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Fixed point theorems and equivalence results for classes of multivalued mappings in modular function spaces

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Abstract. We contribute to the development of equivalence of fixed point iterative sequences for multivalued mappings in modular function spaces, by proving the equivalence of convergence of implicit Mann, implicit Ishikawa, implicit Noor, implicit multistep iterative sequences for multivalued $\rho-$ quasi-contractive-like mapppings in modular function spaces. An example is provided to support the applicability of the results. This work is complementary to equivalence results on normed and metric spaces in the literature.

1 Introduction and preliminaries

The existence and approximation of fixed points for multivalued mappings in modular function spaces abound in the literature. Some of the notable authors

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whose research work are very important in this study are ([8], [9], [10], [11], [12], [14], [15] and [16]).

Let Ω be a nonempty set and $\sum$ be a nontrival $\sigma-$algebra of subsets of Ω. Let P be a $\delta-$ring of subsets of Ω such that $E \cap A \in P$ for any $E \in P$ and $A \in \sum$. Assume there exists an increasing sequence $K_n \in P$ such that $\Omega = \bigcup K_n$. Let I_A represent the characteristic function of the set A in Ω. Let ϵ represent the linear space of all simple functions with supports from P. Let M_∞ represent the space of all extended measurable functions, that is, all functions $f : \Omega \to [-\infty, \infty]$ such that there exist a sequence $\{g_n\} \subset \epsilon, |g_n| \leq |f|$ and $g(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Definition 1 [16] *Let $\rho : M_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if*

(1) $\rho(0) = 0$;

(2) ρ *is monotone, that is,* $|f(\omega)| \leq g|(\omega)|$ *for any* $\omega \in \Omega$ *implies* $\rho(f) \leq \rho(g)$, *where* $f, g \in M_\infty$;

(3) ρ *is orthogonally subadditive, that is,* $\rho(fI_{A\cup B}) \leq \rho(fI_A) + \rho(fI_B)$ *for any* $A, B \in \sum$ *such that* $A \cap B \neq \phi$, $f \in M_\infty$;

(4) ρ *has Fatou property, that is,* $|f_n(\omega)| \uparrow |f(\omega)|$ *for all* $\omega \in \Omega$ *implies* $\rho(f_n) \uparrow \rho(f)$, *where* $f \in M_\infty$;

(5) ρ *is order continuous in* ϵ, *that is,* $g_n \in \epsilon$ *and* $|g_n(\omega)| \downarrow 0$ *for all* $\omega \in \Omega$ *implies* $\rho(g_n) \downarrow 0$.

Definition 2 [8]. *Let ρ be a regular function pseudomodular;*

(a) *we say that ρ is a regular convex function modular if* $\rho(f) = 0$ *implies* $f = 0$ ρ *-a.e.*

(b) *we say that ρ is a regular convex function semimodular if* $\rho(\alpha f) = 0$ *for every* $\alpha > 0$ *implies* $f = 0$ ρ *-a.e.*

ρ *also satisfies the following properties* [10]*:*

(1) $\rho(0) = 0$ *iff* $f = 0$ ρ *-a.e.*

(2) $\rho(\alpha f) = \rho(f)$ *for every scalar* α *with* $|\alpha| = 1$ *and* $f \in M$.

(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ *if* $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ *and* $f, g \in M$, ρ *is called a convex modular if, in addition, the following property is satisfied:*

(4) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ *if* $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ *and* $f, g \in M$.

The class of all nonzero regular convex function modulars on Ω *is denoted by* $\mathfrak{R}$.

Definition 3 [16]. *The modular function space* L_ρ *is defined as:* $L_\rho = \{f \in M : \rho(\lambda f) \to 0$ *as* $\lambda \to 0\}$.

In general terms, the modular ρ *is not subadditive and therefore does not behave as a norm or a distance. Nevertheless, the modular space* L_ρ *can be furnished with an* $F-$*norm defined thus:*

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\Big(\frac{f}{\alpha}\Big) \le \alpha\right\}.$$

In the instance ρ *is convex modular,* $\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \le 1\}$ *defines a norm on the modular space* L_ρ. *This type of norm is known as the Luxemburg norm.*

Definition 4 [16]. *A nonzero regular convex function* ρ *is said to satisfy the* Δ_2- *condition, if* $\sup_{n\ge1}\rho(2f_n, D_k) \to 0$ *as* $k \to \infty$ *whenever* $\{D_k\}$ *decreases to* $\emptyset$ *and* $sup_{n\ge1}\rho(f_n, D_k) \to 0$ *as* $k \to \infty$. *If* ρ *is convex and satisfies* Δ_2-*condition, then* $L_\rho = E_\rho$.

Definition 5 [16] *Let* ρ *be a nonzero regular convex function modular defined on* Ω.

(i) *Let* $r > 0$, $\epsilon > 0$. *Define* $D_1(r,\epsilon) = \{(f,g) : f,g \in L_\rho, \rho(f), \rho(g) \le r, \rho(f-g) \ge \epsilon r\}$. *Suppose,* $\delta_1(r,\epsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f,g) \in D_1(r,\epsilon)\}$, *if* $D_1(r,\epsilon) \ne \emptyset$ *and* $\delta_1(r,\epsilon) = 1$ *if* $D_1(r,\epsilon) = \emptyset$. *We say that* ρ *satisfies* $(UC1)$, *if for every* $r > 0$, $\epsilon > 0$, $\delta_1(r,\epsilon) > 0$. *Observe that for every* $r > 0$, $D_1(r,\epsilon) \ne \emptyset$, $\epsilon > 0$ *small enough.*

(ii) *We say that* ρ *satisfies* $(UUC1)$, *if for every* $s \ge 0$, $\epsilon > 0$, *there exists* $\eta_1(s,\epsilon) > 0$ *depending only on* s *and* ϵ *such that* $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ *for any* $r > s$.

(iii) *Let* $r > 0$, $\epsilon > 0$. *Define* $D_2(r,\epsilon) = \{(f,g) : f,g \in L_\rho, \rho(f), \rho(g) \le r, \rho(\frac{f-g}{2}) \ge \epsilon r\}$. *Suppose,* $\delta_2(r,\epsilon) = inf\,\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f,g) \in D_2(r,\epsilon)\}$ *if* $D_2(r,\epsilon) \ne \emptyset$ *and* $\delta_2(r,\epsilon) = 1$ *if* $D_2(r,\epsilon) = \emptyset$. *We say that* ρ *satisfies* $(UC2)$, *if for every* $r > 0$, $\epsilon > 0$, $\delta_2(r,\epsilon) > 0$. *Observe that for every* $r > 0$, $D_2(r,\epsilon) \ne \emptyset$, $\epsilon > 0$ *small enough.*

(iv) *We say that* ρ *satisfies* $(UUC2)$, *if for every* $s \ge 0$, $\epsilon > 0$, *there exists* $\eta_2(s,\epsilon) > 0$ *depending only on* s *and* ϵ *such that* $\delta_2(r,\epsilon) > \eta_2(s,\epsilon) > 0$ *for any* $r > s$.

(v) *We say that* ρ *is strictly convex* (SC), *if for every* $f, g \in L_p$ *such that* $\rho(f) = \rho(g)$ *and* $\rho(\frac{f+g}{2}) = \frac{\rho(f)+\rho(g)}{2}$, *there holds* $f = g$.

Definition 6 [16]. *Let* L_ρ *be a modular space. The sequence* $\{f_n\} \in L_\rho$ *is called:*

(1) $\rho-$*convergent to* $f \in L_\rho$, *if* $\rho(f_n - f) \to 0$ *as* $n \to \infty$*;*

(2) $\rho-$*Cauchy, if* $\rho(f_n - f_m) \to 0$ *as* $n, m \to \infty$.

Remark 1 $\rho-$*convergent sequence implies* $\rho-$*Cauchy sequence, if and only if* $\rho-$ *satisfies the* Δ_2- *condition. However,* ρ *does not satisfy the triangle inequality.*

Definition 7 [16]. *Let* L_ρ *be a modular space. A subset* $D \subset L_\rho$ *is called:*

(1) $\rho-$*closed, if the* $\rho-$*limit of a* $\rho-$*convergent sequence of* D *always belongs to* D;

(2) $\rho-$*a.e. closed, if the* $\rho-$*a.e. limit of a* $\rho-$*a.e. convergent sequence of* D *always belongs to* D;

(3) $\rho-$*compact, if every sequence in* D *has a* $\rho-$*convergent subsequence in* D;

(4) $\rho-$*a.e. compact, if every sequence in* D *has a* $\rho-$*a.e. convergent subsequence in* D;

Definition 8 [16]. *Let* L_ρ *be a modular space. A function* $f \in L_\rho$ *is called a fixed point of a multivalued mapping* $T : L_\rho \to P_\rho(D)$ *if* $f \in Tf$. *The set of all fixed points of* T *is represented by* $F_\rho(T)$.

The following contractive defintions are useful in stating our definitions in terms of functions in modular function spaces. In 1972, Zamfirescu [22] proved a remarkable generalization of the Banach fixed point theorem by employing the following quasi-contractive mapping:

$$d(Tx, Ty) \leq h \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \quad (1)$$

where $0 \leq h < 1$. In a Normed linear space setting, condition (1) implies

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|, \quad (2)$$

where $0 \leq \delta < 1$ and $\delta = \max\left\{h, \frac{h}{2-h}\right\}$.

In [18], the following contractive definition was used. Let X be a Banach space, for each $x, y \in X$, there exists $\delta \in [0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\|. \tag{3}$$

In [13], the following contractive definition was employed in proving stability results. Let X be a Banach space, for each $x, y \in X$, there exist $\delta \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|). \tag{4}$$

It is important to remark that contractive condition (4) is a generalization of (3) and (2) for single valued map T.

The modified versions of contractive conditions (2)–(4), is hereby presented in a modular function space as follows.

Let L_ρ be a modular space. A set $D \subset L_\rho$ is called $\rho-$proximinal if for each $f \in L_\rho$ there exists an element $g \in D$ such that $\rho(f-g) = \mathtt{dist}_\rho(f, D)$. We represent the family of nonempty $\rho-$bounded $\rho-$proximinal subsets of D by $P_\rho(D)$, the family of nonempty $\rho-$closed $\rho-$bounded subsets of D by $C_\rho(D)$ and the family of $\rho-$compact subsets of D by $K_\rho(D)$. Let $H_\rho(.,.)$ be the $\rho-$Hausdorff distance on $C_\rho(L_\rho)$, that is, $H_\rho(A, B) = \max\{\sup_{f\in A} \mathrm{dist}_\rho(f, B), \sup_{g\in B} \mathrm{dist}_\rho(g, A)\}$, $A, B \in C_\rho(L_\rho)$.

A multivalued map $T : D \to C_\rho(L_\rho)$ is said to be:

(1) $\rho-$contraction mapping, if there exists a constant $\delta \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g), \text{ for all } f, g \in D. \tag{5}$$

(2) $\rho-$Zamfirescu mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + 2\delta\rho(Tf - f), \text{ for all } f, g \in D. \tag{6}$$

(3) $\rho-$quasi- contractive mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + L\rho(Tf - f), \text{ for all } f, g \in D, L \geq 0. \tag{7}$$

(4) $\rho-$quasi-contractive-like mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + \varphi_\rho(\rho(Tf - f)), \text{ for all } f, g \in D. \tag{8}$$

where $\varphi_\rho : R^+ \to R^+$ is a $\rho-$monotone increasing function with $\varphi_\rho(0) = 0$.

Implicit iterations exist in literature and have been proved to have advantage over explicit iterations for nonlinear problems as they provide better approximation of fixed points, and are widely used in many applications, when explicit iterations are ineficient. Approximation of fixed points in computer oriented programs using implicit iterations can reduce the computational cost of the fixed point problems (see [7]). The following implicit iterative sequences in the framework of modular function spaces are hereby presented:

Let L_ρ be a modular space, $D \subset L_\rho$ and $T : D \to P_\rho(D)$ be a multivalued mapping, then the implicit multistep iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n) f_n^1 + \alpha_n u_{n+1}, \\ f_n^i = (1-\beta_n^i) f_n^{i+1} + \beta_n^i u_n^i, \ \ i = 1, 2, ..., k-2 \\ f_n^{k-1} = (1-\beta_n^{k-1}) f_n + \beta_n^{k-1} u_n^{k-1}, \ \ n = 0, 1, 2, ..., \end{cases} \tag{9}$$

where $u_n \in P_\rho^T(f_n)$, $u_n^i \in P_\rho^T(f_n^i)$, $u_n^{k-1} \in P_\rho^T(f_n^{k-1})$, the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0,1)(i = 1, 2, ..., k-1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Noor iterative sequence $\{g_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} g_0 \in D \\ g_{n+1} = (1-\alpha_n) g_n^1 + \alpha_n v_{n+1}, \\ g_n^1 = (1-\beta_n^1) g_n^2 + \beta_n^1 v_n^1, \\ g_n^2 = (1-\beta_n^2) g_n + \beta_n^2 v_n^2, \ \ n = 0, 1, 2, ..., \end{cases} \tag{10}$$

where $v_{n+1} \in P_\rho^T(g_{n+1})$, $v_n^1 \in P_\rho^T(g_n)^1$, $v_n^2 \in P_\rho^T(g_n^2)$, the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0,1)$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Ishikawa iterative sequence $\{h_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} h_0 \in D \\ h_{n+1} = (1-\alpha_n) h_n^1 + \alpha_n s_{n+1}, \\ h_n^1 = (1-\beta_n^1) h_n + \beta_n^1 s_n^1, \ \ n = 0, 1, 2, ..., \end{cases} \tag{11}$$

where $s_{n+1} \in P_\rho^T(h_{n+1})$, $s_n^1 \in P_\rho^T(h_n^1)$, the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty} \subset (0,1)$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Mann iterative sequence $\{g_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} g_0 \in D \\ g_{n+1} = (1-\alpha_n) g_n + \alpha_n v_{n+1}, \ \ n = 0, 1, 2, ..., \end{cases} \tag{12}$$

where $v_{n+1} \in P_\rho^T(g_{n+1})$, the sequence $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The following Lemmas will be needed in proving the main results.

Lemma 1 [10]. *Let* $T : D \to P_\rho(D)$ *be a multivalued mapping and* $P_\rho^T(f) = \{g \in Tf : \rho(f-g) = dist_\rho(f, Tf)\}$. *Then the following are equivalent:*

(1) $f \in F_\rho(T)$, *that is,* $f \in Tf$.

(2) $P_\rho^T(f) = \{f\}$, *that is,* $f = g$ *for each* $g \in P_\rho^T(f)$.

(3) $f \in F(P_\rho^T(f))$, *that is,* $f \in P_\rho^T(f)$. *Further* $F_\rho(T) = F(P_\rho^T(f))$ *where* $F(P_\rho^T(f))$ *represent the set of fixed points of* $P_\rho^T(f)$.

Lemma 2 [6]. *Let* δ *be a real number satisfying* $0 \leq \delta < 1$ *and* $\{\epsilon_n\}_{n=0}^{\infty}$ *a sequence of positive numbers such that* $\lim_{n\to\infty}\epsilon_n = 0$, *then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying* $u_{n+1} \leq \delta u_n + \epsilon_n$, *n=0,1,2,..., we have* $\lim_{n\to\infty} u_n = 0$.

Laudable papers have written by notable researchers on the convergence and the equivalence of convergence of various iterative sequences for single mapping T *on normed and metric spaces. That is, different authors have shown that the convergence of any of the iterative method to the unique fixed point of the contractive operator for single map* T *is equivalent to the convergence of the other iterative sequences. For a look at some of the fine works in this direction, see references:* [1], [2], [5], [6], [7], [19],[20], [21], and [23]. Some results also appear for pair of maps, for example, (see [3], [4], and [17] for details). The new version of equivalence results will now be proved for multivalued $\rho-$quasi-contractive-like mappings in modular function spaces in the following theorems.

2 Main results

Theorem 1 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive-like mapping, satisfying the contractive condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f-g) + \varphi_\rho(\rho(Tf-f)), \tag{13}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$, *where* $\delta \in [0,1)$ *and* $\varphi_\rho : R^+ \to R^+$ *is a* $\rho-$*monotone increasing function with* $\varphi_\rho(0) = 0$. *Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann* (12) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0,1)$ *such that* $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\sum_{n=0}^{\infty}\beta_n^i = \infty$, *for* $i = 1, 2, ..., k-1$. *Then the following are equivalent:*

(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;

(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. Let $p \in F_\rho(T)$, from Lemma 1, $P_\rho^T(p) = \{p\}$ and $F_\rho(T) = F(P_\rho^T)$.

We prove that (i) $\Rightarrow$ (ii). Assume $\lim_{n\to\infty} g_n = p$. Using $\rho-$quasi-contractive-like condition (13), implicit Mann (12) and implicit multistep iterative sequences (9), we obtain the following:

$$\rho(g_{n+1} - f_{n+1}) = \rho[(1-\alpha_n)(g_n - f_n^1) + \alpha_n(v_{n+1} - u_{n+1})]. \tag{14}$$

Using the convexity of ρ in equation (14), we have

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) &= (1-\alpha_n)\rho(g_n - f_n^1) + \alpha_n\rho(v_{n+1} - u_{n+1}) \\ &\leq (1-\alpha_n)\rho(g_n - f_n^1) + \alpha_n(H_\rho(P_\rho^T(g_{n+1}), P_\rho^T(f_{n+1}))).\end{aligned} \tag{15}$$

Using (13), let $f = f_{n+1}$, $g = g_{n+1}$, then, from (15), we get the following:

$$H_\rho(P_\rho^T(g_{n+1}), P_\rho^T(f_{n+1})) \leq \delta\rho(g_{n+1} - f_{n+1}) + (1+\delta)\varphi_\rho(\rho(g_{n+1} - p)). \tag{16}$$

Substituting inequality (16) in inequality (15), we obtain

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) \;\; \leq \;\; & (1-\alpha_n)\rho(g_n - f_n^1) + \delta\alpha_n\rho(g_{n+1} - f_{n+1}) + \\ & (1+\delta)\alpha_n\varphi_\rho(\rho(g_{n+1} - p)).\end{aligned}$$

That is,

$$\rho(g_{n+1} - f_{n+1}) \;\leq\; \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\rho(g_n - f_n^1) + \left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(g_{n+1} - p)). \tag{17}$$

$$\rho(g_n - f_n^1) \;=\; \rho(g_n - ((1-\beta_n^1)f_n^2 + \beta_n^1 u_n^1)). \tag{18}$$

Using the convexity of ρ in equation (18), we have

$$\begin{aligned}\rho(g_n - f_n^1) &\leq (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - u_n^1) \\ &\leq (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - v_n) + \beta_n^1\rho(v_n - u_n^1) \\ &\leq (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - p) \\ &\quad + \beta_n^1\rho(v_n - p) + \beta_n^1\rho(v_n - u_n^1) \\ &\leq (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - p) + \beta_n^1 H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \\ &\quad + \beta_n^1 H_\rho(P_\rho^T(g_n), P_\rho^T(f_n^1)).\end{aligned} \tag{19}$$

Using (13), let $f = p$, $g = g_n$, and also let $f = g_n$, $g = f_n^1$, then, from (19), we get the following:

$$\begin{aligned}\rho(g_n - f_n^1) \leq &\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - f_n^2) + \left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - p)\\ &+\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{20}$$

Substituting inequality (20) in inequality (17), we obtain

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) \leq &\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - f_n^2)\\ &+\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - p)\\ &+\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(g_n - p))\\ &+\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(g_{n+1} - p)).\end{aligned} \tag{21}$$

Similarly, an application of (13) and (9) and (12) give the following

$$\begin{aligned}\rho(g_n - f_n^2) \leq &\left(\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right)\rho(g_n - f_n^3) + \left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\rho(g_n - p)\\ &+\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{22}$$

$$\begin{aligned}\rho(g_n - f_n^3) \leq &\left(\frac{1-\beta_n^3}{1-\delta\beta_n^3}\right)\rho(g_n - f_n^4) + \left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3})\rho(g_n - p\right)\\ &+\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{23}$$

$$\vdots$$

$$\begin{aligned}\rho(g_n - f_n^{k-2}) \leq &\left(\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(g_n - f_n^{k-1}) + \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(g_n - p)\\ &+\left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{24}$$

$$\rho(g_n - f_n^{k-1}) \leq \left(\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - f_n) + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - p) + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(g_n - p)). \tag{25}$$

Substituting inequalities (25), (24), (23), (22) in inequality (21) inductively and simplifying, we obtain

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) \leq & \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{1-\beta_n^3}{1-\delta\beta_n^3}\right)\ldots \\ & \left(\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - f_n) \\ & + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\ldots \\ & \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - p) \\ & + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\ldots \\ & \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(g_n - p)) \\ & + \left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n})\varphi_\rho(\rho(g_{n+1} - p)\right).\end{aligned} \tag{26}$$

Observe that

$$\begin{aligned}\left[\frac{1-\alpha_n}{1-\delta\alpha_n}\right] &\leq 1-\alpha_n+\delta\alpha_n, \\ \left[\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right] &\leq 1-\beta_n^1+\delta\beta_n^1, \\ \left[\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right] &\leq 1-\beta_n^2+\delta\beta_n^2, \ldots, \\ \left[\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right] &\leq 1-\beta_n^{k-2}+\delta\beta_n^{k-2} \text{ and} \\ \left[\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right] &\leq 1-\beta_n^{k-1}+\delta\beta_n^{k-1}.\end{aligned} \tag{27}$$

Applying the inequality (27) in inequality (26) and simplifying, we obtain

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) &\leq (1 - \alpha_n + \delta\alpha_n)(1 - \beta_n^1 + \delta\beta_n^1)(1 - \beta_n^2 + \delta\beta_n^2), ..., \\ &\quad (1 - \beta_n^{k-2} + \delta\beta_n^{k-2})(1 - \beta_n^{k-1} + \delta\beta_n^{k-1}) + e_n \\ &\leq [1 - (1 - \delta)\alpha_n]\rho(g_n - f_n) + e_n,\end{aligned} \tag{28}$$

where,

$$\begin{aligned}e_n &= [1 - (1 - \delta)\alpha_n][1 - (1 - \delta)\beta_n^1][1 - (1 - \delta)\beta_n^2][1 - (1 - \delta)\beta_n^3]... \\ &\quad [1 - (1 - \delta)\beta_n^{k-2}][1 - (1 - \delta)\beta_n^{k-1}]\rho(g_n - p) \\ &\quad + \left(\frac{1 - \alpha_n}{1 - \delta\alpha_n}\right)\left(\frac{(1 + \delta)\beta_n^1}{1 - \delta\beta_n^1}\right)\left(\frac{(1 + \delta)\beta_n^2}{1 - \delta\beta_n^2}\right)\left(\frac{(1 + \delta)\beta_n^3}{1 - \delta\beta_n^3}\right)... \\ &\quad \left(\frac{(1 + \delta)\beta_n^{k-2}}{1 - \delta\beta_n^{k-2}}\right)\left(\frac{(1 + \delta)\beta_n^{k-1}}{1 - \delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(g_n - p)) \\ &\quad + \left(\frac{(1 + \delta)\alpha_n}{1 - \delta\alpha_n})\varphi_\rho(\rho(g_{n+1} - p)\right).\end{aligned}$$

Using the fact that $0 \leq \delta < 1$ and the conditions $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0, 1)(i = 1, 2, ..., k - 1)$ in iterative sequences (9) to (12) in (28), it follows that

$$\lim_{n\to\infty} \rho(g_n - f_n) = 0.$$

Since by assumption $\lim_{n\to\infty} g_n = p$, then $\rho(f_n - p) \leq \rho(g_n - f_n) + \rho(g_n - p) \to 0$ as $n \to \infty$. That is, $\lim_{n\to\infty} f_n = p$.

Next we show that (ii) $\to$ (i). Assume $\lim_{n\to\infty} f_n = p$.

Then using $\rho-$quasi-contractive-like condition (13), implicit multistep (9) and implicit Mann iterative sequences (12), we obtain the following:

$$\rho(f_{n+1} - g_{n+1}) = \rho[(1 - \alpha_n)(f_n^1 - g_n) + \alpha_n(u_{n+1} - v_{n+1})]. \tag{29}$$

Using the convexity of ρ in (29), we have

$$\begin{aligned}\rho(f_{n+1} - g_{n+1}) &= (1 - \alpha_n)\rho(f_n^1 - g_n) + \alpha_n\rho(u_{n+1} - v_{n+1}) \\ &\leq (1 - \alpha_n)\rho(f_n^1 - g_n) + \alpha_n(H_\rho(P_\rho^T(f_{n+1}), P_\rho^T(g_{n+1}))).\end{aligned} \tag{30}$$

Using (13), let $f = f_{n+1}$, $g = g_{n+1}$, then, from (29), we get the following:

$$H_\rho(P_\rho^T(f_{n+1}), P_\rho^T(g_{n+1})) \leq \delta\rho(f_{n+1} - g_{n+1}) + (1 + \delta)\varphi_\rho(\rho(f_{n+1} - p)). \tag{31}$$

Substituting inequality (31) in inequality (30), we obtain

$$\begin{aligned}\rho(f_{n+1}-g_{n+1}) &\leq (1-\alpha_n)\rho(f_n^1-g_n)+\delta\alpha_n\rho(f_{n+1}-g_{n+1})\\ &+(1+\delta)\alpha_n\varphi_\rho(\rho(f_{n+1}-p)).\end{aligned}$$

That is,

$$\rho(f_{n+1}-g_{n+1}) \leq \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\rho(f_n^1-g_n)+\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(f_{n+1}-p)). \quad (32)$$

$$\begin{aligned}\rho(f_n^1-g_n) \leq \left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^2-g_n)+\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^1-p)+\\ \left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(f_n^1-p)).\end{aligned} \quad (33)$$

Substituting inequality (33) in inequality (32), we obtain

$$\begin{aligned}\rho(f_{n+1}-g_{n+1}) \leq& \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^2-g_n)\\ &+\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^1-p)\\ &+\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(f_n^1-p))\\ &+\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(f_{n+1}-p)).\end{aligned} \quad (34)$$

Similarly, an application of (13) and (9) and (12) give the following

$$\begin{aligned}\rho(f_n^2-g_n) \leq& \left(\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right)\rho(f_n^3-g_n)+\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\rho(f_n^2-p)\\ &+\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\varphi_\rho(\rho(f_n^2-p)).\end{aligned} \quad (35)$$

$$\begin{aligned}\rho(f_n^3-g_n) \leq& \left(\frac{1-\beta_n^3}{1-\delta\beta_n^3}\right)\rho(f_n^4-g_n)+\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\rho(f_n^3-p)\\ &+\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\varphi_\rho(\rho(f_n^3-p)).\end{aligned} \quad (36)$$

$$\vdots$$

$$\rho(f_n^{k-2}-g_n) \leq \left(\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(f_n^{k-1}-g_n) + \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(f_n^{k-2}-p) + \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\varphi_\rho(\rho(f_n^{k-2}-p)). \tag{37}$$

$$\rho(f_n^{k-1}-g_n) \leq \left(\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(f_n-g_n) + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(f_n^{k-1}-p) + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(f_n^{k-1}-p)). \tag{38}$$

Substituting inequalities (38), (37), (36), (35) in inequality (32) inductively and simplifying, we obtain

$$\rho(f_{n+1}-g_{n+1}) \leq [1-(1-\delta)\alpha_n]\rho(f_n-g_n)+b_n, \tag{39}$$

where,

$$\begin{aligned} b_n \;=\; & [1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n^1][1-(1-\delta)\beta_n^2][1-(1-\delta)\beta_n^3]... \\ & [1-(1-\delta)\beta_n^{k-2}][1-(1-\delta)\beta_n^{k-1}]\rho(f_n^{k-1}-p) \\ & +\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)... \\ & \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(f_n^{k-1}-p)) \\ & +\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(f_{n+1}-p)). \end{aligned}$$

Using the fact that $0 \leq \delta < 1$ and the conditions $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0,1)(i = 1,2,...,k-1)$ in iterative sequences (9) to (12) in (39), it follows that

$$\lim_{n\to\infty} \rho(f_n-g_n) = 0.$$

Since by assumption $\lim_{n\to\infty} f_n = p$, then
$\rho(g_n-p) \leq \rho(f_n-g_n)+\rho(f_n-p) \to 0$ as $n \to \infty$.
That is, $\lim_{n\to\infty} g_n = p$. This ends the proof. □

Since the implicit Noor (10), the implicit Ishikawa (11) and the implicit Mann (12) iterative sequences are special cases of the implicit multistep iterative sequence (9), then Theorem 1 leads to the following corollary:

Corollary 1 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive-like mapping, satisfying contractive-like condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f-g) + \varphi_\rho(\rho(Tf-f)), \tag{40}$$

for all $g, h, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0, 1)$ *and* $\varphi_\rho : R^+ \to R^+$ *is a* $\rho-$*monotone increasing function with* $\varphi_\rho(0) = 0$*. Let* $g_0, h_0, g_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ *be defined by the implicit Mann* (12)*, implicit Ishikawa* (11) *and implicit Noor iterative sequences* (10) *respectively, where the sequences* $\{\alpha_n\}_{n=0}^\infty, \{\beta_n^1\}_{n=0}^\infty, \{\beta_n^2\}_{n=0}^\infty \subset (0, 1)$ *such that* $\sum_{n=0}^\infty \alpha_n = \infty$, $\sum_{n=0}^\infty \beta_n^i = \infty$*, for* $i = 1, 2$*. Then the following are equivalent:*
a. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T*;*
(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T*.*

b. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T*;*
(ii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T*.*

Proof. The proof of Corollary 1 is similar to that of Theorem 1. □

Corollary 2 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive-like mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f-g) + \varphi_\rho(\rho(Tf-f)), \tag{41}$$

for all $g, h, g, f \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0, 1)$ *and* $\varphi_\rho : R^+ \to R^+$ *is a* $\rho-$*monotone increasing function with* $\varphi_\rho(0) = 0$*. Let* $g_0, h_0, g_0, f_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\}, \{f_n\} \subset D$ *be defined by the implicit Mann* (12)*, implicit Ishikawa* (11)*, implicit Noor* (10)*, implicit multistep* (9) *iterative sequences respectively, where the sequences* $\{\alpha_n\}_{n=0}^\infty, \{\beta_n^i\}_{n=0}^\infty \subset (0, 1)$ *such that* $\sum_{n=0}^\infty \alpha_n = \infty$, $\sum_{n=0}^\infty \beta_n^i = \infty$ *for* $i = 1, 2, ..., k-1$*. Then the following are equivalent:*

(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T*;*

(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T;

(iii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T;

(iv) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Theorem 2 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \le \delta\rho(f - g) + J\rho(Tf - f), \tag{42}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0, 1)$ *and* $J \geq 0$*. Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann iterative sequences* (12) *respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0, 1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2, ..., k - 1$*. Then the following are equivalent:*

(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;

(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. The method of proof of Theorem 2 is similar to that of Theorem 1. The proof is complete. □

Theorem 2 leads to the following corollary:

Corollary 3 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \le \delta\rho(f - g) + J\rho(Tf - f), \tag{43}$$

for all $g, h, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0, 1)$ *and* $J \geq 0$*. Let* $g_0, h_0, g_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ *be defined by the implicit Mann* (12)*, implicit Ishikawa* (11) *and implicit Noor* (10) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0, 1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for*

$i = 1, 2$. *Then the following are equivalent:*
a. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T.

b. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T.

Theorem 3 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that,* P_ρ^T *is a* $\rho-$*Zamfirescu mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + 2\delta\rho(Tf - f), \tag{44}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$, *where* $\delta \in [0, 1)$. *Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann iterative sequences* (12) *respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0, 1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2, ..., k-1$. *Then the following are equivalent:*

(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;

(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. The method of proof of Theorem 3 is similar to that of Theorem 1. The proof is complete. □

Theorem 3 leads to the following corollary:

Corollary 4 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*Zamfirescu mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + 2\delta\rho(Tf - f), \tag{45}$$

for all $g, h, g \in D$ *and* $F_\rho(T) \neq \emptyset$, *where* $\delta \in [0, 1)$. *Let* $g_0, h_0, g_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ *be defined by the implicit Mann* (12), *implicit Ishikawa*

(11) *and implicit Noor* (10) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0,1)$ *such that,* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2$*. Then the following are equivalent:*

a. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T.

b. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T.

Theorem 4 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*contraction mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g), \tag{46}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0, 1)$*. Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann iterative sequences* (12) *respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0,1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2, ..., k-1$*. Then the following are equivalent:*
(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. The method of proof of Theorem 4 is similar to that of Theorem 1. The proof is complete. □

Theorem 4 leads to the following corollary:

Corollary 5 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*contraction mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g), \tag{47}$$

for all $g, h, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0,1)$*. Let* $g_0, h_0, g_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ *be defined by the implicit Mann* (12)*, implicit Ishikawa* (11) *and implicit Noor* (10) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0,1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2$*. Then the following are equivalent:*
a. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T*;*
(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T*.*

b. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T*;*
(ii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T*.*

3 Numerical example

Example 1 [3]*. Let* $M[0,1]$ *be the collection of all real-valued measurable functions on* $[0,1]$ *and* $\rho : M[0,1] \to R$ *a convex function modular defined by* $\rho(f) = \int_0^1 |f| \ \ \forall f \in M[0,1]$*. Let* $D = \{f \in L_\rho : \ 0 \leq f(x) \leq 2 \ \forall x \in [0,1]\}$ *be a subset of the modular function space* $L_\rho = M[0,1]$ *defined by* ρ*.* D *is nonempty, closed, and convex. Define map* $T : D \to P_\rho(D)$ *by* $Tf = \{\delta f\}$*, where* $\delta = 0.9$*.* T *satisfies property (I), has a unique fixed point* $f = 0$ *(since* $0 \in T(0)$*), and* P_ρ^T *is a* ρ*-contraction, with* $P_\rho^T(f) = \{Tf\} \ \forall f \in D$*. In fact,* P_ρ^T *is an* m*-strong* ρ*-strong contraction for all* $m \in N$*, since* $\rho(g) = m\rho(\frac{g}{m})$*.*

We present the results of convergence to $f = 0$ of implicit Mann iterative sequence (12), implicit Ishikawa iterative sequence (11), implicit Noor iterative sequence (10), and implicit multistep iterative sequence (9) using MATLAB. The parameters used are the following: $g_0(x) = h_0(x) = f_0(x) = 0.5x + 0.95 \ \forall x \in [0,1]$, $\alpha_n = \frac{1}{4} + \frac{1}{n+2}$, $\beta_n^i = \frac{1}{n+2}$ for $i = 1, 2, ..., k-1$, where $k = 11$ and $n = 1, 2, ..., 130$.

Table 1. The approximate values of Implicit Mann ($IMA(g_n)$), implicit Ishikawa ($ISH(h_n)$), implicit Noor($INO(g_n)$), implicit multistep ($IMU(f_n)$) iterative sequences.

n	$IMA(g_n)$	$ISH(h_n)$	$INO(g_n)$	$IMU(f_n)$
0	1.2	1.2	1.2	1.2
1	1.5621	1.1465	1.1332	1.1041
⋮	⋮	⋮	⋮	⋮
16	0.1997	0.1898	0.1779	0.1667
17	0.1985	0.1786	0.1646	0.1554
⋮	⋮	⋮	⋮	⋮
24	0.1098	0.1078	0.1064	0.1002
25	0.0989	0.0975	0.0966	0.0945
⋮	⋮	⋮	⋮	⋮
60	0.0216	0.0199	0.0196	0.0192
⋮	⋮	⋮	⋮	⋮
77	0.0189	0.0167	0.0152	0.0102
78	0.0173	0.0154	0.0141	0.0099
79	0.0164	0.0147	0.0135	0.0095
⋮	⋮	⋮	⋮	⋮
101	0.0102	0.0097	0.0087	0.0022
⋮	⋮	⋮	⋮	⋮
110	0.0078	0.0054	0.0043	0.0004
⋮	⋮	⋮	⋮	⋮
112	0.0055	0.0031	0.0023	0.0000
⋮	⋮	⋮	⋮	⋮
120	0.0043	0.0028	0.0000	0.0000
⋮	⋮	⋮	⋮	⋮
124	0.0029	0.0000	0.0000	0.0000
⋮	⋮	⋮	⋮	⋮
130	0.0000	0.0000	0.0000	0.0000

4 Interpretation of result

The unique fixed point value of implicit Mann iterative sequence (12), implicit Ishikawa iterative sequence (11), implicit Noor iterative sequence (10), and implicit multistep iterative sequence (9) from table 1, at iterations 130, 124, 120 and 112 respectively is 0 (zero). This shows that the convergence of the various iterative sequences (12), (11), (10) and (9) are equivalent, since they all converge to the same unique fixed point 0.

5 Conclusion

The convergence of implicit multistep iterative sequence (9) had been shown to be equivalent to the convergences of implicit Noor (10), implicit Ishikawa (11), and implicit Mann (12) iterative sequences for multivalued $\rho-$ quasi-contractive-like mappings in modular function spaces. An example had benn provided in section 3 to demonstrate the applicability of the equivalence results. The various iterative sequences and multivalued mappings considered in this study have good potentials for further studies.

Competing interest

The authors declare that there are no competing interests.

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Preopen sets and prelocally closed sets in generalised topology and minimal structure spaces

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Abstract. The intention of this article is to introduce and characterise the concept of preopen sets and prelocally closed sets in Generalised Topology and Minimal structure spaces.

1 Introduction

The notion of minimal structure space was introduced by Maki et al. [7] in 1999. The concepts of m-preopen sets and m-precontinous functions on minimal spaces was studied by Min and Kim [9]. Boonpok [1] introduced the concept of m-preopen sets and studied the notion of M-continuous and weakly M-continuous functions in Biminimal spaces. Carpintero et al.[4] also studied

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and characterised the concepts of m-preopen sets and their related notions in Biminimal spaces. The concept of Generalised topologies was introduced by Csaszar [5] in 2002. He also introduced the concepts of continuous functions and associated interior and closure operators on generalised neighborhood systems and generalised topological spaces. In particular, he investigated characterisations for the generalised continuous functions by using a closure operator defined on generalised neighborhood systems. The concept of generalised topology and minimal structure(GTMS) spaces was introduced by Buadong et al.[3] in 2011, which is a space with a generalised topology and a minimal structure. In 2013, Zakari [11] studied on some generalisations for closed sets in generalised topology and minimal structure spaces. He also studied *gm*-continuous functions between GTMS spaces in [12]. The idea of a locally closed set in topological space was defined by Kuratowski and Sierpinski [6]. Bourbaki [2] defined this notion in a way that a subset of a space is locally closed if it is the intersection of an open set and a closed set in X. Minimal structure in fuzzy topological spaces has been studied by Tripathy and Debnath [10].

2 Preliminaries

Definition 1 [7] *Let* $\mathsf{P}(\mathsf{X})$ *be the power set of a nonempty set* X*. A subfamily* M_X *of* $\mathsf{P}(\mathsf{X})$ *is called a minimal structure (briefly* $\mathfrak{m}$*-structure) on* X *if* $\emptyset \in \mathsf{M}_\mathsf{X}$ *and* $\mathsf{X} \in \mathsf{M}_\mathsf{X}$*. A set* X *with an* $\mathfrak{m}$*-structure* M_X *is called an* $\mathfrak{m}$*-space and is denoted by* $(\mathsf{X}, \mathsf{M}_\mathsf{X})$*. Each member of* M_X *is said to be an* M_X*-open set and the complement of an* M_X*-open set is said to be* M_X*-closed set.*

Definition 2 [7] *Let* $(\mathsf{X}, \mathsf{M}_\mathsf{X})$ *be an* $\mathfrak{m}$*-space where,* X *is a nonempty set and* M_X *an* $\mathfrak{m}$*-structure on* X*. Let* S *be a subset of* X*, then the* M_X*-closure of* S *and the* M_X*-interior of* S *are defined as follows*

(a) $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Cl}(\mathsf{S}) = \cap\{\mathsf{F} : \mathsf{S} \subset \mathsf{F}, \mathsf{X} \setminus \mathsf{F} \in \mathsf{M}_\mathsf{X}\}$.

(b) $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Int}(\mathsf{S}) = \cup\{\mathsf{G} : \mathsf{G} \subset \mathsf{S}, \mathsf{G} \in \mathsf{M}_\mathsf{X}\}$.

Lemma 1 [7] *Let* $(\mathsf{X}, \mathsf{M}_\mathsf{X})$ *be an* $\mathfrak{m}$*-space where,* X *is a nonempty set and* M_X *an* $\mathfrak{m}$*-structure on* X*. Let* S *and* T *be subsets of* X*, then the following properties hold:*

(a) $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Cl}(\mathsf{X} \setminus \mathsf{S}) = \mathsf{X} \setminus \mathsf{M}_\mathsf{X}\text{-}\mathsf{Int}(\mathsf{S})$ *and* $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Int}(\mathsf{X} \setminus \mathsf{S}) = \mathsf{X} \setminus \mathsf{M}_\mathsf{X}\text{-}\mathsf{Cl}(\mathsf{S})$.

(b) *If* $(\mathsf{X} \setminus \mathsf{S}) \in \mathsf{M}_\mathsf{X}$*, then* $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Cl}(\mathsf{S}) = \mathsf{S}$ *and if* $\mathsf{S} \in \mathsf{M}_\mathsf{X}$*, then* $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Int}(\mathsf{S}) = \mathsf{S}$.

(c) $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Cl}(\emptyset) = \emptyset$*,* $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Cl}(\mathsf{X}) = \mathsf{X}$*,* $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Int}(\emptyset) = \emptyset$ *and* $\mathsf{M}_\mathsf{X}\text{-}\mathsf{Int}(\mathsf{X}) = \mathsf{X}$.

(d) *If* $S \subset T$, *then* $M_X\text{-}Cl(S) \subset M_X\text{-}Cl(T)$ *and* $M_X\text{-}Int(S) \subset M_X\text{-}Int(T)$.

(e) $M_X\text{-}Int(S) \subset S \subset M_X\text{-}Cl(S)$.

(f) $M_X\text{-}Cl(M_X\text{-}Cl(S)) = M_X\text{-}Cl(S)$ *and* $M_X\text{-}Int(M_X\text{-}Int(S)) = M_X\text{-}Int(S)$.

Definition 3 [9] *Let* (X, M_X) *be an* $\mathfrak{m}$*-space where,* X *is a nonempty set and* M_X *an* $\mathfrak{m}$*-structure on* X*. A subset* S *of* X *is said to be* M_X*-preopen set if* $S \subset M_X\text{-}Int(M_X\text{-}Cl(S))$*. The complement of an* M_X*-preopen set is called an* M_X*-preclosed set.*

Definition 4 [5] *Let* X *be a nonempty set and* G_X *a collection of subsets of* X*. Then* G_X *is called a generalised topology (briefly GT) on* X *if and only if* $\emptyset \in G_X$ *and* $G_i \in G_X$ *for* $i \in I \neq \emptyset$ *implies* $\cup_{(i \in I)} G_i \in G_X$*. The pair* (X, G_X) *is called a generalised topological space (briefly GTS) on* X*. The elements of* G_X *are called* G_X*-open sets and the complements are called* G_X*-closed sets.*

The closure of a subset S *in a generalised topological space* (X, G_X)*, denoted by* $G_X\text{-}Cl(S)$ *is the intersection of generalised closed sets including* S *and the interior of* S*, denoted by* $G_X\text{-}Int(S)$*, is the union of generalised open sets contained in* S*.*

Theorem 1 [3] *Let* (X, G_X) *be a generalised topological space and* $S \subseteq X$*. Then*

(a) $G_X\text{-}Cl(S) = X \setminus G_X\text{-}Int(X \setminus S)$.

(b) $G_X\text{-}Int(S) = X \setminus G_X\text{-}Cl(X \setminus S)$.

Proposition 1 [8] *Let* (X, G_X) *be a generalised topological space and* $S \subseteq X$*. Then*

(a) $x \in G_X\text{-}Int(S)$ *if and only if there exists* $V \in G_X$ *such that* $x \in V \subseteq S$*;*

(b) $x \in G_X\text{-}Cl(S)$ *if and only if* $V \cap S \neq \emptyset$ *for every* G_X*-open set* V *containing* x*.*

Proposition 2 [8] *Let* (X, G_X) *be a generalised topological space. Let* S *and* T *be subsets of* X*, then the following properties hold:*

(a) $G_X\text{-}Cl(X \setminus S) = X \setminus G_X\text{-}Int(S)$ *and* $G_X\text{-}Int(X \setminus S) = X \setminus G_X\text{-}Cl(S)$.

(b) *If* $X \setminus S \in G_X$*, then* $G_X\text{-}Cl(S) = S$ *and if* $S \in G_X$*, then* $G_X\text{-}Int(S) = S$.

(c) *If* $S \subseteq T$*, then* $G_X\text{-}Cl(S) \subseteq G_X\text{-}Cl(T)$ *and* $G_X\text{-}Int(S) \subseteq G_X\text{-}Int(T)$.

(d) $S \subseteq G_X\text{-}Cl(S)$ *and* $G_X\text{-}Int(S) \subseteq S$.

