, points spéciaux). — § 4. Représentation géométrique d'un groupe de Coxeter (Résultats réciproques à ceux du dernier paragraphe). — § 5. Invariants dans l'algèbre symétrique (Séries de Poincaré, développement des travaux de Chevalley). — § 6. Transformation de Coxeter (Considérations sur les valeurs propres d'une transformation de Coxeter et les exposants du groupe de Coxeter). — Annexe: Compléments sur les représentations linéaires (Théorème de Maschke).

Chapitre VI: § 1. Systèmes de racines (L'étude des systèmes de racines y est menée en utilisant des chambres; poids, matrices de Cartan). — § 2. Groupe de Weylaffine (Points spéciaux, normalisateur, groupes cristallographiques. Comme application des résultats, on y calcule également l'ordre du groupe de Weyl). — § 3. Invariants exponentiels (On étudei les éléments invariants et anti-invariants de l'algèbre du groupes des poids). — § 4. Classification des systèmes de racines (Groupes de Coxeter finis, graphes de Dynkin, graphes de Dynkin complétés, construction des systèmes de racines). — A la fin de l'ouvrage, on trouve également un Résumé des principales propriétés des systèmes de racines, ainsi que 10 planches qui contiennent des résultats obtenus dans le Ch. VI. § 4. sur les systèmes A-G.

Continuant les traditions de Bourbaki, le présent ouvrage contient une Note historique éclairante et une riche collection d'exercices, où, parmi les exemples et les applications des résultats obtenus dans le texte, on trouve également des résultats inédits.

L. Márki

N. BOURBAKI: Éléments de mathématique, XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie.

— ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES N° 1285. HERMANN, PARIS, 1971.

Ce livre traite de la théorie générale des algèbres de Lie. L'exposition est strictement logique, ce qui est habituel chez Bourbaki. Voici la table des matières:

§ 1. Définition des algèbres de Lie.
§ 2. Algèbre enveloppante d'une algèbre de Lie.
§ 3. Représentations.
§ 4. Algèbres de Lie nilpotentes.
§ 5. Algèbres de Lie résolubles.
§ 6. Algèbres de Lie semi-simples.
§ 7. Théorème d'Ado.

La compréhension du texte est facilitée par des exemples bien choisis dont un grand nombre est destiné a montrer la relation qui existe entre les groupes de Lie et les algèbres de Lie. De nombreux exercices complètent le livre. Ils servent non seulement d'exemples, mais ils touchent également des domaines qui ne sont pas traités dans le texte (par exemple: p-algèbres de Lie, propriétés cohomologiques, etc.).

Ce volume en est à son deuxième tirage. Il ne diffère du premier qu'à peu de choses près (l'index terminologique est élargi; la démonstration du lemme 1, § 5. No. 3. est simplifiée grâce à un résultat d'Alg. chap. VII; le § 6. No. 2. Lemme 1 contient également l'ancien cor. 2 de la prop. 2 du même paragraphe).

L. Márki

W. HAACK—W. WENDLAND: Vorlesungen über partielle und Pfaffsche Differentialgleichungen.

(Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Mathematische Reihe Nr. 39.)

BIRKHÄUSER, BASEL—STUTTGART 1969, 555 S.

In diesem Buch haben die Verfasser für die Behandlung partieller Differentialgleichungen den unkonventionellen Weg gewahlt, dass sie diese als Pfaffsche Gleichungen auffassen. Im Falle linearer partieller Differentialgleichungen zweiter Ordnung ist durch die invariante quadratische Form der Charakteristiken eine Polarverwandtschaft gegeben, die jeder linearen Pfaffschen form im n dimensionalen Raum eine konjugierte Form $(n-1)$-ter Ordnung zuordnet. Eine besonders wichtige Rolle in dieser Theorie spielt die konjugierte Form des Differentials dF einer Funktion F von n Veränderlichen. Mit Hilfe dieser konjugierten Form kann man z. B. die Beltramische Differentialgleichung in einer äusserst bündigen Form angeben. Durch Benutzung des Integralsatzes von Gauss-Cartan kann man dann die Differentialgleichung sofort in eine Integralgleichung überführen.

In dem ersten Teil des Buches werden lineare Differentialgleichungen zweiter Ordnung in zwei Veränderlichen betrachtet. Behandelt werden hier das Randwertproblem der elliptischen, und das Anfangs-Randwertproblem der parabolischen Differentialgleichung, das Cauchysche Problem der hyperbolischen Gleichung, Gleichungen von Laplaceschen Typ und Greensche Funktion.

Gegenstand des zweiten Teiles sind Systeme von Gleichungen erster Ordnung in zwei Veränderlichen. Sie werden wiederum auf Pfaffsche Systeme zurückgeführt. Die Theorie der hyperbolischen Systeme wird eben nur berührt, elliptische Systeme werden aber ausführlich behandelt.

Im dritten und letzten Teil wird die Theorie Pfaffscher Formen von mehr als zwei Veränderlichen geschildert. Hilfsmittel sind das äussere Produkt, die äussere Ableitung, ko- und kontravariante Darstellung linearer Mannigfaltigkeiten. In diesem Teile werden u.a integrierbare und nicht integrierbare Pfaffsche Systeme, der Existenzsatz von E. Cartan, Legendresche und kanonische Transformationen und die Hadamardsche Theorie der hyperbolischen Differentialgleichungen zweiter Ordnung behandelt.

Das Buch zeigt, dass die Verknüpfung von Differentialgleichungen und Pfaffscher Formen zweifellos den Vorteil hat, dass gewisse Sätze einfach behandelt werden können.

E. Makai

FENYŐ, STEFAN: Moderne mathematische Methoden in der Technik. Band 2. (International Series of Numerical Mathematics, Vol. 11)

BIRKHAUSER VERLAG, BASEL UND STUTTGART, 1971. 336 P. 79 FIGURES. — FR. 62.—

In contrast to the first volume, the reviewed one deals with finite methods of applied mathematics. Part 1 is devoted to linear algebra. It contains the elements of matrix algebra and matrix analysis; they are, then, applied to the theory of systems of linear equations, integral equations, systems of differential equations and to special related problems of engineering mechanics and electrical networks.

Part 2 is entitled „the theory of optimization". In fact, it considers the elements of linear programming as well as transport problems solved by means of the „Hungarian method" (König-Egerváry). Convex programming is stressed, too.

Part 3. considers the elements of graph theory, further, problems concerning trees, graphs on surfaces, vector spaces generated by graphs, directed graphs and the corresponding matrices. Finally graph theoretical methods in the investigation of networks and the theorem of Ford and Fulkerson are described.

The separate parts are independent of other parts of the whole work. The treatment is up to date and carefully composed. Everywhere several modern results have been included in the text, too. Numerical methods are avoided.

The material included in the book is considerable. A greater number of worked out examples might, however, have counterbalanced the concise style of the main text. The bibliography contains 25 items, mainly in German; books of Hungarian mathematicans have also been included. — Unfortunately there are several striking misprints in the text, e.g. in the Foreword and the Contents.

P. Medgyessy

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Printed in Hungary

A kiadásért felel az Akadémiai Kiadó igazgatója — Műszaki szerkesztő: Botyánszky Pál
A kézirat nyomdába érkezett: 1972. febr. 8. — Terjedelem: 21 (A/5) ív, 19 ábra

72-659 — Szegedi Nyomda

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Studia Scientiarum Mathematicarum Hungarica

A Magyar Tudományos Akadémia matematikai folyóirata

Szerkesztőség: 1061 Budapest V., Reáltanoda u. 13—15.

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It is published semiannually, making up one volume per year.

Editorial Office: 1061 Budapest V., Reáltanoda u. 13—15, Hungary.
Technical Editor: Gy. Petruska

Subscription rate: $ 16.00 per volume. Orders may be placed with *Kultúra* Trading Co. for Books an Newspapers, Budapest 62, P. O. B. 149 or with its representatives abroad.

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Studia Scientiarum Mathematicarum Hungarica 7 (1972) 239—241

EXTREMAL GRAPHS OF DIAMETER TWO WITH PRESCRIBED MINIMUM DEGREE

J. A. BONDY and U. S. R. MURTY

Let $G_{n,m}$ denote a graph on n vertices and m edges. In [1] ERDŐS, RÉNYI and SÓS considered the problem of determining the least value of m such that there exists a $G_{n,m}$ which is of diameter 2 and in which the maximum degree is k, where n and k are given positive integers, $n>k$. In this note we consider the following problem: what is the least value $f(n, k)$ of m such that there exists a $G_{n,m}$ of diameter 2 and in which the minimum degree is k? When $m=f(n, k)$ such a graph is called an *extremal graph.* If t is an integer the function α is defined by the rule that $\alpha(t)=0$ or 1 according as t is odd or even.

THEOREM 1. *If* $n > k^3+\alpha(n)\cdot\alpha(k)\cdot k+1$ *then*

$$f(n, k) = \left[\frac{(n-1)(k+1)+1}{2}\right]$$

and every extremal graph has a vertex of degree $n-1$.

PROOF. For the sake of simplicity we shall only present the proof in the case when at least one of n and k is odd, that is when $\alpha(n)\cdot\alpha(k) = 0$. The proof of the case when $\alpha(n)\cdot\alpha(k) = 1$ is similar. We shall also assume that $k>1$. The theorem is obvious for $k=1$.

Let H' be a $(k-1)$-regular graph on $n-1$ vertices. Let H be the graph obtained by adding a vertex to H' and joining it to all the vertices of H'. Clearly H is of diameter 2, the minimum degree in H is k and H has $(n-1)(k+1)/2$ edges. We thus have

$$f(n, k) \leqq \frac{(n-1)(k+1)}{2}. \tag{1}$$

Now let $G_{n,f(n,k)}$ be an extremal graph and let $x_1, x_2, \ldots, x_n$ be its vertices, with degrees $d_1, d_2, \ldots, d_n$ such that $d_1 \geqq d_2 \geqq \cdots \geqq d_n = k$. Let N_k denote the number of vertices of degree k. Then $d_1 \leqq N_k$ since

$$d_1+k\cdot N_k+(k+1)(n-N_k-1) \leqq 2f(n, k) \leqq (n-1)(k+1). \tag{2}$$

Now if there is a vertex x_i of degree k not joined to x_1, then the number of vertices at a distance not greater than 2 from x_i is at most

$$\sum_{j=2}^{k+1} d_j+1.$$

Since G is of diameter 2,

$$\sum_{j=2}^{k+1} d_j + 1 \geqq n$$

and hence

$$\sum_{j=1}^{k+1} d_j \geqq \frac{(k+1)(n-1)}{k}.$$

We now have

$$\frac{(k+1)(n-1)}{k} + k(n-k-1) \leqq \sum_{j=1}^{n} d_j \leqq (n-1)(k+1)$$

or

$$n \leqq k^3 + 1,$$

This contradiction establishes that x_1 is joined to all vertices of degree k and hence $d_1 = N_k$. The theorem follows if $N_k = n-1$. But if $N_k < n-1$ then (2) implies that $d_i = k+1$ for all i, $2 \leqq i \leqq n - N_k$. Let x_j be a vertex of degree $k+1$. The degrees of neighbours of x_j are bounded above by $k+1$. It follows from the fact that G is of diameter 2 that

$$k(k+1) \geqq n-1$$

or

$$n \leqq k^2 + k + 1 < k^3 + 1, \quad \text{since} \quad k > 1.$$

This contradiction implies that $N_k = n-1$ and the theorem is proved.

Let $g(n, k)$ $\big(h(n, k)\big)$ denote the least number of edges a k-connected (k-edge-connected) graph of diameter 2 on n vertices can have.

THEOREM 2. *If* $n > k^3 + \alpha(n) \cdot \alpha(k) \cdot k + 1$ *then* $h(n, k) = f(n, k)$ *and, further, if* $k \neq 2$ *then* $g(n, k) = f(n, k)$.

PROOF. The case when $k=1$ is trivial. So we shall assume that $k>1$. Since a k-connected (k-edge-connected) graph has no vertices of degree less than k it follows that $g(n, k) \geqq f(n, k)$ and $h(n, k) \geqq f(n, k)$. When $k=2$ the graph H of theorem 1 is 2-edge connected. If follows that $h(n, 2) = f(n, 2)$. When $k > 2$ H' in theorem 1 can be chosen so that it is $(k-1)$-connected $\big((k-1)$-edge-connected$\big)$. The existence of such a graph follows from HARARY [2]. Now, if H' in theorem 1 is chosen to have the additional property described above, H of theorem 1 will be k-connected (k-edge-connected). The theorem follows.

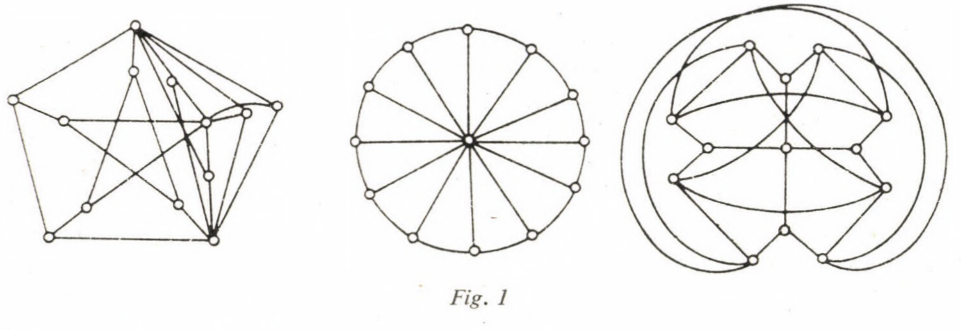

Fig. 1

In [3] MURTY showed that $g(n, 2) = 2n-5$ for $n \geqq 5$ and characterized extremal 2-connected graphs of diameter 2. It follows from theorem 2 that $g(n, 3) = 2n-2$ for $n \geqq 29$. Further, since the only 2-connected 2-regular graphs are cycles, it follows that for $n \geqq 29$ the only extremal 3-connected graphs of diameter 2 are wheels. It may be shown that $g(n, 3) = \left[\frac{3n+1}{2}\right]$ for $n \leqq 10$, $g(11, 3)=18$, $g(12, 3)=21$ and $g(13, 3)=24$. Figure 1 shows the extremal graphs when $n=13$.

By strengthening the arguments in the proof of theorem 1 we can show that $g(n, 3) = 2n-2$ for $n \geqq 18$, but we do not know if there exists an n, $13 < n < 18$ such that $g(n, 3) \leqq 2n-3$. Nor do we know a counter-example to the conjecture that when $n \geqq 14$ the extremal 3-connected graphs of diameter 2 are wheels.

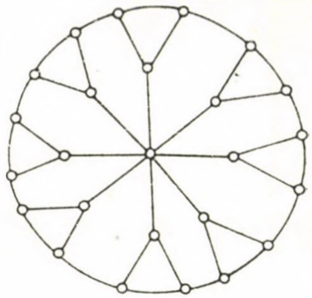
Fig. 2

Let $f(n, k, d)$ denote the least number m such that there exists a $G_{n,m}$ of diameter d in which the minimum degree is k. Is it true that there exists an n_0 such that $f(3n+1, 3, 4) = 5n$ for all $n \geqq n_0$?

The construction illustrated above shows that $f(3n+1, 3, 4) \leqq 5n$. This construction was suggested to us by Professor ERDŐS.

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Studia Scientiarum Mathematicarum Hungarica 7 (1972) 243—249

POISSON PROCESSES DEFINED ON AN ABSTRACT SPACE

by

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We prove two theorems for Poisson type processes defined on an abstract space, and show two applications of these theorems. Observe that theorems 1 and 2 are extensions of two results by A. RÉNYI [1] and the proofs also essentially follow his ideas.

Let X be an arbitrary abstract set, $\mathcal{R}$ a ring of subsets of X, and λ an additiv set function on $\mathcal{R}$. Let $(\Omega, \mathcal{A}, \mathsf{P})$ denote a probability space, and $\xi(A, \omega)$ a non-negative integer valued function, which is, for each $A \in \mathcal{R}$, a random variable on $(\Omega, \mathcal{A}, \mathsf{P})$, and for each $\omega \in \Omega$, additive on $\mathcal{R}$. I.e., for arbitrary $A, B \in \mathcal{R}$ and $A \cdot B = \emptyset$, let

$$\xi(A+B, \omega) = \xi(A, \omega)+\xi(B, \omega). \tag{1}$$

Assume that

I. for each $\varepsilon > 0$ and $A \in \mathbf{R}$, there exists a decomposition $A = \sum_{i=1}^{n} \mathrm{A}_i$ of A with $A_i \in R$ and $\lambda(A_i) < \varepsilon$ $(i=1, 2, \dots, n)$, and

II. for each $A \in \mathcal{R}$,

$$\mathsf{P}(\xi(A)=0) = e^{-\lambda(A)} \tag{2}$$

and

$$\mathsf{P}(\xi(A) \geqq 2) \leqq \lambda(A)\delta(\lambda(A)) \tag{3}$$

where

$$\lim_{x \to 0} \delta(x) = 0.$$

THEOREM 1. *If conditions* I *and* II *are met then, for any disjoint set $A_1, \dots, A_n \in \mathcal{R}$, the variables $\xi(A_1), \dots, \xi(A_n)$ are independent.*

PROOF. Let $B(A)$ be, for each $A \in \mathcal{R}$, the event $\{\xi(A)=0\}$. Let $A = \sum_{i=1}^{n} A_i$, where $A_1, A_2, \dots, A_n \in \mathcal{R}$ are disjoint sets. Then

$$B(A) = \{\xi(A)=0\} = \left\{\sum_{i=1}^{n} \xi(A_i)=0\right\} = \prod_{i=1}^{n} \{\xi(A_i)=0\} = \prod_{i=1}^{n} B(A_i),$$

and

$$\mathsf{P}(B(A)) = \mathsf{P}(\xi(A)=0) = e^{-\lambda(A)} = \prod_{i=1}^{n} e^{-\lambda(A_i)} = \prod_{i=1}^{n} \mathsf{P}(B(A_i)).$$

Hence the events $B(A_1), \dots, B(A_n)$ are independent. Let $J_{\overline{B(A)}}$ denote the indicator of the event $\overline{B(A)}$, i.e.

$$J_{\overline{B(A)}} = \begin{cases} 0, & \text{if } \xi(A)=0 \\ 1, & \text{if } \xi(A)>0 \end{cases}.$$

Then, for disjoint sets $A_1, \dots, A_n \in \mathcal{R}$, the random variables $J_{\overline{B(A_1)}}, \dots, J_{\overline{B(A_n)}}$ are independent.

Let $C, D \in \mathcal{R}$ be two disjoint sets. For any arbitrary $\eta>0$, there exist disjoint decompositions $C = \sum_{i=1}^{n} C_i$ and $D = \sum_{i=1}^{m} D_i$, for which $\lambda(C_i)<\eta$ $(i=1, 2, \dots, n)$ and $\lambda(D_i)<\eta$ $(i=1, \dots, m)$ (see I.).

Then the random variables $J_{\overline{B(C_1)}} \dots J_{\overline{B(C_n)}}, J_{\overline{B(D_1)}} \dots J_{\overline{B(D_m)}}$ and, therefore, the random variables $\sum_{i=1}^{n} J_{\overline{B(C_i)}}$ and $\sum_{i=1}^{m} J_{\overline{B(D_i)}}$ are also independent.

Let us consieder the event

$$\left\{\xi(C) \neq \sum_{i=1}^{n} J_{\overline{B(C_i)}}\right\}.$$

Since

(4) $$\left\{\xi(C) \neq \sum_{i=1}^{n} J_{\overline{B(C_i)}}\right\} = \left\{\xi(C) > \sum_{i=1}^{n} J_{\overline{B(C_i)}}\right\},$$

we have

(5) $$\left\{\xi(C) \neq \sum_{i=1}^{n} J_{\overline{B(C_i)}}\right\} \subset \left\{\max_{1 \leq i \leq n} \xi(C_i) \geq 2\right\}.$$

For each $\varepsilon>0$, let $\eta>0$ be such that $\delta(\eta)<\varepsilon$. Since $\lim_{x \to 0} \delta(x)=0$ there exists such an η. From (5) we have

(6) $$\mathsf{P}\left(\xi(C) \neq \sum_{i=1}^{n} J_{\overline{B(C_i)}}\right) \leq \mathsf{P}\left(\max_{1 \leq i \leq n} \xi(C_i) \geq 2\right) \leq \sum_{i=1}^{n} \mathsf{P}(\xi(C_i) \geq 2) \leq$$

$$\leq \sum_{i=1}^{n} \lambda(C_i)\delta(\lambda(C_i)) \leq \varepsilon\lambda(C)$$

Following the same argument, we have

(7) $$\mathsf{P}\left(\xi(D) \neq \sum_{i=1}^{m} J_{\overline{B(D_i)}}\right) \leq \varepsilon\lambda(D)$$

From the independence of the random variables $\sum_{i=1}^{n} J_{\overline{B(C_i)}}$ and $\sum_{i=1}^{m} J_{\overline{B(D_i)}}$, and from (6) and (7), we have

$$|\mathsf{P}(\xi(C)=k) \cdot \mathsf{P}(\xi(D)=l) - \mathsf{P}(\xi(C)=k, \xi(D)=l)| < 3\varepsilon\lambda(C+D)$$

for any nonnegative k, l integers and any $\varepsilon>0$. If $A_1, \dots, A_n \in \mathcal{R}$ are arbitrary disjoint sets, then for $n=2$ the independence of $\xi(A_1), \dots, \xi(A_n)$, has been already proved. We may extend the proof to $n>2$ by induction.

THEOREM 2. *If conditions* I *and* II *are met then, for each* $A \in \mathcal{R}$ *and nonnegative integer* k, *we have*

$$\mathsf{P}(\xi(A)=k) = \frac{\lambda(A)^k}{k!} e^{-\lambda(A)}$$

PROOF. Let $\varphi_A(t)$ denote, for each $A \in \mathcal{R}$, the characteristic function of the random variable $\xi(A)$, i.e.

$$\varphi_A(t) = \mathsf{M}(e^{it\xi(A)}).$$

Since

$$|\varphi_A(t)| \geqq 2e^{-\lambda(A)} - 1,$$

we have

$$|\varphi_A(t)| \neq 0, \quad \text{for} \quad \lambda(A) < \log 2.$$

Because of condition I, A has a disjoint decomposition $A = \sum_{k=1}^{n} A_k$ with $\lambda(A_k) < \log 2$; i.e. $\varphi_{A_k}(t) \neq 0$ $(k=1, 2, \dots, n)$. But, because of Theorem 1, the random variables $\xi(A_1), \dots, \xi(A_n)$ are independent, hence

$$\varphi_A(t) = \prod_{k=1}^{n} \varphi_{A_k}(t) \neq 0,$$

and we have

$$\log \varphi_A(t) = \sum_{k=1}^{k} \log \varphi_{A_k}(t). \tag{8}$$

From condition II

$$\mathsf{P}(\xi(A_k) \geqq 2) \leqq \lambda(A_k)\delta(\lambda(A_k)) \qquad (k=1, 2 \dots n).$$

From this we get,

$$|\varphi_{A_k}(t) - e^{-\lambda(A_k)} - e^{it}(1-e^{-\lambda(A_k)})| \leqq$$
$$\leqq \left| \sum_{l=2}^{\infty} \mathsf{P}(\xi(A_k)=l)e^{ilt} + \mathsf{P}(\xi(A_k)=1)e^{it} - e^{it}(1-e^{-\lambda(A_k)}) \right| \leqq$$
$$\leqq 2\mathsf{P}(\xi(A_k) \geqq 2) \leqq 2\lambda(A_k)\delta(\lambda(A_k))$$

Hence there exists a complex valued function $\varrho(x)$ of a real variable, for which $|\varrho(x)| \leqq 1$, $\varrho(0)=0$, and

$$\varphi_{A_k}(t) = e^{-\lambda(A_k)} + e^{it}(1-e^{-\lambda(A_k)}) + 2\lambda(A_k)\delta(\lambda(A_k))\varrho(\lambda(A_k)). \tag{9}$$

By (9) we have

$$\lim_{\lambda(A_k)\to 0} \frac{\log \varphi_{A_k}(t)}{\lambda(A_k)} = \frac{d}{dx} \log[e^{-x} + e^{it}(1-e^{-x}) + 2x\delta(x)\varrho(x)]_{x=0} = -1 + e^{it}$$

from which

$$\log \varphi_{A_k}(t) = \lambda(A_k)(e^{it}-1) + \lambda(A_k)\varepsilon(\lambda(A_k)), \tag{10}$$

where

$$\lim_{x \to 0} \varepsilon(x) = 0.$$

Substituting (10) into (8), we have

$$\log \varphi_A(t) = \lambda(A)(e^{it}-1) + \sum_{k=1}^{n} \lambda(A_k)\varepsilon(\lambda(A_k)). \tag{11}$$

It follows from (11), that $\sum_{k=1}^{n} \lambda(A_k)\varepsilon(\lambda(A_k))$ is a constant, and

$$\left| \sum_{k=1}^{n} \varepsilon(\lambda(A_k))\lambda(A_k) \right| \leq \lambda(A) \max_{1 \leq k \leq n} |\varepsilon(\lambda(A_k))| \to 0,$$

for any $\lambda(A_k) \to 0$.

Hence

$$\sum_{k=1}^{n} \lambda(A_n)\varepsilon(\lambda(A_k)) = 0. \tag{12}$$

From (11) and (12) we have

$$\log \varphi_A(t) = \lambda(A)(e^{it}-1)$$

i. e.

$$\varphi_A(t) = e^{\lambda(A)(e^{it}-1)}$$

and

$$\mathsf{P}(\xi(A)=k) = \frac{\lambda(A)^k}{k!} e^{-\lambda(A)}.$$

Let $\xi_1(A, \omega), \xi_2(A, \omega), \ldots, \xi_n(A, \omega)$ take as values nonnegative integers, and be, for all $A \in \mathscr{R}$, random variables on (Ω, A, P), and also additive, for all $\omega \in \Omega$, on $\mathscr{R}$.

THEOREM 3. *Let, for each* $A \in \mathscr{R}$,

$$\mathsf{P}(\xi_i(A)=k) = \frac{\lambda_i(A)^k}{k!} e^{-\lambda(A)}, \qquad (i=1 \ldots n), \tag{13}$$

and the set function $\lambda(A) = \sum_{i=1}^{n} \lambda_i(A)$ *be a measure on* $\mathscr{R}$, *according to condition* I.

If, for each $A \in \mathscr{R}$, *we have*

$$\mathsf{P}\left(\prod_{i=1}^{n} \{\xi_i(A)=k_i\} \right) = \prod_{i=1}^{n} \mathsf{P}(\xi_i(A)=k_i), \tag{14}$$

for all nonnegative integers $k_1, k_2, \ldots k_n$ *for which*

$$\sum_{i=1}^{n} k_i \leq 1,$$

then

$$\mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=k \right) = \frac{\lambda(A)^k}{k!} e^{-\lambda(A)}$$

for any $A \in \mathscr{R}$.

PROOF. Let $A \in \mathcal{R}$ be an arbitrary set, then from (14), we get

$$(15)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=0\right) = \mathsf{P}\left(\prod_{i=1}^{n} \{\xi_i(A)=0\}\right) = \prod_{i=1}^{n} \mathsf{P}(\xi_i(A)=0),$$

on account of (13) and (15) we have

$$(16)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=0\right) = \prod_{i=1}^{n} e^{-\lambda_i(A)} = e^{-\lambda(A)}$$

Since

$$(17)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)\geqq 2\right) = 1-\mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)\leqq 1\right),$$

we have from (16)

$$(18)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)\geqq 2\right) = 1-e^{-\lambda(A)}-\mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=1\right).$$

From (14)

$$(19)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=1\right) = \mathsf{P}\left(\sum_{k=1}^{n} \prod_{i=1}^{n} \{\xi_i(A)=\delta_{ik}\}\right) = \sum_{k=1}^{n} \prod_{j=1}^{n} \mathsf{P}(\xi_i(A)=\delta_{ik}),$$

where δ_{ik} stands for the Kronecker symbol.

By (13) and (19)

$$(20)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=1\right) = \sum_{k=1}^{n} \prod_{i=1}^{n} \lambda_i(A)^{\delta_{ik}} e^{-\lambda_i(A)} = \lambda(A)e^{-\lambda(A)}.$$

From (20) through (18)

$$(21)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)\geqq 2\right) = 1-e^{-\lambda(A)}-\lambda(A)e^{-\lambda(A)}.$$

Because of (16) and (21), the condition II is met, and $\lambda(A)$ meets condition I. Hence from Theorem 2, we get

$$\mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=k\right) = \frac{\lambda(A)^k}{k!} e^{-\lambda(A)}$$

for any $A \in \mathcal{R}$.

THEOREM 4. *Let, for any $A \in \mathcal{R}$,*

$$(22)\qquad \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=k\right) = \frac{\lambda(A)^k}{k!} e^{-\lambda(A)}$$

where $\lambda(A)$ is a measure on $\mathcal{R}$, according to condition I.

If, for all $A \in \mathcal{R}$,

$$(23)\qquad \mathsf{P}\left(\prod_{i=1}^{n} \{\xi_i(A)=0\}\right) = \prod_{i=1}^{n} \mathsf{P}(\xi_i(A)=0)$$

and

$$(24)\qquad \mathsf{P}(\xi_i(A)=0) = a_i^{F(\lambda(A))}, \qquad (i=1, 2 \dots n)$$

$a_i > 1$ *being an arbitrary number, and* $F(x)$ *an arbitrary real valued function, then*

$$\mathsf{P}(\xi_i(A)=k) = \frac{\mu_i(A)^k}{k!} e^{-\mu_i(A)}$$

for

$$u_i(A) = \lambda(A) \frac{\log a_i}{\sum_{l=1}^{n} \log a_l} \qquad (i=1, 2 \dots n).$$

PROOF. Because of (22) and (23)

$$e^{-\lambda(A)} = \mathsf{P}\left(\sum_{i=1}^{n} \xi_i(A)=0\right) = \mathsf{P}\left(\prod_{i=1}^{n} \{\xi_i(A)=0\}\right) = \prod_{i=1}^{n} \mathsf{P}(\xi_i(A)=0). \tag{25}$$

From (24)

$$e^{-\lambda(A)} = \prod_{i=1}^{n} a_i^{F(\lambda(A))}.$$

Hence

$$F(\lambda(A)) = \frac{-\lambda(A)}{\sum_{l=1}^{n} \log a_l}$$

$$\mathsf{P}(\xi_i(A)=0) = e^{-\lambda(A)\frac{\log a_i}{\sum_{l=1}^{n} \log a_l}}, \qquad (i=1, 2 \dots n). \tag{26}$$

Observe that (2), in condition II, and also, trivially, (3) are met, because, for any $A \in \mathcal{R}$,

$$\{\xi_i(A) \geqq 2\} \subset \left\{\sum_{l=1}^{n} \xi_l(A) \geqq 2\right\}, \qquad (i=1, 2 \dots n),$$

and, therefore,

$$\mathsf{P}(\xi_i(A) \geqq 2) \leqq \mathsf{P}\left(\sum_{l=1}^{n} \xi_l(A) \geqq 2\right), \qquad (i=1, 2 \dots n).$$

Condition I is also met for all $i=1, 2, \dots, n$, because, if $\lambda(A)$ meets condition I then, for an arbitrary $C>0$, $C\lambda(A)$ meets also condition I.

Since both conditions in Theorem 2 are met, we have, for any $A \in \mathcal{R}$,

$$\mathsf{P}(\xi_i(A)=k) = \frac{u_i(A)^k}{k!} e^{-u_i(A)} \qquad (i=1, 2 \dots n).$$

Here

$$u_i(A) = \lambda(A) \frac{\log a_i}{\sum_{l=1}^{n} \log a_l}.$$

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(Received February 2, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 249—256

MAXIMA AND MINIMA OF FUNCTIONS AND FUNCTIONALS WITH RESTRICTED ANALYTIC PROPERTIES

by

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1. Introduction

In this paper a method is developed for treating the classical free endpoint problem of the calculus of variations: $I(y) = \int_0^1 f(x, y, y')\,dx$ under hypotheses less then $C^{(3)}$. A method also is presented for determining maxima and minima of real functions $y=f(x)$ subject to the restriction of continuity and isolated extremes. Finally, the problem of determining a continuous function $y=y(x)$ defined on $0 \leqq x \leqq 1$ such that the integral $I(y) = \int_0^1 f(x, y(x))\,dx$ is minimized under mild conditions on the integrand. The technique employed is that of treating the function f as the limit of a family of approximants, each of which has the differentiability properties required by conventional methods of analysis. The optimizing values of functions for these approximants are shown to converge to an optimizing value or function for the original problem for a suitably selected family of approximants. Since not every arbitrarily chosen sequence of approximants to f preserves extremes under limiting processes, the strength of the method lies in the choice of the approximations that are used.

2. Weighting and spline kernels

Definition 2. 1. A weighting kernel $w_\delta(x)$ is defined as a function with the properties:

(1) $w_\delta(x)$ is continuous;
(2) $w_\delta(x) > 0, \ |x| < \delta$;
(3) $w_\delta(x) = 0, \ |x| \geqq \delta$;
(4) $w_\delta(x) = w_\delta(-x)$;
(5) $w_\delta(|x|)$ is monotone decreasing, but not necessarily strictly so;
(6) $\int_{-\delta}^{\delta} w_\delta(\tau)\,d\tau = 1$.

Although the step function $s_\delta(x) = \frac{1}{2\delta}$ for $|x| < \delta$ and $s_\delta = 0$ for $|x| \geqq 0$ does not satisfy condition (1) as a weighting kernel, it has properties that are similar to those of this class of functions, and so useful in the associated analysis that it will be convenient, when so indicated, to include it as a special member of this class. For instance $s_\delta(x)$ corresponds to the spline kernel of degree zero, in the following inductive definition.

Definition 2. 2. A spline kernel of degree zero is the function $k_\delta^{(0)}(x)=s_\delta(x)$; a spline kernel of degree $n+1$ is the function

$$k_\delta^{(n+1)}(x) = \begin{cases} \frac{1}{a} \int\limits_{-\delta}^{x} k_{\delta/2}^{(n)}(\tau+\delta/2)\,d\tau, & (-\delta \leqq x \leqq 0) \\ \frac{1}{2} \int\limits_{-\delta}^{0} k_{\delta/2}^{(n)}(\tau+\delta/2)\,d\tau - \frac{1}{a} \int\limits_{0}^{x} k_{\delta/2}^{(n)}(\tau-\delta/2)\,d\tau, & (0 \leqq x \leqq \delta) \\ 0 & x < -\delta,\ x > \delta, \end{cases}$$

where

$$a = 2 \int\limits_{-\delta}^{0} k_{\delta/2}^{(n)}(\tau+\delta/2)\,d\tau.$$

That spline kernels of degree $n \geqq 1$ are weighting kernels follows readily from the definition. The following important lemma is easily proven by an inductive argument also applied directly to the definition.

LEMMA 2. 1. *Each $k_\delta^{(n)}(x)$ is a function of class $C^{(n-1)}$, for $n \geqq 1$.*

The term: spline, is meaningful since the $k_\delta^{(n)}(x)$ is composed of 2^n polynomial subarcs of degree n which are pieced together, in accordance with the usual concept of a mathematical spline, in such a way that continuity of all the derivatives up to and including those of order $(n-1)$ is preserved at the joints. A function may have an extreme at an endpoint of an interval which may or may not be achieved. Such endpoint extremes are of no interest to us here and are expressly excluded from the following definition.

Definition 2. 3. If $f(x)$ is a continuous function defined on an interval, then the point $(p, f(p))$ is a minimum point of $f(x)$ of class (A) if and only if there is a sufficiently small open interval containing p such that $(p, f(p))$ is a strict local minimum of $f(x)$ relative to this interval and no other point of this interval is a local minimum of $f(x)$.

A similar definition of a maximum point of $f(x)$ of class (A) applies.

3. Approximants

Consider the function $y=|x|$. Lacking adequate differentiability properties, its minimum cannot be determined by solving $\frac{dy}{dx} = 0$. However, a sequence of approximating curves tending to $|x|$ can be constructed, the minima of which can be ascertained by the conventional method and converge to the minimum of $|x|$. This procedure is fraught with dangers for a given sequence of minima of the approximants may converge to a point which is not a minimum of $|x|$, or a sequence of maxima may converge to a minimum. A host of other pathological cases is easily constructed for other functions.

Our purpose is to describe approximating functions $\varphi_\delta(x)$ which have the property that sequences which converge to a limit function, preserve convergence of extremes to an extreme of the limit function. The approximants can be chosen to have differentiability properties of any prescribed order.

Definition 3. 1. Let $w_\delta(x)$ be a weighting kernel, and $f(x)$ an integrable function defined on the interval $I=[0, 1]$. Relative to this $w_\delta(x)$ the function $\varphi_\delta(x)\equiv\varphi_\delta(x; f; w_\delta)$ is defined by:

$$\varphi_\delta(x) = \int_{-\delta}^{\delta} f(x+\tau)w_\delta(\tau)\,d\tau = \int_{x-\delta}^{x+\delta} f(\lambda)w_\delta(\lambda-x)\,d\lambda$$

on the interval $I' = [\delta, 1-\delta]$.

The following lemma has a direct proof which is not given.

LEMMA 3. 1. *If $(p, f(p))$ is an extreme for $f(x)$ of class (A), there is an open interval I containing p such that for all δ sufficiently small any extreme of $\varphi_\delta(x)$ occurs in $J\subset I$, where $J \equiv \{x | p-\delta \leqq x \leqq p+\delta\}$. At least one extreme occurs on J.* This lemma applies for $w_\delta(x)=s_\delta(x)$.

4. Functions of one variable

It was pointed out at the beginning of the previous section that not always can the extreme of a function be determined as the limit of extremes of approximating sequences. In this section we show that for certain types of weighting kernels, viz. spline kernels, this condition can be brought about for the associated sequences of approximants $\varphi_\delta(x)$.

LEMMA 4. 1. *If $f(x)$ has a minimum of class (A) at $(p, f(p))$ and $\varphi_\delta(x)$ is defined for the kernel $s_\delta(x)$, for δ sufficiently small there is an open interval around p such that $\varphi_\delta(x)$ has a unique minimum of class (A) on this interval.*

PROOF. By lemma 3. 1, there is at least one minimum $(q, \varphi_\delta(q))$ of $\varphi_\delta(x)$ on J, and there $\varphi_\delta'(q)=0$. But then

$$\varphi_\delta'(q) = \frac{d}{dh}\varphi_\delta(q+h)|_{h=0} = \frac{1}{2\delta}\frac{d}{dh}\int_{q+h-\delta}^{q+h+\delta} f(\lambda)\,d\lambda = \frac{1}{2\delta}[f(q+\delta)-f(q-\delta)] = 0.$$

By definition of minima of class (A), there can be only one value q for which $f(q+\delta)=f(q-\delta)$ whenever δ is sufficiently small so that $f(q+\delta)$, $f(q-\delta)$ are on those intervals where $f(x)$ strictly decreases or increases.

Corollary: Lemma 4. 1 holds with minima of class (A) replaced by "unique minima on a sufficiently small open interval" in the hypotheses and conclusion.

The preceding lemma is the first case in the inductive proof involved in establishing the following lemma.

LEMMA 4. 2. *If $f(x)$ has a minimum of class (A) at $(p, f(p))$ and $\varphi_\delta(x)$ is defined for the spline kernel $k_\delta^{(n)}(x)$, $n\geqq 1$, for δ sufficiently small there is an open interval around p such that $\varphi_\delta(x)$ has a unique minimum of class (A) on this interval.*

PROOF. We indicate the dependence of $\varphi_\delta(x)$ on n writing $\varphi_\delta(x)\equiv\varphi_\delta(x;n)$, and assume the theorem for $\varphi_\delta(x;n-1)$. Using (3) of definition 2. 1:

$$\frac{d}{dx}\varphi_\delta(x;n) = -\int_{x-\delta}^{x+\delta} f(\lambda)k_\delta'^{(n)}(\lambda-x)\,d\lambda = -\int_{-\delta}^{\delta} f(x+\tau)k_\delta'^{(n)}(\tau)\,d\tau =$$

$$= \frac{-1}{a}\int_{-\delta}^{0} f(x+\tau)k_{\delta/2}^{(n-1)}\left(\tau+\frac{\delta}{2}\right)d\tau + \frac{1}{a}\int_{0}^{\delta} f(x+\tau)k_{\delta/2}^{(n-1)}\left(\tau-\frac{\delta}{2}\right)d\tau$$

where a is the appropriate factor introduced in definition 2. 2. With $\tau = t-\frac{\delta}{2}$, $\tau = t+\frac{\delta}{2}$ in the first and second integrals respectively,

$$\frac{d}{dx}\varphi_\delta(x;n) = \frac{-1}{a}\int_{-\delta/2}^{\delta/2} f\left(x-\frac{\delta}{2}+t\right)k_{\delta/2}^{(n-1)}(t)\,dt + \frac{1}{a}\int_{-\delta/2}^{\delta/2} f\left(x+\frac{\delta}{2}+t\right)k_{\delta/2}^{(n-1)}(t)\,dt$$

$$= \frac{1}{a}\left\{\varphi_{\delta/2}\left(x+\frac{\delta}{2};\,n-1\right)-\varphi_{\delta/2}\left(x-\frac{\delta}{2};\,n-1\right)\right\}.$$

It is easily shown that $\delta_\delta(x;n)$ is at least of class $C^{(n)}$, so that at a minimum $\frac{d}{dx}\varphi_\delta(x;n)=0$. Therefore, if a minimum for $\varphi_\delta(x;n)$ occurs at $x=q$, then $\varphi_{\delta/2}\left(q+\frac{\delta}{2};n-1\right) = \varphi_{\delta/2}\left(q-\frac{\delta}{2};n-1\right)$. The inductive hypothesis asserts that for a sufficiently small $\frac{\delta}{2}$, $\varphi_{\delta/2}(x;n-1)$ has a minimum of class (A) in an open interval containing p. Since this implies that $\varphi_{\delta/2}(x;n-1)$ is strictly decreasing to the left of this minimum point, and strictly increasing to the right, on the interval there is only one value of q at which the above equality can be satisfied.

THEOREM 4. 1. *Let $k_\delta^{(n)}(x)$, $n=0, 1, 2, \ldots$ be any spline kernel; $f(x)$ be a continuous function with a minimum of class (A) at $(p, f(p))$. There exists a constant Δ and a set of points q_δ such that $(q_\delta, \varphi_\delta(q_\delta))$ is a minimum of class (A) for $\varphi_\delta(x)$ and $\lim_{\delta\to 0} q_\delta=p$.*

PROOF. By lemma 3.1 any minimum of $\varphi_\delta(x)$ occurs in $J_\delta\equiv\{x|p-\delta\leqq x\leqq p+\delta\}$, when δ is sufficiently small. By lemmas 4. 1 and 4. 2, there is an open interval I around p such that $\varphi_\delta(x)$ has a unique minimum q_δ of class (A) on this interval again when δ is sufficiently small. Choose Δ so that for $\delta<\Delta$ both conditions prevail and $J_\delta\subset I$. The existence of the set of points q_δ has now been demonstrated; their convergence to p follows from the fact that J_δ reduces to p as $\delta\to 0$.

THEOREM 4. 2. *Let $k_\delta^{(n)}(x)$, $n=0, 1, 2, \ldots$ be any spline kernel; $f(x)$ be a continuous function with minima class (A). Suppose that q_δ is a minimum of $\varphi_\delta(x)$ and that $\lim_{\delta\to 0} q_\delta=p$ exists. Then $(p, f(p))$ is a minimum of class (A) for $f(x)$.*

PROOF. Suppose that $(p, f(p))$ is not a minimum of $f(x)$. By hypothesis, there is an open interval $I\equiv(\alpha, \beta)$ symmetric about p on which $f(x)$ is either strictly monotone increasing or decreasing. The former case only will be considered, as the argument for the latter is similar.

Define $I_\delta \equiv (\alpha+\delta, \beta-\delta)$, $\delta < (\beta-\alpha)/2$, observing that $I_{\delta_1} \subset I_{\delta_2}$, $\delta_2 < \delta_1$ and I_δ becomes I as $\delta \to 0$.

We prove that for all sufficiently small δ, $\varphi_\delta(x)$ is strictly monotone on I_δ, and hence contradict the existence of minima at q_δ such that $\lim_{\delta\to 0} q_\delta = p$.

For $x_1 < x_2$

$$\varphi_\delta(x_2)-\varphi_\delta(x_1) = \int_{x_1+\delta}^{x_2+\delta} f(\lambda)k_\delta^{(n)}(\lambda-x_2)\,d\lambda - \int_{1-x\delta}^{x_2-\delta} f(\lambda)k_\delta^{(n)}(\lambda-x_1)\,d\lambda +$$
$$+\int_{x_2-\delta}^{x_1+\delta} f(\lambda)k_\delta^{(n)}(\lambda-x_2)\,d\lambda - \int_{x_2-\delta}^{x_1+\delta} f(\lambda)k_\delta^{(n)}(\lambda-x_1)\,d\lambda.$$

Properties (4) and (2) of definition 2. 1, as well as the strict monotoneity of $f(x)$ on $I \supset I_\delta$, imply that the sum of the first two terms of the last expression are strictly positive. For the other two terms, let $\bar{x} = (x_1+\delta+x_2-\delta)/2$ and $g(x) = f(x)$, $x_1+\delta \leqq x \leqq \bar{x}$ and $g(x) = f(2\bar{x}-x)$, $\bar{x} \leqq x \leqq x_2-\delta$. Properties (5), (4) and (2) of definition 2. 1, as well as the symmetry of $g(x)$ about $\bar{x}$ show that:

$$\int_{x_2-\delta}^{x_1+\delta} g(x)[k_\delta^{(n)}(\lambda-x_2)-k_\delta^{(n)}(\lambda-x_1)]\,d\lambda = 0.$$

Since $f(x) \geqq g(x)$:

$$\int_{x_2-\delta}^{x_1+\delta} f(x)[k_\delta^{(n)}(\lambda-x_2)-k_\delta^{(n)}(\lambda-x_1)]\,d\lambda = \int_{x_2-\delta}^{x_1+\delta} g(x)[k_\delta^{(n)}(\lambda-x_2)-k_\delta^{(n)}(\lambda-x_1)]\,d\lambda$$
$$+\int_{\bar{x}}^{x_1+\delta} (f(x)-g(x))[k_\delta^{(n)}(\lambda-x_2)-k_\delta^{(n)}(\lambda-x_1)]\,d\lambda \geqq 0.$$

This completes the proof.

Theorem 4. 1 and 4.2 may be restated as in the Corollary of Lemma 4. 1 and the same proofs changed in an obvious way to establish the new propositions.

5. Minimization of

$$\int_0^1 f(x, y)\,dx$$

In this section we briefly discuss the problem of minimizing the integral

$$I(y) = \int_0^1 f(x, y)\,dx$$

on the class of continuous functions $y = y(x)$ defined on the interval [0, 1]. The significance of the analysis lies in the mild conditions [1] imposed on the integrand $f(x, y)$. The domain of definition of $f(x, y)$ is taken to be $J \times R$ where $J \equiv [0, 1]$ and R is any bounded or unbounded open interval of the y-axis.

THEOREM 5. 1. *Let $f(x, y)$ be continuous on $J \times R$ and suppose that as a function of y, for each x, it has a unique minimizing value of class (A), represented as $y=\bar{y}(x)$. Then $y=\bar{y}(x)$ is a continuous function and uniquely minimizes $I(y)$ in the class of $C^{(0)}$ functions.*

PROOF. The $I(y)$ is uniquely minimized by $y=\bar{y}(x)$ is quite clear for if $y=h(x)$ is any function differing from $\bar{y}(x)$ on a set of positive measure, then the uniqueness of the minimum of $f(x, y)$ as a function of y guarantees that $f(x, h(x)) > f(x, \bar{y}(x))$ on this set. Consequently: $I(h) > I(\bar{y})$.

To show that $\bar{y}(x)$ is continuous let $J_\delta(p) \equiv \{x | p-\delta < x < p+\delta\}$ and $R_\varepsilon(q) \equiv \equiv \{y | q-\varepsilon < y < q+\varepsilon\}$. We will show that for given $\varepsilon > 0$, for every $x=p$, there is an associated δ such that $\bar{y}(x) \in R_\varepsilon(\bar{y}(p))$ when $x \in J_\varepsilon(p)$. Choose $x=p$, $y=\bar{y}(p)=q$ and consider $R_\varepsilon(q)$. By the unique minimum property of $f(x, y)$ it is true that $f(p, q+\varepsilon)-f(p, q) > 0$ and $f(p, q-\varepsilon)-f(p, q) > 0$. Denote the lesser of these differences by α. There are three neighborhoods

$$J_{\delta_1}(p) \times R_{\varepsilon_1}(q-\varepsilon), \quad J_{\delta_2}(p) \times R_{\varepsilon_2}(q+\varepsilon), \quad J_{\delta_3}(p) \times R_{\varepsilon_3}(q)$$

such that

$$f(x, y) > f(p, q-\varepsilon)-\alpha/3 \quad \text{on} \quad J_{\delta_1}(p) \times R_{\varepsilon_1}(q-\varepsilon)$$

$$f(x, y) > f(p, q+\varepsilon)-\alpha/3 \quad \text{on} \quad J_{\delta_2}(p) \times R_{\varepsilon_2}(q+\varepsilon)$$

$$f(x, y) < f(p, q)+\alpha/3 \quad \text{on} \quad J_{\delta_3}(p) \times R_{\varepsilon_3}(q)$$

Let $d=\min[\delta, \delta_1, \delta_2, \delta_3]$; then for $x \in J_d(p)$, $f(x, y)$ has larger values for $y \in R_{\varepsilon_1}(q+\varepsilon)$ and $y \in R_{\varepsilon_2}(q-\varepsilon)$ than it has for $y \in R_{\varepsilon_3}(q)$. By the continuity of $f(x, y)$ and its unique minimum property as a function of y, it follows that this minimum must occur for $q-\varepsilon < y < q+\varepsilon$. This implies that for any $x \in J_d(p)$, the function $\bar{y}(x)$ has its values in $R_\varepsilon(q)$. Thus, $\bar{y}(x)$ is indeed continuous.

THEOREM 5. 2. *Let $f(x, y)$ be a continuous function on $I \times R$ with a unique minimum of class (A) as a function of y for each x, given by $y=\bar{y}(x)$. Let $\varphi_\delta(x, y) = \int_{-\delta}^{+\delta} f(x, \lambda+y) k_\delta^{(n)}(\lambda)\, dy$, $n=0, 1, 2, \ldots$. Then $I_\delta(y) = \int_0^1 \varphi_\delta(x, y)\, dx$ for sufficiently small δ has a unique minimizing curve $y=\bar{y}(x)$ in the class of $C^{(0)}$ functions. Moreover, $y_\delta(x)$ converges uniformly to $\bar{y}(x)$ as $\delta \to 0$.*

PROOF. The proof follows directly by applying theorem 4. 1, lemmas 4. 1, 4. 2, 3. 1, and theorem 5. 1.

6. Variational problem: $\int_0^1 f(x, y, y')\, dx$

An investigation is made in this section of a free endpoint variational problem minimizing the integral

$$I(y) = \int_0^1 f(x, y, y')\, dx.$$

The admissible class of variational arcs is taken to be $C^{(1)}$ functions defined on [0, 1]. The interest in the problem lies in the mild conditions imposed on the integrand which are very much less restrictive than those of classical theories of the calculus of variations. Our techniques are those of the previous sections in which $I(y)$ is approximated by other integrals, the optimizing curves for which are shown to tend in the limit to the optimizing curve for $I(y)$. The approximants may be chosen so that their integrands have adequate differentiability properties so that they can be treated by the usual methods of the calculus of variations.

The problem treated is a weak, local, relative minimizing problem in that an open ε-neighborhood of a curve $y=\bar{y}(x)$ is taken to be the set of admissible curves $y=y(x)$ satisfying both inequalities: $|\bar{y}(x)-y(x)|<\varepsilon$, $|\bar{y}'(x)-y'(x)|<\varepsilon$, $0\leqq x\leqq 1$. If $\{y_\delta(x)\}$ is a sequence of functions of the admissible class depending in the parameter δ, we write $y_\delta(x)\underset{\delta}{\to}\bar{y}(x)$, or simply $y_\delta(x)\to\bar{y}(x)$, in case convergence of the sequence occurs in the above topology.

The domain of definition of the integrand: $f(x,y,z)$, regarded as a function of three independent variables, is taken to be D: $0\leqq x\leqq 1$, $-\infty<y<+\infty$, $-\infty<z<+\infty$. It is necessary to extend this function onto a somewhat larger domain D_Δ: $-\Delta\leqq x\leqq 1+\Delta$, $-\infty<y<+\infty$, $-\infty<z<+\infty$ for a suitable constant $\Delta>0$, in order to define our approximants below. This is done by considering the function:

$$F(x,y,z)\equiv\begin{cases} f(x,y,z), & (x,y,z)\in D\\ f(0,y,z), & -\Delta\leqq x<0, \quad y,z\in D\\ f(1,y,z), & 1<x\leqq 1+\Delta, \quad y,z\in D\end{cases}$$

Definition 6. 1. Let $w_\delta(x)$ be a weighting kernel, and $F(x,y,z)$ an integrable function defined on D_Δ. Relative to this kernel the function $\varphi_\delta(x,y,z)\equiv\varphi_\delta(x,y,z;f;w_\delta)$ is defined on D by:

$$\varphi_\delta(x,y,z)=\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}F(x+\varrho,y+\sigma,z+\tau)w_\delta(\varrho)w_\delta(\sigma)w_\delta(\tau)\,d\varrho\,d\sigma\,d\tau=$$

$$=\int_{x-\delta}^{x+\delta}\int_{y-\delta}^{y+\delta}\int_{z-\delta}^{z+\delta}F(\lambda,\mu,\gamma)w_\delta(\lambda-x)w_\delta(\mu-y)w_\delta(\gamma-z)\,d\lambda\,d\mu\,d\gamma$$

It should be observed that if $f(x,y,z)$ is continuous on D then so is $\varphi_\delta(x,y,z)$. If in addition $w_\delta(x)$ is of class $c^{(n)}$, then $\varphi_\delta(x,y,z)$ is of class $c^{(n)}$. Also if $f(x,y,z)$ is continuous on D and $J\equiv\{x|0\leqq x\leqq 1\}$, $K\equiv\{y||y|\leqq k\}$, $L\equiv\{z||z|\leqq l\}$ for any finite constants $k,l>0$, then $\varphi_\delta(x,y,z)$ converges uniformly as $\delta\to 0$, in the conventional sense, to $f(x,y,z)$ on $J\times K\times L$.

Applying the above remarks, one can prove that if $f(x,y,z)$ is continuous and $y(x)$ is of class $C^{(1)}$ then $I_\delta(y)\to I(y)$ as $\delta\to 0$.

Theorem 6. 1. *Let $f(x,y,z)$ be continuous on D and let $I(y)$ be uniquely minimized by $y=\bar{y}(x)$, where $\bar{y}(x)$ is of class $C^{(1)}$. Suppose that $I_\delta(y)$ is minimized by $y=\bar{y}_\delta(x)$ and that $\bar{y}_\delta(x)\to p(x)$, in the sense defined at the outset of this section, as $\delta\to 0$. If $p(x)$ is of class $C^{(1)}$, then $p(x)=\bar{y}(x)$.*

Proof. Let $B=\sup\limits_{0\leqq x\leqq 1}[|\bar{y}(x)|, |\bar{y}'(x)|, |\bar{y}'_+(0)|, |\bar{y}'_-(1)|,$

$$|p(x)|, |p'(x)|, |p'_+(0)|, |p'_-(1)|]$$

and consider the bounded closed cell:

$$R: 0\leqq x\leqq 1, \quad |y|\leqq B+1, \quad |z|\leqq B+1.$$

Writing alternately $\bar{y}'(x)=z(x)$ and $p'(x)=z(x)$ we see that $\bar{y}(x)$ and $p(x)$ are in R. Both functions satisfy the hypotheses of the remarks prior to Theorem 6. 1. Since $\bar{y}_\delta(x)$ converges uniformly, in the sense defined at the outset of this section, to $p(x)$ it follows that for δ sufficiently small, these curves also lie in R.

By the remarks following Definition 6. 1., $\varphi_\delta(x, y, z)$ converges uniformly to $f(x, y, z)$ on R; also, $f(x, y, z)$ is uniformly continuous on R. Therefore,

$$I_\delta(\bar{y}_\delta) = \int_0^1 \varphi_\delta(x, \bar{y}_\delta, \bar{y}_\delta')\,dx \to \int_0^1 f(x, p, p')\,dx = I(p).$$

By the remarks prior to Theorem 6. 1: $I_\delta(\bar{y}) \to I(\bar{y})$ so that for any $\varepsilon>0$, $\exists \Delta$ such that for $\delta<\Delta$

$$I_\delta(\bar{y}) < I(\bar{y})+\varepsilon.$$

But $\bar{y}_\delta(x)$ minimizes $I_\delta(y)$, so that:

$$I_\delta(\bar{y}_\delta) \leqq I_\delta(\bar{y}) < I(\bar{y})+\varepsilon.$$

Passing to the limit as $\varepsilon \to 0$ gives

$$\lim_{\delta\to 0} I_\delta(\bar{y}_\delta) = I(p) \leqq I(\bar{y}).$$

But $\bar{y}(x)$ minimizes $I(y)$, so that $I(p)\geqq I(\bar{y})$; consequently, $I(p)=I(\bar{y})$. The uniqueness of the minimizing curve for I, shows that $p(x)=\bar{y}(x)$.

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(Received February 9, 1971.
Revised March 1, 1972)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 257—266

DETERMINATION OF CONTROLLABLY PERIODIC PERTURBED SOLUTIONS BY POINCARÉ'S METHOD

by

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In paper [1] perturbed systems containing a small parameter have been considered while it was assumed that the unperturbed system is autonomous, has a D-periodic (periodic) solution, the perturbation is nonautonomous and is controllably periodic. Sufficient criteria have been given in this case for the existence, uniqueness and stability of a D-periodic (periodic) solution of the perturbed system. In the present paper Poincaré's method is worked out for the determination of the D-periodic (periodic) solution of the perturbed system in the analytic case (§ 2.), and an effective form is given to the stability criterion (§ 3.). For sake of convenience in § 1. some results of [1] to be used here are summed up in a modified form for the analytic case.

1. Consider the system

$$\frac{dx}{dt} = f(x) + \mu g\left(\frac{t}{\tau}, x, \mu, \tau\right), \tag{1}$$

where x, f, g are n-dimensional vectors, t, μ, τ real scalars and the function on the right hand side

$$F\left(\frac{t}{\tau}, x, \mu, \tau\right) = f(x) + \mu g\left(\frac{t}{\tau}, x, \mu, \tau\right) \tag{2}$$

is analytic in the region $I_t \times \Omega \times I_\mu \times I_\tau$, where $I_t = \{t: -\infty < t < +\infty\}$, Ω an open and connected region in the n-dimensional space of x-s, $I_\mu = \{\mu: |\mu| < \alpha\}$, $\alpha > 0$, $I_\tau = \{\tau: |\tau - \tau_0| < \beta\}$, $0 < \beta < \tau_0$. Assume that the function g and, as a consequence, F is periodic in t with period $\tau \in I_\tau$, i.e.

$$g(s+1, x, \mu, \tau) \equiv g(s, x, \mu, \tau), \tag{3}$$

and that F is also periodic in the vector x with vector period $a\tau$, i. e.

$$F\left(\frac{t}{\tau}, x + a\tau, \mu, \tau\right) \equiv F\left(\frac{t}{\tau}, x, \mu, \tau\right) \tag{4}$$

for all $(t, x, \mu, \tau) \in I_t \times \Omega \times I_\mu \times I_\tau$, where a is some vector which, in general, may depend on μ and τ. However, we shall assume that

$$a\tau = a^0 \tau_0, \tag{5}$$

where a^0 is a constant vector. In case $a = a^0 = 0$, (4) imposes no restriction on (2). If $a^0 \neq 0$, Ω is supposed to have the following property: if $x \in \Omega$, then $x + at \in \Omega$, $-\infty < t < +\infty$, i.e. Ω is a cylinder with axis parallel to a in the n-dimensional space.

Alongside with (1) we consider the unperturbed autonomous system

$$\frac{dx}{dt} = f(x), \tag{6}$$

which we get to if $\mu=0$ is substituted into (2). It will be assumed that (6) has a non-constant D-periodic solution $p(t)$ with period $\tau_0>0$, coefficient vector a^0 (i.e. $p(t)= = v(t)+a^0 t$, where $v(t)$ is τ_0-periodic) and with path contained in Ω for $-\infty< <t<+\infty$. (In the special case $a^0=0$, $p(t)$ is τ_0-periodic in the ordinary sense.) We introduce the notations (dot denotes differentiation with respect to t everywhere)

$$p^0 = p(0) = (p_1^0, p_2^0, \dots, p_n^0),$$
$$\dot p^0 = \dot p(0) = (\dot p_1^0, \dot p_2^0, \dots, \dot p_n^0).$$

As it was shown in [1] all solutions of (1) with $|\mu|$ and $|\tau-\tau_0|$ sufficiently small and with initial values sufficiently close to $t=0$, $x=p^0$ will pass through the hyperplane passing through p^0 and orthogonal to the vector $\dot p^0$ and it can be assumed without loss of generality that $\dot p_1^0\neq 0$, $\dot p_2^0=\dot p_3^0=\dots=\dot p_n^0=0$. Thus, all solutions of (1) with $|\mu|$ and $|\tau-\tau_0|$ sufficiently small and with initial values sufficiently close to $t=0$, $x=p^0$ are taken care of if we denote by

$$x = x(t; \vartheta, p^0+h, \mu, \tau)$$

the solution of (1) for which

$$x(\vartheta; \vartheta, p^0+h, \mu, \tau) = p^0+h \tag{7}$$

holds, where $h = (0, h_2, h_3, \dots, h_n)$.

The first variational system of (6) corresponding to the solution $p(t)$, i.e.

$$\frac{dy}{dt} = f_x'\big(p(t)\big)\cdot y \tag{8}$$

is a linear system with τ_0-periodic coefficients. Here f_x' is the $n\times n$ derivative matrix of the vector f with elements $\frac{\partial f_i}{\partial x_k}$, where f_i and x_k are the coordinates of the vectors f and x, respectively, $i, k=1, 2, \dots, n$. The periodic function $\dot p(t)$ is a solution of (8) and thus, as it is well known, 1 is a characteristic multiplier.

THEOREM 1. *If* 1 *is a simple characteristic multiplier of system* (8), *then to all sufficiently small values of* $|\mu|$ *and* $|\vartheta|$ *there belongs a unique period* $\tau=\tau(\mu,\vartheta)$ *and a unique* $h=h(\mu,\vartheta)$ *such that*

$$\varphi(t; \mu, \vartheta) = x\big(t; \vartheta, p^0+h(\mu,\vartheta), \mu, \tau(\mu,\vartheta)\big) \tag{9}$$

is a D-periodic solution with period $\tau(\mu,\vartheta)$ *and coefficient vector a of the perturbed system* (1), *where also* $\tau=\tau(\mu,\vartheta)$; *the vector a is uniquely determined by condition* (5), *the functions* $\tau(\mu,\vartheta)$, $h(\mu,\vartheta)$ *and* (9) *are analytic in a neighbourhood of* $\mu=\vartheta=0$, *and for all* t, $\tau(0,0)=\tau_0$, $h(0,0)=0$ *and* $\varphi(t;0,0)=p(t)$.

The proof is analogous to that of Theorem 1 of [1] and will therefore be omitted here. The analyticity of the respective functions follows from well known theorems such as the theorem on p. 469 of [2] and (10. 3) p. 44 of [3], respectively.

Keeping to the assumptions of Theorem 1, consider the first variational system of (1) with $\tau=\tau(\mu, \vartheta)$ corresponding to the solution (9):

$$\frac{dy}{dt} = \left[f'_x(\varphi(t;\mu,\vartheta)) + \mu g'_x\left(\frac{t}{\tau(\mu,\vartheta)}, \varphi(t;\mu,\vartheta), \mu, \tau(\mu,\vartheta)\right)\right] y, \tag{10}$$

where g'_x has a meaning similar to f'_x.

Let us denote by $Y(t; \mu, \vartheta)$ the fundamental matrix solution of (10) for which $Y(0; \mu, \vartheta)=U$ (U: the unit matrix) holds. Let us denote the corresponding principal matrix of (10) by

$$C(\mu, \vartheta) = Y(\tau(\mu, \vartheta); \mu, \vartheta). \tag{11}$$

Obviously,

$$C(0, 0) = Y(\tau_0; 0, 0)$$

and 1 is a simple eigenvalue of $C(0, 0)$. Let us denote by $\lambda(\mu, \vartheta)$ that eigenvalue of (11) (the characteristic multiplier of system (10)) for which $\lambda(0, 0)=1$ holds.

There holds the following

THEOREM 2. *Under the assumptions of Theorem 1, there exists* $\varrho_1>0$, $\varrho_2>0$ *such that in the region*

$$|\mu| < \varrho_1, \quad |\vartheta| < \varrho_2 \tag{12}$$

$\lambda(\mu, \vartheta)$ *is a real valued analytic function of its arguments and if the remaining* $n-1$ *characteristic multipliers of system* (8) *are in modulus less than one, then the D-periodic solution* (9) *of the perturbed system* (1) *with* $\tau=\tau(\mu, \vartheta)$ *is asymptotically stable for* μ*-s satisfying also the following condition*

$$\mu\lambda'_\mu(0, 0) < 0. \tag{13}$$

The proof is analogous to that of Theorem 3 of [1]. Analiticity of $\lambda(\mu, \vartheta)$ follows from the analiticity of (11) and from the fact that a simple root of a polynomial is an analytic function of the coefficients in a neighbourhood of the actual coefficients.

Remark. As it was shown in [1], $\lambda'_\vartheta(0, 0)=0$. If also $\lambda'_\mu(0, 0)=0$ then, of course, no μ exists satisfying (13). However, if $\mu=\vartheta=0$ is a maximum point of $\lambda(\mu, \vartheta)$, then (9) is asymptotically stable in the whole region (12), $\mu\neq 0$.

2. Based on Theorem 1 we are going to apply Poincaré's method and show an effective way to the approximate determination of the D-periodic solution (9) and the period $\tau(\mu, \vartheta)$ of the non-autonomous perturbed system (1). Cf. [4] pp. 354—356. Introducing the new independent variable s by

$$t = \vartheta + s\tau(\mu, \vartheta), \tag{14}$$

system (1) and the D-periodic solution (9) assumes the form

$$\frac{dx}{ds} = \tau(\mu, \vartheta)\left[f(x) + \mu g\left(s + \frac{\vartheta}{\tau(\mu, \vartheta)}, x, \mu, \tau(\mu, \vartheta)\right)\right] \tag{15}$$

and

$$\psi(s; \mu, \vartheta) = \varphi(\vartheta + s\tau(\mu, \vartheta); \mu, \vartheta),$$

respectively. ψ is obviously D-periodic with period 1 and coefficient vector $a\tau(\mu, \vartheta) \equiv \equiv a^0\tau_0$. We expand the solution ψ of (15) and the function $\tau(\mu, \vartheta)$ for fixed ϑ in powers of μ:

$$\psi(s; \mu, \vartheta) = \sum_{k=0}^{\infty} \mu^k \psi^k(s, \vartheta), \tag{16}$$

$$\tau(\mu, \vartheta) = \sum_{k=0}^{\infty} \mu^k \tau_k(\vartheta). \tag{17}$$

It is easy to see that

$$\varphi(t; 0, \vartheta) = p(t-\vartheta)$$

and

$$\tau_0(\vartheta) = \tau(0, \vartheta) \equiv \tau_0, \tag{18}$$

hence

$$\psi^0(s, \vartheta) = \psi(s; 0, \vartheta) = \varphi(\vartheta + s\tau_0; 0, \vartheta) = p(s\tau_0). \tag{19}$$

Since the coefficient vector of ψ, namely the vector $a^0\tau_0$ does not depend on μ, ψ's derivatives with respect to μ, i.e. the vectors $\psi^k(s, \vartheta)$, $k=1, 2, \dots$ are simple periodic functions with period 1. Substituting the expansions (16) and (17) into (15) we have the identity

$$\sum_{k=0}^{\infty} \mu^k \frac{d\psi^k(s, \vartheta)}{ds} \equiv \sum_{k=0}^{\infty} \mu^k \tau_k(\vartheta) \left[f(\psi(s; \mu, \vartheta)) + \mu g\left(s + \frac{\vartheta}{\tau(\mu, \vartheta)}, \psi(s; \mu, \vartheta), \mu, \tau(\mu, \vartheta)\right)\right].$$

Expanding the second factor on the right hand side in powers of μ and equating the respective coefficients we have

$$\frac{d\psi^1(s, \vartheta)}{ds} \equiv \tau_0 f'_x(p(s\tau_0))\psi^1(s, \vartheta) + \tau_0 g\left(s + \frac{\vartheta}{\tau_0}, p(s\tau_0), 0, \tau_0\right) + \tau_1(\vartheta) f(p(s\tau_0)),$$

and similar identities for $\psi^k(s, \vartheta)$, $k=2, 3, \dots$. Thus $z=\psi^1(s, \vartheta)$ satisfies the system

$$\frac{dz}{ds} = \tau_0 f'_x(p(s\tau_0))z + \tau_0 g\left(s + \frac{\vartheta}{\tau_0}, p(s\tau_0), 0, \tau_0\right) + \varkappa f(p(s\tau_0)) \tag{20}$$

with $\varkappa = \tau_1(\vartheta)$. ψ^k $k=2, 3, \dots$ satisfies similar linear systems of differential equations. By assumption (7), since $\psi^0(0, \vartheta) = p(0) = p^0$, the first component of the vector ψ^k, $k=1, 2, \dots$ at $s=0$, i.e. at $t=\vartheta$, is zero, in particular

$$\psi_1^1(0, \vartheta) = 0. \tag{21}$$

We are going to show that the conditions given above determine $\psi^1(s, \vartheta)$ and $\tau_1(\vartheta)$ uniquely.

There holds the following

THEOREM 3. *If $z(s)$ is a vector function and $\varkappa(\vartheta)$ a real number such that*

$$z(s+1) \equiv z(s), \tag{22}$$

the first component of $z(0)$ *is zero, i.e.*

(23) $$z_1(0) = 0$$

and $z(s)$ *satisfies* (20) *with* $\varkappa = \varkappa(\vartheta)$, *then* $z(s) \equiv \psi^1(s, \vartheta)$ *and* $\varkappa(\vartheta) = \tau_1(\vartheta)$.

PROOF. Introducing the notations

$$w(s) = \psi^1(s, \vartheta) - z(s), \quad \gamma = \frac{1}{\tau_0}(\tau_1(\vartheta) - \varkappa(\vartheta)),$$

substituting $z = \psi^1(s, \vartheta)$, $\varkappa = \tau_1(\vartheta)$ and $z = z(s)$, $\varkappa = \varkappa(\vartheta)$ into (20), respectively and subtracting the corresponding identities, we get that $w(s)$ is a solution of the inhomogeneous linear system

$$\frac{dw}{ds} = \tau_0 f'_x(p(s\tau_0))w + \gamma\tau_0 f(p(s\tau_0))$$

or, taking into account that $p(t)$ is a solution of (6), of

(24) $$\frac{dw}{ds} = \tau_0 f'_x(p(s\tau_0))w + \gamma \frac{dp(s\tau_0)}{ds}.$$

The corresponding homogeneous system

(25) $$\frac{dv}{ds} = \tau_0 f'_x(p(s\tau_0))v$$

is the first variational system of system

$$\frac{dx}{ds} = \tau_0 f(x)$$

corresponding to the solution $p(s\tau_0)$ of the latter. Thus $\frac{dp(s\tau_0)}{ds}$ is a periodic solution with period 1 of (25) and so is the vector

$$v^1(s) = \frac{1}{\tau_0 \dot{p}_1^0} \frac{dp(s\tau_0)}{ds}.$$

By assumptions on $\dot{p}(0)$ the components of $v^1(0)$ are

(26) $$v_k^1(0) = \delta_{1k}, \qquad k = 1, 2, \dots n$$

where δ_{ik} is the Kronecker symbol. It is easy to see by substitution that $w = \gamma\tau_0\dot{p}_1^0 s v^1(s)$ is a solution of (24), thus the difference $v^2(s) = w(s) - \gamma\tau_0\dot{p}_1^0 s v^1(s)$ is a solution of (25). If we denote by $V(s)$ the fundamental matrix solution of (25) for which $V(0) = U$, the unit matrix, holds, we get that

$$w(s) = v^2(s) + \gamma\tau_0 \dot{p}_1^0 s v^1(s) = V(s)v^2(0) + \gamma\tau_0 \dot{p}_1^0 s v^1(s).$$

Since $w(s)$ is periodic with period 1,

$$0 = w(1) - w(0) = V(1)v^2(0) - Uv^2(0) + \gamma\tau_0 \dot{p}_1^0 v^1(0),$$

where the periodicity of $v^1(s)$ was taken into account. Hence we have

$$(V(1)-U)v^2(0) = -\gamma\tau_0 \dot{p}_1^0 v^1(0).$$

Because of (26), $v^1(0)$ constitutes the first column vector of $V(0)=U$, thus $v^1(s)$ is identically equal to the first column vector of $V(s)$. Taking also into account the periodicity of $v^1(s)$ the last equation assumes the form

$$\sum_{k=2}^{n} c_{1k} v_k^2 = -\gamma\tau_0 \dot{p}_1^0$$

$$\sum_{k=2}^{n} (c_{ik}-\delta_{ik}) v_k^2 = 0, \qquad (i=2, 3, \ldots, n), \tag{27}$$

where c_{ik} and v_k^2 are the components of $V(1)$ and $v^2(0)$, respectively ($c_{i1}=\delta_{i1}$, $i=1, 2, \ldots, n$). The eigenvalues of $V(1)$ are the characteristic multipliers of system (25) which are the same as those of system (8). Since 1 is a simple characteristic multiplier, the determinant of the system consisting of the last $n-1$ equations of (27) is non-zero. Thus $v_2^2=v_3^2=\ldots=v_n^2=0$. As a consequence, from the first equation of (27) follows $\gamma=0$, i.e. $\varkappa(\vartheta)=\tau_1(\vartheta)$. Furthermore, since $v^2(0)$ is a scalar multiple of $v^1(0)$, so is $w(s)\equiv v^2(s)$, i.e. $w(s)\equiv cv^1(s)$. But from (21) and (23) follows that the first component of $w(0)$ is zero, while by (26) that of $v^1(0)$ is one. Thus $c=0$, $w(s)\equiv 0$ and this completes the proof.

It is to be noted that (20) subject to conditions (22) and (23) can be solved by standard methods and $\varkappa=\tau_1(\vartheta)$ determined, provided that a fundamental solution of (8) is known.

It can successively be proved in a similar fashion that $\psi^2(s, \vartheta)$, $\tau_2(\vartheta)$; $\psi^3(s, \vartheta)$, $\tau_3(\vartheta)$; etc. are uniquely determined by the successive systems similar to (20), the condition of periodicity and that analogous to (21).

3. Under the assumptions of Theorems 1 and 2 we are giving a method to check the validity of the stability condition (13). It will be shown that the derivative $\lambda'_\mu(0, 0)$ occurring there can be evaluated provided that a fundamental matrix solution of (8), $\psi^1(s, 0)$ and $\tau_1(0)$ (in expansions (16) and (17), respectively) are known. The results are closely related to those of W. S. Loud ([5], [6]) though the problem and the approach are different.

Using the notations introduced after (10) and in (11) we have the identities

$$\dot{Y}(t; \mu, \vartheta) \equiv \left[f'_x(\varphi(t; \mu, \vartheta)) + \mu g'_x\left(\frac{t}{\tau(\mu, \vartheta)}, \varphi(t; \mu, \vartheta), \mu, \tau(\mu, \vartheta)\right)\right] Y(t; \mu, \vartheta), \tag{28}$$

$$Y(0; \mu, \vartheta) \equiv U. \tag{29}$$

For $\mu=\vartheta=0$ we have that $Y_0(t)=Y(t; 0, 0)$ is a fundamental matrix solution of (8) and, of course, $Y_0(0)=U$. For $\mu=0$, $\varphi(t; 0, \vartheta) = p(t-\vartheta)$ for all ϑ and $Y_0(t-\vartheta)$ is a fundamental matrix solution of

$$\dot{y} = f'_x(p(t-\vartheta))y. \tag{30}$$

Obviously

$$Y(t; 0, \vartheta) \equiv Y_0(t-\vartheta) Y_0^{-1}(-\vartheta). \tag{31}$$

Since the characteristic matrix (11) of system (10) is analytic, it can be expanded for fixed ϑ in powers of μ:

$$C(\mu, \vartheta) = C_0(\vartheta) + \mu C_1(\vartheta) + \mu^2 R(\mu, \vartheta), \tag{32}$$

where R is analytic.

