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Gauss Lucas theorem and Bernstein-type inequalities for polynomials

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Abstract. According to Gauss-Lucas theorem, every convex set containing all the zeros of a polynomial also contains all its critical points. This result is of central importance in the geometry of critical points in the analytic theory of polynomials. In this paper, an extension of Gauss-Lucas theorem is obtained and as an application some generalizations of Bernstein-type polynomial inequalities are also established.

1 Extension of Gauss-Lucas theorem

Let g be a real differential function, then Rolle's theorem guarantees the existence of at least one critical point (zero of its derivative g') between any two real zeros of g. While as, in case of analytic functions of a complex variable, Rolle's theorem does not hold in general. This fact can be realized from the function $g(z) = e^{iz} - 1$ which has zeros at $z = 0$ and $z = 2\pi$, however, its derivative g' has no zeros whatsoever.

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If the idea of a critical point lying between two points on the real line is replaced in the complex plane by the concept of a critical point located in some region containing the zero of the function then following result (see [3, pp. 22], [4, pp. 179]) called as Gauss-Lucas theorem gives a relative location of critical points with respect to the zero for polynomials.

Theorem 1 *The critical points of a non-constant polynomial* f *lie in the convex hull* $\mathcal{H}$ *of the zeros of* f.

Let a circle $\mathcal{C}$ encloses all the zeros of f then by theorem 1, $\mathcal{H} \subset \mathcal{C}$. On the other hand, through each pair of vertices of the polygon $\mathcal{H}$ a family of circles C_δ can be drawn which contains $\mathcal{H}$ and consequently all the zeros and critical points of f. The region $\Gamma = \cap C_\delta$ would also contain all zeros of f and f'. Hence, all the critical points of f must lie in the region common to all possible Γ's, which is equal to $\mathcal{H}$. Thus, an equivalent form of Theorem 1 can be stated as follows.

Theorem 2 *A circle* $\mathcal{C}$ *containing all the zeros of a non-constant polynomial* f *also encloses all the zeros of its derivative* f'.

In literature, there exist different variants of Gauss-Lucas theorem (for references see [3, pp. 22], [4, pp. 180], [5, pp. 71]). In this paper, we first present the following extension of Gauss-Lucas theorem.

Theorem 3 *Let all the zeros of an* n*th degree polynomial* $f(z)$ *lie in* $|z| \leq r$, *then for every* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) \leq \frac{n}{2}$, *the zeros of* $zf'(z) - \alpha f(z)$ *also lie in* $|z| \leq r$.

Proof. Let $P(z) = zf'(z) - \alpha f(z)$ and $w \in \mathbb{C}$ with $|w| > r$. Suppose $z_1, z_2, \ldots, z_n$ be the zeros of $f(z)$, then $|z_\nu| \leq r$ and $|w| - |z_\nu| > 0$ for $\nu = 1, 2, \ldots, n$. Now,

$$\begin{aligned}
\frac{P(w)}{f(w)} &= \frac{wf'(w)}{f(w)} - \alpha = \sum_{\nu=1}^{n} \frac{w}{w - z_\nu} - \alpha \\
&= \frac{1}{2} \sum_{\nu=1}^{n} \frac{(w - z_\nu) + (w + z_\nu)}{w - z_\nu} - \alpha \\
&= \frac{n}{2} - \alpha + \frac{1}{2} \sum_{\nu=1}^{n} \frac{(w + z_\nu)(\bar{w} - \bar{z_\nu})}{|w - z_\nu|^2}.
\end{aligned}$$

This implies

$$\Re\left(\frac{P(w)}{f(w)}\right) = \frac{n}{2} - \Re(\alpha) + \frac{1}{2} \sum_{\nu=1}^{n} \frac{|w|^2 - |z_\nu|^2}{|w - z_\nu|^2}$$

$$\geq \frac{n}{2} - \frac{n}{2} + \frac{1}{2}\sum_{\nu=1}^{n} \frac{|w|^2 - |z_\nu|^2}{|w - z_\nu|^2} > 0.$$

This further implies that $P(w) \neq 0$. Hence, $P(z)$ cannot have a zero in $|z| > r$. Therefore, we conclude that all the zeros of the polynomial $zf'(z) - \alpha f(z)$ lie in $|z| \leq r$. □

Note that the Theorem 2 follows by taking $\alpha = 0$ in Theorem 3.

2 Bernstein-type inequalities

The zero-preserving property of the derivative, which emerge out of Gauss-Lucas theorem, plays an important role in Bernstein-type inequalities for polynomials. A simple proof of the following result using Gauss-Lucas theorem can be found in a comprehensive book of Rahman & Schmeisser [5, pp. 510].

Theorem 4 *Let a polynomial $F(z)$ of degree n has all its zeros in $|z| \leq 1$ and $G(z)$ be a polynomial of degree at most n such that $|G(z)| \leq |F(z)|$ for $|z| = 1$, then*

$$|G'(z)| \leq |F'(z)| \qquad \text{for} \quad |z| \geq 1. \tag{1}$$

The equality holds outside the closed unit disk if and only if $G(z) \equiv e^{i\delta}F(z)$ for some $\delta \in \mathbb{R}$.

By taking $F(z) = Mz^n$, where $M = \max_{|z|=1} |G(z)|$, following sharp estimate for the derivative over closed unit disc, called as Bernstein's inequality [1], follows immediately.

Theorem 5 *Let $G(z)$ be a polynomial of degree at most n, then*

$$\max_{|z|=1} |G'(z)| \leq n \max_{|z|=1} |G(z)|. \tag{2}$$

The equality is attained in (2) if and only if $G(z) = az^n$, $a \neq 0$. Therefore, for the polynomials having zeros away from origin, there is a scope for an improvement in (2). In this regard, Erdös conjectured that if a polynomial $G(z)$ of degree n has no zero in $|z| < 1$, then

$$\max_{|z|=1} |G'(z)| \leq \frac{n}{2} \max_{|z|=1} |G(z)|. \tag{3}$$

This conjecture was later proved by Lax [2].

As an application of Theorem 3, here we next present the following extension of Theorem 4. The proof of this theorem is similar to that of theorem 4 given in [5].

Theorem 6 *Let a polynomial* $F(z)$ *of degree* n *has all its zeros in* $|z| \le 1$ *and* $G(z)$ *be a polynomial of degree at most* n *such that* $|G(z)| \le |F(z)|$ *for* $|z| = 1$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \le n/2$, $\Re(\beta) \le n/2$ *and* $|z| \ge 1$, *we have*

$$|zG'(z) - \alpha G(z)| \le |zF'(z) - \alpha F(z)| \tag{4}$$

and

$$\begin{aligned}|z^2G''(z) + (1-\alpha-\beta)zG'(z) + \alpha\beta G(z)|\\ \le |z^2F''(z) + (1-\alpha-\beta)zF'(z) + \alpha\beta F(z)|.\end{aligned} \tag{5}$$

The bound is sharp and equality holds for some point z in $|z| > 1$ *if and only if* $G(z) = e^{i\delta}F(z)$ *for some* $\delta \in \mathbb{R}$.

Proof. Since the result holds trivially true, if $G(z) = e^{i\delta}F(z)$ for some $\delta \in \mathbb{R}$. Therefore, let $G(z) \ne e^{i\delta}F(z)$. Consider the function $\phi(z) = G^*(z)/F^*(z)$ where $F^*(z) = z^n\overline{F(1/\overline{z})}$ and $G^*(z) = z^n\overline{G(1/\overline{z})}$. Since $F(z)$ has its all zeros in $|z| \le 1$, then $F^*(z)$ has no zero in $|z| < 1$. This implies that the rational function $\phi(z)$ is analytic for $|z| \le 1$. Also, $|G(z)| = |G^*(z)|$ and $|F(z)| = |F^*(z)|$ for $|z| = 1$, therefore, $|\phi(z)| \le 1$ for $|z| = 1$. By invoking maximum modulus theorem, we obtain;

$$|\phi(z)| < 1 \qquad \text{for} \quad |z| < 1.$$

On replacing z by $1/z$ in the above inequality, we get $|G(z)| < |F(z)|$ for $|z| > 1$. It follows by Rouche's theorem that for any $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1$, the polynomial $G(z) - \lambda F(z)$ of degree n has all its zeros in $|z| \le 1$. Applying Theorem 3 to the polynomial $P(z) = G(z) - \lambda F(z)$, for $\alpha \in \mathbb{C}$ with $\Re(\alpha) \le n/2$, we obtain that the polynomial

$$\begin{aligned}zP'(z) - \alpha P(z) &= z(G'(z) - \lambda F'(z)) - \alpha(G(z) - \lambda F(z))\\ &= (zG'(z) - \alpha G(z)) - \lambda(zF'(z) - \alpha F(z))\end{aligned}$$

has all zeros in $|z| \le 1$. This implies

$$|zG'(z) - \alpha G(z)| \le |zF'(z) - \alpha F(z)| \qquad \text{for} \quad |z| > 1. \tag{6}$$

If inequality (6) were not true, then there is some point $w \in \mathbb{C}$ with $|w| > 1$ such that $|wG'(w) - \alpha G(w)| > |wF'(w) - \alpha F(w)|$. By Theorem 3, $wF'(w) - \alpha F(w) \neq 0$. Now, choose $\lambda = -\frac{wG'(w)-\alpha G(w)}{wF'(w)-\alpha F(w)}$ and note that λ is a well defined complex number with modulus greater than 1. So, with this choice of λ, one can easily observe that w is a zero of $zP'(z) - \alpha P(z)$ of modulus greater than one. This is a contradiction, since all the zeros of this polynomial lie in $|z| \leq 1$. Hence, the inequality (6) is true, by continuity (6) also holds for $|z| = 1$. This proves the inequality (4).

Finally, if we take $H(z) = zG'(z) - \alpha G(z)$ and $K(z) = zF'(z) - \alpha F(z)$ with $\Re(\alpha) \leq n/2$, then by inequality (4), $|H(z)| \leq |K(z)|$ for $|z| = 1$. Therefore, by using inequality (4) again, for $\beta \in \mathbb{C}$ with $\Re(\beta) \leq n/2$, we get, $|zH'(z) - \beta H(z)| \leq |zK'(z) - \beta K(z)|$ for $|z| \geq 1$, which is equivalent to (5). This completes the proof of this theorem. □

For $\alpha = 0$ the inequality (4) reduces to (1).

The following result can be deduced from Theorem 6 by taking $F(z) = Mz^n$ where $M = \max_{|z|=1} |G(z)|$.

Corollary 1 *Let* $G(z)$ *be a polynomial of degree* n *and* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$, *then for* $|z| \geq 1$,

$$|zG'(z) - \alpha G(z)| \leq |n - \alpha||z|^n \max_{|z|=1} |G(z)| \tag{7}$$

$$\begin{aligned} &|z^2 G''(z) + (1 - \alpha - \beta) z G'(z) + \alpha\beta G(z)| \\ &\qquad \leq |n(n - \alpha - \beta) + \alpha\beta||z|^n \max_{|z|=1} |G(z)|. \end{aligned} \tag{8}$$

Equality in (7) *and* (8) *hold for* $G(z) = az^n$ *where* $a \neq 0$.

The next corollary follows by taking $\alpha = \beta = n/2$ in (7) and (8).

Corollary 2 *Let* $G(z)$ *be a polynomial of degree* n, *then for* $|z| \geq 1$,

$$\left| zG'(z) - \frac{n}{2} G(z) \right| \leq \frac{n}{2} |z|^n \max_{|z|=1} |G(z)| \tag{9}$$

$$\left| z^2 G''(z) + (1 - n) z G'(z) + \frac{n^2}{4} G(z) \right| \leq \frac{n^2}{4} |z|^n \max_{|z|=1} |G(z)|. \tag{10}$$

The inequalities (9) *and* (10) *are sharp and equality holds for* $G(z) = az^n$, $a \neq 0$

Next, if we take $\alpha = 1$ in (7) and $\beta = 0$ in (8), we obtain the following:

Corollary 3 *Let* $G(z)$ *be a polynomial of degree* $n \geq 2$ *and* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, *then for* $|z| \geq 1$,

$$|zG'(z) - G(z)| \leq (n-1)|z|^n \max_{|z|=1} |G(z)|$$

$$|zG''(z) + (1-\alpha)G'(z)| \leq n|n-\alpha||z|^{n-1} \max_{|z|=1} |G(z)|.$$

The results are best possible and $G(z) = az^n$, $a \neq 0$ *is the extremal polynomial for both the inequalities.*

The Theorem 6 and preceding corollaries can be improved for the class of polynomials having no zero in $|z| < 1$. For that, we require following lemmas.

3 Lemmas

If we are given an nth degree polynomial $f(z)$ which does not vanish for $|z| < 1$, then all the zeros of $q(z) = z^n\overline{f(1/\overline{z})}$ lie in $|z| \leq 1$ and $|f(z)| = |q(z)|$ for $|z| = 1$. Applying Theorem 6 by taking $G(z) = f(z)$ and $F(z) = q(z)$, we get:

Lemma 1 *Let a polynomial* $f(z)$ *of degree* n *has no zero in* $|z| < 1$ *and* $q(z) = z^n\overline{f(1/\overline{z})}$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ *and* $|z| \geq 1$, *we have*

$$|zf'(z) - \alpha f(z)| \leq |zq'(z) - \alpha q(z)| \tag{11}$$

and

$$\begin{aligned}|z^2f''(z) + (1-\alpha-\beta)zf'(z) + \alpha\beta f(z)|\\ \leq |z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)|.\end{aligned} \tag{12}$$

Lemma 2 *Let* $f(z)$ *be a polynomial of degree* n *and* $q(z) = z^n\overline{f(1/\overline{z})}$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ *and* $|z| \geq 1$, *we have*

$$|zf'(z) - \alpha f(z)| + |zq'(z) - \alpha q(z)| \leq (|n-\alpha| + |\alpha|)|z|^n \max_{|z|=1} |f(z)|,$$

and

$$\begin{aligned}&|z^2f''(z) + (1-\alpha-\beta)zf'(z) + \alpha\beta f(z)|\\ &\quad + |z^2q''(z) + (1-\alpha-\beta)zq'(z) + \alpha\beta q(z)|\\ &\qquad \leq |n(n-\alpha-\beta) + \alpha\beta| + |\alpha\beta|)|z|^n \max_{|z|=1} |f(z)|.\end{aligned}$$

Proof. Let $M = \max_{|z|=1} |f(z)|$ then by Rouche's theorem, for every $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$, the polynomial $f(z) + \lambda M$ does not vanish in $|z| < 1$. Applying Lemma 1 to $f(z) + \lambda M$, then for $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ and $|z| \geq 1$, we have

$$|(zf'(z) - \alpha f(z)) + \lambda\alpha M| \leq |(zq'(z) - \alpha q(z)) + \bar{\lambda}(n - \alpha)Mz^n| \tag{13}$$

and

$$\begin{aligned} &|z^2 f''(z) + (1-\alpha - \beta)zf'(z) + \alpha\beta f(z) + \alpha\beta\lambda M| \\ &\qquad \leq |z^2 q''(z) + (1 - \alpha - \beta)zq'(z) + \alpha\beta q(z) \\ &\qquad\qquad + \bar{\lambda}\{n(n - \alpha - \beta) + \alpha\beta\}Mz^n|, \end{aligned} \tag{14}$$

where $q(z) = z^n\overline{f(1/\bar{z})}$. In view of Corollary 1, we can choose argument of λ in (13) and separately in (14) such that

$$|(zq'(z) - \alpha q(z)) + \bar{\lambda}(n - \alpha)Mz^n| = |\bar{\lambda}||n - \alpha||z|^n M - |zq'(z) - \alpha q(z)|$$

and

$$\begin{aligned} &|z^2 q''(z) + (1 - \alpha - \beta)zq'(z) + \alpha\beta q(z)) + \bar{\lambda}\{n(n - \alpha - \beta) + \alpha\beta\}Mz^n| \\ &\quad = |\bar{\lambda}||n(n - \alpha - \beta) + \alpha\beta||z|^n M - |z^2 q''(z) + (1 - \alpha - \beta)zq'(z) + \alpha\beta q(z)| \end{aligned}$$

for $|z| \geq 1$. Using these inequalities in (13) and (14) and taking $|\lambda| = 1$, we conclude for $|z| \geq 1$,

$$|zf'(z) - \alpha f(z)| + |zq'(z) - \alpha q(z)| \leq (|n - \alpha| + |\alpha|)M|z|^n,$$

and

$$\begin{aligned} &|z^2 f''(z) + (1-\alpha - \beta)zf'(z) + \alpha\beta f(z)| \\ &\qquad + |z^2 q''(z) + (1 - \alpha - \beta)zq'(z) + \alpha\beta q(z)| \\ &\qquad\qquad \leq |n(n - \alpha - \beta) + \alpha\beta| + |\alpha\beta|)M|z|^n. \end{aligned}$$

This completes the proof. □

4 Extension of Erdös-Lax theorem

Finally, we prove the following extension of inequality (3) for the class of polynomials having no zero in $|z| < 1$.

Theorem 7 *Let* $G(z)$ *be a polynomial of degree* n *and has no zero in* $|z| < 1$, *then for every* $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, $\Re(\beta) \leq n/2$ *and* $|z| \geq 1$, *we have*

$$|zG'(z) - \alpha G(z)| \leq \frac{|n-\alpha| + |\alpha|}{2} |z|^n \max_{|z|=1} |G(z)| \tag{15}$$

and

$$\begin{aligned} &|z^2 G''(z) + (1-\alpha-\beta) zG'(z) + \alpha\beta G(z)| \\ &\qquad \leq \frac{|n(n-\alpha-\beta) + \alpha\beta| + |\alpha\beta|}{2} |z|^n \max_{|z|=1} |G(z)|. \end{aligned} \tag{16}$$

Equality in (15) *and* (16) *hold for* $G(z) = az^n + b$ *where* $|a| = |b| \neq 0$.

Proof. The proof follows by combining the lemmas 1 and 2. □

Note that inequality (3) follows from (15) by taking $\alpha = 0$.

If we take $\alpha = 1$ in (15) and $\beta = 0$ in (16), we obtain the following:

Corollary 4 *Let a polynomial* $G(z)$ *of degree* $n \geq 2$ *does not vanish for* $|z| < 1$ *and* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) \leq n/2$, *then for* $|z| \geq 1$,

$$|zG'(z) - G(z)| \leq \frac{n}{2} |z|^n \max_{|z|=1} |G(z)| \tag{17}$$

$$|zG''(z) + (1-\alpha) zG'(z)| \leq \frac{n|n-\alpha|}{2} |z|^{n-1} \max_{|z|=1} |G(z)|. \tag{18}$$

These inequalities are sharp.

Remark 1 *A polynomial* $f(z)$ *of degree* n *is said to be self-inversive if* $f(z) = \sigma q(z)$, *where* $q(z) = z^n \overline{f(1/\overline{z})}$ *and* $|\sigma| = 1$. *It is not difficult to prove that the Theorem 7 also holds for self-inversive polynomials as well.*

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References

[1] S. Bernstein, Sur la limitation des dérivées des polynomes, *C. R. Math. Acad. Sci. Paris*, **190** (1930), 338–340.

[2] P. D. Lax, Proof of a conjecture of P. Erdös on the derivative of a polynomial, *Bull. Amer. Math. Soc.*, **50** (1944), 509–513.

[3] M. Marden, *Geometry of polynomials*, American Mathematical Society, Providence, 1985.

[4] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros,* World Scientific Publishing Company, Singapore, 1994.

[5] Q. I. Rahman & G. Schmeisser, *Analytic theory of polynomials*, Claredon Press, Oxford, 2002.

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Two generalizations of dual-complex Lucas-balancing numbers

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Abstract. In this paper, we study two generalizations of dual-complex Lucas-balancing numbers: dual-complex k-Lucas balancing numbers and dual-complex k-Lucas-balancing numbers. We give some of their properties, among others the Binet formula, Catalan, Cassini, d'Ocagne identities.

1 Introduction

The sequence of balancing numbers, denoted by $\{B_n\}$, was introduced by Behera and Panda in [4]. In [9], Panda introduced the sequence of Lucas-balancing numbers, denoted by $\{C_n\}$ and defined as follows: if B_n is a balancing number, the number $C_n = \sqrt{8B_n^2 + 1}$ is called a Lucas-balancing number.

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Recall that a balancing number n with balancer r is the solution of the Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r). \tag{1}$$

The balancing and Lucas-balancing numbers fulfill the following recurrence relations

$$B_n = 6B_{n-1} - B_{n-2} \text{ for } n \geq 2, \text{ with } B_0 = 0, B_1 = 1,$$

$$C_n = 6C_{n-1} - C_{n-2} \text{ for } n \geq 2, \text{ with } C_0 = 1, C_1 = 3.$$

The Table 1 includes initial terms of the balancing and Lucas-balancing numbers for $0 \leq n \leq 7$.

Table 1.

n	0	1	2	3	4	5	6	7
B_n	0	1	6	35	204	1189	6930	40391
C_n	1	3	17	99	577	3363	19601	114243

The Binet type formulas for the balancing and Lucas-balancing numbers have the forms

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$C_n = \frac{\alpha^n + \beta^n}{2},$$

respectively, for $n \geq 0$, where $\alpha = 3 + 2\sqrt{2}$, $\beta = 3 - 2\sqrt{2}$.

The concept of balancing numbers has been extended and generalized by many authors, see [7, 8, 10]. In this paper, we focus our attention on k-Lucas balancing numbers and k-Lucas-balancing numbers and their applications in the theory of dual-complex numbers.

Based on the concept from [6], Özkoç in [7] introduced k-Lucas balancing numbers as follows.

For some positive integer $k \geq 1$ let C_n^k denote the nth k-Lucas balancing number which is the number defined by

$$C_n^k = 6kC_{n-1}^k - C_{n-2}^k$$

for $n \geq 2$, with $C_0^k = 1$, $C_1^k = 3$.

Theorem 1 ([7]) *The Binet type formula for* k*-Lucas balancing numbers is*

$$C_n^k = \frac{(3-\beta_k)\alpha_k^n - (3-\alpha_k)\beta_k^n}{2\sqrt{9k^2-1}}, \tag{2}$$

for $n \geq 0$, $k \geq 1$, *where* $\alpha_k = 3k + \sqrt{9k^2-1}$, $\beta_k = 3k - \sqrt{9k^2-1}$.

Another generalization of the Lucas-balancing numbers was presented in [12]. For integer $k \geq 1$ the sequence of k-Lucas-balancing numbers (written with two hyphens) is defined recursively by

$$C_{k,n} = 6kC_{k,n-1} - C_{k,n-2}$$

for $n \geq 2$, with $C_{k,0} = 1$, $C_{k,1} = 3k$.

Theorem 2 ([13]) *The Binet type formula for* k*-Lucas-balancing numbers is*

$$C_{k,n} = \frac{\alpha_k^n + \beta_k^n}{2} \tag{3}$$

for $n \geq 0$, $k \geq 1$, *where* $\alpha_k = 3k + \sqrt{9k^2-1}$, $\beta_k = 3k - \sqrt{9k^2-1}$.

Note that for $k = 1$ we have $C_n^1 = C_{1,n} = C_n$.

Complex and dual numbers are well known two dimensional number systems. Let $\mathbb{C}$ and $\mathbb{D}$ denote the set of complex numbers with imaginary unit i and the set of dual numbers with nilpotent unit ε, respectively. The set of dual-complex numbers is expressed in the form

$$\mathbb{DC} = \{w = z_1 + \varepsilon z_2 : \ z_1, z_2 \in \mathbb{C}, \ \varepsilon^2 = 0, \ \varepsilon \neq 0\},$$

see [1]. Here if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then any dual-complex number can be written as

$$w = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2. \tag{4}$$

If $w_1 = z_1 + \varepsilon z_2$ and $w_2 = z_3 + \varepsilon z_4$ are any two dual-complex numbers then the equality, the addition, the subtraction, the multiplication by scalar and the multiplication are defined in the natural way:

$$\begin{aligned}
&w_1 = w_2 \ \text{ only if } \ z_1 = z_3, \ z_2 = z_4,\\
&w_1 \pm w_2 = (z_1 \pm z_3) + \varepsilon(z_2 \pm z_4),\\
&\text{for } \ s \in \mathbb{R} : \ sw_1 = sz_1 + \varepsilon s z_2,\\
&w_1 \cdot w_1 = z_1 z_3 + \varepsilon(z_1 z_4 + z_2 z_3).
\end{aligned}$$

If we write the dual-complex numbers using (4) then the multiplication of dual-complex numbers can be made analogously as multiplications of algebraic expressions using Table 2.

Table 2. The dual-complex numbers multiplication

$\cdot$	i	ε	$i\varepsilon$
i	-1	$i\varepsilon$	$-\varepsilon$
ε	$i\varepsilon$	0	0
$i\varepsilon$	$-\varepsilon$	0	0

Balancing and Lucas-balancing numbers are numbers defined by the linear recurrence relation and they are named as numbers of the Fibonacci type. These numbers have many applications in the theory of hypercomplex numbers, for details see [14]. Some interesting properties of dual-complex Fibonacci and dual-complex Lucas numbers we can find in [5]. The dual-complex Pell numbers (quaternions) were introduced quite recently in [3]. In [2], the author investigated one-parameter generalization of dual-complex Fibonacci numbers, called dual-complex k-Fibonacci numbers. Based on these ideas we define and study dual-complex Lucas-balancing numbers and their generalizations.

2 Main results

Let $n \geq 0$ be an integer. The nth dual-complex balancing number $\mathbb{DC}B_n$ and nth dual-complex Lucas-balancing number $\mathbb{DC}C_n$ are defined as

$$\mathbb{DC}B_n = B_n + iB_{n+1} + \varepsilon B_{n+2} + i\varepsilon B_{n+3},$$

$$\mathbb{DC}C_n = C_n + iC_{n+1} + \varepsilon C_{n+2} + i\varepsilon C_{n+3},$$

where B_n is the nth balancing number, C_n is the nth Lucas-balancing number and i, ε, $i\varepsilon$ are dual-complex units.

In the similar way we define the nth dual-complex k-Lucas balancing number $\mathbb{DC}C_n^k$ and the nth dual-complex k-Lucas-balancing number $\mathbb{DC}C_{k,n}$ as

$$\begin{aligned}\mathbb{DC}C_n^k &= C_n^k + iC_{n+1}^k + \varepsilon C_{n+2}^k + i\varepsilon C_{n+3}^k,\\ \mathbb{DC}C_{k,n} &= C_{k,n} + iC_{k,n+1} + \varepsilon C_{k,n+2} + i\varepsilon C_{k,n+3},\end{aligned}$$

respectively.

For $k = 1$ we have $\mathbb{DC}C_n^1 = \mathbb{DC}C_{1,n} = \mathbb{DC}C_n$.

Theorem 3 (*Binet type formulas*) *Let* $n \geq 0$, $k \geq 1$ *be integers. Then*

$$\text{(i)}\quad \mathbb{DC}C_n^k = \frac{(3-\beta_k)\alpha_k^n\widehat{\alpha_k} - (3-\alpha_k)\beta_k^n\widehat{\beta_k}}{2\sqrt{9k^2-1}},$$
$$\text{(ii)}\quad \mathbb{DC}C_{k,n} = \frac{\alpha_k^n\widehat{\alpha_k} + \beta_k^n\widehat{\beta_k}}{2},$$

where

$$\alpha_k = 3k + \sqrt{9k^2-1}, \quad \beta_k = 3k - \sqrt{9k^2-1} \tag{5}$$

and

$$\widehat{\alpha_k} = 1 + i\alpha_k + \varepsilon\alpha_k^2 + i\varepsilon\alpha_k^3, \quad \widehat{\beta_k} = 1 + i\beta_k + \varepsilon\beta_k^2 + i\varepsilon\beta_k^3. \tag{6}$$

Proof. By formula (2) we get

$$\begin{aligned}
\mathbb{DC}C_n^k &= C_n^k + iC_{n+1}^k + \varepsilon C_{n+2}^k + i\varepsilon C_{n+3}^k \\
&= \frac{(3-\beta_k)\alpha_k^n - (3-\alpha_k)\beta_k^n}{2\sqrt{9k^2-1}} + i\frac{(3-\beta_k)\alpha_k^{n+1} - (3-\alpha_k)\beta_k^{n+1}}{2\sqrt{9k^2-1}} \\
&+ \varepsilon\frac{(3-\beta_k)\alpha_k^{n+2} - (3-\alpha_k)\beta_k^{n+2}}{2\sqrt{9k^2-1}} + i\varepsilon\frac{(3-\beta_k)\alpha_k^{n+3} - (3-\alpha_k)\beta_k^{n+3}}{2\sqrt{9k^2-1}}
\end{aligned}$$

and after calculation we obtain (i). By the same method, using (3), we can prove formula (ii). □

For $k = 1$ we obtain the Binet type formula for the dual-complex Lucas-balancing numbers.

Corollary 1 *Let* $n \geq 0$ *be an integer. Then*

$$\mathbb{DC}C_n = \frac{\alpha^n\hat{\alpha} + \beta^n\hat{\beta}}{2},$$

where

$$\begin{aligned}
&\alpha = 3 + 2\sqrt{2}, \quad \beta = 3 - 2\sqrt{2}, \\
&\hat{\alpha} = 1 + i(3+\sqrt{8}) + \varepsilon(17+6\sqrt{8}) + i\varepsilon(99+35\sqrt{8}), \\
&\hat{\beta} = 1 + i(3-\sqrt{8}) + \varepsilon(17-6\sqrt{8}) + i\varepsilon(99-35\sqrt{8}).
\end{aligned} \tag{7}$$

Moreover, by simple calculations, we get

$$\begin{aligned}
\alpha_k + \beta_k &= 6k,\\
\alpha_k - \beta_k &= 2\sqrt{9k^2-1},\\
\alpha_k\beta_k &= 1,\\
(3-\alpha_k)(3-\beta_k) &= 10-18k,\\
\alpha_k^3+\beta_k^3 &= (\alpha_k+\beta_k)^3 - 3\alpha_k\beta_k(\alpha_k+\beta_k) = 216k^3-18k
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\alpha_k}\widehat{\beta_k} &= \left(1+i\alpha_k+\varepsilon\alpha_k^2+i\varepsilon\alpha_k^3\right)\left(1+i\beta_k+\varepsilon\beta_k^2+i\varepsilon\beta_k^3\right)\\
&= i(\alpha_k+\beta_k)+i\varepsilon(\alpha_k^3+\beta_k^3+\alpha_k+\beta_k)\\
&= i(6k)+i\varepsilon(216k^3-12k).
\end{aligned}$$

In particular, for $k=1$, we have

$$\hat{\alpha}\hat{\beta} = 6i + 204i\varepsilon.$$

Now we will give some identities such as Catalan type, Cassini type and d'Ocagne type identities for the dual-complex k-Lucas balancing numbers and dual-complex k-Lucas-balancing numbers. These identities can be proved using the Binet type formulas for these numbers.

Theorem 4 (*Catalan type identity for dual-complex k-Lucas balancing numbers*) *Let $k \geq 1$, $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned}
&\mathbb{DC}C_{n-r}^k \cdot \mathbb{DC}C_{n+r}^k - \left(\mathbb{DC}C_n^k\right)^2 =\\
&= \frac{(3-\beta_k)(3-\alpha_k)}{4(9k^2-1)}\left(2-\left(\frac{\beta_k}{\alpha_k}\right)^r-\left(\frac{\alpha_k}{\beta_k}\right)^r\right)\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}$$

where α_k, β_k and $\widehat{\alpha_k}$, $\widehat{\beta_k}$ are given by (5) and (6), respectively.

Proof. By formula (i) of Theorem 3 we have

$$\begin{aligned}
&\mathbb{DC}C_{n-r}^k \cdot \mathbb{DC}C_{n+r}^k - \left(\mathbb{DC}C_n^k\right)^2\\
&= \frac{-(3-\alpha_k)(3-\beta_k)\alpha_k^{n-r}\beta_k^{n+r}\widehat{\alpha_k}\widehat{\beta_k} - (3-\alpha_k)(3-\beta_k)\alpha_k^{n+r}\beta_k^{n-r}\widehat{\alpha_k}\widehat{\beta_k}}{4(9k^2-1)}\\
&+ \frac{2(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\widehat{\alpha_k}\widehat{\beta_k}}{4(9k^2-1)}\\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\widehat{\alpha_k}\widehat{\beta_k}}{4(9k^2-1)}\left(2-\left(\frac{\beta_k}{\alpha_k}\right)^r-\left(\frac{\alpha_k}{\beta_k}\right)^r\right).
\end{aligned}$$

Using the fact that $\alpha_k\beta_k = 1$, we obtain the desired formula. □

Theorem 5 (*Catalan type identity for dual-complex* k*-Lucas-balancing numbers*) *Let* $k \geq 1$, $n \geq 0$, $r \geq 0$ *be integers such that* $n \geq r$. *Then*

$$\mathbb{DC}C_{k,n-r} \cdot \mathbb{DC}C_{k,n+r} - (\mathbb{DC}C_{k,n})^2 = \frac{1}{4}\left(\left(\frac{\beta_k}{\alpha_k}\right)^r + \left(\frac{\alpha_k}{\beta_k}\right)^r - 2\right)\widehat{\alpha_k}\widehat{\beta_k},$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Proof. By formula (ii) of Theorem 3 we have

$$\begin{aligned}
&\mathbb{DC}C_{k,n-r} \cdot \mathbb{DC}C_{k,n+r} - (\mathbb{DC}C_{k,n})^2 \\
&= \frac{\alpha_k^{n-r}\widehat{\alpha_k}\beta_k^{n+r}\widehat{\beta_k} + \beta_k^{n-r}\widehat{\beta_k}\alpha_k^{n+r}\widehat{\alpha_k} - 2\alpha_k^n\widehat{\alpha_k}\beta_k^n\widehat{\beta_k}}{4} \\
&= \frac{1}{4}\alpha_k^n\beta_k^n\widehat{\alpha_k}\widehat{\beta_k}\left(\left(\frac{\beta_k}{\alpha_k}\right)^r + \left(\frac{\alpha_k}{\beta_k}\right)^r - 2\right).
\end{aligned}$$

Using the fact that $\alpha_k\beta_k = 1$, we obtain the desired formula. □

Note that for $r = 1$ we obtain Cassini type identities for the dual-complex k-Lucas balancing numbers and the dual-complex k-Lucas-balancing numbers.

Corollary 2 (*Cassini type identity for dual-complex* k*-Lucas balancing numbers*) *Let* $k \geq 1$, $n \geq 1$ *be integers. Then*

$$\mathbb{DC}C_{n-1}^k \cdot \mathbb{DC}C_{n+1}^k - \left(\mathbb{DC}C_n^k\right)^2 = (18k - 10)\,\widehat{\alpha_k}\widehat{\beta_k},$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Corollary 3 (*Cassini type identity for dual-complex* k*-Lucas-balancing numbers*) *Let* $k \geq 1$, $n \geq 1$ *be integers. Then*

$$\mathbb{DC}C_{k,n-1} \cdot \mathbb{DC}C_{k,n+1} - (\mathbb{DC}C_{k,n})^2 = \left(9k^2 - 1\right)\widehat{\alpha_k}\widehat{\beta_k},$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Theorem 6 (*d'Ocagne type identity for dual-complex* k*-Lucas balancing numbers*) *Let* $k \geq 1$, $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$\begin{aligned}
&\mathbb{DC}C_m^k \cdot \mathbb{DC}C_{n+1}^k - \mathbb{DC}C_{m+1}^k \cdot \mathbb{DC}C_n^k = \\
&= \frac{(3 - \alpha_k)(3 - \beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)}{2\sqrt{9k^2 - 1}}\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Proof. By formula (i) of Theorem 3 we have

$$
\begin{aligned}
&\mathbb{DCC}_m^k \cdot \mathbb{DCC}_{n+1}^k - \mathbb{DCC}_{m+1}^k \cdot \mathbb{DCC}_n^k \\
&= \frac{-(3-\beta_k)\alpha_k^m \widehat{\alpha_k}(3-\alpha_k)\beta_k^{n+1}\widehat{\beta_k} - (3-\alpha_k)\beta_k^m \widehat{\beta_k}(3-\beta_k)\alpha_k^{n+1}\widehat{\alpha_k}}{4(9k^2-1)} \\
&+ \frac{(3-\beta_k)\alpha_k^{m+1}\widehat{\alpha_k}(3-\alpha_k)\beta_k^n\widehat{\beta_k} + (3-\alpha_k)\beta_k^{m+1}\widehat{\beta_k}(3-\beta_k)\alpha_k^n\widehat{\alpha_k}}{4(9k^2-1)} \\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n+1} + \beta_k^{m-n+1} - \alpha_k\beta_k^{m-n} - \alpha_k^{m-n}\beta_k\right)}{4(9k^2-1)}\widehat{\alpha_k}\widehat{\beta_k} \\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\alpha_k-\beta_k)}{4(9k^2-1)}\widehat{\alpha_k}\widehat{\beta_k} \\
&= \frac{(3-\alpha_k)(3-\beta_k)\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)}{2\sqrt{9k^2-1}}\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}
$$

which ends the proof. □

Theorem 7 (*d'Ocagne type identity for dual-complex* k*-Lucas-balancing numbers*) *Let* $k \geq 1$, $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$
\begin{aligned}
&\mathbb{DCC}_{k,m} \cdot \mathbb{DCC}_{k,n+1} - \mathbb{DCC}_{k,m+1} \cdot \mathbb{DCC}_{k,n} = \\
&= \frac{1}{4}\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\beta_k - \alpha_k)\,\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}
$$

where α_k, β_k *and* $\widehat{\alpha_k}$, $\widehat{\beta_k}$ *are given by* (5) *and* (6), *respectively.*

Proof. By formula (ii) of Theorem 3 we have

$$
\begin{aligned}
&\mathbb{DCC}_{k,m} \cdot \mathbb{DCC}_{k,n+1} - \mathbb{DCC}_{k,m+1} \cdot \mathbb{DCC}_{k,n} \\
&= \frac{\alpha_k^m\widehat{\alpha_k}\beta_k^{n+1}\widehat{\beta_k} + \beta_k^m\widehat{\beta_k}\alpha_k^{n+1}\widehat{\alpha_k} - \alpha_k^{m+1}\widehat{\alpha_k}\beta_k^n\widehat{\beta_k} - \beta_k^{m+1}\widehat{\beta_k}\alpha_k^n\widehat{\alpha_k}}{4} \\
&= \frac{1}{4}\alpha_k^n\beta_k^n\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\beta_k - \alpha_k)\,\widehat{\alpha_k}\widehat{\beta_k} \\
&= \frac{1}{4}\left(\alpha_k^{m-n} - \beta_k^{m-n}\right)(\beta_k - \alpha_k)\,\widehat{\alpha_k}\widehat{\beta_k},
\end{aligned}
$$

which ends the proof. □

For $k = 1$ we obtain the Catalan, Cassini and d'Ocagne identities for the dual-complex Lucas-balancing numbers.

Corollary 4 (*Catalan type identity for dual-complex Lucas-balancing numbers*) *Let* $n \geq 0$, $r \geq 0$ *be integers such that* $n \geq r$. *Then*

$$\mathbb{DC}C_{n-r} \cdot \mathbb{DC}C_{n+r} - (\mathbb{DC}C_n)^2 =$$
$$= -\frac{1}{4}\left(2 - \left(\frac{\beta}{\alpha}\right)^r - \left(\frac{\alpha}{\beta}\right)^r\right)\hat{\alpha}\hat{\beta},$$

where α, β, $\hat{\alpha}$ *and* $\hat{\beta}$ *are given by* (7).

Corollary 5 (*Cassini type identity for dual-complex Lucas-balancing numbers*) *Let* $n \geq 1$ *be an integer. Then*

$$\mathbb{DC}C_{n-1} \cdot \mathbb{DC}C_{n+1} - (\mathbb{DC}C_n)^2 = 8\hat{\alpha}\hat{\beta},$$

where $\hat{\alpha}$ *and* $\hat{\beta}$ *are given by* (7).

Corollary 6 (*d'Ocagne type identity for dual-complex Lucas-balancing numbers*) *Let* $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$\mathbb{DC}C_m \cdot \mathbb{DC}C_{n+1} - \mathbb{DC}C_{m+1} \cdot \mathbb{DC}C_n = -\sqrt{2}\left(\alpha^{m-n} - \beta^{m-n}\right)\hat{\alpha_k}\hat{\beta_k},$$

where α, β, $\hat{\alpha}$ *and* $\hat{\beta}$ *are given by* (7).

3 Concluding Remarks

Cobalancing numbers were defined and introduced in [10] by modification of formula (1). The authors called positive integer number n a cobalancing number with cobalancer r if

$$1 + 2 + \ldots + n = (n + 1) + (n + 2) + \ldots + (n + r).$$

Let b_n denote the nth cobalancing number. The nth Lucas-cobalancing number c_n is defined with $c_n = \sqrt{8b_n^2 + 8b_n + 1}$, see [7, 8].

In [11], we can find some relations of balancing and cobalancing numbers with Pell numbers. Related to these dependences it seems to be interesting to define dual-complex cobalancing numbers, dual-complex Lucas-cobalancing numbers and next to find relations of dual-complex balancing and cobalancing numbers with dual-complex Pell numbers (quaternions). For dual-complex Pell numbers details, see [3].

References

[1] E. Ata and Y. Yaylı, Dual unitary matrices and unit dual quaternions, *Differ. Geom. Dyn. Syst.*, **10** (2008), 1–12.

[2] F. T. Aydın, Dual-complex k-Fibonacci numbers, *Chaos Solitons Fractals*, **115** (2018), 1–6.

[3] F. T. Aydın, Dual Complex Pell Quaternions, *Journal of Computer Science & Computational Mathematics*, June 2020, DOI: 10.20967/jcscm.2020.02.001.

[4] A. Behera and G. K. Panda, On the square roots of triangular numbers, *Fibonacci Quart.*, **37(2)** (1999), 98–105.

[5] M. A. Güngör and A. Z. Azak, Investigation of Dual-Complex Fibonacci, Dual-Complex Lucas Numbers and Their Properties, *Adv. Appl. Clifford Algebr.*, **27** (2017), 3083–3096.

[6] K. Liptai, F. Luca, Á. Pintér and L. Szalay, Generalized balancing numbers, *Indag. Math. (N.S.)*, **20(1)** (2009), 87–100.

[7] A. Özkoç, Tridiagonal matrices via k-balancing number, *British Journal of Mathematics & Computer Science*, **10(4)** (2015), 1–11.

[8] A. Özkoç and A. Tekcan, On k-balancing numbers, *Notes on Number Theory and Discrete Mathematics*, Vol. 23, No. 3 (2017), 38–52.

[9] G. K. Panda, Some fascinating properties of balancing numbers, *Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications, Cong. Numerantium*, **194** (2009), 185–189.

[10] G. K. Panda and P. K. Ray, Cobalancing numbers and cobalancers, *Int. J. Math. Math. Sci.*, **8** (2005), 1189–1200.

[11] G. K. Panda and P. K. Ray, Some links of balancing and cobalancing numbers with Pell and associated Pell numbers, *Bull. Inst. Math. Acad. Sin. (N.S.)*, Vol. 6 (2011), No. 1, 41–72.

[12] B. K. Patel, N. Irmak and P. K. Ray, Incomplete balancing and Lucas-balancing numbers, *Math. Rep. (Bucur.)*, **20(1)** (2018), 59–72.

[13] P. K. Ray, On the properties of k-balancing and k-Lucas-balancing numbers, *Acta Comment. Univ. Tartu. Math.*, Vol. 21, No 2 (2017), 259–274.

[14] A. Szynal-Liana and I. Włoch, *Hypercomplex numbers of the Fibonacci type*, Oficyna Wydawnicza Politechniki Rzeszowskiej, Rzeszów 2019.

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A short note on Layman permutations

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Abstract. A permutation p of $[k] = \{1, 2, 3, \ldots, k\}$ is called Layman permutation iff $i+p(i)$ is a Fibonacci number for $1 \leq i \leq k$. This concept is introduced by Layman in the A097082 entry of the Encyclopedia of Integers Sequences, that is the number of Layman permutations of $[n]$. In this paper, we will study Layman permutations. We introduce the notion of the Fibonacci complement of a natural number, that plays a crucial role in our investigation. Using this notion we prove some results on the number of Layman permutations, related to a conjecture of Layman that is implicit in the A097083 entry of OEIS.

1 Introduction

Sequence $(F_i)_{i=0}^{\infty}$ is the Fibonacci sequence ([9] A000045) defined as $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with $F_0 = 0$ and $F_1 = 1$. We refer to $F_2 < F_3 < F_4 < \ldots$ as Fibonacci numbers. These numbers

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

are the initial object of essential mathematical research. Also, many deep results in mathematics use them to solve central open problems. For example, the solution of Hilbert's tenth problem [8], or designing complex data structures for important algorithms [4] rely on properties of Fibonacci numbers.

Many mathematical concepts are related to Fibonacci numbers. Enumerating special permutations leads to the sequence $(F_i)_{i=0}^{\infty}$: The set of permutations

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with $|\sigma(i) - i| \leq 1$ for all $i = 1, \ldots, n$ is called the set of Fibonacci permutations. Investigating these has proved very fruitful (see for example [1] and [3]). Permutation polynomials can also be linked to Fibonacci numbers (see [2]).

Our motivation is different from the above. Layman introduced a special property of permutations (hereafter referred to as *Layman's property*) which is also related to Fibonacci numbers. Such permutations are henceforth called *Layman permutations.*

Definition 1 (Layman (2004) [7]) *A permutation* p *of* $[k] = \{1, 2, 3, \ldots, k\}$ *is called Layman permutation iff* $i + p(i)$ *is a Fibonacci number for all* $1 \leq i \leq k$.

The following permutations are Layman permutations

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

We use the two-line notation to represent permutations. The last one denotes $\pi : 1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$, $4 \mapsto 4$, i.e. $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 2$, $\pi(4) = 4$. The Layman's property means that the column sums in these permutations are Fibonacci numbers.

Layman also submitted the sequence of "the number of Layman permutations of $[n]$" to OEIS (entry A097082). The first few terms suggest that for all positive integer n the set $[n]$ has Layman permutation. Also, infinitely often $[n]$ has unique Layman permutation. The sequence of these positive integers is submitted as the A097083 entry of OEIS. These entries of the Encyclopedia do not have any mathematical content. The statements in A097083 are all hypothetical ones, they are conjectures.

The main reason for this paper is to establish some mathematical results on these sequences. Our main results are two claims. The first one is an easy observation.

Observation 1 *For all positive natural number* n *the set* $[n]$ *has a Layman permutation.*

For the second results we need to introduce the sequence

$$M_m(n) = \sum_{2 \leq i \leq n, i \equiv n \pmod{m}} F_i = F_n + F_{n-m} + F_{n-2m} + \ldots.$$

We can consider $M_m(2)$ as the initial term of the sequence or start the sequence with $M_m(0) = M_m(1) = 0$ (the value of the empty sum).

$$M_4(n) = \sum_{2 \le i \le n, i \equiv n \pmod 4} F_i = F_n + F_{n-4} + F_{n-8} + \dots$$

plays a very important role in our discussion.

Theorem 1 *If* $n \in \mathbb{N}_+$ *is not in the sequence* $(M_4(k))_{k=2}^{\infty}$ *then* $[n]$ *has at least two Layman permutations.*

The entry A097083 of OEIS suggests the following conjecture.