(e) $G_X\text{-}Cl(G_X\text{-}Cl(S)) = G_X\text{-}Cl(S)$ *and* $G_X\text{-}Int(G_X\text{-}Int(S)) = G_X\text{-}Int(S)$.

(One may refer to Buadong et al. [3] Proposition 2.4)

Definition 5 [3] *Let* X *be a nonempty set and let* G_X *be a generalised topology and* M_X *a minimal structure on* X. *A triple* (X, G_X, M_X) *is called a generalised topology and minimal structure space (briefly GTMS space).*

Let (X, G_X, M_X) *be a GTMS space and* S *be a subset of* X. *The closure and interior of* S *in* G_X *are denoted by* $G_X\text{-Cl}(S)$ *and* $G_X\text{-Int}(S)$, *respectively. The closure and interior of* S *in* M_X *are denoted by* $M_X\text{-Cl}(S)$ *and* $M_X\text{-Int}(S)$, *respectively.*

Definition 6 [3] *Let* (X, G_X, M_X) *be a GTMS space. A subset* S *of* X *is said to be a* (G_X, M_X)*-closed set if* $G_X\text{-Cl}(M_X\text{-Cl}(S)) = S$ *and a subset* S *of* X *is said to be a* (M_X, G_X)*-closed set if* $M_X\text{-Cl}(G_X\text{-Cl}(S)) = S$. *The complement of a* (G_X, M_X)*-closed (resp.* (M_X, G_X)*-closed) set is said to be* (G_X, M_X)*-open set (resp.* (M_X, G_X)*-open set).*

Lemma 2 [3] *Let* (X, G_X, M_X) *be a GTMS space and* $S \subseteq X$. *Then*

(a) S *is* (G_X, M_X)*-closed if and only if* $M_X\text{-Cl}(S) = S$ *and* $G_X\text{-Cl}(S) = S$.

(b) S *is* (M_X, G_X)*-closed if and only if* $M_X\text{-Cl}(S) = S$ *and* $G_X\text{-Cl}(S) = S$.

Proposition 3 [3] *Let* (X, G_X, M_X) *be a GTMS space and* $S \subseteq X$. *Then* S *is* (G_X, M_X)*-closed if and only if* S *is* (M_X, G_X)*-closed.*

Definition 7 [3] *Let* (X, G_X, M_X) *be a GTMS space and* S *be a subset of* X. *Then* S *is said to be a closed set if* S *is* (G_X, M_X)*-closed. The complement of a closed set is an open set.*

Proposition 4 [3] *Let* (X, G_X, M_X) *be a GTMS space. Then* S *is open if and only if* $S = G_X\text{-Int}(M_X\text{-Int}(S))$.

Proposition 5 [3] *Let* (X, G_X, M_X) *be a GTMS space.*

(a) *If* S *and* T *are closed, then* $S \cap T$ *is closed.*

(b) *If* S *and* T *are open, then* $S \cup T$ *is open.*

Definition 8 [3] *Let* (X, G_X, M_X) *be a GTMS space and* T *be a subset of* X. *Then* T *is said to be* s*-closed if* $G_X\text{-Cl}(T) = M_X\text{-Cl}(T)$ *and* T *is said to be* c*-closed if* $G_X\text{-Cl}(M_X\text{-Cl}(T)) = M_X\text{-Cl}(G_X\text{-Cl}(T))$. *The complement of a* s*-closed (resp.* c*-closed) set is called a* s*-open (resp.* c*-open) set.*

Proposition 6 [3] *Let* (X, G_X, M_X) *be a GTMS space and* $T \subseteq X$. *Then*

(a) T *is* s*-open if and only if* $G_X\text{-Int}(T) = M_X\text{-Int}(T)$.

(b) T *is* c*-open if and only if* $G_X\text{-Int}(M_X\text{-Int}(T)) = M_X\text{-Int}(G_X\text{-Int}(T))$

Proposition 7 [3] *Let* (X, G_X, M_X) *be a GTMS space and* $T \subseteq X$.

(a) *If* T *is closed, then* T *is* s*-closed.*

(b) *If* T *is* s*-closed, then* T *is* c*-closed.*

3 (G_X, M_X)-preopen sets

Definition 9 *Let* (X, G_X, M_X) *be a GTMS space. A subset* S *of* X *is said to be a* (G_X, M_X)*-preopen set if* $S \subset G_X\text{-Int}(M_X\text{-Cl}(S))$ *and* (M_X, G_X)*-preopen set if* $S \subset M_X\text{-Int}(G_X\text{-Cl}(S))$. *A subset* S *of* X *is said to be a* (G_X, M_X)*-preclosed set (*(M_X, G_X)*-preclosed set) if the complement* $X \setminus S$ *of* S *is a* (G_X, M_X)*-preopen set (respectively,* (M_X, G_X)*-preopen set). The set of all* (G_X, M_X)*-preopen sets of* (X, G_X, M_X) *is denoted by* $(G_X, M_X)\text{-PO}(X)$ *and the set of all* (G_X, M_X)*-preclosed sets of* (X, G_X, M_X) *is denoted by* $(G_X, M_X)\text{-PC}(X)$.

Remark 1 *A set which is* (G_X, M_X)*-preopen need not be* (M_X, G_X)*-preopen in general as can be seen from the following example.*

Example 1 *Let* $X = \{a, b, c\}$. *We define generalised topology* G_X *and minimal structure space* M_X *on* X *as follows:* $G_X = \{\emptyset, \{a\}, \{a, c\}\}$ *and* $M_X = \{\emptyset, \{b\}, \{b, c\}, X\}$. *Let* $S = \{b\}$ *and* $T = \{c\}$ *then* S *is* (M_X, G_X)*-preopen but is not* (G_X, M_X)*-preopen set whereas* T *is both* (G_X, M_X)*-preopen and* (M_X, G_X)*-preopen.*

Remark 2 *The intersection of two* (G_X, M_X)*-preopen sets need not be a* (G_X, M_X)*-preopen set as the following example illustrates.*

Example 2 *Let* $X = \{a, b, c, d\}$. *We define generalised topology* G_X *and minimal structure space* M_X *on* X *by* $G_X = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ *and* $M_X = \{\emptyset, \{a\}, \{a, c\}, X\}$. *Let* $S = \{a, b\}$ *and* $T = \{b, c\}$. *Then* S *and* T *are* (G_X, M_X)*-preopen sets but* $S \cap T = \{b\}$ *is not a* (G_X, M_X)*-preopen set.*

Definition 10 *Let* (X, G_X, M_X) *be a GTMS space and* S *be a subset of* X. *Then*

(a) *the* (G_X, M_X)*-preclosure of* S *(*(M_X, G_X)*-preclosure of* S*) denoted by* $(G_X, M_X)\text{-}Cl_p(S)$ *(respectively,* $(M_X, G_X)\text{-}Cl_p(S)$*) is defined as the intersection of all* (G_X, M_X)*-preclosed (respectively,* (M_X, G_X)*-preclosed) sets containing* S*.*

b) *the* (G_X, M_X)*-preinterior of* S *(*(M_X, G_X)*-preinterior of* S*) denoted by* $(G_X, M_X)\text{-}Int_p(S)$ *(respectively,* $(M_X, G_X)\text{-}Int_p(S)$*) is defined as the union of all* (G_X, M_X)*-preopen (respectively,* (M_X, G_X)*-preopen) sets contained in* S*.*

Theorem 2 *Let* (X, G_X, M_X) *be a GTMS space and* T *be a subset of* X*. Then*

(a) *if* T *is* s*-closed,* (G_X, M_X)*-preopen set* T *is a* G_X*-preopen set and* (M_X, G_X)*-preopen set* T *is a* M_X*-preopen set.*

(b) *if* T *is* s*-open,* (G_X, M_X)*-preopen set* T *is a* M_X*-preopen set and* (M_X, G_X)*-preopen set* T *is a* G_X*-preopen set.*

Proof.

(a) Let T be a (G_X, M_X)-preopen set. Then $T \subset G_X\text{-}Int(M_X\text{-}Cl(T))$. But T is s-closed, so $G_X\text{-}Cl(T) = M_X\text{-}Cl(T)$. Thus $T \subset G_X\text{-}Int(G_X\text{-}Cl(T))$. Hence T is a G_X-preopen set. Similarly, if T is a (M_X, G_X)-preopen set then T is a M_X-preopen set.

(b) Let T be a (G_X, M_X)-preopen set. Then $T \subset G_X\text{-}Int(M_X\text{-}Cl(T))$. But T is s-open, so $G_X\text{-}Int(T) = M_X\text{-}Int(T)$, by Proposition 6. Thus $T \subset M_X\text{-}Int(M_X\text{-}Cl(T))$. Hence T is a M_X-preopen set. Similarly, if T is a (M_X, G_X)-preopen set then T is a G_X-preopen set.

□

Theorem 3 *Let* (X, G_X, M_X) *be a GTMS space. Then the arbitrary union of* (G_X, M_X)*-preopen (*(M_X, G_X)*-preopen) sets is a* (G_X, M_X)*-preopen (respectively,* (M_X, G_X)*-preopen) set.*

Proof. Let $\{S_\alpha\}_{\alpha\in\Lambda}$ be a family of (G_X, M_X)-preopen sets in (X, G_X, M_X). Since, $S_\alpha \subset G_X\text{-}Int(M_X\text{-}Cl(S_\alpha))\ \forall \alpha \in \Lambda$. So $\cup_{\alpha\in\Lambda} S_\alpha \subset \cup_{\alpha\in\Lambda}\{G_X\text{-}Int(M_X\text{-}Cl(S_\alpha))\} \subset G_X\text{-}Int(\cup_{\alpha\in\Lambda}\{M_X\text{-}Cl(S_\alpha)\}) = G_X\text{-}Int(M_X\text{-}\ Cl\{\cup_{\alpha\in\Lambda} S_\alpha\})$. Thus, $\cup_{\alpha\in\Lambda} S_\alpha$ is a (G_X, M_X)-preopen set in (X, G_X, M_X). □

Theorem 4 *Let* (X, G_X, M_X) *be GTMS space and* S *and* T *be subsets of* X*. Then the following properties hold:*

(a) (G_X, M_X)-$Int_p(S) = \cup\{F : F \subset S$ *and* $F \in (G_X, M_X)$-$PO(X)\}$

(b) (G_X, M_X)-$Int_p(S)$ *is the largest* (G_X, M_X)*-preopen subset of* X *contained in* S.

(c) S *is* (G_X, M_X)*-preopen if and only if* $S = (G_X, M_X)$-$Int_p(S)$

(d) (G_X, M_X)-$Int_p((G_X, M_X)$-$Int_p(S)) = (G_X, M_X)$-$Int_p(S)$

(e) *If* $S \subset T$, *then* (G_X, M_X)-$Int_p(S) \subset (G_X, M_X)$-$Int_p(T)$.

(f) (G_X, M_X)-$Int_p(S) \cup (G_X, M_X)$-$Int_p(T) \subset (G_X, M_X)$-$Int_p(S \cup T)$

(g) (G_X, M_X)-$Int_p(S \cap T) \subset (G_X, M_X)$-$Int_p(S) \cap (G_X, M_X)$-$Int_p(T)$

Proof.

(a) Let $x \in (G_X, M_X)$-$Int_p(S)$. Then there exists $F \in (G_X, M_X)$-$PO(X)$ containing x such that $x \in F \subset S$. So, $x \in \cup\{F : F \subset S$ and $F \in (G_X, M_X)$-$PO(X)\}$ showing that (G_X, M_X)-$Int_p(S) \subset \cup\{F : F \subset S$ and $F \in (G_X, M_X)$-$PO(X)\}$. Let $x \in \cup\{F : F \subset S$ and $F \in (G_X, M_X)$-$PO(X)\}$. Then there exists a $F \in (G_X, M_X)$-$PO(X)$ containing x such that $x \in F \subset S$. Thus $x \in (G_X, M_X)$-$Int_p(S)$ showing that $\cup\{F : F \subset S$ and $F \in (G_X, M_X)$-$PO(X)\} \subset (G_X, M_X)$-$Int_p(S)$. Hence, (G_X, M_X)-$Int_p(S) = \cup\{F : F \subset S$ and $F \in (G_X, M_X)$-$PO(X)\}$.

The proofs of (b) – (e) are evident.

(f) (G_X, M_X)-$Int_p(S) \subset (G_X, M_X)$-$Int_p(S \cup T)$ and (G_X, M_X)-$Int_p(T) \subset (G_X, M_X)$-$Int_p(S \cup T)$. Then by (e) we obtain (G_X, M_X)-$Int_p(S) \cup (G_X, M_X)$-$Int_p(T) \subset (G_X, M_X)$-$Int_p(S \cup T)$.

(g) Since $S \cap T \subset S$ and $S \cap T \subset T$, by (e) we get (G_X, M_X)-$Int_p(S \cap T) \subset (G_X, M_X)$-$Int_p(S)$ and (G_X, M_X)-$Int_p(S \cap T) \subset (G_X, M_X)$-$Int_p(T)$. Thus by (e), (G_X, M_X)-$Int_p(S \cap T) \subset (G_X, M_X)$-$Int_p(S) \cap (G_X, M_X)$-$Int_p(T)$.

□

Theorem 5 *Let* S *and* T *be subsets of* (X, G_X, M_X). *Then the following properties hold:*

(a) (G_X, M_X)-$Cl_p(S) = \bigcap\{G : S \subset G$ *and* $G \in (G_X, M_X)$-$PC(X)\}$.

(b) (G_X, M_X)-$Cl_p(S)$ *is the smallest* (G_X, M_X)*-preclosed subset of* X *containing* S.

(c) S *is* (G_X, M_X)*-preclosed if and only if* $S = (G_X, M_X)$-$Cl_p(S)$.

(d) (G_X, M_X)-$Cl_p((G_X, M_X)$-$Cl_p(S)) = (G_X, M_X)$-$Cl_p(S)$.

(e) *If* $S \subset T$ *then* $(G_X, M_X)\text{-}Cl_p(S) \subset (G_X, M_X)\text{-}Cl_p(T)$.

(f) $(G_X, M_X)\text{-}Cl_p(S) \cup (G_X, M_X)\text{-}Cl_p(T) \subset (G_X, M_X)\text{-}Cl_p(S \cup T)$.

(g) $(G_X, M_X)\text{-}Cl_p(S \cap T) \subset (G_X, M_X)\text{-}Cl_p(S) \cap (G_X, M_X)\text{-}Cl_p(T)$.

Proof.

(a) Suppose that $x \notin (G_X, M_X)\text{-}Cl_p(S)$. Then there exists $V \in (G_X, M_X)\text{-}PO(X)$ containing x such that $V \cap S = \emptyset$. Since $X \setminus V$ is a (G_X, M_X)-preclosed set containing S and $x \notin X \setminus V$, we get $x \notin \cap\{G : S \subset G$ and $G \in (G_X, M_X)\text{-}PC(X)\}$. Conversely, suppose that $x \notin \cap\{G : S \subset G$ and $G \in (G_X, M_X)\text{-}PC(X)\}$. Then there exists $G \in (G_X, M_X)\text{-}PC(X)$ such that $S \subset G$ and $x \notin G$. Since $X \setminus G$ is a (G_X, M_X)-preopen set containing x, we get$(X \setminus G) \cap S = \emptyset$. This shows that $x \notin (G_X, M_X)\text{-}Cl_p(S)$. Thus we get $(G_X, M_X)\text{-}Cl_p(S)=\cap\{G : S \subset G$ and $G \in (G_X, M_X)\text{-}PC(X)\}$.

The proofs of the other parts can be established similarly. □

We formulate the following result, which can be established easily.

Theorem 6 *Let* (X, G_X, M_X) *be a GTMS space and* $S \subset X$. *Then the following properties hold:*

(a) $(G_X, M_X)\text{-}Int_p(X \setminus S) = X \setminus (G_X, M_X)\text{-}Cl_p(S)$

(b) $(G_X, M_X)\text{-}Cl_p(X \setminus S) = X \setminus (G_X, M_X)\text{-}Int_p(S)$.

4 (G_X, M_X)-prelocally closed sets

Definition 11 *A subset* Q *of a GTMS* (X, G_X, M_X) *is said to be*

(a) (G_X, M_X)-*locally closed (in short* $(G_X, M_X)\text{-}L_c$*) if* $Q = R \cap S$*, where* R *is* G_X*-open and* S *is* M_X*-closed.*

(b) (G_X, M_X)-*Prelocally closed (in short* $(G_X, M_X)\text{-}PL_c$*) if* $Q = R \cap S$*, where* R *is* G_X*-preopen and* S *is* M_X*-preclosed.*

(c) (G_X, M_X)-*Prelocally closed* (in short* $(G_X, M_X)\text{-}PL_c^*$*) if* $Q = R \cap S$*, where* R *is* G_X*-preopen and* S *is* M_X*-closed.*

(d) (G_X, M_X)-*Prelocally closed** (in short* $(G_X, M_X)\text{-}PL_c^{**}$*) if* $Q = R \cap S$*, where* R *is* G_X*-open and* S *is* M_X*-preclosed.*

Here we denote the collection of all $(G_X, M_X)\text{-}L_c$*,* $(G_X, M_X)\text{-}PL_c$*,* $(G_X, M_X)\text{-}PL_c^*$ *and* $(G_X, M_X)\text{-}PL_c^{**}$ *sets in* (X, G_X, M_X) *by* $(G_X, M_X)\text{-}L_c(X)$*,* $(G_X, M_X)\text{-}PL_c(X)$*,* $(G_X, M_X)\text{-}PL_c^*(X)$ *and* $(G_X, M_X)\text{-}PL_c^{**}(X)$ *respectively.*

Theorem 7 *If* $R, S \in (G_X, M_X)\text{-}PL_c(X)$, *then* $R \cap S \in (G_X, M_X)\text{-}PL_c(X)$.

Proof. Since $R, S \in (G_X, M_X)\text{-}PL_c(X)$, so $R = V \cap W$ where V is G_X-preopen and W is M_X-preclosed, and $S = A \cap B$ where A is G_X-preopen and B is M_X-preclosed. Now, $R \cap S = (V \cap W) \cap (A \cap B) = (V \cap A) \cap (W \cap B)$. Since $V \cap A$ is G_X-preopen and $W \cap B$ is M_X-preclosed, so $R \cap S \in (G_X, M_X)\text{-}PL_c(X)$. □

Remark 3 *The converse of the Theorem 7 is not necessarily true as shown in the example below.*

Example 3 *Let* $X = \{a, b, c\}$, $G_X = \{\emptyset, \{a\}, \{a, c\}\}$ *and* $M_X = \{\emptyset, \{b\}, \{b, c\}, X\}$. *Then* $(G_X, M_X)\text{-}PL_c(X) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. *We have* $\{a, b\} \cap \{a, c\} = \{a\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{a, b\} \notin (G_X, M_X)\text{-}PL_c(X)$.

Theorem 8 *Let* S *be a subset of* (X, G_X, M_X). *Then*

(a) *If* $S \in (G_X, M_X)\text{-}L_c(X)$, *then* $S \in (G_X, M_X)\text{-}PL_c(X)$, $(G_X, M_X)\text{-}PL_c^*(X)$ *and* $(G_X, M_X)\text{-}PL_c^{**}(X)$.

(b) *If* $S \in (G_X, M_X)\text{-}PL_c^*(X)$, *then* $S \in (G_X, M_X)\text{-}PL_c(X)$.

(c) *If* $S \in (G_X, M_X)\text{-}PL_c^{**}(X)$, *then* $S \in (G_X, M_X)\text{-}PL_c(X)$.

Proof. The above stated results are true in the sense that every G_X-open set is G_X-preopen and every M_X-closed set is M_X-preclosed. □

Remark 4 *The converse of the Theorem 8 is not true in general as the following example illustrates.*

Example 4 *Let* $X = \{a, b, c\}$, $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{b\}, X\}$. *Then*
$(G_X, M_X)\text{-}L_c(X) = \{\emptyset, \{a\}, \{a, c\}, X\}$,
$(G_X, M_X)\text{-}PL_c(X) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$,
$(G_X, M_X)\text{-}PL_c^*(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ *and*
$(G_X, M_X)\text{-}PL_c^{**}(X) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$.
It is clearly seen that

(a) $\{c\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{c\} \notin (G_X, M_X)\text{-}L_c(X)$; $\{a, b\} \in (G_X, M_X)\text{-}PL_c^*(X)$ *but* $\{a, b\} \notin (G_X, M_X)\text{-}L_c(X)$; $\{c\} \in (G_X, M_X)\text{-}PL_c^{**}(X)$ *but* $\{c\} \notin (G_X, M_X)\text{-}L_c(X)$.

(b) $\{c\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{c\} \notin (G_X, M_X)\text{-}PL_c^*(X)$.

(c) $\{a, b\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{a, b\} \notin (G_X, M_X)\text{-}PL_c^{**}(X)$.

Theorem 9 *Let* T *be a subset of* (X, G_X, M_X), *then the following statements are equivalent:*

(a) $T \in (G_X, M_X)\text{-}PL_c(X)$.

(b) $T = Q \cap M_X\text{-}Cl_p(T)$ *for some* G_X*-preopen set* Q.

(c) $M_X\text{-}Cl_p(T) \setminus T$ *is* G_X*-preclosed.*

(d) $T \cup (X \setminus M_X\text{-}Cl_p(T))$ *is* G_X*-preopen*

(e) $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$.

Proof. (a) ⇒ (b) Let $T \in (G_X, M_X)\text{-}PL_c(X)$. Then $T = Q \cap R$, where Q is G_X-preopen and R is M_X-preclosed. Since $T \subset R$ and $M_X\text{-}Cl_p(T)$ is the smallest M_X-preclosed set containing T, so $M_X\text{-}Cl_p(T) \subset R$. Now,

$$T = Q \cap R \supset Q \cap M_X - Cl_p(T) \tag{1}$$

Since $T \subset Q$ and $T \subset M_X\text{-}Cl_p(T)$, so

$$T \subset Q \cap M_X - Cl_p(T) \tag{2}$$

Hence, from (1) and (2), $T = Q \cap M_X\text{-}Cl_p(T)$.

(b) ⇒ (c) Let $T = Q \cap M_X\text{-}Cl_p(T)$ for some G_X-preopen set Q. Now, $M_X\text{-}Cl_p(T) \setminus T = M_X\text{-}Cl_p(T) \setminus (Q \cap M_X\text{-}Cl_p(T)) = (M_X\text{-}Cl_p(T) \setminus Q) \cup (M_X\text{-}Cl_p(T) \setminus M_X\text{-}Cl_p(T)) = M_X\text{-}Cl_p(T) \setminus Q = M_X\text{-}Cl_p(T) \cap (X \setminus Q)$, which is M_X-preclosed, since $X \setminus Q$ is M_X-preclosed.

(c) ⇒ (d) $T \cup (X \setminus M_X\text{-}Cl_p(T)) = X \setminus (M_X\text{-}Cl_p(T) \setminus T)$. By (c), $(M_X\text{-}Cl_p(T) \setminus T$ is G_X-preclosed, so $X \setminus M_X\text{-}C_p(T) \setminus T)$ is G_X-preopen. Hence, $T \cup (X \setminus M_X\text{-}Cl_p(T))$ is G_X-preopen.

(d) ⇒ (e) Let $T \cup (X \setminus M_X\text{-}Cl_p(T))$ be G_X-preopen. Then $T \cup (X \setminus M_X\text{-}Cl_p(T)) = G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$. Hence $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$.

(e) ⇒ (a) By (e) we have $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$. Since, $T \subset M_X\text{-}Cl_p(T)$, so $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)$. Further, $\{G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)\} \subset \{T \cup (X \setminus M_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)\} = \{M_X\text{-}Cl_p(T) \cap T\} \cup \{M_X\text{-}Cl_p(T) \cap (X \setminus M_X\text{-}Cl_p(T))\} = T$. Consequently, $T = G_X\text{-}Int_p(T \cup (X \setminus G_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)$. Since, $G_X\text{-}Int_p(T \cup (X \setminus M_X - Cl_p(T)))$ is G_X-preopen and $M_X\text{-}Cl_p(T)$ is M_X-preclosed, so $T \in (G_X, M_X)\text{-}PL_c(X)$. □

Theorem 10 *If* $S \subset T \subset (X, G_X, M_X)$ *and* $T \in (G_X, M_X)\text{-}PL_c(X)$, *then there exists* $R \in (G_X, M_X)\text{-}PL_c(X)$ *such that* $S \subset R \subset T$.

Proof. Since $T \in (G_X, M_X)$-$PL_c(X)$, by Theorem 9., we have $T = Q \cap M_X$-$Cl_p(T)$ where Q is G_X-preopen. As $S \subset T$ and $T \subset Q$, so $S \subset Q$. Also, $S \subset M_X$-$Cl_p(S)$. Therefore, $S \subset Q \cap M_X$-$Cl_p(S)$. Now, $R = Q \cap M_X$-$Cl_p(S) \subset Q \cap M_X$-$Cl_p(T) = T$ Since, Q is G_X-preopen and M_X-$Cl_p(S)$ is M_X-preclosed, so $R \in (G_X, M_X)$-$PL_c(X)$ such that $S \subset R \subset T$. □

Theorem 11 *Let* S *be a subset of* (X, G_X, M_X). *Then*

(a) $S \in (G_X, M_X)$-$PL_c^*(X)$ *if and only if* $S = T \cap M_X$-$Cl(S)$ *for some* G_X-*preopen set* T.

(b) M_X-$Cl(S) \setminus S$ *is* G_X-*preclosed if and only if* $S \cup (X \setminus M_X$-$Cl(S))$ *is* G_X-*preopen.*

Proof.

(a) Let $S \in (G_X, M_X)$-$PL_c^*(X)$. Then there exists a G_X-preopen set T and a M_X-closed set N such that $S = T \cap N$. Therefore, $S \subset T$. Since $S \subset M_X$-$Cl(S)$, so $S \subset T \cap M_X$-$Cl(S)$. Also, since $S \subset N$ where N is M_X-closed set and M_X-$Cl(S)$ is the smallest M_X-closed set containing S, so M_X-$Cl(S) \subset N$. This implies $T \cap M_X$-$Cl(S) \subset T \cap N = S$. Hence, $S = T \cap M_X$-$Cl(S)$. Conversely, let $S = T \cap M_X$-$Cl(S)$ where T is G_X-preopen set. Since M_X-$Cl(S)$ is M_X-closed, so by definition, $S \in (G_X, M_X)$-$PL_c^*(X)$.

(b) Let M_X-$Cl(S) \setminus S$ be G_X-preclosed. Then $X \setminus M_X - Cl(S) \setminus S$ is G_X-preopen. Now, $X \setminus M_X$-$Cl(S) \setminus S) = S \cup (X \setminus M_X$-$Cl(S))$. Hence, $S \cup (X \setminus M_X$-$Cl(S))$ is G_X-preopen. Conversely, let $S \cup (X \setminus M_X$-$Cl(S))$ be G_X-preopen. We have $X \setminus (S \cup (X \setminus M_X$-$Cl(S)))$ is G_X-preclosed. Now, $X \setminus (S \cup (X \setminus M_X$-$Cl(S))) = M_X$-$Cl(S) \setminus S$. Hence M_X-$Cl(S) \setminus S$ is G_X-preclosed. □

Theorem 12 *Let* S, T *be two subsets of* (X, G_X, M_X). *If* $S \in (G_X, M_X)$-$PL_c(X)$ *and* T *is* G_X-*preopen or* M_X-*preclosed then* $S \cap T \in (G_X, M_X)$-$PL_c(X)$.

Proof. Let $S \in (G_X, M_X)$-$PL_c(X)$. Then $S = Q \cap R$, where Q is G_X-preopen and R is M_X-preclosed. Now, if T is G_X-preopen, then $Q \cap T$ is also G_X-preopen. Also, R is M_X-preclosed and $S \cap T = (Q \cap R) \cap T = (Q \cap T) \cap R$. Hence, $S \cap T \in (G_X, M_X)$-$PL_c(X)$. Again if T is M_X-preclosed, then $T \cap R$ is M_X-preclosed. Now, $S \cap T = (Q \cap R) \cap T = Q \cap (T \cap R)$. Hence, $S \cap T \in (G_X, M_X)$-$PL_c(X)$. □

Theorem 13 *Let* S *and* T *be two subsets of* (X, G_X, M_X). *Then*

(a) *If* $S, T \in (G_X, M_X)$-$PL_c^*(X)$ *then* $S \cap T \in (G_X, M_X)$-$PL_c^*(X)$.

(b) *If* $S, T \in (G_X, M_X)$-$PL_c^{**}(X)$ *then* $S \cap T \in (G_X, M_X)$-$PL_c^{**}(X)$.

Proof.

(a) Let $S, T \in (G_X, M_X)$-$PL_c^*(X)$. Then $S = A \cap B$, where A is G_X-preopen and B is M_X-closed and $T = C \cap D$, where C is G_X-preopen and D is M_X-closed. Now, $S \cap T = (A \cap B) \cap (C \cap D) = (A \cap C) \cap (B \cap D)$. Since, $A \cap C$ is G_X-preopen and $B \cap D$ is M_X-closed, so $S \cap T \in (G_X, M_X)$-$PL_c^*(X)$.

(b) Let $S, T \in (G_X, M_X)$-$PL_c^{**}(X)$. Then $S = A \cap B$, where A is G_X-open and B is M_X-preclosed and $T = C \cap D$, where C is G_X-open and D is M_X-preclosed. Now, $S \cap T = (A \cap B) \cap (C \cap D) = (A \cap C) \cap (B \cap D)$. Since, $A \cap C$ is G_X-open and $B \cap D$ is M_X-preclosed, so $S \cap T \in (G_X, M_X)$-$PL_c^{**}(X)$.

□

Remark 5 *The union of any two* (G_X, M_X)-PL_c^* *(respectively,* (G_X, M_X)-PL_c^{**}*) sets is not necessarily a* (G_X, M_X)-PL_c^* *(respectively,* (G_X, M_X)-PL_c^{**}*) set as shown in the following examples.*

Example 5

(a) *Let* $X = \{a, b, c\}$, $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{a, b\}, \{a, c\}, X\}$. *Then,* $\{b\}, \{c\} \in (G_X, M_X)$-$PL_c^*(X)$ *but* $\{b\} \cup \{c\} = \{b, c\} \notin (G_X, M_X)$-$PL_c^*(X)$.

(b) *Let* $X = \{a, b, c\}$, $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{b, c\}, X\}$. *Then,* $\{b\}, \{c\} \in (G_X, M_X)$-$PL_c^{**}(X)$, *but* $\{b\} \cup \{c\} = \{b, c\} \notin (G_X, M_X)$-$PL_c^*(X)$.

Remark 6 (G_X, M_X)-$PL_c^*(X)$ *and* (G_X, M_X)-$PL_c^{**}(X)$ *are independent as the following example exhibits.*

Example 6 *Let* $X = \{a, b, c\}$, $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{b, c\}, X\}$. *Then*
(G_X, M_X)-$PL_c^*(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ *and*
(G_X, M_X)-$PL_c^{**}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$.
So, (G_X, M_X)-$PL_c^*(X) \neq (G_X, M_X)$-$PL_c^{**}(X)$.

We formulate the following result without proof which can be established easily.

Theorem 14 *If* S *is* (G_X, M_X)-PL_c^{**} *and* T *is either* M_X*-closed or* G_X*-open, then* $S \cap T$ *is* (G_X, M_X)-PL_c^{**}.

5 Conclusion

In this paper, the rudimentary concepts of preopen sets and prelocally closed sets in GTMS spaces have been examined. With the help of these notions, researchers can turn their attention towards generalisations of various types of continuous functions as well as to develop the notion of connectedness of generalised topology by considering minimal structure spaces.

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Declaration

The authors declare that the article does not have any competing interest involved.

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Analysis of a batch arrival multi-server queueing system with waiting servers, synchronous working vacations and impatient customers

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Abstract. This paper is concerned with the analysis of an infinite-capacity batch arrival multi-server queueing system with Bernoulli feedback, synchronous multiple and single working vacation policies, waiting servers, reneging and retention of reneged customers. The steady-state solution of the queueing system is obtained by using probability generating function (PGF). In addition, important performance measures of the queueing system are derived. Then, a cost model is formulated in order to carry out the parameter optimization using genetic algorithm (GA). Finally, numerical study is presented in which various system performance measures are evaluated based on supposed numerical values given to the system parameters.

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Key words and phrases: multi-server queueing systems, synchronous working vacation, batch arrival, reneging, cost model

1 Introduction

In recent past, there has been a growing interest in the analysis of queueing models with working vacation, where during the vacation period, the server serves the arrivals with slower service rate rather than completely stopping the service. This interesting research area has shown a noticeable effect on queueing applications, especially in call centers, computer networks, manufacturing, production systems, etc. Excellent research work on the subject can be found in Servi and Finn [20], Baba [5], Jain and Jain [14], Sudhesh and Raj [23], Kempa and Kobielnik [17], Zhang and Zhou [31] and the references therein.

In queueing models cited above, it is supposed that customers arrive to the system one by one at a time, wherein there are many situations where customers arrive in a group, such examples can be found in digital communication systems, data traffic segmented as packets, and so forth. These queueing systems are known as batch arrival queues. For a comprehensive review of related models, the readers can be referred to Khalaf et al. [16], Baba [6], Baruah et al. [7], Singh et al. [21], Bhagat and Jain [8], Ayyappan and Udayageetha [4], Zhang [30] and the references therein.

Working vacation queueing models with impatient customers have been investigated extensively because of the large application in many real-life problems (cf. Yue et al. [28], Vijaya Laxmi and Jyothsna [24], Sudhesh et al. [22], Bouchentouf and Yahiaoui [12], Vijaya Laxmi and Rajesh [25], and Jain et al. [15]. The analysis of customers' impatience in multi-server vacation queueing models is more complex compared to single-server vacation queueing systems with impatient customers, where the servers may either take the same vacation together (synchronous vacation) or take individual vacations (asynchronous vacations) independently. Thus, a very limited literature is available for these models. The readers can be referred to Altman and Yechiali [1], Yue et al. [27], Altman and Yechiali [2], Yue et al. [29], Majid and Manoharan [18], Yahiaoui et al.[26], and Bouchentouf and Guendouzi [10].

The concept of vacation queues with a waiting server was first introduced by Boxma et al. [13], where once the system is empty, the server waits for a random amount of time before going on vacation. This situation reflects many real life queueing systems, particularly when dealing with human behaviour. For recent research works on the subject, the reader can refer to Padmavathy et al. [19], Ammar [3], Bouchentouf and Guendouzi [9], and Bouchentouf et al. [11].

In this paper, we deal with an infinite-space multi-server queueing system with batch arrival, waiting servers, synchronous multiple and single working vacation policies, Bernoulli feedback, reneging and retention of reneged customers. Our investigation has a great application in many practical life situations, especially when we deal with a human behavior, examples can be found in post offices, banks, hospitals, etc.

The rest of the paper is organized in the following manner. In Section 2, we describe the model. In Section 3, we develop the equations of the steady state probabilities of the model and derive their steady-state solutions, using the probability generating functions (PGFs). Section 4 is devoted to derive various performance measures. In Section 5, we formulate a cost model. Section 6 is consecrated to the numerical analysis.

2 The model

Consider a $M^X/M/c$ queueing system with feedback, waiting servers, both multiple and single working vacation policies, reneging, and retention of reneged customers.

– Customers arrive into the system according to a Poisson process with arrival rate λ. The sizes of successive arriving batches are i.i.d. random variables X_1, X_2,... distributed with probability mass function $P(X = l) = b_l; \; l = 1, 2, 3, ...$

– The service discipline is FCFS, and the system capacity is supposed to be infinite.

– The service time during normal busy period is assumed to be exponentially distributed with mean $1/\mu_1$.

– When the busy period is finished, the servers wait a random duration of time before they switch to a working vacation. This waiting duration is exponentially distributed with mean $1/\varpi$.

– The period of working vacation has an exponential distribution with mean $1/\vartheta$.

– During vacation, servers can provide service to new arrival. The service time during this period is assumed to be exponentially distributed with mean $1/\mu_2$, with $\mu_2 < \mu_1$.

– A synchronous working vacation is considered; once the system is empty, the servers, all together, go on working vacation, and they also return to the system as one at the same time.

– Both single and multiple working vacation are taking into consideration:

• Multiple working vacation policy (MWV); once the system is still empty at the end of a working vacation period, the servers begin another working vacation period. Otherwise, a normal busy period begins.

• Single working vacation policy (SWV); the servers take a working vacation all together and they comeback to the system as one, then wait passively for new arrivals. Otherwise, they start a new normal busy period.

– During working vacation period, the new arrival activates an impatience timer T, which is exponentially distributed with parameter χ. If the customer's service has not been completed before the customer's timer expires, the customer may abandon the queue. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers. Each reneged customer may leave the system with probability α and may be retained with probability $\alpha' = 1 - \alpha$.

– The inter-arrival times, vacation periods and service times during busy and working vacation periods are mutually independent.

– If the customer is unsatisfied with the quality of the service or if he requires another one, he can join the end of the queue with probability β'. Otherwise, he leaves the system definitively with probability β, where $\beta + \beta' = 1$.

It is worth noting that the system is stable under the condition $\lambda E(X) < c\beta\mu_1$, such that $E(X)$ is the mean of a batch of arrivals.

3 Steady-state solution

We present the steady-state probabilities of the system under both single and multiple working vacation policies. Let δ denote the indicator function:

$$\delta = \begin{cases} 1, & \text{for the single working vacation model;} \\ 0, & \text{for the multiple working vacation model.} \end{cases}$$

Let $L(t)$ be the number of customers in the system at time t. Let $J(t)$ denote the state of the servers at time t such that

$$J(t) = \begin{cases} 1, & \text{when the servers are on a normal busy period;} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the process $\{(J(t), L(t)), t \geq 0\}$ is a continuous-time Markov process with state space

$$\Omega = \{(j, n) : j = 0, 1, \ \ n = 0, 1, ...\}.$$

Let $P_{j,n} = \lim_{t\to\infty} P\{J(t) = j, L(t) = n\}, (j, n) \in \Omega$, denote the system state probabilities. The state transition diagram corresponding to our queueing system is illustrated in Figure 1.

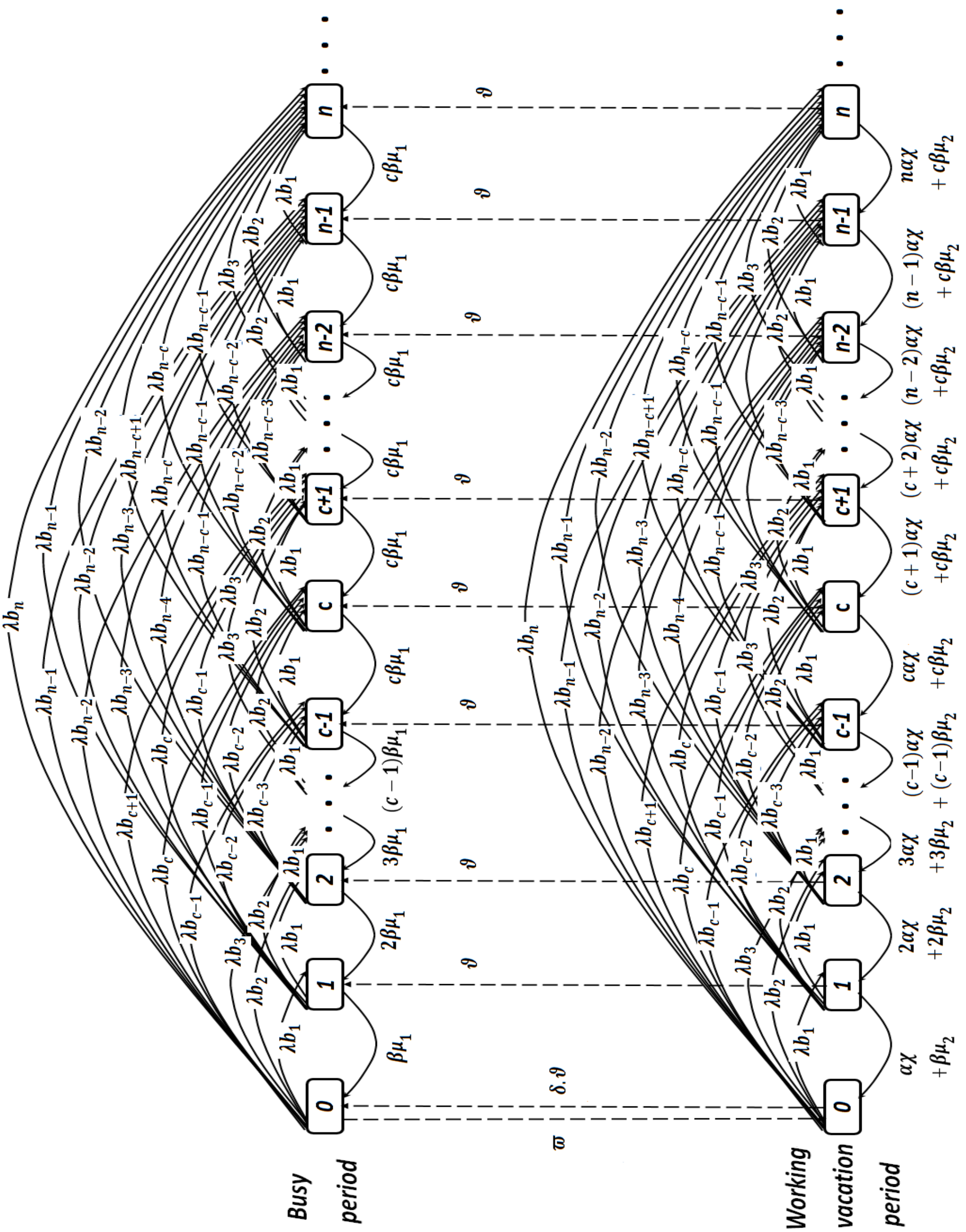

Figure 1: State-transition-rate diagram for SWV ($\delta = 1$) and MWV ($\delta = 0$).