Because of $\tau(0, \vartheta) \equiv \tau_0$ and of (31)

$$C_0(\vartheta) = C(0, \vartheta) = Y(\tau(0, \vartheta); 0, \vartheta) = Y(\tau_0; 0, \vartheta) = Y_0(\tau_0 - \vartheta) Y_0^{-1}(-\vartheta). \tag{33}$$

In particular

$$C_0(0) = C(0, 0) = Y_0(\tau_0). \tag{34}$$

We are going to evaluate the matrix $C_1(\vartheta)$:

$$\begin{aligned} C_1(\vartheta) = C'_\mu(0, \vartheta) &= \dot{Y}(\tau(0, \vartheta); 0, \vartheta)\tau'_\mu(0, \vartheta) + Y'_\mu(\tau(0, \vartheta); 0, \vartheta) = \\ &= \dot{Y}(\tau_0; 0, \vartheta)\tau_1(\vartheta) + Y'_\mu(\tau_0; 0, \vartheta), \end{aligned} \tag{35}$$

where the expansion (17) has been used. For the evaluation of the first term on the right hand side it has to be taken into account that (31) is a solution of (30).

Thus

$$\dot{Y}(\tau_0; 0, \vartheta) = f'_x(p(\tau_0 - \vartheta)) Y_0(\tau_0 - \vartheta) Y_0^{-1}(-\vartheta). \tag{36}$$

For the evaluation of the second term on the right hand side of (35) we differentiate (28) with respect to μ at $\mu = 0$:

$$\begin{aligned} \dot{Y}'_\mu(t; 0, \vartheta) = \left[f''_{xx}(p(t-\vartheta))\varphi'_\mu(t; 0, \vartheta) + g'_x\left(\frac{t}{\tau_0}, p(t-\vartheta), 0, \tau_0\right)\right] Y(t; 0, \vartheta) + \\ + f'_x(p(t-\vartheta)) Y'_\mu(t; 0, \vartheta). \end{aligned}$$

Here f''_{xx} denotes the $n \times n \times n$ "three dimensional matrix" with elements $f''_{ix_k x_l}$ (the second partial derivatives of the components of the vector f; $i, k, l = 1, 2, \ldots, n$) and $f''_{xx}\varphi'_\mu$ denotes the $n \times n$ matrix with elements

$$\sum_{l=1}^{n} f''_{ix_k x_l}\varphi'_{l\mu}, \qquad i, k = 1, 2, \ldots, n,$$

where $\varphi'_{l\mu}$ is the lth component of the vector φ'_μ. $\varphi'_\mu(t; 0, \vartheta)$ can be determined with the help of expansion (16) differentiating

$$\varphi(t; \mu, \vartheta) = \psi\left(\frac{t-\vartheta}{\tau(\mu, \vartheta)}; \mu, \vartheta\right) = \sum_{k=0}^{\infty} \mu^k \psi^k\left(\frac{t-\vartheta}{\tau(\mu, \vartheta)}, \vartheta\right)$$

with respect to μ at $\mu = 0$ and taking into account (19), (18) and (17):

$$\begin{aligned} \varphi'_\mu(t; 0, \vartheta) &= \frac{\partial}{\partial \mu} p\left(\frac{t-\vartheta}{\tau(\mu, \vartheta)} \tau_0\right)_{\mu=0} + \psi^1\left(\frac{t-\vartheta}{\tau_0}, \vartheta\right) = \\ &= \psi^1\left(\frac{t-\vartheta}{\tau_0}, \vartheta\right) - \dot{p}(t-\vartheta)\frac{(t-\vartheta)\tau_1(\vartheta)}{\tau_0}. \end{aligned}$$

Taking into account (31) and introducing the notation

$$(37)\qquad B(t,\vartheta)=\left\{f''_{xx}(p(t-\vartheta))\left[\psi^1\left(\frac{t-\vartheta}{\tau_0},\ \vartheta\right)-\dot p(t-\vartheta)\,\frac{(t-\vartheta)\tau_1(\vartheta)}{\tau_0}\right]+\right.$$

$$\left.+g'_x\left(\frac{t}{\tau_0},\ p(t-\vartheta),0,\tau_0\right)\right\}Y_0(t-\vartheta)\,Y_0^{-1}(-\vartheta),$$

we see that $Y'_\mu(t;0,\vartheta)$ satisfies the inhomogeneous linear matrix differential equation

$$(38)\qquad \dot Y=f'_x(p(t-\vartheta))Y+B(t,\vartheta)$$

and because of $Y(0;\mu,\vartheta)\equiv U$, $Y'_\mu(0;0,\vartheta)=Y'_\mu(0;\mu,\vartheta)=0$, the zero matrix. Since, as it was told in connection with (30), $Y_0(t-\vartheta)$ is a fundamental solution of the homogeneous system corresponding to (38), we get by the method of the variation of constant that

$$Y'_\mu(t;0,\vartheta)=Y_0(t-\vartheta)\int_0^t Y_0^{-1}(u-\vartheta)\,B(u,\vartheta)\,du,$$

i. e.

$$(39)\qquad Y'_\mu(\tau_0;0,\vartheta)=Y_0(\tau_0-\vartheta)\int_0^{\tau_0} Y_0^{-1}(t-\vartheta)\,B(t,\vartheta)\,dt.$$

Substituting (36) and (39) into (35) we get that

(40)

$$C_1(\vartheta)=\tau_1(\vartheta)f'_x(p(\tau_0-\vartheta))\,Y_0(\tau_0-\vartheta)\,Y_0^{-1}(-\vartheta)+Y_0(\tau_0-\vartheta)\int_0^{\tau_0}Y_0^{-1}(t-\vartheta)\,B(t,\vartheta)\,dt.$$

In particular

$$(41)\qquad C_1(0)=\tau_1(0)f'_x(p(\tau_0))\,Y_0(\tau_0)+Y_0(\tau_0)\int_0^{\tau_0}Y_0^{-1}(t)\,B(t,0)\,dt,$$

where

$$(42)\quad B(t,0)=\left\{f''_{xx}(p(t))\left[\psi^1\left(\frac{t}{\tau_0},\ 0\right)-\dot p(t)\,\frac{t\tau_1(0)}{\tau_0}\right]+g'_x\left(\frac{t}{\tau_0},\ p(t),0,\tau_0\right)\right\}Y_0(t).$$

Having determined the first two terms in expansion (32) a suitable expression will be given to the characteristic polynomial of matrix $C(\mu,\vartheta)$, and the derivative with respect to μ at $\mu=0$, $\vartheta=0$ of the eigenvalue $\lambda(\mu,\vartheta)$ (for which $\lambda(0,0)=1$) will be given in an explicit form. The following notations will be used. The characteristic polynomial of matrix $C(\mu,\vartheta)$ will be denoted by

$$(43)\qquad (-1)^n d(\lambda;\mu,\vartheta)=\det\,[C(\mu,\vartheta)-\lambda U],$$

$$(44)\qquad d(\lambda;\mu,\vartheta)=\lambda^n+\alpha_{n-1}\lambda^{n-1}+\dots+\alpha_1\lambda+\alpha_0.$$

Here obviously $\alpha_k(\mu,\vartheta)$, $k=0,1,2,\dots,n-1$, is an analytic function of its arguments and such is the function $d(\lambda;\mu,\vartheta)$, too.

The ith row vector of the matrix $C_0(\vartheta)$, $C_1(\vartheta)$, $R(\mu,\vartheta)$ and U will be denoted by c_i^0, c_i^1, r_i and u_i, respectively, $i=1,2,\dots,n$, and the matrix $C_0(\vartheta)$ by

$$C_0(\vartheta)=\begin{bmatrix}c_1^0\\ \vdots\\ c_n^0\end{bmatrix},\ \text{etc.}$$

c_i^0, c_i^1, $i=1, 2, \ldots, n$ are functions of ϑ but the writing out of the argument will be omitted for the sake saving space.

There holds the following

THEOREM 4. *Under the assumptions of Theorem 1*

$$\lambda'_\mu(0,0) = \frac{(-1)^{n+1}}{d'_\lambda(1;0,0)} \sum_{i=1}^{n} \det \begin{bmatrix} c_1^0(0)-u_1 \\ \cdots\cdots\cdots \\ c_{i-1}^0(0)-u_{i-1} \\ c_i^1(0) \\ c_{i+1}^0(0)-u_{i+1} \\ \cdots\cdots\cdots \\ c_n^0(0)-u_n \end{bmatrix}, \tag{45}$$

where $c_i^0(0)$ and $c_i^1(0)$ are the ith row vectors of matrices $C_0(0)$ and $C_1(0)$, respectively and the i-th term of the sum is got from $\det[C_0(0)-U]$ *replacing the i-th row by $c_i^1(0)$ $i=1, 2, \ldots, n$.*

PROOF. Splitting up the determinant (43) by the terms in its last row, using expansion (32) to a sum of determinants we get

$$(-1)^n d(\lambda;\mu,\vartheta) = \det[C(\mu,\vartheta)-\lambda U] = \det \begin{bmatrix} c_1^0+\mu c_1^1+\mu^2 r_1-\lambda u_1 \\ \cdots\cdots\cdots \\ c_n^0+\mu c_n^1+\mu^2 r_n-\lambda u_n \end{bmatrix} =$$

$$= \det \begin{bmatrix} c_1^0-\lambda u_1+\mu c_1^1+\mu^2 r_1 \\ \cdots\cdots\cdots \\ c_{n-1}^0-\lambda u_{n-1}+\mu c_{n-1}^1+\mu^2 r_{n-1} \\ c_n^0-\lambda u_n \end{bmatrix} + \mu \det \begin{bmatrix} c_1^0-\lambda u_1+\mu c_1^1+\mu^2 r_1 \\ \cdots\cdots\cdots \\ c_{n-1}^0-\lambda u_{n-1}+\mu c_{n-1}^1+\mu^2 r_{n-1} \\ c_n^1 \end{bmatrix} + \mu^2 R_1 .$$

Splitting up the second term in the last expression similarly by the terms in the $(n-1)$-st row, etc. we get

$$(-1)^n d(\lambda;\mu,\vartheta) = \det \begin{bmatrix} c_1^0-\lambda u_1+\mu c_1^1+\mu^2 r_1 \\ \cdots\cdots\cdots \\ c_{n-1}^0-\lambda u_{n-1}+\mu c_{n-1}^1+\mu^2 r_{n-1} \\ c_n^0-\lambda u_n \end{bmatrix} + \mu \det \begin{bmatrix} c_1^0-\lambda u_1 \\ \cdots\cdots \\ c_{n-1}^0-\lambda u_{n-1} \\ c_n^1 \end{bmatrix} + \mu^2 R_2 .$$

Applying the same method in the first term on the right hand side using the $(n-1)$-st, $(n-2)$-nd, ..., first rows we get

$$(-1)^n d(\lambda;\mu,\vartheta) = \det[C_0(\vartheta)-\lambda U] + \mu \sum_{i=1}^{n} \det \begin{bmatrix} c_1^0-\lambda u_1 \\ \cdots\cdots \\ c_i^1 \\ \cdots\cdots \\ c_n^0-\lambda u_n \end{bmatrix} + \mu^2 R_3 . \tag{46}$$

By assumption, 1 is a simple root of the polynomial $d(\lambda; 0, 0)$, i.e. $d(1; 0, 0)=0$ and $d'_\lambda(1; 0, 0)\neq 0$. Hence, λ can be expressed from equation $d(\lambda; \mu, \vartheta)=0$ as a function $\lambda(\mu, \vartheta)$ analytic in a neighbourhood of $(0, 0)$ such that $\lambda(0, 0)=1$, and

$$\lambda'_\mu(0, 0) = -\frac{d'_\mu(1; 0, 0)}{d'_\lambda(1; 0, 0)}. \tag{47}$$

Substituting $\lambda=1$, $\vartheta=0$ into (46) and differentiating (46) with respect to μ at $\mu=0$:

$$d'_\mu(1; 0, 0) = (-1)^n \sum_{i=1}^{n} \det \begin{bmatrix} c_1^0(0)-u_1 \\ \cdots\cdots \\ c_i^1(0) \\ \cdots\cdots \\ c_n^0(0)-u_n \end{bmatrix}.$$

Substituting the last expression into (47) we get the result which was to be proved.

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(Received February 9, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 267—287

ÜBER DIE PARKETTIERUNGSMÖGLICHKEIT DES DREIDIMENSIONALEN HYPERBOLISCHEN RAUMES DURCH KONGRUENTE POLYEDER

von

I. VERMES

Das Problem der Parkettierungsmöglichkeit des Raumes durch kongruente Polyeder ist schon seit langem bekannt. D. HILBERT beschäftigte sich auch mit dieser Frage in seiner Arbeit: „*Mathematische Probleme*" [4]. Das analoge Problem der hyperbolischen Ebene ist in einem Buch von L. FEJES TÓTH [3] gelöst, wo die Existenz der hyperbolischen Mosaike gezeigt wurde, falls die Mosaikelemente reguläre Vielecke bzw. total asymptotische Vielecke sind. Die Parkettierungsmöglichkeit durch nicht-total asymptotische Vielecke[1] ist in [6] und [7] untersucht. H. ZEITLER hat bewiesen [8], daß es vier verschiedene Parkettierungen des dreidimensionalen hyperboli chen Raumes mit regulären Polyedern gibt.

In dieser Arbeit beschäftigen wir uns mit der Parkettierungsmöglichkeit des hyperbolischen Raumes durch kongruente Polyeder. Die in unseren Untersuchungen vorkommenden Polyeder sind sogenannte reguläre asymptotische Pyramide, reguläre Prismen und total-asymptotische Polyeder des hyperbolischen Raumes.

1. Wir verstehen unter einer *regulären asymptotischen Pyramide* ein Polyeder, das mit einem regulären n-Eck — als Grundvieleck — und n kongruenten einfach asymptotischen gleichwinkligen Dreiecken begrenzt ist.

Auf Grund dieser Definition können wir den folgenden SATZ beweisen:

Vier verschiedene Parkettierungen des dreidimensionalen hyperbolischen Raumes können durch reguläre asymptotische Pyramiden gebildet werden. Es gibt — unter den vorkommenden Pyramiden — eine Pyramide, die ein Viereck als Grundvieleck hat und drei Pyramiden, deren Grundvielecke Sechsecke sind.

BEWEIS: Zuerst wollen wir den Beweis kurz aufreißen. Der Beweis besteht aus drei Teilen. In dem ersten Teil zeigen wir, daß nur vier unendliche Tangentenpolyeder um die Grenzkugel konstruiert werden können, deren Seitenflächen reguläre Vielecke und ihre Ecken dreikantige bzw. vierkantige Zentralecken eines entsprechenden regulären Körpers sind. Die Zentralecken eines regulären Körpers sind natürlich durch jene Halbgeraden bestimmt, die aus dem Zentrum des Körpers durch seine Eckpunkte gezogen werden können. Wir betrachten in dem zweiten Teil die Zentralecken eines entsprechenden regulären Körpers, und wir werden je eine Grenzkugel in jede Zentralecke einsetzen. Die oben erwähnten unendlichen Tangentenpolyeder können zu jeder Grenzkugel gebildet werden. Man kann die Umgebung jeder Ecke der Tangentenpolyeder mit ebensolchen Zentralecken ausfüllen, und weitere Grenzkugel können in die leeren Ecken eingesetzt werden, usw. Wir bekommen mit der Fortsetzung dieses Verfahrens einen regulären unendlichen Graphen, dessen Knotenpunkte und Kanten die Eckpunkte bzw. die Kanten aller Tangentenpolyeder sind.

[1] Ein n-Eck ist $(n-k)$ fach asymptotisch, falls es k Ecken im Endlichen hat.

Die Eckpunkte der Tangentenpolyeder werden mit dem Ende der zu dem Tangentenpolyeder gehörenden Grenzkugel verbunden, und so bekommen wir je eine Parkettierung. Die regulären Lagerungen der Grenzkugel können auf Grund der oben erwähnten Konstruktion übersehen werden.

a) Es gibt in der hyperbolischen Geometrie fünf körperliche Ecken, die die Umgebung eines Punktes ausfüllen können, weil es fünf reguläre Mosaike auf der Kugel gibt.[2] Die fünf Parkettierungen bestimmen auf der Kugel Tetraeder, Hexaeder, Oktaeder, Dodekaeder und Ikosaeder. Zu jedem regulären Körper gehören Zentralecken. In einer solchen Zentralecke gibt es eine aus dem Eckpunkt ausgehende Halbgerade, die den gleichen Winkel α mit den Deckflächen dieser Ecke einschließt. Zwei benachbarte Kanten einer Zentralecke bilden den Winkel γ. Die zu den verschiedenen regulären Körpern gehörenden Werte von $\sin\alpha$ und $\operatorname{tg}\frac{\gamma}{2}$ sind in der Tabelle 1. zusammengefaßt.

	$\sin\alpha$	$\operatorname{tg}\frac{\gamma}{2}$
Tetraeder	$\frac{\sqrt{6}}{3}$	$\sqrt{2}$
Hexaeder	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
Oktaeder	$\frac{\sqrt{3}}{3}$	1
Dodekaeder	$\sqrt{\frac{5-\sqrt{5}}{10}}$	$\sqrt{\frac{7-3\sqrt{5}}{2}}$
Ikosaeder	$\sqrt{\frac{3-\sqrt{5}}{6}}$	$\sqrt{\frac{3-\sqrt{5}}{2}}$

Tabelle 1

Betrachten wir eine Zentralecke mit dem Eckpunkt O und die Grenzkugel, die in diese Zentralecke eingesetzt ist. Diese Grenzkugel berührt eine Deckfläche der Zentralecke in dem Punkt P, dessen Abstand a von O das zu dem Winkel α gehörige Parallellot ist (Fig. 1.). So folgt auf Grund der Trigonometrie der hyperbolischen Geometrie[3] die Beziehung:

$$\text{(1)} \qquad \operatorname{ch} a = \frac{1}{\sin \Pi(a)} = \frac{1}{\sin\alpha}$$

[2] S. z. B. [3] S. 94—96.

[3] Die Trigonometrie der hyperbolischen Geometrie s. z. B. in [1]. $\Pi(a)$ bezeichnet den zum Abstand a gehörigen Parallelwinkel.

Betrachten wir anderseits die aus dem Ende der Grenzkugel auf dieselbe gebildete Projektion eines um die Grenzkugel umschriebenen Tangentenpolyeders. Diese Projektion ist eine reguläres Mosaik der Grenzkugel. Die Geometrie der Grenzkugel ist euklidisch[4], folglich können die regulären Mosaike auf ihr nur aus Drei-, Vier-, und Sechsecken aufgebaut werden. Aus dieser Bemerkung folgt, daß nur reguläre Drei-, Vier-, und Sechsecke als Seitenflächen der um die Grenzkugel umschriebenen Tangentenpolyeder existieren können. Ein unendliches reguläres Polyeder um eine der oben betrachtenen, in eine der Zentralecken gesetzten Grenzkugeln kann also dann und nur dann gebildet werden, wenn der Winkel OPT gleich $\frac{\pi}{n}$ $(n=3, 4, 6)$ ist, wobei T der Fußpunkt des aus P auf die Kante der Zentralecke gefällten Lotes ist. Für das Dreieck OPT gilt:

$$(2) \qquad \operatorname{ch} a = \operatorname{ctg}\frac{\gamma}{2} \cdot \operatorname{ctg}\frac{\pi}{n}.$$

Aus (1) und (2) ergibt sich die Gleichung:

$$(3) \qquad \sin\alpha = \operatorname{tg}\frac{\gamma}{2} \cdot \operatorname{tg}\frac{\pi}{n} \qquad (n=3, 4, 6).$$

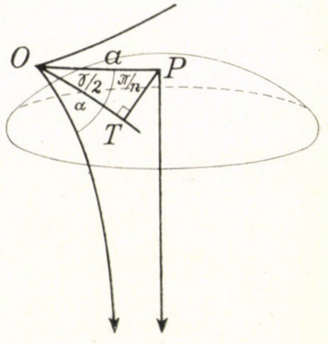

Fig. 1

Man kann leicht einsehen, daß (3) nur in vier Fällen erfüllt ist, und zwar im Falle der Zentralecken des

Tetraeders für $n=6$
Hexaeders für $n=4$
Oktaeders für $n=6$
Ikosaeders für $n=6$.

Die Gleichung (3) gilt im Falle des Dodekaeders nicht.

b) Auf Grund des vorigen Teiles sind die Längen l_m der Kanten der möglichen Tangentenpolyeder eindeutig bestimmt. Bezeichne jetzt A_m den Typus der entsprechenden Zentralecken $(m=1, 2, 3, 4)$. Betrachten wir die Zentralecken vom Typus A_m mit dem Eckpunkt O, die die Umgebung des Punktes O vollständig ausfüllen, und sei je eine Grenzkugel in jede Zentralecke eingesetzt. Bestimmen wir auf den Kanten dieser Zentralecken die Punkte O_{1i} mit dem Abstand l_m von O. Die Punkte O_{1i} können die Eckpunkte weiterer Zentralecken vom Typus A_m sein, die die Umgebung von O_{1i} vollständig ausfüllen und ihre, die Strecken $O_{1i}O$ enthaltende Deckflächen die Grenzkugel berühren. In den Ecken um die Punkte O_{1i}, die die Strecken $O_{1i}O$ auf ihren Kanten nicht enthalten, gibt es keine Grenzkugel. In diese Ecken kann man weitere Grenzkugeln einsetzen, und das Verfahren kann für diese Grenzkugeln ebenso, wie vorhin fortgesetzt werden; die Punkte O_{2ij} ergeben sich aus den Punkten O_{1i}, wie O_{1i} aus O, und so weiter ad infinitum.

Die Punkte $O, O_{1i}, O_{2ij}, \dots, O_{nij\dots k}, \dots$ bilden einen zusammenhängenden regulären unendlichen Graphen.

c) Es ist leicht zu beweisen, daß endliche Knotenpunkte dieses Graphen im Inneren einer Kugel mit dem Mittelpunkt O liegen. Zu diesem Graphen gehören

[4] S. z. B. [2].

unendlich viele Grenzkugeln mit ihren unendlichen Tangentenpolyedern. Falls die Eckpunkte jedes Tangentenpolyeders mit dem Ende der zu ihm gehörenden Grenzkugel verbunden werden, dann bekommen wir einander kongruente asymptotische reguläre Pyramiden. Die Parkettierung wird aus diesen Pyramiden aufgebaut. In der Tabelle 2. sind die Daten der Pyramiden gegeben.

Das Grundvieleck der Pyramide	Der Halbmesser des Umkreises des Grundvielecks
reguläres Sechseck	$\operatorname{ar ch} \frac{\sqrt{6}}{2}$
reguläres Viereck	$\operatorname{ar ch} \sqrt{2}$
reguläres Sechseck	$\operatorname{ar ch} \sqrt{3}$
reguläres Sechseck	$\operatorname{ar ch} \sqrt{\frac{9+3\sqrt{5}}{2}}$

Tabelle 2

2. Unter einer *Abstandsfläche* versteht man die Menge der Punkte, deren Entfernung von einer festen Ebene ein vorgegebener Abstand d ist. Eine Abstandsfläche besteht aus zwei Teilen, die sich in je einem der Halbräume in welche die Ebene *(Grundebene)* den Raum teilt, befinden. Jeder dieser Teile wird *Halbabstandsfläche* der Grundebene genannt.

Hier, bei diesen Untersuchungen kann eine ähnliche Methode wie vorhin angewendet werden. Ein unendliches reguläres Tangentenpolyeders kann um eine Abstandsfläche in folgender Weise gebildet werden:

Man betrachtet ein reguläres Mosaik auf der Grundebene und die Senkrechten auf sie durch die Mittelpunkte der Mosaikelemente. Ferner bestimmt man die Tangentenebenen der Abstandsfläche in den Punkten, in welchen die Senkrechten die Abstandsfläche schneiden. Setzen wir voraus, daß die benachbarten Tangentenebenen sich schneiden und diese Schnittgeraden zwei reguläre unendliche Graphen mit im Endlichen liegenden Knotenpunkten, die in bezug auf die Grundebene symmetrisch sind, bilden. Die Knotenpunkte und die Kanten der Graphen sind die Eckpunkte bzw. die Kanten des unendlichen regulären Tangentenpolyeders.

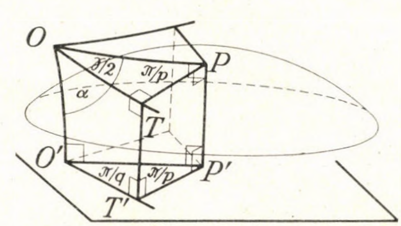

Fig. 2

Jetzt müssen wir untersuchen, ob die Ecken des Tangentenpolyeders den Zentralecken kongruent seien. Zu diesem Zwecke betrachten wir einen Teil eines Tangentenpolyeders und seine rechtwinklige Projektion auf die Grundebene (Fig. 2.). O sei ein solcher Eckpunkt des Tangentenpolyeders, daß die zu dem Punkt O gehörige körperliche Ecke eine Zentralecke sei. Die Abstandsfläche berührt eine Deckfläche der Zentralecke in dem Punkt P, der der Mittelpunkt einer Seitenfläche des Tangentenpolyeders ist. Die Projektion dieser Seitenfläche auf die Grundebene ist auch ein reguläres Vieleck, sein Mittelpunkt sei P'. Es sei T der Fußpunkt eines, aus P auf eine der betreffenden Seitenfläche gehörigen Kante des Tangentenpolyeders gefällten Lotes. Ihre Projektion T' spielt dann in der Projektion des Tangentenpolyeders dieselbe Rolle, wie T im Tangentenpolyeder. So haben wir: $\sphericalangle OPT = \sphericalangle O'P'T' = \frac{\pi}{p}$. Die ganze Zahl p zeigt an, daß das Vieleck ein p-Eck ist. α und $\frac{\gamma}{2}$ bezeichnen die Winkel, die in dem vorigen Abschnitt zur Charakterisierung der Zentralecken gebraucht wurden. Es sei q eine positive ganze Zahl, die der Ungleichung

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}. \tag{4}$$

genügt. Diese Ungleichung ist für die hyperbolischen Mosaike bekannt.[5]
Für das Dreieck OPT erhalten wir:

$$\operatorname{ch} OP = \operatorname{ctg} \frac{\gamma}{2} \cdot \operatorname{ctg} \frac{\pi}{p}, \tag{5}$$

und in ähnlicher Weise für das Dreieck $O'P'T'$:

$$\operatorname{ch} O'P' = \operatorname{ctg} \frac{\pi}{q} \cdot \operatorname{ctg} \frac{\pi}{p}. \tag{6}$$

Es ist bekannt, daß ein Lambertsches Viereck in der Engelschen Zuordnung einem rechtwinkligen Dreieck entspricht, und beide drücken sich durch die gleichen streckentrigonometrischen Formeln aus.[6]

Es ergibt sich für das Lambertsche Viereck $OPO'P'$, daß

$$\operatorname{ch} OP = \operatorname{ch} O'P' \cdot \operatorname{ch} a, \tag{7}$$

gilt, wo a das zu dem Winkel α gehörige Parallellot ist. So kann (7) in der folgenden Form geschrieben werden:

$$\sin \alpha = \frac{\operatorname{ch} O'P'}{\operatorname{ch} OP}. \tag{8}$$

Aus (5), (6) und (8) ergibt sich die Gleichung

$$\operatorname{tg} \frac{\pi}{q} = \frac{\operatorname{tg} \frac{\gamma}{2}}{\sin \alpha}. \tag{9}$$

(9) drückt die hinreichende Bedingung dafür, daß die Ecken des unendlichen Polyeders um die Abstandsfläche einer Zentralecke kongruent seien, aus.

[5] S. [3] S. 94.
[6] S. [5] §§ 10. und 15.

Nach (9) ist $q=3$, falls die Zentralecke zu einem Tetraeder oder Oktaeder oder Ikosaeder gehört, und $q=4$ bzw. $q=5$ gelten im Falle des Hexaeders bzw. Dodekaeders. Diese Werte für q entsprechen der notwendigen Bedingung. Die den unendlichen Tangentenpolyedern entsprechenden hyperbolischen Mosaike auf der Grundebenen müssen auf Grund der Beziehung (4) den folgenden Beschränkungen genügen:

Im Falle

$$q=3, \quad p>6;$$

$$q=4, \quad p>4;$$

$$q=5, \quad p>3.$$

Um den Abstand d der zu einem solchen Tangentenpolyeder gehörenden Abstandsfläche zu bestimmen, schreiben wir für das Lambertsche Viereck $OPO'P'$ die folgende Gleichung auf:

$$\text{(10)} \qquad \operatorname{sh} O'P' = \operatorname{sh} d' \cdot \operatorname{th} a,$$

wo d' und $PP'=d$ komplementäre Strecken[7] sind, und a das zu dem Winkel α gehörige Parallellot bezeichnet. Daraus folgt, daß (10) in der folgenden Form geschrieben werden kann:

$$\text{(11)} \qquad \operatorname{sh} d = \frac{\cos\alpha}{\operatorname{sh} O'P'}.$$

Wir bekommen aus (11) und (5) durch bekannte Identitäten, daß die Beziehung

$$d = \operatorname{ar\,sh} \frac{\cos\alpha \cdot \operatorname{tg}\frac{\pi}{p} \cdot \operatorname{tg}\frac{\pi}{q}}{\sqrt{1-\operatorname{tg}^2\frac{\pi}{p} \cdot \operatorname{tg}^2\frac{\pi}{q}}}$$

gilt. Die Werte von d sind im Falle der verschiedenen Zentralecken in der Tabelle 3 zusammengefaßt.

Die Verbindungsstrecken der Eckpunkte eines, zu einer Abstandsfläche gehörigen, in bezug auf die Grundebene symmetrischen Tangentenpolyeders ergeben eine Zerlegung desselben in kongruente Polyeder. Wir nennen sie — entsprechend der nächstfolgenden Definition — reguläre Prismen. Unter einem regulären Prisma verstehen wir ein konvexes Polyeder, welches man so erhält, daß man auf den durch die Eckpunkte eines regulären Vielecks gehenden und auf die Ebene des Vielecks senkrechten Geraden seine Eckpunkte in gleichen Entfernungen von der Ebene des Vielecks bestimmt.

Betrachten wir kongruente Zentralecken um den Punkt O und die entsprechenden Halbabstandsflächen mit den zugehörigen d aus der Tabelle 3. Zu dem Wert d gehört ein Abstand c_p, der die Kantenlänge des aus p-Ecken bestehenden unendlichen Tangentenpolyeders ist. Bestimmen wir auf den Kanten dieser Zentralecken

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[7] Es ist bekannt, daß d und d' komplementäre Strecken sind, falls $\Pi(d)+\Pi(d')=\frac{\pi}{2}$ ist, wo $\Pi(d)$ den zum Lote d gehörigen Parallelwinkel bezeichnet.

Die Zentralecke gehört zu dem	d
Tetraeder	$\operatorname{ar\,sh}\dfrac{\operatorname{tg}\dfrac{\pi}{p}}{\sqrt{1-3\cdot\operatorname{tg}^2\dfrac{\pi}{p}}}$
Hexaeder	$\operatorname{ar\,sh}\dfrac{\sqrt{2}\cdot\operatorname{tg}\dfrac{\pi}{p}}{2\sqrt{1-\operatorname{tg}^2\dfrac{\pi}{p}}}$
Oktaeder	$\operatorname{ar\,sh}\dfrac{\sqrt{2}\cdot\operatorname{tg}\dfrac{\pi}{p}}{\sqrt{1-3\cdot\operatorname{tg}^2\dfrac{\pi}{p}}}$
Dodekaeder	$\operatorname{ar\,sh}\dfrac{\sqrt{\dfrac{3-\sqrt{5}}{2}}\cdot\operatorname{tg}\dfrac{\pi}{p}}{\sqrt{1-(5-2\sqrt{5})\cdot\operatorname{tg}^2\dfrac{\pi}{p}}}$
Ikosaeder	$\operatorname{ar\,sh}\dfrac{\sqrt{\dfrac{3+\sqrt{5}}{2}}\cdot\operatorname{tg}\dfrac{\pi}{p}}{\sqrt{1-3\cdot\operatorname{tg}^2\dfrac{\pi}{p}}}$

Tabelle 3

die Punkte O_{1i} mit den Abständen c_p von O. Die Punkte O_{1i} können die Eckpunkte weiterer solcher Zentralecken sein, die die Umgebung von O_{1i} vollständig ausfüllen und ihre die Strecken $O_{1i}O$ enthaltenden Deckflächen je eine Halbabstandsfläche berühren. In den Ecken um die Punkte O_{1i}, die die Strecken $O_{1i}O$ auf ihren Kanten nicht enthalten, gibt es keine Halbabstandsfläche. In diese Ecken kann man weitere Halbabstandsflächen setzen, und der Prozeß kann unbegrenzt fortgesetzt werden. Die gegebenen Punkte $O, O_{1i}, O_{2ij}, \ldots, O_{nij\ldots k}, \ldots$ bilden einen zusammenhängenden regulären unendlichen Graphen.

Spiegeln wir gleichzeitig diese Konfiguration an die Grundebenen der vorkommenden Halbabstandsflächen. Nach den Spiegelungen kommen weitere Halbabstandsflächen vor, und in ähnlicher Weise wie vorhin können gleichzeitige Spiegelungen an ihren Grundebenen durchgeführt werden, u. s. w. Wenn man die Zerlegung der zu den Abstandsflächen gehörigen Tangentenpolyeder in kongruente Prismen betrachtet, sieht man, daß man eine Parkettierung des hyperbolischen Raumes vor sich hat.

Somit gilt der SATZ:

Es gibt unendlich viele Parkettierung des dreidimensionalen hyperbolischen Raumes, deren Elemente kongruente reguläre Prismen sind.

Bemerkung: Die regulären Lagerungen der Abstandsflächen können auf Grund der oben erklärten Konstruktion übersehen werden. Durch eine einfache Rechnung bestätigen wir, daß es kein solches reguläres Prisma in den Parkettierungen gibt, welches ein halbregelmäßiges Vielflach wäre bz.w. eine Inkugel hätte.

3. Die regulären Körper können in der hyperbolischen Geometrie den Parkettierungen der Kugel entsprechend in fünf Typen geteilt werden, wo zu jedem Typus eine unendliche Serie der Körper gehört. Jetzt untersuchen wir, welche Parkettierungen des Raumes aus regulären Körpern aufgebaut werden können.

Zu diesem Zweck bestimmen wir die regulären Körper, deren Ecken einer Zentralecke kongruent sind. Die Anzahl der Kanten der Zentralecken, die zu einem Tetraeder, Oktaeder und Ikosaeder gehören, ist gleich der Anzahl, der aus einem Eckpunkt ausgehenden Kanten eines Tetraeders, Hexaeders und Dodekaeders; Laut Winkelsummensätze der Vielecke und in anbetracht der Werte von γ aus der Tabelle 1. kann aber nur die zu einem Oktaeder bzw. Ikosaeder gehörige Zentralecke einer Ecke eines Dodekaeders bzw. einer Ecke eines Hexaeders oder Dodekaeders kongruent sein.

Ebenso erhalten wir, daß die Zentralecke eines Hexaeders bzw. Dodekaeders und eine Ecke eines Oktaeders bzw. Ikosaeders gleiche Anzahl Kanten haben, jedoch nur eine Dodekaederzentralecke einer Ecke des Ikosaeders kongruent sein kann.

Es wird gezeigt, daß die vier Kongruenzmöglichkeiten bei den entsprechenden regulären Körpern verwirklicht werden können. Betrachten wir einen Teil eines regulären Körpers, dessen Inkugel den Punkt C als Zentrum hat (Fig. 3.). O sei ein Eckpunkt des Körpers und P sei ein Berührungspunkt der Inkugel und des Körpers. T sei der Halbierungspunkt einer Kante des Körpers. Für das Dreieck OPT gilt:

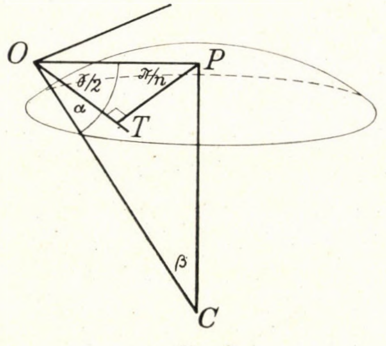

Fig. 3

$$\text{ch}\, OP = \text{ctg}\,\frac{\pi}{n}\cdot \text{ctg}\,\frac{\gamma}{2}. \tag{12}$$

Ferner bekommen wir für das Dreieck OPC, daß

$$\text{ch}\, OP\cdot \sin\alpha = \cos\beta \tag{13}$$

ist. Aus (12) und (13) folgt die Beziehung:

$$\cos\beta = \frac{\sin\alpha}{\text{tg}\,\frac{\gamma}{2}\cdot \text{tg}\,\frac{\pi}{n}}. \tag{14}$$

Diese Gleichung wird in den oben erwähnten Fällen erfüllt, das heißt die Zentralecken und die Ecken der entsprechenden regulären Körper sind einander kongruent. Die Werte von $\sin\alpha$ und $\operatorname{tg}\frac{\gamma}{2}$ stehen in der Tabelle 1. Die Tabelle 4. enthält die Werte von $\cos\beta$ und $\operatorname{tg}\frac{\pi}{n}$ (nur für die notwendigen Fälle).

Die in der Parkettierung vorkommenden regulären Körper	$\cos\beta$	$\operatorname{tg}\frac{\pi}{n}$
Hexaeder	$\frac{\sqrt{3}}{3}$	$1 \quad (n=4)$
Dodekaeder	$\sqrt{\frac{5+2\sqrt{5}}{15}}$	$\sqrt{5-2\sqrt{5}} \quad (n=5)$
Ikosaeder	$\sqrt{\frac{5+2\sqrt{5}}{15}}$	$\sqrt{3} \quad (n=3)$

Tabelle 4

Damit ist der Satz von H. ZEITLER bewiesen: Es gibt in dem hyperbolischen Raum nur vier verschiedene Parkettierungen, die aus regulären Körpern (eine Hexaedern, zwei aus Dodekaedern, eine aus Ikosaedern) aufgebaut werden können.[8]

Bemerkung: In diesem Beweis haben wir den Eulerschen Polyedersatz — im Gegensatz zu H. ZEITLER — nicht benutzt. In diesen Untersuchungen ist es gleichzeitig möglich die regulären Lagerungen kongruenter Kugeln zu übersehen.

4. Wir untersuchen noch die Parkettierungsmöglichkeit des hyperbolischen Raumes durch total-asymptotische Polyeder. Der Begriff der *regulären asymptotischen Zentralecke* spielt in Verbindung hiermit eine wichtige Rolle. Eine solche reguläre asymptotische Zentralecke ergibt sich in folgender Weise: Man betrachtet eine reguläre Parkettierung auf der Grenzkugel und jene Geraden, die die Eckpunkte eines Elementes der Parkettierung mit dem zur Grenzkugel gehörigen Ende verbinden. Die benachbarten Geraden bestimmen paarweise die Deckflächen, die die reguläre asymptotische Zentralecke begrenzen. Aus der Parkettierungsmöglichkeit der Grenzkugel folgt, daß es drei reguläre asymptotische Zentralecken gibt:

1. mit drei Deckflächen; der Winkel zwischen zwei Deckflächen ist gleich $\frac{\pi}{3}$
2. mit vier Deckflächen; der Winkel zwischen zwei Deckflächen ist gleich $\frac{\pi}{2}$

[8] K. BÖRÖCZKY hat mich darauf aufmerksam gemacht, daß auch H. S. M. COXETER sich schon früher in seiner Arbeit: Regular honeycombs in hyperbolic space. Proc. Int. Congress of Math. Amsterdam 1954. Vol. III. 155—169. mit der Lösung des Parkettierungsproblems durch reguläre Polyeder in der hyperbolischen Geometrie beschäftigte.

3. mit sechs Deckflächen; der Winkel zwischen zwei Deckflächen ist gleich $\frac{2\pi}{3}$.

Unter einem regulären asymptotischen Polyeder versteht man einen Körper, dessen „Eckpunkte" die Enden der aus dem Zentrum eines regulären Körpers ausgehenden und durch seine Eckpunkte gezogenen Halbgeraden sind. Die Seitenflächen sind total asymptotische reguläre Vielecke, die Kanten sind Geraden.

Es ist leicht zu beweisen, daß nur die regulären asymptotischen Polyeder, deren „Ecken" reguläre asymptotische Zentralecken sind, eine Parkettierung des Raumes bilden können. Daraus folgt, daß keine Parkettierung aus asymptotischen Ikosaedern aufgebaut werden kann. Im Falle der übrigen vier asymptotischen regulären Polyeder ergibt sich mit einfacher Rechnung, daß die „Ecken" dieser Polyeder reguläre asymptotische Zentralecken sind. Aus der Regelmäßigkeit der Polyeder folgt, daß diese Bedingung gleichzeitig auch hinreichend ist. So kann man den folgenden Satz aussagen: *der hyperbolische Raum kann aus asymptotischen regulären Tetraedern, Hexaedern, Oktaedern und Dodekaedern aufgebaut werden.*

Wir verstehen unter einer regulären *total-asymptotischen Pyramide* ein Polyeder, welches in folgender Weise konstruiert werden kann: man betrachtet ein total-asymptotisches Vieleck mit dem Mittelpunkt P in einer Ebene, die eine Tangentialebene einer Grenzkugel ist, die diese Ebene in dem Punkt P berührt. Dieses Vieleck ist die Grundfläche und die weitere „Ecke" ist das Ende der Grenzkugel.

Drei verschiedene unendliche reguläre Tangentenpolyeder können mit total-asymptotischen Deckflächen konstruiert werden, diese sind die total-asymptotischen Drei-, Vier-, und Sechsecke. Sechs, vier und drei Kanten treffen sich in einer „Ecke" dieser unendlichen regulären Tangentenpolyeder. Es ist leicht zu beweisen, daß die Ecken reguläre asymptotische Zentralecken sind. (Daraus folgt schon, daß eine reguläre total-asymptotische dreiseitige Pyramide ein reguläres total-asymptotisches Tetraeder ist.)

Es ist klar, daß eine zu der Grundfläche gehörende „Ecke" einer regulären total-asymptotischen Pyramide durch kongruente Pyramiden lückenlos umschlossen werden kann. So können wir den folgenden SATZ aussagen:

Der hyperbolische Raum kann aus regulären total-asymptotischen drei-, vier- und sechsseitigen Pyramiden aufgebaut werden. Die Parkettierung aus regulären total-asymptotischen dreiseitigen Pyramiden ist identisch der Parkettierung durch asymptotische reguläre Tetraeder.

Unter einem *regulären total-asymptotischen Prisma* versteht man das Polyeder, welches in folgender Weise konstruiert werden kann. Man betrachtet ein reguläres Vieleck und die Senkrechten durch seine Eckpunkte auf die Ebene des Vieleckes. Die „Ecken" dieses Polyeders seien die Enden der erwähnten Senkrechten. Diese Enden bestimmen die zwei total-asymptotischen regulären Grundvielecke des Prismas und die durch benachbarte Geraden bestimmten total-asymptotischen Vierecke sind seine Deckflächen. Um die unendlichen regulären total-asymptotischen Tangentenpolyeder — deren „Ecken" reguläre asymptotische Zentralecken sind — zu bestimmen, bezeichnen wir durch O in der Fig. 2. die „Ecke" eines solches Tangentenpolyeders und durch P den Mittelpunkt einer Deckfläche. Wenn wir das Zeichensystem der Fig. 2. benutzen, so bekommen wir, daß

$$\Pi(O'P') + \Pi(PP') = \frac{\pi}{2} \tag{15}$$

gilt, also aus dem Dreieck $O'P'T'$ folgt die Beziehung

$$\operatorname{ch} O'P' = \operatorname{ctg}\frac{\pi}{q}\cdot\operatorname{ctg}\frac{\pi}{p}. \tag{16}$$

Aus (16) und (15) ergibt sich die Gleichung

$$\operatorname{th} PP' = \operatorname{tg}\frac{\pi}{q}\cdot\operatorname{tg}\frac{\pi}{p}. \tag{17}$$

Bezeichne ϑ den Winkel $T'TP$ in dem Lambertschen Viereck $PP'T'T$, so ist

$$\operatorname{ctg}\vartheta = \operatorname{sh} PT\cdot\operatorname{th} PP';$$

somit folgt aus (17) und aus der Gleichung

$$\operatorname{sh} PT = \operatorname{ctg}\frac{\pi}{q}$$

die Gleichung:

$$\operatorname{ctg}\vartheta = \operatorname{tg}\frac{\pi}{q}. \tag{18}$$

Ist $q=3$, so ist $\vartheta = \frac{\pi}{6}$

$q=4$, so ist $\vartheta = \frac{\pi}{4}$

$q=6$, so ist $\vartheta = \frac{\pi}{3}$.

Daraus folgt unmittelbar, daß zu den unendlichen Tangentenpolyedern in allen drei möglichen Fällen reguläre asymptotische Zentralecken gehören. Im Folgenden beschäftigen wir uns nur mit solchen Tangentenpolyedern. Die Projektion eines Tangentenpolyeders bildet auf der Grundebene ein reguläres Mosaik. Daraus folgt, daß das unendliche Tangentenpolyeder in kongruente reguläre total-asymptotische Prismen zerlegt werden kann.

Es ist klar, daß eine „Ecke" eines regulären total-asymptotischen Prismas durch kongruente Prismen lückenlos umschlossen werden kann. Man kann auch leicht einsehen, daß *unendlich viele Parkettierungen aus den regulären total-asymptotischen Prismen aufgebaut werden können.*

Sind $q=6$ und $p=4$, so bekommen wir eine Parkettierung aus regulären asymptotischen Hexaedern. In dem Dreieck $O'P'T'$ gilt nähmlich die Beziehung

$$\operatorname{th} T'P' = \operatorname{th} O'P'\cdot\frac{\sqrt{2}}{2}, \tag{19}$$

und auf Grund der Gleichung (17) ist

$$\operatorname{th} PP' = \frac{\sqrt{3}}{3}. \tag{20}$$

Aus (15) und (20) ergibt sich die Gleichung

$$\text{th}\, O'P' = \sqrt{\frac{2}{3}},$$

und so folgt aus (19), daß auch

$$\text{th}\, T'P' = \frac{\sqrt{3}}{3}$$

besteht.

Daraus folgt, daß $T'P' = PP'$ ist; das Prisma hat also eine Inkugel, und damit ist es gezeigt, daß dieses Prisma ein asymptotisches reguläres Hexaeder ist.

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(Eingegangen: 22. Februar, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 279—280

A NOTE ON FACTOR-CRITICAL GRAPHS

by

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A graph[1] will be called *factor-critical* or, shortly, *critical* if removing any vertex of it the remaining graph has a 1-factor. We are going to give a construction which yields every critical graph:

THEOREM. *Every critical graph and only these graphs can be constructed from the one-point graph by the iteration of the following construction: we connect two (not necessarily different) vertices of an (already constructed) critical graph by a suspending arc[2] of odd length.*

PROOF. I. Graphs constructed by this method are critical. We show this by induction on the number of steps of the construction. The one-point graph is evidently critical. Assume G is critical and add a suspending arc P of odd length to it. Let us remove a vertex x of the obtained graph G'. If this vertex is a vertex of G then what remains from G has a 1-factor and this can be extended to a factor of G' by the edges of the new arc. On the other hand, if x lies on the new suspending arc then it is of odd distance from one of the endpoints y of the suspending arc. Removing y from G a graph with a 1-factor arises, and this 1-factor can be extended to a 1-factor of $G'-x$.

II. Let now G be a critical graph, we show is can be built up by the given construction. Removing any vertex x of G there is a 1-factor F_x of the remaining graph. It is easily seen that $F_x \cup F_y$ $(x \neq y)$ contains a path P_{xy} connecting x to y and containing edges of F_x and F_y alternatingly, beginning with an edge of F_y and ending with an edge of F_x.

Let a be a vertex of G. Consider a maximal subgraph G_0 of G such that

(a) G_0 can be built up by the construction of the theorem;

(b) G_0 contains a;

(c) G_0 contains none or both endpoints of any edge of F_a. Such a G_0 exists since the subgraph consisting only of a has these properties. We show now $G_0 = G$.

Obviously, every edge of G connecting two vertices of G_0 belongs to G_0. Since G is, obviously, connected, it is enough to show that there cannot be any edge connecting a vertex x of G_0 to vertex y not belonging to G_0. Assume indirectly there would be such an edge. By property (c) of G_0, $(x, y) \notin F_a$. Consider the path P_{ya}. This connects y to a and thus it has a common point z with G_0 nearest y. Let P be the piece of P_{ya} between y and z. The edge of P incident with y and then every second

[1] We consider finite, undirected graphs without loops and multiple edges.

[2] An arc is a (simple) path or a circuit on which a beginning and endpoint is selected. A suspending arc is an arc whose inner vertices are of valence 2.

edge of it belongs to F_a, as mentioned above. The edge of P incident with z does not belong to F_a by (c). Hence P is of even length. Thus, adding P and (x, y) to G_0 we obtain a subgraph which also has properties (a), (b), (c), a contradiction.

Remarks: 1. This construction is analogue to a construction for graphs with 1-factor, deduced from a theorem of HETYEI [1].

2. One can observe that further requirements concerning the building up the critical graph can be satisfied: e.g. one can start with an odd circuit containing a prescribed edge.

3. Let us call a step of the construction *trivial* if it is the addition of a single edge. One can observe that a graph with odd many vertices is Hamiltonian if and only if it is critical and it can be built up using one non-trivial step only. Thus the minimum number of non-trivial steps would be interesting to be estimated (e.g. by some function of the minimum valence).

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(Received February 25, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 281—283

THE VARIANCE OF THE NUMBER OF SPANNING CYCLES IN A RANDOM GRAPH

by

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1. Introduction. A *graph* G_n is a collection of *nodes* $1, 2, \ldots, n$ some pairs of which are joined by a single *edge*. The complete n-graph $\langle n \rangle$ is the graph G_n in which each pair of distinct nodes i and j is joined by an edge ij. A *path* is a sequence of edges of the type $\{ab, bc, \ldots, ij\}$; we assume the nodes $a, b, \ldots, j$ are distinct except that nodes a and j may be the same in which case we call the path a *cycle*. The *length* of a path or cycle is the number of edges it contains; a *spanning cycle* of G_n, where $n \geqq 3$, is any cycle of length n.

Let $x = x(G_n)$ denote the number of spanning cycles in the graph G_n. If μ denotes the average value of x over all the $2^{\frac{1}{2}n(n-1)}$ graphs G_n with n labelled nodes, then clearly $\mu = (n-1)!/2^{n+1}$. Our object here is to derive a formula for the variance σ^2 of x over all graphs G_n and to show that $\sigma^2 \sim (e^2-1)\mu^2$ as n tends to infinity.

2. Three lemmas. Let $f(m, h)$ denote the number of families of h paths in $\langle m \rangle$ such that each node of $\langle m \rangle$ belongs to precisely one path and each path has length at least one.

LEMMA 1. *If* $2 \leqq 2h \leqq m$, *then*

$$f(m, h) = \frac{m!}{2^h h!} \binom{m-h-1}{h-1}.$$

PROOF. The formula certainly holds when $m \geqq 2h = 2$ and when $m = 2h \geqq 2$. If we classify the admissible families of h paths in $\langle m+1 \rangle$ according as the $(m+1)$-st node does or does not belong to a path of length one, we find that

$$f(m+1, h) = mf(m-1, h-1) + (m+h)f(m, h).$$

The result now follows directly by induction. (See [4; § 5] for a different approach to to a closely related problem.)

If $n \geqq 3$ let $P_l(n)$ denote the number of ordered pairs of spanning cycles in $\langle n \rangle$ that have l or more edges in common; each such pair is to be counted separately for each set of l edges they have in common. Since $\langle n \rangle$ has $(n-1)!/2$ spanning cycles it follows that

$$P_0(n) = (P_n(n))^2 = \left\{\frac{(n-1)!}{2}\right\}^2.$$

LEMMA 2. *If* $1 \leqq l \leqq n-1$, *then*

$$P_l(n) = \frac{n!(n-l-1)!}{n-l} \sum_{h=1}^{l} \binom{n-l}{h} \binom{l-1}{h-1} 2^{h-2}.$$

PROOF. If two spanning cycles in $\langle n\rangle$ have l edges in common then these l edges determine a family of paths in $\langle n\rangle$; if there are h paths of length at least one in this family, then there are $l+h$ nodes that belong to these h paths and $n-l-h$ isolated nodes that don't belong to these paths. There are $\binom{n}{l+h}f(l+h, h)$ ways to form such a family of h nontrivial paths that involve $l+h$ nodes altogether. There are $2^h(n-l-1)!/2=2^{h-1}(n-l-1)!$ spanning cycles in $\langle n\rangle$ that contain any such collection of l edges determining h nontrivial paths; the factor $(n-l-1)!/2$ counts the number of cyclic orderings of the h paths and $n-l-h$ isolated nodes, and the factor 2^h arises from the fact that each of the h nontrivial paths can be transversed in one of two ways in a spanning cycle. It follows, therefore, that

$$P_l(n) = \sum_{h=1}^{l} \binom{n}{l+h} f(l+h, h)2^{2h-2}\{(n-l-1)!\}^2.$$

The required formula for $P_l(n)$ now follows from Lemma 1.

If $n\geqq 3$ let $S_l(n)$ denote the number of ordered pairs of spanning cycles in $\langle n\rangle$ that have exactly l edges in common. PALÁSTI [2, 3] gave an explicit formula for $S_l(n)$, in effect, when $n-4\leqq l\leqq n$ and discussed recurrence relations for $S_0(n)$ and $S_1(n)$. The following result is an immediate consequence of the method of inclusion and exclusion.

LEMMA 3. *If* $0\leqq l\leqq n$ *and* $n\geqq 3$, *then*

$$S_l(n) = \sum_{t=0}^{n-l} (-1)^t \binom{l+t}{l} P_{l+t}(n).$$

3. Main result. We now derive a formula for the variance σ^2 of the number of spanning cycles in a random graph G_n with n labelled nodes in which the probability that any given pair of nodes is joined by an edge is $\frac{1}{2}$.

THEOREM. *If* $n\geqq 3$, *then* $\sigma^2 = \left(\frac{1}{2}\right)^{2n} \sum_{l=1}^{n} P_l(n).$

PROOF. If two spanning cycles in $\langle n\rangle$ have exactly l edges in common, then the probability that both these cycles are present in a random graph G_n is $(1/2)^{2n-l}$. Hence, if μ_2 denotes the expected value of x^2, we find that

$$\mu_2 = \left(\frac{1}{2}\right)^{2n} \sum_{l=0}^{n} S_l(n)2^l = \left(\frac{1}{2}\right)^{2n} \sum_{l=0}^{n} \left\{\sum_{t=0}^{n-l} (-1)^t \binom{l+t}{l} P_{l+t}(n)\right\} 2^l =$$

$$= \left(\frac{1}{2}\right)^{2n} \sum_{l=0}^{n} \left\{\sum_{i=0}^{l} \binom{l}{i} 2^i(-1)^{l-i}\right\} P_l(n) = \left(\frac{1}{2}\right)^{2n} \sum_{l=0}^{n} P_l(n).$$

The required formula now follows from the fact that

$$\sigma^2 = \mu_2-\mu^2 = \mu_2-P_0(n)/2^{2n}.$$

COROLLARY. *If* $n\to\infty$, *then* $\sigma^2 \sim (e^2-1)\mu^2$.

PROOF. If l is any fixed positive integer and $n>l$, then $P_l(n)$ is a polynomial in n; when n is large the leading term arises from the term $h=l$ in the formula in Lemma 2. Consequently,

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^{2n}\frac{P_l(n)}{\mu^2}=\lim_{n\to\infty}\frac{n}{n-l}\cdot\frac{1}{(n-1)_l}\binom{n-l}{l}2^l=\frac{2^l}{l!}$$

for each fixed positive integer l. It is a straightforward exercise to verify that the hypothesis of TANNERY's theorem (see [1; p. 136]) holds here; we may conclude, therefore, that

$$\lim_{n\to\infty}\frac{\sigma^2}{\mu^2}=\sum_{l=1}^{\infty}\frac{2^l}{l!}=e^2-1.$$

Let p denote a constant such that $0<p<1$; if each edge in a random graph G_n is present with probability p, then a slight modification of the preceding argument shows that $\sigma^2\sim(e^{2(1-p)/p}-1)\mu^2$, as $n\to\infty$, where now $\mu=(n-1)!p^n/2$.

The preparation of this paper was assisted by a grant from the National Research Council of Canada.

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(Received March 16, 1971)

ON THE RAREFACTION OF RENEWAL PROCESSES I.

by

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1. We start from the following

THEOREM 1. *Let $\xi_1, \xi_2, \dots$ be a sequence of independent and identically distributed random variables with the common mean-value μ. We suppose that μ is finite. Let further ν_n $(n=1, 2, \dots)$ be a sequence of positive integer-valued random variables, such that ν_n does not depend on $\xi_1, \xi_2, \dots$ and for a sequence $\omega(n)$ $(\omega(n) \to +\infty$, as $n \to +\infty)$*

$$\lim_{n\to+\infty} \mathsf{P}\left(\frac{\nu_n}{\omega(n)} < x\right) = G(x) \qquad (G(+0) = 0)$$

exists, where $G(x)$ is a distribution function. Then

$$\lim_{n\to+\infty} \mathsf{P}\left(\frac{\xi_1+\dots+\xi_{\nu_n}}{\omega(n)} < x\right) = \begin{cases} G\left(\dfrac{x}{\mu}\right), & \text{if } \mu>0 \\ E(x), & \text{if } \mu=0 \\ 1-G\left(-\dfrac{x}{\mu}+0\right), & \text{if } \mu<0, \end{cases}$$

where $E(x)=0$, if $x \leqq 0$ and $E(x)=1$ if $x>0$.

PROOF. Let $f(t)$ denote the common characteristic function of the random variables ξ_i $(i=1, 2, \dots)$ and let us consider the characteristic function of the variable

$$\zeta_n = \frac{\xi_1+\dots+\xi_{\nu_n}}{\nu_n}.$$

This characteristic function is

$$\mathsf{M}\left(f\left(\frac{t}{\nu_n}\right)^{\nu_n}\right),$$

where the symbol $\mathsf{M}(\cdot)$ denotes mathematical expectation. It is known that the sequence $f\left(\frac{t}{n}\right)^n$ converges to $e^{i\mu t}$ as $n\to+\infty$, where i denotes the complex imaginary unit. So $f\left(\frac{t}{\nu_n}\right)^{\nu_n}$ converges in probability to $e^{i\mu t}$. Since we have $\left|f\left(\frac{t}{\nu_n}\right)^{\nu_n}\right| \leqq 1$, we obtain by Lebesgue's convergence theorem that

$$\lim_{n\to+\infty} \mathsf{M}\left(f\left(\frac{t}{\nu_n}\right)^{\nu_n}\right) = e^{i\mu t},$$

which means that ζ_n converges in probability to μ. We can now write

$$\frac{\xi_1+\dots+\xi_{\nu_n}}{\omega(n)} = \zeta_n \cdot \frac{\nu_n}{\omega(n)}.$$

From this by a lemma of H. Cramér we obtain the assertion of our theorem.

2. Let us consider the renewal process $\tau_0 \equiv 0 \leqq \tau_1 \leqq \tau_2 \leqq \cdots$ i.e. a point process, for which the differences $\xi_i = \tau_i - \tau_{i-1}$ $(i=1, 2, \dots)$ are independent, identically distributed, non-negative random variables, with common distribution function $F(x)$. We shall denote by μ the mean-value of the variables ξ_i $(i=1, 2, \dots)$, if it exists. In what follows this process will be called the original renewal process.

Let $\nu_i^{(n)}$ $(n=1, 2, \dots;\ i=1, 2, \dots)$ be a sequence of positive integer-valued random variables which are independent and identically distributed with the common distribution $\{p_k\}$:

$$\mathsf{P}(\nu_i^{(n)}=k) = p_k \qquad (k=1, 2, \dots;\ i, n=1, 2, \dots).$$

We shall suppose that the variables $\nu_i^{(n)}$ are also independent of the original process and that $0<\mathsf{D}^2(\nu_i^{(n)})=\sigma^2<+\infty$, where the symbols $\mathsf{P}(\cdot)$ and $\mathsf{D}^2(\cdot)$ denote probability and variance. The supposition $0<\sigma^2<+\infty$ implies that the variables $\nu_i^{(n)}$ are with probability 1 non-constant: $p_k \neq 1$, $k=1, 2, \dots$. Let $f(s)$ denote the probability generating function of the variables $\nu_i^{(n)}$ and M their common mean-value. It is obvious that the function

$$f(s) = \sum_{k=1}^{\infty} p_k s^k$$

is defined for every complex s on the unit circle $|s| \leqq 1$ and $M>1$. We shall now make the following operations on the original renewal process: in the first step we consider the random variables $\nu_i^{(1)}$ $(i=1, 2, \dots)$ and we put

$$\tau_0^{(1)} \equiv 0$$

further

$$\tau_i^{(1)} = \tau_{\nu_1^{(1)}+\nu_2^{(1)}+\dots+\nu_i^{(1)}} \qquad (i=1, 2, \dots).$$

So, we put that renewal point of the original renewal process the index of which is $\nu_1^{(1)}+\nu_2^{(1)}+\dots+\nu_i^{(1)}$ $(i=1, 2, \dots)$. It can be easily seen that the differences

$$\tau_i^{(1)} - \tau_{i-1}^{(1)} \qquad (i=1, 2, \dots)$$

are non-negative, independent and identically distributed random variables with the common distribution function

$$F_1(x) = \sum_{k=1}^{\infty} p_k F^{(*k)}(x), \tag{1}$$

where $F^{(*k)}(x)$ is the k-fold convolution of $F(x)$ with itself. So the point process $\tau_0^{(1)} \equiv 0 \leqq \tau_1^{(1)} \leqq \tau_2^{(1)} \leqq \cdots$ is again a renewal process with the distribution function (1).

If the mathematical expectation μ of the original process is finite then the expectation of $\tau_i^{(1)}-\tau_{i-1}^{(1)}$ is μM. So we have

$$\mathsf{M}\left(\frac{\tau_i^{(1)}-\tau_{i-1}^{(1)}}{M}\right)=\mu.$$

We shall call $\tau_0^{(1)}\equiv 0\leqq\tau_1^{(1)}\leqq\tau_2^{(1)}\leqq\cdots$ the first step rarefaction of the original renewal process.

If the k-th step rarefaction $\tau_0^{(k)}\equiv 0\leqq\tau_1^{(k)}\leqq\cdots$ of the original renewal process is already defined, the $(k+1)$-th one will be defined in the following manner: we consider the random variables $\nu_i^{(k+1)}$ $(i=1,2,\ldots)$ and put

$$\tau_0^{(k+1)}\equiv 0$$

further

$$\tau_i^{(k+1)}=\tau^{(k)}_{\nu_1^{(k+1)}+\cdots+\nu_i^{(k+1)}}\qquad (i=1,2,\ldots). \tag{2}$$

It is easy to see that the random variables

$$\tau_i^{(k+1)}-\tau_{i-1}^{(k+1)}\qquad (i=1,2,\ldots)$$

are non-negative, independent and identically distributed with the common distribution function

$$F_{k+1}(x)=\sum_{j=1}^{\infty}p_jF_k^{(*j)}(x)\qquad (k=1,2,\ldots)$$

So $\tau_0^{(k+1)}\equiv 0\leqq\tau_1^{(k+1)}\leqq\cdots$ is a renewal process which will be called the $(k+1)$-th step rarefaction of the original renewal process. Moreover, if μ is finite, then

$$\mathsf{M}(\tau_i^{(k+1)}-\tau_{i-1}^{(k+1)})=\mu M^{k+1}.$$

The renewal points of the n-th step rarefaction can be exprimed directly by the renewal points of the original process as follows: let us define recursively the random variables $Z_i^{(n)}$ in the following manner:

$$Z_i^{(1)}=\nu_i^{(1)}\qquad (i=1,2,\ldots),$$

and

$$Z_1^{(n)}=\sum_{l=1}^{\nu_1^{(n)}}Z_l^{(n-1)},\quad Z_2^{(n)}=\sum_{l=\nu_1^{(n)}+1}^{\nu_1^{(n)}+\nu_2^{(n)}}Z_l^{(n-1)},\ \ldots$$

$$Z_i^{(n)}=\sum_{l=\nu_1^{(n)}+\cdots+\nu_{i-1}^{(n)}+1}^{\nu_1^{(n)}+\cdots+\nu_i^{(n)}}Z_l^{(n-1)}\qquad (n\geqq 2). \tag{3}$$

By the aid of these random variables we obtain

$$\tau_i^{(n)}=\tau_{Z_1^{(n)}+\cdots+Z_i^{(n)}}\qquad (i=1,2,\ldots)$$

or in an equivalent form:

$$\tau_i^{(n)}-\tau_{i-1}^{(n)}=\sum_{j=Z_1^{(n)}+\cdots+Z_{i-1}^{(n)}+1}^{Z_1^{(n)}+\cdots+Z_i^{(n)}}\xi_j, \tag{4}$$

where $\xi_j=\tau_j-\tau_{j-1}$.

3. We see by induction that the random variables $Z_i^{(n)}$ take on, as values, the positive integers and for fixed n and varying i they are independent and identically distributed with the common probability generating function

$$f_n(s) = \underbrace{f(f(\ldots(f(s))\ldots))}_{n\text{-times}}. \tag{5}$$

Now we prove the following

LEMMA 1. *Suppose that* $0<\sigma^2<+\infty$. *Then for every fixed* i

$$\lim_{n\to+\infty} \mathsf{P}(Z_i^{(n)}<M^n x) = G(x) \tag{6}$$

exists, where $G(x)$ *is a distribution function with mean-value* 1 *and variance*

$$\frac{\sigma^2}{M^2-M}.$$

Furthermore, $G(x)$ *has probability density function. The limiting distribution* (6) *belongs to the class of the limiting distributions for Galton—Watson processes with* $M>1$.

PROOF. Let us consider that Galton—Watson process for which the generating function of the first generation is equal to $f(s)$, the probability generating function of the random variables $v_i^{(n)}$. Then, as it is well-known, the generating function of the n-th generation is (5). Also, the number in the n-th generation, divided by M^n, converges with probability 1 to a random variable W, having an absolutely continuous distribution $G(x)$. Since now the distribution of $Z_i^{(n)}/M^n$ coincides with that of the number of the n-th generation divided by M^n, we see that

$$\lim_{n\to\infty} \mathsf{P}(Z_i^{(n)}<M^n x) = \mathsf{P}(W<x)$$

(cf. [5] Theorem 8. 1.). Finally, the random variable W has mean-value 1 and variance

$$\frac{\sigma^2}{M^2-M}.$$

On the basis of this assertion we prove

THEOREM 2. *If the mean-value* μ *of the original renewal process is finite, further* $0<\sigma^2<+\infty$, *then for every fixed* i

$$\lim_{n\to+\infty} \mathsf{P}(\tau_i^{(n)}-\tau_{i-1}^{(n)}<\mu M^n x) = G(x),$$

where $G(x)$ *is defined in Lemma 1.*

PROOF. The assertion follows immediately from (4), Lemma 1, and Theorem 1. In fact, by (4) we see that $\tau_i^{(n)}-\tau_{i-1}^{(n)}$ is a sum of independent and identically distributed random variables with mean-value μ, where the number of the summands is $Z_i^{(n)}$.

Now we shall deal with the rate of convergence in Theorem 2.

THEOREM 3. *If*

$$\gamma = \int_0^{+\infty} x^2 \, dF(x) < +\infty,$$

further $0 < \sigma^2 < +\infty$ *and the limiting distribution* $G(x)$, *defined in Lemma 1, has bounded probability density function, then for every fixed i*

$$|\mathsf{P}(\tau_i^{(n)} - \tau_{i-1}^{(n)} < \mu M^n x) - G(x)| \leq K M^{-n/3}, \tag{7}$$

where K is a constant not depending on n.

PROOF. By the supposition that γ is finite it follows that μ is also finite. Let $\varphi(t)$ and $\psi(t)$ denote the characteristic functions of the distribution functions $G(x)$ and $F(x)$, respectively. It is known that $\varphi(t)$ satisfies the so-called Poincaré's equation

$$\varphi(Mt) = f(\varphi(t)),$$

(cf. [5], Theorem 8. 2.). It follows from this that for $n = 1, 2, \ldots$

$$\varphi(t) = f_n\left(\varphi\left(\frac{t}{M^n}\right)\right) \tag{8}$$

holds. On the basis of (4) we see that the characteristic function of

$$\frac{\tau_i^{(n)} - \tau_{i-1}^{(n)}}{\mu M^n} \qquad (i = 1, 2, \ldots)$$

is $f_n\left(\psi\left(\frac{t}{\mu M^n}\right)\right)$. Now

$$\psi\left(\frac{t}{\mu M^n}\right) = 1 + i\frac{t}{M^n} - \frac{\gamma}{2\mu^2 M^{2n}} t^2 + o\left(\frac{t^2}{M^{2n}}\right), \qquad (n \to +\infty), \tag{9}$$

and

$$\varphi\left(\frac{t}{M^n}\right) = 1 + i\frac{t}{M^n} - \frac{\sigma^2 + M^2 - M}{M^2 - M} \frac{t^2}{2M^{2n}} + o\left(\frac{t^2}{M^{2n}}\right), \qquad (n \to +\infty), \tag{10}$$

where i is the complex imaginary unit.

By (8) and Lagrange's theorem

$$\left| f_n\left(\psi\left(\frac{t}{\mu M^n}\right)\right) - \varphi(t) \right| = \left| f_n\left(\psi\left(\frac{t}{\mu M^n}\right)\right) - f_n\left(\varphi\left(\frac{t}{M^n}\right)\right) \right| =$$

$$= \left| f_n'(\vartheta_n(t)) \left(\psi\left(\frac{t}{\mu M^n}\right) - \varphi\left(\frac{t}{M^n}\right)\right) \right| \leq M^n \left| \psi\left(\frac{t}{\mu M^n}\right) - \varphi\left(\frac{t}{M^n}\right) \right|,$$

where $|\vartheta_n(t)| \leq 1$. By (9) and (10)

$$M^n \left| \psi\left(\frac{t}{\mu M^n}\right) - \varphi\left(\frac{t}{M^n}\right) \right| = O\left(\frac{t^2}{M^n}\right) \qquad (n \to +\infty),$$

where the constant in O does not depend on n. So there exists an absolute constant $C>0$ such that

$$\left|\frac{f_n\left(\psi\left(\frac{t}{\mu M^n}\right)\right)-\varphi(t)}{t}\right| \leqq C\frac{|t|}{M^n}.$$

This implies that for every T

$$(11) \qquad I = \int_{-T}^{T} \left|\frac{f_n\left(\psi\left(\frac{t}{\mu M^n}\right)\right)-\varphi(t)}{t}\right| dt \leqq \frac{CT^2}{M^n}.$$

To estimate

$$|\mathsf{P}(\tau_i^{(n)}-\tau_{i-1}^{(n)}<\mu M^n x)-G(x)|,$$

we can now apply Esseen's theorem. (Cf. [6]). It turns out that

$$|\mathsf{P}(\tau_i^{(n)}-\tau_{i-1}^{(n)}<\mu M^n x)-G(x)| \leqq \frac{k}{2\pi} I + c(k)\frac{A}{T},$$

where I is defined by (11) and $A=\sup\limits_x G'(x)$. Here $k>1$ is an arbitrary constant and $c(k)$ depedens only on k. By (11) we have

$$|\mathsf{P}(\tau_i^{(n)}-\tau_{i-1}^{(n)}<\mu M^n x)-G(x)| \leqq \frac{k}{2\pi}\frac{CT^2}{M^n}+c(k)\frac{A}{T}.$$

Put now $T=M^{n/3}$. We obtain the assertion of our theorem by putting

$$K=\frac{k}{2\pi}C+c(k)A.$$

As a special case of theorem 2. we get a result of A. Rényi [2] and others. Let

$$P(\nu_i^{(n)}=k) = q(1-q)^{k-1} \qquad (k=1,2,\ldots;\ i,n=1,2,\ldots).$$

In this case we obtain, that

$$\lim_{n\to+\infty} \mathsf{P}\left(Z_i^{(n)} < \frac{x}{q^n}\right) = 1-e^{-x} \qquad (x\geqq 0).$$

Thus by Theorem 2

$$\lim_{n\to+\infty} \mathsf{P}(q^n(\tau_i^{(n)}-\tau_{i-1}^{(n)})<\mu x) = 1-e^{-x} \qquad (x\geqq 0),$$

for every fixed $i=1,2,\ldots$. In [3] the following estimation of the rate of convergence is given:

$$|\mathsf{P}(q^n(\tau_i^{(n)}-\tau_{i-1}^{(n)})<\mu x)-(1-e^{-x})| \leqq Cq^n \ln\frac{1}{q^n},$$

which is better in this special case than that of our Theorem 3, in the general case.

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(Received March 10, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 293—305

ON THE RAREFACTION OF RENEWAL PROCESSES II.

by

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In this paper the definitions, notions and notations of the previous paper will be used.

4. If the mean-value μ of the original renewal process is infinite then we cannot normalize the variables

$$\tau_i^{(n)} - \tau_{i-1}^{(n)} \qquad (i=1, 2, \dots),$$

as in Theorem 2, to ensure the existence of a limiting distribution. In what follows in this paper we will not suppose the existence of the mean-value of the original renewal process. So, we substitute the normalizing factor $\frac{1}{\mu M^n}$ by a factor δ_n, where $\delta_n \to 0$ as $n \to +\infty$ ($\delta_n > 0$) and we ask for the existence of a limiting distribution as $n \to +\infty$ by choosing suitably δ_n. I.e. we ask under what conditions there exists the limiting distribution of the variables

$$\delta_n(\tau_i^{(n)} - \tau_{i-1}^{(n)})$$

for every fixed $i=1, 2, \dots$ if $n \to +\infty$ and $\delta_n \to 0$ ($\delta_n > 0$).

Following the considerations of [4], for a special case, we give general a necessary and sufficient condition. Let in what follows $\psi(s)$ denote the Laplace transform of $F(x)$, the distribution function of the original renewal process. We suppose that for $s \geqq 0$ the limit

$$\lim_{\delta \to 0} \frac{1-\psi(\delta s)}{1-\psi(\delta)} = a(s)$$

exists, where the limit is taken for $\delta > 0$.

THEOREM 4. *Let us suppose that* $0 < \sigma^2 < +\infty$. *In order that for a suitable choice of the normalizing constants* δ_n *the random variables*

$$\delta_n(\tau_i^{(n)} - \tau_{i-1}^{(n)}) \qquad (i=1, 2, \dots) \tag{12}$$

have a limiting distribution as $n \to +\infty$, *it is necessary and sufficient that for* $s > 0$ *one of the following limit relations*

$$\lim_{n \to +\infty} M^n(1-\psi(\delta_n s)) = 0,$$

or

$$\lim_{n \to +\infty} M^n(1-\psi(\delta_n s)) = Cs^\alpha$$

hold, where $C > 0$ *and* $0 < \alpha \leqq 1$, C *and* α *being constants.*

PROOF. Necessity. Let us denote by $\psi_n(s)$ the Laplace transform of (12). Then as it is easily seen

$$\psi_n(s) = f_n(\psi(\delta_n s)), \tag{13}$$

where $f_n(z)$ is the generating function of $Z_i^{(n)}$. Now for $0<z\leqq 1$ the inverse function $u(z)$ of $f(z)$ exists and it is obvious that the inverse function of $f_n(z)$ is $u_n(z)$, where $u_n(z)$ is the n-th iterate of $u(z)$. Further, $u(z)$ satisfies all the conditions which are necessary to ensure the existence of the limit

$$\chi(z) = \lim_{n\to+\infty} M^n(u_n(z)-1) \qquad z\in(0, 1].$$

(Cf. KUCZMA [1], definitions: pages 19, 20, assertion: page 137.). It is that solution of the so-called Schröder's functional equation

$$\chi(u(z)) = \frac{1}{M}\chi(z)$$

for which $\chi'(1)=1$, $\chi(1)=0$, further $\chi(z)$ is strictly monotonically increasing in (0, 1] and twice continuously differentiable.