Conjecture 1 *For* $n \in \mathbb{N}_+$ *that set* $[n]$ *has a unique Layman permutation if and only if* n *is in the sequence* $(M_4(k))_{k=2}^{\infty}$.

We established one direction of the conjecture.

In section 2 we introduce the notion of the Fibonacci complement of a positive integer. Using the properties of this notion in section 3 we prove our main results.

Throughout the paper the set $\{0, 1, 2, 3, \dots\}$, i.e. the set of natural numbers is denoted as $\mathbb{N}$. $\mathbb{N}_+$ denotes the set of positive integers. The intervals are always intervals of $\mathbb{Z}$, so $]2, 6] = (2, 6] = \{3, 4, 5, 6\}$. $A \dot{\cup} B$ denotes $A \cup B$ and contains the extra information that A and B are disjoint.

2 Fibonacci complement of positive integers

Definition 2 *Let* $n \in \mathbb{N}_+$ *be a positive integer.* $\nu \in \mathbb{N}$ *is the Fibonacci complement of* n *iff* $1 \le \nu \le n$ *and* $n + \nu$ *is a Fibonacci number.*

We will use F-complement as an abbreviation of Fibonacci complement.

Observation 2 *Every positive number has one or two F-complements.*

Proof. Let F_ℓ be the minimal Fibonacci number, that is larger than n: $F_{\ell-1} \le n < F_\ell$. The F-complements are the terms of $F_\ell - n < F_{\ell+1} - n < F_{\ell+2} - n < \dots$, that are at most n.

$$n < F_\ell < F_\ell + (F_{\ell+1} - n) = F_{\ell+2} - n,$$

hence we have only two options left: $F_\ell - n$ and $F_{\ell+1} - n$.

$$F_\ell = F_{\ell-1} + F_{\ell-2} \le n + n,$$

so $F_\ell - n \le n$ is an F-complement indeed. □

Notation 1 *Using the notation of the proof of the above observation we write* $\overline{n}^F$ *for* $F_\ell - n$*, i.e.* $\overline{n}^F$ *is the only F-complement of* n *or the smaller of the two ones.*

The final result of this section describes which case occurs for each natural number n. For this, we need some preparations.

Recall, that

$$M_3(k) = F_k + F_{k-3} + F_{k-6} + \dots .$$

Lemma 1 $M_k(3)$ *is the largest natural number* t*, that satisfies* $2t < F_{k+2}$.

Proof. It is well-known that $2F_k = F_{k+2} - F_{k-1}$ (see [6]). So

$$\begin{aligned} 2M_3(k) =& 2F_k + 2F_{k-3} + 2F_{k-6} + \dots \\ =& (F_{k+2} - F_{k-1}) + (F_{k-1} - F_{k-4}) + (F_{k-4} - F_{k-7}) + \dots . \end{aligned}$$

The last term is $2F_2 = F_4 - 1$ or $2F_3 = F_5 - 1$ or $2F_4 = F_6 - 2$. Depending on the parity of F_{k+2} (F_s is even iff s is divisible by 3, see [6]) we get $F_{k+2} - 1$ (when F_{k+2} is odd) or $F_{k+2} - 2$ (when F_{k+2} is even). After collapsing the telescopic sum we get $F_{k+2} - 1$ or $F_{k+2} - 2$, that proves the claim. $\square$

Recall that

$$M_2(k) = F_k + F_{k-2} + F_{k-4} + \dots = F_{k+1} - 1,$$

where the last equality is a well-known, easy fact on Fibonacci numbers (see [6]). Using Lemma 1 we get the following important claim.

Lemma 2 *For all* $\ell \in \mathbb{N}_+$ *any number* $n \in [M_3(\ell-1)+1, M_2(\ell-1)] = [M_3(\ell-1)+1, F_\ell[$ *has two F-complements. If* $n \in [F_{\ell-1}, M_3(\ell-1)]$ *for any* $\ell \in \mathbb{N}_+$*, then it has exactly one F-complement.*

Note that $[F_{\ell-1}, M_3(\ell-1)]\dot\cup]M_3(\ell-1), F_\ell[$ covers all integers in $[F_{\ell-1}, F_\ell[$, furthermore these intervals partition $\mathbb{N}_+$.

Proof. Take an arbitrary natural number n from $[F_{\ell-1}, F_\ell[$. Note that our notation coincides with the notation of the proof of Observation 2: F_ℓ is the minimal Fibonacci number, that is larger than n: $F_{\ell-1} \le n < F_\ell$.

From the proof of Observation 2 we know that n has two F-complements iff $F_{\ell+1} - n \le n$, i.e. $F_{\ell+1} \le 2n$.

Lemma 1 says that $M_3(\ell-1)$ satisfies $2M_3(\ell-1) < F_{\ell+1}$. Hence the elements of $[F_{\ell-1}, M_3(\ell-1)]$, i.e. $M_3(\ell-1)$ and smaller numbers from our interval have unique F-complement.

Lemma 1 also says that $2(M_3(\ell-1)+1) \ge F_{\ell+1}$. Hence $n \in]M_3(\ell-1), F_\ell[$ has two F-complements. $\square$

3 Layman permutations

Notation 2 *Let $\mathcal{L}_n$ be denote the set of Layman permutations.*

We will use the so-called two-line notation to describe permutations. A $2 \times n$ matrix visualizes the permutation. The Layman property is equivalent to that each column sum is a Fibonacci number. Examples for Layman permutations:

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\2&1&5&4&3\end{pmatrix},$$

$$\begin{pmatrix}1&2&3&4&5&6\\1&6&5&4&3&2\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6\\4&6&5&1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7\\2&1&5&4&3&7&6\end{pmatrix},$$

$$\begin{pmatrix}1&2&3&4&5&6&7\\7&6&5&4&3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\1&3&2&4&8&7&6&5\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8\\4&3&2&1&8&7&6&5\end{pmatrix}.$$

$$\begin{pmatrix}1&2&3&4&5&6&7&8&9&10\\7&1&10&9&8&2&6&5&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7&8&9&10\\2&6&10&9&8&7&1&5&4&3\end{pmatrix}.$$

Observation 3 *For any positive integer n the set $\mathcal{L}_n$ is not empty.*

Our previous examples prove that the claim is true for $n \leq 8$. If $\overline{n}^F = 1$ (equivalently $\overline{n}^{F-} := \overline{n}^F - 1 = 0$) then the reverse permutation exhibits the truth of Observation 3.

One can prove Observation 3 by induction: If $\overline{n}^F - 1 > 0$ take $p \in \mathcal{L}_{\overline{n}^{F-}}$ and extend it with

$$\begin{pmatrix}\overline{n}^F & \overline{n}^F+1 & \dots & n-1 & n\\ n & n-1 & \dots & \overline{n}^F+1 & \overline{n}^F\end{pmatrix}.$$

The argument, proving Observation 3, immediately gives us the following claim.

Observation 4 *If n has two F-complements then $\mathcal{L}_n$ has more than one element.*

Indeed. We have already constructed one. In that we used $\overline{n}^F$ to cut $[n]$ into two blocks and apply induction plus a reverse permutation. We can do the same with a second F-complement.

Observe that $M_4(\ell-1) \in [F_{\ell-1}, M_3(\ell-1)]$, hence $M_4(\ell-1)$ has a unique F-complement.

Also understanding the simple proof leads to the following definition.

Definition 3 *Let* $(s_i^{(n)})$ *the following finite, decreasing sequence of positive integers:* $s_0 = n$*, furthermore if* s_i *exists and* $\overline{s_i}^{F-}$ *is positive then* s_{i+1} *exists too and* $s_{i+1} = \overline{s_i}^{F-}$.

Based on $(s_i^{(n)})_{i=0}^k$ we can explicitly describe the permutation produced by the above recursion: Partition $\{1, 2, \dots, n\}$ into blocks

$$\{1, 2, \dots, s_k\} \cup \{s_k + 1, \dots, s_{k-1}\} \cup \dots \cup \{s_2 + 1, \dots, s_1\} \cup \{s_1 + 1, \dots, s_0\}$$

and reverse the order of each block (note that $\overline{s_k}^F = 1$ and $\overline{s_{i-1}}^F = s_i + 1$ for $i = 1, 2, \dots, k-1$).

Let us see a few examples (each arrow denotes the application of the mapping $x \mapsto \overline{x}^{F-}$):

$$s^{(2021)} : 2021 \to 562 \to 47 \to 8 \to 4,$$

$$s^{(1869)} : 1869 \to 714 \to 272 \to 104 \to 59 \to 13 \to 5 \to 2, s^{(14)} : 14 \to 6 \to 1,$$

$$s^{10} : 10 \to 2, s^{(9)} : 9 \to 3 \to 1, s^{(8)} : 8 \to 4, s^{(7)} : 7, s^{(6)} : 6 \to 1.$$

Corollary 1 *Assume that for* $n \in \mathbb{N}_+$ *in the sequence* $(s_i^{(n)})_{i=0}^k$ *we have the element* 6 *or* 10 *or a number with two F-complements. Then* $\mathcal{L}_n$ *has more than one element.*

For example, $\mathcal{L}_6$ and $\mathcal{L}_{10}$ have more than 1 permutation (see of our previous examples). $\mathcal{L}_7$ has more than 1 permutation since 7 has two F-complements (1 and 6). $\mathcal{L}_{2021}$ has more than one permutation since 47 is in its s-sequence and 47 has two F-complements (8 and 42): For example, we obtain two elements of $\mathcal{L}_{2021}$ we start with two elements of $\mathcal{L}_{47}$ based on the two F-complements of 47 and extend them by

$$\begin{pmatrix} 48 & 49 & \dots & 561 & 562 & 563 & \dots & 2021 \\ 562 & 561 & \dots & 49 & 48 & 2021 & \dots & 563 \end{pmatrix}.$$

Note that in the case of $n = M_4(\ell)$ the corresponding s-sequence is

$$M_4(\ell), M_4(\ell - 2), M_4(\ell - 4), \dots$$

a sequence ending with $M_4(3) = 2$ or with $M_4(2) = 1$. Indeed $M_4(\ell) + M_4(\ell - 2) = M_2(\ell) = F_{\ell+1} - 1$, i.e. $\overline{M_4(\ell)}^{F-} = M_4(\ell - 2)$.

A simple consequence of Corollary 1 is the following Theorem.

Theorem 2 *Assume that* $n \in \mathbb{N}_+$ *is a number not in the form* $M_4(\ell)$*. Then* $\mathcal{L}_n$ *has more than one element.*

Proof. $n \in [F_{\ell-1}, F_\ell[$ for a unique ℓ. We are going to prove our claim, by induction on ℓ.

The claim is easy for $\ell = 3, 4, 5, 6, 7$. For the induction step, assume that

$$n \in [F_{\ell-1}, F_\ell[\setminus\{M_4(\ell-1)\} = [F_{\ell-1}, M_3(\ell-1)]\setminus\{M_4(\ell-1)\}\dot\cup[M_3(\ell-1)+1, M_2(\ell-1)]$$

If $n \in [M_3(\ell-1)+1, M_2(\ell-1)]$, them we are done since $n = s_0^{(n)}$ has two F-Complements. If $k \in [F_{\ell-1}, M_3(\ell-1)]$, then

$$s_1^{(k)} = \overline{k}^{F-} = F_\ell - k - 1 \in [F_\ell - M_3(\ell-1) - 1, F_\ell - F_{\ell-1} - 1].$$

Remember, that $\overline{M_4(\ell-1)}^{F-} = M_4(\ell-3)$.

So if $n \in [F_{\ell-1}, M_3(\ell-1)] \setminus \{M_4(\ell-1)\}$ then

$$s_1^{(n)} = \overline{(n)}^{F-} = F_\ell - n - 1 \in [F_\ell - M_3(\ell-1) - 1, M_4(\ell-3) - 1]\dot\cup[M_4(\ell-3)+1, F_{\ell-2} - 1].$$

Easy to check that $M_4(\ell-4) < F_\ell - M_3(\ell-1) - 1$ hence the right hand side does not contain any number of the form $M_4(m)$. The Theorem is proved. $\square$

The proof really gave us the claim, that if n is not of the form $M_4(\ell)$, then the assumption of Corollary 1 holds.

So the hardness of Layman's conjecture (Conjecture 1) is to prove that for $n = M_4(\ell)$ we have a unique Layman permutation.

4 Conclusion

We consider Conjecture 1 as a nice, important conjecture. It has a graph theoretical interpretation about bipartite graphs with a unique perfect matching. The investigation of bipartite graphs with unique perfect matching ([5]) is independent of our motivation. The conjecture connects two different lines of research. We made the first step to settle the conjecture. We need further effort to understand Layman permutations.

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References

[1] J. -L. Baril and P. T. Do, ECO-Generation for p-generalized Fibonacci and Lucas permutations, *PU.M.A.*, **17**(2006) (1–2), 19–37.

[2] N. Fernando and M. H. Rashid, *Fibonacci Permutation Polynomials*, Manuscript, December 2017, available at https://www.researchgate.net/publication/321994479_Fibonacci_Permutation_Polynomials (accessed at 2022 June).

[3] E. Foundas, C. Patsakis and G. Chondrocoukis, Lucas permutations and some notes on Fibonacci permutations, *Journal of Discrete Mathematical Sciences and Cryptography*, Volume 11, 2008(2), 181–190.

[4] M. L. Fredman, R. E. Tarjan, Fibonacci heaps and their uses in improved network optimization algorithms, *Journal of the Association for Computing Machinery*, **34**(3) (1987), 596–615.

[5] C. D. Godsil, Inverses of trees, *Combinatorica*, **5**(1985), 33–39.

[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications, Volume 1*, Second Edition, John Wiley & Sons, Hoboken NJ, USA, 2018.

[7] J. W. Layman, Entries A097082 and A097083 in OEIS (2004).

[8] Y. V. Matiyasevic, *Hilbert's tenth problem*, Foundations of Computing Series, MIT Press, Cambridge, MA, 1993. Translated from the 1993 Russian original by the author; With a foreword by Martin Davis.

[9] OEIS, The On-Line Encyclopedia of Integer Sequences, https://oeis.org/, Founded in 1964 by N.J.A. Sloan.

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Extremal trees for the Randić index

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Abstract. Graph theory has applications in various fields due to offering important tools such as topological indices. Among the topological indices, the Randić index is simple and of great importance. The Randić index of a graph $\mathcal{G}$ can be expressed as $R(\mathcal{G}) = \sum_{xy \in Y(\mathcal{G})} \frac{1}{\sqrt{\tau(x)\tau(y)}}$, where $Y(\mathcal{G})$ represents the edge set and $\tau(x)$ is the degree of vertex x. In this paper, considering the importance of the Randić index and applications two-trees graphs, we determine the first two minimums among the two-trees graphs.

1 Introduction

Let $\mathcal{G}$ be a simple graph having the vertex set $X = X(\mathcal{G})$ and the edge set $Y(\mathcal{G})$. Moreover, $\upsilon = |X(\mathcal{G})|$ and $m = |Y(\mathcal{G})|$. In this case, we say that $\mathcal{G}$ is a graph of order υ and size m. The open neighborhood of vertex x is defined as $\Omega_{\mathcal{G}}(x) = \Omega(x) = \{y \in X(\mathcal{G}) | xy \in Y(\mathcal{G})\}$ and the degree of x is denoted by $|\Omega(x)| = \tau_{\mathcal{G}}(x) = \tau(x)$. Suppose that $\mathcal{G}$ is a graph with $x \in X(\mathcal{G})$ and $xy \in Y(\mathcal{G})$, then the graphs $G - x$ and $G - xy$ are obtained by removing the vertex x and the edge xy from $\mathcal{G}$, respectively.

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The two-tree graph was first defined in the paper [9] in two steps as follows:
A. If $s = 0$ then $\Theta_0 = K_2$. In this case, we have a two-tree with two vertices.
B. Suppose Θ_s is a two-tree produced during the s-th step. Then, graph Θ_{s+1} is generated during $(s+1)$-step from graph Θ_s by adding a new vertex adjacent to the two end vertices of an edge in Θ_s.

Some examples are shown in Figure 1. They will play an important role in the later development. Two-tree graphs have found many applications in complex networks see [6, 13] and more information see e.g. [5, 7, 8, 12].

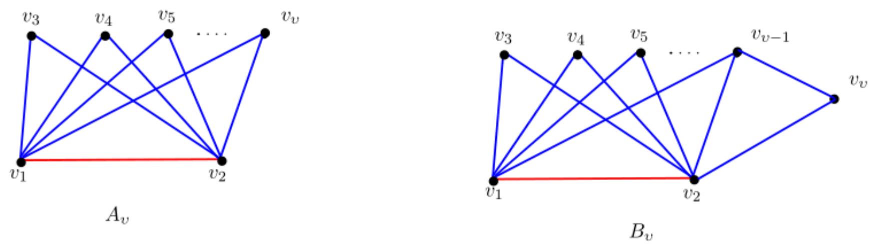

Figure 1: The graphs A_υ and B_υ.

In the past few decades, the topological indices due to their wide applications has been considered by many researchers. One of their oldest applications in chemistry was proposed by Wiener in the paper [11], which gave rise to the Wiener index. Due to the application of topological indices in various fields, many indices have been defined nowadays and their applications have been identified. Among those numerous indices [10], the Randić index was the most successful. The Randić index was first introduced in chemistry by Milan Randić in [4] to obtain the boiling point of paraffin and is defined as follows:

$$R(\mathcal{G}) = \sum_{xy \in Y(\mathcal{G})} \frac{1}{\sqrt{\tau(x)\tau(y)}}.$$

As we mentioned before, due to the importance of the Randić index in other indices, it has been studied by many researchers over the years.

The Randić index has been considered extensively by researchers in the field of mathematics. For example, Lu et al. [3] discussed the Randić index quasi-tree graphs. Bermudo et al. [1] characterized Randić index tree with given domination number. In [2], the authors characterized Randić index for chemical trees. Motivated by the above line of research and the importance of the two-tree graphs, we in this paper intend to discuss the Randić index of two-tree graphs. We establish the Randić index of two-tree graphs and determine the first two minimums among the two-tree graphs.

2 Some lemmas

In this section, we will prove a few lemmas that will help us achieve the main results.

Lemma 1 *Let* $3 \leq g, h \leq \upsilon - 1$*. For the function*

$$\varphi(g,h) = \frac{g-1}{\sqrt{2g}} + \frac{h-1}{\sqrt{2h}} - \frac{g-2}{\sqrt{2(g-1)}} - \frac{h-2}{\sqrt{2(h-1)}} + \frac{1}{\sqrt{gh}} - \frac{1}{\sqrt{(g-1)(h-1)}},$$

we have $\varphi(g,h) \geq \varphi(\upsilon-1, \upsilon-1)$*.*

Proof. By deriving from function $\varphi(g,h)$, we have

$$\begin{aligned}\frac{\partial\varphi(g,h)}{\partial g} =& \frac{g-2}{(2g-2)^{3/2}} - \frac{g-1}{(2g)^{3/2}} \\ & - \frac{1}{\sqrt{2g-2}} + \frac{1}{\sqrt{2g}} + \frac{h-1}{2((g-1)(h-1))^{3/2}} - \frac{h}{2(gh)^{3/2}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial h}\frac{\partial\varphi(g,h)}{\partial g} =& \frac{1}{2((g-1)(h-1))^{3/2}} - \frac{1}{2(gh)^{3/2}} \\ & - \frac{3(g-1)(h-1)}{4((g-1)(h-1))^{5/2}} + \frac{3gh}{4(gh)^{5/2}}.\end{aligned}$$

Note that when $g, h \geq 3$, we get $\frac{\partial}{\partial h}\frac{\partial\varphi(g,h)}{\partial g} < 0$. Therefore, we have

$$\begin{aligned}\frac{\partial\varphi(g,h)}{\partial g} \leq& \frac{\partial\varphi(g,3)}{\partial g} \\ =& \frac{g-2}{(2g-2)^{3/2}} - \frac{g-1}{(2g)^{3/2}} \\ & - \frac{1}{\sqrt{2g-2}} + \frac{1}{\sqrt{2g}} + \frac{1}{(2g-2)^{3/2}} - \frac{3}{2(3g)^{3/2}}.\end{aligned}$$

It is not difficult to see that the above inequality is negative for $g \geq 3$. Hence, we derive $\frac{\partial\varphi(g,h)}{\partial g}, \frac{\partial\varphi(g,h)}{\partial h} < 0$ and that means $\varphi(g,h) \geq \varphi(\upsilon-1, \upsilon-1)$. □

Lemma 2 *Let $3 \leq g, h \leq \upsilon - 2$. For the following function*

$$\psi(g,h) = \frac{1}{\sqrt{gh}} - \frac{1}{\sqrt{(g-1)(h-1)}} + \frac{1}{\sqrt{3g}} - \frac{1}{\sqrt{3(g-1)}} + \frac{1}{\sqrt{3h}} - \frac{1}{\sqrt{3(h-1)}} + \frac{g-3}{\sqrt{2g}} - \frac{g-3}{\sqrt{2(g-1)}} + \frac{h-3}{\sqrt{2h}} - \frac{h-3}{\sqrt{2(h-1)}},$$

we have $\psi(g,h) \geq \psi(\upsilon-2, \upsilon-2)$.

Proof. By deriving from function $\psi(g,h)$, we have

$$\frac{\partial\psi(g,h)}{\partial g} = \frac{g-3}{(2g-2)^{3/2}} - \frac{g-2}{(2g)^{3/2}} - \frac{1}{\sqrt{2g-2}} + \frac{1}{\sqrt{2g}} + \frac{h-1}{2((g-1)(h-1))^{3/2}} - \frac{h}{2(gh)^{3/2}} + \frac{3}{2(3g-3)^{3/2}} - \frac{3}{2(3g)^{3/2}}$$

and

$$\frac{\partial}{\partial h}\frac{\partial\psi(g,h)}{\partial g} = \frac{1}{2((g-1)(h-1))^{3/2}} - \frac{1}{2(gh)^{3/2}} - \frac{3(g-1)(h-1)}{4((g-1)(h-1))^{5/2}} + \frac{3gh}{4(gh)^{5/2}}.$$

Note that when $g, h \geq 3$, we get $\frac{\partial}{\partial h}\frac{\partial\psi(g,h)}{\partial g} < 0$. Hence, we have

$$\begin{aligned}\frac{\partial\psi(g,h)}{\partial g} \leq& \frac{\partial g(g,3)}{\partial g} \\ =& \frac{g-3}{(2g-2)^{3/2}} - \frac{g-2}{(2g)^{3/2}} - \frac{1}{\sqrt{2g-2}} \\ &+ \frac{1}{\sqrt{2g}} + \frac{1}{(2g-2)^{3/2}} + \frac{3}{2(3g-3)^{3/2}} - \frac{3}{(3g)^{3/2}}.\end{aligned}$$

It is not difficult to see that the above inequality is negative for $g \geq 3$. Hence, we obtain $\frac{\partial\psi(g,h)}{\partial g}, \frac{\partial\psi(g,h)}{\partial h} < 0$ and that means $\psi(g,h) \geq \psi(\upsilon-1, \upsilon-1)$. □

Lemma 3 *For $\upsilon > 4$, we have*

$$h(\upsilon) = \frac{2(\upsilon-4)}{\sqrt{2\upsilon-4}} - \frac{(\upsilon-5)}{\sqrt{2\upsilon-6}} - \frac{\upsilon-3}{\sqrt{2\upsilon-2}} + \frac{2}{\sqrt{3\upsilon-6}} - \frac{1}{\sqrt{3\upsilon-9}} - \frac{1}{\sqrt{3\upsilon-3}} + \frac{1}{\upsilon-2} - \frac{1}{\upsilon-3} + \frac{1}{\sqrt{(\upsilon-3)(\upsilon-2)}} - \frac{1}{\sqrt{(\upsilon-2)(\upsilon-1)}} > 0.$$

Proof. We know that

$$\begin{aligned}h(\upsilon) &= \frac{2(\upsilon-4)}{\sqrt{2\upsilon-4}} - \frac{(\upsilon-5)}{\sqrt{2\upsilon-6}} - \frac{\upsilon-3}{\sqrt{2\upsilon-2}} + \frac{2}{\sqrt{3\upsilon-6}} - \frac{1}{\sqrt{3\upsilon-9}} \\ &\quad - \frac{1}{\sqrt{3\upsilon-3}} + \frac{1}{\upsilon-2} - \frac{1}{\upsilon-3} + \frac{1}{\sqrt{(\upsilon-3)(\upsilon-2)}} - \frac{1}{\sqrt{(\upsilon-2)(\upsilon-1)}} \\ &= \frac{2\sqrt{3}\upsilon - 8\sqrt{3} + 2\sqrt{2}}{\sqrt{6(\upsilon-2)}} - \frac{\sqrt{3}\upsilon - 5\sqrt{3} + \sqrt{2}}{\sqrt{6(\upsilon-3)}} - \frac{\sqrt{3}\upsilon - 3\sqrt{3} + \sqrt{2}}{\sqrt{6(\upsilon-1)}} \\ &\quad + \frac{\sqrt{(\upsilon-3)(\upsilon-2)} - 1}{(\upsilon-3)(\upsilon-2)} - \frac{1}{\sqrt{(\upsilon-2)(\upsilon-1)}} > 0.\end{aligned}$$

It is not difficult the above inequality holds for $\upsilon \geq 4$.

□

3 Main results

In this section, we will discuss the Randić index of two-tree graphs and determine the first two minimums among this type of tree.

We start by proving a new result for Randić index.

Theorem 1 *For a two-tree graph $\mathcal{G}$ with $\upsilon \geq 4$ vertices, we have*

$$R(\mathcal{G}) \geq \frac{1}{\upsilon-1} + \frac{2(\upsilon-2)}{\sqrt{2(\upsilon-1)}}.$$

The equality holds if and only if $\mathcal{G} = A_\upsilon$ (see Figure 1).

Proof. We start the proof by inducing on υ. First, we assume that $\mathbb{T}_\upsilon$ is a two-tree of with four vertices. If $\mathcal{G}$ is a graph with four vertices, then this graph can be obtained from the complete graph with four vertices by removing an edge. By applying the definition of the Randić index for this graph, we get $R(\mathbb{T}_4) = 1.9663264951888 = \frac{2\sqrt{6}+1}{3}$. We suppose our result holds for $\upsilon - 1$. Select a vertex of degree two from the graph $\mathbb{T}_\upsilon$, and we call it z. It is not difficult to see that the graph $\mathbb{T}_\upsilon - z$ is a two-tree of $\upsilon - 1$ vertices.

By applying the induction hypothesis, we derive $R(\mathbb{T}_\upsilon - z) \geq R(A_{\upsilon-1})$ and the equality holds if and only if $\mathbb{T}_\upsilon - z \cong A_{\upsilon-1}$. Hence, to complete the proof it suffices to show that $R(\mathbb{T}_\upsilon) \geq R(A_\upsilon)$.

Assume that x and y are two vertices adjacent to the vertex z in $\mathbb{T}_\upsilon$. Let $\tau_{\mathbb{T}_\upsilon}(x) = \vartheta$, $\tau_{\mathbb{T}_\upsilon}(y) = \rho$ and $\Omega_{\mathbb{T}_\upsilon}(x)\backslash\{y,z\} = \{x_1, x_2, \ldots, x_{\vartheta-2}\}$, $\Omega_{\mathbb{T}_\upsilon}(y)\backslash\{x,z\} =$

$\{x_1, x_2, \ldots, x_{\rho-2}\}$. Noth that $3 \le \vartheta, \rho \le \upsilon - 1$. By applying Lemma 1 and the induction hypothesis, we can write that

$$\begin{aligned}
R(\mathbb{T}_\upsilon) =& R(\mathbb{T}_\upsilon - z) + \frac{1}{\sqrt{2\vartheta}} + \frac{1}{\sqrt{2\rho}} + \frac{1}{\sqrt{\vartheta\rho}} - \frac{1}{\sqrt{(\vartheta-1)(\rho-1)}} \\
&+ \sum_{i=1}^{\vartheta-2} \left(\frac{1}{\sqrt{\vartheta\tau(x_i)}} - \frac{1}{\sqrt{(\vartheta-1)\tau(x_i)}} \right) \\
&+ \sum_{j=1}^{\rho-2} \left(\frac{1}{\sqrt{\rho\tau(y_j)}} - \frac{1}{\sqrt{(\rho-1)\tau(y_j)}} \right) \\
\ge& R(A_{\upsilon-1}) + \frac{1}{\sqrt{2\vartheta}} + \frac{1}{\sqrt{2\rho}} + \frac{1}{\sqrt{\vartheta\rho}} - \frac{1}{\sqrt{(\vartheta-1)(\rho-1)}} \\
&+ \sum_{i=1}^{\vartheta-2} \left(\frac{1}{\sqrt{2\vartheta}} - \frac{1}{\sqrt{2(\vartheta-1)}} \right) + \sum_{j=1}^{\rho-2} \left(\frac{1}{\sqrt{2\rho}} - \frac{1}{\sqrt{2(\rho-1)}} \right) \\
=& R(A_{\upsilon-1}) + \frac{\vartheta-1}{\sqrt{2\vartheta}} + \frac{\rho-1}{\sqrt{2\rho}} \\
&- \frac{\vartheta-2}{\sqrt{2(\vartheta-1)}} - \frac{\rho-2}{\sqrt{2(\rho-1)}} + \frac{1}{\sqrt{\vartheta\rho}} - \frac{1}{\sqrt{(\vartheta-1)(\rho-1)}} \\
\ge& R(A_{\upsilon-1}) + \frac{2(\upsilon-2)}{\sqrt{2(\upsilon-1)}} - \frac{2(\upsilon-3)}{\sqrt{2(\upsilon-2)}} + \frac{1}{\upsilon-1} - \frac{1}{\sqrt{(\upsilon-2)(\upsilon-2)}} \\
=& \frac{1}{\upsilon-2} + \frac{2(\upsilon-3)}{\sqrt{2(\upsilon-2)}} + \frac{2(\upsilon-2)}{\sqrt{2(\upsilon-1)}} - \frac{2(\upsilon-3)}{\sqrt{2(\upsilon-2)}} + \frac{1}{\upsilon-1} - \frac{1}{(\upsilon-2)} \\
=& R(A_\upsilon),
\end{aligned}$$

where the last equality is right if and only if $\mathbb{T}_\upsilon - z \cong A_{\upsilon-1}, \vartheta = \rho = \upsilon - 1$ and $d_{\mathbb{T}_\upsilon}(x_i) = 2$ for $i = 1, 2, \ldots, \upsilon - 3$ and that means $\mathbb{T}_\upsilon \cong A_\upsilon$. This completes the proof. □

Theorem 2 *For a two-tree graph $\mathcal{G}$ with $\upsilon \ge 5$ vertices and $\mathcal{G} \ncong A_\upsilon$, we have*

$$\begin{aligned}
R(\mathcal{G}) \ge& \frac{1}{\sqrt{(\upsilon-1)(\upsilon-2)}} + \frac{\upsilon-3}{\sqrt{2(\upsilon-1)}} \\
&+ \frac{\upsilon-4}{\sqrt{2(\upsilon-2)}} + \frac{1}{\sqrt{3(\upsilon-1)}} + \frac{1}{\sqrt{3(\upsilon-2)}} + \frac{\sqrt{6}}{6}.
\end{aligned}$$

The equality holds if and only if $\mathcal{G} \cong B_\upsilon$; see Figure 1.

Proof. We begin the proof by inducing on υ. We assume that $\mathbb{T}_\upsilon$ is a two-tree with n vertices and $\mathbb{T}_\upsilon \ncong A_\upsilon$. First, we assume that $\mathbb{T}_\upsilon$ is a two-tree of with five vertices. We obtain $\mathbb{T}_\upsilon \cong A_\upsilon$ or $\mathbb{T}_\upsilon \cong B_\upsilon$. Since $\mathbb{T}_\upsilon \ncong A_\upsilon$, then $\mathcal{G} \cong B_\upsilon$. Obviously $R(\mathbb{T}_5) = 2.4342869646372 = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} + \frac{\sqrt{6}}{3} + \frac{1}{3}$. Suppose that theorem is true for $\upsilon - 1$. It is clear that a two-tree graph has at least two vertices of degree two. In addition, we have $\mathbb{T}_\upsilon \ncong A_\upsilon$. Hence we choose a vertex of $\mathbb{T}_\upsilon$ with degree two, and call this vertex z. We have $\mathbb{T}_\upsilon - z \ncong A_{\upsilon-1}$. Clearly, the graph $\mathbb{T}_\upsilon - w$ is a two-tree with $\upsilon - 1$ vertices. By applying the induction hypothesis, we get that $R(\mathbb{T}_\upsilon - z) \geq R(B_{\upsilon-1})$ and the equality holds if and only if $\mathbb{T}_\upsilon - z \cong B_{\upsilon-1}$. To complete the proof, it suffices for us to show that $R(\mathbb{T}_\upsilon) \geq R(B_\upsilon)$. Assume that x and y are two vertices adjacent to the vertex z in $\mathbb{T}_\upsilon$. Note that $\upsilon \geq 5$. Given that the graph is a two-tree graph, there should exist a vertex ξ adjacent to two vertices x and y satisfying $\tau_{\mathbb{T}_\upsilon}(\xi) \geq 3$ (Otherwise, $\mathbb{T}_\upsilon - z \ncong A_{\upsilon-1}$). Let $\tau_{\mathbb{T}_\upsilon}(x) = \vartheta$, $\tau_{\mathbb{T}_\upsilon}(y) = \rho$, $\tau_{\mathbb{T}_\upsilon}(\xi) = \zeta$ and $\Omega_{\mathbb{T}_\upsilon}(x) \setminus \{x, z, \xi\} = \{x_1, x_2, \dots, x_{\vartheta-3}\}$, $\Omega_{\mathbb{T}_\upsilon}(y) \setminus \{x, z, \xi\} = \{y_1, y_2, \dots, y_{\rho-3}\}$. Note that $3 \leq \vartheta, \rho, \zeta \leq \upsilon - 1$. For convenience here we assume that $\vartheta \leq \rho$ and $\max\{\vartheta, \rho, \zeta\} = \rho$. Since vertex ξ is not adjacent to the vertex z, $\zeta \leq \upsilon - 2$ and $\vartheta \leq \rho \leq \upsilon - 2$. By applying Lemma 2, Lemma 3 and the induction hypothesis, we obtain

$$\begin{aligned}
R(\mathbb{T}_\upsilon) =& R(\mathbb{T}_\upsilon - z) + \frac{1}{\sqrt{2\vartheta}} + \frac{1}{\sqrt{2\rho}} + \frac{1}{\sqrt{\vartheta\rho}} - \frac{1}{\sqrt{(\vartheta-1)(\rho-1)}} \\
&+ \frac{1}{\vartheta\zeta} - \frac{1}{\sqrt{(\vartheta-1)\zeta}} + \frac{1}{\sqrt{\rho\zeta}} - \frac{1}{\sqrt{(\rho-1)\zeta}} \\
&+ \sum_{i=1}^{\vartheta-3} \left(\frac{1}{\sqrt{\vartheta\tau(x_i)}} - \frac{1}{\sqrt{(\vartheta-1)\tau(x_i)}} \right) \\
&+ \sum_{j=1}^{\rho-3} \left(\frac{1}{\sqrt{\rho\tau(y_j)}} - \frac{1}{\sqrt{(\rho-1)\tau(y_j)}} \right) \\
\geq& R(B_{\upsilon-1}) \frac{1}{\sqrt{2\vartheta}} + \frac{1}{\sqrt{2\rho}} + \frac{1}{\sqrt{\vartheta\rho}} - \frac{1}{\sqrt{(\vartheta-1)(\rho-1)}} \\
&+ \frac{1}{\sqrt{3\vartheta}} - \frac{1}{\sqrt{3(\vartheta-1)}} + \frac{1}{\sqrt{3\rho}} - \frac{1}{\sqrt{3(\rho-1)}} \\
&+ \sum_{i=1}^{\vartheta-3} \left(\frac{1}{\sqrt{2\vartheta}} - \frac{1}{\sqrt{2(\vartheta-1)}} \right) + \sum_{j=1}^{\rho-3} \left(\frac{1}{\sqrt{2\rho}} - \frac{1}{\sqrt{2(\rho-1)}} \right)
\end{aligned}$$

$$
\begin{aligned}
&=R(B_{\upsilon-1})+\frac{1}{\sqrt{\vartheta\rho}}-\frac{1}{\sqrt{(\vartheta-1)(\rho-1)}}\\
&\quad+\frac{1}{\sqrt{3\vartheta}}-\frac{1}{\sqrt{3(\vartheta-1)}}+\frac{1}{\sqrt{3\rho}}-\frac{1}{\sqrt{3(\rho-1)}}\\
&\quad+\frac{\vartheta-2}{\sqrt{2\vartheta}}-\frac{\vartheta-3}{\sqrt{2(\vartheta-1)}}+\frac{\rho-2}{\sqrt{2\rho}}-\frac{\rho-3}{\sqrt{2(\rho-1)}}\\
&\geq R(B_{\upsilon-1})+\frac{2(\upsilon-4)}{\sqrt{2\upsilon-4}}-\frac{2(\upsilon-5)}{\sqrt{2\upsilon-6}}+\frac{2}{\sqrt{3\upsilon-6}}-\frac{2}{\sqrt{3\upsilon-9}}\\
&\quad+\frac{1}{\upsilon-2}-\frac{1}{\upsilon-3}\\
&=\frac{3(\upsilon-4)}{\sqrt{2\upsilon-4}}-\frac{\upsilon-5}{\sqrt{2\upsilon-6}}+\frac{\sqrt{6}}{6}+\frac{3}{\sqrt{3\upsilon-6}}-\frac{1}{\sqrt{3\upsilon-9}}\\
&\quad+\frac{1}{\upsilon-2}-\frac{1}{\upsilon-3}+\frac{1}{\sqrt{(\upsilon-3)(\upsilon-2)}}\\
&=R(B_{\upsilon})+\frac{2(\upsilon-4)}{\sqrt{2\upsilon-4}}-\frac{(\upsilon-5)}{\sqrt{2\upsilon-6}}-\frac{\upsilon-3}{\sqrt{2\upsilon-2}}+\frac{2}{\sqrt{3\upsilon-6}}-\frac{1}{\sqrt{3\upsilon-9}}\\
&\quad-\frac{1}{\sqrt{3\upsilon-3}}+\frac{1}{\upsilon-2}-\frac{1}{\upsilon-3}+\frac{1}{\sqrt{(\upsilon-3)(\upsilon-2)}}-\frac{1}{\sqrt{(\upsilon-2)(\upsilon-1)}}\\
&>R(B_{\upsilon}).
\end{aligned}
$$

Hence, we derive $\rho \leq \upsilon-1$ and $\max\{\vartheta,\zeta\} \leq \upsilon-2$. Otherwise, we have $\mathbb{T}_\upsilon \ncong A_\upsilon$. Again by applying Lemma 2 and the induction hypothesis, we know that

$$
\begin{aligned}
R(\mathbb{T}_\upsilon) =&R(\mathbb{T}_\upsilon - z)+\frac{1}{\sqrt{2\vartheta}}+\frac{1}{\sqrt{2\rho}}+\frac{1}{\sqrt{\vartheta\rho}}-\frac{1}{\sqrt{(\vartheta-1)(\rho-1)}}\\
&+\frac{1}{\sqrt{\vartheta\zeta}}-\frac{1}{\sqrt{(\vartheta-1)\zeta}}+\frac{1}{\sqrt{\rho\zeta}}-\frac{1}{\sqrt{(\rho-1)\zeta}}\\
&+\sum_{i=1}^{\vartheta-3}\left(\frac{1}{\sqrt{\vartheta\tau(x_i)}}-\frac{1}{\sqrt{(\vartheta-1)\tau(x_i)}}\right)\\
&+\sum_{j=1}^{\rho-3}\left(\frac{1}{\sqrt{\rho\tau(y_j)}}-\frac{1}{\sqrt{(\rho-1)\tau(y_j)}}\right)\\
\geq& R(B_{\upsilon-1})\frac{1}{\sqrt{2\vartheta}}+\frac{1}{\sqrt{2\rho}}+\frac{1}{\sqrt{\vartheta\rho}}-\frac{1}{\sqrt{(\vartheta-1)(\rho-1)}}\\
&+\frac{1}{\sqrt{3\vartheta}}-\frac{1}{\sqrt{3(\vartheta-1)}}+\frac{1}{\sqrt{3\rho}}-\frac{1}{\sqrt{3(\rho-1)}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{\vartheta-3}\left(\frac{1}{\sqrt{2\vartheta}}-\frac{1}{\sqrt{2(\vartheta-1)}}\right)+\sum_{j=1}^{\rho-3}\left(\frac{1}{\sqrt{2\rho}}-\frac{1}{\sqrt{2(\rho-1)}}\right)\\
=&R(B_{\upsilon-1})+\frac{1}{\sqrt{\vartheta\rho}}-\frac{1}{\sqrt{(\vartheta-1)(\rho-1)}}\\
&+\frac{1}{\sqrt{3\vartheta}}-\frac{1}{\sqrt{3(\vartheta-1)}}+\frac{1}{\sqrt{3\rho}}-\frac{1}{\sqrt{3(\rho-1)}}\\
&+\frac{\vartheta-2}{\sqrt{2\vartheta}}-\frac{\vartheta-3}{\sqrt{2(\vartheta-1)}}+\frac{\rho-2}{\sqrt{2\rho}}-\frac{\rho-3}{\sqrt{2(\rho-1)}}\\
\geq&R(B_{\upsilon-1})+\frac{\upsilon-3}{\sqrt{2\upsilon-2}}-\frac{\upsilon-5}{\sqrt{2\upsilon-6}}+\frac{1}{\sqrt{3\upsilon-3}}-\frac{1}{\sqrt{3\upsilon-9}}\\
&+\frac{1}{\sqrt{(\upsilon-2)(\upsilon-1)}}-\frac{1}{\sqrt{(\upsilon-3)(\upsilon-2)}}\\
=&\frac{1}{\sqrt{(\upsilon-1)(\upsilon-2)}}+\frac{\upsilon-3}{\sqrt{2(\upsilon-1)}}+\frac{\upsilon-4}{\sqrt{2(\upsilon-2)}}\\
&+\frac{1}{\sqrt{3(\upsilon-1)}}+\frac{1}{\sqrt{3(\upsilon-2)}}+\frac{\sqrt{6}}{6}\\
=&R(B_{\upsilon}).
\end{aligned}
$$

It is easy to check that the last equality holds if and only if $\mathbb{T}_\upsilon - z \cong B_{\upsilon-1}, \vartheta = \upsilon-2, \rho = \upsilon-1, \zeta = 3$ and $\tau_{\mathbb{T}_\upsilon}(x_i) = \tau_{\mathbb{T}_\upsilon}(y_j) = 2$ for $i = 1, 2, \ldots, \upsilon-3$ and $j = 1, 2, \ldots, \upsilon-2$, which means $\mathbb{T}_\upsilon \cong B_\upsilon$. □

4 Concluding Remark

In this article we have studied the Randić index for two-tree graphs and also discussed the first minimum and the second minimum of these graphs. However, discussions about the maximum of these graphs have not yet been resolved and seem to be difficult. In view of this, we make the following conjecture.

Conjecture 1 *If $\mathcal{G}$ is a two-tree graph with $\upsilon \geq 6$ vertices, then $R(\mathcal{G}) \leq \frac{\upsilon}{2} - k$ and the equality holds for graphs shown in Figure 2.*

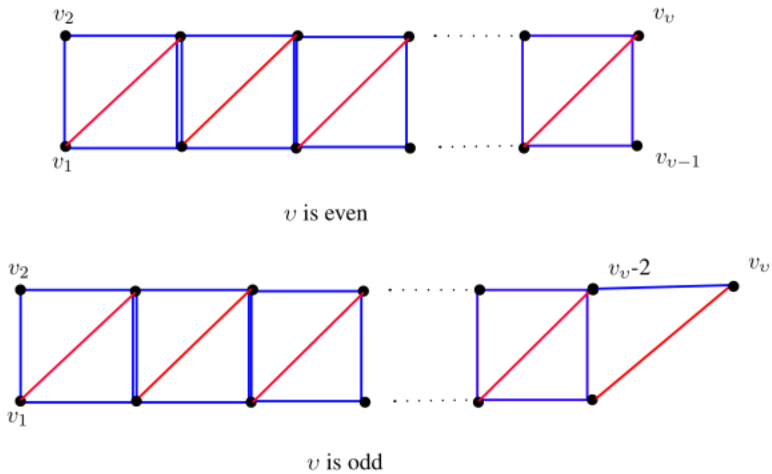

Figure 2: Graph described in conjecture 1 with $\mathsf{R}(\mathcal{G}) = \frac{v}{2} - \mathsf{k}$ where, $\mathsf{k} = 0.0716960995065$.

References

[1] S. Bermudo, J. E. Nápoles JE, J. Rada, Extremal trees for the Randić index with given domination number. *Applied Mathematics and Computation*, **375**, (2020) 125122.

[2] G. Caporossi, I. Gutman, P. Hansen, Variable neighborhood search for extremal graphs: Iv: chemical trees with extremal connectivity index. *Computers & Chemistry*, **23(5)**, (1999) 469–477.

[3] M. Lu, J. Gao, On the Randić index of quasi-tree graphs. *Journal of Mathematical Chemistry*, **42(3)**, (2007) 297–310.

[4] M. Randic, Characterization of molecular branching. *Journal of the American Chemical Society*, **97(23)**, (1975) 6609–6615.

[5] Y. Shang, Mean commute time for random walks on hierarchical scale-free networks, *Internet Mathematics*, **8(4)**, (2012) 321–337.

[6] Y. Shang, Distinct clusterings and characteristic path lengths in dynamic small-world networks with identical limit degree distribution, *Journal of Statistical Physics*, **149(3)**, (2012) 505–518.

[7] Y. Shang, Geometric assortative growth model for small-world networks, *The Scientific World Journal*, **2014**, (2014).

[8] Y. Shang, Sombor index and degree-related properties of simplicial networks. *Applied Mathematics and Computation*, **419**, (2022), 126881.

[9] X. Sun, Y. Gao, J. Du, The harmonic index of two-trees and quasi-tree graphs, *J. Math. Inequalities*, **13**, (2019) 479–493.

[10] R. Todeschini, V. Consonni, *Molecular Descriptors for Chemoinformatics*, Mannhold R, Kubinyi H, Folkers G, Editors, (2009).

[11] H. Wiener, Structural determination of paraffin boiling points. *Journal of the American Chemical Society*, **69(1)**, (1947), 17–20.

[12] Z. Zhang, B. Wu, Y. Lin, Counting spanning trees in a small-world Farey graph, *Physica A: Statistical Mechanics and its Applications*, **391(11)**, (2012), 3342–3349.

[13] Z. Zhang, H. Liu, B. Wu, T. Zou, Spanning trees in a fractal scale-free lattice, *Physical Review E*, **83(1)**, (2011), 016116.

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Partial sums of the Rabotnov function

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Abstract. This article deals with the ratio of normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ and its sequence of partial sums $(\mathbb{R}_{\alpha,\beta})_m(z)$. Several examples which illustrate the validity of our results are also given.