Via Markov chain theory, we obtain the steady-state equations as follows:

$$(\lambda+\delta\vartheta)P_{0,0} = (\alpha\chi+\beta\mu_2)P_{0,1}+\varpi P_{1,0}, \;\; n=0, \tag{1}$$

$$(\lambda+\vartheta+\beta\mu_2+\alpha\chi)P_{0,1} = \lambda b_1 P_{0,0}+2(\beta\mu_2+\alpha\chi)P_{0,2}, \;\; n=1, \tag{2}$$

$$(\lambda+\vartheta+n(\beta\mu_2+\alpha\chi))P_{0,n} = \lambda\sum_{m=1}^{n} b_m P_{0,n-m}+(n+1)(\beta\mu_2+\alpha\chi)P_{0,n+1},$$
$$2\le n\le c-1, \tag{3}$$

$$(\lambda+\vartheta+c\beta\mu_2+n\alpha\chi)P_{0,n} = \lambda\sum_{m=1}^{n} b_m P_{0,n-m}+(c\beta\mu_2+(n+1)\alpha\chi)P_{0,n+1}, \tag{4}$$
$$n\ge c,$$

$$(\lambda+\varpi)P_{1,0} = \delta\vartheta P_{0,0}+\beta\mu_1 P_{1,1} \;\; n=0, \tag{5}$$

$$(\lambda+\beta\mu_1)P_{1,1} = \lambda b_1 P_{1,0}+2\beta\mu_1 P_{1,2}+\vartheta P_{0,1}, \;\; n=1, \tag{6}$$

$$(\lambda+n\beta\mu_1)P_{1,n} = \lambda\sum_{m=1}^{n} b_m P_{1,n-m}+(n+1)\beta\mu_1 P_{1,n+1}+\vartheta P_{0,n},$$
$$2\le n\le c-1, \tag{7}$$

$$(\lambda+c\beta\mu_1)P_{1,n} = \lambda\sum_{m=1}^{n} b_m P_{1,n-m}+c\beta\mu_1 P_{1,n+1}+\vartheta P_{0,n}, n\ge c. \tag{8}$$

The normalizing condition is defined as

$$\sum_{n=0}^{\infty} P_{0,n}+\sum_{n=0}^{\infty} P_{1,n}=1. \tag{9}$$

Let the probability generating functions (PGFs) presented as

$$G_j(z)=\sum_{n=0}^{\infty} z^n P_{j,n}, \;\; j=0,1. \tag{10}$$

Define

$$G_j'(z)=\frac{d}{dz}G_j(z), \;\; j=0,1.$$

The probability generating function(PGF) of the batch size X is given by

$$G(z)=\sum_{i=1}^{\infty} b_i z^i, \;\; |z|\le 1, \;\; G(1)=\sum_{i=1}^{\infty} b_i=1. \tag{11}$$

Multiplying equations (1)-(4) by z^n, and summing all possible values of n, we find

$$\begin{aligned}\alpha\chi z(1-z)G_0'(z) - [\lambda z(1-G(z)) + \vartheta z - c\beta\mu_2(1-z)]G_0(z) = \\ \beta\mu_2(1-z)\Phi_1(z) - \varpi z P_{1,0} - \vartheta(1-\delta)zP_{0,0}.\end{aligned} \tag{12}$$

In the same way, we get from equations (5)-(8),

$$\begin{aligned}(\lambda z(G(z)-1) + c\beta\mu_1(1-z))G_1(z) = -\vartheta z G_0(z) + \varpi z P_{1,0} \\ + \vartheta(1-\delta)zP_{0,0} + \beta\mu_1(1-z)\Phi_2(z),\end{aligned} \tag{13}$$

where

$$\Phi_1(z) = \sum_{n=0}^{c-1}(c-n)z^n P_{0,n}, \text{ and, } \Phi_2(z) = \sum_{n=0}^{c-1}(c-n)z^n P_{1,n}.$$

Next, putting $z = 1$ in equation (12) or equation (13), we find

$$\vartheta G_0(1) = \varpi P_{1,0} + \vartheta(1-\delta)P_{0,0}. \tag{14}$$

For $z \neq 1$, equation (12) can be given as

$$\begin{aligned}G_0'(z) - \left[\frac{\lambda\varphi'(z)}{\alpha\chi} + \frac{\vartheta + c\beta\mu_2}{\alpha\chi(1-z)} - \frac{c\beta\mu_2}{\alpha\chi z(1-z)}\right]G_0(z) = -\frac{\varpi}{\alpha\chi(1-z)}P_{1,0} \\ -\frac{\vartheta(1-\delta)}{\alpha\chi(1-z)}P_{0,0} + \frac{\beta\mu_2}{\alpha\chi z}\Phi_1(z),\end{aligned} \tag{15}$$

with

$$\varphi'(z) = \frac{1-G(z)}{1-z}.$$

Now, multiply both sides of Equation (15) by $e^{-\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{\frac{\vartheta}{\alpha\chi}} z^{\frac{c\beta\mu_2}{\alpha\chi}}$, we obtain

$$\begin{aligned}\frac{d}{dz}\left(e^{\frac{-\lambda\varphi(z)}{\alpha\chi}} z^{\frac{c\beta\mu_2}{\alpha\chi}}(1-z)^{\frac{\vartheta}{\alpha\chi}}G_0(z)\right) = e^{\frac{-\lambda\varphi(z)}{\alpha\chi}} z^{\frac{c\beta\mu_2}{\alpha\chi}}(1-z)^{\frac{\vartheta}{\alpha\chi}}\Bigg[-\frac{\varpi}{\alpha\chi(1-z)}P_{1,0} \\ +\frac{\beta\mu_2}{\alpha\chi z}\Phi_1(z) - \frac{\vartheta(1-\delta)}{\alpha\chi(1-z)}P_{0,0}\Bigg].\end{aligned} \tag{16}$$

Integrating the above equation from 0 to z, it yields

$$G_0(z) = e^{\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{-\frac{\vartheta}{\alpha\chi}} z^{-\frac{c\beta\mu_2}{\alpha\chi}}\left[\beta\mu_2 K_2(z) - \left(\vartheta(1-\delta)P_{0,0} + \varpi P_{1,0}\right)K_1(z)\right], \quad (17)$$

where

$$K_1(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}-1} s^{\frac{c\beta\mu_2}{\alpha\chi}} ds,$$

and

$$K_2(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}} s^{\frac{c\beta\mu_2}{\alpha\chi}-1}\Phi_1(s) ds.$$

To solve the differential equation (12), we must find $\Phi_1(z)$. Recursively, from equations (1)-(3), we obtain

$$P_{0,n} = \gamma_n P_{0,0} + \eta_n P_{1,0}, \quad (18)$$

with

$$\gamma_n = \begin{cases} 1, \ n=0; \\ \dfrac{\lambda+\vartheta\delta}{\beta\mu_2+\alpha\chi}, \ n=1; \\ \omega_{n-1}\gamma_{n-1} - \dfrac{M}{n}\displaystyle\sum_{i=1}^{n-1} b_i\gamma_{n-1-i}, \\ 2 \le n \le c-1, \end{cases} \qquad \eta_n = \begin{cases} 0, \ n=0; \\ \dfrac{-\varpi}{\beta\mu_2+\alpha\chi}, \ n=1; \\ \omega_{n-1}\eta_{n-1} - \dfrac{M}{n}\displaystyle\sum_{i=1}^{n-1} b_i\eta_{n-1-i}, \\ 2 \le n \le c-1, \end{cases}$$

where

$$\omega_n = \frac{\lambda+\vartheta+n(\beta\mu_2+\alpha\chi)}{(n+1)(\beta\mu_2+\alpha\chi)} \quad \text{and} \quad M = \frac{\lambda}{\beta\mu_2+\alpha\chi}.$$

Consequently,

$$G_0(z) = e^{\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{-\frac{\vartheta}{\alpha\chi}} z^{-\frac{c\beta\mu_2}{\alpha\chi}}\left[\left(\beta\mu_2 K_5(z) - \vartheta(1-\delta)K_1(z)\right)P_{0,0} - \left(\varpi K_1(z) - \beta\mu_2 K_4(z)\right)P_{1,0}\right], \quad (19)$$

with

$$K_4(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}} s^{\frac{c\beta\mu_2}{\alpha\chi}-1}\sum_{n=0}^{c-1}(c-n)s^n\eta_n ds,$$

and

$$K_5(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}} s^{\frac{c\beta\mu_2}{\alpha\chi}-1}\sum_{n=0}^{c-1}(c-n)s^n\gamma_n ds.$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $z = 1$ is the root of denominator of the right hand side of Equation (15). Thus, from equation (19), we get

$$P_{1,0} = \rho_0 P_{0,0}, \tag{20}$$

where

$$\rho_0 = \left[\frac{\beta\mu_2 K_5(1) - \vartheta(1-\delta)K_1(1)}{\varpi K_1(1) - \beta\mu_2 K_4(1)}\right].$$

Substituting equation (20) into equation (19), we obtain

$$\begin{aligned} G_0(z) = e^{\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{-\frac{\vartheta}{\alpha\chi}} z^{-\frac{c\beta\mu_2}{\alpha\chi}} \Bigg[\beta\mu_2\Big(K_5(z) + K_4(z)\rho_0\Big) \\ -\Big(\vartheta(1-\delta) + \varpi\rho_0\Big)K_1(z)\Bigg] P_{0,0}. \end{aligned} \tag{21}$$

Since $P_{0,.} = G_0(1) = \sum_{n=0}^{\infty} P_{0,n}$, by substituting equation (20) into (14), we get

$$P_{0,.} = \frac{\varpi\rho_0 + \vartheta(1-\delta)}{\vartheta} P_{0,0}. \tag{22}$$

Now, equation (13) can be written as

$$G_1(z) = \frac{-\vartheta z G_0(z) + \varpi z P_{1,0} + \vartheta(1-\delta)zP_{0,0} + \beta\mu_1(1-z)\Phi_2(z)}{\lambda z(G(z)-1) + c\beta\mu_1(1-z)}. \tag{23}$$

Next, in order to define $G_1(z)$ in terms of $P_{0,0}$, we need to express $P_{1,n}$ in terms of $P_{0,0}$. To this end, we employ the recursive method, then from equations (5)-(7) using equation 18, we get

$$P_{1,n} = \rho_n P_{0,0}, \tag{24}$$

where

$$\rho_n = \begin{cases} \rho_0, \text{ n=0}; \\ \dfrac{(\lambda+\varpi)\rho_0 - \vartheta\delta}{\beta\mu_1}, \text{ n=1}; \\ \omega_{n-1}\rho_{n-1} - \dfrac{\Delta_1}{n}\sum_{i=1}^{n-1} b_i\rho_{n-1-i} - \dfrac{\Delta_2}{n}\xi_{n-1}, \, 2 \leq n \leq c-1, \end{cases}$$

with

$$\xi_n = \gamma_n + \rho_0\eta_n, \;\; \omega_n = \frac{\lambda + (n-1)\beta\mu_1}{n(\beta\mu_1)}, \;\; \Delta_1 = \frac{\lambda}{\beta\mu_1}, \text{ and } \Delta_2 = \frac{\vartheta}{\beta\mu_1}.$$

Next, substituting equation (14) into (23), we have

$$G_1(z) = \frac{\beta\mu_1(1-z)\Phi_2(z) - z\vartheta(G_0(z) - G_0(1))}{\lambda z(G(z)-1) + c\beta\mu_1(1-z)}. \tag{25}$$

Via equation (25), applying l'hospital rule, we get

$$\lim_{z\to 1} G_1(z) = G_1(1) = \frac{\beta\mu_1\Phi_2(1) + \vartheta G_0'(1)}{c\beta\mu_1 - \lambda G'(1)}, \tag{26}$$

where

$$\Phi_2(1) = \sum_{n=0}^{c-1}(c-n)\rho_n P_{0,0}.$$

Now, via equation (15), applying l'hospital rule, it yields

$$\lim_{z\to 1} G_0'(z) = G_0'(1) = \frac{(\lambda G'(1) - c\beta\mu_2)G_0(1) + \beta\mu_2\Phi_1(1)}{\alpha\chi + \vartheta}, \tag{27}$$

where

$$\Phi_1(1) = \sum_{n=0}^{c-1}(c-n)(\gamma_n + \eta_n\rho_0)P_{0,0}.$$

Next, substituting equation (22) in equation (27), it yields

$$G_0'(1) = \frac{(\lambda G'(1) - c\beta\mu_2)(\varpi\rho_0 + \vartheta(1-\delta)) + \beta\mu_2\vartheta H_1(1)}{\vartheta(\alpha\chi + \vartheta)}P_{0,0}, \tag{28}$$

with

$$H_1(1) = \sum_{n=0}^{c-1} (c-n)(\gamma_n + \rho_0 \eta_n).$$

Then, substituting equation (28) into (26), we get $G_1(1)$ in terms of $P_{0,0}$.

Since $P_{1,.} = G_1(1) = \sum_{n=0}^{\infty} P_{1,n} > 0$, we obtain

$$P_{1,.} = R(1) P_{0,0}, \tag{29}$$

where

$$R(1) = \frac{\beta\mu_1 H_2(1)(\alpha\chi + \vartheta) + (\lambda G'(1) - c\beta\mu_2)(\varpi\rho_0 + \vartheta(1-\delta)) + \beta\mu_2\vartheta H_1(1)}{(c\beta\mu_1 - \lambda G'(1))(\alpha\chi + \vartheta)},$$

and

$$H_2(1) = \sum_{n=0}^{c-1} (c-n)\rho_n.$$

Finally, by substituting equations (22) and (29) into (9), we get

$$P_{0,0} = \left(\frac{\varpi\rho_0 + \vartheta(1-\delta)}{\vartheta} + R(1) \right)^{-1}.$$

4 Performance measures

– Let L_{wv} be the system size when the servers are in working vacation period. Then, the mean system size when the servers are in working vacation period is given by

$$\begin{aligned} \mathbb{E}(L_{wv}) &= G_0'(1) = \lim_{z \to 1} G_0'(z) \\ &= \frac{(\lambda G'(1) - c\beta\mu_2)(\varpi\rho_0 + \vartheta(1-\delta)) + \beta\mu_2\vartheta H_1(1)}{\vartheta(\vartheta + \alpha\chi)} P_{0,0}. \end{aligned} \tag{30}$$

– Let L_1 be the system size when the servers are in busy period. Therefore, the mean system size when the servers are in this period is as follows

$$\mathbb{E}(L_1) = G_1'(1) = \lim_{z \to 1} G_1'(z).$$

From equation (25), we get

$$\mathbb{E}(L_1) = \frac{\vartheta}{2(c\beta\mu_1 - \lambda G'(1))} G_0''(1) + \frac{\vartheta(\lambda G''(1) + 2c\beta\mu_1)}{2(c\beta\mu_1 - \lambda G'(1))^2} G_0'(1)$$
$$+ \left[\frac{\beta\mu_1}{c\beta\mu_1 - \lambda G'(1)} H_2'(1) + \frac{\lambda\beta\mu_1(2G'(1) + G''(1))}{2(c\beta\mu_1 - \lambda G'(1))^2} H_2(1)\right] P_{0,0}, \tag{31}$$

with $H_2'(1) = \sum_{n=0}^{c-1} n(c-n)\rho_n$, and $G_0''(1)$ is obtained by differentiating twice $G_0(z)$ at $z = 1$. Via equation (12), we have

$$G_0''(1) = \frac{(2\lambda G'(1) - 2c\beta\mu_2)G_0'(1) + (\lambda G''(1) + 2c\beta\mu_2)G_0(1)}{\alpha\chi + \vartheta}. \tag{32}$$

Then, substituting equation(32) into (31), we find

$$\mathbb{E}(L_1) = \left[\frac{\vartheta(2\lambda G'(1) - 2c\beta\mu_2)}{2(c\beta\mu_1 - \lambda G'(1))(\alpha\chi + \vartheta)} + \frac{\vartheta(\lambda G''(1) + 2c\beta\mu_1)}{2(c\beta\mu_1 - \lambda G'(1))^2}\right] \mathbb{E}(L_{wv})$$
$$+ \left[\frac{\beta\mu_1}{c\beta\mu_1 - \lambda G'(1)} H_2'(1) + \frac{\lambda\beta\mu_1(2G'(1) + G''(1))}{2(c\beta\mu_1 - \lambda G'(1))^2} H_2(1)\right] P_{0,0} \tag{33}$$
$$+ \frac{\vartheta(\lambda G''(1) + 2c\beta\mu_2)}{2(c\beta\mu_1 - \lambda G'(1))(\alpha\chi + \vartheta)} P_{0,\cdot}.$$

– The mean system size: Let L denote the number of customers in the system. Thus

$$\mathbb{E}(L) = \mathbb{E}(L_{wv}) + \mathbb{E}(L_1).$$

– The mean queue length:

$$\mathbb{E}(L_q) = \sum_{n=c+1}^{\infty} (n-c)P_{n,0} + \sum_{n=c+1}^{\infty} (n-c)P_{n,1} = \mathbb{E}(L) - c + \Big(H_1(1) + H_2(1)\Big) P_{0,0}.$$

– The probability that the servers are idle during busy period: From (19), we get

$$P_I = \left[\frac{\beta\mu_2 K_5(1) - \vartheta(1-\delta)K_1(1)}{\varpi K_1(1) - \beta\mu_2 K_4(1)}\right] P_{0,0}.$$

– The probability that the servers are on working vacation period:

$$P_{wv} = G_0(1) = \frac{\varpi\rho_0 + \vartheta(1-\delta)}{\vartheta} P_{0,0}.$$

– The probability that the servers are working (serving customers) during normal busy period:

$$P_B = 1 - P_{wv} - P_I.$$

– The average rate of reneging:

$$R_{ren} = \alpha\xi \sum_{n=1}^{\infty} n P_{0,n} = \alpha\chi\mathbb{E}(L_{wv}).$$

– The average rate of retention of impatient customers:

$$R_{ret} = \alpha'\chi \sum_{n=1}^{\infty} n P_{0,n} = \alpha'\chi\mathbb{E}(L_{wv}).$$

5 Cost model

We present a cost model in order to develop a cost-optimum analysis of the queueing model under consideration. The following cost elements are needed:

- C_1 : Cost per unit time when the servers are working during busy period.
- C_2 : Cost per unit time when the servers are on working vacation.
- C_3 : Cost per unit time when the servers are idle during busy period.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.
- C_5 : Cost per unit time when a customer reneges.
- C_6 : Cost per unit time when a customer is retained.
- C_7 : Cost per service per unit time.
- C_8 : Cost per unit time when a customer returns to the system as a feedback customer.
- C_9 : Fixed server purchase cost per unit.

Let F be the total expected cost per unit time of the system:

$$F = C_1 P_B + C_2 P_{wv} + C_3 P_I + C_4 \mathbb{E}(L_q) + C_5 R_{ren} + C_6 R_{ret} + c\mu_2(C_7 + \beta' C_8) + c\mu_1(C_7 + \beta' C_8) + cC_9.$$

We consider in this investigation the cost optimization problem under a given cost structure via genetic algorithm (GA). A total expected cost function has been developed in order to determine an optimum regular and working service rates μ_1^* μ_2^*, the number of servers in the system c^* as well as the optimum expected cost $F(\mu_1^*, \mu_2^*, c^*)$.

The optimization problem may be illustrated mathematically as:

$$\text{Minimize: } F(\mu_1, \mu_2, c).$$

6 Numerical analysis

This section presents a numerical results conducted by coding computer program in R software in order to show the applicability of the theoretical analysis. We perform an analysis on the optimum values μ_1^*, μ_2^* and c^* based on changes in specific values of the system parameters λ, r, ϖ, ϑ, χ, α and β. For computational aim, we assume that the arrival batch size X follows a geometric distribution with parameter r;

$$P(X = l) = b_l = (1-r)^{l-1}r, \ 0 < r < 1 \ (l = 1, 2, ...),$$

with

$$B(z) = \frac{rz}{1-(1-r)z}, \quad E(X) = B'(1) = \frac{1}{r}, \quad \text{and} \quad E(X^2) = B''(1) = \frac{2(1-r)}{r^2}.$$

The different cost elements are taken as $C_1 = 15$, $C_2 = 10$, $C_3 = 5$, $C_4 = 15$, $C_5 = 25$, $C_6 = 5$, $C_7 = 15$, $C_8 = 10$, and $C_9 = 3$.

The total cost function is presented in Tables 1-4 and plotted (using GA) in Figures 2-3 by varying values of the system parameters. Further, Tables 1-4 depict the optimum values of μ_1, μ_2, c, and the minimum expected cost F^* along with the corresponding performance measures P_{wv}^*, P_I^*, P_B^*, $\mathbb{E}(L_{wv})^*$, $\mathbb{E}(L_1)^*$, R_{ren}^*, and R_{ret}^* for different values of λ, r, ϑ, ϖ, β, χ, and α, where

- Figure 2: $\lambda = 2.0$, $\chi = 1.50$, $\vartheta = 0.40$, $\varpi = 2.8$, $c = 4$, $r = 0.70$, $\beta = 0.75$, and $\alpha = 0.60$.
- Figure 3: $\lambda = 2.0$, $\chi = 1.50$, $\vartheta = 0.40$, $\varpi = 2.8$, $c = 3$, $r = 0.70$, $\beta = 0.75$, and $\alpha = 0.60$.
- Table 1: $\chi = 1.80$, $\vartheta = 0.80$, $\varpi = 1.50$, $\beta = 0.70$, and $\alpha = 0.60$.
- Table 2: $\chi = 1.80$, $\varpi = 1.50$, $r = 0.75$, $\beta = 0.70$, and $\alpha = 0.60$.
- Table 3: $\lambda = 2.10$, $\chi = 1.80$, $\vartheta = 1.70$, $r = 0.75$, and $\alpha = 0.60$.
- Table 4: $\lambda = 2.10$, $\vartheta = 1.20$, $\varpi = 1.40$, $r = 0.75$, and $\beta = 0.8$.

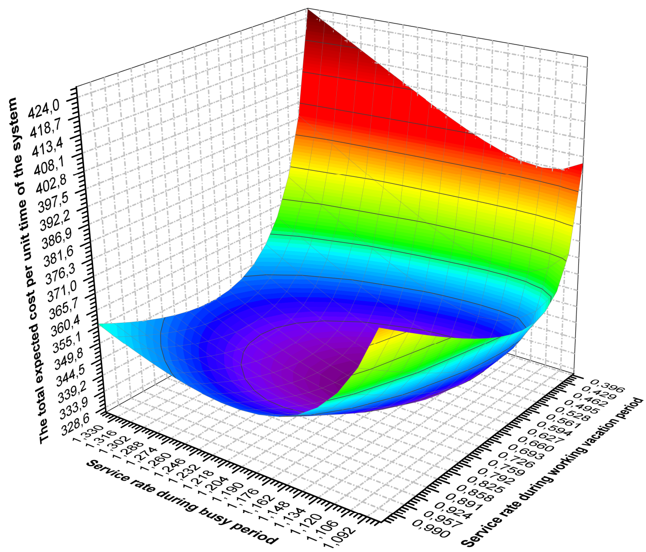

Figure 2: Optimal cost vs. μ_1 and μ_2 in multiple working vacation policy.

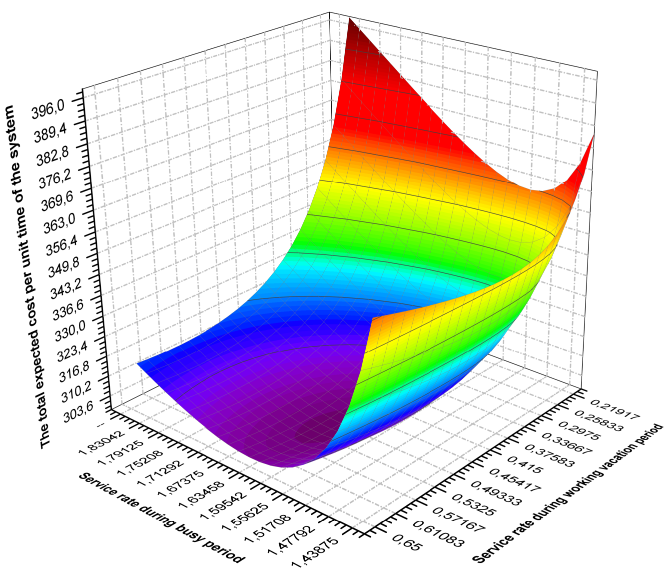

Figure 3: Optimal cost vs. μ_1 and μ_2 in single working vacation policy.

Table 1: Optimum performance measures for different values of λ and r.

	(λ, r)	(2.00,0.40)	(2.50,0.40)	(3.00,0.40)	(2.00,0.80)	(2.50,0.80)	(3.00,0.80)
MWV	μ_1^*	2.502229	2.814931	3.327466	2.308868	2.406292	2.410777
	μ_2^*	1.030319	0.994439	0.992243	0.966170	0.906214	0.895353
	c^*	4	4	4	4	4	4
	F^*	557.2287	629.3045	706.5886	456.5071	470.2396	472.7053
	P_{wv}^*	0.392421	0.272212	0.230084	0.599018	0.456898	0.322744
	P_I^*	0.008850	0.010206	0.013398	0.039825	0.043037	0.038241
	P_B^*	0.598729	0.717582	0.756518	0.361157	0.500066	0.639014
	$\mathbb{E}(L_{wv})^*$	1.991646	1.745720	1.787051	1.748974	1.680051	1.431087
	$\mathbb{E}(L_1)^*$	5.724183	9.696449	11.07218	0.664982	1.236184	2.069782
	R_{ren}^*	2.150977	1.885377	1.930015	1.888891	1.814455	1.545574
	R_{ret}^*	1.433985	1.256918	1.286677	1.259261	1.209637	1.030383
SWV	μ_1^*	2.568105	2.940578	3.443902	2.327490	2.415154	2.477015
	μ_2^*	0.810703	0.807034	0.805419	0.807520	0.804545	0.803916
	c^*	3	4	4	3	3	3
	F^*	466.4389	553.3533	645.1619	315.42420	329.8097	342.7443
	P_{wv}^*	0.168896	0.115348	0.086045	0.255817	0.184708	0.123709
	P_I^*	0.090078	0.061519	0.045891	0.136436	0.098511	0.065978
	P_B^*	0.741026	0.823133	0.868064	0.607747	0.716780	0.810313
	$\mathbb{E}(L_{wv})^*$	0.777926	0.736848	0.668306	0.673716	0.616176	0.499193
	$\mathbb{E}(L_1)^*$	7.314611	8.614643	11.298461	1.667467	2.546185	3.969401
	R_{ren}^*	0.840160	0.795796	0.721771	0.727614	0.665470	0.539129
	R_{ret}^*	0.560107	0.530530	0.481180	0.485076	0.443647	0.359419

Table 2: Optimum performance measures for different values of ϑ and λ.

	(ϑ, λ)	(0.60,2.40)	(1.20,2.40)	(1.80,2.40)	(0.60,2.80)	(1.20,2.80)	(1.80,2.80)
MWV	μ_1^*	2.190517	2.017012	2.015180	2.207845	2.110887	2.105012
	μ_2^*	0.909674	0.808033	0.821407	0.891618	0.847985	0.846099
	c^*	4	4	4	4	4	4
	F^*	431.3379	390.9331	376.1093	456.1155	401.9582	386.6201
	P_{wv}^*	0.418996	0.334364	0.284606	0.353305	0.232393	0.190844
	P_I^*	0.027096	0.037466	0.046395	0.027895	0.033732	0.039143
	P_B^*	0.553908	0.628170	0.668999	0.618799	0.733875	0.770013
	$\mathbb{E}(L_{wv})^*$	1.717204	1.052859	0.743039	1.697097	0.857633	0.583564
	$\mathbb{E}(L_1)^*$	1.840412	1.972379	1.994956	2.371337	2.861829	2.894786
	R_{ren}^*	1.854580	1.137087	0.802482	1.832864	0.926243	0.630249
	R_{ret}^*	1.236387	0.758058	0.534988	1.221910	0.617495	0.420166
SWV	μ_1^*	2.030994	2.083993	2.095927	2.302819	2.328809	2.358547
	μ_2^*	0.803204	0.810854	0.821965	0.819441	0.811570	0.829329
	c^*	3	3	3	3	3	3
	F^*	316.0318	301.9229	295.5010	348.8217	331.6997	326.7999
	P_{wv}^*	0.143504	0.090891	0.064578	0.127889	0.077189	0.057959
	P_I^*	0.057401	0.072713	0.077494	0.051156	0.061751	0.069551
	P_B^*	0.799095	0.836396	0.857928	0.820955	0.861059	0.872489
	$\mathbb{E}(L_{wv})^*$	0.538211	0.253632	0.144440	0.564638	0.253497	0.152763
	$\mathbb{E}(L_1)^*$	4.548054	4.317499	4.412283	5.172256	5.082254	4.873308
	R_{ren}^*	0.581268	0.273922	0.155995	0.609809	0.273777	0.164984
	R_{ret}^*	0.387512	0.182615	0.103996	0.406539	0.182518	0.109990

Table 3: Optimum performance measures for different values of ϖ and β.

	(ϖ,β)	(0.70,0.40)	(1.40,0.40)	(2.10,0.40)	(0.70,0.80)	(1.40,0.80)	(2.10,0.80)
MWV	μ_1^*	2.522527	2.519482	2.507588	2.520041	2.500059	2.494978
	μ_2^*	1.043003	1.074571	1.080151	0.877730	0.822386	0.815934
	c^*	4	4	4	4	4	4
	F^*	473.9346	484.3575	488.3804	429.9572	439.7665	445.7521
	P_{wv}^*	0.238491	0.329456	0.386641	0.388090	0.507123	0.559101
	P_I^*	0.038605	0.028735	0.022758	0.126880	0.079166	0.057835
	P_B^*	0.722904	0.641809	0.590601	0.485030	0.413712	0.383064
	$\mathbb{E}(L_{wv})^*$	0.557301	0.770098	0.903807	0.903750	1.182511	1.303904
	$\mathbb{E}(L_1)^*$	2.857388	2.240101	1.856173	0.874812	0.676071	0.589290
	R_{ren}^*	0.601885	0.831706	0.976112	0.976050	1.277112	1.408216
	R_{ret}^*	0.401257	0.554471	0.650741	0.650700	0.851408	0.938811
SWV	μ_1^*	3.053832	2.996650	2.963749	2.496449	2.487807	2.485160
	μ_2^*	0.822178	0.814544	0.813022	0.837259	0.844172	0.852589
	c^*	3	3	3	3	3	3
	F^*	397.0467	399.3800	402.0786	299.0660	306.4026	312.8086
	P_{wv}^*	0.031943	0.056947	0.078803	0.095088	0.162837	0.214488
	P_I^*	0.077577	0.069150	0.063793	0.230929	0.197730	0.173633
	P_B^*	0.890480	0.873903	0.857405	0.673983	0.639433	0.611879
	$\mathbb{E}(L_{wv})^*$	0.061687	0.109877	0.152020	0.192472	0.329719	0.434485
	$\mathbb{E}(L_1)^*$	5.108708	5.373007	5.512663	1.681255	1.625559	1.573625
	R_{ren}^*	0.066622	0.118667	0.164181	0.207870	0.356096	0.469244
	R_{ret}^*	0.044415	0.079111	0.109454	0.138580	0.237398	0.312829

Table 4: Optimum performance measures for different values of χ and α.

	(χ,α)	(1.40,0.40)	(2.00,0.40)	(2.60,0.40)	(1.40,0.80)	(2.00,0.80)	(2.60,0.80)
MWV	μ_1^*	2.035823	2.006344	2.003822	2.002408	2.001909	2.001728
	μ_2^*	0.652905	0.762774	0.778089	0.786285	0.958438	1.147763
	c^*	4	4	4	4	4	4
	F^*	365.2232	379.3121	397.5228	383.8521	414.1506	439.5431
	P_{wv}^*	0.299538	0.361594	0.456885	0.489400	0.661038	0.795944
	P_I^*	0.073968	0.064365	0.054125	0.049638	0.027214	0.008487
	P_B^*	0.626494	0.574041	0.488990	0.460962	0.311747	0.195570
	$\mathbb{E}(L_{wv})^*$	1.138056	1.166085	1.280413	1.312554	1.406371	1.409206
	$\mathbb{E}(L_1)^*$	1.531777	1.341754	1.068388	0.983563	0.526212	0.206863
	R_{ren}^*	0.637311	0.932868	1.331630	1.470061	2.250193	2.918669
	R_{ret}^*	0.955967	1.399302	1.997445	0.367515	0.562548	0.729667
SWV	μ_1^*	2.012715	2.008421	2.006876	2.003986	2.001375	2.000632
	μ_2^*	0.418530	0.454064	0.528121	0.473229	0.501516	0.556894
	c^*	3	3	3	3	3	3
	F^*	252.1570	256.7518	261.3502	254.0936	257.2435	264.9412
	P_{wv}^*	0.144164	0.151797	0.159100	0.169835	0.183949	0.190602
	P_I^*	0.123569	0.130112	0.136371	0.145573	0.157671	0.163373
	P_B^*	0.732268	0.718091	0.704529	0.684591	0.658381	0.646026
	$\mathbb{E}(L_{wv})^*$	0.481415	0.429367	0.387929	0.391589	0.328053	0.276519
	$\mathbb{E}(L_1)^*$	2.540087	2.433606	2.352216	2.232054	2.075916	1.995080
	R_{ren}^*	0.269592	0.343494	0.403446	0.438580	0.524884	0.575158
	R_{ret}^*	0.404389	0.515241	0.605170	0.109645	0.131221	0.143790

Discussion

1. Figures 2 and 3 describe the impact of μ_1 and μ_2 on the optimal expected cost, for multiple and single working vacation policies, respectively. We clearly see the convexity of the curves, which shows that there exist certain values of the service rates μ_1 and μ_2 that minimize the total expected cost function for the chosen set of model parameters. Further, the optimal expected cost per unit time converges to the solution $F = 327.783374$ at $\mu_1^* = 1.184418$, $\mu_2^* = 0.650655$, and $c^* = 4$, under multiple working vacation, and converges to $F = 299.378265$ at $\mu_1^* = 1.615057$, $\mu_2^* = 0.505152$, and $c^* = 3$, under single working vacation.

2. From Figures 2-3 and Tables 1-4, it is clearly observed that the optimum service rate μ_1^* of multiple vacation model is smaller than that of single vacation model, whereas, the optimum service rate μ_2^*, the minimum expected cost F^*, and the optimum value of the optimum number of servers c^* of multiple vacation model is bigger than that of single vacation model, as intuitively expected.

3. For both SWV and MWV, μ_1^* increases (resp. decrease) with λ (resp. with r), while μ_2^* decreases with λ and r. In view of the stability of the system, this results are quite reasonable. We remark from Table 2 that, μ_2^* increases with λ, this can be due to the choice of ϑ.

4. The parameters μ_1^* and μ_2^* decrease with ϑ in MWV and increase along the increasing of ϑ in SWV. Moreover, a decreasing trend is seen in μ_1^* with ϖ, χ, α and β in both MWV and SWV. While μ_2^* increases with χ and α, under both policies, it decreases with β in MWV and significantly increases along the increasing of β in SWV. These results match with our intuition. Further, μ_2^* is not monotone with ϖ; this is due to the choice of the system parameters.

5. For both SWV and MWV, the optimum expected cost F^* increases with λ, ϖ, χ, and α, while it decreases with ϑ, r and β. This is quite reasonable, λ (resp. r and β) increases (resp. decrease) the mean system size, this results in the increasing (resp. the decreasing) of the minimum expected cost. On the other side, with the increasing of vacation rate, the servers rapidly switch to the busy period at which the customers are served with a large service rate, this implies a decreases in F^*. In addition, the higher the waiting server rate, the greater the probability that the servers go on working vacation and the bigger the average

rate of impatience which results in the increasing of optimal expected cost F^*.

6. For both MWV and SWV models, along the increasing of λ, the characteristics $\mathbb{E}(L_1)^*$ and P_B^* increase, while $\mathbb{E}(L_{wv})^*$, P_{wv}^*, R_{ren}^*, and R_{ret}^* decrease; obviously, the arrival rate increases the system size which, in returns, increases the probability that the servers are serving customers during normal busy servers. On the other side, this implies a decrease in the mean number of customers in the system during vacation period which results in the decreasing of P_{wv}^*, R_{ren}^* and R_{ret}^*. Further, we observe from Table 2 that P_I^* decreases with λ, as it should be, while from Table 1, it increases with the increasing of the arrival rate λ, this can be due to the choice of the system parameters ϑ and r.

7. For both policies, with the increasing of r, $\mathbb{E}(L_1)^*$, $\mathbb{E}(L_{wv})^*$, P_B^*, R_{ren}^*, and R_{ret}^* decrease significantly, whereas P_{wv}^* increases with r, which is coherent with the fact that increasing the batch size r decreases the system size when the servers are in working vacation. Consequently, the probability of working vacation decreases. Further, the probability that the servers are idle during normal busy period P_I^* decreases with r under MWV and increases with the increasing of the batch size r under SWV, as intuitively expected.

8. For both MWV and SWV policies, an increasing trend is observed in $\mathbb{E}(L_1)^*$, P_B^*, and P_I^* with ϑ and a decreasing trend is seen in $\mathbb{E}(L_1)^*$, P_{wv}^*, $\mathbb{E}(L_{wv})^*$, R_{ren}^*, and R_{ret}^* along the increasing of the vacation rate ϑ. This is quite explicable, the higher the vacation rate, the greater the mean system size during normal busy period, and the smaller the mean number of customers in the system during working vacation period, which lead to the decreasing of average rates of reneging and retention.

9. For both MWV and SWV models, the characteristics P_B^* and $\mathbb{E}(L_1)^*$ decrease with χ and α, while P_{wv}^* and R_{ren}^* increase with the increasing of χ and α. Evidently, the impatience rate increases the probability of working vacation, thus significant customers may leave the system which results in the decreasing of the mean number of customers in the system during normal busy period. Consequently, the probability that the servers work during normal busy period decreases significantly. Further, R_{ret}^* increases (resp. decreases) with χ (resp. α) under single and multiple working vacation policies, as it should be. Then, obviously, $\mathbb{E}(L_{wv})^*$

(resp. P_I^*) decreases (resp. increases) with χ and α in SWV model and increases (resp. decreases) along the increasing of χ and α in MWV model.

10. Both $\mathbb{E}(L_{wv})^*$, P_{wv}^*, R_{ren}^*, and R_{ret}^* increase with the increasing of ϖ and β, as intuitively expected. It is quite clear that the probabilities of non-feedback decreases the mean system size. Therefore, the servers switch to working vacation period at which the customers may leave the system because of the impatience phenomenon. Consequently, the average reneging and retention rates increase with β and ϖ. In addition, as the mean waiting time of the servers decreases, the working vacation period increases at which the impatience phenomenon may take place. Thus, R_{ren}^* and R_{ret}^* increase with ϖ. Further, to keep the system size under control and to avoid more reneging of customers, the firm may employ some strategies, which can be increasing the service rates, or engaging some additional service channels. Therefore, the average retention rate increases. Moreover, P_B^* decreases with ϖ and β, while P_I^* increases with β and decreases with ϖ, as intuitively expected. In addition, $\mathbb{E}(L_1)^*$ decreases significantly with β, while it increases with ϖ for $\beta = 0.4$ and decreases along the increasing of ϖ for $\beta = 0.80$. This can be due to the choice of the system parameters.

11. By comparing the two policies, multiple and single working vacations, we observe that

$$\begin{aligned}
P_{wv}^*(\text{single working vacation}) &< P_{wv}^*(\text{multiple working vacation}),\\
R_{ren}^*(\text{single working vacation}) &< R_{ren}^*(\text{multiple working vacation}),\\
R_{ret}^*(\text{single working vacation}) &< R_{ret}^*(\text{multiple working vacation}),\\
\mathbb{E}(L_{wv})^*(\text{single working vacation}) &< \mathbb{E}(L_{wv})^*(\text{multiple working vacation}),
\end{aligned}$$

while

$$\begin{aligned}
\mathbb{E}(L_1)^*(\text{multiple working vacation}) &< \mathbb{E}(L_1)^*(\text{single working vacation}),\\
P_B^*(\text{multiple working vacation}) &< P_B^*(\text{single working vacation}),\\
P_I^*(\text{multiple working vacation}) &< P_I^*(\text{single working vacation}).
\end{aligned}$$

Thus, it can be concluded that the single working vacation model has better performance measures than the multiple working vacations model. This perfectly matches with our expected intuition.

7 Conclusion

In this paper, we considered an infinite-space multi-server queueing system with batch arrival, waiting servers, synchronous multiple and single working vacation policies, Bernoulli feedback, reneging and retention of reneged customers. We developed the equations of the steady state probabilities of the model and obtained their steady-state solutions, using the probability generating functions (PGFs). Further, we derived various performance measures. A cost model has been formulated. In addition, the cost optimization problem under a given cost structure via genetic algorithm (GA) has been done. For further works, it will be interesting to deal with more realistic models including $\mathsf{G/G/c}$ and $\mathsf{G^X/G/c}$ queues with waiting servers, multiple and single working vacation policies, reneging, and retention of reneged customers.

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On tridiagonal matrices associated with Jordan blocks

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Abstract. This paper aims to show how some standard general results can be used to uncover the spectral theory of tridiagonal and related matrices more elegantly and simply than existing approaches. As a typical example, we apply the theory to the special tridiagonal matrices in recent papers on orthogonal polynomials arising from Jordan blocks. Consequently, we find that the polynomials and spectral theory of the special matrices are expressible in terms of the Chebyshev polynomials of second kind, whose properties yield interesting results. For special cases, we obtain results in terms of the Fibonacci numbers and Legendre polynomials.

1 Introduction

In the recent survey [9], we presented important modern applications involving tridiagonal matrices in applied mathematics, physics, and engineering and

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showed how the solutions/spectral theory could be easily obtained from Losonczi's work [13] of nearly thirty years ago. The latter work seems not only to have been neglected, but has also been re-cast or re-discovered in alternative guises since. In fact, many of these papers cite less general work that followed Losonczi a decade or so later. Interestingly, as described in [10], Losonczi's work grew out of the 1928 seminal paper by Egerváry and Szász [5].

In this note we uncover most of the important material in an existing publication in a simple and more elegant manner. Specifically, we aim to study the work of Capparelli and Maroscia [2] on orthogonal polynomials arising from Jordan blocks, which appeared in 2013, more than 20 years after Losonczi's paper. Before doing so, however, we need to present a summary of the main results of [13] in the following section.

2 A general perturbed tridiagonal Toeplitz matrix

The spectral characterization of an n-dimensional tridiagonal Toeplitz matrix possessing perturbations of the form

$$M_n(a,b,c,d) = \begin{pmatrix} a & c & & & & \\ d & 0 & c & & & \\ & d & \ddots & \ddots & & \\ & & \ddots & \ddots & c & \\ & & & d & 0 & c \\ & & & & d & b \end{pmatrix}_{n\times n}$$

has become the focus of much interest or activity over the past thirty years. On this issue the interested reader is advised to consult [9], which also provides an extensive recent list of references to the applications stemming from this activity.

The first major theoretical study dealing with this type of matrix was Losonczi's 1992 paper [13], which, in turn, was apparently motivated by the Rutherford's seminal paper [16] (cf. also [6]). From [13, Theorem 2] we find that the characteristic polynomial of M_n is given by

$$p_n(x) = (\sqrt{cd})^n \left(U_n\left(\frac{x}{2\sqrt{cd}}\right) - \frac{(a+b)}{\sqrt{cd}}\, U_{n-1}\left(\frac{x}{2\sqrt{cd}}\right) + \frac{ab}{cd}\, U_{n-2}\left(\frac{x}{2\sqrt{cd}}\right)\right). \tag{1}$$

The Chebyshev polynomials of the second kind have been studied extensively, e.g. [3], and are known to satisfy the following three-term recurrence relation:

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x)\,, \quad \text{for all } n = 1, 2, \ldots\,, \tag{2}$$

with the initial values being $U_0(x) = 1$ and $U_1(x) = 2x$. Arguably, the most commonly-used form for the Chebyshev polynomials is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad 0 \leqslant \theta < \pi,$$

where $x = \cos\theta$ and n is a non-negative integer. Other standard representations are

$$\begin{aligned} U_n(x) &= \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}} && (3)\\ &= \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{2k+1} (1-x)^k && (4) \end{aligned}$$

Furthermore, the generating function of $U_n(x)$, i.e., where they appear as the coefficients of an infinite power series, is given by

$$\sum_{n=0}^{\infty} U_n(x)\, t^n = \frac{1}{1-2xt+t^2}\,. \tag{5}$$

For more details about these polynomials and their properties, the reader is referred to Chapter 22 of [1]. Note also that Losonczi's paper applies to more general matrices than the above tridiagonal Toeplitz matrix. Nevertheless, there is no loss of generality in computing the eigenvalues of M_n and its characteristic polynomial given by (1) using [13].

As far as the eigenvectors are concerned, if λ is an eigenvalue of $M_{n,1}$ assuming that $cd \neq 0$, then from Section 3 of [13], the eigenvectors corresponding to λ, which can be represented as $u = (u_0, u_1, \ldots, u_{n-1})^T$ with the superscript denoting the transpose, are given by

$$u_\ell = C\left(-\sqrt{\frac{c}{d}}\right)^{\ell} \left(\sin(\ell+1)\,\theta + \frac{a}{b}\sin\ell\theta\right)\,, \quad \text{for } \ell = 0, 1, \ldots, n-1, \tag{6}$$

provided $\lambda = -2\sqrt{cd}\cos\theta$, $\theta \neq m\pi$, $m \in Z$, and the arbitrary constant, $C \neq 0$. Since it is not required here, we do not present the $\theta = m\pi$ result. By presenting the above material, we aim in this note to derive shorter and thus, more elegant, proofs in the study of singular values of Jordan blocks than those presented in [2].

3 Jordan blocks

According to Definition 3.1.1 in [12], a Jordan block $J_n(r)$ is a $n \times n$ matrix composed of zeros everywhere except along the diagonal, which consists of a single number and the superdiagonal, where each entry or element is equal to unity. Thus, it has the following form:

$$J_n(r) = \begin{pmatrix} r & 1 & & & \\ & r & 1 & & \\ & & \ddots & \ddots & \\ & & & r & 1 \\ & & & & r \end{pmatrix}_{n\times n} .$$

The scalar r is generally a complex number, while the determinant of a Jordan block is r^n. However, since we are concerned the singular values of $J_n(r)$, i.e., the square root of the eigenvalues of their product with their transpose, r will be treated here mainly as a real number following [6], although some interesting results will be obtained when $r = i$.

Capparelli and Maroscia [2] study the product of the transpose of a Jordan block with itself, which yields

$$J_n^T(r)J_n(r) = \begin{pmatrix} r^2 & r & & & \\ r & 1+r^2 & r & & \\ & r & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & 1+r^2 \end{pmatrix}_{n\times n} . \tag{7}$$

Their primary aim is to determine the singular values of $J_n(r)$, although bounds for more general cases, where the entries are less uniform, can be found in [6]. Consequently, they focus on the following matrices:

$$\mathcal{T}_n(\alpha, r) = \begin{pmatrix} \alpha & r & & & \\ r & 1+\alpha & r & & \\ & r & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & 1+\alpha \end{pmatrix}_{n\times n} . \tag{8}$$

and

$$\mathcal{U}_n(\alpha, r) = \begin{pmatrix} \alpha & r & & & \\ r & \alpha & r & & \\ & r & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & \alpha \end{pmatrix}_{n\times n}. \tag{9}$$

Since the determinant of the transpose of a square matrix is equal to the determinant of the square matrix, from the above we see that $\det J_n(r) = (\det \mathcal{T}_n(r^2, r))^{1/2}$.

In the remainder of this section we shall consider in the next subsection general results involving α and r for the above tridiagonal matrices, and then special cases in the following subsection.

3.1 General cases

Our first observation is concerned with the relationship between the determinants of both matrices in addition to the derivation of their generating functions. These are encapsulated in the following theorem.

