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REMARKS TO A PAPER OF L. FEJES TÓTH

by

B. BOLLOBÁS

The notion of the permeability of a layer of open convex disks was introduced by FEJES TÓTH [1]. He proved some results about the permeability of a layer of parallelograms and of a layer of circles ([1] and [2]). This paper deals also with the permeability of a layer of parallelograms. It does not contain essentially new results, but the methods are different from those of FEJES TÓTH and the results are a little sharper.

We define a layer as a set of open convex disks contained in a strip. If w denotes the width of the strip, and l the length of the path connecting the edges of the strip in such a way as to evade all disks, then

$$p = \frac{w}{\inf l}$$

is defined to be the *permeability* of the layer, where the infimum extends over all paths of the above kind.

Our definition permits a wider class of configurations than that of FEJES TÓTH in [2], which permitted only a non accumulating set of open disks.

Let $\pi = ABCD$ be a parallelogram, for which $AB = CD = a$, $BC = DA = b$ and α, the angle at A, is not obtuse.

Choose an arbitrary line, whose direction will be called vertical. Let $s(X, Y)$ denote the level difference between the points X and Y ($s(X, Y)$ is positive if X is higher than Y, negative otherwise).

Consider a position of π, in which B is the highest vertex of π, and the slope of AB is not greater than the slope of BC (Fig. 1). *This will be supposed throughout the paper.* It is easily seen that there is exactly one point on the side AB, say X, satisfying

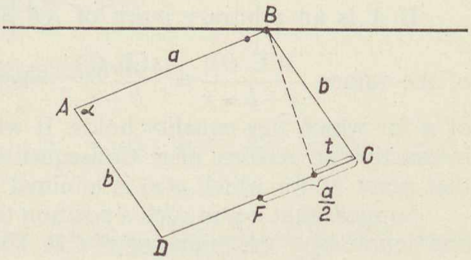

Fig. 1

$$\frac{s(X, D)}{a+b-x} = \frac{s(X, C)}{b+x},$$

where $BX = x$, i.e. such that the ratios of the lengths of the paths and the level differences are equal when we go from X through A to D, and when from X through B to C. It is obvious that the inequality $0 \leqq x \leqq \frac{a}{2}$ must hold.

Put $\bar{p}=\min\frac{s(X,Y)}{bx}$, where the minimum extends over all positions of π, satisfying the conditions. FEJES TÓTH [2] proved that, the permeability of a layer of non accumulating parallelograms similar to π is greater than $\bar{p}$, and on the other hand if $\varepsilon>0$, then there is a layer of translated replicas of π, whose permeability is less than $\bar{p}+\varepsilon$.

The layer mentioned above can be obtained in the following way. Put π into a position, in which $\frac{sX,D}{a+b-x}=\frac{sX,C}{b+x}=\bar{p}$. Denote by Y the point such that $\overrightarrow{YB}=\overrightarrow{AD}$ ($\overrightarrow{UV}$ denotes the vector from U to V). Arrange the translates of π in such a way that the vertices, corresponding to A, form the lattice generated by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AY}$. Taking the parallelograms in a suitably wide horizontal strip, a layer is obtained, whose permeability is less than $\bar{p}+\varepsilon$ (Fig. 2).

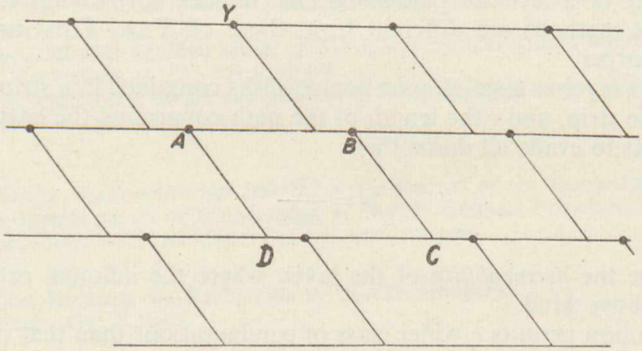

Fig. 2

In what follows we shall show, how the value of p can be determined for a given parallelogram, and then we examine some consequences of the result obtained.

If X is an arbitrary point of AB $\left(0<x\leqq\frac{a}{2}\right)$, denote by $p(x)$ the minimum of the values $\frac{s(X,D)}{a+b-x}=\frac{s(X,C)}{b+x}$, where the minimum is taken over all positions of π for which this equality holds. It will be shown that the minimum is attained in exactly one position of π. Consequently to calculate $\bar{p}$ it is sufficient to determine that point X, for which $p(x)$ is minimal: $p(x)=\bar{p}$.

Suppose that π is in such a position that the equality above holds. Put $t=b\cdot\cos\alpha$ and denote by F the midpoint of CD. Then

$$s(X,F)=\frac{s(X,C)+s(X,D)}{2},$$

consequently (Fig. 3)

$$\frac{s(X,D)}{a+b-x}=\frac{s(X,C)}{b+x}=\frac{s(X,F)}{\frac{1}{2}\{(a+b-x)+(b+x)\}}=\frac{s(X,F)}{\frac{a}{2}+b}$$

Let G be the point of CD satisfying $CG=x$, i.e. $BC \| XG$, furthermore let H be on the line DC beyond C, such that $CH=b$. The line through G parallel to XF meets XH at K. According to our last equality, the similarity of the triangles XFH

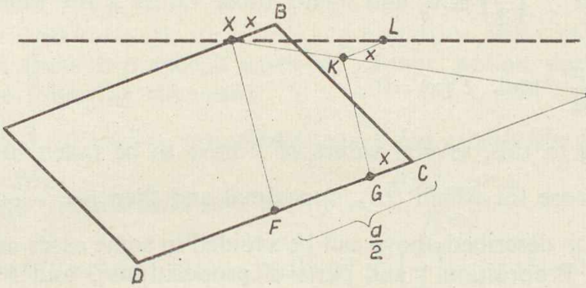

Fig. 3

and KHG implies that $s(K, G) = s(X, C)$. So if $KGCL$ is a parallelogram, then $s(L, C)=s(K, g)=s(X, C)$, i.e. *the line XL is horizontal.* This proves that the equality $\dfrac{s(X, D)}{a+b-x} = \dfrac{s(X, C)}{b+x} = p(x)$ holds in exactly one position of π. Putting $s(X, F) = d(x)$, we have $p(x) = \dfrac{d(x)}{b+\frac{a}{2}}$.

Let (u, v) be a Cartesian coordinate system, whose origin is at F, and $\vec{FC}$ is the direction of the u axis. Then the equation of the line through X and L is the following:

$$\text{(1)} \qquad \frac{2a-4x}{a(a+2b)}u + \frac{2(ab+at+2ax-2x^2-2tx)}{a(a+2b)\sqrt{b^2-t^2}}v = 1.$$

As $d(x)$ is the distance of F from the line defined by (1),

$$\frac{a^2(a+2b)^2(b^2-t^2)}{4} \cdot \frac{1}{d^2} = (ab+at+2ax-2x^2-2tx)^2 + (b^2-t^2)(a-2x)^2 = f(x).$$

If for $0 \leqq x \leqq \frac{a}{2}$ $d(x)$ is defined by the equation $d(x) = \left(b+\frac{a}{2}\right) \cdot p(x)$, then even in the case $x=0$, i.e. when the points X and L coincide, $d(x)$ will be the distance of F from the line defined by (1), since both the line and $d(x)$ are continuous functions of x.

Consequently we should like to find that value of x for which $f(x)$ is maximal. The derivative of $f(x)$ is the following:

$$f'(x) = 4\{4x^3 - 6(a-t)x^2 + (2a^2+2b^2-2ab-6at)x + a(a-b)(b+t)\}.$$

For a given parallelogram, i.e. for a three tuple (a, b, t), the following values of x may give the maximal value of $f(x)$:

$$x=0 \quad \text{if} \quad f'(a) \leqq 0,$$

$$x=\frac{a}{2} \quad \text{if} \quad f'\left(\frac{a}{2}\right) \geqq 0, \text{ and finally those values } x \text{ for which}$$

$$0<x<\frac{a}{2} \quad \text{and} \quad f'(x)=0.$$

If, according to this, several values of x have to be taken into consideration then, we select those for which $f(x)$ is maximal and then $\bar{p}=\dfrac{a^2(b^2-t^2)}{f(x)}$.

The discussion described above can be avoided in some cases and simple results can be obtained. Proposition 1 and parts of propositions 3 and 4 were proved by FEJES TÓTH in [2]. In the proofs below the trivial relations

$$\lim_{x\to-\infty} f'(x) = -\infty, \quad \lim_{x\to+\infty} f'(x) = +\infty \tag{2}$$

will be used several times.

In the following proportions we shall suppose that $\dfrac{s(X, D)}{a+b-x} = \dfrac{s(X, C)}{b+x} = p(x) = \bar{p}$, *i.e. the positions of* π *and* X *are extremal.*

PROPOSITION I. *If* π *is a rectangle* (*i.e.* $t=0$), *then* X *is the midpoint of* AB *and* AB *is horizontal.*

If $t=0$, $f'\left(\frac{a}{2}\right)=0$ and $f''\left(\frac{a}{2}\right) = -4a^2<0$. Then (2) implies that $x=\frac{a}{2}$ is the only root of $f'(x)=0$ in the interval $\left[0, \frac{a}{2}\right]$, and since $f(0)<f\left(\frac{a}{2}\right)$, the maximum of $f(x)$ is attained at $x=\frac{a}{2}$ in the interval $\left[0, \frac{a}{2}\right]$. $\dfrac{s(X, D)}{a+b-\frac{a}{2}} = \dfrac{s(X, C)}{b+\frac{a}{2}}$ gives immediately that AB is horizontal and $\bar{p}=\dfrac{b}{b+\frac{a}{2}}\dfrac{1}{1+\frac{a}{2b}}$.

PROPOSITION 2. *If* π *is a rhombus, then* $x=0$ *if* $a\leqq 3t$ (*i.e.* $\alpha \leqq \arccos \frac{1}{3}$), *and* $x=\dfrac{a-3t}{2}$ *otherwise.*

It follows from the conditions that $f'(0)=0$ and $f'\left(\frac{a}{2}\right)=-6a^2t<0$, so we can disregard the case $x=\frac{a}{2}$. Dividing the equation $f'(x)=0$ by x, a quadratic equation is obtained, which immediately shows that $x=0$ and $x=\dfrac{a-3t}{2}$ can be the roots of $f'(x)=0$ in the interval $\left[0, \frac{a}{2}\right]$. Consequently $x=0$ if $a \leqq 3t$. In the case $a>3t$, $f''(0)>0$, so we can disregard $x=0$ too, and hence $x=\frac{a-3t}{2}$.

It is immediately seen that in the first case (i.e. when $a \leqq 3t$, $\alpha \leqq \arccos \frac{1}{3}$), $\bar{p}-\sin\frac{a}{2}$.

Apart from these two special cases we cannot obtain such simple results, but we have the following statement.

PROPOSITION 3. *If $a>b>t>0$, then $f'(x)=0$ has exactly one solution in $\left[0, \frac{a}{2}\right]$, say x_0; this is the required value of x and* $\bar{p}=\frac{a^2(b^2-t^2)^{\frac{1}{2}}}{f(x_0)}$.

Indeed then $f'(0=4a(a-b)(b+t)>0$ and $f'\left(\frac{a}{2}\right)=-6a^2t<0$ so the maximum of $f(x)$ in $\left[0, \frac{a}{2}\right]$ is attained at an inner point of $\left[0, \frac{a}{2}\right]$. But according to the relations (2), $f'(x)$ vanishes at exactly one point of $\left[0, \frac{a}{2}\right]$, so this is the required value of x.

REMARK. Suppose $6x^2-6ax+a^2-ab \neq 0$. Then $f'(x)=0$ implies

$$t=\frac{4x\ -6ax^2+(2a^2+2b^2-2ab)x+ab(a-b)}{6ax-6x^2-a^2+ab)}. \tag{3}$$

If for a three tuple a, b, x $\left(a>b>0,\ 0<x<\frac{a}{2}\right)$ the value of t, given by (3), satisfies $0<t<b$, then it follows from Proposition 3, that this value of x belongs to the parallelogram determined by the values a, b and t. I.e. for the parallelogram given by a, b and t, $\bar{p}=\frac{a^2(b^2-t^2)}{f(x)}$ holds. This gives us a quick simple method of calculating $\bar{p}$ for a large number of parallelograms and consequently to obtain good approximations of $\bar{p}$ for all parallelograms.

We now prove the converse of proposition 2.

PROPOSITION 4. *If the "shorter" diagonal BD is vertical or the "longer" diagonal AC is horizontal, then π is a rhombus and α is less than* $\arccos \frac{1}{3}$.

The slope of $AB(\geqq BC)$ is less than or equal to the slope of BC, so the shorter diagonal BD is vertical if and only if $a=b$ and the two slopes are equal. According to Proposition 2, the equality of the slopes occurs if and only if $\alpha \leqq \arccos \frac{1}{3}$.

On the other hand if AC is horizontal, the equation $\frac{s(X,D)}{a+b-x}=\frac{s(X,C)}{b+x}$ implies $\frac{c(2a-x)}{a+b-x}=\frac{c(a-x)}{b+x}$ (where c is equal to the sinus of the slope of AB), i.e. $2x^2-4ax+a(a-b)=0$. The roots of this equation are $x=\frac{2a\pm\sqrt{2a^2+2ab}}{2}$.

As x can not be greater than $\frac{a}{2}$, $x=\frac{a}{2}-\sqrt{\frac{a^2+ab}{2}}$ is the only solution. Substitute this value into (3). If $a>b$, it can be seen, that the denominator is positive, and then a rearrangement of the equation shows that t is greater than b. This being impossible, $a=b$ must hold and so $x=0$. But because of Proposition 2 this holds if and only if $\alpha \leqq \arccos \frac{1}{3}$.

Finally we prove the following generalization of the theorem of FEJES TÓTH.

THEOREM. (i) *Let π_0 be a parallelogram, other then a rhombus of acute angle at most* $\arccos \frac{1}{3}$. *If p is the permeability of a layer $\mathcal{R}$ of parallelograms similar to π_0, then $p>\bar{p}$ and* $\inf p=\bar{p}$.

(ii) *Let π_1 be a rhombus, whose acute angle α is at most* $\arccos \frac{1}{3}$. *Then the permeability of a layer of parallelograms similar to π_1 is at least* $\sin \frac{\alpha}{2}$, *and the equality can be attained. The permeability of a layer of non accumulating parallelograms is greater than* $\sin \frac{\alpha}{2}$. *The infimum of the permeability of a layer of translates of* π_1 is also $\sin \frac{\alpha}{2}$.

On account of FEJES TÓTH's theorem, it is sufficient to show that $p>\bar{p}$.

Choose a parallelogram of $\mathcal{R}$. Owing to the conditions of the theorem this parallelogram has a side UV satisfying $\frac{s(U, V)}{UV}>\bar{p}$. Denote by w the width of the layer and by w_1 and w_2 the distances of U and V from the upper and lower edges, respectively. Thus $w_1+s(U, V)+w_2=w$.

We shall prove that there are paths of length l_1, l_2, connecting the upper edge of the strip with U and V with the lower edge of the strip, and such that $\frac{w_1}{l_1}\geqq\bar{p}, \frac{w_2}{l_2}\geqq\bar{p}$.

Combining these paths with the interval UV, the path obtained, of length l_1+UV+l_2 satisfies $\frac{w}{l}>\bar{p}$, and this proves (i).

Because of the symmetry it is sufficient to prove the existence of a path of length at most $\frac{w_1}{p}$, connecting U with the upper edge. Denote by $\mathcal{R}^n$ the layer formed by those parallelograms of $\mathcal{R}$, whose diameter is at least $1/n$. According to FEJES TÓTH's theorem if $n>0$, there is a path connecting U with the upper edge and evading the parallelograms of $\mathcal{R}^n$, for which $g_n \leqq \frac{w_1}{\bar{p}}$, where g_n denotes the length of the path. Join a path of length $\frac{w_1}{\bar{p}}-g_n$ to the endpoint of this path, and denote by h_n the path obtained (Fig. 4.). Fix a coordinate system and let $(x, y)=h_n(t)$ be the parametric form of the path h_n, where the length of the path is the parameter: $0\leqq t\leqq\frac{w_1}{\bar{p}}$. Take the sequence of the functions $h_1(t), h_2(t), h_3(t), \ldots$ The values of these functions are in a compact domain (in the circle of centre U

and radius $\left.\frac{w_1}{p}\right)$, and the functions are equicontinuous since $|h_n(t')-h_n(t'')|<\varepsilon$ if $|t'-t''|<\varepsilon$. On account of the theorem of ARZELA and ASCOLI it is possible to select a uniformly convergent subsequence of this sequence. Denote the limit function by $h(t)$. The curve defined by $(x, y)=h(t)$ is a continuous, rectifiable path, its length is at most w_1/p and it connects U with the upper edge, since every element of the sequence has this property and the convergence is uniform. It is immediately seen that this path evades every parallelogram of $\mathscr{R}$. This proves statement (i).

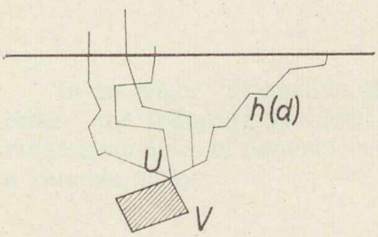

Fig. 4

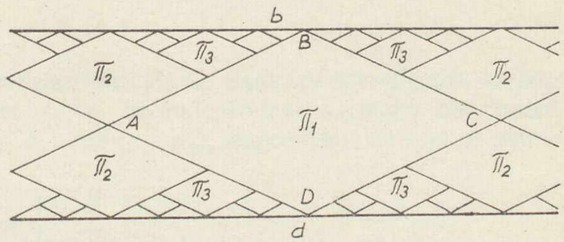

Fig. 5

(ii) As in (i), it can be shown that the permeability of a layer is at least $\bar{p}$, and according to Proposition 2, $\bar{p}=\sin\frac{\alpha}{2}$. The second part of the statement follows from FEJES TÓTH's theorem, so to complete the proof it is sufficient to construct a layer $\mathscr{R}_1$ of parallelograms similar to π_1, whose permeability is $\sin\frac{2}{\alpha}$.

Put the rhombus $\pi_1=ABCD$ in such a position that the diagonal AC be horizontal where the angle α is at A. Reduce π_1 to the $\frac{1}{2^{n-1}}$ and denote it by π_n. The edges of the strip will be the lines b, d, going through B and D and parallel to AC. First place the replicas of π_1 into the strip, translated through $n\cdot\overrightarrow{AC}$ $(n=\pm 1, \pm 2, \ldots)$. Then the remaining parts of the strip are isosceles triangles with angles $\pi-\alpha$ at the apex. Then put translated replicas of π_2 into these triangles. The remaining parts of the strip are again isosceles triangles, which can just contain the translates of π_3, and so on (Fig. 5.). Denote by $\mathscr{R}_1$ the layer obtained in this way. It is immediately seen that a path, connecting b with d and evading all these parallelograms, is the union of some sides and some parts of some sides of the rhombi.

Consequently the permeability of $\mathscr{R}_1$ is $\sin\frac{\alpha}{2}$.

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REMARK ON K. SARKADI'S PAPER ENTITLED: "ESTIMATION AFTER SELECTION"

by
G. TUSNÁDY

In his paper "Estimation after Selection" [1] K. SARKADI investigates, among others, the following problem: Let x_1, x_2 be independent normally distributed random variables of parameters (μ_1, σ_1) and (μ_2, σ_2), respectively, and let us define a variable m by

$$m = \begin{cases} \mu_1 & \text{if } x_1 \leqq x_2 \\ \mu_2 & \text{if } x_1 > x_2 \end{cases}$$

In connection with the estimation of m he shows (see Theorem 3.1) that an estimation $\tau(x_1, x_2)$ of finite variance must be biassed. Our remark is that this is true without supposing the variance being finite.

From his proof we use the fact that it suffices to show the following

THEOREM. *Let x be a normally distributed random variable with parameters $(\mu, 1)$. Then the function* $\mu \int_0^\mu e^{-\frac{t^2}{2}} dt$ *cannot have any unbiassed estimation.*

PROOF. In other words, we have to show that there exists no function $\tau(x)$ such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2}} \tau(x)\, dx$$

(as a LEBESGUE integral) is equal to

$$\mu \int_0^\mu e^{-\frac{t^2}{2}} dt$$

for all real μ. Suppose, on the contrary, the existence of such a $\tau(x)$. By elementary computation the equation becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mu x} e^{-\frac{x^2}{2}} \tau(x)\, dx \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{-\mu x} \vartheta(x)\, dx = \mu e^{\frac{\mu}{2}} \int_0^\mu e^{-\frac{t^2}{2}} dt$$

Now, the left-hand side, as well as the right-hand side of this equation define an entire function of μ and by analytic continuation they coincide for all complex μ. In particular, we have on the imaginary axis

$$\int_{-\infty}^{\infty} e^{-i\nu x}\vartheta(x)dx = i\nu e^{\frac{(i\nu)^2}{2}}\int_0^{i\nu} e^{-\frac{t^2}{2}}dt = -\nu e^{-\frac{\nu^2}{2}}\int_0^{\nu} e^{\frac{t^2}{2}}dt.$$

Owing to RIEMANN's lemma, the left-hand side tends to 0 as $\nu\to\infty$, while

$$\lim_{\nu\to\infty} \nu e^{-\frac{\nu^2}{2}}\int_0^{\nu} e^{\frac{t^2}{2}}dt = \lim_{\nu\to\infty}\frac{\int_0^{\nu} e^{\frac{t^2}{2}}dt}{\nu^{-1}e^{\frac{\nu^2}{2}}} = \lim\frac{e^{\frac{\nu^2}{2}}}{-\nu^{-2}e^{\frac{\nu^2}{2}}+e^{\frac{\nu^2}{2}}} = 1$$

which is a contradiction and the proof is completed.

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ON RATIONAL APPROXIMATION OF ABSOLUTELY CONTINUOUS FUNCTIONS

by

G. FREUD

1. Introduction

To an arbitrary continuous function $f(x)$ $(0 \leqq x \leqq 1)$ we consider its best approximation by rational functions of degree n:

$$R_n(f) = \min_{a_i, b_i} \max_{x \in [0,1]} \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_n x^n}$$

(the existence of this minimum was proved by J. L. WALSH [4]). Let us now consider the class V_1 of functions $f(x)$ $(0 \leqq x \leqq 1)$, which are the indefinite integrals of a function $f'(x)$ of bounded variation $V(f')$. It follows from the very remarkable paper of P. SZÜSZ and P. TURÁN [3], that there exists a decreasing positive sequence $\{\lambda_n\}$, $\lambda_n \to 0$ so that for each $f \in V_1$ we have

$$R_n(f) \leqq \lambda_n V(f'). \tag{1}$$

In what follows let λ_n be the smallest number for which (1) is satisfied.

P. SZÜSZ and P. TURÁN gave for $\{\lambda_n\}$ the estimate $\lambda_n = O\left(\frac{\log^4 n}{n^2}\right)$. This was improved by the author [2] to

$$\lambda_n = O\left(\frac{\log^2 n}{n^2}\right). \tag{2}$$

In the present paper we establish an upper estimate of $R_n(f)$ for absolutely continuous functions f. In our estimate of $R_n(f)$ the number λ_n is involved. We use this fact to prove the lower estimate[1]

$$\lambda_n \geqq \frac{c_1}{n^2} \tag{3}$$

A less precise result of D. NEWMAN stated $\lambda_n \neq o\left(\frac{1}{n^2 \log n}\right)$ only. (See [3].)

It is quite probable, that $\lambda_n = O\left(\frac{1}{n^2}\right)$ holds. Is this the case, we deduce from our theorem a proof of a conjecture of D. NEWMAN (see below).

[1] $c_1, c_2, \ldots$ denote positive absolute constants.

2. The Estimate of $R_n(f)$

Let $f(x)$ be absolutely continuous in [0, 1], let its (almost everywhere existing) derivative be $f'(x) \in L$. We consider the modulus of smothness of f

$$\omega_2(f;\delta) = \max_{\substack{|h| \leq \delta \\ (x-h,\, x+h) \subset [0,\,1]}} |f(x+h) - 2f(x) + f(x-h)|$$

further the expression

$$\int_0^{1-h} |f'(t+h) - f'(t)| dt = \Delta(f';h).$$

From $f' \in L$ we conclude $\lim_{h\to 0} \Delta(f';h) = 0$.

THEOREM 1. *We have for each absolutely continuous f and each positive integer ν*

(4) $$R_n(f) \leq \omega_2(f;\nu^{-1}) + \nu\Delta(f';\nu^{-1})\lambda_n \qquad (\nu = 1, 2, \ldots).$$

REMARK. Let us assume $\lambda_n = O(n^{-2})$ to hold. If $f(x)$ is absolutely continuous and belongs to the ZYGMUND class Z, i.e. $\omega_2(f;\delta) = O(\delta)$ then choosing

$$\nu = [n\{\max_{0 \leq h \leq n^{-1}} \Delta(f';h)\}^{-\frac{1}{2}}] + 1 = \nu_0(n)$$

we obtain from (4)

(5) $$R_n(f) = o\left(\frac{1}{n}\right).$$

It was conjectured by D. NEWMANN[2] that (5) is valid for $f \in \text{Lip } 1$. Now, from $f \in \text{Lip } 1$ follows that $f \in Z$ and that f is absolutely continuous. In this way Newmann's hypothesis follows from the hypothesis $\lambda_n = O(n^{-2})$ which is possibly simpler to prove.

As a second application, let $f(x)$ be absolutely continuous and $\omega_2(f;\delta) \leq \leq \delta \log^2 \frac{1}{\delta}$. In this case we insert in (4) $\nu = \min\{\nu_0(n), n^2\}$. We obtain, using (2), that $R_n(f) = o\left(\frac{\log^2 n}{n}\right)$. This is better than the polynomial approximation of the same class which is $c_2 \frac{\log^2 n}{n}$. This result is new and independent of any hypothesis.

PROOF OF THEOREM 1. As a first step we approximate $f(x)$ by the polygonal line $\psi_\nu(x)$ through the points $\left(\frac{k}{\nu}, f\left(\frac{k}{\nu}\right)\right)$ $(k = 0, 1, \ldots, \nu)$. Then $\psi_\nu \in V_1$ and ψ'_ν is piecewise constant and has the jump equal to $\nu\left[f\left(\frac{k+1}{\nu}\right) - 2f\left(\frac{k}{\nu}\right) + f\left(\frac{k-1}{\nu}\right)\right]$

[2] See the communication of H. S. SHAPIRO, at page 189 in the book *On approximation theory* (edited by P. L. BUTZER and J. I. KOREVAAR S. N. M. Vol. 5. Birkhäuser Verlag, Basel 1964).

at the point $x=\frac{k}{\nu}$, so that

$$V(\psi_n') = \nu \sum_{k=1}^{\nu-1} \left| f\left(\frac{k+1}{\nu}\right) - 2f\left(\frac{k}{\nu}\right) + f\left(\frac{k-1}{\nu}\right) \right| = \nu \sum_{k=1}^{n-1} \left| \int_{\frac{k-1}{\nu}}^{\frac{k}{\nu}} \left[f'\left(t+\frac{1}{\nu}\right) - f'(t) \right] dt \right| \leqq$$

(6)

$$\leqq \nu \int_0^{1-\frac{1}{\nu}} \left| f'\left(t+\frac{1}{\nu}\right) - f'(t) \right| dt = \nu \Delta(f'; \nu^{-1}).$$

By a lemma of H. BURKILL (Lemma 5. 2 in paper [1]) we have

(7) $$|f(x)-\psi_\nu(x)| \leqq \omega_2\left(f; \frac{1}{2\nu}\right).$$

From (6) and (1) we obtain

(8) $$R_n(\psi_n) \leqq \nu \Delta(f'; \nu^{-1}) \lambda_n.$$

Finally, from (7) and (8) it follows (4), Q.e.d.