1 Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathcal{U}$. Consider the function $R_{\alpha,\beta}(z)$ defined by

$$R_{\alpha,\beta}(z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{\beta^n}{\Gamma((n+1)(1+\alpha))} z^{n(1+\alpha)} \tag{2}$$

where Γ stands for the Euler gamma function and $\alpha \geq 0$, $\beta \in \mathbb{C}$ and $z \in \mathcal{U}$. This function was introduced by Rabotnov in 1948 [14] and is therefore known

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as the Rabotnov function.
The function defined by (2) does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization of the Rabotnov function $R_{\alpha,\beta}(z)$: for $z \in \mathcal{U}$,

$$\mathbb{R}_{\alpha,\beta}(z) = \Gamma(1+\alpha) z^{1/(1+\alpha)} R_{\alpha,\beta}\left(z^{1/(1+\alpha)}\right) = \sum_{n=0}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} z^{n+1} \tag{3}$$

where $\alpha \geq 0$ and $\beta \in \mathbb{C}$.
Note that some special cases of $\mathbb{R}_{\alpha,\beta}(z)$ are:

$$\begin{cases} \mathbb{R}_{0,-\frac{1}{3}}(z) = ze^{-\frac{z}{3}} \\ \mathbb{R}_{1,\frac{1}{2}}(z) = \sqrt{2z}\sinh\sqrt{\frac{z}{2}} \\ \mathbb{R}_{1,-\frac{1}{4}}(z) = 2\sqrt{z}\sin\frac{\sqrt{z}}{2} \\ \mathbb{R}_{1,1}(z) = \sqrt{z}\sinh\sqrt{z} \\ \mathbb{R}_{1,2}(z) = \frac{\sqrt{2z}\sinh\sqrt{2z}}{2}. \end{cases} \tag{4}$$

For various interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) refered to the works of Kazımoğlu et al. [7], Çağlar and Orhan [1], Lin and Owa [9], Deniz and Orhan [3, 4], Owa et al. [13], Sheil-Small [17], Silverman [18] and Silvia [20]. Recently, some researchers have studied on partial sums of special functions (see [2, 7, 8, 12, 16, 22]).
In this paper, we investigate the ratio of normalized Rabotnov function $R_{\alpha,\beta}(z)$ and its derivative defined by (3) to their sequences of partial sums

$$\begin{cases} (\mathbb{R}_{\alpha,\beta})_0(z) = z \\ (\mathbb{R}_{\alpha,\beta})_m(z) = z + \sum\limits_{n=1}^{m} A_n z^{n+1}, \ m \in \mathbb{N} = \{1,2,3,\ldots\}, \end{cases} \tag{5}$$

where

$$A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))}, \ \alpha \geq 0 \text{ and } \beta \in \mathbb{C}.$$

We obtain lower bounds on ratios like

$$\Re\left\{\frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)}\right\}, \ \Re\left\{\frac{(\mathbb{R}_{\alpha,\beta})_m(z)}{\mathbb{R}_{\alpha,\beta}(z)}\right\}, \ \Re\left\{\frac{\mathbb{R}'_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})'_m(z)}\right\}, \ \Re\left\{\frac{(\mathbb{R}_{\alpha,\beta})'_m(z)}{\mathbb{R}'_{\alpha,\beta}(z)}\right\}.$$

Several examples will be also given.
Results concerning partial sums of analytic functions may be found in [5, 15].

2 Main results

In order to prove our results we need the following lemma.

Lemma 1 *Let* $\alpha \geq 0$ *and* $\beta \in \mathbb{C}$. *Then the function* $\mathbb{R}_{\alpha,\beta}(z)$ *satisfies the following inequalities:*

$$|\mathbb{R}_{\alpha,\beta}(z)| \leq e^{\frac{|\beta|}{1+\alpha}} \quad (z \in \mathcal{U}) \tag{6}$$

$$\left|\mathbb{R}'_{\alpha,\beta}(z)\right| \leq \left(1 + \frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}} \quad (z \in \mathcal{U}). \tag{7}$$

Proof. By using the inductive method, we easily see that

$$(1+\alpha)^n (n)!\Gamma(1+\alpha) \leq \Gamma((1+\alpha)(n+1))$$

and thus

$$\frac{\Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \leq \frac{1}{(1+\alpha)^n (n)!}, \quad \alpha \geq 0, \ n \in \mathbb{N}. \tag{8}$$

Making use of (8) and also the well-known triangle inequality, for $z \in \mathcal{U}$, we have

$$\begin{aligned} |\mathbb{R}_{\alpha,\beta}(z)| &= \left| z + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} z^{n+1} \right| \leq 1 + \sum_{n=1}^{\infty} \frac{|\beta|^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{|\beta|^n}{(1+\alpha)^n (n)!} = e^{\frac{|\beta|}{1+\alpha}} \end{aligned}$$

and thus, inequality (6) is proved.
To prove (7), using again (8) and the triangle inequality, for $z \in \mathcal{U}$, we obtain

$$\begin{aligned} \left|\mathbb{R}'_{\alpha,\beta}(z)\right| &= \left| 1 + \sum_{n=1}^{\infty} \frac{(n+1)\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} z^n \right| \leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)|\beta|^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)|\beta|^n}{(1+\alpha)^n (n)!} = \left(1 + \frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}} \end{aligned}$$

and thus, inequality (7) is proved. □

Let $w(z)$ be an analytic function in $\mathcal{U}$. In the sequel, we will frequently use the following well-known result:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \ z \in \mathcal{U} \text{ if and only if } |w(z)| < 1, \ z \in \mathcal{U}.$$

Theorem 1 *Let $\alpha \geq 0$ and $0 < |\beta| \leq (1+\alpha)\ln 2$. Then*

$$\Re\left\{\frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)}\right\} \geq 2 - e^{\frac{|\beta|}{1+\alpha}},\ z \in \mathcal{U} \tag{9}$$

and

$$\Re\left\{\frac{(\mathbb{R}_{\alpha,\beta})_m(z)}{\mathbb{R}_{\alpha,\beta}(z)}\right\} \geq e^{\frac{-|\beta|}{1+\alpha}},\ z \in \mathcal{U}. \tag{10}$$

Proof. From inequality (6) we get

$$1 + \sum_{n=1}^{\infty} A_n \leq e^{\frac{|\beta|}{1+\alpha}}, \text{where } A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))}.$$

The last inequality is equivalent to

$$\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1}\right)\sum_{n=1}^{\infty} A_n \leq 1.$$

In order to prove the inequality (9), we consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)} = \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1}\right)\frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)} - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}} - 1} - 1\right)$$

and, thus we have

$$\frac{1+w(z)}{1-w(z)} = \frac{1 + \sum_{n=1}^{m} A_n z^n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}{1 + \sum_{n=1}^{m} A_n z^n}. \tag{11}$$

From (11), we obtain

$$w(z) = \frac{\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}{2 + 2\sum_{n=1}^{m} A_n z^n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n}{2 - 2\sum_{n=1}^{m} A_n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n}.$$

Now, $|w(z)| \leq 1$ if and only if

$$\left(\frac{2}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n \leq 2-2\sum_{n=1}^{m} A_n$$

which is equivalent to

$$\sum_{n=1}^{m} A_n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n \leq 1. \tag{12}$$

To prove (12), it suffices to show that its left-hand side is bounded above by

$$\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=1}^{\infty} A_n$$

which is equivalent to

$$\left(\frac{2-e^{\frac{|\beta|}{1+\alpha}}}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=1}^{m} A_n \geq 0.$$

The last inequality holds true for $0 < |\beta| \leq (1+\alpha)\ln 2$.
We use the same method to prove the inequality (10). Consider the function $w(z)$ given by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}+1\right)\frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)} - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\\ &= \frac{1+\sum_{n=1}^{m} A_n z^n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n z^n}{1+\sum_{n=1}^{m} A_n z^n}.\end{aligned}$$

From the last equality we get

$$w(z) = \frac{-\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}+1\right)\sum_{n=m+1}^{\infty} A_n z^n}{2+2\sum_{n=1}^{m} A_n z^n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}-1\right)\sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}+1\right)\sum_{n=m+1}^{\infty} A_n}{2-2\sum_{n=1}^{m} A_n - \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}-1\right)\sum_{n=m+1}^{\infty} A_n}.$$

Then, $|w(z)| \leq 1$ if and only if

$$\sum_{n=1}^{m} A_n + \left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=m+1}^{\infty} A_n \leq 1. \tag{13}$$

Since the left-hand side of (13) is bounded above by

$$\left(\frac{1}{e^{\frac{|\beta|}{1+\alpha}}-1}\right)\sum_{n=1}^{\infty} A_n,$$

we have that the inequality (10) holds true. Now, the proof of our theorem is completed. □

Theorem 2 *Let* $\alpha \geq 0$ *and* $1 < \left(1+\frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}} \leq 2.$ *Then*

$$\Re\left\{\frac{\mathbb{R}'_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})'_m(z)}\right\} \geq 2-\left(1+\frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}}, \ z \in \mathcal{U} \tag{14}$$

and

$$\Re\left\{\frac{(\mathbb{R}_{\alpha,\beta})'_m(z)}{\mathbb{R}'_{\alpha,\beta}(z)}\right\} \geq \left(\frac{1+\alpha}{1+\alpha+|\beta|}\right) e^{\frac{-|\beta|}{1+\alpha}}, \ z \in \mathcal{U}. \tag{15}$$

Proof. From (7) we have

$$1+\sum_{n=1}^{\infty}(n+1)A_n \leq \mathcal{L}_{\alpha,\beta},$$

where $A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))}$, $\mathcal{L}_{\alpha,\beta} = \left(1+\frac{|\beta|}{1+\alpha}\right) e^{\frac{|\beta|}{1+\alpha}}$, $\alpha \geq 0, \ \beta \in \mathbb{C}$ and $n \in \mathbb{N}$. The above inequality is equivalent to

$$\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=1}^{\infty}(n+1)A_n \leq 1.$$

To prove (14), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{1}{\mathcal{L}_{\alpha,\beta}-1}\frac{\mathbb{R}'_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})'_m(z)} - \frac{2-\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}$$

which gives

$$w(z)=\frac{\frac{1}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n+\frac{1}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)|\leq\frac{\frac{1}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\frac{1}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The condition $|w(z)|\leq 1$ holds true if and only if

$$\sum_{n=1}^{m}(n+1)A_n+\frac{1}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n\leq 1. \tag{16}$$

The left-hand side of (16) is bounded above by

$$\frac{1}{\pounds_{\alpha,\beta}-1}\sum_{n=1}^{\infty}(n+1)A_n$$

which is equivalent to

$$\frac{2-\pounds_{\alpha,\beta}}{\pounds_{\alpha,\beta}-1}\sum_{n=1}^{m}(n+1)A_n\geq 0$$

which holds true for $1<\left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}}\leq 2$.

The proof of (15) follows the same pattern. Consider the function $w(z)$ given by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \frac{\pounds_{\alpha,\beta}}{\pounds_{\alpha,\beta}-1}\frac{\mathbb{R}'_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})'_m(z)}-\frac{1}{\pounds_{\alpha,\beta}-1}\\ &= \frac{1+\sum_{n=1}^{m}(n+1)A_nz^n-\frac{1}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{1+\sum_{n=1}^{\infty}(n+1)A_nz^n}.\end{aligned}$$

Consequently, we have that

$$w(z)=\frac{-\frac{\pounds_{\alpha,\beta}}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n-\frac{2-\pounds_{\alpha,\beta}}{\pounds_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)| \leq \frac{\frac{\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\frac{2-\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The last inequality implies that $|w(z)| \leq 1$ if and only if

$$\frac{2}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n \leq 2-2\sum_{n=1}^{m}(n+1)A_n$$

or equivalently

$$\sum_{n=1}^{m}(n+1)A_n+\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=m+1}^{\infty}(n+1)A_n \leq 1. \tag{17}$$

It remains to show that the left-hand side of (17) is bounded above by

$$\frac{1}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=1}^{\infty}(n+1)A_n.$$

This is equivalent to

$$\frac{2-\mathcal{L}_{\alpha,\beta}}{\mathcal{L}_{\alpha,\beta}-1}\sum_{n=1}^{m}(n+1)A_n \geq 0,$$

which holds true for $1 < \left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}} \leq 2$. Now, the proof of our theorem is completed. □

3 Illustrative examples and image domains

In this section, we present several illustrative examples along with the geometrical descriptions of the image domains of the appropriately chosen disk by the partial sums which we considered in our main theorems in Sections 2. From Theorem 1 and Theorem 2, we obtain the following corollaries for special cases of α and β.

Corollary 1 *If we take $\alpha = 0$ and $\beta = -\frac{1}{3}$, we have*

$$\mathbb{R}_{0,-\frac{1}{3}}(z) = ze^{-\frac{z}{3}}, \quad \mathbb{R}'_{0,-\frac{1}{3}}(z) = -\frac{1}{3}e^{-\frac{z}{3}}(z-3)$$

and for $m = 0$ *we get*

$$\left(\mathbb{R}_{0,-\frac{1}{3}}(z)\right)_0(z) = z, \quad \left(\mathbb{R}'_{0,-\frac{1}{3}}(z)\right)_0(z) = 1,$$

so,

$$\begin{aligned}
\Re\left\{e^{-\frac{z}{3}}\right\} &\geq 2 - e^{\frac{1}{3}} \approx 0.60439, \quad z \in \mathcal{U}, \\
\Re\left\{e^{\frac{z}{3}}\right\} &\geq e^{-\frac{1}{3}} \approx 0.71653, \quad z \in \mathcal{U}, \\
\Re\left\{-\frac{1}{3}e^{-\frac{z}{3}}(z-3)\right\} &\geq 2 - \frac{4}{3}e^{\frac{1}{3}} \approx 0.13918, \quad z \in \mathcal{U}, \\
\Re\left\{-\frac{3e^{\frac{z}{3}}}{z-3}\right\} &\geq \frac{3}{4}e^{-\frac{1}{3}} \approx 0.5374, \quad z \in \mathcal{U}.
\end{aligned}$$

Corollary 2 *For* $\alpha = 1$ *and* $\beta = \frac{1}{2}$, *we obtain*

$$\mathbb{R}_{1,\frac{1}{2}}(z) = \sqrt{2z}\sinh\sqrt{\frac{z}{2}}, \quad \mathbb{R}'_{1,\frac{1}{2}}(z) = \frac{1}{2}\cosh\sqrt{\frac{z}{2}} + \frac{\sinh\sqrt{\frac{z}{2}}}{\sqrt{2z}}$$

and for $m = 0$ *we have*

$$\left(\mathbb{R}_{1,\frac{1}{2}}(z)\right)_0(z) = z, \quad \left(\mathbb{R}'_{1,\frac{1}{2}}(z)\right)_0(z) = 1,$$

so,

$$\begin{aligned}
\Re\left\{\sqrt{\frac{2}{z}}\sinh\sqrt{\frac{z}{2}}\right\} &\geq 2 - e^{\frac{1}{4}} \approx 0.71597, \quad z \in \mathcal{U}, \\
\Re\left\{\sqrt{\frac{z}{2}}\operatorname{csch}\sqrt{\frac{z}{2}}\right\} &\geq e^{-\frac{1}{4}} \approx 0.7788, \quad z \in \mathcal{U}, \\
\Re\left\{\frac{1}{2}\cosh\sqrt{\frac{z}{2}} + \frac{\sinh\sqrt{\frac{z}{2}}}{\sqrt{2z}}\right\} &\geq 2 - \frac{5}{4}e^{\frac{1}{4}} \approx 0.39497, \quad z \in \mathcal{U}, \\
\Re\left\{\frac{2}{\cosh\sqrt{\frac{z}{2}} + \frac{\sqrt{2}\sinh\sqrt{\frac{z}{2}}}{\sqrt{z}}}\right\} &\geq \frac{4}{5}e^{-\frac{1}{4}} \approx 0.62304, \quad z \in \mathcal{U}.
\end{aligned}$$

Setting $m = 0$, $\alpha = 1$ and $\beta = -\frac{1}{4}$ in Theorem 1 and Theorem 2 respectively, we obtain the next result involving the function $\mathbb{R}_{1,-\frac{1}{4}}(z)$, defined by (4), and its derivative.

Corollary 3 *The following inequialities hold true:*

$$\begin{aligned}
\Re\left\{\frac{1}{\sqrt{z}}\sin\frac{\sqrt{z}}{2}\right\} &\geq \frac{2-e^{\frac{1}{8}}}{2} \approx 0.43343, \quad z \in \mathcal{U},\\
\Re\left\{\sqrt{z}\csc\frac{\sqrt{z}}{2}\right\} &\geq 2e^{-\frac{1}{8}} \approx 1.765, \quad z \in \mathcal{U},\\
\Re\left\{\cos\frac{\sqrt{z}}{2}+\frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}\right\} &\geq 4-\frac{9}{4}e^{\frac{1}{8}} \approx 1.4504, \quad z \in \mathcal{U},\\
\Re\left\{\frac{1}{\cos\frac{\sqrt{z}}{2}+\frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}}\right\} &\geq \frac{4}{9}e^{-\frac{1}{8}} \approx 0.39222, \quad z \in \mathcal{U}.
\end{aligned}$$

Example 1 *The image domains of* $f_1(z) = \frac{1}{\sqrt{z}}\sin\frac{\sqrt{z}}{2}$, $f_2(z) = \sqrt{z}\csc\frac{\sqrt{z}}{2}$, $f_3(z) = \cos\frac{\sqrt{z}}{2} + \frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}$ *and* $f_4(z) = \frac{1}{\cos\frac{\sqrt{z}}{2}+\frac{2\sin\frac{\sqrt{z}}{2}}{\sqrt{z}}}$ *are shown in Figure 1.*

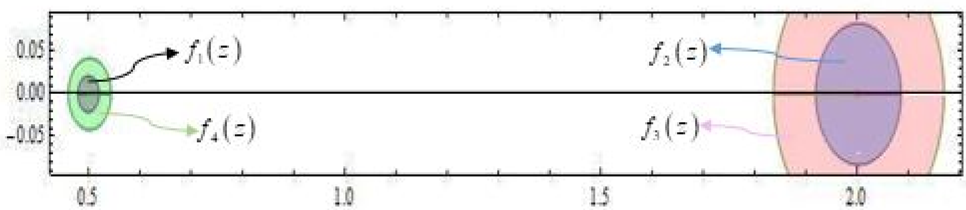

Figure 1.

It is therefore of interest to determine the largest disk $\mathcal{U}_\rho$ in which the partial sums $f_n = z + \sum\limits_{k=1}^{n} a_k z^{k+1}$ of the functions $f \in \mathcal{A}$ are univalent, starlike, convex and close-to-convex. Recently, Ravichandran also wrote a survey [15] on geometric properties of partial sums of univalent functions. By the Noshiro-Warschowski Theorem (see [6]) for $m = 0$ in the inequality (14) of Theorem 2, we conclude that the function $\mathbb{R}_{\alpha,\beta}$ is univalent and also close-to-convex under the condition $1 < \left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}} \leq 2$. Noshiro [11] showed that the radius of starlikeness of f_n partial sums of the functions $f \in \mathcal{A}$ is $1/M$ if satisfies the inequality $|f'(z)| \leq M$. Therefore if we consider the inequality (7) in Lemma 1, we conclude that the radius of starlikeness of $(\mathbb{R}_{\alpha,\beta})_m$ is $\left(\frac{1+\alpha}{1+\alpha+|\beta|}\right)e^{\frac{-|\beta|}{1+\alpha}}$. For functions whose derivatives has positive real part $(\Re(f'(z)) > 0)$, Silverman [19] and Singh [21] proved that f_n is univalent in $|z| < r_n$, where r_n is the smallest positive root of the equation $1 - r - 2r^n = 0$ and convex in $|z| < 1/4$, respectively. In light of these results, for $m = 0$ in the inequality (14) of Theorem 2, $(\mathbb{R}_{\alpha,\beta})_m$ is univalent in $|z| < r_n$ and convex in $|z| < 1/4$. According to the result of Miki [10], from (14), $(\mathbb{R}_{\alpha,\beta})_m$ is close-to-convex in $|z| < 1/4$. The results are all sharp.

References

[1] M. Çağlar and H. Orhan, On neighborhood and partial sums problem for generalized Sakaguchi type functions, *The Scientifc Annals of Al.I. Cuza University of Iasi*, **1** (2014), 17–28.

[2] M. Çağlar and E. Deniz, Partial sums of the normalized Lommel functions, *Math. Inequal. Appl.*, **18** (2015), 1189–1199.

[3] E. Deniz and H. Orhan, Some Properties Of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator, *Czech. Math. J.*, **60** (2010), 699–713.

[4] E. Deniz and H. Orhan, Certain subclasses of multivalent functions defined by new multiplier transformations, *Arab. J. Sci. Eng.*, **36** (2011), 1091–1112.

[5] B. A. Frasin, Generalization of partial sums of certain analytic and univalent functions, *Appl. Math. Lett.*, **21** (2008), 735–741.

[6] A. W. Goodman, Univalent Functions, *Vol I. Mariner Publ. Comp., Tampa*, Florida, 1984.

[7] S. Kazımoğlu, E. Deniz and M. Çağlar, Partial Sums of The Bessel-Struve Kernel Function *3rd International Conference on Mathematical and Related Sciences: Current Trend and Developments*, (2020), 267–275.

[8] S. Kazımoğlu, Partial Sums of The Miller-Ross Function, *Turk. J. Sci.*, **6** (2021), 167–173.

[9] L.J. Lin and S. Owa, On partial sums of the Libera integral operator, *J. Math. Anal. Appl.*, **213** (1997), 444–454.

[10] Y. Miki, A note on close-to-convex functions, *J. Math. Soc. Japan*, **8** (1956), 256–268.

[11] K. Noshiro, On the starshaped mapping by an analytic function, *Proc. Imp. Acad.*, **8** (1932), 275-277.

[12] H. Orhan and N. Yağmur, Partial Sums of generalized Bessel functions, *J. Math. Inequal.*, **8** (2014), 863–877.

[13] S. Owa, H.M. Srivastava and N. Saito, Partial sums of certain classes of analytic functions, *Int. J. Comput. Math.*, **81** (2014), 1239–1256.

[14] Y. N. Rabotnov, Equilibrium of elastic media with an aftereffect, *Prikl. Matem. Mekh.*, **12** (1948), 53–62.

[15] V. Ravichandran, Geometric properties of partial sums of univalent functions, *Math. Newslett.*, **22** (2012), 208–221.

[16] M. S. Rehman, Q. Z. Ahmad, H. M. Srivastava, B. Khan and N. Khan, Partial sums of generalized $q-$Mittag-Leffler functions, *Aims Math.*, **5** (2019), 408–420.

[17] T. Sheil-Small, A note on partial sums of convex schlicht functions, *Bull. London Math. Soc.*, **2** (1970), 165–168.

[18] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, **209** (1997), 221–227.

[19] H. Silverman, Partial sums of a class of univalent functions, *Tamkang J. Math.*, **29** (1998), 171-174.

[20] E.M. Silvia, On partial sums of convex functions of order α, *Houston J. Math.*, **11** (1985), 397–404.

[21] R. Singh, Radius of convexity of partial sums of a certain power series, *J. Austral. Math. Soc.*, **11** (1970), 407-410.

[22] N. Yağmur, H. Orhan, Partial sums of generalized Struve functions, *Miskolc Math. Notes*, **17** (2016), 657–670.

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Grüss-type fractional inequality via Caputo-Fabrizio integral operator

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Abstract. In this article, the main objective is to establish the Grüss-type fractional integral inequalities by employing the Caputo-Fabrizio fractional integral.

1 Introduction

Grüss inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions. In 1935, G. Grüss proved the following well known classical integral inequality, see [24, 27].

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Theorem 1 [27] *Let* $f, g : [a, b] \to R$ *be two integrable functions such that* $\phi \le f(x) \le \Phi$ *and* $\gamma \le g(x) \le \Gamma$ *for all* $x \in [a, b]$*;* ϕ, Φ, γ *and* Γ *are constant, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx\right|\\ &\le \frac{1}{4}(\Gamma-\gamma)(\Phi-\phi),\end{aligned} \tag{1}$$

where the constant $\frac{1}{4}$ *is sharp.*

During the last few years, several numerous generalizations, variants and extensions of the Grüss inequality have appeared in the literature, see [18, 19, 20, 22, 24, 25, 26, 27, 29, 30, 31, 40] and the references cited therein. Chinchane and Pachpatte [10], investigated some new fractional integral inequalities of the Grüss-type by considering the Saigo fractional integral operator. In [1, 21, 34, 35, 36] authors obtained some the Grüss-type inequalities by using different types of fractional integral operators. Fractional calculus is generalization of traditional calculus into non-integer differential and integral order. Fractional calculus is very important due to it's various application in field of science and technology, see [2, 4, 32, 37].

In [5, 6], Caputo and Fabrizio introduced a new fractional derivative and application of new time and spatial fractional derivative with exponential kernels. In literature very little work is reported on fractional integral inequalities using Caputo and Caputo-Fabrizio integral operator. Wang et al. [39] presented some properties of Caputo–Fabrizio fractional integral operator in the setting of-convex function. Recently, Nchama and et al. [28], proposed some fractional integral inequalities using the Caputo-Fabrizio fractional integral.

Recently, many researchers have worked on fractional integral inequalities using the Riemann-Liouville, Hadamard and q-fractional integral, see [3, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 23, 38]. In [16], Dahmani and et al. gave the following fractional integral inequality using the Riemann-Liouville fractional integral.

Motivated from [5, 6, 10, 16, 28, 39], our purpose in this paper is to propose some new results using the Caputo-Fabrizio integral operator. The paper has been organized as follows, in Section 2, we recall some auxiliary results related to the Caputo-Fabrizio integral operator. In Section 3, we investigate the Grüss-type fractional integral inequality using the Caputo-Fabrizio integral operator, in Section 4, we give the concluding remarks.

2 Preliminaries

In this section, we give some auxiliary results of fractional calculus that will be useful in this paper.

Definition 1 [6, 28] *Let* $\alpha \in \mathbb{R}$ *such that* $0 < \alpha < 1$. *The Caputo-Fabrizio fractional integral of order* α *of a function* f *is defined by*

$$\mathcal{I}_{0,x}^{\alpha} f(x) = \frac{1}{\alpha} \int_0^x e^{-(\frac{1-\alpha}{\alpha})(x-s)} f(s) ds. \tag{2}$$

Definition 2 [6, 28] *Let* $\alpha, a \in \mathbb{R}$ *such that* $0 < x < 1$. *The Caputo-Fabrizio fractional derivative of order* α *of a function* f *is defined by*

$$\mathcal{I}_{a,x}^{\alpha} f(x) = \frac{1}{1-\alpha} \int_a^x e^{\frac{-\alpha}{1-\alpha}(x-s)} f'(s) ds. \tag{3}$$

Definition 3 *Let* $\alpha > 0, \beta, \eta \in \mathbb{R}$, *then the Saigo fractional integral* $\mathcal{I}_{0,x}^{\alpha,\beta,\eta}[f(x)]$ *of order* α *for a real valued continuous function* $f(x)$ *is defined by*

$$\mathcal{I}_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{x}) f(\tau) d\tau, \tag{4}$$

where the function $F_1(-)$ *is the Gaussian hypergeometric function defined by*

$$F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(x)^n}{n!},$$

and $(a)_n$ *is the pochhammer symbol*

$$(a)_n = a(a+1)...(a+n-1), (a)_0 = 1.$$

Definition 4 *The Hadamard fractional integral is defined by*

$$^{H}\mathcal{I}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\log \frac{x}{\tau})^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \text{ for } \mathrm{Re}(\alpha) > 0,\ x > 1. \tag{5}$$

Definition 5 *The Riemann-Liouville fractional integral is defined by*

$$\mathcal{I}_{0,x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau. \tag{6}$$

3 Grüss-type fractional integral inequality

In this section, we investigate Grüss-type fractional integral inequalities involving the Caputo-Fabrizio fractional integer operator, for which assume that (H_1) There exist two integrable function $\Phi_1(x)$, $\Phi_2(x)$ on $[0,\infty[$, such that

$$\Phi_1(x) \le u(x) \le \Phi_2(x), \ \ \text{for all} \ \ x \in [0,\infty[.$$

(H_2) There exist two integrable function $\Psi_1(x)$,$\Psi_2(x)$ on $[0,\infty[$, such that

$$\Psi_1(x) \le v(x) \le \Psi_2(x), \ \ \text{for all} \ \ x \in [0,\infty[.$$

Theorem 2 *Suppose that* u *be an integrable function defined on* $[0,\infty[$*, consider the condition* (H_1) *hold. Then for all* $x > 0$*,* $\alpha, \beta > 0$*, we have*

$$\begin{aligned} &\mathcal{I}^{\beta}_{0,t}\Phi_1(x)\mathcal{I}^{\alpha}_{0,t}u(x) + \mathcal{I}^{\alpha}_{0,t}\Phi_2(x)\mathcal{I}^{\beta}_{0,t}u(x) \ge \\ &\mathcal{I}^{\alpha}_{0,t}\Phi_2(x)\mathcal{I}^{\beta}_{0,t}\Phi_1(x) + \mathcal{I}^{\alpha}_{0,t}u(x)\mathcal{I}^{\beta}_{0,t}u(x). \end{aligned} \tag{7}$$

Proof. From condition (H_1), for all $\rho, \sigma \ge 0$, we obtain

$$\left(\Phi_2(\rho) - u(\rho)\right)\left(u(\sigma) - \Phi_1(\sigma)\right) \ge 0. \tag{8}$$

that is

$$\Phi_2(\rho)u(\sigma) - \Phi_2(\rho)\Phi_1(\sigma) - u(\rho)u(\sigma) + u(\rho)\Phi_1(\sigma) \ge 0, \tag{9}$$

which implies that

$$\Phi_2(\rho)u(\sigma) + u(\rho)\Phi_1(\sigma) \ge \Phi_2(\rho)\Phi_1(\sigma) + u(\rho)u(\sigma). \tag{10}$$

Multiplying (10) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho \in (0,x)$, $x > 0$.

$$\begin{aligned} &u(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + \Phi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho) \\ &\ge \Phi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + u(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho). \end{aligned} \tag{11}$$

Now, integrating (11) with respect to ρ from 0 to x, we have

$$\begin{aligned} &\frac{u(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{\Phi_1(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho \\ &\ge \frac{\Phi_1(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{u(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho, \end{aligned} \tag{12}$$

therefore

$$\begin{aligned} & u(\sigma)\mathcal{I}^{\alpha}_{0,t}\Phi_2(x) + \Phi_1(\sigma)\mathcal{I}^{\alpha}_{0,t}u(x) \\ & \geq \Phi_1(\sigma)\mathcal{I}^{\alpha}_{0,t}\Phi_2(x) + u(\sigma)\mathcal{I}^{\alpha}_{0,t}u(x). \end{aligned} \tag{13}$$

Now, multiplying (13) by $\frac{1}{\beta}e^{-(\frac{1-\beta}{\beta})(x-\sigma)}$, which is positive because $\sigma \in (0, x)$, $x > 0$. Then integrating obtained result with respective to σ from 0 to x, we obtain

$$\begin{aligned} & \frac{\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}u(\sigma)d\sigma + \frac{\mathcal{I}^{\alpha}_{0,x}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}\Phi_1(\sigma)d\sigma \\ & \geq \frac{\mathcal{I}^{\alpha}_{0,t}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}\Phi_1(\sigma)d\sigma + \frac{\mathcal{I}^{\alpha}_{0,t}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}u(\sigma)d\sigma. \end{aligned} \tag{14}$$

This completes the proof. □

Remark 1 *If* u *be an integrable function defined on* $[0,\infty[$*, such that* $\gamma \leq u(x) \leq \Gamma$*, for all* $x \in [0,\infty[$ *and* $\gamma, \Gamma \in \mathbb{R}$*. Then for all* $x > 0$ *and* $\alpha, \beta > 0$*, we have*

$$\begin{aligned} & \gamma\left(\frac{1}{1-\beta}\left[1 - e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u(x) + \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\beta}_{0,x}u(x) \\ & \geq \Gamma\gamma\left(\frac{1}{1-\beta}\left[1 - e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\beta}_{0,x}u(x). \end{aligned} \tag{15}$$

Theorem 3 *If* u *and* v *be two integrable functions defined on* $[0,\infty[$*, Suppose that* (H_1) *and* (H_2) *holds. Then for all* $x > 0$*,* $\alpha, \beta > 0$*, the following inequalities satisfied*

$$\begin{aligned} & (h1)\ \mathcal{I}^{\beta}_{0,x}\Psi_1(x)\,\mathcal{I}^{\alpha}_{0,x}u(x) + \mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\,\mathcal{I}^{\beta}_{0,x}v(x) \\ & \geq \mathcal{I}^{\beta}_{0,x}\Psi_1(x)\,\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) + \mathcal{I}^{\alpha}_{0,x}u(x)\,\mathcal{I}^{\beta}_{0,x}v(x). \\ & (h2)\ \mathcal{I}^{\beta}_{0,x}\Phi_1(x)\,\mathcal{I}^{\alpha}_{0,x}v(x) + \mathcal{I}^{\alpha}_{0,x}\Psi_2(x)\,\mathcal{I}^{\beta}_{0,x}u(x) \\ & \geq \mathcal{I}^{\beta}_{0,x}\Phi_1(x)\,\mathcal{I}^{\alpha}_{0,x}\Psi_2(x) + \mathcal{I}^{\beta}_{0,x}u(x)\,\mathcal{I}^{\alpha}_{0,x}v(x). \\ & (h3)\ \mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\,\mathcal{I}^{\beta}_{0,x}\Psi_2(x) + \mathcal{I}^{\alpha}_{0,x}u(x)\,\mathcal{I}^{\beta}_{0,x}v(x) \\ & \geq \mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\,\mathcal{I}^{\beta}_{0,x}v(x) + \mathcal{I}^{\beta}_{0,x}\Psi_2(x)\,\mathcal{I}^{\alpha}_{0,x}u(x). \\ & (h4)\ \mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\,\mathcal{I}^{\beta}_{0,x}\Psi_1(x) + \mathcal{I}^{\alpha}_{0,x}u(x)\,\mathcal{I}^{\beta}_{0,x}v(x) \\ & \geq \mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\,\mathcal{I}^{\beta}_{0,x}v(x) + \mathcal{I}^{\beta}_{0,x}\Psi_1(x)\,\mathcal{I}^{\alpha}_{0,x}u(x). \end{aligned} \tag{16}$$

Proof. To prove (h1), we use condition (H_1) and (H_2), for all $x \in [0, \infty[$, we have

$$(\Phi_2(\rho) - u(\rho))\,(v(\sigma) - \Psi_1(\sigma)) \geq 0, \tag{17}$$

which implies that

$$\Phi_2(\rho)v(\sigma) + u(\rho)\Psi_1(\sigma) \geq \Phi_2(\rho)\Psi_1(\sigma) + u(\rho)u(\sigma). \tag{18}$$

Multiplying (18) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho \in (0, x)$, $x > 0$

$$\begin{aligned} &v(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + \Psi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho) \\ &\geq \Psi_1(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho) + v(\sigma)\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho). \end{aligned} \tag{19}$$

Integrating (19) with respect to ρ from 0 to x, we get

$$\begin{aligned} &\frac{v(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{\Psi_1(\sigma))}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho \\ &\geq \frac{\Psi_1(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}\Phi_2(\rho)d\rho + \frac{v(\sigma)}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}u(\rho)d\rho, \end{aligned} \tag{20}$$

therefore

$$v(\sigma)\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) + \Psi_1(\sigma)\mathcal{I}^{\alpha}_{0,x}u(x) \geq \Psi_1(\sigma)\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) + v(\sigma)\mathcal{I}^{\alpha}_{0,x}u(x). \tag{21}$$

Multiplying both sides of (21) by $\frac{1}{\beta}e^{-(\frac{1-\beta}{\beta})(x-\sigma)}$, which is positive because $\sigma \in (0, x)$, $x > 0$. Then integrating resulting identity with respective σ over 0 to x, we obtain

$$\begin{aligned} &\frac{\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}v(\sigma)d\sigma + \frac{\mathcal{I}^{\alpha}_{0,x}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}\Psi_1(\sigma)d\sigma \\ &\geq \frac{\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}v\Psi_1(\sigma)d\sigma + \frac{\mathcal{I}^{\alpha}_{0,x}u(x)}{\beta}\int_0^x e^{-(\frac{1-\beta}{\beta})(x-\sigma)}v(\sigma)d\sigma. \end{aligned} \tag{22}$$

This gives desired inequality (h1).
To prove (h2)-(h4), we use the following inequalities respectively
$(\Psi_2(\rho) - v(\rho))\,(u(\sigma) - \Phi_1(\sigma)) \geq 0$.
$(\Phi_2(\rho) - u(\rho))\,(v(\sigma) - \Psi_2(\sigma)) \leq 0$.
$(\Phi_2(\rho) - u(\rho))\,(v(\sigma) - \Psi_1(\sigma)) \leq 0$. □

Remark 2 *If* u *and* v *be two integrable function defined on* $[0, \infty[$*, Assume that* (H_3) *There exist real constant* $\Gamma, \gamma, \Gamma^{'}, \gamma^{'}$ *such that*

$$\gamma \leq u(x) \leq \Gamma \text{ and } \gamma^{'} \leq v(x) \leq \Gamma^{'} \ \forall x \in [0, \infty[. \tag{23}$$

Then for all $x > 0$*,* $\alpha, \beta > 0$*, the following inequalities satisfied*

$$\begin{aligned}
&(h_1) \left(\gamma^{'}\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u(x) + \left(\Gamma\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\beta}_{0,x}v(x)\\
&\geq \left(\gamma^{'}\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\Gamma\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\beta}_{0,x}v(x).\\
&(h_2) \left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}v(x) + \left(\Gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\beta}_{0,x}u(x)\\
&\geq \left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\Gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\beta}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}v(x).\\
&(h_3) \left(\Gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\Gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\beta}_{0,x}v(x)\\
&\geq \left(\Gamma\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\beta}_{0,x}v(x) + \left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u(x).\\
&(h_4) \left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\left(\gamma^{'}\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) + \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\beta}_{0,x}v(x)\\
&\geq \left(\gamma\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\beta}_{0,x}v(x) + \left(\gamma^{'}\frac{1}{1-\beta}\left[1-e^{-(\frac{1-\beta}{\beta})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u(x).
\end{aligned} \tag{24}$$

Lemma 1 *If* u *be an integrable function on* $[0, \infty)$*, and* $\Phi_1(x)$*,* $\Phi_2(x)$ *be two integrable functions on* $[0, \infty)$*. Assume that the condition* H_1 *holds. Then for all* $x > 0$*,* $\alpha > 0$*, we have*

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x) - (\mathcal{I}^{\alpha}_{0,x}u(x))^2\\
&= (\mathcal{I}^{\alpha}_{0,x}\Phi_2(x) - \mathcal{I}^{\alpha}_{0,x}u(x))\,(\mathcal{I}^{\alpha}_{0,x}u(x) - \mathcal{I}^{\alpha}_{0,x}\Phi_1(x))\\
&- \left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\left[(\Phi_2(x)-u(x))(u(x)-\Phi_1(x))\right]\\
&+ \left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Phi_1u(x) - \mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\mathcal{I}^{\alpha}_{0,x}u(x)\\
&+ \left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Phi_2u(x) - \mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\mathcal{I}^{\alpha}_{0,x}u(x)
\end{aligned} \tag{25}$$

$$+\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x).$$

Proof. Since u be an integrable function on $[0,\infty)$. For all ρ, $\sigma>0$, we have

$$\begin{aligned}
&(\Phi_2(\sigma)-u(\sigma))\,(u(\rho)-\Phi_1(\rho))+(\Phi_2(\rho)-u(\rho))\,(u(\sigma)-\Phi_1(\sigma))\\
&-(\Phi_2(\rho)-u(\rho))\,(u(\rho)-\Phi_1(\rho))-(\Phi_2(\sigma)-u(\sigma))\,(u(\sigma)-\Phi_1(\sigma))\\
&=u^2(\rho)+u^2(\sigma)-2u(\tau)u(\rho)+\Phi_2(\sigma)u(\rho)+\Phi_1(\rho)u(\sigma)-\Phi_1(\rho)\Phi_2(\sigma)\\
&+\Phi_1(\rho)u(\sigma)+\Phi_1(\sigma)u(\rho)-\Phi_1(\sigma)\Phi_2(\rho)-\Phi_2(\rho)u(\rho)+\Phi_1(\rho)\Phi_2(\rho)\\
&-\Phi_1(\rho)u(\rho)-\Phi_2(\sigma)u(\sigma)+\Phi_1(\sigma)\Phi(\sigma)-\Phi_1(\sigma)u(\sigma).
\end{aligned}\tag{26}$$

Multiplying both sides of (26) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho\in(0,x)$, $x>0$, integrating obtained result with respect to ρ from 0 to x, we have

$$\begin{aligned}
&(\Phi_2(\sigma)-u(\sigma))\,(\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x))\\
&+(\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}u(x))\,(u(\sigma)-\Phi_1(\sigma))\\
&-\mathcal{I}_{0,x}^{\alpha}\left[(\Phi_2(x)-u(x))\,(u(x)-\Phi_1(x))\right]-\frac{(\ln x)^{\alpha}}{\Gamma(\alpha+1)}\,(\Phi_2(\sigma)\\
&-u(\sigma))\,(u(\sigma)-\Phi_1(\sigma))\\
&=\mathcal{I}_{0,x}^{\alpha}u^2(x)+u^2(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&-2u(\sigma)\mathcal{I}_{0,x}^{\alpha}u(x)+\Phi_2(\sigma)\mathcal{I}_{0,x}^{\alpha}u(x)\\
&+u(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)-\Phi_2(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)+u(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\\
&+\Phi_1(\sigma)\mathcal{I}_{0,x}^{\alpha}u(x)-\Phi_1(\sigma)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_2u(x)\\
&+\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1u(x)-\Phi_2(\sigma)u(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&+\Phi_1(\sigma)\Phi_2(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&-\Phi_1(\sigma)u(\sigma)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right).
\end{aligned}\tag{27}$$

Again, multiplying (27) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\sigma)}$, which is positive because $\sigma\in(0,x)$,

$x > 0$, integrating obtained result with respect to ρ from 0 to x, we have

$$\begin{aligned}
&\left(\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}u(x)\right)\left(\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\right)\\
&+\left(\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}u(x)\right)\left(\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\right)\\
&-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\left[\left(\Phi_2(x)-u(x)\right)\left(u(x)-\Phi_1(x)\right)\right]\\
&-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\left[\left(\Phi_2(x)-u(x)\right)\left(u(x)-\Phi_1(x)\right)\right]\\
&=\mathcal{I}_{0,x}^{\alpha}u^2(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+\mathcal{I}_{0,x}^{\alpha}u^2(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&-2\mathcal{I}_{0,x}^{\alpha}u(x)\mathcal{I}_{0,x}^{\alpha}u(x)+\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\mathcal{I}_{0,x}^{\alpha}u(x)+\mathcal{I}_{0,x}^{\alpha}u(x)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\\
&-\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\\
&+\mathcal{I}_{0,x}^{\alpha}u(x)\,{}_{H}D_{1,x}^{-\alpha}\Phi_2(x)+\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\\
&-\mathcal{I}_{0,x}^{\alpha}\Phi_2u(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&-\mathcal{I}_{0,x}^{\alpha}\Phi_1u(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&-\mathcal{I}_{0,x}^{\alpha}\Phi_2u(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&-\mathcal{I}_{0,x}^{\alpha}\Phi_1u(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right),
\end{aligned} \tag{28}$$

which implies that

$$\begin{aligned}
&2\left(\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)-\mathcal{I}_{0,x}^{\alpha}u(x)\right)\left(\mathcal{I}_{0,x}^{\alpha}u(x)-\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\right)\\
&-2\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}_{0,x}^{\alpha}\left[\left(\Phi_2(x)-u(x)\right)\left(u(x)-\Phi_1(x)\right)\right]\\
&=2\mathcal{I}_{0,x}^{\alpha}u^2(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)-2(\mathcal{I}_{0,x}^{\alpha}u(x))^2+2\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\mathcal{I}_{0,x}^{\alpha}u(x)\\
&+2\mathcal{I}_{0,x}^{\alpha}u(x)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)-2\mathcal{I}_{0,x}^{\alpha}\Phi_2(x)\mathcal{I}_{0,x}^{\alpha}\Phi_1(x)\\
&-2\mathcal{I}_{0,x}^{\alpha}\Phi_2u(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)+2\mathcal{I}_{0,x}^{\alpha}\Phi_1\Phi_2(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\\
&-2\mathcal{I}_{0,x}^{\alpha}\Phi_1u(x)\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right).
\end{aligned} \tag{29}$$

This completes the proof. □

If $\Phi_1(x) = \gamma$ and $\Phi_2(x) = \Gamma$; $\gamma, \Gamma \in \mathbb{R}$ for all $x \in [0, \infty)$, then inequality (25) reduces to following lemma.