Theorem 1 *The determinant of the matrix $\mathcal{T}_n(\alpha, r)$ is related to the determinant of $\mathcal{U}_n(\alpha, r)$ according to*

$$\det \mathcal{T}_n(\alpha, r) = \det \mathcal{U}_n(1+\alpha, r) - \det \mathcal{U}_{n-1}(1+\alpha, r), \tag{10}$$

while their generating functions are given by

$$\sum_{n=0}^{\infty} \det \mathcal{U}_n(\alpha, r) t^n = \frac{1}{1 - \alpha t + r^2 t^2}, \tag{11}$$

and

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(\alpha, r) t^n = \frac{1-t}{1 - (1+\alpha)t + r^2 t^2}. \tag{12}$$

Proof. From Theorem 2 in [13], or specifically, (1) with $x = 1 + \alpha$, $a = -1$, $b = 0$, and $c = d = r$, we arrive at

$$\det \mathcal{T}_n(\alpha, r) = r^{n-1} \left(r\, U_n\left(\frac{1+\alpha}{2r}\right) - U_{n-1}\left(\frac{1+\alpha}{2r}\right)\right). \tag{13}$$

On the other hand, the determinant of $\mathcal{U}_n(\alpha, r)$ is found using the same values except that $x = \alpha$ and $a = b = 0$. Then we find that

$$\det \mathcal{U}_n(\alpha, r) = r^n U_n\left(\frac{\alpha}{2r}\right) . \tag{14}$$

Introducing (14) into (13) yields (10).

If one multiplies (14) by t^n and sums from $n = 0$ to ∞, then one obtains

$$\sum_{n=0}^{\infty} \det \mathcal{U}_n(\alpha, r) t^n = \sum_{n=0}^{\infty} U_n\left(\frac{\alpha}{2r}\right)(rt)^n.$$

The right-hand side of the above result is simply the generating function for the Chebyshev polynomials of the second kind. Therefore, putting t and x in (5) equal to rt and $\alpha/2r$, respectively, yields (11).

Adopting the same procedure to (14) yields

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(\alpha, r) t^n = \sum_{n=0}^{\infty} \Big(\det \mathcal{U}_n(1+\alpha, r) - \det \mathcal{U}_{n-1}(1+\alpha, r)\Big) t^n.$$

The first sum on the right-hand side of the above equation can be replaced by the right-hand side (rhs) of (11) with α replaced by $1+\alpha$, while in the second sum n needs to be replaced by $n+1$. Then the second sum is identical to (11) except α is replaced $\alpha + 1$ and it is multiplied by t. Combining the results for both sums yields (12). □

The generating function (11) has been derived in Proposition 1.2 of [2], but as a result of Theorem 2 from [13], which also appears in the previous section, the derivation here is much more succinct. If (3) is introduced into (14), then (1.7) in Proposition 1.3 of [2] follows, which is

$$\det \mathcal{U}_n(\alpha, r) = \frac{1}{\sqrt{\alpha^2 - 4r^2}} \left(\left(\frac{\alpha + \sqrt{\alpha^2 - 4r^2}}{2}\right)^{n+1} - \left(\frac{\alpha - \sqrt{\alpha^2 - 4r^2}}{2}\right)^{n+1} \right).$$

In this instance, however, the condition accompanying the proposition that $\alpha^2 - 4r^2 > 0$ in [2], becomes redundant. Moreover, by introducing the above result into (13), one obtains the corresponding form for $\det \mathcal{T}_n(\alpha, r)$. In addition, inserting (14) into the recurrence relation given by (2) produces the following recurrence relation:

$$\alpha \det \mathcal{U}_n(\alpha, r) = \det \mathcal{U}_{n+1}(\alpha, r) + r^2 \det \mathcal{U}_{n-1}(\alpha, r).$$

Finally, introducing (2) into (10) yields a recurrence relation for $\det \mathcal{T}_n(\alpha, r)$, which is

$$2(1+\alpha) \det \mathcal{T}_n(\alpha, r) = \det \mathcal{T}_{n+1}(\alpha, r) + r^2 \det \mathcal{T}_{n-1}(\alpha, r).$$

3.2 Special cases

By putting $\alpha = r^2$ in (13), we obtain

$$\det \mathcal{T}_n(r^2, r) = r^{n-1}\left(r\,U_n\left(\frac{r}{2}+\frac{1}{2r}\right) - U_{n-1}\left(\frac{r}{2}+\frac{1}{2r}\right)\right).$$

By noting that $\det \mathcal{T}_1(r^2, r) = r^2$ and the recurrence relation (2), one can prove by induction that

$$\det \mathcal{T}_n(r^2, r) = r^{2n}.$$

Hence, we find that $\det J_n(r) = r^n$. Furthermore, if λ denotes the eigenvalues of $\mathcal{T}_n(r^2, r)$, then we can modify the above result to obtain the characteristic equation for the matrix, which is

$$r\,U_n\left(\frac{r^2+1-\lambda}{2r}\right) - U_{n-1}\left(\frac{r^2+1-\lambda}{2r}\right) = 0.$$

The above equation can be solved by using the Solve routine in Mathematica. For example, when $n = 2$, one simply types
Solve[(r ChebyshevU[2, (r^2 + 1 - λ)/(2 r)] - ChebyshevU[2 - 1, (r^2 + 1 - λ)/(2 r)]) == 0, λ],
which yields

$$\lambda_1 = (1 + 2r^2 - \sqrt{1+4r^2})/2, \quad \text{and} \quad \lambda_2 = (1 + 2r^2 + \sqrt{1+4r^2})/2.$$

These results have also been obtained by Capparelli and Maroscia [2].

By following the same procedure for $n = 3$, we find that

$$\lambda_1 = \frac{1}{3}\Big(2 + 3r^2 - \frac{2^{1/3}(1+6r^2)}{p(r)} - 2^{-1/3}p(r)\Big),$$

$$\lambda_2 = \frac{1}{3}\Big(2 + 3r^2 - \frac{2^{1/3}e^{i\pi/3}(1+6r^2)}{p(r)} + 2^{-1/3}e^{-i\pi/3}p(r)\Big),$$

and

$$\lambda_3 = \frac{1}{3}\Big(2 + 3r^2 + \frac{2^{1/3}e^{-i\pi/3}(1+6r^2)}{p(r)} + 2^{-1/3}e^{i\pi/3}p(r)\Big),$$

where

$$p(r) = \left(2 - 9r^2 + 3\sqrt{3}\sqrt{-4r^2 - 13r^4 - 32r^6}\right)^{1/3}.$$

For $n = 4$, the characteristic polynomial becomes

$$\lambda^4 - (4r^2+3)\lambda^3 + (6r^4+6r^2+3)\lambda^2 - (4r^6+3r^4+2r^2+1)\lambda + r^8 = 0,$$

while for $n = 5$, it is given by

$$\lambda^5 - (5r^2+4)\lambda^4 + (10r^4+12r^2+6)\lambda^3 - (10r^6+12r^4+9r^2+4)\lambda^2$$
$$+ \ (5r^8+4r^6+3r^4+2r^2+1)\lambda - r^{10} = 0.$$

Although the above equations can be solved for λ using the Solve routine in Mathematica, the solutions are too cumbersome to display here.

For $\alpha = 1$ and $r = \exp(i\pi/2)$ or i, the right-hand side of (11) becomes the right-hand side of the generating function for the Fibonacci numbers, F_n, divided by t. As a consequence, we observe that

$$\det \mathcal{U}_{n-1}(1, i) = F_n, \tag{15}$$

while from (13), we arrive at

$$F_n = \exp(i(n-1)\pi/2) U_{n-1}(-i/2). \tag{16}$$

See also [7]. This result can also be derived from (5) by setting $x = -i/2$ and $t = it$.

If we put $\alpha = 2x$ and $r = 1$, then we obtain

$$\sum_{n=0}^{\infty} \det \mathcal{U}_n(2x, 1) t^n = \frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} t^n \sum_{j=0}^{n} \mathcal{P}_j(x)\mathcal{P}_{n-j}(x),$$

where $\mathcal{P}_n(x)$ denotes the Legendre polynomial of degree n and is not be confused with $P_n(x)$ in [2] appearing here shortly. By equating like powers of t, we obtain

$$\det \mathcal{U}_n(2x, 1) = \sum_{j=0}^{n} \mathcal{P}_j(x)\mathcal{P}_{n-j}(x). \tag{17}$$

Now introducing (14) into the above equation yields

$$\sum_{j=0}^{n} \mathcal{P}_j(x)\mathcal{P}_{n-j}(x) = U_n(x).$$

Alternatively, with the aid of (16), we arrive at

$$i^n \sum_{j=0}^{n} \mathcal{P}_j(-i/2)\mathcal{P}_{n-j}(-i/2) = F_n,$$

which is an interesting relationship between a sum of products of Legendre polynomials with imaginary arguments and the Fibonacci numbers.

In a similar manner the generating function of $\det \mathcal{T}_n(\alpha, r)$ is found to be

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(\alpha, r) t^n = \frac{1-t}{1-(1+\alpha)t+r^2t^2}. \tag{18}$$

By putting $\alpha = 0$ and $r = i$, we observe that the right-hand side of (12) becomes the difference of two forms of the generating function for the Fibonacci numbers, namely

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(0, i) t^n = \sum_{n=0}^{\infty} (F_{n+1} - F_n)\, t^n.$$

Equating like powers of t yields

$$\det \mathcal{T}_n(0, i) = F_{n-1},$$

while from (15), we see that $\det U_{n-1}(1, i) = \det T_{n+1}(0, i)$.

If we set $\alpha = 1 - x$ and $r = 1$ in (18), then we derive the generating function of the characteristic polynomial of $\mathcal{T}_n(1, 1)$, which is denoted by $P_n(x)$ in [2]. That is, $P_n(x) = \det \mathcal{T}_n(1 - x, 1)$. Note also for the case of $\mathcal{T}_n(1, 1)$, which we discuss in more detail later, the right-hand side of the generating function reduces to $1/(1-t)$. By expanding this term as the geometric series for $|t| < 1$ and equating like powers of t on both sides, one finds that $\det \mathcal{T}_n(1, 1) = 1$. On the other hand, if we replace α by $1 + \alpha$ in (11) and subtract (12) from the resulting equation, then we find that

$$\sum_{n=0}^{\infty} \big(\det \mathcal{U}_n(1+\alpha, r) - \det \mathcal{T}_n(\alpha, r)\big) t^n = \frac{t}{1-(1+\alpha)t+r^2t^2}.$$

Consequently, from the preceding material, we arrive at

$$\det \mathcal{U}_n(1, i) - \det \mathcal{T}_n(0, i) = F_n.$$

Now if we set $\alpha = 1 - x$ and $r = 1$, then we obtain the generating function of the other set of polynomials appearing in Section 2 of [2], denoted by $Q_n(x)$. Thus, we arrrive at

$$Q_n(x) = \det \mathcal{U}_n(2 - x, 1) - \det \mathcal{T}_n(1 - x, 1) = U_{n-1}(1 - x/2).$$

Alternatively, this may be expressed as $Q_n(x)+P_n(x) = \det \mathcal{U}_n(2-x,1)$ or the sum of both polynomials with index n yields the characteristic polynomial of $\mathcal{U}_n(2,1)$. Then using (17), we can express the sum of the polynomials in terms of shifted Legendre polynomials as

$$Q_n(x) + P_n(x) = \sum_{j=0}^{n} \mathcal{P}_j(1-x)\mathcal{P}_{n-j}(1-x).$$

For the case of $\alpha = r = 1$ in $\mathcal{T}_n(\alpha, r)$ or

$$\mathcal{T}_n(1,1) = \begin{pmatrix} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2 \end{pmatrix}_{n\times n}$$

we find that the characteristic polynomial reduces to

$$P_n(x) = U_n\left(\frac{2-x}{2}\right) - U_{n-1}\left(\frac{2-x}{2}\right), \tag{19}$$

using (1) (the reader is also referred to [8]). If we set $x = i+2$ into the above recurrence relation, then with the aid of (16) we find that

$$i^n P_n(i+2) = F_{n+1} - iF_n.$$

On the other hand, by considering (4) and Pascal's formula, which is

$$\binom{n+k+1}{2k+1} - \binom{n+k}{2k+1} = \binom{n+k}{2k},$$

we find that

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} x^k.$$

This is proved in Theorem 2.5 of [2]. On the other hand, the polynomials $Q_n(x)$, which are presented after Corollary 2.6, are even simpler to evaluate since we have already observed that they are equal to $U_{n-1}(1-x/2)$. Introducing (4) into this result yields

$$Q_{n-1}(x) = \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} x^k.$$

By combining (5) with (19), we obtain the generating function of $P_n(x)$ appearing in Proposition 2.1 of [2] or

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1-t}{1-(2-x)t+t^2}.$$

If we put $\alpha = 1-x$ and $r = 1$ in (12) and equate like powers of t, then we arrive at $P_n(x) = \det \mathcal{T}_n(1-x, 1)$, which was derived earlier. Furthermore, by setting $x = 2$, we find that

$$\begin{aligned} P_n(2) &= \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} 2^k \\ &= U_n(0) - U_{n-1}(0) \\ &= \begin{cases} 1 & \text{if } n \equiv 0,3 \pmod 4, \\ -1 & \text{if } n \equiv 1,2 \pmod 4. \end{cases} \end{aligned} \tag{20}$$

Hence it can be seen that

$$\det \mathcal{T}_n(-1, 1) = \begin{cases} 1 & \text{if } n \equiv 0,3 \pmod 4, \\ -1 & \text{if } n \equiv 1,2 \pmod 4. \end{cases}$$

With regard to the question posed after Corollary 2.6 in [2] concerning whether there exists a combinatorial interpretation of the above sequence, the answer is affirmative. To see this, we refer the reader to [17], where it is not only stated that Sequence A087960 is given by

$$P_n(2) = (-1)^{\binom{n+1}{2}},$$

but there are also numerous references to applications. In fact, from the site we observe that $P_n(2)$ represents the coefficient of x^{n+1} in the power series of $(1+x)/(1+x^2) - 1$ and is given by

$$P_n(2) = (-1)^{\binom{n+1}{2}} = \cos\big(n\pi/2\big) - \sin\big(n\pi/2\big).$$

A similar sequence, where each term is shifted by incrementing n by one, also appears as A057077 in [17], again with numerous applications.

It should also be noted that Capparelli and Maroscia were unaware that the properties of the $P_n(x)$ are well-known in the theory of orthogonal polynomials. In particular, these polynomials were first studied by Chihara [4], where they were referred to as co-recursive polynomials. More generally, they are now regarded as a particular case of anti-associated polynomials of a certain family of orthogonal polynomials derived by Ronveaux and van Assche [15].

Finally, we consider the singular values of the Jordan block with $r = 1$. As indicated before, the product of the Jordan block with its transpose yields $\mathcal{T}_n(r^2, r)$ as per (7). Since the transpose has the same characteristic polynomial as the Jordan block, the characteristic polynomial of $\mathcal{T}_n(1,1)$ is the square of the characteristic polynomial of the Jordan block. From (19), $P_n(x)$ represents the characteristic polynomial for $\mathcal{T}_n(1,1)$, whose eigenvalues are obtained by setting the right-hand side of (19) to zero. Thus, we require the solutions of

$$U_n\left(\frac{2-x}{2}\right) = U_{n-1}\left(\frac{2-x}{2}\right).$$

The above equation is solved simply by replacing $1 - x/2$ by $\cos\theta$. After carrying out a little algebra using (2), one arrives at

$$\cos\big((n+1/2)\theta\big)\sin\big(\theta/2\big) = 0,$$

whose solutions are

$$\theta = (2k+1)\pi/(2n+1).$$

As a consequence, the eigenvalues of $\mathcal{T}_n(1,1)$ are given by

$$\begin{aligned} \lambda_k &= 2 - 2\cos\left(\frac{(2k+1)\pi}{2n+1}\right) \\ &= 4\sin^2\left(\frac{(2k+1)\pi}{4n+2}\right), \quad \text{for } k = 0, 1, \ldots, n-1. \end{aligned}$$

From (6) or Theorem 3 in [13], the eigenvectors of $\mathcal{T}_n(1,1)$ denoted by $u = (u_0, u_1, \ldots, u_{n-1})^T$ are simply given by

$$u_k = (-1)^k C\Big(\sin(k+1)\phi + \frac{1}{2}\sin k\phi\Big),$$

and

$$\phi = \arccos\Big(\cos\Big(\frac{(2k+1)\pi}{2n+1}\Big) - 1\Big).$$

Furthermore, taking the square root of the eigenvalues yields the singular values of $J_n(1)$. Hence we have arrived at (3.1) in Theorem 3.3 of [2] with little effort, while Capparelli and Maroscia had to introduce Lemma 3.2. It should also be mentioned that there was no need to prove this lemma since it is a well-known result that appears as No. 17.14.4 on p. 250 of Hansen [11] or as No. I.1.9 on p. 760 of Prudnikov et al. [14], both of which preceded [2] by several decades.

4 Conclusion

In this article we have shown how important theory in Losonczi's article [13] can be used to uncover the spectral theory of tridiagonal and related matrices. As an example, we have demonstrated that most of the interesting results in Capparelli and Maroscia's work [2] on orthogonal polynomials arising from Jordan blocks can be obtained in a simple and elegant manner, often requiring only a few lines. We have achieved this by applying the results in [13] to the matrix types, $\mathcal{T}_n(\alpha, r)$ and $\mathcal{U}_n(\alpha, r)$, given by (8) and (9), respectively. The first type of matrix is related to the product of a Jordan block and its tranpose, which also enabled us to derive the spectral theory of these matrices from [13]. Furthermore, we were able to show that the orthogonal polynomials, denoted by $P_n(x)$ and $Q_n(x)$ in [2], were expressible in terms of the second kind of Chebyshev polynomials. In carrying out this work, we also answered the question after Corollary 2.6 in [2] whether there is a natural combinatorial interpretation of (20) or (2.9) in [2]. This was shown to be minus one raised to a specific binomial coefficient. Finally, we found that the spectral theory for imaginary elements/entries in special cases of $\mathcal{U}_n(\alpha, r)$ and $\mathcal{T}_n(\alpha, r)$, can be expressed in terms of the Fibonacci numbers and Legendre polynomials.

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Study of periodic solutions for third-order iterative differential equations via Krasnoselskii-Burton's theorem

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Abstract. This paper studies the existence of periodic solutions of a third order iterative differential equation. The main tool used here is Krasnoselskii-Burton's fixed point theorem dealing with a sum of two mappings, one is a large contraction and the other is compact.

1 Introduction

Delay or iterative differential equations have attracted considerable attention in mathematics during recent years since these equations have been showed to be valuable tools in the modeling of many phenomena in various fields of science, physics, chemistry and engineering, etc. In particular, periodicity, positivity and stability of solutions for delay or iterative differential equations has been studied extensively by many authors, see the references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. For example in [9], the third-order iterative differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = x(t)\sum_{k=1}^{n} c_k(t)x^{[k]}(t),$$

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has been investigated. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, Bouakkaz et al. obtained the existence, uniqueness and continuous dependence of periodic solution. Inspired and motivated by the references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], we study the existence of periodic solutions for the third order iterative differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)h(x(t)) = x(t)\sum_{k=1}^{n} c_k(t)x^{[k]}(t), \quad (1)$$

where $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t))$, ..., $x^{[n]}(t) = x^{[n-1]}(x(t))$, p, q, r and c_k, $k = \overline{1,n}$ are continuous real-valued functions. Our purpose here is to use Krasnoselskii-Burton's fixed point theorem to prove the existence of periodic solutions for (1). To prove the existence of periodic solutions, we transform (1) into an equivalent integral equation and then use Krasnoselskii-Burton's fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a large contraction and the other is compact.

2 Preliminaries

For $T > 0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

and for $N, K > 0$, let

$$P_T(N,K) = \{x \in P_T, \ \|x\| \le N, \ |x(t_2) - x(t_1)| \le K|t_2 - t_1|, \forall t_1, t_2 \in \mathbb{R}\},$$

be a closed convex and bounded subset of P_T.

Throughout this paper, we assume that

(H1) There exist two differentiable positive T-periodic functions a_1, a_2 and a positive real constant ρ such that

$$\begin{cases} a_1(t) + \rho = p(t), \\ a_1'(t) + a_2(t) + \rho a_1(t) = q(t), \\ a_2'(t) + \rho a_2(t) = r(t). \end{cases}$$

(H2) $p, q, r \in P_T$ and

$$\int_0^T p(s)\,ds > \rho T \text{ and } \int_0^T q(s)\,ds > 0.$$

Now, we consider the equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = e(t), \tag{2}$$

where e is a continuous T-periodic function. It is easy to see that by virtue of (H1) and (H2), the above equation can be transformed into the following system

$$\begin{cases} y'(t) + \rho y(t) = e(t), \\ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t). \end{cases}$$

Lemma 1 **([5])** *If* $y, e \in P_T$*, then* y *is a solution of the equation*

$$y'(t) + \rho y(t) = e(t),$$

if and only if

$$y(t) = \int_t^{t+T} G_1(t,s)\,e(s)\,ds, \tag{3}$$

where

$$G_1(t,s) = \frac{\exp(\rho(s-t))}{\exp(\rho T) - 1}. \tag{4}$$

Corollary 1 **([14])** *Green's function* G_1 *satisfies the following property*

$$m_1 \le G_1(t,s) \le M_1,$$

where

$$m_1 = \frac{1}{\exp(\rho T) - 1}, \; M_1 = \frac{\exp(\rho T)}{\exp(\rho T) - 1}.$$

Lemma 2 **([13])** *Suppose that (H1), (H2) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T a_1(v)\,dv\right) - 1\right]}{Q_1 T} \ge 1, \tag{5}$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp\left(\int_t^s a_1(v)\,dv\right)}{\exp\left(\int_0^T a_1(v)\,dv\right) - 1} a_2(s)\,ds \right|,$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T a_1(v)\,dv\right)\right)^2 R_1^2.$$

Then, there are continuous and T*-periodic functions* a *and* b *such that*

$$b(t) > 0, \quad \int_0^T a(v)dv > 0,$$

and

$$a(t) + b(t) = a_1(t), \quad \frac{d}{dt}b(t) + a(t)b(t) = a_2(t) \text{ for all } t \in \mathbb{R}.$$

Lemma 3 ([17]) *Suppose the conditions of Lemma 2 hold and* $y \in P_T$*. Then the equation*

$$\frac{d^2}{dt^2}x(t) + a_1(t)\frac{d}{dt}x(t) + a_2(t)x(t) = y(t),$$

has a T*-periodic solution. Moreover, the periodic solution can be expressed as*

$$x(t) = \int_t^{t+T} G_2(t,s)\,y(s)\,ds, \tag{6}$$

where

$$G_2(t,s) = \frac{\int_t^s \exp\left[\int_t^v b(u)\,du + \int_v^s a(u)\,du\right] dv}{\left[\exp\left(\int_0^T a(v)\,dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)\,dv\right) - 1\right]} + \frac{\int_s^{t+T} \exp\left[\int_t^v b(u)\,du + \int_v^{s+T} a(u)\,du\right] dv}{\left[\exp\left(\int_0^T a(v)\,dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)\,dv\right) - 1\right]}. \tag{7}$$

Lemma 4 ([13]) *Let* $A = \int_0^T a_1(v)\,dv$ *and* $B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(a_2(v))\,dv\right)$*. If*

$$A^2 \geq 4B, \tag{8}$$

then

$$\min\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) := l,$$

and

$$\max\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right) := L.$$

Corollary 2 ([14]) *Green's function* G_2 *satisfies the following properties*

$$m_2 \leq G_2(t,s) \leq M_2$$

where

$$m_2 = \frac{T}{\left(e^L - 1\right)^2} \text{ and } M_2 = \frac{T \exp\left(\int_0^T a_1(v)\,dv\right)}{\left(e^l - 1\right)^2}.$$

Lemma 5 ([8]) *For any* $t_1, t_2 \in \mathbb{R}$

$$\int_{t_1}^{t_1+T} |G_2(t_2,s) - G_2(t_1,s)|\,ds \leq \mu\,|t_2 - t_1|,$$

where

$$\mu = Te^{2L}\eta\left[T\lambda_2\gamma\left(2e^{2L}+1\right) + e^L + 1\right],$$

and

$$\eta = \frac{1}{\left[\exp\left(\int_0^T a(v)\,dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)\,dv\right) - 1\right]},$$

$$\lambda_2 = \max_{t\in[0,T]} |b(t)|, \quad \gamma = \exp\left(-\int_0^T b(v)\,dv\right).$$

Lemma 6 ([11]) *Suppose the conditions of Lemma 2 hold and* $e \in P_T$*. Then the equation*

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = e(t),$$

has a T*-periodic solution. Moreover, the periodic solution can be expressed by*

$$x(t) = \int_t^{t+T} G(t,s)\,e(s)\,ds, \tag{9}$$

where

$$G(t,s) = \int_t^{t+T} G_2(t,\sigma)\,G_1(\sigma,s)\,d\sigma. \tag{10}$$

***Corollary 3** ([14]) Green's function* G *satisfies the following property*

$$m \leq G(t,s) \leq M,$$

where

$$m = \frac{T^2}{\left(e^L-1\right)^2\left(\exp(\rho T) - 1\right)} \text{ and } M = \frac{T^2\left(\rho T + \exp\left(\int_0^T a(v)\,dv\right)\right)}{\left(e^l-1\right)^2\left(\exp(\rho T) - 1\right)}.$$

Lemma 7 ([20]) *For any* $\varphi, \psi \in P_T(L, K)$,

$$\left\|\varphi^{[i]} - \psi^{[i]}\right\| \leq \sum_{j=0}^{i-1} K^j \left\|\varphi - \psi\right\|, \ i = 1, 2, ...$$

Lemma 8 ([19]) *It holds*

$$P_T(N, K) = \{x \in P_T, \ \|x\| \leq N, \ |x(t_2) - x(t_1)| \leq K|t_2 - t_1|, \ \forall t_1, t_2 \in [0, T]\}.$$

Lemma 9 *Suppose (H1), (H2) and (5) hold. The function* $x \in P_T(N, K)$ *is a solution of (1) if and only if*

$$x(t) = \int_t^{t+T} r(s) H(x(s)) G(t, s) ds + \sum_{i=1}^{n} \int_t^{t+T} c_i(s) x(s) x^{[i]}(s) G(t, s) ds, \tag{11}$$

where

$$H(x) = x - h(x). \tag{12}$$

Proof. Let $x \in P_T(N, K)$ be a solution of (1). Rewrite (1) as

$$\begin{aligned} &x'''(t) + p(t) x''(t) + q(t) x'(t) + r(t) x(t) \\ &= r(t) H(x(t)) + x(t) \sum_{k=1}^{n} c_k(t) x^{[k]}(t). \end{aligned}$$

From Lemma 6, we have

$$x(t) = \int_t^{t+T} G(t, s) \left[r(s) H(x(s)) + x(s) \sum_{k=1}^{n} c_k(s) x^{[k]}(s) \right] ds.$$

The proof is completed. □

Definition 1 (Large contraction [10]) *Let* $(\mathbb{M}, d)$ *be a metric space and consider* $\mathcal{B} : \mathbb{M} \to \mathbb{M}$. *Then* $\mathcal{B}$ *is said to be a large contraction if given* $\phi, \varphi \in \mathbb{M}$ *with* $\phi \neq \varphi$ *then* $d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq d(\phi, \varphi)$ *and if for all* $\varepsilon > 0$, *there exists a* $\delta \in (0, 1)$ *such that*

$$[\phi, \varphi \in \mathbb{M}, \ d(\phi, \varphi) \geq \varepsilon] \Rightarrow d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq \delta d(\phi, \varphi).$$

Theorem 1 (Krasnoselskii-Burton [10]) *Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,

(ii) $\mathcal{A}$ is compact and continuous,

(iii) $\mathcal{B}$ is a large contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

We will use this theorem to show the existence of periodic solutions for (1).

Theorem 2 ([1]) *Let $\|.\|$ be the supremum norm, $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le N\}$ where N is a positive constant. Suppose that h is satisfying the following conditions*

(I) $h : \mathbb{R} \to \mathbb{R}$ *is continuous on* $[-N, N]$ *and differentiable on* $(-N, N)$,

(II) *the function* h *is strictly increasing on* $[-N, N]$,

(III) $\sup_{t\in(-N,N)} h'(t) \le 1$.

Then the mapping H define by (12) is a large contraction on the set $\mathbb{M}$.

3 Existence of periodic solutions

To apply the Theorem 1 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a compact and the other is a large contraction. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and

$$\begin{aligned}\mathbb{M} &= P_T(N, K) \\ &= \{\varphi \in P_T,\ \|\varphi\| \le N,\ |\varphi(t_2) - \varphi(t_1)| \le K|t_2 - t_1|,\ \forall t_1, t_2 \in [0, T]\},\end{aligned} \tag{13}$$

with $N, K > 0$. Define a mapping $\mathcal{S} : \mathbb{M} \to P_T$ by

$$\begin{aligned}(\mathcal{S}\varphi)(t) = &\int_t^{t+T} r(s) H(\varphi(s)) G(t,s)\, ds \\ &+ \sum_{i=1}^{n} \int_t^{t+T} c_i(s)\, \varphi(s)\, \varphi^{[i]}(s)\, G(t,s)\, ds.\end{aligned}$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A}, \mathcal{B} : \mathbb{M} \to P_T$ are given by

$$(\mathcal{A}\varphi)(t) = \sum_{i=1}^{n} \int_{t}^{t+T} c_i(s)\,\varphi(s)\,\varphi^{[i]}(s)\,G(t,s)\,ds, \tag{14}$$

and

$$(\mathcal{B}\varphi)(t) = \int_{t}^{t+T} r(s)\,H(\varphi(s))\,G(t,s)\,ds, \tag{15}$$

where c_i in P_T, $i = \overline{1,n}$.

Remark 1 *The compactness of* $P_T(N,K)$ *results immediately from the application of Lemma 8 and Ascoli-Arzela theorem.*

We need the next lemma in our next results. This lemma and its proof can be found in [9].

Lemma 10 *For any* $\varphi, \psi \in \mathbb{M}$,

$$\left\| \varphi\varphi^{[i]} - \psi\psi^{[i]} \right\| \leq N \left(1 + \sum_{j=0}^{i-1} K^j \right) \|\varphi - \psi\|, \ i = 1, 2, \ldots.$$

We will show set of preparatory lemmas to use them in the proof of the main existence results.

Lemma 11 *Suppose that (H1), (H2), (5) hold and* $c_i \in P_T(N_{c_i}, K_{c_i})$, $i = \overline{1,n}$. *If*

$$J\left(TMN^2 \sum_{i=1}^{n} N_{c_i} \right) \leq N, \tag{16}$$

and

$$J\left(\mu M_1 T^2 + 2M_1 M_2 T + 2M \right) N^2 \sum_{i=1}^{n} N_{c_i} \leq K, \tag{17}$$

hold, where J *is a positive constant with* $J \geq 3$. *Then the operator* $\mathcal{A}$ *defined by (14) is continuous and compact on* $\mathbb{M}$.

Proof. Let $\varphi \in \mathbb{M}$. For having $\mathcal{A}\varphi \in \mathbb{M}$ we show that $\mathcal{A}\varphi \in P_T$, $\|\mathcal{A}\varphi\| \leq N$ and $|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0,T]$. First it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. That is, if $\varphi \in P_T$ then $\mathcal{A}\varphi$ is periodic with period T. By using Corollary 2 and (16), we obtain

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| &\leq \sum_{i=1}^{n} \int_{t}^{t+T} |c_i(s)| |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t,s)| \, ds \\
&\leq TMN^2 \sum_{i=1}^{n} N_{c_i} \leq \frac{N}{J} \leq N.
\end{aligned}$$

So, we get

$$\|\mathcal{A}\varphi\| \leq N.$$

Second we prove that, for any $\varphi \in \mathbb{M}$ the function $\mathcal{A}\varphi$ is K-Lipschitzian. Let $\varphi \in \mathbb{M}$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\leq \Bigg| \sum_{i=1}^{n} \int_{t_2}^{t_2+T} c_i(s)\,\varphi(s)\,\varphi^{[i]}(s)\,G(t_2,s)\,ds \\
&- \sum_{i=1}^{n} \int_{t_1}^{t_1+T} c_i(s)\,\varphi(s)\,\varphi^{[i]}(s)\,G(t_1,s)\,ds \Bigg| \\
&\leq \sum_{i=1}^{n} \int_{t_2}^{t_1} |c_i(s)| |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t_2,s)| \, ds \\
&+ \sum_{i=1}^{n} \int_{t_1+T}^{t_2+T} |c_i(s)| |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t_2,s)| \, ds \\
&+ \sum_{i=1}^{n} \int_{t_1}^{t_1+T} |c_i(s)| |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t_2,s) - G(t_1,s)| \, ds.
\end{aligned}$$

It follows from Corollaries 2, 2 and Lemma 5 that

$$\begin{aligned}
&|G(t_2,s) - G(t_1,s)| \\
&= \left| \int_{t_2}^{t_2+T} G_2(t_2,\sigma)\,G_1(\sigma,s)\,d\sigma - \int_{t_1}^{t_1+T} G_2(t_1,\sigma)\,G_1(\sigma,s)\,d\sigma \right| \\
&\leq \int_{t_2}^{t_1} |G_2(t_2,\sigma)| |G_1(\sigma,s)| \, d\sigma + \int_{t_1+T}^{t_2+T} |G_2(t_2,\sigma)| |G_1(\sigma,s)| \, d\sigma \\
&+ \int_{t_1}^{t_1+T} |G_1(\sigma,s)| |G_2(t_2,\sigma) - G_2(t_1,\sigma)| \, d\sigma.
\end{aligned}$$

So

$$|G(t_2,s) - G(t_1,s)| \le (2M_1M_2 + \mu TM_1)|t_2 - t_1|. \tag{18}$$

Using Corollary 3 and (18), we get

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\le 2MN^2\left(\sum_{i=1}^{n} N_{c_i}\right)|t_2 - t_1| + TN^2(2M_1M_2 + \mu TM_1)\left(\sum_{i=1}^{n} N_{c_i}\right)|t_2 - t_1| \\
&= N^2\left(2M + 2TM_1M_2 + \mu T^2M_1\right)\left(\sum_{i=1}^{n} N_{c_i}\right)|t_2 - t_1| \\
&\le \frac{K}{J}|t_2 - t_1| \le K|t_2 - t_1|.
\end{aligned}$$

So, we have

$$|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \le K|t_2 - t_1|.$$

which shows $\mathcal{A} : \mathbb{M} \to \mathbb{M}$.

Now, For $\varphi, \psi \in \mathbb{M}$, $c_i \in P_T(N_{c_i}, K_{c_i})$, $i = \overline{1,n}$, and from Corollary 3, we obtain

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \\
&\le \sum_{i=1}^{n}\int_{t}^{t+T}|c_i(s)||G(t,s)|\left|\varphi(s)\varphi^{[i]}(s) - \psi(s)\psi^{[i]}(s)\right| ds \\
&\le M\sum_{i=1}^{n}\int_{t}^{t+T}|c_i(s)|\left|\varphi(s)\varphi^{[i]}(s) - \psi(s)\psi^{[i]}(s)\right| ds.
\end{aligned}$$

Using Lemma 10, we get

$$|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \le NMT\sum_{i=1}^{n} N_{c_i}\left(1 + \sum_{j=0}^{i-1} K^j\right)\|\varphi - \psi\|.$$

This implies the continuity of $\mathcal{A}$. We use Remark 1 and the fact that continuous operators maps compact sets into compact sets we deduce that $\mathcal{A}$ is a compact operator. □

The next result proves the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions. Assume that

$$\theta MT \le 1, \tag{19}$$

$$\max\left(|H(-N)|, |H(N)|\right) \leq \frac{(J-1)}{J} N, \tag{20}$$

and

$$\left(2M + 2TM_1M_2 + \mu T^2 M_1\right)\theta N \leq K, \tag{21}$$

where $\theta = \max_{t\in[0,T]} |r(t)|$.

Lemma 12 *Let $\mathcal{B}$ be defined by (15). Suppose (H1), (H2), ((5)), (19)–(21) and all conditions of Theorem 2 hold. Then $\mathcal{B}$ is a large contraction on $\mathbb{M}$.*

Proof. Let $\mathcal{B}$ be defined by (15). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$. For having $\mathcal{B}\varphi \in \mathbb{M}$ we will show that $\|\mathcal{B}\varphi\| \leq N$ and $|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0,T]$. Let $\varphi \in \mathbb{M}$, by (19) and (20) we get

$$\begin{aligned}
|(\mathcal{B}\varphi)(t)| &\leq \int_t^{t+T} |G(t,s)|\,|r(s)|\,|H(\varphi(s))|\,ds \\
&\leq \theta MT \max\{|H(-N)|, |H(N)|\} \leq \frac{(J-1)N}{J} \leq N.
\end{aligned}$$

Then, for any $\varphi \in \mathbb{M}$, we have

$$\|\mathcal{B}\varphi\| \leq N.$$

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned}
&|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)| \\
&\leq \left| \int_{t_2}^{t_2+T} G(t_2,s)\,r(s)\,H(\varphi(s))\,ds - \int_{t_1}^{t_1+T} G(t_1,s)\,r(s)\,H(\varphi(s))\,ds \right| \\
&\leq \int_{t_2}^{t_1} |G(t_2,s)|\,|r(s)|\,|H(\varphi(s))|\,ds \\
&+ \int_{t_1+T}^{t_2+T} |G(t_2,s)|\,|r(s)|\,|H(\varphi(s))|\,ds \\
&+ \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\,|r(s)|\,|H(\varphi(s))|\,ds.
\end{aligned}$$

Using Corollary 3 and (18), we have

$$|(\mathcal{B}\varphi)(t_2) - (\mathcal{B}\varphi)(t_1)|$$

$$\leq 2M\theta\frac{(J-1)N}{J}|t_2-t_1|+\theta\frac{(J-1)N}{J}T(2M_1M_2+\mu TM_1)|t_2-t_1|$$
$$=\left(2M+2TM_1M_2+\mu T^2M_1\right)\theta\frac{(J-1)N}{J}|t_2-t_1|.$$

From (21), we obtain

$$\begin{aligned}|(\mathcal{B}\varphi)(t_2)-(\mathcal{B}\varphi)(t_1)| &\leq \frac{(J-1)K}{J}|t_2-t_1|\\ &\leq K|t_2-t_1|.\end{aligned}$$

Consequently, we have $\mathcal{B}:\mathbb{M}\to\mathbb{M}$.

It remains to show that $\mathcal{B}$ is a large contraction. By Theorem 2 H is large contraction on $\mathbb{M}$, then for any $\varphi,\psi\in\mathbb{M}$, with $\varphi\neq\psi$ we have

$$\begin{aligned}&|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)|\\ &\leq\left|\int_t^{t+T}G(t,s)r(s)[H(\varphi(s))-H(\psi(s))]ds\right|\\ &\leq\theta MT\|\varphi-\psi\|\leq\|\varphi-\psi\|.\end{aligned}$$

Then $\|\mathcal{B}\varphi-\mathcal{B}\psi\|\leq\|\varphi-\psi\|$. Now, let $\varepsilon\in(0,1)$ be given and let $\varphi,\psi\in\mathbb{M}$, with $\|\varphi-\psi\|\geq\varepsilon$, from the proof of Theorem 2, we have found a $\delta\in(0,1)$, such that

$$|(H\varphi)(t)-(H\psi)(t)|\leq\delta\|\varphi-\psi\|.$$

Thus,

$$\begin{aligned}&|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)|\\ &\leq\left|\int_t^{t+T}G(t,s)r(s)[H(\varphi(s))-H(\psi(s))]ds\right|\\ &\leq\theta MT\delta\|\varphi-\psi\|\leq\delta\|\varphi-\psi\|.\end{aligned}$$

The proof is complete. □

Theorem 3 *Suppose the hypotheses of Lemmas 11, 12 hold. Let $\mathbb{M}$ defined by (13). Then (1) has a T-periodic solution in $\mathbb{M}$.*

Proof. By Lemmas 11, $\mathcal{A}:\mathbb{M}\to\mathbb{M}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 12, the mapping $\mathcal{B}:\mathbb{M}\to\mathbb{M}$ is a large contraction. Next, we show that if $\varphi,\psi\in\mathbb{M}$, we have $\|\mathcal{A}\varphi+\mathcal{B}\psi\|\leq N$ and

$|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. Let $\varphi, \psi \in \mathbb{M}$. By (16), (19) and (20), we obtain

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq TMN^2 \sum_{i=1}^{n} N_{c_i} + \frac{(J-1)N}{J} \leq \frac{N}{J} + \frac{(J-1)N}{J} = N.$$

Now, let $\varphi, \psi \in \mathbb{M}$ and $t_1, t_2 \in [0, T]$. By (17), (21), we get

$$\begin{aligned}
&|(\mathcal{A}\varphi + \mathcal{B}\psi)(t_2) - (\mathcal{A}\varphi + \mathcal{B}\psi)(t_1)| \\
&\leq |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| + |(\mathcal{B}\psi)(t_2) - (\mathcal{B}\psi)(t_1)| \\
&\leq \frac{K}{J}|t_2 - t_1| + \frac{(J-1)K}{J}|t_2 - t_1| \\
&\leq K|t_2 - t_1|.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 9 this fixed point is a solution of (1). Hence (1) has a T-periodic solution. □

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Existence results of infinitely many weak solutions of a singular subelliptic system on the Heisenberg group

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Abstract. This article shows the existence and multiplicity of weak solutions for the singular subelliptic system on the Heisenberg group

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + a(\xi)\frac{u}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{H}^n} v + b(\xi)\frac{v}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

The approach is based on variational methods.

1 Introduction

The aim of this article is to establish the existence and multiplicity of weak solutions for the singular subelliptic system

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + a(\xi)\frac{u}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{H}^n} v + b(\xi)\frac{v}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{H}^n, n \geq 1$, is an open, bounded subset containing the origin with smooth boundary. $\lambda > 0$, $a, b \in L^\infty(\Omega)$ such that $ess_\Omega \inf a > 0$ and $ess_\Omega \inf b > 0$, $F : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(\cdot, s_1, s_2)$ is continuous in $\bar{\Omega}$, for all $(s_1, s_2) \in \mathbb{R}^2$ and $F(\xi, \cdot, \cdot)$ is C^1 in $\mathbb{R}^2$ for every $\xi \in \Omega$, and F_u, F_v denote the partial derivatives of F, with respect to u, v respectively.

Singular elliptic problems have been intensively studied in the last decades, in [6, 16] the authors investigated infinitely many solutions for singular elliptic problems.In [1] and [5] some problems which depend on continuous component of time like coherent states in quantum optics are probed. These problems are studied in a space which have a component of time and are known as Heisenberg group. Important topics where the Heisenberg group reveals itself as an essential factor are quantum mechanics, ergodic theory, representation theory of nilpotent Lie group, foundation of abelian harmonic analysis, and the theory of partial differential equations. We are now interested in the last one.

Recently the existence of radial solutions of Neumann problem on Heisenberg group is studied (see [10, 11, 12, 13]).

Here we recall some definitions and results on Heisenberg group (see [2, 8, 9, 15]). The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ\ )$ is the space $\mathbb{R}^{2n+1}$ with the noncommutative law of product

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(\langle y, x'\rangle - \langle x, y'\rangle)),$$

where $x, x', y, y' \in \mathbb{R}^n,\ t, t' \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. This operation endows $\mathbb{H}^n$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t},\ \ X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t},\ \ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t},\ \ i = 1, 2, 3, \cdots, n.$$

These generators satisfy the noncommutative formula

$$[X_i, Y_j] = -4\delta_{ij} T, \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

Let $z = (x, y) \in \mathbb{R}^{2n}$ and $\xi = (z, t) \in \mathbb{H}^n$. The parabolic dilation $\delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)$, satisfies $\delta_\lambda(\xi_0 \circ \xi) = \delta_\lambda \xi\ \circ\ \delta_\lambda \xi_0$, and $|\xi|_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$, is a norm with respect to the parabolic dilation which is known as Korányi gauge norm $N(z, t)$. The Heisenberg distance between two points (z, t) and (z', t') is given by $\rho(z, t; z', t') = |(z', t')^{-1} \circ (z, t)|_{\mathbb{H}^n}$. Clearly, the vector fields $X_i, Y_i, i = 1, 2, \cdots, n$, are homogeneous of degree 1 under the norm $|.|_{\mathbb{H}^n}$ and T is homogenous of degree 2. The Korányi ball of center ξ_0 and radius r is defined by

$$B_{\mathbb{H}^n}(\xi_0, r) = \{\xi : |\xi^{-1} \circ \xi_0|_{\mathbb{H}^n} \leq r\},$$

and it satisfies $|B_{\mathbb{H}^n}(\xi_0, r)| = |B_{\mathbb{H}^n}(0, r)| = r^{2n+2}|B_{\mathbb{H}^n}(0, 1)|$. The Heisenberg gradient and the Kohn-Laplacian (the Heisenberg Laplacian) operator on $\mathbb{H}^n$ are given by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n) \text{ and } \Delta_{\mathbb{H}^n} = \sum_{i=1}^{n} X_i^2 + Y_i^2,$$

respectively. We define the associated Sobolev space as following:

$$H^1(\Omega, \mathbb{H}^n) := \{u \in L^2(\Omega) : \ X_i u, Y_i u \in L^2(\Omega), i = 1, 2, \cdots, n\},$$

and $H_0^1(\Omega, \mathbb{H}^n)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega, \mathbb{H}^n)$ with respect to the norm

$$\|u\|_{H^1(\Omega,\mathbb{H}^n)} = \left(\int_\Omega (|\nabla_{\mathbb{H}^n} u|^2 + |u|^2) d\xi \right)^{\frac{1}{2}},$$

where $u : \Omega \subset \mathbb{H}^n \to \mathbb{R}$. A norm on $H_0^1(\Omega, \mathbb{H}^n)$ is

$$\|u\|_{H_0^1(\Omega,\mathbb{H}^n)} = \left(\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi \right)^{\frac{1}{2}},$$

which is equivalent to the standard one. The dual space of $H_0^1(\Omega, \mathbb{H})$ is denoted by $H^{-1}(\Omega, \mathbb{H})$. Here we recall Hardy's inequality and some results on the Heisenberg group.