Now by (13)

$$u_n(\psi_n(s)) = \psi(\delta_n s) \qquad (s>0)$$

and from this

$$M^n(u_n(\psi_n(s))-1) = M^n(\psi(\delta_n s)-1) \qquad (s>0). \tag{14}$$

Now by our supposition $\psi_n(s)$ converges to the Laplace transform $\varphi(s)$ of the limiting distribution and

$$\lim_{n\to+\infty} M^n(u_n(\varphi(s))-1) = \chi(\varphi(s));$$

hence we obtain by an easy argumentation that

$$\lim_{n\to+\infty} M^n(u_n(\psi_n(s))-1) = \chi(\varphi(s)).$$

This by (14) implies

$$\lim_{n\to+\infty} M^n(\psi(\delta_n s)-1) = \chi(\varphi(s)) \qquad (s>0). \tag{15}$$

We have to distinguish two cases. In the first one we consider the possibility $\varphi(s)\equiv 1$ $(s>0)$. In this case $\chi(\varphi(s))\equiv\chi(1)=0$, and so by (15)

$$\lim_{n\to+\infty} (\psi(\delta_n s)-1)M^n = 0;$$

which proves in case $\varphi(s)\equiv 1$ the necessity of the first condition of our assertion. In the other case there exists a positive s_0 such that $\varphi(s_0)<1$. Since $\chi(\varphi(s))$ is not positive and for increasing s it decreases, we see for $s=s_0$ that

$$\lim_{n\to+\infty} M^n(\psi(\delta_n s_0)-1) = \chi(\varphi(s_0)) < 0. \tag{16}$$

Thus by (15) and (16) the limit

$$(17)\qquad \lim_{n\to+\infty}\frac{1-\psi(\delta_n s)}{1-\psi(\delta_n s_0)}=\lim_{n\to+\infty}\frac{1-\psi\left(\delta_n s_0\dfrac{s}{s_0}\right)}{1-\psi(\delta_n s_0)}=\frac{\chi(\varphi(s))}{\chi(\varphi(s_0))}$$

exists. So, the conditions of Lemma 1 of Feller's book [3], page 335, are satisfied for the function

$$U(s)=1-\psi(s).$$

Consequently

$$(18)\qquad \lim_{n\to+\infty}\frac{1-\psi(\delta_n s)}{1-\psi(\delta_n s_0)}=\left(\frac{s}{s_0}\right)^{\alpha},$$

where $-\infty<\alpha<+\infty$. This and the limit relation (17) shows that

$$(19)\qquad \chi(\varphi(s))=\chi(\varphi(s_0))\left(\frac{s}{s_0}\right)^{\alpha}=C^*s^{\alpha},$$

where $C^*=\chi(\varphi(s_0))/s_0^{\alpha}<0$. This implies

$$(20)\qquad \varphi(s)=\chi^{-1}(C^*s^{\alpha}),\qquad (s\geqq 0),$$

where $\chi^{-1}(\cdot)$ denotes the inverse function of $\chi(\cdot)$. $\varphi(s)$ is the Laplace transform of the limit distribution and so it is monotonically decreasing for s increasing. Thus the case $\alpha<0$ is impossible, because otherwise C^*s^{α} increases. The case $\alpha=0$ is also impossible. In fact, if $\alpha=0$, then by (20)

$$\varphi(s)=\chi^{-1}(C^*).$$

But $C^*<0$ and $\chi^{-1}(z)$ strictly monotonically increases. These imply $\chi^{-1}(C^*)<1$, which contradicts to the fact that $\varphi(s)$ is a Laplace transform.

We show now that $\alpha>1$ cannot occur. Put on the basis of (20)

$$(21)\qquad \varphi'(s)=\frac{C^*\alpha s^{\alpha-1}}{\chi'(\varphi(s))}.$$

If we would have $\alpha>1$ then this fact would imply $\varphi'(0)=0$. This means that $\varphi(s)\equiv 1$, which contradicts to the fact that for $s=s_0$ we have $\varphi(s_0)<1$. So, we have

$$0<\alpha\leqq 1.$$

Let us denote by C the value $-C^*$. Then by (15)

$$(22)\qquad \lim_{n\to+\infty}M^n(1-\psi(\delta_n s))=Cs^{\alpha},\qquad (s>0),$$

where C and α are constants, $C>0$ and $0<\alpha\leqq 1$. This proves the necessity part of the assertion.

Sufficiency. If (22) holds, then

$$\psi(\delta_n s)=1-\frac{Cs^{\alpha}}{M^n}(1+o(1)),\qquad (n\to+\infty).$$

For fixed $s \geqq 0$ we have

$$\left| 1 - \frac{Cs^\alpha}{M^n}(1+o(1)) - e^{-\frac{Cs^\alpha}{M^n}(1+o(1))} \right| = o\left(\frac{1}{M^n}\right),$$

from which

$$\psi(\delta_n s) = e^{-\frac{Cs^\alpha}{M^n}(1+o(1))} + o\left(\frac{1}{M^n}\right),$$

and so the Laplace transform of (12) is of the form

$$\psi_n(s) = f_n(\psi(\delta_n s)) = f_n\left(e^{-\frac{Cs^\alpha}{M^n}(1+o(1))} + o\left(\frac{1}{M^n}\right)\right).$$

By Lagrange's theorem

$$\left| f_n\left(e^{-\frac{Cs^\alpha}{M^n}(1+o(1))} + o\left(\frac{1}{M^n}\right)\right) - f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right) \right| =$$

$$= \left| f_n'(\vartheta_n(s))\left[e^{-\frac{Cs^\alpha}{M^n}}\left(e^{-\frac{Cs^\alpha}{M^n}o(1)} - 1\right) + o\left(\frac{1}{M^n}\right)\right] \right| \leqq M^n\left(\frac{o(1)}{M^n} + o\left(\frac{1}{M^n}\right)\right) = o(1),$$

$$(|\vartheta_n(s)| \leqq 1).$$

Thus, if the limit

$$\lim_{n\to+\infty} f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right) \tag{23}$$

exists, then there exists also the limit of the Laplace transform $\psi_n(s)$ of the random variables (12). Let us consider for this purpose the random variable

$$Z_i^{(n)}/M^n$$

The Laplace transform of this at the point Cs^α is exactly

$$f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right)$$

By our assumption, concerning the variables $v_i^{(n)}$, Lemma 1 is applicable. Henceforth (23) exists:

$$\lim_{n\to+\infty} f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right) = \varphi(s)$$

and it is the Laplace transform of some probability distribution. In fact, if $g(s)$ denotes the Laplace transform of the probability distribution function $G(x)$, defined in Lemma 1, then we can write

$$\varphi(s) = g(Cs^\alpha).$$

Our argumentations would be similar in the case when

$$\lim_{n\to+\infty} M^n(1-\psi(\delta_n s)) = 0.$$

After some calculations we would get $\varphi(s) \equiv 1$. This proves the theorem.

From this we obtain immediately

Corollary 1. If for a suitable choice of the normalizing factors δ_n the random variables (12) have a limiting distribution, as $n \to +\infty$ then this distribution is either degenerate or his Laplace transform is of the form

$$\varphi(s) = g(Cs^\alpha),$$

where $C>0$ and $0<\alpha\leq 1$ and $g(s)$ is the Laplace transform of the distribution function $G(x)$, defined in Lemma 1. The distribution corresponding to $g(Cs^\alpha)$ has mean-value if and only if $\alpha=1$.

In the sequel we shall deal with the domain of attraction of the possible limiting distributions and with the question of choice of the normalizing factors in (12). We call that the original renewal process belongs to the domain of attraction of the limiting distribution characterized by the Laplace transform $g(Cs^\alpha)$ if its distribution function $F(x)$ is such that for a suitable choice of the normalizing factors δ_n, the limiting distribution of (12) has Laplace transform $g(Cs^\alpha)$.

Consider first the case $0<\alpha<1$ and put

$$U(x) = \int_0^x (1-F(z))\,dz,$$

which is a monotonic function. The Laplace transform of $U(x)$ is

$$\frac{1-\psi(s)}{s}.$$

By (18)

$$\lim_{n\to+\infty} \frac{\dfrac{1-\psi(\delta_n s)}{\delta_n s}}{\dfrac{1-\psi(\delta_n s_0)}{\delta_n s_0}} = \left(\frac{s}{s_0}\right)^{\alpha-1}.$$

Consequently, by our supposition

$$\frac{1-\psi\left(\dfrac{s}{s_0}\right)}{\dfrac{s}{s_0}} \sim \left(\frac{s}{s_0}\right)^{\alpha-1} L\left(\frac{s_0}{s}\right), \qquad s\to 0,$$

where $L(x)$ is a slowly oscillating function in the neighborhood of $x=+\infty$. Applying a Tauberian theorem (cf. [3], Chapter XIII., §.5., Theorem 4) we deduce from this

$$1-F(x) \sim \frac{1}{\Gamma(1-\alpha)}\, x^{-\alpha} L(x), \qquad x\to+\infty.$$

So, for an arbitrary $k>0$

$$\lim_{x\to+\infty} \frac{1-F(kx)}{1-F(x)} = k^{-\alpha}.$$

Conversely, let us suppose that this limit relation holds. Then by some argumentations, similar to those of [3], Chapter XIII, page 511, we obtain

$$1-F(x) \sim x^{-\alpha} L(x), \qquad x\to+\infty,$$

and this implies by the above used Tauberian theorem that

$$\frac{1-\psi(s)}{s} \sim \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} L\left(\frac{1}{s}\right), \qquad s \to 0.$$

From this, for any arbitrary sequence δ_n, such that $\delta_n \to 0$, we have

$$\lim_{n\to+\infty} \frac{1-\psi(\delta_n s)}{1-\psi(\delta_n)} = s^{\alpha}.$$

Let us choose the sequence δ_n such that

$$\frac{M^n}{C} = \frac{1}{1-\psi(\delta_n)} \qquad (C>0) \tag{24}$$

hold. Then finally we obtain

$$\lim_{n\to+\infty} M^n\big(1-\psi(\delta_n s)\big) = Cs^{\alpha} \qquad (0<\alpha<1).$$

Thus we have the following

THEOREM 5. *In order that for a suitable choice of the normalizing factors the original renewal process belong to the domain of attraction of the distribution function characterized by the Laplace transform* $g(Cs^{\alpha})$ $(C>0, 0<\alpha<1)$ *it is necessary and sufficient that for the distribution function* $F(x)$ *of the original process the limit relation*

$$\lim_{n\to+\infty} \frac{1-F(kx)}{1-F(x)} = k^{-\alpha}$$

hold for every fixed $k>0$.

Let us now deal with the case $\alpha=1$. (Cf. also [4].) Now by (18)

$$\lim_{n\to+\infty} \frac{\dfrac{1-\psi(\delta_n s)}{\delta_n s}}{\dfrac{1-\psi(\delta_n s_0)}{\delta_n s_0}} = 1.$$

It follows by our supposition that

$$\frac{1-\psi\left(\dfrac{s}{s_0}\right)}{\dfrac{s}{s_0}} \sim L\left(\frac{s_0}{s_0}\right) \qquad s \to 0,$$

where $L(x)$ is slowly oscillating function in the neighborhood of $x=+\infty$. From this

$$\frac{1-\psi(z)}{z} \sim L\left(\frac{1}{z}\right), \qquad z \to 0.$$

Thus the function

$$U(x) = \int_0^x \big(1-F(z)\big)\,dz$$

is slowly oscillating. (Cf. [3] Chapter XIII, §. 5, Theorem 2.) Let now $t>1$ be a fixed number. Then

$$0=\lim_{x\to+\infty}\left(\frac{U(tx)}{U(x)}-1\right)=\lim_{x\to+\infty}\frac{\int_x^{tx}(1-F(z))\,dz}{U(x)}\geqq$$

$$\geqq\frac{t-1}{t}\varlimsup_{x\to+\infty}\frac{U(tx)}{U(x)}\frac{tx(1-F(tx))}{\int_0^{tx}(1-F(z))\,dz}\geqq 0.$$

This gives that for $y\to+\infty$

$$\lim_{y\to+\infty}\frac{y(1-F(y))}{\int_0^{y}(1-F(z))\,dz}=0$$

For $0<t<1$ we can make a similar argument. Conversely, let us now suppose that the last limit relation holds. Then

$$0\leqq\lim_{x\to+\infty}\left|1-\frac{U(bx)}{U(x)}\right|=\lim_{x\to+\infty}\frac{\left|\int_{bx}^{x}(1-F(z))\right|dz}{\int_0^{x}(1-F(z))\,dz}\leqq|1-b|\lim_{x\to+\infty}\frac{x(1-F(kx))}{U(kx)}\frac{U(kx)}{U(x)},$$

where $k=1$, if $b>1$ and $k=b$, if $b<1$. Since the limit of the right-hand side is zero, the function

$$U(x)=\int_0^{x}(1-F(z))\,dz$$

is slowly oscillating. From this we deduce by the above Tauberian theorem that

$$1-\psi(s)\sim sU\left(\frac{1}{s}\right),\qquad s\to 0.$$

So, for any arbitrary sequence $\delta_n>0$, such that $\delta_n\to 0$, we have

$$\frac{\dfrac{1-\psi(\delta_n s)}{\delta_n s}}{\dfrac{1-\psi(\delta_n)}{\delta_n}}\to 1,\qquad (n\to+\infty).$$

Let us now choose the normalizing factors δ_n such that

$$\frac{M^n}{C}=\frac{1}{1-\psi(\delta_n)},\qquad C>0 \tag{25}$$

hold. Then from the preceding limit relation we obtain

$$\lim_{n\to+\infty} M^n(1-\psi(\delta_n s)) = Cs.$$

We have thus proved the following

THEOREM 6. *In order that for a suitable choice of the normalizing factors* δ_n *the original renewal process belong to the domain of attraction of the distribution function, characterized by the Laplace transform* $g(s)(C>0)$, *it is necessary and sufficient that for the distribution function of the original process the limit relation*

$$\lim_{x\to+\infty} \frac{x(1-F(x))}{\int_0^x (1-F(z))\,dz} = 0$$

hold.

Now let us consider the problem of choice of the normalizing factors. By (24) and (25) we can write

$$\delta_n = \psi^{-1}\left(1-\frac{C}{M^n}\right);$$

here ψ^{-1} denotes the inverse of the Laplace transform $\psi(s)$ $(s\geqq 0)$. In that case the existence of the limit distribution is ensured if the distribution function of the original renewal process belongs to the corresponding domain of attraction.

As an example we show the following: if the mean-value μ of the original renewal process is finite, then the above choice of δ_n gives

$$\delta_n = \frac{C}{\mu M^n} + o\left(\frac{1}{M^n}\right)$$

in accordance with Theorem 2. In fact, let us expand $\psi^{-1}(x)$ at the point $x=1$ in a Taylor series:

$$\psi^{-1}(x) = -\frac{1}{\mu}(x-1)+o(x-1), \qquad x\to 1,$$

and put $x = 1-\dfrac{C}{M^n}$. We obtain

$$\delta_n = \psi^{-1}\left(1-\frac{C}{M^n}\right) = \frac{C}{\mu M^n}+o\left(\frac{1}{M^n}\right).$$

We show also the converse statement: if we choose $\delta_n = \dfrac{C}{M^n}$, then the existence of a limiting distribution implies the finiteness of the mean-value of the original renewal process. Namely, the following assertion holds:

THEOREM 7. *If we prescribe for* δ_n *to be* $\dfrac{1}{M^n}$ *in* (12), *then for the existence of the limit distribution of the random variables*

$$\frac{1}{M^n}(\tau_i^{(n)}-\tau_{i-1}^{(n)}) \qquad (i=1, 2, \ldots, \textit{fixed})$$

it is necessary and sufficient that the mean-value of the distribution function $F(x)$ of the original process exists.

PROOF. If

$$\mu = \int_0^{+\infty} x\,dF(x) < +\infty$$

then Theorem 2 shows that the limit distribution of

(26) $$\frac{1}{M^n}\left(\tau_i^{(n)} - \tau_{i-1}^{(n)}\right)$$

exists and equals $G\left(\frac{x}{\mu}\right)$. Conversely, suppose that (26) has a non-degenerate limiting distribution. Applying Theorem 4, we see that in this case

$$\lim_{n\to+\infty} M^n\left(1-\psi\left(\frac{s}{M^n}\right)\right) = Cs^{\alpha} \qquad (s>0)$$

holds, where $C>0$, $0<\alpha\leqq 1$. This limit relation can be written in the form

$$\lim_{n\to+\infty}\int_0^{+\infty} \frac{1-e^{-\frac{s}{M^n}x}}{\frac{s}{M^n}}\,dF(x) = Cs^{\alpha-1} \qquad (s>0).$$

For a fixed $s>0$, the sequence

$$f_n(x) = \frac{1-e^{-\frac{s}{M^n}x}}{\frac{s}{M^n}}$$

is such that $f_n(x)\geqq 0$ and $\lim\limits_{n\to+\infty} f_n(x)=x$. So, by Fatou's lemma, we get

$$\int_0^{+\infty} x\,dF(x) \leqq \lim_{n\to+\infty}\int_0^{+\infty}\frac{1-e^{-\frac{s}{M^n}x}}{\frac{s}{M^n}}\,dF(x) = Cs^{\alpha-1} < +\infty,$$

which proves our assertion.

Let us now return to the determination of the factors δ_n in the general case. Our aim is to determine δ_n by the aid of the distribution function $F(x)$ of the original renewal process.

If $0<\alpha<1$ then the limit relation

$$\lim_{x\to+\infty}\frac{1-F(kx)}{1-F(x)} = k^{-\alpha}$$

holds. From this we deduce that

$$1-F(x) \sim \frac{L(x)}{x^{\alpha}}, \qquad x\to+\infty,$$

where $L(x)$ is a slowly oscillating function. From this

$$1-\psi(\delta_n)\sim\Gamma(1-\alpha)\,\delta_n^\alpha L(\delta_n^{-1}).$$

On the other hand by Theorem 4

$$1-\psi(\delta_n)\sim\frac{C}{M^n},\qquad n\to+\infty.$$

Confering these asymptotic formulas we see

$$\Gamma(1-\alpha)\,\delta_n^\alpha L(\delta_n^{-1})\sim\frac{C}{M^n}$$

Since $\delta_n^\alpha L(\delta_n^{-1})\sim 1-F(\delta_n^{-1})$, we have

$$\Gamma(1-\alpha)\left(1-F(\delta_n^{-1})\right)\sim\frac{C}{M^n},$$

and so

$$\delta_n\sim\frac{1}{F^{-1}\left(1-\dfrac{C}{\Gamma(1-\alpha)M^n}\right)},\qquad n\to+\infty.$$

Let now $\alpha=1$. Then

$$1-\psi(\delta_n)\sim\delta_n U\left(\frac{1}{\delta_n}\right)$$

and on the other hand

$$1-\psi(\delta_n)\sim\frac{C}{M^n}$$

From these relations the formula

$$\frac{M^n}{C}\,\delta_n U\left(\frac{1}{\delta_n}\right)\sim 1,\qquad n\to+\infty,$$

follows, where

$$U(x)=\int_0^x\left(1-F(z)\right)dz$$

The last asymptotic formula determines δ_n.

We have proved

THEOREM 8. *In order that the random variables* (12) *have a limiting distribution, the Laplace transform of which is* $g(Cs^\alpha)$ $(C>0,\ 0<\alpha\leqq 1)$, *where* $g(s)$ *is the Laplace transform of* $G(x)$ *defined in Lemma 1, we can determine the constants* δ_n *according to the formula*

$$\delta_n\sim\psi^{-1}\left(1-\frac{C}{M^n}\right)\qquad(n\to+\infty),$$

where $\psi^{-1}(x)$ *is the inverse function of* $\psi(s)$, *the Laplace transform of* $F(x)$.

By means of $F(x)$ *the constants* δ_n *can be determined as follows: for* $C>0$ *and* $0<\alpha<1$

$$\delta_n \sim \frac{1}{F^{-1}\left(1-\dfrac{C}{\Gamma(1-\alpha)M^n}\right)}, \qquad n\to+\infty,$$

where $F^{-1}_{(x)}(x)$ *is the inverse of* $F(x)$;

for $C>0$ *ane* $\alpha=1$ *the constants* δ_n *can be determined by the asymptotic formula*

$$\frac{M^n}{C}\delta_n U\left(\frac{1}{\delta_n}\right)\sim 1, \qquad n\to+\infty,$$

where

$$U(x)=\int_0^x (1-F(z))\,dz.$$

Remark. If we want that the limit distribution be degenerate, we have to determine the constants δ_n according to the asymptotic formula

$$\delta_n \sim \psi^{-1}\left(1+o\left(\frac{1}{M^n}\right)\right), \qquad n\to+\infty,$$

We call that the original renewal process belongs to the normal domain of attraction of the possible limiting distributions, defined by Theorems 5 and 6, if the normalizing factors δ_n are determined by the equality

$$\delta_n \sim \frac{1}{M^{n/\alpha}} \qquad (0<\alpha\leqq 1).$$

Theorem 7 shows that for $\alpha=1$ the original renewal process belongs to the normal domain of attraction of the limit distribution, the Laplace transform of which is $g(Cs)$, if and only if its mean-value is finite.

For $0<\alpha<1$ we have the following

THEOREM 7′. *The original renewal process belongs to the normal domain of attraction of the limiting distribution, the Laplace transform of which is* $g(Cs^\alpha)$, $0<\alpha<1$, *if and only if the distribution function* $F(x)$ *of the original process is of the form*

$$F(x)=1-\frac{1}{x^\alpha}\left(\frac{C}{\Gamma(1-\alpha)}+a(x)\right) \qquad (x>0),$$

where

$$\lim_{x\to+\infty} a(x)=0.$$

PROOF. Sufficiency. For fixed $k>0$ we have

$$\lim_{x\to+\infty}\frac{1-F(kx)}{1-F(x)}=\lim_{x\to+\infty}\frac{1}{k^\alpha}\,\frac{\dfrac{C}{\Gamma(1-\alpha)}+a(kx)}{\dfrac{C}{\Gamma(1-\alpha)}+a(x)}=k^{-\alpha},$$

by the supposition. Thus the possible limit distribution has the Laplace transform $g(Cs^{\alpha})$. In this case the normalizing factors δ_n can be chosen, according to Theorem 8, as follows:

$$\delta_n \sim \frac{1}{F^{-1}\left(1-\frac{C}{\Gamma(1-\alpha)M^n}\right)} \qquad (n\to+\infty).$$

This means that

$$F\left(\frac{1}{\delta_n}\right) = 1-\frac{C}{\Gamma(1-\alpha)M^n}.$$

On the other hand, by our supposition

$$F\left(\frac{1}{\delta_n}\right) = 1-\left(\frac{C}{\Gamma(1-\alpha)}+a\left(\frac{1}{\delta_n}\right)\right)\delta_n^{\alpha}.$$

Comparing these we obtain

$$\frac{C}{\Gamma(1-\alpha)M^n} \sim \left(\frac{C}{\Gamma(1-\alpha)}+a\left(\frac{1}{\delta_n}\right)\right)\delta_n^{\alpha},$$

from which

$$\delta_n \sim \frac{1}{M^{n/\alpha}}.$$

Necessity. Now

$$\delta_n \sim \frac{1}{M^{n/\alpha}}$$

and we have for every $k>0$

$$\frac{1-F(kM^{n/\alpha})}{1-F(M^{n/\alpha})} = k^{-\alpha}(1+o(1)) \qquad (n\to+\infty).$$

In the proof of Theorem 8 we have seen that

$$1-F(\delta_n^{-1}) = 1-F(M^{n/\alpha}) \sim \frac{C}{\Gamma(1-\alpha)M^n}.$$

Confering these we obtain

$$\frac{1-F(kM^{n/\alpha})}{C}\Gamma(1-\alpha)M^n = k^{-\alpha}(1+o(1)).$$

Let now $y=kM^{n/\alpha}$. Then

$$\frac{1-F(y)}{C}\Gamma(1-\alpha)\frac{y^{\alpha}}{k^{\alpha}} = \frac{1+o(1)}{k^{\alpha}}.$$

This implies that

$$1-F(y) = \frac{1}{y^{\alpha}}\left(\frac{C}{\Gamma(1-\alpha)}+o(1)\right)$$

which means our assertion.

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(Received March 10, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 307—316

ON HERMITE—FEJÉR INTERPOLATION PROCESSES

by

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I. Introduction

Let all the zeros x_{kn} of $\omega_n(x) = c(x-x_{1n})(x-x_{2n})\dots(x-x_{nn})$ fall in $[-1, 1]$. The polynomial π_{2n-1} of degree $2n-1$ at most, attaining the value y_{kn} at x_{kn} and having there the derivative y'_{kn}, is given by the formula

(1)
$$\pi_{2n-1}(x) = H_n(\omega_n; y_{kn}; x) + \mathfrak{H}_n(\omega_n; y'_{kn}; x)$$
where

(2)
$$H_n(\omega_n; y_{kn}; x) = \sum_{k=1}^{n} y_{kn} v_{kn}(x) l_{kn}^2(x)$$
and

(3)
$$\mathfrak{H}_n(\omega_n; y'_{kn}; x) = \sum_{k=1}^{n} y'_{kn}(x-x_{kn}) l_{kn}^2(x).$$

Here the expressions $l_{kn}(x)$ $(k=1, 2, \dots, n)$ are the fundamental polynomials of Lagrange interpolation. The study of the polynomials (1) for $n\to\infty$, provided that $y_{kn}=f(x_{kn})$ where $f(x)$ is some fixed function, and the y'_{kn} are subjected to some restrictions (e.g. concerning their order of magnitude), were first studied by L. FEJÉR [3], [7]; they are called Hermite—Fejér interpolation polynomials. It seems to be of particular interest to study the case, if $\omega_n(x)=p_n(w; x)$ where $\{p_n(w; x)\}$ are the orthogonal polynomials belonging to the nonnegative weight $w(x)$ with support $[-1, 1]$. For $\omega(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, i.e. for the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ the problem was completely settled by L. FEJÉR [5] and G. SZEGŐ [14]. Further cases were studied by J. BALÁZS [1] and G. FREUD [8]. In our present paper we give a convergence theorem for such weights, that $(1-x)^\alpha w(x)$ is nondecreasing and $(1+x)^\beta w(x)$ is nonincreasing for some real α, β.

It is interesting from a methodological point of view, that our argument is based entirely on the orthogonality property of the $p_n(w)$, while in [3] and [13] (as well as in [1]) an essential use was made of the Sturm—Liouville type differential equations satisfied by Jacobi polynomials. This meant that there seemed to be no easy generalizations of these results. The first theorem on convergence of the Hermite—Fejér interpolation polynomials over the zeros of orthogonal polynomials, where only some sort of a general condition is imposed on the weight, was proved by the author [8]. We make use of a relation proved there.

In Part II of this paper we prove a monotonity property of Christoffel functions, which is the fundament of our treatment. In part III we are going to formulate and prove our convergence theorems.

At the end of our paper we make a remark concerning "normal point groups" of interpolation. They were introduced by L. FEJÉR [5], [6]. He proved that the zeros of $P_n^{(\alpha,\beta)}$ for $-1<\alpha,\ \beta\leqq 0$ form a normal point system. Later G. GRÜNWALD [12] proved that the Hermite—Fejér interpolation process converges for every continuous function, provided that the point system is normal. Unfortunately, no further explicit example of normal point system was known beside FEJÉR's. We are going to prove in part IV, that the zeros of $P_n(x)+AP_{n-1}(x)$ (P=Legendre polynomial of degree n) form a normal point system for $|A|\leqq 1$.

II. Monotonity properties of Christoffel numbers

Let $W(x)$ be a nonnegative weight function and let $\{p_n(W;x)\}$ be the corresponding normed orthogonal polynomials. Using the same notations as in our book [10], we consider the Christoffel function

$$\lambda_n(W;x)=\left[\sum_{k=0}^{n-1} p_k^2(W;x)\right]^{-1}. \tag{4}$$

LEMMA 1. *Let $W(x)=0$ for $x<0$. If for some real ϱ $x^\varrho W(x)$ is a nonincreasing function, then $x^{\varrho-1}\lambda_n(W;x)$ $(n=2,3,\ldots)$ are decreasing functions for $x>0$.*

PROOF. Let $u>1$; then we have by assumption $(ux)^\varrho W(ux)\leqq x^\varrho W(x)$ $(-\infty<x<\infty)$ so that

$$W_u(x)\overset{\text{def}}{=\!=}u^\varrho W(ux)\leqq W(x)\qquad(-\infty<x<\infty). \tag{5}$$

By Theorem I. 4. 2 of [10] we have

$$\lambda_n(W_u;x)\leqq\lambda_n(W;x). \tag{6}$$

We conclude from the definition of W_u that

$$p_k(W_u;x)=u^{(1-\varrho)/2}p_k(W;ux)\qquad(k=0,1,\ldots)$$

so that by (4) we have $\lambda_n(W_u;x)=u^{\varrho-1}\lambda_n(W;ux)$;
inserting this in (6), we obtain

$$(ux)^{\varrho-1}\lambda_n(W;ux)\leqq x^{\varrho-1}\lambda_n(W;x),\qquad(x\geqq 0).$$

Since $u\geqq 1$ was arbitrary, we conclude that $x^{\varrho-1}\lambda_n(w;x)$ is nonincreasing. We see from (4) that $\lambda_n^{-1}(w;x)$ is a nonvanishing polynomial of exact degree $2n-2$ so that there can not exist any interval, where $x^{\varrho-1}\lambda_n(w;x)$ would be constant. It follows that this function must be decreasing. Q.e.d.

We consider now the formula (2) for $\omega_n(x)=p_n(w;x)$.

LEMMA 2. *Let $w(x)\in\mathcal{L}$ be a nonnegative weight function with support $[-1,1]$, let further $(1+x)^\beta w(x)$ be nonincreasing and $(1-x)^\alpha w(x)$ be nondecreasing; then for every natural $n>1$ and $1\leqq k\leqq n$ we have*

$$v_{kn}(-1)\geqq\beta\quad\text{and}\quad v_{kn}(1)\geqq\alpha. \tag{7}$$

PROOF. Applying Lemma 1 for $W(x)=w(-1+x)$ resp. to $W(x)=w(1-x)$, we get that $(1+t)^{\beta-1}\lambda_n(w;t)$ is decreasing and $(1-t)^{\alpha-1}\lambda_n(w;t)$ is increasing for $-1\leqq t\leqq 1$. By differentiation we obtain

$$(8)\qquad 1+(1-t)\frac{\lambda_n'(w;t)}{\lambda_n(w;t)}\geqq\alpha,\quad 1-(1+t)\frac{\lambda_n'(w;t)}{\lambda_n(w;t)}\geqq\beta\qquad(-1\leqq t\leqq 1)$$

for $n\geqq 2$. Now we proved in our previous paper [8] (see (4) and (19)) that in case $\omega_n=p_n(w)$ we have

$$(9)\qquad v_{kn}(x)=1+\frac{\lambda_n'(w;x_{kn})}{\lambda_n(w;x_{kn})}(x-x_{kn}).$$

We see that (7) follows from (8) and (9) provided that $n\geqq 2$. For $n=1$ $\omega_n''=p_n''(w)\equiv 0$, so that $v_{kn}(x)\equiv 1$; this completes the proof.

LEMMA 3. *If $w(x)$ satisfies the same assumptions as in Lemma 2, then*

$$(10)\qquad |p_n(w;x)|\leqq c_1(\delta)\qquad(-1+\delta\leqq x\leqq 1-\delta;\ n=0,1,\dots).$$

Here $c_1(\delta)$, and in the following $c_2(\delta),\dots$ denote positive numbers depending only on δ and the choice of $w(x)$. (Resp. it depends only on w, if no dependence on δ is indicated.)

PROOF. We may suppose without loss of generality that $n\geqq 2$. We have seen that as a consequence of our assumptions $(1-t)^{\alpha-1}\lambda_\nu(w;t)$ is increasing and $(1+t)^{\beta-1}\lambda_\nu(w;t)$ is decreasing for $\nu\geqq 2$, $-1\leqq t\leqq 1$. Let $x\in[-1+\delta,1-\delta]$ and let x_{in} be the zero of $p_n(w)$ which is nearest to x. By (4) we have

$$\lambda_{n+1}(w;x_{in})=\lambda_n(w;x_{in}).$$

If $x<x_{in}$, we conclude

$$\lambda_{n+1}^{-1}(w;x)\leqq\left(\frac{1+x_{in}}{1+x}\right)^{1-\beta}\lambda_{n+1}^{-1}(w;x_{in})=$$

$$=\left(\frac{1+x_{in}}{1+x}\right)^{1-\beta}\lambda_n^{-1}(w;x_{in})\leqq\left(\frac{1+x_{in}}{1+x}\right)^{1-\beta}\left(\frac{1-x}{1-x_{in}}\right)^{1-\alpha}\lambda_n^{-1}(w;x)$$

and similarly, we have for $x>x_{in}$

$$\lambda_{n+1}^{-1}(w;x)\leqq\left(\frac{1-x_{in}}{1-x}\right)^{1-\alpha}\left(\frac{1+x}{1+x_{in}}\right)^{1-\beta}\lambda_n^{-1}(w;x).$$

In both cases we get, taking (4) in consideration,

$$(11)$$
$$p_n^2(w;x)=\lambda_{n+1}^{-1}(w;x)-\lambda_n^{-1}(w;x)\leqq c_2(\delta)|x-x_{in}|\lambda_n^{-1}(w;x)\qquad(x\in[-1+\delta,1-\delta]).$$

By standard theorems on orthogonal polynomials

$$(12)\qquad |x-x_{in}|\leqq c_3(\delta)n^{-1},\quad c_4(\delta)n\leqq\lambda_n^{-1}(w;x)\leqq c_5(\delta)n$$

are valid. (See Theorem III. 5. 1 resp. Theorem III. 3. 3 in our book [10]. The first estimate was found by P. ERDŐS—P. TURÁN [2] and the second by G. SZEGŐ [13].)

By inserting (12) and the second half of (13) in our estimate (11), we get (10), Q.e.d.

III. Convergence of the Hermite—Fejér interpolation process

In what follows, let $f(x)$ be a bounded real-valued function defined in $[-1, 1]$, let x_{kn} be the zeros of $p_n(w; x)$, and let

$$H_n(w;f;x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} f(x_{kn}) v_{kn}(x) l_{kn}^2(x). \tag{14}$$

LEMMA 4. *Under the conditions of Lemma 2 we have*

$$\lim_{n\to\infty} \sum_{k=1}^{n} (x-x_{kn})^2 |v_{kn}(x)| l_{kn}^2(x) = 0 \tag{15}$$

uniformly in $x \in [-1+\delta, 1-\delta]$.

PROOF. The linear function $v_{kn}(x)$ attains for $x = x_{kn} \in (-1, 1)$ the value 1 and satisfies (7), so that the slope of $v_{kn}(x)$ is greater than $\dfrac{\alpha-1}{1-x_{kn}}$ and smaller than $\dfrac{1-\beta}{1+x_{kn}}$; it follows

$$|v_{kn}(x)| \leqq \frac{c_7}{1-x_{kn}^2} \qquad (k=1, 2, \dots n). \tag{16}$$

We apply the formula

$$l_{kn}(x) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \lambda_n(w; x_{kn}) \frac{p_{n-1}(w; x_{kn})}{x-x_{kn}} p_n(w; x) \tag{17}$$

proved in [9] (see also formula III. (6. 3) in [10])[1]. Here $\gamma_n(w)$ is the leading coefficient of $p_n(w)$, and we have by a well-known lemma, used first by G. ALEXITS (see e.g. Lemma I. 7. 2. in [10])

$$0 < \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \leqq 2. \tag{18}$$

Combining (16), (17) and (18) and making use of Lemma 3

$$\sum_{k=1}^{n} |v_{kn}(x)| l_{kn}^2(x) (x-x_{kn})^2 \leqq \tag{19}$$

$$\leqq 4c_7 p_n^2(w; x) \sum_{k=1}^{n} \lambda_n^2(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1-x_{kn}^2)^{-1} \leqq$$

$$\leqq c_8(\delta) \left[\max_{1\leqq k\leqq n} \lambda_n(w; x_{kn})\right] \sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1-x_{kn}^2)^{-1}$$

$$(x \in [-1+\delta, 1-\delta]).$$

[1] In the notations of our book [10] we have

$$l_{kn}(x) = l_n(w; x; x_{kn}) = l_n(x; x_{kn}).$$

Since $w(x)$ is vanishing outside $[-1, 1]$, we have for every fixed ξ

$$\lim_{n\to\infty} \lambda_n(w; \xi) = 0 \tag{20}$$

(see [10], §II. 2). By (4) $\lambda_n(w; \xi) \geqq \lambda_{n+1}(w; \xi)$ and all functions $\lambda_n(w; \xi)$ $(n=1, 2, \ldots)$ are continuous, so that by Dini's theorem, (20) holds uniformly for $\xi \in [-1, 1]$. So the factor before the Σ sign in (19) tends to zero for $n \to \infty$. To prove lemma 4, it remains to show that

$$\sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1-x_{kn}^2)^{-1} = O(1). \tag{21}$$

This is shown as follows: By the Lagrange interpolation formula and (17) we have

$$p_{n-1}(w; 1) = \sum_{k=1}^{n} l_{kn}(1) p_{n-1}(w; x_{kn}) =$$

$$= \frac{\gamma_{n-1}(w)}{\gamma_n(w)} p_n(w; 1) \sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1-x_{kn})^{-1}$$

and similarly

$$p_{n-1}(w; -1) = -\frac{\gamma_{n-1}(w)}{\gamma_n(w)} p_n(w; -1) \sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1+x_{kn})^{-1}.$$

From these two formulae we obtain

$$\begin{aligned} &\sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1-x_{kn}^2)^{-1} = \\ &= \frac{1}{2} \sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1-x_{kn})^{-1} + \\ &+ \frac{1}{2} \sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1+x_{kn})^{-1} = \\ &= \frac{1}{2} \frac{\gamma_n(w)}{\gamma_{n-1}(w)} \left[\frac{p_{n-1}(w; 1)}{p_n(w; 1)} - \frac{p_{n-1}(w; -1)}{p_n(w; -1)} \right]. \end{aligned} \tag{22}$$

Using again that $w(x)$ is vanishing outside $[-1, 1]$ we have by Lemma 4 of our paper [11]

$$0 \leqq \frac{p_{n-1}(w; 1)}{p_n(w; 1)} \leqq 2 \frac{\gamma_n(w)}{\gamma_{n-1}(w)}, \quad 0 \leqq -\frac{p_{n-1}(w; -1)}{p_n(w; -1)} \leqq 2 \frac{\gamma_n(w)}{\gamma_{n-1}(w)} \tag{23}$$

so that from (22) and (23)

$$\sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn}) (1-x_{kn}^2)^{-1} \leqq 2 \left[\frac{\gamma_n(w)}{\gamma_{n-1}(w)} \right]^2. \tag{24}$$

(We remind the reader that (24) was deduced from the single assumption that $w(x)=0$ for $x \notin [-1, 1]$; this fact seems to be of independent interest.)

From the assumptions of Lemma 2 follows that $(1-x^2)^{-1/2}\log w(x)\in\mathcal{L}$. By a celebrated theorem of G. SZEGŐ (see e.g. [10], Table V. A, first row) we have

(25) $$\lim_{n\to\infty}\frac{\gamma_n(w)}{\gamma_{n-1}(w)}=2.$$

From (24) and (25) we infer that (21) is valid, and our proof is finished.

LEMMA 5. *Under the assumptions of Lemma 2 we have*

(26) $$\sum_{k=1}^{n}|v_{kn}(x)|\,l_{kn}^2(x)\leqq c_{10}(\delta)\qquad (x\in[-1+\delta,\ 1-\delta]).$$

PROOF. By a previous result of the author (see [8], Lemma 1) it follows from

$$c_{11}(\delta)\leqq w(x)\leqq c_{12}(\delta)\qquad \left(|x|\leqq 1-\frac{\delta}{3}\right)$$

that

$$\sum_{|x_{kn}|\leqq 1-\delta/2} l_{kn}^2(x)\leqq c_{13}(\delta)\qquad (|x|\leqq 1-\delta).$$

As a consequence of our assumptions, (16) is valid, so that

(27) $$\sum_{|x_{kn}|\leqq 1-\delta/2}|v_{kn}(x)|\,l_{kn}^2(x)\leqq c_{14}(\delta)\qquad (x\in[-1+\delta,\ 1-\delta]).$$

Now, (26) is a consequence of (27) and Lemma 4. Q.e.d.

THEOREM I. *Let the support of* $0\leqq w(x)\in\mathcal{L}$ *be* $[-1,1]$ *and for a pair* α,β *of reals let* $(1-x)^{\alpha}w(x)$ *be nondecreasing and* $(1+x)^{\beta}w(x)$ *be nonincreasing. If* $f(x)$ *is bounded for* $x\in[-1,1]$ *and continuous at the point* $\xi\in(-1,1)$ *then we have*[2]

(28) $$\lim_{n\to\infty}H_n(w;f;\xi)=f(\xi).$$

If $f(x)$ *is continuous in* $[a,b]\subset(-1,1)$, *then* (28) *holds uniformly in* $\xi\in[a,b]$.

PROOF. The relation (28) is a consequence of the following three facts:

A) We have

$$\sum_{k=1}^{n}v_{kn}(x)\,l_{kn}^2(x)\equiv 1;$$

B)

$$\sum_{k=1}^{n}|v_{kn}(x)|\,l_{kn}^2(x)\leqq c_{10}(\delta)\qquad (x\in[-1+\delta,\ 1-\delta]);$$

C) for every fixed $\varrho>0$ we have uniformly in $x\in[-1+\delta,1-\delta]$

$$\lim_{n\to\infty}\sum_{|x-x_{kn}|>\varrho}|v_{kn}(x)|\,l_{kn}^2(x)=0.$$

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[2] See (14).

We obtain A), by inserting in (1), (2) and (3) $\pi_{2n-1}(x) \equiv 1$. B) was proved as Lemma 5, and C) is a consequence of Lemma 4. The deduction of (28) from A), B) and C) is standard and may be left to the reader.

THEOREM II. *Let $w(x)$ be a weight function with support in $[-1, 1]$ satisfying*

$$0 < m \leqq w(x) \leqq M \qquad (x \in [a_1, b_1] \subset (-1, 1)) \tag{29}$$

and

$$|p_n(w; x)| \leqq K \qquad (x \in [a, b] \subset (a_1, b_1)) \tag{30}$$

(m, M, K are independent of x resp. of x and n).

Let further $\varepsilon_n = o(1)$ be a positive null sequence and let

$$\{y'_{kn}\} \qquad (k = 1, 2, \dots n;\ n = 1, 2, \dots)$$

be a triangular matrix for which the conditions

$$|y'_{kn}| \leqq \varepsilon_n \frac{n}{\log n} \qquad (x_{kn} \in [a_1, b_1]) \tag{31}$$

and

$$(1 - x_{kn}^2)\lambda_n(w; x_{kn})|y'_{kn}| \leqq \varepsilon_n \qquad (x_{kn} \in [-1, 1]) \tag{32}$$

are satisfied; under the above listed conditions we have

$$\left|\mathfrak{H}_n(p_n(w); y'_{kn}; x)\right| \leqq \sum_{k=1}^{n} |y'_{kn}|\,|x - x_{kn}|\,l_{kn}^2(x) \leqq \mathscr{L}\varepsilon_n \to 0 \tag{33}$$

with an $\mathscr{L}$ independent of x and n.

Remark. If

$$w(x) \leqq M \qquad (x \in [-1, 1]) \tag{34}$$

(which is certainly the case under the assumptions of Theorem I) we have

$$\lambda_n(w; \xi) \leqq 100M\left(\frac{\sqrt{1-\xi^2}}{n} + \frac{1}{n^2}\right)$$

(see [8], Lemma II). Hence if we assume (34), then (32) is satisfied provided that

$$\left[\frac{(1 - x_{kn}^2)^{3/2}}{n} + \frac{1 - x_{kn}^2}{n^2}\right]|y'_{kn}| \leqq \varepsilon'_n = o(1). \tag{35}$$

In this way we can replace in Theorem II the condition (31) by (34) and (35); this is an improvement of the results of the author in [8], where

$$\left[\frac{(1 - x_{kn}^2)^{1/2}}{n} + \frac{1}{n^2}\right]|y'_{kn}| \leqq \varepsilon'_n = o(1)$$

was assumed, as well as some previous results of L. FEJÉR [4].

PROOF of Theorem II. We have proved in our paper [8] (see Lemma IV) that for fixed $\varrho>0$

$$\sum_{|x-x_{kn}|<\varrho} |x-x_{kn}|\, l_{kn}^2(x) = O\left(\frac{\log n}{n}\right).$$

Our proof will be complete, if we show that

$$\sum_{k=1}^{n} (1-x_{kn}^2)^{-1} \lambda_n^{-1}(w; x_{kn})\,(x-x_{kn})^2\, l_{kn}^2(x) \leqq \mathscr{L}_1(w), \tag{36}$$

with an $\mathscr{L}_1(w)$ not depending on n. To prove (36), we infer from (17) with the aid of the estimates (18) and (21), that we have for every weight w with support in $[-1, 1]$

$$\sum_{k=1}^{n} (1-x_{kn}^2)^{-1} \lambda_n^{-1}(w; x_{kn})\,(x-x_{kn})^2\, l_{kn}^2(x) = \tag{37}$$

$$= \frac{\gamma_{n-1}^2(w)}{\gamma_n^2(w)}\, p_n^2(w; x) \sum_{k=1}^{n} \lambda_n(w; x_{kn}) p_{n-1}^2(w; x_{kn})\,(1-x_{kn}^2)^{-1} \leqq 2c_9\, p_n^2(w; x).$$

Now (36) follows from (37) and (30). Q.e.d.

IV. On normal point systems

Let $x_{kn}\in[-1, 1]$ $(k=1, 2, \ldots, n;\ n=1, 2, \ldots)$ be a triangular matrix, and we form the corresponding Hermite—Fejér interpolation polynomials. The linear functions $v_{kn}(x)$ have (at most) one zero X_{kn}; this X_{kn} were called by L. FEJÉR [5] the points conjugate to x_{kn}; further he called the point system $\{x_{kn}\}$ normal, if all conjugate points are situated outside $(-1, 1)$. In what follows we give a new one-parameter family of normal points systems. Up to now, only the two-parameter point system of the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}$, $-1<\alpha, \beta\leqq 0$ was known to be normal. (See L. FEJÉR [5], [6].)

LEMMA 6. *Let* $\omega_n(x) = p_n(w; x)+Ap_{n-1}(w; x)$, *and let* ξ *be one of the roots of* ω_n. *Then we have*

$$\frac{\omega_n''(\xi)}{\omega_n'(\xi)} = -\frac{\lambda_n'(w; \xi)}{\lambda_n(w; \xi)}. \tag{38}$$

PROOF. $\omega_n(x)$ is a constant multiple of

$$p_{n-1}(w; \xi)p_n(w; x)-p_n(w; \xi)p_{n-1}(w; x),$$

so that

$$\frac{\omega_n''(\xi)}{\omega_n'(\xi)} = \frac{p_{n-1}(w; \xi)p_n''(w; \xi)-p_n(w; \xi)p_{n-1}''(w; \xi)}{p_{n-1}(w; \xi)p_n'(w; \xi)-p_n(w; \xi)p_{n-1}'(w; \xi)}.$$

In turn, we have (see [10], § I. 4)

$$\lambda_n^{-1}(w; \xi) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)}\,[p_{n-1}(w; \xi)p_n'(w; \xi)-p_n(w; \xi)p_{n-1}'(w; \xi)].$$

(38) is a consequence of these two formulae.

Let us denote — as usual — by $P_n(x)$ the n'th Legendre polynomial.

THEOREM III. *The zeros of* $\omega_n(x) = P_n(x) + AP_{n-1}(x)$ *form a normal point system in the sense of* L. FEJÉR *if* $|A| \leqq 1$.

PROOF. All zeros of $\omega_n(x)$ are real and simple (see e.g. [10], Theorem I. 3. 1) and all these zeros with the possible exception of one of them, are inside $(-1, 1)$. Since $P_\gamma(1)=1$ and $P_\gamma(-1)=(-1)^\gamma$, the zero of $P_n - P_{n-1}$ and of $P_n + P_{n-1}$ outside of $(-1, 1)$ is $\xi'=1$ and $\xi''=-1$, resp. Consequently, $P_n(x) \pm P_{n-1}(x)$ does not vanish for $|x|>1$. Since, comparing degrees, $P_{n-1}(x) = o\{|P_n(x)|\}$ for $|x| \to \infty$, we have

$$|P_{n-1}(x)| < |P_n(x)| \qquad (|x|>1).$$

We conclude that all zeros of $\omega_n(x)$ are situated in $[-1, 1]$. Since the conditions of Lemma 2 are satisfied with $\alpha=\beta=0$, we obtain from (8) and Lemma 6, that for the zeros of $\omega_n(x)$

$$v_{kn}(1) \geqq 0, \quad v_{kn}(-1) \geqq 0$$

hold. Since $v_{kn}(x)$ is linear, it can not vanish in $(-1, 1)$. Q.e.d.

THEOREM IV. *If* x_{kn} $(k=1, 2, \dots, n)$ *are zeros of* $P_n(x) + A_n P_{n-1}(x)$ *where* $|A_n| \leqq 1$ $(n=1, 2, \dots)$ *and the numbers* y'_{kn} *are uniformly bounded, then for an arbitrary function* $f(x)$ *continuous in* $[-1, 1]$ *we have uniformly with respect to* x *in every* $[a, b \subset \subset (-1, 1)$

$$\lim_{n\to\infty} \{H_n(\omega_n; f(x_{kn}); x) + \mathfrak{H}_n(\omega_n; y'_{kn}; x)\} = f(x).$$

PROOF. This follows from Theorem III and G. GRÜNWALD's theorem (see [12]).

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(Received March 24, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 317—319

ON FREE SUBSEMIGROUPS OF A FREE SEMIGROUP

by

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In the present note we give some criteria for a subset S of a free semigroup to be a system of free generators of a (free) subsemigroup. This question has a significance in coding theory where unique decodability depends on it. There exists the SARDINAS—PATTERSON method [2] for deciding whether S has the required property or not; all our conditions can be regarded as more theoretical versions of the result of SARDINAS and PATTERSON rendering possible to provide a shorter proof of the last.

Let F be a free semigroup. Given two subsets S and T of F, we use the notation

$$ST^{[-1]} = \{x : xt \in S \text{ for some } t \in T\},$$

$$T^{[-1]}S = \{x : tx \in S \text{ for some } t \in T\}.$$

Let F' denote the subsemigroup generated by S. We shall make use of the following lemma ([1], p. 119):

A subsemigroup F' of the free semigroup F is itself free iff $ax=by$ $(a, b, x, y \in F')$ implies $a=b$ or $au=b$ or $a=bu$ with $u \in F'$.

First of all we have:

CRITERION A. *S is a system of free generators of F' iff*

$$(F'^{[-1]}F') \cdot (F'F'^{[-1]}) \cap S = \emptyset. \tag{1}$$

PROOF. If F' is not free then an identity of the form

$$s_1 \ldots s_k = s_1' \ldots s_l' \qquad (k, l \geqq 1;\ s_i, s_j' \in S \text{ for } i=1, \ldots, k \text{ and } j=1, \ldots, l)$$

holds. Now let $s_1=s_1', \ldots, s_{i-1}=s_{i-1}'$, $s_i \neq s_i'$. Then $s_1 \ldots s_i \neq s_1' \ldots s_i'$ and therefore there exists a $u \in F$ such that either $s_1 \ldots s_i = s_1' \ldots s_i' u$ or $s_1 \ldots s_i u = s_1' \ldots s_i'$. Suppose e.g. that the first of them is valid. Since F is cancellative, we have $s_i = s_i' u$ and $us_{i+1} \ldots \ldots s_k = s_{i+1}' \ldots s_l'$, that is $u \in F'F'^{[-1]}$. On the other hand, $s_i' \in F' \subseteqq F'^{[-1]}F'$; thus $s_i \in (F'^{[-1]}F') \cdot (F'F'^{[-1]})$.

Conversely, let F' be free and $s \in (F'^{[-1]}F') \cdot (F'F'^{[-1]})$ for some $s \in S$. More concretely, let $s=xy$, $ax=b$, $yc=d$ for some $x, y \in F$, $a, b, c, d \in F'$. Then $bd=asc$ and thus $a=b$ or $au=b$ or $a=bu$ with u in F'. Hence $x=u \in F'$ and, similarly, $y \in F'$. Thus $s=xy$ cannot be a free generator.

From the proof it turns out that we could replace (1) by $S \cdot (F'F'^{[-1]}) \cap S = \emptyset$ or, equivalently, we have

CRITERION B. *S is a system of free generators of F' iff*

$$F'F'^{[-1]}\cap S^{[-1]}S=\emptyset. \tag{2}$$

Of course, instead of (2) we may take the dual relation $F'^{[-1]}F'\cap SS^{[-1]}=\emptyset$. Construct a series of subsets $S_i\subseteq F$ as follows. Put

$$S_1=S^{[-1]}S,\quad S_{i+1}=S_i^{[-1]}F'\cup F'^{[-1]}S_i.$$

Denote $\bigcup_{i=1}^{\infty}S_i$ by T_1. Remark that, obviously,

$$T_1^{[-1]}F'\cup F'^{[-1]}T_1\subseteq T_1. \tag{3}$$

Further, $T_1\subseteq F'^{[-1]}F'$. Indeed, this inclusion holds for S_1 on the place of T_1. Suppose $S_i\subseteq F'^{[-1]}F'$ and let $x\in S_{i+1}$. Then either $ax\in S_i\subseteq F'^{[-1]}F'$, $bax\in F'$ for some $a,b\in F'$ or $tx=a\in F'$ for some $t\in S_i$. In the latter case there exists an element $b\in F'$ such that $bt\in F'$ and thus $bt\cdot x=ba\in F'$. In both cases x turned out to be an element of $F'^{[-1]}F'$.

Now we are going to show:

CRITERION C. *S is a system of free generators of F' iff*

$$F'\cap T_1=\emptyset. \tag{4}$$

PROOF. Assume $F'\cap T_1\neq\emptyset$, choose the least integer i such that $F'\cap S_i\neq\emptyset$ and put $a\in F'\cap S_i$. Then either $i=1$, i.e. $a\in S^{[-1]}S$, $sa\in S$ for some $s\in S$ and S cannot be a system of free generators, or $i>1$, $a\in S_{i-1}^{[-1]}F'\cup F'^{[-1]}S_{i-1}$. However, $a\notin F'^{[-1]}S_{i-1}$ (otherwise there would exist an element b such that $ba\in F'\cap S_{i-1}$ which contradicts the minimality of i), so that $a\in S_{i-1}^{[-1]}F'$ and $ta\in F'$ for a $t\in S_{i-1}$. Hence $t\in F'F'^{[-1]}$. But $t\in T_1\subseteq F'^{[-1]}F'$, $t\notin F'$ by the choice of i and thus $F'^{[-1]}F'\cap\cap F'F'^{[-1]}\nsubseteq F'$. The last is equivalent to F' being not free (see [3]).

Conversely, if S is not free, then, by virtue of (2), there exists an element $x\in F'F'^{[-1]}\cap S^{[-1]}S\subseteq F'F'^{[-1]}\cap T_1$ and $xa=b$ for some $a,b\in F'$. Hence $a\in T_1^{[-1]}F'\subseteq T_1$ by (3) and thus $F'\cap T_1\neq\emptyset$.

One can see by an easy induction that every set T which contains S_1 and has the property (3) must contain each S_i and, consequently, contains T_1. Thus, from (3) follows

CRITERION C'. *S is a system of free generators of F' iff there exists a set $T\subseteq F$ such that*

$$T\supseteq S^{[-1]}S\cup T^{[-1]}F'\cup F'^{[-1]}T \tag{5}$$

and $F'\cap T=\emptyset$.

Criterion C is very similar to the result of SARDINAS and PATTERSON. Namely, put

$$S_1'=S_1=S^{[-1]}S,\quad S_{i+1}'=S_i'^{[-1]}S\cup S^{[-1]}S_i'$$

and denote $\bigcup_{i=1}^{\infty}S_i'$ by T_2. Then, in the same way as we obtained (3), we have

$$T_2^{[-1]}S\cup S^{[-1]}T_2\subseteq T_2 \tag{3'}$$

and we can prove

CRITERION D (SARDINAS—PATTERSON [2]). *S is a system of free generators of F' iff*

$$S \cap T_2 = \emptyset. \tag{6}$$

PROOF. It is easy to see by induction that $S_i' \subseteqq S_i$. Thus, $T_2 \subseteqq T_1$, $S \subset F'$ and the necessity of (6) follows from that of (4).

In order to prove the sufficiency assume that (6) holds but (2) does not. Remark that $F'^{[-1]}T_2 \subseteqq T_2$. Indeed, if $ax \in T_2$, $a \in F'$ and $a = s_1' \dots s_l'$ $(s_j' \in S)$ then one can see by induction that $s_2' \dots s_l' x \in T_2, \dots, s_l' x \in T_2$, $x \in T_2$. Hence $s_k \in T_2$ if $s_1 \dots s_k \in T_2$ and (6) implies $T_2 \cap F' = \emptyset$.

Let $t \in F'F'^{[-1]} \cap S^{[-1]}S$, $ta = b = s_1' \dots s_l'$, $a = s_1 \dots s_k$, $(a, b \in F',\ s_i, s_j' \in S)$. Then $t \in S^{[-1]}S \subseteqq T_2$. Let $k+l$ be minimal for $t \notin F'F'^{[-1]} \cap T_2$. If $l=1$ then $ts_1 \dots s_k = s_1'$ implies $s_1 \dots s_k \in T_2^{[-1]} S \subseteqq T_2$ by (3′) which contradicts (6) as we have already seen. If $l>1$ then either $t = s_1' u$, $u \in S^{[-1]}T_2 \subseteqq T_2$ and (by the cancellativity of F) $us_1 \dots s_k = s_2' \dots s_l'$ or $s_1' = tu$, $u \in T_2^{[-1]} S \subseteqq T_2$ and $s_1 \dots s_k = us_2' \dots s_l'$, both in contradiction with the minimality of $k+l$, which completes the proof.

In an analogous way as we have gotten C′ from C we can deduce from D

CRITERION D′. *S is a system of free generators of F' iff there exists a set $T \subseteqq F$ such that $T \supseteqq S^{[-1]}S \cup T^{[-1]}S \cup S^{[-1]}T$ and $T \cap S = \emptyset$.*

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(Received February 4, 1971)

ON THE STABILITY OF INTERPOLATION

by

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1. Given n distinct points

$$+1 \geqq x_1 > x_2 > \cdots > x_{n-1} > x_n \geqq -1 \tag{1.1}$$

in $[-1, 1]$ and arbitrary numbers y_ν $(\nu=1, 2, \ldots, n)$, we consider the interpolation process

$$R_n(x) = \sum_{\nu=1}^{n} y_\nu r_\nu(x) \tag{1.2}$$

where the fundamental polynomials $r_\nu(x)$ satisfy the requirements

$$r_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \phantom{\text{for }} j = \nu. \end{cases} \tag{1.3}$$

Let the degree of the process be defined by

$$\deg R_n = \sum_{\nu=1}^{n} \text{degree of } r_\nu(x). \tag{1.4}$$

In [2] EGERVÁRY and TURÁN have investigated the process which is stable with respect to the interval $[-1, 1]$. The process (1.2) is generally called stable if for arbitrary real y_ν and y_ν^* we have for $-1 \leqq x \leqq 1$

$$\min_{\nu=1,2,\ldots,n} (y_\nu - y_\nu^*) \leqq \left[\sum_{\nu=1}^{n} y_\nu r_\nu(x) - \sum_{\nu=1}^{n} y_\nu^* r_\nu(x)\right] \leqq \max_{\nu=1,\ldots,n} (y_\nu - y_\nu^*) \tag{1.5}$$

and is called 'most economical stable' if it is stable and the $\deg R_n$ in the sense of (1.4) is minimal.

2. The above quoted authors have determined the 'most economical' stable interpolation process and have shown that such a process is possible if and only if $x_1 = 1$, $x_n = -1$ and x_ν $(\nu = 2, 3, \ldots, n-1)$ are the zeros of $P_{n-2}(x)$ where $P_{n-2}(x)$ is the $(n-2)$th Legendre polynomial with $P_{n-2}(1) = 1$. They have further shown that the resulting sequence of interpolatory polynomials converges uniformly to $f(x)$ in $[-1, 1]$ if $f(x)$ is continuous for $-1 \leqq x \leqq 1$ and $y_\nu = f(x_\nu)$ $(\nu = 1, 2, \ldots, n)$ and $n \to \infty$. The classical examples of (1.2) are the Lagrange interpolation

$$L_n(x) = \sum_{\nu=1}^{n} y_\nu l_\nu(x) \tag{2.1}$$

and the Hermite—Fejér interpolation

(2. 2)
$$H_n(x) = \sum_{\nu=1}^{n} y_\nu h_\nu(x) \text{ [1]}.$$

In the former, with $\omega(x) = c(x-x_1)(x-x_2)\dots(x-x_n)$, ($c$ is a non-zero constant)

$$r_\nu(x) = l_\nu(x) = \frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)} \qquad (\nu = 1, 2, \dots, n)$$

i.e. all fundamental functions are of degree $n-1$, and in the other

$$r_\nu(x) = h_\nu(x) = \left[1 - \frac{\omega''(x_\nu)}{\omega'(x_\nu)}(x-x_\nu)\right] l_\nu^2(x) \qquad (\nu = 1, 2, \dots, n)$$

i.e. all fundamental functions are of degree $2n-1$. FEJÉR proved that at a certain choice of the points x_ν, the interpolatory polynomials $H_n(x)$ in (2. 2) have a stability property. The degree $H_n(x) = n(2n-1) > (n-1)(2n-1)$. The polynomials (2. 1), according to a theorem of G. FABER, never have such a stability property.

P. SZÁSZ [5] has considered another type of interpolation which he calls quasi-Hermite—Fejér interpolation. Here for

$$+1 = \xi_1 > \xi_2 > \dots > \xi_n = -1$$

and

$$\omega^*(x) = c^*(x-\xi_2)(x-\xi_3)\dots(x-\xi_{n-1}), \qquad (c^* \text{ — a non-zero constant})$$

$$r_1(x) = \frac{1+x}{2} \cdot \frac{\omega^*(x)^2}{\omega^*(1)^2}, \quad r_n(x) = \frac{1-x}{2} \cdot \frac{\omega^*(x)^2}{\omega^*(-1)}$$

and for $\nu = 2, 3, \dots, n-1$,

$$r_\nu(x) = \frac{1-x^2}{1-\xi_\nu^2}\left[1 + \left\{\frac{2\xi_\nu}{1-\xi_\nu^2} - \frac{\omega^{*\prime\prime}(\xi_\nu)}{\omega^{*\prime}(\xi_\nu)}\right\}(x-\xi_\nu)\right]\left[\frac{\omega^*(x)}{\omega^{*\prime}(\xi_\nu')(x-\xi_\nu)}\right]^2.$$

The interpolatory polynomials

(2. 3)
$$S(x) = y_1 \frac{1+x}{2} P_{n-2}^2(x) + y_n \frac{1-x}{2} P_{n-2}(x)^2 + \sum_{\nu=2}^{n-1} y_\nu \frac{1-x^2}{1-\xi_\nu^2}\left[\frac{P_{n-2}(x)}{P_{n-2}'(\xi_\nu)(x-\xi_\nu)}\right]^2 \text{ [2]}$$

constructed for $\omega^*(x) = P_{n-2}(x)$, $P_{n-2}(x)$ being the $(n-2)$th Legendre polynomial with $P_{n-2}(1) = 1$, are the same as those obtained by EGERVÁRY and TURÁN [2]. Still another type of interpolation considered by P. SZÁSZ [6] is, as he calls, almost-Hermite—Fejér interpolation. Here for

$$+1 = \zeta_1 > \zeta_2 > \dots > \zeta_n > -1$$

and $\quad \omega^{**}(x) = c^{**}(x-\zeta_2)(x-\zeta_3)\dots(x-\zeta_n), \qquad (c^{**}$ — a non-zero constant)

$$r_1(x) = \frac{\omega^{**}(x)^2}{\omega^{**}(1)^2}$$

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[1] Called step parabolas of FEJÉR.
[2] Called the quasi-step parabolas.

and

$$r_\nu(x) = \frac{1-x}{1-\zeta_\nu}\left[1+\left\{\frac{1}{1-\zeta_\nu}-\frac{\omega^{**''}(\zeta_\nu)}{\omega^{**'}(\zeta_\nu)}\right\}(x-\zeta_\nu)\right]\left[\frac{\omega^{**}(x)}{\omega^{**'}(\zeta_\nu)(x-\zeta_\nu)}\right]^2, \quad (\nu=2,3,\ldots,n).$$

The interpolatory polynomials

$$(2.4)\quad S^*(x) = y_1\frac{\omega^{**}(x)^2}{\omega^{**}(1)^2}+\sum_{\nu=2}^{n} y_\nu \frac{1-x}{1-\zeta_\nu}\left[1+\left\{\frac{1}{1-\zeta_\nu}-\frac{\omega^{**''}(\zeta_\nu)}{\omega^{**'}(\zeta_\nu)}\right\}(x-\zeta_\nu)\right]\times \times\left[\frac{\omega^{**}(x)}{\omega^{**'}(\zeta_\nu)(x-\zeta_\nu)}\right]^2,$$

are called the almost step parabolas. We mention here a paper of A. SHARMA [7] where he consideres a similar interpolation.

3. In this paper we investigate the problem of 'most economical' stable interpolation in the interval $(-1, 1]$ i.e. we determine the polynomials

$$(3.1)\qquad \Lambda_n(x) = \sum_{\nu=1}^{n} y_\nu \lambda_\nu(x)$$

equal to given real values y_ν $(\nu=1, 2, \ldots, n)$ on the points

$$(3.2)\qquad 1 = x_1 > x_2 > \cdots > x_n > -1$$

and satisfying in the interval $-1<x\leqq 1$ the following two conditions

$$(3.3)\qquad 0 \leqq (1+x)\left[\sum_{\nu=1}^{n} y_\nu\lambda_\nu(x)-\sum_{\nu=1}^{n} y_\nu^*\lambda_\nu(x)\right] \leqq \max_{\nu=1,\ldots,n}\{|y_\nu-y_\nu^*|(1+x_\nu)\}$$

where y_ν and y_ν^* $(\nu=1, 2, \ldots, n)$ are arbitrary;

$$(3.4)\qquad \text{the deg}\, \Lambda_n(x) \stackrel{\text{def}}{=} \sum_{\nu=1}^{n}\{\text{degree of } \lambda_\nu(x)\}$$

being minimal.

We shall prove the following

THEOREM 1. *The minimal degree, in the sense of* (3. 4), *of interpolation polynomial* $\Lambda_n(x)$ *in* (3. 1), *stable in the sense of* (3. 2) *and* (3. 3), *is* $(n-1)(2n-1)$ *which is attained if and only if* $x_1=1$ *and* x_ν $(\nu=2, 3, \ldots, n)$ *are the zeros of* $(n-1)$th *Legendre polynomial* $P_{n-1}(x)$ *with* $P_{n-1}(1)=1$.

The expression for $\Lambda_n(x)$ in this case is given by

$$(3.5)\quad \bar{\Lambda}_n(x) = y_1 P_{n-1}(x)^2+\sum_{\nu=2}^{n} y_\nu \frac{1-x}{1-x_\nu}\left[\frac{P_{n-1}(x)}{P'_{n-1}(x_\nu)(x-x_\nu)}\right]^2 = \sum_{\nu=1}^{n} y_\nu \bar{\lambda}_\nu(x).$$

We shall further prove

THEOREM 2. *Let* $f(x)$ *be a function continuous for* $-1\leqq x\leqq 1$ *and* $y_1=f(x_1)=f(1)$, $y_\nu=f(x_\nu)$ *in the points* x_ν $(\nu=2, 3, \ldots, n)$ *which are the zeros of* $P_{n-1}(x)$, *then* $\bar{\Lambda}_n(x)$ *in* (3. 5) *converges to* $f(x)$ *in* $-1<x\leqq 1$. *The convergence being uniform in each subinterval* $-1+\delta\leqq x\leqq 1$, $(0<\delta<2)$.

4. The proof of theorem 1 follows on the same pattern as that of Theorem I of EGERVÁRY and TURÁN [2]. Let k be an integer with $1 \leqq k \leqq n$ and

$$y_\nu = \begin{cases} 1 & \text{for } \nu = k \\ 0 & \text{for } \nu \neq k \end{cases} \quad (1 \leqq \nu \leqq n),$$

and

$$y_\nu^* = 0 \quad \text{for} \quad \nu = 1, 2, \dots, n.$$

Then (3. 3) gives for $k = 1, 2, \dots, n$ and $-1 < x \leqq 1$, the inequality

$$0 \leqq \left(\frac{1+x}{1+x_k}\right) \lambda_k(x) \leqq 1. \tag{4. 1}$$

We also note that $\lambda_\nu(x)$ in (3. 1) satisfy

$$\lambda_\nu(x_j) = \begin{cases} 1 & \text{for } \nu = j \\ 0 & \text{for } \nu \neq j \end{cases} \quad (\nu = 1, 2, \dots, n). \tag{4. 2}$$

Thus if we suppose that

$$1 > x_1 > x_2 > \cdots > x_n > -1, \tag{4. 3}$$

then (4. 1) and (4. 2) give

$$\lambda_k(x_\nu) = \lambda_k'(x_\nu) = 0, \qquad \nu \neq k. \tag{4. 4}$$

(4. 1) and (4. 2) further give owing to

$$\left[\frac{1+x}{1+x_k} \lambda_k(x)\right]_{x = x_k} = 1,$$

$$\left[\frac{1+x}{1+x_k} \lambda_k(x)\right]'_{x = x_k} = 0,$$

$$\lambda_k'(x_k) + \frac{1}{1+x_k} \lambda_k(x_k) = 0$$

i.e.

$$\lambda_k'(x_k) = -\frac{1}{1+x_k} \qquad (k = 1, 2, \dots, n). \tag{4. 5}$$

Let us denote

$$\omega_n(x) = \prod_{\nu=1}^{n} (x - x_\nu) \tag{4. 6}$$

where x_ν $(\nu = 1, 2, \dots, n)$ are given by (4. 3), then (4. 4) means that $\lambda_k(x)$ is divisible by

$$\left[\frac{\omega_n(x)}{(x - x_k)}\right]^2.$$

In this case

$$\deg \Lambda_n(x) \geqq n(2n-2) > (n-1)(2n-1).$$

If we put

$$1 = x_1 > x_2 > \cdots > x_n > -1$$

and repeat the same reasoning we get for $2 \leqq k \leqq n$ and $2 \leqq \nu \neq k \leqq n$

$$\lambda_k(x_\nu) = \lambda_k'(x_\nu) = 0, \tag{4.7}$$

further

$$\lambda_k(x_1) = \lambda_k(1) = 0$$

and

$$\lambda_k'(x_k) = -\frac{1}{1+x_k}.$$

Also for $\nu = 2, 3, \ldots, n$

$$\lambda_1(x_\nu) = \lambda_1'(x_\nu) = 0. \tag{4.8}$$

So if we introduce

$$\omega_{n-1}^*(x) = \prod_{\nu=2}^{n} (x - x_\nu),$$

(4.8) means that $\lambda_1(x)$ is divisible by $\omega_{n-1}^*(x)^2$ and for $k \geqq 2$, $\lambda_k(x)$ by

$$(1-x)\left[\frac{\omega_{n-1}^*(x)}{x-x_k}\right]^2.$$

Hence

$$\deg \Lambda_n \geqq (2n-2)+(n-1)(2n-3) = (n-1)(2n-1)$$

and equality holds only if

$$\lambda_1(x) = \left[\frac{\omega_{n-1}^*(x)}{\omega_{n-1}^*(1)}\right]^2 \tag{4.9}$$

and for $k = 2, 3, \ldots, n$

$$\lambda_k(x) = \frac{1-x}{1-x_k}\left[\frac{\omega_{n-1}^*(x)}{\omega_{n-1}^{*\prime}(x_k)(x-x_k)}\right]^2 \tag{4.10}$$

where in the last case also

$$\lambda_k'(x_k) = -\frac{1}{1+x_k} \tag{4.11}$$

must be fulfilled. Thus on differentiating three times the identity (4. 10) and using (4. 11) we have

$$\frac{\omega_{n-1}^{*\prime\prime}(x_k)}{\omega_{n-1}^{*\prime}(x_k)} = \frac{2x_k}{1-x_k^2} \qquad (k = 2, 3, \ldots, n)$$

i.e.

$$(1-x_k^2)\,\omega_{n-1}^{*\prime\prime}(x_k) - 2x_k\,\omega_{n-1}^{*\prime}(x_k) = 0$$

which means that for a suitable constant c

$$(1-x^2)\,\omega_{n-1}^{*\prime\prime}(x) - 2x\,\omega_{n-1}^{*\prime}(x) + c\,\omega_{n-1}^*(x) = 0.$$

As well-known, then $c = n(n-1)$ and $\omega_{n-1}^*(x) = P_{n-1}(x)$, where $P_{n-1}(x)$ is the $(n-1)$th Legendre polynomial. Hence to $\deg \Lambda_n(x) = (n-1)(2n-1)$ it is necessary that

$$\bar{\Lambda}_n(x) = y_1 P_{n-1}(x)^2 + \sum_{\nu=2}^{n} y_\nu \frac{1-x}{1-x_\nu}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P_{n-1}'(x_\nu)}\right]^2 = \sum_{\nu=1}^{n} y_\nu \bar{\lambda}_\nu(x). \tag{4.12}$$

Here it is interesting to compare the almost step parabolas introduced by P. SZÁSZ in (2. 4) constructed on the zeros of $P_{n-1}(x)$, i.e. with

$$q^*(x) = y_1 P_{n-1}(x)^2 + \sum_{\nu=2}^{n} y_\nu \frac{1-x^2}{1-x_\nu^2}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2.$$

To show that the interpolation polynomial (4. 12) satisfies the stability condition (3. 3), we shall need the following

LEMMA 1. *We have for* $-1 \leqq x \leqq 1$

$$(4.13)\qquad 1-\frac{1+x}{2} P_{n-1}(x)^2 - \sum_{\nu=2}^{n} \frac{1-x^2}{1-x_\nu^2}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2 \geqq 0.$$

PROOF. The following identity is due to EGERVÁRY and TURÁN [2]

$$(4.14)\qquad P_{n-1}(x)^2 + \sum_{\nu=2}^{n} \frac{1-x^2}{1-x_\nu^2}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2 = 1.$$

Since

$$1-\frac{1+x}{2} P_{n-1}^2(x) - \sum_{\nu=2}^{n} \frac{1-x^2}{1-x_\nu^2}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2 =$$

$$= 1-P_{n-1}^2(x) - \sum_{\nu=2}^{n} \frac{1-x^2}{1-x_\nu^2}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2 + \frac{1-x}{2} P_{n-1}^2(x) = \frac{1-x}{2} P_{n-1}^2(x)$$

by virtue of (4. 14). Thus (4. 13) is proved.

The assertion of the lemma can be written as

$$\sum_{\nu=1}^{n} \frac{1+x}{1+x_\nu} \bar{\lambda}_\nu(x) \leqq 1, \qquad -1 < x \leqq 1.$$

Since the fundamental functions $\bar{\lambda}_\nu(x)$ are non negative and we have for real y_ν and y_ν^*

$$(1+x)\left\{\sum_{\nu=1}^{n} y_\nu \bar{\lambda}_\nu(x) - \sum_{\nu=1}^{n} y_\nu^* \bar{\lambda}_\nu(x)\right\} = \sum_{\nu=1}^{n} \{(y_\nu - y_\nu^*)(1+x_\nu)\}\left\{\frac{1+x}{1+x_\nu}\bar{\lambda}_\nu(x)\right\},$$

the proof of the theorem is completed.

5. To prove Theorem 2 we shall need three simple lemmas. We shall denote by ε the arbitrary positive number as small as we please and by $c_1, c_2, c_3, \ldots$ the positive numerical constants.

LEMMA 5. 1. *We have*

$$(5.1)\qquad \sum_{\nu=2}^{n} \frac{1-x^2}{1-x_\nu^2} |x-x_\nu| \left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2 \leqq 2\varepsilon$$

uniformly for $-1 \leqq x \leqq 1$.

This fact has been proved by P. SZÁSZ [5].

LEMMA 5. 2. *We have for* $-1+\delta \leqq x \leqq 1$ $(0<\delta<2)$ *the estimation*

$$\left| \sum_{\nu=1}^{n} \bar{\lambda}_\nu(x) - 1 \right| \leqq c_1 \varepsilon. \tag{5.2}$$

PROOF. We have from (4. 12)

$$\sum_{\nu=1}^{n} \bar{\lambda}_\nu(x) - 1 = P_{n-1}(x)^2 + \sum_{\nu=2}^{n} \frac{1-x}{1-x_\nu} \left[\frac{P_{n-1}(x)}{(x-x_\nu) P'_{n-1}(x_\nu)} \right]^2 - 1$$

which on account of the identity (4. 14) gives

$$1 - \sum_{\nu=1}^{n} \bar{\lambda}_\nu(x) = \sum_{\nu=2}^{n} \left\{ \frac{1-x^2}{1-x_\nu^2} - \frac{1-x}{1-x_\nu} \right\} \left[\frac{P_{n-1}(x)}{(x-x_\nu) P'_{n-1}(x_\nu)} \right]^2 = \tag{5.3}$$

$$= \sum_{\nu=2}^{n} \frac{1-x}{1-x_\nu^2} (x-x_\nu) \left[\frac{P_{n-1}(x)}{P'_{n-1}(x_\nu)(x-x_\nu)} \right]^2$$

Since

$$\frac{1}{1+x} < \frac{1}{\delta} \quad \text{for} \quad -1+\delta \leqq x \leqq 1,$$

we have in the same interval

$$\left| \sum_{\nu=2}^{n} \frac{1-x}{1-x_\nu^2} (x-x_\nu) \left[\frac{P_{n-1}(x)}{(x-x_\nu) P'_{n-1}(x_\nu)} \right]^2 \right| \leqq$$

$$\leqq \frac{1}{\delta} \sum_{\nu=2}^{n} \frac{1-x^2}{1-x_\nu^2} |x-x_\nu| \left[\frac{P_{n-1}(x)}{(x-x_\nu) P'_{n-1}(x_\nu)} \right]^2 \leqq c_1 \varepsilon,$$

using Lemma 5. 1.