3. Lower Estimate of λ_n

Let

(9) $$f_1(x) = \sum_{\nu=1}^{\infty} \frac{T_{9^{2\nu}}(x-\frac{1}{2})}{9^{3\nu}},$$

where $T_n(x) = \cos n\,(\arc\cos x)$ is the Čebyšev's polynomial. Taking the partial sums of this series, it is easy to see that the n-th degree polynomial approximation of $f_1(x)$ in $[-\frac{1}{2}, \frac{3}{2}]$ is $O(n^{-3/2})$. It follows that in $[0, 1]$ $f_1(x)$ has a continuous derivative $f_1'(x)$ which belongs to the class Lip $\frac{1}{2}$, so that

(10) $$\omega_2(f_1; h) \leqq c_3 h^{3/2} \quad \text{and} \quad \Delta(f_1'; h) \leqq c_4 h^{1/2}.$$

Let N be a positive integer. At the points

$$x_k = \frac{1}{2} + \cos\frac{k\pi}{9^{2(N+1)}} \qquad (3\cdot 9^{2N+1} \leqq k \leqq 6\cdot 9^{2N+1})$$

which are more than $2.\,9^{2N}+2$ in number, the difference

$$f_1(x) - \sum_{\nu=1}^{N} \frac{T_{9^{2\nu}}(x-\frac{1}{2})}{9^{3\nu}}$$

attains the extremal value

$$\pm \sum_{\nu=N+1}^{\infty} \frac{1}{9^{3\nu}}$$

with alternating signe. By Čebyšev's theorem we have

$$R_{9^{2N}}(f_1)=\sum_{\nu=N+1}^{\infty}\frac{1}{9^{3\nu}}>\frac{1}{9^{3(N+1)}}. \tag{11}$$

Now we insert $f=f_1$ and $n=9^{2N}$ in (4), and take (10) and (11) in consideration:

$$\frac{1}{9^{3(N+1)}}<R_{9^{2N}}(f_1)\leqq c_3\nu^{-3/2}+c_4\nu^{1/2}\lambda_{9^{2N}}. \tag{12}$$

Let finally r be a fixed integer for which $c_3 r^{-3/2}<\frac{1}{2\cdot 9^3}$ is satisfied. Taking $\nu=r9^{2N}$ we obtain from (12)

$$\lambda_{9^{2N}}>c_4^{-1}r^{1/2}9^N\left[\frac{1}{9^{3(N+1)}}-c_3r^{-3/2}\frac{1}{9^{3N}}\right]>c_4^{-1}r^{1/2}9^N\cdot\frac{1}{2}\frac{1}{9^{3(N+1)}}>\frac{c_5}{9^{2N}}. \tag{13}$$

From (13) and the monotonity of λ_n follows (3). Q.e.d.

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ON THE DETERMINANT OF RANDOM MATRICES

by

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§ 1.

a) In a previous paper [2] we proved the following theorem:

If A_n denotes the number of singuler $n \times n$ matrices with elements 0, 1 (i.e. the number of $n \times n$ (0, 1) matrices having determinant 0), then

$$\lim_{n=+\infty} \frac{A_n}{2^{n^2}} = 0.$$

This theorem is equivalent with the following:

If $\varepsilon_{i,j} (i, j = 1, 2, \dots)$ are independent random variables with common distribution

$$\mathsf{P}(\varepsilon_{1,1}=1)=\mathsf{P}(\varepsilon_{1,1}=0)=\tfrac{1}{2},$$

then

$$\lim_{n=+\infty} \mathsf{P}\left(\begin{vmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \cdots & \varepsilon_{1,n} \\ \varepsilon_{2,1} & \varepsilon_{2,2} & \cdots & \varepsilon_{2,n} \\ & \cdots & & \\ \varepsilon_{n,1} & \varepsilon_{n,2} & \cdots & \varepsilon_{n,n} \end{vmatrix} = 0 \right) = 0.$$

b) In this paper we give the following generalization of the above theorem:

THEOREM 1. Let $\xi_{i,j}$ $(i, j = 1, 2, \dots)$ *be independent random variables with common non-degenerated distribution function* $F(x)$. *Then*

$$\lim_{n=+\infty} \mathsf{P}\left(\begin{vmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,n} \\ \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,n} \\ & \cdots & & \\ \xi_{n,1} & \xi_{n,2} & \cdots & \xi_{n,n} \end{vmatrix} = 0 \right) = 0.$$

c) In course of the proof of Theorem 1 we use the following theorem that is of independent interest:

THEOREM 2. *Let* $\xi_1, \xi_2, \dots, \xi_k, \dots$ *be independent random variables with common non-degenerated distribution.*

Let $\delta(a_{n,1}; a_{n,2}; \dots; a_{n,n})$ *denote the greatest jump of the distribution function of the random variable* $\sum_{k=1}^{n} a_{n,k}\xi_k$ *where* $a_{n,1}; a_{n,2}; \dots; a_{n,n}$ *are arbitrary real numbers different from* 0, *i.e.*

$$\delta(a_{n,1}; a_{n,2}; \dots; a_{n,n}) = \max_{-\infty < x < +\infty} \mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}\xi_k = x \right).$$

Let us put

$$c_n = \sup_{\{a_{n,k}\}_{k=1}^n} \delta(a_{n,1}; a_{n,2}; \ldots; a_{n,n})$$

where the suprema is extended over all possible n-tuples of non-zero real numbers $a_{n,k}$.
Then

$$\lim_{n=+\infty} c_n = 0.$$

This theorem has been proved jointly with I. CSISZÁR.

d) As an example we mention the following known corollary of Theorem 2:

COROLLARY. *Let* $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ *be independent random variables with common non-degenerated distribution and*

$$\mathsf{E}(\xi_1) = 0,$$

$$\mathsf{E}(\xi_1^2) < +\infty.$$

Let $c_1, c_2, \ldots, c_k, \ldots$ *be real numbers for those* $\sum_{k=1}^{n} c_k^2 < +\infty$ *holds, and let us put* $\xi = \sum_{k=1}^{\infty} c_k \xi_k$ *(*ξ *is defined with probability* 1).

Then ξ *has a continuous distribution.*

Putting in Theorem 2 $a_{n,k} = c_k$ $(n = 1, 2, \ldots;\ k = 1, 2, \ldots, n)$ we obtain the Corollary.

(On the other hand the distribution of the limit ξ needs not be absolutely continuous. E.g. if

$$\mathsf{P}(\xi_1 = 1) = \mathsf{P}(\xi_1 = -1) = \tfrac{1}{2}$$

and

$$c_k = \frac{1}{3^k}$$

then the range of ξ is the Cantor-set, i.e. the distribution is concentrated in a 0-set and so it is not absolutely continuous.)

e) In § 2 and § 3 we give the proofs of Theorem 2 and Theorem 1, respectively.

§ 2. Proof of Theorem 2

a) If the distribution function $F(x)$ of the variables is continuous then $c_k = 0$ $(n = 1, 2, \ldots)$.

If it has only one jump $p < 1$ then $c_n = p^n$ $(n = 1, 2, \ldots)$, and hence $\lim_{n=+\infty} c_n = 0$.

That is we can suppose that $F(x)$ has at least two jumps. Let $p_1, p_2, \ldots$ be the jumps of $F(x)$, i.e.

$$\mathsf{P}(\xi_1 = x_k) = p_k > 0 \qquad (k = 1, 2, \ldots).$$

Let us put

$$p = \min(p_1, p_2).$$

The distribution function of $\sum_{k=1}^{n} a_{n,k}\xi_k$ is the convolution of the distribution functions of $\sum_{k=1}^{n-1} a_{n,k}\xi_k$ and $a_{n,n}\xi_n$, further the maximal jump of a convolution is not greater than the maximal jump of any factor of the convolution, we have

$$c_n \geqq c_{n+1} \qquad (n=1, 2, \dots)$$

and so there exists

$$\lim_{n=+\infty} c_n = c.$$

We have to prove that $c=0$.

On the contrary assume that $c>0$, that will lead to a contradiction.

b) Putting

$$\varepsilon_0(n) = \frac{1}{n}$$

define

$$\varepsilon_{r+1}(n) = \frac{1}{p}\varepsilon_r(n+1) + \frac{1}{p}(c_n - c)$$

for $r=0, 1, 2, \dots$.

Clearly for any fix r

$$\varepsilon_r(n) > 0 \qquad (n=1, 2, \dots)$$

and

$$\lim_{n=+\infty} \varepsilon_r(n) = 0.$$

We prove by induction on r that for any r and n $(r=0, 1, 2, \dots;\ n=1, 2, \dots)$ there exist an n-tuple $\{a_{n,k}^{(r)}\}_{k=1}^{n}$ of non-zero real numbers and an $(r+1)$-tuple $\{y_{n,t}^{(r)}\}_{t=1}^{r+1}$ of different real numbers such that

$$\text{(0)} \qquad \mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r)}\xi_k = y_{n,t}^{(r)}\right) > c - \varepsilon_r(n) \qquad (t = 1, 2, \dots, r+1)$$

hold.

That is, we prove that for any r and n the variables $\xi_1, \xi_2, \dots, \xi_n$ have such a linear combination that has $(r+1)$ jumps greater than $c-\varepsilon_r(n)$.

But it presents a contradiction, since putting $r=\left[\frac{1}{c}\right]$ and such a great natural number m for that

$$\varepsilon_r(m) < c - \frac{1}{r+1}$$

we get

$$1 \geqq \sum_{t=1}^{r+1} \mathsf{P}\left(\sum_{k=1}^{m} a_{m,k}^{(r)}\xi_k = y_{m,t}^{(r)}\right) > \sum_{t=1}^{r+1} \left(c - \varepsilon_r(m)\right) > \sum_{t=1}^{r+1} \frac{1}{r+1} = 1,$$

that is absurd.

c) Now let us go to prove the above statement.

Because of the definition of c_n for all natural number n there exist $\{a_{n,k}^{(0)}\}_{k=1}^{n}$ non-zero numbers and $y_{n,1}^{(0)}$ for that

$$\mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(0)}\xi_k = y_{n,1}^{(0)}\right) > c_n - \frac{1}{n} \geqq c - \frac{1}{n} = c - \varepsilon_0(n)$$

and it is the relation (0) in the case $r=0$.

Assumed that the above statement has already been proved for some r (and all n) let us prove it for $r+1$ (and all n).

So we know that for any n there exist $\{a_{n+1,k}^{(r)}\}_{k=1}^{n+1}$ non-zero numbers and $\{y_{n+1,t}^{(r)}\}_{t=1}^{r+1}$ different numbers such that

$$\mathsf{P}\left(\sum_{k=1}^{n+1} a_{n+1,k}^{(r)}\xi_k = y_{n+1,t}^{(r)}\right) > c - \varepsilon_r(n+1) \qquad (t = 1, 2, \ldots, r+1).$$

Let us pus

$$a_{n,k}^{(r+1)} = \frac{a_{n+1,k}^{(r)}}{a_{n+1,n+1}^{(r)}} \qquad (k = 1, 2, \ldots, n)$$

and

$$\frac{y_{n+1,t}^{(r)}}{a_{n+1,n+1}^{(r)}} = w_t \qquad (t = 1, 2, \ldots, r+1).$$

So we have

$$\mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k + \xi_{n+1} = w_t\right) > c - \varepsilon_r(n+1)$$

$$(t = 1, 2, \ldots, r+1).$$

$\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k$ and ξ_{n+1} are independent, therefore

$$c - \varepsilon_r(n+1) < \mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k + \xi_{n+1} = w_t\right) =$$

$$= \sum_{\nu} \mathsf{P}(\xi_{n+1} = x_\nu)\mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k = w_t - x_\nu\right) =$$

$$= \sum_{\nu} p_\nu \mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k = w_t - x_\nu\right) \leqq p_\delta \mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k = w_t - x_\delta\right) + (1-p_\delta)c_n$$

for $t = 1, 2, \ldots, r+1;\quad \delta = 1, 2$.

Hence we get

$$p_\delta \mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k = w_t - x_\delta\right) > c - \varepsilon_r(n+1) - c_n + c_n p_\delta \geqq$$

$$\geqq (c - c_n) - \varepsilon_r(n+1) + c p_\delta \qquad (t = 1, 2, \ldots, r+1;\ \delta = 1, 2)$$

and divided by p_δ

$$\mathsf{P}\left(\sum_{k=1}^{n} a_{n,k}^{(r+1)}\xi_k = w_t - x_\delta\right) > c - \frac{1}{p_\delta}[(c_n - c) + \varepsilon_r(n+1)] \geqq$$

$$\geqq c - \frac{1}{p}[(c_n - c) + \varepsilon_r(n+1)] = c - \varepsilon_{r+1}(n) \qquad (t = 1, 2, \ldots, r+1;\ \delta = 1, 2).$$

As the numbers $w_1, w_2, \ldots, w_{r+1}$ are different and so are x_1 and x_2, among the numbers $w_t - x_\delta$ $(t=1, 2, \ldots, r+1; \delta=1, 2)$ at least $r+2$ are different ones, these are the numbers

$$\{y_{n,t}^{(r+1)}\}_{t=1}^{r+2} \qquad \text{Q, e. d.}$$

§ 3. Proof of Theorem 1

a) Let us consider an $m \times n$ matrix and one of its row vectors. We have two possibilities: either this row is a linear combination of the remaining ones or its is independent of them. In the second case we say that the row degree of this row is $+\infty$.

In the first case consider a basis of the remaining vectors. The row is a uniquely determined linear combination of this basis.

The number of non-zero coefficients at this linear combination is called the degree of the row vector with respect to this basis.

The row degree of the row is the leat one of its row degrees with respect to the all basises composable from the remaining vectors.

The definitions of the column degrees are analogous.

In what follows $m \times n$ matrix means a random $m \times n$ matrix, i.e. the matrix

$$\begin{pmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,n} \\ \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,n} \\ & & \cdots & \\ \xi_{m,1} & \xi_{m,2} & \cdots & \xi_{m,n} \end{pmatrix}$$

where $\xi_{i,j}$ $(i, j=1, 2, \ldots)$ are independent random variables with common non-degenerated distribution function $F(x)$, q denotes the greater jump of $F(x)$ and $\varepsilon(n) = c_n$ where c_n is defined in Theorem 2.

Instead of the above matrix we often write simply $M(m, n)$, and $R(m, n)$ denotes the rank of $M(m, n)$.

b) LEMMA 1. *Let us consider an $m \times n$ matrix $(n < m^2)$. The probability of the event $B_1(m, n)$ that there is a column of $M(m, n)$ the column degree of that is at most $\sqrt{m}$ — is less than $c_1 q^{m/3}$ for some suitable absolute constant c_1.*

PROOF. Let $A(i_1, i_2, \ldots, i_t)$ (for $i_1 < i_2 < \ldots < i_t$) denote the event that the i_1-th, i_2-th, ..., i_t-th columns of the matrix are independent.

$$\text{Then } \mathsf{P}(B_1(m,n)) \leqq n \sum_{\substack{i_1<i_2<\ldots<i_t\\ t\leqq\sqrt{m}}} \mathsf{P}(A(i_1, i_2, \ldots, i_t))[1-\mathsf{P}(A(i_1, i_2, \ldots, i_t, n))] =$$

$$= n \sum_{t\leqq\sqrt{m}} \binom{n-1}{t} \mathsf{P}(A(1, 2, \ldots, t))[1-\mathsf{P}(A(1, 2, \ldots, t, n))].$$

If the n-th column is a linear combination of the linear independent 1-th, 2-th, ..., t-th columns $(t \leqq \sqrt{m})$ then the rank of the matrix

$$(*) \qquad \begin{pmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,t} & \xi_{1,n} \\ \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,t} & \xi_{2,n} \\ & & \cdots & & \\ \xi_{m,1} & \xi_{m,2} & \cdots & \xi_{m,t} & \xi_{m,n} \end{pmatrix}$$

is equal to t.

Let the j_1-th, j_2-th, ..., j_t-th rows of the matrix $M(m, t)$ be the linear independent ones (i.e. a basis of the rows of $M(m, t)$).

Let $x_{1,j}; x_{2,j}; \ldots; x_{t,j}$ be the coefficients of the j-th row of $M(m, t)$ with respect to the above basis of the rows.

(The numbers j_k and $x_{k,r}$ are random variables.)

So it holds

$$\xi_{j,s} = \sum_{\nu=1}^{t} x_{\nu,j}\,\xi_{j_\nu,s} \qquad \begin{pmatrix} j = 1, 2, \ldots, m \\ s = 1, 2, \ldots, t \end{pmatrix}$$

Clearly the j_1-th, j_2-th, ..., j_t-th row vectors of the matrix $(*)$ are also linearly independent and as the rank of this matrix is t, they compose a basis of this matrix.

It is clear as well that the coefficients of the j-th row of this matrix with respect to this basis are the numbers

$$x_{1,j};\ x_{2,j};\ \ldots;\ x_{t,j}.$$

Hence

$$\xi_{j,n} = \sum_{\nu=1}^{t} x_{\nu,j}\,\xi_{j_\nu,n} \qquad (j = 1, 2, \ldots, m).$$

So we get

$$\mathsf{P}(A(1, 2, \ldots, t))[1-\mathsf{P}(A(1, 2, \ldots, t, n))] \leqq$$

$$\leqq \mathsf{P}\left(\xi_{j,n} = \sum_{\nu=1}^{t} x_{\nu,j}\,\xi_{j_\nu,n} \quad \text{for} \quad j = 1, 2, \ldots, m\right) \leqq$$

$$\leqq \mathsf{P}\left(\xi_{j,n} = \sum_{\nu=1}^{t} x_{\nu,j}\,\xi_{j_\nu,n} \quad \text{for} \quad j = 1, 2, \ldots, m \quad \text{but} \quad j \neq j_\mu \quad (\mu = 1, 2, \ldots, t)\right).$$

But the variables $\xi_{j,n}$ are independent of the variables $\sum_{\nu=t}^{t} x_{\nu,j}\xi_{j_\nu,n}$ for $j\neq j_\mu$ that implies that[1]

$$\mathsf{P}\left(\xi_{j,n}=\sum_{\nu=1}^{t} x_{\nu,j}\,\xi_{j_\nu,n} \quad \text{for} \quad j\neq j_\mu\right)=\prod_{j\neq j_\mu}\mathsf{P}\left(\xi_{j,n}=\sum_{\nu=1}^{t} x_{\nu,j}\,\xi_{j_\nu,n}\right)=$$

$$=\prod_{j\neq j_\mu}\mathsf{E}\left(\mathsf{P}\left(\xi_{j,n}=\sum_{\nu=1}^{t} x_{\nu,j}\,\xi_{j_\nu,n}\,\middle|\,M(m,t);\xi_{j_1,n};\xi_{j_2,n};\dots;\xi_{j_t,n}\right)\right)\leqq$$

$$\leqq \prod_{j\neq j_\mu}\mathsf{E}(q)=\prod_{j\neq j_\mu} q = q^{m-t}\leqq q^{m-\sqrt{m}}<q^{m/2}.$$

So we get

$$\mathsf{P}\big(B_1(m,n)\big)\leqq n\sum_{t\leqq\sqrt{m}}\binom{n-1}{t}q^{m/2}<n\cdot\sqrt{m}\,n^{\sqrt{m}}q^{m/2}<m^{2\sqrt{m}+2,5}\,q^{m/2}<c_1\,q^{m/3}$$

for some suitable $c_1>0$. Q.e.d.

In a similar way we obtain that if $m<n^2$ then the probability of the event $B_2(m,n)$ that there is a row of $M(m,n)$ the row degree of that is at most $\sqrt{n}$—is less than $c_1q^{n/3}$.

From now on we suppose that $m<(n-1)^2$ and $n<(m-1)^2$.

c) Similarly as at the proof of Lemma 1, we estimate the probability of the event $C_1(m,n)$ that the last column of a random $m\times n$ matrix is a linear combination of the other ones.

So $C_1(m,n)$ means that $R(m,n)=R(m,n-1)$.

If the rank of the matrix $M(m,n-1)$ is m, then

$$\mathsf{P}\big(C_1(m,n)\big)=1.$$

We prove that

$$\lim_{n=+\infty}\ \max_{\sqrt{n}+1<m<(n-1)^2}\ \mathsf{P}\big(C_1(m,n),\,R(m,n-1)<m\big)=0^2$$

that is

$$\lim_{n=+\infty}\ \max_{\sqrt{n}+1<m<(n-1)^2}\ \mathsf{P}\big(R(m,n)=R(m,n-1),\,R(m,n-1)<m\big)=0.$$

$$\mathsf{P}\big(C_1(m,n),\,R(m,n-1)<m\big)=\mathsf{P}\big(C_1(m,n),\,R(m,n-1)<m,\,B_2(m,n-1)\big)+$$

$$+\mathsf{P}\big(C_1(m,n),\,R(m,n-1)<m,\,\overline{B_2(m,n-1)}\big)<c_1q^{\frac{n-1}{3}}+$$

$$+\mathsf{P}\big(C_1(m,n),\,R(m,n-1)<m,\,\overline{B_2(m,n-1)}\big).$$

The event $C=C_1(m,n)\cap\{R(m,n-1)<m\}\cap\overline{B_2(m,n-1)}$, means that the rows of $M(m,n-1)$ are not linearly independent, the last column is a linear combination of the other ones and the row degree of each row of $M(m,n-1)$ is at least $\sqrt{n}$.

[1] $\mathsf{E}(\xi|\eta_1,\dots,\eta_k)$ denotes the conditional expectation of ξ with respect to the smallest σ-algebra with respect to that $\eta_1\dots,\eta_k$ are measurable. If A is an event, $\mathsf{P}(A|\eta_1\dots,\eta_k)=$ $=\mathsf{E}(\alpha|\eta_1,\dots,\eta_k)$, where α is the indicator function of A.

[2] $\mathsf{P}(A,B,C,\dots)$ denotes $\mathsf{P}(A\cap B\cap C\cap\dots)$.

Let the j_1-th, j_2-th, ..., j_t-th $(t<m)$ rows of the matrix $M(m, n-1)$ compose a basis of the rows of $M(m, n-1)$ and $x_{1,j}; x_{2,j}; \dots; x_{t,j}$ the coefficients of the j-th row of $M(m, n-1)$ with respect to this basis.

I.e.

$$\xi_{j,s} = \sum_{\nu=1}^{t} x_{\nu,j}\, \xi_{j_\nu,s} \qquad \begin{pmatrix} j = 1, 2, \dots, m \\ s = 1, 2, \dots, n-1 \end{pmatrix}.$$

Clearly, the j_1-th, j_2-th, ..., j_t-th rows of $M(m, n)$ are also linearly independent and because of $R(m, n) = R(m, n-1)$, they compose a basis of the rows of $M(m, n)$.

So

$$\xi_{j,n} = \sum_{\nu=1}^{t} x_{\nu,j}\, \xi_{j_\nu,n} \qquad (j=1, 2, \dots, m).$$

Let $1 \leqq j_0 \leqq m$ be a natural number different from $j_1, j_2, \dots, j_t$.

(As well as $j_1, j_2, \dots, j_t$ — j_0 is a random variable.)

$$\xi_{j_0,n} = \sum_{\nu=1}^{t} x_{\nu,j_0}\, \xi_{j_\nu,n},$$

i.e.

$$\sum_{\nu=0}^{t} x_{\nu,j_0}\, \xi_{j_\nu,n} = 0 \quad \text{where} \quad x_{0,j_0} = -1.$$

As the variables j_ν and $x_{\nu,j}$ are determined by $M(m, n-1)$,

(1) $$\mathsf{P}(C|M(m, n-1)) \leqq \mathsf{P}\left(\sum_{\nu=0}^{t} x_{\nu,j_0}\, \xi_{j_\nu,n} = 0,\ \overline{B_2(m, n-1)} \middle| M(m, n-1) \right) \leqq$$

$$\leqq \mathsf{P}\left(\sum_{\nu=0}^{t} x_{\nu,j_0}\, \xi_{j_\nu,n} = 0,\ \begin{Bmatrix} \text{the number of non zero } x_{\nu,j_0}\text{'s is} \\ \text{at least } \sqrt{n} \end{Bmatrix} \middle| M(m, n-1) \right) \leqq$$

$$\leqq \varepsilon([\sqrt{n}]) \text{ for all } \omega \in \Omega.$$

Hence

$$\mathsf{P}(C) = \mathsf{E}(\mathsf{P}(C|M(m, n-1))) \leqq \mathsf{E}(\varepsilon([\sqrt{n}])) = \varepsilon([\sqrt{n}]).$$

So we have

$$\mathsf{P}(C_1(m, n),\ R(m, n-1) < m) = \mathsf{P}(R(m, n) = R(m, n-1),\ R(m, n-1) < m) <$$

$$< c_1 q^{\frac{n-1}{3}} + \varepsilon([\sqrt{n}]) = \varphi(n) \xrightarrow[n]{} 0$$

(for all m satisfying $m < (n-1)^2$, $n < (m-1)^2$).

Similarly, if $C_2(m, n)$ denotes the event that the last row of a random $m \times n$ matrix is a linear combination of the other ones, then

$$\mathsf{P}(C_2(m, n),\ R(m-1, n) < n) = \mathsf{P}(R(m, n) = R(m-1, n),\ R(m-1), n) < n) <$$

$$< \varphi(m) \xrightarrow[m]{} 0,$$

and if D denotes the event

$$D = C_2(m, n) \cap \{R(m-1, n) < n\} \cap \overline{B_1(m-1, n)}$$

then

$$\mathsf{P}(D|M(m-1,n)) \leqq \varepsilon([\sqrt{m}]) \quad \text{for all} \quad \omega \in \Omega. \tag{2}$$

d) Let us consider an $(n+1)\times(n+1)$ matrix.

Since $R(n,n+1)<n+1$, by (2) we get

$$\mathsf{P}(R(n+1,n+1) = R(n,n+1),\ \overline{B_1(n,n+1)}|M(n,n+1)) =$$

$$= \mathsf{P}(R(n+1,n+1) = R(n,n+1),\ R(n,n+1) < n+1,\ \overline{B_1(n,n+1)}|M(n,n+1)) =$$

$$= \mathsf{P}(D|M(n,n+1)) \leqq \varepsilon([\sqrt{n+1}]) < \varphi(n) \tag{3}$$

in almost all $\omega\in\Omega$, and

$$\mathsf{P}(R(n+1,n+1) = R(n,n+1)) = \mathsf{E}(\mathsf{P}(D|M(n,n+1)))+$$

$$\mathsf{E}(\mathsf{P}(R(n+1,n+1) = R(n,n+1),\ B_1(n,n+1)|M(n,n+1))) \leqq \tag{α}$$

$$\leqq \varepsilon([\sqrt{n+1}])+c_1 q^{n/3} < \varphi(n).$$

$$\mathsf{P}(R(n+1,n+1) = R(n,n) = n) \leqq \mathsf{P}(R(n+1,n+1) = R(n,n+1)) < \varphi(n). \tag{β}$$

Similarly we get

$$\begin{aligned} &\mathsf{P}(R(n,n+1) = R(n,n),\ R(n,n) < n,\ \overline{B_2(n,n)}|M(n,n)) = \\ &= \mathsf{P}(C|M(n,n)) \leqq \varepsilon([\sqrt{n+1}]) < \varphi(n) \quad \text{in all} \quad \omega\in\Omega, \end{aligned} \tag{4}$$

and

$$\mathsf{P}(R(n,n+1) = R(n,n),\ R(n,n)<n)<\varphi(n). \tag{γ}$$

Applying (3) and (4) we have (using the fact that $R(n+1,n+1)<R(n,n)+2$ can occur only if $R(n+1,n+1) = R(n,n+1)$ or $R(n,n+1) = R(n,n)$ and $\mathsf{P}(A|M(n,n+1))<c$ implies that $\mathsf{P}(A|M(n,n))<c$)

$$\mathsf{P}(R(n+1,n+1) < R(n,n)+2,\ R(n,n) < n,\ \overline{B_2(n,n)},\ \overline{B_1(n,n+1)}|M(n,n)) < 2\varphi(n) \tag{5}$$

on all $\omega\in\Omega$ and

$$\begin{aligned} &\mathsf{P}(R(n+1,n+1) = R(n,n) = n,\ \overline{B_2(n,n)},\ \overline{B_1(n,n+1)}|M(n,n)) \leqq \\ &\leqq \mathsf{P}(R(n+1,n+1) = R(n,n+1),\ \overline{B_1(n,n+1)}|M(n,n)) \leqq \varphi(n) < 2\varphi(n) \end{aligned} \tag{5'}$$

in almost all $\omega\in\Omega$.