Lemma 1 [7] *For* $n \geq 1$ *and for any* $u \in H_0^1(\Omega, \mathbb{H}^n)$*, we have*

$$\int_\Omega \frac{|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} \, d\xi \leq \left(\frac{n+1}{n^2} \right)^2 \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi.$$

For convenience, we write the above inequality by

$$\int_\Omega \frac{|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} \, d\xi \leq \frac{1}{C_n} \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi,$$

where $C_n = (\frac{n^2}{n+1})^2$.

Lemma 2 [14] *Let $\Omega \subset \mathbb{H}^n$ be a bounded open set. Then the compact embedding $H^1_0(\Omega, \mathbb{H}^n) \subset\subset L^p(\Omega)$ for $1 \leq p < Q^*$ is satisfied, where $Q^* = \frac{2Q}{Q-2}$ is critical exponent of $Q = 2n + 2$, which is homogeneous dimension of $\mathbb{H}^n$.*

We denote the Sobolev embedding constant of the above compact embedding by $S_p > 0$, i.e.

$$\|u\|_{L^p(\Omega)} \leq S_p \, \|u\|_{H^1_0(\Omega,\mathbb{H}^n)} \quad \text{for all } u \in H^1_0(\Omega, \mathbb{H}^n), \; 1 \leq p < Q^*. \tag{2}$$

In the sequel, X will denote the space $H^1_0(\Omega, \mathbb{H}^n) \times H^1_0(\Omega, \mathbb{H}^n)$, which is a reflexive Banach space endowed with the norm

$$\|(u, v)\| = \|u\|_{H^1_0(\Omega,\mathbb{H}^n)} + \|v\|_{H^1_0(\Omega,\mathbb{H}^n)}.$$

We use the following multiple critical points theorem due to G. Bonanno [3].

Theorem 1 *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strong continuous, sequentially weakly lower semi-continuous, coercive and Ψ is sequentially weakly upper-semi-continuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{(\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then

(a) *If $\gamma < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\gamma}\right)$, the following alternative holds: either*

(a1) *$I_\lambda := \Phi - \lambda\Psi$ possesses a global minimum, or*

(a2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\delta < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\delta}\right)$, the following alternative holds: either*

(b1) *there is a global minimum of Φ that is a local minimum of I_λ, or*

(b2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that weakly converges to a global minimum of Φ.*

2 Weak solutions

In this section we prove the existence of weak solutions for the system (1). Du to do this, we the definition of the weak solution.

Definition 1 *One says that* $(u,v)\in X$ *is a weak solution to the system* (1) *if*

$$\int_\Omega \nabla_{\mathbb{H}^n} u\cdot\nabla_{\mathbb{H}^n}\varphi\, d\xi+\int_\Omega \frac{u\varphi}{(|z|^4+t^2)^{\frac{1}{2}}}a(\xi)\,d\xi-\lambda\int_\Omega F_u(\xi,u,v)\varphi\, d\xi$$
$$+\int_\Omega \nabla_{\mathbb{H}^n} v\cdot\nabla_{\mathbb{H}^n}\psi\, d\xi+\int_\Omega \frac{v\psi}{(|z|^4+t^2)^{\frac{1}{2}}}b(\xi)\,d\xi-\lambda\int_\Omega F_v(\xi,u,v)\psi\, d\xi=0,$$

for every $(\varphi,\psi)\in X$.

Define the functional $I_\lambda : X\to\mathbb{R}$ by

$$I_\lambda(u,v)=\Phi(u,v)-\lambda\Psi(u,v),$$

for all $(u,v)\in X$, where

$$\Phi(u,v)=\frac{1}{2}\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)}+\frac{1}{2}\int_\Omega\frac{|u|^2}{(|z|^4+t^2)^{\frac{1}{2}}}a(\xi)\,d\xi$$
$$+\frac{1}{2}\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)}+\frac{1}{2}\int_\Omega\frac{|v|^2}{(|z|^4+t^2)^{\frac{1}{2}}}b(\xi)\,d\xi,$$

and $\Psi(u,v)=\int_\Omega F(\xi,u,v)\,d\xi$.
Lemma 1 implies

$$\begin{aligned}&\tfrac{1}{2}\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)}+\tfrac{1}{2}\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)}<\Phi(u,v)\\&<(\tfrac{C_n+\|a\|_\infty}{2C_n})\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)}+(\tfrac{C_n+\|b\|_\infty}{2C_n})\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)},\end{aligned}$$

which implies that Φ is coercive. Moreover, from the weakly lower semicontinuity of norm, we known that Φ is sequentially weakly lower semicontinuous. Notice that functionals Φ,Ψ are well defined and continuously Gâteaux differentiable functionals whose derivatives at the point $(u,v)\in X$ are the functionals $\Phi'(u,v)$ and $\Psi'(u,v)$ given by

$$\langle\Phi'(u,v),(\varphi,\psi)\rangle=\int_\Omega \nabla_{\mathbb{H}^n} u\cdot\nabla_{\mathbb{H}^n}\varphi\, d\xi+\int_\Omega \frac{u\varphi}{(|z|^4+t^2)^{\frac{1}{2}}}a(\xi)\,d\xi$$
$$+\int_\Omega \nabla_{\mathbb{H}^n} v\cdot\nabla_{\mathbb{H}^n}\psi\, d\xi+\int_\Omega \frac{v\psi}{(|z|^4+t^2)^{\frac{1}{2}}}b(\xi)\,d\xi,$$

$$\langle\Psi'(u,v),(\varphi,\psi)\rangle=\int_\Omega F_u(\xi,u,v)\varphi\, d\xi+\int_\Omega F_v(\xi,u,v)\psi\, d\xi,$$

for all $(\varphi, \psi) \in X$.
$\Psi' : X \to X^*$ is a compact operator, indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $(u, v) \in X$, let $(u_k, v_k) \to (u, v)$ weakly in X as $k \to \infty$, by the Lemma 2, we deduce that $(u_k, v_k) \to (u, v)$ in $L^p(\Omega) \times L^p(\Omega)$, therefore, $(u_k(\xi), v_k(\xi)) \to (u(\xi), v(\xi))$ for a.e $\xi \in \Omega$. Since $F(\xi, \cdot, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $\xi \in \Omega$, so $\Psi'(\xi, u_k, v_k) \to \Psi'(\xi, u, v)$ strongly as $k \to +\infty$. Thus we have that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2, [17], it follows that Ψ is sequentially weakly continuous.

Here is an example to show a function F with the conditions defined in (1) can exists.

Example 1 *It could be possible to consider the same example given in* [4]*. Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *containing the origin and with smooth boundary* $\partial\Omega$*. Consider the increasing sequence of positive real numbers given by*

$$a_1 := 2, \qquad a_{n+1} := n! a_n^2 + 2,$$

for every $n \geq 1$*. Define the* C^1*-function* $F : \mathbb{R}^2 \to \mathbb{R}$ *as follows*

$$F(s_1, s_2) = \begin{cases} (a_{n+1})^4 e^{1 - \frac{1}{1 - [\|s_1 - a_{n+1}\|^p + \|s_2 - a_{n+1}\|^p]}} & (s_1, s_2) \in \cup_{n \geq 1} B\big((a_{n+1}, a_{n+1}), 1\big), \\ 0 & \textit{otherwise}, \end{cases}$$

where $B\big((a_{n+1}, a_{n+1}), 1\big)$ *is an open unit ball of center* (a_{n+1}, a_{n+1})*.*

Due to study the existence of infinitely many weak solutions, suppose there exist $R_0 > 0$ such that $R_0 < \mathrm{dist}(0, \partial\Omega)$ and $\zeta \in (0, 1)$. Set

$$\begin{aligned} L_a &:= \frac{R_0^2(1-\zeta)^2}{8 S_p\left(\frac{C_n + \|a\|_\infty}{2C_n}\right)\left(1 - \frac{t^2}{r^2}\right)(1 + 4t^2)\omega_{2n+1} R_0^{2n+1}(1 - \zeta^{2n+1})} \\ L_b &:= \frac{R_0^2(1-\zeta)^2}{8 S_p\left(\frac{C_n + \|b\|_\infty}{2C_n}\right)\left(1 - \frac{t^2}{r^2}\right)(1 + 4t^2)\omega_{2n+1} R_0^{2n+1}(1 - \zeta^{2n+1})}, \end{aligned} \tag{3}$$

where w_n denotes the volume of the n-dimensional unit ball in $\mathbb{R}^n$ and S_p is given by (2).

Theorem 2 *Assume that*
(i$_1$) $F(\xi, s_1, s_2) \geq 0$ *for every* $(\xi, s_1, s_2) \in \Omega \times \mathbb{R}^2_+$*,*

(i$_2$) *There exist* $R_0 > 0$ *such that* $R_0 < \mathrm{dist}(0, \partial\Omega)$ *and* $\zeta \in (0,1)$, *given by* (3). *Assume* $A < LB$, *where*

$$\begin{aligned} A &:= \liminf_{\sigma\to+\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}\le\sigma} F(\xi, s_1, s_2)d\xi}{\sigma^2},\\ B &:= \limsup_{s_1,s_2\to+\infty} \frac{\int_{B_{2n+1}(0,\zeta R_0)} F(\xi, s_1, s_2)d\xi}{s_1^2+s_2^2},\\ L &:= \min\{L_a, L_b\}. \end{aligned}$$

Then for each $\lambda \in \Lambda := \frac{1}{8S_p}\left(\frac{1}{LB}, \frac{1}{A}\right)$, *problem* (1) *has an unbounded sequence of weak solutions in* X.

Proof. We apply the part (a) of Theorem 1. Certainly, the weak solutions of the problem (1) are exactly the solutions of the equation $I'_\lambda(u,v) = 0$. The functional Φ and Ψ satisfy the assumptions of Theorem 1. Now we show that $\gamma < +\infty$. Since X compactly embedded in $L^p(\Omega) \times L^p(\Omega)$, and from (2) one has

$$\|u\|_{L^p(\Omega)} \le S_p\|u\|_{H^1_0(\Omega,\mathbb{H}^n)} \quad \text{and} \quad \|v\|_{L^p(\Omega)} \le S_p\|v\|_{H^1_0(\Omega,\mathbb{H}^n)},$$

for all $(u,v) \in X$. Thus

$$\frac{1}{2}\|u\|^2_{L^p(\Omega)} + \frac{1}{2}\|v\|^2_{L^p(\Omega)} < S_p\Big(\frac{1}{2}\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)} + \frac{1}{2}\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)}\Big).$$

So, for each $r > 0$

$$\begin{aligned} \Phi^{-1}(]-\infty, r[) :&= \{(u,v) \in X : \Phi(u,v) < r\}\\ &= \{(u,v) \in X : \frac{1}{2}\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)} + \frac{1}{2}\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)} < r\}\\ &\subseteq \{(u,v) \in X : \frac{1}{2}\|u\|^2_{L^p(\Omega)} + \frac{1}{2}\|v\|^2_{L^p(\Omega)} < S_p r\}, \end{aligned} \tag{4}$$

and it follows that

$$\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v) < \sup_{\{(u,v)\in X: \frac{1}{2}\|u\|_{L^p(\Omega)}+\frac{1}{2}\|v\|_{L^p(\Omega)}<S_p r\}} \int_\Omega F(\xi, u, v)d\xi.$$

Note that $\Phi(0,0) = 0$ and $\Psi(0,0) \geq 0$. Therefore, for every $r > 0$,

$$\begin{aligned}\varphi(r) := &\inf_{\Phi(u,v)<r} \frac{\left(\sup_{(u',v')\in\Phi^{-1}(]-\infty,r[)} \Psi(u',v')\right) - \Psi(u,v)}{r - \Phi(u,v)}\\ &\leq \frac{\sup_{\Phi^{-1}(]-\infty,r[)} \Psi}{r}\\ &\leq \frac{1}{r} \sup_{\frac{1}{2}\|s_1\|_{L^p(\Omega)}+\frac{1}{2}\|s_2\|_{L^p(\Omega)}<S_p r} \int_\Omega F(\xi, s_1, s_2)d\xi.\end{aligned}$$

Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\lim_{k\to+\infty} \sigma_k = +\infty$ and

$$\lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}\leq\sigma_k} F(\xi, s_1, s_2)d\xi}{\sigma_k^2} = A < +\infty. \tag{5}$$

Set $r_k = \frac{1}{2S_p}(\frac{\sigma_k}{2})^2$, from (4), one has

$$\frac{1}{2}\|u\|^2_{L^p(\Omega)} + \frac{1}{2}\|v\|^2_{L^p(\Omega)} < S_p r_k, \text{ for all } \xi \in \Omega.$$

So,

$$\|u\|_{L^p(\Omega)} \leq \sqrt{2S_p r_k} \quad \text{and} \quad \|v\|_{L^p(\Omega)} \leq \sqrt{2S_p r_k},$$

thus, for each $k \in \mathbb{N}$ large enough

$$\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \leq \sqrt{2S_p r_k} + \sqrt{2S_p r_k} = 2\sqrt{2S_p r_k} = \sigma_k.$$

Hence,

$$\begin{aligned}\varphi(r_k) &\leq \frac{\sup_{\{(u,v)\in X:\|u\|_{L^p(\Omega)}+\|v\|_{L^p(\Omega)}<\sigma_k\}} \int_\Omega F(\xi, u, v)d\xi}{\frac{1}{2S_p}(\frac{\sigma_k}{2})^2}\\ &\leq 8S_p \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}<\sigma_k} F(\xi, s_1, s_2)d\xi}{\sigma_k^2}.\end{aligned} \tag{6}$$

Hence, from (5) and (6), one has

$$\begin{aligned}\gamma &\leq \liminf_{k\to+\infty} \varphi(r_k) \leq 8S_p \lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}<\sigma_k} F(\xi, s_1, s_2)d\xi}{\sigma_k^2}\\ &= 8S_p A < +\infty.\end{aligned}$$

This implies

$$\gamma \leq 8S_p A < \frac{1}{\lambda}.$$

We conclude that $\Lambda \subseteq]0, \dfrac{1}{\gamma}[$. For $\lambda \in \Lambda$, we show that the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Indeed, since $\dfrac{1}{\lambda} < 8S_pLB$, we can consider two positive real sequences $\{\eta_{i,k}\}_{i=1}^2$ and $\theta > 0$ such that $\sqrt{\sum_{i=1}^2 \eta_{i,k}} \to +\infty$ as $k \to \infty$ and

$$\frac{1}{\lambda} < \theta < 8S_pL\frac{\int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi}{\eta_{1,k}^2 + \eta_{2,k}^2}, \tag{7}$$

for k large enough. Suppose $u_k(\xi) = (u_{1k}(\xi), u_{2k}(\xi))$ be a sequence in X defined by

$$u_{ik}(\xi) = \begin{cases} 0 & \text{if } \xi \in \mathbb{H}^n \setminus B_{2n+1}(0, R_0) \\ \eta_{i,k} & \text{if } \xi \in B_{2n+1}(0, \zeta R_0) \\ \dfrac{\eta_{i,k}}{R_0(1-\zeta)}(R_0 - |\xi|) & \text{if } \xi \in B_{2n+1}(0, R_0) \setminus B_{2n+1}(0, \zeta R_0), \end{cases} \tag{8}$$

for $i = 1, 2$, where $B_n(0, r)$ denotes the n-dimensional open ball with center 0 and radius $r > 0$.
Bearing (3) in mind, we have

$$\begin{aligned} \Phi(u_{1k}, u_{2k}) &= \frac{1}{2}\int_\Omega |\nabla_{\mathbb{H}^n} u_{1k}(\xi)|^2 d\xi + \frac{\|a\|_\infty}{2}\int_\Omega \frac{|u_{1k}(\xi)|^2}{(|z|^4 + t^2)^{\frac{1}{2}}}d\xi \\ &+ \frac{1}{2}\int_\Omega |\nabla_{\mathbb{H}^n} u_{2k}(\xi)|^2 d\xi + \frac{\|b\|_\infty}{2}\int_\Omega \frac{|u_{2k}(\xi)|^2}{(|z|^4 + t^2)^{\frac{1}{2}}}d\xi \\ &\leq \left(\frac{C_n + \|a\|_\infty}{2C_n}\right)\int_{B_{2n+1}(0,R_0)\setminus B_{2n+1}(0,\zeta R_0)} |\nabla_{\mathbb{H}^n} u_{1k}(\xi)|^2 d\xi \\ &+ \left(\frac{C_n + \|b\|_\infty}{2C_n}\right)\int_{B_{2n+1}(0,R_0)\setminus B_{2n+1}(0,\zeta R_0)} |\nabla_{\mathbb{H}^n} u_{2k}(\xi)|^2 d\xi \\ &= \left(\frac{C_n + \|a\|_\infty}{2C_n}\right)\frac{\eta_{1,k}^2}{R_0^2(1-\zeta)^2}\left(1 - \frac{t^2}{r^2}\right)(1 + 4t^2)\omega_{2n+1}R_0^{2n+1}(1 - \zeta^{2n+1}) \\ &+ \left(\frac{C_n + \|b\|_\infty}{2C_n}\right)\frac{\eta_{2,k}^2}{R_0^2(1-\zeta)^2}\left(1 - \frac{t^2}{r^2}\right)(1 + 4t^2)\omega_{2n+1}R_0^{2n+1}(1 - \zeta^{2n+1}) \\ &= \frac{1}{8S_p}\left(\frac{\eta_{1,k}^2}{L_a} + \frac{\eta_{2,k}^2}{L_b}\right). \end{aligned} \tag{9}$$

On the other hand, by assumption that $F(\xi, s_1, s_2) \geq 0$, we have

$$\Psi(u_{1k}, u_{2k}) = \int_\Omega F(\xi, u_{1k}, u_{2k})d\xi \geq \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi. \qquad (10)$$

So, it follows from (7), (9) and (10) that

$$\begin{aligned} I_\lambda(u_{1k}, u_{2k}) &= \Phi(u_{1k}, u_{2k}) - \lambda\Psi(u_{1k}, u_{2k}) \\ &\leq \frac{1}{8S_pL}(\eta_{1,k}^2 + \eta_{2,k}^2) - \lambda \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi \\ &\leq \frac{(1-\lambda\theta)}{8S_pL}(\eta_{1,k}^2 + \eta_{2,k}^2), \end{aligned}$$

for k large enough, so $\lim_{k\to+\infty} I_\lambda(u_{1k}, u_{2k}) = -\infty$, and hence, the claim follows.

The alternative of Theorem 1 case (a) assures the existence of unbounded sequence $\{u_k = (u_{1k}, u_{2k})\} \subset X$ of critical points of the functional I_λ and the proof of Theorem 2 is complete. □

Theorem 3 *Assume that* (i$_1$) *holds and*

(i$_3$) $F(\xi, 0, 0) = 0$ *for every* $\xi \in \Omega$.

(i$_4$) *There exist* $R_0 > 0$ *such that* $R_0 < \mathrm{dist}(0, \partial\Omega)$ *and* $\zeta \in (0, 1)$, *given by* (3). *Assume* $A' < LB'$ *where*

$$\begin{aligned} A' &:= \liminf_{\sigma\to0^+} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}\leq\sigma} F(\xi, s_1, s_2)d\xi}{\sigma^2}, \\ B' &:= \limsup_{s_1,s_2\to0^+} \frac{\int_{B_{2n+1}(0,\zeta R_0)} F(\xi, s_1, s_2)d\xi}{s_1^2 + s_2^2}, \\ L &:= \min\{L_a, L_b\}. \end{aligned}$$

Then for each $\lambda \in \Lambda := \frac{1}{8S_p}\left(\frac{1}{LB'}, \frac{1}{A'}\right)$, *problem* (1) *admits a sequence of weak solutions which converges to* 0.

Proof. Note that $\Phi(0, 0) = 0$ and $\Psi(0, 0) = 0$. Therefore, for every $r > 0$, Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\sigma_k \to 0^+$ as $k \to +\infty$ and

$$\lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}\leq\sigma_k} F(\xi, s_1, s_2)d\xi}{\sigma_k^2} = A' < +\infty. \qquad (11)$$

Put $r_k = \frac{1}{2S_p}(\frac{\sigma_k}{2})^2$ for all $k \in \mathbb{N}$, and $\delta := \liminf_{r\to 0^+} \varphi(r)$. Hence, from (6) and (11), one has

$$\begin{aligned}\delta &\le \liminf_{k\to+\infty} \varphi(r_k)\\ &\le 8S_p \lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}<\sigma_k} F(\xi, s_1, s_2)d\xi}{\sigma_k^2}\\ &= 8S_pA' < +\infty.\end{aligned}$$

We conclude that $\Lambda \subseteq]0, \frac{1}{\delta}[$. For $\lambda \in \Lambda$, we show that the functional I_λ is unbounded from below. Indeed, since

$$\frac{1}{\lambda} < 8S_pLB',$$

we can consider two positive real sequences $\{\eta_{i,k}\}_{i=1}^2$ and $\theta > 0$ such that $\sqrt{\sum_{i=1}^2 \eta_{i,k}} \to 0$ as $k \to \infty$ and

$$\frac{1}{\lambda} < \theta < 8S_pL\frac{\int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi}{\eta_{1,k}^2 + \eta_{2,k}^2}, \tag{12}$$

for k large enough. Let (u_k) be the sequence defined in (8). By combining (9), (10) and (12), we obtain

$$\begin{aligned}I_\lambda(u_{1k}, u_{2k}) &= \Phi(u_{1k}, u_{2k}) - \lambda\Psi(u_{1k}, u_{2k})\\ &\le \frac{1}{8S_pL}(\eta_{1,k}^2 + \eta_{2,k}^2) - \lambda \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi\\ &\le \frac{(1-\lambda\theta)}{8S_pL}(\eta_{1,k}^2 + \eta_{2,k}^2),\end{aligned}$$

for k large enough, so $\lim_{k\to+\infty} I_\lambda(u_{1k}, u_{2k}) = -\infty$, and hence, the claim follows.
The alternative of Theorem 1 case (b) assures the existence of sequence (u_k) of pairwise distinct critical points of I_λ which weakly converges to 0. This completes the proof of Theorem 3. □

Remark 1 *We observe that, if $F_{t_i}(\xi, 0, 0) \neq 0$, then, by Theorem 2 and 3 we obtain the existence of infinitely many non-trivial weak solutions.*

Corollary 1 *Let* Ω *be a bounded open subset of* $\mathbb{H}^n$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function. Suppopse* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *is defined by* $F(\xi, s) := \int_0^s f(\xi, \iota)d\iota$, *for all* $\xi \in \Omega$ *and* $s \in \mathbb{R}$, *and satisfying*

i) $F(\xi, s) > M|s|^2$,

ii) $|F(\xi, s)| < C\|s\|^2_{L^p(\Omega)}$,

for some positive constants M *and* C. *Let* $2s_pC|\Omega| < L'M\zeta^{2n+1}$, *where*

$$L' := \frac{2C_n}{(C_n + \|a\|_\infty)} \frac{R_0^2(1-\zeta)^2}{\left(1 - \frac{t^2}{r^2}\right)(1+4t^2)},$$

then for each $\lambda \in (\frac{1}{L'M\zeta^{2n+1}}, \frac{1}{2s_pC|\Omega|})$, *the problem*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + a(\xi)\frac{u}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

has an unbounded sequence of weak solutions in $H_0^1(\Omega, \mathbb{H}^n)$.

Proof. We apply Theorem 1 part of (a). The strategy of the proof is similar to Theorem 2, hence, we omit the details. Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\lim_{k\to+\infty} \sigma_k = +\infty$. Set $r_k = \frac{1}{2S_p}{\sigma_k}^2$. By (ii) we have

$$\begin{aligned} \gamma &\le \liminf_{k\to+\infty} \varphi(r_k) \\ &\le 2S_p \lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s\|_{L^p(\Omega)}<\sigma_k} F(\xi, s)d\xi}{\sigma_k^2} \\ &= 2S_pC|\Omega| < +\infty. \end{aligned}$$

This implies $\gamma \le 2S_pC|\Omega| < \dfrac{1}{\lambda}$. Let (u_k) be the sequence defined in (8). Then,

$$\begin{aligned} \Phi(u_k) &< \left(\frac{C_n + \|a\|_\infty}{2C_n}\right) \int_{B_{2n+1}(0,R_0)\setminus B_{2n+1}(0,\zeta R_0)} |\nabla_{\mathbb{H}^n} u_k(\xi)|^2 d\xi \\ &\le \frac{(C_n + \|a\|_\infty)}{2C_n} \frac{\eta_k^2}{R_0^2(1-\zeta)^2}\left(1 - \frac{t^2}{r^2}\right)(1+4t^2)\omega_{2n+1}R_0^{2n+1}(1-\zeta^{2n+1}) \\ &= \frac{\eta_k^2 \omega_{2n+1} R_0^{2n+1}}{L'}. \end{aligned} \tag{14}$$

On other hand, by (i), we have

$$\Psi(u_k) = \int_\Omega F(\xi, u_k)d\xi \geq \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_k)d\xi > M\eta_k^2 \zeta^{2n+1} \omega_{2n+1} R_0^{2n+1}. \tag{15}$$

So, it follows from (14) and (15) that

$$\begin{aligned} I_\lambda(u_k) &= \Phi(u_k) - \lambda\Psi(u_k) \\ &\leq \eta_k^2 \omega_{2n+1} R_0^{2n+1} \left(\frac{1}{L'} - \lambda M \zeta^{2n+1} \right), \end{aligned}$$

for k large enough, so $\lim_{k\to+\infty} I_\lambda(u_k) = -\infty$, and hence, the claim follows. The alternative of Theorem 1 case (a) assures the existence of unbounded sequence $\{u_k\}$ of critical points of the functional I_λ, and the proof is complete. □

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A power of a meromorphic function sharing a set with its higher order derivative

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Abstract. In this paper, we deduce the form of a nonconstant meromorphic function f when some power of f shares certain set counting multiplicities in the weak sense with the k-th derivative of the power. The results of this paper generalize the results due to Lahiri and Zeng [Afr. Mat. 27 (2016), 941-947].

1 Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions in the complex plane and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f - a$ and $g - a$ coincide both in locations and multiplicities then we say that f and g share the value a CM (counting multiplicities) and if they coincide only in locations (may or may not have the same multiplicities) then we say that f and g share the value a IM (ignoring multiplicities). For a meromorphic function f in the complex plane, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r outside a possible exceptional set of finite logarithmic measure. Throughout this paper, we adopt the standard notations of Nevanlinna Theory as described in [1] and [8]. We now recall the following definitions.

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Definition 1 [3] *For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f| = 1)$ *the counting function of simple* a*-points of* f*. For a positive integer* m *we denote by* $N(r, a; f| \leq m)$ *(*$N(r, a; f| \geq m)$*) the counting function of those* a*-points of* f *whose multiplicities are not greater (less) than* m *where each* a*-point is counted according to its multiplicity.*

$\overline{N}(r, a; f| \leq m)$ *(*$\overline{N}(r, a; f| \geq m)$*) are defined analogously, where in counting the* a*-points of* f *we ignore the multiplicities.*

Definition 2 [2] *Let* a *be any value in the extended complex plane, and let* p *be an arbitrary nonnegative integer. We denote by* $N_p(r, a; f)$ *the counting function of* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$*. Then*

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \ \ldots \ + \overline{N}(r, a; f \mid \geq p).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$*.*

In 1983, Mues and Steinmetz [7] proved the following result.

Theorem A *Let* f *be a nonconstant meromorphic function and* a*,* b *be two distinct finite complex numbers. If* f *and* f' *share* a*,* b *CM, then* $f = ce^z$*, where* c *is a nonzero constant.*

In 2004, Lin and Huang [6] proved the following result considering certain power of a meromorphic function.

Theorem B *Let* f *be a nonconstant meromorphic function,* $n(\geq 8)$ *be an integer and* a *be a nonzero complex number. If* f^n *and* $(f^n)'$ *share the value* a *CM, then* $f = ce^{\frac{z}{n}}$*, where* c *is a nonzero constant.*

In 2008, Lei, Fang, Yang and Wang [5] improved Theorem B by relaxing the lower bound of n and proved the result for $n \geq 4$.

For $a \in \mathbb{C} \cup \{\infty\}$, let $E(a, f)$ denote the set of all a-points of f where an a-point is counted according to its multiplicity and $\overline{E}(a, f)$ denote the set of distinct a-points of f. If $S \subset \mathbb{C} \cup \{\infty\}$, then we define $E(S, f) = \cup_{a \in S} E(a, f)$. We say that f and g share the set S counting multiplicities (CM) if $E(S, f) = E(S, g)$. Similarly we define $\overline{E}(S, f) = \cup_{a \in S} \overline{E}(a, f)$.

Let $a \in \mathbb{C} \cup \{\infty\}$ and $B \subset \mathbb{C} \cup \{\infty\}$. We denote by $\overline{E}_B(a; f, g)$ the set of all those distinct a-points of f which are b-points of g with same multiplicities for some $b \in B$ and $\overline{E}_B(A; f, g) = \cup_{a \in A} \overline{E}_B(a; f, g)$ for $A \subset \mathbb{C} \cup \{\infty\}$.

For $S \subset \mathbb{C} \cup \{\infty\}$, we now put $Y = \{\overline{E}(S, f) \cup \overline{E}(S, g)\} \setminus \overline{E}_S(S; f, g)$. We say that f and g share the set S counting multiplicities in the weak sense or WCM if $\overline{N}_Y(r, a; f) = S(r, f)$ and $\overline{N}_Y(r, a; g) = S(r, g)$ for every $a \in S$, where $\overline{N}_Y(r, a; f)$ denotes the reduced counting function of those a-points of f which lie in the set Y (see [4]). Intuitively, sharing WCM is little less than sharing CM by an unimportant error term. We also see that f and g share the set S CM if and only if $Y = \phi$. Further, WCM value sharing is same as "CM" value sharing when $S = \{a\}$(p. 226, [8]).

In 2016, using the concept of WCM value sharing of a set, Lahiri and Zeng [4] proved the following theorems which improve Theorem B.

Theorem C Let f be a nonconstant meromorphic function, m, $n(\geq 4)$ be positive integers and $S = \{a_1, a_2, ..., a_m\} \subset \mathbb{C}\setminus\{0\}$ be a set of distinct numbers. If f^n and $(f^n)'$ share the set S WCM, then $f = ce^{\frac{\omega z}{n}}$, where $c(\neq 0)$, ω are constants and $\omega^m = 1$. Further $f = ce^{\frac{z}{n}}$ if either $\sum_{i=1}^{m} a_i \neq 0$ or m is prime and $S \neq \{az : z^m = 1\}$, where a is any nonzero number.

Remark 1 [4] *If $\sum_{i=1}^{m} a_i = 0$, then ω may not be equal to 1. For example, let $S = \{1, -1, 2, -2\}$ and $f = ce^{\frac{-z}{4}}$, where c is a nonzero constant.*

Remark 2 [4] *If $S = \{az : z^m = 1\}$, then ω may not be equal to 1 even if m is prime. For, let $S = \{2, 2\omega, 2\omega^2\}$ and $f = ce^{\frac{\omega z}{4}}$, where c is a nonzero constant and ω is an imaginary cube root of unity.*

Remark 3 [4] *If m is not a prime, then ω may not be equal to 1 even if $S \neq \{az : z^m = 1\}$, where a is any nonzero constant. The example in Remark 1 makes it evident.*

Theorem D *Let f be a nonconstant meromorphic function, $m(\geq 2)$, $n(\geq 3)$ be positive integers and $S = \{a_1, a_2, ..., a_m\} \subset \mathbb{C}\setminus\{0\}$ be a set of distinct numbers such that $\sum_{i=1}^{m} a_i = 0$. If f^n and $(f^n)'$ share the set S WCM, then $f = ce^{\frac{\omega z}{n}}$, where $c(\neq 0)$, ω are constants and $\omega^m = 1$.*

Regarding Theorems C and D, it is natural to ask the following question which is the motive of this paper.

Question 1 *What happens if the function f^n share the set S WCM with its k-th derivative in Theorems C and D?*

In this paper, we find possible answer to the above question and prove the following theorems.

Theorem 1 *Let* f *be a nonconstant meromorphic function,* m, n, $k(\geq 1)$ *be positive integers satisfying* $n \geq k+1+\sqrt{k+2}$ *and* $S=\{a_1, a_2, ..., a_m\} \subset \mathbb{C}\backslash\{0\}$ *be a set of distinct complex numbers. If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM then either* $f = ce^{\frac{\omega^{\frac{1}{k}}}{n}\nu z}$, *where* $c(\neq 0)$, ω *and* ν *are constants with* $\omega^m = 1$ *and* $\nu^k = 1$ *or* f^n *is a linear combination of* $e^{\omega^{\frac{1}{k}}\nu_1 z}$, $e^{\omega^{\frac{1}{k}}\nu_2 z}$, $\ldots$, $e^{\omega^{\frac{1}{k}}\nu_k z}$, *where* ν_i*'s are the distinct* k*-th roots of unity. Further, if either* $\sum_{i=1}^{m} a_i \neq 0$ *or* m *is prime and* $S \neq \{az : z^m = 1\}$, *where* a *is any nonzero number, then* $\omega = 1$.

Theorem 2 *Let* f *be a nonconstant meromorphic function,* $m(\geq 2)$, n, k *be positive integers satisfying* $n > \frac{3(k+1)+\sqrt{k^2+10k+17}}{4}$ *and* $S = \{a_1, a_2, ..., a_m\} \subset \mathbb{C}\backslash\{0\}$ *be a set of distinct complex numbers such that* $\sum_{i=1}^{m} a_i = 0$. *If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM, then either* $f = ce^{\frac{\omega^{\frac{1}{k}}}{n}\nu z}$, *where* $c(\neq 0)$, ω *and* ν *are constants with* $\omega^m = 1$ *and* $\nu^k = 1$ *or* f^n *is a linear combination of* $e^{\omega^{\frac{1}{k}}\nu_1 z}$, $e^{\omega^{\frac{1}{k}}\nu_2 z}$, $\ldots$, $e^{\omega^{\frac{1}{k}}\nu_k z}$, *where* ν_i*'s are the distinct* k*-th roots of unity.*

Remark 4 *Theorems C and D can be obtained by putting* $k = 1$ *in Theorems 1 and 2, as in this case, we obtain* $\nu = 1$.

2 Lemmas

Let a, a_1, a_2, $\ldots$, a_m be distinct finite complex numbers. We put $z_i = a - a_i$ for $i = 1, 2, ..., m$ and $\sigma_0 = 1$, $\sigma_1 = \sum_{i=1}^{m} z_i$, $\sigma_2 = \sum_{1\leq i<j\leq m} z_i z_j$, $\ldots$, $\sigma_m = z_1 z_2 \ldots z_m$. We say that a complex number C satisfies the property (A) if $\sigma_i(C^i - 1) = 0$ and a complex number K satisfies the property (B), if $K^i\sigma_{m-i} = \sigma_i\sigma_m$, $i = 1, 2, 3, ..., m$ (see [8], p.482).

Now we state some lemmas which will be needed in the sequel.

Lemma 1 *Let* f *be a nonconstant meromorphic function and* $S = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{C}$ *be a set of distinct complex numbers. Further suppose that* $N(r, a; f) + N(r, a; f^{(k)}) + N(r, \infty; f) = S(r, f)$ *for some* $a \in \mathbb{C}\backslash S$. *If* f *and* $f^{(k)}$ *share the set* S *WCM, then either* $f^{(k)} - a \equiv C(f - a)$ *or* $(f^{(k)} - a)(f - a) \equiv K$, *where* C *satisfies the property (A) and* K *satisfies the property (B).*

Proof. Clearly $N(r, a; f) = N(r, a; f^{(k)}) = N(r, \infty; f) = S(r, f)$.

If z_0 is a pole of f of order l then z_0 is a pole of $f^{(k)}$ of order $l + k$. Now, $l + k \leq (k + 1)l$, therefore $N(r, \infty; f^{(k)}) \leq (k + 1)N(r, \infty; f)$, which implies

$N(r,\infty;f^{(k)}) = S(r,f)$. Thus, using Lemma 3.8 of [8] (p.193) we deduce that

$$\delta(a,f) = \delta(\infty,f) = \delta(a,g) = \delta(\infty,g) = 1,$$

where $g = f^{(k)}$. The rest of the proof can be completed in the line of Theorem 10.26 of [8], (p. 482). □

Lemma 2 [8]*(Theorem 1.24, p.39) Let* f *be a nonconstant meromorphic function and* k *be a positive integer. Then*

$$N(r,0;f^{(k)}) \leq N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 3 [9] *Let* f *be a nonconstant meromorphic function and* p, k *be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 4 *Let* f *be a nonconstant meromorphic function,* m, k, $n(> k+1)$ *be positive integers and* $S = \{a_1, a_2, ..., a_m\}$ *be a set of distinct nonzero complex numbers. If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM, then one of the following holds:*

(i) $N(r,0;f) \leq \frac{1}{n-k-1}\overline{N}(r,\infty;f) + S(r,f)$;

(ii) $(f^n)^{(k)} \equiv \omega f^n$, *where* $\omega^m = 1$.

Proof. Let $g = f^n$. Put

$$\phi = \sum_{i=1}^{m} \frac{g'}{g-a_i} - \sum_{i=1}^{m} \frac{g^{(k+1)}}{g^{(k)}-a_i}. \tag{1}$$

Now we consider the following cases.

Case 1. Let $\phi \not\equiv 0$. Then $m(r,\phi) = S(r,g) = S(r,f)$. If z_0 is a zero of f with multiplicity l, then z_0 is a zero of ϕ with multiplicity at least $l(n-k-1)$. Since g and $g^{(k)}$ share S WCM, from (1) we get $N(r,\infty;\phi) \leq \overline{N}(r,\infty;f) + S(r,f)$. Therefore

$$\begin{aligned} N(r,0;f) &\leq \frac{1}{n-k-1}N(r,0;\phi) \\ &\leq \frac{1}{n-k-1}T(r,\phi) + O(1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n-k-1} N(r, \infty; \phi) + S(r, f) \\
&\leq \frac{1}{n-k-1} \overline{N}(r, \infty; f) + S(r, f).
\end{aligned}
$$

Case 2. Let $\phi \equiv 0$. Then

$$
\sum_{i=1}^{m} \frac{g'}{g - a_i} \equiv \sum_{i=1}^{m} \frac{g^{(k+1)}}{g^{(k)} - a_i}.
$$

Integrating,

$$
\prod_{i=1}^{m} (g - a_i) \equiv c \prod_{i=1}^{m} (g^{(k)} - a_i), \tag{2}
$$

where c is a nonzero constant.

If $N(r, 0; f) = S(r, f)$, then (i) holds. So we assume that $N(r, 0; f) \neq S(r, f)$. If z_0 is a zero of f with multiplicity l, then z_0 is a zero of g and $g^{(k)}$ of multiplicities nl and $nl - k$ respectively. So from (2) we see that $c = 1$. Also we have $g^{(nl)}(z_0) \neq 0$. Thus from (2) we obtain

$$
\begin{aligned}
& g^m + \sum_{i=1}^{m} (-a_i) g^{m-1} + \sum_{1 \leq i < j \leq m} (a_i a_j) g^{m-2} + \ldots + \sum_{i=1}^{m} (-1)^{m-1} \frac{a_1 a_2 \ldots a_m}{a_i} g \\
& \equiv (g^{(k)})^m + \sum_{i=1}^{m} (-a_i)(g^{(k)})^{m-1} + \sum_{1 \leq i < j \leq m} (a_i a_j)(g^{(k)})^{m-2} \\
& + \ldots + \sum_{i=1}^{m} (-1)^{m-1} \frac{a_1 a_2 \ldots a_m}{a_i} g^{(k)}.
\end{aligned} \tag{3}
$$

If $m = 1$, then $(f^n)^{(k)} = f^n$. Let $m \geq 2$. We differentiate (3) $nl - k$ times and put $z = z_0$ to obtain

$$
\sum_{i=1}^{m} (-1)^{m-1} \frac{a_1 a_2 \ldots a_m}{a_i} = 0.
$$

Hence from (3) we get

$$\begin{aligned} & g^m+\sum_{i=1}^{m}(-a_i)g^{m-1}+\sum_{1\le i<j\le m}(a_ia_j)g^{m-2}+\ldots+\sum_{i=1}^{m}(-1)^{m-2}\frac{a_1a_2...a_m}{a_ia_j}g^2 \\ & \equiv (g^{(k)})^m+\sum_{i=1}^{m}(-a_i)(g^{(k)})^{m-1}+\sum_{1\le i<j\le m}(a_ia_j)(g^{(k)})^{m-2} \\ & +\ldots+\sum_{i=1}^{m}(-1)^{m-2}\frac{a_1a_2...a_m}{a_ia_j}(g^{(k)})^2. \end{aligned} \tag{4}$$

Differentiating both sides of (4) $2(nl-k)$ times and putting $z=z_0$, we get

$$\sum_{1\le i<j\le m}\frac{a_1a_2...a_m}{a_ia_j}=0.$$

Proceeding similarly, we get

$$\sum_{i=1}^{m}a_i=\sum_{1\le i<j\le m}a_ia_j=\ \ldots\ =0.$$

Hence from (3) we get $g^m\equiv(g^{(k)})^m$ and so $(f^n)^{(k)}\equiv\omega f^n$, where $\omega^m=1$. This proves the lemma. □

Lemma 5 *Let* f *be a nonconstant meromorphic function,* m, $n(\ge 2)$ *be positive integers and* $S=\{a_1,a_2,...,a_m\}$ *be a set of distinct nonzero complex numbers. If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM, then*

$$N(r,\infty;f)\le\frac{k+2}{n-1}\overline{N}(r,0;f)+\frac{k}{n-1}\overline{N}(r,\infty;f)+S(r,f).$$

Proof. Let $g=f^n$. We put

$$\phi=\frac{mg'}{g}-\sum_{i=1}^{m}\frac{g'}{g-a_i}-\frac{mg^{(k+1)}}{g^{(k)}}+\sum_{i=1}^{m}\frac{g^{(k+1)}}{g^{(k)}-a_i}. \tag{5}$$

Casa 1: Let $\phi\not\equiv 0$. Then $m(r,\phi)=S(r,g)=S(r,f)$. We can write (5) as

$$\phi\ =\ \frac{g'}{g\prod_{i=1}^{m}(g-a_i)}\left[\sum_{i=1}^{m}(-a_i)g^{m-1}+P_{m-2}(g)\right]$$

$$-\frac{g^{(k+1)}}{g^{(k)}\prod_{i=1}^{m}(g^{(k)}-a_i)}\left[\sum_{i=1}^{m}(-a_i)(g^{(k)})^{m-1}+P_{m-2}(g^{(k)})\right], \tag{6}$$

where $P_{m-2}(z)$ is a polynomial of degree at most $m-2$ if $m \geq 2$ and $P_{-1}(z) \equiv 0$.

If z_0 is a pole of f with multiplicity l then z_0 is a zero of ϕ with multiplicity at least $(n-1)l$. Since g and $g^{(k)}$ share the set S WCM, using Lemma 3 we see that

$$\begin{aligned} N(r,\infty;\phi)=\overline{N}(r,\infty;\phi) &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g^{(k)})+S(r,f) \\ &\leq \overline{N}(r,0;f)+k\overline{N}(r,\infty;f)+N_{k+1}(r,0;f^{n})+S(r,f) \\ &\leq (k+2)\overline{N}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Hence we obtain

$$\begin{aligned} N(r,\infty;f) &\leq \frac{1}{n-1}N(r,0;\phi) \\ &\leq \frac{1}{n-1}T(r,\phi)+S(r,f) \\ &= \frac{1}{n-1}N(r,\infty;\phi) \\ &\leq \frac{k+2}{n-1}\overline{N}(r,0;f)+\frac{k}{n-1}\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Case 2: Let $\phi \equiv 0$. Then integrating (5) we have,

$$g^{m}\prod_{i=1}^{m}(g^{(k)}-a_i) \equiv c(g^{(k)})^{m}\prod_{i=1}^{m}(g-a_i), \tag{7}$$

where $c(\neq 0)$ is a constant.