LEMMA 5. 3. *We have*

$$\left| \sum_{\nu=1}^{n} [f(x) - f(x_\nu)] \bar{\lambda}_\nu(x) \right| \leqq c_2 \varepsilon \tag{5.4}$$

uniformly in $-1+\delta \leqq x \leqq 1$.

PROOF. From (4. 12) we have

$$\sum_{\nu=1}^{n} [f(x) - f(x_\nu)] \bar{\lambda}_\nu(x) = [f(x) - f(1)] P_{n-2}^2(x) +$$

$$+ \sum_{\nu=2}^{n} [f(x) - f(x_\nu)] \frac{1-x}{1-x_\nu} \left[\frac{P_{n-1}(x)}{(x-x_\nu) P'_{n-1}(x_\nu)} \right]^2 = I_1 + I_2.$$

Now

$$|I_1| = |[f(x) - f(1)] P_{n-1}(x)^2| \leqq c_3 \varepsilon \tag{5.5}$$

uniformly in $-1+\delta \leqq x \leqq 1$ $(0<\delta<2)$ since it vanishes at $x=1$ and is uniformly continuous for $-1 \leqq x \leqq 1$ while $P_{n-1}(x) \leqq c_4 n^{-\frac{1}{2}}$ uniformly for $-1+\delta \leqq x \leqq 1-\delta$ $(0<\delta<1)$ and is bounded in $[-1, 1]$.

We now estimate I_2. We have then

$$|I_2| \leqq \Big|\sum_{|x_\nu - x|\leqq\delta}\Big| + \Big|\sum_{|x_\nu - x|>\delta}\Big| = \sum\nolimits_1 + \sum\nolimits_2$$

Since the continuous function $f(x)$ is uniformly continuous in $-1 \leqq x \leqq 1$, for any $\varepsilon > 0$ there exists a positive number δ such that

$$|f(x)-f(y)| \leqq \varepsilon \quad \text{when} \quad |x-y| < \delta.$$

(5. 6)
$$\sum\nolimits_1 \leqq \varepsilon \sum_{|x_\nu - x|\leqq\delta} \frac{1-x}{1-x_\nu}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2 \leqq$$
$$\leqq \frac{2\varepsilon}{\delta} \sum_{|x_\nu - x|\leqq\delta} \frac{1-x^2}{1-x_\nu^2}\left[\frac{P_{n-1}(x)}{(x-x_\nu)P'_{n-1}(x_\nu)}\right]^2 \leqq c_5\,\varepsilon.$$

This follows from the fact that

$$\left|\frac{1+x_\nu}{1+x}\right| < \frac{2}{\delta} \quad \text{for} \quad -1+\delta \leqq x \leqq 1$$

and the identity (4. 14).

Let $|f(x)| \leqq M$ for $-1 \leqq x \leqq 1$. Then

$$\sum\nolimits_2 \leqq \frac{2M}{\delta^2}(1-x)P_{n-1}^2(x) \sum_{\nu=2}^{n} \frac{1}{1-x_\nu}\,\frac{1}{P'_{n-1}(x_\nu)^2} \leqq$$
$$\leqq \frac{4M}{\delta^2}(1-x)P_{n-1}(x)^2 \sum_{\nu=2}^{n} \frac{1}{(1-x_\nu^2)P'^{\,2}_{n-1}(x_\nu)} = \frac{4M}{\delta^2}(1-x)P_{n-1}(x)^2.$$

This follows from another identity proved by EGERVÁRY and TURÁN [2]:

(5. 7)
$$\sum_{\nu=2}^{n} \frac{1}{(1-x_\nu^2)P'^{\,2}_{n-1}(x_\nu)} = 1.$$

Again using Bernstein inequality

$$|P_{n-1}(x)| < \frac{c_6}{\sqrt{n-1}}\,\frac{1}{\sqrt[4]{1-x^2}}, \qquad -1 < x < 1$$

we have

(5. 8)
$$\sum\nolimits_2 \leqq \frac{4M}{\delta^2}\,\frac{c_6}{n-1}\,\frac{1}{1+x} < \frac{4M}{\delta^3}\,\frac{c_7}{n-1} < c_8\,\varepsilon \quad \text{for} \quad -1+\delta \leqq x \leqq 1.$$

Thus (5. 5), (5. 6) and (5. 8) complete the proof of the lemma.

6. We now turn to the proof of Theorem 2. Let

(6. 1)
$$|f(x)| \leqq M \quad \text{for} \quad -1 \leqq x \leqq 1.$$

We have

(6. 2)
$$|\bar{A}_n(x)-f(x)| = \left|\sum_{\nu=1}^{n}[f(x_\nu)-f(x)]\bar{\lambda}_\nu(x) - f(x)\left[1-\sum_{\nu=1}^{n}\bar{\lambda}_\nu(x)\right]\right| \leqq$$
$$\leqq \left|\sum_{\nu=1}^{n}[f(x_\nu)-f(x)]\bar{\lambda}_\nu(x)\right| + |f(x)|\left|1-\sum_{\nu=1}^{n}\bar{\lambda}_\nu(x)\right|.$$

Let now $-1+\delta \leqq x \leqq 1$, then Lemma 5. 2 and (6. 1) give

$$|f(x)| \left| 1-\sum_{\nu=1}^{n} \bar{\lambda}_\nu(x) \right| < Mc_1 \varepsilon, \tag{6. 3}$$

and Lemma 5. 3 gives for such $x's$

$$\left| \sum_{\nu=1}^{n} [f(x_\nu)-f(x)] \bar{\lambda}_\nu(x) \right| \leqq c_9 \varepsilon. \tag{6. 4}$$

Thus (6. 2), (6. 3) and (6. 4) complete the proof of Theorem 2.

Remark. The problem of 'most economical' stable interpolation in the interval $[-1, 1)$ can similarly be done with the similar results. We may mention here that the same problem in the interval $(-1, 1)$ has been solved by J. BALÁZS [1].

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(Received March 25, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 331—336

B-ALGEBRA VALUED SOLUTION OF THE COSINE EQUATION

by

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1. Introduction

The present paper deals with the functional equation

$$F(xy)+F(xy^{-1}) = 2F(x)F(y) \tag{I}$$

where F is a function on a group $\mathcal{G}$ and the range of F is a Banach algebra $\mathcal{A}$. KANNAPPAN [1] solved the equation for the case when $\mathcal{A}$ is the set of complex numbers. He proved that every solution of (I) on $\mathcal{G}$ has the form

$$F(x) = \frac{E(x)+E(x^{-1})}{2} \tag{II}$$

provided

$$F(xyz)=F(xzy). \tag{III}$$

Here E is a homomorphism of $\mathcal{G}$ onto the multiplicative group of nonzero complex numbers. M. HOSSZÚ [2] investigated the solutions of (I), the range of which is in a commutative ring.

If $E(x)$ is a homomorphism of $\mathcal{G}$ onto a multiplicative subgroup of $\mathcal{A}$, i.e. $E(xy)=E(x)E(y)$ satisfying (III), then $F(x)$ defined by (II) is a solution of (I). We shall prove that under certain conditions the solution of (I) has the form (II).

A $\mathcal{B}$-algebra may or may not have a unit element. If it has, we can suppose without loss of generality that its norm is equal to 1. (See [3] p. 5.)

2. Some properties of the solutions

From now on a solution of (I) means an $\mathcal{A}$-valued function $F(x)$, on an arbitrary group $\mathcal{G}$, where $\mathcal{A}$ is a Banach-algebra and $F(x)\not\equiv\mathbf{0}$, where $\mathbf{0}$ is the zero element of $\mathcal{A}$. We shall denote by e the unit element of $\mathcal{G}$.

Trivial consequence of condition (III) is that $F(xy)=F(yx)$, because $F(xy)=$ $=F(exy)=F(eyx)=F(yx)$.

LEMMA 2. 1. *Let $F(x)$ be a solution of* (I) *satisfying* (III), *not identically zero and $F(x)=F(x^{-1})$ for every $x\in\mathcal{G}$. Then the closed subalgebra $\mathcal{A}_F$ generated by $\{F(x):x\in\mathcal{G}\}$ is commutative with unit element $F(e)$.*

PROOF. To prove that $\mathcal{A}_F$ is commutative we change x and y in (I):

$$F(yx)+F(yx^{-1}) = 2F(y)F(x)$$

Subtracting it from (I) using (III) and the condition $F(y)=F(y^{-1})$ we get

(2. 1) $$F(x)F(y)=F(y)F(x)$$

for every $x, y \in \mathcal{G}$.

Putting $y=e$ in (I) we have

$$F(x) = F(x)F(e),$$

Since $\mathcal{A}_F$ is commutative

(2. 2) $$F(e) = I_F$$

is the unit element in $\mathcal{A}_F$.

Remark 2. 2: It is easy to verify that for the commutative solutions of (I) — i.e. $\mathcal{A}_F$ is commutative — satisfying (III) $F(x)=F(x^{-1})$ holds for every $x \in \mathcal{G}$.

LEMMA 2. 3. *For every solution of* (I) *satisfying* (III) *and* $F(x)=F(x^{-1})$ *for every* $x \in \mathcal{G}$, *holds:*

(2. 3) $$F(x^2)+I_F = 2F^2(x)$$

(2. 4) $$F(x^2)+F(y^2) = 2F(xy)F(xy^{-1})$$

(2. 5) $$[F(xy)+F(xy^{-1})]^2 = 4[F^2(x)-I_F][F^2(y)-I_F]$$

(2. 6) $$[F(xy)-F(x)F(y)]^2 = [F^2(x)-I_F][F^2(y)-I_F].$$

PROOF. Putting $y=x$ in (I) we get (2. 3). If we replace x by xy and y by xy^{-1} in (I) using (III) we get (2. 4). In similar manner as in [1] using (I) (III) (2. 3) and (2. 4) we get (2. 5). Using $F(xy^{-1}) = 2F(x)F(y)-F(xy)$ in (2. 5) we obtain (2. 6).

Remark 2. 4: If $\mathcal{A}$ is the B-algebra of the continuous operators of a Banach-space $\mathcal{B}$ then $I_F=F(e)$ is a projection. Hence $\mathcal{B}=\mathcal{B}_1+\mathcal{B}_2$ where $\mathcal{B}_1$, $\mathcal{B}_2$ are subspaces, and $I_F h_1=\mathbf{0}$ for every element h_1 of $\mathcal{B}_1$ and $I_F h_2=h_2$ for every element h_2 of $\mathcal{B}_2$, further every element h of $\mathcal{B}$ can be written uniquely as a sum $h = h_1+h_2$ where $h_1 \in \mathcal{B}_1$, $h_2 \in \mathcal{B}_2$. (See [4] p. 480.)

In this special case $\mathcal{B}_1 \subseteqq \operatorname{Ker} F(x)$ for every $x \in \mathcal{G}$, where $h \in \operatorname{Ker} F(x)$ if and only if $F(x)h=\mathbf{0}$. Naturally $\mathcal{B}_1 \subseteqq \bigcap_{x \in \mathcal{G}} \operatorname{Ker} F(x)$. If $\mathcal{A}$ is the Banach algebra of the continuous operators of a Hilbert space $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal subspaces if and only if $I_F=I_F^*$ (see [4] p. 482).

3. Form of the solution

THEOREM 3. 1. *Let* $E(x)$ *be a homomorphism of* $\mathcal{G}$ *onto* $\mathcal{A}$, *and let* $E(xy)=E(yx)$ *for every* $x, y \in \mathcal{G}$. *Then the function defined by*

$$F(x) = \frac{E(x)+E(x^{-1})}{2}$$

is a solution of (I).

PROOF.

$$2F(x)F(y) = \tfrac{1}{2}[E(x)E(y)+E(x)E(y^{-1})+E(x^{-1})E(y)+E(x^{-1})E(y^{-1})] =$$
$$= \tfrac{1}{2}[E(xy)+E(xy^{-1})+E(x^{-1}y)+E(x^{-1}y^{-1})] =$$
$$= \tfrac{1}{2}\big[E(xy)+E((xy)^{-1})\big]+\tfrac{1}{2}\big[E(xy^{-1})+E((xy^{-1})^{-1})\big] = F(xy)+F(xy^{-1}).$$

LEMMA 3. 2. *Let $F(x)$ be a solution of* (I) *satisfying* (III). *$F(x)=F(x^{-1})$ for every $x\in\mathcal{G}$, $F(e)\neq\mathbf{0}$ and $F(x)$ assumes the values $\pm I_F$ only. Then $F(x)$ has the form* (II).

PROOF. $F^2(x)=I_F$ for all $x\in\mathcal{G}$. It is a consequence of (2. 6), that

$$[F(xy)-F(x)F(y)]^2 = \mathbf{0}.$$

All possible values of the function $F(xy)-F(x)F(y)$ are $\mathbf{0}$; $2I_F$; $-2I_F$. Because of $I_F=I_F^2$, $I_F\neq\mathbf{0}$, we have

$$F(xy) = F(x)F(y) \quad \text{for every} \quad x, y\in\mathcal{G},$$

i.e. $F(x)$ is a homomorphism of $\mathcal{G}$ onto $\mathcal{A}$, and

$$F(x) = \frac{F(x)+F(x^{-1})}{2}.$$

We shall denote by $\sigma(A)$ the spectrum of an element A of $\mathcal{A}$.

THEOREM 3. 3. *Let $F(x)$ be a solution of* (I), *satisfying the next conditions:*

(i) $$F(xyz) = F(xzy)$$

(ii) $$F(x) = F(x^{-1}) \quad \textit{for every} \quad x\in\mathcal{G}$$

(iii) $$F(e) = I_F \neq \mathbf{0}$$

(iv) *there exists a Jordan curve $\mathcal{J}$ connecting the points ± 1 and a point x_0 in $\mathcal{G}$ such that $\mathcal{J}\cap\sigma(F(x_0)) = \varnothing$.*

Then there exists a homomorphism of $\mathcal{G}$ onto $\mathcal{A}$ for which

$$F(x) = \frac{E(x)+E(x^{-1})}{2}.$$

PROOF. Let us consider the closed subalgebra A_F generated by $\{F(x): x\in\mathcal{G}\}$. $\mathcal{A}_F$ is commutative by Lemma 2. 1, (i) and (ii). Then because of condition (iv) there exists a $B\in\mathcal{A}_F$ for which

(3. 1) $$B^2 = A^2 - I_F$$

(see [5] p. 167) where $A=F(x_0)$.

A consequence of condition (iv) is that B is an invertable element of $\mathcal{A}_F$. Let

(3. 2) $$E(x) = F(x)+B^{-1}[F(xx_0)-F(x_0)F(x)] = B^{-1}[F(x)(B-A)+F(xx_0)].$$

Using (I), (3. 1) (3. 2) we have

$$F(x)=\frac{E(x)+E(x^{-1})}{2}.$$

By the help of (I) (2. 3) (3. 1) and (3. 2) it is easy to verify that

$$E(x)E(x^{-1})=I_F$$

which means that $E(x^{-1})=E^{-1}(x)$, hence

$$F(x)=\frac{E(x)+E^{-1}(x)}{2}$$

($E^{-1}(x)$ is the inverse of $E(x)$ in the closed sub-algebra $\mathscr{A}_F$).
The consequence of this formula is the fact that $E(x)\neq\mathbf{0}$ for every $x\in\mathscr{G}$. Now we shall prove that $E(x)$ is a homomorphism so that $E(xy)=E(x)E(y)$ for every $x,y\in\mathscr{G}$.

$$E(x)E(y)=$$

$$=(B^{-1})^2\{F(xx_0)F(yx_0)+(B-A)[F(xx_0)F(y)+F(yx_0)F(x)]+(B-A)^2F(x)F(y)\}.$$

We need to observe the following expressions:

$$2[F(xx_0)F(y)+F(yx_0)F(x)]=F(xx_0y)+F(xx_0y^{-1})+F(yx_0x)+F(yx_0x^{-1})=$$

applying condition (1) we get:

$$=2F(x_0xy)+F\big(x_0(x^{-1}y)^{-1}\big)+F(x_0x^{-1}y)=2F(x_0xy)+2F(x_0)F(x^{-1}y).$$

Hence

$$F(xx_0)F(y)+F(yx_0)F(x)=F(x_0xy)+A[2F(x)F(y)-F(xy)]. \tag{3. 3}$$

$$2F(xx_0)F(yx_0)=F(xx_0yx_0)+F\big(xx_0(yx_0)^{-1}\big)=F(xx_0x_0y)+F(xx_0x_0^{-1}y^{-1})=$$

$$=F(x_0x_0yx)+F(xy^{-1})=2F(x_0)F(x_0yx)-F\big(x_0(x_0yx)^{-1}\big)+2F(x)F(y)-F(xy)=$$

$$=2F(x_0)F(x_0xy)-F\big((yx)^{-1}\big)+2F(x)F(y)-F(xy)=$$

using (ii) we get

$$=2F(x_0)F(x_0xy)+2F(x)F(y)-2F(xy).$$

Hence

$$F(xx_0)F(yx_0)=F(x)F(y)-F(xy)+AF(x_0xy). \tag{3. 4}$$

Using (3. 1) (3. 3) and (3. 4) we have

$$E(x)E(y)=(B^{-1})^2\big\{F(x)F(y)-F(xy)+AF(x_0xy)+$$

$$+(B-A)\big[F(x_0xy)+A\big(2F(x)F(y)-F(xy)\big)\big]+(B-A)^2F(x)F(y)\big\}=$$

$$=(B^{-1})^2\{F(x)F(y)[I_F+2A(B-A)+(B-A)^2]+BF(x_0xy)-$$

$$-F(xy)[I_F+A(B-A)]\}=(B^{-1})^2\{BF(x_0xy)+B(B-A)F(xy)\}=$$

$$=B^{-1}\{F(xy)(B-A)+F(xyx_0)\}$$

which proves the theorem.

4. Further properties of the solution

Let $F(x)$ be a solution of (I) of form (II). What can we say about the uniqueness of the homomorphism $E(x)$? The following example shows that two different homomorphism may determine the same solution.

Example: Let $\mathcal{G}$ be the additive group of real numbers, and $\mathcal{A}$ the Banach algebra of the 2 by 2 matrices

$$E_1(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad E_2(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix}.$$

In both cases

$$F(t) = \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix}.$$

Let $R(h)$ denote the spectral radius of an element h of $\mathcal{A}$.

THEOREM 4. 1. *Let $F(x)$ be a solution of* (I) *satisfying* (III), $F(x)=F(x^{-1})$, *and $\|F(x)\| \leqq K$ for every element x of $\mathcal{G}$. Then $R(F(x)) \leqq 1$ for every $x \in \mathcal{G}$.*

PROOF. Suppose that $R(F(y))>1$ for some $y \in \mathcal{G}$. Then there exists a complex number λ_0, $|\lambda_0|>1$ for which the inverse of the element $(F(y)-\lambda_0 I_F)$ does not exist. (2. 3) implies that $\lambda_1 = 2\lambda_0^2-1$ belongs to the spectrum of $F(y^2)$ and $\lambda_2 = 2\lambda_1^2-1$ belongs to the spectrum of $F(y^{2^2})$ etc. $\lambda_{n+1} = 2\lambda_n^2-1$ belongs to the spectrum of $F(y^{2^{n+1}})$. It is wellknown that $\|F(x)\| \geqq R(F(x)) = \sup |\lambda|$ where the supremum is taken for all λ belonging to the spectrum of $F(x)$. $|\lambda_0|>1$ implies

$$\lim_{n\to\infty} |\lambda_n| = +\infty$$

which contradicts the boundedness of $\|F(x)\|$.

THEOREM 4. 2. *Let $F(x)$ be a solution of* (I), $\|F(x)\|<K$ *for $x \in \mathcal{G}$ and $F(x)$ satisfying the conditions* (i)—(iii) *of Theorem 3. 3 and let there exist an $x_0 \in \mathcal{G}$ such that $\sigma(F(x_0))$ does not contain the points ± 1. Then $F(x)$ has the form* (ii).

PROOF. It is a consequence of Theorem 4. 1 that $\sigma(F(x))$ is in the closed unit disc for every element x of $\mathcal{G}$. Hence the points ± 1 can be connected by a Jordan curve, disjoint from $\sigma(F(x_0))$.

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Mathematical Institute of the Hungarian Academy of Sciences
(Received April 15, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1962) 337—342

ON WEIGHTED SIMULTANEOUS POLYNOMIAL APPROXIMATION

by

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I. Introduction

In the present paper we are giving an extension of a previous result (J. CZIPSZER—G. FREUD [1]), concerning trigonometric approximation[1] to polynomial approximation on $(-\infty, \infty)$ with the weight $\varrho(x)=e^{-x^2/2}$. Let $\|f\|_p^*=\|f\varrho\|_p$, where $\|\cdot\|_p$ is the usual $L_p(-\infty, \infty)$ norm $(1\leqq p\leqq\infty)$, and let Π_n denote the set of polynomials of degree at most n. Let us further define for an f with $\|f\|_p^*<\infty$

$$\varepsilon_n^{(p)*}(f) = \inf_{\varphi_n\in\pi_n} \|f-\varphi_n\|_p^*.$$

In what follows let us denote by $a_1, a_2, \dots$ positive absolute constants.

The main result of the present paper reads as follows:

THEOREM 1. *Let* $\|f\|_p^*<\infty$ *and* $F(t) = \int^t f(\tau)\,d\tau$, *further let* $r_n\in\Pi_n$ *and*

$$\|F-r_n\|_p^* \leqq \varepsilon \tag{1}$$

for a $p\in[1, \infty]$ *then we have*

$$\|f-r_n'\|_p^* \leqq a_1\varepsilon n^{1/2}+a_2\varepsilon_{n-1}^{(p)*}(f). \tag{2}$$

Let us note the following corollary of our theorem:

THEOREM 2.

$$\|F-r_n\|_p^* \leqq \theta\varepsilon_n^{(p)*}(F) \qquad (\theta\geqq 1) \tag{3}$$

implies

$$\|f-r_n'\|_p^* \leqq a_3\theta\varepsilon_{n-1}^{(p)*}(f). \tag{4}$$

In fact, Theorem 2 is a consequence of Theorem 1 and the relation

$$\varepsilon_n^{(p)*}(F) \leqq a_4 n^{-1/2}\varepsilon_{n-1}^{(p)*}(f). \tag{5}$$

The inequality (5) was proved by the author in his paper [4].

Since an additive constant has no influence either on the degree of polynomial approximation, or on $f=F'$ we can take $F(t) = \int_0^t f(\tau)\,d\tau$.

[1] In [3] we extended our result to approximation by rational polynomials in the norm of $C(-1, 1)$.

II. On the arithmetic means of expansions in Hermite polynomials[2]

Let us denote by $H_n(t)$ the orthogonal Hermite polynomial of degree n (see e.g. G. SZEGŐ [6]). Clearly $H_n \in L_p^*$ $(1 \leqq p \leqq \infty)$, so that if $f \in L_p^*$ for some $p \in [1, \infty]$, the Fourier—Hermite coefficients

$$c_k(f) = \frac{1}{[\|H_k\|_2^*]^2} \int_{-\infty}^{\infty} f(t) H_k(t) e^{-t^2} \, dt \tag{6}$$

exist, and the orthogonal expansion

$$f(t) \sim \sum_{k=0}^{\infty} c_k(f) H_k(t) \tag{7}$$

has a meaning.

We are going to consider the partial sums

$$s_n(f; t) = \sum_{k=0}^{n} c_k(f) H_k(t) \tag{8}$$

and the arithmetical means

$$\sigma_n(f; t) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(f; t) \tag{9}$$

of (7).

LEMMA 1. *If for the functions $f(t)$ and $g(t)$ the Fourier—Hermite coefficients $c_i(f), c_i(g)$ $(i=0, 1, \ldots)$ (see (6)) exist, then we have*

$$\int_{-\infty}^{\infty} \sigma_n(f; t) g(t) e^{-t^2} \, dt = \int_{-\infty}^{\infty} f(t) \sigma_n(g; t) e^{-t^2} \, dt. \tag{10}$$

PROOF. From (6) and (7) we get

$$\int_{-\infty}^{\infty} s_k(f; t) g(t) e^{-t^2} \, dt = \int_{-\infty}^{\infty} \sum_{i=0}^{k} c_i(f) H_i(t) g(t) e^{-t^2} \, dt = \tag{11}$$

$$= \sum_{i=0}^{k} [\|H_i\|_2^*]^2 c_i(f) c_i(g) = \int_{-\infty}^{\infty} s_k(g; t) f(t) e^{-t^2} \, dt.$$

Taking the average of (11) for $k=0, 1, \ldots, n$, (see (9)), we obtain (10). Q.e.d.

LEMMA 2. *We have for every $1 \leqq p \leqq \infty$*

$$\|\sigma_n(f)\|_p^* \leqq a_6 \|f\|_p^* \qquad (n=0, 1, \ldots). \tag{12}$$

PROOF. For $p=\infty$ this was proved in Part 5 of the paper G. FREUD—S. KNAPOWSKI [2][3] (see also G. FREUD [3]). For $p=1$ the proof runs as follows: By Lemma 1

$$\|\sigma_n(f)\|_1^* = \sup_{\|g\|_\infty^* \leqq 1} \int_{-\infty}^{\infty} \sigma_n(f; t) g(t) e^{-t^2} \, dt = \sup_{\|g\|_\infty^* \leqq 1} \int_{-\infty}^{\infty} \sigma_n(g; t) f(t) e^{-t^2} \, dt.$$

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[2] An outline of the main result of this chapter was given (in an other context) in the author's paper [4].

[3] Correction: In formula (5. 3) of [2] $(1-n^{-1})$ should be replaced by $(1-n^{-1})^{-n}$.

Applying now the already proved case $p=\infty$ of (12), we infer

$$\|\sigma_n(f)\|_1^* \leqq a_6 \int_{-\infty}^{\infty} |f(t)| e^{-t^2/2}\, dt = a_6 \|f\|_1^*.$$

In this way (12) holds for $p=1$ as well as for $p=\infty$, i.e. the linear operators $A_n(g, x)=e^{-x^2/2}\sigma_n[e^{t^2/2}g(t); x]$ are uniformly bounded (L_∞, L_∞) and (L_1, L_1). By the Riesz—Thorin interpolation theorem, they are also uniformly bounded (L_p, L_p) so that (12) is valid. (See e.g. A. ZYGMUND [7] Theorem XII. 1. 11.)

III. On differentiation of Hermite expansions

In this section we prove that by differentiating term by term the expansion in Hermite polynomials of $F(t) = \int_0^t f(\tau)\, d\tau$ $(f \in L_p^*)$, we obtain the Hermite polynomial expansion of $f(t)$. The emphasis is on the circumstance that beside the natural assumption: $f \in L_p^*$ for some $1 \leqq p \leqq \infty$, no further assumptions whatever are needed.

LEMMA 3. *Setting* $F(t) = \int_0^t f(\tau)\, d\tau$, *we have for every* $1 \leqq p \leqq \infty$ *and every* $f \in L_p^*$

$$\|F\|_p^* \leqq a_7 \|f\|_p^*. \tag{13}$$

The PROOF is divided in three parts.

a) $p=\infty$. Let $h(t) = \int_0^t e^{\tau^2/2}\, d\tau \in L_\infty^*$ then

$$|F(t)e^{-t^2/2}| = \left|e^{-t^2/2} \int_0^t [f(\tau)e^{-\tau^2/2}] e^{\tau^2/2}\, d\tau\right| \leqq$$

$$\leqq \|f\|_\infty^* \left|e^{-t^2/2} \int_0^t e^{\tau^2/2}\, d\tau\right| \leqq \|h\|_\infty^* \|f\|_\infty^*$$

i.e.

$$\|F\|_\infty^* \leqq a_8 \|f\|_\infty^*. \tag{14}$$

b) $p=1$. We have

$$\|F\|_1^* = \sup_{\|g\|_\infty^* \leqq 1} \int_{-\infty}^{\infty} F(t)g(t)e^{-t^2}\, dt \leqq \tag{15}$$

$$\leqq \sup_{\|g\|_\infty^* \leqq 1} \int_0^{\infty} F(t)g(t)e^{-t^2}\, dt + \sup_{\|g\|_\infty^* \leqq 1} \int_{-\infty}^{0} F(t)g(t)e^{-t^2}\, dt.$$

We turn to the first of these expressions. As a consequence of $\|f\|_1^* < \infty$ we have

$$F(t) = o(e^{t^2/2}), \qquad t \to \infty \tag{16}$$

Let now

$$\Gamma(t) = \int_t^{\infty} g(\tau)e^{-\tau^2}\, d\tau$$

From $\|g\|_\infty^* \leqq 1$ we obtain

$$|\Gamma(t)| \leqq \int_t^\infty e^{-\tau^2/2}\,d\tau \leqq \max(\pi^{1/2}, |t|^{-1}e^{-t^2/2}) \leqq a_9 e^{-t^2/2} \tag{17}$$

so that by partial integration, using (16) and (17)

$$\int_0^\infty F(t)g(t)e^{-t^2}\,dt = -[F\Gamma]_0^\infty + \int_0^\infty f(t)\Gamma(t)\,dt =$$

$$= \int_0^\infty f(t)\Gamma(t)\,dt \leqq a_9 \int_0^\infty |f(t)|\,e^{-t^2/2}\,dt.$$

Similarly

$$\int_{-\infty}^0 F(t)g(t)e^{-t^2}\,dt \leqq a_9 \int_{-\infty}^0 |f(t)|\,e^{-t^2/2}\,dt.$$

From (15) and the last two inequalities we get

$$\|F\|_1^* \leqq 4a_9\|f\|_1^*. \tag{18}$$

c) $1<p<\infty$. We see from (14) and (18), that, taking $a_7 = a_8+4a_9$, (13) is satisfied for $p=\infty$ and $p=1$.

Applying the Riesz—Thorin interpolation theorem, to the linear operator

$$B(g;x) = e^{-x^2/2}\int_0^x g(t)e^{t^2/2}\,dt$$

we conclude that (13) holds also for $1<p<\infty$; Lemma 3 is proved.

LEMMA 4. *If $f\in L_p^*$ for some $1\leqq p\leqq\infty$ then we have*

$$s_n'(F;x) = s_{n-1}(f;x). \tag{19}$$

PROOF. By Lemma 3: $F\in L_p^*$. We are going to prove

$$c_k(F)H_k'(x) = c_{k-1}(f)H_{k-1}(x) \qquad (k=1,2,\ldots). \tag{20}$$

The relation (19) is a clear consequence of (20). We refer to G. SZEGŐ [6], (5. 5. 10), (5. 5. 8) and (5. 5. 1) for the identities

$$H_n'(x) = 2nH_{n-1}(x) \tag{21}$$

$$[e^{-x^2}H_n(x)]' = e^{-x^2}[H_n'(x)-2xH_n(x)] = -e^{-x^2}H_{n+1}(x) \tag{22}$$

and

$$[\|H_k\|_2^*]^2 = \sqrt{\pi}\,2^k k!, \tag{23}$$

For $1<p<\infty$ let $p^{-1}+q^{-1} = 1$, then by Hölder's inequality

$$|F(x)| = \left|\int_0^x f(t)\,dt\right| \leqq e^{x^2/2}\left|\int_0^x f(t)e^{-t^2/2}\,dt\right| \leqq e^{x^2/2}|x|^{1/q}\|f\|_p^* \tag{24}$$

and this is verified directly for $p=1$, $q=\infty$, and $p=\infty$, $q=1$, resp. It follows that the partial integration in the following calculation based on (22), is legitimate:

$$(25) \qquad (\|H_{k-1}\|_2^*c_{k-1}(f) = \int_{-\infty}^{\infty} f(x)H_{k-1}(x)e^{-x^2}\,dx =$$

$$= [F(x)H_{k-1}(x)e^{-x^2}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F(x)[H_{k-1}(x)e^{-x^2}]'\,dx =$$

$$= \int_{-\infty}^{\infty} F(x)H_k(x)e^{-x^2}\,dx = (\|H_k\|_2^*)^2\,c_k(F).$$

We see from (21), (23) and (25) that (20) is satisfied, and this proves our Lemma 4.

IV. Proof of the main theorems

We consider the de la Vallée Poussin-type means

$$(26) \qquad V_n(F;x) = \frac{1}{n}\sum_{\nu=n}^{2n-1} s_\nu(F;x) = 2\sigma_{2n-1}(F;x) - \sigma_{n-1}(F;x).$$

By Lemma 4 we have

$$(27) \qquad V_n'(F;x) = \frac{1}{n}\sum_{\nu=n-1}^{2n-2} s_\nu(f;x) = \frac{2n-1}{n}\sigma_{2n-2}(f;x) - \frac{n-1}{n}\sigma_{n-2}(f;x).$$

Let now $P_n \in \Pi_n$, thus $s_\nu(P_n) \equiv P_n$ ($\nu = n, n+1, \ldots$) hence from (26) we have

$$\|V_n(F;x) - F(x)\|_p^* = \|V_n(F-P_n;x) + P_n(x) - F(x)\|_p^* \leqq \|V_n(F-P_n)\|_p^* + \|(F-P_n)\|_p^* \leqq$$

$$\leqq 2\|\sigma_{2n-1}(F-P_n)\|_p^* + \|\sigma_{n-1}(F-P_n)\|_p^* + \|F-P_n\|_p^*$$

and by Lemma 2

$$\|V_n(F) - F\|_p^* \leqq a_{10}\|F-P_n\|_p^* \qquad (P_n \in \pi_n);$$

consequently

$$(28) \qquad \|V_n(F) - F\|_p^* \leqq a_{10}\,\varepsilon_n^{(p)*}(F).$$

Similarly, let $p_{n-1} \in \pi_{n-1}$, thus $s_\nu(p_{n-1}) \equiv p_{n-1}$ ($\nu = n-1, n, \ldots$). From (27) we get

$$V_n'(F;x) - p_{n-1}(x) = \frac{1}{n}\sum_{\nu=n-1}^{2n-2}[s_\nu(f;x) - s_\nu(p_{n-1};x)] =$$

$$= \frac{1}{n}\sum_{\nu=n-1}^{2n-2} s_\nu(f-p_{n-1};x) = \frac{2n-1}{n}\sigma_{2n-2}(f-p_{n-1};x) - \frac{n-1}{n}\sigma_{n-2}(f-p_{n-1};x).$$

Hence using again Lemma 2

$$\|V_n'(F) - f\|_p^* \leqq \|V_n'(F) - p_{n-1}\|_p^* + \|f-p_{n-1}\|_p^* \leqq$$

$$\leqq \frac{2n-1}{n}\|\sigma_{2n-2}(f-p_{n-1})\|_p^* + \frac{n-1}{n}\|\sigma_{n-2}(f-p_{n-1})\|_p^* + \|f-p_{n-1}\|_p^* \leqq$$

$$\leqq a_{11}\|f-p_{n-1}\|_p^* \qquad (p_{n-1} \in \pi_{n-1}).$$

thus

(29) $$\|V_n'(F)-f\|_p^* \leqq a_{11}\varepsilon_n^{(p)*}(f).$$

PROOF of Theorem 1. From (1) and (28) we have

$$\|r_n - V_n(F)\|_p^* \leqq \varepsilon + a_{10}\varepsilon_n^{(p)*}(F).$$

Since $r_n - V_n(F) \in \Pi_{2n-1}$, we obtain using our inequality proved in [5]

(30) $$\|r_n' - V_n'(F)\|_p^* \leqq a_{12}n^{1/2}[\varepsilon + a_{10}\varepsilon_n^{(p)*}(F)].$$

From (29) and (30) we obtain, considering (5)

$$\|f-r_n'\|_p^* \leqq \|f-V_n'(F)\|_p^* + \|r_n - V_n'(F)\|_p^* \leqq$$

$$\leqq a_{11}\varepsilon_{n-1}^{(p)*}(f) + a_{10}a_{12}n^{1/2}\varepsilon_n^{(p)*}(F) + a_{12}n^{1/2}\varepsilon \leqq a_{12}n^{1/2}\varepsilon + a_{13}\varepsilon_{n-1}^{(p)*}(f)$$

and this is the required inequality (2).

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(Received May 13, 1971.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 343—353

ASYMPTOTIC ENUMERATION OF REGULAR MATRICES

by

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Introduction

Let us consider linear graphs having m "red" and n "blue" *labelled* vertices. Connecting edges (unlabelled and uncoloured) are permitted only between vertices of different colour. Such a bicoloured graph may be called *regular* if every vertex of the same colour has the same degree, i.e. the red and blue vertices are incident to the same number of edges — p and q each, say, — the total number of edges being $mp=nq=N$. Multiple edges, connecting the same vertices, are permitted here Let $D_{p,q}^{(m,n)}$ be the number of such regular graphs. They have been enumerated (among other enumeration problems) by R. C. READ [1].

Labelled bicoloured graphs can uniquely be represented by matrices of size $m\times n$ having non-negative integer elements. Regularity of the graphs means that the row-sums of the corresponding matrices are equal, as well as the column-sums. Also such matrices may be called *regular*.

Let, furthermore $C_{p,q}^{(m,n)}$ be the number of regular bocoloured graphs *without multiple edges* or, in the matrix-terminology, let $C_{p,q}^{(m,n)}$ be the number of regular zero-one matrices.

In § 1 we discuss the enumeration problem for both types of regular matrices.

In § 2 we discuss a more general problem: the asymptotic behaviour of the number of matrices (of 0—1 or non-negative integer elements) with arbitrarily given row- and column-sums.

THEOREM 1. *If* $mp=nq=N\to\infty$, *whereas* $p=\text{const.}$, $q=\text{const.}$, *then*

$$D_{p,q}^{(m,n)} \sim \frac{N!}{(p!)^m (q!)^n}\, e^{(p-1)(q-1)/2}, \tag{1}$$

$$C_{p,q}^{(m,n)} \sim \frac{N!}{(p!)^m (q!)^n}\, e^{-(p-1)(q-1)/2}. \tag{2}$$

It is evident that the enumeration function for labelled bicoloured graphs of given partition (for matrices with prescribed row-sums and column-sums) can be written as

$$H(x_1, \dots, x_m; y_1, \dots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1-x_i y_j} = \tag{3}$$

$$= \exp\left\{\sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{m} x_i^k \sum_{j=1}^{n} y_j^k\right\} = \prod_k \sum_{r_k=0}^{\infty} \frac{1}{k^{r_k}\cdot r_k!} \left(\sum x_i^k\right)^{r_k} \left(\sum y_j^k\right)^{r_k}$$

if multiple edges are permitted unrestrictedly, but if multiple edges are to be excluded then the enumeration function is

$$(4)\qquad G(x_1, \dots, x_m; y_1, \dots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1+x_i y_j) =$$

$$= \exp\left\{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{i=1}^{m} x_i^k \sum_{j=1}^{n} y_j^k\right\} = \prod_k \sum_{r_k=0}^{\infty} \frac{(-1)^{(k+1)r_k}}{k^{r_k} \cdot r_k!} \left(\sum x_i^k\right)^{r_k} \left(\sum y_j^k\right)^{r_k}.$$

It follows then from the polynomial theorem that

$$(5)\qquad C_{p,q}^{(m,n)} = \sum \frac{(-1)^{r_2+r_4+\dots}}{1^{r_1} 2^{r_2} \dots p^{r_p}} \cdot \frac{\prod_{k=1}^{p} r_k!}{\prod_{k=1}^{p} \prod_{i=1}^{m} s_{ki}! \prod_{k=1}^{p} \prod_{j=1}^{n} t_{kj}!}$$

where the sum runs over all non-negative integral values of r_k, s_{ki}, t_{kj} satisfying

$$(6)\qquad \begin{cases} \sum_{k=1}^{p} k s_{ki} = p & i=1,2,\dots,m \\ \sum_{k=1}^{p} k t_{kj} = q & j=1,2,\dots,n \end{cases}$$

$$(7)\qquad \sum s_{ki} = \sum t_{kj} = r_k \qquad k=1,2,\dots,p,$$

supposed $p \leqq q$ that does not affect generality.

Let P be the number of all different partitions of p, let them be denoted by

$$(u_{1l}, u_{2l}, \dots, u_{pl}) \qquad l=1,2,\dots,P,$$

and indexed such a way that

$$(8)\qquad \begin{aligned} &u_{11}=p, \quad u_{21}=\dots=u_{p1}=0, \\ &u_{12}=p-2, \quad u_{22}=1, \quad u_{32}=\dots=u_{p2}=0. \end{aligned}$$

Similarly, let Q be the number of the (restricted) partitions of q, denoted by

$$(v_{1l}, v_{2l}, \dots, v_{pl}) \qquad l=1,2,\dots,Q$$

and

$$(9)\qquad \begin{aligned} &v_{11}=q, \quad v_{21}=\dots=v_{p1}=0, \\ &v_{12}=q-2, \quad v_{22}=1, \quad v_{32}=\dots=v_{p2}=0. \end{aligned}$$

Suppose that for each l the l-th partition of p and q occurs σ_l and τ_l times respectively. Summing up the terms in (5) that belong to the same partitions we have

$$(10)\qquad C_{p,q}^{(m,n)} = \sum \frac{(-1)^{r_2+r_4+\dots}}{\prod_{k=1}^{p} k^{r_k}} \prod_{k=1}^{p} r_k! \frac{m!}{\prod_{l=1}^{P} \sigma_l!} \frac{n!}{\prod_{l=1}^{Q} \tau_l!} \frac{1}{\prod_{l=1}^{P}\left(\prod_{k=1}^{p} u_{kl}!\right)^{\sigma_l} \prod_{l=1}^{Q}\left(\prod_{k=1}^{p} v_{kl}!\right)^{\tau_l}}$$

where now the sum is extended over non-negative values of r_k σ_l and τ_l restricted by

$$\sum_{l=1}^{P} \sigma_l = m, \quad \sum_{l=1}^{Q} \tau_l = n, \tag{11}$$

and — according to (7) —

$$\sum_{l=1}^{P} \sigma_l u_{kl} = \sum_{l=1}^{Q} \tau_l v_{kl} = r_k, \qquad k=1, 2, \ldots, p. \tag{12}$$

It is clear that a formula similar to (10) holds for $D_{p,q}^{(m,n)}$, but without the $(-1)^{r_2+r_4+\cdots}$ sign factors in the terms.

The structure of terms in (10) has shown us that, from the point of view of the $N\to\infty$ asymptotics, only the first two partitions are important.

From (10) the following integral formula can be deduced

$$C_{p,q}^{(m,n)} = \int_{x_1=0}^{\infty} \cdots \int_{x_p=0}^{\infty} \left(\frac{1}{2\pi i}\right)^p \oint_{(0)} \cdots \oint_{(0)} \frac{e^{-x_1-x_2-\cdots-x_p}}{y_1 y_2 \cdots y_p} \times \tag{13}$$

$$\times \left\{ \sum_{l=0}^{P} \alpha_l \prod_{k=1}^{p} \left[\frac{(-1)^{k+1} x_k}{k y_k}\right]^{u_{kl}} \right\}^m \left\{ \sum_{l=0}^{Q} \beta_l \prod_{k=1}^{p} y^{v_{kl}} \right\}^n dy_p \ldots dy_1,$$

where

$$\alpha_l = \frac{1}{\prod_{k=1}^{p} u_{kl}!} \qquad l=1, 2, \ldots, P,$$

$$\beta_l = \frac{1}{\prod_{k=1}^{p} v_{kl}!} \qquad l=1, 2, \ldots, Q.$$

In particular, according to (8) and (9),

$$\alpha_1 = \frac{1}{p!}, \quad \alpha_2 = \frac{1}{(p-2)!}, \qquad \beta_1 = \frac{1}{q!}, \quad \beta_2 = \frac{1}{(q-2)!}.$$

(10) can easily be verified by integrating term-by-term and keeping in mind that

$$\int_0^{\infty} x^r e^{-x} = r!, \qquad \frac{1}{2\pi i} \oint_{(0)} y^r \, dy = \begin{cases} 1 & \text{if } r=-1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying the idea of Laplace's asymptotic method, one can deduce (2) from (13) and (1) can be obtained in a similar way.

We do not go into details here, since in § 2 we are going to give a proof for the more general case. This latter proof is of combinatorial type, and we mentioned the previous sketched proof only for showing two quite different interesting approaches. (We should also mention that while having worked on this problem the analytic proof was the first one we have obtained.)

§ 2

1) To consider the general case, we make the following model.

We have n urns and l_1 balls labelled 1, l_2 balls labelled 2, ..., l_m balls labelled m. We consider the balls (even the ones of the same label) distinguishable.

We put k_1 balls to the first urn, one by one, ..., k_n balls to the n-th urn, one by one. This can be interpreted as a permutation of the $\sum_{i=1}^{n} k_i = \sum_{j=1}^{m} l_j = N$ (distinguishable) balls. The first k_1 elements of a permutation will be called the first block, the next k_2 elements the second block etc. Thus we have $N!$ permutations, and we define a matrix for each permutation: the element a_{ij} of this matrix is the number of balls labelled j in the i-th block. Thus the row-sums of this matrix are $k_1, k_2, \dots, k_n$ and the column-sums are $l_1, l_2, \dots, l_m$. Let us call a permutation simple, if elements in the same block are of different labels. This means that the elements of the corresponding matrix are 0 and 1 only.

2) Let $\mathcal{M} = \mathcal{M}(k_1, \dots, k_n; l_1, \dots, l_m)$ denote the class of matrices of non-negative integers with row-sums $k_1, \dots, k_n$ and column-sums $l_1, \dots, l_m$, and $\mathcal{M}_0$ the subclass with 0—1 elements. M and M_0 denote the number of matrices in $\mathcal{M}$ and $\mathcal{M}_0$, respectively.

We are going to examine the asymptotic behaviour of M and M_0 for $1 \leq k_i \leq K$, $i = 1, \dots, n$; $1 \leq l_j \leq L$, $j = 1, \dots, m$, K, L are kept fixed and $N \to \infty$ (that is equivalent to $n \to \infty$ or $m \to \infty$).

A 0—1 matrix corresponds to exactly $\prod_{i=1}^{n} k_i! \prod_{j=1}^{m} l_j!$ (simple) permutations, thus M_0 is equal to the number P_0 of simple permutations divided by $\Pi k_i! \Pi l_j!$. We introduce a few more notations:

$$\bar{k} = \frac{\sum_{i=1}^{n} k_i}{n} = \frac{N}{n}, \quad \bar{l} = \frac{\sum_{j=1}^{m} l_j}{m} = \frac{N}{m},$$

$$\overline{\binom{k}{2}} = \frac{\sum_{i=1}^{n} \binom{k_i}{2}}{n}, \quad \overline{\binom{l}{2}} = \frac{\sum_{j=1}^{m} \binom{l_j}{2}}{m},$$

$$\alpha = 2 \cdot \frac{\left(\sum \binom{k_i}{2}\right)\left(\sum \binom{l_j}{2}\right)}{(\sum k_i)(\sum l_j)} = 2 \cdot \frac{\overline{\binom{k}{2}}\,\overline{\binom{l}{2}}}{\bar{k}\bar{l}}.$$

Note that $\alpha \leq 2\binom{K}{2}\binom{L}{2}$, i.e. α is bounded, and in the case $k_1 = \dots = k_n = k$, $l_1 = \dots = l_m = l$ we have

$$\alpha = \frac{(k-1)(l-1)}{2}.$$

We formulate our results in two theorems:

THEOREM 2.

$$M_0 \sim \frac{N!}{\prod k_i! \prod l_j!} e^{-\alpha} \tag{14}$$

$$M \sim \frac{N!}{\prod k_i! \prod l_j!} e^{\alpha}. \tag{15}$$

The symbol $\sim$ obviously means (e.g. in (14))

$$1-\varepsilon < \frac{M_0}{\dfrac{N!}{\prod k_i! \prod l_j!} e^{-\alpha}} < 1+\varepsilon \quad \text{for} \quad N > N_0(\varepsilon)$$

whatever are the numbers $n, m, k_i, l_j, N, \alpha$ (K and L are fixed!).

Actually, we are going to obtain a remainder term:

$$M_0 = \frac{N!}{\prod k_i! \prod l_j!} e^{-\alpha}\left(1+O\left(\frac{\log N}{\sqrt{N}}\right)\right),$$

where the symbol $O(\cdot)$ stands for a bound depending on K and L only. Probably the right remainder term is $O\left(\frac{1}{N}\right)$.

Theorem 1 is a special case of Theorem 2.

3) Let $M_2 = M_2(k_1, \dots, k_n; l_1, \dots, l_m)$ denote the number of matrices having 0, 1 and 2 elements only (and given row-sums k_i and column-sums l_j) not having two 2's in the same row or column, $M_{2,t}$ the number of those of them which have exactly t 2's. Thus

$$M_2 = \sum_{t=0} M_{2,t} \quad \text{and} \quad M_0 = M_{2,0}.$$

We are going to prove the surprising statement that this very particular class of matrices represents the majority of the class $\mathcal{M}$, i.e. in "almost all" matrices (with given row- and column-sums) there is no element greater than 2.

THEOREM 3.

$$M_2 \sim M \tag{16}$$

and for fixed t

$$M_{2,t} \sim M \frac{(2\alpha)^t}{t!} e^{-2\alpha}. \tag{17}$$

Formulae (16) and (17) say that the number of elements greater than 1 is of a Poisson distribution.

First we will prove (14), i.e. the relation

$$P_0 \sim N! e^{-\alpha}, \tag{18}$$

then (16), finally (17), which implies (15) (as the special case $t=0$).

In the proofs we will use the following simple inequality for positive numbers a_i:

(19) $$\frac{1}{r!}\left(\sum a_i\right)^r\left(1-\binom{r}{2}\frac{\sum a_i^2}{\left(\sum a_i\right)^2}\right) \leqq \sum_{i_1<i_2<\dots<i_r} a_{i_1}a_{i_2}\dots a_{i_r} \leqq \frac{1}{r!}\left(\sum a_i\right)^r.$$

4) The proof of relation (18).

If in a block of a permutation there are two elements of the same label, we say that a duplication occurs. (Simple permutation is one with no duplication).

Let $A_r(i_1, \dots, i_r)$ denote the number of permutations in which there are duplications in the i_1-th, ..., i_r-th blocks, and put

$$A_r = \sum_{i_1<\dots<i_r} A_r(i_1, \dots, i_r).$$

By the exclusion-inclusion formula

$$P_0 = \sum_{r=0} (-1)^r A_r.$$

Since

$$A_r(i_1, \dots, i_r) \leqq \binom{k_{i_1}}{2}\dots\binom{k_{i_r}}{2}\left(\sum_{j_1,\dots,j_r=1}^{m}\binom{l_{j_1}}{2}\dots\binom{l_{j_r}}{2}2^r\right)(N-2r)!,$$

using (19) we get

$$A_r \leqq \frac{N!}{r!}\frac{\left(\sum\binom{k_i}{2}\right)^r\left(\sum\binom{l_j}{2}\right)^r 2^r}{N^{2r}}\frac{1}{\left(1-\frac{2r}{N}\right)^{2r}} = \frac{N!}{r!}\alpha^r\frac{1}{\left(1-\frac{2r}{N}\right)^{2r}},$$

and applying the Bernuolli inequality we have for $r \leqq \frac{\sqrt{N}}{3}$

(20) $$A_r \leqq \frac{N!}{r!}\alpha^r\left(1+\frac{8r^2}{N}\right).$$

For estimating $A_r(i_1, \dots, i_r)$ from below, we take into account only the number of permutations where in the i_1-th, ..., i_r-th block there is one duplication and all the other elements of these blocks are different:

$$A_r(i_1, \dots, i_r) \geqq$$

$$\geqq \binom{k_{i_1}}{2}\dots\binom{k_{i_r}}{2}\left(\sum_{\substack{j_1,\dots,j_r\\ \text{are different}}}\binom{l_{j_1}}{2}\dots\binom{l_{j_r}}{2}2^r\right)(N-rKL)^{k_{i_1}+\dots+k_{i_r}-2r}(N-k_{i_1}-\dots-k_{i_r})! \geqq$$

$$\geqq \binom{k_{i_1}}{2}\dots\binom{k_{i_r}}{2}\left(2\sum\binom{l_j}{2}\right)^r\left[1-\binom{r}{2}\frac{\sum\binom{l_j}{2}^2}{\left(\sum\binom{l_j}{2}\right)^2}\right]\cdot\frac{N!}{N^{2r}}\left(1-\frac{rKL}{N}\right)^{rK}.$$

Put

$$s = \min\left(\sum_{k_i\geqq 2} 1, \sum_{l_j\geqq 2} 1\right).$$

Applying (19) again,

$$A_r \geqq \frac{N!}{r!}\alpha^r\left(1-\frac{rKL}{N}\right)^{rK}\left[1-\binom{r}{2}\frac{\sum\binom{l_j}{2}^2}{\left(\sum\binom{l_j}{2}\right)^2}\right]\left[1-\binom{r}{2}\frac{\sum\binom{k_i}{2}^2}{\left(\sum\binom{k_i}{2}\right)^2}\right],$$

and since $\sum\binom{k_i}{2} \geqq s,\ \sum\binom{l_j}{2} \geqq s,$
we obtain (applying the Beronulli-inequality)

$$A_r \geqq \frac{N!}{r!}\alpha^r\left(1-\frac{2r^2K^2L^2}{s}\right). \tag{21}$$

Combining (20) and (21) we get

$$\left|A_r-\frac{N!}{r!}\alpha^r\right| \leqq \frac{8r^2K^2L^2}{s}\cdot\frac{N!}{r!}\alpha^r. \tag{22}$$

Put

$$R = 2[\log s].$$

(22) implies

$$\left|\sum_{r=0}^{R}(-1)^r A_r - N!\sum_{r=0}^{R}(-1)^r\frac{\alpha^r}{r!}\right| \leqq \frac{8R^2K^2L^2}{s}N!e^{\alpha}.$$

Since R is even, we have

$$\sum_{r=0}^{R+1}(-1)^r A_r \leqq P_0 \leqq \sum_{r=0}^{R}(-1)^r A_r,$$

that implies for $N > N_0$

$$\left|P_0 - N!\sum_{r=0}^{R}(-1)^r\frac{\alpha^r}{r!}\right| \leqq \left(\frac{2\alpha^{R+1}}{(R+1)!}+\frac{8R^2K^2L^2e^{\alpha}}{s}\right)N!$$

and since α is bounded by $\frac{K^2L^2}{2}$

$$|P_0 - N!e^{-\alpha}| \leqq N!\left(\frac{3\alpha^{R+1}}{(R+1)!}+\frac{8R^2K^2L^2e^{\alpha}}{s}\right) \leqq N!32e^{K^2L^2}\frac{\log^2 s}{s} \leqq$$

$$\leqq N!\cdot 32e^{K^2L^2}\frac{\log N}{\sqrt{N}}$$

for

$$s \geqq \sqrt{N}\log N, \qquad N > N_0.$$

On the other hand, $A_0 = N!$ and $A_1 \leqq \alpha N!\left(1+\frac{8}{N}\right) \leqq 2\alpha N!$ for $N \geqq 8$, thus $N!(1-2\alpha) \leqq P_0 \leqq N!$ and hence

$$|P_0 - N!e^{-\alpha}| \leqq 2\alpha N!$$

for $\alpha \leqq \frac{1}{2}$. From the definition of α and s we have $\alpha \leqq K^2L^2\frac{s}{N}$, therefore for $s \leqq \sqrt{N}\log N$ and $N > N_0$ we have

$$|P_0 - N!e^{-\alpha}| \leqq 2K^2L^2\frac{\log N}{\sqrt{N}}\cdot N! \leqq N!32e^{K^2L^2}\frac{\log N}{\sqrt{N}}.$$

Thus we have proved for $N > N_0$ the relation

$$|P_0 - N!e^{-\alpha}| \leqq 32e^{K^2L^2}\frac{\log N}{\sqrt{N}}N!, \tag{23}$$

which proves (18), (since α is bounded).

5) The proof of (16).

First we prove that for $N > eKL$

$$M \leqq C\cdot\frac{N!}{\prod k_i!\prod l_j!}, \tag{24}$$

where the constant $C = C(K, L)$ does not depend on m, n, N, k_i, l_j. (We will find $C = e^{e^2K^3L^3}$).

As $P_0 \leqq N!$, we have

$$M_0 \leqq \frac{N!}{\prod k_i!\prod l_j!}. \tag{25}$$

First we estimate the number of matrices containing exactly t elements greater than 1, and then sum over t.

We can choose the t positions in $\binom{mn}{t}$ ways, and the elements in these positions in at most $(\max(K, L)-2)^t \leqq (KL)^t$ ways. We map the matrices with given elements in given t positions and 0—1's in other places to 0—1 matrices by substituting the t elements by 0, thus making a one-to-one mapping from these matrices to a subset of 0—1 matrices with given new row- and column-sums, where these new sums are obtained by subtracting our t elements from the corresponding row- and column-sums.

(E.g. the set of matrices with $a_{11} = a$, $a_{ij} = 0, 1$ for $(i, j) \neq (1, 1)$ with row sums $k_1, \ldots, k_n$, column-sums $l_1, \ldots, l_m$ is mapped to the matrices with $a_{11} = 0$, $a_{ij} = 0, 1$, with row-sums $k_1 - a, k_2, \ldots, k_n$ and column-sums $l_1 - a, l_2, \ldots, l_m$, thus the number of the above matrices is at most

$$M_0(k_1 - a, k_2, \ldots, k_n; l_1 - a, l_2, \ldots, l_m)).$$

Thus the number of these matrices is at most

$$\frac{N'!}{\prod k_i'!\prod l_j'!},$$

where k_i', l_j', N' are the new row-, column-, and total sums. Since the elements in these t positions were at least 2, we have $N' \leqq N-2t$, whence we get

$$\frac{N'!}{\prod k_i'! \prod l_j'!} \leqq \frac{N'!}{\prod k_i! \prod l_j!} K^{\Sigma(k_i-k_i')} L^{\Sigma(l_j-l_j')} =$$

$$= \frac{N!}{\prod k_i! \prod l_j!} \cdot \frac{N'!}{N!} (KL)^{N-N'} \leqq \frac{N!}{\prod k_i! \prod l_j!} \left(\frac{e}{N} KL\right)^{N-N'} \leqq$$

$$\leqq \frac{N!}{\prod k_i! \prod l_j!} \left(\frac{e}{N} KL\right)^{2t}.$$

Thus we have obtained the upper bound

$$M \leqq \sum_{t=0} \binom{mn}{t} (KL)^t \frac{N!}{\prod k_i! \prod l_j!} \left(\frac{e}{N} KL\right)^{2t} \leqq$$

$$\leqq \frac{N!}{\prod k_i! \prod l_j!} \sum_{t=0}^{\infty} \frac{(e^2 K^3 L^3)^t}{t!} = \frac{N!}{\prod k_i! \prod l_j!} e^{e^2 K^3 L^3} = C \frac{N!}{\prod k_i! \prod l_j!}.$$

We use the same type of argument for proving (16).

The number of matrices with row- and column-sums k_i, l_j in which there is an element greater than 2, is at most

$$\sum_{i,j} \sum_{a=3} M(k_1, \ldots, k_i - a, \ldots, k_n; l_1, \ldots, l_j - a, \ldots, l_m) \leqq C \sum_{i,j} \sum_{a=3} \frac{(N-a)!(KL)^a}{\prod k_i! \prod l_j!} \leqq$$

$$\leqq C \frac{N!}{\prod k_i! \prod l_j!} \sum_{a=3} \left(\frac{eKL}{N}\right)^a mn \leqq \frac{N!}{\prod k_i! \prod l_j!} \cdot \frac{2Ce^3 K^3 L^3}{N}$$

for $N > 2eKL$.

The relation

$$0 \leqq M - M_2 \leqq \frac{2Ce^3 K^3 L^3}{N} \cdot \frac{N!}{\prod k_i! \prod l_j!}$$

proves (16) if we mention that for $N > N_0$

$$M > c \frac{N!}{\prod k_i! \prod l_j!}$$

where $c = c(K, L)$ depends only on K, L. Indeed, for $N > N_0$

$$M \geqq M_0 \geqq \frac{1}{2} \frac{N!}{\prod k_i! \prod l_j!} e^{-\alpha} \geqq \frac{1}{2} e^{-K^2 L^2/2} \frac{N!}{\prod k_i! \prod l_j!}.$$

6) The proof of (17).

If we subtract the 2's, we obtain a mapping to some 0—1 matrices whose new row- and column-sums are obtained by subtracting 2's from the corresponding sums. For given t rows and columns there are $t!$ possible positions for the t 2's. Thus

$$M_{2,t}(k_1, \ldots, k_n; l_1, \ldots, l_m) \leqq t! \sum M_0(k_1', \ldots, k_n'; l_1', \ldots, l_m') = t! \sum,$$

where the sum is extended over such $m+n$ tuples

$$(k'_1, \dots, k'_n; l'_1, \dots, l'_m)$$

in which for t of the indices i

$$k'_i = k_i - 2$$

and for the remaining $n-t$ indices i

$$k'_i = k_i,$$

and the same holds for the numbers l'_j. Conversely, if we add 2 to t elements (not two in a row or column) of a 0—1 matrix, we obtain matrices in $\mathcal{M}_{2,t}$ or matrices in $\mathcal{M} - \mathcal{M}_2$, therefore, for the above sum $\sum$

$$t! \sum \leqq M_{2,t} + (M - M_2).$$

Thus, the inequality

$$t! \sum - (M - M_2) \leqq M_{2,t} \leqq t! \sum \tag{26}$$

shows that for proving (17) we must show that

$$\sum \sim M \frac{(2\alpha)^t}{(t!)^2} e^{-2\alpha} \tag{27}$$

(since $M - M_2 = o(M)$).

Putting

$$\alpha' = 2 \frac{\sum \binom{k'_i}{2} \sum \binom{l'_j}{2}}{\sum k'_i \sum l'_j}, \qquad N' = \sum k'_i = N - 2t$$

we have

$$\sum \sim \sum_{k'_i, l'_j} \frac{N'!}{\prod k'_i! \prod l'_j} e^{-\alpha'}$$

the sum being extended over the above conditions, (We can apply the assymptotic formula for M_0 term by term, since it did not depend on the particular choice of k_i, l_j, s, only on K, L) whence

$$\sum \sim \frac{N!}{\prod k_i! \prod l_j!} e^{-\alpha} \sum_{k'_i, l'_j} \frac{N'!}{N!} \prod \frac{k'_i!}{k_i!} \prod \frac{l'_j!}{l_j!} e^{\alpha - \alpha'} \sim M_0 \frac{1}{N^{2t}} \sum_{k'_i, l'_j} \prod \frac{k'_i!}{k_i!} \prod \frac{l'_j!}{l_j!} e^{\alpha - \alpha'}.$$

It is easy to see that $\alpha - \alpha' = o(1)$ (i.e. $|\alpha - \alpha'| < \varepsilon$ for $N > N_0(\varepsilon)$, whatever are the numbers m, n, k_i, l_j), therefore

$$\sum \sim M_0 \frac{1}{N^{2t}} \left(\sum_{i_1 < \dots < i_t} \binom{k_{i_1}}{2} \cdots \binom{k_{i_t}}{2} 2^t \right) \sum_{j_1 < \dots < j_t} \binom{l_{j_1}}{2} \cdots \binom{l_{j_t}}{2} 2^t \sim M_0 \frac{(2\alpha)^t}{(t!)^2}.$$

It remains to show that $M \sim M_0 e^{2\alpha}$. For fixed T

$$\sum_{t=0}^{T} M_{2,t} \sim M_0 \sum_{t=0}^{T} \frac{(2\alpha)^t}{t!}$$

and by (26)

$$\sum_{t=T+1}^{n} M_{2,t} < \binom{m}{t}\binom{n}{t} \sum_{t=T+1}^{n} t! \frac{(N-2t)!}{\prod k_i! \prod l_j!} K^{2t} L^{2t} \leqq$$

$$\leqq \frac{N!}{\prod k_i! \prod l_j!} \sum_{t=T+1}^{\infty} \frac{1}{t!} \left(\frac{mne^2 K^2 L^2}{N^2} \right)^t \leqq \frac{N!}{\prod k_i! \prod l_j!} \sum_{t=T+1}^{\infty} \frac{(e^2 K^2 L^2)^t}{t!} < \varepsilon$$

for $T > T_0(\varepsilon)$.

Therefore

$$M_2 \sim M_0 e^{2\alpha}$$

whence

$$M \sim M_0 e^{2\alpha}.$$

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(Received May 27, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 355—362

A NOTE ON THE ROBBINS—MONRO METHOD

by

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Let ξ and η be two random variables with conditional distribution function

$$H(y|x) = \mathsf{P}(\eta < y | \xi = x)$$

further denote the regression function by

$$y = M(x) = \int_{-\infty}^{+\infty} y\, dH(y|x).$$

For the sake of simplicity we assume that $M(x)$ is strictly increasing. ROBBINS and MONRO, in their paper [1] gave a statistical method to find the root of the equation

(1) $$\alpha = M(x)$$

provided that the root exists. Their method is the following: they choose an arbitrary number x_1 and define a sequence of random variables recursively by

(2) $$x_{n+1} = x_n + a_n(\alpha - y_n)$$

where y_n is a random variable distributed according to $H(y|x_n)$ and $\{a_n\}$ is a sequence of positive numbers[1].

Several authors have proved (under different conditions) that x_n is going to the root of the equation (1). (See for example [2], [3], [4]). All of these authors assumed that

(i) $$\sum_{n=1}^{\infty} a_n = \infty$$

(ii) $$\sum_{n=1}^{\infty} a_n^2 < \infty.$$

It is quite clear that condition (i) is really necessary. The aim of the present paper is to investigate condition (ii). Our conditions on $H(y|x)$ (and on $M(x)$) will be much stronger than those of the mentioned authors, but condition (ii) will be replaced by a much weaker condition, in fact we give in some sense the best possible condition.

[1] In fact it will be assumed that

$$\mathsf{E}(y_n | x_1, x_2, \ldots, x_n) = M(x_n)$$

with probability 1.

THEOREM. *Let η be a bounded random variable, i.e. we assume that there exist $-\infty<a<b<+\infty$ such that*

$$H(a|x)=0, \quad H(b|x)=1$$

for every x, and let

$$M(x)=\int_a^b y\,dH(y|x)$$

be a strictly increasing function. Assume that the equation $\alpha=M(x)$ has a root θ (i.e. $\alpha=M(\theta)$). Further let $a_1, a_2, \ldots$ be a nonincreasing sequence of positive numbers obeying the following two conditions:

Condition 1.

$$\sum_{n=1}^{\infty} a_n=\infty,$$

Condition 2. There exists a sequence $\{\omega(n)\}$ of positive number for which

$$\lim_{n\to\infty}\omega(n)=\infty \quad \text{and} \quad a_n \leqq \frac{1}{\omega(n)\log n}.$$

Then the sequence $x_1, x_2, \ldots$ (defined by (2)) is tending to θ with probability 1.

Remark. Without the loss of generality we can assume that $\alpha=\theta=0$. The proof will be given in this case.

Before the proof of this theorem some lemmas are given.

LEMMA 1. *Let $\xi_0=0$, $\xi_1, \xi_2, \ldots$ be a sequence of uniformly bounded random variables for which*

$$\varrho_{k+1}=\mathsf{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \ldots, \xi_0) \leqq -m<\infty \qquad (k=0, 1, 2, \ldots)$$

(with probability 1) further let $a_1, a_2, \ldots$ be a bounded sequence of positive numbers for which

$$\sum_{n=1}^{\infty} a_n=\infty.$$

Then

$$\sum_{n=1}^{\infty}\frac{a_n(\xi_n-\varrho_n)}{\sum_{k=1}^{n} a_k} \tag{3}$$

is convergent with probability 1, further

$$\mathsf{P}\left\{\lim_{n\to\infty}\frac{\sum_{k=1}^{n} a_k(\xi_k-\varrho_k)}{\sum_{k=1}^{n} a_k}=0\right\}=1, \tag{4}$$

$$\mathsf{P}\left\{\overline{\lim_{n\to\infty}}\frac{\sum_{k=1}^{n} a_k(\xi_k+m)}{\sum_{k=1}^{n} a_k}\leqq 0\right\}=1 \tag{4$^+$}$$

and

$$\mathsf{P}\left\{\lim_{n\to\infty}\sum_{k=0}^{n} a_k\xi_k = -\infty\right\} = 1. \tag{5}$$

PROOF. Our statement (3) is a straight consequence of the convergence theorem of Loéve (see [5], p 387) and the fact that

$$\sum_{n=1}^{\infty}\frac{a_n^2}{\left(\sum_{k=1}^{n} a_k\right)^2} < \infty.$$

(4) follows from (3) and the well-known Kronecker lemma (see e.g. [6], p. 35,) (4^+) immediately follows from (4) and (5) is a straight consequence of (4^+).

LEMMA 2. *Let $\xi_0=0, \xi_1, \xi_2, \dots$ be a sequence of uniformly bounded random variables with bound K (i.e. $|\xi_i|\leqq K$ $(i=1, 2, \dots)$) such that*

$$\mathsf{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \dots, \xi_0) \leqq -m < 0 \qquad (k=0, 1, 2, \dots)$$

with probability 1. Further let

$$\mu_{k+1}(t) = \mathsf{E}(e^{\xi_{k+1}t}|\xi_k, \xi_{k-1}, \dots, \xi_0) \qquad (k=0, 1, 2, \dots).$$

Then there exists a $t_0>0$ such that

$$\mathsf{P}\{\mu_k(t) \leqq 1 \ \ (k=0, 1, 2, \dots;\ 0\leqq t\leqq t_0)\} = 1.$$

PROOF. Clearly we have (with probability 1)

$$\mu_k(t) = 1+t\mathsf{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \dots, \xi_0)+\frac{t^2}{2}\mathsf{E}(\xi_{k+1}^2|\xi_k, \xi_{k-1}, \dots, \xi_0)+\dots \leqq$$

$$\leqq 1-mt+\frac{t^2}{2}K^2+\frac{t^3}{3!}K^3+\dots = 1-mt+t^2K^2\left(\frac{1}{2}+\frac{1}{3!}tK+\dots\right) \leqq$$

$$\leqq 1-mt+t^2K^2e^{Kt}$$

what proves our lemma.

LEMMA 3. *Let $\xi_0=0, \xi_1, \xi_2, \dots$ be a sequence of random variables and let*

$$\eta_n = \frac{e^{t\sum_{k=1}^{n}\xi_k}}{\prod_{k=0}^{n-1}\mathsf{E}(e^{\xi_{k+1}t}|\xi_k, \xi_{k-1}, \dots, \xi_0)}.$$

Then $\eta_1, \eta_2, \dots$ is a martingale with[2]

$$\mathsf{E}(\eta_n) = 1.$$

The PROOF is trivial.

[2] The existence of the occurring expectations is assumed.

LEMMA 4. *Let $\xi_0=0, \xi_1, \xi_2, \dots$ be a sequence of uniformly bounded random variables with bound K such that*

$$\mathsf{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \dots, \xi_0) \leqq -m < 0 \qquad (k=0, 1, 2, \dots)$$

(with probability 1), further let $a_1, a_2, \dots$ be a sequence of nonincreasing positive numbers. Then

$$\mathsf{P}\left\{\sup_{n\geqq 1} \sum_{k=1}^{n} a_k \xi_k \geqq \varepsilon\right\} \leqq e^{-\frac{\varepsilon}{t_0 a_1}}$$

where t_0 is defined in Lemma 2.

PROOF. By Lemma 2 if $ta_1 \leqq t_0$ then

$$\frac{\exp\left(t \sum_{k=1}^{n} a_k \xi_k\right)}{\prod_{k=0}^{n-1} \mathsf{E}(e^{ta_{k+1}\xi_{k+1}}|\xi_k, \xi_{k-1}, \dots, \xi_0)} \geqq \exp\left(t \sum_{k=1}^{n} a_k \xi_k\right).$$

Hence by the martingale-inequality (for any N) we have

$$\mathsf{P}\left(\sup_{1\leqq m\leqq N} \sum_{k=1}^{m} a_k \xi_k \geqq \varepsilon\right) = \mathsf{P}\left(\sup_{1\leqq m\leqq N} \exp\left(\frac{t_0}{a_1} \sum_{k=1}^{m} a_k \xi_k\right) \geqq e^{\varepsilon\frac{t_0}{a_1}}\right) \leqq$$

$$\leqq \mathsf{P}\left(\sup_{1\leqq m\leqq N} \frac{\exp\left(\frac{t_0}{a_1} \sum_{k=1}^{m} a_k \xi_k\right)}{\prod_{k=0}^{m-1} \mathsf{E}(e^{ta_{k+1}\xi_{k+1}}|\xi_k, \xi_{k-1}, \dots, \xi_0)} \geqq e^{\varepsilon\frac{t_0}{a_1}}\right) \leqq e^{-\varepsilon\frac{t_0}{a_1}}$$

what proves our Lemma 4.

LEMMA 5. *Define the random variables $\xi_0, \xi_1=\xi_1(N, \varepsilon), \xi_2=\xi_2(N, \varepsilon), \dots$ as follows:*

$$\xi_0=0$$

$$\xi_k = \begin{cases} -y_{N+k-1} & \text{if } x_{N+k-1} > \varepsilon \\ -M(\varepsilon) & \text{if } x_{N+k-1} \leqq \varepsilon \end{cases} \quad k=1, 2, \dots$$

Then

$$\mathsf{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \dots, \xi_0) \leqq -M(\varepsilon) \qquad (k=0, 1, 2, \dots) \tag{6}$$

with probability 1.

PROOF. Let

$$\mathscr{F} = B(\xi_0, \xi_1, \xi_2, \dots, \xi_k)$$

be the smallest σ-algebra with respect to which the random variables $\xi_0, \xi_1, \dots, \xi_k$ are measurable. We will prove that if $B \in \mathscr{F}$ then

$$\int_B \mathsf{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \dots, \xi_0)\, d\mathsf{P} \leqq -M(\varepsilon)\mathsf{P}(B) \tag{7}$$

what clearly implies (6). We have

$$\int_B \mathsf{E}(\xi_{k+1}|\xi_k, \xi_{k-1}, \dots, \xi_0) = \int_B \xi_{k+1} = \int_{B\cap\{x_{N+k}>\varepsilon\}} \xi_{k+1} + \int_{B\cap\{x_{N+k}\leqq\varepsilon\}} \xi_{k+1} =$$

$$= -\int_{B\cap\{x_{N+k}>\varepsilon\}} y_{N+k} - \int_{B\cap\{x_{N+k}\leqq\varepsilon\}} M(\varepsilon) =$$

$$= -M(\varepsilon)\mathsf{P}(B\cap\{x_{N+k}\leqq\varepsilon\}) - \int_{B\cap\{x_{N+k}>\varepsilon\}} \mathsf{E}(y_{N+k}|x_1, x_2, \dots, x_{N+k}) =$$

$$= -M(\varepsilon)\mathsf{P}(B\cap\{x_{N+k}\leqq\varepsilon\}) - \int_{B\cap\{x_{N+k}>\varepsilon\}} M(x_{N+k}) \leqq -M(\varepsilon)\mathsf{P}(B).$$

LEMMA 6. *Under the conditions of our Theorem we have*

$$\mathsf{P}(\varliminf_{n\to\infty} x_n \leqq \varepsilon) = 1 \tag{8}$$

for every $\varepsilon > 0$.

PROOF. Making use of the notations of Lemma 5 we have

$$\mathsf{P}\{\inf_{n\geqq N} x_n > \varepsilon\} \leqq \mathsf{P}\{x_N > \varepsilon, x_{N+1} > \varepsilon, \dots, x_{N+n} > \varepsilon\} =$$

$$= \mathsf{P}\left\{x_N - \sum_{k=N}^{N+n-1} a_k y_k > \varepsilon, x_N > \varepsilon, x_{N+1} > \varepsilon, \dots, x_{N+n-1} > \varepsilon\right\} \leqq$$

$$\leqq \mathsf{P}\left\{x_N + \sum_{k=1}^{n} a_{N+k-1}\xi_k > \varepsilon\right\}$$

Hence by (5) of Lemma 1 we have (8).