Lemma 2 *If $\gamma, \Gamma \in \mathbb{R}$, and $u(x)$ be an integrable function on $[0, \infty)$ and satisfying the condition $\gamma \le u(x) \le \Gamma$. Then for all $x > 0$, $\alpha > 0$, we have*

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x) - (\mathcal{I}^{\alpha}_{0,x}u(x))^2 \\
&= \left(\Gamma\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right) - \mathcal{I}^{\alpha}_{0,x}u(x)\right) \times \\
&\left(\mathcal{I}^{\alpha}_{0,x}u(x) - \left(\gamma\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\right) \\
&- \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\left((\Gamma - u(x))(u(x) - \gamma)\right).
\end{aligned} \tag{30}$$

Theorem 4 *Let u and v be two integrable functions on $[0, \infty)$, and $\Phi_1(x)$, $\Phi_2(x), \Psi_1(x)$ and $\Psi_2(x)$ are four integrable functions on $[0, \infty)$ satisfying the conditions H_1 and H_2 on $[0, \infty)$. Then for all $x > 0$, $\alpha > 0$, we have*

$$\begin{aligned}
&\left|\left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}uv(x) - \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}v(x)\right| \\
&\le \sqrt{R(u, \Phi_1(x), \Phi_2(x))R(v, \Psi_1(x), \Psi_2(x))}.
\end{aligned} \tag{31}$$

where $R(a, b, c)$ is defined by

$$\begin{aligned}
R(a, b, c) = &\left(\mathcal{I}^{\alpha}_{0,x}c(x) - \mathcal{I}^{\alpha}_{0,x}a(x)\right)\left(\mathcal{I}^{\alpha}_{0,x}a(x) - \mathcal{I}^{\alpha}_{0,x}b(x)\right) \\
&+ \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}ba(x) - \mathcal{I}^{\alpha}_{0,x}b(x)\mathcal{I}^{\alpha}_{0,x}a(x) \\
&+ \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}ca(x) - \mathcal{I}^{\alpha}_{0,x}c(x)\mathcal{I}^{\alpha}_{0,x}a(x) \\
&+ \mathcal{I}^{\alpha}_{0,x}b(x)\mathcal{I}^{\alpha}_{0,x}c(x) + \left(\frac{1}{1-\alpha}\left[1 - e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}bc(x).
\end{aligned} \tag{32}$$

Proof Let u and v be two functions defined on $[0, \infty)$ satisfying the condition H_1 and H_2. Define

$$H(\rho, \sigma) := \left(u(\rho) - u(\sigma)\right)\left(v(\rho) - v(\sigma)\right); \rho, \sigma \in (0, x), x > 0, \tag{33}$$

it follows that

$$H(\rho,\sigma) := u(\rho)v(\rho) - u(\rho)v(\sigma) - u(\sigma)v(\rho) + u(\sigma)v(\sigma). \tag{34}$$

Now, multiplying (34) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}$, which is positive because $\rho \in (0,x)$, $x>0$, integrating obtained result with respect to ρ from 0 to x, we have

$$\begin{aligned}&\frac{1}{\alpha}\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}H(\rho,\sigma)d\rho\\&= \mathcal{I}^{\alpha}_{0,x}uv(x) - u(\tau)\mathcal{I}^{\alpha}_{0,x}u(x) - u(\sigma)\mathcal{I}^{\alpha}_{0,x}v(x) + u(\sigma)v(\sigma)\mathcal{I}^{\alpha}_{0,x}(1).\end{aligned} \tag{35}$$

Multiplying (35) by $\frac{1}{\alpha}e^{-(\frac{1-\alpha}{\alpha})(x-\sigma)}$, which is positive because $\sigma \in (0,x)$, $x>0$, integrating obtained result with respect to σ from 0 to x, we have

$$\begin{aligned}&\frac{1}{\alpha^2}\int_0^t\int_0^t e^{-(\frac{1-\alpha}{\alpha})(x-\rho)}e^{-(\frac{1-\alpha}{\alpha})(x-\sigma)}H(\rho,\sigma)d\rho d\sigma\\&=2\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}uv(x) - \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}u(x)\right).\end{aligned} \tag{36}$$

Applying the Cauchy-Schwarz inequality to (36), we have

$$\begin{aligned}&\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}uv(x) - \mathcal{I}^{\alpha}_{0,x}u(x)\mathcal{I}^{\alpha}_{0,x}v(x)\right)^2 \leq\\&\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x) - (\mathcal{I}^{\alpha}_{0,x}u(x))^2\right)\times\\&\left(\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}v^2(x) - \mathcal{I}^{\alpha}_{0,x}v(x)^2\right),\end{aligned} \tag{37}$$

since $(\Phi_2(x)-u(t))(u(t)-\Phi_1(x)) \geq 0$ and $(\Psi_2(x)-v(t))(v(t)-\Psi_1(x)) \geq 0$, we have

$$\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}(\Phi_2(x)-u(t))(u(t)-\Phi_1(x)) \geq 0, \tag{38}$$

and

$$\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}(\Psi_2(x)-v(t))(v(t)-\Psi_1(x)) \geq 0. \tag{39}$$

Thus, we have

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x)-(\mathcal{I}^{\alpha}_{0,x}u(x))^2\\
&\leq(\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)-\mathcal{I}^{\alpha}_{0,x}u(x))\,(\mathcal{I}^{\alpha}_{0,x}u(x)-\mathcal{I}^{\alpha}_{0,x}\Phi_1(x))\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Phi_1u(x)-\mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\mathcal{I}^{\alpha}_{0,x}u(x)\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Phi_2u(x)-\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)\mathcal{I}^{\alpha}_{0,x}u(x)\\
&+\mathcal{I}^{\alpha}_{0,x}\Phi_1(x)\mathcal{I}^{\alpha}_{0,x}\Phi_2(x)-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Phi_1\Phi_2(x)\\
&=R(u,\Phi_1,\Phi_2),
\end{aligned}\tag{40}$$

and

$$\begin{aligned}
&\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}u^2(x)-(\mathcal{I}^{\alpha}_{0,x}u(x))^2\\
&\leq(\mathcal{I}^{\alpha}_{0,x}\Psi_2(x)-\mathcal{I}^{\alpha}_{0,x}u(x))\,(\mathcal{I}^{\alpha}_{0,x}u(x)-\mathcal{I}^{\alpha}_{0,x}\Psi_1(x))\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Psi_1u(x)-\mathcal{I}^{\alpha}_{0,x}\Psi_1(x)\mathcal{I}^{\alpha}_{0,x}u(x)\\
&+\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Psi_2u(x)-\mathcal{I}^{\alpha}_{0,x}\Psi_2(x)\mathcal{I}^{\alpha}_{0,x}u(x)\\
&+\mathcal{I}^{\alpha}_{0,x}\Psi_1(x)\mathcal{I}^{\alpha}_{0,x}\Psi_2(x)-\left(\frac{1}{1-\alpha}\left[1-e^{-(\frac{1-\alpha}{\alpha})x}\right]\right)\mathcal{I}^{\alpha}_{0,x}\Psi_1\Psi_2(x)\\
&=R(u,\Psi_1,\Psi_2).
\end{aligned}\tag{41}$$

Combining (37), (40) and (41), we get (31).

4 Concluding Remarks

Nchama et al. [28], investigated some integral inequalities by considering Caputo-Fabrizio fractional integral operator. In [6] Caputo and Farbrizio introduced a new fractional differential and integral operator. Motivated by the above work, here we studied Grüss-type inequalities and other fractional inequalities by considering Caputo-Fabrizio fractional integral operator. By the help of this study we establish more general inequalities than in the classical cases. The inequalities investigated in this paper give some contribution to the fields of fractional calculus and Caputo-Fabrizio fractional integral operator.

These inequalities are expected to lead to some application for finding bounds and uniqueness of solutions in fractional differential equations.

References

[1] T. A. Aljaaidi and D. B. Pachpatte,Some Grüss-type inequalities using generalized Katugampola fractional integral, *AIMS Mathematics*, **2** (5), (2020), 1011–1024. doi: 10.3934/math.2020070.

[2] D. Baleanu, J. A. T. Machado and C. J. Luo, *Fractional Dynamic and Control,* Springer, 2012, pp 159–171.

[3] S. Belarbi and Z. Dahmani,On some new fractional integral inequality, *J. Inequal. Pure and Appl. Math.*, **10** (3) (2009), Art.86, 5 pp.

[4] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Fractional Calculus in the Mellin setting and Hadardmard-type fractional integrals, *J. Math. Anal. Appl.*, (2002), 1–27.

[5] M. Caputo and M. Fabrizio, A new defination of fractional derivative without singular kernel, *Progr., Fract., Differ., Appl.*, **1** (2) (2015), 73–85.

[6] M. Caputo and M. Fabrizio, Applications of new time and spatial fractional derivative with exponential kernels, *Progr., Fract., Differ., Appl.*, **2** (2) (2016), 7–8.

[7] V. L. Chinchane and D. B. Pachpatte, A note on some integral inequalities via Hadamard integral, *J. Fractional Calculus Appl.*, **4** (11) (2013), 1–5.

[8] V. L. Chinchane and D. B. Pachpatte, On some integral inequalities using Hadamard fractional integral, *Malaya J. Math.*, **1** (1) (2012), 62–66.

[9] V. L. Chinchane and D. B. Pachpatte, Some new integral inequalities using Hadamard fractional integral operator, *Adv. Inequal. Appl.*, 2014, 2014:12.

[10] V. L. Chinchane and D. B. Pachpatte, On some Grüss-type fractional inequalities using Saigo fractional integral operator, *Journal of Mathematics*, vol. 2014, Article ID 527910, 9 pages, 2014. https://doi.org/10.1155/2014/527910.

[11] Z. Dahmani, On Minkowski and Hermit-Hadamard integral inequalities via fractional integration, *Ann. Funct. Anal.*, **1** (1) (2010), 51–58.

[12] Z. Dahmani, New inequalities in fractional integrals, *Int. J. Nonlinear Sci.*, **9**, 4(2010), 493–497.

[13] Z. Dahmani, The Riemann-Liouville operator to generate some new inequalities, *Int. J. Nonlinear Sci.*, **12** (4) (2011), 452-455.

[14] Z. Dahmani, Some results associate with fractional integrals involving the extended Chebyshev, *Acta Univ. Apulensis Math. Inform.*, **27** (2011), 217–224.

[15] Z. Dahmani and A. Benzidane, New inequalities using Q-fractional theory, *Bull. Math. Anal. Appl.*, **4** (1) (2012), 190–196.

[16] Z. Dahmani, L. Tabharit and S. Taf, New generalisation of Grüss inequality using Riemann-Liouville fractional integrals, *Bull. Math. Anal. Appl.*, **2** (3)(2012), 92–99.

[17] Z. Denton and A. S. Vatsala, Fractional integral inequalities and application, *Comput. Math. Appl.*, **59** (2010), 1087–1094.

[18] S. S. Dragomir, Some integral inequalities of Grüss type, *Indian J. Pur. Appl. Math.*, **31** (4) (2002), 397–415.

[19] S. S. Dragomir, A companion of the Grüss inequality and applications, *Appl. Math. Lett.*, **17** (2004), no. 4, 429–435.

[20] S. S. Dragomir and S. Wang, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for home numerical quadrature rules, *Comput. Math. Appl.*, **33** (1997), no. 11, 15–20.

[21] G. Farid, A. U. Rehman, V. N. Mishra, S. Mehmood, Fractional integral inequalities of Gruss type via generalized Mittag-Leffler function, *Int. J. Anal. Appl.*, **17** (4) (2019), 548–558.

[22] N. Elezovic, L. Marangunic and J. Pecaric, Some improvements of Grüss type inequality, *J. Math. Inequal.*, **1** (2007), no. 3, 425–436.

[23] A. A. George, *Fractional Differentiation Inequalities,* Springer Publishing Company, New York, 2009.

[24] G. Grüss, Uber das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)$ $g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.*, **39** (1935), 215–226.

[25] A. M. Mercer, P. Mercer, New proofs of the Grüss inequality, *Aust. J. Math. Anal. Appl.*, **1** (2) (2004), Art. 12.

[26] A. M. Mercer, An improvement of the Grüss inequality, *JIPAM. J. Inequal. Pure Appl. Math.*, **6** (2005), no. 4, Article 93.

[27] Yu Miao, Fei Han and Jianyong Mu, A new Ostrowski-Grüss type inequality, *Kragujevac Journal of Mathematics*, **37** (2) (2013), Pages 307–317.

[28] G. A. M. Nchama, A. L. Mecias and M. R. Richard , The Caputo-Fabrizio fractional integral to generate some new inequalities, *Inf. Sci. Lett.*, **2** (8), (2019), Pages 73–80.

[29] B. G. Pachpatte, On multidimensional Grüss type inetegral inequalities, *J. Inequal. Pure and Appl. Math.*, Vol 03, Iss. 2, Art. 27, (2002), 1–7.

[30] B. G. Pachpatte, A note on Chebyshev-Grüss inequalities for differential equations, *Tamsui Oxf. J. Math. Sci.*, **22** (1), (2006), 29–36.

[31] G. H. Peng and Y. Miao, A note on Grüss type inequality, *Appl. Math. Sci.*, (Ruse) **3** (2009), no. 8, 399–402.

[32] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives to method of their solution and some of their application,* Acadamic Press, San Diego CA, 1999.

[33] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Application of Fractional Differential Equations,* Elsevier, Amersterdam, 2006.

[34] G. Rahman, K. S. Nisar, S. Rashid and T. Abdeljawad, Certain Grüss-type inequalities via tempered fractional integrals concerning another function, *J. Inequal. Appl.*, 2020, 147 (2020). https://doi.org/10.1186/s13660-020-02420-x.

[35] S. Rashid, F. Jarad and M. A. Noor, Grüss-type integrals inequalities via generalized proportional fractional operators, *RACSAM*, 114, 93 (2020). https://doi.org/10.1007/s13398-020-00823-5.

[36] E. Set, M. Tomar and M. Z. Sarikaya, On generalized Grüss type inequalities for k-fractional integrals, *Applied Mathematics and Computation*, 269 (2015), 29–34. DOI: 10.1016/j.amc.2015.07.026.

[37] S. G. Somko, A. A. Kilbas and O. I. Marichev, *Fractional Integral and Derivative Theory and Application,* Gordon and Breach, Yverdon, Switzerland, 1993.

[38] W. Yang, Some new fractional quantum integral inequalities, *Appl. Math. Latt.*, 25, 6 (2012), 963–969.

[39] X. Wang, M. S. Saleem, K. N. Aslam, X. Wu and T. Zhou, On Caputo–Fabrizio fractional integral inequalities of Hermite–Hadamard type for modified -convex Functions, *Journal of Mathematics*, vol. 2020, Article ID 8829140, 17 pages, 2020. https://doi.org/10.1155/2020/8829140.

[40] C. Zhu, W. Yang and Q. Zhao, Some new fractional q- integral Grüss-type inequalities and other inequalities, *J. Inequal. Appl.*, 2012:299.

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On unique and non-unique fixed point in parametric $\mathcal{N}_{\mathfrak{b}}-$metric spaces with application

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Abstract. We propose $\mathcal{SA}$, $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$, and $\mathcal{SA}_{\eta,\delta,\zeta}-$contractions and notions of $\eta-$admissibility type $\mathfrak{b}$ and $\eta_{\mathfrak{b}}-$regularity in parametric $\mathcal{N}_{\mathfrak{b}}$-metric spaces to determine a unique fixed point, a unique fixed circle, and a greatest fixed disc. Further, we investigate the geometry of non-unique fixed points of a self mapping and demonstrate by illustrative examples that a circle or a disc in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space is not necessarily the same as a circle or a disc in a Euclidean space. Obtained outcomes are extensions, unifications, improvements, and generalizations of some of the well-known previous results. We provide non-trivial illustrations to exhibit the importance of our explorations. Towards the end, we resolve the system of linear equations to demonstrate the significance of our contractions in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space.

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1 Introduction and preliminaries

Banach [2] stated the first metric fixed point result in 1922. After this, enormous generalizations and extensions of Banach's result have been announced ([1], [4], [7], [13], [17], [23], [27] -[30], and so on). These essentially centred around two components: (i) by changing the structure and (ii) by changing the conditions on the mapping under consideration. One such interesting structure, parametric $\mathcal{N}_{\mathfrak{b}}$-metric spaces is recently introduced by Tas and Özgür [21]. It generalizes the metric space (Fréchet [5]), $\mathfrak{b}-$metric space (Bakhtin [1] and Czerwik [4]), $\mathcal{S}-$metric space (Sedghi et al. [17]), $\mathcal{S}_{\mathfrak{b}}-$metric space (Souayah and Mlaiki [19] and Sedghi et al. [16]), parametric $\mathcal{S}$-metric space (Tas and Özgür [20]), $A_{\mathfrak{b}}$-metric space (Ughade et al. [30]) and so on. It is worth to mention that Souayah et al. [19] used the symmetry condition, in addition to other conditions used by Sedghi et al. [16]. Motivated by the fact that the equations, obtained on modeling real-world problems may be solved using the fixed point technique and geometry of nonunique fixed points, we familiarize $\mathcal{SA}$, $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$, $\mathcal{SA}_{\eta,\delta,\zeta}-$contractions and the notions of η-admissibility of type $\mathfrak{b}$ and $\eta_{\mathfrak{b}}-$ regularity in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space to establish a unique fixed point, a unique fixed circle, and a greatest fixed disc. In the sequel, with the help of examples and remarks, we demonstrate that our contractions are incomparable over each one of those contractions wherein the continuity of mapping is presumed for the survival of a fixed point. Further, we investigate the geometry of non-unique fixed points in reference to fixed circle or greatest fixed disc problems and demonstrate by illustrative examples that a circle or a disc in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space is not necessarily the same as a circle or a disc in a Euclidean space. We conclude the paper by resolving the system of linear equations to demonstrate the significance of our proposed contractions in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space.

We denote $\mathcal{N}(\mathfrak{x},\mathfrak{x},\cdots,(\mathfrak{x})_{\mathfrak{n}-1},\mathfrak{y},\mathfrak{t})$ by $\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}$.

Definition 1 [21] *Let $\mathcal{X}\neq\emptyset$, $\mathfrak{b}\geq 1$ be a given real number, $\mathfrak{n}\in\mathbb{N}$. A distance function $\mathcal{N}:\mathcal{X}^{\mathfrak{n}}\times(0,\infty)\to[0,\infty)$ is a parametric $\mathcal{N}_{\mathfrak{b}}-$metric if*

(N1) $\mathcal{N}(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}},\mathfrak{t})=0$ *iff* $\mathfrak{x}_1=\mathfrak{x}_2=\cdots=\mathfrak{x}_{\mathfrak{n}-1}=\mathfrak{x}_{\mathfrak{n}}$;

(N2) $\mathcal{N}(\mathfrak{x}_1,\mathfrak{x}_2,\cdots,\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}},\mathfrak{t})\leq\mathfrak{b}[\mathcal{N}(\mathfrak{x}_1,\mathfrak{a},\mathfrak{t})+\mathcal{N}(\mathfrak{x}_2,\mathfrak{a},\mathfrak{t})+\cdots+\mathcal{N}(\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{a},\mathfrak{t})+\mathcal{N}(\mathfrak{x}_{\mathfrak{n}},\mathfrak{a},\mathfrak{t})]$,

$\mathfrak{t}>0$, *for every* $\mathfrak{a}$, $\mathfrak{x}_{\mathfrak{i}}\in\mathcal{X}$ *and* $\mathfrak{i}=1,2,\cdots,\mathfrak{n}$.

Example 1 [21] *Let* $\mathcal{X} = \{\mathcal{S}|\mathcal{S} : (0,\infty) \to \mathbb{R}\}$ *be the set of functions and* $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ *be*

$$\mathcal{N}(\mathcal{S}t, \mathcal{T}t, \mathcal{J}t, t) = \frac{1}{9}\big(|\mathcal{S}t - \mathcal{T}t| + |\mathcal{S}t - \mathcal{J}t| + |\mathcal{T}t - \mathcal{J}t|\big)^2,$$

$t > 0$*, for every* $\mathcal{S}$*,* $\mathcal{T}$*,* $\mathcal{J} \in \mathcal{X}$*. Noticeably,* $(\mathcal{X},\mathcal{N})$ *is a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space with* $\mathfrak{n} = 3$ *and* $\mathfrak{b} = 4$*.*

Remark 1 *Noticeably, a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric is an improvement of a parametric* $\mathcal{S}-$*metric* [20] *because every parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric, for* $\mathfrak{b} = 1$ *and* $\mathfrak{n} = 3$*, is a parametric* $\mathcal{S}-$*metric. However, one may verify that a parametric* $\mathcal{S}-$*metric need not essentially be a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric.*

Lemma 1 [21] *In a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X},\mathcal{N})$*,*

(i) $\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \mathfrak{b}\mathcal{N}_{\mathfrak{y},\mathfrak{x},t}$ and $\mathcal{N}_{\mathfrak{y},\mathfrak{x},t} \leq \mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}$,

(ii) $\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \mathfrak{b}[(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x},\mathfrak{z},t} + \mathcal{N}_{\mathfrak{y},\mathfrak{z},t}]$ *and* $\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \mathfrak{b}[(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x},\mathfrak{z},t} + \mathfrak{b}\mathcal{N}_{\mathfrak{z},\mathfrak{y},t}]$,

$t > 0$ *and* $\mathfrak{x}$ $\mathfrak{y} \in \mathcal{X}$*.*

Definition 2 [21] *Let* $\{\mathfrak{x}_{\mathfrak{k}}\}$ *be a sequence in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X},\mathcal{N})$*, then*

(1) $\{\mathfrak{x}_{\mathfrak{k}}\}$ *converges to* $\mathfrak{x}$*, if for* $\epsilon > 0$*, there exists an* $\mathfrak{n}_o \in \mathbb{N}$ *so that, we attain* $\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x},t} \leq \epsilon$*, i.e.,* $\lim_{k\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x},t} = 0$*,* $\mathfrak{l} \geq \mathfrak{n}_0$*. It is denoted* $\lim_{\mathfrak{k}\to\infty} \mathfrak{x}_{\mathfrak{k}} = \mathfrak{x}$*;*

(2) $\{\mathfrak{x}_{\mathfrak{k}}\}$ *is a Cauchy sequence, if for each* $\epsilon > 0$*, there exists an* $\mathfrak{n}_o \in \mathbb{N}$ *so that, we attain* $\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{l}},t} \leq \epsilon$*, i.e.,* $\lim_{k\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{l}},t} = 0$*,* $\mathfrak{k}$*,* $\mathfrak{l} \geq \mathfrak{n}_o$*;*

(3) $(\mathcal{X},\mathcal{N})$ *is complete if every Cauchy sequence in* $(\mathcal{X},\mathcal{N})$ *converges to a point in it.*

Lemma 2 [21] *If* $\{\mathfrak{x}_{\mathfrak{k}}\}$ *and* $\{\mathfrak{y}_{\mathfrak{k}}\}$ *are two sequences in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X},\mathcal{N})$ *that converge to* $\mathfrak{x}$ *and* $\mathfrak{y}$ *respectively in* $\mathcal{X}$ *then:*

(i) $\mathfrak{x}$ *is unique,*

(ii) $\{\mathfrak{x}_{\mathfrak{k}}\}$ *is a Cauchy sequence,*

(iii) $\frac{1}{\mathfrak{b}^2}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} \leq \lim_{k\to\infty} \inf \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{y}_{\mathfrak{k}},t} \leq \lim_{k\to\infty} \sup \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{y}_{\mathfrak{k}},t} \leq \mathfrak{b}^2\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}$.

Lemma 3 [21] *If two sequences* $\{\mathfrak{x}_{\mathfrak{k}}\}$ *and* $\{\mathfrak{y}_{\mathfrak{k}}\}$ *in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X},\mathcal{N})$ *are such that*

$$\lim_{k\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{y}_{\mathfrak{k}},t} = 0,$$

when $\{\mathfrak{x}_{\mathfrak{k}}\}$ *is convergent and* $\lim_{k\to\infty} \mathfrak{x}_{\mathfrak{k}} = \mathfrak{x}_o$*,* $\mathfrak{x}_o \in \mathcal{X}$*, then* $\lim_{k\to\infty} \mathfrak{y}_{\mathfrak{k}} = \mathfrak{x}_o$*.*

2 Main results

I. Existence of a single fixed point

We define $\mathcal{SA}$, $\eta - \mathcal{SA}$, $\eta - \mathcal{SA}_{\min}$, and $\mathcal{SA}_{\eta,\delta,\zeta}-$contractive conditions in a parametric $\mathcal{N}_{\mathfrak{b}}$-metric space to prove a fixed point.

Definition 3 *A self mapping $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ in a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space $(\mathcal{X},\mathcal{N})$ with $\mathfrak{b}\geq 1$ is called an $\mathcal{SA}-$contraction if*

$$\begin{aligned}\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},t} \leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}}\\ + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}},\end{aligned} \tag{1}$$

where, $\sum_{i=1}^{5}\mathfrak{a}_i < 1$ and $\mathfrak{a}_1+\mathfrak{a}_3\mathfrak{b} < 1$, ($\mathfrak{a}_i$, $i = 1$ to 5, are non-negative constants).

Definition 4 *A self mapping $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ in a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b}\geq 1$ is called an $\eta-\mathcal{SA}-$contraction if*

$$\begin{aligned}\eta(\mathfrak{x},\mathfrak{y},t)\,\mathcal{N}_{\mathcal{S}x,\mathcal{S}\mathfrak{y},t} \leq \mathfrak{a}_1\,\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} + \mathfrak{a}_2\,\phi\left(\max\begin{Bmatrix}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t},\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t},\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t},\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t},\\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t},\ \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}+\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{2}\end{Bmatrix}\right)\\ + \mathfrak{a}_3\,\phi\left(\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}\left[1+\sqrt{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}}\right]^2}{(1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},t})^2}\right),\end{aligned} \tag{2}$$

$\mathfrak{x}$, $\mathfrak{y}\in\mathcal{A}$, $\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3\geq 0$, $\mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3<1$, $\mathfrak{a}_1+\mathfrak{b}\mathfrak{a}_2+\mathfrak{a}_3<1$, $0\leq\mathfrak{a}_2<\frac{1-\mathfrak{a}_1-\mathfrak{a}_3}{\mathfrak{b}^2+\mathfrak{b}(n-1)}$, η, $\phi:[0,\infty)\to[0,\infty)$ are increasing functions and $\phi(t)<t$, $t>0$.

Definition 5 *A self mapping $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ in a parametric $\mathcal{N}_{\mathfrak{b}}$-metric space $(\mathcal{X},\mathcal{N})$ with $\mathfrak{b}\geq 1$ is called an $\eta-\mathcal{SA}_{\min}$*

$$\begin{aligned}\eta(\mathfrak{x},\mathfrak{y},t)\,\phi(\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},t}) \leq \mathfrak{a}_1\,\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} + \mathfrak{a}_2\,\phi\left(\min\begin{Bmatrix}\mathcal{N}_{\mathfrak{x},\mathfrak{y},t},\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t},\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t},\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t},\\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t},\ \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}+\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{2}\end{Bmatrix}\right)\\ + \mathfrak{a}_3\,\phi\left(\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}\left[1+\sqrt{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}}\right]^2}{(1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},t})^2}\right),\end{aligned} \tag{3}$$

$\mathfrak{x}$, $\mathfrak{y}\in\mathcal{X}$ and $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3\geq 0$ with $\mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3<1$, η, $\phi:[0,\infty)\to[0,\infty)$ are increasing functions and $\phi(t)<t$, $t>0$.

Definition 6 *A self mapping $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ in a parametric $\mathcal{N}_{\mathfrak{b}}$-metric space $(\mathcal{X},\mathcal{N})$ with $\mathfrak{b}\geq 1$ is called an $\mathcal{SA}_{\eta,\delta,\zeta}-$ contraction if*

$$[\eta(\mathfrak{x},\mathfrak{y},t)-1+\delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},t}}\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x},\mathfrak{y},t})\left(\max\left\{\mathcal{N}_{\mathfrak{x},\mathfrak{y},t},\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t},\mathrm{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t},\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t},\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}+\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{2}\right\}\right)}, \tag{4}$$

$\mathfrak{x}$, $\mathfrak{y}\in\mathcal{X}$, a constant $\delta\geq 1$, $\zeta:[0,\infty)\to[0,\frac{1}{\mathfrak{b}}]$ and $\eta:[0,\infty)\to[0,\infty)$ is a non-decreasing function.

Next, we prove our first main result for an $\mathcal{SA}-$contraction.

Theorem 1 *Let $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ be an $\mathcal{SA}-$contraction* (1) *in a complete parametric $\mathcal{N}_{\mathfrak{b}}$-metric space $(\mathcal{X},\mathcal{N})$. Then, $\mathcal{S}$ has a unique fixed point in $\mathcal{X}$.*

Proof. Let us assume that $\mathfrak{x}_o\in\mathcal{X}$. Let a sequence $\{\mathfrak{x}_n\}$ be constructed as $\mathfrak{x}_{n+1}=\mathcal{S}\mathfrak{x}_n$. If, we have $\mathfrak{x}_{n_o}=\mathfrak{x}_{n_o+1}$ then $\mathfrak{x}_{n_o}=\mathfrak{x}_{n_o+1}=\mathcal{S}\mathfrak{x}_{n_o}$, $n_o\in\mathbb{N}$, i.e., we infer that $\mathfrak{x}_{n_o}$ is a fixed point of $\mathcal{S}$.
Let $\mathfrak{x}_{n_o}\neq\mathfrak{x}_{n_o+1}$, $n_0\in\mathbb{N}$. Using inequality (1), we attain

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} &= \mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,t}\\
&\leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+a_2\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_3\frac{\mathcal{N}_{x_{n-1},\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&\quad+a_4\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_5\frac{\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&= a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+a_2\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_3\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&\quad+a_4\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}+a_5\frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}\\
&\leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+a_2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}+a_4[\mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}].
\end{aligned}$$

It follows that

$$(1-a_2-\mathfrak{b}^2a_4)\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\leq\big(a_1+\mathfrak{b}(n-1)a_4\big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}. \tag{5}$$

Again using inequality (1), we obtain

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_{n+1},\mathfrak{x}_n,t} &= \mathcal{N}_{\mathcal{S}\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}\\
&\leq a_1\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}+a_2\frac{\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}}+a_3\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}}
\end{aligned}$$

$$
\begin{aligned}
&+ \mathfrak{a}_4 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}}{1+\mathrm{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} + \mathfrak{a}_5 \frac{\mathrm{N}_{\mathfrak{x}_{n-1},\mathrm{S}\mathfrak{x}_{n-1},t}\mathrm{N}_{\mathfrak{x}_{n-1},\mathrm{S}\mathfrak{x}_n,t}}{1+\mathrm{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}}\\
&= \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t} + \mathfrak{a}_2 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} + \mathfrak{a}_3 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}}\\
&+ \mathfrak{a}_4 \frac{\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_n,t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}} + \mathfrak{a}_5 \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}}{1+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n-1},t}}\\
&\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + \mathfrak{a}_2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} + \mathfrak{a}_5[\mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}]. \quad (6)
\end{aligned}
$$

It follows that

$$
(1-\mathfrak{a}_2-\mathfrak{b}^2\mathfrak{a}_5)\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \big(\mathfrak{a}_1+\mathfrak{b}(n-1)\mathfrak{a}_5\big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}. \quad (7)
$$

Adding inequalities (5) and (7), we obtain

$$
\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \left(\frac{2\mathfrak{a}_1+\mathfrak{b}(n-1)(\mathfrak{a}_4+\mathfrak{a}_5)}{2-2\mathfrak{a}_2-\mathfrak{b}^2(\mathfrak{a}_4+\mathfrak{a}_5)}\right)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.
$$

Let $\frac{2\mathfrak{a}_1+\mathfrak{b}(n-1)(\mathfrak{a}_4+\mathfrak{a}_5)}{2-2\mathfrak{a}_2-\mathfrak{b}^2(\mathfrak{a}_4+\mathfrak{a}_5)} = h$. In view of $\sum_1^5 \mathfrak{a}_i < 1$, $h \in (0,1)$. Then,

$$
\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.
$$

Similarly,

$$
\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.
$$

So,

$$
\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^2\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.
$$

Following the same pattern, we attain

$$
\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_o,\mathfrak{x}_1,t}.
$$

Since, $h \in (0,1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$
\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \quad (8)
$$

Then, for $\mathfrak{l} > \mathfrak{k}$, $\mathfrak{k}$, $\mathfrak{l} \in \mathbb{N}$, using equation (8), condition (N2) and Lemma 1, we obtain

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} &\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+1},t} \leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_\mathfrak{l},t}\\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^3\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_\mathfrak{l},t}\\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^4\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+2},t}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^5(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} \\
&\quad + \mathfrak{b}^5\mathcal{N}_{\mathfrak{x}_{\mathfrak{l}},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} \\
&\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^5(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} \\
&\quad + \mathfrak{b}^7\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{x}_{\mathfrak{k}+4},\mathfrak{t}} + \cdots .
\end{aligned}
$$

Letting $\mathfrak{k}$, $\mathfrak{l} \to \infty$, we obtain

$$\lim_{\mathfrak{k},\, \mathfrak{l}\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{l}},\mathfrak{t}} = 0,$$

i.e., $\{\mathfrak{x}_{\mathfrak{n}}\}$ is a Cauchy sequence. Using the completeness of the space, $\lim_{\mathfrak{k},\, \mathfrak{l}\to\infty} \mathfrak{x}_{\mathfrak{k}} = \mathfrak{x}, \mathfrak{x} \in \mathcal{X}$.

Assume $\mathfrak{x}$ is not a fixed point of $\mathcal{S}$. Applying inequality (1), we obtain

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathcal{S}\mathfrak{x},\mathfrak{t}} = \mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}} &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} \\
&\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{x,\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} \\
&= \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} \\
&\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}}. \qquad (9)
\end{aligned}
$$

As $\mathfrak{k} \to \infty$, using condition (N1), we get $\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} \leq 0$, i.e., $\mathcal{S}\mathfrak{x} = \mathfrak{x}$.

Presume that $\mathfrak{y}$ is one more fixed point of $\mathcal{S}$, then $\mathcal{S}\mathfrak{x} = \mathfrak{x}$ and $\mathcal{S}\mathfrak{y} = \mathfrak{y}$. Using inequality (1), we obtain

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} = \mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\
&\quad + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} = (\mathfrak{a}_1 + \mathfrak{a}_3\mathfrak{b})\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \text{ a contradiction.}
\end{aligned}
$$

Thus, $\mathfrak{x} = \mathfrak{y}$, i.e., a fixed point of $\mathcal{S}$ is unique. □

Next, we furnish a non-trivial illustration to exhibit the validity of the above outcome.

Example 2 *Let* $\mathcal{X} = \mathbb{R}^+ \cup \{0\}$. *Let a function* $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ *be*

$$\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \begin{cases} 0, & \text{if } \mathfrak{x}=\mathfrak{y}=\mathfrak{z}; \\ \mathfrak{t}^2\max\{\mathfrak{x},\mathfrak{y},\mathfrak{z}\}, & \text{otherwise,} \end{cases}$$

for each $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z} \in \mathcal{X}$ *and* $\mathfrak{t} > 0$. *Then,* $(\mathcal{X}, \mathcal{N})$ *is a complete parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space for* $\mathfrak{b} = 2$ *and* $\mathfrak{n} = 3$. *Define* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *as*

$$\mathcal{S}\mathfrak{x} = \begin{cases} \frac{\mathfrak{x}^2}{16}, & \text{if } \mathfrak{x} \in [0, \mathfrak{a}); \\ \frac{\mathfrak{x}}{15}, & \text{if } \mathfrak{x} \in [\mathfrak{a}, \infty), \end{cases}, \quad \mathfrak{x} \in \mathcal{X}$$

with $\mathfrak{a} \in (\frac{1}{4}, 1)$. *Taking* $\mathfrak{a}_1 = \frac{1}{5} = \mathfrak{a}_2 = \mathfrak{a}_3 = \mathfrak{a}_4$ *and* $\mathfrak{a}_5 = \frac{1}{10}$, $\mathcal{S}$ *verifies the hypotheses of Theorem 1 and has a unique fixed point at* $\mathfrak{x} = 0$.

For $\mathfrak{a}_1 \in [0, 1)$ and $\mathfrak{a}_2 = \mathfrak{a}_3 = \mathfrak{a}_4 = \mathfrak{a}_5 = 0$, Theorem 1 is an extension and an improvement of Banach [2] to a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space wherein the involved mapping is not necessarily continuous.

Following Sintunavarat [18], we familiarize $\eta-$admissibility of type $\mathfrak{b}$ and $\eta_{\mathfrak{b}}-$regularity to determine a fixed point in a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space $(\mathcal{X}, \mathcal{N})$.

Definition 7 *A self mapping* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *is called* $\eta-$*admissible of type* $\mathfrak{b}$ *if there exists an* $\eta : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ *so that* $\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \geq \mathfrak{b}$ *implies that* $\eta(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{t} > 0$ *and* $\mathfrak{x}$, $\mathfrak{y} \in \mathcal{X}$.

Example 3 *Let* $\mathcal{X} = \{(0, 0), (1, 0), (1, 2), (1, 3), (1, 4)\}$ *be a subset of* $\mathbb{R}^2$. *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be*

$$\mathcal{S}\mathfrak{x} = \begin{cases} (1, 2), & \text{if } \mathfrak{x} \in \mathcal{X} \backslash \{(1, 4)\} \\ (1, 3), & \text{if } \mathfrak{x} = (1, 4). \end{cases}$$

Now, define an $\eta : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ *as*

$$\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \begin{cases} (1, 0), & \text{if } \mathfrak{x}, \ \mathfrak{y} \in \mathcal{X} \backslash \{(1, 4)\} \\ \frac{3}{2}, & \text{if } \mathfrak{x} = (1, 4). \end{cases}$$

In case $\mathfrak{x}$, $\mathfrak{y} \in \mathcal{X} \backslash \{(1, 4)\}$, *then* $\eta(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}) = \eta((1, 2),\ (1, 2),\ \mathfrak{t}) = (1, 0)$. *If* $\mathfrak{x} = \mathfrak{y} = (1, 3)$, *then* $\eta(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}) = \eta\big(\mathcal{S}(1, 4), \mathcal{S}(1, 4), \mathfrak{t})\big) = \eta((1, 3),\ (1, 3), \mathfrak{t})) = (1, 0)$, $\mathfrak{t} > 0$. *Hence,* $\mathcal{S}$ *is* $\eta-$*admissible of type* $\mathfrak{b}$. *One may verify that* $\mathcal{S}$ *is neither an* $\alpha-$*admissible* [14] *nor an* $\alpha-$*admissible type* $\mathcal{S}$ [18].

Definition 8 *Let* $\{\mathfrak{x}_{\mathfrak{n}}\}$ *be a sequence in* $\mathcal{X}$ *so that* $\eta(\mathfrak{x}_{\mathfrak{n}}, \mathfrak{x}_{\mathfrak{n}+1}, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{n} \in \mathbb{N} \cup \{0\}$, $\mathfrak{t} > 0$ *and* $\lim_{\mathfrak{n} \longrightarrow \infty} \mathfrak{x}_{\mathfrak{n}} = \mathfrak{x} \in \mathcal{X}$, *then* $\mathcal{X}$ *is called* $\eta_{\mathfrak{b}}-$*regular if* $\eta(\mathfrak{x}_{\mathfrak{n}}, \mathfrak{x}, \mathfrak{t}) \geq \mathfrak{b}$.

Theorem 2 *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be an* $\eta - \mathcal{S}\mathcal{A}-$ *contraction* (2) *in a complete parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X}, \mathcal{N})$ *with* $\mathfrak{b} \geq 1$ *satisfying*

(i) $\mathcal{S}$ *is* $\eta-$*admissible of type* $\mathfrak{b}$*;*

(ii) $\mathcal{X}$ *is* $\eta_{\mathfrak{b}}-$*regular;*

(iii) *There exists a* $\mathfrak{x}_o \in \mathcal{X}$ *so that* $\eta(x_0, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$*, for* $\mathfrak{t} > 0$*.*

Then, $\mathcal{S}$ *has a unique fixed point.*

Proof. Consider $\mathfrak{x}_o \in \mathcal{X}$ so that $\eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{t} > 0$. Let a sequence $\{\mathfrak{x}_\mathfrak{n}\}$ be constructed as $\mathfrak{x}_{\mathfrak{n}+1} = \mathcal{S}\mathfrak{x}_\mathfrak{n}$, $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. Since, $\eta(\mathfrak{x}_o, \mathfrak{x}_1, \mathfrak{t}) = \eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$ and $\eta(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{t}) = \eta(\mathcal{S}x_0, \mathcal{S}\mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, using $(\mathfrak{ii})$. Following this pattern, we attain $\eta(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1}, \mathfrak{t}) \geq \mathfrak{b}$. In case, $\mathfrak{x}_\mathfrak{n} = \mathfrak{x}_{\mathfrak{n}+1}$, we conclude that $\mathfrak{x}_\mathfrak{n}$ is a fixed point of $\mathcal{S}$. Let $\mathfrak{x}_\mathfrak{n} \neq \mathfrak{x}_{\mathfrak{n}+1}$. Using inequality (2), we obtain

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}} &= \mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{n}-1},\mathcal{S}\mathfrak{x}_\mathfrak{n},\mathfrak{t}} \leq \eta(\mathfrak{x}_{\mathfrak{n}-1}, \mathfrak{x}_\mathfrak{n}, \mathfrak{t})\mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{n}-1},\mathcal{S}\mathfrak{x}_\mathfrak{n},\mathfrak{t}} \\
&\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathfrak{a}_2\, \phi\Bigg(\max \left\{ \begin{array}{l} \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathcal{S}\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathcal{S}\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathcal{S}\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathcal{S}\mathfrak{x}_\mathfrak{n},\mathfrak{t}}}{2} \end{array} \right\} \Bigg) \\
&\quad + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{y},\mathfrak{t}}\big[1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{t}}}\big]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}})^2} \right) \\
&= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathfrak{a}_2\, \phi\Bigg(\max \left\{ \begin{array}{l} \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}}{2} \end{array} \right\} \Bigg) \\
&\quad + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{y},\mathfrak{t}}\big[1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}}\big]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}})^2} \right) \\
&= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathfrak{a}_2\, \phi\Bigg(\max \left\{ \begin{array}{l} \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \\ \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}}{2} \end{array} \right\} \Bigg) \\
&\quad + \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{y},\mathfrak{t}}\big[1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}}\big]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}})^2} \right) \\
&\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathfrak{a}_2\, \phi\Bigg(\max \left\{ \begin{array}{l} \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \\ \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}}{2} \end{array} \right\} \Bigg) \\
&\quad + \mathfrak{a}_3\, \phi(\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}). \qquad (10)
\end{aligned}
$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}}}{2}\} = \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}$. Then, inequality (10) becomes

$$\mathcal{N}_{\mathfrak{x}_\mathfrak{n},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}} \leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}} + \mathfrak{a}_2\phi(\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}) + \mathfrak{a}_3\phi(\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_\mathfrak{n},\mathfrak{t}}).$$

Since, $\phi(t) < t$, we attain

$$\mathcal{N}_{x_n,x_{n+1},t} < (a_1 + a_2 + a_3)\mathcal{N}_{x_{n-1},x_n,t}.$$

Let $a_1 + a_2 + a_3 = h$, we have

$$\mathcal{N}_{x_n,x_{n+1},t} < h\mathcal{N}_{x_{n-1},x_n,t}.$$

Similarly,

$$\mathcal{N}_{x_{n-1},x_n,t} < h\mathcal{N}_{x_{n-2},x_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{x_n,x_{n+1},t} < h^2\mathcal{N}_{x_{n-2},x_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{x_n,x_{n+1},t} \leq h^n\mathcal{N}_{x_o,x_1,t}.$$

Since, $h \in (0, 1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{x_n,x_{n+1},t} = 0. \tag{11}$$

If $\max\{\mathcal{N}_{x_{n-1},x_n,t}, \mathcal{N}_{x_n,x_{n+1},t}, \mathcal{N}_{x_{n-1},x_{n+1},t}, \frac{\mathcal{N}_{x_{n-1},x_n,t}+\mathcal{N}_{x_n,x_{n+1},t}}{2}\} = \mathcal{N}_{x_n,x_{n+1},t}$. Then, inequality (10) becomes

$$\mathcal{N}_{x_n,x_{n+1},t} \leq a_1\mathcal{N}_{x_{n-1},x_n,t} + a_2\phi(\mathcal{N}_{x_n,x_{n+1},t}) + a_3\phi(\mathcal{N}_{x_{n-1},x_n,t}).$$

Since, $\phi(t) < t$, we attain

$$\mathcal{N}_{x_n,x_{n+1},t} \leq \Big(\frac{a_1 + a_3}{1 - a_2}\Big)\mathcal{N}_{x_{n-1},x_n,t}.$$

Let $\left(\frac{a_1+a_3}{1-a_2}\right) = h$, we have

$$\mathcal{N}_{x_n,x_{n+1},t} \leq h\mathcal{N}_{x_{n-1},x_n,t}.$$

Similarly,

$$\mathcal{N}_{x_{n-1},x_n,t} \leq h\mathcal{N}_{x_{n-2},x_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{x_n,x_{n+1},t} \leq h^2\mathcal{N}_{x_{n-2},x_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{x_n,x_{n+1},t} \leq h^n\mathcal{N}_{x_o,x_1,t}.$$

Since, $h \in (0,1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{12}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, N_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+N_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}$. Then, inequality (10) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}) + a_3\phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}).$$

Utilizing Lemma 1 and the definition of ϕ , we attain

$$\begin{aligned}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} &\leq (a_1+a_3)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\phi\big(b(n-1)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + b^2\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}\big)\\ &\leq \Big(\frac{a_1+a_3+a_2b(n-1)}{1-b^2a_2}\Big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.\end{aligned}$$

Let $\frac{a_1+a_3+a_2b(n-1)}{1-b^2a_2} = h$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^2\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $0 \leq a_2 < \frac{1-a_1-a_3}{b^2+b(n-1)}$, $h \in (0,1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{13}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}$. Then, inequality (10) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\Big(\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\Big) + a_3\phi(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}).$$

Since, $\phi(t) < t$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq a_1\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + a_2\Big(\frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\Big) + a_3(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t})$$

$$\leq \Big(\frac{2\mathfrak{a}_1 + \mathfrak{a}_2 + 2\mathfrak{a}_3}{2 - \mathfrak{a}_2}\Big)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Let $\left(\frac{2\mathfrak{a}_1+\mathfrak{a}_2+2\mathfrak{a}_3}{2-\mathfrak{a}_2}\right) = h$, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h\mathcal{N}_{x_{n-1},x_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq h\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^2\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following this pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq h^n\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $h \in (0, 1)$, letting $n \to \infty$, $h^n \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{14}$$

Then, for $\mathfrak{k}$, $\mathfrak{l} \in \mathbb{N}$ so that $\mathfrak{l} > \mathfrak{k}$, using equation (14), condition (N2), and Lemma 1, we obtain

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} &\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+1},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_\mathfrak{l},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^3\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_\mathfrak{l},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^4\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+2},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^5(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},t} \\
&\quad + \mathfrak{b}^5\mathcal{N}_{\mathfrak{x}_\mathfrak{l},\mathfrak{x}_{\mathfrak{k}+3},t} \\
&\leq \mathfrak{b}(n-1)\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_{\mathfrak{k}+1},t} + \mathfrak{b}^3(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},t} + \mathfrak{b}^5(n-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},t} \\
&\quad + \mathfrak{b}^7\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{x}_{\mathfrak{k}+4},t} + \cdots .
\end{aligned}$$

Letting $\mathfrak{k}$, $\mathfrak{l} \to \infty$, we get $\lim_{\mathfrak{k},\ \mathfrak{l}\to\infty} \mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathfrak{x}_\mathfrak{l},t} = 0$, i.e., $\{\mathfrak{x}_n\}$ is a Cauchy sequence. Using the completeness hypotheses, $\lim_{\mathfrak{k},\mathfrak{l}\to\infty} \mathfrak{x}_\mathfrak{k} = \mathfrak{x}$, $\mathfrak{x} \in \mathcal{X}$.