Now (7) can be rewritten in concise form as

$$\prod_{i=1}^{m}\left(1-\frac{a_i}{g^{(k)}}\right)=c\prod_{i=1}^{m}\left(1-\frac{a_i}{g}\right).$$

From the above we note that if f has a pole at $z=z_0$, say, then $c=1$. Hence from (7) we get

$$\begin{aligned} &\left(-\sum_{i=1}^{m}a_i\right)g^{m}\left(g^{(k)}\right)^{m-1}+g^{m}Q_{m-2}\left(g^{(k)}\right) \\ &=\left(-\sum_{i=1}^{m}a_i\right)\left(g^{(k)}\right)^{m}g^{m-1}+\left(g^{(k)}\right)^{m}Q_{m-2}(g), \end{aligned} \tag{8}$$

where $Q_{m-2}(z)$ is a polynomial of degree at least $m-2$ if $m \geq 2$ and $Q_{-1}(z) \equiv 0$.

Let $\sum_{i=1}^{m} a_i \neq 0$. If z_0 is a pole of f with multiplicity l, then z_0 is a pole of multiplicity $2mnl + mk - nl - k$ of the left hand side of (8) and a pole of multiplicity $2mnl + mk - nl$ of the right hand side of the same, which can not happen. Now we assume $\sum_{i=1}^{m} a_i = 0$. If z_0 is a pole of f with multiplicity l, then z_0 is a pole of multiplicity $2mnl + mk - 2k - 2nl$ of the left hand side of (8) and a pole of multiplicity $2mnl + mk - 2nl$ of the right hand side of the same, which is impossible. Thus f has no pole in both the cases and hence the lemma. □

Lemma 6 *Let f be a nonconstant meromorphic function, $m(\geq 2)$, n be integers and $S = \{a_1, a_2, ..., a_m\}$ be a set of distinct nonzero complex numbers with $\sum_{i=0}^{m} a_i = 0$. If f^n and $(f^n)^{(k)}$ share the set S WCM, then*

$$N(r, \infty; f) \leq \frac{k+2}{2n-1}\overline{N}(r, 0; f) + \frac{k}{2n-1}\overline{N}(r, \infty; f) + S(r, f).$$

Proof. The lemma can be proved in a similar way as in Lemma 5 noting that if z_0 is a pole of f with multiplicity l, then it is a zero of ϕ with multiplicity at least $(2n-1)l$. □

3 Proof of theorems

Proof. [Proof of Theorem 1] First, we suppose that $(f^n)^{(k)} \not\equiv \omega f^n$ for any constant ω satisfying $\omega^m = 1$. Then using (i) of Lemma 4 and Lemma 5 we have

$$\begin{aligned} N(r, \infty; f) &\leq \frac{k+2}{n-1}\overline{N}(r, 0; f) + \frac{k}{n-1}\overline{N}(r, \infty; f) + S(r, f) \\ &\leq \frac{k+2}{n-1}N(r, 0; f) + \frac{k}{n-1}\overline{N}(r, \infty; f) + S(r, f) \\ &\leq \frac{1}{n-1}\left(\frac{k+2}{n-k-1} + k\right)\overline{N}(r, \infty; f) + S(r, f). \end{aligned} \tag{9}$$

Since $n > k + 1 + \sqrt{k+2}$, from (9) it is clear that $N(r, \infty; f) = S(r, f)$, $N(r, 0; f) = S(r, f)$ and hence $N(r, 0; f^{(k)}) = S(r, f)$, by Lemma 2.

Now,

$$\begin{aligned} T\big(r, (f^n)^{(k)}\big) &= m\big(r, (f^n)^{(k)}\big) + N\big(r, \infty; (f^n)^{(k)}\big) \\ &\leq m(r, f^n) + N(r, \infty; f^n) + k\overline{N}(r, \infty; f) + S(r) \\ &= T(r, f^n) + S(r), \end{aligned} \tag{10}$$

and

$$\begin{aligned} T(r,f^n) &\le T\big(r,(f^n)^{(k)}\big) + T\left(r, \frac{f^n}{(f^n)^{(k)}}\right) + S(r) \\ &\le T\big(r,(f^n)^{(k)}\big) + N\left(r,\infty; \frac{(f^n)^{(k)}}{f^n}\right) + S(r) \\ &\le T\big(r,(f^n)^{(k)}\big) + N\big(r,\infty;(f^n)^{(k)}\big) + N(r,0;f^n) + S(r) \\ &\le T\big(r,(f^n)^{(k)}\big) + n(k+1)N\big(r,\infty;f\big) + nN(r,0;f) + S(r) \\ &= T\big(r,(f^n)^{(k)}\big) + S(r), \end{aligned} \tag{11}$$

where $S(r) = \max\{S(r,f), S(r,f^n), S\big(r,(f^n)^{(k)}\big)\}$.

From (10) and (11) we obtain $T\big(r,(f^n)^{(k)}\big) = T(r,f^n)+S(r) = nT(r,f)+S(r)$ and therefore $S(r,f) = S(r,f^n) = S\big(r,(f^n)^{(k)}\big)$. Also by Lemma 2 we see that

$$N(r,0;f^n) + N\big(r,0;(f^n)^{(k)}\big) + N(r,\infty;f^n) \le 2nN(r;0;f) \\ +k\overline{N}(r,\infty;f) + nN(r,\infty;f) = S(r,f).$$

So by Lemma 1, we obtain either $(f^n)^{(k)} \equiv Cf^n$ or $(f^n)^{(k)}f^n \equiv K$, where C and K satisfy properties (A) and (B) respectively as given earlier with $a = 0$.

As $\sigma_m(C^m - 1) = 0$ and $\sigma_m \ne 0$, we get $C = \omega$, where $\omega^m = 1$. Therefore, $(f^n)^{(k)} \equiv \omega f^n$ where ω is a constant satisfying $\omega^m = 1$, a contradiction with our assumption. Therefore $(f^n)^{(k)}f^n \equiv K$, where $K^m = (\sigma_m)^2 \ne 0$. From this it follows that f is an entire function having no zero. Thus we may put $f^n = e^{\alpha}$, where α is a nonconstant entire function. So from above we get $e^{2\alpha}P(\alpha', \ldots, \alpha^{(k)}) \equiv K$, where $P(\alpha', \ldots, \alpha^{(k)})$ is a differential polynomial in $\alpha', \alpha'', \ldots, \alpha^{(k)}$. Since α is an entire function, we have $T(r,\alpha^{(j)}) = S(r,f)$ for $j \in \{1,2, \ldots, k\}$, and hence $T(r,P) = S(r,f) = S(r,e^{\alpha})$. Thus, we obtain

$$2T(r,e^{\alpha}) = T(r,P) + O(1) = S(r,e^{\alpha}),$$

a contradiction.

Hence we must have $(f^n)^{(k)} \equiv \omega f^n$ for some constant ω satisfying $\omega^m = 1$. On solving this k-th order differential equation for f, we obtain either $f = ce^{\frac{\omega^{\frac{1}{k}}}{n}\nu z}$, where $c(\ne 0)$ and ν are constants with $\nu^k = 1$ or f^n is a linear combination of $e^{\omega^{\frac{1}{k}}\nu_1 z}, e^{\omega^{\frac{1}{k}}\nu_2 z}, \ldots, e^{\omega^{\frac{1}{k}}\nu_k z}$, where ν_i's are the distinct k-th roots of unity. The rest of the proof can be completed in a similar way as done in the last part of the proof of Theorem 1.1 in [4]. □

Proof. [Proof of Theorem 2] Using Lemma 6 instead of Lemma 5, this theorem can be proved in the line of Theorem 1. Here we omit the details. □

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Norm attaining bilinear forms on the plane with the l_1-norm

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Abstract. For given unit vectors $x_1, \cdots, x_n$ of a real Banach space E, we define

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n) = \{T \in \mathcal{L}(^nE) : |T(x_1, \cdots, x_n)| = \|T\| = 1\},$$

where $\mathcal{L}(^nE)$ denotes the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \ldots, x_n)|$.

In this paper, we classify $NA(\mathcal{L}(^2l_1^2))((x_1, x_2), (y_1, y_2))$ for unit vectors $(x_1, x_2), (y_1, y_2) \in l_1^2$, where $l_1^2 = \mathbb{R}^2$ with the l_1-norm.

1 Introduction

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a real Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \ldots, x_n)|$. The subspace of all continuous symmetric n-linear forms on E is denoted by $\mathcal{L}_s(^nE)$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists $T \in \mathcal{L}(^nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed

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with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

Elements $x_1, \ldots, x_n \in E$ is called *norming points* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$. In this case, T is called a *norm attaining* n-linear form at $x_1, \ldots, x_n$. Similarly, an element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. In this case, P is called a *norm attaining* n-homogeneous polynomial at x. Let $X = \mathcal{L}(^nE)$ or $\mathcal{L}_s(^nE)$. For $x, x_1, \cdots, x_n \in S_E$, we define

$$NA(X)(x_1, \cdots, x_n) = \{T \in X : |T(x_1, \cdots, x_n)| = \|T\| = 1\}$$

and

$$NA(\mathcal{P}(^nE))(x) = \{P \in \mathcal{P}(^nE) : |P(x)| = \|P\| = 1\}.$$

Notice that

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}(^nE))(\pm x_1, \cdots, \pm x_n),$$

$$NA(\mathcal{L}_s(^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}_s(^nE))(\pm x_{\sigma(1)}, \cdots, \pm x_{\sigma(n)})$$

and

$$NA(\mathcal{P}(^nE))(x) = NA(\mathcal{P})(^nE))(-x)$$

for all $x, x_1, \cdots, x_n \in S_E$ and for all permutation σ on $\{1, \ldots, n\}$.

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jimenez-Sevilla and Paya [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

It seems to be natural and interesting to study about

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n), NA(\mathcal{L}_s(^nE))(x_1, \cdots, x_n) \text{ and } NA(\mathcal{P}(^nE))(x)$$

for $x, x_1, \cdots, x_n \in S_E$. Kim [6] classified $NA(\mathcal{P}(^2l_p^2))((x_1, x_2))$ for $(x_1, x_2) \in S_{l_p^2}$ and $p = 1, 2, \infty$, where $l_p^2 = \mathbb{R}^2$ with the l_p-norm.

In this paper, we classify $NA(\mathcal{L}(^2l_1^2))((x_1, x_2), (y_1, y_2))$ for $(x_1, x_2), (y_1, y_2) \in S_{l_1^2}$.

2 Results

Let $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2l_1^2)$ for some $a, b, c, d \in \mathbb{R}$. For simplicity, we denote $T = (a, b, c, d)$.

Theorem 1 *Let* $T = (a, b, c, d) \in \mathcal{L}(^2l_1^2)$ *for some* $a, b, c, d \in \mathbb{R}$. *Then,*

$$\|T\| = \max\{|a|, |b|, |c|, |d|\}.$$

Proof. Let $M := \max\{|a|, |b|, |c|, |d|\}$. Let $(x_j, y_j) \in S_{l_1^2}$ for $j = 1, 2$. It follows that

$$\begin{aligned}
|T((x_1, y_1), (x_2, y_2))| &\leq |a|\ |x_1x_2| + |b|\ |y_1y_2| + |c|\ |x_1y_2| + |d|\ |x_2y_1| \\
&\leq M\ (|x_1x_2| + |y_1y_2| + |x_1y_2| + |x_2y_1|) \\
&= M(|x_1| + |y_1|)(|x_2| + |y_2|) = M \\
&= \max\{|T(1, 0), (1, 0)|, |T(0, 1), (0, 1)|, |T(1, 0), (0, 1)|, \\
&\quad |T(0, 1), (1, 0)|\} \leq \|T\|.
\end{aligned}$$

Therefore, $\|T\| = M$. □

Notice that if $\|T\| = 1$, then $|a| \leq 1, |b| \leq 1, |c| \leq 1$ and $|d| \leq 1$.

Lemma 1 *Let* $T = (a, b, c, d) \in \mathcal{L}(^2l_1^2)$ *for some* $a, b, c, d \in \mathbb{R}$. *The following are equivalent: let* $(x_1, y_1), (x_2, y_2) \in S_{l_1^2}$.

(a) $T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$;
(b) $T_1 := (b, a, d, c) \in NA(\mathcal{L}(^2l_1^2))((y_1, x_1), (y_2, x_2))$;
(c) $T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2))$;
(d) $T_3 := (-a, -b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((-x_1, -y_1), (x_2, y_2))$;
(e) $T_4 := (a, b, d, c) \in NA(\mathcal{L}(^2l_1^2))((x_2, y_2), (x_1, y_1))$;
(f) $T_5 := (a, -b, -c, d) \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, -y_2))$.

The following theorem classifies $NA(\mathcal{L}(^2l_1^2))((x_1, x_2), (y_1, y_2))$ for unit vectors $(x_1, x_2), (y_1, y_2) \in l_1^2$.

Theorem 2 *Let* $(x_1, y_1), (x_2, y_2) \in S_{l_1^2}$. *Then the following statements holds:*

Case 1. $x_j y_j \neq 0$ *for all* $j = 1, 2$.

If $x_j y_j > 0$ *for all* $j = 1, 2$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, 1, 1)\}.$$

If $x_j y_j < 0$ *for all* $j = 1, 2$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, -1, -1)\}.$$

If $x_1 y_1 > 0$ *and* $x_2 y_2 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, -1, -1, 1)\}.$$

If $x_1 y_1 < 0$ *and* $x_2 y_2 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, -1, 1, -1)\}.$$

Case 2. $x_1 y_1 = 0$ *and* $x_2 y_2 \neq 0$

If $x_1 = 0$ *and* $x_2 y_2 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}.$$

If $x_1 = 0$ *and* $x_2 y_2 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, -1) : |a| \leq 1, |c| \leq 1\}.$$

If $y_1 = 0$ *and* $x_2 y_2 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, 1, d) : |b| \leq 1, |d| \leq 1\}.$$

If $y_1 = 0$ *and* $x_2 y_2 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, -1, d) : |b| \leq 1, |d| \leq 1\}.$$

Case 3. $x_2 y_2 = 0$ *and* $x_1 y_1 \neq 0$

If $x_2 = 0$ *and* $x_1 y_1 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, 1, d) : |a| \leq 1, |d| \leq 1\}.$$

If $x_2 = 0$ *and* $x_1y_1 < 0$, *then*

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, -1, d) : |a| \le 1, |d| \le 1\}.$$

If $y_2 = 0$ *and* $x_1y_1 > 0$, *then*

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, 1) : |b| \le 1, |c| \le 1\}.$$

If $y_2 = 0$ *and* $x_1y_1 < 0$, *then*

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, -1) : |b| \le 1, |c| \le 1\}.$$

Case 4. $x_1y_1 = x_2y_2 = 0$

If $x_1 = x_2 = 0$, *then*

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, d) : |a| \le 1, |c| \le 1, |d| \le 1\}.$$

If $x_1 = y_2 = 0$, *then*

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, c, 1) : |a| \le 1, |b| \le 1, |c| \le 1\}.$$

If $x_2 = y_1 = 0$, *then*

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, 1, d) : |a| \le 1, |b| \le 1, |d| \le 1\}.$$

If $y_1 = y_2 = 0$, *then*

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, d) : |b| \le 1, |c| \le 1, |d| \le 1\}.$$

Proof. Let $(x_1, y_1), (x_2, y_2) \in S_{l_1^2}$. Let $T = (a, b, c, d) \in NA(\mathcal{L}(^2l_\infty^2))((x_1, y_1), (x_2, y_2))$ for some $a, b, c, d \in \mathbb{R}$. By Theorem 1, $|a| \le 1, |b| \le 1, |c| \le 1$ and $|d| \le 1$. By Lemma 1, we may assume that $a \ge 0$. We consider four cases.

Case 1. $x_jy_j \neq 0$ for all $j = 1, 2$.

Suppose that $x_jy_j > 0$ for all $j = 1, 2$.

Claim. $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, 1, 1)\}$.

It is obvious that $\{\pm(1, 1, 1, 1)\} \subseteq NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$. It follows that

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1| \\ &\le a\,|x_1x_2| + |b|\,|y_1y_2| + |c|\,|x_1y_2| + |d|\,|x_2y_1| \end{aligned}$$

$$\begin{aligned} &\leq |x_1x_2| + |y_1y_2| + |x_1y_2| + |x_2y_1| \\ &= (|x_1| + |y_1|)(|x_2| + |y_2|) = 1, \end{aligned}$$

which shows that $a = |b| = |c| = |d| = 1$. We will show that $b = 1$. Assume that $b = -1$. Then

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 - y_1y_2 + cx_1y_2 + dx_2y_1| \\ &= |x_1(x_2 + cy_2) + y_1(dx_2 - y_2)| \\ &\leq |x_1|\ |x_2 + cy_2| + |y_1|\ |dx_2 - y_2| \\ &\leq |x_1| + |y_1| = 1, \end{aligned}$$

which shows that

$$|x_2 + cy_2| = |dx_2 - y_2| = 1$$

because $|x_1| > 0$ and $|y_1| > 0$. Since $x_2y_2 > 0$, $c = 1, d = -1$. Hence,

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 - y_1y_2 + x_1y_2 - x_2y_1| \\ &= |x_1 - y_1|\ |x_2 + y_2| = |x_1 - y_1| < 1, \end{aligned}$$

which is a contradiction. Therefore, $b = 1$. It follows that

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 + y_1y_2 + cx_1y_2 + dx_2y_1| \\ &\leq |x_1|\ |x_2 + cy_2| + |y_1|\ |dx_2 + y_2| \\ &= |x_1| + |y_1| = 1, \end{aligned}$$

which shows that

$$|x_2 + cy_2| = |dx_2 + y_2| = 1$$

because $|x_1| > 0$ and $|y_1| > 0$. Hence, $c = d = 1$. Therefore, $T = (1, 1, 1, 1)$, which concludes $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) \subseteq \{\pm(1, 1, 1, 1)\}$. Therefore, we have shown the claim.

Suppose that $x_jy_j < 0$ for all $j = 1, 2$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)).$$

Since $x_j(-y_j) > 0$ for all $j = 1, 2$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)) = \{\pm(1, 1, 1, 1)\}.$$

Hence,

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, y_2)) = \{\pm(1, 1, -1, -1)\}.$$

Suppose that $x_1y_1 > 0$ and $x_2y_2 < 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_5 := (a, -b, -c, d) \in NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, -y_2)).$$

Since $x_1y_1 > 0$ and $x_2(-y_2) > 0$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, -y_2)) = \{\pm(1, 1, 1, 1)\}.$$

Hence,

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, y_2)) = \{\pm(1, -1, -1, 1)\}.$$

Suppose that $x_1y_1 < 0$ and $x_2y_2 > 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)).$$

Since $x_1(-y_1) > 0$ and $x_2(-y_2) < 0$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)) = \{\pm(1, -1, -1, 1)\}.$$

Hence,

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, y_2)) = \{\pm(1, -1, 1, -1)\}.$$

Case 2. $x_1y_1 = 0$ and $x_2y_2 \neq 0$

Suppose that $x_1 = 0$ and $x_2y_2 > 0$.

Claim. $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}$.

It is obvious that $\{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\} \subseteq NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$. Since $x_1 = 0$, $|y_2| = 1$ and

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |by_1y_2 + dx_2y_1| \\ &\leq |dx_2 + by_2| \leq |d|\,|x_2| + |b|\,|y_2| \\ &\leq |x_2| + |y_2| = 1, \end{aligned}$$

which shows that $|dx_2 + by_2| = 1 = |b| = |d|$. Since $x_2y_2 > 0$, $b = d = 1$ or $b = d = -1$. Hence, $T = \pm(a, 1, c, 1)$ for some $|a| \leq 1, |c| \leq 1$. Therefore, we have shown the claim.

Suppose that $x_1 = 0$ and $x_2y_2 < 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)).$$

Since $x_1(-y_1) = 0$ and $x_2(-y_2) < 0$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}.$$

Hence,

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, y_2)) = \{\pm(a, 1, c, -1) : |a| \leq 1, |c| \leq 1\}.$$

Suppose that $y_1 = 0$ and $x_2y_2 > 0$.
Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_1 := (b, a, d, c) \in NA(\mathcal{L}(^2l_1^2))((y_1, x_1), (y_2, x_2)).$$

By the above claim,

$$NA(\mathcal{L}(^2l_1^2))((y_1, x_1), (y_2, x_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}.$$

Hence, $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, 1, d) : |b| \leq 1, |d| \leq 1\}$.

Suppose that $y_1 = 0$ and $x_2y_2 < 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)).$$

Since $-y_1 = 0$ and $x_2(-y_2) > 0$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)) = \{\pm(1, b, 1, d) : |b| \leq 1, |d| \leq 1\}.$$

Hence,

$$NA(\mathcal{L}(^2l_1^2))((x_1,y_2),(x_2,y_2)) = \{\pm(1,b,-1,d) : |b| \leq 1, |d| \leq 1\}.$$

Case 3. $x_2y_2 = 0$ and $x_1y_1 \neq 0$

Lemma 1 implies that $T \in NA(\mathcal{L}(^2l_1^2))((x_1,y_1),(x_2,y_2))$ if and only if $T_1 := (b,a,d,c) \in NA(\mathcal{L}(^2l_1^2))((y_1,x_1),(y_2,x_2))$. By Case 2, the assertions of Case 3 hold.

Case 4. $x_1y_1 = x_2y_2 = 0$

Suppose that $x_1 = x_2 = 0$.

Claim. $NA(\mathcal{L}(^2l_1^2))((x_1,y_1),(x_2,y_2)) = \{\pm(a,1,c,d) : |a| \leq 1, |c| \leq 1, |d| \leq 1\}$.

It is obvious that

$$\{\pm(a,1,c,d) : |a| \leq 1, |c| \leq 1, |d| \leq 1\} \subseteq NA(\mathcal{L}(^2l_1^2))((x_1,y_1),(x_2,y_2)).$$

Since $x_1 = x_2 = 0$, $|y_1| = |y_2| = 1$ and

$$1 = |T((x_1,y_1),(x_2,y_2))| = |by_1y_2| = |b|$$

which shows that $|b| = 1$. Hence, $T = \pm(a,1,c,d)$ for some $|a| \leq 1, |c| \leq 1, |d| \leq 1$. Therefore, we have shown the claim.

If $x_1 = y_2 = 0$, then $|y_1| = |x_2| = 1$ and

$$1 = |T((x_1,y_1),(x_2,y_2))| = |dx_2y_1| = |d|$$

which shows that $|d| = 1$. Hence, $T = \pm(a,b,c,1)$ for some $|a| \leq 1, |b| \leq 1, |c| \leq 1$. Hence, $NA(\mathcal{L}(^2l_1^2))((x_1,y_1),(x_2,y_2)) = \{\pm(a,b,c,1) : |a| \leq 1, |b| \leq 1, |c| \leq 1\}$.

If $x_2 = y_1 = 0$, then $|x_1| = |y_2| = 1$ and

$$1 = |T((x_1,y_1),(x_2,y_2))| = |cx_1y_2| = |c|$$

which shows that $|c| = 1$. Hence, $T = \pm(a,b,1,d)$ for some $|a| \leq 1, |b| \leq 1, |d| \leq 1$. Hence, $NA(\mathcal{L}(^2l_1^2))((x_1,y_1),(x_2,y_2)) = \{\pm(a,b,1,d) : |a| \leq 1, |b| \leq 1, |d| \leq 1\}$.

If $y_1 = y_2 = 0$, then $|x_1| = |x_2| = 1$ and

$$1 = |T((x_1,y_1),(x_2,y_2))| = |ax_1x_2| = |a| = a$$

which shows that $a = 1$. Hence, $T = \pm(1,b,c,d)$ for some $|b| \leq 1, |c| \leq 1, |d| \leq 1$. Hence, $NA(\mathcal{L}(^2l_1^2))((x_1,y_1),(x_2,y_2)) = \{\pm(1,b,c,d) : |b| \leq 1, |c| \leq 1, |d| \leq 1\}$. Therefore, we complete the proof. $\square$

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Continuous dependence for double diffusive convection in a Brinkman model with variable viscosity

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Abstract. This current work is presented to deal with the model of double diffusive convection in porous material with variable viscosity, such that the equations for convective fluid motion in a Brinkman type are analysed when the viscosity varies with temperature quadratically. Hence, we carefully find a priori bounds when the coefficients depend only on the geometry of the problem, initial data, and boundary data, where this shows the continuous dependence of the solution on changes in the viscosity. A convergence result is also showen when the variable viscosity is allowed to tend to a constant viscosity.

1 Introduction

Studies in the exploration of double-diffusive convection topic in a fluid-saturated porous layer have been an active field for a long time making this topic closely

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related to many other research papers. The various ways of getting the heat and the mass combined to transfer, can be seen in a lot of real life problems.

In fact, the importance of the continuous dependence on changes in the boundary conditions, initial conditions, coefficients, or even in the system of the equations, has been increasingly recognized. This aspect of continuous dependence, or stability, is what we refer to as structural stability, cf. [1], and in many ways is more important than the classical idea of stability, or continuous dependence on the initial data. Continuous dependence on modeling in elasticity has been shown to be of considerable importance in a seminal paper, see [2].

The system of equations which explains the double diffusive convective flow in a porous medium using a Brinkman model has been proposed in [3].In addition, [4, 5, 6, 7] have presented nonlinear stability analyses for a model which does not employ a Brinkman term but instead includes a Forchheimer term. Moreover, a recent study which includes both Brinkman and Forchheimer is suggested in [8]. Brinkman model with a viscosity which depends linearly on temperature is introduced in [9]. Early studies dealing with structural stability issues in porous flows (cf. [10], [11]), have recently developed for porous flow model which has a viscosity depends on concentration [12]. In this paper we continue the work of Payne et al. [12] who study the continuous dependence Brinkman and Forchheimer models when the viscosity is linear function for concentration. However, we study the double diffusive convection in a Brinkman model when the viscosity is linear function for temperature.

The layout of this paper is constructed as follows. In the next section, we will present mathematical formulas of the system. In Section 3, we develop a priori bounds. The goal of Sections 4 and 5 is to demonstrate continuous dependence on changes in the viscosity coefficients. Finally, the convergence to the constants viscosity solution will be establish in Sections 6 and 7.

2 Basic equations

The momentum equation for flow in a porous saturated material of Brinkman type may be taken as

$$-\Delta u_i + (1 + \alpha T + \beta T^2) u_i = -\frac{\partial p}{\partial x_i} + g_i T + \mathcal{I}_i C, \tag{1}$$

where, α and β are constants, and u_i, T, C and p are velocity, temperature, concentration and pressure, respectively. g_i and $\mathcal{I}_i$ are vectors incorporating the gravity field which take $|g_i| \leq 1$ and $|\mathcal{I}_i| \leq 1$. The balance of mass equation is

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2}$$

Furthermore, the temperature and concentration equations, respectively, have the following forms

$$\begin{aligned} \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} &= \Delta T, \\ \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} &= \Delta C. \end{aligned} \tag{3}$$

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. Thus, Equ. (1)-(3) are defined on $\Omega \times (0, \mathcal{T})$, for $\mathcal{T} < \infty$, and the boundary conditions

$$u_i = f_i(x, t), \quad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \tag{4}$$

and

$$T(x, t) = h(x, t), \;\; C(x, t) = k(x, t), \;\; x \text{ on } \; \partial\Omega, \;\; t \in (0, \mathcal{T}), \tag{5}$$

where h and k are prescribed functions and n is the unit outward normal to $\partial\Omega$, and the initial data for the temperature and concentration is given as

$$T(x, 0) = T_0(x), \quad C(x, 0) = C_0(x), \quad x \in \Omega, \tag{6}$$

where T_0 and C_0 are prescribed functions.

3 A priori bounds

In this section, we derive bounds for various norms of u_i, T and C, in terms of data. These bounds will be used in the next sections in the continuous dependence and converges proof. To develop a priori bounds, we introduce the functions $G(x, t)$, $K(x, t)$, $F(x, t)$ and $H(x, t)$ as solutions to the boundary value problems

$$\begin{aligned} \Delta G(x, t) &= 0, && \text{in} \;\; \Omega, \\ G(x, t) &= h(x, t), && \text{on} \;\; \partial\Omega, \end{aligned} \tag{7}$$

$$\begin{aligned} \Delta K(x, t) &= 0, && \text{in} \;\; \Omega, \\ K(x, t) &= k(x, t), && \text{on} \;\; \partial\Omega, \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta F(x, t) &= 0, && \text{in} \;\; \Omega, \\ F(x, t) &= h^{2r-1}(x, t), && \text{on} \;\; \partial\Omega, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \Delta H(x,t) &= 0, && \text{in } \Omega, \\ H(x,t) &= k^{2r-1}(x,t), && \text{on } \partial\Omega, \end{aligned} \tag{10}$$

where r is a positive integer. We commence with deriving a bound for $\|\mathbf{u}\|$, and we let b_i solve the Stokes flow problem in Ω, namely

$$\begin{aligned} \Delta b_i &= \rho_{,i}, \quad \frac{\partial b_i}{\partial x_i} && \text{in } \Omega, \\ b_i &= f_i, && \text{on } \partial\Omega, \end{aligned} \tag{11}$$

where ρ is a pressure term. By the triangle inequality,

$$\|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{b}\| + \|\mathbf{b}\|. \tag{12}$$

Next, we employ (1) and (11) to derive

$$\begin{aligned} &\|\nabla(\mathbf{u}-\mathbf{b})\|^2 + \int_\Omega (1+\alpha T+\beta T^2)(u_i-b_i)(u_i-b_i)\,d\mathbf{x} \\ &= -\int_\Omega (1+\alpha T+\beta T^2)(u_i-b_i)a\,d\mathbf{x} \\ &\quad + \int_\Omega g_i T(u_i-b_i)\,d\mathbf{x} + \int_\Omega \mathcal{I}_i C(u_i-b_i)\,d\mathbf{x}. \end{aligned} \tag{13}$$

The Cauchy-Schwarz inequality together with the arithmetic-geometric mean and Sobolev inequalities are used on the right-hand side to find

$$\begin{aligned} &\|\nabla(\mathbf{u}-\mathbf{b})\|^2 + \frac{1}{2}\int_\Omega (1+\alpha T+\beta T^2)(u_i-b_i)(u_i-b_i)\,d\mathbf{x} \\ &\leq \frac{3}{2}\int_\Omega (1+\alpha T+\beta T^2)b_i b_i\,d\mathbf{x} + \frac{3}{2}(\|T\|^2+\|C\|^2) \\ &\leq \frac{3}{2}(\|T\|^2+\|C\|^2) + \frac{3}{2}\|\mathbf{b}\|^2 + \frac{3}{2}\alpha\|T\|\|\mathbf{b}\|_4^2 + \frac{3}{2}\beta\|T\|^2\|\mathbf{b}\|^2 \\ &\leq \frac{3}{2}(\|T\|^2+\|C\|^2) + \frac{3}{2}(1+\beta\|T\|^2)\|\mathbf{b}\|^2 + \frac{3}{2}\mathcal{C}\alpha\|T\|(\mathbf{b}\|^2+\|\nabla\mathbf{b}\|^2), \end{aligned} \tag{14}$$

here $\mathcal{C}$ is a constant in the Sobolev inequality. As proposed in [14], we can see that

$$\|\mathbf{b}\|^2 \leq 6d\oint_{\partial\Omega} f_i f_i\,dA + 4d^2\int_\Omega (b_{i,j}-b_{j,i})(b_{i,j}-b_{j,i})\,d\mathbf{x}$$

$$\leq (6d + 4d^2\bar{k}_1)\oint_{\partial\Omega} f_i f_i dA + 4d^2\bar{k}_2 \oint_{\partial\Omega} |\nabla_s \mathbf{f}|^2 dA, \tag{15}$$

where d is the radius of the smallest circumscribed ball for $D, \bar{k}_1$ and $\bar{k}_2$ are a priori constants given in [14] and ∇_s denotes the tangential derivative. Furthermore, we have that

$$\begin{aligned}
\|\nabla \mathbf{b}\|^2 &= \frac{1}{2}\int_\Omega (b_{i,j} - b_{j,i})(b_{i,j} - b_{j,i})d\mathbf{x} + \int_\Omega b_{i,j} b_{j,i} d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (b_{i,j} - b_{j,i})(b_{i,j} - b_{j,i})d\mathbf{x} + \oint_{\partial\Omega} (b_{i,j} b_j - b_{j,i} b_i) dA \\
&= \frac{1}{2}\int_\Omega (b_{i,j} - b_{j,i})(b_{i,j} - b_{j,i})d\mathbf{x} + \oint_{\partial\Omega} n^i s^j (b_j \nabla_s b_i - b_i \nabla_s b_j) dA \\
&\leq \frac{1}{2}\bar{k}_1 \oint_{\partial\Omega} f_i f_i dA + \frac{1}{2}\bar{k}_2 \oint_{\partial\Omega} |\nabla_s \mathbf{f}|^2 dA + \oint_{\partial\Omega} n^i s^j (f_j \nabla_s f_i - f_i \nabla_s f_j) dA,
\end{aligned} \tag{16}$$

where s^j denotes a tangential vector.
Let us consider the right-hand sides of (3) and (16) by $D_1^2(t)$ and $2D_2^2(t)$, respectively. Observe that D_1 and D_2 are data terms. Then from (14)-(16), yields

$$\begin{aligned}
&\|\nabla(\mathbf{u} - \mathbf{b})\|^2 + \frac{1}{2}\int_\Omega (1 + \alpha T + \beta T^2)(u_i - b_i)(u_i - b_i) d\mathbf{x} \\
&\leq \frac{3}{2}(\|T\|^2 + \|C\|^2) + \frac{3}{2}(1 + \beta\|T\|^2)D_1^2 + 3\mathcal{C}\alpha\|T\|D_2^2 \\
&\leq \frac{3}{2}(\|T\| + \|C\|)^2 + 3(\|T\| + \|C\|)D_3 + \frac{3}{2}D_3^2 + \frac{3}{2}\beta\|T\|^2 D_3^2 \\
&\leq \frac{3}{2}\Big((\|T\| + \|C\|)D_3\Big)^2 + \frac{3}{2}\beta\Big((\|T\| + \|C\|)D_3\Big)^2,
\end{aligned} \tag{17}$$

where $D_3 = D_1$ if $D_1 \geq \mathcal{C}\alpha D_2^2$, otherwise $D_3 = \mathcal{C}\alpha D_2^4$. Then,

$$\|\mathbf{u} - \mathbf{b}\| \leq \sqrt{3(1 + \beta)}(\|T\| + \|C\|)D_3. \tag{18}$$

Subsequently, from (12)

$$\|\mathbf{u}\| \leq \sqrt{3(1 + \beta)}(\|T\| + \|C\|)D_3 + D_1, \tag{19}$$

and from (19), we conclude

$$\|\mathbf{u}\|^2 \leq 12(1 + \beta)(\|T\|^2 + \|C\|^2)D_3^2 + 2D_1^2. \tag{20}$$

Now, adopting the following expressions

$$\int_0^t \int_\Omega (T-G)\left(\frac{\partial T}{\partial s} + u_i \frac{\partial T}{\partial x_i} - \Delta T\right) \mathbf{dx}ds = 0, \tag{21}$$

and

$$\int_0^t \int_\Omega (C-K)\left(\frac{\partial C}{\partial s} + u_i \frac{\partial C}{\partial x_i} - \Delta C\right) \mathbf{dx}ds = 0, \tag{22}$$

where t is some number such that $0 \le t \le \mathcal{T}$. Next, integrating by part in (21) and employing the boundary condition $(7)_2$, to see that

$$\|T\|^2 + 2\int_0^t \|\nabla T\|^2 ds \le \|T_0\|^2 + 2(G,T) + 2\left|(G_0, T_0)\right| + 2\left|\int_0^t (G_{,s}, T)ds\right|$$

$$+2\int_0^t \int_\Omega G u_i T_{,i} \mathbf{dx}ds + 2\int_0^t \oint_{\partial\Omega} h\left(\frac{\partial G}{\partial n}\right) dAds + \int_0^t \oint_{\partial\Omega} |f| h^2 dAds,$$

by using Cauchy-Schwarz and arithmetic-geometric mean inequalities in above inequality, we have

$$\begin{aligned}
\frac{1}{2}\|T\|^2 + \int_0^t \|\nabla T\|^2 ds &\le 2\|T_0\|^2 + 2\|G\|^2 + \|G_0\|^2 + \int_0^t \|G_{,s}\|^2 ds \\
&+ G_m^2 \int_0^t \|\mathbf{u}\|^2 ds + \int_0^t \|T\|^2 ds + \int_0^t \oint_{\partial\Omega} h^2 dAds \\
&+ \int_0^t \oint_{\partial\Omega} \left(\frac{\partial G}{\partial n}\right)^2 dAds + \int_0^t \oint_{\partial\Omega} |f| h^2 dAds,
\end{aligned} \tag{23}$$

with inserting (20) in (23), yields

$$\begin{aligned}
\|T\|^2 + 2\int_0^t \|\nabla T\|^2 ds &\le 4\|T_0\|^2 + 4\|G\|^2 + 2\|G_0\|^2 + 2\int_0^t \|G_{,s}\|^2 ds \\
&+ 2\left(1 + 6G_m^2[1+\beta]D_3^2\right)\int_0^t \|T\|^2 ds \\
&+ 2G_m^2 \int_0^t \left(6[1+\beta]\|C\|^2 D_3^2 + D_1^2\right) ds \\
&+ 2\int_0^t \oint_{\partial\Omega} \left(\frac{\partial G}{\partial n}\right)^2 dAds + 2\int_0^t \oint_{\partial\Omega} (1+|f|) h^2 dAds.
\end{aligned} \tag{24}$$

Next, we return to the Equ. (22) and realize integral by parts with the aid of the Cauchy-Schwarz and arithmetic-geometric mean inequalities, to find

$$\begin{aligned}\frac{1}{4}\|C\|^2+\frac{3}{4}\int_0^t\|\nabla C\|^2ds &\le \|C_0\|^2+\frac{1}{2}\|K_0\|^2+\|K\|^2+\frac{1}{2}\int_0^t\|C\|^2ds\\&+K_m^2\int_0^t\|\mathbf{u}\|^2ds+\frac{1}{2}\int_0^t\oint_{\partial\Omega}(1+|f|)k^2dAds\\&+\frac{1}{2}\int_0^t\|K_{,s}\|^2ds+\frac{1}{2}\int_0^t\oint_{\partial\Omega}\left(\frac{\partial K}{\partial n}\right)^2dAds,\end{aligned}\tag{25}$$

from (20), we derive (25) into

$$\begin{aligned}\|C\|^2+3\int_0^t\|\nabla C\|^2ds &\le 4\|C_0\|^2+2\|K_0\|^2+4\|K\|^2\\&+2\left(1+12K_m^2[1+\beta]D_3^2\right)\int_0^t\|C\|^2ds\\&+4K_m^2\int_0^t\left(6[1+\beta]\|T\|^2D_3^2+D_1^2\right)ds+2\int_0^t\oint_{\partial\Omega}(1+|f|)k^2dAds\\&+2\int_0^t\|K_{,s}\|^2ds+2\int_0^t\oint_{\partial\Omega}\left(\frac{\partial K}{\partial n}\right)^2dAds,\end{aligned}\tag{26}$$

where G_m and K_m are the maximum value of G and K, respectively, on $\partial\Omega\times(0,\mathcal{T})$. Next, combining (24) and (26), yields

$$\begin{aligned}&\|T\|^2+\|C\|^2+2\int_0^t\|\nabla T\|^2ds+3\int_0^t\|\nabla C\|^2ds\\&\le 2\left(1+6G_m^2[1+\beta]D_3^2+12K_m^2[1+\beta]D_3^2\right)\int_0^t\|T\|^2ds\\&\quad+2\left(1+6G_m^2[1+\beta]D_3^2+12K_m^2[1+\beta]D_3^2\right)\int_0^t\|C\|^2ds+E(t)\\&=\lambda\int_0^t\left(\|T\|^2+\|C\|^2\right)ds+E(t),\end{aligned}\tag{27}$$

where

$$\lambda=2\left(1+6G_m^2[1+\beta]D_3^2+12K_m^2[1+\beta]D_3^2\right),$$

and $E(t)$ is given by

$$\begin{aligned} E(t) = {} & 4\|T_0\|^2 + 2\|G_0\|^2 + 4\|C_0\|^2 + 2\|K_0\|^2 + 4\|G\|^2 + 4\|K\|^2 \\ & + 2\left(G_m^2 + 2K_m^2\right)\int_0^t D_1^2 ds + 2\int_0^t \|G_{,s}\|^2 ds + 2\int_0^t \|K_{,s}\|^2 ds \\ & + 2\int_0^t \oint_{\partial\Omega} (1+|f|)h^2 dA ds + 2\int_0^t \oint_{\partial\Omega} (1+|f|)k^2 dA ds \\ & + 2\int_0^t \oint_{\partial\Omega} \left(\frac{\partial G}{\partial n}\right)^2 dA ds + 2\int_0^t \oint_{\partial\Omega} \left(\frac{\partial K}{\partial n}\right)^2 dA ds. \end{aligned} \tag{28}$$

For a function ϕ, which satisfies [15]

$$\begin{aligned} \Delta\phi &= 0, \quad \text{in } \Omega, \\ \phi &= M, \quad \text{on } \partial\Omega, \end{aligned} \tag{29}$$

then one may use a Rellich identity, [13], to denote c_1 and c_2 such that

$$\|\nabla\phi\|^2 + c_1 \oint_{\partial\Omega} \left(\frac{\partial\phi}{\partial n}\right)^2 dA \le c_2 \oint_{\partial\Omega} |\nabla_s M|^2 dA, \tag{30}$$

where ∇_s refers to the surface gradient over $\partial\Omega$. Also observe that

$$2(\psi\nabla\phi, \nabla\phi) + \|\phi\|^2 \le \psi_1 \oint_{\partial\Omega} M^2 dA, \tag{31}$$

where

$$\psi_1 = \max_{\partial\Omega} \left|\frac{\partial\psi}{\partial n}\right|,$$

with solving the boundary value problem,

$$\begin{aligned} \Delta\psi &= -1, \quad \text{in } \Omega, \\ \psi &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{32}$$

Thus, (31) and (32) lead to bounds for $E(t)$ in terms of data. In fact, one may show

$$E(t) \le \tilde{D}(t), \tag{33}$$

so that

$$\begin{aligned}D(t) = 4\|T_0\|^2 + 4\|C_0\|^2 + 2\left(G_m^2 + 2K_m^2\right)\int_0^t D_1^2 ds + 2\psi_1 \oint_{\partial\Omega} h_0^2 dA \\ + 2\psi_1 \oint_{\partial\Omega} k_0^2 dA + 4\psi_1 \oint_{\partial\Omega} h^2 dA + 4\psi_1 \oint_{\partial\Omega} k^2 dA \\ + 2\psi_1 \int_0^t \oint_{\partial\Omega} h_{,\eta}^2 dA d\eta + 2\psi_1 \int_0^t \oint_{\partial\Omega} k_{,\eta}^2 dA d\eta \\ + 2\int_0^t \oint_{\partial\Omega} (1+|f|)h^2 dA d\eta + 2\int_0^t \oint_{\partial\Omega} (1+|f|)k^2 dA d\eta \\ + \frac{2c_2}{c_1}\int_0^t \oint_{\partial\Omega} |\nabla_s h|^2 dA d\eta + \frac{2c_2}{c_1}\int_0^t \oint_{\partial\Omega} |\nabla_s k|^2 dA d\eta.\end{aligned} \tag{34}$$

From (28) which leads us to

$$\mathcal{F}' - \lambda\mathcal{F} \leq \tilde{D}(t), \tag{35}$$

where we have introduced the function $\mathcal{F}$, which is defined by

$$\mathcal{F}(t) = \int_0^t \left(\|T\|^2 + \|C\|^2\right) ds.$$

Upon assuming

$$\tilde{D}_1(t) = \int_0^t \tilde{D}(s) e^{\lambda(t-s)} ds, \tag{36}$$

one integrates (35) to show

$$\mathcal{F}(t) \leq \tilde{D}_1(t). \tag{37}$$

Furthermore, setting $\tilde{D}_2 = \lambda\tilde{D}_1 + \tilde{D}$, one uses (3) to find

$$\|T\|^2 + \|T\|_4^4 + \|C\|^2 \leq \tilde{D}_2(t). \tag{38}$$

Then, (27), (37) and (38) give

$$\begin{gathered}\int_0^t \|T\|^2 ds \leq \tilde{D}_1, \quad \int_0^t \|C\|^2 ds \leq \tilde{D}_1, \\ \|T\|^2 \leq \tilde{D}_2, \quad \|C\|^2 \leq \tilde{D}_2, \\ \int_0^t \|\nabla T\|^2 ds \leq \frac{1}{2}\tilde{D}_2, \quad \int_0^t \|\nabla C\|^2 ds \leq \frac{1}{3}\tilde{D}_2.\end{gathered} \tag{39}$$

The next step is to derive a bound for $\sup_{\Omega\times[0,T]}|T|$, from

$$\int_0^t\int_\Omega (T^{2r-1}-F)\left(\frac{\partial T}{\partial s}+u_i\frac{\partial T}{\partial x_i}-\Delta T\right)dxds=0. \tag{40}$$