Now we can turn to the

PROOF of the Theorem. Introduce the following notations.

$$A_n = \bigcup_{m=n+1}^{\infty} \{\omega : x_n \leqq \varepsilon/2, x_{n+1} \geqq \varepsilon/2, x_{n+2} \geqq \varepsilon/2, \dots, x_{m-1} \geqq \varepsilon/2, x_m \geqq \varepsilon\}$$

$$A = \{\omega : \varlimsup_{n\to\infty} x_n \geqq \varepsilon\}$$

Because of (8)

$$\mathsf{P}(A) = \mathsf{P}(\varlimsup_{n\to\infty} A_n).$$

Making use of the notations of Lemma 5 since $x_n \leqq \varepsilon/2$ implies $x_{n+1} \leqq 3/4\varepsilon$ for n large enough we have:

$$\mathsf{P}(A_n) = \mathsf{P}\left\{\bigcup_{m=n+1}^{\infty}\left[x_n \leqq \varepsilon/2, x_{n+1} \geqq \varepsilon/2, \dots, x_{m-1} \geqq \varepsilon/2, x_{n+1} - \sum_{k=n+1}^{m-1} a_k y_k \geqq \varepsilon\right]\right\} \leqq$$

$$\leqq \mathsf{P}\left\{\bigcup_{m=n+1}^{\infty}\left[x_n \leqq \varepsilon/2, x_{n+1} \geqq \varepsilon/2, \dots, x_{m-1} \geqq \varepsilon/2, \sum_{k=1}^{m-n-1} a_{n+k}\xi_k(n+1, \varepsilon/2) \geqq \varepsilon/4\right]\right\} \leqq$$

$$\leqq \mathsf{P}\left(\sup_{M\geqq 1} \sum_{k=1}^{M} a_{n+k}\xi_k \geqq \varepsilon/4\right) \leqq e^{-\frac{C}{a_{n+1}}}$$

if n is big enough. The Borel—Cantelli lemma implies that among the events A_n only finitely many can occur (with probability 1) i.e. $\mathsf{P}(A)=0$ which proves our Theorem.

LEMMA[3] 7. *Let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables with distribution*

$$\mathsf{P}(\xi_i=0)=\mathsf{P}(\xi_i=1)=1/2 \qquad (i=1,2,\dots)$$

further let

$$A_n=\{\omega: \sup_{0\le i\le n-l(n)} (\xi_{i+1}+\xi_{i+2}+\dots+\xi_{i+l(n)})=l(n)\}$$

where $l(n)=[\log n]$ and log *is meant with base* 2.

Then

$$\mathsf{P}\left\{\prod_{n=1}^{\infty}\sum_{k=n}^{\infty} A_k\right\}=1$$

i.e. among the events $A_1, A_2, \dots$ infinitely many will occur with probability 1.

PROOF. Introduce the following notations

$$n_k=[2k\log k]$$

$$B_k=\{\omega: \sup_{n_k\le i\le n_{k+1}-l(n_{k+1})} [\xi_{i+1}+\xi_{i+2}+\dots+\xi_{i+l(n_{k+1})}]=l(n_{k+1})\}.$$

To see that the definition of B_k is not meaningless, we have to show that

$$n_{k+1}-l(n_{k+1})>n_k$$

but it clearly holds if k is big enough.

Investigate now the probability of the event

$$\bar{A}_{n_k}\bar{A}_{n_{k+1}}\dots\bar{A}_{n_j} \qquad (j>k).$$

Obviously we have

$$\mathsf{P}(\bar{A}_{n_k}\bar{A}_{n_{k+1}}\dots\bar{A}_{n_j})=\mathsf{P}(\bar{A}_{n_k}\bar{A}_{n_{k+1}}\dots\bar{A}_{n_{j-1}})[1-\mathsf{P}(A_{n_j}|\bar{A}_{n_{j-1}}\dots\bar{A}_{n_k})].$$

Since $B_k\subset A_{n_{k+1}}$ and hence

$$\mathsf{P}(A_{n_j}|\bar{A}_{n_{j-1}}\dots\bar{A}_{n_k})\ge \mathsf{P}(B_{j-1}|\bar{A}_{n_{j-1}}\dots\bar{A}_{n_k})=\mathsf{P}(B_{j-1})\ge\frac{1}{2^{l(n_j)}}\ge\frac{1}{n_j},$$

we have

$$\mathsf{P}(\bar{A}_{n_k}\bar{A}_{n_{k+1}}\dots\bar{A}_{n_j})\le \mathsf{P}(\bar{A}_{n_k}\dots\bar{A}_{n_{j-1}})\left(1-\frac{1}{n_j}\right)$$

and by induction

$$\mathsf{P}(\bar{A}_{n_k}\dots\bar{A}_{n_j})\le\left(1-\frac{1}{n_k}\right)\dots\left(1-\frac{1}{n_j}\right). \tag{9}$$

[3] This lemma is essentially due to ERDŐS and RÉNYI ([7]) however the present formulation is a little bit different from theirs.

Now our proof is going straight:

$$\mathsf{P}\left(\prod_{n=1}^{\infty}\sum_{k=n}^{\infty}A_k\right)=1-\mathsf{P}\left(\sum_{n=1}^{\infty}\prod_{k=n}^{\infty}\bar{A}_k\right)\geqq 1-\sum_{n=1}^{\infty}\mathsf{P}\left(\prod_{k=n}^{\infty}\bar{A}_k\right)\geqq 1-\sum_{n=1}^{\infty}\mathsf{P}(\prod_{n_l\geqq n}\bar{A}_{n_l})$$

but by (9)

$$\mathsf{P}(\prod_{n_l\geqq n}\bar{A}_{n_l})=0,$$

hence our lemma is proved.

Now by an example we can show that our Theorem is the best possible.

Let the distribution of ξ be a uniform distribution in $(-1, +1)$ i.e.

$$\mathsf{P}(\xi<x)=\begin{cases}0 & \text{if } \quad x\leqq -1\\ \dfrac{x+1}{2} & \text{if } -1\leqq x\leqq +1\\ 1 & \text{if } \quad x\geqq +1\end{cases}$$

and let

$$\mathsf{P}(\eta=\xi+1)=\mathsf{P}(\eta=\xi-1)=1/2.$$

Then clearly

$$M(x)=\mathsf{E}(\eta|\xi=x)=x \qquad (-1\leqq x\leqq +1)$$

Now construct the sequence $x_1, x_2, \ldots$ where x_1 is an arbitrary point of $(-1, +1)$,

$$x_{n+1}=x_n-\frac{C}{\log n}y_n$$

and y_n is distributed according to

$$\mathsf{P}(y_n=x_n+1)=\mathsf{P}(y_n=x_n-1)=1/2.$$

We will show that in this case

$$\mathsf{P}(\lim_{n\to\infty}x_n=0)=0. \tag{10}$$

Proof. Let

$$A_n(\varepsilon)=\bigcup_{k=n+1}^{2n}\{|x_k|\geqq\varepsilon\} \qquad \left(0<\varepsilon<\frac{C}{C+2}\right)$$

By Lemma 7 the sequence $y_{n+1}, y_{n+2}, \ldots, y_{2n}$ has a subsequence (with probability 1 for infinitely many n)

$$y_k, y_{k+1}, \ldots, y_{k+l(n)} \qquad (n\leqq k<k+l(n)\leqq 2n)$$

such that

$$y_k=x_k+1,\ y_{k+1}=x_{k+1}+1, \ldots, y_{k+l(n)}=x_{k+l(n)}+1$$

what shows that among the events A_n infinitely many will be occured i.e. (10) is proved.

The author is indebted to E. Csáki for his valuable remarks, especially the fundamental idea of the proof of Lemma 4 is due to him.

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(Received June 10, 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 363—374

ON A COMBINATORIAL PROBLEM I.

by

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1. Introduction

The aim of this paper is to investigate an instance of a general problem stated by P. ERDŐS in [1]. We only state the special case we deal with.

Let H be a set of $n+m$ elements and let $\mathcal{N}=\{N_1, \dots, N_k\}$ be a system of n-element subsets of H. Assume that

(i) $\bigcap_{i=1}^{k} N_i = \emptyset$ and

(ii) for each $M \subset H$, $|M| = n+1$ there is a triplet $X \subset M$, $|X|=3$ such that $X \not\subset N_i$ for $i=1, \dots, k$, i.e. no N_i covers X.

Given n and k these conditions obviously impose a condition on m. We will give a lower estimation of the function

$m(n,k) = \min \{m$: there are H and $\mathcal{N}$ satisfying the above conditions$\}$.

Note that, replacing in (ii) $|X|=3$ by $|X|=r$ for $r \geqq 2$ we get a more general problem and we are to return to it in a forthcoming paper. The case $r=2$ has been investigated by FOLKMAN and HAJNAL ([2], [3]) and following ERDŐS's paper by L. SURÁNYI, E. MILNER and N. SAUER, and the first author ([4], [5], [6]).

Most of the first results concern the case $r=2$, and the results for $r>2$ must be of different character since these problems involve hypergraphs instead of ordinary graphs.

We are going to prove, that

$$m(n,k) \geqq \frac{1}{2\sqrt{2}}\sqrt{n}$$

and as far as we know this is the first sharp result for the cases $r>2$.

Note that MILNER and SAUER proved independently the following weaker result:

$$m(n,k) \geqq \log_3 n.$$

Throughout this paper the cardinality of the set X will be denoted by $|X|$. If $\mathcal{U}=\{N_1, \dots, N_k\}$ is a system of subsets of H and $H' \subset H$, we briefly say that H' is covered t-times by $\mathcal{U}$ if $H' \subset N_i$ holds for *at least* t different suffices i. On the other hand the expressions "not covered" or "single covered" mean that H' is contained in no or exactly one member of $\mathcal{U}$ respectively.

We remark that our result is best possible in the following sense:

(iii) $m(n, n-1) \leqq \sqrt{2}\sqrt{n}$ for infinitely many n.

2. Proof of (iii)

Let $m \geqq 3$ be an integer and let

$$n-1 = \binom{m+1}{2} = \frac{1}{2}(m^2+m). \tag{1}$$

Consider now a set $H = H_1 \cup H_2$, $|H| = m+n$, $|H_1| = n-1$, $|H_2| = m+1$. According to (1) we may denote the elements of H_1 by

$$x_1, \ldots, x_{n-1}$$

and the (unordered) pairs of H_2 by

$$p_1, \ldots, p_{n-1}.$$

Put

$$N_i = (H_1 - \{x_i\}) \cup p_i \quad \text{for} \quad i=1, \ldots, n-1,$$

and

$$\mathcal{N} = \{N_1, \ldots, N_{n-1}\}.$$

For every $x \in H$ there is $N \in \mathcal{N}$ with $x \notin N$. In fact, if $x \in H_1$ that is $x = x_j$ then $x \notin N_j$; if $x \in H_2$ then we pick a pair p_l with $x \notin p_l$ and then $x \notin N_l$. This gives the property (i). To verify (ii) let $M \subset H$ be an arbitrary $(n+1)$-tuple (generally, k-tuple means simply a k-element set). $|H_2 \cap M| = t \geqq 2$, and this intersection contains $\binom{t}{2}$ pairs. Using

$$|H_1 - M| = |H_1| - |H_1 \cap M| = n-1-(n+1-t) = t-2$$

and

$$\binom{t}{2} = \frac{t}{2}(t-1) \geqq t-1 > t-2$$

we find that the number of pairs contained in $H_2 \cap M$ exceeds the number of elements in $H_1 - M$. This implies the existence of a pair p_j in $H_2 \cap M$ such that $x_j \in H_1 \cap M$. That is the triplet $\{x_j\} \cup p_j$ in M is not covered by $\mathcal{N}$. Our example provides

$$m(n, n-1) \leqq \sqrt{2}\sqrt{n} - O(n^{1/4})$$

and

$$m(n, k) \geqq \sqrt{2}\sqrt{n} - O(n^{1/4})$$

seems to be a reasonable conjecture for every k. Moreover we believe that, the construction described above gives the extremal configuration.

3. Lemmas

First we remark, that

a) $m(n, 2) = n$;

b) $m(n, 3) \geqq \frac{n}{2}$, more precisely $m(n, 3) = \frac{n}{2}$ if n is even and $m(n, 3) = \left[\frac{n}{2}\right] + 1$ if n is odd number. Assertion a) does not require proof and b) is to be seen as follows. Let $H = n+m$ and denote $\{N_1, N_2, N_3\}$ a system satisfying (i) and (ii). According to (i)

$$(H-N_1)\cup(H-N_2)\cup(H-N_3) = H,$$

therefore $3m \geqq n+m$, i.e. $m \geqq \frac{n}{2}$. To establish the equality let H_1, H_2, H_3 be pairwise disjoint sets each containing $\frac{n}{2}$ elements (we prove only the case when n is even number) and define

$$N_i = H_j \cup H_k, \qquad 1 \leqq i, j, k \leqq 3, \ i \neq j \neq k \neq i,$$

$$\mathcal{N} = \{N_1, N_2, N_3\}.$$

Property (i) is trivial and to check (ii) it is enough to observe that every $(n+1)$-tuple M meets each N_i and the triplet $\{x_1, x_2, x_3\}$ is not covered by $\mathcal{N}$ if $x_i \in N_i \cap M$.

In what follows we will assume $k \geqq 4$.

We need some simple standard lemmas.

LEMMA 1. *Let $\mathcal{U} = \{U_1, U_2, \ldots, U_\nu\}$ be an arbitrary system of n-tuples in H, $|H| = n+m$, and let α_s denote the number of points of H covered at most s-times by $\mathcal{U}$. Then*

$$\alpha_s \leqq m \frac{\nu}{\nu - s}. \tag{2}$$

PROOF. Let c_i denote the covering number of $x_i \in H$ with respect to $\mathcal{U}$. Counting the points with multiplicity

$$\sum_{i=1}^{n+m} c_i = \nu n.$$

Hence

$$s\alpha_s + \nu(n+m-\alpha_s) \geqq \nu n,$$

and this gives (2). In the special case $s = \frac{\nu+1}{2}$

$$\alpha_{(\nu+1)/2} \leqq m \frac{2\nu}{\nu-1} \tag{3}$$

holds.

LEMMA 2. *Let $\mathcal{U}$ denote the same as above, and suppose that both points $x, y \in H$ are covered $\frac{\nu+2}{2}$-times. Then the pair $p = \{x, y\}$ is double covered.*

PROOF. Denote by n_x the number of n-tuples containing x but omitting y. n_y is defined similarly and let n_{xy} denote the number of n-tuples covering p. According to our assumption

$$n_x + n_{xy} \geqq \frac{\nu+2}{2},$$

$$n_y + n_{xy} \geqq \frac{\nu+2}{2},$$

thus

$$n_x + n_y + 2n_{xy} \geqq v + 2.$$

On the other hand

$$n_x + n_y + n_{xy} \leqq v,$$

hence we obtain

$$n_{xy} \geqq 2.$$

LEMMA 3. *If a system of n-tuples* $\mathcal{N} = \{N_1, \ldots, N_k\}$, $N_i \subset H$, $|H| = n+m$ *satisfies condition* (i) (see in the introduction), *then*

(3a) $$k \geqq \frac{n+m}{m}.$$

PROOF. (i) implies

$$\bigcup_{i=1}^{k} (H - N_i) = H,$$

hence

$$km \geqq m + n,$$

that is

$$k \geqq \frac{m+n}{m}.$$

LEMMA 4. *Suppose that the system of n-tuples* $\mathcal{N} = \{N_1, \ldots, N_k\}$, $N_i \subset H$, $|H| = n+m$ *satisfies* (i) *and* (ii) *and moreover it is minimal with respect to* (i), *i.e.:*

(iv) $$\bigcap_{i \neq j} N_i \neq \emptyset, \qquad j = 1, \ldots, k.$$

Then each N_i *contains a pair single covered by* $\mathcal{N}$.

PROOF. By (iv) choose $x_j \in \bigcap_{i \neq j} N_i$ $j = 1, \ldots, k$, and let

$$A = \{x_1, \ldots, x_k\}.$$

It is important to observe that by (i)

(4) $$x_j \notin N_j, \qquad j = 1, \ldots, k,$$

but

(5) $$x_j \in N_i \quad \text{if} \quad i \neq j.$$

This implies

$$x_i \neq x_j \qquad (i \neq j).$$

In fact,

$$x_j \notin N_j \quad \text{and} \quad x_i \in N_j$$

hold simultaneously. In particular, we have

(6) $$|A| = k,$$

(7) $$N_i \cap A = A - \{x_i\}, \qquad i = 1, \ldots, k,$$

(8) $$N_i \cup A = N_i \cup \{x_i\}, \qquad i = 1, \ldots, k.$$

Now, if all the pairs, say in N_1 were double covered, then the $(n+1)$-tuple $M = N_1 \cup A$ would contradict (ii). In fact, if a triplet $\tau \subset M$ does not contain x_1, then using (8) $\tau \subset N_1$; in the other case $\tau = \{x_1, y, z\}$, and the pair $\{y, z\}$ is double covered, that is $\{y, z\} \subset N_i$, $i \neq 1$. By (5) $x_1 \in N_i$, thus we have $\tau \subset N_i$. This completes the proof.

4. Estimation of $m(n, k)$

For the convenience of the reader first we prove the weaker inequality $m(n, k) \geqq \frac{1}{2} n^{1/3}$ to elucidate the idea of our proof without giving the technicalities needed for the improvement.

THEOREM 1. *If* $|H| = n+m$ *and a system* $\mathcal{N}$ *of n-tuples* $N_1, \dots, N_k$ *in H satisfies* (i) *and* (ii), *then*

$$m(n, k) \geqq \frac{1}{2} n^{1/3}.$$

PROOF. Without any loss of generality we may also suppose that $\mathcal{N}$ is minimal with respect to (i), i.e. (iv) is also fulfilled. We can assume

$$\bigcup_{i=1}^{k} N_i = H$$

as well, and recall that, to avoid the trivialities $k \geqq 4$ is always assumed. Our trick is to find an upper estimate for the number of n-tuples containing single covered pairs. By lemma 4 *all* the n-tuples contain a single covered pair, therefore lemma 3 yields a lower estimation for m.

Let $y_1, \dots, y_\sigma$ denote the single covered elements of H (it is assumed that every element is contained in at least one n-tuple). Of course, all the pairs of type $\{y_i, z\}$ are single covered, too. Let $p = \{u, z\}$ be an arbitrary single covered pair. By lemma 2 we may suppose that u is covered at most $\frac{k+1}{2}$-times by $\mathcal{N}$. All these imply that any single covered pair can be represented either as

$$p = \{y_i, z\}$$

or as

$$p = \{u, z\}$$

where u is double covered but at most $\frac{k+1}{2}$-times covered by $\mathcal{N}$. Denoting by δ the number of elements u satisfying these conditions we get from lemma 1 the inequalities

$$\delta \leqq m \frac{2k}{k-1} \tag{9}$$

and

$$\sigma \leqq m \frac{k}{k-1}. \tag{10}$$

Consider now an element u appearing in a representation of second type. The number of single covered pairs represented by the aid of the same fixed u is at most $2m$. In fact, u is double covered, say $u \in N' \cap N''$. If the pair $p=\{u, z\}$ is single covered, then the relation $z \notin N' \cap N''$ holds. But the number of possibilities to choose such z is at most

$$|H-(N' \cap N'')| = |(H-N') \cup (H-N'')| \leqq 2m.$$

Hence the total number of pairs having a representation of second type is at most $2m\delta$. The number of n-tuples covering single covered pairs is at most as large as the number of such pairs itself. That is, $\sigma + 2m\delta$ is an upper estimation for the number of n-tuples containing single covered pairs. Finally (9), (10) and lemma 4 yield

$$k \leqq m\frac{k}{k-1} + 2m \cdot m\frac{2k}{k-1}$$

and then by lemma 3

$$\frac{m+n}{m} \leqq m\frac{k}{k-1} + 2m^2\frac{2k}{k-1},$$

thus using $k \geqq 4$

$$\frac{n}{m} \leqq 3m^2\frac{8}{3}$$

i. e.

$$m \geqq \frac{1}{2}n^{1/3}.$$

5. Proof of the main result

In the proof of theorem 1 the set A (defined in lemma 4) played a role of fundamental importance. The trick to get the sharper estimation is to iterate the previous method repeatedly. Recall that

$$A = \{x_1, \ldots, x_k\}$$

where

$$x_i \in \bigcap_{j \neq i} N_j, \qquad i=1, \ldots, k.$$

Now put

$$\mathcal{N}^{(0)} = \mathcal{N}$$

$$N_i^{(0)} = N_i, \qquad i=1, \ldots, k,$$

$$k_0 = k,$$

$$A_0 = \{x_1^{(0)}, \ldots, x_{k_0}^{(0)}\} = A,$$

$$\overline{N}_i^{(0)} = N_i^{(0)} - A_0, \qquad i=1, \ldots, k.$$

Suppose that for some integer j

$$k_j \quad (k_j \geqq 4), \qquad \mathcal{N}^{(j)} = \{N_1^{(j)}, \ldots, N_{k_j}^{(j)}\},$$

$$A_j, \quad \bar{N}_i^{(j)} = N_i^{(j)} - \bigcup_{\nu=0}^{j} A_\nu, \qquad i=1, \ldots, k_j$$

all have already been defined, and the system

$$R^{(j)} = \{\bar{N}_0^{(j)}, \ldots, \bar{N}_{k_j}^{(j)}\}$$

satisfies (i). Consider subsystems in $R^{(j)}$ which are minimal systems with respect to (i). If no such subsystem contains more than three sets, we stop. In the other case let

$$\mathcal{N}^{(j+1)} = \{N_1^{(j+1)}, \ldots, N_{k_{j+1}}^{(j+1)}\} \subset \mathcal{N}^{(j)}$$

be chosen so that $k_{j+1} \geqq 4$ and the corresponding remainders

$$\left\{N_i^{(j+1)} - \bigcup_{\nu=0}^{j} A_\nu : i=1, \ldots, k_{j+1}\right\}$$

form a subsystem in $R^{(j)}$ minimal with respect to (i). By the minimality there exist elements

$$x_i^{(j+1)} \in \bigcap_{\substack{l=1 \\ l \neq i}}^{k_{j+1}} \left\{N_l^{(j+1)} - \bigcup_{\nu=0}^{j} A_\nu\right\}, \qquad i=1, \ldots, k_{j+1},$$

and let

$$A_{j+1} = \{x_1^{(j+1)}, \ldots, x_{k_{j+1}}^{(j+1)}\}.$$

Then

$$k_j, \quad A_j, \quad \mathcal{N}^{(j)}$$

are defined for $0 \leqq j \leqq t$ for some t. That is the length of the induction process is t. It is obvious from the construction that

(11) $$A_j \text{ are pairwise disjoint } \quad (0 \leqq j \leqq t),$$

(12) $$|A_j| \geqq 4, \qquad 0 \leqq j \leqq t,$$

(13) $$|N_i^{(j)} \cap A_\nu| = k_\nu - 1, \qquad \nu \leqq j,\ i=1, \ldots, k_j,\ j=0, 1, \ldots, t.$$

We state some simple lemmas.

LEMMA 5. $t < m$.

PROOF. If $N \in \mathcal{N}^{(t)}$ is an arbitrary n-tuple, then by (13) all the sets A_j have exactly one element not belonging to N. Hence H must contain at least $n+t+1$ elements, i.e. $n+t+1 \leqq n+m$, thus we have

$$t \leqq m-1.$$

Since $\mathcal{N} = \{N_1, \ldots, N_k\}$ was an arbitrary enumeration of $\mathcal{N}$ we may as well assume that

$$\mathcal{N}^{(j)} = \{N_1, \ldots, N_{k_j}\} \quad \text{for} \quad j \leqq t.$$

This makes possible to put simply

$$N_i^{(j)} = N_i^{(0)} = N_i \quad \text{for} \quad j \leqq t.$$

Of course when emphasis is needed to indicate the relation $N \in \mathcal{N}^{(j)}$ we prefer to use the notation $N = N_i^{(j)}$.

LEMMA 6.

$$k_0 + k_1 + \dots + k_t \geqq n - 2m. \tag{14}$$

PROOF. Put $s = k_0 + \dots + k_t$ and let $N \in \mathcal{N}^{(t)}$, then

$$\left| N - \bigcup_{\nu=0}^{t} A_\nu \right| = n - (k_0 - 1) - \dots - (k_t - 1) = n - s + t + 1.$$

By our assumption in the system

$$\left\{ N_i - \bigcup_{\nu=0}^{t} A_\nu : 1 \leqq i \leqq k_t \right\}$$

(satisfying (i)) the (i)-minimal subsystems consist of at most three sets. Recall that $m(n, 2) = n$ and $m(n, 3) \geqq \frac{n}{2}$. Thus we have

$$\left| \bigcup_{i=1}^{k_t} \left(N_i - \bigcup_{\nu=0}^{t} A_\nu \right) \right| \geqq \frac{3}{2}(n - s + t + 1).$$

On the other hand

$$\bigcup_{\nu=0}^{t} A_\nu \cup \bigcup_{i=1}^{k_t} \left(N_i - \bigcup_{\nu=0}^{t} A_\nu \right) \subset H,$$

therefore

$$s + \frac{3}{2}(n - s + t + 1) \leqq m + n,$$

that is

$$n - 2m + 3t + 3 \leqq s.$$

Deleting the term $3t + 3 \geqq 0$ we obtain (14).

To prepare our next lemma we define an important class of $(n+1)$-tuples. They will play the same role as the $(n+1)$-tuples defined in (8) in the proof of lemma 4. Let $0 \leqq j \leqq t$ be fixed. By the definition of the sets A_ν $(\nu \leqq j)$

$$A_\nu - N_i^{(j)} = \{x_i^{(\nu)}\}, \quad i \leqq k_\nu, \quad (N_i^{(j)} = N_i^{(\nu)}).$$

Therefore the set

$$A_0 \cup \dots \cup A_j \cup N_i^{(j)} = A_0 \cup \dots \cup A_j \cup \overline{N}_i^{(j)} = N_i^{(j)} \cup \{x_i^{(0)}, \dots, x_i^{(j)}\}$$

is an $(n+j+1)$-tuple. Deleting an arbitrary j-tuple from the remainder

$$\overline{N}_i^{(j)} = N_i^{(j)} - \bigcup_{\nu=0}^{j} A_\nu$$

the union

$$A_0 \cup \dots \cup A_j \cup (\overline{N}_i^{(j)} - \{y_1, \dots, y_j\}), \quad (y_r \in N_i^{(j)},\ 1 \leqq r \leqq j)$$

yields always an $(n+1)$-tuple. $(n+1)$-tuples of such type will be called $(n+1)$-tuples of order j and denoted by $M^{(j)}$.

The next result is analogous to lemma 4.

LEMMA 7. a) *Every n-tuple* $N_i \in \mathcal{N}^{(j)}$ *contains at least* $j+1$ *pairs single covered with respect to* $\mathcal{N}^{(j)}$, $0 \leqq j \leqq t$;

b) *every pair* $p \subset N_i^{(j)} \cap \bigcup_{\nu=0}^{j} A_\nu$ *is double covered by* $\mathcal{N}^{(j)}$.

PROOF. First we prove assertion b). Let $p=\{z_1, z_2\}$, $z_1 \in A_\nu$, $z_2 \in A_\mu$ ($\nu=\mu$ may happen). $k_j \geqq 4$ implies the existence of an n-tuple $N_l^{(j)} \neq N_i^{(j)}$ such that

$$A_\nu - N_l^{(j)} = A_\nu - N_l^{(\nu)} = x_l^{(\nu)} \neq z_1,$$

and

$$A_\mu - N_l^{(j)} = A_\mu - N_l^{(\mu)} = x_l^{(\mu)} \neq z_2.$$

So we have $p \subset N_l^{(j)}$.

Now suppose assertion a) to be false: one can find

$$N_i^{(j)} \in \mathcal{N}^{(j)}$$

such that N_i contains at most j single covered pairs (covering is understood with respect to the system $\mathcal{N}^{(j)}$). By the previously proved assertion b) all these single covered pairs meet the remainder

$$\overline{N}_i^{(j)} = N_i - \bigcup_{\nu=0}^{j} A_\nu.$$

Cancelling j points in a suitable way from $\overline{N}_i^{(j)}$ all the remaining pairs in N_i are double covered. Let $M^{(j)}$ denote the corresponding $(n+1)$-tuple of order j. Now we check property (ii) on $M^{(j)}$. Chosen an arbitrary triplet

$$\{z_1, z_2, z_3\} \subset M^{(j)}$$

we distinguish two cases. First suppose the existence of a pair $\{z_\alpha, z_\beta\} \subset \{z_1, z_2, z_3\}$ covered by N_i, say

$$\{z_1, z_2\} \subset N_i \cap M^{(j)}.$$

By the definition of $M^{(j)}$ the pair $\{z_1, z_2\}$ is double covered, i.e.

$$\{z_1, z_2\} \subset N_i \cap N_l^{(j)}, \qquad i \neq l, \; N_l^{(j)} = N_l$$

and either N_i or $N_l^{(j)}$ contains z_3 and thus it contains $\{z_1, z_2, z_3\}$. In the second case

$$|\{z_1, z_2, z_3\} - N_i| \geqq 2.$$

We may suppose

$$\{z_1, z_2\} \subset \bigcup_{\nu=0}^{j} A_\nu - N_i, \qquad z_1 = x_q^{(\alpha)}, \; z_2 = x_r^{(\beta)}, \; \alpha, \beta \leqq j.$$

If $z_3 \in \bigcup_{\nu=0}^{j} A_\nu$ holds as well, then

$$\{z_1, z_2, z_3\} \subset N_l^{(j)}$$

for some l, since each z_τ ($\tau=1, 2, 3$) belongs to all but one element of $\mathcal{N}^{(j)}$, and by $k_j \geqq 4$ we can find an n-tuple from $\mathcal{N}^{(j)}$ which contains them simultaneously. Thus we can assume

$$z_3 \in \overline{N}_i^{(j)} \cap M^{(j)}.$$

If z_3 were single covered by $\mathcal{N}^{(j)}$, all the pairs incident to z_3 would be single covered as well. This is impossible, because the pairs in $N_i \cap M^{(j)}$ are double covered. Hence z_3 is double covered and we can select an n-tuple $N_l^{(j)}$, $l \neq i$ with $z_3 \in N_l^{(j)}$. But $N_l^{(j)} = N_l^{(\alpha)} = N_l^{(\beta)}$, thus

$$x_q^{(\alpha)} \in N_l^{(\alpha)}, \quad x_r^{(\beta)} \in N_l^{(\beta)}$$

because of $l \neq i$. Thus $\{z_1, z_2, z_3\} \subset N_l^{(j)}$. It follows that all the triplets in $M^{(j)}$ are covered, a contradiction to (ii). The proof of the lemma is complete.

LEMMA 8. *There exists a set of pairs* $P = \bigcup_{j=0}^{t} P_j$ *such that*

(*) $P_{j_1} \cap P_{j_2} = \emptyset$ *if* $j_1 \neq j_2$;

(* *) $p \in \bigcup_{\nu=0}^{i} P_\nu$ *is at most single covered by* $\mathcal{N}^{(i)}$, $i = 0, \dots, t$;

(* * *) $|P_j| = k_j$ *and* $|P_j \cap (N \times N)| = 1$ *for every* $N \in \mathcal{N}^{(j)}$, $j = 0, 1, \dots, t$.

PROOF. The sets P_j $(0 \leqq j \leqq t)$ will be defined by induction on j. According to lemma 7, every n-tuple $N \in \mathcal{N}^{(0)}$ contains a pair single covered by $\mathcal{N}^{(0)}$. Choose such a pair from every n-tuple. These pairs being single covered they are different. Denoting by P_0 the set of the selected pairs, (* *) and (* * *) are satisfied by definition.

Consider now $\mathcal{N}^{(1)}$. Making use of lemma 7 again all the n-tuples $N \in \mathcal{N}^{(1)}$ contain at least two pairs single covered by $\mathcal{N}^{(1)}$, and the pairs in $N_\nu^{(1)}$, certainly differ from those in $N_\mu^{(1)}$. Pick out from each $N \in \mathcal{N}^{(1)}$ a pair not yet included in P_0 and single covered by $\mathcal{N}^{(1)}$. Denoting the set of the selected pairs by P_1 the procedure continues in an obvious manner, and the starred properties are trivially fulfilled.

LEMMA 9. *Given* $x \in H$ *the number of pairs from* P *incident to* x *is at most* $3m$.

PROOF. Suppose that

$$Q = \{\{x, y_1\}, \dots, \{x, y_s\}\}$$

are all the pairs containing x and belonging to $P = \bigcup_{\nu=0}^{t} P_\nu$. Take the maximal index α such that P_α contains at least two different pairs from Q (put $\alpha = -1$ if $|Q \cap P_\nu| \leqq 1$, $\nu = 0, \dots, t$). By the choice of α, $|Q \cap P_\nu| \leqq 1$, if $\nu > \alpha$, hence

$$|Q \cap \bigcup_{\nu > \alpha} P_\nu| = \sum_{\nu > \alpha} |Q \cap P_\nu| \leqq \sum_{\nu=0}^{t} 1 = t + 1 \leqq m.$$

By the property (* * *) there exist $N_i^{(\alpha)} \in \mathcal{N}^{(\alpha)}$, $N_j^{(\alpha)} \in \mathcal{N}^{(\alpha)}$, $i \neq j$, with $x \in N_i^{(\alpha)} \cap N_j^{(\alpha)}$. Referring to (* *) the pairs in $Q \cap \bigcup_{\nu \leqq \alpha} P_\nu$ are at most single covered by $\mathcal{N}^{(\alpha)}$. That is, if $\{x, y\} \in Q \cap \bigcup_{\nu \leqq \alpha} P_\nu$, then

$$y \in H - (N_i^{(\alpha)} \cap N_j^{(\alpha)}).$$

This yields at most

$$|H-(N_i^{(\alpha)}\cap N_j^{(\alpha)})| = |(H-N_i^{(\alpha)})\cup(H-N_j^{(\alpha)})| \leqq 2m$$

possibilities to find y. Finally

$$|Q| = |Q\cap \bigcup_{\nu\leqq\alpha} P_\nu| + |Q\cap \bigcup_{\nu>\alpha} P_\nu| \leqq 2m+m = 3m.$$

Now we state the main result of the paper.

THEOREM 2.

$$m \geqq \frac{1}{2\sqrt{2}}\sqrt{n}-\frac{1}{8}.$$

PROOF. There exists a set $S\subset H$ such that S meets every pair from P (defined in the above lemma) and $|S|\leqq\frac{8}{3}m$. In fact, the elements of P are at most single covered by the system $\mathcal{U}=\mathcal{N}^{(t)}$ and by lemma 2

$$S = \left\{x\in H,\ x \text{ is at most } \frac{k_t+1}{2}\text{-times covered by } \mathcal{N}^{(t)}\right\}$$

satisfies the required properties: lemma 1 yields

$$|S| \leqq m\frac{2k_t}{k_t-1} \text{ and by } k_t\geqq 4 \text{ we have } |S| \leqq \frac{8}{3}m.$$

Now applying lemma 9 to the points of S we get the estimation

$$|P| \leqq \frac{8}{3}m\cdot 3m = 8m^2.$$

Using lemma 6

$$n-2m \leqq k_0+\dots+k_t = |P| \leqq 8m^2,$$

that is

$$n \leqq 8m^2+2m < \left(\sqrt{8}\,m+\frac{1}{\sqrt{8}}\right)^2,$$

hence

$$m > \frac{1}{2\sqrt{2}}\sqrt{n}-\frac{1}{8}.$$

Remark. The method with a slight modification would easily give $m\geqq\left(\frac{1}{\sqrt{6}}-\varepsilon\right)\sqrt{n}$ as well.

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(Received May 3, 1971)

KONVERGENZ VON QUADRATURVERFAHREN VOM GAUSSSCHEN TYP

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I Einleitung

Es sei $a \leqq y_1 < y_2 < \cdots < y_m \leqq b$, $x_{k,n} \in [a, b]$, $k = 1, 2, \ldots n$,

$$x_{k,n} \neq x_{j,n} k \neq j, \quad \Omega(x) = \prod_{i=1}^{m} (x - y_i)^{\alpha_i}, \qquad \alpha_i \geqq 1, \ \alpha_i \in \mathbf{N}, \text{ und } \Omega(x_{k,n}) \neq 0.$$

$f(x) \in C[a, b]$ besitze in jedem Punkt y_i eine Peano Ableitung der Ordnung α_i, $f^{[\alpha_i]}(y_i)$, dann existieren bekanntlich auch die Peano-Ableitungen der Ordnung

$$f^{[r_i]}(y_i) \qquad 1 \leqq r_i \leqq \alpha_i - 1, \ i = 1, 2, \ldots m \ \text{([1] S. 200)}.$$

Zur Annäherung an das Riemann—Stieltjes Integral $\int_a^b f d\alpha(x)$, $\alpha \in BV[a, b]$, betrachten wir Gaußsche Quadraturformeln des Typs

$$(1.\,1) \qquad G_n(f) = \sum_{i=1}^{m} \sum_{k=0}^{\alpha_i - 1} B_{k,i}^n f^{[k]}(y_i) + \sum_{k=1}^{n} A_{k,n} f(x_{k,n}).$$

Ist $\alpha(x)$ monoton wachsend in $[a, b]$ mit unendlich vielen Wachstumspunkten und $\Omega(x) \geqq 0$ $x \in [a, b]$, dann läßt sich die Existenz der obigen Formel (1. 1) beweisen ([7], [8] S. 9 f). Sie ist dann die eindeutig bestimmte Formel des Typs (1. 1), die Polynome bis zum

Grad $N + 2n$, $\sum_{i=1}^{m} \alpha_i = N + 1$, exakt integriert.

Sonderfälle von (1. 1) sind die bekannten Radau-Formeln, Lobatto-Formeln ([2], S. 39, 40), verallgemeinerte Radau- und Lobatto-Formeln ([3], [4]).

Im zweiten Abschnitt wird eine Methode zur Aufstellung der Gaußschen Formeln angegeben, die es erlaubt, ihre Bestimmungsstücke aus denen von bekannten Gauß—Jacobi Quadraturverfahren zu ermitteln.

Im dritten Abschnitt zeigen wir die Konvergenz der Gaußschen Formeln des Typs 1. 1 für jedes $f \in C[a, b]$, das in den Punkten y_i eine Peano Ableitung der Ordnung α_i besitzt. Die Idee zu dem einfachen Beweis von (3. 2) verdanke ich einer persönlichen Mitteilung von Herrn Prof. G. FREUD.

II Konstruktion

Sei $C_{y_i}^{[\alpha_i]}[a, b]$ die Menge aller auf $[a, b]$ stetigen Funktionen, die an jedem Punkt y_i, $i=1, 2, \dots, m$, eine Peano Ableitung der Ordnung α_i besitzen.

Sei $f \in C_{y_i}^{[\alpha_i]}[a, b]$ und $Pf = H_N(f; x)$ das eindeutig bestimmte Hermite-Interpolationspolynom vom Grad $\leqq N$, $N+1 = \sum_{i=1}^{m} \alpha_i$, mit

(2. 1) $$H_N(f; y_i)^{(k)} = f^{[k]}(y_i) \qquad k=0, 1, \dots (\alpha_i - 1), \; i=1, 2, \dots m. \text{ *)}$$

Mit $Q = I - P$ (I=Identität) gilt $f = Pf + Qf$. Sei nun $\tilde{Q}_n g$ eine Quadraturformel der Form

$$\tilde{Q}_n g = \sum_{k=1}^{n} A_{k,n} g(x_{k,n}),$$

dann setzen wir

(2. 2) $$\tilde{G}_n f = \int_a^b Pf \, d\alpha + \tilde{Q}_n(Qf).$$

$\tilde{G}_n f$ ist eine Quadraturformel der Form (1. 1), und es gilt

$$\tilde{G}_n f = \int_a^b f \, d\alpha \quad \text{für} \quad f \in \mathbf{P}_N. \text{ **)}$$

Es ist klar, daß (2. 2) genau dann die Gaußsche Formel des Typs (1. 1) ist, falls

(2. 3) $$\tilde{Q}_n f = \int_a^b f \, d\alpha \quad \text{für jedes } f \text{ der Form } f = \Omega(x) P_{2n-1},$$

oder äquivalent

(2. 4) $$\tilde{Q}_n(\Omega(x) P_{2n-1}) = \int_a^b P_{2n-1} \Omega(x) \, d\alpha \quad \text{für jedes} \quad P_{2n-1} \in \mathbf{P}_{2n-1}.$$

Unter den eingangs getroffenen Voraussetzungen über $\Omega(x)$ und $\alpha(x)$ ist bekanntlich (2. 4) die definierende Eigenschaft des Gauß—Jacobi Quadraturverfahrens zur Belegung $d\beta = \Omega(x) d\alpha$ ([6] S. 100, [5] S. 26 f.). Daher gilt

LEMMA 1. *Sind $\{x_{k,n}\}_{k=1}^{n}$ die Nullstellen des Orthogonalpolynoms $p_n(d\beta, x)$, und $A_{k,n} = \frac{1}{\Omega(x_{k,n})} C_{k,n}$, wobei die $C_{k,n}$ die Gewichte der Gauß—Jacobi Quadraturformel zur Belegung $d\beta$ sind, dann lautet die Gaußsche Formel des Typs* (1. 1)

(2. 5) $$G_n(f) = \int_a^b Pf \, d\alpha + Q_n(Qf), \qquad f \in C_{y_i}^{[\alpha_i]}[a, b],$$

mit

$$Q_n(Qf) = \sum_{k=1}^{n} A_{k,n}(Qf)(x_{k,n}).$$

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*) $f_{(y_i)}^{[0]} = f(y_i)$.

**) $\mathbf{P}_N$ = Menge aller algebraischen Polynome vom Grad $\leqq N$.

III. Konvergenz

Sei $f \in C_{y_i}^{[\alpha_i]}[a, b]$, dann gilt wegen

$$\int_a^b f\,d\alpha = \int_a^b Pf\,d\alpha + \int_a^b Qf\,d\alpha$$

und (2. 5)

(3. 1)
$$\lim_{n \to \infty} G_n(f) = \int_a^b f\,d\alpha$$

genau dann, falls

(3. 2)
$$\lim_{n \to \infty} Q_n(Qf) = \int_a^b Qf\,d\alpha \quad \text{für jedes} \quad f \in C_{y_i}^{[\alpha_i]}[a, b],$$

LEMMA 2. *Sei* $f \in C_{y_i}^{[\alpha_i]}[a, b]$, *dann gilt*

(3. 3)
$$\lim_{x \to y_i} \frac{\alpha_i!}{(x-y_i)^{\alpha_i}} \{f(x) - H_N(f; x)\} = f^{[\alpha_i]}(y_i) - H_N(f, y_i)^{(\alpha_i)} \qquad i = 1, 2, \dots m$$

BEWEIS. $H_N(f; x) \in \mathbf{P}_N$; daher ist nach der Taylorformel

$$H_N(f; x) = \sum_{k=0}^{\alpha_i - 1} \frac{H_N(f; y_i)^{(k)}}{k!} (x-y_i)^k + \frac{H_N(f; \xi_i)^{(\alpha_i)}}{\alpha_i!} \cdot (x-y_i)^{\alpha_i}$$

$$\min\{x, y_i\} < \xi_i < \max\{x; y_i\}.$$

Wegen (2. 1) folgt

$$H_N(f; x) = \sum_{k=0}^{\alpha_i - 1} \frac{f^{[k]}(y_i)}{k!} (x-y_i)^k + \frac{H_N(f; \xi_i)^{(\alpha_i)}}{\alpha_i!} \cdot (x-y_i)^{\alpha_i}.$$

$$\frac{\alpha_i!}{(x-y_i)^{\alpha_i}} \{f(x) - H_N(f; x)\} = \frac{\alpha_i!}{(x-y_i)^{\alpha_i}} \left\{ f(x) - \sum_{k=0}^{\alpha_i - 1} \frac{f^{[k]}(y_i)}{k!} (x-y_i)^k \right\} - H_N(f; \xi_i)^{(\alpha_i)}.$$

Da $f \in C_{y_i}^{[\alpha_i]}[a, b]$, folgt daraus die Behauptung.

SATZ 1. *Sei* $f \in C_{y_i}^{[\alpha_i]}[a, b]$, *dann gilt für die Gaußschen Formeln* (1. 1)

$$\lim_{n \to \infty} G_n(f) = \int_a^b f\,d\alpha.$$

BEWEIS. Sei $Q_n^*(f)$ die Gauß—Jacobi Quadraturformel zur Belegung $d\beta = \Omega(x)\,d\alpha$.
Nach Lemma 2 ist

$$\varphi(x) = \begin{cases} \Omega^{-1}(x)\{f(x) - H_N(f; x)\} & x \neq y_i \\ \lim_{x \to y_i} \Omega^{-1}(x)\{f(x) - H_N(f; x)\} & x = y_i \end{cases}$$

eine auf $[a, b]$ stetige Funktion; daher ist ***)

$$\lim_{n\to\infty} Q_n^*(\varphi) = \int_a^b \varphi\, d\beta = \int_a^b Q f\, d\alpha.$$

Aber

$$Q_n^*(\varphi) = \sum_{k=1}^{n} \frac{1}{\Omega(x_{k,n})} C_{k,n}(Qf)(x_{k,n}) = Q_n(Qf)$$

wegen Lemma 1. Damit ist alles bewiesen.

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(Eingegangen: 6 Juli, 1971)

***) Gauß—Jacobi Quadraturverfahren konvergieren bekanntlich für jede stetige Funktion (z. B. [6] S. 106).

НЕКОТОРЫЕ СЛУЧАИ, КОГДА ТРИГОНОМЕТРИЧЕСКОЕ ИНТЕРПОЛИРОВАНИЕ ДАЕТ ПРИБЛИЖЕНИЕ, ПОРЯДКА НАИЛУЧШЕГО

G. P. NÉVAI (Г. П. НЕВАИ)

Пусть 2π-периодическая функция $f(x)$ r-раз непрерывно дифференцируема. Хорошо известно, что отклонение частных сумм ряда Фурье функции $f(x)$ от самой функции $f(x)$ имеет порядок $\log n \cdot n^{-r}\omega(f^{(r)}; n^{-1})$. Г. И. Натансон в работе [1] доказал, что при некоторых ограничениях, наложенных на функцию $f(x)$, множитель $\log n$ может быть опущен, т. е. суммы Фурье дают приближение, порядка наилучшего. В этой работе мы покажем, что аналогичный результат имеет место в случае интерполяции.

В § 1. сформулируем, а в § 2. докажем основную теорему. В § 3. будем доказывать вспомогательные леммы.

Выражаю глубокую благодарность моему учителю, Г. И. Натансону, за постановку проблемы и за ценные указания, без которых эта работа не была бы выполнена.

§ 1.

В работе используем обозночения книга А. Ф. Тимана [2].

Функция f называется q-монотонной на отрезке $[a, b]$, если для всех x и $\delta \geqq 0$, для которых $x, x+q\delta \in [a, b]$, $\Delta_\delta^q f(x)$ сохраняет знак.

Класс тех 2π-периодических функций f, для которых существует такое разбиение $\varrho = \varrho(f)$ отрезка $[-\pi, \pi]$ на N частей

$$\varrho: -\pi = C_0 < C_1 < \cdots < C_N = \pi, \tag{1}$$

что на каждом отрезке $[c_k, c_{k+1}]$ $(k=0, 1, \ldots, N-1)$ функция f есть q-монотонна, обозначим через $M(N, q)$.

Будем говорить, что мажоранта $\omega(\delta)$ удовлетворяет условию $(*)$, и писать $\omega(\delta) \in (*)$, если существуют $A > 0$, $\mu \in (0, 1)$, такие, что

$$\omega(n\delta) \leqq An^{\mu}\omega(\delta), \tag{$*$}$$

при $0 < \delta < \pi$ и $n = 1, 2, \ldots$.

Пусть $f(x)$ есть функция периода 2π и $D_n(x)$-ядро Дирихле, т. е.

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}. \tag{2}$$

Тогда тригонометрический интерполяционный полином n-го порядка, совподающий с f точках

$$x_k^{(n)} = \frac{2\pi k}{2n+1}, \qquad k=0, \pm 1, \pm 2, \dots \tag{3}$$

выражается формулой

$$\tilde{S}_n(x, f) = \frac{2}{2n+1} \sum_{k=-n}^{n} f(x_k^{(n)}) D_n(x - x_k^{(n)}). \tag{4}$$

Теорема. *Пусть r неотрицательное, целое число.*

А. *Если $f \in W^{(r)} H_\omega \cap M(N, r+1)$, и $\omega(\delta) \in (*)$, то тогда*

$$|f(x) - \tilde{S}_n(x, f)| = O\left(\frac{\log N}{n^r}\,\omega\left(\frac{\left|\sin\left(n+\frac{1}{2}\right)x\right|}{n}\right)\right), \tag{5}$$

где константа в „O"-члене зависит от r и $\omega(\delta)$.

Б. *Если $f \in W^{(r)} H_\omega \cap M(N, r+2)$, то тогда справедлива* (5), *но в этот раз константа, входящая в „O"-член зависит только от r.*

Замечание 1. Условие кусочной $r+1$-монотонности функции f нельзя заменить условием отраниченности вариации $f^{(r)}$.

Замечание 2. Условие $(*)$, наложенное на мажоранту $\omega(\delta)$ существенно.

§ 2.

Под h будем понимать выражение $\frac{2\pi}{2n+1}$. В этом и следующем параграфах будем пользоваться модулями гладкости различных порядков, определения и свойства которых даны в книге А. Ф. Тимана [2] (стр. 115).

Возьмем произвольную точку $x = x_0$, и зафиксируем ее. Всюду в дальнейшем будем считать $0 < x_0 \leqq \frac{\pi}{2n+1}$, $f(x_0) = 0$ (см. [3]).

Лемма 1. *Пусть $\{y^i\}_{i=0}^r$ произвольная последовательность $(r+1)$ чисел, x_0 любая точка, $\tau = \left[\frac{r+1}{2}\right]$. Тогда существует единственный тригонометрический полином $H_\tau(x)$ порядка τ, удовлетворяющий условию*

$$H_\tau(x)^{(i)}|_{x=x_0} = y^i, \qquad i = 0, 1, \dots, r, \tag{6}$$

если при нечетном r свободный член в $H_\tau(x)$ равняется нулю. Далее

$$\max_{-\pi \leqq x \leqq \pi} |H_\tau(x)| \leqq C(r) \max_{0 \leqq i \leqq r} |y^i| \; *. \tag{7}$$

* Все константы обозначим через C, и в скобках укажем, от чего они зависят.

Построим теперь такой $H_\tau(x)$, что

$$H_\tau(x)^{(i)}|_{x=x_0} = f(x)^{(i)}|_{x=x_0}, \qquad i=0, 1, \ldots, r. \tag{8}$$

Пусть

$$g(x) = f(x) - H_\tau(x). \tag{9}$$

Функция $g(x)$ обладает следующим важным свойством

$$g(x)^{(i)}|_{x=x_0} = 0, \qquad i=0, 1, \ldots, r. \tag{10}$$

Пусть $n > \tau$. Тогда $\tilde{S}_n(x_0, f) - f(x_0) = \tilde{S}_n(x_0, g)$. Значит мы должны оценивать $|\tilde{S}_n(x_0, g)|$.

Определим функции $A_k^\nu(x)$ ($k = 0, \pm 1, \ldots$; $\nu=0, 1, \ldots$) следующим образом. Положим

$$A_k^0(x) = g(x_k^{(n)}) D_n(x - x_k^{(n)}).$$

Если же $\nu > 0$, то положим

$$A_k^\nu(x) = \frac{1}{2}[A_k^{\nu-1}(x) + A_{k+1}^{\nu-1}(x)]. \tag{11}$$

Лемма 2.

$$A_k^\nu(x) = \frac{1}{2^\nu} \sum_{i=0}^{\nu} C_\nu^i \Delta_h^{\nu-i} g(x_k^{(n)}) D_n^i(x - x_{k+\nu-i}^{(n)}), \tag{12}$$

где

$$\Delta_h^j g(x_k^{(n)}) = \sum_{\mu=0}^{j} (-1)^{\mu-j} C_j^\mu g(x_{k+\mu}^{(n)}), \tag{13}$$

$$D_n^i(x - x_k^{(n)}) = \sum_{\mu=0}^{i} C_i^\mu D_n(x - x_{k+\mu}^{(n)}), \tag{14}$$

далее

$$\tilde{S}_n(x_0, g) = \frac{2}{2n+1} \sum_{k=-n}^{n} A_k^\nu(x_0), \qquad \nu=0, 1, \ldots \tag{15}$$

Лемма 2 доказывается с помощью индукции по ν. Отметим, что (15) есть собственно говоря преобразование Ньютона формулы (4).

Зафиксируем число ν. Пусть $\nu=R$ (ниже увидим, что нам годится $R = r+1$ или $R = r+2$. Будем считать, что $n > R+1$. Выделим те значения k, для которых в (14) значение аргумента D_n^i будет близко к нулю. Точнее запишем

$$\tilde{S}_n(x_0, g) = \tilde{S}_n^1(x_0, g) + \tilde{S}_n^2(x_0, g),$$

$$\tilde{S}_n^1(x_0, g) = \frac{2}{2n+1} \sum_{k=-R}^{0} A_k^R(x_0), \tag{16}$$

$$\tilde{S}_n^2(x_0, g) = \frac{2}{2n+1} \left(\sum_{k=-n}^{-R-1} + \sum_{k=1}^{n} \right) A_k^R(x_0). \tag{17}$$

Лемма 3.

$$|\tilde{S}_n^1(x_0, g)| = O\left(n^{-r}\omega\left(\frac{\left|\sin\left(n+\frac{1}{2}\right)x_0\right|}{n}\right)\right), \tag{18}$$

константа в „0"-члене зависит от R и r.

Ввиду (18) мы должны оценивать только (17). По (12) и (17) имеем

$$\tilde{S}_n^2(x_0, g) = \sum_{i=1}^{3} \tilde{S}_n^2(x_0, g)_i,$$

где

$$\tilde{S}_n^2(x_0, g)_1 = \frac{2}{2n+1}\frac{1}{2^R}\left(\sum_{k=-n}^{-R-1} + \sum_{k=1}^{n}\right) \Delta_h^R g(x_k^{(n)}) D_h^0(x_0 - x_{k+R}^{(n)}), \tag{19}$$

$$\tilde{S}_n^2(x_0, g)_2 = \frac{2}{2n+1}\frac{R}{2^R}\left(\sum_{k=-n}^{-R-1} + \sum_{k=1}^{n}\right) \Delta_h^{R-1} g(x_k^{(n)}) D_n^1(x_0 - x_{k+R-1}^{(n)}), \tag{20}$$

$$\tilde{S}_n^2(x_0, g)_3 = \frac{2}{2n+1}\frac{1}{2^R}\sum_{i=2}^{R} C_R^i \left[\left(\sum_{k=-n}^{-R-1} + \sum_{k=1}^{n}\right) \Delta_h^{R-i} g(x_k^{(n)}) D_n^i(x_0 - x_{k+R-i}^{(n)})\right]. \tag{21}$$

Лемма 4. *Если $R = r+1$ и $\omega(\delta) \in (*)$, или же $R = r+2$, то тогда*

$$|\tilde{S}_n^2(x_0, g)_j| = O\left(n^{-r}\omega\left(\frac{\sin\left(n+\frac{1}{2}\right)x_0}{n}\right)\right), \qquad j=2, 3, \tag{22}$$

где при $R = r+1$ константа в „0"-члене зависит от r и $\omega(\delta)$, а во втором случае только от r.

В силу (22) в дальнейшем мы должны оценивать (19). Теперь мы добавим и вычтем из (19) выражение

$$\tilde{S}_n^2(x_0, H_\tau)_1 = \frac{2}{2n+1}\frac{1}{2^R}\left(\sum_{k=-n}^{-R-1} + \sum_{k=1}^{n}\right) \Delta_h^R H_\tau(x_k^{(n)}) D_n^0(x_0 - x_{k+R}^{(n)}). \tag{23}$$

По (9)

$$\tilde{S}_n^2(x_0, g)_1 = \tilde{S}_n^2(x_0, f)_1 - \tilde{S}_n^2(x_0, H_\tau)_1, \tag{24}$$

где

$$\tilde{S}_n^2(x_0, f)_1 = \frac{2}{2n+1}\frac{1}{2^R}\left(\sum_{k=-n}^{-R-1} + \sum_{k=1}^{n}\right) \Delta_h^R f(x_k^{(n)}) D_n^0(x_0 - x_{k+R}^{(n)}). \tag{25}$$

Лемма 5. *Если $R = r+1$, $\omega(\delta) \in (*)$ или $R = r+2$, то*

$$|\tilde{S}_n^2(x_0, H_\tau)_1| = O\left(n^{-r}\omega\left(\frac{\left|\sin\left(n+\frac{1}{2}\right)x_0\right|}{n}\right)\right), \tag{26}$$

при $R = r+1$ *константа в „0"-члене зависит от* r *и* $\omega(\delta)$, *при* $R = r+2$ *только от* r.

По (24) и (26) наша цель оценивать (25). Теперь положим $R = r+1$ или $R = r+2$, в зависимости от того, что функция f кусочно $(r+1)$ или $(r+2)$-монотонна. Наничая с этого места будем использовать тот факт, что $f \in M(N, R)$. Прежде всего определим последовательность $\{k_\nu\}$ вида

$$(27) \qquad -n = k_{-Q_1} < k_{-Q_1+1} < \cdots < k_{-1} = -R-1 < 0 < k_1 = 1 < \cdots < k_{Q_2} = n$$

$(Q_1 + Q_2 = Q)$ следующим образом. Рассмотрим (1). Пусть

$$C_i^* = \begin{cases} C_i - Rh, & \text{если } C_i - Rh > C_{i-1} \\ C_{i-1}, & \text{если } C_i - Rh \leqq C_{i-1} \end{cases} \qquad (i = 1, \ldots, N).$$

Обозначим через k_μ те индексы k, для которых

$$x_k^{(n)} \in \bigcup_{i=1}^{N} [C_i^*, C_i] \cup \{x_{-n}^{(n)}, x_{-R-1}^{(n)}, x_1^{(n)}\}.$$

Таким образом мы получили множество $\{k_\mu\}$. Из этого выбросим те k_μ, для которых $-R \leqq k_\mu \leqq 0$. Полученное $\{k_\mu\}'$ очевидно можно снабжать индексами ν так, чтобы $\{k_\mu\}$ было вида (27). Какими свойствами, важными для дальнейшего, обладает $\{k_\mu\}$? Ясно, что

$$(28) \qquad Q \leqq (r+6)N$$

Далее, если только $k_\nu < k < k_{\nu+1}$ $(\nu \neq -1)$, тогда

$$(29) \qquad \Delta_h^R f(x_k^{(n)})$$

сохраняет знак. Это вытекает из того, что $[x_{k_\nu+1}^{(n)}, x_{k_{\nu+1}-1+R}^{(n)}]$ полностью содержится в одном из отрезков $[c_i, c_{i+1}]$ $(i=0, 1, \ldots, N-1)$. Теперь (25) можем переписать в виде

$$(30) \qquad \tilde{S}_n^2(x_0, f)_1 = \frac{2}{2n+1} \frac{1}{2^R} \sum_{\nu=-Q_1}^{-2} \sum_{k=k_\nu+1}^{k_{\nu+1}-1} \Delta_h^R f(x_k^{(n)}) D_n^0(x_0 - x_{k+R}^{(n)}) +$$

$$+ \frac{2}{2n+1} \frac{1}{2^R} \sum_{\nu=-Q_1}^{-1} \Delta_h^R f(x_{k_\nu}^{(n)}) D_n^0(x_0 - x_{k_\nu+R}^{(n)}) +$$

$$+ \frac{2}{2n+1} \frac{1}{2^R} \sum_{\nu=1}^{Q_2-1} \sum_{k=k_\nu+1}^{k_{\nu+1}-1} \Delta_h^R f(x_k^{(n)}) D_n^0(x_0 - x_{k+R}^{(n)}) +$$

$$+ \frac{2}{2n+1} \frac{1}{2^R} \sum_{\nu=1}^{Q_2} \Delta_h^R f(x_{k_\nu}^{(n)}) D_n^0(x_0 - x_{k_\nu+R}^{(n)}) = \sum_{i=1}^{4} \tilde{S}_n^2(x_0, f)_1^i.$$

Лемма 6. *При $R=r+1$ или $R=r+2$ имеет место оценка*

$$(31)\qquad |\tilde S_n^2(x_0,f)_1^i| = O\left(\frac{\log N}{n^r}\,\omega\left(\frac{\left|\sin\left(n+\frac{1}{2}\right)x_0\right|}{n}\right)\right),\qquad i=2,4,$$

константа, входящая в „0"-член зависит только от r.

Доказательство. Ограничимся случаем $i=4$. В силу (14) и (27) имеем следующую цепочку неравеств

$$|\tilde S_n^2(x_0,f)_1^4| = \frac{2}{2n+1}\,\frac{1}{2^R}\left|\sum_{\nu=1}^{Q_2}\Delta_h^R f(x_{k_\nu}^{(n)})D_n(x_0-x_{k_\nu+R}^{(n)})\right| \leqq$$

$$\leqq \frac{\omega_R(f;h)}{2^R}\,\frac{2}{2n+1}\sum_{\nu=1}^{Q_2}|D_n(x_0-x_{k_\nu+R}^{(n)})| \leqq \frac{\omega_R(f;h)}{2^R}\,\frac{2}{2n+1}\sum_{\nu=1}^{Q_2}|D_n(x_0-x_k^{(n)})|.$$

Напомним, что $0<x_0\leqq\dfrac{\pi}{2n+1}$ и $f\in W^{(r)}H_\omega$. Значит

$$|\tilde S_n^2(x_0,f)_1^4| = O\left(h^r\omega(h)\log Q_2\left|\sin\left(n+\frac{1}{2}\right)x_0\right|\right).$$

Утверждение леммы немедленно получается из (28) и неравенства

$$(32)\qquad \left|\sin\left(n+\frac{1}{2}\right)x_0\right|\cdot\omega\left(\frac{1}{n}\right)\leqq 2\omega\left(\frac{\left|\sin\left(n+\frac{1}{2}\right)x_0\right|}{n}\right).$$

Для того, чтобы доказать основную теорему, нам осталось оценивать $\tilde S_n^2(x_0,f)_1^i$ $(i=1,3)$. Следующие две леммы по существу доказал Г. И. Натансон в работе [1], но для полноты изложения мы их теперь докажем.

Лемма 7. *Пусть функция f удовлетворяет всем условиям пункта* A. *теоремы и $R=r+1$. Тогда*

$$(33)\qquad |\tilde S_n^2(x_0,f)_1^i| = O\left(\frac{\log N}{n^r}\,\omega\left(\frac{\left|\sin\left(n+\frac{1}{2}\right)x_0\right|}{n}\right)\right),\qquad i=1,\ 3,$$

константа в „0"-члене зависит от r и $\omega(\delta)$.

Доказательство. Рассмотрим только случай $i=3$. В силу неравенства $\operatorname{cosec} x\leqq\dfrac{\pi}{2}\cdot\dfrac{1}{x}\left(0<x<\dfrac{\pi}{2}\right)$ и по (2), (3), (14), (27), (29), (30) очевидно имеем

$$(34)\qquad |\tilde S_n^2(x_0,f)_1^3| \leqq \frac{\sin\left(1+\frac{1}{2}\right)x_0}{2^{R+1}}\sum_{\nu=1}^{Q_2-1}\left|\sum_{k=k_\nu+1}^{k_{\nu+1}-1}\frac{\Delta_h^R f(x_k^{(n)})}{k-1}\right|.$$

В силу (32) достаточно показать, что сумма в (34) имеет порядок $n^{-r}\log N\cdot\omega(n^{-1})$. Но

$$\sum_{\nu=1}^{Q_2-1}\left|\sum_{k=k_\nu+1}^{k_{\nu+1}-1}\frac{\Delta_h^R f(x_k^{(n)})}{k-1}\right| = \sum_{\nu=1}^{Q_2-1}\left|\sum_{k=k_\nu+2}^{k_{\nu+1}-1}\frac{\Delta_h^{R-1} f(x_k^{(n)})-\Delta_h^{R-1} f(x_{k_\nu+1}^{(n)})}{(k-2)(k-1)}+\right.$$

$$\left.+\frac{\Delta_h^R f(x_{k_\nu+1}^{(n)})-\Delta_h^{R-1} f(x_{k_\nu+1}^{(n)})}{k_{\nu+1}-2}\right| \leqq h^r\sum_{\nu=1}^{Q_2-1}\sum_{k=k_\nu+2}^{k_{\nu+1}-1}\frac{\omega(f^{(r)};(k-k_\nu-1)h)}{(k-2)(k-1)}+$$

$$+h^r\sum_{\nu=1}^{Q_2-1}\frac{\omega(f^{(r)};(k_{\nu+1}-k_\nu-1)h)}{k_{\nu+1}-2}\leqq Ah^r\omega(h)\sum_{k=3}^{\infty}\frac{k^\mu}{(k-2))k-1)}+$$

$$+Ah^r\omega(h)\sum_{\nu=1}^{Q_2-1}\frac{(k_{\nu+1}-k_\nu-1)^\mu}{k_{\nu+1}-2}=O\left(n^{-r}\omega\left(\frac{1}{n}\right)\sum_{\nu=1}^{Q_2-1}\frac{(k_{\nu+1}-k_\nu-1)^\mu}{k_{\nu+1}-2}\right).$$

Последняя сумма, как показано в работе [1], имеет порядок $\log Q_2$, значит в силу (28) лемма доказана.

Лемма 8. *Пусть функция f удовлетворяет всем условиям пункта* Б. *теоремы и $R=r+2$. Тогда имеет место* (33), *и константа, входящая в „0”-член зависит только от r.*

Доказательство. Ограничимся случаем $i=3$. Ввиду (32) достаточно показать, что сумма в (34) имеет порядок $n^{-r}\log N\cdot\omega(n^{-1})$. Очевидно имеем следующую цепочку неравенств

$$\sum_{\nu=1}^{Q_2-1}\left|\sum_{k=k_\nu+1}^{k_{\nu+1}-1}\frac{\Delta_h^R f(x_k^{(n)})}{k-1}\right| =$$

$$=\sum_{\nu=1}^{Q_2-1}\left|\frac{-\Delta_h^{R-1} f(x_{k_\nu+1}^{(n)})}{k_\nu}+\sum_{k=k_\nu+2}^{k_{\nu+1}-1}\frac{\Delta_h^{R-1} f(x_k^{(n)})}{(k-2)(k-1)}+\frac{\Delta_h^{R-1} f(x_{k_\nu+1}^{(n)})}{k_{\nu+1}-2}\right|\leqq$$

$$\leqq\omega_{R-1}(f;h)\left(\sum_{\nu=1}^{Q_2-1}\frac{1}{k_\nu}+\sum_{k=3}^{\infty}\frac{1}{(k-2)(k-1)}+\sum_{\nu=2}^{Q_2}\frac{1}{k_\nu-2}\right)\leqq$$

$$\leqq h^r\omega(h)(2\log Q_2+C)\leqq C(r)n^{-r}\log N\omega\left(\frac{1}{n}\right).$$

Последние три леммы доказывают основную теорему.

§ 3.

Доказательство леммы 1. (см. (6) и (7))

а) Пусть $r=2\tau$, $H_\tau(x)=\frac{a_0}{2}+\sum_{k+1}^{\tau}(a_k\cos kx+b_k\sin kx)$. Тогда (6) можем рассматривать, как систему линейных уравнений относительно коэффициентов $H_\tau(x)$. (6) имеет единственное решение тогда и только тогда, если ей соответствующая однородная система имеет лишь тривиальное решение. А это последнее очевидным образом выполняется, так как в этом случае точка x_0 будет нулем $H_\tau(x)$ кратности $2\tau+1$.

б) Пусть $r=2\tau-1$. Будем искать $\tilde{H}_\tau(x)$ удовлетворяющий условиям

$$\tilde{H}_\tau(x_0)=0,\quad \tilde{H}_\tau(x)^{(i)}|_{x=x_0}=y^{i-1},\qquad i=1,2,\dots,r+1. \tag{35}$$

Теперь имеем $r+2=2\tau+1$ условий. По пункту а) существует единственный $\tilde{H}_\tau(x)$, удовлетворяющий (35). Положим $H_\tau(x)=\tilde{H}_\tau(x)'$. Это и будет решением нашей задачи.

в) Переходим к доказательству (7). Для простоты считаем $r=2\tau-1$. Обозначим матрицу системы (6) через $V(x_0)$. Тогда имеет место следующее неравенство (см. [5] стр. 411)

$$\sum_{k=1}^{\tau}(|a_k|+|b_k|)\leqq 2\tau\max_{0\leqq k\leqq 2\tau-1}\sum_{i=0}^{2\tau-1}|\alpha_{ik}|\cdot\max_{0\leqq i\leqq r}|y^i|, \tag{36}$$

где a_k, b_k коэффициенты $H_\tau(x)$, α_{ik} элементы $V(x_0)^{-1}$. Значит $\alpha_{ik}=(\det V(x_0))^{-1}A_{ik}$, где A_{ik} есть алгебраическое дополнение элемента V_{ik} матрицы $V(x_0)$. A_{ik} есть определитель матрицы порядка $2\tau-1$, все элементы которой по абсолютной величине не превосходят τ^r. Значит $|A_{ik}|\leqq r!\,\tau^{r^2}$. Отсюда $|\alpha_{ik}|\leqq C(r)(|\det V(x_0)|)^{-1}$. Но $\det V(x)$ есть непрерывная функция от x, $\det V(x)\neq 0$ при всех x, значит $|\det V(x_0)|^{-1}\leqq C(r)$. Отсюда по (36)

$$\sum_{k=1}^{\tau}(|a_k|+|b_k|)\leqq C(r)\max_{0\leqq i\leqq r}|y^i|.$$

Лемма 9. *Рассмотрим функцию $g(x)$, определенную в (9). Тогда*

$$\omega(g^{(r)};\delta)\leqq C(r)\omega(\delta). \tag{37}$$

Доказательство. В силу (7) и (8)

$$\omega(g^{(r)};\delta)\leqq\omega(f^{(r)};\delta)+\omega(H_\tau^{(r)};\delta)\leqq$$

$$\leqq\omega(f^{(r)};\delta)+\delta\max_{-\pi\leqq x\leqq\pi}|H_\tau^{(r+1)}(x)|\leqq\omega(f^{(r)};\delta)+\delta\tau^{r+1}\max_{-\pi\leqq x\leqq\pi}|H_\tau(x)|\leqq$$

$$\leqq\omega(f^{(r)};\delta)+C(r)\delta\max_{\substack{0\leqq i\leqq r\\ -\pi\leqq x\leqq\pi}}|f^{(i)}(x)|.$$

Не трудно показать, что

$$\max_{\substack{0\leqq i\leqq r\\ -\pi\leqq x\leqq\pi}}|f^{(i)}(x)|\leqq(2\pi)^r\omega(2\pi). \tag{38}$$

Эти неравенства и доказывают лемму.

Доказательство леммы 3. (см. (16) и (18)). В силу (11) имеем

$$|\tilde{S}_n^1(x_0,g)|\leqq\frac{2}{2n+1}|g(0)|\,|D_n(x_0)|+\frac{2}{2n+1}\sum_{|k|=1}^{R}|g(x_k^{(n)})|\,|D_n(x_0-x_k^{(n)})|.$$

Ввиду того, что $0 \leqq x_0 \leqq \dfrac{\pi}{2n+1}$ и по (2), (10), (37)

$$|\tilde{S}_n^1(x_0, g)| \leqq$$

$$\leqq \left(\frac{h}{2}\right)^r \omega(g^{(r)}; x_0) + \frac{2R}{2n+1}\left(h(R+1)\right)^r \omega\left(g^{(r)}; h(R+1)\right) \frac{\sin\left(n+\frac{1}{2}\right)x_0}{\sin\frac{\pi}{2n+1}} \leqq$$

$$\leqq C(r, R) n^{-r} \left(\omega(x_0) + \sin\left(n+\frac{1}{2}\right) x_0 \cdot \omega\left(\frac{1}{n}\right)\right).$$

В силу (32) и неравенства $x_0 < \dfrac{\pi}{2} \cdot \dfrac{\sin(n+\frac{1}{2})x_0}{n}$ лемма доказана.

Для доказательства леммы 4. нам понадобится

Лемма 10. *Пусть R и i целые числа, такие, что $R>0$ и $1 \leqq i \leqq R$. Тогда*

$$(39) \qquad |D_n^i(x_0 - x_{k+R-i}^{(n)})| \leqq C(R) n \sin\left(n+\frac{1}{2}\right) \begin{cases} \dfrac{1}{|k-1|^{i+1}}, & -n \leqq k \leqq -R-1 \\ \dfrac{1}{(k+R)^{i+1}}, & 1 \leqq k \leqq n, \end{cases}$$

где функция D_n^i определена в (14).

Доказательство. В силу (2), (3), (14)

$$|D_n^i(x_0 - x_{k+R-i}^{(n)})| = \frac{\sin\left(n+\frac{1}{2}\right)x_0}{2} \left| \Delta_{h/2}^i \frac{1}{\sin\dfrac{x_{k+R-i}^{(n)} - x_0}{2}} \right|.$$

Легко показать, что $2^{-1}(x_{k+R-i}^{(n)} - x_0) + \dfrac{\mu h}{2} \in \left[\dfrac{x_{k-(1/2)}^{(n)}}{2}, \dfrac{x_{k+R}^{(n)}}{2}\right]$ при $\mu = 0, 1, \ldots, i$.

Отсюда

$$|D_n^i(x_0 - x_{k+R-i}^{(n)})| \leqq \frac{\sin\left(n+\frac{1}{2}\right)x_0}{2} \omega_i\left(\frac{1}{\sin t}; \frac{x_{k-(1/2)}^{(n)}}{2}, \frac{x_{k+R}^{(n)}}{2}; \frac{h}{2}\right).$$

Значит

$$|D_n^i(x_0 - x_{k+R-i}^{(n)})| \leqq \frac{\sin\left(n+\frac{1}{2}\right)x_0}{2} \left(\frac{h}{2}\right)^i \max_{\frac{x_{k-(1/2)}^{(n)}}{2} \leqq t \leqq \frac{x_{k+R}^{(n)}}{2}} |(\operatorname{cosec} t)^{(i)}|.$$

В силу неравенства $|(\operatorname{cosec} x)^{(i)}| \leqq (2i-1)!!|\operatorname{cosec}^{i+1} x|$, на доказательстве которого не останавливаемся, имеем

$$|D_n^i(x_0 - x_{k+R-i}^{(n)})| \leqq \frac{(2R-1)!!}{2} \sin\left(n+\frac{1}{2}\right)x_0 \left(\frac{h}{2}\right)^i \max_{\frac{x_{k-(1/2)}^{(n)}}{2} \leqq t \leqq \frac{x_{k+R}^{(n)}}{2}} |\operatorname{cosec}^{i+1} t|.$$

Ясно, что при $-n \leqq k \leqq -R-1$ $\max |\operatorname{cosec}^{i+1} t|$ достигается в точке $t = 2^{-1}x_{k-R}^{(n)}$, если $1 \leqq k \leqq n-R$, тогда в точке $t = 2^{-1}x_{k-\frac{1}{2}}^{(n)}$, если же $n-R+1 \leqq k \leqq n$, то тогда этот максимум не превосходит $\left|\operatorname{cosec}^{i+1} \frac{x_{n-R}^{(n)}}{2}\right|$. (Считаем, что $n > R$.) Далее в интервалах $\left[-\frac{\pi}{2}, \delta\right], \left[\delta, \frac{\pi}{2}\right]$ $(\delta > 0)$ имеет место оценка $|\operatorname{cosec} t| \leqq \frac{\pi}{2}\frac{1}{|t|}$. Отсюда заключаем, что

$$|D_n^i(x_0 - x_{k+R-i}^{(n)})| \leqq$$

$$\leqq \frac{(2R-1)!!}{2} \sin\left(n+\frac{1}{2}\right)x_0\left(\frac{h}{2}\right)^i \frac{\pi}{2} \begin{cases} \dfrac{1}{\left|\dfrac{x_{k+R}^{(n)}}{2}\right|^{i+1}}, & -n \leqq k \leqq -R-1 \\[2ex] \dfrac{1}{\left(\dfrac{x_{k-(1/2)}^{(n)}}{2}\right)^{i+1}}, & 1 \leqq k \leqq n-R \\[2ex] \dfrac{1}{\left(\dfrac{x_{n-R}^{(n)}}{2}\right)^{i+1}}, & n-R+1 \leqq k \leqq n, \end{cases}$$

или

$$(40) \qquad |D_n^i(x_0 - x_{k+R-i}^{(n)})| \leqq C(R) n \sin\left(n+\frac{1}{2}\right)x_0 \begin{cases} \dfrac{1}{|k+R|^{i+1}}, & -n \leqq k \leqq -R-1 \\[2ex] \dfrac{1}{\left(K-\dfrac{1}{2}\right)^{i+1}}, & 1 \leqq k \leqq n-R \\[2ex] \dfrac{1}{(n-R)^{i+1}}, & n-R+1 \leqq k \leqq n. \end{cases}$$

Если покажем, что $\frac{|k-1|}{|k+R|} \leqq C(R)(-n \leqq k \leqq -R-1)$, $\frac{k+R}{k-\frac{1}{2}} \leqq C(R)(1 \leqq k \leqq n-R)$, $\frac{k+R}{n-R} \leqq C(R)$ $(n-R+1 \leqq k \leqq n)$, то (40) и докажет лемму. А эти утверждения очевидным образом выполняются.