Assume $\mathfrak{x}$ is not a fixed point of $\mathcal{S}$. Since, $\mathcal{X}$ is $\eta_\mathfrak{b}$−regular, then $\mathfrak{x}_n \to \mathfrak{x}$ as $n \to \infty$ and $\eta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, t) \geq \mathfrak{b}$, which implies that $\eta(\mathfrak{x}_n, \mathfrak{x}, t) \geq \mathfrak{b}$, $n \in \mathbb{N} \cup \{0\}$. Using inequality (2), we attain

$$\mathcal{N}_{\mathfrak{x}_\mathfrak{k},\mathcal{S}\mathfrak{x},t} = \mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},t} \leq \eta(\mathfrak{x}_{\mathfrak{k}-1}, \mathfrak{x}, t)\mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},t}$$

$$\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{2} \end{Bmatrix} \right)$$
$$+ \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} \left[1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}}\right]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}})^2} \right)$$
$$= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}} + \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{2} \end{Bmatrix} \right)$$
$$+ \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} \left[1 + \sqrt{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x}_{\mathfrak{k}},\mathfrak{t}}}\right]^2}{(1 + \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}})^2} \right). \tag{15}$$

As $\mathfrak{k} \to \infty$, using Lemma 1 and condition $(\mathsf{N1})$, we get $\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} \leq 0$ which implies that $\mathcal{S}\mathfrak{x} = \mathfrak{x}$.
Let $\mathcal{S}$ has one more fixed point, i.e., $\mathcal{S}\mathfrak{x} = \mathfrak{x}$ and $\mathcal{S}\mathfrak{y} = \mathfrak{y}$, $(\mathfrak{x} \neq \mathfrak{y})$. Applying inequality (2), we obtain

$$\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \leq \eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) \mathsf{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} \leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} + \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{2} \end{Bmatrix} \right)$$
$$+ \mathfrak{a}_3\, \phi\left(\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \left[1 + \sqrt{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}\right]^2}{(1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}})^2} \right)$$
$$\leq \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\left(\max \begin{Bmatrix} \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, 0, \mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \\ \frac{\mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}{2} \end{Bmatrix} \right) + \mathfrak{a}_3\, \phi(\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}})$$
$$= \mathfrak{a}_1 \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\, \phi\big(\mathfrak{b}\mathcal{N}_{\mathfrak{x},\mathfrak{v},\mathfrak{t}}\big) + \mathfrak{a}_3\, \phi(\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}).$$

Since, $\phi(\mathfrak{t}) < \mathfrak{t}$, $\mathfrak{t} > 0$, $\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \leq (\mathfrak{a}_1 + \mathfrak{b}\mathfrak{a}_2 + \mathfrak{a}_3)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}$, which is a contradiction. Thus, $\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} = 0$, i.e., $\mathfrak{x} = \mathfrak{y}$. So, a fixed point of $\mathcal{S}$ is unique. □

The next example is provided to justify Theorem 2.

Example 4 *Let* $\mathcal{X} = \mathbb{R}^+ \cup \{0\}$ *and function* $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ *be given by* $\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \frac{1}{2}(|\mathfrak{x}-\mathfrak{y}| + |\mathfrak{x}-\mathfrak{z}| + |\mathfrak{y}-\mathfrak{z}|)^2$, *for every* $\mathfrak{t} > 0$*and* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z} \in \mathcal{X}$*. Then,* $(\mathcal{X},\mathcal{N})$ *is a complete parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space with* $\mathfrak{b} = 2$ *and* $\mathfrak{n} = 3$*. Define* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *as*

$$\mathcal{S}\mathfrak{x} = \begin{cases} \frac{\mathfrak{x}^2}{16}, & \textit{if } \mathfrak{x} \in [0,\mathfrak{a}); \\ \frac{\mathfrak{x}}{15}, & \textit{if } \mathfrak{x} \in [\mathfrak{a},\infty), \end{cases}$$

$\mathfrak{x} \in \mathcal{X}$*,* $\mathfrak{a} \in (\frac{1}{4}, 1)$*. Now, define an* $\eta : \mathcal{X} \times \mathcal{X} \times (0,\infty) \to (0,\infty)$ *as*

$$\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = \begin{cases} 1, & \textit{if } \mathfrak{x},\ \mathfrak{y} \in \mathcal{X}; \\ \frac{3}{2}, & \textit{if otherwise} \end{cases}$$

and $\phi(\mathfrak{s}) = \frac{1}{2}\mathfrak{s}$. *Taking* $\mathfrak{a}_1 = \frac{1}{10} = \mathfrak{a}_2$ *and* $\mathfrak{a}_3 = \frac{1}{15}$, $\mathcal{S}$ *verifies the hypotheses of Theorem 2 and has a unique fixed point at* $\mathfrak{x} = 0$.

For $\mathfrak{a}_2 \in [0, 1)$, $\mathfrak{a}_1 = \mathfrak{a}_3 = 0$ and $\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 1$, Theorem 2 is an extension and an improvement of Ćirić [3] to a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space wherein the involved mapping is not necessarily continuous.

The next result is slightly more interesting as here the max term is replaced by the min term in Theorem 2.10.

Theorem 3 *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be an* $\eta - \mathcal{SA}_{\min}$ *contraction* (3)) *in a complete parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space* $(\mathcal{X}, \mathcal{N})$ *with* $\mathfrak{b} \geq 1$ *satisfying*

(i) $\mathcal{S}$ *is* $\eta-$*admissible of type* $\mathfrak{b}$*;*

(ii) $\mathcal{X}$ *is* $\eta_{\mathfrak{b}}-$*regular;*

(iii) *There exists a* $\mathfrak{x}_o \in \mathcal{X}$ *so that* $\eta(\mathfrak{x}_o, \mathfrak{f}\mathfrak{x}_o, \mathfrak{t}) \geq 1$, $\mathfrak{t} > 0$.

Then, $\mathcal{S}$ *has a unique fixed point in* $\mathcal{X}$.

Proof. The proof is easy and follows the pattern of Theorem 2. □

The following result is more interesting as a weaker control function ϕ is used with the $\eta-$admissibility of type $\mathfrak{b}$ function, without exploiting $\eta_{\mathfrak{b}}-$regularity, for a more general contractivity condition involving rational and irrational terms to establish a fixed point of discontinuous mapping.

Theorem 4 *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be a* $\mathcal{SA}_{\eta,\delta,\zeta}-$ *contraction* (4) *in a complete parametric* $\mathsf{N}_{\mathfrak{b}}$*-metric space* $(\mathcal{X}, \mathcal{N})$ *satisfying*

(i) $\mathcal{S}$ *is* $\eta-$*admissible of type* $\mathfrak{b}$*;*

(ii) *There exists a* $\mathfrak{x}_o \in \mathcal{X}$ *so that* $\eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{t} > 0$.

Then, $\mathcal{S}$ *has a unique fixed point.*

Proof. Consider $\mathfrak{x}_o \in \mathcal{X}$ so that $\eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$, $\mathfrak{t} > 0$. Let a sequence $\{\mathfrak{x}_n\}$ be constructed as $\mathfrak{x}_{n+1} = \mathcal{S}\mathfrak{x}_n$, $n \in \mathbb{N} \cup \{0\}$. As $\eta(\mathfrak{x}_o, \mathfrak{x}_1, \mathfrak{t}) = \eta(\mathfrak{x}_o, \mathcal{S}\mathfrak{x}_o, \mathfrak{t}) \geq \mathfrak{b}$ and $\eta(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{t}) = \eta(\mathcal{S}\mathfrak{x}_o, \mathcal{S}\mathcal{S}\mathfrak{x}_o) \geq \mathfrak{b}$, using (ii). Following the same pattern, we attain $\eta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}) \geq \mathfrak{b}$. If $\mathfrak{x}_n = \mathfrak{x}_{n+1}$, then we conclude that $\mathfrak{x}_n$ is a fixed point of $\mathcal{S}$.

Let $\mathfrak{x}_n \neq \mathfrak{x}_{n+1}$. Utilizing $\mathcal{SA}_{\eta\delta\zeta}-$contraction (4), we attain

$$\delta^{\mathcal{N}_{\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}}} = \delta^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1}, \mathcal{S}\mathfrak{x}_n, \mathfrak{t}}} \leq [\eta(\mathfrak{x}_{n-1}, \mathfrak{x}_n, \mathfrak{t}) - 1 + \delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{n-1}, \mathcal{S}\mathfrak{x}_n, \mathfrak{t}}}$$

$$\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}) \max\left\{\begin{array}{l}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}, \mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_{n-1},t}, \\ \mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathcal{S}\mathfrak{x}_{n-1},t}+\mathcal{N}_{\mathfrak{x}_n,\mathcal{S}\mathfrak{x}_n,t}}{2}\end{array}\right\}}$$

$$= \delta^{\frac{1}{\mathfrak{b}} \max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\}}$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}} \max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} + \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\}. \tag{16}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}$. Then, inequality (16) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t} \leq \frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}^2}\mathcal{N}_{\mathfrak{x}_{n-2},\mathfrak{x}_{n-1},t}.$$

Following the same pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}^n}\mathcal{N}_{\mathfrak{x}_0,\mathfrak{x}_1,t}.$$

Since, $\mathfrak{b} \geq 1$, letting $n \to \infty$, $\frac{1}{\mathfrak{b}^n} \to 0$, we attain

$$\lim_{n\to\infty} \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0. \tag{17}$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}$. Then, inequality (16) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \text{ a contradiction.}$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} = 0.$$

If $\max\{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}, \mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}, \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}\} = \frac{\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}+\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t}}{2}$. Then, inequality (16) becomes

$$\mathcal{N}_{\mathfrak{x}_n,\mathfrak{x}_{n+1},t} \leq \left(\frac{1}{2\mathfrak{b}-1}\right)\mathcal{N}_{\mathfrak{x}_{n-1},\mathfrak{x}_n,t}.$$

Let $\left(\frac{1}{2\mathfrak{b}-1}\right) = \mathfrak{h}$, we attain

$$\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}} \leq \mathfrak{h}\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}},\mathfrak{t}}.$$

Similarly,

$$\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{x}_{\mathfrak{n}},\mathfrak{t}} \leq \mathfrak{h}\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-2},\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{t}}.$$

Therefore,

$$\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}} \leq \mathfrak{h}^2\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}-2},\mathfrak{x}_{\mathfrak{n}-1},\mathfrak{t}}.$$

Following the same pattern, we attain

$$\mathcal{N}_{\mathfrak{x}_{\mathfrak{n}},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}} \leq \mathfrak{h}^{\mathfrak{n}}\mathcal{N}_{\mathfrak{x}_{0},\mathfrak{x}_{1},\mathfrak{t}}.$$

Since, $\mathfrak{h} \in (0,1)$, letting $\mathfrak{n} \to \infty$, $\mathfrak{h}^{\mathfrak{n}} \to 0$, we attain

$$\lim_{\mathfrak{n}\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{n}},\mathfrak{x}_{\mathfrak{n}+1},\mathfrak{t}} = 0. \tag{18}$$

Now, claim that $\{\mathfrak{x}_{\mathfrak{n}}\}$ is a Cauchy sequence. Then, for $\mathfrak{k}$, $\mathfrak{l} \in \mathbb{N}$ so that $\mathfrak{l} > \mathfrak{k}$, using equation (18), condition (N2), and Lemma 1, we have

$$\begin{aligned}
\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{l}},\mathfrak{t}} &\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}\mathcal{N}_{\mathfrak{x}_{\mathfrak{l}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} \leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^2\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{l}},\mathfrak{t}} \\
&\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^3\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{l}},\mathfrak{t}} \\
&\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^4\mathcal{N}_{\mathfrak{x}_{\mathfrak{l}},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} \\
&\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^5(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} \\
&\quad + \mathfrak{b}^5\mathcal{N}_{\mathfrak{x}_{\mathfrak{l}},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} \\
&\leq \mathfrak{b}(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}} + \mathfrak{b}^3(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{t}} + \mathfrak{b}^5(\mathfrak{n}-1)\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+2},\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{t}} \\
&\quad + \mathfrak{b}^7\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}+3},\mathfrak{x}_{\mathfrak{k}+4},\mathfrak{t}} + \cdots .
\end{aligned}$$

Letting $\mathfrak{k}$, $\mathfrak{l} \to \infty$, we obtain

$$\lim_{\mathfrak{k},\mathfrak{l}\to\infty} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathfrak{x}_{\mathfrak{l}},\mathfrak{t}} = 0.$$

Therefore, $\{\mathfrak{x}_{\mathfrak{n}}\}$ is a Cauchy sequence. Using the completeness hypotheses, $\lim_{\mathfrak{k}\to\infty} \mathfrak{x}_{\mathfrak{k}} = \mathfrak{x} \in \mathcal{X}$.

Assume $\mathfrak{x}$ is not a fixed point of $\mathcal{S}$. Applying inequality (4), we obtain

$$\begin{aligned}
\delta^{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}},\mathcal{S}\mathfrak{x},\mathfrak{t}}} = \delta^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}} &\leq [\eta(\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}) - 1 + \delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x},\mathfrak{t}}} \\
&\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}) \max\left\{\begin{matrix} \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}, \mathcal{N}_{x,\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathcal{S}\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{t}}+\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{2} \end{matrix}\right\}}
\end{aligned}$$

$$= \delta^{\frac{1}{\mathfrak{b}} \max\left\{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x}_{\mathfrak{k}-1},\mathfrak{x},\mathfrak{t}}+\mathcal{N}_{\mathfrak{x},\mathfrak{x}_{\mathfrak{k}+1},\mathfrak{t}}}{2}\right\}}. \tag{19}$$

As $\mathfrak{k} \to \infty$, using Lemma 1 and condition (N1), we get $\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} \leq 0$, i.e., $\mathcal{S}\mathfrak{x} = \mathfrak{x}$.

Let $\mathcal{S}$ has one more fixed point, i.e., $\mathcal{S}\mathfrak{y} = \mathfrak{y}$, $(\mathfrak{x} \neq \mathfrak{y})$. Applying inequality (4), we obtain

$$\begin{aligned}
\delta^{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} &= \delta^{\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}} \leq [\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) - 1 + \delta]^{\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}} \\
&\leq \delta^{\zeta(\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}) \max \left\{ \begin{matrix} \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}, \\ \mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}+\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{2} \end{matrix} \right\}} \\
&= \delta^{\frac{1}{\mathfrak{b}} \max\left\{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}, \frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}+\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{2}\right\}} \\
&= \delta^{\max\left\{\frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \left(\frac{1+\mathfrak{b}}{2\mathfrak{b}}\right)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}\right\}}.
\end{aligned}$$

Therefore,

$$\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} \leq \max\left\{\frac{1}{\mathfrak{b}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}, \left(\frac{1+\mathfrak{b}}{2\mathfrak{b}}\right)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}\right\}, \text{ a contradiction.}$$

Thus, $\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} = 0$, i.e., $\mathfrak{x} = \mathfrak{y}$. Hence, a fixed point of $\mathcal{S}$ is unique. □

Next, we provide examples to demonstrate the authenticity of Theorem 4 besides exhibiting its supremacy over prior related outcomes.

Example 5 *Let $\mathcal{X}$ be the set of Lebesgue measurable functions on $[0,1]$ so that $\int_0^1 |\mathfrak{x}(\mathfrak{t})|\partial\mathfrak{t} < 1$. Let $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \to [0,\infty)$ be*

$$\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \frac{1}{3}\int_0^1 (|\mathfrak{x}-\mathfrak{y}| + |\mathfrak{x}-\mathfrak{z}| + |\mathfrak{y}-\mathfrak{z}|)^2 \partial\mathfrak{t}, \quad \mathfrak{t} > 0 \text{ and } \mathfrak{x},\ \mathfrak{y},\ \mathfrak{z} \in \mathcal{X}.$$

Then, $(\mathcal{X},\mathcal{N})$ is a complete parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b} = 2$ and $\mathfrak{n} = 3$. Define $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ so that $\mathcal{S}\mathfrak{x} = \sin\mathfrak{x}$, $\mathfrak{x} \in \mathcal{X}$. Define $\eta : \mathcal{X}\times\mathcal{X}\times(0,\infty) \to (0,\infty)$ as $\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = e^{\mathfrak{x}+\mathfrak{y}+\mathfrak{t}}$. Let $\delta = 2$ and $\zeta : [0,\infty) \to (0,\frac{1}{2}]$ be given by $\zeta(\mathfrak{s}) = \frac{1}{2}$. Take $\mathfrak{x} = \frac{1}{2} = \mathfrak{y} = \mathfrak{t}$. Applying inequality (4), we get

$$\begin{aligned}
&[e^{\frac{3}{2}} - 1 + 2]^{\mathcal{N}_{\sin\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}} = [e^2 - 1 + 2]^0 = 1 \\
&\leq 2^{\zeta(\mathcal{N}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}})\left(\max\left\{\mathcal{N}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}, \mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}, \mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}, \mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}, \frac{\mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}+\mathcal{N}_{\frac{1}{2},\sin\frac{1}{2},\frac{1}{2}}}{2},\right\}\right)}.
\end{aligned}$$

Since, $\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = e^{\mathfrak{x}+\mathfrak{y}+\mathfrak{t}} > \mathfrak{b}$ implies that $\eta(\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}) = e^{\mathcal{S}\mathfrak{x}+\mathcal{S}\mathfrak{y}+\mathfrak{t}} = e^{\sin\mathfrak{x}+\sin\mathfrak{y}+\mathfrak{t}} > \mathfrak{b}$. Therefore, $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is $\eta-$admissible type $\mathfrak{b}$. Hence, $\mathcal{S}$ verifies the hypotheses of Theorem 4 and has a unique fixed point at $\mathfrak{x} = 0$. Clearly, $\mathcal{N}$ is not a parametric $\mathcal{S}-$metric.

Example 6 *Let $\mathcal{X} = \mathbb{R}^+ \cup \{0\}$. Let a function $\mathcal{N} : \mathcal{X}^3 \times (0, \infty) \to [0, \infty)$ be*

$$\mathcal{N}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{t}) = \begin{cases} 0, & \text{if } \mathfrak{x} = \mathfrak{y} = \mathfrak{z}; \\ \frac{1}{3}, & \text{otherwise}, \end{cases},$$

for each $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z} \in \mathcal{X}$ and $\mathfrak{t} > 0$. Then, $(\mathcal{X}, \mathcal{N})$ is a complete parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b} = 2$ and $\mathfrak{n} = 3$. Define $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{S}\mathfrak{x} = \begin{cases} \frac{1}{3}, & \textit{if } \mathfrak{x} \in [0, \frac{1}{4}); \\ \frac{2}{3}, & \textit{if } \mathfrak{x} \in [\frac{1}{4}, \infty), \end{cases}$$

$\mathfrak{x} \in \mathcal{X}$. Define $\eta : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ as $\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 40\mathfrak{x} + \mathfrak{y} + \mathfrak{t}$, $\delta = 2$ and $\zeta : [0, \infty) \to (0, \frac{1}{2}]$ be given by $\zeta(\mathfrak{s}) = \frac{1}{2}$.

Case I. *If $\mathfrak{x} \in [0, \frac{1}{4})$ and $\mathfrak{x} = \mathfrak{y} = \mathfrak{t}$. Take $\mathfrak{x} = \frac{1}{5} = \mathfrak{y} = \mathfrak{t}$. Applying inequality (4), we get*

$$\begin{aligned} &[40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} - \imath + 2]^{\mathcal{N}_{\frac{1}{3},\frac{1}{3},\frac{1}{3}}} = [40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} + \imath]^0 = 1 \\ &\leq 2^{\zeta\left(\mathrm{N}_{\frac{1}{5},\frac{1}{5},\frac{1}{5}}\right)\left(\max\left\{\mathcal{N}_{\frac{1}{5},\frac{1}{5},\frac{1}{5}}, \mathcal{N}_{\frac{1}{5},\frac{1}{3},\frac{1}{5}}, \mathcal{N}_{\frac{1}{5},\frac{1}{3},\frac{1}{5}}, \mathcal{N}_{\frac{1}{5},\frac{1}{3},\frac{1}{5}}, \frac{\mathcal{N}_{\frac{1}{5},\frac{1}{3},\frac{1}{5}} + \mathcal{N}_{\frac{1}{5},\frac{1}{3},\frac{1}{5}}}{2}\right\}\right)}. \end{aligned}$$

Case II. *If $\mathfrak{x} \in [\frac{1}{4}, \infty)$ and $\mathfrak{x} = \mathfrak{y} = \mathfrak{t}$. Take $\mathfrak{x} = \frac{1}{4} = \mathfrak{y} = \mathfrak{t}$. Applying inequality (4), we get*

$$\begin{aligned} &[40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} - 1 + 2]^{\mathcal{N}_{\frac{2}{3},\frac{2}{3},\frac{2}{3}}} = [40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} + 1]^0 = 1 \\ &\leq 2^{\zeta\left(\mathcal{N}_{\frac{2}{3},\frac{2}{3},\frac{2}{3}}\right)\left(\max\left\{\mathcal{N}_{\frac{1}{4},\frac{1}{4},\frac{1}{4}}, \mathcal{N}_{\frac{1}{4},\frac{2}{3},\frac{1}{4}}, \mathcal{N}_{\frac{1}{4},\frac{2}{3},\frac{1}{4}}, \mathcal{N}_{\frac{1}{4},\frac{2}{3},\frac{1}{4}}, \frac{\mathcal{N}_{\frac{1}{4},\frac{2}{3},\frac{1}{4}} + \mathcal{N}_{\frac{1}{4},\frac{2}{3},\frac{1}{4}}}{2}\right\}\right)}. \end{aligned}$$

Case III. *If $\mathfrak{x} \in [0, \frac{1}{4})$, $\mathfrak{y} \in [\frac{1}{4}, \infty)$ and $\mathfrak{x} \neq \mathfrak{y} \neq \mathfrak{t}$. Take $\mathfrak{x} = \frac{1}{10}$, $\mathfrak{y} = \frac{1}{4}$ and $\mathfrak{t} = \frac{1}{9}$. Applying inequality (4), we get*

$$\begin{aligned} &[40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} - 1 + 2]^{\mathcal{N}_{\frac{1}{3},\frac{2}{3},\frac{1}{9}}} \\ &\leq 2^{\zeta\left(\mathcal{N}_{\frac{1}{10},\frac{1}{4},\frac{1}{9}}\right)\left(\max\left\{\mathcal{N}_{\frac{1}{10},\frac{1}{4},\frac{1}{9}}, \mathcal{N}_{\frac{1}{10},3,\frac{1}{9}}, \mathcal{N}_{\frac{1}{4},\frac{1}{3},\frac{1}{9}}, \mathcal{N}_{\frac{1}{4},\frac{2}{3},\frac{1}{9}}, \frac{\mathcal{N}_{\frac{1}{10},\frac{1}{3},\frac{1}{9}} + \mathcal{N}_{\frac{1}{4},\frac{2}{3},\frac{1}{9}}}{2}\right\}\right)}. \end{aligned}$$

Now, $\eta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 40\mathfrak{x} + \mathfrak{y} + \mathfrak{t} > \mathfrak{b}$ implies that $\eta\big(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, \mathfrak{t}\big) = 40\mathcal{S}\mathfrak{x} + \mathcal{S}\mathfrak{y} + \mathfrak{t} > \mathfrak{b}$. Therefore, $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is an $\eta-$admissible of type $\mathfrak{b}$. Hence, $\mathcal{S}$ verifies the hypotheses of Theorem 4 and has a unique fixed point at $\mathfrak{x} = \frac{2}{3}$.

For $\eta(\mathfrak{x},\mathfrak{y},\mathfrak{t})=1$, Theorem 4 is an extension and an improvement of Ćirić [3] to a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space wherein the continuity of mapping is not essentially required.

Remark 2 (i) *If we take $\mathfrak{n}=3$, $\mathfrak{b}=1$ in Theorems 1, 2, 3 and 4, we get results in a parametric $\mathcal{S}-$metric space. Consequently, our outcomes generalize, improve, unify, and extend the known outcomes, choosing suitably the values of constants $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, the functions ϕ and η (for instance: Bakhtin [1], Banach [2], Ćirić [3], Czerwik [4], Samet et al. [14], Sedghi et al. [15]-[17], Tas and Özgür [20, 21], Ughade et al. [30]). It is interesting to see that parametric $\mathcal{N}_{\mathfrak{b}}-$metric space is essentially greater, improved, and distinct than that of parametric $\mathcal{S}-$metric spaces or metric spaces due to the fact that it is defined on a domain with $\mathfrak{n}$ dimensions .*

(ii) *Clearly, $\mathcal{N}_{\mathfrak{b}}$ is not a parametric $\mathcal{S}-$metric and an underlying function is discontinuous in nature in the above Examples 2.6, 2.11, and 2.15. Consequently, our examples are not applicable to the recent and celebrated results existing in the literature wherein continuity of mapping is an essential condition and the underlying metric is other than the parametric $\mathcal{N}_{\mathfrak{b}}-$metric.*

(iii) *Theorems 1, 2, 3, and 4 along with the supporting Examples 2, 4, and 6, assert that continuity of self mapping is not a significant requirement for the survival of a unique fixed point of a $\mathcal{SA}$, $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$, or $\mathcal{SA}_{\eta,\delta,\zeta}-$ contraction mapping in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space. It is worth mentioning here that the continuity of a self mapping is an indispensable condition for proving a fixed point in most of the theorems existing in the literature (For a detailed discussion on the continuity, refer to Tomar and Karapinar [22]). Consequently, our outcomes reveal the prominence of novel contractions and mark supremacy.*

II. Existence of a unique fixed circle/fixed disc

Following Özgür [12], we introduce notions of the disc and fixed disc in parametric $\mathcal{N}_{\mathfrak{b}}-$ metric spaces and then apply our contractions to obtain a unique fixed circle/fixed disc. It is worth mentioning here that a fixed point of mapping is not always unique and the set of non-unique fixed points may form some geometrical shape like a circle or a disc or an ellipse or an elliptic disc. For more work on geometry, we may refer to [6]-[10], [25]-[26]. In the following, $(\mathcal{X},\mathcal{N})$ denotes the parametric $\mathcal{N}_{\mathfrak{b}}-$metric space.

Definition 9 [21] *A circle centred at $\mathfrak{x}_o$ having a radius $\mathfrak{r}$ in $(\mathcal{X},\mathcal{N})$ is*

$$\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}} = \{\mathfrak{x} \in \mathcal{X} : \mathcal{N}_{\mathfrak{x},\mathfrak{x}_o,\mathfrak{t}} = \mathfrak{r}\}.$$

Definition 10 *We define a disc centred at $\mathfrak{x}_o$ having a radius $\mathfrak{r}$ in $(\mathcal{X},\mathcal{N})$ as*

$$\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}} = \{\mathfrak{x} \in \mathcal{X} : \mathcal{N}_{\mathfrak{x},\mathfrak{x}_o,\mathfrak{t}} \leq \mathfrak{r}\}.$$

Definition 11 *For a self-mapping $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ in $(\mathcal{X},\mathcal{N})$, if $\mathcal{S}\mathfrak{x} = \mathfrak{x}$, $\forall\ \mathfrak{x} \in \mathcal{C}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_o,r}/\mathcal{D}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$, then $\mathcal{C}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}/\mathcal{D}^{\mathcal{X}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ is called a fixed circle/fixed disc of $\mathcal{S}$.*

Example 7 *Let $\mathcal{X} = \mathbb{R}^2$ and for $\mathfrak{n} = 3$, $\mathcal{N} : \mathcal{X}^3 \times (0,\infty) \longrightarrow \mathbb{R}^+$ be*

$$\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \mathfrak{t}^3(|\mathfrak{x}-\mathfrak{y}| + |\mathfrak{y}-\mathfrak{z}| + |\mathfrak{z}-\mathfrak{x}|)^2,$$

where $\mathfrak{x} = (\mathfrak{x}_1,\mathfrak{x}_2)$, $\mathfrak{y} = (\mathfrak{y}_1,\mathfrak{y}_2)$, $\mathfrak{z} = (\mathfrak{z}_1,\mathfrak{z}_2)$ and $|\mathfrak{x}-\mathfrak{y}| = |\mathfrak{x}_1-\mathfrak{y}_1| + |\mathfrak{x}_2-\mathfrak{y}_2|$. Obviously, $(\mathcal{X},\mathcal{N})$ is a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b} = 4$. Then, a circle centred at $\mathfrak{x}_o = (0,0)$ having a radius $\mathfrak{r} = 32$ is

$$\begin{aligned}
\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}} &= \{\mathfrak{x} \in \mathcal{X} : \mathcal{N}(\mathfrak{x},\mathfrak{x},\mathfrak{x}_o,\mathfrak{t}) = 32\} \\
&= \{\mathfrak{x} \in \mathcal{X} : \mathfrak{t}^3(|\mathfrak{x}-\mathfrak{x}| + |\mathfrak{x}-\mathfrak{x}_o| + |\mathfrak{x}_o-\mathfrak{x}|)^2 = 32\} \\
&= \{\mathfrak{x} \in \mathcal{X} : 4\mathfrak{t}^3(|\mathfrak{x}-\mathfrak{x}_o|)^2 = 32\} \\
&= \{4\mathfrak{t}^3(|\mathfrak{x}_1| + |\mathfrak{x}_2|)^2 = 32\} \\
&= \{(|\mathfrak{x}_1| + |\mathfrak{x}_2|)^2 = \frac{8}{\mathfrak{t}^3}\}.
\end{aligned}$$

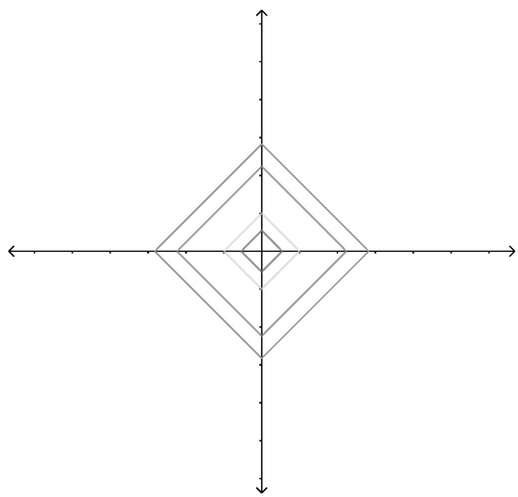

Fig. 1

Fig. 1 *Circles centred at* $(0,0)$ *with radius* 32 *for* $\mathfrak{t} = 1,\ 1.18,\ 2,\ 3$ *are shown by the red, the green, the pink and the orange lines respectively.*

Similarly, a disc $\mathcal{D}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *centred at* $\mathfrak{x}_0 = (0,0)$ *having radius* $\mathfrak{r} = 32$ *is*

$$\mathcal{D}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}} = \{(|\mathfrak{x}_1| + |\mathfrak{x}_2|)^2 \leq \frac{8}{\mathfrak{t}^3}\}$$

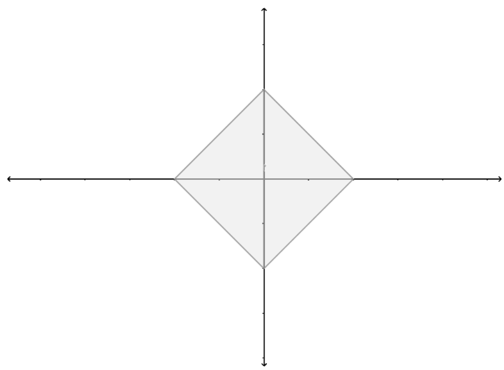

Fig. 2

Fig. 2 *Disc centred at* $(0,0)$ *with radius* 32 *for* $\mathfrak{t} = 2$ *is shown by the pink shaded region.*

Now, we establish a unique fixed circle as an application of the $\mathcal{SA}-$contractive condition.

Theorem 5 *Let* $\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *be a circle in* $(\mathcal{X},\mathcal{N})$. *Define* $\zeta : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ *as:*

$$\zeta(\mathfrak{x}) = \begin{cases} \mathfrak{x} - \mathfrak{r}, & \mathfrak{x} > 0 \\ 0, & \mathfrak{x} = 0 \end{cases}. \tag{20}$$

If a self mapping $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ *verifies*

(i) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathfrak{x}_0,\mathfrak{t}} = \mathfrak{r}$,

(ii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} > \mathfrak{r},\ \mathfrak{x} \neq \mathfrak{y}$,

(iii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} \leq \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} - \zeta(\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}),\ \mathfrak{x},\ \mathfrak{y} \in \mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$,

then $\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *is a fixed circle of* $\mathcal{S}$. *Further if* $\mathcal{SA}-$*contractive condition* (1) *holds for* $\mathfrak{x} \in \mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *and* $\mathfrak{y} \in \mathcal{X} \setminus \mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$, *then* $\mathcal{C}^{\mathcal{N}_\mathfrak{b}}_{\mathfrak{x}_0,\mathfrak{r}}$ *is a unique fixed circle of* $\mathcal{S}$.

Proof. Let $\mathfrak{x} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$ be an arbitrary point. Using (i), $\mathcal{S}\mathfrak{x} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$. Now, we establish that $\mathfrak{x}$ is a fixed point of $\mathcal{S}$. Consider $\mathcal{S}\mathfrak{x} \neq \mathfrak{x}$. Taking $\mathfrak{y} = \mathcal{S}\mathfrak{x}$ in (ii)

$$\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}^2\mathfrak{x},\mathfrak{t}} > \mathfrak{r}. \tag{21}$$

Now, using (iii)

$$\begin{aligned}\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}^2\mathfrak{x},\mathfrak{t}} &\leq \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} - \zeta(\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}) \\ &= \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} - \mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}} + \mathfrak{r} \\ &= \mathfrak{r},\end{aligned} \tag{22}$$

a contradiction. So a self mapping $\mathcal{S}$ fixes the circle $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$, i.e., a set of non-unique fixed points of $\mathcal{S}$ includes a circle.

Let there exist two fixed circles $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}_0}$ and $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$ ($\mathfrak{r}_0 \neq \mathfrak{r}_1$) of $\mathcal{S}$, i.e., $\mathcal{S}$ satisfies the conditions (i) to (iii) for each of the circles $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}_0}$ and $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$. Let $\mathfrak{x} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}$ and $\mathfrak{y} \in \mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$. Using (iv),

$$\begin{aligned}\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} = \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{x,y,\mathfrak{t}}} \\ &\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\ &= \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\ &\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\ &< \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_3\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}} \\ &< (\mathfrak{a}_1 + \mathfrak{b}\mathfrak{a}_3)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}},\end{aligned}$$

a contradiction. Thus, $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}_0}$ is a unique fixed circle of $\mathcal{S}$. □

Example 8 *Let $\mathcal{X} = \mathbb{R}^2$ and for $\mathfrak{n} = 3$, $\mathcal{N} : \mathcal{X}^3 \times (0, \infty) \longrightarrow \mathbb{R}^+$ be*

$$\mathcal{N}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{t}) = \mathfrak{t}^2\left(\left|\sin^{-1}\mathfrak{x} - \sin^{-1}\mathfrak{y}\right|^2 + \left|\sin^{-1}\mathfrak{y} - \sin^{-1}\mathfrak{z}\right|^2 + \left|\sin^{-1}\mathfrak{z} - \sin^{-1}\mathfrak{x}\right|^2\right),$$

where $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2)$, $\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2)$, $\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2)$ and

$$\left|\sin^{-1}\mathfrak{x} - \sin^{-1}\mathfrak{y}\right|^2 = \left|\sin^{-1}\mathfrak{x}_1 - \sin^{-1}\mathfrak{y}_1\right|^2 + \left|\sin^{-1}\mathfrak{x}_2 - \sin^{-1}\mathfrak{y}_2\right|^2.$$

Clearly, $(\mathcal{X},\mathcal{N})$ *is a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space with* $\mathfrak{b}=4$. *Then, a circle centred at* $\mathfrak{x}_o=(0,0)$ *having radius* $\mathfrak{r}=8$ *is*

$$\begin{aligned}
\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}} &= \{\mathfrak{x}\in\mathcal{X}:\mathcal{N}(\mathfrak{x},\mathfrak{x},\mathfrak{x}_o,\mathfrak{t})=8\}\\
&= \Big\{\mathfrak{x}\in\mathcal{X}:\mathfrak{t}^2\Big(\Big|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}\Big|^2+\Big|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}_0\Big|^2\\
&\quad +\Big|\sin^{-1}\mathfrak{x}_0-\sin^{-1}\mathfrak{x}\Big|^2\Big)=8\Big\}\\
&= \Big\{2\mathfrak{t}^2\Big(\Big|\sin^{-1}\mathfrak{x}_1\Big|^2+\Big|\sin^{-1}\mathfrak{x}_2\Big|^2\Big)=8\Big\}\\
&= \Big\{\Big|\sin^{-1}\mathfrak{x}_1\Big|^2+\Big|\sin^{-1}\mathfrak{x}_2\Big|^2=\frac{4}{\mathfrak{t}^2}\Big\}.
\end{aligned}$$

For $\mathfrak{t}=2$,

$$\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}}=\{\mathfrak{x}\in\mathcal{X}:\Big|\sin^{-1}\mathfrak{x}_1\Big|^2+\Big|\sin^{-1}\mathfrak{x}_2\Big|^2=1\}. \tag{23}$$

Define a self mapping $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ *as* $\mathcal{S}\mathfrak{x}=\begin{cases}\mathfrak{x}, & \mathfrak{x}\in\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}\\ (0,0.84), & \textit{otherwise}\end{cases}$. *Then, a self mapping* $\mathcal{S}$ *verifies all the postulates of Theorem 5 and fixes a unique circle* $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$, *i.e., the set of non-unique fixed points of a self mapping* $\mathcal{S}$ *contains a unique fixed circle* $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$.

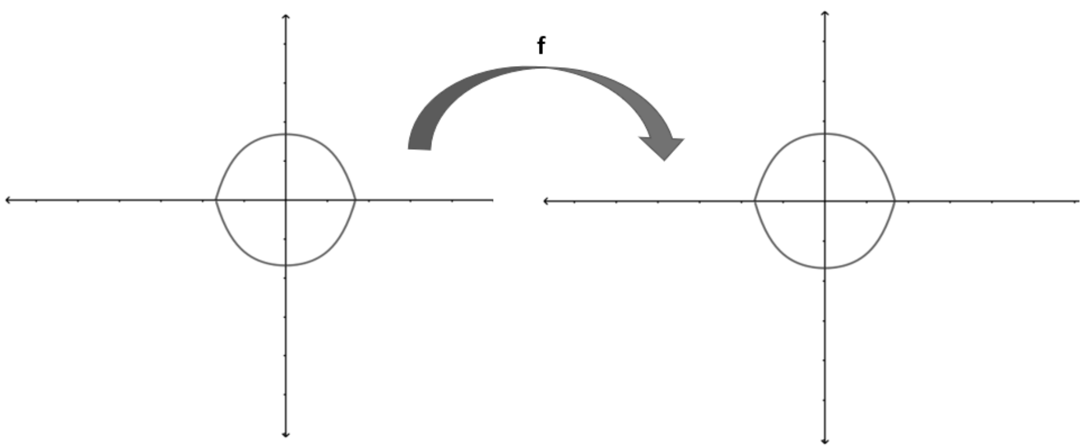

Fig. 3

Fig. 3 *The blue lines demonstrate a circle 23 which is fixed by a function* $\mathcal{S}$.

Theorem 6 *Let $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ be a disc in $(\mathcal{X},\mathcal{N})$. Define $\zeta : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ as in Equation (20). If a self mapping $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ verifies*

(i) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathfrak{x}_o,\mathfrak{t}} \leq \mathfrak{r}$,

(ii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} > \mathfrak{r}, \ \mathfrak{x} \neq \mathfrak{y}$,

(iii) $\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} \leq \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} - \zeta(\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}), \ \mathfrak{x}, \ \mathfrak{y} \in \mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$,

then $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ is a fixed disc of $\mathcal{S}$.

(iv) *Further if, $\mathcal{SA}$−contractive condition (1) holds for $\mathfrak{x} \in \mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ and $\mathfrak{y} \in \mathcal{X} \setminus \mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$, then $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ is a disc of maximum radius $\mathfrak{r}$, i.e., there is no fixed disc $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ of $\mathcal{S}$ having a radius greater than $\mathfrak{r}$.*

Proof. Following the pattern of Theorem 5, we can easily show that $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ is a fixed disc of $\mathcal{S}$.
Let there exist two fixed discs $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ and $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$; $\mathfrak{r}_o < \mathfrak{r}_1$ of $\mathcal{S}$; i.e., $\mathcal{S}$ satisfies the conditions $(\mathfrak{i})$ to $(\mathfrak{iii})$ for each of the discs $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ and $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$. Let $\mathfrak{x} \in \mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ and $\mathfrak{y} \in \mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_1,\mathfrak{r}_1}$ such that $\mathfrak{y} \notin \mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$. Using $(\mathfrak{iv})$,

$$\begin{aligned}
\mathcal{N}_{\mathcal{S}\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}} = \mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{x,y,\mathfrak{t}}} + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\
&\quad + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\
&= \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathfrak{x},\mathfrak{t}}\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\
&\quad + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathfrak{y},\mathfrak{t}}\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}}}{1+\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}}} \\
&< \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}} + \mathfrak{a}_3\mathcal{N}_{\mathfrak{y},\mathfrak{x},\mathfrak{t}} \\
&< (\mathfrak{a}_1 + \mathfrak{b}\mathfrak{a}_3)\mathcal{N}_{\mathfrak{x},\mathfrak{y},\mathfrak{t}},
\end{aligned}$$

a contraction. Hence, $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}_o}$ is a fixed disc of $\mathcal{S}$ having a maximum radius $\mathfrak{r}$. □

Example 9 *If in Example 8, a disc centred at $\mathfrak{x}_o = (0,0)$ having radius $\mathfrak{r} = 8$*

is

$$\begin{aligned}\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}} &= \{\mathfrak{x}\in\mathcal{X}:\mathcal{N}(\mathfrak{x},\mathfrak{x},\mathfrak{x}_o,\mathfrak{t})\leq 8\}\\ &= \Big\{\mathfrak{x}\in\mathcal{X}:\mathfrak{t}^2\Big(\Big|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}\Big|^2+\Big|\sin^{-1}\mathfrak{x}-\sin^{-1}\mathfrak{x}_0\Big|^2\\ &\quad+\Big|\sin^{-1}\mathfrak{x}_0-\sin^{-1}\mathfrak{x}\Big|^2\Big)\leq 8\Big\}\\ &= \Big\{2\mathfrak{t}^2\Big(\Big|\sin^{-1}\mathfrak{x}_1\Big|^2+\Big|\sin^{-1}\mathfrak{x}_2\Big|^2\Big)\leq 8\Big\}\\ &= \Big\{\Big|\sin^{-1}\mathfrak{x}_1\Big|^2+\Big|\sin^{-1}\mathfrak{x}_2\Big|^2\leq\frac{4}{\mathfrak{t}^2}\Big\}.\end{aligned}$$

For $\mathfrak{t}=2$,

$$\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x}\in\mathcal{X}:\Big|\sin^{-1}\mathfrak{x}_1\Big|^2+\Big|\sin^{-1}\mathfrak{x}_2\Big|^2\leq 1\}. \tag{24}$$

Define a self mapping $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ *as* $\mathcal{S}\mathfrak{x}=\begin{cases}\mathfrak{x}, & \mathfrak{x}\in\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}\\ (0,0.84), & otherwise\end{cases}$. *Then, a self mapping* $\mathcal{S}$ *verifies all the postulates of Theorem 6 except (iv) and fixes a disc* $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$, *i.e., the set of non-unique fixed points of a self mapping* $\mathcal{S}$ *contains a fixed disc* $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$.

Remark 3 (i) *Following a similar pattern, we may establish a unique fixed circle (greatest fixed disc) using* $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$ *and* $\mathcal{SA}_{\eta,\delta,\zeta}-$ *contractions.*

(ii) *It is fascinating to see that the shape of a circle or a disc may change on changing the radius, the centre, or the involved metric (refer to figures 1 and 3).*

(iii) *It is not necessary that a circle or a disc in a parametric* $\mathcal{N}_{\mathfrak{b}}-$*metric space is the same as a circle or a disc in a Euclidean space.*

(iv) *Noticeably, the radius of a fixed circle or a fixed disc does not depend on a centre and may not be maximal.*

(v) $\mathcal{S}\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}=\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *or* $\mathcal{S}\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}=\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *does not imply that* $\mathcal{C}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *or* $\mathcal{D}^{\mathcal{N}_{\mathfrak{b}}}_{\mathfrak{x}_o,\mathfrak{r}}$ *is a fixed circle or a fixed disc of* $\mathcal{S}$.

3 An application

Motivated by the fact that the theory of linear systems is the foundation of numerical linear algebra, which performs a significant role in chemistry, physics,

computer science, engineering, and economics, we resolve the system of linear equations in parametric $\mathcal{N}_{\mathfrak{b}}-$metric space using $\mathcal{SA}-$contraction condition (1).
Let $\mathcal{X} = \mathbb{R}^m$and $\mathcal{N} : \mathcal{X}^m \times (0, \infty) \to [0, \infty)$ be

$$\mathcal{N}(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, t) = t^3\left(\Sigma_{i=1}^m |\mathfrak{x}_i - \mathfrak{y}_i| + \Sigma_{i=1}^m |\mathfrak{y}_i - \mathfrak{z}_i| + \Sigma_{i=1}^m |\mathfrak{z}_i - \mathfrak{x}_i|\right)^2,$$

where $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_m)$, $\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_m)$, $\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_m) \in \mathbb{R}^m$. Obviously, $(\mathcal{X}, \mathcal{N})$ is a parametric $\mathcal{N}_{\mathfrak{b}}-$metric space with $\mathfrak{b} = 4$, $n = 3$.

Theorem 7 *The system of linear equations*

$$\begin{aligned} \mathfrak{c}_{11}\mathfrak{x}_1 + \mathfrak{c}_{12}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{1m}\mathfrak{x}_m &= \mathfrak{d}_1 \\ \mathfrak{c}_{21}\mathfrak{x}_1 + \mathfrak{c}_{22}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{2m}\mathfrak{x}_m &= \mathfrak{d}_2 \\ &\cdots \\ \mathfrak{c}_{m1}\mathfrak{x}_1 + \mathfrak{h}_{m2}\mathfrak{x}_2 + \cdots + \mathfrak{c}_{mm}\mathfrak{x}_m &= \mathfrak{d}_m, \end{aligned} \tag{25}$$

where $\mathfrak{c}_{ij}$, $\mathfrak{d}_i \in \mathbb{R}$, $i, j = 1, 2, \ldots, m$, *have a unique solution if* $\max_{j=1}^m \left(\Sigma_{i=1}^m |\mathfrak{c}_{ij}|\right)^2 < \lambda < 1$.