By α) and γ) we obtain the important relation

$$\mathsf{P}(R(n+1,n+1)<R(n,n)+2,\ R(n,n)<n)<2\varphi(n). \tag{δ}$$

(This last relation shows why is the rank of a matrix equal to its order with probability near to 1 for great order. Viz. according to δ) if we increase the order by one, the rank grows by two with probability tending to 1. So the rank will "overtake" the order.)

e) Let us put $\mathsf{P}(R(n,n)=n) = \mathsf{P}(\mathrm{Det}\, M(n,n)\neq 0) = P_n$. (Our aim is to prove that $\lim\limits_{n=+\infty} P_n=1$.)

Putting $S_n = \mathsf{E}(R(n,n))$ and using relations β) and δ) we compute S_{n+1} as follows:

$$S_{n+1} = \int_{\Omega} R(n+1,n+1)d\mathsf{P} = \int_{\{R(n,n)<n\}} R(n+1,n+1)d\mathsf{P} +$$

$$+ \int_{\{R(n,n)=n\}} R(n+1,n+1)d\mathsf{P} = \int_{\{R(n,n)<n,\ R(n+1,n+1)<R(n,n)+2\}} R(n+1,n+1)d\mathsf{P} +$$

$$+ \int_{\{R(n,n)<n,\ R(n+1,n+1)=R(n,n)+2\}} R(n+1,n+1)d\mathsf{P} +$$

$$+ \int_{\{R(n,n)=R(n+1,n+1)=n\}} R(n+1,n+1)d\mathsf{P} +$$

$$+ \int_{\{R(n,n)=n,\ R(n+1,n+1)=R(n,n)+1} R(n+1,n+1)\,d\mathsf{P} \geqq$$

$$\geqq \int_{\{R(n,n)<n,\ R(n+1,n+1)<R(n,n)+2\}} R(n,n)d\mathsf{P} +$$

$$+ \int_{\{R(n,n)<n,\ R(n+1,n+1)=R(n,n)+2\}} (R(n,n)+2)\,d\mathsf{P} + \int_{\{R(n,n)=R(n+1,n+1)=n\}} R(n,n)d\mathsf{P} +$$

$$+ \int_{\{R(n,n)=n,\ R(n+1,n+1)=R(n,n)+1\}} (R(n,n)+1)d\mathsf{P} = \mathsf{E}(R(n,n)) +$$

$$+2\mathsf{P}(R(n,n) < n,\ R(n+1,n+1) = R(n,n)+2) +$$

$$+\mathsf{P}(R(n,n) = n,\ R(n+1,n+1) = R(n,n)+1) = \mathsf{E}(R(n,n)) + 2\mathsf{P}(R(n,n) < n) -$$

$$-2\mathsf{P}(R(n,n) < n,\ R(n+1,n+1) < R(n,n)+2) + \mathsf{P}(R(n,n) = n) -$$

$$-\mathsf{P}(R(n,n) = n,\ R(n+1,n+1) = R(n,n)) > \mathsf{E}(R(n,n)) + 2\mathsf{P}(R(n,n) < n) -$$

$$-4\varphi(n) + \mathsf{P}(R(n,n) = n) - \varphi(n) = S_n + 2 - P_n - 5\varphi(n).$$

Hence

$$S_{n+1} - S_n > 2 - P_n - 5\varphi(n).$$

If for $n > n_0$

$$P_n \leqq 1 - 5\varphi(n) - \frac{1}{n}$$

held, then we should get

$$S_{N+1} > S_{N+1} - S_{n_0} = \sum_{k=n_0}^{N} (S_{k+1} - S_k) > \sum_{k=n_0}^{N} (2 - P_k - 5\varphi(k)) \geqq$$

$$\geqq \sum_{k=n_0}^{N} \left(2 - \left(1 - 5\varphi(k) - \frac{1}{k}\right) - 5\varphi(k)\right) = \sum_{k=n_0}^{N} \left(1 + \frac{1}{k}\right) >$$

$$> N - n_0 + \sum_{k=n_0}^{N} \frac{1}{k} \quad \text{for all} \quad N > n_0.$$

But it N_0 is sufficiently large, then

$$\sum_{k=n_0}^{N_0} \frac{1}{k} > n_0+1$$

and so we should get

$$S_{N_0+1} > N_0+1$$

and it is a contradiction.

We obtained

LEMMA 2. *There exists a sequence* $n_1, n_2, \ldots, n_k, \ldots$ *of natural numbers such that*

$$P_{n_k} > 1 - 5\varphi(n_k) - \frac{1}{n_k}$$

i.e.

$$\limsup_{n=+\infty} P_n = 1.$$

f) Now we are going to prove that

$$\lim_{n=+\infty} P_n = 1$$

that is the statement of Theorem 1.

In order to prove this relation we are making use of the following Lemma that is proved in [2] as Lemma 7:

LEMMA 7. *Let* $f(x, y)$ *be a function defined for all pair* (x, y) *if* $x \geqq y$) *of non-negative integers with the following properties:*

There exists a natural number n and a real number $0 < c < 1$ *such that*

1° $$f(x, y) \geqq 0$$

2° $$f(x, x) = 1$$

3° $$f(x, y+1) \geqq f(x, y)$$

4° $$f(n, n-1) < c$$

5° $$f(m+1, k) \leqq cf(m, k) + (1-c)f(m, k-2) + d_m$$

for all $m \geqq n$ and $0 \leqq k \leqq m$, where $\{d_m\}$ is a sequence of positive numbers.

Then

$$f(m, m-1) < 2c + \sum_{s=n}^{\infty} d_s$$

for all $m \geqq n$.

Now let $f(x, y)$ define as

$$f(x, y) = \mathsf{P}(R(x, x) \leqq y).$$

Clearly 1°, 2°, 3° hold.

Using relation (5) we estimate $f(m+1,k)$ for $k \leqq m$. For $k<m$

$$f(m+1,k) = \mathsf{P}(R(m+1,m+1) \leqq k) = \mathsf{P}(R(m+1,m+1) \leqq k,\ R(m,m) \leqq k-2)+$$
$$+\mathsf{P}(R(m+1,m+1) \leqq k,\ k-2 < R(m,m) \leqq k) \leqq f(m,k-2)+$$
$$+\mathsf{P}(R(m+1,m+1) \leqq k,\ k-2 < R(m,m) \leqq k) \leqq f(m,k-2)+\mathsf{P}(B_1(m,m+1))+$$
$$+\mathsf{P}(B_2(m,m))+$$
$$+\mathsf{P}(R(m+1,m+1) \leqq k,\ k-2 < R(m,m) \leqq k,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}) \leqq$$
$$\leqq f(m,k-2)+2c_1 q^{m/3} + \int\limits_{\{k-2<R=(m,m)\leqq k\}} \mathsf{P}(R(m+1,m+1) < R(m,m)+2,\ R(m,m) <$$
$$< m,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}|M(m,m))\,d\mathsf{P} \leqq f(m,k-2)+2c_1 q^{m/3}+$$
$$+ \int\limits_{\{k-2<R(m,m)\leqq k\}} 2\varphi(m)\,d\mathsf{P} = f(m,k-2)+2\varphi(m)(f(m,k)-f(m,k-2))+2c_1 q^{m/3} \leqq$$
$$\leqq f(m,k-2)+c(f(m,k)-f(m,k-2))+2c_1 q^{m/3} = cf(m,k)+$$
$$+(1-c)f(m,k-2)+2c_1 q^{m/3}$$

where c is an arbitrary real number for that $c \geqq 2\varphi(m)$ holds.

For $k=m$ we get the same estimate, because the only difference is that instead of the above term

$$\mathsf{P}(R(m+1,m+1) \leqq k,\ k-2 < R(m,m) \leqq k,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}) =$$
$$= \int\limits_{\{k-2<R(m,m)\leqq k\}} \mathsf{P}(R(m+1,m+1) < R(m,m)+2,\ R(m,m) <$$
$$< m,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}|M(m,m))\,d\mathsf{P} \leqq 2\varphi(m)\mathsf{P}(k-2 < R(m,m) \leqq k)$$

we have now (using (5′))

$$\mathsf{P}(R(m+1,m+1) \leqq m,\ m-2 < R(m,m) \leqq m,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}) =$$
$$= \mathsf{P}(R(m+1,m+1) \leqq m,\ R(m,m) = m-1,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)})+$$
$$+\mathsf{P}(R(m+1,m+1) \leqq m,\ R(m,m) = m,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}) \leqq$$
$$\leqq \int\limits_{\{R(m,m)-m-1\}} \mathsf{P}(R(m+1,m+1) < R(m,m)+2,\ R(m,m) <$$
$$< m,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}|M(m,m))\,d\mathsf{P} + \int\limits_{\{R(m,m)=m\}} \mathsf{P}(R(m+1,m+1) =$$
$$= R(m,m) = m,\ \overline{B_1(m,m+1)},\ \overline{B_2(m,m)}|M(m,m))\,d\mathsf{P} \leqq$$
$$\leqq 2\varphi(m)[\mathsf{P}(R(m,m) = m-1)+\mathsf{P}(R(m,m) = m)] =$$
$$= 2\varphi(m)\mathsf{P}(m-2 < R(m,m) \leqq m).$$

g) Let ε be an arbitrary positive number. Let the integer N be so large that for the N-th element of the sequence n_k (defined in Lemma 2) holds

$$5\varphi(n_N)+\frac{1}{n_N}<\frac{\varepsilon}{3}$$

$\left(\text{that implies } P_{n_N}>1-\frac{\varepsilon}{3}\right)$ and

$$\sum_{s=n_N}^{\infty} c_1 q^{s/3}<\frac{\varepsilon}{3}.$$

Putting in Lemma 7 $n=n_N$, $c=\frac{\varepsilon}{3}$ and $d_m=c_1q^{m/3}$, the conditions 4° and 5° hold for $m \geqq n_N$ (because $f(n, n-1) = 1-P_n$) and so by Lemma 7

$$f(m, m-1) = 1-P_m < 2\cdot\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

for $m \geqq n_N$. Q.e.d.

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SOLID CIRCLE-PACKINGS AND CIRCLE-COVERINGS

by

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To Professor H. Hadwiger on his sixtieth birthday

§ 1. Introduction

In a certain sens, which must be specified separately in the spherical, Euclidean and hyperbolic geometry, the face-incircles of the regular trihedral tessellations form a densest packing, and the face-circumcircles of these tessellations form a thinnest covering. It may be conjectured that these statements continue to hold for the trihedral Archimedean tessallations. The aim of this paper is to support this conjecture by proving its validity in several cases.

In § 2 we introduce the notion of the solidity of a packing and covering, which admits a unique formulation of the results. § 3 contains simple proofs of the solidity of the circle-packings and circle-coverings generated from the regular trihedral tessellations. This paragraph is included for the sake of completenes. The results itselves are not new [1]. In § 4 we deal with some Archimedean packings and coverings.

§ 2. Definitions

We shall operate in a two-dimensional shperical, Euclidean or hyperbolic space, briefly in a *plane*. By a *circle* we mean an open or a closed circular disc, according as we consider a packing or a covering. A *circle-packing* is defined as a set of circles such that each point of the plane belongs at most to one circle. Analogously, a set of circles is said to form a *circle-covering*, if each point of the plane belongs at least to one circle.

A *circle-packing* is said to be *solid* if no finite subset of the circles can be re-arranged so as to obtain a packing not congruent to the original one. In a quite similar way, a *circle-covering* is said to be *solid* if no finite subset of the circles can be rearranged so as to obtain a covering not congruent to the original one.

On the sphere the density of a set of circles is defined as the quotient of the total area of the circles and the surface-area of the sphere. Thus here a solid packing of n congruent circles is a densest packing of n congruent circles, but not vice versa. For example, in a densest packing of 5 congruent circles two circles are centered at the poles and the rest are centered in three points of the equator under the single condition that their mutual distane is $\geqq \pi/2$. Therefore these packings are not solid. Similarly, on the sphere a solid covering of n congruent circles is always a thinnest covering of n congruent circles, but not vice versa. For the sphere cannot be covered by 3 circles less than a hemisphere. But a covering of 3 hemispheres is not solid. Besides the cases when $n=5$ and $n=3$, respectively, no further counter-examples are known.

In the Euclidean plane the density is defined by a limiting value which may be interpreted as the quotient of the total area of the circles and the area of the plane. Here too, a solid packing of congruent circles is a densest packing of congruent circles, and a solid covering of congruent circles is a thinnest covering of congruent circles. For otherwise the circles lying is a sufficiently big circle could be rearranged so as to be packed in a smaller circle, or so as to cover a bigger circle, respectively, which contradicts the solidity. But a densest packing or a thinnest covering is not necessarily solid, because the density remains invariant by removing from a packing or by adding to a covering a finite number of circles.

In the hyperbolic geometry the definition of a satisfactory density notion involves considerable difficulties, as pointed out by K. BÖRÖCZKY (see [2]). In some cases these difficulties can be evaded by using a certain kind of standard triangulation. If, for example, in a packing of congruent circles the circle-density has in each triangle the same constant value c, and it can be proved that in any other packing of these circles the circle-density is in all triangles $\leqq c$, then we can say that the first packing constitutes a densest packing. But if we have, say, two kinds of circles, we must face the further problem of defining their frequency. Thus in the hyperbolic plane the simple notion of the solidity has particular advantages.

A *tessellation* is a set of polygons joining along whole sides so as to fill the plane without intersteces and without overlapping. A tessellation with regular faces and equivalent vertices is said to be *uniform.* If all faces of a uniform tessallation are congruent, the tessallation is *regular.* In the opposite case the tessallation is called *semi-regular* or *Archimedean.*

A regular tessallation with p-gonal faces, q at each vertex, is denoted by the symbol $\{p, q\}$. For any integers $p \geqq 2$ and $q \geqq 2$ there is a $\{p, q\}$ which is spherical, Euclidean or hyperbolic according as

$$\frac{1}{p}+\frac{1}{q} \gtreqless \frac{1}{2},$$

respectively. We shall be concerned also with the truncated $\{p, q\}$, symbolically $t\{p, q\}$, which exists for any $p \geqq 2$ and $q \geqq 3$. It is a uniform tessellation having two $2p$-gonal faces and one q-gonal face at each vertex. Observe that $t\{p, 2p\} = \{2p, 3\}$.

Now we can specify the conjectures mentioned in § 1 as follows.

CONJECTURE p. The face-incircles of a trihedral Archimedean tessellation always form a solid packing.

CONJECTURE c. The face-circumcircles of a trihedral Archimedean tessellation always form a solid covering.

We complete these conjectures by the further conjectures that the face-incircles of a more than trihedral uniform tessellation never form a solid packing and the face-circumcircles of such a tessellation never form a solid covering.

In what follows we denote a domain and its area by the same symbol.

§ 3. Regular Packings and Coverings

We shall prove the following theorems.

THEOREM *rp*. *For any integer $p \geqq 2$ the face-incircles of $\{p, 3\}$ form a solid packing.*

THEOREM *rc*. *For any integer $p > 2$ the face-circumcircles of $\{p, 3\}$ form a solid covering.*

The proofs rest on a series of lemmas most of which are well known. In order to make this paper as selfcontained as possible we shall give simple proofs of these lemmas.

LEMMA 1. Of all p-gons ($p \geqq 2$) containing a circle of radius r the regular p-gon of inradius r has the least area.

Obviously, we may suppose that the p-gon, U, is circumscribed about the circle. Let $V_1, \ldots, V_p$ be the vertices of U in this cyclic order. Let T_i be the point of tangency of the side $V_i V_{i+1}$ and the circle ($i=1, \ldots, n$; $V_{p+1} = V_1$). Let O be the center of the circle and a_i the area of the triangle $V_i O V_{i+1}$. Then $a_i = a(\alpha_i) + a(\alpha_{i+1})$, where $\alpha_i = \sphericalangle V_i O T_i$, $\alpha_{i+1} = T_i O V_{i+1}$ and the fonction $a(x)$ is defined by

$$a(x) = \begin{cases} x - \arcsin(\cos r \sin x), & 0 < x < \pi \\ \dfrac{r^2}{2} \tan x, & 0 < x < \pi/2 \\ \arcsin(\cosh r \sin x) - x, & 0 < x < \arcsin 1/\cosh r \end{cases}$$

according to the three types of geometries.

We claim that $a(x)$ is a convex function. In the Euclidean case this is obvious, and in the non-Euclidean cases it follows from

$$a''(x) = \frac{\sin x \cos r \sin^2 r}{(1 - \cos^2 r \sin^2 x)^{3/2}} > 0$$

and

$$a''(x) = \frac{\sin x \cosh r \sinh^2 r}{(1 - \cosh^2 r \sin^2 x)^{3/2}} > 0,$$

respectively. Therefore

$$U = a_1 + \ldots + a_p = 2a(\alpha_1) + \ldots + 2a(\alpha_p) \geqq 2n\, a(\pi/p),$$

with equality if and only if $\alpha_1 = \ldots = \alpha_p = \pi/p$.

LEMMA 2. Of all p-gons ($p \geqq 3$) contained in a circle of radius r the regular p-gon of circumradius r has the greatest area.

Retaining the above notations, we obviously may suppose that U is inscribed in the circle in such a way that it contains O. Now we have $a_i = 2b(\beta_i)$, where $0 < 2\beta_i = \sphericalangle V_i O V_{i+1}$ and the function $b(x)$ is defined by

$$b(x) = \begin{cases} x - \arctan(\cos r \tan x), & 0 < x \leqq \pi/2 \\ \dfrac{r^2}{4} \sin 2x, & 0 < x \leqq \pi/2 \\ \arctan(\cosh r \tan x) - x, & 0 < x \leqq \pi/2. \end{cases}$$

We claim that $b(x)$ is a concave function. In the Euclidean case this is obvious and in the non-Euclidean cases it follows from

$$b''(x) = -\frac{\sin 2x \cos r \sin^2 r}{(1-\sin^2 r \sin^2 x)^2} < 0$$

and

$$b''(x) = -\frac{\sin 2x \cos r \sinh^2 r}{(1+\sinh^2 r \sin^2 x)^2} < 0,$$

respectively. Thus

$$U = a_1 + \ldots + a_p = 2b(\beta_1) + \ldots + 2b(\beta_p) \leqq 2n\, b(\pi/p).$$

Equality holds only if $\beta_1 = \ldots = \beta_p = \pi/p$.

Lemma 3. If $f(x)$ is a continuous convex function defined for $x > 0$, then so is the function $g(x) = x f(1/x)$.

For, we have

$$\frac{g(u)+g(v)}{2} = \frac{u+v}{2}\,\frac{uf(1/u)+vf(1/v)}{u+v} \geqq \frac{u+v}{2} f\left(\frac{2}{u+v}\right) = g\left(\frac{u+v}{2}\right)$$

for any $u > 0$ and $v > 0$, as stated.

Let $A(p, r)$ and $B(p, r)$ be the area of a regular p-gon circumscribed about and inscribed in a circle of radius r, respectively. We extend the definition of these functions for non-integral values of p by linear interpolation. Observe that in the hyperbolic goemetry $A(p, r)$ is defined only for $p \geqq p_0$, where p_0 is the least positive integer such that $\cosh r \sin \pi/p_0 < 1$. Otherwise $A(p, r)$ and $B(p, r)$ are defined for any $p \geqq 3$, except the spherical case, where $A(p, r)$ is defined for $p \geqq 2$.

The convexity of $a(x)$ and the concavity of $b(x)$ along with Lemma 3 imply the convexity of $x\, a(\pi/x)$ and the concavity of $x\, b(\pi/x)$. Since for whole values of x we have $A(x, r) = 2x\, a(\pi/x)$ and $B(x, r) = 2x\, b(\pi/x)$, we have proved the following lemmas.

Lemma 4. $A(x, r)$ is a convex function of x.

Lemma 5. $B(x, r)$ is a concave function of x.

We still shall need

Lemma 6. Let $p_1, \ldots, p_n$ be the number of sides of n non-overlapping convex polygons joining along whole sides and having a connected pointset union U. Then

$$p_1 + \ldots + p_n \leqq 6(n+h-2) + f - b,$$

where h is the number of the connected components of which the complementary of U consists and f and b are the numbers of vertices lying on the boundary of U at which only two and more than two edges meet, respectively. Equality holds only if at each vertex at most three edges meet.

Let v_i be the number of vertices at which i edges meet. Then

$$v_2 + v_3 + \ldots = v$$

and

$$2v_2 + 3v_3 + \ldots = 2e,$$

where v and e are the total number of vertices and edges, respectively. On the other hand, we have

$$p_1+\ldots+p_n \leqq 2e-f-b.$$

Combining these equalities with Euler's formula,

$$n+h+v = e+2,$$

and observing that $v_2=f$, we obtain

$$p_1+\ldots+p_n \leqq 6(n+h-2)+f-b-2(v_4+2v_5+\ldots) \leqq 6(n+h-2)+f-b,$$

as stated.

Now we are ready to prove the above theorems.

Let $d_1, d_2, \ldots$ be the faces of $\{p, 3\}$, S the set of the incircles $c_1, c_2, \ldots$ of $d_1, d_2, \ldots$ and r the radius of these circles. Let $\{c_1, \ldots, c_m\}$ be a subset of S and let $c_1', \ldots, c_m'$ be m circles of radius r which together with $c_{m+1}, c_{m+2}, \ldots$ form a packing S'. Observe that for $p<6$ the set $\{c_{m+1}, c_{m+2}, \ldots\}$ may be empty. We must show that S' and S are congruent.

In a set of congruent circles we define the *Dirichlet cell* of a circle as the set of points whose distance from the center of the respective circle is less than from the center of any other circle. Let $d_1', d_2', \ldots$ be the Dirichlet cells of the circles of S'. For a suitable indexing of the circles, we can find an integer n not less than m such that the pointset union of $d_1', \ldots, d_n'$ is connected and that d_i' and d_i are identical for any $i>n$. Applying Lemma 6, on the one hand, to the polygons $d_1, \ldots, d_n$, on the other hand, to the polygons $d_1', \ldots, d_n'$, we find that

$$np=6(n+h-2)+f-b$$

and

$$p_1+\ldots+p_n \leqq 6(n+h-2)+f-b,$$

whence

$$p_1+\ldots+p_n \leqq np.$$

Since each polygon d_i' contains a circle of radius r, we have, by Lemma 1, $d_i' \geqq A(p_i, r)$. Thus, in view of Lemma 4,

$$d_1'+\ldots+d_n' \geqq A(p_1, r)+\ldots+A(p_n, r) \geqq n\,A(p, r) = d_1+\ldots+d_n.$$

Since $d_1'+\ldots+d_n' = d_1+\ldots+d_n$, in both inequalities equality must hold, showing that the d_i''s are regular p-gons with inradius r.

This completes the proof of Theorem *rp*.

The proof of Theorem *rc* is quite similar. We must only replace some terms as "incircle", "contains", etc. by "circumcircle", "is contained in", etc. Then refering instead of Lemma 1 and 4 to Lemma 2 and 5, we have

$$d_1'+\ldots+d_n' \leqq B(p_1, r)+\ldots+B(p_n, r) \leqq n\,B(p, r) = d_1+\ldots+d_n$$

which again implies the regularity of the d_i''s.

§ 4. Semi-regular Packings and Coverings

Since the face-incircles of $\{p, 3\}$ constitute a solid packing, so do the face-incircles of $t\{p, 3\}, p \geqq 2$. Apart of these trivial cases, the proof of the solidity of an Archimedean packing or covering requires finer considerations. We restrict ourselves to some Archimedean packings and coverings consisting of two kinds of circles.

THEOREM *sp. Let a trihedral Archimedean tessellation consist of p-gonal and P-gonal faces with inradii r and R, respectively. If*

$$A(p+1, r)+A(P-1, R)>A(p, r)+A(P, R)$$

and

$$A(p-1, r)+A(P+1, R)>A(p, r)+A(P, R),$$

then the face-incircles of the tessellation form a solid packing.

THEOREM *sc. Let a trihedral Archimedean tessellation consist of p-gonal and P-gonal faces with circumradii r and R, respectively. If*

$$B(p+1, r)+B(P-1, R)<B(p, r)+B(P, R)$$

and

$$B(p-1, r)+B(P+1, R)<B(p, r)+B(P, R),$$

then the face-circumcircles of the tessellation form a solid covering.

In order to prove these theorems we introduce a notion which will enable us to define the Dirichlet cells of incongruent circles. Let c be a circle with radius r and center O. In the Euclidean plane the *power* u of a point P with respect to c is defined by $u = t^2-r^2$, where $t=\overline{PO}$. This is an increasing analytic function of t which depends for $t>r$ only on the distance $\overline{PQ}=s$, where Q is a point on the boundary of c such that the line PQ is a tangent of c. We want to define u in the non-Euclidean geometries so as to preserve these properties. Since on the sphere $\cos r \cos s = \cos t$ and in the hyperbolic plane $\cosh r \cosh s = \cosh t$, we define u by $u = -\cos t/\cos r$ and $u = \cosh t/\cosh r$, respectively.

Let c_1 and c_2 be two circles with radii r_1 and r_2 and centers O_1 and O_2, respectively. Let L be the locus of the points having equal powers with respect to c_1 and c_2. In the Euclidean plane L is a straight line. We shall show that this well known fact remains valid in the non Euclidean geometries. Since the proofs are very similar in the spherical and in the hyperbolic geometries, we restrict ourselves to the hyperbolic plane.

Let P be a point of L and F the foot of the perpendicular drawn from P to the line O_1O_2. Write $\overline{O_1F}=x_1$, $\overline{O_2F}=x_2$, $\overline{PF}=y$, $\overline{PO_1}=t_1$ and $\overline{PO_2}=t_2$. Since $\cosh t_i = \cosh x_i \cosh y$, $i=1, 2$, the equality

$$\frac{\cosh t_1}{\cosh r_1} = \frac{\cosh t_2}{\cosh r_2}$$

implies that

$$\frac{\cosh x_1}{\cosh r_1} = \frac{\cosh x_2}{\cosh r_2},$$

and vice versa. Consequently, the line through F perpendicular to O_1O_2 belongs to L. The fact that apart from this line L has no further points, follows from the monotonity of the power as a function of t.

We now define in a set of circles the *Dirichlet cell* of a circle as the set of points which have a smaller power with respect to the respective circle than with respect to any other circle. As the intersection of half-planes, this is a convex polygon. These polygons join along whole sides and fill the plane without overlapping and without intersteces.

Now we turn to the proof of the above theorems. Because of the complete analogy of the proofs, we restrict ourselves to Theorem sp.