Доказательство леммы 4. (см. (20), (21), (22))

а) Пусть $j=2$. В (20) имеются две суммы. Ограничимся рассмотрением первой из них. В силу (39)

$$\left|\frac{2}{2n+1}\frac{R}{2^R}\sum_{k=-n}^{-R-1}\Delta_h^{R-1}g(x_k^{(n)})D_n^1(x_0-x_{k+R-1}^{(n)})\right| \leqq$$

$$\leqq C(R)\sin\left(n+\frac{1}{2}\right)x_0\sum_{k=-n}^{-R-1}\omega_{R-1}(g;x_k^{(n)},x_{k+R-1}^{(n)};h)\frac{1}{|k-1|^2}\leqq$$

$$\leqq C(R)\frac{\sin\left(n+\frac{1}{2}\right)x_0}{n^r}\sum_{k=-n}^{-R-1}\omega_{R-1-r}(g^{(r)};x_k^{(n)},x_{k+R-1}^{(n)};h)\frac{1}{|k-1|^2}.$$

Если $R=r+2$, то по (32) и (37) пункт а) доказан. Если же $R=r+1$ и $\omega(\delta)\in(*)$ то последнее выражение не превосходит

$$C(R)\frac{\sin\left(n+\frac{1}{2}\right)x_0}{n^r}\sum_{k=-n}^{-R-1}\max_{x_k^{(n)}\leqq t\leqq x_0}|g^{(r)}(t)|\frac{1}{|k-1|^2}\leqq$$

$$\leqq C(R)\frac{\sin\left(n+\frac{1}{2}\right)x_0}{n^r}\sum_{k=-n}^{-R-1}\omega(|k-1|h)\frac{1}{|k-1|^2}\leqq$$

$$\leqq AC(R)\frac{\sin\left(n+\frac{1}{2}\right)x_0}{n^r}\omega\left(\frac{1}{n}\right)\sum_{k=-n}^{-R-1}\frac{1}{|k-1|^{2-\mu}}.$$

А это последнее в силу (32) имеет требуемый порядок.

б) Пусть $j=3$. Будем оценивать только вторую сумму, стоящую в (21). В следующей цепочке неравенств мы используем (10), (37), (39), свойства модулей гладкости, далее то, что при $i\geqq 2$ $f^{(R-i)}$ непрерывна.

$$\left|\frac{2}{2n+1}\frac{1}{2^R}\sum_{i=2}^{R}C_R^i\sum_{k=1}^{n}\Delta_h^{R-i}g(x_k^{(n)})D_n^i(x_0-x_{k+R-i}^{(n)})\right|\leqq$$

$$\leqq C(R)\sin\left(n+\frac{1}{2}\right)x_0\sum_{i=2}^{R}\sum_{k=1}^{n}\omega_{R-i}(g;x_k^{(n)},x_{k+R-i}^{(n)};h)\frac{1}{(k+R)^{i+1}}\leqq$$

$$\leqq C(R)\sin\left(n+\frac{1}{2}\right)x_0\sum_{i=2}^{R}h^{R-i}\sum_{k=1}^{n}\max_{x_k^{(n)}\leqq\alpha\leqq x_{k+R-i}^{(n)}}|g^{(R-i)}(\alpha)|\frac{1}{(k+R)^{i+1}}\leqq$$

$$\leqq C(R)\sin\left(n+\frac{1}{2}\right)x_0\sum_{i=2}^{R}h^{R-i}\sum_{k=1}^{n}\max_{x_0\leqq\alpha\leqq x_{k+R}^{(n)}}|g^{(R-i)}(\alpha)|\frac{1}{(k+R)^{i+1}}\leqq$$

$$\leqq C(R)\sin\left(n+\frac{1}{2}\right)x_0\sum_{i=2}^{R}h^{R-i}\sum_{k=1}^{n}\big(h(k+R)\big)^{i-R+r}\max_{x_0\leqq\alpha\leqq x_{k+R}^{(n)}}|g^{(r)}(\alpha)|\frac{1}{(k+R)^{i+1}}\leqq$$

$$\leqq C(R)h^r\sin\left(n+\frac{1}{2}\right)x_0\sum_{k=1}^{n}\frac{\omega\big(h(k+R)\big)}{(k+R)^{1+R-r}}.$$

В силу (32) очевидно уже, что и при $k = r+1$, $\omega(\delta)\in(*)$ и при $R = r+2$ (21) имеет нам нужный порядок.

Доказательство леммы 5. (см. (23), (26)). Очевидно имеем следующее неравенство

$$|\tilde{S}_n^2(x_0, H_\tau)_1| \leqq \frac{\omega_R(H_\tau; h)}{2^R} \frac{2}{2n+1} \sum_{|k|=1}^{n} |D_n(x_0 - x_k^{(n)})|.$$

Как показано в работе [4], это не превосходит

$$\frac{\omega_R(H_\tau; h)}{2^R} \sin\left(n+\frac{1}{2}\right)x_0 \left(\frac{2}{\pi}\log n + C\right).$$

В силу (6), (7), (8), (38) и неравеиства Бернштейна имеем

$$|\tilde{S}_n^2(x_0, H_\tau)_1| \leqq C(R) \frac{\sin\left(n+\frac{1}{2}\right)x_0}{n^R} \log n \max_{-\pi\leqq x\leqq\pi} |H_\tau^{(R)}(x)| \leqq$$

$$\leqq C(R, r) \frac{\sin\left(n+\frac{1}{2}\right)x_0}{n^R} \log n \max_{-\pi\leqq x\leqq\pi} |H_\tau(x)| \leqq$$

$$\leqq C(R, r) \sin\left(n+\frac{1}{2}\right)x_0 \frac{\log n}{n^R} \omega(2\pi).$$

Так как $R = r+1$ и $\omega(\delta)\in(*)$, или $R = r+2$, то в силу (32) лемма доказана.

Таким образом мы доказали все вспомогательные леммы.

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(Поступило 13-ого апреля 1971 г.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 391—394

АСИМПТОТИЧЕСКАЯ ФОРМУЛА ДЛЯ ОТКЛОНЕНИЯ ТРИГОНОМЕТРИЧЕСКИХ ИНТЕРПОЛЯЦИОННЫХ СУММ

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Пусть $H(V, \omega)$-класс 2π-периодических непрерывных функций, вариация которых по периоду не превосходит заданного числа $V (0 < V \leqq \infty)$, а модуль непрерывности — заданной мажоранты модулей непрерывности $\omega(\delta)$. Пусть тригонометрический полином n-ого порядка $\tilde{S}_n(x, f)$ совпадает с $f(x)$ в узлах

$$x_k = \frac{2k\pi}{2n+1} \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

т.е.

$$\tilde{S}_n(x, f) = \frac{2}{2n+1} \sum_{k=-n}^{n} f(x_k) D_n(x - x_k),$$

где $D_n(x)$-ядро Дирихле. В настоящей работе мы докажем следующую теорему.

Теорема. *Справедливо асимптотическое равенство*

$$(1) \qquad \sup_{f \in H(V, \omega)} |f(x) - \tilde{S}_n(x, f)| =$$

$$= \frac{1}{\pi} \left| \sin\left(n + \frac{1}{2}\right) x \right| \log\left(1 + \min\left\{n, \frac{V}{2\omega\left(\frac{2\pi}{2n+1}\right)}\right\}\right) \omega\left(\frac{2\pi}{2n+1}\right) +$$

$$+ O\left(\omega\left(\frac{\left|\sin\left(n + \frac{1}{2}\right) x\right|}{n}\right)\right),$$

где константа, входящая в „0"-член абсолютная.

Если $\omega(\delta) = \frac{V}{2} \delta \, (0 < v < \infty)$ или же $V = \infty$, то наш результат совпадает с результатом С. Н. Никольского для классов $\frac{V}{2} \operatorname{Lip} 1$ и H_ω соответственно (см. [1] стр. 45). Аналогичную задачу рассматривал В. Г. Коминар для сумм Фурье (см. [2], стр. 197—201).

Лемма. *Пусть* $0<\hat{x}\leqq\frac{\pi}{2n+1}$, $\Delta f(x)=f(x)-f\left(x+\frac{2\pi}{2n+1}\right)$. *Тогда*

(2)

$$\tilde{S}_n(\hat{x},f)-f(\hat{x})=\frac{1}{2n+1}\left(\sum_{k=-n}^{-1}+\sum_{k=1}^{n}\right)\Delta f(x_k)D_n(\hat{x}-x_k)+O\left(\omega\left(f;\frac{\left|\sin\left(n+\frac{1}{2}\right)\hat{x}\right|}{n}\right)\right),$$

где константа, входящая в „0"-член абсолютная.

Доказательство Леммы. Пусть $g(x)=f(x)-f(\hat{x})$. Непосредственный подсчет показывает, что

$$\tilde{S}_n(\hat{x},f)-f(\hat{x})=\frac{1}{2}\tilde{S}'_n(\hat{x},\Delta f)+\frac{1}{4}\left[\tilde{S}'_n(\hat{x},\Delta f)+\tilde{S}'_n\left(\hat{x}+\frac{2\pi}{2n+1},\Delta f\right)\right]+ \tag{3}$$

$$+\frac{1}{4}\left[\tilde{S}'_n(\hat{x},g)+2\tilde{S}'_n\left(\hat{x}+\frac{2\pi}{2n+1},g\right)+\tilde{S}'_n\left(\hat{x}+\frac{4\pi}{2n+1},g\right)\right]+\frac{2}{2n+1}g(0)D_n(\hat{x}),$$

где $\tilde{S}'_n$ означает, что в соответствующей интерполяционной сумме не участвует слагаемое с множителем $D_n(\hat{x}-x_0)=D_n(\hat{x})$. Написав явный вид второго и третьего слагаемых в (3) и используя очевидные соотношения

$$|\Delta f(x_k)|=O\left(\omega\left(f;\frac{1}{n}\right)\right),\quad |g(x_k)|=O\left(\omega\left(f;\frac{|k|}{n}\right)\right),$$

$$|D(\hat{x}-x_k)+D_n(\hat{x}-x_{k+1})|=O\left(\left|\sin\left(n+\frac{1}{2}\right)\hat{x}\right|\frac{n}{k^2}\right)\qquad(k\neq 0,-1),$$

$$|D_n(\hat{x}-x_k)+2D_n(\hat{x}-x_{k+1})+D_n(\hat{x}-x_{k+2})|=O\left(\left|\sin\left(n+\frac{1}{2}\right)\hat{x}\right|\frac{n}{|k|^3}\right)$$

$$(k\neq 0,-1,-2).$$

$$\left|\sin\left(n+\frac{1}{2}\right)\hat{x}\right|\omega\left(f;\frac{1}{n}\right)=O\left(\omega\left(f;\frac{\left|\sin\left(n+\frac{1}{2}\right)\hat{x}\right|}{n}\right)\right),$$

будет ясно, что они имеют порядок остаточного члена в (2). Четвертое же слагаемое в (3) по абсолютной величине не превосходит $\omega(f;x)\leqq\omega\left(f;\frac{|\sin(n+\frac{1}{2})\hat{x}|}{n}\right)$. Лемма доказана.

Доказательство Теоремы. Возьмем произвольную точку $x=\hat{x}$ и ее зафиксируем. Всюду в дальнейшем будем считать $0<\hat{x}\leqq\frac{\pi}{2n+1}$ (см. [3]). В силу

Леммы мы должны оценивать суммы, стоящие в (2). Будем рассматривать только вторую из них. Она не превосходит

$$\frac{1}{2n+1}\sum_{k=1}^{n}|\Delta f(x_k)|\,|D_n(\hat{x}-x_k)|. \tag{4}$$

Так как мы интересуемся наибольшим значением (4), то мы в праве считать, что $|\Delta f(x_k)|$, убывает с возрастанием k (см. [4], стр. 184). Отсюда заключаем, что $|\Delta f(x_k)|\leqq\frac{V}{2k}$. Обозначим

$$N=\min\left\{n,\left[\frac{V}{2\omega\left(\frac{2\pi}{2n+1}\right)}\right]\right\}. \tag{5}$$

Тогда (4) можем переписать в виде

$$\frac{1}{2n+1}\left(\sum_{k=1}^{N}+\sum_{k=N+1}^{n}\right)|\Delta f(x_k)|\,|D_n(\hat{x}-x_k)|\leqq$$

$$\leqq\frac{1}{2n+1}\,\omega\left(\frac{2\pi}{2n+1}\right)\sum_{k=1}^{N}|D_n(\hat{x}-x_k)|+\frac{V}{2(2n+1)}\sum_{k=N+1}^{n}\frac{|D_n(\hat{x}-x_k)|}{k}\leqq$$

$$\leqq\frac{1}{2\pi}\,\omega\left(\frac{2\pi}{2n+1}\right)\left|\sin\left(n+\frac{1}{2}\right)\hat{x}\right|\left(\log(N+1)+O(1)\right)+O\left(\left|\sin\left(n+\frac{1}{2}\right)\hat{x}\right|V\sum_{k=N+1}^{n}\frac{1}{k^2}\right).$$

Ввиду (5) последнее слагаемое раняется нулю, либо имеет порядок $\omega\left(\frac{|\sin(n+\frac{1}{2})\hat{x}|}{n}\right)$ в зависимости от того, что $N=n$ или $N<n$ соответственно. Таким образом мы доказали неравенство в (1). Нам осталось построить такую функцию $f(t)=f(t,n,\hat{x})$, которая (1) обращает в равенство. Пусть $\alpha=\frac{2\pi}{2n+1}$, t_0 корень уравнения $\omega(\alpha)=2\omega(t)$. Положим

$$f(t)=\begin{cases}0, & 0\leqq t\leqq\alpha,\\ -\frac{1}{2}\omega(t-\alpha), & \alpha\leqq t\leqq 2\alpha,\\ (-1)^{k+1}\{\frac{1}{2}\omega(\alpha)-\omega(t-k\alpha)\}, & k\alpha\leqq t\leqq(k+1)\alpha,\ k=2,3,\dots,N-1,\\ (-1)^{N+1}\{\frac{1}{2}\omega(\alpha)-\omega(t-N\alpha)\}, & N\alpha\leqq t\leqq N\alpha+t_0,\\ 0, & N\alpha+t_0\leqq t\leqq\pi,\\ f(-t+\alpha), & -\pi\leqq t\leqq 0.\\ f(t+2\pi)=f(t). & \end{cases}$$

Такого вида функции впервые рассматривались С. М. Никольским (см. [1] стр. 49). Как показано в работе [5] (стр. 493) $\omega(f;\delta)\leqq\omega(\delta)$. Ввиду (5) ясно, что $\operatorname*{var}_{[-\pi,\pi]} f\leqq V$. Значит $f\in H(V,\omega)$. Не трудно убедиться в том, что функция $f(t)$ есть искомая экстремальная функция. Теорема полностью доказана.

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(Поступило 9-ого мая 1971 г.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 395—409

ÜBER EINEN SATZ VON V. F. GAPOSCHKIN

von

K. TANDORI

1. V. F. GAPOSCHKIN [1] hat die folgende Behauptung mitgeteilt.

A. *Es sei* $\{\varphi_n(x)\}_1^\infty$ *ein im Grundintervall* (a, b) *orthonormiertes Funktionensystem und* $\{M_n\}_1^\infty$ *eine Folge von positiven Zahlen mit*

$$\sum_{n=1}^{\infty} \operatorname{mes}\big((a, b) \cap E(|\varphi_n(x)| > M_n)\big) < \infty.$$

Gilt

$$\sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n} |a_k|\, M_k + 1\right) < \infty, \tag{1}$$

dann konvergiert die Reihe

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{2}$$

in (a, b) *fast überall.*

(V. F. GAPOSCHKIN hat diese Behauptung in einer ziemlich allgemeineren Form — für nach einem beliebigen Mass orthonormiertes System — ausgesprochen. Diese Allgemeinheit ist aber für die Nachfolgenden unwesentlich.)

Aus der Behauptung *A* folgt unmittelbar der folgende Satz.

B. *Ist die Bedingung*

$$\sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n} |a_k| + 1\right) < \infty \tag{3}$$

erfüllt, dann konvergiert die Reihe (2) *für jedes in* (a, b) *gleichmässig beschränkte orthonormierte System* $\{\varphi_n(x)\}_1^\infty$ *in* (a, b) *fast überall.*

V. F. GAPOSCHKIN hat noch die Frage aufgeworfen, ob die Bedingung (3) notwendig dafür ist, dass die Reihe (2) für jedes in (a, b) gleichmässig beschränkte orthonormierte System $\{\varphi_n(x)\}_1^\infty$ in (a, b) fast überall konvergiert.

In dieser Note werden wir eine schärfere Behauptung beweisen.

SATZ. *Ist* (3) *erfüllt, dann konvergiert die Reihe* (2) *für jedes orthonormierte System* $\{\varphi_n(x)\}_1^\infty$ *in Orthogonalitätsintervall fast überall.*

(Im folgenden verstehen wir unter $\log \alpha$ das Logarithmus von α mit der Basis 2.)

In der Arbeit [1] sind mehrere Folgerungen der Behauptung *B* erwähnt, die für gleichmässig beschränkte orthonormierte Systeme gültig sind. Nach unserem Satz sind diese Folgerungen auch für beliebige orthonormierte Systeme richtig; diese Folgerungen werden wir aber hier nicht ausführlich zitieren.

Im Punkt 3 dieser Arbeit werden wir zeigen, dass die Bedingung (3) nicht notwendig dafür ist, dass die Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}_1^\infty$ im Orthogonalitätsintervall fast überall konvergiert. Endlich, in Punkt 4 werden wir die Verhältnisse der Bedingung (2) zu den anderen bekannten Konvergenzkriterien untersuchen.

2. BEWEIS des Satzes. Für eine Koeffizientenfolge $\{a_n\}_1^\infty$ setzen wir

$$\|\{a_n\}_1^\infty\|^2 = \sup_{\{\varphi_n\}} \int_a^b \sup_{1\leq i\leq j} (a_i\varphi_i(x)+\dots+a_j\varphi_j(x))^2\,dx,$$

wobei $\sup\limits_{\{\varphi_n\}}$ bedeutet, dass das Supremum für jedes in (a, b) orthonormierte System $\{\varphi_n(x)\}_1^\infty$ gebildet ist. In der Arbeit [4] haben wir gezeigt, dass aus

$$\|\{a_n\}_1^\infty\| < \infty \tag{4}$$

die Konvergenz fast überall in (a, b) der Reihe (2) bei jedem in (a, b) orthonormierten System $\{\varphi_n(x)\}_1^\infty$ sich ergibt. Zum Beweis von (4) ist es offensichtlich genüngend zu zeigen, dass

$$\int_a^b \sup_{1\leq n}(a_1\varphi_1(x)+\dots+a_n\varphi_n(x))^2\,dx < C_1 \tag{5}$$

für ein beliebiges in (a, b) orthonormiertes System $\{\varphi_n(x)\}_1^\infty$ mit einer positiven Konstante C_1 gilt. (Im folgenden bezeichnen $C_1, C_2, \dots$ positive Konstanten, die nur von der Folge $\{a_n\}_1^\infty$ abhängen können.) Wir werden also zeigen, dass aus (3) die Abschätzung (5) sich ergibt.

Nehmen wir an, dass (3) erfüllt ist. Dann gilt

$$\sum_{n=1}^\infty a_n^2 < \infty$$

offensichtlich, und so konvergiert die Folge $\{a_n\}_1^\infty$ zu 0. Es sei $\{a_{n_k}\}_1^\infty$ eine Anordnung der Folge $\{a_n\}_1^\infty$, für die $|a_{n_1}|\geq\dots\geq|a_{n_k}|\geq\dots$ erfüllt ist. Es sei Z_l die in der natürlichen Anordnung geordnete Menge der Indizes n_k mit $2^{2^l}<k\leq 2^{2^{l+1}}$ $(l=1, 2, \dots)$, weiterhin sei Z_0 die in der natürlichen Anordnung geordnete Menge der Indizes $n_1, \dots, n_4$.

Es seien l und i natürliche Zahlen. Wir bezeichnen mit $Z_l^*(i)$ die Menge der Indizes $n\in Z_l$, für die

$$\sum_{\substack{k\\ k\in Z_{l+i};\ k\leq n}} |a_k| \geq (2^{2^{l+i}})^{1/4}$$

ist. Für jede natürliche Zahl l gibt es offensichtlich eine natürliche Zahl i_0 derart, dass für $i>i_0$ $Z_l^*(i)=\emptyset$ ist. Dann setzen wir

$$Z_l^{**}(i) = \emptyset \quad (i>i_0), \qquad Z_l^{**}(i_0) = Z_l^*(i_0),$$

$$Z_l^{**}(i) = Z_l^*(i)\setminus \bigcup_{k=i+1}^\infty Z_l^*(k) \qquad (i=1, \dots, i_0-1),$$

$$Z_l^{**}(0) = Z_l\setminus \bigcup_{k=1}^\infty Z_l^{**}(k).$$

Offensichtlich sind

$$Z_l = \bigcup_{i=0}^{\infty} Z_l^{**}(i) \quad (l=1, 2, \ldots); \qquad Z_l^{**}(i) \cap Z_l^{**}(j) = \emptyset \quad (l=1, 2, \ldots; \ i \neq j),$$

und aus der Definition von $Z_l^{**}(i)$ folgen

$$\text{(6)} \qquad \sum_{n \in A} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \geqq C_2 \Big(\sum_{n \in A} a_n^2 \Big) (2^{l+i})^2 \qquad (A \subseteqq Z_l^{**}(i); \ l, i = 1, 2, \ldots),$$

$$\text{(7)} \qquad \sum_{\substack{k \\ k \in Z_{l+i+s}; k \leqq n}} |a_k| < (2^{2^{l+i+s}})^{1/4} \qquad (n \in Z_l^{**}(i); \ l, s = 1, 2, \ldots; \ i = 0, 1, \ldots).$$

Es gilt noch offensichtlich, dass aus $n \in Z_l^{**}(i)$, $m \in Z_l^{**}(j)$ $(l=1, 2, \ldots; \ 0 \leqq i < j)$ die Ungleichung $n < m$ folgt.

Es seien $l(\geqq 1)$, $i(\geqq 0)$ beliebige ganze Zahlen. Für jedes s $(s=1, 2, \ldots)$ definieren wir die Menge $A(l, i, s)$ folgenderweise. $A(l, i, s)$ sei die Menge der Indizes $m \in Z_{l+i+s}$, für die $m < n$ mit einem Element n von $Z_l^{**}(i)$ erfüllt ist. Aus (7) folgt

$$\text{(8)} \qquad \sum_{k \in A(l, i, s)} |a_\kappa| < (2^{2^{l+i+s}})^{1/4}. \qquad (l, s = 1, 2, \ldots; \ i = 0, 1, \ldots).$$

Es kann leicht erreicht werden, dass für jedes $l(=2, 3, \ldots)$ die Mengen $A(\lambda, i, s)$ $(\lambda, s = 1, 2, \ldots; \ i = 0, 1, \ldots; \ \lambda + i + s = l)$ paarweise disjunkt sind. Offensichtlich gilt

$$A_l = \bigcup_{\substack{\lambda, s \geqq 1, \ i \geqq 0 \\ \lambda + i + s = l}} A(\lambda, i, s) \subseteqq Z_l \qquad (l = 2, 3, \ldots).$$

Wir setzen für jedes $l(\geqq 2)$

$$Z_l(i) = Z_l^{**}(i) \setminus A_l \qquad (i = 0, 1, \ldots),$$

weiterhin sei

$$Z_1(i) = Z_1^{**}(i) \qquad (i = 0, 1, \ldots).$$

Nach obigen sind die Mengen $Z_0, Z_l(i)$ $(l=1, 2, \ldots; \ i=0, 1, \ldots)$, $A(l, i, s)$ $(l, s=1, 2, \ldots; \ i=0, 1, \ldots)$ paarweise disjunkt, und ist

$$Z_0 \cup \left(\bigcup_{l=1}^{\infty} \bigcup_{i=0}^{\infty} Z_l(i) \right) \cup \Big(\bigcup_{l, s \geqq 1, \ i \geqq 0} A(l, i, s) \Big)$$

mit der Menge der natürlichen Zahlen gleich. Die Mengen $Z_l(i)$ $(l=1, 2, \ldots; \ i=0, 1, \ldots)$, $A(l, i, s)$ $(l, s=1, 2, \ldots; \ i=0, 1, \ldots)$ seien geordnete Mengen, wobei ihre Elemente in der natürlichen Anordnung angestellt sind. Aus (6) folgt

$$\text{(9)} \qquad \sum_{n \in Z_l(i)} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \geqq C_2 \Big(\sum_{n \in Z_l(i)} a_n^2 \Big) (2^{l+i})^2 \qquad (l, i = 1, 2, \ldots).$$

Um

$$\sum_{n \in Z_l(0)} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \qquad (l = 1, 2, \ldots)$$

zu abschätzen, nehmen wir

$$\infty > C_3 = \sum_{n=1}^{\infty} a_n^2 = \sum_{l=0}^{\infty} \sum_{n \in Z_l} a_n^2 \geqq \sum_{l=1}^{\infty} \sum_{n \in Z_l} a_n^2 \geqq C_4 \sum_{l=1}^{\infty} \min_{k \in Z_l} a_k^2 \, 2^{2^{l+1}},$$

woraus

$$\max_{k \in Z_{l+1}} |a_k| \leqq \min_{k \in Z_l} |a_k| \leqq \frac{C_5}{\sqrt{2^{2^{l+1}}}} \qquad (l=1, 2, \ldots) \tag{10}$$

sich ergibt. Es ist weiterhin

$$\min_{k \in Z_0} |a_k| \leqq \frac{C_6}{\sqrt{2^2}}. \tag{11}$$

Ist für ein $l(\geqq 1)$

$$\sum_{n \in Z_l(0)} |a_n| < (2^{2^l})^{1/4},$$

dann gilt auf Grund von (10) und (11)

$$\sum_{n \in Z_l(0)} a_n^2 (2^l)^2 \leqq \sum_{n \in Z_l(0)} |a_n| (2^l)^2 \min_{k \in Z_{l-1}} |a_k| \leqq C_7 \frac{(2^l)^2}{(2^{2^l})^{1/4}}. \tag{12}$$

Nehmen wir an, dass für ein $l(\geqq 1)$

$$\sum_{n \in Z_l(0)} |a_n| \geqq (2^{2^l})^{1/4}$$

gilt. Dann sei $\mathfrak{Z}_s(l)$ $(s=1, \ldots, 2^{l+1})$ diejenige Untermenge von $Z_l(0)$, die folgenderweise definiert ist: $n \in \mathfrak{Z}_s(l)$ gilt dann und nur dann, wenn $n \in Z_l(0)$ und

$$\frac{\max_{k \in Z_l(0)} |a_k|}{2^s} < |a_n| \leqq \frac{\max_{k \in Z_l(0)} |a_k|}{2^{s-1}}$$

erfüll sind. Weiterhin sei

$$\mathfrak{Z}_{2^{l+2}}(l) = Z_l(0) \setminus \bigcup_{s=1}^{2^{l+1}} \mathfrak{Z}_s(l).$$

Dann ist auf Grund der Definition von $\mathfrak{Z}_s(l)$:

$$\sum_{n \in Z_l(0)} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) = \sum_{s=1}^{2^{l+2}} \sum_{n \in \mathfrak{Z}_s(l)} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \geqq \tag{13}$$

$$\geqq \sum_{s=1}^{2^{l+2}} \sum_{n \in \mathfrak{Z}_s(l)} a_n^2 \log^2 \Big(\sum_{k \in \mathfrak{Z}_s(l)} |a_k| + 1 \Big) \geqq C_8 \sum_{s=1}^{2^{l+2}} \alpha_s^2 \overline{\overline{\mathfrak{Z}}}_s(l) \log^2 (\alpha_s \overline{\overline{\mathfrak{Z}}}_s(l) + 1),$$

wobei $\alpha_s = \max_{k \in Z_l(0)} |a_k|/2^s$ ist, und $\overline{\overline{\mathfrak{Z}}}_s(l)$ die Mäshtigkeit der Menge $\mathfrak{Z}_s(l)$ bezeichnet.

Ist für ein s $(1 \leqq s \leqq 2^{l+1})$

$$\alpha_s \overline{\overline{\mathfrak{Z}}}_s(l) \geqq (2^{2^l})^{1/4},$$

dann gilt

$$\alpha_s^2 \overline{\overline{\mathfrak{Z}}}_s(l) \log^2 (\alpha_s \overline{\overline{\mathfrak{Z}}}_s(l) + 1) \geqq C_9 \Big(\sum_{n \in \mathfrak{Z}_s(l)} a_n^2 \Big) (2^l)^2. \tag{14}$$

Ist aber für ein s $(1 \leqq s \leqq 2^{l+1})$

$$\alpha_s \overline{\overline{\mathfrak{Z}}}_s(l) < (2^{2^l})^{1/4},$$

dann gilt auf Grund von (10) und (11)

$$(15)\qquad \Big(\sum_{n \in \mathfrak{Z}_s(l)} a_n^2\Big)(2^l)^2 \leqq C_{10}\, \alpha_s \overline{\overline{\mathfrak{Z}}}_s(l) \min_{k \in Z_{l-1}} |a_k| (2^l)^2 \leqq C_{11} \frac{(2^{2^l})^{1/4}}{(2^{2^l})^{1/2}} (2^l)^2.$$

Weiterhin ist

$$(16)\qquad \Big(\sum_{n \in \mathfrak{Z}_{2^{l+2}}(s)} a_n^2\Big)(2^l)^2 \leqq C_{12} \frac{\min\limits_{k \in Z_{l-1}} a_k^2}{2^{2^{l+1}}} 2^{2^{l+1}} (2^l)^2 \leqq C_{13} \frac{(2^l)^2}{2^{2^l}}.$$

Aus (13), (14), (15) und (16) ergibt sich

$$\Big(\sum_{n \in Z_l(0)} a_n^2\Big)(2^l)^2 \leqq C_{14} \sum_{n \in Z_l(0)} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1\right) + C_{15} \frac{(2^l)^3}{(2^{2^l})^{1/4}}.$$

Daraus und aus (12) erhalten wir also

$$(17)\qquad \Big(\sum_{n \in Z_l(0)} a_n^2\Big)(2^l)^2 \leqq C_{16} \left(\sum_{n \in Z_l(0)} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1\right) + \frac{(2^l)^3}{(2^{2^l})^{1/4}}\right).$$

Aus der Definition von $A(l, i, s)$ und aus (8) folgt endlich

$$(18)\qquad \Big(\sum_{n \in A(l,i,s)} a_n^2\Big)(2^{l+i+s})^2 \leqq C_{17} \frac{(2^{2^{l+i+s}})^{1/4}}{(2^{2^{l+i+s}})^{1/2}} (2^{l+i+s})^2 = C_{17} \frac{(2^{l+i+s})^2}{(2^{2^{l+i+s}})^{1/4}}$$

$$(l, s = 1, 2, \ldots;\ i = 0, 1, \ldots).$$

Weiterhin gilt offensichtlich

$$(19)\qquad \Big(\sum_{n \in Z_0} a_n^2\Big) 2^2 \leqq C_{18}.$$

Nach diesen Vorbereitungen können wir unseren Satz beweisen. Wir setzen die Mengen

$$E = \bigcup_{l=1}^{\infty} \bigcup_{i=0}^{\infty} Z_l(i), \quad F = Z_0 \cup \Big(\bigcup_{l,s \geqq 1;\, i \geqq 0} A(l, i, s)\Big).$$

Nach obigen sind E und F disjunkte Mengen, und ihre Vereinigung ist mit der Menge von natürlichen Zahlen gleich. Die Elemente von E bezeichnen wir mit $\nu_1, \nu_2, \ldots$ $(\nu_1 < \nu_2 < \cdots)$.

Es sei $\{\varphi_n(x)\}_1^\infty$ ein beliebiges orthonormierte System in (a, b). Wir bilden die Orthogonalreihen

$$\sum\nolimits_1 = \sum_{n \in E} a_n \varphi_n(x) = \sum_{k=1}^{\infty} a_{\nu_k} \varphi_{\nu_k}(x),$$

$$\sum\nolimits_2 = \sum_{n \in F} a_n \varphi_n(x).$$

Aus einem bekannten Resultat (s. z. B. [4]) folgt

$$(20)\qquad \int_a^b \sup_k (a_{\nu_1}\varphi_{\nu_1}(x)+\dots+a_{\nu_k}\varphi_{\nu_k}(x))^2\,dx \leqq C_{19}\sum_{k=1}^{\infty} a_{\nu_k}^2 \log^2(k+1).$$

Es sei k eine beliebege natürliche Zahl. Dann gibt es ganze Zahlen $l(\geqq 1)$, $i(\geqq 0)$ mit $\nu_k \in Z_l(i)$. Auf Grund deı Definition der Mengen $Z_l(i)$ gilt aber

$$k \leqq C_{20}(2^{2^{l+i}})^{3/4},$$

und so ist

$$\log^2(k+1) \leqq C_{21}(2^{l+i})^2.$$

Aus (3), (9), (17) und (20) erhalten wir

$$(21)\qquad \int_a^b \sup_k (a_{\nu_1}\varphi_{\nu_1}(x)+\dots+a_{\nu_k}\varphi_{\nu_k}(x))^2\,dx \leqq$$

$$\leqq C_{22}\left(\sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n}|a_k|+1\right)+\sum_{l=1}^{\infty}\frac{(2^l)^3}{(2^{2^l})^{1/4}}\right) = C_{23} < \infty.$$

Im folgenden benützen wir das Lemma von D. E. MENCHOFF und H. RADEMACHER ([2], [3]).

Es seien $c_1, \dots, c_N$ $(N \geqq 1)$ reelle Zahlen und $\{\psi_n(x)\}_1^N$ ein in (a, b) orthonormiertes System. Dann gibt es eine Funktion $\delta(x)$ mit folgenden Eigenschaften: es gelten

$$\max_{1\leqq n\leqq N} |c_1\psi_1(x)+\dots+c_n\psi_n(x)| \leqq \delta(x) \qquad (a \leqq x \leqq b),$$

$$\int_a^b \delta^2(x)\,dx \leqq C_{24}\log^2(N+1)\sum_{n=1}^{N} c_n^2.$$

Die ν-te Partialsumme der Reihe $\sum_2$ bezeichnen wir mit $s_\nu(x)$. Auf Grund der Definition der Menge F gilt die folgende Darstellung:

$$s_\nu(x) = s_{\nu_0}^{(0)}(x)+\sum_{l=1}^{\infty}\sum_{i=0}^{\infty}\sum_{s=1}^{\infty} s_{\nu(l,i,s)}^{(l,i,s)}(x),$$

wobei $s_{\nu_0}^{(0)}(x)$, bzw. $s_{\nu(l,i,s)}^{(l,i,s)}(x)$ gewisse Partialsumme der Summe

$$\sum\nolimits_0 = \sum_{n\in Z_0} a_n\varphi_n(x), \quad \text{bzw.} \quad \sum\nolimits_{l,i,s} = \sum_{n\in A(l,i,s)} a_n\varphi_n(x)$$

$(l, s=1, 2, \dots;\ i=0, 1, \dots)$ bezeichnet. Wir bezeichnen mit $\delta_0(x)$, bzw. mit $\delta_{l,i,s}(x)$ diejenige Funktion, die nach dem vorigen Lemma zu der Summe $\sum_0$, bzw. zu der Summe $\sum_{l,i,s}$ gehört. Dann gilt nach der Cauchyschen Ungleichung

$$\sup_\nu |s_\nu(x)| \leqq \delta_0(x)+\sum_{l=1}^{\infty}\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\delta_{l,i,s}(x) \leqq$$

$$\leqq \left(1+\sum_{l=1}^{\infty}\sum_{i=0}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{l+i+s}}\right)^{1/2}\left(\delta_0^2(x)+\sum_{l=1}^{\infty}\sum_{i=0}^{\infty}\sum_{s=1}^{\infty}2^{l+i+s}\delta_{l,i,s}^2(x)\right)^{1/2}.$$

Nach Integrieren, auf Grund von (18) und (19) bekommen wir

$$(22)\qquad \int_a^b \sup_\nu s_\nu^2(x)\,dx \leqq C_{25}\left[\Big(\sum_{n\in Z_0} a_n^2\Big)2^2 + \sum_{l=1}^{\infty}\sum_{i=0}^{\infty}\sum_{s=1}^{\infty} 2^{l+i+s}\Big(\sum_{n\in A(l,i,s)} a_n^2\Big)(2^{l+i+s})^2\right]$$

$$\leqq C_{26}\left(1+\sum_{l=1}^{\infty}\sum_{i=0}^{\infty}\sum_{s=1}^{\infty}\frac{(2^{l+i+s})^3}{(2^{2^{l+i+s}})^{1/4}}\right) = C_{27} < \infty.$$

Auf Grund der Definition der Summen $\sum_1$, und $\sum_2$ erhalten wir aus (21) und (22)

$$\int_a^b \sup_n (a_1\varphi_1(x)+\cdots+a_n\varphi_n(x))^2\,dx \leqq$$

$$\leqq 2\int_a^b \sup_k (a_{\nu_1}\varphi_{\nu_1}(x)+\cdots+a_{\nu_k}\varphi_{\nu_k}(x))^2\,dx + 2\int_a^b \sup_\nu s_\nu^2(x)\,dx \leqq 2C_{23}+2C_{27} < \infty.$$

Damit haben wir (5) bewiesen. So haben wir gezeigt, dass aus (3) die fast überall Konvergenz im Orthogonalitätsintervall der Reihe (2) bei jedem orthonormierten System $\{\varphi_n(x)\}_1^\infty$ folgt.

Es kann auch die folgende stärkere Behauptung gezeigt werden.

Für jede Folge $\{a_n\}_1^\infty$ *gilt*

$$(23)\qquad \|\{a_n\}_1^\infty\| \leqq C\left\{\sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n}|a_k|+1\right)\right\}^{1/2}$$

mit einer positiven absoluten Konstante C.

Es genügt diese Ungleichung im Falle

$$(24)\qquad \sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n}|a_k|+1\right) < \infty$$

zu beweisen.

Nehmen wir an, dass (23) mit einer positiven absoluten Konstante C nicht für jede Folge $\{a_n\}_1^\infty$ mit (24) erfüllt wird. Dann gibt es offensichtlich für jedes l eine Folge $\{a_n(l)\}_1^\infty$ mit

$$(25)\qquad \sum_{n=1}^{\infty} a_n^2(l) \log^2\left(\sum_{k=1}^{n}|a_k(l)|+1\right) \leqq \frac{1}{2^l}$$

und

$$(26)\qquad \|\{a_n(l)\}_1^\infty\| > 4^l \qquad (l=1,2,\ldots).$$

Es sei n_l die kleinste natürliche Zahl mit

$$\sum_{n=1}^{n_l} |a_n(l)| \geqq 1.$$

Wir setzen

$$c_n(l) = a_{n+n_l-1}(l) \qquad (n=1,2,\ldots;\ l=1,2,\ldots).$$

Aus (25) folgt

$$(27)\qquad \sum_{n=1}^{\infty} c_n^2(l)\log^2\left(\sum_{k=1}^{n}|c_k(l)|+1\right) \leqq \frac{1}{2^l} \qquad (l=1,2,\ldots).$$

In der Arbeit [4] haben wir gezeigt, dass für beliebige Folgen $\{a_n\}_1^\infty$, $\{b_n\}_1^\infty$

$$(28)\qquad \|\{a_n\}_1^\infty\| \leqq \sum_{n=1}^{\infty} |a_n|,$$

$$(29)\qquad \|\{a_n+b_n\}_1^\infty\| \leqq \|\{a_n\}_1^\infty\| + \|\{b_n\}_1^\infty\|$$

bestehen. Aus (26), (28), (29) und aus der Definition von n_l erhalten wir

$$(30)\qquad \|\{c_n(l)\}_1^\infty\| \geqq \|\{a_n(l)\}_1^\infty\| - \sum_{k=1}^{n_l-1} |a_k(l)| \geqq 4^l - 1 > 3^l \qquad (l=1, 2, \dots).$$

Aus (25) und aus der Definition von n_l erhalten wir

$$(31)\qquad \sum_{n=1}^{\infty} c_n^2(l) \leqq \frac{1}{2^l} \qquad (l=1, 2, \dots).$$

Auf Grund der Definition von $\|\{a_n(l)\}_1^\infty\|$ und von (30) können wir für jedes l einen Indizes N_l derart angeben, dass

$$(32)\qquad \|\{c_n(l)\}_1^{N_l}\| > 2^l \qquad (l=1, 2, \dots)$$

besteht. Auf Grund von (28), (30), (31) und (32) können wir auch

$$(33)\qquad 3\cdot 2^l \geqq \sum_{n=1}^{N_l} |c_n(l)| \geqq 2 \sum_{n=1}^{N_{l-1}} |c_n(l-1)| \geqq 2 \qquad (l=2, 3, \dots)$$

annehmen.

Wir bilden die Folge $\{a_n\}_1^\infty$ folgenderweise: es sei

$$a_n = c_{n-(N_1+\dots+N_{l-1})}(l) \qquad (n=1, \dots, N_l;\ l=1, 2, \dots).$$

Dann gelten auf Grund von (31), (32) und (33)

$$(34)\qquad \sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n} |a_k|+1\right) = \sum_{l=1}^{\infty}\sum_{n=1}^{N_l} c_n^2(l)\log^2\left(\sum_{s=1}^{l-1}\sum_{k=1}^{N_s} |c_k(s)| + \sum_{k=1}^{n} |c_k(l)|+1\right) \leqq$$

$$\leqq 2\sum_{l=2}^{\infty}\frac{1}{2^l}\log^2 6(1+\dots+2^{l-1}) + 2\sum_{l=1}^{\infty}\sum_{n=1}^{N_s} c_n^2(l)\log^2\left(\sum_{k=1}^{n}|c_k(l)|+1\right) < \infty$$

und

$$(35)\qquad \sum_{l=1}^{\infty} \|\{a_n\}_{N_1+\dots+N_{l-1}+1}^{N_1+\dots+N_l}\|^2 \geqq \sum_{l=1}^{\infty} 2^l = \infty,$$

wobei $\{a_n\}_{N_1+\dots+N_{l-1}+1}^{N_1+\dots+N_l}$ die Folge

$$\{\underbrace{0, \dots, 0}_{N_1+\dots+N_{l-1}}, a_{N_1+\dots+N_{l-1}+1}, \dots, a_{N_1+\dots+N_l}, 0, \dots\}$$

bedeutet.

In der Arbeit [4] haben wir bewiesen, dass für jede Folge $\{a_n\}_1^\infty$ und für jede Indexfolge $(0=)\, n_0 < \dots < n_l < \dots$

$$\sum_{l=0}^{\infty} \|\{a_n\}_{n_l+1}^{n_{l+1}}\|^2 \leqq \|\{a_n\}_1^\infty\|^2$$

besteht. Aus (35) folgt

$$\|\{a_n\}_1^\infty\| = \infty. \tag{36}$$

In der Arbeit [4] haben wir weiterhin gezeigt, dass im Falle (36) ein im Grundintervall (0, 1) orthonormiertes System $\{\psi_n(x)\}_1^\infty$ existiert, für welches die Reihe

$$\sum_{n=1}^\infty a_n \psi_n(x) \tag{37}$$

in (0, 1) fast überall divergiert. Auf Grund von (34), aus unserem Satz folgt, dass die Reihe (37) in (0, 1) fast überall konvergiert. Damit haben wir (23) gezeigt.

3. In diesem Punkt werden wir zeigen, dass die Bedingung (3) nicht notwendig dafür ist, dass die Reihe (2) bei jedem orthonormierten System $\{\varphi_n(x)\}_1^\infty$ im Orthogonalitätsintervall fast überall konvergiert.

Wir setzen nämlich mit einem $\varepsilon(>0)$

$$a_1 = 0, \quad a_n = \frac{1}{\sqrt{n \log^{1+\varepsilon} n}} \qquad (n=2, 3, \dots;\ n \neq 2^{2^k};\ k=2, 3, \dots)$$

und

$$a_{2^{2^k}} = \frac{1}{\sqrt{k \log^{3+\varepsilon} k}} \qquad (k=2, 3, \dots).$$

Dann ist

$$\sum_{k=1}^n a_k \geqq C_{29} \frac{n}{\sqrt{n \log^{3+\varepsilon} n}} \qquad (n=2,3,\dots),$$

und so gilt

$$\sum_{n=1}^\infty a_n^2 \log^2 \left(\sum_{k=1}^n |a_k| + 1 \right) \geqq \sum_{k=2}^\infty a_{2^{2^k}}^2 \log^2 \left(\sum_{l=1}^{2^{2^k}} |a_l| + 1 \right) \geqq C_{30} \sum_{k=2}^\infty \frac{2^k}{k \log^{3+\varepsilon} k} = \infty.$$

Für die Folge $\{a_n\}_1^\infty$ ist also (3) nicht erfüllt. Es gelten aber

$$\sum_{n \neq 2^{2^k};\ k=2,3,\dots} a_n^2 \log^2 n \leqq \sum_{n=2}^\infty \frac{\log^2 n}{n \log^{3+\varepsilon} n} = \sum_{n=2}^\infty \frac{1}{n \log^{1+\varepsilon} n} < \infty,$$

$$\sum_{k=2}^\infty a_{2^{2^k}}^2 \log^2 k = \sum_{k=2}^\infty \frac{1}{k \log^{3+\varepsilon} k} \log^2 k = \sum_{k=2}^\infty \frac{1}{k \log^{1+\varepsilon} k} < \infty.$$

Ist $\{\varphi_n(x)\}_1^\infty$ ein beliebiges orthonormiertes System, dann folgt aus dem Menchoff—Rademacherschen Satz ([2], [3]), dass die Orthogonalreihen

$$\sum_{\substack{n=1 \\ n \neq 2^{2^k};\ k=2,3,\dots}}^\infty a_n \varphi_n(x), \quad \sum_{k=2}^\infty a_{2^{2^k}} \varphi_{2^{2^k}}(x)$$

im Orthogonalitätsintervall fast überall konvergieren. Daraus folgt, dass mit dieser

Koeffizientenfolge $\{a_n\}_1^\infty$ die Reihe (2) bei jedem orthonormierten System $\{\varphi_n(x)\}_1^\infty$ im Orthogonalitätsintervall fast überall konvergiert.

4. Nach dem Menchoff—Rademacherschen Satz ist die Bedingung

$$\sum_{n=1}^{\infty} a_n^2 \log^2 n < \infty \tag{38}$$

genügend dafür, dass die Reihe (2) bei jedem orthonormierten System im Orthogonalitätsintervall fast überall konvergiert.

Aus (38) *folgt* (3). Ist nähmlich (38) erfüllt, dann gilt $\sum_{n=1}^{\infty} a_n^2 = C_{31} < \infty$. Durch Anwendung der Cauchyschen Ungleichung ergibt sich

$$\sum_{k=1}^{n} |a_k| + 1 \leqq C_{32} \sqrt{n},$$

und so ist

$$\sum_{n=1}^{\infty} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \leqq C_{33} \sum_{n=1}^{\infty} a_n^2 \log^2 n < \infty.$$

Im allgemeinen folgt aber aus (3) *die Bedingung* (38) *nicht*. (3) und (38) sind also im allgemeinen nicht äquivalent.

Es sei nähmlich

$$a_n = \frac{1}{n^2} \quad (n = 1, 2, \ldots;\ n \neq 2^k;\ k = 1, 2, \ldots), \qquad a_{2^k} = \frac{1}{k^{3/4}} \quad (k = 1, 2, \ldots).$$

Dann ist

$$\sum_{n=1}^{\infty} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \leqq$$

$$\leqq \sum_{n=1}^{\infty} \frac{1}{n^4} \log^2 \left(\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{\substack{k \\ 2^k \leqq n}} \frac{1}{k^{3/4}} \right) + \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \log^2 \left(\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{\substack{k \\ 2^k \leqq n}} \frac{1}{k^{3/4}} \right) \leqq$$

$$\leqq C_{34} \sum_{k=1}^{\infty} \frac{1}{k^4} \log^2 k + C_{35} \sum_{k=1}^{\infty} \frac{\log^2 k}{k^{3/2}} < \infty,$$

und

$$\sum_{n=1}^{\infty} a_n^2 \log^2 n \geqq \sum_{k=1}^{\infty} \frac{k^2}{k^{3/2}} = \infty.$$

Aus (3) folgt (38) nicht.

Im Falle $|a_1| \geqq \cdots \geqq |a_n| \geqq \cdots$ folgt aber aus (3) die Bedingung (38) auch. *Für im absoluten Betrag monoton abnehmende Folge* $\{a_n\}_1^\infty$ *sind* (3) *und* (38) *äquivalent.* (Diese Tatsache wurde von V. F. Gaposchkin [1] ohne Beweis erwähnt.) Die Menge der natürlichen Zahlen teilen wir in zwei disjunkte Teilmenge I_1, I_2 folgenderweise ein: ist $|a_n| \geqq 1/n^{3/4}$, dann sei $n \in I_1$, und es sei $n \in I_2$ im entgegengesetzten Falle. Dann ist

$$\sum_{n \in I_2} a_n^2 \log^2 n \leqq 3 \sum_{n \in I_2} \frac{1}{n^{3/2}} \log^2 n \leqq \sum_{n=2}^{\infty} \frac{\log^2 n}{n^{3/2}} < \infty.$$

Weiterhin gilt

$$\sum_{n \in I_1} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \geqq \sum_{n \in I_1} a_n^2 \log^2 \left(n \frac{1}{n^{3/4}} \right) \geqq C_{36} \sum_{n \in I_1} a_n^2 \log^2 n.$$

Aus (3) folgt also in diesem Falle die Bedingung (38).

In den Arbeiten [5] und [6] haben wir gezeigt, dass

(39) $$\sum_{n=2}^{\infty} a_n^2 \log n \log_+ \frac{1}{a_n^2} < \infty$$

und

(40) $$S = \sum_{\nu=0}^{\infty} \sqrt{\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}} a_n^{*2} \log^2 n} < \infty$$

voneinander unabhängige, hinreichende Bedingungen dafür sind, dass die Reihe (2) bei jedem orthonormierten System $\{\varphi_n(x)\}_1^\infty$ im Orthogonalitätsintervall fast überall konvergiert. (Aus (40) folgt auch, dass die Reihe (2) bei jedem orthonormierten System $\{\varphi_n(x)\}_1^\infty$ im Orthogonalitätsintervall fast überall unbedingt konvergiert.) Hier ist

$$\log_+ \alpha = \begin{cases} \log \alpha, & \alpha \geqq 2, \\ 1 & \text{sonst}, \end{cases}$$

weiterhin für eine Folge $\{a_n\}_1^\infty$ mit

$$\sum_{n=1}^{\infty} a_n^2 < \infty$$

bedeutet $\{a_n^*\}_1^\infty$ eine Anordnung der Folge $\{a_n\}_1^\infty$, für die $|a_1^*| \geqq \cdots \geqq |a_n^*| \geqq \cdots$ besteht; im Falle

$$\sum_{n=1}^{\infty} a_n^2 = \infty$$

soll man $S = \infty$ setzen.

Wir zeigen, dass *die Bedingungen* (39) *und* (3), *bzw.* (40) *und* (3) *voneinander unabhängig sind.*

Es sei

$$a_{2^k} = \frac{1}{k^{3/4}} \quad (k = 1, 2, \ldots), \qquad a_n = \frac{1}{n^2} \quad (n = 1, 2, \ldots;\ n \neq 2^k;\ k = 1, 2, \ldots).$$

Dann ist

$$\sum_{n=1}^{\infty} a_n^2 \log n \log_+ \frac{1}{a_n^2} \geqq \sum_{k=1}^{\infty} \frac{k}{k^{3/2}} \log k^{3/4} = \infty,$$

und

$$\sum_{n=1}^{\infty} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \leqq$$

$$\leqq \sum_{n=1}^{\infty} \frac{1}{n^4} \log^2 \left(\sum_{k=1}^{\infty} \frac{1}{n^2} + \sum_{\substack{k \geqq 1 \\ 2^k \leqq n}} \frac{1}{k^{3/4}} \right) + \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \log^2 \left(\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{l=1}^{k} \frac{1}{l^{3/4}} \right) \leqq$$

$$\leqq C_{37} \left(\sum_{n=1}^{\infty} \frac{1}{n^4} \log^2 (\log n) + \sum_{k=1}^{\infty} \frac{\log^2 k}{k^{3/2}} \right) < \infty.$$

Aus (3) folgt also (39) nicht.

Es sei nun mit $\varepsilon > 0$

$$a_1 = 0, \quad a_n = \frac{1}{\sqrt{n \log^{3+\varepsilon} n}} \qquad (n=2, 3, \ldots;\ n \neq 2^k;\ k=2, 3, \ldots),$$

$$a_{2^k} = \frac{1}{k \sqrt{\log^{2+\varepsilon} k}} \qquad (k=2, 3, \ldots).$$

Dann ist

$$\sum_{n=1}^{\infty} a_n^2 \log n \log_+ \frac{1}{a_n^2} =$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log^{3+\varepsilon} n} \log n \log (n \log^{3+\varepsilon} n) + \sum_{k=2}^{\infty} \frac{1}{k^2 \log^{2+\varepsilon} k} k \log (k^2 \log^{2+\varepsilon} k) \leqq$$

$$\leqq C_{38} \left(\sum_{n=2}^{\infty} \frac{1}{n \log^{1+\varepsilon} n} + \sum_{k=2}^{\infty} \frac{1}{k \log^{1+\varepsilon} k} \right) < \infty,$$

$$\sum_{n=1}^{\infty} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \geqq \sum_{k=2}^{\infty} a_{2^k}^2 \log^2 \left(\sum_{l=1}^{2^k} |a_l| + 1 \right) \geqq$$

$$\geqq \sum_{k=2}^{\infty} \frac{1}{k^2 \log^{2+\varepsilon} k} \log^2 \left(\sum_{l=2}^{2^k} \frac{1}{\sqrt{n \log^{3+\varepsilon} n}} \right) \geqq$$

$$\geqq C_{39} \sum_{k=2}^{\infty} \frac{1}{k^2 \log^{2+\varepsilon} k} \log^2 2^k \geqq C_{40} \sum_{k=2}^{\infty} \frac{1}{\log^{2+\varepsilon} k} = \infty.$$

Aus (39) folgt also (3) nicht. Die Bedingungen (39) und (3) sind voneinander unabhängig.

Es sei mit $\varepsilon > 0$

$$a_1 = a_2 = 0, \quad a_n = \frac{1}{\sqrt{n \log^3 n (\log \log n)^{1+\varepsilon}}} \qquad (n=2, 3, \ldots).$$

Dann ist

$$\sum_{n=1}^{\infty} a_n^2 \log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \leqq C_{40} \sum_{n=3}^{\infty} \frac{1}{n \log^3 n (\log \log n)^{1+\varepsilon}} \log^2 n =$$

$$= C_{40} \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^{1+\varepsilon}} < \infty,$$

während

$$\sum_{\nu=0}^{\infty}\sqrt{\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}} a_n^{*2}\log^2 n} \geqq \sum_{\nu=1}^{\infty}\sqrt{\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}} \frac{1}{n\log n\,(\log\log n)^{1+\varepsilon}}} \geqq$$

$$\geqq C_{41}\sum_{\nu=1}^{\infty}\sqrt{\frac{1}{\nu^{1+\varepsilon}}} = \infty.$$

Aus (3) folgt also (40) nicht.

Es sei endlich mit $\varepsilon(>0)$

$$a_1 = a_2 = 0,\quad a_n = \frac{1}{\sqrt{n\log^3 n(\log\log n)^{2+\varepsilon}}}\qquad (n=3, 4, \dots;\ n\neq 2^{2^k};\ k=4, 5, \dots),$$

$$a_{2^{2^k}} = \frac{1}{\sqrt{k\log^3 k(\log\log k)^{2+\varepsilon}}}\qquad (k=4, 5, \dots).$$

Dann ist

$$\sum_{n=1}^{\infty} a_n^2\log^2\left(\sum_{k=1}^{n}|a_k|+1\right) \geqq C_{42}\sum_{k=4}^{\infty}\frac{1}{k\log^3 k(\log\log k)^{2+\varepsilon}}\log^2(2^{2^k}/2^{3k}k^{2+\varepsilon}) \geqq$$

$$\geqq C_{43}\sum_{k=4}^{\infty}\frac{2^{2k}}{k\log^3 k(\log\log k)^{2+\varepsilon}} = \infty,$$

und

$$\sum_{\nu=0}^{\infty}\sqrt{\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}} a_n^{*2}\log^2 n} \leqq C_{44}\sum_{\nu=0}^{\infty}\sqrt{\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}} \frac{\log^2 n}{n\log^3 n(\log\log n)^{2+\varepsilon}}} \leqq$$

$$\leqq C_{43}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{1+\varepsilon/2}} < \infty.$$

Aus (40) folgt (3) auch nicht. Die Bedingungen (40) und (3) sind also voneinander unabhängig.

In der Arbeit [7] haben wir gezeigt, dass

$$\sum_{n=1}^{\infty} a_n^2\log_+^2\frac{1}{a_n^2} < \infty \tag{41}$$

notwendig ist dafür, dass die Reihe (2) bei jedem orthonormierten System $\{\varphi_n(x)\}_1^\infty$ in Orthogonalitätsintervall fast überall konvergiert. *Auf Grund unseres Satzes folgt also* (41) *aus* (3).

Diese Behauptung kann man leicht unmittelbar zeigen. Nehmen wir also an, dass die Bedingung (3) erfüllt ist. Ohne Beschränkung der Allgemeinheit können wir

$$a_n^2 \leqq \frac{1}{2}\qquad (n=1, 2, \dots)$$

voraussetzen. Für jedes s $(s=1, 2, \dots)$ sei I_s die Menge diejenigen Indizes n, für die

$$\frac{1}{2^{s+1}} < a_n^2 \leqq \frac{1}{2^s}$$

erfüllt wird. Nehmen wir an, dass die Menge I_s $(s=1, 2, \ldots)$ geordnet ist, wobei die Elemente von I_s in der natürlichen Anordnung gestellt sind. Die Elemente von I_s bezeichnen wir mit $n_1(s), \ldots, n_\varrho(s)$ $\big(n_1(s)<\cdots<n_{\varrho(s)}(s)\big)$, wobei $\varrho(s)$ die Mächtigkeit von I_s bezeichnet $(s=1, 2, \ldots)$. (Wegen (3) besteht $\sum_{n=1}^{\infty} a_n^2<\infty$, so gilt $a_n \to 0$ $(n\to\infty)$, und darum ist jede Menge I_s endlich.)

Es sei

$$r(s) = n_{[\varrho(s)/2]}(s), \quad \text{im Falle} \quad \varrho(s) \geqq 2$$

($[\varrho(s)/2]$ bezeichnet den ganzen Teil von $\varrho(s)/2$), und sei $r(s) = n_1(s)-1$, im Fall $\varrho(s)=1$. Dann gilt die Abschätzung

$$(42) \qquad \sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n} |a_k|+1\right) = \sum_{s=1}^{\infty} \sum_{n\in I_s} a_n^2 \log^2\Big(\sum_{\substack{k\\ k=1}}^{n} |a_k|+1\Big) \geqq$$

$$\geqq \sum_{s=1}^{\infty} \sum_{n\in I_s} a_n^2 \log^2\Big(\sum_{\substack{k\\ k\in I_s;\, k\leqq n}} |a_k|+1\Big) \geqq C_{44} \sum_{s=1}^{\infty} \sum_{\substack{n\in I_s\\ n>r(s)}} a_n^2 \log^2\Big(\sum_{\substack{k\\ k\in I_s;\, k\leqq r(s)}} |a_k|+1\Big) \geqq$$

$$\geqq C_{45} \sum_{s=1}^{\infty} \Big(\sum_{n\in I_s} a_n^2\Big) \log^2\Big(\sum_{k\in I_s} |a_k|+1\Big) \geqq C_{46} \sum_{s=1}^{\infty} \frac{1}{2^s}\, \varrho(s) \log^2\left(\frac{\varrho(s)}{\sqrt{2^s}}\right).$$

Ist $\varrho(s)/\sqrt{2^s} \geqq (2^s)^{1/4}$, dann gilt

$$(43) \qquad \frac{1}{2^s}\, \varrho(s) \log^2\left(\frac{\varrho(s)}{\sqrt{2^s}}\right) \geqq C_{43} \frac{1}{2^s}\, \varrho(s) \log 2^s \geqq C_{48} \sum_{n\in I_s} a_n^2 \log_+^2 \frac{1}{a_n^2}.$$

Ist aber $\varrho(s)/\sqrt{2^s}<(2^s)^{1/4}$, dann gilt $\varrho(s)<(2^s)^{3/4}$, und so besteht

$$(44) \qquad \sum_{n\in I_s} a_n^2 \log_+^2 \frac{1}{a_n^2} \leqq C_{49} \frac{1}{2^s} (2^s)^{3/4} \log (2^s)^{3/4} \leqq C_{50} \frac{s^2}{(2^s)^{1/4}}.$$

Aus (3) folgt also (41).

Wir bemerken noch eine Tatsache. In einer vorigen Arbeit [8] haben wir ein Verfahren für die Abschätzung von $\|\{a_n\}_1^\infty\|$ angegeben, und damit haben wir eine hinreichende Bedingung dafür erreicht, dass die Reihe (2) bei jedem orthonormierten System $\{\varphi_n(x)\}_1^\infty$ im Orthogonalitätsintervall fast überall konvergiert. Man kann leicht zeigen, dass diese Bedingung schwächer ist, als (3); d.h. folgt aus (3), dass diese Bedingung für jede Folge $\{a_n\}_1^\infty$ erfüllt ist. Den ausführlichen Beweis dieser Behauptung werden wir aber hier nicht diskutieren.

5. Da zum Beweis unseres Satzes benützte Ergebnisse für beliebiges in einem Massraum (X, S, μ) nach dem Mass μ orthonormierte System $\{\varphi_n(x)\}_1^\infty$ gültig sind, gilt auch unser Satz auch in diesem Fall.

Als Anwendung unseres Satzes erwähnen wir ein neue Form des Gesetztes des grossen Zahlen für orthogonaler zufällige Grössen.

SATZ II. *Es sei* $\{\xi_n\}_1^\infty$ *eine Folge von zufälligen Grössen mit*

$$\mathsf{M}(\xi_n) = 0 \quad (n=1, 2, \ldots); \qquad \mathsf{M}(\xi_m \xi_n) = 0 \quad (m, n=1, 2, \ldots).$$

Ist

$$\sum_{n=1}^{\infty} \frac{\mathsf{D}^2(\xi_n)}{n^2} \log^2 \left(\sum_{k=1}^{n} \frac{\mathsf{D}(\xi_k)}{k} + 1 \right) < \infty,$$

dann gilt

$$\frac{\xi_1 + \cdots + \xi_n}{n} \to 0$$

mit Wahrscheinlichkeit 1.

Aus dem Satz I folgt nähmlich, dass die Reihe

$$\sum_{n=1}^{\infty} \frac{\xi_n}{n} = \sum_{n=1}^{\infty} \frac{\mathsf{D}(\xi_n)}{n} \frac{\xi_n}{\mathsf{D}(\xi_n)}$$

mit Wahrscheinlichkeit 1 konvergiert, und daraus, durch Anwendung des Kroneckerschen Lemmas ergibt sich

$$\frac{\xi_1 + \cdots + \xi_n}{n} \to 0$$

mit Wahrscheinlichkeit 1.

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(Eingegangen: 6. Juli 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 411—416

ALGEBRAIC LOGARITHM AND THE CONTINUOUS ENDOMORPHISMS OF THE OPERATOR FIELD

by

A. BLEYER

In the study of the linear operator transformations the notion of the weakly exponential function (definition 3) has proved very useful (see [2], [3]). However, the investigation of the solution of equation (1), which defines the weakly exponential function partly, leads to planty of open questions. It is an open problem to prove that $f'(x)=0$ implies $f(x)\equiv c\in M$ where the differentiation is taken in the sense of definition 2. Corollary 3 gives the answer in a special case when $f(x)$ is a weakly exponential function. In complete generality it is still an open problem. The present paper proves connection between the algebraic and the weakly exponential function. In fact, this result, theorem 3, is a generalization of MIKUSINSKI's theorem ([8]) which states the similar relation between exponential function (in his sense, see [7], [9]) and algebraically exponential function (see [4], [8]).

In case of the exponential function of MIKUSINSKI's type the result of theorem 3 and corollary 1 up to 3 can be found in any standard book of operational calculus (see [7], [9]). Finally we shall apply our result to the integral representation of sequentially continuous endomorphisms (see [1], [2], [3]). Generalizing the representation theorem of [2] and [6] we shall give an algebraic characterization of sequentially continuous (therefore also strongly continuous) endomorphisms.

Throughout the paper we shall use standard notations and notions which the reader is familiar with and they can be found in the book of J. MIKUSINSKI [7] (or A. ERDÉLYI [9]). The notations and notions which are different to those will be stated here.

Definition 1. A linear operator transformation is said to be sequentially continuous if $x_n \overset{M}{\to} x$ implies $F(x_n)\overset{M}{\to} F(x)$ where $\overset{M}{\to}$ means the usual first type convergence of the operator field M.

(See [1], [2].) In his study of operator transformations E. GESZTELYI used another type of continuity for operator transformations, we shall call it *strong continuity*, (see [6]). The strong continuity implies continuity by a theorem of GESZTELYI (see [6]).

Definition 2. An operator function $f(\lambda)$ is sequentially differentiable at the point λ_0 and its derivative is $f'(\lambda_0)$ if

$$\frac{f(\lambda_n)-f(\lambda_0)}{\lambda_n-\lambda_0} \overset{M}{\to} f'(\lambda_0)$$

whenever $\lambda_n \to \lambda_0$.

This notion of the derivative is weaker than the MIKUSINSKI's type but it is consistent with that in the following sense: if $f(\lambda)$ is a differentiable operator function in the sense of MIKUSINSKI then it is differentiable in the sense of definition 2. (See [9].)

Definition 3. Let us consider the operator function equation

$$E(x)E(y) = E(x+y)E(0), \tag{1}$$

where x, y are of a fixed interval $[a, b]\ni 0$. $E(x)$ is a non-trivial differentiable solution of (1) *where the differentiation should be taken in the sense of definition 2, then $E(x)$ is called weakly exponential function and the operator $w = -\frac{E'(0)}{E(0)}$ is said to be weak logarithm.*

Some of the important properties of weakly exponential function have been inverstigated in [2] therefore we state only one here which will be used. *Under requirement of sequential continuity and if* (1) *is fulfilled for all real x, y, $E(x)$ can be extended uniquely. $E(x)$ satisfies the differential equation*

$$\frac{dE(x)}{dx} + wE(x) = 0, \tag{2}$$

where the differentiation should be taken in the sense of definition 2. For proof we refer to [2].

To prove our main result we need the notions of algebraic logarithm, algebraic integration and algebraically exponential function. They are due to MIKUSINSKI [8] and E. GESZTELYI [4]; we state here the definitions because of their importance. D will stand for algebraic derivation. (See [4], [8].)

Definition 4. If for a given operator a there exists and operator b such that $Db = a$ then a is called algebriacally integrable operator. In such case the notation

$$\int a = b$$

will be used. Denote the set of operators of the form $\int \frac{Da}{a}$ by W. For $w \in W$ there is an operator a such that $w = \int \frac{Da}{a}$ and then under the symbol $w = \log a$ this relation will be understood. This correspondence also will be denoted by the notation

$$a = \varepsilon^{w}$$

and w is said to be algebraic logarithm. The algebraically exponential function $E(\lambda)$ is defined as a solution of the algebraic differential equation

$$D(E(\lambda)) = \lambda E(\lambda) D(w). \tag{3}$$

THEOREM 1. *If there exists* $\log a$ *and a^{λ} is a sequentially continuous operator function satisfying the properties*

a) *a^{λ} is defined for every real λ,*

b) $a^{\lambda} a^{\mu} = a^{\lambda+\mu}$,

c) $a^{0} = 1$,

then $\log a^{\lambda}$ *exists for every real number λ and* $\log a^{\lambda} = \lambda \log a$.

The theorem has been proved actually by GESZTELYI. In [4] he stated a similar theorem with the continuity in the sense of MIKUSINSKI instead of sequential continuity. However, his proof has used only the properties of the sequential continuity.

THEOREM 2. *If a^λ is a sequentially continuous power function (i.e. it satisfies all conditions of theorem 1 from a) up to c)) and $a=\varepsilon^w$ then $a^\lambda=\varepsilon^{\lambda w}$.*

PROOF. This is an obvious consequence of theorem 1 and definition 4.

Now we are in the position to formulate and prove the main result of the present paper giving *connection between algebraic and weak logarithm.*

THEOREM 3. *If $E(\lambda)$ is a weakly exponential function and $w=-\dfrac{E'(0)}{E(0)}$ then w is an algebraic logarithm.*

PROOF. Set $\tilde{E}(\lambda)=E(\lambda)[E(0)]^{-1}$ (we can suppose $E(0)\neq 0$ since the other case $E(x)\equiv 0$ evidently holds by (1)). We obtain

$$\tilde{E}(\lambda)\tilde{E}(\mu)=\tilde{E}(\lambda+\mu). \tag{4}$$

Taking the algebraic derivative of (4)

$$D(\tilde{E}(\lambda))\tilde{E}(\mu)+D(\tilde{E}(\mu))\tilde{E}(\lambda)=D(\tilde{E}(\lambda+\mu)). \tag{5}$$

Dividing (5) by (4)

$$\frac{D(\tilde{E}(\lambda))}{\tilde{E}(\lambda)}+\frac{D(\tilde{E}(\mu))}{\tilde{E}(\mu)}=\frac{D(\tilde{E}(\lambda+\mu))}{\tilde{E}(\lambda+\mu)}. \tag{6}$$

Setting $g(\lambda)=\dfrac{D(\tilde{E}(\lambda))}{\tilde{E}(\lambda)}$ we have a Cauchy type equation for operator function $g(\lambda)$:

$$g(\lambda)+g(\mu)=g(\lambda+\mu). \tag{7}$$

To solve (7) put $g(1)=c$ and then $g\left(\dfrac{p}{q}\right)=\dfrac{p}{q}c$ follows immediately from (7) for every rational number $\dfrac{p}{q}$. From the sequential continuity of $E(\lambda)$ and the strong continuity of D (see [6]) it follows that $g(\lambda)$ is second type sequentially continuous, i.e.: if $\lambda_n\to\lambda_0$ then $g(\lambda_n)\xrightarrow{\Pi-M}g(\lambda_0)$, where $\xrightarrow{\Pi-M}$ means the second type convergence of operators (see [2], [6], [7]). By this $g(\lambda)=c\lambda$ follows for every real number λ. In fact, the solution $g(\lambda)=c\lambda$ is continuous in the sense of MIKUSINSKI*. However, by virtue of (2)

$$\tilde{E}'(\lambda)=-w\tilde{E}(\lambda) \tag{8}$$

and taking the algebraic derivative of (8)

$$D(\tilde{E}'(\lambda))=-D(w)\tilde{E}(\lambda)-wD(\tilde{E}(\lambda)). \tag{9}$$

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* We could observe the same fact at the logarithm of a sequentially continuous power function, since $\log a^\lambda=\lambda\log a$ is always continuous in the sense of Mikusinski.

But, by (7) $D(\tilde{E}(\lambda))=g(\lambda)\tilde{E}(\lambda)=c\lambda\tilde{E}(\lambda)$ whence

$$D(\tilde{E}'(\lambda)) = -\tilde{E}(\lambda)\,[D(w)+cw\lambda]. \tag{9}$$

Since D is strongly continuous operator transformation (see [6]) D commutes with the differentiation (see [2], [6]) hence we obtain

$$D(\tilde{E}'(\lambda)) = (D(\tilde{E}(\lambda)))' = (c\lambda\tilde{E}(\lambda))' = c\tilde{E}(\lambda)+c\lambda\tilde{E}'(\lambda).$$

Now by virtue of (8)

$$D(\tilde{E}'(\lambda)) = \tilde{E}(\lambda)\,[c-wc\lambda]. \tag{10}$$

Therefore, by (10) and (9)

$$c = -D(w). \tag{11}$$

Since $D(\tilde{E}(\lambda))=c\lambda\tilde{E}(\lambda)$, by definition 4 and by theorem 2 $\tilde{E}(\lambda)=\varepsilon^{-\lambda w}$ follows from (11) which proves the theorem.

COROLLARY 1. *If $E(\lambda)$ is a weakly exponential function and $E(0)=1$ then $E(\lambda)$ is an algebraically exponential function.*

Remark. Corollary 1 is a generalization of a theorem of J. MIKUSINSKI stating that every exponential function in the sense of MIKUSINSKI is an algebraically exponential function (see [8]).

COROLLARY 2. *If $E(\lambda)$ is a weakly exponential function with properties $E(0)=a$ and $E'(0)=-wa$ then $E(\lambda)$ is the unique solution of* (1).

PROOF. If $E(0)=a=0$ then $E(\lambda)\equiv 0$. (See [2].) If $a\neq 0$ then we need the following result of E. GESZTELYI (see [4]): if $c, d\in M$ and $\log c=\log d$ then $c=\gamma d$ where γ is number. Now by virtue of theorem 3 $E(\lambda)=\gamma\varepsilon^{-\lambda w}a$ where γ is a number. However, $E(0)=a\gamma\varepsilon^{0}=a$ which means $\gamma=1$.

COROLLARY 3. *If $E(\lambda)$ is a weakly exponential function with $E'(0)=0$ then $E(\lambda)\equiv \equiv E(0)$.*

In fact, corollary 3 gives the answer of question related to sequential differentiation (and stated in [2]) what we mentioned in the introduction.

Application to the theory of sequentially continuous linear operator transformations.

The first study of (strongly) continuous linear operator transformations is due to GESZTELYI [6]. He gave a nice characterization of strongly continuous endomorphism there. In [2] we have generalized his integral representation theorem, more precisely we proved

THEOREM 4. *If F is a sequentially continuous endomorphism of the operator field M then $F(s)$ is a weak logarithm and*

$$F(u) = \int_{-\infty}^{\infty} u(\lambda)\,\mathrm{Exp}\,(-\lambda F(s))\,d\lambda \tag{12}$$

for all $u \in CU$ *where* $\operatorname{Exp}(-\lambda F(s))$ *is a weakly exponential function. Moreover, assuming* $F(s)$ *to be a logarithm of Mikusinski's type*

$$F(u) = \int_{-\infty}^{\infty} u(\lambda) \exp(-\lambda F(s))\, d\lambda \tag{13}$$

holds instead of (12) *for all* $u \in CU$.

Here we apply theorem 3 and corollary 1. By virtue of those we obtain a nice algebrai characterization of (sequentially or strongly) continuous endomorphisms.

THEOREM 5. *If* F *is a sequentially continuous endomorphism of the operator field,* $F: M \to M$, *then* $F(s)$ *is an algebraic logarithm, the algebraically exponential function* $\varepsilon^{-\lambda F(s)}$ *is sequentially continuous and*

$$F(u) = \int_{-\infty}^{\infty} u(\lambda) \varepsilon^{-\lambda F(s)}\, d\lambda \tag{14}$$

for all $u \in CU$.

COROLLARY 4. *Let* F *be a sequentially continuous endomorphism of* M. *If* w *is a logarithm (of weak or Mikusinski type) then* $F(w)$ *is an algebraic logarithm,* $\varepsilon^{-\lambda F(w)}$ *does exist and is a sequentially continuous operator function, moreover*

$$F(\operatorname{Exp}(-\lambda w)) = \varepsilon^{-\lambda F(w)} \tag{15}$$

If F *is strongly continuous and* w *is a Mikusinski logarithm then*

$$\varepsilon^{-\lambda F(w)} = \exp(-\lambda F(w)) \tag{16}$$

If F *is continuous (in any of the above two sense) and* w *is an algebraic logarithm then*

$$F(\varepsilon^{-\lambda w}) = \varepsilon^{-\lambda F(w)} \tag{17}$$

provided $\varepsilon^{-\lambda w}$ *is sequentially differentiable and satisfies all conditions of theorem 1.*

PROOF. It is an evident consequence of theorem 4, corollary 1 and the well-known properties of the exponential function $\exp(-\lambda w)$ (see [7], [9]).

Lastly, we give a new proof of a commutation theorem of operator transformations. For strongly continuous endomorphisms E. GESZTELYI proved first [6], and for sequentially continuous endomorphisms we generalized in [3]. The present proof is the simplest.

COROLLARY 5. *F is a sequentially continuous endomorphism and commutes with D if and only if* $F = T^{\alpha}$.

PROOF. Indeed, by corollary 4 and the rules of algebraic derivative (see [4], [8]).

$$D(F(\exp(-\lambda s))) = D(\varepsilon^{-\lambda F(s)}) = \lambda D(F(s)) \varepsilon^{-\lambda F(s)} \tag{18}$$

and

$$F(D(\exp(-\lambda s))) = F(-\lambda \exp(-\lambda s)) = -\lambda \varepsilon^{-\lambda F(s)}. \tag{19}$$

Since D commutes with F by (18) and (19) $D(F(s))=1$. By virtue of algebraic integrations (see [4]) $F(s) = s+\alpha$ where α is a number. Therefore, by theorem 4 (or 5) we obtain $F(x)=T^{\alpha}(x)$ for every $x \in M$. The proof is complete.