Integrating by parts, we see that

$$\begin{aligned}\int_\Omega T^{2r}dx+\frac{2(2r-1)}{r}\int_0^t\int_\Omega \nabla T^r\nabla T^r dxds &= \int_\Omega T_0^{2r}dx+2r(T,F)-2r(T_0,F_0)\\ &-2r\int_0^t\int_F TF_{,s}dxds+2r\int_0^t\int_\Omega T_{,i}Fu_i dxds\\ &+2r\int_0^t\oint_{\partial\Omega} h\frac{\partial F}{\partial n}dAds-\int_0^t\oint_{\partial\Omega} fT^{2r}dAds\\ \le \int_\Omega T_0^{2r}dx+2r\left(\int_0^t\|F_{,s}\|^2ds\int_0^t\|T\|^2ds\right)^{1/2}&+2r(\|T\|\|F\|+\|T_0\|\|F_0\|)\\ +4rh_m^{2r-1}\left(\int_0^t\left[6[1+\beta](\|T\|^2+\|C\|^2)D_3^2+D_1^2\right]ds\int_0^t\|\nabla T\|^2ds\right)^{1/2}&\\ +2r\left(\int_0^t\oint_{\partial\Omega}h^2dAds\int_0^t\oint_{\partial\Omega}\left[\frac{\partial F}{\partial n}\right]^2dAds\right)^{1/2}&+\int_0^t\oint_{\partial\Omega}|f|h^{2r}dAds.\end{aligned} \tag{41}$$

Using arithmetic-geometric mean inequality and (30), (31) with (39), yield

$$\begin{aligned}\int_\Omega T^{2r}dx &\le \int_\Omega T_0^{2r}dx+2r(\sqrt{\tilde{D}_2}+\|T_0\|)\left(\psi_1\oint_{\partial\Omega}h^{4r-2}dA\right)^{1/2}\\ &+2r\left(\tilde{D}_1\psi_1\int_0^t\oint_{\partial\Omega}\left[h_{,\eta}^{2r-1}\right]^2dAd\eta\right)^{1/2}+rh_m^{2r-1}\left(2\int_0^t D_1^2d\eta+24[1+\beta]\tilde{D}_1+2\tilde{D}_2\right)\\ &+2r\left(\frac{c_2}{c_1}\int_0^t\oint_{\partial\Omega}\left[\nabla_\eta h\right]^2dAd\eta\int_0^t\oint_{\partial\Omega}h^2dAd\eta\right)^{1/2}+\int_0^t\oint_{\partial\Omega}|f|h^{2r}dAd\eta.\end{aligned} \tag{42}$$

Then, from further application for Cauchy-Schwarz, we get

$$\left(\oint_{\partial\Omega}h^{4r-2}dA\right)^{1/2}\le h_m^{2r-1}\left(\oint_{\partial\Omega}dA\right)^{1/2}=\frac{h_m^{2r}}{h_m}\sqrt{[m(\partial\Omega)]}, \tag{43}$$

$$\left(\int_0^t\oint_{\partial\Omega}h^{4r-4}h_{,\eta}^2dAd\eta\right)^{1/2}\le\frac{h_m^{2r}}{h_m^2}\left(\int_0^t\oint_{\partial\Omega}h_{,\eta}^2dAd\eta\right)^{1/2}, \tag{44}$$

and

$$\left(\int_0^t \oint_{\partial\Omega} h^{4r-4}\left[\nabla_\eta h\right]^2 dA d\eta\right)^{1/2} \leq \frac{h_m^{2r}}{h_m^2}\left(\int_0^t \oint_{\partial\Omega}\left[\nabla_\eta h\right]^2 dA d\eta\right)^{1/2}, \tag{45}$$

where $m(\partial\Omega)$ is the surface measure of $\partial\Omega$. Employing (43) − (45) in (3), lead to

$$\begin{aligned}
\int_\Omega T^{2r} dx \leq & \int_\Omega T_0^{2r} dx + \frac{2rh_m^{2r}}{h_m}(\sqrt{\tilde{D}_2} + \|T_0\|)\sqrt{\psi_1[m(\partial\Omega)]} \\
& + \frac{2r(2r-1)h_m^{2r}}{h_m}\left(\tilde{D}_1\psi_1 \int_0^t \oint_{\partial\Omega} h_{,\eta}^2 dA d\eta\right)^{1/2} \\
& + \frac{2rh_m^{2r}}{h_m}\left(\int_0^t D_1^2 d\eta + 12[1+\beta]\tilde{D}_1 + \tilde{D}_2\right) + h_m^{2r}\int_0^t \oint_{\partial\Omega} |f| dA d\eta \\
& + \frac{2r(2r-1)h_m^{2r}}{h_m^2}\left(\frac{c_2}{c_1}\int_0^t \oint_{\partial\Omega}\left[\nabla_\eta h\right]^2 dA d\eta \int_0^t \oint_{\partial\Omega} h^2 dA d\eta\right)^{1/2}.
\end{aligned} \tag{46}$$

After taking the power $1/2r$ of (46), we obtain

$$\|T\|_{2r} \leq \left(\|T_0\|_{2r}^{2r} + h_m^{2r}\sum_{i=1}^{5}\gamma_i\right)^{1/2r}, \tag{47}$$

where γ_i may be obtained from (46), here

$$h_m = \max_{\partial\Omega\times[0,\mathcal{T}]} |h|.$$

Taking the limit $r \to \infty$, yields a priori bound

$$\sup_{\Omega\times[0,\mathcal{T}]} |T| \leq \max\{|T_0|_m, \sup_{[0,\mathcal{T}]} h_m\}, \tag{48}$$

where

$$|T_0|_m = \max_\Omega |T_0|.$$

Finally, we have to find a bound for $\sup_{\Omega\times[0,\mathcal{T}]} |C|$. Now, form the expression

$$\int_0^t \int_\Omega (C^{2r-1} - H)\left(\frac{\partial C}{\partial s} + u_i\frac{\partial C}{\partial x_i} - \Delta C\right) dx ds = 0. \tag{49}$$

Following the same manner in (40)-(48), we have that

$$\sup_{\Omega\times[0,\mathcal{T}]} |C| \leq \max\{|C_0|_m, \sup_{[0,\mathcal{T}]} k_m\}, \tag{50}$$

where

$$|C_0|_m = \max_\Omega |C_0|.$$

4 Continuous dependence on α

To investigate continuous dependence on the viscosity coefficient α in (1), we let (u_i, T, C_1, p) and (v_i, S, C_2, q) be solutions to (1) − (6) for the same data functions f, h and T_0, but for different viscosity coefficients, α_1 and α_2, respectively. Define the difference solution (w_i, θ, ϕ, π) by

$$w_i = u_i - v_i, \quad \theta = T - S, \quad \phi = C_1 - C_2, \quad \pi = p - q, \quad \alpha = \alpha_1 - \alpha_2. \tag{51}$$

Then from (1) − (6), this solution satisfies the boundary-initial-value problem

$$\begin{gathered}
-\triangle w_i + (1 + \alpha_2 S + \beta S^2) w_i = -\alpha_2 \theta u_i - \alpha T u_i - \beta (T+S)\theta u_i - \pi_{,i} + g_i \theta + \mathcal{I}_i \phi, \\
w_{i,i} = 0, \\
\frac{\partial \theta}{\partial t} + w_i \frac{\partial S}{\partial x_i} + u_i \frac{\partial \theta}{\partial x_i} = \Delta T, \\
\frac{\partial \phi}{\partial t} + w_i \frac{\partial C_2}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_i} = \Delta C_2, \\
w_i = \theta = \phi = 0 \quad \text{on} \quad \partial\Omega \times [0, \mathcal{T}], \\
\theta(x, 0) = \phi(x, 0) = 0, \quad x \in \Omega.
\end{gathered} \tag{52}$$

Next, we multiply Equ. $(52)_1$ by w_i and integrate over Ω.

$$\begin{aligned}
\|\nabla \mathbf{w}\|^2 + \int_\Omega (1 + \alpha_2 S + \beta S^2) w_i w_i \, d\mathbf{x} &= -\alpha \int_\Omega T u_i w_i \, d\mathbf{x} - \alpha_2 \int_\Omega \theta u_i w_i \, d\mathbf{x} \\
&- \beta \int_\Omega (T+S)\theta u_i w_i \, d\mathbf{x} + g_i(\theta, w_i) + \mathcal{I}_i(\phi, w_i) \\
&\leq \alpha T_m \|\mathbf{u}\| \|\mathbf{w}\| + \|\theta\| \|\mathbf{w}\| + \|\phi\| \|\mathbf{w}\| \\
&+ \Big(\alpha_2 + \beta[T_m + S_m]\Big) \|\theta\| \Big(\int_\Omega u_i w_i u_j w_j \, d\mathbf{x} \Big)^{1/2},
\end{aligned} \tag{53}$$

where, T_m and S_m are the maximum value of T and S, respectively. The last term in (53) is bounded as the following, [14], i.e.

$$\begin{aligned}
\int_\Omega u_i w_i u_j w_j \, d\mathbf{x} &\leq \frac{2}{\pi} \Big(\|\nabla \mathbf{w}\|^2 \oint_{\partial\Omega} f_i f_i \, dA + \|\nabla \mathbf{u}\|^2 \|\mathbf{w}\| \|\nabla \mathbf{w}\| \Big) \\
&\leq \frac{2}{\pi} \|\nabla \mathbf{w}\|^2 \Big(\oint_{\partial\Omega} f_i f_i \, dA + \kappa^{-1/2} \|\nabla \mathbf{u}\|^2 \Big),
\end{aligned} \tag{54}$$

where κ is the Poincaré constant for Ω.
To employ (53) and (54) we need data bounds for $\|\mathbf{u}\|$ and $\|\nabla \mathbf{u}\|$, thus, from

the triangle inequality

$$\|\nabla \mathbf{u}\| \leq \|\nabla(\mathbf{u}-\mathbf{b})\| + \|\nabla \mathbf{b}\|,$$

and then from the inequality before (14) and (16) we find

$$\|\nabla \mathbf{u}\|^2 \leq \left(\sqrt{6(1+\beta)\tilde{D}_2 D_3} + \sqrt{2} D_2\right)^2. \tag{55}$$

Hence, return to (53) we conclude that

$$\begin{aligned}
&\|\nabla \mathbf{w}\|^2 + \int_\Omega (1 + \alpha_2 S + \beta S^2) w_i w_i \, d\mathbf{x} \\
&\leq \alpha T_m D_4 \|\mathbf{w}\| + \|\theta\|\|\mathbf{w}\| + \|\phi\|\|\mathbf{w}\| + D_6 \|\theta\|\|\nabla \mathbf{w}\|,
\end{aligned} \tag{56}$$

where

$$D_4 = \sqrt{6(1+\beta)\tilde{D}_2 D_3} + D_1, \quad D_5 = \sqrt{6(1+\beta)\tilde{D}_2 D_3} + \sqrt{2} D_2,$$

and

$$D_6 = \sqrt{\frac{2}{\pi}} \left(\alpha_2 + \beta[T_m + S_m]\right) \left(\oint_{\partial\Omega} f_i f_i \, dA + \kappa^{-1/2} D_5\right)^{1/2}.$$

Thus, from (56), we may derive

$$\begin{aligned}
\|\nabla \mathbf{w}\|^2 + \mu^* \|\mathbf{w}\|^2 &\leq \frac{3\alpha^2 T_m^2 D_4^2}{\mu^*} + \frac{3}{\mu^*}(\|\theta\|^2 + \|\phi\|^2) + D_6^2 \|\theta\|^2 \\
&\leq \frac{3\alpha^2 T_m^2 D_4^2}{\mu^*} + \left(\frac{3}{\mu^*} + D_6^2)(\|\theta\|^2 + \|\phi\|^2\right),
\end{aligned} \tag{57}$$

where

$$0 < \mu^* \leq 1 + \alpha_2 S + \beta S^2.$$

Moreover, multiplying $(52)_3$ by θ and $(52)_4$ by ϕ, with integrating over Ω, we can see that

$$\frac{d}{dt}\|\theta\|^2 \leq \frac{S_m^2}{2}\|\mathbf{w}\|^2, \tag{58}$$

and

$$\frac{d}{dt}\|\phi\|^2 \leq \frac{C_{2m}^2}{2}\|\mathbf{w}\|^2. \tag{59}$$

Employing (58) and (59) with integrating the result, yields

$$\begin{aligned}\|\theta\|^2+\|\phi\|^2 &\le \frac{1}{2}(S_m^2+C_{2m}^2)\int_0^t \|\mathbf{w}\|^2 ds\\ &\le \frac{1}{2\mu^*}(S_m^2+C_{2m}^2)\int_0^t (\|\nabla\mathbf{w}\|^2+\mu^*\|\mathbf{w}\|^2)ds,\end{aligned} \tag{60}$$

where C_{2m} is the maximum value of C_2. Next, substituting (60) in (57), we have

$$\|\nabla\mathbf{w}\|^2+\mu^*\|\mathbf{w}\|^2 \le \frac{3\alpha^2T_m^2D_4^2}{\mu^*}+\mathcal{J}\int_0^t (\|\nabla\mathbf{w}\|^2+\mu^*\|\mathbf{w}\|^2)ds, \tag{61}$$

here

$$\mathcal{J}=\frac{1}{2\mu^*}(S_m^2+C_{2m}^2)(\frac{3}{\mu^*}+D_6^2).$$

By integration (61), we find

$$\int_0^t (\|\nabla\mathbf{w}\|^2+\mu^*\|\mathbf{w}\|^2)ds \le \frac{3\alpha^2T_m^2D_4^2}{\mu^*}t+\mathcal{J}\int_0^t (t-s)(\|\nabla\mathbf{w}\|^2+\mu^*\|\mathbf{w}\|^2)ds, \tag{62}$$

thus, from (62) we obtain

$$\int_0^t (t-s)(\|\nabla\mathbf{w}\|^2+\mu^*\|\mathbf{w}\|^2)ds \le \alpha^2\mathcal{J}_2(t), \tag{63}$$

and

$$\int_0^t (\|\nabla\mathbf{w}\|^2+\mu^*\|\mathbf{w}\|^2)ds \le \alpha^2\mathcal{J}_3(t), \tag{64}$$

where

$$\mathcal{J}_2(t)=\int_0^t \mathcal{J}_1(s)e^{\mathcal{J}(t-s)}ds,\;\; \mathcal{J}_1(t)=\frac{3T_m^2D_4^2}{\mu^*}t \;\text{ and }\; \mathcal{J}_3(t)=\mathcal{J}_1+\mathcal{J}\mathcal{J}_2.$$

Finally, inserting (64) in (60) we also find

$$\|\theta\|^2+\|\phi\|^2 \le \frac{1}{2\mu^*}\alpha^2\mathcal{J}_3(S_m^2+C_{2m}^2). \tag{65}$$

Inequality (65) yields continuous dependence on the viscosity coefficient α and it is truly a priori such that the coefficients of α^2 depend only on boundary and initial conditions.

5 Continuous dependence on β

In this section, we study the continuous dependence on the coefficient β. We first, let (u_i, T, C_1, p) and (v_i, S, C_2, q) be solutions to (1) − (6) for the same data functions f, h and T_0, but for different viscosity coefficients, β_1 and β_2, respectively. Hence, define the difference solution (w_i, θ, ϕ, π) as

$$w_i = u_i - v_i, \quad \theta = T - S, \quad \phi = C_1 - C_2, \quad \pi = p - q, \quad \beta = \beta_1 - \beta_2. \tag{66}$$

Thus, from (1) − (6) this solution satisfies the boundary-initial-value problem

$$\begin{gathered}
-\triangle w_i + (1 + \alpha S + \beta_2 S^2) w_i = -\alpha\theta u_i - \beta S^2 u_i - \beta_1 (T + S)\theta u_i - \pi_{,i} + g_i\theta + \mathcal{I}_i\phi, \\
w_{i,i} = 0, \\
\frac{\partial \theta}{\partial t} + w_i \frac{\partial S}{\partial x_i} + u_i \frac{\partial \theta}{\partial x_i} = \Delta T, \\
\frac{\partial \phi}{\partial t} + w_i \frac{\partial C_2}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_i} = \Delta C_2, \\
w_i = \theta = \phi = 0 \quad \text{on} \quad \partial\Omega \times [0, \mathcal{T}], \\
\theta(x, 0) = \phi(x, 0) = 0, \quad x \in \Omega.
\end{gathered} \tag{67}$$

Now, multiply Equ. $(67)_1$ by w_i and integrate over Ω, we have

$$\begin{aligned}
\|\nabla \mathbf{w}\|^2 + \int_\Omega (1 + \alpha S + \beta_2 S^2) w_i w_i \, d\mathbf{x} &= -\beta \int_\Omega S^2 u_i w_i \, d\mathbf{x} - \beta_1 \int_\Omega (T + S)\theta u_i w_i \, d\mathbf{x} \\
&\quad -\alpha \int_\Omega \theta u_i w_i \, d\mathbf{x} + g_i(\theta, w_i) + \mathcal{I}_i(\phi, w_i) \\
&\le \beta S_m^2 \|\mathbf{u}\| \|\mathbf{w}\| + \|\theta\| \|\mathbf{w}\| + \|\phi\| \|\mathbf{w}\| \\
&\quad + \Big(\alpha + \beta_1 [T_m + S_m]\Big) \|\theta\| \left(\int_\Omega u_i w_i u_j w_j \, d\mathbf{x} \right)^{1/2}.
\end{aligned} \tag{68}$$

We are now performing a similar manner starting from (54), to obtain

$$\|\nabla \mathbf{w}\|^2 + \nu^* \|\mathbf{w}\|^2 \le \frac{3\beta^2 S_m^4 D_4^2}{\nu^*} + \frac{1}{2\nu^*} (S_m^2 + C_{2\,m}^2)\left(\frac{3}{\nu^*} + D_7^2\right) \int_0^t (\|\nabla \mathbf{w}\|^2 + \nu^* \|\mathbf{w}\|^2) \, ds, \tag{69}$$

where

$$0 < \nu^* \le 1 + \alpha S + \beta_2 S^2$$

and

$$D_7 = \sqrt{\frac{2}{\pi}}\Big(\alpha + \beta_1[T_m + S_m]\Big)\Big(\oint_{\partial\Omega} f_i f_i dA + \kappa^{-1/2} D_5\Big)^{1/2}.$$

By integrating (69), we can find

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \nu^*\|\mathbf{w}\|^2) ds \le \beta^2 \mathcal{K}_1 t + \mathcal{K}_2 \int_0^t (t-s)(\|\nabla \mathbf{w}\|^2 + \nu^*\|\mathbf{w}\|^2) ds, \quad (70)$$

where

$$\mathcal{K}_1 = \frac{3S_m^4 D_4^2}{\nu^*} \quad \text{and} \quad \mathcal{K}_2 = \frac{1}{2\nu^*}(S_m^2 + C_{2\,m}^2)(\frac{3}{\nu^*} + D_7^2).$$

Thus, from (70) we have

$$\int_0^t (t-s)(\|\nabla \mathbf{w}\|^2 + \nu^*\|\mathbf{w}\|^2) ds \le \beta^2 \int_0^t \mathcal{K}_1 t e^{\mathcal{K}_2(t-s)} ds, \quad (71)$$

and

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \nu^*\|\mathbf{w}\|^2) ds \le \beta^2 \Big(\mathcal{K}_1 t + \mathcal{K}_2 \int_0^t \mathcal{K}_1 t e^{\mathcal{K}_2(t-s)} ds\Big). \quad (72)$$

Further, (71) leads to

$$\|\theta\|^2 + \|\phi\|^2 \le \frac{1}{2\nu^*}\beta^2(S_m^2 + C_{2\,m}^2)\Big(\mathcal{K}_1 t + \mathcal{K}_2 \int_0^t \mathcal{K}_1 t e^{\mathcal{K}_2(t-s)} ds\Big). \quad (73)$$

Obviously, (73) demonstrates continuous dependence on the viscosity coefficient β and it is actually a priori such that the coefficients of β^2 depend only on boundary and initial data

6 Convergence to the constant viscosity α

Let now (u_i, T, C_1, p) and (v_i, S, C_2, q) be the solutions that satisfy the following boundary-initial-value problems:

$$\begin{aligned} -\Delta u_i + (1 + \alpha T + \beta T^2) u_i &= -\frac{\partial p}{\partial x_i} + g_i T + \mathcal{I}_i C_1, \\ u_{i,i} &= 0, \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} &= \Delta T, \\ \frac{\partial C_1}{\partial t} + u_i \frac{\partial C_1}{\partial x_i} &= \Delta C_1, \end{aligned} \quad (74)$$

in $\Omega \times (0,\mathcal{T})$, with

$$\begin{aligned} u_i &= f(x,t), \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ T(x,t) = h(x,t), &\quad C_1(x,t) = k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0,\mathcal{T}), \end{aligned} \tag{75}$$

$$\begin{aligned} -\Delta v_i + (1+\beta S^2)v_i &= -\frac{\partial q}{\partial x_i} + g_i S + \mathcal{I}_i C_2, \\ v_{i,i} &= 0, \\ \frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} &= \Delta S, \\ \frac{\partial C_2}{\partial t} + v_i \frac{\partial C_2}{\partial x_i} &= \Delta C_2, \end{aligned} \tag{76}$$

in $\Omega \times (0,\mathcal{T})$, and

$$\begin{aligned} v_i &= f(x,t), \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ S(x,t) = h(x,t), &\quad C_2(x,t) = k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0,\mathcal{T}]. \end{aligned} \tag{77}$$

The variables w_i, θ, ϕ and π were introduced in (51) which satisfy the boundary-initial-value problem

$$\begin{aligned} -\triangle w_i + (1+\beta S^2)w_i &= -\pi_{,i} + g_i\theta + \mathcal{I}_i\phi - \alpha T u_i - \beta(T+S)\theta u_i, \\ w_{i,i} &= 0, \\ \frac{\partial \theta}{\partial t} + w_i \frac{\partial S}{\partial x_i} + u_i \frac{\partial \theta}{\partial x_i} &= \Delta T, \\ \frac{\partial \phi}{\partial t} + w_i \frac{\partial C_2}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_i} &= \Delta C_2, \\ w_i = \theta = \phi &= 0 \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ \theta(x,0) = \phi(x,0) &= 0, \quad x \in \Omega. \end{aligned} \tag{78}$$

Next, we start with multiplying $(78)_1$ by w_i and integrating over Ω, with employing the Cauchy-Schwarz and arithmetic-geometric-mean inequalities, to derive

$$\begin{aligned} \|\nabla \mathbf{w}\|^2 + \int_\Omega (1+\beta S^2) w_i w_i \mathbf{dx} &\le \alpha T_m \|\mathbf{w}\|\|\mathbf{u}\| + \|\theta\|\|\mathbf{w}\| + \|\phi\|\|\mathbf{w}\| \\ &+ \beta(T_m + S_m)\|\theta\| \bigg(\int_\Omega u_i w_i u_j w_j \mathbf{x} \bigg)^{1/2}, \end{aligned} \tag{79}$$

Upon using (54) with data bounds for $\|\mathbf{u}\|$ and $\|\nabla \mathbf{u}\|$, yields

$$\|\nabla \mathbf{w}\|^2 + \gamma^* \|\mathbf{w}\|^2 \le \alpha T_m D_4 \|\mathbf{w}\| + \|\theta\|\|\mathbf{w}\| + \|\phi\|\|\mathbf{w}\| + D_8 \|\theta\|\|\nabla \mathbf{w}\|, \tag{80}$$

where

$$0 < \gamma^* \leq 1 + \beta S^2 \quad \text{and} \quad D_8 = \sqrt{\frac{2}{\pi}}\beta(T_m + S_m)\left(\oint_{\partial\Omega} f_i f_i dA + \kappa^{-1/2} D_5\right)^{1/2}.$$

Thus, from (80), after employing arithmetic-geometric mean inequality, we conclude

$$\|\nabla \mathbf{w}\|^2 + \gamma^*\|\mathbf{w}\|^2 \leq \frac{3\gamma^2 T_m D_4^2}{\gamma^*} + (\frac{3}{\gamma^*} + D_8^2)(\|\theta\| + \|\phi\|). \tag{81}$$

Multiply $(78)_3$ and $(78)_4$ by θ and ϕ, respectively with integrating over Ω, to derive

$$\|\theta\|^2 + \|\phi\|^2 \leq \frac{1}{2\alpha^*}(S_m^2 + C_{2\,m}^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \gamma^*\|\mathbf{w}\|^2)ds. \tag{82}$$

Inserting (82) in (81), gives

$$\|\nabla \mathbf{w}\|^2 + \|\mathbf{w}\|^2 \leq \frac{3\alpha^2 T_m D_4^2}{\gamma^*} + \frac{1}{2\gamma^*}(\frac{3}{\gamma^*} + D_8^2)(S_m^2 + C_{2\,m}^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \gamma^*\|\mathbf{w}\|^2)ds. \tag{83}$$

Finally, integration (83), yields

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \gamma^*\|\mathbf{w}\|^2)ds \leq \frac{6\gamma^*\alpha^2 T_m^2 D_4^2 \exp\left(\frac{1}{2\gamma^*}(\frac{3}{\gamma^*} + D_8^2)(S_m^2 + C_{2\,m}^2)t\right)}{(\frac{3}{\gamma^*} + D_8^2)(S_m^2 + C_{2\,m}^2)}. \tag{84}$$

Evidently, (84) demonstrates a convergence of u_i to v_i as $\alpha \to 0$, in the indicated measure. By combining (84) and (83), we also obtain a convergence of w_i in $L^2(\Omega)$ and $H^1(\Omega)$ norms , and from (82) we may obtain a convergence of θ and ϕ in the $L^2(\Omega)$ norm.

7 Convergence to the constant viscosity β

Let (u_i, T, C_1, p) and (v_i, S, C_2, q) be solutions that satisfy the following boundary-initial-value problems:

$$\begin{aligned}
-\Delta u_i + (1 + \alpha T + \beta T^2)u_i &= -\frac{\partial p}{\partial x_i} + g_i T + \mathcal{I}_i C_1, \\
u_{i,i} &= 0, \\
\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} &= \Delta T, \\
\frac{\partial C_1}{\partial t} + u_i \frac{\partial C_1}{\partial x_i} &= \Delta C_1,
\end{aligned} \tag{85}$$

in $\Omega \times (0, \mathcal{T})$, with

$$\begin{aligned} u_i = f(x,t), \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ T(x,t) = h(x,t), \quad C_1(x,t) = k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0,\mathcal{T}), \end{aligned} \tag{86}$$

$$\begin{aligned} -\Delta v_i + (1+\alpha S)v_i = -\frac{\partial q}{\partial x_i} + g_i S + \mathcal{I}_i C_2, \\ v_{i,i} = 0, \\ \frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} = \Delta S, \\ \frac{\partial C_2}{\partial t} + v_i \frac{\partial C_2}{\partial x_i} = \Delta C_2, \end{aligned} \tag{87}$$

in $\Omega \times (0, \mathcal{T})$, with

$$\begin{aligned} v_i = f(x,t), \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ S(x,t) = h(x,t), \quad C_2(x,t) = k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0,\mathcal{T}). \end{aligned} \tag{88}$$

The variables w_i, θ, ϕ and π were introduced in (51) which satisfy the boundary-initial-value problem

$$\begin{aligned} -\triangle w_i + (1+\alpha S)w_i = -\pi_{,i} + g_i\theta + \mathcal{I}_i\phi - \beta T^2 u_i - \alpha\theta u_i, \\ w_{i,i} = 0, \\ \frac{\partial \theta}{\partial t} + w_i \frac{\partial S}{\partial x_i} + u_i \frac{\partial \theta}{\partial x_i} = \Delta T, \\ \frac{\partial \phi}{\partial t} + w_i \frac{\partial C_2}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_i} = \Delta C_2. \\ w_i = \theta = \phi = 0 \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ \theta(x,0) = \phi(x,0) = 0, \quad x \in \Omega. \end{aligned} \tag{89}$$

Now, we multiply $(89)_1$ by w_i and integrate over Ω, with the aid of the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \|\nabla \mathbf{w}\|^2 + \int_\Omega (1+\alpha S) w_i w_i \mathbf{dx} \le \beta T_m^2 \|\mathbf{w}\| \|\mathbf{u}\| + \|\theta\| \|\mathbf{w}\| \\ + \|\phi\| \|\mathbf{w}\| + \alpha \|\theta\| \left(\int_\Omega u_i w_i u_j w_j \mathbf{x} \right)^{1/2}, \end{aligned} \tag{90}$$

Then, with further employing for (54) and data bounds for $\|\mathbf{u}\|$ and $\|\nabla \mathbf{u}\|$ with application of the arithmetic-geometric mean inequality to see that

$$\|\nabla \mathbf{w}\|^2 + \delta^* \|\mathbf{w}\|^2 \le \frac{3\beta^2 T_m^4 D_4^2}{\delta^*} + (\frac{3}{\delta^*} + D_9^2)(\|\theta\|^2 + \|\phi\|^2), \tag{91}$$

here

$$0 < \delta^* \leq 1 + \alpha S \quad \text{and} \quad D_9 = \alpha\sqrt{\frac{2}{\pi}}\left(\oint_{\partial\Omega} f_i f_i \, dA + \kappa^{-1/2} D_5\right)^{1/2}.$$

Since

$$\|\theta\|^2 + \|\phi\|^2 \leq \frac{1}{2\delta^*}(S_m^2 + C_{2\,m}^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2)\,ds. \tag{92}$$

Substituting (92) in (91) we observe that

$$\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2 \leq \frac{3\beta^2 T_m^4 D_4^2}{\delta^*} + \frac{1}{2\delta^*}(3 + D_9^2)(S_m^2 + C_{2\,m}^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2)\,ds. \tag{93}$$

Finally, integrating (93), yields

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2)\,ds \leq \frac{6\delta^*\beta^2 T_m^4 D_4^2 \exp\left(\frac{1}{2\delta^*}(\frac{3}{\delta^*} + D_9^2)(S_m^2 + C_{2\,m}^2)t\right)}{(\frac{3}{\delta^*} + D_9^2)(S_m^2 + C_{2\,m}^2)}. \tag{94}$$

We can see that in (94), the convergence is demonstrated with u_i to v_i as $\beta \to 0$, in the indicated measure. By combining (93) and (94), we also obtain a convergence of w_i in $L^2(\Omega)$ and $H^1(\Omega)$ norm, and from (91) we may obtain a convergence of θ and ϕ in the $L^2(\Omega)$ norm.

8 Conclusions

In this current paper, the problem of double diffusive convection in a Brinkman model has been considered when the viscosity varies with temperature. Specifically, in this work we presented a priori bounds with coefficients that depend only on boundary data, initial data and we demonstrated that the solution depends continuously on changes in the viscosity coefficients α and β, respectively. Moreover, the convergence result is established on Brinkman model when the variable viscosity is allowed to approach to a constant viscosity.

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On a metric topology on the set of bivariate means

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Abstract. In this paper, we define a distance d on the set $\mathcal{M}$ of bivariate means. We show that $(\mathcal{M}, d)$ is a bounded complete metric space which is not compact. Other algebraic and topological properties of $(\mathcal{M}, d)$ are investigated as well.

1 Introduction

A bivariate mean m, as proposed by Cauchy in [5], is a map from $(0, \infty) \times (0, \infty)$ into $(0, \infty)$ satisfying the following condition

$$\forall x, y > 0 \qquad \min(x, y) \le m(x, y) \le \max(x, y). \tag{1}$$

The two maps $(x, y) \longmapsto \min(x, y)$ and $(x, y) \longmapsto \max(x, y)$, will be denoted by min and max, are trivial means called the lower mean and the upper mean, respectively. The property in (1) is called by Audenaert in [1], the in-betweenness property of the mean m. Some other standard examples of means are given in

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the following:

$$\begin{cases} A := A(x,y) = & \dfrac{x+y}{2} \\ G := G(x,y) = & \sqrt{xy} \\ H := H(x,y) = & \dfrac{2xy}{x+y} \end{cases} \tag{2}$$

and are known, in the literature, as the arithmetic mean, geometric mean and harmonic mean, respectively. These three means are included in the following families of means, with $t \in [0,1]$,

$$\begin{cases} A_t := A_t(x,y) = (1-t)x + ty, \\ G_t := G_t(x,y) = x^{1-t}y^t, \\ H_t := H_t(x,y) = \Big((1-t)x^{-1} + ty^{-1}\Big)^{-1}. \end{cases} \tag{3}$$

These are called weighted arithmetic mean, weighted geometric mean and weighted harmonic mean, respectively. Another example of classical means is the Heron mean He_t defined by, [8], $He_t = tG+(1-t)A$ with $t \in [0,1]$. Clearly, $He_0 = A$ and $He_1 = G$. For further examples of means, we refer the reader to [4, 3, 9] for instance and the related references cited therein.

Otherwise, it is not hard to check that the following relationship

$$A_t(x,y) - H_t(x,y) = t(1-t)\frac{(x-y)^2}{tx + (1-t)y} \tag{4}$$

holds for any $x, y > 0$ and $t \in [0,1]$.

We define symmetric (resp. homogeneous, monotone) means in the habitual way. The weighted means (3) are homogeneous, monotone, not symmetric unless $t = 1/2$, case for which they coincide with A, G and H, respectively. In the literature, see [9] for instance, we can find a lot of symmetric, homogeneous monotone means. For example, the following

$$L(x,y) := \frac{x-y}{\log(x/y)},\ x \neq y;\ \ \text{with } L(x,x) = x \tag{5}$$

is known as the logarithmic mean of $x > 0$ and $y > 0$.

As example of mean which is not monotone, we can mention the contra-harmonic mean defined for all $x, y > 0$ by, $C(x,y) = \dfrac{x^2+y^2}{x+y}$. It is well known that the following inequalities, [9]

$$H(x,y) \leq G(x,y) \leq L(x,y) \leq A(x,y) \leq C(x,y) \tag{6}$$

hold for any $x, y > 0$.

The mean C is a particular case of the so-called Lehmer power mean defined for $p \in \mathbb{R}$ by

$$L_p(x, y) = \frac{x^p + y^p}{x^{p-1} + y^{p-1}}.$$

It is easy to see that $L_1 = A$, $L_0 = H$ and $L_2 = C$. We can also check that the equality

$$L_p(x, y) - A(x, y) = \frac{(x - y)(x^{p-1} - y^{p-1})}{2(x^{p-1} + y^{p-1})} \tag{7}$$

holds for any $x, y > 0$ and $p \in \mathbb{R}$.

The set of all (bivariate) means will be denoted by $\mathcal{M}$. We also denote by $\mathcal{M}_s$, resp. $\mathcal{M}_h$, the set of all symmetric means, resp. homogeneous means. As pointed out in [2], the sets $\mathcal{M}$, $\mathcal{M}_s$ and $\mathcal{M}_h$ are convex.

In Section 2 below, we will define a metric d on the set $\mathcal{M}$ and we study its algebraic properties as well as some examples for computations of $d(m_1, m_2)$ when $m_1, m_2 \in \mathcal{M}$. Afterwards, Section 3 is devoted to investigate some topological properties of the metric space $\big(\mathcal{M}, d\big)$.

2 Metric topology on $\mathcal{M}$

Let $m_1, m_2 \in \mathcal{M}$. From (1), we immediately deduce that, for all $x, y > 0$, we have

$$|m_1(x, y) - m_2(x, y)| \le \max(x, y) - \min(x, y) = |x - y|. \tag{8}$$

We can then put the following definition.

Definition 1 *Let* $m_1, m_2 \in \mathcal{M}$. *For all* $x, y > 0$, *we define*

$$\mathcal{T}\big(m_1, m_2\big)(x, y) = \begin{cases} \dfrac{m_1(x, y) - m_2(x, y)}{x - y} & \textit{if } x \neq y, \\ 0 & \textit{if } x = y \end{cases} \tag{9}$$

Following (8), $\mathcal{T}\big(m_1, m_2\big)$ is a map from $(0, \infty) \times (0, \infty)$ into $[-1, 1]$, i.e.

$$\forall x, y > 0 \qquad |\mathcal{T}\big(m_1, m_2\big)(x, y)| \le 1. \tag{10}$$

Further, if m_1 and m_2 are both symmetric and homogenous then we have,

$$\mathcal{T}\big(m_1, m_2\big)(x, y) = \mathcal{T}\big(m_1, m_2\big)(y, x) \ \text{ and}$$

$$\mathcal{T}(m_1, m_2)(tx, ty) = \mathcal{T}(m_1, m_2)(x, y),$$

for all $x, y > 0$ and $t > 0$.

Now, we are in a position to state the following definition.

Definition 2 *For $m_1, m_2 \in \mathcal{M}$, we set*

$$d(m_1, m_2) = \sup_{x,y>0} \left|\mathcal{T}(m_1, m_2)(x, y)\right|. \tag{11}$$

It is clear that if $m_1, m_2 \in \mathcal{M}_h$ then we have

$$d(m_1, m_2) = \sup_{0<x} |\mathcal{T}(m_1, m_2)(x, 1)|. \tag{12}$$

Proposition 1 *$(\mathcal{M}, d)$ is a bounded metric space.*

Proof. It is easy to check that for all $m_1, m_2, m_3 \in \mathcal{M}$ the next properties are satisfied:

- $d(m_1, m_2) = d(m_2, m_1)$.
- $d(m_1, m_2) = 0 \Longleftrightarrow m_1 = m_2$.
- $d(m_1, m_3) \le d(m_1, m_2) + d(m_2, m_3)$.

These confirm that d establishes a distance on $\mathcal{M}$. Otherwise, from inequality (10), we immediately deduce that $(\mathcal{M}, d)$ is bounded. □

Remark 1 i) *In the particular case of symmetric means defined on a symmetric domain in $\mathbb{R}$, Farhi gave in [7] the following formula*

$$d(m_1, m_2) = \sup_{0<x,y} \left(\frac{1}{e^{f_{m_1}(x,y)} + 1} - \frac{1}{e^{f_{m_2}(x,y)} + 1}\right) \tag{13}$$

where for a mean $m \in \mathcal{M}_s$, $f_m(x, y)$ is defined for all $0 < x,\ y$ by

$$\begin{cases} f_m(x, y) = \log\left(-\dfrac{x - m(x, y)}{y - m(x, y)}\right), & \text{for } x \neq y \\ f_m(x, x) = 0 \end{cases} \tag{14}$$

which can be useful in a computational point of view as well as the relation (12) as we will see later.

ii) *If the means m_1 and m_2 are symmetric then the distance $d(m_1, m_2)$ can be also defined by the next formula*

$$d(m_1, m_2) = \sup_{0<x<y} \left|\mathcal{T}(m_1, m_2)(x, y)\right|.$$

Now we will state some important examples where distances between some special means are determined with respect to the metric d.

Example 1 *Simple computations lead to the following:*

(i) $d(A_{t_1}, A_{t_2}) = |t_1 - t_2|, \forall t_1, t_2 \in [0,1]$. *In particular,* $d(A, A_t) = |t - 1/2|$, $\forall t \in [0,1]$.

(ii) $d(A, H) = d(A, G) = d(A, C) = 1/2$.

(iii) $d(H, C) = d(\min, C) = 1,\ d(G, C) = 1$.

(iv) $d(A, He_t) = t/2$ *with* $t \in [0,1]$. *In particular* $d\big(A, \frac{A+G}{2}\big) = 1/4$, *since* $He_{1/2} = \frac{A+G}{2}$.

Remark 2 *One can check that, if* m, m_1 *and* m_2 *are three means such that* $m \le m_1 \le m_2$, *(resp.* $m_1 \le m_2 \le m$*), then we have*

$$d(m, m_1) \le d(m, m_2), \quad (\textit{resp. } d(m, m_1) \ge d(m, m_2)).$$

This, with the arithmetic-geometric-harmonic mean inequality, namely $H \le G \le A$, *yields the following:*
- *If* $m \in \mathcal{M}$ *is such that* $m \le H$ *then* $d(m, H) \le d(m, G) \le d(m, A)$.
- *If* $m \in \mathcal{M}$ *is such that* $m \ge A$ *then* $d(m, A) \le d(m, G) \le d(m, H)$.

We now state the following proposition which contains more examples giving the computation of $d(m_1, m_2)$ when m_1 and m_2 are among the previous standard bivariate means.

Proposition 2 *The following equalities hold,*

(i) *For any* $m \in \mathcal{M}$ *we have* $d(m, A) \le 1/2$.

(ii) $d\big(A_t, H_t\big) = \max(t, 1 - t)$ *for any* $t \in (0,1)$.

(iii) $d\big(A_t, G_t\big) = \max(t, 1 - t)$ *for all* $t \in (0,1)$.

(iv) $d\big(L_p, A\big) = 1/2$ *for any real number* $p \ne 1$.

(v) $d(A, L) = 1/2$.

Proof. (i) Let $m \in \mathcal{M}$ and $x, y > 0$. We have

$$|m(x,y) - A(x,y)| \leq \max\big(x - A(x,y), y - A(x,y)\big) = \max\Big(\frac{x-y}{2}, \frac{y-x}{2}\Big), \tag{15}$$

which yields $d(m, A) \leq 1/2$.

Since the means A_t, G_t,H_t,L_p and L are homogenous, then it suffices to use the formulae (12) for establishing the equalities (ii)-(v).

(ii) Let $x > 0$. According to (4) with $y = 1$, we obtain

$$\big|\mathcal{T}\big(A_t, H_t\big)(x,1)\big| = t(1-t)\frac{|1-x|}{1-t+tx}.$$

Studying the variations of the function $\phi : x \longmapsto \dfrac{|1-x|}{1-t+tx}$, defined for $x \in (0,\infty)$, we conclude that it decreases on $(0,1]$ and increases on $[1,\infty)$. Then we have

$$\sup_{x\in(0,\infty)} \phi(x) = \max\big(\phi(0^+), \lim_{x\to\infty}\phi(x)\big) = \max\left(\frac{1}{t}, \frac{1}{1-t}\right).$$

It follows that

$$d\big(A_t, H_t\big) = \sup_{0<x}\big|\mathcal{T}\big(A_t, H_t\big)(x,1)\big| = \max(t, 1-t).$$

(iii) Now, we have for any $x \in (0,\infty)$ with $x \neq 1$

$$\big|\mathcal{T}\big(A_t, G_t\big)(x,1)\big| = \left|\frac{(1-t)x + t - x^{1-t}}{1-x}\right|.$$

If for $x \in (0,1)$ we set $u_t(x) = \dfrac{(1-t)x + t - x^{1-t}}{1-x}$ then simple computation leads to

$$u_t'(x) = \frac{1-(1-t)x^{-t} - tx^{1-t}}{(1-x)^2} = \frac{v_t(x)}{(1-x)^2},$$

where

$$v_t(x) = 1 - (1-t)x^{-t} - tx^{1-t} \;\text{ and }\; v_t'(x) = t(1-t)x^{-t}(x^{-1}-1) \geq 0.$$

It follows that v_t is a strictly increasing function on $(0,1)$ and so $v_t(x) \leq v_t(1^-) = 0$ for any $x \in (0,1)$. We then deduce that u_t is a strictly decreasing function on $(0,1)$. Hence,

$$\sup_{0<x<1}\big|\mathcal{T}\big(A_t, G_t\big)(x,1)\big| = u_t(0) = t.$$

With similar computations we can prove that,

$$\sup_{x>1} \left|\mathcal{T}\big(A_t, G_t\big)(x,1)\right| = 1 - t.$$

So we get $d\big(A_t, G_t\big) = \max(t, 1-t)$.
(iv) Let $x \in (0,\infty)$. By virtue of (7) we have

$$\mathcal{T}(L_p, A)(x,1) := \frac{|x^{p-1} - 1|}{2(x^{p-1}+1)}.$$

The case $p = 1$ is trivial, since $L_1 = A$ and so $d(L_1, A) = 0$. Assume that $p \neq 1$. Setting $x^{p-1} = z$ and using elementary techniques of real analysis, we find $d\big(L_p, A\big) = 1/2$.

(v) As previously, with the help of (5) and (6), we have for any $x \in (0,1)$ (after a simple reduction)

$$\mathcal{T}(L, A)(x,1) := \frac{(x+1)\log(x) - 2(x-1)}{2|1-x|\,\log(x)}.$$

We first show that

$$\forall x \in (0,1) \qquad \frac{(x+1)\log(x) - 2(x-1)}{2(1-x)\log(x)} \leq 1/2. \tag{16}$$

After simple manipulations, (16) is reduced to the following one

$$\forall x \in (0,1) \qquad x\log(x) - x + 1 \geq 0,$$

which is so easy to show by similar way as previously.
For $x > 1$ the inequality,

$$\frac{(x+1)\log(x) - 2(x-1)}{2(x-1)\log(x)} \leq 1/2. \tag{17}$$

is equivalent to

$$\log(x) - x + 1 \leq 0,$$

which can be simply confirmed.
The results (16) and (21) enable us to write for all $x > 0$ that,

$$\frac{(x+1)\log(x) - 2(x-1)}{2|1-x|\,\log(x)} \leq \frac{1}{2}. \tag{18}$$

Now, by l'Hopital's rule we can check that

$$\lim_{x\to 1}\frac{(x+1)\log(x)-2(x-1)}{(1-x)\log(x)}=1.$$

This, when combined with (18), yields the desired result. □

Remark 3 *According to the preceding proposition, the relationship* $d(L_p, A) = 1/2$ *shows that the map* $p \longmapsto d(L_p, A)$ *is discontinuous at* $p = 1$*, since* $L_1 = A$ *and so* $d(L_1, A) = 0$*.*

Proposition 3 *The distance between two harmonic weighted means* H_{t_1} *and* H_{t_2} *is given by*

$$d\big(H_{t_1}, H_{t_2}\big)=\frac{|t_1-t_2|\theta(t_1,t_2)}{2(1-t_1)(1-t_2)+(t_1+t_2-2t_1t_2)\theta(t_1,t_2)},$$

where we set

$$\theta(t_1,t_2):=\sqrt{\frac{(1-t_1)(1-t_2)}{t_1t_2}}.$$

Proof. We also use (12). For $x \in (0, \infty)$, simple computation leads to

$$\big|\mathcal{T}\big(H_{t_1},H_{t_2}\big)(x,1)\big|=\frac{|t_1-t_2|x}{(1-t_1+t_1x)(1-t_2+t_2x)}.$$

Let us set

$$\forall x\in(0,\infty)\qquad g(x):=\frac{x}{(1-t_1+t_1x)(1-t_2+t_2x)}.$$

By computing the derivative of g we easily obtain

$$\forall x\in(0,\infty)\qquad g^{'}(x)=\frac{-t_1t_2x^2+(1-t_1)(1-t_2)}{(1-t_1+t_1x)^2(1-t_2+t_2x)^2}.$$

We then deduce that g attains its maximum point at $\theta(t_1, t_2)$. Computing $g\big(\theta(t_1, t_2)\big)$ we find the desired result after simple reductions. The proof is completed. □

Another result of interest is recited in the following.