In the tessellation let the p-gons and P-gons be $d_1, d_2, \dots$ and $D_1, D_2, \dots$. Let the incircles of the faces be $c_1, c_2, \dots$ and $C_1, C_2, \dots$, respectively. Let S be the set of all these circles and $\{c_1, \dots, c_m, C_1, \dots, C_M\}$ a subset of S. Consider m circles $c'_1, \dots, c'_m$ of radius r and M circles $C'_1, \dots, C'_M$ of radius R which together with $c_{m+1}, \dots$ and $C_{M+1}, \dots$ form a packing S'. Let $d'_1, \dots$ and $D'_1, \dots$ be the Dirichlet cells of the circles of S'. Let n and N be integers not less than m and M, respectively, such that the pointset union of $d'_1, \dots, d'_n, D'_1, \dots, D'_N$ is connected and that for any $i > n$ d'_i and d_i are identical and for any $I > N$ D'_I and D_I are identical. Let $p_1, \dots, p_n$ and $P_1, \dots, P_N$ be the number of sides of $d'_1, \dots, d'_n$ and $D'_1, \dots, D'_N$, respectively. The argument used in the proof of Theorem rp shows that $k + K \leqq g$, where $k = p_1 + \dots + p_n$, $K = P_1 + \dots + P_N$ and $g = np + NP$. Thus

$$T' = d'_1 + \dots + d'_n + D'_1 + \dots + D'_N \geqq A(p_1, r) + \dots + A(p_n, r) +$$
$$+ A(P_1, R) + \dots + A(P_N, R) \geqq nA\left(\frac{k}{n}, r\right) + NA\left(\frac{K}{N}, R\right) \geqq$$
$$\geqq nA\left(\frac{k}{n}, r\right) + NA\left(\frac{g-k}{N}, R\right) = C(k).$$

We claim that

$$C(k) \geqq C(np) = n\,A(p, r) + N\,A(P, R).$$

To see this observe that, as a sum of two convex functions, $C(k)$ is a convex function of k. Thus the inequality $C(k) \geqq C(np)$ will be proved by showing that $C(np-1) > C(np)$ and $C(np+1) > C(np)$. Since between consecutive integers $A(x, r)$ and $A(x, R)$ are linear functions of x, we have

$$C(np \pm 1) = nA\left(p \pm \frac{1}{n}, r\right) + NA\left(P \mp \frac{1}{N}, R\right) =$$
$$= (n-1)A(p, r) + A(p \pm 1, r) + (N-1)A(P, R) + A(P \mp 1, R) =$$
$$= C(np) + A(p \pm 1, r) + A(P \mp 1, R) - A(p, r) - A(P, R),$$

which is, by assumption, really greater than $C(np)$. Therefore

$$T' \geqq n\,A(p, r) + N\,A(P, R) = d_1 + \dots + d_n + D_1 + \dots + D_N = T.$$

Since $T' = T$, in each inequality involving T' equality must hold. It follows that the d'_i's and D'_i's are regular p-gons and P-gons with inradius r and R, respectively. This proves the solidity of the packing S.

We continue to consider some applications of Theorem *sp* and *sc*. In the tessellation $t\{p, q\}$ let the inradii and circumradii of the $2p$-gonal and q-gonal faces be R, $\bar{R}$, r and $\bar{r}$, respectively. Throwing a glance to a characteristic triangle of $\{p, q\}$, these radii can easily be computed. For instance, in the spherical case, we have

$$\cos R = \cos\frac{\pi}{q}/\sin\frac{\pi}{p}, \quad \cos(R+r) = \cot\frac{\pi}{p}\cot\frac{\pi}{q},$$

$$\tan\bar{R} = \tan R/\cos\frac{\pi}{2q}, \quad \tan\bar{r} = \tan r/\cos\frac{\pi}{q}.$$

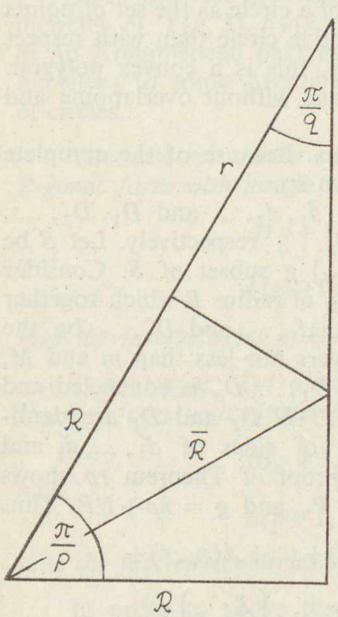


Fig. 1

We are looking for those values of p and q which satisfy, on the one hand, the inequalities

(a)

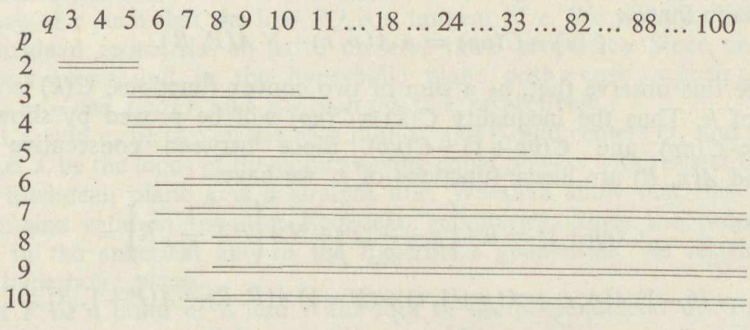
$$A(2p, R) + A(q, r) - A(2p \pm 1, R) - A(q \mp 1, r) < 0,$$

on the other hand, the inequalities

(b)

$$B(2p, \bar{R}) + B(q, \bar{r}) - B(2p \pm 1, \bar{R}) - B(q \mp 1, \bar{r}) > 0.$$

The left sides of (a) and (b) have been computerized for $2 \leqq p \leqq 10$, $3 \leqq q \leqq 100$. It turned out that (b) was never satisfied without (a) having been satisfied too. Indicating by a single line that only (a) is satisfied, and by a double line that both (a) and (b) are fulfilled, the results may be tabulated as follows:



For example, Theorem *sp* implies that the face-incircles of $t\{3, 10\}$ or $t\{3, 11\}$ form a solid packing but it leaves the question open whether tha same is true, say for $t\{3, 12\}$. Again, Theorem *sc* implies that the face-circumcircles of $t\{5, 6\}$ form a solid covering but it does not decide the question whether the same is true, say, for $t\{5, 5\}$.

For $t\{4, 4\}$ the packing problem deserves special attention. To put the problem in a new light, we emphasize the following special case of Conjecture *p*: The face-incircles of a $\{p, q\}$ will always be solidified by inscribing a circle into each gap. In view of Theorem *rp*, this conjecture turns out to be right for $\{3, 3\}$. Theorem *sp* shows its correctnes for $\{5, 5\}$, $\{6, 6\}$, $\{7, 7\}$, $\{8, 8\}$, $\{9, 9\}$ and $\{10, 10\}$, and it seems to be a question of computation to verify the conjecture for $\{11, 11\}, \dots$. This is a challenge to prove the conjecture for $\{4, 4\}$, all the more because the conjecture is obviously true for the two remaining regular Euclidean tessellations, namely for $\{3, 6\}$ and $\{6, 3\}$.

We conclude with two remarks showing how tantalizing this problem is. Consider a $\{4, 4\}$ with $R=1$. Then $r=\sqrt{2}-1$ and $A(8, R)+A(4, r)=4$. On the other hand, we have $A(7, R)+A(5, r)\approx 3.9943$, which is only by some 0.14% less than 4. Since $A(6, R)+A(6, r)\approx 4.0584>4$, there is no other possibility of rearranging a finite number of the face-incircles of $t\{4, 4\}$ but by means of nearly regular septagonal and pentagonal Dirichlet cells. But it is intuitively obvious that this is impossible.

Our second remark concerns the density of the face-incircles of $t\{4, 4\}$. This density equals 0.92015... . On the other hand, it is known [3, 4] that the density of a packing of any kind of circles with radii not less than $\sqrt{2}-1$ and not greater than 1 cannot exceed 0.92084... . This is the density of two small circles and one big circle all touching one another in the triangle determined by the centers of the circles. However, in this triangle the sectors of the small circles have a total angle which is more than 4-times greater than the angle of the sector of the big circle. But since, on the one hand, among the face-incircles of $t\{4, 4\}$ there are as many big circles as small ones and, on the other hand, the above density bound cannot be attained even for a packing of arbitrary circles with radii lying between $\sqrt{2}-1$ and 1, practically it can be taken for granted that the face-incircles of $t\{4, 4\}$ form a densest packing.

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ON THE SECOND ORDER HALF-LINEAR DIFFERENTIAL EQUATION

by

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1. Introduction

By a second order half-linear differential equation an equation is meant within which the dependence on the function and its derivative — i.e. the restoring force — is homogeneous of the first degree. The equation in question has the form as follows

$$(p(t)u')' + q(t)f(u, p(t)u') = 0, \qquad t \in (a, b) = I \tag{1.1}$$

where

1. $p(t) > 0$ and $g(t)$ are continuous in I,

2. $f(u, v)$ is continuous for every u, v and satisfies some condition assuring the uniqueness for given initial conditions of the certainly existing solution of (1.1). (E.g. $f(u, v) \in \mathrm{Lip}\,(1)$ in any bounded and closed subset of the u, v plane.)

3. $f(\lambda u, \lambda v) = \lambda f(u, v)$ for every λ, u, v,

4. $sg\, f(u, v) = sg\, u$.

E.g. $f(u, v)$ may be chosen as

$$\frac{u^3}{u^2+v^2} \quad \text{or} \quad \frac{uv^2}{u^2+v^2}, \quad \text{etc.}$$

The properties of the solutions of equation (1.1) are in much similar to that of the corresponding linear equation. However, these solutions have also quite different properties.

A lot of these properties has been shown in papers [1—5]. Now let these be shortly recalled. (S. 1°—16°)

1° For given initial conditions $u(t_0) = u_0$, $u'(t_0) = u_0'$, $t_0 \in (a, b)$ (u_0, u_0' are arbitrary) (1.1) has a unique solution $u(t)$ existing in the whole interval I.

2° If $u(t)$ is a solution $cu(t)$ ($c = \mathrm{const}$) is another one.

3° If the solutions $u(t)$ and $v(t)$ have a zero ($\in I$) in common, then all their zeros are in common and one of the solutions is a constant multiple of the other (they depend linearly). Linearly independent solutions cannot have zeros in common.

4° If $u_1(t) = ku_2(t)$, $u_1'(t) = ku_2'(t)$ ($k = \mathrm{const} \neq 0$) hold at $t = t_0$, then it is valid for every $t \in I$. Consequently the Wronskian $W = u_1'u_2 - u_1u_2'$ vanishes everywhere or nowhere. On the other hand it cannot be stated that $W = \mathrm{const}$.

5° In consequence of 4° the zeros of the linearly independent solutions separate each other.

6° For a solution $u(t)\neq 0$ in I the function $z=\frac{pu'}{u}$ satisfies the Riccati-type equation

$$z'+\frac{1}{p}z^2+qg(z)=0,\quad g(z)=f(1,z)>0 \tag{1.2}$$

and reversely, i.e. (1.1) and (1.2) are equivalent concerning such a solution.

7° Equation (1.2) can have solutions of the form $u=e^{\lambda t}$ provided p, q are constant. Let e.g. be $p=1$, $q=\pm k^2=$const, then putting $u=e^{\lambda t}$ in equation

$$u''\pm k^2 f(u,u')=0 \tag{1.3}$$

and cancelling it we have

$$\lambda^2\pm k^2 g(\lambda)=0 \tag{1.4}$$

as the „characteristic equation" corresponding to (1. 3), and (1. 4) can have real or complex zeros and equation (1. 3) solutions of the form

$$e^{\alpha t},\ e^{i\alpha t},\ e^{\alpha t+i\beta t}\qquad (\alpha,\beta\ \text{real})$$

However, the separated real and imaginar parts of a complex solution are not solutions of (1. 3). The complex valued solution is periodic with the period $\frac{2\pi}{\alpha}$ and $e^{\alpha t+i\beta t}$ is a „damped complex oscillation".

Conditions may be given assuring the existence of (real or complex) roots of (1. 4), moreover in a way that (1. 4) has roots in finite number or in infinite number, but not clustering provided I is infinite.

8° Assume $p=1$, $q=\pm k^2$, then (1. 2) may be written as

$$\frac{dz}{dt}=-\left(z^2\pm k^2 g(z)\right)$$

for a non-vanishing solution $u(t)$. If, in addition $\lambda=z=\frac{u'}{u}$ does not satisfy (1. 4), (i.e. u is not of the form $ce^{\lambda t}$; suppose the existence of *real* z_0 not fulfilling (1.4)), then

$$\int_{z_0}^{z}\frac{ds}{s^2\pm k^2 g(s)}=t_0-t\qquad (t\in I) \tag{1.5}$$

which is surely invertable for the sign $+$.

The case where $z=\lambda$ satisfies (1. 4) was treated in 7° and gave $z=\frac{u'}{u}=\lambda=$const, $u=Ce^{\lambda t}$.

For the linear equation $u''\pm k^2u=0$ the characteristic equation $z^2\pm k^2=0$ $(z=\lambda)$ gives the exponential solutions while the other *non-vanishing* solutions are given by (1. 5) $\left(g(s)\equiv 1\right)$. Namely (for the sign $-$) $u=C\,\mathrm{ch}\,k(t+\alpha)=C_1e^{kt}+C_2e^{-kt}$ where $C_1=\frac{C}{2}e^{k\alpha}$, $C_2=\frac{C}{2}e^{-k\alpha}$ are not arbitrary (C and α are). However u is a solution of (1. 3) for arbitrary C_1 and C_2 too, moreover it is the general solution.

9° Every solution of equation (1. 3), (taking the sign $+$), (existing for all t) is oscillatory and assuming

$$f(u, -v) = f(u, v) \tag{C}$$

they are periodic and their successive quarter waves [an arc between a zero and the immediately following (previous) extremum-point] are congruent (s. [1]).

10° If $p=1$, $q \geqq 0$, $\int^{\infty} q(t)\,dt = +\infty$, $I=(0, +\infty)$, then every solution of (1.1) is oscillatory in I. Without the assumption $q \geqq 0$ one has only the alternative statement: either $u(t) \to 0$ as $t \to \infty$, or $u(t)$ is oscillatory (s. [4]).

11° The analogous of the Sturmian comparison theorems hold (s. [1]). If in equations

$$(p_i u')' + q_i f(u, p_i u') = 0, \qquad i=1,2, \ t \in I, \ I: t_0 \leqq t \leqq t^\circ \tag{1. 5_i}$$

we have

$$0 < p_2 \leqq p_1, \quad q_1 \leqq q_2, \quad t \in I$$

equation (1. 5_2)* will be called a Sturm majorant of (1. 5_1) and if in addition

$$q_1 < q_2$$

or

$$p_1 > p_2 > 0 \quad \text{and} \quad q_2(t) \neq 0$$

hold at some point of I, then (1. 5_2) is called a strict majorant of (1. 5_1) on I.

The statements of the Sturm theorems are well known, however, let they be recalled here for the sake of the sequel.

1. Let (1. 5_2) be a Sturm majorant of (1. 5_1) on I and $u_1 \not\equiv 0$ a solution of (1. 5_1) having exactly n ($n \geqq 1$) zeros $t_1 < t_2 < \ldots < t_n$ on $t_0 < t \leqq t^\circ$ (t_0 may or may not be a zero), and $u_2 \not\equiv 0$ a solution of (1. 5_2) satisfying

$$\frac{p_1 u_1'}{u_1} \geqq \frac{p_2 u_2'}{u_2} \tag{1. 6}$$

at $t = t_0$. (A fraction with vanishing denominator is regarded to be $+\infty$.) In particular, (1. 6) holds at $t = t_0$, if $u_1(t_0) = 0$. Then u_2 has at least n zeros on $t_0 < t \leqq t^\circ$. Furthermore u_2 has at least n zeros on $t_0 < t < t^\circ$, if either the sign $>$ holds in (1. 6) or (1. 5_2) is a strict Sturm majorant of (1. 5_1) on $t_0 \leqq t \leqq t_n$.

2. Assume the conditions of the first part of 1. and that u_2 has exactly n zeros on $t_0 < t \leqq t^\circ$. Then (1. 6) holds at $t = t^\circ$ (now a fraction in (1. 6) with vanishing denominator is to be taken as $-\infty$), moreover with the sign $<$ if so is at $t = t_0$ or (1. 5_2) is a strict majorant of (1. 5_1).

Corollary of 1. Sturm's separation theorem. Let (1. 5_2) be a Sturm majorant of (1. 5_1) on I and u_1 vanish at $t = t_1, t_2$ ($t_1 < t_2$) of I. Then u_2 has at least one zero on $[t_1, t_2]$. In particular, if $p_1 \equiv p_2$, $q_1 \equiv q_2$, u_1 and u_2 are linearly independent solutions of (1. 5_1) $\equiv$ (1. 5_2), then the zeros of u_1 separate and are separated by those of u_2.

* This means the second equation of (1. 5).

12° If $p=1$, $q\geqq 0$ and q is increasing, then provided condition (C) a quarter wave can be put in the former one (by reflection on an ordinate or rotation around a zero), consequently the quarter-wave-length and the absolute value of the extrema decrease (Consequence of 11°; s. [1]).

13° These comparison theorems can be extended on some half-linear first order systems (s. [1]).

14° The eigenvalue problem can be formulated and solved too, but the eigenfunctions are not orthogonal and there is no possibility of a development in series as for the linear equations (s. [1]).

15° Certain pair of functions built up from a solution and its derivative or from two linearly independent solutions behave like these independent solutions themselves concerning the separation of their zeros (s. [2] which involves also conditions for non-oscillation).

16° Every solution $u(t)$ of

$$u''+u+\varrho(t)f(u,u') = 0, \quad \varrho(t) = O\left(\frac{1}{t^2}\right), \qquad t\to\infty$$

has an asymptotic form

$$u = a_\infty \sin(t+\delta_\infty)+O\left(\frac{1}{t}\right), \qquad t\to\infty$$

(S. [3]).

17° Let $q(t)$ be periodic with the period T and its half periods (half waves) congruent. Then the constants α and β in the equation

$$u''+[\alpha+\beta g(t)]f(u,u') = 0 \tag{1.7}$$

may be chosen in such a way that (1. 7) has as well solutions with period T as solutions with period $2T$, the half periods of which consist of 2, 4, 6, ... or 2, 6, 10, ... quarter waves, respectively (s. [5]).

2. Further results. Comparison of the solutions of linear and half-linear differential equations

THEOREM. *Let us regard the following two equations*

$$(2.1)\qquad \begin{aligned}(pu')'+qf(u,pu') &= 0\\ (pu')'+qu &= 0\end{aligned}$$

with their non-trivial solutions $u_1(t)$ *and* $u_2(t)$ *respectively and assume* $q\geqq 0$, $q\neq 0$ *in some points of* $t_0\leqq t\leqq t^\circ$ *and*

$$f(1,z)\leqq 1 \tag{2.2}$$

for arbitrary z what is satisfied by the given examples.

Suppose again the existence of the zeros $t_1<t_2<\ldots<t_n$ *of* u_1 *in* $t_0<t\leqq t^\circ$ *and the validity of* (1. 6) $\left(i.e.\ \frac{u_1'}{u_1}\geqq\frac{u_2'}{u_2}\ at\ t=t_0\right)$, *then* u_2 *has at least n zeros* $\bar{t}_i$ *on* $t_0<t\leqq t^\circ$, *moreover on* $t_0<t<t^\circ$, *if either in* (1. 6) *or in* (2. 2) *the inequality holds.*

PROOF. By the substitutions

$$\operatorname{tg} \varphi_1 = \frac{u_1}{pu_1'}, \quad \operatorname{tg} \varphi_2 = \frac{u_2}{pu_2'} \tag{2.3}$$

we have for φ_1 and φ_2 the equations

$$\begin{aligned} \varphi_1' &= \frac{1}{p} \cos^2 \varphi_1 + q \sin^2 \varphi_1 f(1, \operatorname{tg} \varphi_1) \\ \varphi_2' &= \frac{1}{p} \cos^2 \varphi + q \sin^2 \varphi_2 \end{aligned} \tag{2.4}$$

On account of (1. 6) it is possible to choose the values of the functions φ_1, φ_2 to be continuous and satisfying

$$0 \leqq \varphi_1(t_0) \leqq \varphi_2(t_0) < \pi \tag{2.5}$$

By a known way it can be easily shown that φ_i $(i = 1, 2)$ passes at the k^{th} zero of u_i the value $k\pi$ $(k = 1, 2, \ldots, n)$ increasingly, hence just once. Denoting the right members of (2. 4) by $F_1(t, \varphi)$ and $F_2(t, \varphi)$ we have by (2. 2)

$$F_1(t, \varphi) \leqq F_2(t, \varphi) \qquad \text{(for every } t \text{ and } \varphi) \tag{2.6}$$

hence on account of a familiar theorem

$$\varphi_1(t) \leqq \varphi_2(t), \quad t_0 \leqq t \leqq t^\circ$$

which involves the validity of (1. 6) at $t = t^\circ$ too, and that of $\bar{t}_i \leqq t_i$.

If in (1. 6), i.e. in (2. 5), the inequality sign holds at $t = t_0$, then we take the solution φ_{20} of (2. 4_2) with $\varphi_{20}(t_0) = \varphi_1(t_0)$ and by the just proved assertion we have

$$\varphi_1(t) \leqq \varphi_{20}(t) < \varphi_2(t), \quad t_0 \leqq t \leqq t^\circ$$

since $\varphi_{20}(t)$ and $\varphi_2(t)$ cannot be equal anywhere.

On the other hand, if in (2. 5) the sign = holds, and in (2. 2) the sign $<$ is valid (for every z) and if our assertion were false, then we have

$$\varphi_1(t^\circ) = \varphi_2(t^\circ)$$

consequently

$$\varphi_1(t) = \varphi_2(t), \quad t_0 \leqq t \leqq t^\circ \tag{2.7}$$

(otherwise $\varphi_1(\bar{t}) < \varphi_2(\bar{t})$ in a point $\bar{t}$ with $t_0 < \bar{t} < t^\circ$ implies $\varphi_1(t^\circ) < \varphi_2(t^\circ)$), and

$$\varphi_1'(t) = \varphi_2'(t), \quad t_0 \leqq t \leqq t^\circ \tag{2.8}$$

But

$$\varphi_2' = \frac{1}{p} \cos^2 \varphi_2 + q \sin^2 \varphi_2 f(1, \operatorname{tg} \varphi_2) + \varepsilon(t)$$

where

$$\varepsilon(t) = g \sin^2 \varphi_2 [1 - f(1, \operatorname{tg} \varphi_2)]$$

Then (2. 7)—(2. 8) imply

$$\varepsilon(t) = 0, \quad t_0 \leqq t \leqq t^\circ.$$

Now equation $\sin\varphi_2=0$ can hold just in a point where $u_2=0$. But there is a point in the neighbourhood of which $q\neq 0$ and when there are here points where u_2 or u_2' vanishes, there are also points (in this neighbourhood) where $u_2\neq 0$, $u_2'\neq 0$. Here

$$q\neq 0,\quad \sin\varphi_2\neq 0,\quad \operatorname{tg}\varphi_2\neq\infty$$

consequently

$$1-f(1,\operatorname{tg}\varphi)>0$$

and so

$$\varepsilon(t)\neq 0$$

which means a contradiction.

REMARK. If $q(t)\leqq 0$, then the converse statement holds provided $\frac{u_1'}{u_1}\leqq\frac{u_2'}{u_2}$ at $t=t_0$. Namely in (2. 6) the sign $\leqq$ must be replaced by the sign $\geqq$.

Comparison of the solutions of two half-linear equations

If in equations

$$(p_iu')'+q_if_i(u,p_iu')=0\qquad (i=1,2)$$

the assumptions of 11° or

$$f_1(1,z)<f_2(1,z)\qquad\text{(for every } z)$$

are satisfied then the assertions of 11° and **2.** are valid.

3. Decrease of the amplitudes

The „amplitude" of a solution of (1.1) is defined by

$$A(t)=u^2+\frac{(pu')^2}{pq}$$

With regard to (1.1)

$$A'(t)=-pu'^2\left[2f\left(\frac{u}{pu'},1\right)-2\frac{u}{pu'}+p\frac{(pq)'}{(pq)^2}\right]$$

Putting here $pu'=\varrho\cos\varphi$, $u=\varrho\sin\varphi$

$$A'(t)=-\frac{\varrho^2}{p}\cos^2\varphi\left\{2[f(\operatorname{tg}\varphi,1)-\operatorname{tg}\varphi]+p\frac{(pq)'}{(pq)^2}\right\}\tag{3.1}$$

whence it is obvious the truth of the following assertion:

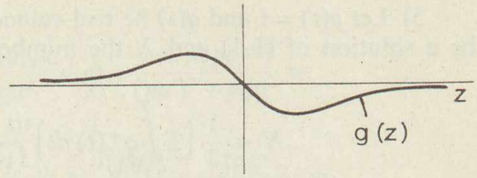
If in (1.1) $p>0$ for $t>t_0$ and

$$2[f(z,1)-z]+p\frac{(pq)'}{(pq)^2}>0\tag{3.2}$$

for every z and $t\geqq t_0$, then $A(t)$ is decreasing for whatever solution of (1.1).

Taking example $f(u, v) = \dfrac{u^3}{u^2+v^2}$

$$g(z) = f(z, 1) - z = -\frac{z}{z^2+1}$$



which has a maximum in absolute value.

As a consequence the extrem a — if they exist — (i.e. $|u|$, where $u'=0$) decrease, but this is true without condition (3. 2) too (s. 12°).

In linear case (3. 2) reduces to $(pq)' > 0$.

4. Distribution of the zeros of the solutions

A lot of results known in real domain may be extended to (1.1).

1) Suppose $f(1, z)$ has (a positive) maximum or a least upper bound and this is (for the sake of simplicity) 1

$$f(1, z) \leqq 1 \qquad (z \text{ arbitrary}).$$

Let $q(t)$ be real and continuous and $m(t) \geqq 0$ a continuous function in $a \leqq t \leqq b$ and

$$\gamma_m = \inf \frac{m(t)}{(t-a)(b-t)} \qquad a < t < b$$

Let (1.1) (with $p=1$) have a solution with two zeros (at least) in $[a, b]$ then

$$(4.1) \qquad \int_a^b m(t) q^+(t)\, dt > \gamma_m (b-a)$$

where $q^+(t) = \max\big(q(t), 0\big)$ (s. [6], p. 345).

The proof proceeding along the same line as in linear case may be omitted.

Let some immediate consequences of (4.1) mentioned here without proof (s. loc. cit.)

2) $\displaystyle\int_a^b (t-a)(b-t) q^+(t)\, dt > b-a$

3) $\displaystyle\int_a^b (t-a) q^+(t)\, dt > 1$

4) $\displaystyle\int_a^b q^+(t)\, dt > \frac{4}{b-a}$

Further (less immediate) consequences (s. loc. cit.) are as follows.

5) Let $p(t)=1$ and $q(t)$ be real-valued and continuous for $0\leqq t\leqq T$. Let $u(t)\not\equiv 0$ be a solution of (1.1) and N the number of its zeros in $0<t\leqq T$. Then

$$N<\frac{1}{2}\left(T\int_0^T q^+(t)\,dt\right)^{1/2}+1 \qquad \text{(consquence of 4)}$$

6) The number N also satisfies

$$N<\int_0^T tq^+(t)\,dt+1 \qquad \text{(consequence of 3)}$$

7) Now we regard the extension of a result of WINTNER and HARTMANN (s. loc. cit. p. 347).