Finally, I should like to express my deepest thank to Professor E. GESZTELYI, who suggested to find connection between algebraic and weak logarithm and helped me with several valuable remarks when this paper was prepared.

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(Received July 20, 1971)
(Revised November 30. 1971)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 417—421

A NOTE ON D. L. BERMAN'S THEOREM ON THE DIVERGENCE OF HERMITE—FEJÉR INTERPOLATION

by

R. B. SAXENA

Let $\{x_k^{(n)}\}_1^n$, $n=1, 2, \ldots$ be a triangular matrix whose nth row is

$$(1) \qquad 1 > x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} > -1.$$

Consider the matrix whose nth row consists of the nodes

$$(2) \qquad x_0^{(n)} = 1 > x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} > -1$$

or the nodes

$$(3) \qquad 1 > x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} > x_{n+1}^{(n)} = -1$$

which is obtained from (1) by adding the point $x_0^{(n)}=1$ or the point $x_{n+1}^{(n)} = -1$.

Let

$$(4) \qquad \omega_n(x) = \prod_{k=1}^{n} (x - x_k^{(n)})$$

where $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ satisfy (2). Then the Hermite—Fejér interpolation polynomial $H_n(f, x)$ of degree $2n+1$ satisfying the property

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}),\ H_n'(f, x_k^{(n)}) = 0, \qquad k=0, 1, 2, \ldots, n$$

is given by

$$(5) \qquad H_n(f, x) = f(1)\left[1 + \frac{2\omega_n'(1)}{\omega_n(1)}(1-x)\right]\frac{\omega_n^2(x)}{\omega_n^2(1)} +$$

$$+ \sum_{k=1}^{n} f(x_k)\left(\frac{1-x}{1-x_k^{(n)}}\right)^2 \left[1 + \left\{\frac{2}{1-x_k^{(n)}} - \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}\right\}(x - x_k^{(n)}\right] [l_k^{(n)} x]^2, \qquad n=1, 2, \ldots$$

where

$$(6) \qquad l_k^{(n)}(x) = \frac{\omega_n(x)}{(x - x_k^{(n)})\,\omega_n'(x_k^{(n)})}, \qquad k=1, 2, \ldots n.$$

In a recent paper, D. L. BERMAN ([2], Theorem 2, p. 14) has proved the

THEOREM. The sequence of polynomials in (5) constructed for the function $f(x)=|x|$ on the nodes $x_k^{(n)} = \cos\left(k - \frac{1}{2}\right)\frac{\pi}{n}$, $k=1, 2, \ldots, n$ satisfies the equality

$$(7) \qquad \overline{\lim_{n\to\infty}}\, H_n(|x|, 0) = \infty,$$

where for $n \equiv 0 \pmod 4$ the order growth of the number $H_n(|x|, 0)$ exceeds $2n^2$.

The bound in (7) is incorrect as shown by the following

THEOREM. *The sequence of polynomials in* (5) *constructed for the function* $f(x) = |x|$ *on the nodes* $x_k^{(n)} = \cos\left(k-\frac{1}{2}\right)\frac{\pi}{n}$, $n \equiv 0 \pmod 4$, $k=1, 2, \ldots, n$ *satisfies the equality*

(8) $$\lim_{n\to\infty} H_n(|x|, 0) = 3.$$

Similar modifications are valid in Theorem 1 of BERMAN [2] corresponding to the matrix (3).

For the proof of the theorem we shall need the

LEMMA. *The following identities hold:**

(a) $$\sum_{k=1}^{2n} \frac{1}{\sin^2 \frac{\theta_k}{2}} = 4n^2$$

(b) $$\sum_{k=1}^{2n} \frac{1}{\sin^4 \frac{\theta_k}{2}} = \frac{16}{3}n^4 + \frac{8}{3}n^2, \qquad \theta_k = \left(k-\frac{1}{2}\right)\frac{\pi}{n}.$$

(c) $$\sum_{k=1}^{n} \frac{1}{x_k^2} = n^2, \qquad x_k = \cos\left(k-\frac{1}{2}\right)\frac{\pi}{n}.$$

PROOF. The identities (a) and (b) have been proved by D. L. BERMAN [1]. We prove (c).

The Hermite—Fejér interpolation polynomial $H_n(f, x)$ of degree $\leqq 2n-1$ constructed for the function $f(x)$ over the nodes $x_k = \cos\left(k-\frac{1}{2}\right)\frac{\pi}{n}$, $k=1, 2, \ldots, n$ satisfying the conditions

$$H_n(f, x_k) = f(x_k), \quad H_n'(f, x_k) = f'(x_k), \qquad k=1, 2, \ldots, n$$

is given by

(9)
$$H_n(f, x) = \sum_{k=1}^{n} f(x_k)(1-xx_k)\left[\frac{T_n(x)}{n(x-x_k)}\right]^2 + \sum_{k=1}^{n} f'(x_k)(1-x_k^2)(x-x_k)\left[\frac{T_n(x)}{n(x-x_k)}\right]^2.$$

Since $H_n(1, x)=1$, putting $f(x)=1$ in the above formula we get the identity

$$\sum_{k=1}^{n} (1-xx_k)\left[\frac{T_n(x)}{n(x-x_k)}\right]^2 \equiv 1.$$

Hence for $x=0$ we find that

$$\sum_{k=1}^{n} \frac{1}{x_k^2} = n^2.$$

* In the following we shall write x_k for $x_k^{(n)}$.

PROOF of Theorem. The points $x_k^{(n)}$ in the theorem form the zeros of the Tchebycheff polynomial of first kind given by $T_n(x)=\cos(n \arccos x)$. Writing $T_n(x)$ for $\omega_n(x)$ in (5) and using the well-known results of $T_n(x)$, we have after simple calculations:

(9)
$$H_n(f, x) =$$
$$= f(1)[1+2n^2(1-x)]T_n^2(x)+\sum_{k=1}^{n} f(x_k)\left(\frac{1-x}{1-x_k}\right)^2 \frac{(1-x_k^2)+(2+x_k)(x-x_k)}{n^2(x-x_k)^2} T_n^2(x).$$

Let $n=4p$ and $f(x)=|x|$, then owing to the property

$$x_{4p+1-k} = -x_k, \qquad k=1, 2, \dots, 2p,$$

we have for $x=0$

(10)
$$H_n(|x|, 0) = (2n^2+1)+\frac{1}{n^2}\sum_{k=1}^{2p}\left[\frac{(1-x_k^2)-x_k(2+x_k)}{x_k(1-x_k)^2}+\frac{(1-x_k^2)+x_k(2-x_k)^2}{x_k(1+x_k)^2}\right]=$$
$$= (2n^2+1)+\frac{2}{n^2}\sum_{k=1}^{2p}\left[\frac{1}{x_k}+\frac{3x_k}{1-x_k^2}-\frac{6x_k}{(1-x_k^2)^2}\right]=$$
$$= (2n^2+1)+\frac{2}{n^2}\sum_{k=1}^{2p}\left[\frac{1}{x_k}+\frac{3}{2(1-x_k)}-\frac{3}{2(1+x_k)}-\frac{3}{2(1-x_k)^2}+\frac{3}{2(1+x_k)^2}\right].$$

On putting $x_k = \cos\theta_k$ where $\theta_k = \left(k-\frac{1}{2}\right)\frac{\pi}{n}$, $n = 4p$, we get

(11)
$$H_n(|x|, 0) = (2n^2+1)+\frac{2}{n^2}\sum_{k=1}^{2p}\frac{1}{x_k}+\frac{3}{n^2}\sum_{k=1}^{2p}\frac{1}{2\sin^2\frac{\theta_k}{2}}-\frac{3}{n^2}\sum_{k=1}^{2p}\frac{1}{1+x_k}-$$
$$-\frac{3}{n^2}\sum_{k=1}^{2p}\frac{1}{4\sin^4\frac{\theta_k}{2}}+\frac{3}{n^2}\sum_{k=1}^{2p}\frac{1}{(1+x_k)^2}=$$
$$= (2n^2+1)+\frac{2}{n^2}\sum_{k=1}^{2p}\frac{1}{x_k}+\frac{3}{2n^2}\sum_{k=1}^{4p}\frac{1}{\sin^2\frac{\theta_k}{2}}-\frac{3}{2n^2}\sum_{k=2p+1}^{4p}\frac{1}{\sin^2\frac{\theta_k}{2}}-$$
$$-\frac{3}{n^2}\sum_{k=1}^{2p}\frac{1}{1+x_k}-\frac{3}{4n^2}\sum_{k=1}^{4p}\frac{1}{\sin^4\frac{\theta_k}{2}}+\frac{3}{4n^2}\sum_{k=2p+1}^{4p}\frac{1}{\sin^4\frac{\theta_k}{2}}+\frac{3}{n^2}\sum_{k=1}^{2p}\frac{1}{(1+x_k)^2}.$$

Now

(12)
$$\sum_{k=2p+1}^{4p}\frac{1}{\sin^2\frac{\theta_k}{2}}=\sum_{k=2p+1}^{4p}\frac{1}{\sin^2\left(k-\frac{1}{2}\right)\frac{\pi}{8p}}=\sum_{k=1}^{2p}\frac{1}{\sin^2\left(2p+k-\frac{1}{2}\right)\frac{\pi}{8p}}=$$
$$=\sum_{k=1}^{2p}\frac{1}{\cos^2\left(k-\frac{1}{2}-2p\right)\frac{\pi}{8p}}=\sum_{k=1}^{2p}\frac{1}{\cos^2\left(k-\frac{1}{2}\right)\frac{\pi}{8p}}=\sum_{k=1}^{2p}\frac{1}{\cos^2\frac{\theta_k}{2}}.$$

Similarly

$$(13)\qquad \sum_{k=2p+1}^{4p} \frac{1}{\sin^4 \frac{\theta_k}{2}} = \sum_{k=1}^{2p} \frac{1}{\cos^4 \frac{\theta_k}{2}}.$$

From (11) on using (12) and (13) we get

$$(14)\qquad H_n(|x|, 0) = (2n^2+1) + \frac{2}{n^2} \sum_{k=1}^{2p} \frac{1}{x_k} + \frac{3}{2n^2} \sum_{k=1}^{4p} \frac{1}{\sin^2 \frac{\theta_k}{2}} - \frac{3}{2n^2} \sum_{k=1}^{2p} \frac{1}{\cos^2 \frac{\theta_k}{2}} -$$

$$-\frac{3}{n^2} \sum_{k=1}^{2p} \frac{1}{1+x_k} - \frac{3}{4n^2} \sum_{k=1}^{4p} \frac{1}{\sin^4 \frac{\theta_k}{2}} + \frac{3}{4n^2} \sum_{k=1}^{4p} \frac{1}{\cos^4 \frac{\theta_k}{2}} + \frac{3}{n^2} \sum_{k=1}^{2p} \frac{1}{(1+x_k)^2} =$$

$$= (2n^2+1) + \frac{2}{n^2} \sum_{k=1}^{2p} \frac{1}{x_k} + \frac{3}{2n^2} \sum_{k=1}^{4p} \frac{1}{\sin^2 \frac{\theta_k}{2}} - \frac{6}{n^2} \sum_{k=1}^{2p} \frac{1}{1+x_k} -$$

$$-\frac{3}{4n^2} \sum_{k=1}^{4p} \frac{1}{\sin^4 \frac{\theta_k}{2}} + \frac{6}{n^2} \sum_{k=1}^{2p} \frac{1}{(1+x_k)^2}.$$

On using the identities (a) and (b) (14) gives

$$(15)\qquad H_n(|x|, 0) = (2n^2+1) + \frac{3}{2n^2} 2n^2 - \frac{3}{4n^2} \left(\frac{8}{3} n^4 + \frac{4}{3} n^2 \right) +$$

$$+\frac{2}{n^2} \sum_{k=1}^{2p} \frac{1}{x_k} - \frac{6}{n^2} \sum_{k=1}^{2p} \frac{1}{1+x_k} + \frac{6}{n^2} \sum_{k=1}^{2p} \frac{1}{(1+x_k)^2} =$$

$$= 3 + \frac{2}{n^2} \sum_{k=1}^{2p} \frac{1}{x_k} - \frac{6}{n^2} \sum_{k=1}^{2p} \frac{1}{1+x_k} + \frac{6}{n^2} \sum_{k=1}^{2p} \frac{1}{(1+x_k)^2}.$$

From (15) using Schwartz inequality and the identity (c) we have

$$H_n(|x|, 0) \leqq 3 + \frac{2}{n^2} \left(\sum_{k=1}^{2p} \frac{1}{x_k^2} \right)^{1/2} \left(\sum_{k=1}^{2p} 1 \right) - \frac{6}{n^2} \frac{1}{2} 2p + \frac{6}{n^2} 2p =$$

$$= 3 + \frac{2}{n^2} \frac{n}{\sqrt{2}} \sqrt{\frac{n}{2}} + \frac{3}{n^2} \frac{n}{2} = 3 + \frac{1}{\sqrt{n}} + \frac{3}{2n}.$$

Therefore

$$(16)\qquad \lim_{n\to\infty} H_n(|x|, 0) \leqq 3.$$

Further from (15) we have

$$H_n(|x|, 0) \geqq 3 - \frac{6}{n^2} \sum_{k=1}^{2p} \frac{1}{1+x_k} = 3 - \frac{6}{n^2} 2p = 3 - \frac{3}{n}.$$

So that

$$\lim_{n\to\infty} H_n(|x|, 0) \geqq 3. \tag{17}$$

Hence from (16) and (17) we have

$$\lim_{n\to\infty} H_n(|x|, 0) = 3.$$

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(Received August 10, 1971.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 423—424

ON CENTROSYMMETRIC CONVEX DOMAINS WITH A PACKING DENSITY INDEPENDENT OF THE DIRECTION

by

E. MAKAI, Jr.

Let D be a convex domain, e a vector, for which D and $D+e$ (the domain obtained from D by translation through e) touch each other. Let f be another vector, for which $D+f$ touches both D and $D+e$. Then the domains $D+me+nf$ $(m, n=0, \pm 1, \pm 2, \ldots)$ form a densest lattice-packing belonging to the direction of e. If this density is independent from the direction, then D is said to have a packing density independent of the direction.

L. FEJES TÓTH [1] pointed out, that the domains of constant width, their affine images and the parallelograms have this property, and he raised the problem of finding all convex domains with a packing density independent of the direction. In a personal conversation he drew my attention to the centrosymmetric domains. We show that besides the convex domains mentioned above there are others with a packing density independent of the direction.

Let D be a centrosymmetric convex domain with a packing density independent of the direction.

Case 1. Let D have a chord PQ passing through the center O of D and let the perimeter contain a straight line segment RS parallel to PQ such that $RS \geqq \frac{1}{2} PQ$. The density of the densest lattice packing of D belonging to the direction of PQ is the ratio of the area of D to the area of the parallelogram containing D all the sides being on supporting lines of D, two being parallel to PQ, two containing P, Q. If D is not such a paralellogram, it can be placed in a centrosymmetric hexagon obtained by deleting two triangles at the opposite vertices of such a parallelogram, of which we make a lattice packing of density one. The domains D placed in these hexagons (in the above way) form a lattice packing with greater density than the above packing.

Case 2. Let D be not of the above type. Let D' be a translate of D in a direction a touching D; D'' a translate of D touching both D and D', their centers being O, O', O''. O'' is the intersection of the boundaries of the domains obtained from D and D' by magnification in ratio 2 from O and O', respectively. Since we have excluded Case 1, on one side of the line OO' O'' is uniquely determined and varies continuously with a. The endpoints of the vectors $v_1=\frac{1}{2}\overrightarrow{OO'}$, $v_2=\frac{1}{2}\overrightarrow{OO''}$, $v_3=\frac{1}{2}\overrightarrow{O'O''}$, $v_{j+3}=-v_j$, $j=1, 2, 3$ issuing from O form an affine regular hexagon inscribed in D. Vice versa, to any such hexagon we can associate two domains D' and D'' in the above manner. We suppose that $v_1, v_2, \ldots, v_6$ follow each other in the positive sense of rotation. By rotating v_1 all the v_j rotate in the same sense.

We consider the above set of affine regular hexagons inscribed in D. Without loss of generality we can suppose that the vertices of one of these hexagons are the sixth roots of unity. Then an arbitrary hexagon of this set has vertices of form $v_j = r_j(\vartheta_1)e^{i\vartheta_j}$, $0 \leqq \vartheta_1 \leqq 60°$, where $\vartheta_j = \vartheta_j(\vartheta_1)$ $(j=2, \dots, 6)$ are increasing functions of ϑ_1. Furthermore we have the following relations: $r_j(0) = r_j(60°) = 1$, $\vartheta_j(0) = (j-1) \cdot 60°$, $\vartheta_j(60°) = j \cdot 60°$, $\sqrt{3}/2 \leqq r_j$, $r_1(\vartheta_1) r_2(\vartheta_2) \cdot \sin(\vartheta_2(\vartheta_1) - \vartheta_1) = \sqrt{3}/2$.

If the boundary of D belongs to C^2 (i.e. in a neighbourhood of any of its points it has an equation of the form $y=f(x)$ or $x=f(y)$ in Cartesian coordinates, where f is twice continuously differentiable), then $\vartheta_j(\vartheta_1)$, $j=2, 3$, their inverses as well as $r_1(\vartheta_1)$ are twice continuously differentiable. Thus, in view of the monotony of $\vartheta_j(\vartheta_1)$, $\vartheta_j'(\vartheta_1) \geqq c > 0$, $j=2, 3$. Again, the condition of the convexity of D is equivalent to

$$r_j^2 + 2\left(\frac{dr_j}{d\vartheta_j}\right)^2 - r_j \frac{d^2 r_j}{d\vartheta_j^2} \geqq 0,$$

$$\frac{d^k r_j}{d\vartheta_j^k}(60°) = \frac{d^k r_{j+1}}{d\vartheta_{j+1}^k}(0), \qquad k = 1, 2.$$

The case $r_1(\vartheta_1) = 1$, $\vartheta_2(\vartheta_1) = \vartheta_1 + 60°$ corresponds to the circle. Since for these functions the left hand sides of the last two inequalities are equal to 1, any functions sufficiently near to the above ones in the C^2-metric (the C^2-distance of twice continuously differentiable functions f, g is $\sup|f-g| + \sup|f'-g'| + \sup|f''-g''|$) having the same k-th derivatives ($k=0, 1, 2$) in 0 and 60° correspond to a centrosymmetric convex domain, not belonging to Case 1, with a packing density independent of the direction.

Note. After completing this paper the author was informed by a letter of DON CHAKERIAN, written to L. FEJES TÓTH, that the convex domains with a packing density independent of the direction are precisely those, whose difference sets are the irreducible domains of MAHLER (see C. G. LEKKERKERKER, Geometry of numbers, Groningen, 1969, p. 225). MAHLER proved the existence of such domains other than the ellipse and parallelogram (Indag. Math., 9 (1947)). CHAKERIAN also points out that a construction for a large class of such domains is given in the above book of LEKKERKERKER, p. 228. It seems that the considerations given in this paper differ from those of LEKKERKERKER.

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(Received August 25, 1971.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 425—428

MAXIMA IN RANDOM WALK AND RELATED RANK ORDER STATISTICS

by

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1. Introduction. Consider $2n$ independent, identically distributed random variables x_k and y_k $(k=1, 2, \ldots, n)$ with the same continuous distribution function. Let ζ_j $(j=1, 2, \ldots, 2n)$ be the values of the combined set of x_k and y_k arranged in order of magnitude and let $\zeta_0=-\infty$; then from the independence and continuity assumptions, it follows that the probability of two ζ_j being equal is zero and we may assume that $\zeta_1<\zeta_2<\cdots<\zeta_{2n}$. On replacing each x_k by -1 and each y_k by $+1$ in this ordered set, we get a sequence of rank order indicators whose suitable functions called rank order statistics can be studied in terms of $H_n(u)\equiv H_n(u+)=n[F_n(u)-G_n(u)]$, where $F_n(u)$ $=$(number of $x_k\leqq u)/n$ is the empirical distribution function of the x_k and $G_n(u)$ is the corresponding one for y_k. Still again, the sequence $\{W_i\}$ of independent random variables having the common distribution $\mathsf{P}(W_i=+1)=p$ and $\mathsf{P}(W_i=-1)=1-p=q$, represents a sequence of rank order indicators and we study here the joint distributions of certain rank order statistics related to the maxima in the random walk $\{S_j\}$, $S_j=\sum_{i=1}^{j} W_i$, $S_0=0$ extending the results due to DWASS and others [1, 2, 3]. The basic results used for the purpose are now quoted from DWASS [1]. The assumption $p<\frac{1}{2}$ implies that the random walk $\{S_j\}$ is transient so that with probability one, $S_j=0$ for only finitely many j. Let T be the largest j for which $S_j=0$ in a given realization $\{S_j\}$, then a function U of S_j's is said to satisfy assumption A when its value is completely determined by $W_1, W_2, \ldots, W_T$. When T is specified to be $2n$, say, then

$$U(W_1, W_2, \ldots, W_T)=U_n, \text{ say}$$

is a rank order statistic and a theorem by DWASS [1] needed in the sequel is

THEOREM 1. *Suppose U_n is a rank order statistic for every n and U is the related function satisfying assumption A, then the following expansion in powers of pq holds for $0<p=1-q<\frac{1}{2}$:*

$$\frac{E(U)}{(1-2p)}=\sum_{n=0}^{\infty} E(U_n)\binom{2n}{n}(pq)^n. \tag{1}$$

In the random walk $\{S_j\}$ with last return to origin occuring at $j=2n$, i.e. $S_{2n}=0$, let D_n^+ = the one-sided maximum positive deviation,

$$=\max_{-\infty<u<\infty} H_n(u)=\max_{0\leqq j\leqq 2n} H_n(\zeta_j),$$

Q_n = the number of times D_n^+ is achieved,

and

$R_n(i)$ = the index for which D_n^+ is achieved for the ith time, $(1 \leqq i \leqq Q_n)$.

The corresponding statistics of the U-type (i.e. for the infinite random walk without the restriction $S_{2n}=0$) satisfying assumption A are D^+, Q and $R(i)$.

2. Some Relevant Generating Functions (GF)

THEOREM 2.

$$\mathsf{E}(t^{R(i)}; D^+ = r, Q = l)/(1-2p) = \left[\frac{1}{2}\psi(t)\right]^{r+i-1} p^{r+l-i}(pqt)^{-r} \tag{2}$$

where $\psi(t) = 1-(1-4pqt^2)^{\frac{1}{2}}$ *is the GF of the recurrence time from level* 0 *to* 0.

[If $p_{m,r,l} \equiv \mathsf{P}\{R(i)=m, D^+ = r, Q = l\}$, $\mathsf{E}(t^{R(i)}; D^+ = r, Q = l)$ is to be interpreted as $\sum_m p_{m,r,l}t^m$].

PROOF. A path $\{S_j\}$ contributing to (2) touches the maximum level r precisely l times and comprises $l+1$ segments as below:

Segment no. 1: A first passage through r with its length having $\mathrm{GF} = [\psi(t)/2qt]^r$.

Segments no. 2 to i: $(i-1)$ successive returns from level r or r from below forming $(i-1)$ negative waves* each one having a length with $\mathrm{GF} = \frac{1}{2}\psi(t)$.

Segments no. $(i+1)$ *to l:* Further $(l-i)$ returns from level r or r forming $(l-i)$ negative waves each occurring with probability $(q \cdot p/q) = p$.

Segment no. $(l+1)$: From level r downwards with no further return to the level r, the probability of which equals $(1-2p)$. Hence

$$\mathsf{E}(t^{R(i)}; D^+ = r, Q = l) = \left[\frac{\psi(t)}{2qt}\right]^r \left[\frac{1}{2}\psi(t)\right]^{i-1} p^{l-i}(1-2p)$$

which leads to (2).

Particular cases. (i) The GF of the index of the first maximum (or that of the last one) when the maximum D_n^+ equals r and the number of maxima Q_n equals l can be obtained from (2) by putting $i=1$ (or $=l$).

(ii) Summation of (2) over l yields for the GF of the index of the ith maximum when the maximum equals r as

$$\frac{\mathsf{E}(t^{R(i)}; D^+ = r)}{(1-2p)} = \left[\frac{1}{2}\psi(t)\right]^{r+i-1} p^{r+2-i}(pq)^{-(r+1)}t^{-r} \tag{3}$$

which for $i=1$ is in agreement with DWASS [1].

(iii) Summation of (2) over r yields the GF of the index $R(i)$ of the ith maximum when the number of maxima equals l as

$$\frac{\mathsf{E}(t^{R(i)}; Q = l)}{(1-2p)} = 2^{2-i}\psi^{i-1}(t)\frac{p^{l-i}qt}{(2qt-\psi(t))}. \tag{4}$$

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* Equivalent to sojourns in DWASS [1].

(iv) Summation of (3) over r (or of (4) over l) yields the GF of the index of the ith maximum as

$$\frac{\mathsf{E}(t^{R(i)})}{(1-2p)} = 2\psi^{i-1}(t)(2p)^{1-i}\frac{t}{(2qt-\psi(t))}. \tag{5}$$

3. Probability Distributions. For λ, μ positive integers and $0<p=1-q<\frac{1}{2}$, the following expansions, in powers of pq, which follow from (16) in [1] are needed in the sequel for deriving the probability distributions of various rank order statistics listed in section 1.

$$\psi^{\lambda}(t) = [1-(1-4pqt^2)^{1/2}]^{\lambda} = 2^{\lambda}\sum_{k=\lambda}^{\infty}\frac{\lambda}{2k-\lambda}\binom{2k-\lambda}{k-\lambda}(pqt^2)^k, \tag{6}$$

and

$$p^{\mu} = \sum_{j=\mu}^{\infty}\frac{\mu}{2j-\mu}\binom{2j-\mu}{j-\mu}(pq)^j, \tag{7}$$

whence the coefficient of $(pq)^n$ in the expansion of $p^{\mu}(pq)^{-\nu}$ is

$$\frac{\mu}{2n+2\nu-\mu}\binom{2n+2\nu-\mu}{n+\nu-\mu}, \tag{8}$$

and the coefficient of $t^g(pq)^n$ in the expansion of $[\frac{1}{2}\psi(t)]^{\lambda}p^{\mu}(pq)^{-\nu}t^{-h}$ on using (6), (7) and (8) is

$$\frac{\lambda}{g+h-\lambda}\binom{g+h-\lambda}{g/2+h/2-\lambda}\frac{\mu}{2n+2\nu-g-h-\mu}\binom{2n+2\nu-g-h-\mu}{n+\nu-g/2-h/2-\mu}. \tag{9}$$

(i) By theorem 1, the coefficient of $t^g(pq)^n$ in (2) expanded with the aid of (6), (7) and on using (9) leads to

$$\mathsf{P}(R_n(i)=g,\ D_n^+=r,\ Q_n=l) = \tag{10}$$

$$= \frac{r+i-1}{g-i+1}\binom{g-i+1}{g/2+r/2}\frac{r+l-i}{2n-g-l+i}\binom{2n-g-l+i}{n-g/2+r/2}\bigg/\binom{2n}{n}.$$

Again, on putting $t=1$ in (2), we get

$$\frac{\mathsf{P}(D^+=r,\ Q=l)}{(1-2p)} = p^{2r+l-1}(pq)^{-r}, \tag{11}$$

a result in agreement with DWASS [1].

This on using theorem 1 and results (7) and (8) gives

$$\mathsf{P}(D_n^+=r,\ Q_n=l) = \frac{2r+l-1}{2n+1-l}\binom{2n+1-l}{n+r}\bigg/\binom{2n}{n}, \tag{12}$$

verifying a known result [1].

(ii) Again by theorem 1, the coefficient of $t^g(pq)^n$ in (3) with the aid of (6), (7) and (9) leads to

$$(13)\quad \mathsf{P}(R_n(i)=g,\, D_n^+=r) = \frac{r+i-1}{g-i+1}\binom{g-i+1}{g/2+r/2}\frac{r+1}{2n+1-g}\binom{2n+1-g}{n-g/2-r/2}\bigg/\binom{2n}{n},$$

which for $i=1$ verifies another known result [1].

(iii) Summations of (10) and (13) over r yields $\mathsf{P}(R_n(i)=g,\ Q_n=l)$ and $\mathsf{P}(R_n(i)=g)$.

4. Acknowledgement. The authors are highly indebted to Dr. H. C. GUPTA, Professor of Mathematical Statistics, University of Delhi for his guidance and to the referee for his suggestions.

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(Received April 15, 1971.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 429—435

ÜBER n-ECKE

von

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Sei $\mathcal{O}$ der Operator, der jedem n-Eck A im n-dimensionalen Raum das n-Eck zuordnet, dessen Ecken die Seitenmitten von A sind. In dieser Arbeit geben wir einen durchsichtigen algebraischen Beweis für folgenden Satz: Bei Anwendung von $\mathcal{O}^k$, $k \to \infty$, nähert sich das beliebige n-Eck A immer mehr einem affinen Bild des ebenen regelmäßigen n-Ecks an. Anschließend wird untersucht, wann A „nicht-ausgeartet" ist, d.h. wann dieses affine Bild des regelmäßigen n-Ecks nicht ausgeartet ist. Der Fall eines beliebigen n-Ecks läßt sich dabei auf den Fall eines ebenen n-Ecks zurückführen. Schließlich wird eine Klasse „nichtausgearteter" ebener n-Ecke angegeben. Diese Klasse umfaßt die konvexen n-Ecke, was eine zuerst von H. REICHARDT ([2]) bewiesene Vermutung von FEJES TÓTH erneut bestätigt.

Der Grundgedanke der Überlegung ist, dem R_n die Struktur einer Algebra zu geben, diese Algebra in ihre irreduziblen Bestandteile zu zerlegen und zu untersuchen, wie sich die n-Ecke bei dieser Zerlegung verhalten. Dabei machen wir uns außerdem die Tatsache zunutze, daß ein beliebiges n-Eck affines Bild eines festen, von den Basisvektoren des R_n aufgespannten n-Ecks ist.

Der Verfasser hat sich bemüht, die Arbeit so zu schreiben, daß sie auch für den algebraisch nicht versierten Leser, gegebenenfalls unter Zuhilfenahme eines beliebigen in die Algebra einführenden Lehrbuches (z. B. [1]), zu verstehen ist. Zu diesem Zweck wird in Punkt **2** ein Satz über die Zerlegung der Algebra $K[x]/f(x)$, $K[x]$ der Polynombereich in einer Unbestimmten über dem Körper K, bewiesen. Die im Anschluß an den Beweis dieses Satzes betrachteten Beispiele, insbesondere auch die Relationen (3) und (4), werden in Punkt **3**, dem eigentlichen Kernstück der Arbeit, angewendet.

Der Verfasser möchte dem Referenten sehr herzlich für die zahlreichen Hinweise danken, die die Verständlichkeit der Arbeit erhöhten.

1. Sei V ein Vektorraum, n eine natürliche Zahl und $\tilde{V}$ die n-fache direkte Summe von V mit sich selbst. Sei auf $\tilde{V}$ der Operator $\mathcal{L}$ definiert als

$$\mathcal{L}(a_0, a_1, \ldots, a_{n-1}) = (a_1, \ldots, a_{n-1}, a_0).$$

Es ist $\mathcal{L}^n = \mathcal{E}$ (Einheitsoperator). $\mathcal{L}$ ist linear und vertauschbar mit jeder linearen Abbildung von $\tilde{V}$, die man erhält, wenn man eine lineare Abbildung von V auf jeden direkten Summanden einzeln anwendet. Dieselben Eigenschaften hat dann auch eine beliebige Linearkombination von Potenzen von $\mathcal{L}$.

Die Elemente von $\tilde{V}$ seien interpretiert als die durch die Ortsvektoren ihrer Ecken definierten n-Ecke von V.

2. Sei K ein Körper, $K[x]$ der Polynombereich in einer Unbestimmten x über K. Weiter sei $q(x) \in K[x]$ ein Polynom vom Grade g ohne mehrfache Nullstellen, $(q(x))$

sei das von $q(x)$ erzeugte Ideal in $K[x]$, und $q(x)$ zerlege sich in das Produkt von Primpolynomen

$$q(x) = \prod_i p_i(x).$$

ξ_i sei eine Nullstelle von $p_i(x)$ aus einem algebraischen Abschluß von K. Dann gilt:

(I) $$B = K[x]/(q(x))$$

ist eine g-dimensionale Algebra über K mit der K-Basis $\varepsilon, b, \dots, b^{g-1}$, *wobei* ε *das Bild der Eins und b das Bild von x beim natürlichen Homomorphismus* $K[x] \to B$ *ist.*
(II) *B zerfällt in die direkte Summe*

(+) $$B = \sum_i B_i$$

mit der Eigenschaft $B_i \cong K(\xi_i)$. *Ist in der Zerlegung* (+)

$$\varepsilon = \sum \varepsilon_i \quad \textit{und} \quad b = \sum b_i,$$

so erhält man eine K-Isomorphismus von B_i *auf den Körper* $K(\xi_i)$ *durch die Zuordnung*

$$\varepsilon_i \to 1 \ (= \textit{Eins von } K)$$

$$b_i \to \xi_i.$$

Beweis. (I) folgt aus dem bekannten Homomorphiesatz für Ringe. (II) soll im Folgenden kurz bewiesen werden. Wir konstruieren dazu in B die orthogonalen Idempotente $\varepsilon_i \in B$. Sei $f_i(x) = q(x)/p_i(x)$ und $h_i(x)$ das Polynom kleinsten Grades mit der Eigenschaft

$$h_i(x) f_i(x) \equiv 1 \bmod p_i(x).$$

Wir definieren

$$\varepsilon_i = f_i(b) h_i(b).$$

Die Relation $\varepsilon_i \varepsilon_j = 0$, für $i \neq j$, folgt sofort. Weiter ist

$$h_i(x) f_i(x) = 1 + \lambda_i(x) p_i(x),$$

woraus folgt

$$h_i(x) q(x) = p_i(x) + \lambda_i(x) p_i^2(x).$$

Andererseits ist

$$\big(h_i(x) f_i(x)\big)^2 = 1 + 2\lambda_i(x) p_i(x) + \lambda_i^2(x) p_i^2(x) =$$

$$= 1 + \lambda_i(x)\big[p_i(x) + \big(p_i(x) + \lambda_i(x) p_i^2(x)\big)\big] \equiv 1 + \lambda_i(x) p_i(x) \bmod \big(q(x)\big).$$

Damit gilt $\varepsilon_i^2 = \varepsilon_i$. Außerdem ist $\sum \varepsilon_i = \varepsilon$, da $\sum h_i(x) f_i(x) \equiv 1 \mod q(x)$ ist.

Der Leser verifiziert nun ohne Schwierigkeit die direkte Zerlegung

$$B = \sum_i B\varepsilon_i \qquad (B\varepsilon_i = B_i).$$

Weiter finden wir

$$p_i(b_i) = p_i(b\varepsilon_i) = p_i(b)\varepsilon_i = p_i(b) f_i(b) h_i(b) = q(b) h_i(b) = 0,$$

b_i ist also Nullstelle von $p_i(x)$, letzteres aufgefaßt als Polynom über dem zu K isomorphen Körper $K\varepsilon_i$. Der zweite Teil der Behauptung (II) folgt nun aus elementaren Sätzen der Körpertheorie.

Beispiele **A** *und* **K**.

Sei $n>2$ eine feste natürliche Zahl, **R** der Körper der reellen Zahlen, **C** der Körper der komplexen Zahlen, $\zeta = e^{2\pi i/n}$.

(1) Wir betrachten die Algebra

$$\mathbf{A} = \mathbf{R}[x]/(x^n - 1)\,.$$

A ist n-dimensional über R, eine Basis sei $\mathbf{1}, \mathbf{e}, \mathbf{e}^2, \ldots, \mathbf{e}^{n-1}$. Gemäß der Zerfällung in $\mathbf{R}[x]$

$$x^n - 1 = \prod_j g_j(x), \quad g_j(\zeta^j) = 0, \qquad j \in J,$$

$J = \{\iota : 0 \leqq \iota \leqq n/2,\ \iota \text{ ganz}\}$, zerlegt sich **A** in die direkte Summe

$$\mathbf{A} = \sum_j S_j, \quad \mathbf{1} = \sum_j 1_j, \quad \mathbf{e} = \sum_j e_j$$

und es existieren R-Isomorphismen

$$\varkappa_j : S_j \to \begin{cases} \mathbf{C}, & 1 \leqq j < n/2 \\ \mathbf{R}, & j = 0,\ n/2 \end{cases}, \quad j \in J,$$

mit der Eigenschaft

$$\varkappa_j 1_j = 1, \quad \varkappa_j e_j = \zeta^j.$$

(2) Wir betrachten die Algebra

$$\mathbf{K} = \mathbf{C}[x]/(x^n - 1)\,.$$

K ist n-dimensional über **C** und hat die **C**-Basis $\mathbf{1}, \mathbf{e}, \ldots, \mathbf{e}^{n-1}$. **A** ist in **K** eingebettet, die Elemente von **A** sind die Linearkombinationen mit reellen Koeffizienten. **K** zerfällt gemäß der Zerlegung

$$x^n - 1 = \prod_{l=1}^{n} (x - \zeta^l)$$

in die direkte Summe

$$\mathbf{K} = \sum_l J_l, \quad \mathbf{1} = \sum_l t_l,$$

die direkten Komponenten von **e** sind

$$\mathbf{e} t_l = \zeta^l t_l,$$

da $\mathbf{e}t_l$ beim durch $t_l \to 1$ definierten **C**-Isomorphismus $T_l \to \mathbf{C}$ in ζ^l übergehen muß.

Wir wollen jetzt die orthogonalen Idempotente t_l berechnen. Es ist $\mathbf{e} = \sum_l \zeta^l t_l$, folglich $\mathbf{e}^\nu = \sum_l \zeta^{l\nu} t_l$. Wir betrachten die direkte Zerlegung des Vektors

$$\mathbf{w}_\lambda = \sum_{\nu=0}^{n-1} \zeta^{\lambda\nu} \mathbf{e}^\nu, \qquad 0 \leqq \lambda < n.$$

Es ist

$$\mathbf{w}_\lambda = \sum_\nu \zeta^{\lambda\nu} \mathbf{e}^\nu = \sum_\nu \zeta^{\lambda\nu} \sum_l \zeta^{l\nu} t_l = \sum_l \Big(\sum_\nu \zeta^{(l+\lambda)\nu}\Big) t_l.$$

Nun gilt

$$\sum_{\nu} \zeta^{(l+\lambda)\nu} = \begin{cases} n \text{ für } l=n-\lambda \\ 0 \text{ für } l\neq n-\lambda. \end{cases}$$

Also ist $\mathbf{w}_\lambda = nt_{n-\lambda}$. Folglich haben wir

$$t_l = \frac{1}{n} \sum_{\nu} \zeta^{(n-l)\nu} \mathbf{e}^{\nu}. \tag{3}$$

Aufgrund der Einbettung von S_j in $T_j + T_{n-j} = \mathbf{C}[x]/(g_j(x))$, $1 \leqq j \leqq n/2$, bzw. in T_j, $j=0, n/2$, erhalten wir außerdem

$$1_j = \begin{cases} t_j + t_{n-j}, & 1 \leqq j < n/2 \\ t_j, & j=0, \ n/2 \end{cases}, \quad j \in J. \tag{4}$$

Schließlich sehen wir, daß für $1 \leqq j < n/2$ t_j und t_{n-j} konjugiert komplex sind und $\{t_j + t_{n-j}, i(t_j - t_{n-j})\}$ eine $\mathbf{R}$-Basis von S_j bildet.

3. Die Algebra $\mathbf{A}$ (vgl. Punkt **2**, Beispiel (1)) wollen wir interpretieren als affinen n-dimensionalen Vektorraum R_n. $\tilde{\mathbf{A}}$ ist dann die Gesamtheit der n-Ecke im R_n (siehe Punkt **1**). Sei π_j die Projektion von $\mathbf{A}$ auf S_j. Das n-Eck

$$E = (\mathbf{1}, \mathbf{e}, \mathbf{e}^2, \ldots, \mathbf{e}^{n-1})$$

wird durch $\varkappa_j \, \pi_j$ abgebildet auf das n-Eck in der komplexen Zahlenebene

$$C_j = (1, \zeta^j, \zeta^{2j}, \ldots, \zeta^{(n-1)j}).$$

C_1 ist das regelmäßige n-Eck, $C_j, j>1$, bei $(j, n)=1$ eine einmal durchlaufene sternförmige Figur und bei $(j, n)>1$ eine mehrfach durchlaufene Figur, C_0 ist ein einziger n-fach überdeckter Punkt.

$\tilde{\mathbf{A}}$ hat die Struktur eines $\mathbf{A}$-Modulus, wenn man die Multiplikation mit elementen von $\mathbf{A}$ komponentenweise (gemäß Punkt **1**) erklärt. In $\tilde{\mathbf{A}}$ betrachten wir den Operator

$$\mathcal{U} = \frac{1}{m}(\mathcal{E} + \mathcal{L}), \qquad m = |1+\zeta|,$$

(vgl. Punkt **1**). Der Operator $\mathcal{U}$ ist von dem Operator $\mathcal{O}$ (siehe Einleitung) nicht wesentlich verschieden; der Normierungsfaktor $1/m$ ist gerade so gewählt, daß das regelmäßige n-Eck C_1 von $\mathcal{U}$ (bis auf eine Drehung) in sich übergeführt wird. ($\mathcal{O}$ hat, wie man leicht sieht, die Gestalt $\frac{1}{2}(\mathcal{E} + \mathcal{L})$.)

Es ist $\mathcal{U}E = \mathbf{d}E$, $\mathbf{d} = (\mathbf{1}+\mathbf{e})/m$, also

$$\mathcal{U}^k E = \mathbf{d}^k E, \qquad k = 1, 2, \ldots$$

Wie verhält sich $\mathcal{U}^k E$ für $k \to \infty$? Wir untersuchen die direkten Komponenten des Vektors $\mathbf{d}^k$ (vgl. Punkt **2**, (1))

$$\varkappa_j \pi_j \mathbf{d}^k = \left(\frac{1+\zeta^j}{m} \right)^k = c_j^k$$

und unterscheiden 3 Fälle:

1) $j=1: c_1^k = (e^{2\pi i/2n})^k, =1$ für $k \equiv 0 \bmod 2n$.

2) $j>1: |c_j| = \left|\frac{1+\zeta^j}{1+\zeta}\right| < 1$, folglich ist $\lim_{k\to\infty} c_j^k = 0$.

3) $j=0$. Diesen Fall können wir außer acht lassen, da C_0 nur ein einziger Punkt ist, die S_0-Komponente folglich nur eine Verschiebung der n-Ecke bewirkt.

Für das Folgende nehmen wir $k \equiv 0 \bmod 2n$ an und beachten die Vertauschbarkeit einer linearen Abbildung mit der Limes-Bildung und dem Operator $\mathcal{U}^k$ (vgl. Punkt **1**). Zunächst gilt

$$\lim_{k\to\infty} \mathcal{U}^k E = \varkappa_1^{-1} C_1.$$

Sei schließlich ein beliebiges n-Eck $A=(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{n-1})$ gegeben und Φ die durch

$$(*) \qquad \Phi \mathbf{e}^l = \mathbf{a}_l, \qquad l=0, 1, \dots, n-1,$$

definierte Affinität. Dann ist

$$\lim_{k\to\infty} \mathcal{U}^k A = \Phi \varkappa_1^{-1} C_1.$$

Beachten wir noch, daß $\varkappa_1^{-1}$ eine nichtausgeartete Affinität ist, so haben wir den

SATZ. *Jedes n-Eck A im R_n nähert sich bei Anwendung von $\mathcal{U}^k$, $k\to\infty$, immer mehr einem affinen Bild des zwei-dimensionalen regelmäßigen n-Ecks an. Dieses affine Bild ist dann und nur dann ausgeartet, wenn die durch A definierte Affinität $(*)$ auf S_1 ausgeartet ist.*

A heiße *nichtausgeartet*, falls ΦS_1 zweidimensional, *vollkommen ausgeartet*, falls $\Phi S_1 = 0$ ist.

Es ist $\Phi S_1 = 0$ dann und nur dann, wenn Φ eine R-Basis von S_1 in Null überführt. Aus der letzten Bemerkung in Punkt **2** folgt sofort: Es ist $\Phi S_1 = 0$ genau dann, wenn $\Phi t_1 = 0$ ist, und genau dann, wenn $\Phi t_{n-1} = 0$ ist. Zum Beweis haben wir nur noch zu beachten, daß Φ die Basis auf reelle Vektoren abbildet. Durch Einsetzen von t_{n-1} (Punkt **2**, (3)) erhalten wir den

SATZ. *A ist dann und nur dann vollkommen ausgeartet, wenn*

$$\sum_{\nu=0}^{n-1} \zeta^\nu \mathbf{a}_\nu = 0$$

ist.

Die Relation $\sum_\nu \zeta^\nu \mathbf{a}_\nu = 0$ ist äquivalent zu den beiden reellen Relationen

$$\sum_\nu \sin\frac{2\pi\nu}{n} \mathbf{a}_\nu = 0, \quad \sum_\nu \cos\frac{2\pi\nu}{n} \mathbf{a}_\nu = 0.$$

Weiter stellen wir fest:

(1) Das n-Eck A ist genau dann nichtausgeartet, wenn es eine Parallelprojektion ψ von A auf eine Ebene gibt, so daß ψA nichtausgeartet ist.

(2) Das ebene n-Eck A ist genau dann nichtausgeartet, wenn für alle Parallelprojektionen φ der Ebene auf die Gerade $\sum_\nu \zeta^\nu \varphi \mathbf{a}_\nu \neq 0$ ist.

(3) Das ebene n-Eck ist nichtausgeartet, wenn es für jede Richtung r eine zu r parallele Gerade R gibt mit den Eigenschaften:
  (a) R schneidet das Polygon A in genau 2 Punkten.
  (b) Die Anzahl der in jeder der beiden durch R definierten offenen Halbebenen liegenden Ecken von A ist $\leqq n/2$.

(4) Jedes konvexe n-Eck ist nichtausgeartet.

BEWEIS von (1). Die durch das n-Eck ψA definierte Affinität $(*)$ ist gerade $\psi\Phi$. Ist $\psi\Phi$ auf S_1 nicht ausgeartet, dann kann es auch Φ nicht sein. Sei umgekehrt Φ auf S_1 nicht ausgeartet; dann ist ΦS_1 also genau wie S_1 ein zweidimensionaler Unterraum des R_n. Wir stellen den R_n dar als direkte Summe von ΦS_1 und einem Unterraum S und definieren ψ als die Parallelprojektion länge S auf die Ebene ΦS_1. $\psi\Phi$ ist dann auf S_1 nicht ausgeartet.

BEWEIS von (2). Ist A nichtausgeartet, so stimmt ΦS_1 mit der Ebene überein, auf der A liegt. Diese wird von φ stets in eine Gerade, also nie in 0 übergeführt. Ist A dagegen ausgeartet, so ist ΦS_1 ein höchstens eindimensionaler Unterraum dieser Ebene, der von einer Parallelprojektion φ in 0 übergeführt wird.

BEWEIS von (3). Wir betrachten A in einem ebenen Koordinatensystem, dessen y-Achse die Gerade R ist, und numerieren die Ecken $\mathbf{a}_\nu$ von A, indem wir vom unteren Schnittpunkt P von R mit A ausgehen und im Gegenuhrzeigersinn auf A entlanglaufen. Ist P eine Ecke, so beginnen wir mit dem Zählen entweder bei der ersten von P verschiedenen Ecke oder bei P, je nachdem in der rechten offenen Halbebene $[n/2]$ oder weniger als $[n/2]$ Ecken liegen. φ sei die Parallelprojektion längs R. Dann sind die Summanden in

$$(**) \qquad \sum_\nu \sin \frac{2\pi\nu}{n} \varphi \mathbf{a}_\nu$$

alle nichtnegativ. Da nicht alle gleich Null sein können, muß die Summe ungleich Null sein.

Es sei darauf hingewiesen, daß die Forderung (3b) für jedes ebene n-Eck erfüllt ist.

BEWEIS von (4). Ein konvexes n-Eck A läßt sich dadurch charakterisieren, daß jede Gerade, die A echt schneidet, das Polygon A in genau 2 Punkten schneidet. Damit erfüllt jedes konvexe n-Eck das Kriterium (3).

Die Klasse der durch (3) charakterisierten ebenen n-Ecke ist echt größer als die Klasse der konvexen n-Ecke; man nehme als einfachstes Beispiel einen regelmäßigen, doppelpunktfreien Stern. Es wird nicht schwer sein, (3) noch zu verschärfen. Anhaltspunkt hierfür ist die Summe $(**)$, in der man ja auch „kleine" negative Summanden zulassen kann.

Eine markante Klasse von n-Ecken sind die doppelpunktfreien. Es gibt ausgeartete ebene doppelpunktfreie n-Ecke: Man gehe z. B. für $n=5$ aus von C_2, belasse die x-Koordinaten und wähle die y-Koordinaten so, daß eine doppelpunktfreie Figur entsteht. Jedoch könnte es stimmen, daß ein ebenes doppelpunktfreies n-Eck nicht vollkommen ausgeartet sein kann.

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(Eingegangen: 10. September, 1971.)

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 437—449

ON A GENERALIZATION OF ABEL—POISSON'S SINGULAR INTEGRAL HAVING KERNELS OF FINITE OSCILLATION

by

EBERHARD L. STARK

Dedicated to Professor Fritz Reutter on the occasion of his 60 th birthday on August 26, 1971.

1. Introduction. The starting point of this investigation is the singular integral of Abel—Poisson

(1) $$I_r(\mathsf{p};f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)p_r(t)\,dt \qquad (r\to 1-),$$

where $f(x)$ is a function of period 2π which is assumed to be continuous ($f\in C_{2\pi}$) or integrable to the pth power ($f\in L^p_{2\pi}$, $1\leqq p<\infty$). The kernel $p_r(x)$ is either given in its closed form

(2) $$p_r(x) = \frac{1}{2}\,\frac{1-r^2}{1-2r\cos x+r^2} > 0 \qquad (0\leqq r<1)$$

or in its Fourier series representation

(3) $$\text{(i)}\quad p_r(x) = \frac{1}{2}+\sum_{k=1}^{\infty} \varrho_{k,r}(\mathsf{p})\cos kx, \qquad \text{(ii)}\quad \varrho_{k,r}(\mathsf{p}) = r^k.$$

It is a known fact (see e.g. [2] and the literature cited there) that the limit

(4) $$\text{(i)}\quad \lim_{r\to 1-}\frac{1-\varrho_{k,r}(\mathsf{p})}{1-r} = \psi(\mathsf{p};k), \qquad \text{(ii)}\quad \psi(\mathsf{p};k) = k \qquad (k=1,2,3,\dots)$$

completely determines the saturation behaviour of (1). The purpose of this paper is to generalize the kernel (2) by modifying the convergence factors (3, ii) in such a fashion that the saturation order of $O(1-r)$, $r\to 1-$, in (4) is improved. Furthermore, the new saturation function ψ is to be treated in connection with the theory of singular integrals having kernels of finite oscillation (cf. [2—4], [13], [9], [17, 18]). Here, an even kernel is said to be of finite oscillation of degree $2m$ if there exist exactly m changes of sign (zeros of odd multiplicity) in the interval $(0, \pi)$, the number of these zeros being independent of the approximation parameter. A discussion of representative examples will show that these new kernels cover some important singular integrals which have been considered in connection with e.g. the solution of partial differential equations.

2. A generalization of Abel—Poisson's singular integral

We set up

$$\varrho_{k,r}(\mathsf{h}) = r^k\{1+k(1-r)h(r)\} \tag{5}$$

with a suitable variation function $h(r)$ defined on $0<r_0\leqq r\leqq 1$; the consistence with the original factors (3, ii) is given by $h(r)\equiv 0$.

LEMMA 1. *If $h(r)$ is continuously differentiable for $r_0\leqq r\leqq 1$ with $h(1)=1$, then*

$$\lim_{r\to 1-}\frac{1-\varrho_{k,r}(\mathsf{h})}{(1-r)^2} = \psi(\mathsf{h};k), \tag{6}$$

$$\psi(\mathsf{h};k) = \frac{1}{2}k\left\{k+2\left[h'(1)+\frac{1}{2}\right]\right\} \qquad (k=0,1,2,\ldots). \tag{7}$$

The PROOF follows immediately by applying de L'Hospital's rule twice, giving

$$\lim_{r\to 1-}\frac{1-r^k\{1+k(1-r)h(r)\}}{(1-r)^2} = \frac{k}{2}\lim_{r\to 1-} r^{k-1}\left\{[kh(r)+rh'(r)]+\frac{1-rh(r)}{1-r}\right\} =$$

$$= \frac{k}{2}\left\{k+h'(1)+\lim_{r\to 1-}\frac{1-rh(r)}{1-r}\right\} = \frac{k}{2}\{k+h'(1)+[1+h'(1)]\}.$$

Thus, if (5) may be interpreted as convergence factors and hence (6) as the saturation limit of an approximation process which is generated via $h(r)$, the saturation order would be improved up to $O([1-r]^2)$, $r\to 1-$. In comparison with (4, ii), the saturation function (7) which only depends upon the single value $h'(1)$ of the variation function is now always a second degree algebraic polynomial of the integer variable k. (In the particular case $h'(1)=-1/2$ the saturation function (7) reduces to $\psi(\mathsf{h};k)=k^2/2$; this plays an extraordinary rôle for positive kernels. For detailed literature see [8].)

Therefore it remains to show that the factors (5) with functions $h(r)$ of Lemma 1 indeed produce kernels, i.e., sequences of functions belonging to $L^1_{2\pi}$, uniformly for $r_0\leqq r<1$, $r\to 1-$.

In this respect we establish the closed representation of these functions. We have

$$p_r(\mathsf{h};x) \equiv \frac{1}{2}+\sum_{k=1}^{\infty}\varrho_{k,r}(\mathsf{h})\cos kx = p_r(x)+(1-r)h(r)z_r(x) \tag{8}$$

with [16, p. 220 (94)]

$$z_r(x) \equiv \sum_{k=1}^{\infty} kr^k\cos kx = r\frac{(1+r^2)\cos x-2r}{(1-2r\cos x+r^2)^2} = \tag{9}$$

$$= \frac{4r(1+r^2)}{(1-r^2)^2}\left(\cos x-\frac{2r}{1+r^2}\right)p_r^2(x) \qquad (0\leqq r\leqq 1).$$

In order to illuminate the background of the initial construction (5) we consider briefly the graph of (9). Apart from $z_r(x)=z_r(-x)$ and $\int_0^\pi z_r(t)\,dt=0$ for all r, this sequence is characterized by the fact that there are two symmetrical changes of sign tending to zero for $r\to 1$, given via

$$\cos x_0=\frac{2r}{1+r^2},\quad \lim_{r\to 1-}\frac{1-\cos x_0}{(1-r)^2}=\frac{1}{2},\quad x_0(r)=(1-r)+o(1-r).$$

Moreover, (9) attains a (negative) minimum at $x_1>x_0$, and

$$z_r'(x_1)=0,\quad z_r(x_1)=-\frac{1}{8}\frac{(1+r^2)^2}{(1+r)^2}\frac{1}{(1-r)^2}$$

$$\cos x_1=\frac{8r^2-(1+r^2)^2}{2r(1+r^2)},\quad \lim_{r\to 1-}\frac{1-\cos x_1}{(1-r)^2}=\frac{3}{2},\quad x_1(r)=\sqrt{3}\,(1-r)+o(1-r).$$

The relative maxima are given by

$$z_r(0)=\frac{r}{(1-r)^2},\quad z_r(\pi)=\frac{r}{(1+r)^2}. \tag{10}$$

Concerning the order of growth at these points and having in mind the factor $(1-r)$ on the right-hand side of (8), (10) may be compared with Harnack's inequality (cf. [2, p. 53], [5]) for the Abel—Poisson kernel (actually strictly positive and strictly decreasing on $[0,\pi]$)

$$\frac{1+r}{2(1-r)}=p_r(0)>p_r(x)>p_r(\pi)=\frac{1-r}{2(1+r)}\qquad (0<x<\pi). \tag{11}$$

compare also with (18), (19).

Using (9) the series (8) passes into the suitable closed expression

(12)
$$p_r(\mathrm{h};x)=p_r^2(x)\frac{4r(1+r^2)}{(1-r)(1+r)^2}\left(h(r)-\frac{1+r}{1+r^2}\right)\left(\cos x-\frac{4r^2h(r)-(1+r)(1+r^2)}{2r[(1+r^2)h(r)-(1+r)]}\right).$$

We note that (12) is not indeterminated in case the denominator on the far right-hand side vanishes; for this particular variation function a direct calculation yields

$$h_D(r)=\frac{1+r}{1+r^2},\quad p_r(\mathrm{h}_D;x)=2\frac{1-r^2}{1+r^2}p_r^2(x),\quad \psi(\mathrm{h}_D;k)=\frac{1}{2}k^2, \tag{13}$$

obviously a positive function, and in view of the normalization in (8) actually a positive kernel.

Now, we shall show that the Lebesgue constants of (8), i.e.,

$$L_r(\mathrm{h})\equiv\int_0^\pi |p_r(\mathrm{h};t)|\,dt\qquad (r_0\le r<1,\ r\to 1-), \tag{14}$$

are bounded for functions $h(r)$ of Lemma 1. But this follows immediately from the representation

$$p_r(\mathrm{h}; x) = p_r(x)\left\{1+\frac{2r}{1+r}h(r)\left[\frac{(1-r)^2}{1-2r\cos x+r^2}-\frac{2(1+r^2)\sin^2(x/2)}{(1-r)^2+4r\sin^2(x/2)}\right]\right\}$$

using the inequalities (for the first one see (11))

$$\frac{(1-r)^2}{1-2r\cos x+r^2}\leqq 1, \qquad \frac{2(1+r^2)\sin^2(x/2)}{(1-r)^2+4r\sin^2(x/2)}\leqq\frac{1}{r};$$

hence

$$L_r(\mathrm{h})\leqq\big(1+2|h(r)|\big)\frac{\pi}{2}.$$

Later on — see (36) — we shall establish asymptotic expansions for this quantity. Here we made no use of the fact that for certain $h(r)$ the functions $p_r(\mathrm{h}; x)$ are in fact positive, as is already seen from (13), for example. In these cases, obviously $L_r(\mathrm{h})\equiv\pi/2$ for all r by the normalized series representation (8). Thus, kernels which are not strictly positive are of main interest.

The next step is concerned with conditions upon the variation function revealing the distribution of signs of $p_r(\mathrm{h}; x)$.

LEMMA 2. *Let $h(r)$ satisfy the conditions of Lemma 1. If*

$$(15)\qquad -\frac{1+r}{2r}<h(r)\leqq\frac{1+r}{2r}\qquad (r_0\leqq r<1),$$

then it generates positive kernels $p_r(\mathrm{h}; x)\geqq 0$. If

$$(16)\qquad h(r)>\frac{1+r}{2r}\qquad (r_0\leqq r<1),$$

then the kernels $p_r(\mathrm{h}; x)$ have two symmetrical changes of sign at $\pm x_0(r)$ with $0<x_0(r)<\pi$, determined explicitly by

$$(17)\qquad \cos x_0(r)=\frac{4r^2h(r)-(1+r)(1+r^2)}{2r[(1+r^2)h(r)-(1+r)]}.$$

This lemma is easily verified using the representation (12) of $p_r(\mathrm{h}; x)$. — Furthermore, the left-hand side of (15) is a basic condition in order that $p_r(\mathrm{h}; 0)>0$ as is seen from

$$(18)\qquad p_r(\mathrm{h}; 0)=\frac{r}{(1-r)}\left\{\frac{1+r}{2r}+h(r)\right\};$$

this means that $p_r(\mathrm{h}; x)$ should have the usual peaking property of a kernel at the origin (on the other hand, in view of Lemma 1, $p_r(\mathrm{h}; x)$ cannot vanish for $x=0$ since then $h(1)=-1$). A first hint indicating (16) can be read off from the behaviour of $p_r(\mathrm{h}; x)$ at the test point $x=\pi$, namely

$$(19)\qquad p_r(\mathrm{h}; \pi)=\frac{r(1-r)}{(1+r)^2}\left\{\frac{1+r}{2r}-h(r)\right\}.$$

In view of (15) and (16), the particular separation function

$$h_S(r) = \frac{1+r}{2r}, \tag{20}$$

which delivers a positive kernel with one fixed zero (of multiplicity two) at $x_0=\pi$, namely

$$p_r(h_S; r) = 4\frac{1-r}{1+r} p_r^2(x)\cos^2\frac{x}{2}, \quad \psi(h_S; k) = \frac{1}{2}k^2, \tag{21}$$

is of fundamental interest in characterizing the inner structure of the general positive kernels (12) as well as of the oscillating kernels with regard to the distribution of the corresponding zeros $x_0(r)$. In this respect, the differentiability properties of (20), namely

$$\text{(i)}\ \ h_S'(1) = -\frac{1}{2}; \qquad \text{(ii)}\ \ h_S''(1) = 1; \qquad \text{(iii)}\ \ h_S^{(\nu)}(1) = (-1)^\nu \frac{1}{2}\nu! \qquad (\nu=1, 2, 3, \dots) \tag{22}$$

and thus the Taylor series expansion

$$h_S(r) = 1+\frac{1}{2}\sum_{\nu=1}^{\infty}(1-r)^\nu \qquad (0<r\leqq 1) \tag{23}$$

(all coefficients being equal and positive), illuminate the different types of oscillating kernels to be described below. There are three cases.

LEMMA 3. *Let $h(r)$ of Lemma 1 satisfy condition* (16).

(a) *If $h'(1) < -1/2$, then the zeros of the kernel* (12) *approach the origin according to*

$$x_0(r) = \frac{\sqrt{2}\sqrt{1-r}}{\sqrt{-[h'(1)+1/2]}} + o(\sqrt{1-r}) \qquad (r\to 1-). \tag{24}$$

(b) *If, in addition, $h(r)$ is twice differentiable with*

$$h(1) = 1, \quad h'(1) = -\frac{1}{2}, \quad h''(1) > 1,$$

then the positive zero of (12) *tends to a fixed zero $\bar{x}_0+$, $0<\bar{x}_0<\pi$, according to*

$$\lim_{r\to 1-} \cos x_0(r) = 1-\frac{2}{h''(1)} \equiv \cos\bar{x}_0. \tag{25}$$

(c) *If $h(r)$ possesses the particular decomposition*

$$h(r) = \frac{1+r}{2r}\,\frac{(1+r^2)-2\alpha r}{2r-\alpha(1+r^2)} \qquad (-1\leqq\alpha<1), \tag{26}$$

then the associated kernel has fixed zeros determined via

$$\cos x_0 = \alpha \qquad (0<x_0\leqq\pi). \tag{27}$$

Indeed, concerning the proof of (a), under the given hypotheses the following limit holds for (17):

$$\lim_{r\to 1-} \frac{1-\cos x_0(r)}{1-r} = \frac{-1}{h'(1)+1/2}.$$

This immediately gives (24). In case (b), starting again from (17) it follows that

$$\lim_{r\to 1-} \frac{4r^2h(r)-(1+r)(1+r^2)}{2r[(1+r^2)h(r)-(1+r)]} = \frac{h''(1)-2}{h''(1)} = \text{const};$$

here the limit satisfies the inequalities

$$-1 < \frac{h''(1)-2}{h''(1)} < 1.$$

This yields the assertion. Part (c) is inferred from (17) by extracting $h(r)$ in accordance with (b), since for $-1<\alpha<1$

$$h(1) = 1, \quad h'(1) = -\frac{1}{2}, \quad h''(1) = 1+\frac{1+\alpha}{1-\alpha} = \frac{2}{1-\alpha} > 1.$$

The value $\alpha=-1$ again delivers $h_S(r)$ of (20), and this for the point $x_0=\pi$.

Further detailed information on the structure of these kernels may again be taken from the test point $x=\pi$; from (19) it follows for $h'(1)\neq -1/2$ that

$$(28) \qquad p_r(\mathsf{h};\pi) = \frac{1}{4}\left[h'(1)+\frac{1}{2}\right](1-r)^2+o([1-r]^2) \qquad (r\to 1-).$$

This characterizes the order of growth at $x=\pi$ for oscillating kernels of Lemma 3(a), thus with changes of sign tending to zero. For $h'(1)=-1/2$, $h''(1)\neq 1$ one has an improved order, namely

$$(29) \qquad p_r(\mathsf{h};\pi) = \frac{1}{8}[1-h''(1)](1-r)^3+o([1-r]^3) \qquad (r\to 1-),$$

clearing up the situation for kernels of Lemma 3(b), (c) with zeros which do not approach the origin.

On the other hand, (28) and (29) may also be used in analyzing the growth of positive kernels at $x=\pi$; here the connection of the generating $h(r)$ which satisfies (15) with (22) and the expansion (23) of $h_S(r)$, respectively, is obvious: the longer the expansion of $h(r)$ coincides with (23), the higher the order of $p_r(\mathsf{h};\pi)$.

We note that, in contrast to these different cases, the behaviour at the peaking point $x=0$ is asymptotically the same for all kernels; this is a consequence of (18).

Thus, the graph of these kernels is essentially influenced by the higher order derivatives of $h(r)$ at $r=1$, whilst the saturation function (7) is completely determined merely by the first derivative $h'(1)$.

3. Asymptotic expansions of Lebesgue constants, main theorem

As a final observation in this respect we shall establish asymptotic expansions for the Lebesgue constants (14). An easy calculation shows that

$$(30)\quad L_r(\mathrm{h}) = \Big(\int_0^{x_0(r)} - \int_{x_0(r)}^{\pi}\Big) p_r(\mathrm{h};t)\,dt = 2\int_0^{x_0(r)} p_r(\mathrm{h};t)\,dt - \frac{\pi}{2} \qquad (0 < x_0(r) \leq \pi)$$

and

$$(31)\quad \int_0^{x_0(r)} p_r(\mathrm{h};t)\,dt = \frac{1}{2}x_0(r) + \sum_{k=1}^{\infty}\frac{1}{k}r^k\sin kx_0(r) + (1-r)h(r)\sum_{k=1}^{\infty} r^k \sin kx_0(r) =$$

$$= \frac{1}{2}x_0(r) + \arctan\frac{r\sin x_0(r)}{1-r\cos x_0(r)} + (1-r)h(r)\frac{r\sin x_0(r)}{1-2r\cos x_0(r)+r^2}.$$

Here we used [15, p. 39], [16, p. 211 (1; 5), p. 220 (90)]

$$\sum_{k=1}^{\infty} r^k \sin kx = \frac{r\sin x}{1-2r\cos x + r^2} \qquad (|r|<1),$$

$$\sum_{k=1}^{\infty}\frac{1}{k}r^k\sin kx = \arctan\frac{r\sin x}{1-r\cos x} \qquad (0<x<2\pi;\ r^2\leq 1).$$

This, together with (17), yields for (30)

$$(32)\quad L_r(\mathrm{h}) = x_0(r) - \frac{\pi}{2} + 2\arctan\frac{\sqrt{4r^2h^2(r)-(1+r)^2}}{2h(r)-(1+r)} + \frac{\sqrt{4r^2h^2(r)-(1+r)^2}}{(1+r)} \qquad (0<r<1);$$

for the particular case (26), (27) it follows from (31) that

$$(33)\quad L_r(\mathrm{h}) = \arccos\alpha - \frac{\pi}{2} + 2\arctan\frac{r\sqrt{1-\alpha^2}}{1-\alpha r} + (1-r)\frac{(1+r)\sqrt{1-\alpha^2}}{2r-\alpha(1-r^2)} \qquad (-1\leq\alpha<1;\ 0<r<1).$$

Concerning (32) and $h'(1) < -1/2$, thus kernels of Lemma 3(a), the following expansions hold $(r \to 1-)$:

$$(34)\quad \frac{\sqrt{4r^2h^2(r)-(1+r)^2}}{2h(r)-(1+r)} = \frac{\sqrt{2}\sqrt{-[h'(1)+1/2]}}{1/2-h'(1)}\frac{1}{\sqrt{1-r}} + o([1-r]^{-1/2}),$$

$$(35)\quad \frac{\sqrt{4r^2h^2(r)-(1+r)^2}}{1+r} = \sqrt{2}\sqrt{-[h'(1)+1/2]}\sqrt{1-r} + o([1-r]^{1/2}).$$

But, (34) confines that there exists r_0, $0<r_0<1$, such that the left-hand side is greater than 1 for r with $r_0<r<1$; thus, using ([15, p. 48])

$$\arctan x = \frac{\pi}{2} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{2j+1}x^{-(2j+1)} \qquad (x>1)$$

as well as (24), (34), (35), the expansion of (32) for variation functions of Lemma 3(a) reduces to

$$(36) \qquad L_r(\mathsf{h}) = \frac{\pi}{2} + o\left(\sqrt{1-r}\right) \qquad (r_0 < r < 1;\ r \to 1-).$$

This delivers a final fundamental characterization for such kernels of finite oscillation of second degree with zeros approaching the origin, namely

$$(37) \quad \begin{cases} \text{(i)} & \int_0^\pi p_r(\mathsf{h};t)\,dt = \frac{\pi}{2} \qquad (0<r<1), \\ \text{(ii)} & \int_0^\pi |p_r(\mathsf{h};t)|\,dt \to \frac{\pi}{2} \qquad (r \to 1-). \end{cases}$$

Although the graph of $p_r(\mathsf{h};x)$ is negative on $x_0(r) < x \leqq \pi$, thus on almost the whole interval $(0, \pi]$ for $r \to 1$, the positive portion of the area under the graph is condensed to the remaining neighbourhood of the peaking point $x=0$ such that the Lebesgue constants, i.e. (36), (37, ii), tend to the original normalization constant of (37, i). Thus, these kernels may be regarded as "nearly positive" ones.

A similar analysis may be carried out for (32) in case $h'(1) = -1/2$, $h''(1) > 1$, as well as for (33).

Summarizing, we state the

THEOREM. *If the convergence factors* $\varrho_{k,r}(\mathsf{p}) = r^k$ *of the strictly positive kernel of Abel—Poisson are generalized to*

$$\varrho_{k,r}(\mathsf{h}) = r^k\{1 + k(1-r)h(r)\} \qquad (0 < r_0 \leqq r < 1),$$

then the variation function $h(r)$ *generates a class of kernels* $p_r(\mathsf{h};x)$ *such that the corresponding singular integral* $(f \in C_{2\pi}, L^p_{2\pi})$

$$(38) \qquad I_r(\mathsf{h};f;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t)\,p_r(\mathsf{h};t)\,dt \qquad (r \to 1-)$$

has the saturation limit (6) *with improved order of approximation* $O([1-r]^2)$ *(Lemma 1). Depending upon the graph of* $h(r)$, *the kernels are positive or of finite oscillation with two symmetrical changes of sign (Lemma 2). Higher order differentiability properties of* $h(r)$ *at* $r=1$ *lead to three different cases for the distribution of these zeros (Lemma 3).*

We conclude this section with some additional remarks concerning a comparison with polynomial kernels. Special attention is called to the fact that parts (b), (c) of Lemma 3 produce a series of kernels which are not (strictly) positive and have saturation function $\psi(\mathsf{h};k) = k^2/2$. Whereas this function (apart from the constant factor which is not essential here) is familiar for positive kernels, there are only a few examples of oscillating kernels such as that of ROGOSINSKI (cf. [2]) which, however, is of *infinite* oscillation. — The new kernel (21), having only one (double) zero at $x=\pi$, may be compared with the kernel of DE LA VALLÉE POUSSIN $V_n(x) = c_n \cos^{2n}(x/2)$ which also has no zeros apart from a zero (of multiplicity $2n$) at this point; for polynomial kernels, $V_n(x)$ seems to be the only example of this kind. —

Finally, it should be noted that the distribution of the zeros of the oscillating kernels elucidated by the above construction is quite different from the corresponding polynomial case: for polynomial kernels of finite oscillation a necessary condition for the improvement of the saturation order in comparison with the generating (positive) factor kernel is that the changes of sign tend to zero, and this with a precise fixed order (cf. [3], [17]).

4. Illustrative examples

The next section is devoted to a collection of representative examples; first we shall give some comments concerning the variation functions and kernels of the *Table*.

The first example ($j=1$) is again the basic separation function* $h_S(r)$ of (22) with $p_r(\mathrm{h}_S;\pi)=0$. — For $j=2$, $h_S(r)$ is, in return, approximated by the positive $h_2(r)$ using the first terms (including $v=2$) of the Taylor series expansion (23), thus with $p_r(\mathrm{h}_2;\pi)=O([1-r]^4)$. — $h_3(r)$ is an example of functions (26) with $\alpha=0$, thus with fixed zeros at $\pm\pi/2$. — $h_4(r)$ is, in turn, a Taylor approximation of $h_3(r)$; in view of (25), the positive zero now tends to $\frac{\pi}{2}+$. $p_r(\mathrm{h}_4;x)$ is non-negative for $r\leq 1/2$ with zero at $x=\pi$ for $r=1/2$. — $h_5(r)$ is again the peculiar function $h_D(r)$ of (13); the associated kernel may be built up from results of M. GHERMANESCO [6, p. 176 ff] (cf. also [7]) of 1932, where generalizations of Abel—Poisson's singular integral are also treated. — For $j=6$, the singular integral (38) with kernel $p_r(\mathrm{h}_6;x)$, which has a seemingly somewhat intricate closed representation, may be rewritten in the following obvious "real line" form (at first sight a positive operator)

$$(39)\qquad I_r(\mathrm{h}_6;f;x)=\frac{2}{\pi}\int_{-\infty}^{\infty}f\left(x+t\log\frac{1}{r}\right)\frac{dt}{(1+t^2)^2}=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x-t)\varrho\chi(\varrho t)\,dt$$

* It might be of interest that there is another somewhat strange connection (see [20, p. 133]) of this function with a certain quantity of the Abel—Poisson kernel (2), namely with the first algebraic moment of its derivative

$$\frac{1}{h_s(r)}=\frac{2r}{1+r}=-\frac{1}{\pi}\int_{-\pi}^{\pi}tp_r'(t)\,dt;$$

from this we may infer

$$\frac{1}{\pi}\int_{-\pi}^{\pi}t\sin tp_r^2(t)\,dt=\frac{(1-r)^2}{2(1+r)}$$

which, in turn, since $p_r^2(t)$ is an essential factor of $p_r(\mathrm{h};t)$ should be compared with the second trigonometric moment (compare [17]) of (8), namely

$$T(\mathrm{h};2;r)\equiv\frac{1}{\pi}\int_{-\pi}^{\pi}\left(2\sin\frac{t}{2}\right)^2p_r(\mathrm{h};t)\,dt=2(1-\varrho_{1,r}(\mathrm{h}))=$$

$$=2(1-r)(1-rh(r))=2(1-r)^2+o([1-r]^2)\qquad(r\to 1-).$$

Since this quantity is intimately related to the saturation order of the corresponding singular integral, $h_s(r)$ is, in a roundabout way, connected with the saturation order of (38).

j	$h_j(r)$	$h'_j(1)$	$h''_j(1)$	$\psi(\mathrm{h}_j; k)$	$p_r(\mathrm{h}_j; x)$
1	$\frac{1+r}{2r}$	$-\frac{1}{2}$	1	$\frac{1}{2}k^2$	$p_r^2(x)\,\frac{2(1-r)}{1+r}\,\{\cos x+1\}$
2	$\frac{1}{2}(r^2-3r+4)$	$-\frac{1}{2}$	1	$\frac{1}{2}k^2$	$p_r^2(x)\,\frac{2r(r^2-r+1)(1-r)}{(1+r)^2}\left\{\cos x-\frac{2r^3-3r-1}{r(r^2-r+2)}\right\}$
3	$\frac{1+r}{2r}\,\frac{1+r^2}{2r}$	$-\frac{1}{2}$	2	$\frac{1}{2}k^2$	$p_r^2(x)\,\frac{1-r^2}{r}\cos x$
4	$\frac{1}{2}(2r^2-5r+5)$	$-\frac{1}{2}$	2	$\frac{1}{2}k^2$	$p_r^2(x)\,\frac{2r(2r^2-r+3)(1-r)}{(1+r)^2}\left\{\cos x-\frac{(1-r)(4r+1)}{r(2r^2-r+3)}\right\}$
5	$\frac{1+r}{1+r^2}$	$-\frac{1}{2}$	0	$\frac{1}{2}k^2$	$p_r^2(x)\,\frac{2(1-r^2)}{1+r^2}$
6	$\frac{\log 1/r}{1-r}$	$-\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}k^2$	$p_r^2(x)\,\frac{4r}{(1+r)^2}\,\frac{(1+r^2)\log(1/r)-(1-r^2)}{(1-r)^2}\left\{\cos x-\frac{1}{2r}\,\frac{4r^2\log(1/r)-(1-r^4)}{(1+r^2)\log(1/r)-(1-r^2)}\right\}$
7	$\frac{1+r}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}k(k+2)$	$p_r^2(x)\,2(1-r\cos x)$
8	1	0	0	$\frac{1}{2}k(k+1)$	$p_r^2(x)\,\frac{4r^2}{(1+r)^2}\left\{\frac{1+2r-r^2}{2r^2}-\cos x\right\}$
9	r	1	0	$\frac{1}{2}k(k+3)$	$p_r^2(x)\,\frac{4r(r^2+r+1)}{(1+r)^2}\left\{\frac{3r^2+2r+1}{2r(2r^2+r+1)}-\cos x\right\}$
10	$2r-1$	2	0	$\frac{1}{2}k(k+5)$	$p_r^2(x)\,\frac{4r(2r^2+r+2)}{(1+r)^2}\left\{\frac{7r^2+2r+1}{2r(2r^2+r+2)}-\cos x\right\}$
11	$\frac{1+r}{2r^2}$	$-\frac{3}{2}$	4	$\frac{1}{2}k(k-2)$	$p_r^2(x)\,\frac{2}{r}(\cos x-r)$
12	$\frac{1}{r}$	-1	2	$\frac{1}{2}k(k-1)$	$p_r^2(x)\,\frac{4}{(1+r)^2}\left\{\cos x-\frac{1}{2}(r^2+2r-1)\right\}$

with kernel (on the real line)

$$\chi(x) = \frac{2}{(1+x^2)^2}, \quad \varrho = \frac{1}{\log \frac{1}{r}}.$$

This is a singular integral of Fejér's type (for the importance of these approximation processes see, for instance, [2]), just as the original Abel—Poisson integral (1), e.g. written as [1, p. 139]

$$I_r(\mathsf{p}; f; x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f\left(x + t \log \frac{1}{r}\right) \frac{dt}{1+t^2}.$$

This significant example (39) is due to T. ANGHELUTZA (1924) in his fundamental note [1] on the approximation of functions of class Lip α by means of the Abel—Poisson operator. — With index $j=7$, the following kernel is connected with solutions of partial differential equations; whereas (1) solves Dirichlet's (ordinary potential) problem for the unit disc (see e.g. [2]), the singular integral $I_r(\mathsf{h}_7; f; x)$ turns out to be a solution of the corresponding biharmonic potential problem

$$\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) U(x, y) = 0, \quad U(x, y)_{|r=1} = f(r), \quad \frac{\partial}{\partial r} U(x, y) = 0.$$

Investiagations concerned with the saturation problem and with the Nikolskiĭ constants of the measure of approximation of this operator have been carried out (1963—1968) in a series of papers [10—12], [14] by S. KANIEV and P. PYCH; cf. also [19]. — Examples 8—10 show very simple variation functions generating positive kernels; $h_{10}(r)$ is partly negative, cf. (15). — The most remarkable illustration here is $p_r(\mathsf{h}_{11}; x)$, a kernel of finite oscillation with two zeros determined via $\cos x_0(r) = r$ with $0 < x_0(r) < \pi/2$ for $1 > r > 0$. It is also hidden in a mass of general formulae in M. GHERMANESCO's paper (second part, 1933) [7, p. 68 ff]. Thus his purpose of improving the approximation order of known operators was unintentionally achieved by means of finite oscillation kernels — long ago before this approach was recognized and first formulated by P. P. KOROVKIN [13] in 1962. For detailed 'historical' remarks on this subject the reader is referred to the relevant section in [17], cf. [4].