Proof. Define a self mapping $\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{X}$ as $\mathcal{S}\mathfrak{x} = \mathcal{C}\mathfrak{x} + \mathfrak{d}$, $\mathfrak{x}$, $\mathfrak{d} \in \mathbb{R}^m$ and $\mathcal{C} = [\mathfrak{c}_{ij}]_{m\times m}$. First, we show that the self-mapping S satisfies Theorem 1. Then, the unique fixed point of the operator $\mathcal{S}$ is the unique solution of a system of linear equations (25). For $\mathfrak{x}$, $\mathfrak{y} \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{N}(\mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{x}, \mathcal{S}\mathfrak{y}, t) &= t^3\left(\Sigma_{i=1}^m |\mathcal{S}\mathfrak{x}_i - \mathcal{S}\mathfrak{x}_i| + \Sigma_{i=1}^m |\mathcal{S}\mathfrak{x}_i - \mathcal{S}\mathfrak{y}_i| + \Sigma_{i=1}^m |\mathcal{S}\mathfrak{y}_i - \mathcal{S}\mathfrak{x}_i|\right)^2 \\ &= 4t^3\left(\Sigma_{i=1}^m |\mathcal{S}\mathfrak{x}_i - \mathcal{S}\mathfrak{y}_i|\right)^2 \\ &= 4t^3(\Sigma_{i=1}^m |\Sigma_{j=1}^m \mathfrak{c}_{ij}(\mathfrak{x}_j - \mathfrak{y}_j)|)^2 \\ &\leq 4t^3\left(\Sigma_{i=1}^m(\Sigma_{j=1}^m |\mathfrak{c}_{ij}|^2 |\mathfrak{x}_j - \mathfrak{y}_j|^2)\right) \\ &\leq 4t^3\left(\max_{j=1}^m \Sigma_{i=1}^m |\mathfrak{c}_{ij}|^2\right)\left(\Sigma_{j=1}^m |\mathfrak{x}_j - \mathfrak{y}_j|^2\right) \\ &< 4t^3\Sigma_{j=1}^m |\mathfrak{x}_j - \mathfrak{y}_j|^2 \\ &= t^3\mathcal{N}(\mathfrak{x}, \mathfrak{x}, \mathfrak{y}, t) \\ &\leq \mathfrak{a}_1\mathcal{N}_{\mathfrak{x},\mathfrak{y},t} + \mathfrak{a}_2\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} + \mathfrak{a}_3\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} \\ &\quad + \mathfrak{a}_4\frac{\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{x},t}\mathcal{N}_{\mathfrak{x},\mathcal{S}\mathfrak{y},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}} + \mathfrak{a}_5\frac{\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{y},t}\mathcal{N}_{\mathfrak{y},\mathcal{S}\mathfrak{x},t}}{1 + \mathcal{N}_{\mathfrak{x},\mathfrak{y},t}}, \end{aligned}$$

then $\mathcal{S}$ satisfies $\mathcal{SA}$−contraction (1) for $\mathfrak{a}_1 = \mathfrak{a}_3 = \mathfrak{a}_5 = \frac{1}{5}$ and $\mathfrak{a}_2 = \mathfrak{a}_4 = \frac{1}{10}$. Thus, Theorem 1 is verified and consequently, $\mathcal{S}$ has a unique fixed point. Hence, a system of linear equations (25) has a unique solution. □

Remark 4 *Similarly, we may apply* $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$ *and* $\mathcal{SA}_{\eta,\delta,\zeta}$*−contractions to resolve a system of linear equations arising from modeling real-world problems. It is worth mentioning here that to model real-life or scientific problems by means of algebra we transform the known situation into mathematical assertions so that it evidently explains the correlation between the unknowns and the known information.*

4 Conclusion

We have established a unique fixed point, a unique fixed circle, and a greatest fixed disc for the $\mathcal{SA}$, $\eta-\mathcal{SA}$, $\eta-\mathcal{SA}_{\min}$, and $\mathcal{SA}_{\eta,\delta,\zeta}-$ contractions in parametric $\mathcal{N}_{\mathfrak{b}}$−metric spaces, which is fascinating, generalized, and distinct than a usual metric space due to the fact that it is defined on a domain with $\mathfrak{n}$ dimensions. In the sequel, we have explored a new direction in the geometry of non-unique fixed points of discontinuous mapping in parametric $\mathcal{N}_{\mathfrak{b}}$−metric spaces. It is interesting to mention here that a circle or a disc in parametric $\mathcal{N}_{\mathfrak{b}}$−metric space changes its shape by changing the centre, the radius, or the metric under consideration. Our theorems are refined and extended variants of the well-known results. The examples furnished display an interesting characteristic of novel contractions that continuity of mappings is not mandatory for the survival of a fixed point. The paper is concluded by resolving the system of linear equations as an application to demonstrate the significance of our contractions in parametric $\mathcal{N}_{\mathfrak{b}}$−metric space. Essentially, these investigations unlock a distinct era in parametric $\mathcal{N}_{\mathfrak{b}}$−metric spaces.

References

[1] I. A. Bakhtin, The contraction mapping principle in almost metric space, *Functional analysis*, **30** (Russian), Ulyanovsk. Gos. Ped. Inst., Ulyanovsk, (1989), 26–37.

[2] S. Banach, Surles opérations dans les ensembles abstraits etleur applicationaux équations intégrales, *Fundamenta Mathematicae*, **3**, (1922), 133181.

[3] L. B. Ćirić, Generalised contractions and fixed-point theorems, *Publ. Inst. Math.*, **12** (26), (1971), 9–26.

[4] S. Czerwik, Contraction mappings in $\mathfrak{b}$-metric spaces, *Acta Math. Inform., Univ. Ostrav*, **1**, (1993), 5–11.

[5] M. Fréchet, *Sur quelques points du calcul fonctionnel*, palemo, (**30** via Ruggiero), (1906).

[6] M. Joshi, A. Tomar, and S. K. Padaliya, Fixed point to fixed ellipse in metric spaces and discontinuous activation function, *Appl. Math. E-Notes*, **21** (2021), 225–237.

[7] M. Joshi, A. Tomar, On unique and nonunique fixed points in metric spaces and application to Chemical Sciences, *J. Funct. Spaces*, 2021. `https://doi.org/10.1155/2021/5525472`

[8] M. Joshi, A. Tomar, H. A. Nabwey and R. George, On unique and nonunique fixed points and fixed circles in $\mathcal{M}_v^{\mathfrak{b}}$-metric space and application to cantilever beam problem, *J. Function Spaces*, 2021: 15 p., (2021). `https://doi.org/10.1155/2021/6681044`

[9] M. Joshi, A. Tomar, and S. K. Padaliya, On geometric properties of non-unique fixed points in $\mathfrak{b}-$metric spaces, Chapter in the book "Fixed Point Theory and its Applications to Real World Problem" *Nova Science Publishers*, New York, USA., (2021), 33–50. `ISBN:978-1-53619-336-7`

[10] M. Joshi, A. Tomar, and S. K. Padaliya, Fixed point to fixed disc and application in partial metric spaces, Chapter in the book "Fixed point theory and its applications to real world problem" Nova Science Publishers, New York, USA., (2021), 391–406. `ISBN:978-1-53619-336-7`

[11] M. S. Khan, M. Swaleh, and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, **30** (1), (1984), 1–9.

[12] N. Y. Özgür, Fixed-disc results via simulation functions, *Turk. J. Math.*, **43**, (2019), 2795–2805.

[13] B. Said, A. Tomar, A coincidence and common fixed point theorem for subsequentially continuous hybrid pairs of maps satisfying an implicit relation, *Math. Morav.* **21** (2), (2017), 15–25.

[14] B. Samet, C. Vetro, and P. Vetro, Fixed point theorems for $(\alpha - \psi)-$contractive type mappings, *Nonlinear Anal.*, **75**, (2012), 2154–2165.

[15] S. Sedghi, N. Shobe, and T. Dosenovic, Fixed point results in $\mathcal{S}-$metric spaces, *Nonlinear Funct. Anal. Appl.*, **20** (1), (2015), 55–67.

[16] S. Sedghi, A. Gholidahneh, T. Dosenovic, J. Esfahani, and S. Radenovic, Common fixed point of four maps in $\mathcal{S}_{\mathfrak{b}}-$metric spaces, *J. Linear Topol. Algebra*, **5**(2016), 93–104.

[17] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in $\mathcal{S}-$metric spaces, *Mat. Vesnik*, **64**, (2012), 258–266.

[18] W. Sintunavarat, Nonlinear integral equations with new admissibilty types in $\mathfrak{b}-$metric spaces, *J. Fixed Point Theory Appl.*, doi10.1007/s11784-015-0276-6, (2017).

[19] N. Souayah, N. Mlaiki, A fixed point theorem in $\mathcal{S}_{\mathfrak{b}}-$metric space, *J. Math., Computer Sci.*, **16**, (2016), 131–139.

[20] Tas, N. and Özgür, N.Y., On parametric $\mathcal{S}-$metric spaces and fixed point type theorems for expansive mappings, *J. Math.*, (2016).

[21] N. Taş, and N. Y. Özgür, Some fixed point results on parametric $\mathcal{N}_{\mathfrak{b}}-$metric spaces, *Commu. Korean Math. Soc.*, 943–960, **33** (3), (2018), 943–960.

[22] A. Tomar, and E. Karapinar, On variants of continuity and existence of fixed point via Meir-Keeler contractions in MC-spaces, *J. Adv. Math. Stud.*, **9** (2), (2016), 348–359.

[23] A. Tomar, R. Sharma, S. Beloul, and A. H. Ansari, $\mathcal{C}-$class functions in generalized metric spaces and applications, *J. Anal.*, (2019), 1–18.

[24] A. Tomar, M. Joshi, S. K. Padaliya, B. Joshi, and A. Diwedi, Fixed point under set-valued relation-theoretic nonlinear contractions and application, *Filomat*, **33** (14), (2019), 4655–4664.

[25] A. Tomar, M. Joshi, M. and S. K. Padaliya, Fixed point to fixed circle and activation function in partial metric space, *J. Appl. Anal.*, **28** (1), (2021), 2021–2057. https://doi.org/10.1515/jaa

[26] A. Tomar and M. Joshi, Near fixed point, near fixed interval circle and near fixed interval disc in metric interval space, Chapter in the book “Fixed Point Theory and its Applications to Real World Problem” *Nova Science Publishers*, New York, USA., 131–150, 2021. ISBN: 978-1-53619-336-7

[27] A. Tomar, S. Beloul, R. Sharma, S. Upadhyay, Common fixed point theorems via generalized condition (B) in quasi-partial metric space and applications, *Demonstr. Math.*, **50** (1), (2017), 278–298.

[28] A. Tomar, Giniswamy, C. Jeyanthi, P. G. Maheshwari, On coincidence and common fixed point of six maps satisfying $\mathsf{F}-$contractions, *TWMS J. App. Eng. Math.*, **6** (2), (2016), 224–231.

[29] A. Tomar, and R. Sharma, Almost alpha-Hardy-Rogers-F-contractions and applications, *Armen. J. Math.*, **11** (11), (2019), 1–19.

[30] M. Ughade, D. Turkoglu, S. K. Singh, R. D. Daheriya, Some fixed point theorems in $\mathcal{A}_{\mathfrak{b}}-$metric space, *British J. Math. & Computer Science* **19** (6), (2016), 1–24.

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Co-unit graphs associated to ring of integers modulo n

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Abstract. Let R be a finite commutative ring. We define a co-unit graph, associated to a ring R, denoted by $G_{nu}(R)$ with vertex set $V(G_{nu}(R))$ $= U(R)$, where $U(R)$ is the set of units of R, and two distinct vertices x, y of $U(R)$ being adjacent if and only if $x + y \notin U(R)$. In this paper, we investigate some basic properties of $G_{nu}(R)$, where R is the ring of integers modulo n, for different values of n. We find the domination number, clique number and the girth of $G_{nu}(R)$.

1 Introduction

A graph $G = (V, E)$ consists of the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(G)$. Further, $|V(G)| = n$ is the *order* and $|E(G)| = m$ is the *size* of G. The *degree* of a vertex v, denoted by $d_G(v)$ (we simply write d_v) is the number of edges incident on the vertex v.

A path of length n is denoted by P_n and a cycle of length n is denoted by C_n. A graph G is connected if their is at least one path between every pair of distinct vertices, otherwise disconnected. As usual, K_n denotes a complete

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graph with n vertices and $K_{a,b}$ denotes a complete bipartite graph with $a+b$ vertices. Also a graph G is said to be $k-regular$ if degree of every vertex of G is k. The $girth$ of a graph G, denoted by $gr(G)$, is the length of the shortest cycle contained G. In G, an independent set is a subset S of the vertex set $V(G)$ if no two vertices of S are adjacent. The $independence\ number$ of G, denoted by $\alpha(G)$, is defined as $\alpha(G) = max\{|S| : S\ is\ an\ independent\ set\ of\ G\}$. Two graphs G_1 and G_2 are said to be isomorphic if there exists a bijection between vertices and edges so that the incidence relationship is preserved and is written as $G_1 \cong G_2$. A subset D of $V(G)$ is called a $dominating\ set$ of G if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. A dominating set of minimum cardinality is called a $\gamma - set$ of G. The $domination\ number$ of G, denoted by $\gamma(G)$, is the cardinality of a $\gamma - set$ of G.

Let R be a finite commutative ring and let $U(R)$ be the set of units of R. Let $R \cong R_1 \times R_2 \times ... \times R_n$ be the direct product of the finite rings R_i. If a_i is a unit in R_i, where $1 \leq i \leq n$, then $(a_1, a_2, a_3, \ldots, a_n)$ is the unit element of $R_1 \times R_2 \times ... \times R_n$.

Let n be a positive integer and let $\mathbb{Z}_n$ be the ring of integers modulo n. Grimaldi [4] defined the unit graph $G(\mathbb{Z}_n)$ whose vertex set is the set of elements of $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if and only if $x+y$ is a unit of $\mathbb{Z}_n$. Ashrafi et. al [2] extended the concept of $G(\mathbb{Z}_n)$ to $G(R)$, where R is any arbitrary associative ring with nonzero identity. More literature on this can be seen in [1, 5, 6, 13, 15, 14].

We define a *co-unit graph* associated to a ring R, denoted by $G_{nu}(R)$, with vertex set as the set $U(R)$ and two vertices $x, y \in U(R)$ are adjacent if and only if $x+y \notin U(R)$. We observe that $G_{nu}(R)$ is an empty graph when R is the ring of real numbers or the ring of rational numbers. More generally, if $\mathbb{R}$ is a field, then $G_{nu}(\mathbb{R})$ is an empty graph. Also, for the ring of integers $\mathbb{Z}$, $G_{nu}(\mathbb{Z}) \cong K_2$, since $U(\mathbb{Z}) = \{-1, 1\}$ and $-1+1 = 0 \notin U(\mathbb{Z})$ which implies that the vertex corresponding to the unit -1 is adjacent to the vertex corresponding to the unit 1 and hence becomes K_2.

In Section 2, we characterize the graphs $G_{nu}(\mathbb{Z}_n)$, for different values of n. Also, we find the domination number, clique number and the girth of $G_{nu}(\mathbb{Z}_n)$.

2 On graphs $G_{nu}(\mathbb{Z}_n)$ associated to the ring $\mathbb{Z}_n$

Definition 1 *The Euler's phi function $\phi(n)$, where n is positive integer, is defined as the number of non-negative integers less than n that are relatively prime to n. If $n \geq 2$ and p is prime, then $\phi(p^n) = p^n - p^{n-1}$.*

We begin with the following observation.

Theorem 1 *The graph* $G_{nu}(\mathbb{Z}_p) \cong (\frac{p-1}{2})K_2$*, where* $p \geq 3$ *is a prime number.*

Proof. Since all nonzero elements of the ring $\mathbb{Z}_p$ are units, so the vertex set of $G_{nu}(\mathbb{Z}_p)$ is V= $\{1, 2, 3, \ldots, , p-1\}$. Partition the vertex set V into two disjoint subsets V_1 and V_2, where $V_1 = \{1, 2, \ldots, \frac{p-1}{2}\}$ and $V_2 = \{\frac{p+1}{2}, \ldots, p-1\}$. Let x and y be any two elements in V_1. Then, clearly $x+y \leq p-2$, implying that the sum of any two elements in V_1 is a unit. Therefore, no two vertices in V_1 are adjacent. Now, let x and y be any elements in V_2. Clearly, $p+2 \leq x+y \leq 2p-3$, so that $x+y$ is a unit, which implies that there is no edge in V_2. Also, for every element $k \in V_1$, $1 \leq k \leq \frac{p-1}{2}$, there is exactly one element $p-k$ in V_2 such that $k+p-k = p$ is a non unit. Hence the graph $G_{nu}(\mathbb{Z}_p)$ is bipartite and contains $\frac{p-1}{2}$ copies of K_2, that is $G_{nu}(\mathbb{Z}_p) \cong (\frac{p-1}{2})K_2$. $\square$

Remark. From Theorem 1, we observe that the independence number of $G_{nu}(\mathbb{Z}_p)$ is equal to $\frac{\phi(p)}{2}$.

Theorem 2 *For prime* $p \geq 5$ *and* $n \geq 2$*, the graph* $G_{nu}(\mathbb{Z}_{p^n})$ *is* p^{n-1}*- regular graph.*

Proof. By Euler's ϕ- function , $\phi(p^n) = p^n - p^{n-1}$. So the order of the graph $G_{nu}(\mathbb{Z}_{p^n})$ is $\phi(p^n)$. Let $V = \{1, 2, \ldots, p-1, p+1, \ldots, 2p-1, 2p+1, \ldots, p^n-1\}$ be the vertex set of $G_{nu}(\mathbb{Z}_{p^n})$. It is clear that V has no vertex of the type np^α. As $|V| = \phi(p^n)$, so the number of non units in $\mathbb{Z}_{p^n}$ is $p^n - \phi(p^n) = p^n - p^n + p^{n-1} = p^{n-1}$. Let $D = \{np^\alpha : n, \alpha \in \mathbb{N}\}$ be the set of non units in $G_{nu}(\mathbb{Z}_{p^n})$, so that $|D| = p^{n-1}$. Consider the set $S = \{np^\alpha - k : k \in V\}$. Clearly, each vertex of V is adjacent to every vertex of S, since for every fixed $k \in V$ and $np^\alpha - k \in S$, we have $k + np^\alpha - k = np^\alpha \notin U(\mathbb{Z}_{p^n})$. Define a mapping $f : D \to S$ by $f(np^\alpha) = np^\alpha - k$. Clearly, f is bijective, so it follows that $|S| = p^{n-1}$. As each vertex of $G_{nu}(\mathbb{Z}_{p^n})$ is adjacent to every vertex of S, so degree of every vertex of $v \in V = |S|$. Thus, $G_{nu}(\mathbb{Z}_{p^n})$ is $p^{n-1} - \mathrm{regular}$. $\square$

Example 1 *Let* $p = 5$ *and* $n = 2$*. The graph* $G_{nu}(\mathbb{Z}_{5^2})$ *is* $5^{2-1} = 5$*—regular, as shown in Figure* 1*.*

Theorem 3 *The graph* $G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)$ *is* $(\phi(p) + \phi(q) - 1)-$ *regular, where both* p *and* q *are distinct odd primes with* $p < q$*. Further, the domination number* $\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = \phi(p)$ *and* $gr(\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = 3$*.*

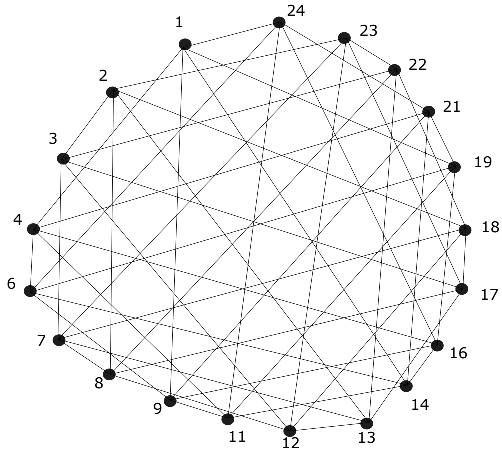

Figure 1: $G_{nu}(\mathbb{Z}_{5^2})$

Proof. Since p and q are odd primes with $p < q$, the number of units in the rings $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are $\phi(p)$ and $\phi(q)$, respectively. So the order of the graph $G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)$ is $\phi(p)\phi(q)$. Now, let

$$\begin{aligned} V =\{&(u_1, v_1), (u_1, v_2), \dots (u_1, v_{q-1}), (u_2, v_1), \dots (u_2, v_{q-1}), \dots \\ &(u_{p-1}, v_1), (u_{p-1}, v_2), \dots, (u_{p-1}, v_{q-1})\} \end{aligned}$$

be the vertex set of $G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)$, where $\{u_i : 1 \leq i \leq (p-1)\}$ and $\{v_j : 1 \leq j \leq (q-1)\}$ are the set of units in $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively. Partition vertex set V into $\phi(p)$ disjoint subsets, each having cardinality $\phi(q)$, which are given by

$$B_1 = \{(u_1, v_1), (u_1, v_2), (u_1, v_3), \dots, (u_1, v_{q-1})\}$$

$$B_2 = \{(u_2, v_1), (u_2, v_2), (u_2, v_3), \dots, (u_2, v_{q-1})\}$$

$$B_3 = \{(u_3, v_1), (u_3, v_2), (u_3, v_3), \dots, (u_3, v_{q-1})\}$$

$$\vdots$$

$$B_{\phi(p)} = \{(u_{p-1}, v_1), (u_{p-1}, v_2), (u_{p-3}, v_3), \dots, (u_{p-1}, v_{q-1})\}.$$

Choose some arbitrary subset, say B_i, $1 \leq i \leq \phi(p)$. We show that each vertex in B_i has degree $\phi(p) + \phi(q) - 1$. Let $(i, x) \in B_i$ be an arbitrary vertex, where $1 \leq x \leq \phi(q)$. Obviously, (i, x) is adjacent to every vertex in $B_{\phi(p)+1-i}$, and (i, x) is adjacent to exactly one vertex in the remaining subsets. So the degree of (i, x) is $\phi(p) + \phi(q) - 1$, proving first part of the result.

The vertices of V are $\{(u_1, v_1), (u_1, v_2), (u_1, v_3), \ldots, (u_1, v_{q-1}), (u_2, v_1), (u_2, v_2), (u_2, v_3), \ldots, (u_2, v_{q-1}), \ldots (u_{p-1}, v_1), (u_{p-1}, v_2), (u_{p-3}, v_3), \ldots, (u_{p-1}, v_{q-1})$. The vertices of the type (u_1, v_i), where $1 \leq i \leq q-1$, are adjacent to the vertex (u_{p-1}, v_1). Similarly, the vertices of the type (u_2, v_i), where $1 \leq i \leq q-1$, are adjacent to the vertex (u_{p-2}, v_1). In this way, the vertices (u_{p-1}, v_i), where $1 \leq i \leq q-1$, are adjacent to the vertex of the type (u_1, v_1). Now, form the subset of the vertex set V, say D, where D contains vertices of the type $\{(u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_{p-1}, v_1)\}$. We have $|D| = p - 1 = \phi(p)$. Also, each vertex of $V \setminus D$ is adjacent to at least one vertex of D. We show that D is minimal with the above conditions. From D, if we remove any number of the vertices of the type (u_x, v_1), where $1 \leq x \leq p-1$, then there exist vertices of the type (u_{p-x}, v_i) in $V \setminus D$, which are not adjacent to any vertex in D. It follows that D is a minimal dominating set and $\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = |D| = p - 1 = \phi(p)$. The subset $\{(u_{p-1}, v_1), (u_1, v_{q-1}), (u_{p-1}, v_{q-1})\}$ of the vertex set V, forms an induced subgraph which is complete. Hence it follows that $gr(\gamma(G_{nu}(\mathbb{Z}_p \times \mathbb{Z}_q)) = 3$. $\square$

Theorem 4 *If* $n = 2m$*, then the graph* $G_{nu}(\mathbb{Z}_{2m})$ *is complete.*

Proof. We know that the number of units of the ring $\mathbb{Z}_{2m}$ is $\phi(2m)$. Also, if $a \in \mathbb{Z}_{2m}$ is a unit then $(a, 2m) = 1$. So the vertex set of $G_{nu}(\mathbb{Z}_{2m})$ contains only odd integers, while as all even integers are nonunits. Let $V = \{v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}, \ldots, v_{\alpha_{\phi(2m)}}\}$ be the vertex set of $G_{nu}(\mathbb{Z}_{2m})$, where the set $\{v_{\alpha_i} | i = 1, 2, 3, \ldots, \phi(2m)\}$ is the set of units of $\mathbb{Z}_{2m}$. Clearly, every v_{α_i} in V is an odd integer. As sum of two odd integers is even, therefore, every two vertices in V are adjacent. Thus, $G_{nu}(\mathbb{Z}_{2m})$ forms a complete graph. $\square$

Theorem 5 *Let* $R \cong \mathbb{Z}_3 \times \mathbb{Z}_p$*, where* p *is odd prime. Then the graph* $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ *is a connected* $p-$*regular graph and* $\gamma(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = \frac{\phi(p)}{2}$, $gr(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = 3$, $cl(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = 4$.

Proof. As the units of the ring $\mathbb{Z}_3$ are $\{1, 2\}$ and units of the ring $\mathbb{Z}_p$ are $\{1, 2, 3, 4, \ldots, p-1\}$, so the vertex set for the graph $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ is

$$V = \{(1,1), (1,2), \ldots, (1, p-1), (2,1), (2,2), \ldots, (2, p-1)\}.$$

Partition vertex set V into two disjoint sets V' and V'' such that $V' = \{(1, i) | 1 \leq i \leq p-1\}$ and $V'' = \{(2, j) | 1 \leq j \leq p-1\}$. Then $|V'| = p - 1$ and $|V''| = p - 1$. By definition, each vertex of V of the type $(2, u)$ is adjacent to every vertex of V of the type $(1, v)$, where u and v are units in the ring

$\mathbb{Z}_p$, since $(1,v)+(2,u)=(3,u')\notin U(\mathbb{Z}_3\times\mathbb{Z}_p)$, where $u'=u+v\in\mathbb{Z}_p$. It follows that $K_{p-1,p-1}$ is an induced subgraph of $G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p)$. Also, in V', corresponding to each vertex of the type $(1,u_r)$, there exists a unique vertex $(1,u_{p-r})$ in V' such that $(1,r)\sim(1,p-r)$. The same argument holds for V''. Therefore, the degree of each vertex in both the sets V' and V'' is equal to $p-1+1=p$. Hence the graph $G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p)$ is p-regular.

Every vertex of V' is adjacent to every vertex of V'', since $(1,u)+(2,u)=(3,u')\notin U(\mathbb{Z}_3\times\mathbb{Z}_p)$. So there exists a path between every pair of vertices $\{(1,u),(2,v)\}$, where $u,v\in\mathbb{Z}_p$. From the above discussion, for partite sets V' and V'', vertices of the type $(1,u_{p-r})$ are adjacent to the vertices of the type $(1,u_r)$ in V', and vertices of the type $(2,u_{p-r})$ are adjacent to vertices of the type $(2,u_r)$ in V''. So, it follows that there is a path between every pair of vertices in V, see Figure 2. Thus, $G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p)$ is connected.

As each vertex of the type $(2,u_i)$ is adjacent to the vertex $(1,1)$, and the vertex $(1,u_{p-1})$ is adjacent to $(1,1)$, so the remaining vertices are of the type $\{(1,u_j)|2\le j\le p-2\}$. Now, corresponding to each vertex of the type $(1,u_j)$ in V, where $2\le j\le p-2$, there exist a vertex of the type $(1,u_{p-j})$, where $2\le j\le p-2$, such that $(1,u_j)+(1,u_{p-j})\notin U(\mathbb{Z}_3\times\mathbb{Z}_p)$. Let D be a subset of the vertex set V defined as $D=\{(1,u_j):1\le j\le\frac{p-1}{2}\}$. Now, each vertex of $V\setminus D$ is adjacent to at least one vertex of D, since each vertex of the type $(2,u_i)$ is adjacent to every vertex of the type $(1,u_i)$. Also, half of the vertices of the type $(1,u_i)$, where $1\le i\le\frac{p-1}{2}$, are adjacent to other half of the vertices of the type $(1,u_k)$, where $\frac{p+1}{2}\le k\le p-1$. From D, if we remove vertices of the type $\{(1,u_r)\}$, where $1\le r\le\frac{p-1}{2}$, then those vertices go to set $V\setminus D$. Therefore, there exist vertices in $V\setminus D$ of the type $(1,u_{p-r})$, $r=1,2,3,\dots\frac{p-1}{2}$, which are not adjacent to any vertex in D. So $D\setminus\{(1,u_r)\}$ does not form a dominating set for $G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p)$. Therefore, it follows that D is a dominating set for $G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p)$ and $|D|=\frac{p-1}{2}=\frac{\phi(p)}{2}$. So $\gamma(G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p))=\frac{\phi(p)}{2}$.

Let $S=\{(1,u_1),(1,u_{p-1}),(2,u_i)\}$ be the subset of V, where u_i, $1\le i\le p-1$, are units in the ring $\mathbb{Z}_p$. The induced subgraph $<S>$ is a cycle of length 3, so it follows that $gr(G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p))=3$. Again, let $S'=\{(1,1),(1,p-1),(2,1),(2,p-1)\}$ be the subset of the vertex set V. Then the induced subgraph $<S'>$ is a complete subgraph. This induced subgraph is maximal, in $G_{nu}(\mathbb{Z}_3\times\mathbb{Z}_p)$. To see this, if we add any vertex either of the type $(1,u_i)$, where $u_i\ne 1,p-1$, or of the type $(2,u_j)$, where $u_j\ne 1,p-1$, to S', then the following three possibilities arise. (i) If we add vertex $(1,u_i)$, where $2\le i\le p-2$, to S', then this vertex is not adjacent to the vertices $(1,1),(1,p-1)$. So the induced subgraph $<S'+(1,u_i)>$ is not complete. (ii) If we add vertex $(2,u_i)$

to S', where $2 \leq i \leq p-2$, then this vertex is not adjacent to the vertices $(2,1),(2,p-1)$. So the induced subgraph $< S' + (2, u_i) >$ is not complete. (iii) If we add both types of vertices as in (i) and (ii), then in this case the induced subgraph $< S' + (1,u_i) + (2,u_i) >$ is not complete. Thus the subgraph induced by $< S' >$ forms a clique in $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ and therefore $cl(G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)) = |S'| = 4$. The graph $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$ can be seen in Figure 2. □

Theorem 6 *If* $R \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_m$*, then the graph* $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$ *is complete.*

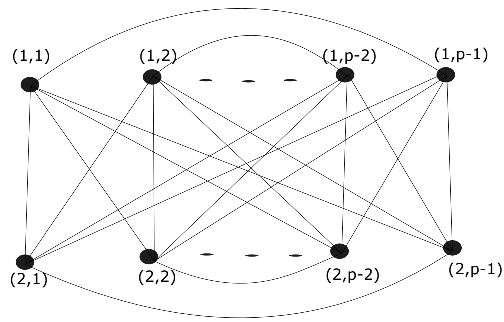

Figure 2: $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_p)$

Proof. It is easy to see that the set of units for the ring $\mathbb{Z}_{2^n}$ are $\{2k+1; k \in \mathbb{Z}\}$. Let V be the vertex set for the graph $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$. Then, $V = \{(x_i, y_j) : 1 \leq i \leq \phi(2^n), 1 \leq j \leq \phi(m)\}$, where x_i and y_i are units in the rings $\mathbb{Z}_{2^n}$ and $\mathbb{Z}_m$, respectively. Since each x_i, $1 \leq i \leq \phi(2^n)$, is an odd integer, therefore, $(x_i, y_j) + (x_r, y_s) \notin U(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$, as $x_i + x_r$ is always even. Thus, each vertex of V is adjacent to every vertex of V. Thus the graph $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_m)$ is a complete graph. □

Theorem 6 can be generalized as follows, the proof of which is similar to that of Theorem 6.

Theorem 7 *If* $R \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_{\alpha_1} \times \mathbb{Z}_{\alpha_2} \times \cdots \times \mathbb{Z}_{\alpha_m}$*, then the graph* $G_{nu}(\mathbb{Z}_{2^n} \times \mathbb{Z}_{\alpha_1} \times \mathbb{Z}_{\alpha_2} \times \cdots \times \mathbb{Z}_{\alpha_m})$ *is complete.*

Theorem 8 *Let* $n \in \mathbb{N}$ *and* p *be a prime. Then* $G_{nu}(\mathbb{Z}_{p^n})$ *is complete if and only if* $p = 2$ *and* $G_{nu}(\mathbb{Z}_{p^n})$ *is complete bipartite if and only if* $p = 3$*. Moreover, if* $p > 3$*, then* $G_{nu}(\mathbb{Z}_{p^n})$ *has* $\dfrac{p-1}{2}$ *components each being a complete bipartite graph isomorphic to* $K_{r,r}$*, where* $r = p^n - 1$*.*

Proof. Partition vertex set $V(G_{nu}((\mathbb{Z}_{p^n})))$ into subsets V_1, V_2, $\ldots$, V_{p-1}, where $V_i = \{pk - i : k \in \mathbb{N}$ and $pk - i < p^n\}$, $1 \leq i \leq p - 1$. Then, as $|V(G_{nu}((\mathbb{Z}_{p^n})))| = (p-1)p^{n-1}$, we have $|V_i| = p^{n-1}$, for each i. Moreover, each V_i is an independent set for all $p \geq 3$.
If $p = 2$, then $V(G_{nu}((\mathbb{Z}_{2^n}))) = V_1 = \{1, 3, 5, \ldots, 2^n - 1\}$ and so $G_{nu}((\mathbb{Z}_{2^n}))$ is complete. For $p = 3$, $V(G_{nu}((\mathbb{Z}_{3^n}))) = V_1 \cup V_2$, where $V_1 = \{3k - 1 : k \in \mathbb{N}$ and $3k - 1 < 3^n\}$ and $V_2 = \{3k - 2 : k \in \mathbb{N}$ and $3k - 2 < 3^n\}$. Then, for any $x \in V_1$ and $y \in V_2$, we have $x + y \notin U(\mathbb{Z}_{3^n})$. Thus, $G_{nu}(\mathbb{Z}_{3^n})$ is isomorphic to $K_{3^{n-1},3^{n-1}}$. Now, for $p > 3$, let $x \in V_t$ and $y \in V_s$, $1 \leq t, s \leq p - 1$. Then x and y are adjacent in $G_{nu}(\mathbb{Z}_{p^n})$ if and only if $t + s = p$. Thus, we partition the set $\{V_1, V_2, \ldots, V_{p-1}\}$ into the $(p - 1)/2$ sets, namely, $V_{j,p-j} = \{V_j, V_{p-j}\}$, $1 \leq j \leq (p - 1)/2$. Then each $V_{j,p-j}$ induces a complete bipartite graph $K_{r,r}$, where $r = |V_i| = p^{n-1}$. □

Conclusion For a finite commutative ring R we associated a co-unit graph, denoted by $G_{nu}(R)$, with vertex set $V(G_{nu}(R)) = U(R)$, where $U(R)$ is the set of units of R, and two distinct vertices x, y of $U(R)$ being adjacent if and only if $x + y \notin U(R)$. We investigated some basic properties of $G_{nu}(R)$, where R is the ring of integers modulo n, for different values of n. We obtained the domination number, the clique number and the girth of $G_{nu}(R)$. For the future work, we need to investigate several other graph invariants of $G_{nu}(R)$, for any ring R. Also, there is scope to study the line graph of the co-unit graph, in analogy to the line graph of the unit graph, see [10]. Further directions to study in co-unit graphs can be metric dimension and spectra, for instance like in [3, 9, 10, 11, 12].

References

[1] M. Afkhami and F. Khosh-Ahang, Unit graphs of rings of polynomials and power series, *Arab. J. Math.* **2,3** (2013), 233–246.

[2] N. Ashrafi, H. R. Maimani, M. R. Pournaki and S. Yassemi, Unit graphs associated to rings, *Comm. Algebra* **38,8** (2010), 2851–2871.

[3] M. Imran Bhat and S. Pirzada, On strong metric dimension of zero divisor graphs of rings, *The Korean J. Math.* **27,3** (2019), 563–580.

[4] R. P. Grimaldi, Graphs from rings, Proceedings of the 20th Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989). Congr. Numer. **71** (1990), 95–103.

[5] S. Kiani, H. R. Maimani, M. R. Pournaki and S. Yassemi, Classification of rings with unit graphs having domination number less than four, *Rend. Sem. Mat. Univ. Padova* **133** (2015), 173–195.

[6] H. R. Maimani, M. R. Pournaki and S. Yassemi, Necessary and sufficient conditions for unit graphs to be Hamiltonian, *Pacific J. Math.* **249,2** (2011), 419–429.

[7] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Orient Blakswan, Hyderabad (2012).

[8] S. Pirzada and Aaqib Altaf, Line graphs of unit graphs associated with the direct product of rings, *The Korean J. Math.* **30,1**, (2022) 65-72.

[9] S. Pirzada, M. Aijaz and Shane Redmond, Upper dimension and bases of zero divisor graphs of commutative rings, *AKCE International J. Graphs Comb.* **17,1**, (2020) 168–173.

[10] S. Pirzada and M. Aijaz, Metric and upper dimension of zero divisor graphs associated to commutative rings, *Acta Univ. Sapientiae, Informatica* **12,1** (2020), 84–101.

[11] S. Pirzada and M. Imran Bhat, Computing metric dimension of compressed zero divisor graphs associated to rings, *Acta Univ. Sap. Mathematica* **10,2** (2018), 298–318.

[12] S. Pirzada and Rameez Raja, On the metric dimension of a zero divisor graph, *Communications Algebra* **45,4** (2017), 1399–1408.

[13] H. Su, G. Tang and Y. Zhou, Rings whose unit graphs are plannar, *Publ. Math. Debrecen* **86,3-4** (2015), 363–376.

[14] H. Su and Y. Wei, The diameter of unit graphs of rings, *Taiwanese J. Math.* **23,1** (2019), 1–10.

[15] H. Su and Y. Zhou, On the girth of the unit graph of a ring, *J. Algebra Appl.* **13,2** (2014), 1350082, 12 pp.

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Some results on Caristi type coupled fixed point theorems

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Abstract. In this work we define the concepts of the coupled orbit and coupled orbitally completeness. After then, using the method of Bollenbacher and Hicks [8], we prove some Caristi type coupled fixed point theorems in coupled orbitally complete metric spaces for a function $P : E \times E \to E$. We also give two examples that support our results.

1 Introduction and preliminaries

In the litareture concerning the fixed point theory, one of the most interesting and useful results is the Caristi's fixed point theorem [9], which is equivalent to Ekeland's variational principle [12] and is also a generalization of the famous Banach contraction principle.

In 1976, Caristi proved in [9] that "if S is a self mapping of a complete metric space (E, ρ) such that there is a lower semi-continuous function ψ from E into $[0, \infty)$ satisfying

$$\rho(u, Su) \leq \psi(u) - \psi(Su)$$

for all $u \in E$, then S has a fixed point".

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In this theorem, saying that "ψ is lower semi-continuous at u if for any sequence $\{u_n\} \subset E$, we have $\lim u_n = u$ implies $\psi(u) \leq \liminf \psi(u_n)$".

Several authors have obtained various extensions and generalizations of Caristi's theorem by considering Caristi type mappings on many different spaces. For example [1, 2, 3, 4, 8, 10, 14, 15, 16, 17, 18, 19, 20, 28, 29], and others.

In this paper, by using the method in [8], we give some Caristi type coupled fixed point theorems for a function P from a product space $E \times E$ to E.

The idea of the coupled fixed point was given first by Opoitsev [22, 23] and Opoitsev and Khurodze [24] and then by Guo and Lakhsmikantham in [13]. The first coupled fixed point theorems under the contractive conditions were studied by Bhaskar and Lakhsmikantham, see [7]. Since then various authors have obtained several important, useful and interesting results for the coupled fixed points under different condition [5, 6, 11, 21, 25, 26, 27].

We now give some basic definitions and notions.

Definition 1 ([7]) *Let E be a nonempty set and $P : E \times E \to E$ be a mapping. An element $(u, v) \in E \times E$ is said to be a coupled fixed point of mapping P if $u = P(u, v)$ and $v = P(v, u)$.*

Definition 2 *Let E be a nonempty set and $P : E \times E \to E$ be a mapping. Let u_0 and v_0 are arbitrary two points in E. Consider the sequences $\{u_n\}$ and $\{v_n\}$ by*

$$u_n = P(u_{n-1}, v_{n-1}),\ v_n = P(v_{n-1}, u_{n-1}) \tag{1}$$

for $n = 1, 2, 3, \ldots$.

Then the sets

$$O_P(u_0, \infty) = \{u_0, u_1, u_2, \ldots\} \quad \text{and} \quad O_P(v_0, \infty) = \{v_0, v_1, v_2, \ldots\}$$

are called the coupled orbit of $(u_0, v_0) \in E \times E$.

Now let (E, ρ) be a metric space. If every Cauchy sequence in $O_P(u_0, \infty)$ and $O_P(v_0, \infty)$ converges to a point in E, for some $(u_0, v_0) \in E \times E$, then the (E, ρ) metric space is said to be coupled orbitally complete.

Note that a complete metric space (E, ρ) clearly coupled orbitally complete, but a coupled orbitally complete metric space (E, ρ) does not necessarily complete as in shown by Example 1.

Definition 3 *Let* (E, ρ) *be a metric space,* $P : E \times E \to E$ *a mapping and* $u_0, v_0 \in E$*. A real-valued function* $B : E \times E \to [0, \infty)$ *is said to be* $((u_0, v_0), P)$*–coupled orbitally weak lower semi-continuous (c.o.w.l.s.c.) at* $(u, v) \in E \times E$ *iff* $\{u_n\}$ *and* $\{v_n\}$ *are sequences in* $O_P(u_0, \infty)$ *and* $O_P(v_0, \infty)$ *respectively and*

$$u_n \to u,\ v_n \to v \quad \textit{implies} \quad B(u, v) \leq \lim_{n\to\infty} \sup B(u_n, v_n)$$

(See [10]).

2 Main results

The following theorem is a version of Caristi's theorem, which was proved by Bollenbacher and Hicks (See [8]).

Theorem 1 *Let* (E, ρ) *be a metric space. Suppose* $S : E \to E$ *and* $\psi : E \to [0, \infty)$*. Suppose there exists an* u *such that*

$$\rho(v, Sv) \leq \psi(v) - \psi(Sv)$$

for every $v \in O_S(u, \infty)$*, and any Cauchy sequence in* $O_S(u, \infty)$ *converges to a point in* E*. Then:*

(a) $\lim S^n u = u'$ *exists,*

(b) $\rho(S^n u, u') \leq \psi(S^n u)$,

(c) $Su' = u'$ *iff* $B(u) = \rho(u, Su)$ *is* S*-orbitally lower semi-continuous at* u*,*

(d) $\rho(S^n u, u) \leq \psi(u)$ *and* $\rho(u', u) \leq \psi(u)$*.*

Now we prove the following coupled fixed point theorem for a function P on the product space $E \times E$.

Theorem 2 *Let* (E, ρ) *be a metric space,* $P : E \times E \to E$ *and* $\psi : E \to [0, \infty)$*. Suppose there exist* $u_0, v_0 \in E$ *such that* (E, ρ) *is coupled orbitally complete and*

$$\max\{\rho(u, P(u, v)), \rho(v, P(v, u))\} \leq \psi(u) + \psi(v) - \psi(P(u, v)) - \psi(P(v, u)) \quad (2)$$

for all $u \in O_P(u_0, \infty)$ *and* $v \in O_P(v_0, \infty)$*. Then:*

(a) $\lim u_n = \lim P(u_{n-1}, v_{n-1}) = u'$ *and* $\lim v_n = \lim P(v_{n-1}, u_{n-1}) = v'$ *exist, where the sequences* $\{u_n\}$ *and* $\{v_n\}$ *are defined as in* (1)*,*

(b) $\max\{\rho(u_n, u'), \rho(v_n, v')\} \leq \psi(u_n) + \psi(v_n)$,

(c) (u', v') *is a coupled fixed point of* P *if and only if* $B(u, v) = \rho(P(u, v), u)$ *is* $((u_0, v_0), P)$*– c.o.w.l.s.c. at* (u', v') *and* (v', u'),

(d) $\max\{\rho(u_n, u_0), \rho(v_n, v_0)\} \leq \psi(u_0) + \psi(v_0)$ *and* $\max\{\rho(u', u_0), \rho(v', v_0)\} \leq \psi(u_0) + \psi(v_0)$.

Proof. (a) Using inequality (2) we have

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\} \\
&= \sum_{k=0}^{n} \max\{\rho(u_k, P(u_k, v_k)), \rho(v_k, P(v_k, u_k))\} \\
&\leq \sum_{k=0}^{n} [\psi(u_k) + \psi(v_k) - \psi(P(u_k, v_k)) - \psi(P(v_k, u_k))] \\
&= \sum_{k=0}^{n} [\psi(u_k) - \psi(u_{k+1}) + \psi(v_k) - \psi(v_{k+1})] \\
&= \psi(u_0) - \psi(u_{n+1}) + \psi(v_0) - \psi(v_{n+1}) \\
&\leq \psi(u_0) + \psi(v_0).
\end{aligned}$$

Hence $\{S_n\}$ is bounded above and also non-decreasing, and so convergent.

Now let m, n be any positive integers with $m > n$. Then from triangle inequality of ρ, we have

$$\begin{aligned}
\max\{\rho(u_n, u_m), \rho(v_n, v_m)\} &\leq \max\Big\{\sum_{k=n}^{m-1} \rho(u_k, u_{k+1}), \sum_{k=n}^{m-1} \rho(v_k, v_{k+1})\Big\} \\
&\leq \sum_{k=n}^{m-1} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\}. \qquad (3)
\end{aligned}$$

Since $\{S_n\}$ is convergent, for every $\varepsilon > 0$, we can find a sufficiently large positive integer N such that

$$\sum_{k=n}^{\infty} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\} < \varepsilon$$

for all $n \geq N$. Thus, we get from (3) that

$$\max\{\rho(u_n, u_m), \rho(v_n, v_m)\} < \varepsilon$$

for all $m, n \geq N$, and so $\{u_n\}$ and $\{v_n\}$ are two Cauchy sequences in $O_P(u_0, \infty)$, and $O_P(v_0, \infty)$ respectively. Since (E, ρ) is coupled orbitally complete,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} P(u_{n-1}, v_{n-1}) = u' \text{ and } \lim_{n\to\infty} v_n = \lim_{n\to\infty} P(v_{n-1}, u_{n-1}) = v'$$

exist.

(b) Let m, n be any positive integers with $m > n$. Using inequalities (2) and (3) we have

$$\begin{aligned}
\max\{\rho(u_n, u_m), \rho(v_n, v_m)\} &\leq \sum_{k=n}^{m-1} \max\{\rho(u_k, u_{k+1}), \rho(v_k, v_{k+1})\} \\
&= \sum_{k=n}^{m-1} \max\{\rho(u_k, P(u_k, v_k)), \rho(v_k, P(v_k, u_k))\} \\
&\leq \sum_{k=n}^{m-1} [\psi(u_k) + \psi(v_k) - \psi(u_{k+1}) - \psi(v_{k+1})] \\
&= \psi(u_n) - \psi(u_m) + \psi(v_n) - \psi(v_m) \\
&\leq \psi(u_n) + \psi(v_n).
\end{aligned}$$

Letting m tend to infinity, we have from (a)

$$\max\{\rho(u_n, u'), \rho(v_n, v')\} \leq \psi(u_n) + \psi(v_n).$$

(c) Assume that $u' = P(u', v')$, $v' = P(v', u')$ and $\{u_n\}$, $\{v_n\}$ are sequences in $O_P(u_0, \infty)$ and $O_P(v_0, \infty)$ respectively with $u_n \to u'$, $v_n \to v'$. Then we have,

$$\begin{aligned}
B(u', v') = \rho(P(u', v'), u') = 0 &\leq \limsup \rho(P(u_n, v_n), u_n) \\
&= \limsup B(u_n, v_n)
\end{aligned}$$

and

$$\begin{aligned}
B(v', u') = \rho(P(v', u'), v') = 0 &\leq \limsup \rho(P(v_n, u_n), v_n) \\
&= \limsup B(v_n, u_n)
\end{aligned}$$

and so B is $((u_0, v_0), P)$–c.o.w.l.s.c. at (u', v') and (v', u').

Now let $u_n = P(u_{n-1}, v_{n-1})$, $v_n = P(v_{n-1}, u_{n-1})$ and B is $((u_0, v_0), P)$–c.o.w.l.s.c. at (u', v') and (v', u'). Then from (a) we have

$$0 \leq \rho(P(u', v'), u') = B(u', v') \leq \limsup B(u_n, v_n)$$

$$= \limsup \rho(P(u_n, v_n), u_n) = 0$$

and

$$\begin{aligned} 0 \le \rho(P(v', u'), v') = B(v', u') &\le \limsup B(v_n, u_n) \\ &= \limsup \rho(P(v_n, u_n), v_n) = 0. \end{aligned}$$

Thus $u' = P(u', v')$ and $v' = P(v', u')$.