If in equation (1.1) $p=1$, $q>0$ are continuous and of bounded variation on $0\leqq t\leqq T$ and for some $\delta>0$

$$\left|z\frac{f(z,\sqrt{q})-z}{1+z^2}\right|\leqq\delta \qquad \text{for every } z \text{ and } 0\leqq t\leqq T \tag{4.2}$$

$$\text{or}\quad \frac{z^2}{1+z^2}\left|f\left(1,\frac{\sqrt{q}}{z}\right)-1\right|\leqq\delta \qquad \text{(e. g. } f(1,v)-1 \text{ is bounded for every } v)$$

then

$$\left|N\pi-\int_0^T q^{1/2}(t)\,dt\right|\leqq\frac{\pi}{4}+\frac{1}{4}\int_0^T\frac{|dq(t)|}{q(t)}+\delta\int_0^T q^{1/2}(t)\,dt$$

PROOF. At first let $u(t)$ be a solution of the (general) equation (1.1) and introduce the following functions

$$\varphi=\varphi(t)=\operatorname{arctg}\frac{q^{1/2}u}{p^{1/2}u'}=\operatorname{arctg}\frac{(pq)^{1/2}u}{pu'}$$

which differ slightly from those used previously. Then

$$d\operatorname{tg}\varphi=(1+\operatorname{tg}^2\varphi)\,d\varphi=\frac{(pu')^2+pqu^2}{(pu')^2}\,d\varphi=$$

$$=\frac{d[(pq)^{1/2}u]\cdot pu'-(pu')'\,dt\,(pq)^{1/2}u}{(pu')^2}=$$

$$=\frac{puu'\,d(pq)^{1/2}+(pq)^{1/2}[pu'^2+quf(u,pu')]\,dt}{(pu')^2}$$

whence

$$d\varphi=\left(\frac{q}{p}\right)^{1/2}\frac{(pu')^2+pquf(u,pu')}{(pu')^2+pqu^2}\,dt+\frac{puu'\,d(pq)^{1/2}}{(pu')^2+pqu^2}=$$

$$=\left(\frac{q}{p}\right)^{1/2}\frac{1+\operatorname{tg}\varphi f\left(\operatorname{tg}\varphi,\sqrt{pq}\right)}{1+\operatorname{tg}^2\varphi}\,dt+\frac{d(pq)^{1/2}}{(pq)^{1/2}}\,\frac{\operatorname{tg}\varphi}{1+\operatorname{tg}^2\varphi}= \tag{4. 3}$$

$$=\left(\frac{q}{p}\right)^{1/2}dt+\left(\frac{q}{p}\right)^{1/2}\operatorname{tg}\varphi\,\frac{f\left(\operatorname{tg}\varphi,\sqrt{pq}\right)-\operatorname{tg}\varphi}{1+\operatorname{tg}^2\varphi}\,dt+\frac{1}{4}\sin 2\varphi\,\frac{d(pq)}{pq}$$

Now take $p\equiv 1$. (4. 3) exhibits that φ passes every zero of u increasingly and so just once. Therefore, if $0\leqq\varphi(0)<\pi$ then

$$N\pi\leqq\varphi(T)<(N+1)\pi$$

So

$$N\pi\leqq\varphi(T)=\varphi(0)+\int_0^T q^{1/2}\,dt+\int_0^T q^{1/2}\operatorname{tg}\varphi\,\frac{f\left(\operatorname{tg}\varphi,\sqrt{q}\right)-\operatorname{tg}\varphi}{1+\operatorname{tg}^2\varphi}\,dt+$$

$$+\frac{1}{4}\int_0^T\sin 2\varphi\,\frac{dq}{q}<(N+1)\pi$$

Therefore

$$N\pi\leqq\pi+\int_0^T q^{1/2}\,dt+\delta\int_0^T q^{1/2}\,dt+\frac{1}{4}\int_0^T\frac{|dq|}{q}$$

and

$$N\pi>-\pi+\int_0^T q^{1/2}\,dt-\delta\int_0^T q^{1/2}\,dt-\frac{1}{4}\int_0^T\frac{|dq|}{q}$$

Now these two inequalities give the above assertion.

In example $f(u,v)=\dfrac{u^3}{u^2+v^2}$ condition (4. 2) reads as $F(z,q)\leqq\delta$ where

$$F(z,q)=-q\,\frac{z^2}{(1+z^2)(q+z^2)}$$

The function $q(t)$ is bounded on $[0, T]$

$$0\leqq k_1\leqq q\leqq k_2$$

and $F(z,q)$ as a function of z remains between $\dfrac{k_2}{k_1}F(z,k_1)$ and $\dfrac{k_1}{k_2}F(z,k_2)$. Therefore $\max|F(z,q)|$ exists.

COROLLARY. If $\int_0^\infty q^{1/2}(t)\,dt = +\infty$ and q is of bounded variation in $[0, \infty]$ so that

$$\int_0^T \frac{|dq|}{q} = o\left(\int_0^T q^{1/2}(t)\,dt\right) \quad \text{as} \quad T\to\infty \tag{4. 4}$$

then from the above formulae

$$1-\delta \leqq \lim_{T=\infty} \frac{N\pi}{\int_0^T q^{1/2}(t)\,dt} \leqq 1+\delta$$

For instance (4. 4) is satisfied when $q \in C_1$ and

$$q'(t) = o\big((q^{3/2}(t)\big), \qquad t\to+\infty$$

8) If in (1) $p\equiv 1$, $q\geqq 0$ continuous and

$$\sup |f(1,z)-1| = A \qquad \text{(for every } z)$$

exists as a finite number, then

$$|N\pi - T| \leqq \pi + \int_0^T |1-q(t)|\,dt + A\int_0^T q(t)\,dt$$

(s. loc. cit. p. 348).

PROOF. The function $\varphi = \text{arc tg}\,\frac{u}{u'}$ satisfies

$$\varphi' = \cos^2\varphi + q\sin^2\varphi f(1, \text{cotg}\,\varphi) = 1+\sin^2\varphi[qf(1, \text{ctg}\,\varphi)-1]$$

Suppose $0 \leqq \varphi(0) < \pi$, then

$$N\pi \leqq \varphi(T) = \varphi(o) + T + \int_0^T \sin^2\varphi[q(t)f(1, \text{ctg}\,\varphi)-1]\,dt < (N+1)\pi$$

But

$$qf(1,z)-1 = q-1+q[f(1,z)-1]$$

So

$$N\pi - T \leqq \pi + \int_0^T |q-1|\,dt + A\int_0^T q\,dt$$

$$N\pi - T > -\pi - \int_0^T |q-1|\,dt - A\int_0^T q\,dt$$

which gives our assertion.

5. Asymptotic behaviour of the solutions
I. Non-oscillatory equations with constant coefficients

5.1. *Pure exponential solutions*

If $p=1$, $q=-\omega^2$ equation (1.1) can have such type of solutions and it has really if e.g.

$$f(u,v)=\frac{u^3}{u^2+\varepsilon v^2} \qquad (\varepsilon>0)$$

as shown in 7° of 1. Now this will be found for ε small enough by another way too, namely by the analogous of the method given in [7].

As a comparison with the corresponding linear equation shows the solutions of the equation

$$u''-\omega^2\frac{u^3}{u^2+\varepsilon u'^2}=0 \tag{5.1.1}$$

are not oscillatory, consequently $u>0$ may be assumed for $t>t_0$ with a certain $t_0=$const. Suppose that $z=u'/u$ remains bounded for $t>t_0$, then $\varepsilon z^2<1$ for an ε small enough and so

$$f(1,z)=\frac{1}{1+\varepsilon z^2}=1-\varepsilon z^2+\varepsilon^2 z^4-+\dots$$

The Riccati-type equation corresponding to (5. 1. 1) will have the form

$$z'+z^2-\omega^2\frac{1}{1+\varepsilon z^2}=z'-\omega^2+(1+\varepsilon\omega^2)z^2-\varepsilon^2\omega^2 z^4+-\dots=0 \tag{5.1.2}$$

Let us assume its solution in the form

$$u=a(t)e^{\psi(t)} \quad \text{or} \quad z=\frac{u'}{u}=\frac{a'}{a}+\psi'=b+\psi', \quad b=a\frac{a'}{a}$$

Suppose b' and ψ' have the expansions in powers of ε in the form

$$b'=\varepsilon A_1(b)+\varepsilon^2 A_2(b)+\dots$$

$$\psi'=\pm\omega+\varepsilon B_1(b)+\varepsilon^2 B_2(b)+\dots$$

where $A_i(b)$, $B_i(b)$ $(i=1,2,\dots)$ are unknown functions to be determined of b. Then

$$z=b+\psi'=(\pm\omega+b)+\varepsilon B_1+\varepsilon^2 B_2+\dots$$

$$z^2=(\pm\omega+b)^2+\varepsilon\cdot 2(\pm\omega+b)B_1+\varepsilon^2[B_1^2+2(\pm\omega+b)B_2]+\dots$$

$$z^4=(\pm\omega+b)^4+\varepsilon\cdot 4(\pm\omega+b)B_1+\dots$$

$$z'=b'+\psi''=b'+(\varepsilon\dot{B}_1+\varepsilon^2\dot{B}_2+\dots)b'=$$

$$=(1+\varepsilon\dot{B}_1+\varepsilon^2\dot{B}_2+\dots)(\varepsilon A_1+\varepsilon^2 A_2+\dots)=$$

$$=A_1\varepsilon+(A_1\dot{B}_1+A_2)\varepsilon^2+\dots \qquad \left(\cdot=\frac{d}{db}\right)$$

Putting all these in (5.1.2) and equating the coefficients of the powers of ε with zero

a) $$(\pm\omega+b)^2-\omega^2=0$$

b) $$(\pm\omega+b)[2B_1+\omega^2(\pm\omega+b)]+A_1=0$$

c) $$A_1\dot{B}_1+A_2+\dot{B}_1+2(\pm\omega+b)B_2+2\omega^2(\pm\omega+b)B_1-\omega^2(\pm\omega+b)^4=0$$

Now we have two cases:

1° From a) $\pm\omega+b=\pm\omega$, whence $b=0$, $a=\text{const}$ and A_i, B_i $(i=1,2,\ldots)$ are constant. Furthermore

$$0=b'=\varepsilon A_1+\varepsilon^2A_2+\ldots$$

gives $A_i=0$ $(i=1,2,\ldots)$. Then b) gives $B_1=\mp\dfrac{\omega^3}{2}$ and

$$\psi'=\frac{u'}{u}=\pm\omega\left(1-\varepsilon\frac{\omega^2}{2}\right)$$

$$u=Ce^{\pm\omega\left(1-\varepsilon\frac{\omega^2}{2}\right)t}. \quad (C \text{ an arbitrary constant})$$

which is the first approximation.

Continuing the process c) gives $B_2=\mp\dfrac{1}{8}\omega^5$ and

$$u=Ce^{\pm\omega\left(1-\varepsilon\frac{\omega^2}{2}-\frac{1}{8}\varepsilon^2\omega^4\right)t}$$

as second approximation.

2° If $\pm\omega+b=\mp\omega$, then $b=\mp2\omega=\dfrac{a'}{a}$, $a=Ce^{\mp2\omega t}$ and $b'=0$ implies $A_i=0$ $(i=1,2,\ldots)$ now too. Thus $B_1=\mp\dfrac{\omega^3}{2}$

$$\psi'=\pm\omega\pm\varepsilon\frac{\omega^3}{2}$$

$$\frac{u'}{u}=\mp2\omega\pm\omega\pm\frac{\omega^3}{2}=\mp\omega\left(1-\varepsilon\frac{\omega^2}{2}\right)$$

$$u=Ce^{\mp\omega\left(1-\varepsilon\frac{\omega^2}{2}\right)t}$$

This first approximation and the second one so obtainable are identical to that received in 1°.

The result can be checked by comparing to 7° in 1. The characteristic equation

$$\lambda^2-\omega^2\frac{1}{1+\varepsilon\lambda^2}=0 \quad \text{or} \quad \varepsilon\lambda^4+\lambda^2-\omega^2=0$$

have a *real* solution which expanded in powers of ε reads as

$$\frac{u'}{u} = \lambda = \pm\omega\left(1-\varepsilon\frac{\omega^2}{2}+\frac{7}{8}\varepsilon^2\omega^4+\dots\right)$$

which agrees in first approximation with the above result.

5. 2. *The general solution of* (5.1.1)

Now let us determine all the solutions of (5.1.1) (all nonvanishing, moreover the vanishing too).

As a generalization of the method of [7] the solution process will be formed as follows. Assume $u(t)$ in the form

$$u(t) = a(t)\,\mathrm{ch}\,\psi(t)+\varepsilon u_1(a,\psi)+\varepsilon^2 u_2(a,\psi)+\dots \tag{5.2.1}$$

where

$$a' = \varepsilon A_1(a)+\varepsilon^2 A_2(a)+\dots$$

$$\psi' = \pm\omega+\varepsilon B_1(a)+\varepsilon^2 B_2(a)+\dots \tag{5.2.2}$$

Determine the functions u_i, A_i, B_i $(i=1,2,\dots)$ in such a way that (5.2.1) be the solution of (5.1.1).

By (5.2.2) we get

$$u' = \pm a\omega\,\mathrm{sh}\,\psi+\varepsilon\left(A\,\mathrm{ch}\,\psi+aB_1\,\mathrm{sh}\,\psi\pm\omega\frac{\partial u_1}{\partial\psi}\right)+$$

$$+\varepsilon^2\left(A_2\,\mathrm{ch}\,\psi+aB_2\,\mathrm{sh}\,\psi+A_1\frac{\partial u_1}{\partial a}+B_1\frac{\partial u_1}{\partial\psi}\pm\omega\frac{\partial u_2}{\partial\psi}\right)+\varepsilon^3\dots$$

$$u'' = a\omega^2\,\mathrm{ch}\,\psi+\varepsilon\left(\pm 2\omega A_1\,\mathrm{sh}\,\psi\pm 2\omega aB_1\,\mathrm{ch}\,\psi+\omega^2\frac{\partial^2 u_1}{\partial\psi^2}\right)+\varepsilon^2\dots$$

Now we expand the function

$$f(\varepsilon,u,u') = \frac{u^3}{u^2+\varepsilon u'^2}$$

in Taylor series in the neighbourhood of $\varepsilon=0$, $u=a\,\mathrm{ch}\,\psi$, $u'=\pm a\omega\,\mathrm{sh}\,\psi$. Obtaining

$$f(\varepsilon,u,u') = f(0,a\,\mathrm{ch}\,\psi,\ \pm a\omega\,\mathrm{ch}\,\psi)+f_\varepsilon(\dots)\varepsilon+f_u(\dots)(\varepsilon u_1+\varepsilon^2u_2+\dots)+$$

$$+f_{u'}(\dots)\left[\varepsilon\left(A_1\,\mathrm{sh}\,\psi+aB_1\,\mathrm{sh}\,\psi\mp\omega\frac{\partial u_1}{\partial\psi}\right)\right]+\dots=$$

$$=\underbrace{f(\dots)}_{=a\,\mathrm{ch}\,\psi}+\varepsilon\left[f_\varepsilon(\dots)+u_1f_u(\dots)+f_{u'}(\dots)\left(A_1\,\mathrm{ch}\,\psi+aB_1\,\mathrm{sh}\,\psi\pm\omega\frac{\partial u_1}{\partial\psi}\right)\right]+\varepsilon^2\dots .$$

So we hawe

$$u''-\omega^2 f(\varepsilon,u,u') = \varepsilon\Bigg[\pm 2\omega A_1 \operatorname{sh}\psi \pm 2\omega a B_1 \operatorname{ch}\psi + \omega^2\frac{\partial^2 u_1}{\partial\psi^2} -$$

$$-\omega^2 f_\varepsilon(\dots)-\omega^2 f_u(\dots)u_1-\omega^2 f_{u'}(\dots)\left(A_1\operatorname{ch}\psi + aB_1\operatorname{sh}\psi \pm \omega\frac{\partial u_1}{\partial\psi}\right)\Bigg]+\varepsilon^2\dots = 0$$

By equating the coefficient of ε to zero we have for the first approximation

$$\omega^2\left[\frac{\partial^2 u_1}{\partial\psi^2} - f_u(\dots)u_1\right] = \omega^2 f_\varepsilon(\dots)+\omega^2 f_{u'}(\dots)\left(A_1\operatorname{ch}\psi + aB_1\operatorname{sh}\psi\pm\omega\frac{\partial u_1}{\partial\psi}\right)\mp$$

$$\mp 2\omega A_1\operatorname{sh}\psi \mp 2\omega a B_1\operatorname{ch}\psi$$

In the present case

$$f_\varepsilon = -\frac{u^3u'^2}{(u^2+\varepsilon u'^2)^2},\quad f_\varepsilon(0,\dots) = -a\omega^2\frac{\operatorname{sh}^2\psi}{\operatorname{ch}\psi}$$

$$f_u = \frac{u^4+\varepsilon 3u^2u'^2}{(u^2+\varepsilon u'^2)^2},\quad f_u(0,\dots) = 1$$

$$f_{u'} = -\frac{2\varepsilon u^3 u'}{(u^2+\varepsilon u'^2)^2},\quad f_{u'}(0,\dots) = 0$$

and we get

$$\omega^2\left(\frac{\partial^2 u_1}{\partial\psi^2}-u_1\right) = -a\omega^4\frac{\operatorname{sh}^2\psi}{\operatorname{ch}\psi}\mp 2\omega A_1\operatorname{sh}\psi\mp 2\omega a B_1\operatorname{ch}\psi \tag{5. 2. 3}$$

Here

$$\frac{\operatorname{sh}^2\psi}{\operatorname{ch}\psi} = \frac{(e^{2\psi}-1)^2}{2e^{\psi}(e^{2\psi}+1)} = \frac{(1-z^2)^2}{2z(1+z^2)} = \frac{1}{2}\left(\frac{1}{z}-3z+4z^3-4z^5+4z^7-+\dots\right)$$

where $e^{\Psi}=z=\operatorname{ch}\psi+\operatorname{sh}\psi$ and $|z|<1$, $z\neq 0$, consequently $\psi<0$. Assume $u_1(a,\psi)$ can be expanded as

$$u_1(a,\psi) = v_0(a)+\sum_{n=1}^{\infty}(v_n(a)\operatorname{ch}n\psi + w_n(a)\operatorname{sh}n\psi) \tag{5. 2. 4}$$

Then

$$\frac{\partial^2 u_1}{\partial\psi^2}-u_1 = -v_0(a)+\sum_{n=2}^{\infty}(n^2-1)(v_n\operatorname{ch}n\psi + w_n\operatorname{sh}n\psi) \tag{5. 2. 5}$$

Putting all these expressions in (5. 2. 3) and comparing the coefficients of $\operatorname{ch} n\psi$, $\operatorname{sh} n\psi$ on both sides we have

$$v_0 = 0,\quad v_n = w_n = \begin{cases}\pm\dfrac{4}{n^2-1}, & n = 3,5,7,\dots\\[2mm] 0, & n = 2,4,6,\dots\end{cases}$$

$$A_1 = \pm a\omega^3,\quad B_1 = \pm\frac{\omega^3}{2}$$

and v_1, w_1 remain undetermined without a further demand. Suppose $v_1 = w_1 = 0$. So $u_1(a, \psi)$ does not contain a term with $\operatorname{ch}\psi$ or $\operatorname{sh}\psi$ influencing the leading term $a \operatorname{ch}\psi$.

Thus

$$a = Ce^{\pm \varepsilon\omega^3 t}, \quad \psi = \pm\omega\left(1+\varepsilon\frac{\omega^2}{z}\right)t + \text{const}$$

But $\psi < 0$ involves

$$\psi = -\omega\left(1+\varepsilon\frac{\omega^2}{2}\right)t - \alpha, \quad (\alpha > 0), \qquad a = Ce^{-\varepsilon\omega^3 t}$$

$$u_1 = \frac{1}{2}e^{3\psi}\left(1-\frac{1}{3}e^{2\psi}+\frac{1}{6}e^{4\psi}-\frac{1}{10}e^{6\psi}+-\ldots\right)$$

and the first approximation is as follows

$$u = a\operatorname{ch}\psi = Ce^{-\varepsilon\omega^3 t}\operatorname{ch}\left[\omega\left(1+\varepsilon\frac{\omega^2}{2}\right)t+\alpha\right] =$$

(5. 2. 6)

$$= C_1 e^{\omega\left(1-\varepsilon\frac{\omega^3}{2}\right)t} + C_2 e^{-\omega\left(1+\varepsilon\frac{3}{2}\omega^2\right)t}$$

where

$$C_1 = \frac{C}{2}e^{\alpha t}, \quad C_2 = \frac{C}{2}e^{-\alpha t}$$

Remark. Here C and $\alpha > 0$ are arbitrary constant, but C_1 and C_2 are not. If $u \not\equiv 0$, they have common signs. For α large we have the exponential solution $C_1 e^{\omega\left(1-\varepsilon\frac{\omega^2}{2}\right)t}$, but α cannot be a large negative number. This agrees with the fact that $C_2 e^{-\omega\left(1+\frac{3}{2}\varepsilon\omega^2\right)t}$ is not a solution (in first approximation) of (5. 1. 1). The vanishing solutions including the other exponential solution

$$e^{-\omega\left(1-\varepsilon\frac{\omega^2}{2}\right)t}$$

can be obtained if we start with $u = a\operatorname{sh}\psi + \varepsilon u_1 + \ldots$. A „corrected first approximation" is given by

$$u = a\operatorname{ch}\psi + \varepsilon u_1(a, \psi) \tag{5. 2. 7}$$

The process can be continued to obtain a second approximation. The difference of the first approximation and the correct solution is of order ε^2 in an interval of order $\frac{1}{\varepsilon}$ (s. [7]).

(5. 2. 6) is a nonvanishing solution (if $C \neq 0$) and (5. 2. 7) too, but they are not pure exponential.

5. 3. *Perturbation of the linear non-oscillatory equation by a half-linear term*

Equation in question reads as

(5. 3. 1) $$u''-\omega^2 u=\varepsilon f(u,u') \qquad (|\varepsilon|\ll 1)$$

With the assumptions (5. 2. 1)—(5. 2. 2) we have

$$u''-\omega^2 u=\varepsilon\left(\pm 2\omega A_1 \operatorname{sh}\psi \pm 2\omega a B_1 \operatorname{ch}\psi+\omega^2\frac{\partial^2 u_1}{\partial\psi^2}-\omega^2 u_1\right)+\varepsilon^2\ldots$$

and by Taylor-expansion

$$\varepsilon f(u,u')=\varepsilon f(a\operatorname{ch}\psi,\ \pm a\omega\operatorname{sh}\psi)+\varepsilon^2\ldots$$

By putting all these expressions in (5. 3.1) we have for the first and second approximations

(5. 3. 2.) $$\begin{aligned}\omega^2\left(\frac{\partial^2 u_1}{\partial\psi^2}-u_1\right)&=f_0(a,\psi)\mp 2\omega A_1\operatorname{sh}\psi\mp 2\omega a B_1\operatorname{ch}\psi\\ \omega^2\left(\frac{\partial^2 u_2}{\partial\psi^2}-u_2\right)&=f_1(a,\psi)\mp 2\omega A_2\operatorname{sh}\psi\mp 2\omega a B_2\operatorname{ch}\psi\end{aligned}$$

where

$$f_0(a,\psi)=f(a\operatorname{ch}\psi,\ \pm a\omega\operatorname{sh}\psi)=af(\operatorname{ch}\psi,\ \pm\omega\operatorname{sh}\psi)$$

$$f_1(a,\psi)=u_1 f_u(\ldots)+\left(A_1\operatorname{ch}\psi+aB_1\operatorname{sh}\psi\pm\omega\frac{\partial u_1}{\partial\psi}\right)f_{u'}(\ldots)-(A_1\dot{A}_1+aB_1^2\operatorname{ch}\psi)-$$
$$-(2A_1B_1+A_1\dot{B}_1 a)\operatorname{sh}\psi\mp 2\omega A_1\frac{\partial^2 u_1}{\partial a\,\partial\psi}\mp 2\omega B_1\frac{\partial^2 u_1}{\partial\psi^2}$$

Now let us assume (for the sake of simplicity) $f(u,v)=\dfrac{u^3}{u^2+v^2}$ and $\omega=\pm 1$. Then

$$f_0(a,\psi)=a\frac{\operatorname{ch}^3\psi}{\operatorname{ch}^2\psi+\operatorname{sh}^2\psi}=\frac{a}{4}\frac{(1+e^{2\psi})^3}{e^{\psi}(1+e^{4\psi})}$$

With $z=e^{\Psi}=\operatorname{ch}\psi+\operatorname{sh}\psi$ we get

$$R=\frac{(1+z^2)^3}{z(1+z^4)}=\frac{1}{z}+3z+2(z^3-z^5-z^7+z^9+z^{11}--++\ldots)$$

$$|z|<1, z\neq 0, \qquad \psi<0$$

By the use of (5. 2. 4)—(5. 2. 5) and comparison of the coefficients

$$v_0=0,\quad v_n=w_n=\begin{cases}\pm\dfrac{a}{2}\dfrac{1}{n^2-1}, & n=3,5,7,\ldots\\ 0, & n=2,4,6,\ldots\end{cases}$$

$$A_1=\pm\frac{3}{8}a,\quad B_1=\pm\frac{3}{8}$$

Choose the values of the indetermined v_1 and w_1 as zero. Then we have in first approximation

$$a = Ce^{\pm\frac{3}{8}\varepsilon t}, \quad \psi = \pm\left[\left(1+\frac{3}{8}\varepsilon\right)t+\alpha\right]$$

But $\psi < 0$ implies

$$a = Ce^{-\frac{3}{8}\varepsilon t}, \quad \psi = -\left[\left(1+\frac{3}{8}\varepsilon\right)+\alpha\right], \qquad (\alpha > 0)$$

where C and α are arbitrary. In first approximation all this results in

$$u = C_1 e^t + C_2 e^{-\left(1+\frac{3}{4}\varepsilon\right)t}, \quad C_1 = \frac{C}{2}e^{\alpha}, \quad C_2 = \frac{C}{2}e^{-\alpha}$$

and

$$u_1 = \frac{a}{16}e^{3\psi}\left(1-\frac{1}{3}e^{2\psi}-\frac{1}{6}e^{4\psi}+\frac{1}{10}e^{6\psi}+\frac{1}{15}e^{8\psi}- - + + \ldots\right)$$

The Remark at the end of 5. 2 is valid here too.

5. 4. *II. Oscillatory (periodic) solutions. (Constant coefficients)*

Consider equation

(5. 4. 1) $$u'' + \omega^2 \frac{u^3}{u^2+\varepsilon u'^2} = 0, \qquad (0<\varepsilon\ll 1)$$

all the real solutions of which are periodic (s. 9° of 1.). The substitution $u = e^{\lambda t}$ gives the complex periodic solutions only and equation (1. 2) furnishes the non-vanishing pieces of the real periodic (oscillatory) solutions. Let us determine them as a whole for $t \geqq 0$ expanded in powers of ε. Corresponding to the method of [7] assume $u(t)$ in the form

(5. 4. 2) $$u(t) = a(t)\cos\psi(t) + \varepsilon u_1(a, \psi) + \varepsilon^2 u_2(a, \psi) + \ldots$$

where

(5. 4. 3) $$\begin{aligned} a' &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots \\ \psi' &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots \end{aligned}$$

and u_i are periodic in ψ with period 2π. Determine u_i, A_i, B_i $(i = 1, 2, \ldots)$ in such a way that $u(t)$ given by (5. 4. 2) be a solution of (5. 4. 1).

The first and corrected first approximation (viz. $u_1(a, \psi) \equiv 0$) is as follows

$$u = a\cos(\bar{\omega}t+\alpha), \quad \bar{\omega} = \omega\left(1-\varepsilon\frac{\omega}{6}\right)$$

where a and α are arbitrary constants. In first approximation the term $\varepsilon u'^2$ in (5. 4. 1) causes a slight change of the frequency only.

5. 5. *Perturbation of the linear oscillatory equation by a half-linear term (constant coefficients)*

As an example let us regard equation

$$u''+\omega^2 u = \varepsilon\frac{u^3}{u^2+u'^2}, \quad (|\varepsilon|\ll 1) \tag{5. 5. 1}$$

and take for the sake of simplicity $\omega=1$. Then the method just used leads to the following corrected first approximation

$$u = a\cos(\bar{\omega}t+\alpha)-\frac{1}{32}\varepsilon a\cos 3(\bar{\omega}t+\alpha), \quad \bar{\omega} = 1-\frac{3}{8}\varepsilon \tag{5. 5. 2}$$

where a and α are arbitrary constants.

In this approximation the perturbation involves the emerging of the third harmonics and a change in the frequency.

The second approximation reads as

$$u = a\cos(\tilde{\omega}t+\alpha)-\frac{1}{32}\varepsilon a\cos 3(\tilde{\omega}t+\alpha)$$

$$\left(\tilde{\omega} = 1-\frac{3}{8}\varepsilon-\frac{17}{128}\varepsilon^2\right)$$

Here a and α are arbitrary.