Apart from the *Table* let us finally discuss a peculiarity of the saturation limit (6) with the help of two particular sequences of variation functions which generate kernels of finite oscillation. This phaenomenon intrinsically describes these kernels, since it follows that a characterization of kernels of finite oscillation with saturation function (41), that is with integer m, is natural by means of the extraordinary behaviour of the mth Fourier coefficient.

For the first place, let

$$h_m(r) = \frac{r^m - 1}{m(r-1)r^m} = \frac{1}{mr^m} \sum_{\nu=0}^{m-1} r^\nu \qquad (m = 1, 2, 3 \ldots); \tag{40}$$

this gives $h_m(1)=1$, $h'_m(1) = -(m+1)/2 \leqq -1$, $h''_m(1) = (m+1)(m+2)/3 \geqq 2$ so that the saturation function reads

$$(41) \qquad \psi(\mathsf{h}_m; k) = \frac{1}{2} k(k-m) \qquad (k=0, 1, 2, \ldots).$$

In view of (24) the (positive) zeros of the respective kernels may be expanded as

$$x_{0,m}(r) = \frac{2}{\sqrt{m}} \sqrt{1-r} + o(\sqrt{1-r}) \qquad (r \to 1-)$$

such that increasing m effects a decrease of the leading coefficient. The functions (40) have been calculated from (5) requiring that for the convergence factors relation

$$(42) \qquad \varrho_{m,r}(\mathsf{h}_m) \equiv 1$$

should be valid (independently of r). This means, in other words, that the mth Fourier coefficient of $p_r(\mathsf{h}_m; x)$ equals 1 for all r, whereas, in view of (41) and (6), for all Fourier coefficients with $k \neq m$ only

$$(43) \qquad \varrho_{k,r}(\mathsf{h}_m) = 1 + O([1-r]^2) \qquad (r \to 1-)$$

holds.

On the other hand, for

$$(44) \qquad h_m(r) = r^{-(m+1)/2} \qquad (m=1, 2, 3, \ldots)$$

one has $h_m(1)=1$, $h_m'(1) = -(m+1)/2 \leqq -1$, $h_m''(1) = (m+1)(m+3)/4 \geqq 2$ so that the same saturation function as (41) appears. But, in contrast to (42), merely the limit relation

$$(45) \qquad \lim_{r \to 1-} \frac{1-\varrho_{m,r}(\mathsf{h}_m)}{(1-r)^3} = \frac{1}{24} m(m^2-1) \qquad (m=2, 3, 4, \ldots)$$

holds. For the index $m=1$ both functions (40) and (44) coincide (see example 12 of the *Table*). Thus, for the corresponding kernels, (44) reveals that $\varrho_{m,r}(\mathsf{h}_m)$ again approaches 1 more rapidly than $\varrho_{k,r}(\mathsf{h}_m)$, $k \neq m$, for which (43) also holds.

A further generalization of the construction in this paper which starts off with additional terms in (5) giving a saturation order of $O([1-r]^q)$, $q>2$, is in preparation. A connection with the conjugate of the Abel—Poisson kernel and with perturbed Taylor expansions will be seen to be of general importance.

Finally, the author would like to thank Prof. P. L. BUTZER and Dr. W. TREBELS for their permanent attention and encouragement.

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(Received September 13, 1971)

* FdM means the respective review in „*Fortschritte der Mathematik*".

Studia Scientiarum Mathematicarum Hungarica 7 (1972) 451—458

ON THE SERIES $\sum c_k \varphi_k$

by

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An orthogonal system $\{\varphi_k\}$ is called a *convergence system* if the convergence of

$$\sum_{k=1}^{\infty} c_k^2 \tag{1}$$

implies the almost everywhere convergence of

$$\sum_{k=1}^{\infty} c_k \varphi_k. \tag{2}$$

If (1) implies that (2) converges in every arrangement of its terms, φ_k is called an unconditional convergence system.

The following sufficient condition was given by G. ALEXITS:

THEOREM A ([1]). *Let $\{\varphi_i\}$ be a uniformly bounded multiplicative system*, i.e. a sequence of measurable functions defined on a measure space $\{X, \mathcal{S}, \mu\}$ of finite measure for which*

$$|\varphi_i| \leqq K \qquad (i=1, 2, \dots)$$

$$\int \varphi_{i_1} \varphi_{i_2} \cdots \varphi_{i_k} = 0 \qquad (k=1, 2, \dots;\ i_1 < i_2 < \dots < i_k).$$

Then $\{\varphi_i\}$ is an unconditional convergence system.

He also proved for such systems that the almost everywhere convergence of (2) implies the convergence of (1).

A simpler proof of Theorem A was found by C. I. PRESTON ([4]).

Another sufficient condition was given by V. F. GAPOSHKIN ([5]), who has proved

THEOREM B. *Let $\{\varphi_i\}$ be a sequence of measurable functions defined on a measure space $\{X, \mathcal{S}, \mu\}$ of finite measure for which*

$$\int \varphi_i^4 \leqq K \qquad (i=1, 2, \dots) \tag{3}$$

$$\int \varphi_i \varphi_j \varphi_k \varphi_l = 0 \tag{4}$$

$$\int \varphi_i^2 \varphi_j \varphi_k = 0, \tag{5}$$

where i, j, k, l are different integers. Then $\{\varphi_i\}$ is an unconditional convergence system.

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* The concept itself was introduced by G. ALEXITS ([2], [3]).

In a joint paper P. Révész and the author proved that condition (5) of Theorem B can be omitted, i.e.

Theorem C ([6]). *Let* $\{\varphi_i\}$ *be a sequence of measurable functions*[1] *satisfying conditions* (3) *and* (4). *Then* $\{\varphi_i\}$ *is an unconditional convergence system.*

In the following Theorem 1 we give a slight generalization of Theorem C for functions multiplicative of order ν (rather than 4) by substituting conditions (3) and (4) with the following two conditions:

$$\int \varphi_i^\nu \leqq K \qquad (i=1,2,\ldots) \tag{3'}$$

$$\int \varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_\nu}=0 \qquad (i_1<i_2<\ldots<i_\nu). \tag{4'}$$

Theorem 1. *Let* $\{\varphi_k\}$ *be a sequence of measurable functions and suppose that there exists an even integer* ν *greater than* 2, *with which* $\{\varphi_k\}$ *satisfies* (3′) *and* (4′). *Then* $\{\varphi_k\}$ *is an unconditional convergence system.*

We also prove that under conditions (3) and (4) the convergence of (1) is also necessary to the almost everywhere convergence of (2). More precisely we have

Theorem 2. *If the sequence* $\{\varphi_k\}$ *satisfies* (3) *and* (4), $\mu(X)<\infty$, *further we have*

$$\int \varphi_k^2 \geqq c>0 \qquad (k=1,2,\ldots),$$

then the almost everywhere convergence of series (2) *implies the convergence of series* (1).

Remark 1. Actually we will prove that this holds on every set E of finite positive measure (even if $\mu(X)=\infty$), on which the square integrals $\int_E \varphi_k^2$ are bounded from below. In particular, if for each set E of positive measure we have

$$\varliminf_{k\to\infty}\int_E \varphi_k^2>0, \text{ then} \tag{6}$$

series (2) cannot be convergent on any positive measured set, unless (1) is convergent; thus we have the following

Corollary. *If the sequence* $\{\varphi_k\}$ *satisfies* (3) *and* (4), *and* (6) *holds on every set of positive measure, then the series* $\sum c_k\varphi_k$ *is almost everywhere convergent or divergent (in every arrangement of its terms), according to whether the series* $\sum c_k^2$ *is convergent or divergent.*

The proof of Theorem 1 is based on the following result of Stečkin:

Theorem D. (see [7]). *Let* $\{\varphi_k\}$ *be a sequence of measurable functions and suppose that there exists an even integer* $\nu>2$, *such that for arbitrary sequence* $\{c_k\}$ *and integer* N *we have*

$$\int\left(\sum_{k=1}^N c_k\varphi_k\right)^\nu \leqq A\left(\sum_{k=1}^N c_k^2\right)^{\nu/2} \tag{7}$$

(A is a positive constant). Then $\{\varphi_k\}$ *is an unconditional convergence system.*

[1] From this point on, measurable function means one defined on a measure space $\{X,\mathcal{S},\mu\}$ of possibly infinite measure, unless otherwise stated.

Making use of Theorem D, in order to prove Theorem 1 it is enough to prove that conditions (3′)—(4′) imply (7).

This is expressed by our

MOMENT ESTIMATION. *Let* $\{\varphi_k\}$ *be a sequence of measurable functions obeying conditions* (3′) *and* (4′) *with some even integer* $\nu > 2$.

Then

$$\int \left(\sum_{k=1}^{N} c_k \varphi_k \right)^{\nu} \leqq A_\nu K \left(\sum_{k=1}^{N} c_k^2 \right)^{\nu/2}, \tag{8}$$

where A_ν *is a positive absolute constant and* K *is defined by* (3′).

We remark that in case $c_1 = \dots = c_N = 1$ (8) boils down to

$$\int \left(\sum_{k=1}^{N} \varphi_k \right)^{\nu} \leqq A_\nu K N^{\nu/2},$$

which estimation was formulated by R. I. SERFLING ([8]), but his proof is not complete.

PROOF. Suppose that the functions $f_1, f_2, \dots, f_N$ satisfy (4′) for some even $\nu > 2$. Put

$$S_i = \sum_{k=1}^{N} f_k^i.$$

We want to estimate

$$\int \left(\sum f_k\right)^{\nu} = \int S_1^{\nu}.$$

S_1^{ν} can be written (in a unique way) as

$$S_1^{\nu} = \nu! \sum_{i_1 < \dots < i_\nu} f_{i_1} \dots f_{i_\nu} + \sum_{(\sigma_1, \dots, \sigma_\nu) \in A} \alpha_{\sigma_1, \dots, \sigma_\nu} S_1^{\sigma_1} \dots S_\nu^{\sigma_\nu},$$

where A is the set of ν-tuples $(\sigma_1, \dots, \sigma_\nu)$ of non-negative integers satisfying

1° $\quad (\sigma_1, \sigma_2, \dots, \sigma_\nu) \neq (\nu, 0, \dots, 0),$

2° $\quad \sigma_1 + 2\sigma_2 + \dots + \nu\sigma_\nu = \nu,$

and the α's are absolute constants.

Using (4′) we have

$$\int S_1^{\nu} = \sum_{(\sigma_1, \dots, \sigma_\nu) \in A} \alpha_{\sigma_1, \dots, \sigma_\nu} \int S_1^{\sigma_1} \dots S_\nu^{\sigma_\nu}. \tag{9}$$

Put

$$M = \max_{(\sigma_1, \dots, \sigma_\nu) \in A} \int |S_1^{\sigma_1} \dots S_\nu^{\sigma_\nu}|$$

and

$$\alpha = \sum_{(\sigma_1, \dots, \sigma_\nu) \in A} |\alpha_{\sigma_1, \dots, \sigma_\nu}|.$$

(9) implies the inequality

$$\int S_1^{\nu} \leqq \alpha M. \tag{10}$$

We apply the following inequality (that can easily be obtained from the Hölder inequality):

$$\left|\int h_1^{p_1} \dots h_n^{p_n}\right| \leqq \left(\int |h_1|^\nu\right)^{p_1/\nu} \dots \left(\int |h_n|^\nu\right)^{p_n/\nu}$$

for non-negative p_i; $p_1+\dots+p_n = \nu$. Putting

$$h_i = |S_i|^{1/i}$$

we get

$$M = \int |S_1^{\sigma_1} \dots S_\nu^{\sigma_\nu}| \leqq \prod_{i=1}^{\nu} \left(\int |S_i|^{\nu/i}\right)^{i\sigma_i/\nu}.$$

Note that the sequence $|S_i|^{\nu/i}$ is monotone decreasing in i, in fact, for $i \geqq 2$ we have

$$|S_i|^{\nu/i} = \left|\sum_{k=1}^{N} f_k^i\right|^{\nu/i} \leqq \left(\max_{1\leqq k\leqq N} |f_k|^{i-2} \sum_{k=1}^{N} f_k^2\right)^{\nu/i} = \left((\max f_k^2)^{(i/2)-1} \sum f_k^2\right)^{\nu/i} \leqq$$

$$\leqq \left(\sum f_k^2\right)^{\nu/2} = S_2^{\nu/2}.$$

Thus, we have

$$M \leqq \left(\int S_1^\nu\right)^{\sigma_1/\nu} \left(\int S_2^{\nu/2}\right)^{1-(\sigma_1/\nu)}$$

and (using (10)) this yields to the estimation

$$\int S_1^\nu \leqq \alpha^{\nu/\nu-\sigma_1} \int S_2^{\nu/2} \leqq \alpha^{\nu/2} \int S_2^{\nu/2} = A_\nu \int S_2^{\nu/2}. \tag{11}$$

So far we have used only condition (4'). Now, if we apply (11) with

$$f_k = c_k \varphi_k, \quad \text{where the sequence} \quad \{\varphi_k\}$$

satisfies (3') and (4'), and use the obvious relation

$$\int \left(\sum c_k^2 \varphi_k^2\right)^{\nu/2} \leqq K\left(\sum c_k^2\right)^{\nu/2},$$

we get (8).

For proving Theorem 2 we will need some lemmas.

Lemma 1*. *Let f be a non-negative measurable function defined on a measure space $(X, \mathcal{S}, \mu)$ of finite measure. Then*

$$\mu\{x : f(x) > \varepsilon\} \geqq \frac{\left(\int f - \varepsilon\mu(X)\right)^2}{\int f^2}$$

for

$$0 < \varepsilon < \frac{\int f}{\mu(X)}.$$

Proof. By the Cauchy—Schwarz inequality we have

$$\int f = \int_{\{f\leqq\varepsilon\}} f + \int_{\{f>\varepsilon\}} f \leqq \varepsilon\mu(X) + [\mu\{x : f(x) > \varepsilon\}]^{1/2} \left(\int f^2\right)^{1/2},$$

whence the statement follows.

* This lemma is analogous to that of Paley and Zygmund (see [9], [10]).

Now we prove a Bessel-type inequality

LEMMA 2. *If the sequence* $\{\varphi_i\}$ *satisfies conditions* (3), (4) *and* f *is a square-integrable function, then*

$$\sum_{i<k} \left(\int f \varphi_i \varphi_k\right)^2 \leqq (2\sqrt{A_4}\, K) \int f^2,$$

where K *is defined in* (3), *and the absolute constant* A_4 *is that of the Moment estimation.*

The proof is similar to that of the Bessel inequality. We start out with the inequality

$$0 \leqq \int \Big(cf - \sum_{i<k\leqq n} \varphi_i \varphi_k \int f \varphi_i \varphi_k\Big)^2 =$$

$$= c^2 \int f^2 - 2c \sum_{i<k\leqq n} \left(\int f \varphi_i \varphi_k\right)^2 + \sum_{\substack{i<k\leqq n \\ l<m\leqq n}} \int \varphi_i \varphi_k \varphi_l \varphi_m \left(\int f \varphi_i \varphi_k\right)\left(\int f \varphi_l \varphi_m\right).$$

Denoting the last sum by S and applying (3), (4) and the Cauchy—Schwarz inequality we have

$$S = \sum_{i\leqq n} \sum_{\substack{k,m\leqq n \\ k,m\neq i}} \int \varphi_i^2 \varphi_k \varphi_m \left(\int f \varphi_i \varphi_k\right)\left(\int f \varphi_i \varphi_m\right) - \sum_{i<k\leqq n} \int \varphi_i^2 \varphi_k^2 \left(\int f \varphi_i \varphi_k\right)^2 \leqq$$

$$\leqq \sum_{i\leqq n} \int \varphi_i^2 \Big(\sum_{\substack{k\leqq n \\ k\neq i}} \varphi_k \int f \varphi_i \varphi_k\Big)^2 \leqq \sum_{i\leqq n} \sqrt{K} \sqrt{\int \Big(\sum_{\substack{k\leqq n \\ k\neq i}} \varphi_k \int f \varphi_i \varphi_k\Big)^4}\,.$$

Now applying the *Moment estimation* with

$$c_k = \int f \varphi_i \varphi_k$$

we get

$$S \leqq \sqrt{K} \sum_{i\leqq n} \sum_{\substack{k\leqq n \\ k\neq i}} \left(\int f \varphi_i \varphi_k\right)^2 \sqrt{A_4 K} = 2\sqrt{A_4}\, K \sum_{i<k\leqq n} \left(\int f \varphi_i \varphi_k\right)^2.$$

Thus we have

$$0 \leqq c^2 \int f^2 - (2c - 2\sqrt{A_4}\, K) \sum_{i<k\leqq n} \left(\int f \varphi_i \varphi_k\right)^2,$$

whence, choosing

$$c = 2\sqrt{A_4}\, K,$$

we obtain

$$\sum_{i<k\leqq n} \left(\int f \varphi_i \varphi_k\right)^2 \leqq 2\sqrt{A_4}\, K \int f^2.$$

Since it is true for all n, the statement of the lemma follows.

LEMMA 3. *If the sequence* $\{\varphi_k\}$ *satisfies* (3), (4) *and* (6) *for a set* E, *then for some* $c>0$ *and positive integer* n_0 *we have*

$$\int_E \left(\sum_{k=n_0}^{n} c_k\varphi_k\right)^2 \geqq c \sum_{k=n_0}^{n} c_k^2,$$

whatever the numbers n, $\{c_k\}$ are.

The proof will show that c can be taken e.g. $\frac{1}{2}\varliminf_{k\to\infty}\int_E \varphi_k^2$.

PROOF. By (6) there is a $c>0$ and a positive integer n_1 for which

$$\int_E \varphi_k^2 \geqq 2c \quad \text{for} \quad k\geqq n_1.$$

By Lemma 2 there exists an integer n_2 for which

$$\sum_{n_2\leqq i<k} \left(\int_E \varphi_i\varphi_k\right)^2 = \sum_{n_2\leqq i<k} \left(\int \chi_E\varphi_i\varphi_k\right)^2 < \frac{c^2}{4}$$

(χ_E denotes the indicator function of the set E). Put $n_0=\max(n_1, n_2)$. Then we have for any $n\geqq n_0$ and numbers c_k

$$\int_E \left(\sum_{k=n_0}^{n} c_k\varphi_k\right)^2 = \sum_{k=n_0}^{n} c_k^2 \int_E \varphi_k^2 + 2 \sum_{n_0\leqq i<k\leqq n} c_i c_k \int_E \varphi_i\varphi_k \geqq$$

$$\geqq 2c \sum_{k=n_0}^{n} c_k^2 - 2\sqrt{\sum_{n_0\leqq i<k\leqq n} c_i^2 c_k^2}\sqrt{\sum_{n_0\leqq i<k\leqq n}\left(\int_E \varphi_i\varphi_k\right)^2} \geqq$$

$$\geqq 2c \sum_{k=n_0}^{n} c_k^2 - 2\sum_{k=n_0}^{n} c_k^2 \sqrt{\frac{c^2}{4}} = c \sum_{k=n_0}^{n} c_k^2.$$

Now we are ready for the

PROOF of theorem 2. Suppose that (6) holds for a set E of finite measure, and put $c=\frac{1}{2}\varliminf_{k\to\infty}\int_E \varphi_k^2$. Suppose further that $\sum_{k-1}^{\infty} c_k^2=\infty$. We establish the following estimation

$$\mu\{x: x\in E,\ \sum c_k\varphi_k(x) \text{ is divergent}\} \geqq \frac{c^2}{A_4 K}$$

(K is defined in (3) and A_4 by the *Moment estimation*. This obviously proves theorem 2 in the stronger form of Remark 1.

Let ε be any positive number for which

$$0<\varepsilon<\frac{c}{\mu(E)}.$$

We have

$$m = \mu\{x \in E : \sum c_k \varphi_k \text{ is divergent}\} \geqq$$

$$\geqq \mu \left\{ x \in E : \frac{\left(\sum_{k=n_0}^{n} c_k \varphi_k\right)^2}{\sum_{k=n_0}^{n} c_k^2} \geqq \varepsilon \text{ for infinitely many } n \right\} \geqq$$

$$\geqq \varlimsup_{n \to \infty} \mu \left\{ x \in E : \frac{\left(\sum_{k=n_0}^{n} c_k \varphi_k\right)^2}{\sum_{k=n_0}^{n} c_k^2} \geqq \varepsilon \right\},$$

where we choose n_0 the number defined in Lemma 3.

According to Lemma 3 and the *Moment estimation*

$$\int_E \frac{\left(\sum_{k=n_0}^{n} c_k \varphi_k\right)^2}{\sum_{k=n_0}^{n} c_k^2} \geqq c$$

and

$$\int_E \left(\frac{\left(\sum_{k=n_0}^{n} c_k \varphi_k\right)^2}{\sum_{k=n_0}^{n} c_k^2} \right)^2 \leqq A_4 K,$$

thus we can apply Lemma 1 with

$$f = \frac{\left(\sum_{k=n_0}^{n} c_k \varphi_k\right)^2}{\sum_{k=n_0}^{n} c_k^2}$$

(and the measure restricted to E):

$$m \geqq \varlimsup_{n \to \infty} \mu \left\{ x \in E : \frac{\left(\sum_{k=n_0}^{n} c_k \varphi_k\right)^2}{\sum_{k=n_0}^{n} c_k^2} \geqq \varepsilon \right\} \geqq \frac{(c - \varepsilon \mu(E))^2}{A_4 K}.$$

Since it is true for all positive (small enough) ε we have

$$m \geqq \frac{c^2}{A_4 K}.$$

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(Received October 8, 1971)

NULLSTELLENEINSCHLIESSUNGEN UND LANDAU—FEJÉR—MONTEL PROBLEM

von

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1. MONTEL [7] bewies den folgenden Satz, der Ergebnisse von LANDAU [4] und FEJÉR [3] verallgemeinert.

SATZ A. *Jedes komplexe Polynom*

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k} \tag{1}$$

mit

$$1 \leqq p < n_1 < n_2 < \cdots < n_k \quad \textit{und} \quad a_p \neq 0$$

besitzt mindestens p Nullstellen in einem Kreisbereich $|z| \leqq R$, *dessen Radius* $R = R(a_0, a_1, \ldots, a_p, k)$ *nur von den ersten* $p+1$ *Koeffizienten und der Anzahl k der folgenden Glieder abhängt.*

Schranken für R wurden von FEJÉR [3], BIERNACKI [1; 2] und MARDEN [6] angegeben. Wir wollen das in Satz A gelöste Einschließungsproblem das LANDAU—FEJÉR—MONTEL*sche* (p, k)*-Problem für die Basis* $\{z^\nu : \nu \in \mathsf{N}_0\}$ *(Taylorentwicklung)* nennen[1].

TURÁN [13] bemerkte, daß für manche Untersuchungen andere Entwicklungen eines Polynom zweckmäßiger sind als die Taylorentwicklung. Zusammen mit MAKAI [5] erhielt er das folgende Resultat, das eine Lösung des (1, 1)-Problems für die Hermiteentwicklung angibt.

SATZ B. *Es existiert eine Konstante K, so daß jedes aus den Hermiteschen Polynomen*

$$H_n(z) := (-1)^n e^{z^2} \frac{d^n}{dz^n}(e^{-z^2}) \quad (n \in \mathsf{N}_0)$$

gebildete Trinom

$$H_0(z) + H_1(z) + \zeta H_n(z) \quad \textit{mit} \quad n \geqq 2 \quad \textit{und} \quad \zeta \in \mathsf{C}$$

mindestens eine Nullstelle im Streifen $|\mathrm{Im}\, z| \leqq K$ *besitzt.*

Der optimale numerische Wert für K wurde später in [10] gefunden. In [9, S. 27—35] wurde schließlich gezeigt:

SATZ C. *Der Satz A bleibt richtig (natürlich mit anderen numerischen Werten für R), wenn man in* (1) *für alle* $n \in \mathsf{N}_0$ *die Potenzen* z^n *durch Polynome*

$$u_n(z) = \lambda_n \prod_{\nu=1}^{n} (z - \zeta_{n\nu}) \quad (\lambda_n \neq 0) \tag{2}$$

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[1] N, R, und C bezeichnen die natürlichen, reellen und komplexen Zahlen; $\mathsf{N}_0 := \mathsf{N} \cup \{0\}$.

ersetzt, für die

$$\limsup_{n\to\infty}\left|\frac{u_n(z)}{\lambda_n}\right|^{\frac{1}{n}} < \infty \quad \text{für} \quad |z| < \infty \tag{3}$$

gilt.

Schranken für R wurden ebenfalls in [9] angegeben. Ein System von Polynomen (2), das die Bedingung (3) erfüllt, wurde in [9] ein *schwach wachsendes System* (abgekürzt *SW-System*) genannt. Obwohl der Satz C für eine umfangreiche Klasse von Basen das LANDAU—FEJÉR—MONTELsche (p, k)-Problem löst, so ist doch der in Satz B behandelte Spezialfall nicht darin enthalten, da die Hermiteschen Polynome kein SW-System bilden.

In der vorliegenden Arbeit wollen wir in Abschnitt 2 und 3 die Methode von MAKAI und TURÁN auf das $(p, 1)$-Problem für eine Klasse von Basen ausdehnen, die eine gewisse Klasse von SW-Systemen umfaßt und das System der Hermiteschen und Laguerreschen Polynome enthält. Die Abschnitte 4 und 5 bringen Ergebnisse zur Abgrenzung der Lösbarkeit von LANDAU—FEJÉR—MONTEL-Problemen.

2. Der folgende Satz liefert für eine etwas indirekt definierte Klasse von Basen eine Lösung des $(p, 1)$-Problems.

SATZ 1. *Es sei $U = \{u_n(z) : n \in \mathsf{N}_0\}$ ein System von Polynomen (2) mit lauter reellen Nullstellen, wobei jeweils z_n eine von maximalem Betrag bezeichne. Für jedes $p \in \mathsf{N}$ gebe es $D > 0$, $E > 0$ und eine Folge von einfach zusammenhängenden Bereichen B_n $(n \in \mathsf{N})$ mit stückweise glattem Rand C_n und den folgenden Eigenschaften:*

(i) *Für alle $z \in B_n$ gilt $|\mathrm{Im}\, z| \leqq D$ und $|z| \leqq E|z_n|$.*

(ii) *Für hinreichend großes n liegen mindestens p Nullstellen von $u_n(z)$ in B_n.*

(iii) *Bezeichnet M_n das Minimum von $|u_n(z)|$ auf C_n, so besteht für jede Teilfolge $\mathsf{N}' \subseteqq \mathsf{N}$ mit*

$$\lim_{n\in \mathsf{N}'\, n\to\infty} \frac{1}{z_n} = 0 \tag{4}$$

die Aussage

$$\lim_{n\in \mathsf{N}'\, n\to\infty}\left|\frac{u_n(z) z_n^p}{M_n}\right| = 0 \quad \text{für} \quad |z| < \infty.$$

Unter diesen Voraussetzungen besitzt jedes komplexe Polynom

$$f(z) = a_0 u_0(z) + a_1 u_1(z) + \cdots + a_p u_p(z) + \zeta u_n(z) \qquad (1 \leqq p < n,\ a_p \neq 0) \tag{5}$$

entweder mindestens p Nullstellen in einem Kreisbereich $|z| \leqq R$, dessen Radius R bei fest vorgegebenem System U nur von $a_0, a_1, \ldots, a_p$ abhängt, oder es liegen mindestens p Nullstellen in B_n.

BEWEIS. Es sei

$$h(z) := a_0 u_0(z) + a_1 u_1(z) + \cdots + a_p u_p(z).$$

Wir nehmen an, es gäbe zu jedem $m \in \mathsf{N}$ ein Polynom

$$f_m(z) := h(z) + \zeta_m u_{n_m}(z) \qquad (\zeta_m \in \mathsf{C}),$$

das sowohl im B_{n_m} als auch im Kreisbereich $|z| \leqq m$ höchstens $p-1$ Nullstellen besitzt und setzen $N' := \{n_m : m \in N\} \cap N$. Da man eine endliche Menge von Polynomen (2) stets zu einem SW-System auffüllen kann, ist N' wegen Satz C eine unendliche Teilfolge der natürlichen Zahlen. Falls

$$\liminf_{n \in N' \, n \to \infty} |z_n|^{-1} > 0,$$

so können die Polynome $u_n(z)$ für $n \in N'$ etwa durch die Festsetzung

$$v_n(z) := \begin{cases} u_n(z) & \text{für} \quad n \in N' \\ z^n & \text{für} \quad n \in N_0 \setminus N' \end{cases}$$

in ein SW-System $V = \{v_n(z) : n \in N_0\}$ eingebettet werden. Satz C liefert dann einen Widerspruch.

Es gelte nun (4). Das Polynom $h(z)$ besitzt alle seine Nullstellen in einem Kreisbereich $|z| \leqq R_0 > 0$, dessen Radius R_0 bei fest vorgegebenem System U nur von $a_0, a_1, \dots, a_p$ abhängt. Auf dem Kreis

$$|z| = 2R_0 = : R_1$$

gilt

$$|h(z)| \geqq |\lambda_p a_p| R_0^p.$$

Bezeichnet τ_n einen Punkt dieses Kreises mit

$$|u_n(\tau_n)| = \max_{|z| = R_1} |u_n(z)|,$$

so folgt nach dem Satz von ROUCHÉ, daß $f(z)$ für

$$\text{(6)} \qquad |\zeta| < \left| \frac{\lambda_p a_p R_0^p}{u_n(\tau_n)} \right|$$

mindestens p Nullstellen im Kreisbereich $|z| < R_1$ besitzt. Für $z \in C_n$ gilt

$$|h(z)| \leqq |\lambda_p a_p| (R_0 + E|z_n|)^p.$$

Ist nun

$$\text{(7)} \qquad |\zeta| > \frac{|\lambda_p a_p| (R_0 + E|z_n|)^p}{M_n},$$

so liegen nach (ii) und dem Satz von ROUCHÉ für hinreichend großes n mindestens p Nullstellen von $f(z)$ im Bereich B_n. Wegen (iii) gilt für alle hinreichend großen $n \in N'$ zudem

$$\frac{|u_n(\tau_n) z_n^p|}{M_n} \leqq \left(\frac{E}{R_0} + \frac{1}{|z_n|} \right)^{-p}.$$

Dann erfüllt jedes $\zeta \in C$ wenigstens eine der beiden Ungleichungen (6) und (7). Damit bekommen wir einen Widerspruch zur Annahme über die Lage der Nullstellen von $f_m(z)$ $(m \in N)$.

Es stellt sich nun die Frage nach der Gestalt der in Satz 1 zulässigen Polynomsysteme U. Betrachten wir zunächst ein Beispiel.

Es sei $\{c_n \neq 0\}$ eine reelle Nullfolge, $\{k_n\}$ eine unendliche Folge von natürlichen Zahlen mit $k_n \leqq n$, $u_0(z) \equiv 1$ und

$$u_n(z):=z^{n-k_n}\left(z-\frac{1}{c_n}\right)^{k_n} \quad \text{für} \quad n \in \mathbf{N}.$$

Man prüft leicht nach, daß $U=\{u_n(z): n \in \mathbf{N}_0\}$ genau dann ein SW-System ist, wenn

$$\limsup_{n\to\infty}\left|\frac{1}{c_n}\right|^{\frac{k_n}{n}}<\infty.$$

Dagegen erfüllt U die Voraussetzungen von Satz 1, wenn nur

$$\limsup_{n\to\infty}\frac{k_n}{n}<\frac{1}{2}.$$

Der folgende Satz zeigt allgemein, daß man die Klasse der in Satz 1 zulässigen Polynomsysteme als Erweiterung einer gewissen Klasse von SW-Systemen auffassen kann. Beim Übergang von SW-Systemen zu den Systemen von Satz 1 kann also die Wachstumsbedingung abgeschwächt werden, jedoch kennt man nur noch für das $(p, 1)$-Problem eine Lösung, und zwar nur noch Schranken für den Imaginärteil von p Nullstellen.

SATZ 2. *Es sei* $U=\{u_n(z): n \in \mathbf{N}_0\}$ *ein SW-System von Polynomen* (2) *mit lauter reellen Nullstellen, wobei jeweils* z_n *eine von maximalem Betrag bezeichne. Es gebe zu jedem* $p \in \mathbf{N}$ *ein* $d>0$ *und ein* $0 \leqq q<1$, *so daß* $u_n(z)$ *für hinreichend großes* n *mindestens* p *Nullstellen im Intervall* $I_n:=\{x \in \mathbf{R}: |x-z_n| \leqq \max(d, q|z_n|)\}$ *besitzt. Dann erfüllt U die Voraussetzungen von Satz 1.*

BEWEIS. O. B. d. A. sei $z_n>0$.[2] Wie der Nullstellenverteilungssatz für SW-Systeme (siehe [9, Satz 4. 1]) zeigt, gibt es eine Konstante $c>0$, so daß jedes $u_n(z)$ für $n \geqq 2p+1$ mindestens $k:=\left[\frac{n}{2}\right]+p+1$ Nullstellen im Intervall $J:=[-c, c]$ besitzt. Wir setzen nun

$$v_n(z):=u_n(z)\frac{1}{\lambda_n}\prod_{\varkappa=1}^{k}(z-\zeta_{n\varkappa})^{-1},$$

wobei $\zeta_{n\varkappa}$ für $1 \leqq \varkappa \leqq k$ in J liegende Nullstellen von $u_n(z)$ bezeichne. Nach [9, Lemma 2. 1] gibt es im Abstand kleiner oder gleich 4 bezüglich I_n eine Distanzlinie C_n auf der $|v_n(z)| \geqq 2$ ist. Der durch seinen Rand C_n eindeutig bestimmte, beschränkte Bereich B_n besitzt die Eigenschaften (i) und (ii) von Satz 1. Unter der Voraussetzung (4) gilt für $z \in C_n$ und hinreichend großes $n \in N'$

$$|u_n(z)|=|\lambda_n v_n(z)|\prod_{\varkappa=1}^{k}|z-\zeta_{n\varkappa}| \geqq 2|\lambda_n|((1-q)z_n-4-c)^k.$$

[2] Für $|z_n| \leqq 1$ setzen man etwa $B_n:=\{z:|z| \leqq |z_n|\}$; für $z_n<-1$ betrachte man $u_n(-z)$

Somit bekommen wir für beliebiges $z\in C$

$$\left|\frac{u_n(z)z_n^p}{M_n}\right| \leqq \frac{|u_n(z)|z_n^p}{2|\lambda_n|((1-q)z_n-4-c)^k} \leqq A\left(\frac{\left|\frac{u_n(z)}{\lambda_n}\right|^{\frac{1}{n}}}{\left|\frac{1-q}{2}z_n\right|^{\frac{1}{2}}}\right)^n$$

mit einer geeigneten Konstanten A. Wegen (3) und (4) gilt deshalb auch (iii).

3. Wir wollen nun Entwicklungen nach den Hermiteschen und den Laguerreschen Polynomen untersuchen. Zu $\alpha\in(-1,\infty)$ werden die Laguerreschen Polynome $L_n^{(\alpha)}(x)$ gegeben durch (siehe [12, S. 100])

$$e^{-x}x^\alpha L_n^{(\alpha)}(x):=\frac{1}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}).$$

Für die Hermiteschen Polynome gilt

$$H_{2n}(x)=(-1)^n 2^{2n} n!\, L_n^{\left(-\frac{1}{2}\right)}(x^2) \tag{8}$$

und

$$H_{2n+1}(x)=(-1)^n 2^{2n+1} n!\, xL_n^{\left(\frac{1}{2}\right)}(x^2). \tag{9}$$

Wir benötigen einige asymptotische Abschätzungen.

Lemma 1. *Für $z\in C$ bestehen die Ungleichungen*

$$|L_n^{(\alpha)}(z)| \leqq g_1(z)n^{\frac{2\alpha-1}{4}}e^{2|nz|^{\frac{1}{2}}}$$

und

$$|H_n(z)| \leqq g_2(z)2^n\left[\frac{n}{2}\right]!\,e^{2|z|\left[\frac{n}{2}\right]^{\frac{1}{2}}},$$

wobei die $g_i(z)$ $(i=1,2)$ von n unabhängige positive Funktionen bezeichnen.

Beweis. Die Abschätzung für $|L_n^{(\alpha)}(z)|$ gewinnt man leicht aus der asymptotischen Darstellung von Perron (siehe [12, S. 197]) und findet dann unter Beachtung von (8) und (9) diejenige für $|H_n(z)|$.

Die Differentialgleichung

$$\frac{d^2y}{dx^2}+\frac{1}{3}xy=0$$

besitzt bis auf einen konstanten Faktor eine einzige nichttriviale Lösung $y(x)$, die für $x\to-\infty$ beschränkt bleibt (siehe [12, S. 18]). Fordert man $y(0)=\frac{\pi}{3\Gamma(2/3)}$, so wird diese Lösung gemeinhin als Airyfunktion Ai (x) bezeichnet. Ai (x) ist eine ganze Funktion, die keine negativen, aber unendlich viele positive Nullstellen besitzt.

Für die Hermiteschen und die Laguerreschen Polynome bestehen in einer Umgebung ihrer größten Nullstelle die folgenden auf PLANCHEREL und ROTACH zurückgehenden asymptotischen Formeln:

LEMMA 2 (*siehe* [12, *S.* 198—199]). *Für* $z = 4n+2\alpha+2-2\left(\frac{2n}{3}\right)^{\frac{1}{3}} t$ *mit komplexem, beschränktem t gilt*

$$L_n^{(\alpha)}(z) = \frac{(-1)^n}{2^\alpha} \frac{e^{\frac{z}{2}}}{\pi} \left(\frac{3}{2n}\right)^{\frac{1}{3}} (\mathrm{Ai}(t) + O(n^{-\frac{2}{3}})).$$

Für $z = (2n+1)^{\frac{1}{2}} - 2^{-\frac{1}{2}} 3^{-\frac{1}{3}} n^{-\frac{1}{6}} t$ *mit komplexem, beschränktem t gilt*

$$H_n(z) = e^{\frac{z^2}{2}} 3^{\frac{1}{3}} \pi^{-\frac{3}{4}} 2^{\frac{n}{2}+\frac{1}{4}} (n!)^{\frac{1}{2}} n^{-\frac{1}{12}} (\mathrm{Ai}(t) + O(n^{-\frac{2}{3}})).$$

Schließlich benötigen wir das

LEMMA 3 (*siehe* [12, *S.* 130—131]). *Für die größte Nullstelle* z_n *des Laguerreschen Polynoms* $L_n^{(\alpha)}(z)$ *gilt*

$$z_n < 4n+2\alpha+2.$$

Für die größte Nullstelle z_n *des Hermiteschen Polynoms* $H_n(z)$ *gilt*

$$z_n < (2n+1)^{\frac{1}{2}}.$$

Mit diesen Hilfsmitteln bekommen wir folgende Lösung des $(p, 1)$-Problems für Laguerre- und Hermiteentwicklungen:

SATZ 3. *Zu jedem komplexen Polynom*

$$f(z) = a_0 L_0^{(\alpha)}(z) + a_1 L_1^{(\alpha)}(z) + \cdots + a_p L_p^{(\alpha)}(z) + \zeta L_n^{(\alpha)}(z)$$
$$(1 \leqq p < n, \quad a_p \neq 0)$$

und jeder Konstanten $1 \leqq Q < e$ *gibt es eine nur von* $a_0, a_1, \ldots, a_p$ *und* Q *abhängende Zahl* R, *so daß der Bereich*

$$B := \left\{ z\colon \begin{array}{lll} |\mathrm{Im}\, z| \leqq RQ^{-\frac{\mathrm{Re}\, z}{2p}} & \textit{für} & \mathrm{Re}\, z \geqq 0, \\ |z| \leqq R & \textit{für} & \mathrm{Re}\, z < 0 \end{array} \right\} \tag{10}$$

mindestens p Nullstellen von $f(z)$ *enthält.*

SATZ 4. *Zu jedem komplexen Polynom*

$$f(z) = a_0 H_0(z) + a_1 H_1(z) + \ldots + a_p H_p(z) + \zeta H_n(z) \qquad (1 \leqq p < n, \ a_p \neq 0)$$

und jeder Konstanten $1 \leqq Q < e$ *gibt es eine nur von* $a_0, a_1, \ldots, a_p$ *und* Q *abhängende Zahl* R, *so daß der Bereich*

$$B := \left\{ z\colon |\mathrm{Im}\, z| \leqq RQ^{-\frac{(\mathrm{Re}\, z)^2}{2p}} \right\}$$

mindestens p Nullstellen von $f(z)$ *enthält.*

BEWEIS. Wir beweisen zunächst Satz 3. Es bezeichne

$$\zeta_{n1} < \zeta_{n2} < \cdots < \zeta_{nn} = : z_n$$

die der Größe nach geordneten Nullstellen von $L_n^{(\alpha)}(z)$ und

$$\tau_1 < \tau_2 < \cdots < \tau_p < \cdots$$

die Nullstellen der Airyfunktion. Ferner sei t_1 diejenige Stelle aus $[\tau_p, \tau_{p+1}]$, für die $|\mathrm{Ai}\,(t)|$ ein lokales Maximum annimmt. Wir setzen

$$\text{(11)} \qquad x_1 : = 4n+2\alpha+2-2\left(\frac{2n}{3}\right)^{\frac{1}{3}} t_1$$

und erklären als B_n denjenigen abgeschlossenen, beschränkten Bereich, dessen Rand C_n von der Geraden

$$\mathrm{Re}\, z = x_1,$$

den beiden Kurven

$$|\mathrm{Im}\, z| = Q^{-\frac{\mathrm{Re}\, z}{2p}}$$

und der Geraden

$$\mathrm{Re}\, z = 4n+2\alpha+2 = : x_2$$

erzeugt wird. Man sieht leicht ein, daß B_n die Eigenschaft (i) von Satz 1 besitzt. Wegen Lemma 2 liegen für hinreichend großes n (etwa $n \geqq n_0$) genau p Nullstellen von $L_n^{(\alpha)}(z)$ im Intervall $[x_1, x_2]$ und damit in B_n. Wir wollen nun die Gültigkeit von (iii) zeigen.

Auf der Geraden $\mathrm{Re}\, z = x_1$ nimmt $|L_n^{(\alpha)}(z)|$ sein Minimum in x_1 an. Nach Lemma 2 gilt für hinreichend großes n (etwa $n \geqq n_1$)

$$\text{(12)} \qquad |L_n^{(\alpha)}(x_1)| \geqq c_1 e^{\frac{x_1}{2}} n^{-\frac{1}{3}}$$

mit einer durch α und p bestimmten Konstanten $c_1 > 0$. Auf dem restlichen Teil von C_n besteht die Ungleichung

$$\text{(13)} \qquad \begin{aligned} |L_n^{(\alpha)}(z)| &\geqq |L_n^{(\alpha)}(x_1)| \prod_{\nu=n-p+1}^{n} \left| \frac{z-\zeta_{n\nu}}{x_1-\zeta_{n\nu}} \right| \geqq \\ &\geqq |L_n^{(\alpha)}(x_1)| Q^{-(2n+\alpha+1)} \prod_{\nu=n-p+1}^{n} |x_1-\zeta_{n\nu}|^{-1}. \end{aligned}$$

Da nach Lemma 3

$$\prod_{\nu=n-p+1}^{n} |x_1-\zeta_{n\nu}| \leqq |x_1-(4n+2\alpha+2)|^p = \left(2\left(\frac{2n}{3}\right)^{\frac{1}{3}} t_1\right)^p$$

gilt, bekommen wir aus (12) und (13) für $z \in C_n$ und $n \geqq \max\,(n_0, n_1)$ die Abschätzung

$$|L_n^{(\alpha)}(z)| \geqq c_2 e^{\frac{x_2}{2}} n^{-\frac{p+1}{3}} Q^{-(2n+\alpha+1)}$$

mit einer durch α und p bestimmten Konstanten $c_2 > 0$. Die damit gewonnene untere

Schranke für M_n liefert für jedes $z \in \mathbf{C}$ unter Beachtung von (11) zusammen mit Lemma 1 und 3

$$\left| \frac{L_n^{(\alpha)}(z)}{M_n} z_n^p \right| \leqq \frac{g_1(z)}{c_3} n^{c_4} e^{c_5 n^{\frac{1}{3}}} e^{2|nz|^{\frac{1}{2}}} \left(\frac{Q}{e}\right)^{2n+\alpha+1},$$

wobei c_i $(3 \leqq i \leqq 5)$ gewisse höchstens von α und p abhängende positive Zahlen bezeichnet. Wegen $1 \leqq Q < e$ verschwindet die rechte Seite für $n \to \infty$. Somit sind alle Voraussetzungen von Satz 1 erfüllt.

Die so erreichten Einschließungsbereiche für p Nullstellen kann man durch einen Bereich der Gestalt (10) überdecken. Der Beweis von Satz 4 verläuft ganz analog.

4. Es sei $U = \{u_n(z) : n \in \mathbf{N}_0\}$ eine Folge von Polynomen (2) mit lauter reellen Nullstellen, die in jedem kompakten Teilbereich von $\mathbf{C}$ gleichmäßig gegen eine ganze Funktion $F(z) \not\equiv 0$ mit unendlich vielen reellen Nullstellen konvergiert (zur Existenz siehe z. B. [8, Théorèmes de PÓLYA]). Man überlegt sich leicht, daß eine derartige Basis U niemals ein SW-System ist; sie erfüllt auch nicht notwendig die Voraussetzungen von Satz 1. Dennoch existieren Einschließungen zum $(p, 1)$-Problem, und zwar sogar Schranken für die Beträge von mindestens p Nullstellen, wie der folgende Satz lehrt:

SATZ 5. *Für jede fest vorgegebene Basis U mit den obigen Eigenschaften besitzt jedes komplexe Polynom*

$$f(z) = a_0 u_0(z) + a_1 u_1(z) + \cdots + a_p u_p(z) + \zeta u_n(z) \quad (1 \leqq p < n,\ a_p \neq 0)$$

mindestens p Nullstellen in einem Kreisbereich $|z| \leqq R$, *dessen Radius R nur von* $a_0, a_1, \ldots$ $\ldots, a_p$ *abhängt.*

Interessant ist dabei, daß man mit den hier verwendeten Basen schon Schranken für das (p, k)-Problem mit $k > 1$ bekommt, wenn man nur die auftretenden Koeffizienten einer gewissen asymptotischen Bedingung unterwirft.

Es sei $B(\delta)$ ein Teilbereich von $\mathbf{C}$, der sich asymptotisch wie ein Sektor mit Öffnungswinkel $\delta < \pi$ verhält, d.h. für jedes $\varepsilon > 0$ existiert ein $K > 0$, so daß für alle $z \in B(\delta)$ mit $|z| > K$ die Bedingung

$$|\arg z - \alpha| \leqq \frac{\delta + \varepsilon}{2}$$

mit einem festen α erfüllt ist. Dann gilt der

SATZ 6. *Es seien eine Basis U und ein Bereich* $B(\delta)$ *vorgegeben, die beide obigen Voraussetzungen genügen. Dann besitzt jedes komplexe Polynom*

$$f(z) = a_0 u_0(z) + a_1 u_1(z) + \cdots + a_p u_p(z) + \gamma_1 u_{n_1}(z) + \gamma_2 u_{n_2}(z) + \cdots + \gamma_k u_{n_k}(z)$$

mit

$$1 \leqq p < n_1 < n_2 < \cdots < n_k, \quad a_p \neq 0 \quad \textit{und} \quad \gamma_\varkappa \in B(\delta) \quad (1 \leqq \varkappa \leqq k)$$

mindestens p Nullstellen in einem Kreisbereich $|z| \leqq R$, *dessen Radius R nur von* $a_0, a_1, \ldots$ $\ldots, a_p$ *und k abhängt.*

BEWEIS. Der Beweis von Satz 6 wird uns gleichzeitig die Richtigkeit von Satz 5 zeigen.

Es seien

$$h(z) := a_0 u_0(z) + a_1 u_1(z) + \cdots + a_p u_p(z) \quad \text{und} \quad k \in N$$

fest vorgegeben. Wir nehmen an, daß es zu jedem $m \in N$ Indizes $p < n_{1m} < n_{2m} < \cdots < n_{km}$ und komplexe Zahlen $\gamma_{\varkappa m} \in B(\delta)$ $(1 \leqq \varkappa \leqq k)$ gibt, für die

$$f_m(z) := h(z) + \gamma_{1m} u_{n_{1m}}(z) + \gamma_{2m} u_{n_{2m}}(z) + \ldots + \gamma_{km} u_{n_{km}}(z)$$

höchstens $p-1$ Nullstellen in $|z| \leqq m$ besitzt. Gilt $\limsup_{m \to \infty} n_{km}^{-1} \neq 0$, so lassen sich für eine unendliche Teilfolge $N' \subseteqq N$ alle für $m \in N'$ in der Entwicklung von $f_m(z)$ vorkommenden Polynome $u_\nu(z) \in U$ in ein SW-System einbetten. Satz C liefert dann einen Widerspruch.

Es sei nun $\varkappa_1$ die kleinste natürliche Zahl $1 \leqq \varkappa_1 \leqq k$ mit

$$\lim_{m \to \infty} \frac{1}{n_{\varkappa_1 m}} = 0.$$

Falls $\varkappa_1 > 1$, so existiert für eine unendliche Teilfolge $N' \subseteqq N$ $l := \max_{m \in N'} n_{\varkappa_1 - 1 m}$, andernfalls sei $N' := N$ und $l := p$. O. B. d. A. können wir annehmen, daß

$$\tau_m := \max_{1 \leqq \varkappa \leqq k} |\gamma_{\varkappa m}| \neq 0$$

ist und setzen

$$\sigma_m := \begin{cases} \tau_m^{-1} & \text{falls} \quad \tau_m > 1, \\ 1 & \text{sonst.} \end{cases}$$

Das Polynom $\hat{f}_m(z) := \sigma_m f_m(z)$ besitzt die gleichen Nullstellen wie $f_m(z)$; die Beträge seiner Koeffizienten bleiben für $m \in N'$ beschränkt. Deshalb existiert eine unendliche Teilfolge $N'' \subseteqq N'$, so daß gleichmäßig in jedem kompakten Teilbereich von C

$$\lim_{m \in N'' \, m \to \infty} \hat{f}_m(z) = g(z) + cF(z) \tag{14}$$

gilt, wobei $c \in C$ und $g(z)$ ein Polynom vom Grad $q \leqq l$ ist. Falls $q < p$, so folgt $\lim_{m \in N'' \, m \to \infty} \sigma_m = 0$. Dann ist $g(z)$ das Nullpolynom, und es gilt ferner für hinreichend großes $m \in N''$

$$\sigma_m^{-1} = \max_{\varkappa_1 \leqq \varkappa \leqq k} |\gamma_{\varkappa m}|.$$

Wegen $\gamma_{\varkappa m} \in B(\delta)$ $(1 \leqq \varkappa \leqq k)$ bekommt man

$$c = \lim_{m \in N'' \, m \to \infty} \sigma_m \sum_{\varkappa = \varkappa_1}^{k} \gamma_{\varkappa m} \neq 0.$$

Deshalb können wir (14) schreiben als

$$\lim_{m \in N'' \, m \to \infty} \hat{f}_m(z) = c_1 g(z) + c_2 F(z) =: G(z),$$

wobei $c_i \in C$ $(i = 1, 2)$, $|c_1| + |c_2| \neq 0$ und $g(z)$ ein Polynom vom Grad q mit $p \leqq q \leqq l$ ist.

Ist $c_1 c_2 = 0$, so besitzt $G(z)$ mindestens p Nullstellen. Nach dem Satz von HURWITZ bleiben dann für $m \in N''$ auch mindestens p Nullstellen von $f_m(z)$ beschränkt, im Widerspruch zur Annahme.

Ist $c_1 c_2 \neq 0$, so besitzt $G(z)$ nach dem Satz von HADAMARD unter Berücksichtigung der von PÓLYA entdeckten Eigenschaften von $F(z)$ (siehe [8]) die Darstellung

$$G(z) = P(z) e^{az^2 + bz}, \tag{15}$$

wobei a und $b \in C$ und $P(z) \not\equiv 0$ ein Polynom vom Grad $r \leqq p-1$ bezeichnet. Für die $(l+1)$-te Ableitung findet man

$$G^{(l+1)}(z) = c_2 F^{(l+1)}(z).$$

Da die rechte Seite nach dem Satz von ROLLE unendlich viele Nullstellen besitzt, bekommt man einen Widerspruch zur Darstellung (15).

Wie der Beweis erkennen läßt, darf man im Falle $k=1$ den Bereich $B(\delta)$ durch C ersetzen. Damit gilt auch Satz 5.

Läßt man die Bedingung $\gamma_\varkappa \in B(\delta)$ $(1 \leqq \varkappa \leqq k)$ für $k > 1$ fallen, so wird der Satz 6 falsch; in allgemeinen existieren dann nicht einmal Schranken für die Beträge der Imaginärteile von Nullstellen, wie das folgende Beispiel zeigt.

Es seien

$$u_0(z) \equiv 1$$

und für $n \in N$

$$u_{2n-1}(z) := z \prod_{\nu=1}^{n-1} \left(1 - \frac{z^2}{\nu^2}\right),$$

$$u_{2n}(z) := u_{2n-1}(z) + c_n \left(1 + \frac{z^2}{n^2}\right)^n, \quad c_n > 0.$$

Für hinreichend kleines c_n liegt nach dem Satz von ROUCHÉ in jedem der $2n-1$ Kreisbereiche $|z \pm \nu| < 1/3$ $(\nu = 0, 1, \ldots, n-1)$ genau eine reelle Nullstelle von $u_{2n}(z)$. Da ein reelles Polynom niemals genau eine nichtreelle Nullstelle besitzen kann, so sind alle Nullstellen von $u_{2n}(z)$ reell. Bildet $\{c_n\}$ eine Nullfolge, so gilt gleichmäßig in jedem kompakten Teilbereich von

$$\lim_{n\to\infty} u_{2n-1}(z) = \lim_{n\to\infty} u_{2n}(z) = \frac{\sin \pi z}{\pi}.$$

Für geeignet gewähltes c_n besitzt deshalb $U := \{u_n(z) : n \in N_0\}$ die in Satz 5 und 6 vorausgesetzten Eigenschaften, wobei wir

$$c_n = o(2^n n^{-n-p}) \tag{16}$$

fordern dürfen.

Ist nun $h(z)$ ein Polynom vom Grad p und

$$f_n(z) := h(z) + \frac{1}{c_n^2}\left(u_{2n-1}(z) - u_{2n}(z)\right) = h(z) - \frac{1}{c_n}\left(1 + \frac{z^2}{n^2}\right)^n,$$

so zeigt der Satz von ROUCHÉ, daß wegen (16) für hinreichend großes n genau n Nullstellen von $f_n(z)$ in jedem der beiden Kreisbereiche $|z \pm in| < 1$ liegen.

5. Die Sätze 1 und 5 lassen vermuten, daß für eine Basis U von Polynomen mit lauter reellen Nullstellen schon unter sehr schwachen Voraussetzungen Einschließungen zum $(p, 1)$-Problem existieren. Eine Abgrenzung der Lösbarkeit des $(1, 1)$-Problems für eine gewisse Klasse von Basen liefert der folgende

SATZ 7. *Es sei U ein System von Polynomen*

$$u_n(z) := (z-c_n)^n \quad (c_n \in \mathrm{R}, n \in \mathrm{N}_0).$$

Zum $(1, 1)$-*Problem existiert genau dann eine Schranke für den Imaginärteil einer Nullstelle, wenn*

$$\limsup_{n\to\infty} \left|\frac{c_n}{n}\right| < \infty. \tag{17}$$

BEWEIS. a) Es sei (17) erfüllt, wobei wir uns wegen Satz C auf den Fall $|c_n| \to \infty$ für $n \to \infty$ beschränken dürfen. Wir betrachten das Polynom

$$f(z) = a_0 + a_1 z + \zeta (z-c_n)^n \quad (a_1 \neq 0,\ n > 1)$$

und setzen

$$a_0 + a_1 c_n = : -\varrho_n e^{i\chi_n} \quad (\varrho_n > 0,\ \chi_n \in [0, 2\pi)).$$

Die Transformation

$$t = e^{-i\chi_n} a_1 (z-c_n) \tag{18}$$

führt die Nullstellen von $f(z)$ über in die des Polynoms

$$g(t) = -\varrho_n + t - re^{i\varphi} t^n,$$

wobei $r>0$ und $\varphi \in [0, 2\pi)$ geeignete von ζ abhängende Größen sind. Es sei nun φ fest. Für hinreichend kleines r besitzt $g(t)$ eine Nullstelle im Kreisbereich

$$K := \{t : |t-\varrho_n| \leqq 1\}.$$

Eine Umrechnung mit Hilfe von (18) zeigt, daß dann $f(z)$ eine Nullstelle im Streifen

$$|\mathrm{Im}\, z| \leqq \frac{1+|a_0|}{|a_1|}$$

besitzt. Es sei nun r^* so gewählt, daß für alle $0 < r \leqq r^*$ eine Nullstelle von $g(t)$ in K liegt, die für $r = r^*$ ein Randpunkt $t^* = \varrho^* e^{i\alpha}$ von K ist. Der Satz von ROUCHÉ zeigt

$$r^* \geqq (\varrho_n + 1)^{-n}.$$

Setzen wir

$$h_1(t) = -\varrho_n + t \quad \text{und} \quad h_2(t) = e^{i\varphi} t^n,$$

so gilt modulo 2π

$$\psi^* := \arg h_1(t^*) = \arg h_2(t^*).$$

Ist $\psi^* = \pi$, so liegt für jedes $r \in [r^*, \infty)$ eine Nullstelle von $g(t)$ in $[0, \varrho_n - 1]$, da $g(0) < 0$ und $g(\varrho_n - 1) \geqq 0$ ist. Anderfalls können wir o. B. d. A. $0 \leqq \psi^* < \pi$ voraussetzen. Da $h_2(t)$ im Sektor

$$S := \left\{t : \alpha \leqq \arg t \leqq \alpha + \frac{2\pi}{n}\right\}$$

jeden Argumentwert modulo 2π annimmt, zeigt eine einfache geometrische Überlegung (Schnitt zweier Geradenscharen), daß zu jedem $\psi\in[\psi^*,\pi)$ ein $\tau:=\tau(\psi)\in S$ existiert, für das

$$\psi=\arg h_1(\tau)=\arg h_2(\tau)$$

gilt. Für

$$r:=r(\psi):=\frac{h_1(\tau)}{h_2(\tau)}$$

ist dann τ Nullstelle von $g(t)$; r hängt dabei stetig von ψ ab. Da $r(\psi^*)=r^*$ und $r(\psi)\to\infty$ für $\psi\to\pi$, besitzt $g(t)$ für jedes $r\in[r^*,\infty)$ eine Nullstelle in S.

Nach einer bekannten Nullstellenschranke (siehe [11, S. 19]) liegen für $r\geqq r^*$ alle Nullstellen von $g(t)$ in

$$B:=\left\{t:|t|\leqq 2\max\left(\left(\frac{\varrho_n}{r^*}\right)^{\frac{1}{n}},\left(\frac{1}{r^*}\right)^{\frac{1}{n-1}}\right)\leqq 2\varrho_n\bigl(1+o(1)\bigr)\right\},$$

also liegt mindestens eine in $S\cap B$. Rücktransformation in die z-Ebene mit Hilfe von (18) zeigt, daß $f(z)$ eine Nullstelle im Streifen

$$|\operatorname{Im} z|\leqq\frac{2}{|a_1|}\left(1+|a_0|+2\pi\frac{\varrho_n}{n}\right)\cdot\bigl(1+o(1)\bigr)$$

besitzt. Die rechte Seite bleibt wegen (17) für $n\to\infty$ beschränkt.

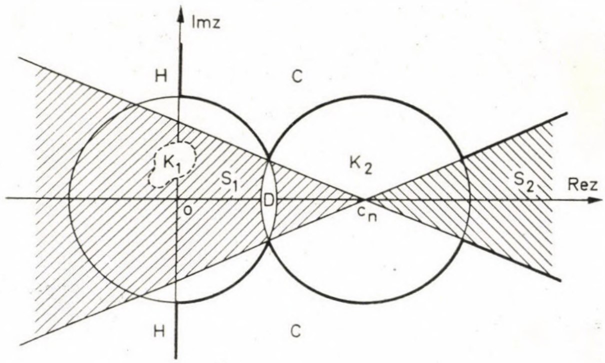

Fig. 1

b) Für $n\geqq 2$ seien (vgl. Fig. 1)

$$f(z):=z-e^{i\frac{\pi}{2}}\left(\frac{2\cos\frac{\pi}{4n}}{c_n}\right)^{n-1}(z-c_n)^n=:z-h(z),$$

$$S_1:=\left\{z:\pi\left(1-\frac{1}{4n}\right)\leqq\arg(z-c_n)\leqq\pi\left(1+\frac{1}{4n}\right)\right\},$$

$$S_2:=\left\{z:-\frac{\pi}{4n}\leqq \arg(z-c_n)\leqq\frac{\pi}{4n}\right\},$$

$$K_1:=\left\{z:|z|\leqq\frac{c_n}{2\cos\frac{\pi}{4n}}\right\},\quad K_2:=\left\{z:|z-c_n|\leqq\frac{c_n}{2\cos\frac{\pi}{4n}}\right\},$$

$$D:=K_1\cap K_2 \quad\text{und}\quad H:=\{z:\operatorname{Re} z\leqq 0\},$$

wobei wir o. B. d. A. $c_n>0$ voraussetzen.

Für $z\in K_1\setminus D$ folgt $|z|<|h(z)|$, für $z\in K_2\setminus D$ folgt $|z|>|h(z)|$, für $z\in S_2$ folgt $\arg z\neq\arg h(z)$, für $z\in D$ folgt $z\in S_1$ und damit ebenfalls $\arg z\neq\arg h(z)$, für $z\in H$ folgt

$$|h(z)|\geqq\left(2\cos\frac{\pi}{4n}\right)^{n-1}|z-c_n|>|z-c_n|>|z|.$$

Deshalb liegen alle Nullstellen von $f(z)$ im Komplement C von $H\cup K_1\cup K_2\cup S_2$, also jedenfalls im Bereich

$$|\operatorname{Im} z|\geqq\frac{c_n}{2}\tan\frac{\pi}{4n}=\frac{\pi}{8}\frac{c_n}{n}\left(1+O\left(\frac{1}{n}\right)\right).$$

Ist (17) verletzt, so bleibt demnach für keine Nullstelle der Imaginärteil beschränkt.

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(Eingegangen: 15. September, 1972.)

BOOK REVIEW

N. BOURBAKI: Éléments de mathématique, fasc. XXXVI. Variétés différentielles et analytiques.

FASCICULE DE RÉSULTATS, PARAGRAPHES 8 À 15. HERMANN, PARIS, 1971. PP. 99.

Comme le titre du fascicule l'indique, cette pièce de la série de Bourbaki continue d'énumérer les définitions et théorèmes de la théorie des variétés différentielles et analytiques, par conséquent c'est un ouvrage de référence sans démonstrations. Nous attendons d'un livre de ce genre qu'il rassemble les notions fondamentales et les résultats principaux de la théorie. Grâce à son contenu, à sa construction et à sa présentation typographique le fascicule satisfait à ces demandes c'est pourquoi il est aussi une pièce profitable de la série. Il est en relation organique avec le fascicule paru en 1967 qui contient les résultats des §§ 1—7 car les notations et conventions des ces paragraphes y restent aussi en vigueur. Les nouvelles conventions sont fait savoir aux lecteurs au commencement du livre et les notations spéciales à chaque paragraphe sont indiquées en tête de celui-ci. Bien que la discussion de la matière d'un livre de tel caractère ne serait pas le devoir d'un compte-rendu je dois rendre compte au moins du contenu du fascicule car — comme ce n'est pas arrivé la première fois dans la vie de Bourbaki — le fascicule de résultats a précédé le volume même.

Le § 8 contient les notions importantes du calcul différentiel d'ordre 1, notamment les définitions et les théorèmes concernant le fibré tangent, les champs de vecteurs, les formes différentielles, la différentiation extérieure, les transformations infinitésimales, le crochet, les relèvements, l'affaiblissement de structure et, finalement, les variétés complexes et presque complexes. — Le § 9 est consacré aux équations différentielles et aux feuilletages. On y trouve la notion des courbes intégrales, des feuilletages, des sous-fibrés intégrables et des fibrés intégrables en caractéristique $p \neq 0$. — Dans le § 10 Bourbaki s'occupe des mesures définies par des formes différentielles, c'est-à-dire de mesure module d'une forme différentielle, des orientations, des formes différentielles *M*-tordues, des mesures associées à des formes différentielles tordues. — C'est l'intégration sur les fibres et la formule de Stokes qui sont l'objets du § 11. — Les notions principales groupées dans le § 12 sont celles des jets d' applications, des jets de sections, des repères et fibrations principales, des jets de sections d'un fibré vectoriel. Dans le § 13 il s'agit des distributions ponctuelles, des produits tensoriels, des coproduits et des distributions à support fini. Le § 14 est consacré à la théorie des opérateurs différentiels, le § 15 traite des variétés d'applications. Le fascicule est complété par un index des notations et par un index terminologique.

J. Merza

L. Chambadal: Les ensembles. (Collection Bordas-Connaissance. Série UNIVERSITÉ. 91.)

BORDAS, PARIS—MONTRÉAL, 1971. XIII + 142 PP.

This booklet of the well-known series published by *Bordas* gives a rigorous, abstract summary of the principles of modern set and function theory. The naive viewpoint in the treatment is avoided consequently, in favour of the axiomatic one. — The chapters are entitled: Sets and relations. Correspondences. Laws of composition. Binary relations. Ordinal and cardinal numbers. Axioms of set theory. — The style is lucid and that of the special monographs. Several recent results have been mentioned, too. — A well-annotated bibliography closes the book.

P. Medgyessy
(Budapest)

I. A. IBRAGIMOV and YU. V. LINNIK: Independent and stationary sequences of random variables.

WOLTERS—NOORDHOFF PUBLISHING CO. GRONINGEN, 1971. 443 PAGES. DFL. 94,50.

This is a carefully edited translation of the book entitled in Russian „Nezavisimye i stacionarno svjazannye veličiny", published in 1965 at „Nauka", Moscow; it may be considered, in some sense, the continuation of the monograph "Limit distributions for sums of independent random variables" by B. V. Gnedenko and A. N. Kolmogorov, published (in Russian) in 1949. — Chapter 1 introduces basic concepts up to infinitely divisible distributions. Chapters 2—5 deal with analytical properties and domains of attraction of stable distributions (including the proof of unimodality of stable distributions), refinements of the limit theorems for normal convergence, local limit theorems for normal convergence, local limit theorems and limit theorems in L_p spaces. Chapters 6—14 are concerned with limit theorems for large deviations, Richter's local theorems and Bernstein's inequality, Cramér's integral theorem and its refinement by Petrov, monomial zones of local normal attraction, monomial zones of local attraction to Cramér's system of limiting tails, narrow zones of normal attraction, wide monomial zones of integral normal attraction, monomial zones of integral attraction to Cramér's system of limiting tails, integral theorems holding on the whole line. Chapter 15 deals with the approximation of distributions of sums of independent components by infinitely divisible distributions. Chapters 16—17 summarize some results from the theory of stationary processes and conditions of weak dependence for stationary processes. Chapter 18 treats of the central limit theorem for stationary processes. Chapter 19 consists of examples and addenda, while Chapter 20 presents some unsolved problems (in the meanwhile, some of them have been solved). Then Appendices on slowly varying functions, theorems on Fourier transforms and a theorem on convergence of conditional expectations follow. Notes and a valuable survey on some contributions of recent years (by I. A. Ibragimov and V. V. Petrov), not contained in the Russian original close the text. A main bibliography (193 items) and a bibliography to the mentioned survey (60 items) are also given by the authors.

Like the book of B. V. Gnedenko and A. N. Kolmogorov, also the reviewed monograph gives a well-readable and research-stimulating account of the investigations concerning certain types of sums of random variables. With a moderate preliminary knowledge in probability theory and analysis any reader is able to study the wealth of beantiful results (due in a great part to the authors) collected in the book and learn, simultaneously, a lot of analytical techniques also. — The importance of relevant investigations is shown the best by that a book of V. V. Petrov, entitled „Summy nezavisimyh slučajnyh veličin" („Nauka", Moskva) has also been published recently.

P. Medgyessy
(Budapest)

FENYŐ, STEFAN: Moderne mathematische Methoden in der Technik. Band 2. (International Series of Numerical Mathematics, Vol. 11.)

BIRKHÄUSER VERLAG, BASEL UND STUTTGART, 1971. 336 P. 79 FIGURES. FR. 62.—

In contrast to the first volume, the reviewed one deals with finite methods of applied mathematics. — Part 1 is devoted to linear algebra. It contains the elements of matrix algebra and matrix analysis; they are, then, applied to the theory of systems of linear equations, integral equations, systems of differential equations and to special related problems of engineering mechanics and electrical networks. — Part 2 is entitled "the theory of optimization". In fact, it considers the elements of linear programming as well as transport problems solved by means of the "Hungarian method" (Kőnig—Egerváry). Convex programming is stressed, too. — Part 3 considers the elements of graph theory, further, problems concerning trees, graphs on surfaces, vector spaces generated by graphs, directed graphs and the corresponding matrices. Finally graph theoretical methods in the investigation of networks and the theorem of Ford and Fulkerson are described.

The separate parts are independent of other parts of the whole work. The treatment is up to date and carefully composed. Everywhere several modern results have been included in the text, too. Numerical methods are avoided.

The material included in the book is considerable. A greater number of worked out examples might, however, have counterbalanced the concise style of the main text. The bibliography contains 25 items, mainly in German; books of Hungarian mathematicians have also been included. — Unfortunately, there are several striking misprints in the text, e.g. in the Foreword and the Contents.

P. Medgyessy
(Budapest)

K. KURATOWSKI: Introduction to Set Theory and Topology. (Containing a Supplement on Elements of Algebraic Topology by R. Engelking.)

COMPLETELY REVISED SECOND ENGLISH EDITION.
349 PP., PERGAMON PRESS, OXFORD—PWN, WARSZAWA, 1972.

The first English edition of this Introduction appeared in 1962. It became an immediate success, owing to the judicious choice of material and to the high quality of the exposition.

In this second edition, the rapid development of set theory and topology during the last few years has fully been taken into account, and the whole text has been made to conform to the most modern requirements.

Although the most essential changes concern the second part of the book (devoted to topology), there are also changes worth noticing in the first part (on set theory). The concepts of inverse limit, of lattice, of ideal, of filter, of a commutative diagram, and of a cartesian product of an arbitrary number of factors are considered. Also, the presentation gives now a slightly deeper insight into the axioms of set theory. In particular, the notion of class (in the sense of Bernays) is mentioned and later applied, mainly in connection with the concept of category used in the Supplement. — In the theory of ordering relations, more emphasis was put on what was previously called partial ordering. This is now called, in conformity with modern usage, ordering.

The changes in the second part of the book are more essential. In the first edition, this part of the book was chiefly devoted to the study of metric spaces. In this second edition, general topological spaces form its main subject. Consequently, more than half of the second part had to be written anew. It contains new topics which were not considered in the first edition, such as cartesian products of topological spaces, the Čech—Stone compactification, quotient spaces, completely regular spaces, quasi-components; in the chapter on simplexes more material can now be found on simplicial mappings and on the nerve of a cover. Finally, the rather short chapter of the first edition, devoted to complexes, chains and homologies, has been replaced by a much more extensive Supplement on the elements of algebraic topology. Written by professor R. Engelking, this Supplement constitutes an excellent introduction to the basic ideas of homology theory.

Having surveyed in brevity the changes which distinguish this second edition of the book from the first one, let us now give a more precise idea about its contents by listing the chapter headings:

I. Propositional calculus. — II. Algebra of sets. Finite operations. — III. Propositional functions. Cartesian products. — IV. The mapping concept. Infinite operations. Families of sets. — V. The concept of the power of a set. Countable sets. — VI. Operations on cardinal numbers. — VII. Order relations. — VIII. Well ordering. — IX. Metric spaces. Euclidean spaces. — X. Topological spaces. — XI. Basic topological concepts. — XII. Continuous mappings. — XIII. Cartesian products. — XIV. Spaces with a countable base. — XV. Complete spaces. — XVI. Compact spaces. — XVII. Connected spaces. — XVIII. Locally connected spaces. — XIX. The concept of dimension. — XX. Simplexes and their properties. — XXI. Cuttings of the plane. — Supplement: Elements of algebraic topology.

Written by one of the foremost authorities on the subject, this book is ideally suited to serve as a first introduction to set theory and to topology (both general and algebraic). The reviewer feels sure that it will gain well-merited popularity with both students and mature mathematicians.

S. Gacsályi

ADDENDUM

To the publication list

Alfréd Rényi's work

published in Studia Scientiarum Mathematicarum Hungarica 6 (1971) 3—22 the following item, having been omitted regrettably during the printing procedure, is to be added:

1963

20. Asymmetric graphs (with P. Erdős). Acta Math. Acad. Sci. Hung. 14 (1963) 295—315.

P. Medgyessy

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Printed in Hungary

A kiadásért felel az Akadémiai Kiadó igazgatója – Műszaki szerkesztő: Botyánszky Pál
A kézirat nyomdába érkezett: 1972. XI. 20. – Terjedelem: 21,25 (A/5) ív, 8 ábra

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Le prix de l'abonnement: $ 16.00 par an (volume). On s'abonne chez *Kultúra*, Société pour le Commerce de Livres et Journaux (Budapest 62, P. O. B. 149) ou chez ses représentants à l'étranger.

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Подписная цена на год (за однии том): $ 16.00. Подписка на журнал принимается Внешнеторговым предприятием „Култура" (Budapest 62, P. O. B. 149) или его представительствами за границей.

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31. XII. 1973. **Index: 26 748**