(d) Using triangle inequality of ρ and inequaliy (2) we have

$$\begin{aligned} \max\{\rho(u_n, u_0), \rho(v_n, v_0)\} &\le \max\left\{\sum_{k=1}^{n} \rho(u_k, u_{k-1}), \sum_{k=1}^{n} \rho(v_k, v_{k-1})\right\} \\ &\le \sum_{k=1}^{n} \max\{\rho(u_k, u_{k-1}), \rho(v_k, v_{k-1})\} \\ &= \sum_{k=1}^{n} \max\{\rho(u_{k-1}, P(u_{k-1}, v_{k-1})), \rho(v_{k-1}, P(v_{k-1}, u_{k-1}))\} \\ &\le \sum_{k=1}^{n} [\psi(u_{k-1}) + \psi(v_{k-1}) - \psi(u_k) - \psi(v_k)] \\ &= \psi(u_0) - \psi(u_n) + \psi(v_0) - \psi(v_n) \\ &\le \psi(u_0) + \psi(v_0). \end{aligned}$$

Letting n tend to infinity, we have from (a)

$$\max\{\rho(u', u_0), \rho(v', v_0)\} \le \psi(u_0) + \psi(v_0).$$

□

We now prove the following theorem.

Theorem 3 *Let* (E, ρ) *be a metric space,* $P : E \times E \to E$ *and* $\psi : E \to [0, \infty)$. *Suppose there exist* $u_0, v_0 \in E$ *such that* (E, ρ) *is coupled orbitally complete and*

$$\rho(u, P(u, v)) + \rho(v, P(v, u)) \le \psi(u) + \psi(v) - \psi(P(u, v)) - \psi(P(v, u)) \quad (4)$$

for all $u \in O_P(u_0, \infty)$ *and* $v \in O_P(v_0, \infty)$. *Then:*

(a) $\lim u_n = \lim P(u_{n-1}, v_{n-1}) = u'$ *and* $\lim v_n = \lim P(v_{n-1}, u_{n-1}) = v'$ *exist, where the sequences* $\{u_n\}$ *and* $\{v_n\}$ *are defined as in* (1),

(b) $\rho(u_n, u') + \rho(v_n, v') \leq \psi(u_n) + \psi(v_n)$,

(c) (u', v') *is a coupled fixed point of* P *if and only if* $B(u, v) = \rho(P(u, v), u)$ *is* $((u_0, v_0), P)$– *c.o.w.l.s.c. at* (u', v') *and* (v', u'),

(d) $\rho(u_n, u_0) + \rho(v_n, v_0) \leq \psi(u_0) + \psi(v_0)$ *and* $\rho(u', u_0) + \rho(v', v_0) \leq \psi(u_0) + \psi(v_0)$.

Proof. We have

$$\begin{aligned} \max\{\rho(u, P(u, v)), \rho(v, P(v, u))\} &\leq \rho(u, P(u, v)) + \rho(v, P(v, u)) \\ &\leq \psi(u) + \psi(v) - \psi(P(u, v)) - \psi(P(v, u)). \end{aligned}$$

The results (a) and (c) of this theorem follow immediately from Theorem 2.

(b) Let m, n be any positive integers with $m > n$. Using triangle inequality of ρ and inequality (4), we have

$$\begin{aligned} \rho(u_n, u_m) + \rho(v_n, v_m) &\leq \sum_{k=n}^{m-1} [\rho(u_k, u_{k+1}), +\rho(v_k, v_{k+1})] \\ &= \sum_{k=n}^{m-1} [\rho(u_k, P(u_k, v_k)) + \rho(v_k, P(v_k, u_k))] \\ &\leq \sum_{k=n}^{m-1} [\psi(u_k) + \psi(v_k) - \psi(u_{k+1}) - \psi(v_{k+1})] \\ &= \psi(u_n) - \psi(u_m) + \psi(v_n) - \psi(v_m) \\ &\leq \psi(u_n) + \psi(v_n). \end{aligned}$$

Letting m tend to infinity, we have from (a)

$$\rho(u_n, u') + \rho(v_n, v') \leq \psi(u_n) + \psi(v_n).$$

(d) Using triangle inequality of ρ and inequaliy (4) we have

$$\begin{aligned} \rho(u_n, u_0) + \rho(v_n, v_0) &\leq \sum_{k=1}^{n} [\rho(u_k, u_{k-1}) + \rho(v_k, v_{k-1})] \\ &= \sum_{k=1}^{n} [\rho(u_{k-1}, P(u_{k-1}, v_{k-1})) + \rho(v_{k-1}, P(v_{k-1}, u_{k-1}))] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{n}[\psi(u_{k-1})+\psi(v_{k-1})-\psi(u_k)-\psi(v_k)] \\
&= \psi(u_0)-\psi(u_n)+\psi(v_0)-\psi(v_n) \\
&\leq \psi(u_0)+\psi(v_0).
\end{aligned}$$

Letting n tend to infinity, we have from (a)

$$\rho(u',u_0)+\rho(v',v_0)\leq \psi(u_0)+\psi(v_0).$$

□

Finally, we prove the following theorem.

Theorem 4 *Let* (E,ρ) *be a metric space,* $P: E\times E\to E$ *and* $\psi: E\to[0,\infty)$. *Suppose there exist* $u_0,v_0\in E$ *such that* (E,ρ) *is coupled orbitally complete and*

$$\rho(u,P(u,v)) \quad \leq \psi(u)-\psi(P(u,v)), \tag{5}$$
$$\rho(v,P(v,u)) \quad \leq \psi(v)-\psi(P(v,u)) \tag{6}$$

for all $u\in O_P(u_0,\infty)$ *and* $v\in O_P(v_0,\infty)$. *Then*:

(a) $\lim u_n=\lim P(u_{n-1},v_{n-1})=u'$ *and* $\lim v_n=\lim P(v_{n-1},u_{n-1})=v'$ *exist, where the sequences* $\{u_n\}$ *and* $\{v_n\}$ *are defined as in* (1),

(b) $\rho(u_n,u')\leq\psi(u_n)$ *and* $\rho(v_n,v')\leq\psi(v_n)$,

(c) (u',v') *is a coupled fixed point of* P *if and only if* $B(u,v)=\rho(P(u,v),u)$ *is* $((u_0,v_0),P)$*– c.o.w.l.s.c. at* (u',v') *and* (v',u'),

(d) $\rho(u_n,u_0)\leq\psi(u_0)$ *and* $\rho(u',u_0)\leq\psi(u_0)$, $\rho(v_n,v_0)\leq\psi(v_0)$ *and* $\rho(v',v_0)\leq\psi(v_0)$.

Proof. From inequalities (5) and (6) we have

$$\rho(u,P(u,v))+\rho(v,P(v,u))\leq\psi(u)+\psi(v)-\psi(P(u,v))-\psi(P(v,u)).$$

The results (a) and (c) of this theorem follow immediately from Theorem 3.

(b) Let m,n be any positive integers with $m>n$. Using triangle inequality of ρ and inequality (5) we get

$$\rho(u_n,u_m) \quad \leq \quad \sum_{k=n}^{m-1}\rho(u_k,u_{k+1})=\sum_{k=n}^{m-1}\rho(u_k,P(u_k,v_k))$$

$$\leq \sum_{k=n}^{m-1} [\psi(u_k) - \psi(u_{k+1})] = \psi(u_n) - \psi(u_m) \leq \psi(u_n).$$

Letting m tend to infinity, we have from (a)

$$\rho(u_n, u') \leq \psi(u_n).$$

Similarly, using triangle inequality of ρ and inequality (6) we get

$$\rho(v_n, v') \leq \psi(v_n).$$

(d) Using triangle inequality of ρ and inequality (5) we have

$$\begin{aligned} \rho(u_n, u_0) &\leq \sum_{k=1}^{n} \rho(u_k, u_{k-1}) = \sum_{k=1}^{n} \rho(u_{k-1}, P(u_{k-1}, v_{k-1})) \\ &\leq \sum_{k=1}^{n} [\psi(u_{k-1}) - \psi(u_k)] \\ &= \psi(u_0) - \psi(u_n) \leq \psi(u_0). \end{aligned}$$

Letting n tend to infinity, we have from (a)

$$\rho(u', u_0) \leq \psi(u_0).$$

Similarly, it can be proved that

$$\rho(v_n, v_0) \leq \psi(v_0) \quad \text{and} \quad \rho(v', v_0) \leq \psi(v_0).$$

□

3 Some Examples

We now give two examples which illustrate our results.

Example 1 *Let* $E = [0, 1)$ *with Euclidean metric* ρ.
Define $P : E \times E \longrightarrow E$ *by* $P(u, v) = u/2$ *for all* (u, v) *in* $E \times E$ *and also define* $\psi : E \longrightarrow [0, \infty)$ *by* $\psi(u) = 2u$ *for all* u *in* E.

Let u_0 *and* v_0 *are arbitrary two points in* E. *Then we have*

$$O_P(u_0, \infty) = \left\{ u_0, \frac{u_0}{2}, \frac{u_0}{2^2}, \ldots, \frac{u_0}{2^n}, \ldots \right\} \quad \textit{and}$$

$$O_P(v_0,\infty) = \left\{v_0, \frac{v_0}{2}, \frac{v_0}{2^2}, \ldots, \frac{v_0}{2^n}, \ldots\right\}.$$

Clearly, (E,ρ) *is coupled orbitally complete as it is not complete. Further, for all* u *in* $O_P(u_0,\infty)$ *and* v *in* $O_P(v_0,\infty)$*, we have*

$$\begin{aligned}\max\{\rho(u,P(u,v)),\rho(v,P(v,u))\} &= \max\{|u-u/2|,|v-v/2|\} = \max\{u/2,v/2\}\\ &\le u+v = \psi(u)+\psi(v)-\psi(P(u,v))-\psi(P(u,v)).\end{aligned}$$

Thus P *satisfies inequality* (2) *with* $\psi(u)=2u$ *and so the conditions of Theorem 2 are satisfied and* $\lim P(u_{n-1},v_{n-1}) = \lim P(v_{n-1},u_{n-1}) = 0$*. Further,* $(0,0)$ *is a coupled fixed point of* P *and* $B(u,v)=\rho(P(u,v),u)$ *is c.o.w.l.s.c. at* $(0,0)$*.*

Example 2 *Let* $E=[0,\infty)$ *with Euclidean metric* ρ *and define*

$$P: E\times E \longrightarrow E \quad by \quad P(u,v) = \begin{cases} 0 & \text{if } u<v \\ 2 & \text{if } u\ge v \end{cases}.$$

for all (u,v) *in* $E\times E$*. If we take* $u_0=2$ *and* $v_0=2$*, then*

$$O_P(2,\infty)=\{2,2,2,\ldots.\} \quad and \quad O_P(2,\infty)=\{2,2,2,\ldots.\}.$$

Clearly, (E,ρ) *is coupled orbitally complete and* P *satisfies inequality* (2) *for all* u *in* $O_P(2,\infty)$ *and* v *in* $O_P(2,\infty)$ *with* $\psi(u)=u$*. So the conditions of Theorem 2 are satisfied and* $\lim P(u_{n-1},v_{n-1}) = \lim P(v_{n-1},u_{n-1}) = 2$*. Further,* $(2,2)$ *is a coupled fixed point of* P *and* $B(u,v)=\rho(P(u,v),u)$ *is c.o.w.l.s.c. at* $(2,2)$*.*

Similarly, if we take $u_0=0$ *and* $v_0=2$*, then*

$$O_P(0,\infty)=\{0,0,0,\ldots.\} \quad and \quad O_P(2,\infty)=\{2,2,2,\ldots.\}.$$

Clearly, (E,ρ) *is coupled orbitally complete and* P *satisfies inequality* (2) *for all* u *in* $O_P(0,\infty)$ *and* v *in* $O_P(2,\infty)$ *with* $\psi(u)=u$*. So the conditions of Theorem 2 are satisfied and* $\lim P(u_{n-1},v_{n-1})=0, \lim P(v_{n-1},u_{n-1})=2$*. Further,* $(0,2)$ *is a coupled fixed point of* P *and* $B(u,v)=\rho(P(u,v),u)$ *is c.o.w.l.s.c. at* $(0,2)$ *and* $(2,0)$*.*

This shows that the coupled fixed point of P *is not unique.*

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References

[1] T. Abdeljawad, E. Karapınar, Quasicone metric spaces and generalizations of Caristi Kirk's theorem, *Fixed Point Theory Appl.*, **2009**, Article ID 574387, 9 pages.

[2] Ö. Acar, İ. Altun, S. Romaguera, Caristi's type mappings on complete partial metric spaces, *Fixed Point Theory*, **14** (1) (2013), 3–10.

[3] R. P. Agarwal, M. A Khamsi, Extension of Caristi's fixed point theorem to vector valued metric spaces. *Nonlinear Anal. TMA*, **74** (1) (2011), 141–145.

[4] J. S. Bae, E. W. Cho, S. H. Yeom, A generalization of the Caristi-Kirk fixed point theorem and its application to mapping theorems, *J. Kor. Math. Soc.* **31** (1) (1994), 29–48.

[5] V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, *Nonlinear Anal. TMA*, **74** (18), (2011), 7347–7355.

[6] V. Berinde, M. Păcurar, A constructive approach to coupled fixed point theorems in metric spaces, *Carpathian J. Math.*, **31** (3) (2015), 277– 287.

[7] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal. TMA*, **65** (7) (2006), 1379–1393.

[8] A. Bollenbacher, T. L. Hicks, A fixed point theorem revisited, *Proc. Amer. Math. Soc.*, **102** (4) (1988), 898–900.

[9] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Amer. Math. Soc.* **215** (1976), 241–251.

[10] L. Ćirić, Periodic and fixed point theorems in a quasi-metric space, *J. Austral. Math. Soc. Ser. A*, **54** (1) (1993), 80–85.

[11] D. Dorić, Z. Kadelburg, S. Radenović, Coupled fixed point results for mappings without mixed monotone property, *Applied Mathematics Letters*, **25** (11) (2012), 1803–1808.

[12] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (2) (1974), 324–353.

[13] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal. TMA*, **11** (5) (1987), 623–632.

[14] T. L. Hicks, Fixed point theorems for quasi-metric spaces, *Math. Japon.*, **33** (1988), 231–236.

[15] T. L. Hicks, Fixed point theorems for **d**-complete topological spaces I, *Internat. J. Math. Math. Sci.*, **15** (3) (1992), 435–440.

[16] J. R. Jachymski, Caristi's fixed point theorem and selection of set-valued contractions, *J. Math. Anal. Appl.*, **227** (1) (1998), 55–67.

[17] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.*, **44** (1996), 381–391.

[18] H. Karayılan, M. Telci, Caristi type fixed point theorems in fuzzy metric spaces, *Hacettepe Journal of Mathematics and Statistics*, **48** (1) (2019), 75–86.

[19] M. A. Khamsi, Remarks on Caristi's fixed point theorem, *Nonlinear Anal. TMA*, **71** (1-2) (2009), 227–231.

[20] F. Khojasteh, E. Karapinar, H. Khandani, Some applications of Caristi's fixed point theorem in metric spaces, *Fixed Point Theory Appl.*, **2016** (16) (2016), DOI 10.1186/s13663-016-0501-z, 10 pages.

[21] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal. TMA*, **70** (12) (2009), 4341–4349.

[22] V. I. Opoitsev, Heterogeneous and combined concave operators, *Siberian Math. J.*, **16** (4) (1975), 597–605 (in Russian).

[23] V. I. Opoitsev, Dynamics of collective behavior. III. Heterogenic system, *Avtomat. i Telemekh.*, **36** (1) (1975), 124–138 (in Russian).

[24] V. I. Opoitsev, T. A. Khurodze, Nonlinear operators in space with a cone. *Tbilis. Gos. Univ. Tbilisi*, **271** (1984), (in Russian)

[25] S. Radenović, Coupled fixed point theorems for monotone mappings in partially ordered metric spaces, *Kragujevac J. Math.*, **38** (2) (2014), 249–257.

[26] W. Shatanawi, Coupled fixed point theorems in generalized metric spaces. *Hacet. J. Math. Stat.*, **40** (3) (2011), 441–447.

[27] W. Shatanawi, A. Pitea, Some coupled fixed point theorems in quasi-partialmetric spaces, *Fixed Point Theory Appl.*, **2013** (153) (2013) DOI:10.1186/1687-1812-2013-153, 15 pages.

[28] T. Suzuki, Generalized Caristi's fixed point theorems by Bae and others, *J. Math. Anal. Appl.* **302** (2)(2005), 502–508.

[29] K. Włodarczyk, R. Plebaniak, Maximality principle and general results of Ekeland and Caristi types without lower semicontinuity assumptions in cone uniform spaces with generalized pseudodistances, *Fixed Point Theory and Appl.*, **2010**, Article ID 175453, 35 pages.

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On Chern classes of the tensor product of vector bundles

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Abstract. We present two formulas for Chern classes (polynomial) of the tensor product of two vector bundles. In the first formula the Chern polynomial of the product is expressed as determinant of a polynomial in a matrix variable involving the Chern classes of the first bundle with Chern classes of the second bundle as coefficients. In the second formula the total Chern class of the tensor product is expressed as resultant of two explicit polynomials. Finally, formulas for the total Chern class of the second symmetric and the second alternating products are deduced.

1 Introduction

One associates a series of cohomological (characteristic) classes $c_i(\mathcal{E}) \in H^{2i}(M)$ called the i^{th} *Chern class* of $\mathcal{E}$, for any $i = 1, \dots, r$, with a complex vector bundle $\mathcal{E}$ of rank r over a manifold M (cf. [9, Ch. IV] or [3, Ch. I, §4]). One can arrange these classes into a polynomial $c(\mathcal{E}; t) = 1 + c_1(\mathcal{E})t + \cdots + c_r(\mathcal{E})t^r$, called the *Chern polynomial*. Its value $c(\mathcal{E}) = c(\mathcal{E}; 1) = 1 + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E})$ at $t = 1$ is the *total Chern class* of $\mathcal{E}$.

We recall some basic properties of the Chern classes. If $\mathcal{E}$ and $\mathcal{F}$ are two complex vector bundles over the same manifold then $c(\mathcal{E} \oplus \mathcal{F}; t) = c(\mathcal{E}; t) \cdot c(\mathcal{F}; t)$ by

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the *Whitney product formula* (cf. [9, (20.10.3)]). Computing with Chern classes one can pretend using the *Splitting Principle* (cf. [9, Ch. IV, §21]) that the bundle $\mathcal{E}$ of rank r splits into direct sum of r complex line bundles and the first Chern classes $\alpha_1, \dots, \alpha_r$ of these hypothetical line bundles are the so-called *Chern roots* of $\mathcal{E}$. Hence, by the Whitney product formula we have $c(\mathcal{E}; t) = \prod_{i=1}^{r}(1+\alpha_i t)$, thus $c_k(\mathcal{E}) = e_k(\alpha) = e_k(\alpha_1, \dots, \alpha_r) = \sum_{1 \le i_1 < \dots < i_k \le r} \alpha_{i_1} \cdots \alpha_{i_k}$ for any $k = 1, \dots, r$, i.e. the Chern classes are *elementary symmetric polynomials* of the Chern roots. The dual bundle $\mathcal{E}^*$ has opposite Chern roots to $\mathcal{E}$, hence its Chern polynomial equals $c(\mathcal{E}^*, t) = \prod_{i=1}^{r}(1 - \alpha_i t) = c(\mathcal{E}, -t)$.

The Chern polynomial does not behave so well for the tensor product like for the direct sum. Nevertheless, for complex line bundles $\mathcal{L}$ and $\mathcal{L}'$ we have $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$ (cf. [9, (20.1)]). Hence, if $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_q$ are Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively, then $\alpha_i + \beta_j$, $i = 1, \dots, r$, $j = 1, \dots, q$ are the Chern roots of the tensor product $\mathcal{E} \otimes \mathcal{F}$ and the Chern polynomial of the tensor product equals

$$c(\mathcal{E} \otimes \mathcal{F}; t) = \prod_{i=1}^{r} \prod_{j=1}^{q} (1 + \alpha_i t + \beta_j t). \tag{1}$$

Our goal is to express (1) in terms of Chern classes of $\mathcal{E}$ and $\mathcal{F}$, or equivalently in terms of elementary symmetric polynomials of the Chern roots $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_q$, respectively.

There are several approaches to compute the Chern classes of the tensor product. We mention the four approaches compared in [4]. The first method computes the Chern classes of the tensor product by eliminating the Chern roots $\alpha_1, \dots, \alpha_r$, $\beta_1, \dots, \beta_q$ from $c(\mathcal{E} \otimes \mathcal{F}) = \prod_{i=1}^{r} \prod_{j=1}^{q} (1 + \alpha_i + \beta_j)$ using relations $c_i(\mathcal{E}) = e_i(\alpha_1, \dots, \alpha_r)$ and $c_j(\mathcal{F}) = e_j(\beta_1, \dots, \beta_q)$ for $i = 1, \dots, r$ and $j = 1, \dots, q$. The second approach uses the multiplicativity of the Chern character (cf. [3, Ch. III, §10.1]) and Newton's identities (cf. [7, (2.11′)]). The third uses Lascoux's formula [6] which expresses the Chern classes of the tensor product as linear combination of products of Schur polynomials of Chern classes of $\mathcal{E}$ and $\mathcal{F}$. The last approach is Manivel's formula [8], which has the same form as Lascoux's formula, but computes the coefficients differently. These methods have been implemented in the library CHERN.LIB [5] for the computer algebra system SINGULAR [2].

2 Chern polynomial of the tensor product: first approach

Lemma 1 *Let* $u_1, \dots, u_r, v_1, \dots, v_q$ *be formal variables. We consider elementary symmetric polynomials* $e_k(u) = \sum_{1\le i_1<\dots<i_k\le r} u_{i_1}\cdots u_{i_k}$ *for any* $k = 1, \dots, r$ *and we set* $e_0(u) = 1$. *We associate with the list* $(e_1(v), \dots, e_q(v))$ *the following matrix*

$$\Lambda(e(v)) = \begin{pmatrix} e_1(v) & -1 & & \\ \vdots & & \ddots & \\ e_{q-1}(v) & & & -1 \\ e_q(v) & & & \end{pmatrix} \tag{2}$$

(*it has non-zero entries only in the first column and above the diagonal*). *Then we have*

$$\prod_{i=1}^{r}\prod_{j=1}^{q}(1+u_i+v_j) = \det\left(\sum_{k=0}^{r} e_k(u)[I+\Lambda(e(v))]^{r-k}\right),$$

where $I = I_q$ *is the* q*-by-*q *identity matrix.*

Proof. First, we diagonalize the matrix $\Lambda(e(v))$. Therefore, we consider the q-by-q matrix $E = E(v_1, \dots, v_q) = \left[e_{i-1}(v_1, \dots, \widehat{v}_j, \dots, v_q)\right]_{i,j=1}^{q}$, where $\widehat{v}_j$ means that the term v_j is omitted. We show that E is non-singular by computing its determinant as follows. We subtract the first column from the other columns, then we raise a $(v_1 - v_j)$-factor from columns $j = 2, \dots, q$, respectively. Expanding the resulting determinant by the first row we get the recurrent relation $\det E(v_1, \dots, v_q) = \prod_{j=2}^{q}(v_1 - v_j)\det E(v_2, \dots, v_q)$, hence $\det E = \prod_{1\le i<j\le q}(v_i - v_j) \neq 0$. Moreover, $\Lambda(e(v))E = E\,\mathrm{diag}(v_1, \dots, v_q)$ by relations $e_i(v_1, \dots, v_q) = e_i(v_1, \dots, \widehat{v}_j, \dots, v_q) + v_j e_{i-1}(v_1, \dots, \widehat{v}_j, \dots, v_q)$, hence $\Lambda(e(v)) = E\mathrm{diag}(v_1, \dots, v_q)E^{-1}$. Furthermore,

$$I + \Lambda(e(v)) = I + E\mathrm{diag}(v_1, \dots, v_q)E^{-1} = E\mathrm{diag}(1+v_1, \dots, 1+v_q)E^{-1}$$

is also diagonalizable with eigenvalues $1 + v_1, \dots, 1 + v_q$. Finally,

$$\prod_{j=1}^{q}\prod_{i=1}^{r}(1+u_i+v_j) = \prod_{j=1}^{q}\sum_{k=0}^{r} e_k(u)(1+v_j)^{r-k} =$$

$$= \det\left(E\mathrm{diag}\left(\sum_{k=0}^{r} e_k(u)(1+v_1)^{r-k}, \dots, \sum_{k=0}^{r} e_k(u)(1+v_q)^{r-k}\right)E^{-1}\right) =$$

$$= \det\left(\sum_{k=0}^{r} e_k(u)E\mathrm{diag}\big(1+v_1,\ldots,1+v_q\big)^{r-k}E^{-1}\right)$$

$$= \det\left(\sum_{k=0}^{r} e_k(u)[I+\Lambda(e(v))]^{r-k}\right).$$

□

Theorem 1 *Let $\mathcal{E}$ and $\mathcal{F}$ be two complex vector bundles of rank r and q, respectively over the same manifold. Then the Chern polynomial of the tensor product $\mathcal{E}\otimes\mathcal{F}$ equals*

$$c(\mathcal{E}\otimes\mathcal{F};t) = \det\left(\sum_{k=0}^{r} c_k(\mathcal{E})t^k[I+\Lambda(c(\mathcal{F});t)]^{r-k}\right),$$

where $c_0(\mathcal{E})=1$ and $\Lambda(c(\mathcal{F});t)$ is the matrix (2) with $c_1(\mathcal{F})t,\ldots,c_q(\mathcal{F})t^q$ in the first column.

Proof. Let $\alpha_1,\ldots,\alpha_r$ and $\beta_1,\ldots,\beta_q$ be the Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then it is enough to show that

$$\prod_{i=1}^{r}\prod_{j=1}^{q}(1+\alpha_i t+\beta_j t) = \det\left(\sum_{k=0}^{r} e_k(\alpha)t^k[I+\Lambda(e(\beta t))]^{r-k}\right),$$

where $\Lambda(e(\beta t))$ equals the matrix $\Lambda(c(\mathcal{F});t)$ only replacing Chern classes $c_j(\mathcal{F})$ by elementary symmetric polynomials $e_j(\beta)=e_j(\beta_1,\ldots,\beta_q)$ of Chern roots for all $j=1,\ldots,q$. Finally, substituting $u_1=\alpha_1 t,\ldots,u_r=\alpha_r t$, $v_1=\beta_1 t,\ldots,v_q=\beta_q t$ in Lemma 1 we get the desired relation. □

3 Resultant and Chern classes of the tensor product: second approach

The second approach uses the resultant of two polynomials. This will lead us to a determinantal formula for Chern classes of the second alternating and the second symmetric products of a vector bundle.

Let $A(t) = a_r + a_{r-1}t + \cdots + a_0t^r = a_0\prod_{i=1}^{r}(t-\alpha_i)$ and $B(t) = b_q + b_{q-1}t + \cdots + b_0t^q = b_0\prod_{j=1}^{q}(t-\beta_j)$ be two polynomials in variable t with roots $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_q$, respectively. The *resultant* of polynomials A and B with respect to t is given by

$$\operatorname{res}(A(t), B(t), t) = a_0^q b_0^r \prod_{i=1}^{r}\prod_{j=1}^{q}(\alpha_i - \beta_j) = \begin{vmatrix} a_0 & & & b_0 & & \\ a_1 & \ddots & & b_1 & \ddots & \\ \vdots & & a_0 & \vdots & & b_0 \\ a_r & & \vdots & b_q & & \vdots \\ & \ddots & a_{r-1} & & \ddots & b_{q-1} \\ & & a_r & & & b_q \end{vmatrix},$$

where the first q columns contain the coefficients of A, while the last r columns contain the coefficients of B and empty spaces contain zeroes (cf. [1, Ch. III]).

Instead of the Chern polynomial $c(\mathcal{F}; t) = 1 + c_1(\mathcal{F})t + \cdots + c_q(\mathcal{F})t^q$ of the rank q vector bundle $\mathcal{F}$ we consider the polynomial with coefficients in reverse order

$$C(\mathcal{F}; t) = \sum_{k=0}^{q} c_k(\mathcal{F})t^{q-k} = c_q(\mathcal{F}) + c_{q-1}(\mathcal{F})t + \cdots + c_1(\mathcal{F})t^{q-1} + t^q. \qquad (3)$$

They are related by $C(\mathcal{F}; t) = t^q c(\mathcal{F}; t^{-1})$ and moreover, we can recover the total Chern class by substituting $t = 1$, i.e. $c(\mathcal{F}) = C(\mathcal{F}; 1)$. Furthermore, if $\beta_1, \dots, \beta_q$ are Chern roots of $\mathcal{F}$ then $C(\mathcal{F}; t) = \prod_{j=1}^{q}(t + \beta_j)$, i.e. the opposite of Chern roots of $\mathcal{F}$ are roots of the polynomial $C(\mathcal{F}; t)$. We note that for the dual bundle $\mathcal{F}^*$ we have $C(\mathcal{F}^*; t) = (-1)^q C(\mathcal{F}; -t)$.

Lemma 2 *If $\alpha_1, \dots, \alpha_r$ are the Chern roots of the complex vector bundle $\mathcal{E}$ of rank r then $\prod_{i=1}^{r}(t-s-\alpha_i) = (-1)^r C(\mathcal{E}; s-t) = \sum_{k=0}^{r}(-1)^k d_k(\mathcal{E}; s)t^{r-k}$ with coefficients $d_k(\mathcal{E}; s) = \binom{r}{k}s^k + \binom{r-1}{k-1}c_1(\mathcal{E})s^{k-1} + \cdots + c_k(\mathcal{E}) = \sum_{i=0}^{k}\binom{r-i}{k-i}c_i(\mathcal{E})s^{k-i}$.*

Proof. Indeed, $\prod_{i=1}^{r}(t - s - \alpha_i) = (-1)^r \prod_{i=1}^{r}(s - t + \alpha_i) = (-1)^r C(\mathcal{E}; s - t)$ and moreover, $\prod_{i=1}^{r}(t - s - \alpha_i) = \sum_{k=0}^{r}(-1)^k e_k(s + \alpha_1, \dots, s + \alpha_r)t^{r-k}$, where

$$e_k(s + \alpha_1, \dots, s + \alpha_r) = \sum_{1 \le i_1 < \cdots < i_k \le r}(s + \alpha_{i_1}) \cdots (s + \alpha_{i_k}) =$$

$$= \sum_{1 \le i_1 < \cdots < i_k \le r}\left[s^k + e_1(\alpha_{i_1}, \dots, \alpha_{i_k})s^{k-1} + \cdots + e_k(\alpha_{i_1}, \dots, \alpha_{i_k})\right] =$$

$$= \binom{r}{k}s^k + \binom{r-1}{k-1}e_1(\alpha_1, \dots, \alpha_r)s^{k-1} + \cdots + \binom{r-k}{0}e_k(\alpha_1, \dots, \alpha_r) =$$

$$= \sum_{i=0}^{r} \binom{r-i}{k-i} e_i(\alpha_1, \dots, \alpha_r) s^{k-i} = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(\mathcal{E}) s^{k-i} = d_k(\mathcal{E}; s).$$

□

In the next theorem we express $C(\mathcal{E} \otimes \mathcal{F}; s)$ as resultant of polynomials $(-1)^r C(\mathcal{E}; s-t)$ and $C(\mathcal{F}; t)$. We can also get a formula for the total Chern class of the tensor product by substituting $s = 1$.

Theorem 2 *If $\mathcal{E}$ and $\mathcal{F}$ are two complex vector bundles of rank r and q, respectively over the same manifold then*

$$C(\mathcal{E} \otimes \mathcal{F}; s) = \mathrm{res}((-1)^r C(\mathcal{E}; s-t), C(\mathcal{F}; t), t), \tag{4}$$

where the polynomial C is defined by (3). Substituting $s = 1$ yields the total Chern class of the tensor product $c(\mathcal{E} \otimes \mathcal{F}) = \mathrm{res}((-1)^r C(\mathcal{E}; 1-t), C(\mathcal{F}; t), t)$. Moreover, the top Chern classes of the tensor product equals

$$c_{rq}(\mathcal{E} \otimes \mathcal{F}) = (-1)^{rq} \mathrm{res}(c(\mathcal{E}; -t), c(\mathcal{F}; t), t),$$

while the top Chern classes of the $\mathsf{Hom}(\mathcal{E}, \mathcal{F})$ bundle equals

$$c_{rq}(\mathsf{Hom}(\mathcal{E}, \mathcal{F})) = (-1)^{rq} \mathrm{res}(c(\mathcal{E}; t), c(\mathcal{F}; t), t).$$

Proof. Denote $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_q$ the Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then

$$C(\mathcal{E} \otimes \mathcal{F}; s) = \prod_{i=1}^{r} \prod_{j=1}^{q} (s + \alpha_i + \beta_j) = \prod_{i=1}^{r} \prod_{j=1}^{q} (s + \alpha_i - (-\beta_j)),$$

hence $C(\mathcal{E} \otimes \mathcal{F}; s)$ is the resultant of polynomials $\prod_{j=1}^{q}(t + \beta_j) = C(\mathcal{F}; t)$ and $\prod_{i=1}^{r}(t - s - \alpha_i) = (-1)^r C(\mathcal{E}; s-t)$ with respect to the variable t, i.e. $C(\mathcal{E} \otimes \mathcal{F}; s) = \mathrm{res}((-1)^r C(\mathcal{E}; s-t), C(\mathcal{F}; t), t)$.

To obtain the top Chern class of the tensor product we substitute $s = 0$ into (4), thus $c_{rq}(\mathcal{E} \otimes \mathcal{F}; t) = \mathrm{res}((-1)^r C(\mathcal{E}; -t), C(\mathcal{F}; t), t)$. The coefficients of polynomials $C(\mathcal{F}; t)$ and $c(\mathcal{F}; t)$ are in reverse order, and similarly the coefficients of polynomials $(-1)^r C(\mathcal{E}; -t)$ and $c(\mathcal{E}; -t)$ are also in reverse order. Hence we get $\mathrm{res}((-1)^r C(\mathcal{E}; -t), C(\mathcal{F}; t), t) = (-1)^{rq} \mathrm{res}(c(\mathcal{E}; -t), c(\mathcal{F}; t), t)$ by reversing the order of rows, the order of the first q columns and last r columns in the defining determinant (3) of the resultant.

Finally, the top Chern class of the $\mathsf{Hom}(\mathcal{E}, \mathcal{F})$-bundle $c_{rq}(\mathsf{Hom}(\mathcal{E}, \mathcal{F}); t) = c_{rq}(\mathcal{E}^* \otimes \mathcal{F}; t) = (-1)^{rq} \mathrm{res}(c(\mathcal{E}^*; -t), c(\mathcal{F}; t), t) = (-1)^{rq} \mathrm{res}(c(\mathcal{E}; t), c(\mathcal{F}; t), t)$. □

4 Chern classes the second alternating product $\wedge^2\mathcal{E}$ and the second symmetric product $S^2\mathcal{E}$

We give a different version of Theorem 2, which leads to determinantal formulas for total Chern classes of the second alternating and symmetric products.

Theorem 3 *If $\mathcal{E}$ and $\mathcal{F}$ are two complex vector bundles of rank r and q, respectively over the same manifold, then*

$$C(\mathcal{E}\otimes\mathcal{F};s) = \operatorname{res}((-1)^r C\left(\mathcal{E};\frac{s}{2}-t\right), C\left(\mathcal{F};\frac{s}{2}+t\right), t),$$

By substituting $s=1$ we get $c(\mathcal{E}\otimes\mathcal{F}) = \operatorname{res}((-1)^r C\left(\mathcal{E};\frac{1}{2}-t\right), C\left(\mathcal{F};\frac{1}{2}+t\right), t)$.

Proof. If $\alpha_1,\dots,\alpha_r$ and $\beta_1,\dots,\beta_q$ are the Chern roots of $\mathcal{E}$ and $\mathcal{F}$, respectively, then

$$C(\mathcal{E}\otimes\mathcal{F};s) = \prod_{i=1}^{r}\prod_{j=1}^{q}(s+\alpha_i+\beta_j) = \prod_{i=1}^{r}\prod_{j=1}^{q}\left(\frac{s}{2}+\alpha_i-\left(-\frac{s}{2}-\beta_j\right)\right),$$

hence $C(\mathcal{E}\otimes\mathcal{F};s)$ is the resultant of $\prod_{i=1}^{q}\left(t+\frac{s}{2}+\beta_j\right) = C\left(\mathcal{F};\frac{s}{2}+t\right)$ and $\prod_{i=1}^{r}\left(t-\frac{s}{2}-\alpha_i\right) = (-1)^r C(\mathcal{E};\frac{s}{2}-t)$. □

If $\alpha_1,\dots,\alpha_r$ are the Chern roots of the vector bundle $\mathcal{E}$ then the total Chern classes of the second alternating and the second symmetric bundles

$$c(\wedge^2\mathcal{E}) = \prod_{1\le i<j\le r}(1+\alpha_i+\alpha_j),$$

$$c(S^2\mathcal{E}) = \prod_{1\le i\le j\le r}(1+\alpha_i+\alpha_j) = c(\mathcal{E};2)c(\wedge^2\mathcal{E}),$$

hence their corresponding C polynomials

$$C(\wedge^2\mathcal{E};s) = \prod_{1\le i<j\le r}(s+\alpha_i+\alpha_j),$$

$$C(S^2\mathcal{E};s) = \prod_{1\le i\le j\le r}(s+\alpha_i+\alpha_j) = 2^r C\left(\mathcal{E};\frac{s}{2}\right)C(\wedge^2\mathcal{E};s).$$

Theorem 4 *Let* $\bar{d}_k = d_k(\mathcal{E}; \frac{s}{2}) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(\mathcal{E}) (\frac{s}{2})^{k-i}$ *for* $k = 0, 1, \ldots, r$ *and* $\bar{d}_k = 0$ *otherwise. With these notations we have*

$$C(\wedge^2\mathcal{E}; s) = \det\left(\left[d_{2i-j}\left(\mathcal{E}; \frac{s}{2}\right)\right]_{i,j=1}^{r-1}\right) = \begin{vmatrix} \bar{d}_1 & 1 & & & & \\ \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & 1 & & \\ \bar{d}_5 & \bar{d}_4 & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \bar{d}_r & \bar{d}_{r-1} & \bar{d}_{r-2} & \bar{d}_{r-3} \\ & & & & \bar{d}_r & \bar{d}_{r-1} \end{vmatrix}. \tag{5}$$

By substituting $s = 1$ *we get* $c(\wedge^2\mathcal{E}) = \det\left(\left[d_{2i-j}(\mathcal{E}; \frac{1}{2})\right]_{i,j=1}^{r-1}\right)$.

Proof. By Theorem 3 we have $C(\mathcal{E}\otimes\mathcal{E}; s) = \mathrm{res}((-1)^r C(\mathcal{E}; \frac{s}{2} - t), C(\mathcal{E}; \frac{s}{2} + t), t)$. Note that $(-1)^r C(\mathcal{E}; \frac{s}{2} - t) = \sum_{k=0}^{r} (-1)^k d_k(\mathcal{E}; \frac{s}{2}) t^{r-k} = \sum_{k=0}^{r} (-1)^k \bar{d}_k t^{r-k}$ and $C(\mathcal{E}; \frac{s}{2} + t) = \sum_{k=0}^{r} d_k(\mathcal{E}; \frac{s}{2}) t^{r-k} = \sum_{k=0}^{r} \bar{d}_k t^{r-k}$, hence

$$C(\mathcal{E}\otimes\mathcal{E}; s) = \begin{vmatrix} 1 & & & 1 & & \\ -\bar{d}_1 & \ddots & & \bar{d}_1 & \ddots & \\ \bar{d}_2 & \ddots & 1 & \bar{d}_2 & \ddots & 1 \\ \vdots & & -\bar{d}_1 & \vdots & & \bar{d}_1 \\ (-1)^r \bar{d}_r & & \vdots & \bar{d}_r & & \vdots \\ & \ddots & (-1)^{r-1}\bar{d}_{r-1} & & \ddots & \bar{d}_{r-1} \\ & & (-1)^r \bar{d}_r & & & \bar{d}_r \end{vmatrix} \tag{6}$$

We add the $(r + i)^{\text{th}}$ column to the i^{th} column, then we subtract the $\frac{1}{2}$ of the i^{th} column from the $(r + i)^{\text{th}}$ column for all $i = 1, \ldots, r$. This results the determinant on the left hand side of (7). From the first r columns we raise a 2^r factor. Then we switch the $(2i)^{\text{th}}$ and $(r+2i)^{\text{th}}$ columns for all $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$. This yields the determinant on the right hand side of (7), which has zeroes in the even and odd rows of the first and last r columns, respectively.

$$\begin{vmatrix} 2 & & & 0 & & \\ 0 & 2 & & \bar{d}_1 & 0 & \\ 2\bar{d}_2 & 0 & \ddots & 0 & \bar{d}_1 & \ddots \\ 0 & 2\bar{d}_2 & \ddots & \bar{d}_3 & 0 & \ddots \\ \vdots & 0 & \ddots & \vdots & \bar{d}_3 & \ddots \\ & \vdots & & & \vdots & \\ & & \ddots & & & \ddots \end{vmatrix} = (-1)^{\lfloor \frac{r}{2} \rfloor} 2^r \begin{vmatrix} 1 & & & & 0 & & & \\ 0 & 0 & & & \bar{d}_1 & 1 & & \\ \bar{d}_2 & \bar{d}_1 & 1 & & 0 & 0 & 0 & \\ 0 & 0 & 0 & \ddots & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & \ddots \\ \vdots & \bar{d}_3 & \bar{d}_2 & \ddots & \vdots & 0 & 0 & \ddots \\ & \vdots & 0 & \ddots & & \vdots & \bar{d}_3 & \ddots \\ & & \vdots & \ddots & & & \vdots & \ddots \end{vmatrix}. \tag{7}$$

Moving the odd rows up and the even rows down yields a 2-by-2 block determinant with zeroes in the off-diagonal blocks and a $(-1)^{r(r-1)/2}$-sign, which cancels the existing $(-1)^{\lfloor r/2 \rfloor}$-sign. We expand this determinant with respect to the first and last rows. These rows contain only zeroes, except the first row has 1 in the first column and the last row has $\bar{d}_r$ in the last column. After expansion the two diagonal blocks become identical, hence

$$2^r \begin{vmatrix} 1 & & & & & 0 & 0 & 0 & \cdots & 0 \\ \bar{d}_2 & \bar{d}_1 & 1 & & & 0 & & & & 0 \\ \bar{d}_4 & \bar{d}_3 & \bar{d}_2 & \ddots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \bar{d}_{r-3} & 0 & & & & 0 \\ & & & & \bar{d}_{r-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{d}_1 & 1 & & & \\ 0 & & & & 0 & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & & \\ \vdots & & & & \vdots & \vdots & \bar{d}_4 & \bar{d}_3 & \ddots & \vdots \\ 0 & & & & 0 & & \vdots & \vdots & \ddots & \bar{d}_{r-2} \\ 0 & 0 & 0 & \cdots & 0 & & & & & \bar{d}_r \end{vmatrix} =$$

$$= 2^r \bar{d}_r \begin{vmatrix} \bar{d}_1 & 1 & & & & \\ \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & 1 & & \\ \bar{d}_5 & \bar{d}_4 & \bar{d}_3 & \bar{d}_2 & \bar{d}_1 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & \bar{d}_r & \bar{d}_{r-1} & \bar{d}_{r-2} & \bar{d}_{r-3} \\ & & & & \bar{d}_r & \bar{d}_{r-1} \end{vmatrix}^2 .$$

Note that $C(\mathcal{E};\frac{s}{2}) = \bar{d}_r = d_r(\mathcal{E};\frac{s}{2})$. Finally, by the relation

$$C(\mathcal{E}\otimes\mathcal{E};s) = C(\wedge^2\mathcal{E}\oplus S^2\mathcal{E};s) = C(\wedge^2\mathcal{E};s)C(S^2\mathcal{E};s) = 2^r C\left(\mathcal{E};\frac{s}{2}\right)C(\wedge^2\mathcal{E};s)^2$$

we are able to identify the $C(\wedge^2\mathcal{E};s)$-part in $C(\mathcal{E}\otimes\mathcal{E};s)$ to be (5). □

Remark 1 *We can also compute* $C(\wedge^{r-2}\mathcal{E};s)$ *from* $C(\wedge^2\mathcal{E};s)$ *by the duality*

$$C(\wedge^{r-2}\mathcal{E};s) = \prod_{1\le i_1<\cdots<i_{r-2}\le r}(s+\alpha_{i_1}+\cdots+\alpha_{i_{r-2}})$$
$$= \prod_{1\le j_1<j_2\le r}(s+c_1(\mathcal{E})-\alpha_{j_1}-\alpha_{j_2}) = (-1)^{\frac{r(r-1)}{2}}C(\wedge^2\mathcal{E};-(s+c_1(\mathcal{E}))).$$

References

[1] D. Cox, J. Little, D. OShea, *Using algebraic geometry*, Graduate Texts in Mathematics, vol. **185**, Springer, New York, 2005.

[2] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, *Singular 4-1-2 — A computer algebra system for polynomial computations*, http://www.singular.uni-kl.de, 2019.

[3] F. Hirzebruch, A. Borel, and RLE Schwarzenberger, *Topological methods in algebraic geometry*, Reprint of the 1978 ed., Springer, 1978.

[4] O. Iena, *On different approaches to compute the Chern classes of a tensor product of two vector bundles*, http://orbilu.uni.lu/handle/10993/27418, 2016.

[5] O. Iena, *CHERN.LIB: a SINGULAR library for symbolic computations with Chern classes*, http://hdl.handle.net/10993/21949, June 2015.

[6] A. Lascoux, *Classes de Chern d'un produit tensoriel*, CR Acad. Sci. Paris **286** (1978), 385–387.

[7] I.G. Macdonald, *Symmetric functions and Hall functions*, Second ed., Oxford University Press, Oxford, 1995.

[8] L. Manivel, *Chern classes of tensor products*, International Journal of Mathematics **27** (2016), no. 10, 1650079.

[9] L.W. Tu, R. Bott, *Differential forms in algebraic topology*, Springer-Verlag, 1982.

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On existence of fixed points and applications to a boundary value problem and a matrix equation in C^*-algebra valued partial metric spaces

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Abstract. We utilize Hardy-Rogers contraction and $\mathsf{CJM}-$contraction in a C^*-algebra valued partial metric space to create an environment to establish a fixed point.

Next, we present examples to elaborate on the novel space and validate our result. We conclude the paper by solving a boundary value problem and a matrix equation as applications of our main results which demonstrate the significance of our contraction and motivation for such investigations.

1 Introduction and preliminaries

Recently Chandok et al. [3] acquainted with the C^*-algebra valued partial metric combining the notions of partial metric (Matthews [12]) and C^*-algebra valued metric (Ma et al. [10]). Tomar and Joshi [17] pointed out, by giving explanatory examples that functions have different natures in different spaces

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and the consequences in C^*-algebra valued metric space can not be reduced to their metric counterparts unless unital C^*-algebra, $\mathbb{A} = \mathbb{R}$. Further, Tomar et al. [16] familiarised contractiveness and expansiveness in a newly introduced space to establish a fixed point and utilized these to solve an integral equation and an operator equation.