5. 6. *Perturbation of the linear oscillatory equation by a half-linear periodic term*

A) *Case of non-resonance*

Let us regard equation

$$u''+u+\varepsilon\sin\nu t\frac{u^3}{u^2+u'^2} = 0 \quad (|\varepsilon|\ll 1) \tag{5. 6. 1}$$

where ν is not an integral number (an integral multiple of the eigenfrequency of the linear equation).

The first approximation of the solution of (5. 6. 1) given by the method of [7] is as follows

$$u = a\cos(t+\alpha)+\varepsilon\frac{a}{4}\left\{\frac{\sin[(3+\nu)t+3\alpha]}{1-(\nu+3)^2}-\frac{\sin[(3-\nu)t+3\alpha]}{1-(\nu-3)^2}\right\}$$

Here a and α are arbitrary constants. The appearing of combination frequencies are to be observed here. The application of the method is significantly facilitated by the homogeneity in u and u' of the perturbator term.

On the other hand for the equation

$$v''+v-\varepsilon\frac{v^3}{v^2+v'^2} = E\sin\omega t$$

which by the transformation

$$v = u + U \sin \nu t, \quad U = \frac{E}{1-\nu^2}$$

assumes the form

$$u'' + u - \varepsilon \frac{(u+U\sin \nu t)^3}{(u+U\sin \nu t)^2 + (u' + U\nu \cos \nu t)^2} = 0$$

the method involves no simplification.

B) *Case of resonance*

By taking in (5. 6. 1) $\nu = \omega = 1$ we have as first (virtually corrected first) approximation

$$u = a\cos(t+\alpha) + \varepsilon \frac{a}{8}\left[-3\sin\alpha - \sin(2t+\alpha) + \frac{1}{3}\sin(2t+3\alpha) - \frac{1}{15}\sin(4t+3\alpha)\right]$$

where a and α are arbitrary constants.

The general case involving resonance, transition to nonresonance and nonresonance may be treated too. Also a second approximation can be determined.

REMARK. The case of „large parameter" may be reduced to that of small parameters. E.g. a term

$$\frac{u^3}{u^2 + \lambda u'^2}$$

with large $\lambda \gg 1$ by the substitution $\lambda = \frac{1}{\varepsilon}$ $(\varepsilon \ll 1)$ can be written as

$$\frac{\varepsilon u^3}{\varepsilon u^2 + u'^2}$$

or a term

$$\lambda \frac{u^3}{u^2 + u'^2} \quad \text{as} \quad \frac{u^3}{\varepsilon u^2 + \varepsilon u'^2}$$

and the corresponding equation can be treated on previous lines.

This fact is a significant favour of these equations opposed to other types of nonlinear equations.

Otherwise the experience and a new deduction of some equations of physics can only decide on the applicability of these equations. Many traditional deductions must be revised in this respect.

6. Half-linear systems of the first order

6.1. EXAMPLES. First let us regard a few examples.

1°

$$u' = \frac{u^3}{u^2 + u'^2}$$

Assume $u = Ce^{\lambda t}$, obtaining $\lambda^3 + \lambda - 1 = 0$ which has one unique real $0,6 < \lambda_1 < 0,7$ root. So the general (real) solution is $u = Ce^{\lambda_1 t}$.

2°

$$u' = \frac{uu'^2}{\varepsilon u^2 + u'^2}.$$

Then

$$\lambda = \frac{1 \pm \sqrt{1-4\varepsilon}}{2} \quad \text{and} \quad \lambda = 0$$

The solution is real if $\varepsilon \leqq \frac{1}{4}$ (2 or 1 non-constant exponential solutions respectively).

3° The system (e.g. the parametric equation of a movement)

$$u' = \frac{u^3}{u^2 + u'^2} \qquad v' = \frac{v^3}{v^2 + v'^2}$$

has the general solution

$$u = C_1 e^{\lambda_1 t}, \quad v = C_2 e^{\lambda_1 t}$$

where λ_1 is the value in 1°. Hence $v = ku$ (k = const). The „paths" are straight lines.

4° Combining 1° and 2° *b*)

$$u' = \frac{u^3}{u^2 + u'^2}, \quad v' = \frac{v^2 + 4v'^2}{4v}$$

$$u = C_1 e^{\lambda_1 t}, \quad v = C_2 e^{\lambda_2 t} \quad \left(\lambda_2 = \frac{1}{2}\right)$$

and the movement takes place on the „parabolic" paths

$$\frac{u^{\lambda_2}}{v^{\lambda_1}} = \text{const}$$

6. 2. *The general form*

$$u' = af(u, u') + bF(v, v')$$

$$v' = cg(u, u') + dG(v, v')$$

of these systems, where f, g, F, G are half-linear terms, a, b, c, d functions of t, may be treated in the same way provided a, b, c, d are constant. The signs of f, g, F, G are not necessarily prescribed. The substitution

$$u = Ae^{\lambda t}, \quad v = Be^{\lambda t}$$

gives the homogeneous linear equation system

$$\lambda A = aAf(1, \lambda) + bBF(1, \lambda)$$

$$\lambda B = cAg(1, \lambda) + dBG(1, \lambda)$$

for A and B which has a non-trivial solution if and only if $K(\lambda)=0$ (characteristic equation) where

$$K(\lambda)=\begin{vmatrix} af(1,\lambda)-\lambda & bF(1,\lambda) \\ cg(1,\lambda) & dG(1,\lambda)-\lambda \end{vmatrix}$$

EXAMPLES.

1) $$f=g=\frac{u^3}{u^2+u'^2},\quad F=G=\frac{v'^3}{v^2+v'^2}$$

Then

$$K(\lambda)=\begin{vmatrix} a-\lambda(1+\lambda^2) & b\lambda^3 \\ c & d\lambda^3-\lambda(1+\lambda^2) \end{vmatrix}$$

and $K(\lambda)=0$ is an equation of the sixth degree.

2) If $f=g=F=G$ then

$$K(\lambda)=\begin{vmatrix} a-\mu & b \\ c & d-\mu \end{vmatrix},\quad \mu=\frac{\lambda}{f(1,\lambda)}\qquad (f(1,\lambda)\neq 0)$$

and μ is the characteristic root of the corresponding linear system. E.g. for $f(u,v)=$ $=\frac{u^3}{u^2+v^2}$ we have $\mu=\lambda(1+\lambda^2)$ which has always a real root for both (real) values of μ.

6. 3. There is *another generalization* of the linear system, too. (s. [1]). Namely

$$\text{(6. 3. 1)}\qquad \begin{aligned} u' &= af(u,v)+bF(u,v), \quad \operatorname{sg} f=\operatorname{sg} g=\operatorname{sg} u \\ v' &= cg(u,v)+dG(u,v), \quad \operatorname{sg} F=\operatorname{sg} G=\operatorname{sg} v \end{aligned}$$

The exponential substitution (taking a, b, c, d constant)

$$\text{(6. 3. 2)}\qquad u=Ae^{\lambda t},\quad v=Be^{\lambda t}$$

leads now to the system

$$\text{(6. 3. 3)}\qquad \begin{aligned} A\lambda &= af(A,B)+bF(A,B) \\ B\lambda &= cg(A,B)+dG(A,B) \end{aligned}$$

where A, B, λ are to be determined and which in general is not linear in A, B. Since $A^2+B^2>0$, suppose e.g. $A\neq 0$ then

$$\frac{B}{A}=\frac{cg\left(1,\frac{B}{A}\right)+dG\left(1,\frac{B}{A}\right)}{af\left(1,\frac{B}{A}\right)+bF\left(1,\frac{B}{A}\right)}=H\left(\frac{B}{A}\right),\quad H(z)=\frac{cg(1,z)+dG(1,z)}{af(1,z)+bF(1,z)}$$

i.e. equation $z=H(z)$ must be satisfied by a value $z=\frac{B}{A}$ giving the above exponential functions. One of A and B may be chosen arbitrarily, then (6. 3. 3) gives λ provided z or λ exist at all. The paths are the straight lines $\frac{v}{u}=\text{const.}$

For solutions where $u=u(t)$ is monotone the system (6. 3.1) is equivalent to the equation

(6. 3. 4)
$$\frac{dv}{du} = H\left(\frac{v}{u}\right)$$

which by the substitution $z=\frac{v}{u}$ can be turned into

(6. 3. 5)
$$u\frac{dz}{du} = H(z)-z$$

For an interval (or a set) where $H(z)\neq z$ (i.e. for z not corresponding to the above values B/A) we have

$$\frac{dz}{H(z)-z} = \frac{du}{u}$$

and $z=\frac{v}{u}$ can be determined. This process gives the non-exponential solutions (if they exist).

REMARK. In linear case (6. 3. 3) reads as

$$\lambda A = aA+bB$$
$$\lambda B = cA+dB$$

and $z=\frac{B}{A}$ satisfies

$$z = \frac{c+dz}{a+bz} \quad \text{oz} \quad bz^2+(a-d)z-c = 0$$

whence

$$z = \frac{B}{A} = \frac{-(a-d)\pm\sqrt{(a+d)^2-4\Delta}}{2b}, \quad \Delta = ad-bc$$

Real solution exists if $(a+d)^2\geqq 4\Delta$, etc.

An alternative well-known way (followed usually) consists of determining λ from

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

etc.

6. 4. *Distribution of the zeros of the solution of* (6. 3.1)

According to the comparison theorems (s. [1]) system (6. 3.1) is oscillatory for $t\geqq 0$ provided e.g. $b(t)\geqq b=\text{const}>0$, $c(t) \leqq -c=\text{const}<0$, $a(t)=d(t)\equiv 0$ for $t\geqq 0$.

First regard the linear system

(6. 4. 1)
$$u' = a(t)u+b(t)v$$
$$v' = c(t)u+d(t)v$$

which is then also oscillatory. By the substitution

$$u = \varrho\sin\varphi, \quad v = \varrho\cos\varphi$$

we have for the continuous (even more continuously derivable) function $\varphi(t)$

$$\varphi'(t) = b\cos^2\varphi + \frac{1}{2}(a-d)\sin 2\varphi - c\sin^2\varphi.$$

which can be made univalent by (e.g.) $0 \leqq \varphi(0) < \pi$.

For a solution $u(t), v(t)$ resp. $\varrho(t), \varphi(t)$ it holds the assertion: $\varphi(t)$ passes the zeros of $u(t)$ increasingly consequently just once and has the value $k\pi$ at the k-th zero for $t \geqq 0$ of $u(t)$. Therefore denoting the number of zeros of $u(t)$ in $0 < t \leqq T$ by N we have

$$N\pi \leqq \varphi(T) = \varphi(0) + \frac{1}{2}\int_0^T (a-d)\sin 2\varphi\, dt + \int_0^T (b+c)\cos^2\varphi\, dt -$$

$$-\int_0^T c\, dt < (N+1)\pi$$

whence

$$N\pi \leqq \pi + \int_0^T \left(\frac{1}{2}|a-d| + |b+c| + |c|\right) dt$$

Or

$$\varphi' = b - (c+b)\sin^2\varphi + \frac{1}{2}(a-d)\sin 2\varphi$$

and

$$N\pi \leqq \varphi(T) = \varphi(0) + \int_0^T b(t)\, dt - \int_0^T (c+b)\sin^2\varphi\, dt$$

$$+\frac{1}{2}\int_0^T (a-d)\sin 2\varphi\, dt < (N+1)\pi$$

which gives

$$\left| N\pi - \int_0^T b(t)\, dt \right| \leqq \pi + \int_0^T |c+b|\, dt + \frac{1}{2}\int_0^T |a-d|\, dt$$

what is an immediate generalization of a known result (s. [6], p. 348, and its extension in 8) of 4. in the present paper).

Take now the system (6. 3.1). From

$$\varphi' = bF(\sin\varphi, \cos\varphi)\cos\varphi + af(\dots)\cos\varphi - dG(\dots)\sin\varphi - cg(\dots)\sin\varphi =$$

$$= bF(\operatorname{tg}\varphi, 1)\cos^2\varphi + [af(1, \operatorname{ctg}\varphi) - dG(\operatorname{tg}\varphi, 1)]\frac{1}{2}\sin 2\varphi - cg(1, \operatorname{ctg}\varphi)\sin^2\varphi =$$

$$= b\cos^2\varphi + \frac{1}{2}(a-d)\sin 2\varphi - c\sin^2\varphi +$$

$$+ b[F(\operatorname{tg}\varphi, 1) - 1]\cos^2\varphi - c[g(1, \operatorname{ctg}\varphi) - 1]\sin^2\varphi +$$

$$+\frac{1}{2}(a-d)[f(1, \operatorname{ctg}\varphi) - 1]\sin 2\varphi + \frac{1}{2}d[f(1, \operatorname{ctg}\varphi) - G(\operatorname{tg}\varphi, 1)]\sin 2\varphi$$

Suppose the expressions in absolute values in the square brackets are always less than A, then we have in the same way as in the above linear case

$$\left|N\pi-\int_0^T b(t)\,dt\right| \leqq \pi+\int_0^T\left(\frac{1}{2}|a-d|+|c+b|\right)dt+$$

$$+A\int_0^T\left(b+|c|+\frac{1}{2}|a-d|+\frac{1}{2}|d|\right)dt$$

REMARK 1. The zeros of u and v separate each other (s. [1]). By following the line of 2. one can easily construct a comparison theorem concerning a linear and a half-linear system.

REMARK 2. For the linear system (6. 4. 1) it is valid the

THEOREM. *If a, b, c, d are continuous for $t\geqq t_0$ and*

$$-\infty<\int_0^T a(t)\,dt<\infty,\quad \int^\infty c(t)\,dt=-\infty,\quad a(t)=d(t),\quad b(t)\geqq\delta=\text{const}>0$$

for $t\geqq t_0$, then every solution $u(t), v(t)$ of (6. 4. 1) *is oscillatory for $t\geqq t_0$* (s. [8]).

PROOF. In the opposite case $u(t)\neq 0$ and e.g. >0 for $t>t_1$ with some $t_1\geqq t_0$. Then $z=\frac{u}{v}$ satisfies the Riccati-type equation

$$z'=c-bz^2,\quad t>t_1$$

whence

$$c\geqq z'\quad\text{or}\quad \int^t c(t)\,dt\geqq z(t)+\text{const}$$

i.e. $z=\frac{v}{u}\to-\infty$ as $t\to\infty$ and $v<0$ for $t>t_2$ $(t_2\geqq t_1)$.

By $(6.\,4.\,1)_1$

$$\frac{u'}{u}=a+b\frac{v}{u},\quad u=u(t_1)e^{\int_{t_1}^t a\,dt}\cdot e^{\int_{t_1}^t b\frac{v}{u}dt}\to 0\quad\text{as}\quad t\to\infty$$

Take now two linearly independent solutions u_1, v_1 and u_2, v_2. On account of the comparison theorems both u_1 and u_2 are non-oscillatory, consequently $u_i\to 0$ $(i=1, 2)$ as $t\to\infty$. The Wronskian

$$W=\begin{vmatrix} u_1 & u_2\\ v_1 & v_2\end{vmatrix}\neq 0$$

satisfies $W'=2aW$ which implies

$$W\leqq W_0 e^{2\int_{t_0}^t a(t)\,dt}$$

and so $|W(t)|\geqq\alpha$ for $t\geqq t_0$ for some $\alpha=\text{const}>0$.

Introduce the functions $\varrho=\varrho(t)$, $\varphi=\varphi(t)$ by

$$u_1 = \varrho \sin \varphi, \quad u_2 = \varrho \cos \varphi, \quad \operatorname{tg} \varphi = \frac{u_1}{u_2}, \quad \varrho^2 = u_1^2 + u_2^2$$

then

$$\left(\frac{u_1}{u_2}\right)' = -\frac{bW}{u_2^2} = -\frac{bW}{\varrho^2 \cos^2 \varphi} = \frac{1}{\cos^2 \varphi} \varphi' \quad \text{or} \quad \varphi' = -\frac{bW}{\varrho^2}$$

whence $\varphi' \to \infty$, $\varphi \to \infty$ as $t \to \infty$ in contradiction with the non-oscillatory character of u_1 and u_2.

This theorem has no simple extension to half-linear systems.

6. 5. *Asymptotic behaviour of the solutions*

(Case of a small parameter)

The facility of the method will be exhibited on an example only where a linear system is perturbed by half-linear terms

(6. 5. 1)
$$\begin{aligned} u' &= v + \varepsilon f(u, v) \\ v' &= -u + \varepsilon g(u, v) \end{aligned} \qquad (|\varepsilon| \ll 1)$$

Let us assume $u(t)$ and $v(t)$ in the forms

(6. 5. 2)
$$\begin{aligned} u &= a(t) \cos \psi(t) + \varepsilon u_1(a, \psi) + \varepsilon^2 u_2(a, \psi) + \dots \\ v &= -a(t) \sin \psi(t) + \varepsilon v_1(a, \psi) + \varepsilon^2 v_2(a, \psi) + \dots \end{aligned}$$

$(u_i, v_i$ periodic)

Suppose furthermore

(6. 5. 3)
$$\begin{aligned} a' &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots \\ \psi' &= 1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \dots \end{aligned}$$

Then

$$u' = -a \sin \psi + \varepsilon \left(A_1 \cos \psi - a B_1 \sin \psi + \frac{\partial u_1}{\partial \psi} \right) + \varepsilon^2 \dots$$

$$v' = -a \cos \psi + \varepsilon \left(-A_1 \sin \psi - a B_1 \cos \psi + \frac{\partial v_1}{\partial \psi} \right) + \varepsilon^2 \dots$$

Developing $f(u, v)$ in Taylor series

$$f(u, v) = f(a \cos \psi, \ -a \sin \psi) + f_u(\dots)(\varepsilon u_1 + \varepsilon^2 u_2 + \dots) + f_v(\dots)(\varepsilon v_1 + \varepsilon v_1^2 + \dots) + \dots$$

Take now

$$f(u, v) = \frac{u^3}{u^2 + v^2}, \quad g(u, v) = \frac{v^3}{u^2 + v^2}$$

then

$$\varepsilon f(u, v) = \varepsilon a \cos^3 \psi + \varepsilon^2 \dots, \quad \varepsilon g(u, v) = -\varepsilon a \sin^3 \psi + \varepsilon^2 \dots$$

Putting all these in (6. 5.1) and comparing the coefficients of the same powers of ε on both sides we have

$$\frac{\partial u_1}{\partial \psi}-v_1=-A_1\cos\psi+aB_1\sin\psi+\frac{a}{4}(3\cos\psi+\cos 3\psi)$$

$$\frac{\partial v_1}{\partial \psi}+u_1=A_1\sin\psi+aB_1\cos\psi-\frac{a}{4}(3\sin\psi-\sin 3\psi) \tag{6. 5. 4}$$

. . . .

. . . .

Assume u_1 and v_1 in the forms

$$u_1=\sum_{n=0}^{\infty}(\alpha_n(a)\cos n\psi+\beta_n(a)\sin n\psi)$$

$$v_1=\sum_{n=0}^{\infty}(\gamma_n(a)\cos n\psi+\delta_n(a)\sin n\psi)$$

and put them in (6. 5. 4). The comparison of the coefficients of $\cos n\psi$ and $\sin n\psi$ gives

$$\alpha_n=\beta_n=\gamma_n=\delta_n=0,\qquad n=2,4,5,6,7,\ldots$$

$$\alpha_3=\delta_3=0,\quad \beta_3=-\gamma_3=\frac{a}{16}$$

$$\beta_1=\gamma_1,\quad \alpha_1=-\delta_1$$

and

$$B_1=0,\quad A_1=\frac{3a}{4}$$

In order that $a(t)$ be the complete amplitude of the basic harmonic take (the othervise indefinite) $\alpha_1,\beta_1,\gamma_1,\delta_1$ as zero. Then *in first approximation*

$$a=Ce^{3/4\,\varepsilon t},\quad \psi=t+\alpha$$

where C and α are arbitrary and so

$$u=Ce^{3/4\,\varepsilon t}\cos(t+\alpha),\quad v=-Ce^{3/4\,\varepsilon t}\sin(t+\alpha)$$

$$u_1=\frac{a}{16}\sin 3\psi=\frac{C}{16}e^{3/4\,\varepsilon t}\sin 3(t+\alpha)$$

$$v_1=-\frac{a}{16}\cos 3\psi=-\frac{C}{16}e^{3/4\,\varepsilon t}\cos 3(t+\alpha)$$

The „corrected first approximation" is as follows

$$u=Ce^{3/4\,\varepsilon t}\left[\cos(t+\alpha)+\frac{\varepsilon}{16}\sin 3(t+\alpha)\right]$$

$$v=-Ce^{3/4\,\varepsilon t}\left[\sin(t+\alpha)+\frac{\varepsilon}{16}\cos 3(t+\alpha)\right]$$

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ON THE AUTOMORPHISM GROUP OF A COMPOSITE GRAPH

by

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Many important binary operations have been defined for graphs; see [4]. By a „composite graph" we mean a graph which can be represented as a combination of other graphs with respect to such operations. The question then arises: How can the (automorphism) group of a composite graph be expressed as a permutation group in terms of the groups of its various constituents?

We emphasize that we are interested here in the group of a graph considered as a permutation group rather than just an abstract group. If two permutation groups A and B acting on object sets X and Y respectively are *isomorphic* (as abstract groups), we write $A \cong B$. Further, if they are not only abstractly isomorphic via a mapping $\sigma: A \to B$, but in addition there exists a 1—1 correspondence $\varrho: X \to Y$ such that $\varrho\alpha x = \sigma\alpha\varrho x$, for each $\alpha \in A$ and $x \in X$, then we write $A \equiv B$ and say that A and B are *identical* permutation groups. Our notation and terminology for operations on permutation groups follows that used in [3].

The purpose of this note is to point out that, as a consequence of a theorem of SABIDUSSI [5], the group of the cartesian product $G \times \ldots \times G$ of n disjoint copies of a connected, cartesian-prime graph G is the exponentation group $[\Gamma(G)]^{S_n}$, introduced by HARARY [2]. Further, the group of any connected graph is a direct product of exponentation groups. Thus the group of any graph is a direct sum of wreath products of direct products of exponentiation groups. Of course this is most noticeable when a graph is expressible as the union of several graphs which are in turn the cartesian product of other graphs.

We first consider the group of the union $nG = G \cup \ldots \cup G$ of n disjoint copies of a connected graph G. As observed by FRUCHT [1], this group is simply the wreath product $S_n[\Gamma(G)]$ of the symmetric group S_n around $\Gamma(G)$, the group of the graph G.

THEOREM 1. *If G is a connected graph, then*

$$\Gamma(nG) \equiv S_n[\Gamma(G)].$$

If G and H are graphs which have no common components, then $\Gamma(G \cup H) \equiv \Gamma(G) + \Gamma(H)$, the direct sum of the groups $\Gamma(G)$ and $\Gamma(H)$. Any graph can be expressed as a union of graphs which have no components in common. Combining these facts with Theorem 1, we have the following more general result for an arbitrary graph.

COROLLARY 1. If $G_1, G_2, \ldots, G_n$ are pairwise non-isomorphic, connected graphs

and
$$G = \bigcup_{k=1}^{n} m_k G_k,$$

then

$$\Gamma(G) = \sum_{k=1}^{n} S_{m_k}[\Gamma(G_k)].$$

It follows that the group of any (disconnected) graph may be expressed as the sum of symmetric wreath products.

Now let G_1 and G_2 be graphs whose sets of points are V_1 and V_2 respectively. The *cartesian product* of G_1 and G_2, denoted $G_1 \times G_2$, is defined in [5] as follows. The set of points of $G_1 \times G_2$ is the set $V_1 \times V_2$. Two points (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent whenever $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1. A *cartesian-prime* graph is nontrivial and is not the cartesian product of two nontrivial graphs. SABIDUSSI [5] has shown that every graph (hence in particular, every connected graph) can be factored uniquely (up to order) as a product of prime graphs.

Furthermore, the following theorem of [5] characterizes the group of the cartesian product of prime graphs.

THEOREM 2. *Let $G_1, G_2, \ldots, G_n$ be disjoint connected, cartesian-prime graphs. Then the group of the union $G_1 \cup G_2 \cup \ldots \cup G_n$ and the group of the product $G_1 \times G_2 \times \ldots \times G_n$ are isomorphic: $\Gamma(G_1 \cup \ldots \cup G_n) \cong \Gamma(G_1 \times \ldots \times G_n)$.*

We denote the cartesian product $G \times \ldots \times G$ of n copies of a connected cartesian-prime graph G by G^n. It is easy to see that the exponentiation group $[\Gamma(G)]^{S_n}$ is identical to a subgroup of $\Gamma(G^n)$. From Theorem 2 it quickly follows that this subgroup is $\Gamma(G^n)$ itself. Thus, corresponding to Theorem 2, we have the following result.

COROLLARY 2. If G is a connected, cartesian-prime graph, then

$$\Gamma(G^n) = [\Gamma(G)]^{S_n}.$$

It also follows from Theorem 2 that if G and H are connected graphs which are relatively prime with respect to the cartesian product, then $\Gamma(G \times H) \equiv \equiv \Gamma(G) \times \Gamma(H)$, the cartesian product of the groups $\Gamma(G)$ and $\Gamma(H)$ (see [2]). Thus the more general result for arbitrary, connected graphs can be stated as follows. Note that we have used Π for the cartesian product of permutation groups or of graphs, depending on the context.

COROLLARY 3. If $G_1, G_2, \ldots, G_n$ are non-isomorphic, connected, cartesian-prime graphs then

$$\Gamma\left(\prod_{k=1}^{n} G_k^{m_k}\right) \equiv \prod_{k=1}^{n} [\Gamma(G_k)]^{S_{m_k}}.$$

Combining Corollaries 1 and 3, we find that to express the group of a composite graph G in terms of operations on permutation groups, we may follow the procedure:

1. First construct the complement $\bar{G}$ and then choose from among G and $\bar{G}$ a graph with the maximum number of components. Since it is well known that $\Gamma(\bar{G}) \equiv \Gamma(G)$, this is admissible.
2. Apply Corollary 1 to express $\Gamma(G)$ in terms of the groups of these components.
3. If one can factor each component into the product of cartesian-prime graphs, we can then apply Corollary 3.

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ON L_p-APPROXIMATION OF FUNCTIONS WHOSE m^{th} DERIVATIVE IS OF BOUNDED VARIATION

by

RONALD DEVORE

1. Introduction. Let f be an element of $L_p[-1,1]$, $1 \leqq p \leqq \infty$, and S and T be subsets of $L_p[-1,1]$. Then $E_p(f,T) = \inf_{g \in T} \|f-g\|_p$ is the error in approximating f by elements of T in the $L_p[-1,1]$ norm and $E_p(S,T) = \sup_{f \in S} E_p(f,T)$ is the error in approximating elements of S by elements of T in the $L_p[-1,1]$ norm.

A problem of particular interest is the determination of $E_p(S,T)$ when S is characterized by some structural property of its elements and T is one of the classes $\mathbf{P}_n$ of algebraic polynomials of degree $\leqq n$ or $\mathbf{T}_n$ of trigonometric polynomials of degree $\leqq n$. The first result of this type is the classical result of J. FAVARD [1] which for the interval $[-\pi, \pi]$ instead of $[-1,1]$ can be stated as follows:

If $\mathbf{W}_m$ is the class of those 2π periodic functions f for which $f^{(m-1)}$ is absolutely continuous and $|f^{(m)}(x)| \leqq 1$ a.e., then

$$E_\infty(\mathbf{W}_m, \mathbf{T}_n) = \frac{K_m}{n^m}, \quad \text{where} \quad K_m = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^{(m+1)j}}{(2j+1)^{m+1}}.$$

Another result in this direction, for L_1 approximation, was obtained by S. NIKOLSKI [2]. Let $\mathbf{A}_m$ be the class of those functions f for which $f^{(m-1)}$ is absolutely continuous on $[-1,1]$ and $f^{(m)}$ is equivalent to a function g whose total variation on $[-1,1]$ is $\leqq 1$. Then for $n \geqq m$, $m=1,2,\ldots$

$$E_1(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_1(m,\mathbf{P}_n)}{m!} E_1(x_+^m, \mathbf{P}_n) \tag{1.1}$$

where

$$(x-a)_+^m = \begin{cases} (x-a)^m & \text{for} \quad x > a \\ 0 & \text{for} \quad x \leqq a \end{cases} \tag{1.2}$$

and

$$M_p(m,\mathbf{P}_n) = \sup_{|\alpha| \leqq 1} \frac{E_p((x-a)_+^m, \mathbf{P}_n)}{E_p(x_+^m, \mathbf{P}_n)} \tag{1.3}$$

Nikolski [3] has also studied the functions $M_1(m,\mathbf{P}_n)$ and $E_1(x_+^m, \mathbf{P}_n)$. He has determined $E_1(x_+^m, \mathbf{P}_n)$ explicitly when m is an odd positive integer and has proved the following assymptotic formulae.

$$E_1(x_+^m, \mathbf{P}_n) = \frac{c_m}{n^{m+2}} + O\left(\frac{\log n}{n^{m+2}}\right) \qquad (n \to \infty) \tag{1.4}$$

$$M_1(m, \mathbf{P}_n) = 1 + O\left(\frac{\log n}{n}\right) \qquad (n \to \infty). \tag{1.5}$$

In this paper, we shall study the L_p approximation $(1 \leqq p \leqq \infty)$ of the class $\mathbf{A}_m$ by $\mathbf{P}_n$. First of all, we shall show that the result of NIKOLSKI (1.1) is valid for $1 \leqq p \leqq \infty$. More precisely, in Theorem 1 we show that for $1 \leqq p \leqq \infty$ and $n \geqq m$, $m = 1, 2, \ldots$

$$E_p(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n).^*$$

Next, we shall solve explicitly the problem of best L_1 approximation to the functions $x_+^m, m = 1, 2, \ldots,$ on the interval $[-1, 1]$ by algebraic polynomials by means of more general results which are interesting in themselves. In Theorem 2, we shall determine the polynomials of best L_1 approximation on $[-1, 1]$ of degree $\leqq 2n-2, 2n-1$ to even functions of the form $h(x^2)$ with $h^{(n)}$ of constant sign on $(0, 1)$.

A similar result for odd functions is Theorem 3 which determines the polynomials of best L_1 approximation on $[-1, 1]$ of degree $\leqq 2n-1, 2n$ to functions of the form $xh(x^2)$ with $h^{(n)}$ of constant sign on $(0, 1)$.

Finally, using the constructive method employed in the proof of Theorem 1 and the explicit determination of polynomials of best L_1 approximation to the functions x_+^m given by Theorems 2 and 3, we shall show that for each function f in $\mathbf{A}_m$ there is a polynomial

$$P_n(x) = A_0(n) + A_1(n)x + \ldots + A_n(n)x^n$$

satisfying:

$$1^\circ \quad \|f - P_n\|_1 \leqq \frac{C_1}{n^{m+1}}$$

$$2^\circ \quad |A_k(n)| \leqq C_2 3^n \qquad k = 0, 1, \ldots, n$$

where C_1 and C_2 depend only on m.

The same result was obtained by J. KOREVAAR [4] for the class $\mathbf{K}_m$ of those functions f for which $f^{(m-1)}$ is absolutely continuous on $[-1, 1]$ and $f^{(m)}$ is continuous except for a finite number of jump discontinuities on $[-1, 1]$.

2. 1. Approximation of $\mathbf{A}_m$ by $\mathbf{P}_n$ in the $\mathbf{L}_p$ norm.

THEOREM 1. *If* $1 \leqq p \leqq \infty$, *then for* $n \geqq m$, $m = 1, 2, \ldots$

$$E_p(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n).$$

PROOF. Let $f \in \mathbf{A}_m$ and $g = f^{(m)}$ a.e. where the variation of g on $[-1, 1]$ is $\leqq 1$. If $\varepsilon > 0$, there is a partition $-1 = x_0 < x_1 < \ldots < x_r = 1$ for which

$$\int_{-1}^{1} |g(x) - l_0(x)|\, dx \leqq \frac{\varepsilon}{2^{\frac{1}{p}+m-1}}$$

* The assymptotic behaviour of $M_p(m, \mathbf{P}_n)$ and $E_p(x_+^m, \mathbf{P}_n)$ has been given by RAICIN in *Dokl. Trans. A. M. S.* **6** (1965) 1171—1174.

where l_0 is the step function which has the value $g(x_k)$ on $(x_k, x_{k+1}]$. We have

$$l_0(x) = g(x_0) + \sum_{k=1}^{r} (g(x_k) - g(x_{k-1}))(x - x_k)_+^0 .$$

We recursively define for $j = 1, 2, \ldots, m$

$$l_j(x) = \int_{-1}^{x} l_{j-1}(t)\, dt + f^{(m-j)}(-1).$$

Then for $x \in [-1, 1]$, we have

$$|f^{(m-j)}(x) - l_j(x)| = \left| \int_{-1}^{x} (f^{(m-j+1)}(t) - l_{j-1}(t))\, dt \right| \leqq$$

$$\leqq \int_{-1}^{1} |f^{(m-j+1)}(t) - l_{j-1}(t)|\, dt \leqq \frac{\varepsilon}{2^{\frac{1}{p}+m-j}}$$

for $j = 1, 2, \ldots, m$. Thus, $\|f - l_m\|_p \leqq \varepsilon$.

We have

$$l_m(x) = P(x) + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1}))(x - x_k)_+^m \tag{2.1}$$

where P is a polynomial of degree $\leqq m$. Let $P_k \in \mathbf{P}_n$ satisfy

$$\|(x - x_k)_+^m - P_k(x)\|_p = E_p((x - x_k)_+^m, \mathbf{P}_n).$$

Then for $Q = P + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1}))\, P_k$, we have $Q \in \mathbf{P}_n$ and

$$\|l_m - Q\|_p \leqq \frac{1}{m!} \sum_{k=1}^{r} |g(x_k) - g(x_{k-1})| \quad \|(x - x_k)_+^m - \mathbf{P}_k(x)\|_p$$

and so

$$\|l_m - Q\|_p \leqq \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n).$$

Thus,

$$\|f - Q\|_p \leqq \|f - l_m\|_p + \|l_m - Q\|_p \leqq \varepsilon + \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n).$$

Since ε is arbitrarily small and f is any function in $\mathbf{A}_m$, we have

$$E_p(\mathbf{A}_m, \mathbf{P}_n) \leqq \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n).$$

The functions $\frac{(x-a)_+^m}{m!}$ are in $\mathbf{A}_m$ and

$$E_p\left(\frac{(x-a)_+^m}{m!}, \mathbf{P}_n\right) = \frac{E_p((x-a)_+^m, \mathbf{P}_n)}{m!}.$$

Therefore,

$$E_p(\mathbf{A}_m, \mathbf{P}_n) = \frac{M_p(m, \mathbf{P}_n)}{m!} E_p(x_+^m, \mathbf{P}_n)$$

and the theorem is proved.

2. 2. L_1 approximation. We now establish two theorems on L_1 approximation which are of interest in themselves and give in particular the value of $E_1(x_+^m, \mathbf{P}_n)$. We denote by $L(f, x_1, x_2, \ldots, x_n, x)$ the Lagrange interpolation polynomial which interpolates the function f at the points $x_1, x_2, \ldots, x_n$.

THEOREM 2. *Let $f(x) = h(x^2)$ where $h^{(n)}(x) > 0$ $(h^{(n)}(x) < 0)$ on $(0, 1)$. Then the polynomial of best L_1 approximation to f on $[-1, 1]$ of degree $\leqq 2n-2, 2n-1$ is $L(f, t_1, t_2, \ldots, t_{2n}, x)$ where*

$$t_k = -\cos\left(\frac{k\pi}{2n+1}\right) \qquad k = 1, 2, \ldots, 2n.$$

Also,

$$E_1(f, \mathbf{P}_{2n-2}) = E_1(f, \mathbf{P}_{2n-1}) = \left| \int_{-1}^{1} f(x) \operatorname{sgn} U_{2n}\, dx \right|$$

where U_{2n} is the Čebyšev polynomial of the second kind.

THEOREM 3. *Let $f(x) = xh(x^2)$ where $h^{(n)}(x) > 0$ $(h^{(n)}(x) < 0)$ on $(0, 1)$. Then the polynomial of best L_1 approximation to f on $[-1, 1]$ of degree $\leqq 2n-1, 2n$ is $L(f, t_1, t_2, \ldots, t_{2n+1}, x)$ where*

$$t_k = -\cos\left(\frac{k\pi}{2n+1}\right) \qquad k = 1, 2, \ldots, 2n+1.$$

Also,

$$E_1(f, \mathbf{P}_{2n-1}) = E_1(f, \mathbf{P}_{2n}) = \left| \int_{-1}^{1} f(x) \operatorname{sgn} U_{2n+1}\, dx \right|.$$

PROOFS. The proofs are similar and only that of Theorem 2 will be given. From the theorem of S. N. BERNSTEIN [5, p.p. 330—332], it is sufficient to show that $f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)$ changes sign at $t_1, t_2, \ldots, t_{2n}$ and only these points on $[-1, 1]$. Let $Q(x) = L(h, t_1^2, t_2^2, \ldots, t_n^2, x)$. Then the degree of Q is $\leqq n-1$ and from Cauchy's remainder formula for Lagrange interpolation we have that for each $x \in (0, 1)$ there is a $\xi_x \in (0, 1)$ such that

$$h(x) - Q(x) = \frac{h^{(n)}(\xi_x)}{n!}(x - t_1^2) \ldots (x - t_n^2).$$

So that,

$$f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x) = h(x^2) - Q(x^2) = \frac{h^{(n)}(\xi_{x^2})}{n!}(x - t_1) \ldots (x - t_{2n})$$

for $x \in (-1, 1)$ $x \neq 0$. Thus $f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)$ changes sign at $t_1, t_2, \ldots, t_{2n}$ and only these points on $[-1, 1] \setminus \{0\}$. Since the function $f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)$ is even, it does not change sign at 0. Finally, since the degree of $Q(x^2)$ is $2n-2$, the theorem is proved.

If we consider the function $f(x)=|x|^s$ $(s>-1)$, Theorem 2 gives the following corollary which was proved by NIKOLSKI using a different method based on Descartes' rule of signs.

COROLLARY 1. *The polynomial of best* L_1 *approximation to* $|x|^s$ $(s>-1)$ *on* $[-1,1]$ *of degree* $\leqq 2n-2, 2n-1$ *is* $L(|x^s|, t_1, t_2, \dots, t_{2n}, x)$ *where*

$$t_k = -\cos\left(\frac{k\pi}{2n+1}\right) \qquad k = 1,2,\dots,2n.$$

Also,

$$E_1(|x|^s, \mathbf{P}_{2n-2}) = E_1(|x^s|, \mathbf{P}_{2n-1}) = \frac{2}{s+1}\left|2\sum_{k=1}^{n}(-1)^k\left(\cos\left(\frac{k\pi}{2n+1}\right)\right)^{s+1}+1\right|.$$

Let us now consider the function $f(x)=x^{m-1}|x|$ when m is an integer $\geqq -1$. Since $x_+^m=\frac{1}{2}(|x|x^{m-1}+x^m)$, we have $E_1(x_+^m, \mathbf{P}_n) = \frac{1}{2}E_1(f, \mathbf{P}_n)$ for $n\geqq m$. Therefore, we have the following two corollaries to Theorems 2 and 3.

COROLLARY 2. *For* m *an odd positive integer and* $n\geqq m$

$$E_1(x_+^m, \mathbf{P}_n) = \frac{1}{m+1}\left|1+2\sum_{k=1}^{[\frac{1}{2}n]+1}(-1)^k\left(\cos\left(\frac{k\pi}{2[\frac{1}{2}n]+1}\right)\right)^{m+1}\right|.$$

COROLLARY 3. *For* m *an even non negative integer and* $n\geqq m$

$$E_1(x_+^m, \mathbf{P}_n) = \frac{1}{m+1}\left|1+2\sum_{k=1}^{[\frac{1}{2}(n+1)]+1}(-1)^k\left(\cos\left(\frac{k\pi}{2[\frac{1}{2}(n+1)]+1}\right)\right)^{m+1}\right|.$$

NIKOLSKI [3] has shown that $M_1(m, \mathbf{P}_n) = 1+O\left(\frac{\log n}{n}\right)$. Thus in the case $p=1$, Theorem 1 becomes:

COROLLARY 4. *For* $n\geqq m, m=1,2,\dots$

$$E_1(\mathbf{A}_m, \mathbf{P}_n) = \left(1+O\left(\frac{\log n}{n}\right)\right)\frac{E_1(x_+^m, \mathbf{P}_n)}{m!}$$

where $E_1(x_+^m, \mathbf{P}_n)$ *is given in corollaries* 2 *and* 3.

3. Estimates on the coefficients of polynomial approximations to functions in $\mathbf{A}_m$

The principal result of this section is the following theorem.

THEOREM 4. *If* $f\in\mathbf{A}_m$, *there is a polynomial*

$$P_n(x) = A_0(n)+A_1(n)x+\dots+A_n(n)x^n$$

satisfying

$$1^\circ \quad \int_{-1}^{1}|f(x)-P_n(x)|\,dx \leqq \frac{C_1}{n^{m+1}}$$

$$2^\circ \quad |A_k(n)|\leqq C_2 3^n \qquad k=0,1,\dots,n$$

where C_1 *and* C_2 *depend only on* m.

PROOF. We consider only the case when both m and n are odd. Other cases are handled in a similar manner. We can also assume that $n \geqq m$. Let Q_n denote the polynomial of best L_1 approximation to x_+^m on $[-1, 1]$ of degree $\leqq n$ which is given by Theorem 2. Then for $|a| \leqq 1$

$$\int_{-1}^{1} \left| (x-a)_+^m - 2^m Q_n\left(\frac{x-a}{2}\right) \right| dx = \int_{-1-a}^{1-a} \left| x_+^m - 2^m Q_n\left(\frac{x}{2}\right) \right| dx \leqq \tag{3.1}$$

$$\leqq \int_{-2}^{2} \left| x_+^m - 2^m Q_n\left(\frac{x}{2}\right) \right| dx = 2^{m+1} E_1(x_+^m, \mathbf{P}_n).$$

Let $f \in \mathbf{A}_m$. Using the notation introduced in the proof of Theorem 1, for a suitable partition $-1 = x_0 < x_1 < \ldots < x_r = 1$, we have

$$\int_{-1}^{1} |f(x) - l_m(x)|\, dx \leqq \varepsilon \leqq E_1(x_+^m, \mathbf{P}_n). \tag{3.2}$$

We define

$$P_n(x) = P(x) + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1})) 2^m Q_n\left(\frac{x-x_k}{2}\right) = \tag{3.3}$$

$$= A_0(n) + A_1(n)x + \ldots + A_n(n)x^n.$$

Using (2.1), (3.1), and (3.3), we have

$$\int_{-1}^{1} |l_m(x) - P_n(x)|\, dx \leqq \tag{3.4}$$

$$\leqq \frac{1}{m!} \sum_{k=1}^{r} \left(|g(x_k) - g(x_{k-1})| \int_{-1}^{1} \left| (x-x_k)_+^m - 2^m Q_n\left(\frac{x-x_k}{2}\right) \right| dx \right) \leqq \frac{2^{m+1}}{m!} E_1(x_+^m, \mathbf{P}_n).$$

Thus, by (3.2), (3.4), and (1.4)

$$\int_{-1}^{1} |f(x) - P_n(x)|\, dx \leqq \left(\frac{2^m}{m!} + 1 \right) E_1(x_+^m, \mathbf{P}_n) \leqq \frac{C_1}{n^{m+1}}.$$

We now estimate the coefficients of P_n. From Theorem 2 and the Lagrange interpolation formula, it follows that

$$Q_n\left(\frac{x-a}{2}\right) = L\left(|x|_+^m, t_1, t_2, \ldots, t_{n+1}, \frac{x-a}{2} \right) = \sum_{t_k > 0} \frac{t_k^m\, U_{n+1}\left(\frac{x-a}{2}\right)}{U'_{n+1}(t_k)\left(\frac{x-a}{2} - t_k\right)},$$

i. e.

$$(3.5)\qquad Q_n\left(\frac{x-a}{2}\right)=\sum_{k=\frac{n+1}{2}+1}^{n+1}\frac{(-1)^{k+1}(n+2)\left(-\cos\left(\frac{k\pi}{n+2}\right)\right)^m U_{n+1}\left(\frac{x-a}{2}\right)}{\sin^2\left(\frac{k\pi}{n+2}\right)\left(\frac{x-a}{2}-t_k\right)}=$$

$$=B_0(n,a)+B_1(n,a)x+\ldots+B_n(n,a)x^n.$$

Thus, by Cauchy's inequality and the maximum modulus principle we have for $|a|\leqq 1$

$$|B_k(n,a)|\leqq\sup_{|z|=1}\left|Q_n\left(\frac{z-a}{2}\right)\right|\leqq\sup_{|z|=1}|Q_n(z)|\leqq$$

$$\leqq\left(\frac{(n+2)^2}{2}\frac{1}{\sin^2\left(\frac{\pi}{n+2}\right)}\max_k\left\{\sup_{|z|=1}\left|\frac{U_{n+1}(z)}{z-t_k}\right|\right\}\right).$$

Since

$$\sin^{-2}\left(\frac{\pi}{n+2}\right)\leqq\frac{1}{4}(n+2)^2\quad\text{and for}\quad |z|=1$$

$$|z-t_k|\geqq 1-|t_k|\geqq 1-\cos\left(\frac{\pi}{n+2}\right)\geqq\frac{2}{(n+2)^2},$$

we have for $|a|\leqq 1$ and $k=0, 1, \ldots, n$

$$(3.6)\qquad |B_k(n,a)|\leqq\frac{(n+2)^6}{16}\sup_{|z|=1}|U_{n+1}(z)|.$$

The leading coefficient of U_{n+1} is 2^{n+1} and thus

$$\sup_{|z|=1}|U_{n+1}(z)|=2^{n+1}\sup_{|z|=1}|(z-t_1)(z-t_2)\ldots(z-t_{n+1})|=$$

$$=2^{n+1}\sup\left|(z^2-t_1^2)(z^2-t_2^2)\ldots\left(z^2-t_{\frac{n+1}{2}}^2\right)\right|\leqq 2^{n+1}(1+t_1^2)(1+t_2^2)\ldots\left(1+t_{\frac{n+1}{2}}^2\right).$$

Since

$$(1+t_1^2)(1+t_2^2)\ldots\left(1+t_{\frac{n+1}{2}}^2\right)\leqq\exp\left(\sum_{k=1}^{\frac{n+1}{2}}\cos^2\left(\frac{k\pi}{n+2}\right)\right)=\exp\left(\frac{n+2}{4}\right)$$

it follows that

$$\sup_{|z|=1}|U_{n+1}(z)|\leqq 2^{n+1}\exp\left(\frac{1}{4}(n+2)\right).$$

Using (3. 6), we find finally, that

$$(3.7)\qquad |B_k(n,a)|\leqq(n+2)^6 2^n\exp\left(\frac{n+2}{4}\right)\leqq C_0 3^n$$

where C_0 is a constant independent of n, and $k=0, 1, \ldots, n$.

Next, from (3. 3) and (3. 5), it follows that

$$A_j(n) = a_j + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1})) 2^m B_j(n, a)$$

where a_j is the j^{th} coefficient of P. Hence, from (3. 7) it follows that

$$|A_j(n)| \leqq |a_j| + \frac{2^m}{m!} \sum_{k=1}^{r} |g(x_k) - g(x_{k-1})| C_0 3^n \leqq |a_j| + C_0 3^n \leqq C_2 3^n$$

for $j = 0, 1, \ldots, n$ and the theorem is proved.

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ON SEQUENCES OF EQUIVALENT EVENTS AND THE COMPOUND POISSON PROCESS

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1. A limit distribution theorem on equivalent events

Let $\{\Omega, \mathcal{A}, \mathsf{P}\}$ be a probability space. Let $A_1, A_2, A_3, \ldots$ be a finite or infinite sequence of events ($A_1, A_2, \ldots$ are subsets of Ω belonging to the σ-algebra $\mathcal{A}$). The events $\{A_n\}$ are called equivalent (see RÉNYI—RÉVÉSZ [1]) if the probability

$$\mathsf{P}(A_{i_1} A_{i_2} \ldots A_{i_k}) = W_k \tag{1.1}$$

depends only on k and it does not depend on the indices

$$i_1 < i_2 < \ldots < i_k.$$

The numbers W_k are called the „*de Finetti constants*" of the sequence of events $\{A_n\}$.

Let $\{\Omega_\nu, \mathcal{A}_\nu, \mathsf{P}_\nu\}$ be a probability space for $\nu = 1, 2, 3, \ldots$ and $A_{1\nu}, A_{2\nu}, \ldots$ be a finite or infinite sequence of equivalent events in the probability space $\{\Omega_\nu, \mathcal{A}_\nu, \mathsf{P}_\nu\}$; with the corresponding de Finetti constant $W_{k,\nu}$. We shall study the limit distribution of random variables X_ν ($\nu = 1, 2, 3, \ldots$) defined by

$$X_\nu = \sum_{i=1}^{\nu} X_{i,\nu} \tag{1.2}$$

where $X_{i,\nu}$ is the indicator of the set $A_{i,\nu}$ (i.e. $X_{i,\nu} = 1$ on $A_{i,\nu}$ and $X_{i,\nu} = 0$ on the complementary set $\bar{A}_{i,\nu}$.)

In 1965 I have proved the following

THEOREM 1. *If for every $j = 1, 2, 3, \ldots$ the limit*

$$\lim_{\nu \to \infty} W_{j,\nu} \nu^j = \lambda_j \tag{1.3}$$

exists and for every $r = 0, 1, 2, \ldots$ the series

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \lambda_{k+r}$$

is convergent, then

$$\lim_{\nu \to \infty} \mathsf{P}(X_\nu = r) = \frac{1}{r!} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \lambda_{k+r}.$$

D. G. KENDALL proved [2] that in case (1.3) holds for $j = 1$ and $j = 2$ with $\lambda_2 = \lambda_1^2$, then the limit distribution of X_ν is a Poisson-distribution. Thus he proved the following

THEOREM 2. *If for every* $\nu=1,2,\ldots$ *the sequence of events* $A_{1,\nu}, A_{2,\nu}, \ldots$ *is an infinite sequence of equivalent events having de Finetti constants* $W_{k,\nu}$, *such that one has*

$$\lim_{\nu\to\infty} W_{1,\nu}\nu=\lambda \quad \text{and} \quad \lim_{\nu\to\infty} W_{2,\nu}\nu^2=\lambda^2,$$

then $\lim_{\nu\to\infty} \mathsf{P}(X_\nu=r)=\frac{1}{r!}e^{-\lambda}\lambda^r$, *for* $r=0,1,2,\ldots$.

In KENDALL's paper [2] there are given some examples which show, that the conditions of Theorem 1. and 2. are not necessary. By using KENDALL's method, we can give a necessary and sufficient condition for the existence of the limit distribution of X_ν, if all the sequences $\{A_{n,\nu}\}$ are infinite.

THEOREM 3. *Let* $A_{1,\nu}, A_{2,\nu}, \ldots$ *be an infinite sequence of equalent events for every* $\nu=1,2,3,\ldots$ *and let* $\alpha_\nu(\omega)$ $(\omega\in\Omega)$ *denote the local density of the sequence* $\{A_{n,\nu}\}$ *considered as a stable sequence of events* (see RÉNYI [4]). *The limit distribution of* X_ν *defined by* (1. 2) *exists if and only if the distribution functions* $G_\nu(x)$ *of the random variables* $\nu\alpha_\nu$ *tend to some distribution function* $G(x)$. *Then the limit distribution is a mixed Poisson-distribution given by*

$$\lim_{\nu\to\infty} P(X_\nu=r)=\int_0^\infty \frac{1}{r!}e^{-x}x^r\,dG(x) \tag{1.3}$$

where $G(x)$ *is a distribution function in* $[0,+\infty)$.

PROOF. We shall use the theorem of RÉNYI and RÉVÉSZ (see [1]). Let $A_1, A_2, \ldots$ be an infinite sequence of equivalent events. Let $\alpha(\omega)$ be the local density of the sequence $\{A_n\}$ considered as a stable sequence. The theorem of RÉNYI and RÉVÉSZ asserts that

$$\mathsf{P}(A_{i_1}A_{i_2}\ldots A_{i_k}|\alpha(\omega))=\alpha^k(\omega) \tag{1.4}$$

with probability 1 for $k=1,2,3,\ldots$ and $i_1<i_2<\ldots<i_k$. By using (1. 4) we can get the distribution of X_ν:

$$\begin{aligned}\mathsf{P}(X_\nu=s)&=\binom{\nu}{s}\mathsf{P}(A_{1,\nu}A_{2,\nu}\ldots A_{s,\nu}\bar{A}_{s+1,\nu}\ldots\bar{A}_{\nu,\nu})=\\ &=\binom{\nu}{s}\int_0^1 \mathsf{P}(A_{1,\nu}A_{2,\nu}\ldots A_{s,\nu}\bar{A}_{s+1,\nu}\ldots\bar{A}_{\nu,\nu}|\alpha_\nu=x)\,dF_\nu(x)=\\ &=\binom{\nu}{s}\int_0^1(1-x)^{\nu-s}x^s\,dF_\nu(x),\end{aligned} \tag{1.5}$$

where $F_\nu(x)$ is the distribution function of $\alpha_\nu(\omega)$. Let us determine the generating function $Q_{X_\nu}(z)$ of X_ν.

$$\begin{aligned}Q_{X_\nu}(z)&=\sum_{i=0}^{\nu}\mathsf{P}(X_\nu=i)z^i=\sum_{i=0}^{\nu}\binom{\nu}{i}z^i\int_0^1(1-x)^{\nu-i}x^i\,dF_\nu(x)=\\ &=\int_0^1(zx+1-x)^\nu\,dF_\nu(x) \quad \text{where} \quad 0\leq z\leq 1.\end{aligned}$$

Let us make the substitution $y = \nu x$; we get

$$Q_{X_\nu}(z) = \int_0^\nu \left(1 - \frac{(1-z)y}{\nu}\right)^\nu dG_\nu(y) \tag{1.6}$$

where

$$G_\nu(y) = F_\nu\left(\frac{y}{\nu}\right) = \mathsf{P}(\nu\alpha_\nu < y).$$

From the inequality

$$\left(1 - \frac{u}{v}\right)^v \leqq e^{-u} \quad \text{for} \quad 0 \leqq u \leqq v$$

it follows that

$$\lim_{\nu\to\infty} Q_{X_\nu}(z) = \int_0^\infty e^{-(1-z)y}\, dG(y). \tag{1.8}$$

The right side of (1. 8) is just the generating function of the mixed Poisson-distribution (1. 3). From the continuity theorem of generating functions it follows, that (1. 3) holds.

The necessity of the conditions follows from the well known fact that the limit distribution of X_ν exists if and only if $\lim_{\nu\to\infty} Q_{X_\nu}(z) = Q(z)$ exists and $Q(z)$ is a generating function.

As a special case we obtain

THEOREM 4. *Let $A_{1,\nu}, A_{2,\nu}, \ldots$ be an infinite sequence of equivalent events with a local density $\alpha_\nu(\omega)$ for every $\nu = 1, 2, \ldots$. The limit distribution of X_ν is a Poisson-distribution if and only if the distribution functions $G(x)$ of $\nu\alpha_\nu(\omega)$ tend to a degenerated distribution function.*

2. An application of the theory of equivalent events to some stochastic point processes

We shall investigate point processes on the line $-\infty < t < +\infty$ satisfying the properties 1 and 2 given below. Let us denote by $\omega(I)$ the number of points of a point process lying in the interval $I = [a, b)$ $(a < b)$. Then $\omega(I)$ is a random variable on the space of the realizations of the process.