In the present work, we familiarize Hardy-Rogers contraction [6] and CJM$-$contraction [5]. The basic idea comprises utilizing the non-negative elements of an unital C^*-algebra ($\mathbb{A}$) as an alternative to a set of real numbers. Our outcomes are improvements and extensions of the existing results in metric spaces. Further, we provide illustrative examples to validate our result. Applications to a Boundary Value problem and a matrix equation conclude the paper.

Definition 1 [3] *A C^*-algebra valued partial metric is a function $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ on a non-empty set $\mathcal{M}$ if:*

(i) $\theta \preceq p(\mathfrak{w}, \mathfrak{v})$ *and* $p(\mathfrak{w}, \mathfrak{w}) = p(\mathfrak{v}, \mathfrak{v}) = p(\mathfrak{w}, \mathfrak{v})$ *if and only if* $\mathfrak{w} = \mathfrak{v}$, θ *is zero element of* $\mathbb{A}$;

(ii) $p(\mathfrak{w}, \mathfrak{w}) \preceq p(\mathfrak{w}, \mathfrak{v})$;

(iii) $p(\mathfrak{w}, \mathfrak{v}) = p(\mathfrak{v}, \mathfrak{w})$;

(iv) $p(\mathfrak{w}, \mathfrak{v}) \preceq p(\mathfrak{w}, \mathfrak{z}) + p(\mathfrak{z}, \mathfrak{v}) - p(\mathfrak{z}, \mathfrak{z})$, $\mathfrak{w}, \mathfrak{v}, \mathfrak{z} \in \mathcal{M}$.

Here, $(\mathcal{M}, \mathbb{A}, p)$ *is a* C^*-*algebra valued partial metric space.*
One may refer to [13] *and* [19], *to study in detail on* C^*-*algebra.*

The following example is given by Tomar et al. [16].

Example 1 *Let* $F(\mathcal{M})$ *be a collection of balls such that* $B(\mathfrak{w}_0, \rho) = \{\mathfrak{v} : |\mathfrak{w}_0 - \mathfrak{v}| \leq \rho,\ \rho > 0\}$ *and* $\mathbb{A} = M_n(\mathbb{C})$ *be the* C^*-*algebra of complex matrices. If* $A = [a_{ij}] \in \mathbb{A}$, *then* $A^* = [\bar{a_{ji}}]$ *is a non-zero element of* $\mathbb{A}$. *Norm is defined as,* $\|A\| = \sup\{\|A\alpha\|_2 : \alpha \in \mathbb{C}^n, \|\alpha\|_2 \leq 1\}$, *where* $\|.\|_2$ *is the usual* l^2-*norm on* $\mathbb{C}^n$. *Define* $p : F(\mathcal{M}) \times F(\mathcal{M}) \longrightarrow \mathbb{A}$ *by* $p[B(\mathfrak{w}_0, \rho), B(\mathfrak{v}_0, \sigma)] = |\mathfrak{w}_0 - \mathfrak{v}_0|AA^* + \max\{\rho, \sigma\}I$. *Then* p *is a* C^*-*algebra valued partial metric however, it is neither a* C^*-*algebra valued metric nor a standard partial metric, since* $p[B(\mathfrak{w}_0, \rho), B(\mathfrak{w}_0, \rho)] = \rho \neq \theta$ *and*

$$\begin{aligned}
p[B(\mathfrak{w}_0, \rho), B(\mathfrak{v}_0, \tau)] &= |\mathfrak{w}_0 - \mathfrak{v}_0|AA^* + \max\{\rho, \tau\}\, I \\
&\preceq [|\mathfrak{w}_0 - \mathfrak{z}_0| + |\mathfrak{z}_0 - \mathfrak{v}_0|]AA^* + [\max\{\rho, \sigma\} + \max\{\sigma, \tau\} - \sigma]I \\
&= p[B(\mathfrak{w}_0, \rho), B(\mathfrak{z}_0, \sigma)] + p[B(\mathfrak{z}_0, \sigma), B(\mathfrak{v}_0, \tau)] - p[B(\mathfrak{z}_0, \sigma), B(\mathfrak{z}_0, \sigma)].
\end{aligned}$$

The C^*-*algebra valued partial metric reduces to the standard partial metric on taking* $\mathbb{A} = \mathbb{R}$*. For detailed discussions on* C^*-*algebra-valued metric spaces, one may refer to Tomar and Joshi* [17]*. Tomar et al.* [16] *discussed the convergence of the sequence when it converges to a zero element of* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *and introduced the following definitions to create an environment to establish a fixed point in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$*.*

Definition 2 [16]

(i) *A sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *is called a Cauchy sequence in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *if* $\lim_{n,m\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_m)$ *exists with respect to* $\mathbb{A}$ *and is finite.*

(ii) $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is complete if every Cauchy sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *converges with respect to* $\mathbb{A}$ *in* $\mathcal{M}$*, to a point* $\mathfrak{w} \in \mathcal{M}$ *and satisfy*

$$\lim_{n,m\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_m) = \lim_{n\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_n) = \mathfrak{p}(\mathfrak{w}, \mathfrak{w}).$$

(iii) *The sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ $\theta-$*converges to a point* $\mathfrak{w} \in \mathcal{M}$ *if*

$$\lim_{n\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}) = \lim_{n\longrightarrow\infty} \mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_n) = \mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \theta.$$

(iv) *A sequence* $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ *is* $\theta-$*Cauchy if* $\lim_{n,m\rightarrow\infty} \mathfrak{p}(\mathfrak{w}_m, \mathfrak{w}_n) = \theta$*,* θ *is the zero element of* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$*.*

(v) $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is called* $\theta-$ *complete if every* $\theta-$*Cauchy sequence converges to a point* $\mathfrak{w} \in \mathcal{M}$ *and* $\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \theta$*.*

Example 2 (Example 3.5**—Tomar et al.** [16]**)** *Let*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{cases} I, & \text{if } \mathfrak{w} = \mathfrak{v} \\ \mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = 2I, & \text{otherwise.} \end{cases}$$

If $\mathcal{M}$ *is a Hausdorff space and* $B(\mathcal{M})$ *is the set of all bounded functions, then* $B(\mathcal{M})$ *becomes a* C^*-*algebra with* $\|f(\mathfrak{w})\| = \sup_{\mathfrak{w}\in\mathcal{M}} |f(\mathfrak{w})|$*. Here, the sequence* $\{\mathfrak{w}_n\} = \mathfrak{a}$*,* $n \geq 1$ *is not* $\theta-$*Cauchy as it converges to* $\mathfrak{a}$*. However,* $\{\mathfrak{w}_n\}$ *is a Cauchy sequence. Implying thereby that every* $\theta-$*Cauchy sequence in* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is a Cauchy sequence. However, the reverse implication is not necessarily true.*

Remark 1 [16] *It is worth mentioning here that if a sequence* $\theta-$*converges to some point then its self-distance, as well as the self-distance of that point, is equal to zero element of* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$*.*

2 Main results

In the following, $\mathbb{A}^+$ denotes a set of self-adjoint (positive) operators of $\mathbb{A}$. Now, following Ma et al. [10], we introduce a Hardy - Rogers contraction and a $\mathsf{CJM}-$contraction, then utilize these to establish a fixed point.

Definition 3 *A self-map $\mathcal{T}$ of $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is called a C^*-algebra valued Hardy-Roger contractive map if*

$$\mathfrak{p}(\mathcal{T}\mathfrak{w}, \mathcal{T}\nu) \preceq \mathcal{A}\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) + \mathcal{B}\mathfrak{p}(\mathfrak{w}, \mathcal{T}\mathfrak{w}) + \mathcal{C}\mathfrak{p}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathcal{D}\mathfrak{p}(\mathfrak{v}, \mathcal{T}\mathfrak{w}) + \mathcal{E}\mathfrak{p}(\mathfrak{w}, \mathcal{T}\mathfrak{v}), \quad (1)$$

$$\forall\ \mathfrak{w},\ \mathfrak{v} \in \mathcal{M}, \quad \|\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E}\| \leq 1 \text{ and } \mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D},\ \mathcal{E} \in \mathbb{A}^+.$$

Example 3 *Let $\mathcal{M} = \mathbb{C}$ and $\mathbb{A}$ =Collection of all scalar matrices on $\mathbb{C}$. Let $\mathfrak{p} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ be defined as,*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} \max\{|\mathfrak{w}|, |\mathfrak{v}|\} & 0 \\ 0 & \max\{|\mathfrak{w}|, |\mathfrak{v}|\} \end{bmatrix}.$$

So $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is a C^*-algebra valued partial metric space and

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} |\mathfrak{w}| & 0 \\ 0 & |\mathfrak{w}| \end{bmatrix} \neq \theta.$$

A function $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ defined as

$$\mathcal{T}\mathfrak{w} = \begin{cases} \frac{\mathfrak{w}}{4}, & \mathfrak{w} \text{ is even} \\ \frac{\mathfrak{w}-1}{5}, & \mathfrak{w} \text{ is odd} \\ 0, & \text{otherwise} \end{cases},$$

is a C^*-algebra valued Hardy-Roger contraction for $\theta \preceq \mathcal{A} = \mathcal{D} = \mathcal{E} \prec \frac{\mathrm{I}}{7}$ and $\theta \preceq \mathcal{B} = \mathcal{C} \prec \frac{\mathrm{I}}{9}$.

It is fascinating to see here that, $\mathcal{T}$ is not a Hardy-Roger contraction [6] as a space under consideration is not a standard metric space.

Definition 4 *A self map $\mathcal{T}$ in $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is called a C^*-algebra valued $\mathsf{CJM}-$contraction, if*

(a) *for each $\varepsilon \succ \theta$ there exists a number $\delta \succ \theta$ satisfying*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) \prec \varepsilon + \delta \Longrightarrow \mathfrak{p}(\mathcal{T}\mathfrak{w}, \mathcal{T}\mathfrak{v}) \prec \varepsilon,$$

(b) $\mathfrak{w} \neq \mathfrak{v} \Longrightarrow \mathfrak{p}(\mathcal{T}\mathfrak{w}, \mathcal{T}\mathfrak{v}) \prec \mathfrak{p}(\mathfrak{w}, \mathfrak{v}), \quad \mathfrak{w}, \mathfrak{v} \in \mathcal{M}.$

Example 4 *Let* $\mathcal{M} = \{0,\ 1\} \cup \{2n : n \in \mathbb{N}\} \cup \{\frac{2n-1}{2} + \frac{1}{2n-1} : n \in \mathbb{N}\}$ *and* $\mathbb{A}$ =*Collection of complex diagonal matrices defined on* $\mathcal{M}$. *Let* $\mathfrak{p} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ *be defined as,*
$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} |\mathfrak{w} - \mathfrak{v}| + \max\{\mathfrak{w}, \mathfrak{v}\}, & 0 \\ 0, & \alpha(|\mathfrak{w} - \mathfrak{v}| + \max\{\mathfrak{w}, \mathfrak{v}\}) \end{bmatrix}$. *So* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ *is a* C^*-*algebra valued partial metric space and* $\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} \mathfrak{w}, & 0 \\ 0, & \mathfrak{w} \end{bmatrix}$. *A function* $\mathcal{T} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}$ *be defined as* $\mathcal{T}\mathfrak{w} = \begin{cases} \frac{2n-1}{2} + \frac{1}{2n-1}, & \mathfrak{w} = 2n \\ 0, & \text{otherwise} \end{cases}$, *is a* C^*-*algebra valued* $CJM-$*contraction for* $\varepsilon,\ \delta > \theta$.

It is fascinating to see here that, $\mathcal{T}$ is not a $CJM-$ contraction [5] as a space under consideration is not a standard metric space.

Now, we establish our result for C^*-algebra valued Hardy-Rogers contraction.

Theorem 1 *If a self map* $\mathcal{T}$ *is a continuous* C^*-*algebra valued Hardy-Rogers contractive map* (1) *of a* $\theta-$*complete* C^*-*algebra valued partial metric space* $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$, *then* $\mathcal{T}$ *has a unique fixed point* $\mathfrak{z} \in \mathcal{M}$ *and* $\mathfrak{p}(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{z}) = \theta = \mathfrak{p}(\mathfrak{z}, \mathfrak{z})$.

Proof. Starting from the given element $\mathfrak{w}_0 \in \mathcal{M}$, form the sequence $\{\mathfrak{w}_n\}$, where $\mathfrak{w}_n = \mathcal{T}\mathfrak{w}_{n-1}, n \in \mathbb{N}$. If $\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = \theta$, for some $n \geq 0$, then $\mathcal{T}\mathfrak{w}_n = \mathfrak{w}_{n+1} = \mathfrak{w}_n$ and $\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_n) = \theta$ and this completes the proof.

Further, take $\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \succ \theta$, $n \geq 0$. For $\mathfrak{w} = \mathfrak{w}_{n+1}, \mathfrak{v} = \mathfrak{w}_{n+2}$, in condition (1),

$$\begin{aligned} \mathfrak{p}(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}) &= \mathfrak{p}(\mathcal{T}\mathfrak{w}_n, \mathcal{T}\mathfrak{w}_{n+1}) \\ &\preceq \mathcal{A}\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_{n+1}) + \mathcal{B}\mathfrak{p}(\mathfrak{w}_n, \mathcal{T}\mathfrak{w}_n) + \mathcal{C}\mathfrak{p}(\mathfrak{w}_{n+1}, \mathcal{T}\mathfrak{w}_{n+1}) \\ &\quad + \mathcal{D}\mathfrak{p}(\mathfrak{w}_{n+1}, \mathcal{T}\mathfrak{w}_n) + \mathcal{E}\mathfrak{p}(\mathfrak{w}_n, \mathcal{T}\mathfrak{w}_{n+1}) \\ &\preceq \mathcal{A}\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_{n+1}) + \mathcal{B}\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_{n+1}) + \mathcal{C}\mathfrak{p}(\mathfrak{w}_{n+1}, w_{n+2}) \\ &\quad + \mathcal{D}\mathfrak{p}(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+1}) + \mathcal{E}[\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_{n+1}) + \mathfrak{p}(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}) \\ &\quad - \mathfrak{p}(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+1})] \\ &= (\mathcal{A} + \mathcal{B} + \mathcal{E})\mathfrak{p}(\mathfrak{w}_n, \mathfrak{w}_{n+1}) + (\mathcal{C} + \mathcal{E})\mathfrak{p}(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}) \\ &\quad + (\mathcal{D} - \mathcal{E})\mathfrak{p}(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+1}), \end{aligned} \tag{2}$$

and

$$\begin{aligned}
p(\mathfrak{w}_{n+2}, \mathfrak{w}_{n+1}) &= p(\mathcal{T}\mathfrak{w}_{n+1}, \mathcal{T}\mathfrak{w}_n) \\
&\preceq \mathcal{A}p(\mathfrak{w}_{n+1}, \mathfrak{w}_n) + \mathcal{B}p(\mathfrak{w}_{n+1}, \mathcal{T}\mathfrak{w}_{n+1}) + \mathcal{C}p(\mathfrak{w}_n, \mathcal{T}\mathfrak{w}_n) \\
&\quad + \mathcal{D}p(\mathfrak{w}_{n,\mathcal{T}\mathfrak{w}_{n+1}}) + \mathcal{E}p(\mathfrak{w}_{n+1}, \mathcal{T}\mathfrak{w}_n) \\
&\preceq \mathcal{A}p(\mathfrak{w}_{n+1}, \mathfrak{w}_n) + \mathcal{B}p(\mathfrak{w}_{n+2}, \mathfrak{w}_{n+1}) + \mathcal{C}p(\mathfrak{w}_{n+1}, \mathfrak{w}_n) \\
&\quad + \mathcal{D}[p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) + p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}) - p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+1})] \\
&\quad + \mathcal{E}p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+1}) \\
&= (\mathcal{A} + \mathcal{C} + \mathcal{D})p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) + (\mathcal{B} + \mathcal{D})p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}) \\
&\quad + (\mathcal{E} - \mathcal{D})p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+1}).
\end{aligned} \tag{3}$$

Adding (2) and (3)

$$\begin{aligned}
2p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2})) &\preceq (2\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E})P(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \\
&\quad + (\mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E})p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}),
\end{aligned}$$

that is,

$$(2-\mathcal{B} - \mathcal{C} - \mathcal{D} - \mathcal{E})p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2})) \preceq (2\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E})p(\mathfrak{w}_n, \mathfrak{w}_{n+1}),$$

that is,

$$p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2})) \preceq \frac{2\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E}}{2 - \mathcal{B} - \mathcal{C} - \mathcal{D} - \mathcal{E}} p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \preceq \xi\, p(\mathfrak{w}_n, \mathfrak{w}_{n+1}),$$

where, $\xi = \frac{2\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}+\mathcal{E}}{2-\mathcal{B}-\mathcal{C}-\mathcal{D}-\mathcal{E}}$ and $0 \leq \|\ \xi\|\ < 1$.
Now, for $n > m$,

$$\begin{aligned}
p(\mathfrak{w}_n, \mathfrak{w}_m) &\preceq p(\mathfrak{w}_n, \mathfrak{w}_{n-1}) + p(\mathfrak{w}_{n-1}, \mathfrak{w}_{n-2}) + \ldots + p(\mathfrak{w}_{m+1}, \mathfrak{w}_m) \\
&\quad - p(\mathfrak{w}_{n-1}, \mathfrak{w}_{n-1}) - p(\mathfrak{w}_{n-2}, \mathfrak{w}_{n-2}) - \ldots - p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+1})p(\mathfrak{w}_n, \mathfrak{w}_m) \\
&\preceq p(\mathfrak{w}_n, \mathfrak{w}_{n-1}) + p(\mathfrak{w}_{n-1}, \mathfrak{w}_{n-2}) + \ldots + p(\mathfrak{w}_{m+1}, \mathfrak{w}_m) \\
&\preceq (\xi^{n-1} + \xi^{n-2} + \cdots + \xi^m)p(\mathfrak{w}_0, \mathfrak{w}_2),
\end{aligned}$$

and hence $\lim_{n,m\longrightarrow\infty} p(\mathfrak{w}_n, \mathfrak{w}_m) = \theta$, that is, $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{M}, \mathbb{A}, p)$.
Using $\theta-$completeness of $(\mathcal{M}, \mathbb{A}, p)$, we have $\mathfrak{z} \in \mathcal{M}$ so that $\mathfrak{w}_n \longrightarrow \mathfrak{z}$ in $(\mathcal{M}, \mathbb{A}, p)$ and $p(\mathfrak{z}, \mathfrak{z}) = \theta$.
Now,

$$\begin{aligned}
p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) &\preceq p(\mathfrak{z}, \mathfrak{w}_{n+1}) + p(\mathfrak{w}_{n+1}, \mathcal{T}\mathfrak{z}) - p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+1}) \\
&\preceq p(\mathfrak{z}, \mathfrak{w}_{n+1}) + p(\mathcal{T}\mathfrak{w}_n, \mathcal{T}\mathfrak{z}) - p(\mathfrak{w}_{n+1}, w_{n+1}).
\end{aligned}$$

Since $\mathcal{T}$ is continuous, $n \longrightarrow \infty$ implies that,

$$p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) \preceq (B + C + D + E)p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) \prec p(\mathfrak{z}, \mathcal{T}\mathfrak{z}),$$

a contradiction, so $p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) = \theta$.
Thus, $p(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{z}) = p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) = p(\mathfrak{z}, \mathfrak{z}) = \theta$, that is, $\mathfrak{z}$ is a fixed point of $\mathcal{T}$.
To conclude the theorem, suppose $\mathfrak{z}$ and $\mathfrak{w}$ are two different fixed points of $\mathcal{T}$, so

$$\begin{aligned} p(\mathfrak{z}, \mathfrak{w}) = p(\mathcal{T}\mathfrak{z}, \mathcal{T}w) &\preceq \mathcal{A}p(\mathfrak{z}, \mathfrak{w}) + \mathcal{B}p(\mathfrak{z}, \mathcal{T}\mathfrak{z}) + \mathcal{C}p(\mathfrak{w}, \mathcal{T}\mathfrak{w}) \\ &\quad + \mathcal{D}p(\mathfrak{w}, \mathcal{T}\mathfrak{z}) + \mathcal{E}p(\mathfrak{z}, \mathcal{T}\mathfrak{w}), \\ &\preceq (\mathcal{A} + \mathcal{D} + \mathcal{E})p(\mathfrak{z}, \mathfrak{w}) \\ &\prec (\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E})p(\mathfrak{z}, \mathfrak{w}) \\ &\prec p(\mathfrak{z}, \mathfrak{w}), \end{aligned}$$

a contradiction. So, $p(\mathfrak{z}, \mathfrak{w}) = \theta$. Hence, $\mathfrak{z} = \mathfrak{w}$. □

Next, an example is provided to validate Theorem 1.

Example 5 *Let $\mathcal{M} = \mathbb{C}$ and $\mathbb{A} = M_3(\mathbb{C})$ be the set of complex matrices. Let, for $a > b > c > 0$, $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ be defined as,*

$$p(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} af(\mathfrak{w}, \mathfrak{v}) & 0 & 0 \\ 0 & bf(\mathfrak{w}, \mathfrak{v}) & 0 \\ 0 & 0 & cf(\mathfrak{w}, \mathfrak{v}) \end{bmatrix},$$

where, $f(\mathfrak{w}, \mathfrak{v}) = \max\{\|\mathfrak{w}\|, \|\mathfrak{v}\|\}$. *So $(\mathcal{M}, \mathbb{A}, p)$ is a complete C^*-algebra valued partial metric space and*

$$p(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} a\|\mathfrak{w}\| & 0 & 0 \\ 0 & b\|\mathfrak{w}\| & 0 \\ 0 & 0 & c\|\mathfrak{w}\| \end{bmatrix} \neq \theta.$$

A continuous function $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ defined as $\mathcal{T}\mathfrak{w} = \frac{\mathfrak{w}}{2}$, is a C^*-algebra valued Hardy-Roger contraction for $\theta \preceq \mathcal{A} = \mathcal{B} = \mathcal{C} \prec \frac{I}{6}$, $\theta \preceq \mathcal{D} = \mathcal{E} \prec \frac{I}{8}$. Consequently, postulates of Theorem 1 are verified and $\mathcal{T}$ has a unique fixed point at $\mathfrak{w} = 0$.

Remark 2

(i) *Conclusion of Theorem 1 continues to be true if $\mathcal{B} = \mathcal{C} = \mathcal{D} = \mathcal{E} = 0$ and we get an extension of Banach [2], to C^*-algebra-valued partial metric spaces.*

(ii) *Conclusion of Theorem 1 continues to be true if* $\mathcal{A} = \mathcal{D} = \mathcal{E} = 0$ *and* $\mathsf{B} = \mathsf{C}$ *and we get an extension of Kannan* [8] *to* C^*-*algebra-valued partial metric spaces.*

(iii) *Conclusion of Theorem 1 continues to be true if* $\mathcal{A} = \mathcal{B} = \mathcal{C} = 0$ *and* $\mathcal{D} = \mathcal{E}$*, we get an extension of Chatterjea* [4] *to* C^*-*algebra-valued partial metric spaces.*

(iv) *Conclusion of Theorem 1 continues to be true if* $\mathcal{D} = \mathcal{E} = 0$*, we get an extension of Reich* [14] *to* C^*-*algebra-valued partial metric spaces.*

Now, we establish our next result for C^*-algebra valued $CJM-$contraction.

Theorem 2 *Theorem 1 continues to be true if* (1) *is replaced by* C^*-*algebra valued* CJM*-contractive map.*

Proof. Define a Picard sequence $\{\mathfrak{w}_n\} \subseteq \mathcal{M}$, $\mathfrak{w}_{n+1} = \mathcal{T}\mathfrak{w}_n$, $n \in \mathbb{N}_0$. If $p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = \theta$ for some $n \geq 0$, then $\mathcal{T}\mathfrak{w}_n = \mathfrak{w}_{n+1} = \mathfrak{w}_n$ and $p(\mathfrak{w}_n, \mathfrak{w}_n) = \theta$ and the proof is complete.
Now, let for all $n \in \mathbb{N}_0$, $p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \succ \theta$. Using (b), we get
$p(\mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}) = p(\mathcal{T}\mathfrak{w}_n, \mathcal{T}\mathfrak{w}_{n+1}) \prec p(\mathfrak{w}_n, \mathfrak{w}_{n+1})$,
that is, the sequence $\{p(\mathfrak{w}_n, \mathfrak{w}_{n+1})\}$ is bounded below and decreasing. Thus, it is convergent and
$\lim_{n\longrightarrow\infty} p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = \varepsilon \succeq \theta$. If $\varepsilon \succ \theta$, then $\varepsilon \prec p(\mathfrak{w}_n, \mathfrak{w}_{n+1})$, for $n \geq m$ or

$$\varepsilon \prec p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \prec \varepsilon + \delta(\varepsilon),\ n \geq m,$$

which contradicts condition (a). Thus, $\lim_{n\longrightarrow\infty} p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = \theta$.
Now, we demonstrate that $\{p(\mathfrak{w}_n, \mathfrak{w}_{n+1})\}$ is a Cauchy sequence. Fix an $\varepsilon \succ \theta$, we may consider $\delta = \delta(\varepsilon) \prec \varepsilon$. Since $\{p(\mathfrak{w}_n, \mathfrak{w}_{n+1})\}$ is monotonically decreasing to θ, there exists $m \in \mathbb{N}$, $n \geq m$ satisfying $p(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \prec \frac{\delta}{s}$.
We shall use the principle of mathematical induction to demonstrate that for $l \in \mathbb{N}$

$$p(\mathfrak{w}_m, \mathfrak{w}_{m+l}) \prec \frac{\varepsilon}{s} + \frac{\delta}{s} \prec \varepsilon + \delta. \tag{4}$$

Clearly, Equation (4) holds for $l = 1$. Suppose Equation (4) holds for some l. We shall prove it for $l + 1$. By the property (iv), we have

$$p(\mathfrak{w}_m, \mathfrak{w}_{m+l+1}) \preceq p(\mathfrak{w}_m, \mathfrak{w}_{m+1}) + p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+l+1}) - p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+1}).$$

It is enough to show that $p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+l+1}) \prec \frac{\varepsilon}{s}$. By the induction hypothesis, $p(\mathfrak{w}_m, \mathfrak{w}_{m+l}) \prec \frac{\varepsilon}{s} + \frac{\delta}{s} \prec \frac{\varepsilon}{s} + \delta$. So using (a), $p(\mathfrak{w}_{m+1}, \mathfrak{w}_{m+l+1}) \prec \frac{\varepsilon}{s}$.
Hence, Equation (4) implies that $\{w_n\}$ is a Cauchy sequence in $\mathcal{M}$.

Using $\theta-$completeness of $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$, there exists $\mathfrak{z} \in \mathcal{M}$ so that $\mathfrak{w}_n \longrightarrow \mathfrak{z}$ in $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ and $\mathfrak{p}(\mathfrak{z}, \mathfrak{z}) = \theta$.

Since $\mathcal{T}$ is continuous, $\mathfrak{w}_{n+1} = \mathcal{T}\mathfrak{w}_n \longrightarrow \mathcal{T}\mathfrak{z}$.

Hence, $\mathcal{T}\mathfrak{z} = \mathfrak{z}$, that is, $\mathfrak{z}$ is a fixed point of $\mathcal{T}$.

To conclude the proof, let $\mathfrak{z}$ and $\mathfrak{w}$ be two different fixed points of $\mathcal{T}$.

$$\mathfrak{p}(\mathfrak{z}, \mathfrak{w}) = \mathfrak{p}(\mathcal{T}\mathfrak{z}, \mathcal{T}\mathfrak{w}) \prec \mathfrak{p}(\mathfrak{z}, \mathfrak{w}),$$

a contradiction. So, $\mathfrak{p}(\mathfrak{z}, \mathfrak{w}) = \theta$.

Hence, $\mathfrak{z} = \mathfrak{w}$. □

Next, an example is provided to validate Theorem 2.

Example 6 *Let $\mathcal{M} = \mathbb{C}$ and $\mathbb{A} = M_2(\mathcal{M})$ be the set of complex matrices. Let, for $\alpha > 0$, $\mathfrak{p} : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{A}$ be,*

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{v}) = \begin{bmatrix} |\mathfrak{w} - \mathfrak{v}| + \max\{|\mathfrak{w}|,\ |\mathfrak{v}|\} & 0 \\ 0 & \alpha(|\mathfrak{w} - \mathfrak{v}| + \max\{|\mathfrak{w}|,\ |\mathfrak{v}|\}) \end{bmatrix}.$$

So, $(\mathcal{M}, \mathbb{A}, \mathfrak{p})$ is a complete C^*-algebra valued partial metric space and

$$\mathfrak{p}(\mathfrak{w}, \mathfrak{w}) = \begin{bmatrix} |\mathfrak{w}| & 0 \\ 0 & \alpha|w| \end{bmatrix} \neq \theta.$$

A continuous function $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ given by $\mathcal{T}\mathfrak{w} = \frac{\mathfrak{w}}{7}$, is a C^*-algebra valued $CJM-$contraction. Hence, all the postulates of Theorem 2 are verified and $\mathcal{T}$ has a unique fixed point at $\mathfrak{w} = 0$.

It is interesting to see that Examples 5 and 6 can not be covered by any function in a standard metric space, a partial metric space, or a C^*-algebra valued metric space in the context of Hardy and Roger [6] and Górnicki [5]. Consequently, C^*-algebra-valued partial metric space is an improved version of existing spaces wherein unital C^*-algebra $(\mathbb{A})$ is exploited as an alternative to a set of real numbers and the results in this space are genuine generalizations / improvements / extensions of the corresponding outcomes in the literature in standard metric spaces. Further, the results of C^*-algebra-valued partial metric spaces do not coincide with / derived from the results in other related spaces.

3 Application

Now, we utilize Theorem 1, to solve a boundary value problem.

Theorem 3 *Consider a boundary value problem*

$$\frac{d^2\mathfrak{w}}{dt^2} = -\phi(t, \mathfrak{w}(t)), \ t \in I = [-1, 1] \text{ and } \phi \in C(I, \mathbb{R}) \tag{5}$$

with two-point boundary condition $\mathfrak{w}(-1) = 0, \mathfrak{w}(1) = 0$.
Assume the following:

(i) $\phi : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a Lipschitz continuous relative to* $\mathfrak{w}$ *for Lipschitz constant value* $0 \leq \|\xi\| \leq \frac{1}{3}, \forall\ t \in I, \mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{R}$ such that $\|\phi(t, \mathfrak{w}_1) - \phi(t, \mathfrak{w}_2)\| \leq \xi(t)\|\mathfrak{w}_1 - \mathfrak{w}_2\|$ *and function* ξ *is continuous on* I.

(ii) $|\phi(t, \mathfrak{w})| \leq \mu(t)\ |\mathfrak{w}|$ *, where,* $0 \leq \|\mu\| \leq \frac{1}{3}$ *and function* μ *is continuous on* I.

Then, the differential equation has exactly one solution $\mathfrak{w}^* \in C(I, \mathbb{R})$.

Proof. The problem in equation (5) may be rewritten as

$$\mathfrak{w}(t) = \int_{-1}^{1} \mathcal{G}(t, u)\phi(u, \mathfrak{w}(u))du, \text{ for } t \in I, \tag{6}$$

and the Green function $\mathcal{G}(t, u) = \begin{cases} (1-t)(1+u), -1 \leq u \leq t \leq 1 \\ (1-u)(1+t), -1 \leq t \leq u \leq 1 \end{cases}$.
Now, if $\mathfrak{w} \in C^2(I, \mathbb{R})$, then $\mathfrak{w}$ is the solution of (5) if and only if it is the solution of (6).
$\mathcal{M} = C(I)$, the set of a continuous function on I forms a C^*-algebra with pointwise operation with $\|\mathfrak{w}\|_\infty = \max_{t\in I} |\mathfrak{w}|$, $\mathfrak{w} \in \mathcal{M}$.
Define $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ by $p(\mathfrak{w}, \mathfrak{v}) = [\|\mathfrak{w} - \mathfrak{v}\| + \|\mathfrak{w}\| + \|\mathfrak{v}\|]f$ is a C^*-algebra valued partial metric space, where, f is the self-adjoint element of $\mathcal{M}$.
Define a self map $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$\mathcal{T}\mathfrak{w}(t) = \int_{-1}^{1} \mathcal{G}(t, u)\phi(u, \mathfrak{w}(u))du, \tag{7}$$

for all $w \in \mathcal{M}$ and $t \in I$. Now, our problem (5) may be expressed as determining a fixed point of $\mathcal{T}$. So

$$\begin{aligned}|\mathcal{T}\mathfrak{w}(t)-\mathcal{T}\mathfrak{v}(t)| &= \left|\int_{-1}^{1}\mathcal{G}(t,u)(\phi(u,\mathfrak{w}(u))-\phi(u,\mathfrak{v}(u)))du\right|,\\ &\preceq \int_{-1}^{1}\mathcal{G}(t,u)\,|\phi(u,\mathfrak{w}(u))-\phi(u,\mathfrak{v}(u))|\,du,\\ &\preceq \int_{-1}^{1}\mathcal{G}(t,u)\xi\,|\mathfrak{w}(u)-\mathfrak{v}(u)|\,du\\ &\preceq \xi\,\|\mathfrak{w}(u)-\mathfrak{v}(u)\|_{\infty}\sup_{t\in I}\int_{-1}^{1}\mathcal{G}(t,u)du.\end{aligned}$$

Therefore,

$$\|\mathcal{T}\mathfrak{w}(t)-\mathcal{T}\mathfrak{v}(t)\| \leq \|\xi\|\,\|\mathfrak{w}(u)-\mathfrak{v}(u)\|_{\infty}. \tag{8}$$

Since, $\int_{-1}^{1}\mathcal{G}(t,u)du = 1-t^2$ and $\sup_{t\in I}\int_{-1}^{1}\mathcal{G}(t,u)du = 1$.
Now,

$$\begin{aligned}|\mathcal{T}\mathfrak{w}(t)| &= \left|\int_{-1}^{1}\mathcal{G}(t,u)\phi(u,\mathfrak{w}(u))du\right|,\\ &\preceq \int_{-1}^{1}\mathcal{G}(t,u)\,|\phi(u,\mathfrak{w}(u))|\,du,\\ &\preceq \int_{-1}^{1}\mu\,|\mathfrak{w}(u)|\,\mathcal{G}(t,u)du,\\ &\preceq \mu\,\|\mathfrak{w}\|_{\infty}\int_{-1}^{1}\mathcal{G}(t,u)du.\end{aligned}$$

Therefore,

$$\|\mathcal{T}\mathfrak{w}(t)\|_{\infty} \leq \|\mu\|\,\|\mathfrak{w}\|_{\infty}, \tag{9}$$

and also

$$\|\mathcal{T}\mathfrak{v}(t)\|_{\infty} \leq \|\mu\|\,\|\mathfrak{v}\|_{\infty}. \tag{10}$$

Now,

$$\begin{aligned}p(\mathcal{T}\mathfrak{w},\mathcal{T}\mathfrak{v}) &= [\|\mathcal{T}\mathfrak{w}-\mathcal{T}\mathfrak{v}\|_{\infty}+\|\mathcal{T}\mathfrak{w}\|_{\infty}+\|\mathcal{T}\mathfrak{v}\|_{\infty}]f\\ &\preceq [\xi\,\|\mathfrak{w}-\mathfrak{v}\|_{\infty}+\mu\,\|\mathfrak{w}\|_{\infty}+\mu\,\|\mathfrak{v}\|_{\infty}]f\\ &\preceq (\xi+2\mu)(\|\mathfrak{w}-\mathfrak{v}\|_{\infty}+\|\mathfrak{w}\|_{\infty}+\|\mathfrak{v}\|_{\infty}]f)\\ &= (\xi+2\mu)p(\mathfrak{w},\mathfrak{v}) \preceq \mathcal{A}p(\mathfrak{w},\mathfrak{v})+\mathcal{B}p(\mathfrak{w},\mathcal{T}\mathfrak{w})+\mathcal{C}p(\mathfrak{w},\mathcal{T}\mathfrak{v})\\ &\quad +\mathcal{D}p(\mathfrak{v},\mathcal{T}\mathfrak{w})+\mathcal{E}p(\mathfrak{w},\mathcal{T}\mathfrak{v}).\end{aligned}$$

Taking $\mathcal{A} = \xi$, $\mathcal{B} = \mathcal{C} = \mathcal{D} = \mathcal{E} = \frac{\mu}{2}$, we may observe that postulates of Theorem 1 are verified, and so $\mathcal{T}$ has only one fixed point $\mathfrak{w}^* \in \mathcal{M}$, that is, boundary value problem (5) has only one solution $\mathfrak{w}^* \in \mathcal{M}$. $\square$

Now, we make use of Theorem 2, to solve a matrix equation to demonstrate the applicability of C^*-algebra valued $CJM-$contraction map. In the following, the symbol $\|.\|$ is the spectral norm of a matrix $\mathcal{P} = [p_{ij}]_{n\times n}$, that is, $\|\mathcal{P}\| = \sqrt{\lambda^+(\mathcal{P}^*\mathcal{P})}$, $\lambda^+(\mathcal{P}^*\mathcal{P})$ is the largest eigenvalue of $\mathcal{P}^*\mathcal{P}$, where $\mathcal{P}^*$ is the conjugate transpose of $\mathcal{P}$. Further, $\|.\|_{tr}$ denotes the trace norm of $\mathcal{P}$ and $\|\mathcal{P}\|_{tr} = \sqrt{\Sigma_{i=1}^n \Sigma_{j=1}^n |p_{ij}|^2} = \sqrt{tr(\mathcal{P}^*\mathcal{P})} = \sqrt{\Sigma_{i=1}^n \sigma_i^2(\mathcal{P})}$, $\sigma_i(\mathcal{P})$, $i = 1, 2, \ldots, n$, denotes largest singular values of $\mathcal{P} \in M_n(\mathbb{C})$. The set of all Hermitian matrices of order n, $H_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$, induced by this trace norm, is a Banach space.

Theorem 4 *Let a non-linear matrix equation be*

$$\mathcal{W} = \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W}) \mathcal{P}_i, \tag{11}$$

where, the C^*-*algebra of complex matrices of order* n, $\mathcal{M} = M_n(\mathbb{C})$, $\mathcal{P}_i \in M_n(\mathbb{C})$ *is an arbitrary matrix of order* n. *Let* $f : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ *be a continuous self map satisfying* $f(\theta) = \theta$ *and*

(i) $\max\{|tr(f\mathcal{W})|,\ |tr(f\mathcal{V})|\}I \preceq \frac{\eta}{2} \max\{|tr(\mathcal{W})|,\ |tr(\mathcal{V})|\}I_n$,

(ii) $|tr(\mathcal{T}\mathcal{W} - \mathcal{T}\mathcal{V})|I_n \preceq \frac{\eta}{2}|tr(\mathcal{W} - \mathcal{V})|I_n$,

(iii) $tr(\mathcal{W}\mathcal{V}) \leq \|\mathcal{W}\| tr(\mathcal{V})$, $\mathcal{W} \in M_n(\mathbb{C})$,

(iv) $\Sigma_{i=1}^n \mathcal{P}_i^* \mathcal{P} \preceq \xi I_n$, *where identity matrix of order* n, $I_n \in M_n(\mathbb{C})$ *and* $\eta \neq 0$.

Then the matrix equation (11) *has exactly one solution* $\mathcal{W}^* \in \mathcal{M}$. *Further, the iteration* $\mathcal{W}_n = \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W}) \mathcal{P}_i$, $\mathcal{W}_0 \in M_n(\mathbb{C})$ *such that* $\mathcal{W}_0 \preceq \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W}) \mathcal{P}_i$, *converges to* $\mathcal{W}^* \in \mathcal{M}$ *satisfying the nonlinear matrix equation* (11).

Proof. Let a map $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ be defined as

$$\mathcal{T}(\mathcal{W}) = \Sigma_{i=1}^n \mathcal{P}_i^* f(\mathcal{W}) \mathcal{P}_i \tag{12}$$

and a C^*-algebra valued partial metric $p : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ be

$$p(\mathcal{W}, \mathcal{V}) = \left[\max\{|tr\mathcal{W}|, |tr\mathcal{V}|\} + |tr(\mathcal{W} - \mathcal{V})|\right] I_n.$$

Noticeably, a fixed point of $\mathcal{T}$ is a solution of a matrix equation (11).

$$\begin{aligned}
p(\mathcal{T}\mathcal{W},\mathcal{T}\mathcal{V}) &= [\max\{tr|\mathcal{T}\mathcal{W}|, tr|\mathcal{T}\mathcal{V}|\} + |tr(\mathcal{T}\mathcal{W} - \mathcal{T}\mathcal{V})|]I_n \\
&= \Big[\max\{|tr(\Sigma_{i=1}^{n}\mathcal{P}_i^* f(\mathcal{W})\mathcal{P}_i)|,\ |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^* f(\mathcal{V})\mathcal{P}_i)|\} \\
&\quad + |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^*(f(\mathcal{W}) - f(\mathcal{V})\mathcal{P}_i))|\Big] I_n \\
&= \Big[\max\{|tr(\Sigma_{i=1}^{n}\mathcal{P}_i^*\mathcal{P}_i f(\mathcal{W}))|,\ |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^*\mathcal{P}_i f(\mathcal{V}))|\} \\
&\quad + |tr(\Sigma_{i=1}^{n}\mathcal{P}_i^*\mathcal{P}_i f(\mathcal{W}) - f(\mathcal{V}))|\Big] I_n \\
&\preceq \|\Sigma_{i=1}^{n}\mathcal{P}_i^*\mathcal{P}_i\| \Big[\max\{|tr(f\mathcal{W})|,\ |tr(f\mathcal{V})|\} + |f\mathcal{W} - f\mathcal{V}|\Big] I_n \\
&\preceq \|\eta I\| \frac{1}{2\eta}[\max\{|tr(\mathcal{W})|,\ |tr(\mathcal{V})|\} + |tr(f\mathcal{W} - f\mathcal{V})|]I_n \\
&= \frac{1}{2}\Big[\max\{|tr(\mathcal{W})|,\ |tr(\mathcal{V})|\} + |tr(f\mathcal{W} - f\mathcal{V})|\Big] I_n \\
&\prec p(\mathcal{W},\mathcal{V}).
\end{aligned}$$

Taking $\varepsilon = \frac{1}{2}\big[\max\{|tr(\mathcal{W})|,\ |tr(\mathcal{V})|\} + |tr(f\mathcal{W} - f\mathcal{V})|\big] I_n$ and $\delta = \frac{3}{2}\varepsilon$, $p(\mathcal{W},\mathcal{V}) \prec \varepsilon + \delta \implies p(\mathcal{T}\mathcal{W},\mathcal{T}\mathcal{V}) \prec \varepsilon$ and $\mathcal{W} \neq \mathcal{V} \implies p(\mathcal{T}\mathcal{W},\mathcal{T}\mathcal{V}) \prec p(\mathcal{W},\mathcal{V})$.

We may observe that postulates of Theorem 2 are verified, and $\mathcal{T}$ has only one fixed point $\mathcal{W}^* \in \mathcal{M}$, that is, matrix equation (11) has only one solution $\mathcal{W}^* \in \mathcal{M}$. □

4 Conclusion

Acknowledging the C^*-algebra valued partial metric space, we have familiarized Hardy-Roger contraction [6] and CJM$-$contraction [5] in it to elicit the fixed point theorems in the most generalized environment. From our results, we have deduced results for a C^*-algebra valued variants of Kannan contraction[8], Chatterjee contraction [4], Reich contraction [14] and Banach contraction [2]. Further, we have solved a boundary value problem using C^*-algebra valued Hardy-Roger contraction and a matrix equation using C^*-algebra valued CJM$-$ contraction. The motivation behind using this space is its application in quantum field theory and statistical mechanics. It is worth to mention that there may be some circumstances when it is possible to apply C^*-algebra valued partial metric results, however it is not possible to apply standard metric results . These novel ideas promote further examinations and applications.

References

[1] Hamed H Alsulami, Ravi P Agarwal, E. Karapınar and F. Khojasteh, A short note on C*−valued contraction mappings, *J. Inequal. Appl.*, **2016**(1) (2016), 1–3.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux équation intégrales, *Fund. Math.*, **3** (1922), 133–181.

[3] S. Chandok, D. Kumar and C. Park, C*−algebra valued partial metric space and fixed point theorems, *J. Ind. Acad. Math.*, **129**(3) (2019), 1–9.

[4] S. K. Chatterjee, Fixed point theorem, *C. R. Acad., Bulgar Sci.*, **25** (1972), 727–730.

[5] J. Górnicki, Remarks on contractive type mappings, *Fixed Point Theory Appl.*, **2017**(1) (2016), 1–12.

[6] G. E. Hardy and T. D. Roger, A generalization of a fixed point theorem of Reich, *Cand. Math. Bull.*, **16**(2), (1973), 201–206.

[7] Z. Kadelburg and S. Radenovi, Fixed point results in C*−algebra-valued metric spaces are direct consequences of their standard metric counterparts, *Fixed Point Theory Appl.*, **2016**(1) (2016), 1–6.

[8] R. Kannan, Some results on fixed points-II, *Amer. Math. Monthly*, **76**(4) (1969), 405–408.

[9] Xiaoyan Lv and Yuqiang Feng, Some fixed point theorem for Reich type contraction in generalized metric spaces, *J. Math. Anal. Appl.*, **9**(5) (2018), 80–88.

[10] Z. Ma, L. Jiang and H. Sun, C*−algebra-valued metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **206** (2014).

[11] Z. Ma and L. Jiang, C*−Algebra-valued b−metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **222**, (2015).

[12] S. G. Matthews, Partial metric topology, General Topology and its Applications, Proc. 8th Summer Conf., Queens College, 1992. *Ann. New York Acad. Sci.*, **728** (1994), 183–197.

[13] G. J. Murphy, C*−*algebra and operator theory*, Academic Press, London, (1990).

[14] S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14**, (1971), 121-124.

[15] T. Senapati and L. Kanta Dey, Remarks on common fixed point results in C*-algebra-valued metric spaces, *J. Inform. Math. Sci.*, **10**(1 & 2) (2018), 333–337.

[16] A. Tomar, M. Joshi, and A. Deep, Fixed points and its applications in C^*-algebra-valued partial metric space, *TWMS J. App. and Eng. Math.*, **11**(2) (2021), 329–340.

[17] A. Tomar and M. Joshi, Note on results in C^*-algebra-valued metric spaces, *Electron. J. Math. Analysis Appl.*, **9**(2) (2021), 262–264. https://doi.org/10.1007/s41478-019-00204-frac.org/Journals/EJMAA/

[18] Q. Xin, L. Jiang and Z. Ma, Common fixed point theorems in C^*-algebra-valued metric spaces, J. *Nonlinear Sci. Appl.*, **9**(9) (2016), 4617–4627.

[19] Q. H. Xu, T. E. D. Bieke, and Z. Q. Chen, Introduction to operator algebras and non commutative $Lp-$spaces, Science Press, Beijing, (2010) (In Chinese).

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