PROPERTY 1. *The events $\omega(I_1) = 0$, $\omega(I_2) = 0, \ldots$ are equivalent, if the lengths $\lambda(I_j)$ of the intervals I_j $(j = 1, 2, \ldots)$ are equal and $I_j \cap I_k = \emptyset$ for $j \neq k$.*

PROPERTY 2. *The process is thin, that is*

$$\frac{\mathsf{P}(\omega(I) \geqq 2)}{\lambda(I)} \to 0 \quad \text{if} \quad \lambda(I) \to 0, \text{ uniformly in } I, \text{ with } \lambda(I) \text{ given.} \tag{2.1}$$

We shall prove that a point process, for which the Properties 1. and 2. are valid, is a mixed Poisson-process, i.e.

$$(2.2)\qquad \mathsf{P}\left(\prod_{i=1}^{r}\{\omega(I_i)=k_i\}\right)=\int_0^1\prod_{i=1}^{r}\left[e^{-\lambda(I_i)\log\frac{1}{\alpha}}\frac{\left(\lambda(I_i)\log\frac{1}{\alpha}\right)^{k_i}}{k_i!}\right]dF(\alpha)$$

provided that $I_i\cap I_j=\emptyset$ if $i\neq j$ and α is the local density of the sequence of equivalent events $\omega([0,1))=0$, $\omega([1,2))=0, \dots$ and $F(\alpha)$ is its distribution function.

A) First we shall prove that the local density of a sequence of equivalent events belonging to a sequence of intervals $I_1, I_2, \dots$ with length $\lambda(I_i)=\lambda$ is equal to α^λ, namely

$$(2.3)\qquad \alpha(\lambda)=\left(\alpha(1)\right)^\lambda=\alpha^\lambda \quad \text{with probability } 1.$$

Let $I_1, I_2, \dots$ and $J_1, J_2, \dots$ be two sequences of intervals for which $I_j\cap I_k=\emptyset$ and $J_j\cap J_k=\emptyset$ for $j\neq k$ and $\lambda(I_j)=\lambda(J_j)=\lambda$ for $j=1,2,\dots$. So we have two sequences of equivalent events:

$$\left\{A_n=\{\omega(I_n)=0\}\right\}_{n=1}^\infty \quad\text{and}\quad \left\{B_n=\{\omega(J_n)=0\}\right\}_{n=1}^\infty$$

with local density δ and β. We can choose the subsequences $I_{n_1}, I_{n_2}, \dots$ and $J_{m_1}, J_{m_2}, \dots$ so, that $I_{n_j}\cap J_{m_k}=\emptyset$ for every $j=1,2,\dots$ and $k=1,2,\dots$. So the sequence of events $A_{n_1}, B_{m_1}, A_{n_2}, B_{m_2}, \dots$ is a sequenec of equivalent events with local density γ.

It follows from the definition of the local density of a stable sequence (RÉNYI [4]) that if $\{C_n\}_{n=1}^\infty$ is a stable sequence of events with the local density α, the subsequence $\{C_{n_k}\}_{k=1}^\infty$ is also a stable sequence with a local density equal to α with probability 1. Therefore the local density of the sequence $\{A_{n_k}\}_{k=1}^\infty$ equals to δ and to γ with probability 1, so

$$\delta=\gamma \text{ with probability } 1.$$

In the same way $\beta=\gamma$ with probability 1, so

$$\delta=\beta \text{ with probability } 1.$$

This means that the local density of sequences of equivalent events belonging to some sequences of disjoint intervals with lengths λ can be denoted by $\alpha(\lambda)$.

Now let $I_1, I_2, \dots$ be a sequence of disjoint intervals with length $\lambda(I_j)=\lambda$. Let us dissect the interval I_j into k disjoint subintervals $I_{j,1}, I_{j,2}, \dots, I_{j,k}$ for which $\lambda(I_{j,i})=\frac{\lambda}{k}$ for every $j=1,2,\dots$ and $i=1,2,\dots,k$. So we have two sequences of equivalent events, namely $\{A_n:\{\omega(I_n)=0\}\}_{n=1}^\infty$ and $\{A_{n,i}:\{\omega(I_{n,i})=0\}\}_{n=1,\ 1\le i\le k}^\infty$

As

$$\{\omega(I_j)=0\}=\prod_{i=1}^{k}\{\omega(I_{j,i})=0\},$$

so

$$\mathsf{P}\left(A_j|\alpha\left(\frac{\lambda}{k}\right)\right)=\mathsf{P}\left(\prod_{i=1}^{k}A_{j,i}|\alpha\left(\frac{\lambda}{k}\right)\right)=\left[\alpha\left(\frac{\lambda}{k}\right)\right]^k.$$

Thus

$$\mathsf{P}\left(\prod_{j=1}^{n} A_{n_j}|\alpha\left(\frac{\lambda}{k}\right)\right) = \mathsf{P}\left(\prod_{j=1}^{n}\prod_{i=1}^{k} A_{n_j,i}|\alpha\left(\frac{\lambda}{k}\right)\right) = \left[\alpha\left(\frac{\lambda}{k}\right)\right]^{kn}.$$

This means, that the events $\{A_n\}_{n=1}^{\infty}$ are independent under the condition that the value of $\alpha\left(\frac{\lambda}{k}\right)$ is fixed, and their conditional probability is $\left[\alpha\left(\frac{\lambda}{k}\right)\right]^k$. From this it follows (see RÉNYI [1]) that the local density of $\{A_n\}_{n=1}^{\infty}$ is equal to $\left[\alpha\left(\frac{\lambda}{k}\right)\right]^k$ with probability 1, so

$$\alpha(\lambda) = \left[\alpha\left(\frac{\lambda}{k}\right)\right]^k.$$

If $\lambda > 0$ is rational, so from

$$\alpha\left(\frac{1}{q}\right)^q = \alpha(1),\ \alpha\left(\frac{p}{q}\right) = \alpha\left(\frac{1}{q}\right)^p \quad \text{and} \quad \alpha\left(\frac{p}{q}\right) = [\alpha(1)]^{\frac{p}{q}}$$

if p and q are integers, it follows (2. 3).

Let now $\lambda > 0$ be irrational, and $I_1, I_2, \ldots$ a sequence of intervals for which $\lambda(I_j) = \lambda$ and the distance of two intervals I_j and I_k is larger than $\varepsilon > 0$ for every $j \neq k$. Let us choose two sequences of rational numbers $s_1, s_2, \ldots$ and $r_1, r_2, \ldots$ for which $\lambda - \frac{\varepsilon}{2} < r_1 < r_2 < \ldots < \lambda < \ldots < s_2 < s_1 < \lambda + \frac{\varepsilon}{2}$ and $\lim_{n\to\infty} (s_n - r_n) = 0$. Let be given the intervals $I_j^{r_i} \subset I_j$ and $I_j^{s_i} \supset I_j$ for every j so that $\lambda(I_j^{r_i}) = r_i$ and $\lambda(I_j^{s_i}) = s_i$ for every i and j. These conditions assure that $I_j^{r_i} \cap I_k^{r_i} = \emptyset$ and $I_j^{s_i} \cap I_k^{s_i} = \emptyset$ for every $j \neq k$ and $i = 1, 2, \ldots$. As $\{\omega(I_j^{s_i}) = 0\} \subset \{\omega(I_j) = 0\} \subset \{\omega(I_j^{r_i}) = 0\}$ for every i and j, we get

$$[\alpha(1)]^{s_i} \leqq \alpha(\lambda) \leqq [\alpha(1)]^{r_i} \quad \text{with probability 1.}$$

If now $i \to \infty$, we get (2. 3)

B) Now it will be proved that under condition $\alpha(\omega) = \alpha$ the events $\omega(I_1) = 0$, $\omega(I_2) = 0, \ldots, \omega(I_k) = 0$ are independent and

$$\mathsf{P}\left(\prod_{i=1}^{k} \{\omega(I_i) = 0\}|\alpha\right) = \alpha^{\sum_{i=1}^{k} \lambda(I_i)} \quad \text{(with probability 1)}$$

if $I_j \cap I_i = \emptyset$ for $i \neq j$.

Let us put

$$I_j = \left[\bigcup_{i=1}^{n_j} I_{j,i}^n\right] \cup I_{j,n_j+1}^n \quad \text{where}$$

$\lambda(I_{j,i}^n) = \frac{1}{n}$ for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, k$, and $\lambda(I_{j,n_j+1}^n) \leqq \frac{1}{n}$ for $j = 1, 2, \ldots, k$.

So

$$A_j = [\omega(I_j) = 0] \subset \prod_{i=1}^{n_j} \{\omega(I_{j,i}^n) = 0\} = A_{j,n}$$

for $j=1, 2, \ldots, k$ and consequently

$$\prod_{j=1}^{k} A_j \subset \prod_{j=1}^{k} A_{j,n}.$$

Obviously $\prod_{n=1}^{\infty} A_{j,n}=A_j$ for $j=1, 2, \ldots, k$ and

$$\prod_{n=1}^{\infty}\left[\prod_{j=1}^{k} A_{j,n}\right] = \prod_{j=1}^{k} A_j,$$

therefore

$$\lim_{n\to\infty} \mathsf{P}(A_{j,n}|\alpha) = \mathsf{P}(A_j|\alpha)$$

and

$$\lim_{n\to\infty} \mathsf{P}\left(\prod_{j=1}^{k} A_{j,n}|\alpha\right) = \mathsf{P}\left(\prod_{j=1}^{k} A_j|\alpha\right).$$

As

$$\mathsf{P}\left(\prod_{j=1}^{k} A_{j,n}|\alpha\right) = \mathsf{P}\left(\prod_{j=1}^{k}\prod_{i=1}^{nj}[\omega(I^n_{j,i})=0]|\alpha\right) = \prod_{j=1}^{k}\prod_{i=1}^{nj}\alpha^{\frac{1}{n}} = \prod_{j=1}^{k}\mathsf{P}(A_{j,n}|\alpha)$$

and

$$\lim_{n\to\infty}\prod_{i=1}^{nj}\alpha^{\frac{1}{n}} = \lim_{n\to\infty}\alpha^{\frac{nj}{n}} = \alpha^{\lambda(I_j)},$$

the proof can be easily finished.

C) Now we are in the position to prove (2. 2). We have to determine the probability of the event

$$A = \prod_{i=1}^{r}\{\omega(I_i)=k_i\} \quad \text{where} \quad I_j\cap I_i=\emptyset \quad \text{for} \quad j\neq i.$$

Let us split the interval I_j for every $j=1, 2, \ldots, r$ into n disjoint subintervals $I^n_{j,1}, I^n_{j,2}, \ldots, I^n_{j,n}$ for which

$$\bigcup_{j=1}^{n} I^n_{j,i} = I_j \text{and} \quad \lambda(I^n_{j,i}) = \frac{\lambda(I_j)}{n} \quad \text{for} \quad i=1,2,\ldots,n$$

Let us denote by H_n the event that at least one of the events $\omega(I^n_{j,k})\geqq 2$ occurs. In consequence of Property 2

$$\mathsf{P}(H_n) \leqq \sum_{j=1}^{r}\sum_{i=1}^{n}\mathsf{P}(\omega(I^n_{j,i})\geqq 2) \leqq \sum_{j=1}^{r}\sum_{i=1}^{n}\varepsilon\lambda(I^n_{j,i}) = \varepsilon\sum_{j=1}^{r}\lambda(I_j)$$

if $n>n_0(\varepsilon)$. So

$$\lim_{n\to\infty}\mathsf{P}(H_n)=0. \tag{2. 4}$$

Let us denote by $A_{j,n}$ the event that the number of intervals $I^n_{j,i}$ $(i=1, 2, \ldots, n)$ for which $\omega(I^n_{j,i})=0$ is realised, is $n-k_j$.

Then

$$\lim_{n\to\infty} \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n}\right) = \mathsf{P}(A). \tag{2.5}$$

Namely $A \supset \prod_{j=1}^{r} A_{j,n} - H_n$ and $A \subset \prod_{j=1}^{r} A_{j,n} \cup H_n$ and therefore

$$\begin{aligned} \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n}\right) - \mathsf{P}(H_n) &\leqq \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n} - H_n\right) \leqq \mathsf{P}(A) \leqq \\ &\leqq \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n} \cup H_n\right) \leqq \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n}\right) + \mathsf{P}(H_n). \end{aligned} \tag{2.6}$$

In consequence of (2. 4) and (2. 6) the equality (2. 5) is valid. The value of $\lim_{n\to\infty} \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n}\right)$ can be calculated as follows. We use the formula

$$\mathsf{P}\left(\prod_{j=1}^{r} A_{j,n}\right) = \int_0^1 \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n} | a = x\right) dF(x). \tag{2.7}$$

In consequence of the result in part B)

$$\mathsf{P}\left(\prod_{j=1}^{r} A_{j,n} | \alpha\right) = \prod_{j=1}^{r} \binom{n}{k_j} \left(1 - \alpha^{\frac{\lambda(I_j)}{n}}\right)^{k_j} \left(1 - \left(1 - \alpha^{\frac{\lambda(I_j)}{n}}\right)\right)^{n-k_j}. \tag{2.8}$$

The limit of the right side of (2. 8) for $n \to \infty$ is

$$\prod_{j=1}^{r} \left[e^{-\lambda(I_j) \log \frac{1}{\alpha}} \frac{\left(\lambda(I_j) \log \frac{1}{\alpha}\right)^{k_j}}{k_j!} \right]$$

unless if $\alpha = 0$. As if $\mathsf{P}(\alpha = 0) > 0$ we have a contradiction to Property 2, and the integrand in (2. 7) is at most 1,

$$\begin{aligned} \lim_{n\to\infty} \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n}\right) &= \int_0^1 \lim \mathsf{P}\left(\prod_{j=1}^{r} A_{j,n} | \alpha = x\right) dF(x) = \\ &= \int_0^1 \prod_{j=1}^{r} \left[e^{-\lambda(I_j) \log \frac{1}{x}} \frac{\left[\lambda(I_j) \log \frac{1}{x}\right]^{k_j}}{k_j!} \right] dF(x) \end{aligned}$$

This proves our theorem.

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ON RANDOM MATRICES II

by

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§ 0. Introduction

This paper is a continuation of our paper [1]. Let $\mathcal{M}(n)$ denote the set of all n by n zero-one matrices; let us denote the elements of a matrix $M_n \in \mathcal{M}(n)$ by ε_{jk} $(1 \leq j \leq n;\ 1 \leq k \leq n)$. Let p denote an arbitrary permutation $p = (p_1, p_2, \dots, p_n)$ of the integers $(1, 2, \dots, n)$ and Π_n the set of all $n!$ such permutations. Let us put for each $p \in \Pi_n$

$$\varepsilon(p) = \varepsilon_{1p_1} \cdot \varepsilon_{2p_2} \cdots \varepsilon_{np_n}. \tag{0. 1}$$

Thus the permanent $\operatorname{perm}(M_n)$ of M_n can be written in the form

$$\operatorname{perm}(M_n) = \sum_{p \in \Pi_n} \varepsilon(p) \tag{0. 2}$$

Thus each $\varepsilon(p)$ $(p \in \Pi_n)$ is a term of the expansion of $\operatorname{perm}(M_n)$.

Let us call two permutations $p' = (p_1', \dots, p_n')$ and $p'' = (p_1'', \dots, p_n'')$ $(p' \in \Pi_n, p'' \in \Pi_n)$ *disjoint* if $p_k' \neq p_k''$ for $k = 1, 2, \dots, n$. Let now define (for each $M_n \in \mathcal{M}(n)$) $v = v(M_n)$ as the largest number of pairwise disjoint permutations $p^{(1)}, \dots, p^{(v)}$ such that $\varepsilon(p^{(i)}) = 1$ $(i = 1, 2, \dots, v)$. Clearly

$$\operatorname{perm}(M_n) \geq v(M_n) \tag{0. 3}$$

thus $v(M_n) \geq 1$ is equivalent to $\operatorname{perm}(M_n) > 0$.

Let us denote by $\mathcal{M}(n, N)$ the set of those n by n zero-one matrices, among the n^2 elements of which exactly N elements are equal to 1 and the remaining $n^2 - N$ to 0 $(0 < N < n^2)$. Let us choose at random a matrix $M_{n,N}$ from the set $\mathcal{M}(n, N)$ with uniform distribution, i.e. so that each of the $\binom{n^2}{N}$ elements of $\mathcal{M}(n, N)$ has the same probability $\binom{n^2}{N}^{-1}$ to be chosen.

Let us denote by $P(n, N, r)$ the probability of the event

$$v(M_{n,N}) \geq r \qquad (r = 1, 2, \dots).$$

Clearly $P(n, N, 1)$ is the probability of the event $\operatorname{perm}(M_{n,N}) > 0$.

In [1] we have shown that if

$$N_1(n) = n \log n + cn + o(n) \tag{0. 4}$$

where c is any fixed real number, one has

$$\lim_{n \to \infty} \mathsf{P}(n, N_1(n), 1) = e^{-2e^{-c}}. \tag{0. 5}$$

This implies that if $\omega(n)$ tends arbitrarily slowly to $+\infty$ for $n \to +\infty$ and

$$N_1^*(n) = n \log n + \omega(n) n \tag{0. 6}$$

then

$$\lim_{n \to \infty} \mathsf{P}(n, N_1^*(n), 1) = 1. \tag{0. 7}$$

In the present paper we shall extend this result, and prove the following

THEOREM 1. *For any fixed natural number r, if*

$$N_r^*(n) = n \log n + (r-1) n \log \log n + n\omega(n) \tag{0. 8}$$

where $\omega(n)$ tends arbitrarily slowly to $+\infty$ for $n \to +\infty$, we have

$$\lim_{n \to +\infty} \mathsf{P}(n, N_r^*(n), r) = 1. \tag{0. 9}$$

Clearly (0. 7) is the special case $r=1$ of (0. 9). (0. 5) can be generalized in a similar way (see Theorem 2). Evidently, the interesting case is when $\omega(n)$ tends slower to $+\infty$ than $\log \log n$.

The method of the proof of Theorem 1 and 2 follows the same pattern as that in [1].

In § 2 we formulate — similarly as in [1] — an analogous result for random zero-one matrices with independent elements, while in § 3 we add some remarks and mention some related open problems.

§ 1. Random matrices with a prescribed number of zeros and ones

We prove in this § Theorem 1. We suppose $r \geqq 2$ as the theorem was proved for $r=1$ in [1].

Suppose that M is an n by n zero-one matrix belonging to the set $\mathcal{M}(n, N_r^*(n))$ where $N_r^*(n)$ is defined by (0. 8), and suppose that $\nu(M) \leqq r-1$.

Clearly we can delete from each row and column of such a matrix $r-1$ suitably selected ones so that the permanent of the remaining matrix M' should be equal to 0. As regards the matrix M' we distinguish two cases: either the deletion can be made so that M' contains a row or a column which consists of zeros only, or not. Let us denote by $Q_1(n, r)$ the probability of the first case, and by $Q_2(n, r)$ the probability of the second case. Clearly if a row (column) of M' consists of zeros only, the corresponding row (column) of M contains at most $r-1$ ones. Conversely, if M contains such a row or column, then clearly $\nu(M) \leqq r-1$. Thus $Q_1(n, r)$ is equal to the probability of the event that in M there is at least one row or column which contains at most $r-1$ ones. Thus we have

$$Q_1(n, r) \leqq 2n \sum_{j=0}^{r-1} \binom{n}{j} \frac{\binom{n^2-n}{N_r(n)-j}}{\binom{n^2}{N_r(n)}} = O(e^{-\omega(n)}) = o(1). \tag{1. 1}$$

Let us pass now to the second case. Let k be the least number such that one can find in M' either k columns and $n-k-1$ rows, or k rows and $n-k-1$ columns, which contain all the ones of M'; according to the theorem of Frobenius (see [2] and [3]) as $\operatorname{perm}(M')=0$, such a k exists, and $1 \leqq k \leqq \left[\frac{n-1}{2}\right]$ because the case $k=0$ has already been taken into account (this was our first case). We may suppose that all ones of M' are covered by k columns and $n-k-1$ rows (the probability of the other case when the ones of M' are covered by k rows and $n-k-1$ columns being the same by symmetry). It follows — as in [1] — that M' contains a submatrix C' consisting of $k+1$ rows and k columns, such that each column of C' contains at least two ones. Let C be the corresponding submatrix of M. It follows that

$$Q_2(n,r) \leqq 2 \sum_{k=1}^{\left[\frac{n-}{2}\right]} q_k \tag{1.2}$$

where $q_k \left(1 \leqq k \leqq \left[\frac{n-1}{2}\right]\right)$ is the probability of the event that M contains a $k+1$ by k submatrix C such that each column of C contain at least two ones, and the submatrix D of M formed by the same rows as C and by those columns which do not intersect C, contains at most $r-1$ ones in each row. Evidently

$$q_k \leqq \binom{n}{k}\binom{n}{k+1}\binom{k+1}{2}^k \sum_{j=0}^{(k+1)(r-1)} \frac{\binom{(k+1)(n-k)}{j}\binom{n(n-k-1)+k(k-1)}{N_r^*-2k-j}}{\binom{n^2}{N_r^*}}. \tag{1.3}$$

It follows from (1. 2) and by an asymptotic evaluation of the expression at the right hand side of (1. 3) that

$$Q_2(n,r)=o(1). \tag{1.4}$$

As

$$1-\mathrm{P}(n,N_r^*(n),r) = Q_1(n,r)+Q_2(n,r) \tag{1.5}$$

it follows in view of (1.1) and (1. 4) that (0. 9) holds. Thus Theorem 1 is proved.

By the same method we can prove the following result, which generalizes (0. 5) for $r \geqq 2$.

THEOREM 2. *If*

$$N_r(n) = n\log n+(r-1)n\log\log n+cn+o(n) \tag{1.6}$$

where $r \geqq 1$ *is an integer and* c *is any real number, we have*

$$\lim_{n\to+\infty} \mathrm{P}(n,N_r(n),r) = e^{-\frac{2e^{-c}}{(r-1)!}}. \tag{1.7}$$

§ 2. Random zero-one matrices with independent elements

Similarly as in [1] let us consider now random n by n matrices $M=(\varepsilon_{ij})$ $(1\leqq i, j\leqq n)$ such that the ε_{ij} are independent random variables which take on the values 1 and 0 with probabilities p_n and $(1-p_n)$. It can be shown that the following result is valid:

THEOREM 3. *For any fixed natural number r, put*

$$p_n = \frac{\log n + (r-1)\log\log n + \omega(n)}{n} \tag{2.1}$$

where $\omega(n)$ tends arbitrarily slowly to $+\infty$ and let M be an n by n random matrix the elements of which are independent random variables, taking on the values 1 *and* 0 *with probability p_n and $1-p_n$ respectively. Then the probability of $\nu(M)\geqq r$ tends to* 1 *for $n\to+\infty$.*

Note that the special case $r=1$ of Theorem 3 is contained in Theorem 2 of our previous paper [1].

As the idea of the proof is essentially the same as that of (0. 9), and the computation even somewhat simpler, we omit the proof of Theorem 3. Theorem 3 can be sharpened in the same way as Theorem 2 sharpens Theorem 1.

§ 3. Remarks and open problems

Let us put

$$\mu(n,k) = \min_{\substack{\nu(M_n)=k \\ M_n\in\mathcal{M}(n)}} (\operatorname{perm}(M_n)). \tag{3.1}$$

Clearly $\mu(n,1)=1$ and $\mu(n,2)=2$; however,for $k\geqq 3$ the question concerning the value of $\mu(n,k)$ is open. We have clearly $\mu(k,k)=k!$ and

$$\mu(k,\kappa-1) = k!\left(\frac{1}{2!}-\frac{1}{3!}+\dots+\frac{(-1)^k}{k!}\right) \tag{3.2}$$

but the value of $\mu(n,k)$ for $n\geqq k+2$ is not known. Clearly for determining $\mu(n,k)$ it is sufficient to consider those matrices M_n which contain exactly k ones in each row and in each column. As each such matrix is the sum of k disjoint permutation matrices, i.e. for such a matrix we have $\nu(M_n)=k$, thus the problem of determining $\mu(n,k)$ is the same as the problem raised by RYSER (see [7], p. 77) concerning the minimum of the permanent of n by n zero-one matrices having exactly k ones in each row and each column. Of course for particular values of n and k one can determine $\mu(n,k)$ (e.g. $\mu(5,3)=12$), but what would be of real interest is the asymptotic behaviour of $\mu(n,k)$ for fixed $k\geqq 3$ and $n\to+\infty$.

Let us put

$$\liminf_{n\to\infty}\sqrt[n]{\mu(n,k)} = \mu_k. \tag{3.3}$$

It seems likely that $\mu_k > 1$ for $k \geqq 3$. One reason for this conjecture is that if the conjecture of VAN DER WAERDEN is true, we have

$$\mu(n,k) \geqq \frac{k^n n!}{n^n} \geqq \left(\frac{k}{e}\right)^n \tag{3.4}$$

i.e. $\mu_k \geqq \frac{k}{e} > 1$ for $k \geqq 3$. We guess that μ_k is even larger than $\frac{k}{e}$.

If in particular M_n is the matrix defined by $\varepsilon_{j,j} = \varepsilon_{j,j+1} = \varepsilon_{j,j-1} = 1$ (we put $\varepsilon_{j,m} = \varepsilon_{j,m-n}$ for $m > n$) and $\varepsilon_{jl} = 0$ if $|l-j| \geqq 2$, then it can be easily shown that $\operatorname{perm}(M_n) = L_n + 2$ where L_n is the n-th LUCAS number, i.e. the n-th term of the Fibonacci-type sequence

$$1, 3, 4, 7, 11, 18, \ldots \tag{3.5}$$

and

$$\lim_{n\to\infty} \sqrt[n]{L_n} = \frac{\sqrt{5}+1}{2} > \frac{3}{e}. \tag{3.6}$$

As regards $\mu(n,k)$, at present it is known only that

$$\lim_{n\to+\infty} \mu(n,3) = +\infty. \tag{3.7}$$

This was conjectured by MARSHALL HALL and proved by R. SINKHORN [8]. As a matter of fact, SINKHORN proved $\mu(n,3) \geqq n$ for all $n \geqq 3$. Of course (3. 7) implies $\lim\limits_{n\to+\infty} \mu(n,k) = +\infty$ for $k = 4, 5, \ldots$ too.

An interesting open problem is the following: evaluate asymptotically $\mathsf{P}(n, n\log n + (r-1)n\log\log n, r)$ if r is not constant, but increases together with n.

There is a striking analogy between Theorem 1 and the following well known result (see e.g. [4]): If $N_r^*(n)$ balls are placed at random into n urns, and $N_r^*(n)$ is given by (0. 8) (with $\omega(n) \to +\infty$) then the probability of each urn containing at least r balls, tends to 1 for $n \to +\infty$. The relation between this problem and that of § 1 is made clear by the following remark. If we interpret the rows (columns) of M as urns and the ones as balls, then there are n urns, and each of the $N_r^*(n)$ „balls" falls with the same probability $1/n$ in any of the „urns".

In another paper ([5]) we have proved the following theorem (Theorem 1 of [5]): a random graph $\Gamma(n,N)$ with n vertices where n is even and $N = \frac{1}{2} n\log n + n\,\omega(n)$ edges where $\omega(n) \to +\infty$ for $n \to +\infty$, contains a factor of degree one with probability tending to 1 for $n \to +\infty$.

Theorem 1 of the present paper suggests the following problem: does a random graph $\Gamma(n,N)$ where n is even and

$$N = \frac{1}{2} n\log n + \frac{r-1}{2} n\log\log n + \omega(n)n$$

where $\omega(n) \to +\infty$, contain at least r *disjoint* factors of degree one with probability tending to 1 for $n \to \infty$? To prove this, besides the method of [5] the results of [6] have to be used.

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INDEX

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