Proposition 4 *The map* $(m_1, m_2) \longmapsto d(m_1, m_2)$ *is jointly convex and separately convex. That is, the two following inequalities*

$$d\Big((1-t)m_1 + tm_3, (1-t)m_2 + tm_4\Big) \leq (1-t)d(m_1, m_2) + td(m_3, m_4) \quad (19)$$

$$d\big((1-t)m_1 + tm_3, m_2\big) \leq (1-t)d(m_1, m_2) + td(m_3, m_2) \quad (20)$$

hold for any $m_1, m_2, m_3, m_4 \in \mathcal{M}$ *and* $t \in [0, 1]$.

Proof. Let $m_1, m_2, m_3, m_4 \in \mathcal{M}$ and $t \in [0, 1]$. For the sake of simplicity, we set

$$\delta := d\Big((1-t)m_1 + tm_3, (1-t)m_2 + tm_4\Big).$$

Then we have

$$\begin{aligned}
\delta &= \sup_{0<x,y} \left| \frac{\big((1-t)m_1 + tm_3\big)(x,y) - \big((1-t)m_2 + tm_4\big)(x,y)}{x-y} \right| \\
&= \sup_{0<x,y} \left| \frac{(1-t)\big(m_1(x,y) - m_2(x,y)\big) + t\big(m_3(x,y) - m_4(x,y)\big)}{x-y} \right| \\
&\leq \sup_{0<x,y} \left| \frac{(1-t)\big(m_1(x,y) - m_2(x,y)\big)}{x-y} \right| + \sup_{0<x,y} \left| \frac{t\big(m_3(x,y) - m_4(x,y)\big)}{x-y} \right| \\
&= (1-t)d(m_1, m_2) + td(m_3, m_4)
\end{aligned}$$

Every jointly convex map is separately convex. That is, (20) follows from (19) when we take $m_4 = m_2$. The proof is finished. □

Remark 4 (i) *According to Example 1,* (iii) *the relation* $d\big(A, \frac{A+G}{2}\big) = 1/4$ *shows that the convexity of the map* $(m_1, m_2) \longmapsto d(m_1, m_2)$ *is not strict.*
(ii) *From the preceding proposition we immediately deduce that, every ball (closed or open) of* $(\mathcal{M}, d)$ *is convex.*

Corollary 1 *The next inequalities hold*

(i) $d(A, \lambda A_t + (1-\lambda)L_p) \leq \lambda|t - 1/2| + \dfrac{1-\lambda}{2}$, *for* $t, \lambda \in [0, 1]$ *and* $p \in \mathbb{R}$.

(ii) $d(A, \lambda A_t + (1-\lambda)He_\alpha) \leq \lambda|t - 1/2| + \dfrac{\alpha(1-\lambda)}{2}$, *for* $t, \lambda, \alpha \in [0, 1]$.

(iii) $d(A, \lambda L_p + (1-\lambda)He_\alpha) \leq \dfrac{\lambda}{2} + \dfrac{\alpha(1-\lambda)}{2}$, *for* $t, \lambda, \alpha \in [0, 1]$ *and* $p \in \mathbb{R}$.

Proof. According to (20), with the help of Example 1 and Proposition 2, we easily obtain the desired inequalities. The details are simple and therefore omitted here for the reader. □

3 Topological properties of $(\mathcal{M}, d)$

We preserve the same notations as in the previous sections.

Proposition 5 *$(\mathcal{M}, d)$ coincides with its closed ball of center A, the arithmetic mean, and radius $1/2$.*

Proof. According to (15) we have $d(m, A) \leq 1/2$ for any $m \in \mathcal{M}$. Inversely, assume that m is a binary map satisfying (15). This is equivalent to

$$|m(x,y) - A(x,y)| \leq \frac{|x-y|}{2} \tag{21}$$

and so,

$$\min(x,y) := \frac{x+y-|x-y|}{2} \leq m(x,y) \leq \frac{x+y+|x-y|}{2} := \max(x,y).$$

The desired result follows and the proof is finished. □

Remark 5 *In a geometrical point of view, the inequality (21) implies that every mean $m(x,y)$ lies on the sphere centered at the arithmetic mean and with radius equal to the half of the euclidian distance between x and y. So, according to the definition given by Dinh et al [6], every bivariate mean satisfies the in-sphere property.*

An important topological property for $(\mathcal{M}, d)$ is quoted in the following theorem.

Theorem 1 *The metric space $(\mathcal{M}, d)$ is complete.*

Proof. Let (m_p) be a Cauchy sequence in $(\mathcal{M}, d)$. Let $a, b > 0$ with $a \neq b$ be fixed. For a given $\epsilon > 0$ there is $\eta_\epsilon \in \mathbb{N}$ such that,

$$p, q \geq \eta_\epsilon \Longrightarrow d(m_p, m_q) \leq \frac{\epsilon}{|a-b|}$$

or equivalently

$$p, q \geq \eta_\epsilon \Longrightarrow \sup_{0<x,y} \left| \frac{m_p(x,y) - m_q(x,y)}{x-y} \right| \leq \frac{\epsilon}{|a-b|}.$$

We then deduce that

$$p, q \geq \eta_\epsilon \Longrightarrow \left| \frac{m_p(a,b) - m_q(a,b)}{a-b} \right| \leq \frac{\epsilon}{|a-b|}.$$

This latter inequality holds for each $a, b > 0$ with $a \neq b$. It follows that, for any $x, y > 0$ there exists an integer N_ϵ such that we get

$$p, q \geq N_\epsilon \Longrightarrow |m_p(x, y) - m_q(x, y)| \leq \epsilon,$$

which means that $(m_p(x, y))_p$ is a Cauchy sequence in $\mathbb{R}$. By completeness of $\mathbb{R}$, for the standard metric, the sequence $\big(m_n(x, y)\big)_n$ is convergent in $\mathbb{R}$ and we put $\lim_{p\uparrow\infty} m_p(x, y) = m(x, y)$ for any $x, y > 0$.

Since $\min(x, y) \leq m_n(x, y) \leq \max(x, y)$ we then deduce, when $p \uparrow \infty$, that $\min(x, y) \leq m(x, y) \leq \max(x, y)$. So we can confirm that $m \in \mathcal{M}$ and it remains to prove that $(m_p)_p$ converges to m in $(\mathcal{M}, d)$.

Since $(m_p(x, y))_p$ is a Cauchy sequence in $\mathbb{R}$ then, for any $x, y > 0$ with $x \neq y$, $\big(m_p(x, y)/(x - y)\big)_p$ is also a Cauchy sequence in $\mathbb{R}$. This means that,

$$\forall \epsilon > 0 \quad \exists N_\epsilon \in \mathbb{N} \quad \forall p, q \geq N_\epsilon \quad \forall x, y > 0,\ x \neq y \quad \left| \frac{m_p(x, y) - m_q(x, y)}{x - y} \right| \leq \epsilon.$$

By letting q to ∞ in this latter inequality we obtain,

$$\left| \frac{m_p(x, y) - m(x, y)}{x - y} \right| \leq \epsilon \ \text{ for all } x, y > 0,\ x \neq y$$

and so $d(m_p, m) \leq \epsilon$ which gives the convergence of $(m_p)_p$ to m in $(\mathcal{M}, d)$. □

In the aim to give more topological properties for $(\mathcal{M}, d)$, we need the following lemma.

Lemma 1 *For every* $p \geq 1$*, we have the following*

$$d\big(L_p, \max\big) = 1.$$

Proof. The means L_p and max are homogenous so we can use the formulae (12). It is easy to check that

$$\forall x \in (0, 1) \qquad \mathcal{T}(L_p, \max)(x, 1) = \frac{x^{p-1}}{1 + x^{p-1}}. \tag{22}$$

For $p = 1$, there is nothing to prove. For $p \neq 1$, we set $z = x^{p-1}$ and the right hand-side of (22) remains a simple homographic function in z which is so monotone for $z \in (0, +\infty)$. For both cases $p > 1$ or $p < 1$, the variable z describes the interval $(0, +\infty)$. Passing to the supremum over z in (22) we obtain the desired result, by simple arguments of real analysis. □

Now, we are in a position to state the following result.

Theorem 2 *The space* $(\mathcal{M}, d)$ *is not compact.*

Proof. Since $(\mathcal{M}, d)$ is a metric space, it suffices to show that there exists a sequence which have no convergent subsequence.
Let's at first recall that if a sequence (m_p) converges in $(\mathcal{M}, d)$ to a mean m then we have $\lim_{p\to+\infty} m_p(x, y) = m(x, y)$ for any $x, y > 0$.
On one hand, by Lemma 1, we have $d(L_p, \max) = 1$ for all integer $p > 1$, and so the sequence $(L_p)_{p>1}$ does not converge to max in $(\mathcal{M}, d)$.
On the other hand, it is not hard to see that $\lim_{p\to+\infty} L_p(a, b) = \max(a, b)$, see [2] for instance. This shows that we can not extract a convergent subsequence from the bounded sequence $(L_p)_{p>1}$. The proof of the theorem is completed.□

Proposition 6 $\mathcal{M}_s$ *and* $\mathcal{M}_h$ *are closed in* $(\mathcal{M}, d)$.

Proof. We show that $\mathcal{M}_s$ is closed. For this, let $(m_n)_n$ be a sequence of symmetric means converging to a mean m in $(\mathcal{M}, d)$. As already mentioned before, for any $x, y > 0$, the two sequences $\big(m_n(x, y)\big)_n$ and $\big(m_n(y, x)\big)_n$ converge in $\mathbb{R}$ to $m(x, y)$ and $m(y, x)$, respectively. Since $m_n(x, y) = m_n(y, x)$ then by letting $n \uparrow \infty$ we obtain $m(x, y) = m(y, x)$ and so $m \in \mathcal{M}_s$.
In a similar way, we prove the closeness of $\mathcal{M}_h$. □

Another topological property of $\big(\mathcal{M}, d\big)$ is recited in the following result.

Theorem 3 *The metric space* $\big(\mathcal{M}, d\big)$ *is path-wise connected and so, it is connected.*

Proof. Let $m_1, m_2 \in \mathcal{M}$ and consider the map $f : [0, 1] \longrightarrow \mathcal{M}$ such that,

$$\forall t \in [0, 1] \quad f(t) = (1 - t)m_1 + tm_2.$$

Since $\mathcal{M}$ is convex then f is well defined. We will show that f is a path (i.e. continuous function) with endpoints m_1 and m_2. Indeed, for all $t_1, t_2 \in [0, 1]$ we have

$$\begin{aligned} d\big(f(t_1), f(t_2)\big) &= \sup_{0<x,y} \left| \frac{f(t_1)(x, y) - f(t_2)(x, y)}{x - y} \right| \\ &= \sup_{0<x,y} \left| \frac{(t_1 - t_2)(m_1(x, y) - m_2(x, y))}{x - y} \right| \\ &= |t_1 - t_2| d(m_1, m_2) \\ &\le |t_1 - t_2| \end{aligned}$$

We then infer that f is uniformly continuous on $[0, 1]$ and so, it is continuous. Moreover, $f(0) = m_1$ and $f(1) = m_2$. In summary, f is a path with endpoints m_1 and m_2. The proof is finished. □

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On logarithm of circular and hyperbolic functions and bounds for $\exp(\pm x^2)$

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Abstract. We show that certain known or new inequalities for the logarithm of circular hyperbolic functions imply bounds for $\exp(\pm x^2)$ proved in [1].

1 Introduction

In the recent paper [1], the following sharp bounds for $\exp(\pm x^2)$ are proved (see Theorem 1, resp. Theorem 2 of [1]).

For $x \in (0, \pi/2)$ one has

$$\left(\frac{1+\cos x}{2}\right)^a < \exp(-x^2) < \left(\frac{1+\cos x}{2}\right)^b \tag{1}$$

where $a = 4$, $b = \pi^2/4\ln 2$ are best possible; and

$$\left(\frac{2+\cos x}{3}\right)^c < \exp(-x^2) < \left(\frac{2+\cos x}{3}\right)^d, \tag{2}$$

with $c = \pi^2/4\ln(3/2)$, $d = 6$ best possible;

$$\left(\frac{1+\cosh x}{2}\right)^\alpha < \exp(x^2) < \left(\frac{1+\cosh x}{2}\right)^\beta \tag{3}$$

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where $\alpha = 4$, $\beta = \pi^2/4\ln[(1+\cosh(\pi/2))/2]$ are best possible;

$$\left(\frac{2+\cosh x}{3}\right)^{\theta} < \exp(x^2) < \left(\frac{2+\cosh x}{3}\right)^{\gamma} \tag{4}$$

with $\theta = 6$ and

$$\gamma = \pi^2/4\ln[(2+\cosh(\pi/2))/3]$$

are best possible.

We first want to point out that relations (1), (2) and (3) are essentially known (but not stated explicitely), and that (4) can be deduced in a similar way, using the Jensen integral inequality.

2 Proofs

First remark that, as

$$\frac{1+\cos x}{2} = \cos^2\frac{x}{2} \quad \text{and} \quad \frac{1+\cosh x}{2} = \cosh^2\frac{x}{2};$$

by letting $\frac{x}{2} = t$, where $t \in (0, \pi/4)$, to prove (1) it is sufficient to show that the function

$$f_1(t) = \frac{\ln(\cos t)}{t^2} \tag{5}$$

is strictly decreasing. As

$$t^3 f_1'(t) = -t \cdot \tan t + 2\ln(1/\cos t),$$

this follows by the inequality

$$\ln\left(\frac{1}{\cos t}\right) < \frac{t}{2} \cdot \tan t. \tag{6}$$

This is proved in [2] (see Corollary 3.8, right side of (1)).

Now, relation (1) with best possible a and b follow by

$$f_1(0+) > f_1(t) > f_1(\pi/4).$$

In a similar manner for (3) it is sufficient to prove that the function

$$f_2(t) = \frac{\ln(\cosh t)}{t^2} \tag{7}$$

is strictly decreasing. As

$$t_3 f_2'(t) = t \cdot \tanh t - 2\ln(\cosh t),$$

this follows by the inequality

$$\ln(\cosh t) > \frac{t}{2} \cdot \tanh t, \tag{8}$$

proved in [5] and [2] (see Lemma 2.1 in [5] and Corollary 3.8, left side of (2) in [2]).

For improvements of (8) and related inequalities, see [9].

Now (3), with best possible α and β follow from

$$f_2(0+) > f_2(t) > f_2(\pi/4).$$

We note that inequalities (6) and (8) are simple consequences of the Jensen integral inequality ([3]):

$$\int_a^b F(x)dx \underset{(>)}{<} (b-a)\left[\frac{F(a)+F(b)}{2}\right] \tag{9}$$

with inequality $<$ when $F(x)$ is strictly convex, and $(>)$, when $F(x)$ is strictly concave on $[a, b]$. Inequality (9) is called also as one of the Hermite-Hadamard inequalities. By letting the convex function $F_1(x) = \tan x$ and $[a, b] = [0, t]$, we get (6). Similarly, by letting $F_2(x) = \tanh x$ and $[a, b] = [0, t]$, we get relation (8).

Now, let

$$F_3(x) = \frac{\sinh t}{2 + \cosh t},$$

remarking that

$$\int_0^t F_3(x)dx = \ln\left(\frac{\cosh x + 2}{3}\right).$$

It is immediate that

$$F_3'(x) = \frac{1 + 2\cosh x}{(\cosh x + 2)^2}$$

and

$$F_3''(x) \cdot (\cosh x + 2) = 2(\sinh x) \cdot (1 - \cosh x) < 0,$$

we get that $F_3(x)$ is strictly concave. Thus, by (9) we get the inequality

$$\ln\left(\frac{\cosh t + 2}{3}\right) > \frac{t}{2} \cdot \frac{\sinh t}{\cosh t + 2}. \tag{10}$$

Similarly, by remarking that

$$\int_0^t \frac{\sin x}{2+\cos x}dx = \ln\left(\frac{3}{2+\cos x}\right)$$

and that the function

$$F_4(x) = \frac{\sin x}{2+\cos x}$$

is strictly concave, we can deduce the inequality

$$\ln\frac{3}{2+\cos t} > \frac{t}{2}\cdot\frac{\sin t}{\cos t+2}. \tag{11}$$

This inequality, with another proof, appears also in [8] (see relation (2.2)).

Now, to prove (2), let

$$f_3(t) = \frac{\left[\ln\left(\frac{\cos t+2}{3}\right)\right]}{t^2}.$$

It is immediate that

$$t^3 f_3'(t) = -\frac{\sin t}{2+\cos t}\cdot t + 2\ln\left(\frac{3}{\cos t+2}\right) > 0$$

by (11). Thus $f_3(t)$ is strictly increasing, and (2) with best possible c and d follow by $f_3(0+) < f_3(t) < f_3(\pi/2)$.

Finally, inequality (4) follows in the same manner by considering

$$f_4(t) = \left[\ln\left(\frac{\cosh t+2}{3}\right)\right]/t^2,$$

and applying inequality (10).

Remarks 1) In [1], the L'Hospital's rule of monotonicity, as well as the following lemma is used:

$$\frac{\sin x}{x} > \frac{1+2\cos x}{2+\cos x}, \quad x\in(0,\pi/2) \tag{12}$$

and

$$\frac{x}{\sinh x} + \cosh x > 2, \quad x > 0. \tag{13}$$

We note that relation (12), with strong improvements, appears in paper [7] (see relation (4.7) and Theorem 4.2). We point out also, that (13) is weaker than the Neumann-Sándor inequality [4]:

$$\frac{\sinh x}{x} < \frac{2 + \cosh x}{3}. \tag{14}$$

Indeed, as

$$\frac{x}{\sinh x} > \frac{3}{2 + \cosh x},$$

and letting $u = \text{ch}x$, one has

$$u + \frac{3}{2 + u} > 2.$$

Indeed, this is equivalent to $u^2 + 2u + 3 > 4 + 2u$, or $u > 1$, which is true. Therefore, one has

$$\frac{x}{\sinh x} + \cosh x > \frac{3}{2 + \cosh x} + \cosh x > 2. \tag{15}$$

2) Other inequalities for the logarithm of circular and hyperbolic functions can be found in papers [2, 5, 6, 8, 9]. For various applications in the theory of means, see [10].

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Analysis of semigroups with soft intersection ideals

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Abstract. In this paper, semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, semilattice of left (right) simple semigroups, semilattice of left (right) groups and semilattice of groups are characterized in terms of soft intersection semigroups, soft intersection ideals of semigroups. Moreover, soft normal semigroups are defined and some characterizations of semigroups with soft normality are given.

1 Introduction

In 1999, the concept of soft sets was introduced by Molodtsov [31] for modeling vagueness and uncertainty. Many complex problems of social science and science involve uncertainties. To be able to deal with these uncertainties and incomplete information, some theories have been proposed such as the theory of probability, as is well known, the most successful theoretical approaches are undoubtedly fuzzy set [1] and interval mathematics [2] . Despite all these

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developments, Molodtsov [3] pointed out that each of these theories have inherent limitations in insufficient parameterization tools, and he introduced the soft set theory for modeling vagueness and uncertainty. Since the soft set theory is very convenient and easy to apply in practice, researches focused on soft sets that have been growing rapidly, and which has some potential applications in many different fields; such as extended theories [4], combination forecast [5], data mining [6], medical diagnosis [7] and decision making [8] . Meanwhile, many related concepts with soft sets, especially soft set operations, have recently undergone tremendous studies. Maji et al. [30] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [12] introduced several operations of soft sets and Sezgin and Atagün [36] and Ali et al. [13] studied on soft set operations as well. Soft set theory have found its wide-ranging applications in the mean of algebraic structures such as groups [11, 37], semirings [18], rings [9], BCK/BCI-algebras [24, 25, 26], BL-algebras [42], near-rings [35] and soft substructures and union soft substructures [14, 38], hemirings [29, 43] and so on [18, 19, 21].

In [20], Feng et al. applied soft relations to semigroups. In [39], Sezer et al. made a new approach to the classical semigroup theory via soft set theory with the concept of soft intersection semigroups. They defined soft intersection semigroups, soft intersection left (right, two-sided) ideals and bi-ideals and soft semiprime ideals of semigroups and obtained their basic properties. As a following study of [39], Sezer et al. [40] defined soft intersection interior ideals, quasi-ideals, generalized bi-ideals and investigate the interrelations of them. Moreover, they characterized regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups by the properties of these ideals in [39, 40].

In this paper, certain classes of semigroups, such as semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, semilattice of left (right) simple semigroups, semilattice of left (right) groups and semilattice of groups in terms of soft intersection ideals, bi-ideals, interior ideals, quasi-ideals, generalized bi-ideals are characterized. Moreover, soft normal semigroups are defined and discussed on the relation of this concept with semigroups.

2 Preliminaries

In this section, some notions relevant to semigroups and soft sets are recalled. A *semigroup* S is a nonempty set with an associative binary operation. Throughout this paper, S denotes a semigroup. A nonempty subset A of S is called a *right ideal* of S if $AS \subseteq A$ and is called a *left ideal* of S if $SA \subseteq A$. By *two-sided ideal* (or simply *ideal*), we mean a subset of S, which is both a left and right ideal of S. A subsemigroup X of S is called a *bi-ideal* of S if $XSX \subseteq X$. A nonempty subset A of S is called an *interior ideal* of S if $SAS \subseteq A$. A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$.

We denote by $L[a](R[a], J[a], B[a]Q[a], I[a])$, the principal left (right, two-sided, bi-ideal, quasi-ideal, interior ideal) of a semigroup S generated by $a \in S$, that is,

$$\begin{aligned} L[a] &= \{a\} \cup Sa, \\ R[a] &= \{a\} \cup aS, \\ J[a] &= \{a\} \cup Sa \cup aS \cup SaS \\ Q[a] &= \{a\} \cup (aS \cap Sa) \\ I[a] &= \{a\} \cup \{a^2\} \cup SaS. \end{aligned}$$

A semigroup S is called *regular* if for every element a of S, there exists an element x in S such that $a = axa$ or equivalently $a \in aSa$. An element a of S is called a *completely regular* if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup S is called *completely regular* if every element of S is completely regular. A semigroup S is called *left (right) regular* if for each element a of S, there exists an element $x \in S$ such that $a = xa^2$ $(a = a^2x)$. A semigroup is called *left (right) regular* if for each element a of S, there exists an element $x \in S$ such that

$$a = xa^2 \ (a = a^2x).$$

A *semilattice* is a structure $S = (S, .)$, where "." is an infix binary operation, called the *semilattice operation*, such that "." is associative, commutative and idempotent. For all undefined concepts and notions about semigroups, see [22, 33].

Definition 1 [15, 31] *A soft set* f_A *over* U *is a set defined by*

$$f_A : E \to P(U) \text{ such that} f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A *is also called an approximate function. A soft set over* U *can be represented by the set of ordered pairs*

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

Definition 2 [15] *Let* $f_A, f_B \in S(U)$*. Then,* f_A *is called a soft subset of* f_B *and denoted by* $f_A \tilde{\subseteq} f_B$*, if* $f_A(x) \subseteq f_B(x)$ *for all* $x \in E$.

Definition 3 [15] *Let* $f_A, f_B \in S(U)$*. Union of* f_A *and* f_B*, denoted by* $f_A \tilde{\cup} f_B$*, is defined as* $f_A \tilde{\cup} f_B = f_{A\tilde{\cup}B}$*, where* $f_{A\tilde{\cup}B}(x) = f_A(x) \cup f_B(x)$ *for all* $x \in E$*. Intersection of* f_A *and* f_B*, denoted by* $f_A \tilde{\cap} f_B$*, is defined as* $f_A \tilde{\cap} f_B = f_{A\tilde{\cap}B}$*, where* $f_{A\tilde{\cap}B}(x) = f_A(x) \cap f_B(x)$ *for all* $x \in E$*.*

Definition 4 [39] *Let* S *be a semigroup and* f_S *and* g_S *be soft sets over the common universe* U*. Then, soft intersection product* $f_S \circ g_S$ *is defined by*

$$(f_S \circ g_S)(x) = \begin{cases} \bigcup_{x=yz}\{f_S(y) \cap g_S(z)\}, & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $x \in S$*.*

Definition 5 [39] *Let* X *be a subset of* S*. We denote by* $\mathcal{S}_X$ *the soft characteristic function of* X *and define as*

$$\mathcal{S}_X(x) = \begin{cases} U, & \text{if } x \in X, \\ \emptyset, & \text{if } x \notin X \end{cases}$$

Definition 6 [39] *Let* S *be a semigroup and* f_S *be a soft set over* U*. Then,* f_S *is called a soft intersection semigroup of* S*, if*

$$f_S(xy) \supseteq f_S(x) \cap f_S(y)$$

for all $x, y \in S$*.*

Definition 7 [39] *A soft set over* U *is called a soft intersection left (right) ideal of* S *over* U *if*

$$f_S(ab) \supseteq f_S(b) \ (f_S(ab) \supseteq f_S(a))$$

for all $a, b \in S$*. A soft set over* U *is called a soft intersection two-sided ideal (soft intersection ideal) of* S *if it is both soft intersection left and soft intersection right ideal of* S *over* U*.*

Definition 8 [39] *A soft intersection semigroup* f_S *over* U *is called a soft intersection bi-ideal of* S *over* U *if*

$$f_S(xyz) \supseteq f_S(x) \cap f_S(z)$$

for all $x, y, z \in S$*.*

Definition 9 [40] *A soft set over* U *is called a soft intersection interior of* S *over* U *if* $f_S(xyz) \supseteq f_S(y)$, *soft intersection generalized bi-ideal of* S *over* U *if* $f_S(xyz) \supseteq f_S(x) \cap f_S(z)$ *for all* $x, y, z \in S$.

For the sake of brevity, soft intersection semigroup, soft intersection right (left, two-sided, interior, generalized bi-) ideal are abbreviated by SI-semigroup, SI-right (left, two-sided, quasi, generalized bi-) ideal, respectively.

It is easy to see that if $f_S(x) = U$ for all $x \in S$, then f_S is an SI-semigroup (right ideal, left ideal, ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of S over U. We denote such a kind of SI-semigroup (right ideal, left ideal, ideal, bi-ideal) by $\widetilde{\mathbb{S}}$ [39].

Definition 10 [40] *A soft set over* U *is called a soft intersection quasi-ideal of* S *over* U *if*

$$(f_S \circ \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \circ f_S) \widetilde{\subseteq} f_S.$$

Definition 11 [39] *A soft set* f_S *over* U *is called soft semiprime if for all* $a \in S$,

$$f_S(a) \supseteq f_S(^2).$$

Theorem 1 [39, 40] *Let* X *be a nonempty subset of a semigroup* S. *Then,* X *is a subsemigroup (left, right, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of* S *if and only if* $\mathcal{S}_X$ *is an* SI*-semigroup (left, right, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of* S.

Proposition 1 [39, 40] *Let* f_S *be a soft set over* U. *Then,*

i) f_S *is an* SI*-semigroup over* U *if and only if* $f_S \circ f_S \widetilde{\subseteq} f_S$.

ii) f_S *is an* SI*-left (right) ideal of* S *over* U *if and only if* $\widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$ $(f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S)$.

iii) f_S *is an* SI*-bi-ideal of* S *over* U *if and only if* $f_S \circ f_S \widetilde{\subseteq} f_S$ *and* $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$.

iv) f_S *is an* SI*-interior ideal of* S *over* U *if and only if* $\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S$.

v) f_S *is an* SI*-generalized bi-ideal of* S *over* U *if and only if* $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$.

Theorem 2 [39] *Every* SI*-left (right, two sided) ideal of a semigroup* S *over* U *is an* SI*-bi-ideal of* S *over* U.

Proposition 2 [40] *For a semigroup* S, *the following conditions are equivalent:*

1) *Every* SI*-ideal of a semigroup* S *over* U *is an* SI*-interior ideal of* S *over* U*.*

2) *Every* SI*-quasi ideal of* S *is an* SI*-semigroup of* S*.*

3) *Every one-sided* SI*-ideal of* S *is an* SI*-quasi-ideal of* S*.*

4) *Every* SI*-quasi-ideal of* S *is an* SI*-bi-ideal of* S*.*

Theorem 3 [39] *For a semigroup* S *the following conditions are equivalent:*

1) S *is regular.*

2) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI*-right ideal* f_S *of* S *over* U *and* SI*-left ideal* g_S *of* S *over* U*.*

Theorem 4 [39] *For a semigroup* S *the following conditions are equivalent:*

1) S *is regular.*

2) *For every* SI*-quasi-ideal of* S*,* $f_S = f_S \circ \widetilde{\mathbb{S}} \circ f_S$*.*

Theorem 5 [40] *Let* f_S *be a soft set over* U*, where* S *is a regular semigroup. Then, the following conditions are equivalent:*

1) f_S *is an* SI*-ideal of* S *over* U*.*

2) f_S *is an* SI*-interior ideal of* S *over* U*.*

Theorem 6 [39] *For a left regular semigroup* S*, the following conditions are equivalent:*

1) *Every left ideal of* S *is a two-sided ideal of* S*.*

2) *Every* SI*-left ideal of* S *is an* SI*-ideal of* S*.*

For more on soft intersection semigroups and ideals, we refer [39, 40].

3 Semisimple semigroups

In this section, semisimple semigroups with respect to SI-ideals of semigroups are characterized. A semigroup S is called *semisimple* if $J^2 = J$ holds for every ideal J of S, that is, every ideal of S is idempotent.

Proposition 3 [41] *For a semigroup S, the following conditions are equivalent:*

1) S *is semisimple.*

2) $a \in (SaS)(SaS)$ *for every element* a *of* S*, that is, there exist elements* $x, y, z \in S$ *such that* $a = xayaz$.

Proposition 4 *Every* SI*-interior ideal of a semisimple semigroup* S *is an* SI*-ideal of* S.

Proof. Let f_S be an SI-interior ideal of S. Let a and b be any elements of S. Then, since S is semisimple, there exist elements $x, y, z \in S$ such that

$$a = xayaz.$$

Thus,

$$f_S(ab) = f_S((xayaz)b) = f_S(xay)a(zb)) \supseteq f_S(a)$$

Hence, f_S is an SI-right ideal of S. Similarly, one can prove that f_S is an SI-left ideal of S. Thus, f_S is an SI-ideal of S. □

Now a characterization of a semisimple semigroup by SI-ideals is given.

Theorem 7 *For a semigroup S, the following conditions are equivalent:*

1) S *is semisimple.*

2) $f_S \circ f_S = f_S$ *for every* SI*-ideal* f_S *of* S*. (That is, every* SI*-ideal is idempotent).*

3) $f_S \circ f_S = f_S$ *for every* SI*-interior* f_S *of* S*. (That is, every* SI*-interior ideal is idempotent).*

4) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-ideals* f_S *and* g_S *of* S.

5) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-ideal* f_S *and every* SI*-interior ideal* g_S *of* S.

6) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-interior ideal* f_S *and every* SI*-ideal* g_S *of* S.

7) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-interior ideals* f_S *and* g_S *of* S.

8) *The set of all* SI*-ideals of a semisimple semigroup* S *is a semilattice under the soft intersection product, that is,* $f_S \circ (g_S \circ h_S) = f_S \circ (g_S \circ h_S)$, $f_S \circ g_S = g_S \circ f_S$ *and* $f_S \circ f_S = f_S$ *for all* SI*-ideals* f_S *and* g_S *of* S.

9) *The set of all* SI*-interior ideals of a semisimple semigroup* S *is a semilattice under the soft intersection product.*

Proof. First assume that (1) holds. Let f_S and g_S be any SI-interior ideals of S. Since, $\widetilde{\mathbb{S}}$ itself is an SI-interior ideal of S and since f_S is an SI-ideal of S by Proposition 4:

$$f_S \circ g_S \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S \quad \text{and} \quad f_S \circ g_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ g_S \widetilde{\subseteq} g_S.$$

Thus, $f_S \circ g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$.

Now, let a be any element of S. Since there exist elements $x, y, z, w \in S$ such that

$$a = (xay)(zaw),$$

$$(f_S \circ g_S)(a) \neq \emptyset.$$

And since f_S and g_S are SI-interior ideals of S,

$$\begin{aligned} (f_S \circ g_S)(a) &= \bigcup_{a=pq} (f_S(p) \cap g_S(q)) \\ &\supseteq f_S(xay) \cap g_S(zaw) \\ &\supseteq f_S(a) \cap g_S(a) \\ &= (f_S \widetilde{\cap} g_S)(a) \end{aligned}$$

and so $f_S \circ g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. Hence,

$$f_S \circ g_S = f_S \widetilde{\cap} g_S.$$

So, (1) implies (7). (7) implies (6), (6) implies (4), (7) implies (5), (5) implies (4), (4) implies (2), (7) implies (3), (3) implies (2) and (7) implies (9), (9) implies (8), (8) implies (2).

Assume that (2) holds. Let a be any element of S. Since the soft characteristic function $\mathcal{S}_{J[a]}$ of the principal ideal $J[a]$ of S is an SI-ideal of S,

$$\mathcal{S}_{J[a]J[a]}(a) = (\mathcal{S}_{J[a]} \circ \mathcal{S}_{J[a]})(a) = \mathcal{S}_{J[a]}(a) = U$$

and so,

$$\begin{gathered} a \in J[a]J[a] = (\{a\} \cup aS \cup Sa \cup SaS)(\{a\} \cup aS \cup Sa \cup SaS) = \\ \{a^2\} \cup a^2S \cup aSa \cup aSaS \cup aSa \cup aSaS \cup aSSa \cup aSSaS \cup Sa^2 \cup Sa^2S \cup \\ SaSa \cup SaSaS \cup SaSa \cup SaSaS \cup SaSSa \cup SaSSaS \subseteq (SaS)(SaS) \end{gathered}$$

Hence, S is semisimple and so, (2) implies (1). □

4 Regular duo semigroups

In this section, a left (right) duo semigroup in terms of SI-ideals is charachterized. A semigroup S is called *left (right) duo* if every left (right) ideal of S is a two-sided ideal of S. A semigroup S is *duo* if it is both left and right duo.

Definition 12 *A semigroup S is called soft left (right) duo if every SI-left (right) ideal of S is an SI-ideal of S and is called soft duo, if it is both soft left and soft right duo.*

Theorem 8 *For a regular semigroup S, the following conditions are equivalent:*

1) S *is left (right) duo.*

2) S *is soft left (right) duo.*

Proof. First assume that S is left duo. Let f_S be any SI-left ideal of S and a and b be any elements of S. It is known that Sa is a left-ideal of S. And so, by hypothesis, it is a two-sided ideal of S. Since S is regular,

$$ab \in (aSa)b \subseteq (Sa)S \subseteq Sa$$

This implies that there exists an element $x \in S$ such that

$$ab = xa.$$

Thus, since f_S is an SI-left ideal of S,

$$f_S(ab) = f_S(xa) \supseteq f_S(a)$$

This means that f_S is an SI-right ideal of S and so f_S is an SI-ideal of S. Thus, S is soft left duo and (1) implies (2).

Conversely, assume that S is soft left duo. Let A be any left ideal of S. Then, the soft characteristic function $\mathcal{S}_A$ of A is an SI-left ideal of S. By assumption, $\mathcal{S}_A$ is an SI-ideal of S and so A is a two-sided ideal of S. Thus, S is left duo and (2) implies (1). The right dual of the proof can be seen similarly. So, the proof is completed. □

Theorem 9 *For a regular semigroup S, the following conditions are equivalent:*

1) S *is duo.*

2) *S is soft duo.*

Every SI-right (left) ideal of S is an SI-bi-ideal of S ([39]). Moreover, we have the following:

Theorem 10 *Let S be a regular duo semigroup. Then, every SI-bi-ideal of S is an SI-ideal of S.*

Proof. Let f_S be any SI-bi-ideal of S and a, b be any elements of S. It is known that Sa is a left ideal of S. Since S is a duo semigroup, Sa is a right ideal of S. And since S is regular,

$$ab \in (aSa)b \subseteq a((Sa)S) \subseteq aSa$$

This implies that there exists an element $x \in S$ such that

$$ab = axa.$$

Then, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(axa) \supseteq f_S(a) \cap f_S(a) = f_S(a).$$

This means that f_S is an SI-right ideal of S. It can be seen in a similar way that f_S is an SI-left ideal of S. Therefore, f_S is an SI-ideal of S. This completes the proof. □

Theorem 11 [17, 32] *For a semigroup S, the following conditions are equivalent:*

1) *S is a regular duo semigroup.*

2) *$A \cap B = AB$ for every left ideal A and every right ideal B of S.*

3) *$Q^2 = Q$ for every quasi-ideal of S. (That is, every quasi-ideal is idempotent.)*

4) *$EQE = E \cap Q \cap E$ for every ideal E and every quasi-ideal Q of S.*

Theorem 12 *For a semigroup S, the following conditions are equivalent:*

1) *S is a regular duo semigroup.*

2) *S is a regular soft duo semigroup.*

3) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*bi-ideals* f_S *and* g_S *of* S.

4) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*bi-ideal* f_S *and for all* SI-*quasi-ideal* g_S *of* S.

5) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*bi-ideal* f_S *and and for all* SI-*right ideal* g_S *of* S

6) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*quasi-ideal* f_S *and for all* SI-*bi-ideal* g_S *of* S.

7) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*quasi-ideals* f_S *and* g_S *of* S.

8) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*quasi-ideal* f_S *and for all* SI-*right ideal* g_S *of* S.

9) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*left ideal* f_S *and for all* SI-*bi-ideal* g_S *of* S.

10) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for all* SI-*left ideal* f_S *and for all* SI-*right ideal* g_S *of* S.

11) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *and* $h_S \circ k_S = h_S \tilde{\cap} k_S$ *for all* SI-*right ideals* f_S *and* g_S *of* S *and for all* SI-*left ideal* h_S *and* k_S *of* S.

12) *Every* SI-*quasi-ideal of* S *is idempotent.*

Proof. The equivalence of (1) and (2) follows from Theorem 9. Assume that (2) holds. Let f_S and g_S be any SI-bi-ideals of S. Then, by Theorem 10, f_S is an SI-right ideal of S and g_S is an SI-left ideal of S. Since S is regular, it follows by Theorem 3 that

$$f_S \circ g_S = f_S \tilde{\cap} g_S$$

Thus, (2) implies (3). It is clear that (3) implies (4), (4) implies (5), (5) implies (8), (8) implies (11), (11) implies (3), (3) implies (6), (6) implies (7), (7) implies (8) and (6) implies (9), (9) implies (10), (10) implies (11).

Assume that (11) holds. Let A and B be any left ideal and right ideal of S, respectively. Let a be any element of $A \cap B$. Then, $a \in A$ and $b \in B$ and so,

$$\mathcal{S}_A(a) = \mathcal{S}_B(a) = U.$$

Since $\mathcal{S}_A$ and $\mathcal{S}_B$ is an SI-left ideal and SI-right ideal of S, respectively, by assumption

$$\mathcal{S}_{AB}(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = (\mathcal{S}_A \tilde{\cap} \mathcal{S}_B)(a) = \mathcal{S}_A(a) \cap \mathcal{S}_B(a) = U,$$

so $a \in AB$. Thus, $A \cap B \subseteq AB$. For the converse inclusion, let a be any element of AB. Thus,

$$\mathcal{S}_{A \cap B}(a) = (\mathcal{S}_A \tilde{\cap} \mathcal{S}_B)(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = \mathcal{S}_{AB}(a) = U$$

This implies that $a \in A \cap B$ and that $AB \subseteq A \cap B$. Thus, $AB = A \cap B$. It follows by Theorem 11 that S is a regular duo semigroup. Thus (11) implies (1). It is clear that (7) implies (12) by taking $g_S = f_S$.

Conversely, assume that (12) holds. Let Q be any quasi-ideal of S and a be any element of Q. Then, $\mathcal{S}_Q$ is an SI-quasi-ideal of S. Then,

$$\mathcal{S}_{Q^2}(a) = (\mathcal{S}_Q \circ \mathcal{S}_Q)(a) = \mathcal{S}_Q(a) = U$$

Thus, $a \in Q^2$ and $Q \subseteq Q^2$. Since the converse inclusion always holds, $Q = Q^2$. It follows by Theorem 11 that S is a regular duo semigroup and that (12) implies (1). This completes the proof. □

Theorem 13 *For a semigroup S, the following conditions are equivalent:*

1) S *is a regular duo semigroup.*

2) $f_S \circ g_S \circ f_S = f_S \widetilde{\cap} g_S$ *for every* SI*-ideal* f_S *and every* SI*-bi-ideal* g_S *of* S.

3) $f_S \circ g_S \circ f_S = f_S \widetilde{\cap} g_S$ *for every* SI*-ideal* f_S *and every* SI*-quasi-ideal* g_S *of* S.

Proof. First assume that (1) holds. Let f_S and g_S be any SI-bi-ideal and any SI-ideal of S, respectively. Then,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} (f_S \circ \widetilde{\mathbb{S}}) \circ \widetilde{\mathbb{S}} = f_S \circ (\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}) \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S$$

On the other hand, since S is regular and duo, f_S is an SI-ideal of S by Theorem 10. Hence,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} (\widetilde{\mathbb{S}} \circ g_S) \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} g_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} g_S$$

and so

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$$

In order to show the converse inclusion, let a be any element of S. Then, since S is regular, there exists an element x in S such that

$$a = axa = (axa)xa$$

Thus,

$$\begin{aligned} (f_S \circ g_S \circ f_S)(a) &= [f_S \circ (g_S \circ f_S)](a) \\ &= \bigcup_{a=pq} [f_S(a) \cap (g_S \circ f_S)(q)] \end{aligned}$$

$$
\begin{aligned}
&\supseteq f_S(ax) \cap (g_S \circ f_S)(axa) \\
&= f_S(ax) \cap \{ \bigcup_{axa=bc} [g_S(b) \circ f_S(c)]\} \\
&\supseteq f_S(ax) \cap (g_S(a) \cap f_S(xa)) \\
&\supseteq f_S(a) \cap (g_S(a) \cap f_S(a)) \\
&= f_S(a) \cap g_S(a) \\
&= (f_S \tilde{\cap} g_S)(a)
\end{aligned}
$$

and so $f_S \circ g_S \circ f_S \tilde{\supseteq} f_S \tilde{\cap} g_S$ Thus,

$$f_S \circ g_S \circ f_S = f_S \tilde{\cap} g_S.$$

Hence, (1) implies (2). It is clear that (2) implies (3).

Assume that (3) holds. Let E and Q any two-sided ideal and quasi-ideal of S, respectively and a be any element of $E \cap Q$. Then,

$$\mathcal{S}_E(a) = \mathcal{S}_Q(a) = U.$$

Since $\mathcal{S}_E$ and $\mathcal{S}_Q$ is an SI-ideal and an SI-quasi-ideal of S, respectively,

$$\mathcal{S}_{EQE}(a) = (\mathcal{S}_E \circ \mathcal{S}_Q \circ \mathcal{S}_E)(a) = (\mathcal{S}_E \tilde{\cap} \mathcal{S}_Q)(a) = \mathcal{S}_E(a) \cap \mathcal{S}_Q(a) = U$$

and so $a \in EQE$. Thus, $E \cap Q \subseteq EQE$. For the converse inclusion, let a be any element of EQE. Thus,

$$\mathcal{S}_{E \cap Q}(a) = (\mathcal{S}_E \tilde{\cap} \mathcal{S}_Q)(a) = (\mathcal{S}_E \circ \mathcal{S}_Q \circ \mathcal{S}_E)(a) = \mathcal{S}_{EQE}(a) = U$$

and so $a \in EQE$. Thus, $EQE \subseteq E \cap Q$ and so $EQE = E \cap Q$. It follows from Proposition 11 that S is regular duo. Hence, (3) implies (1). This completes the proof. □

5 Right (left) zero semigroup

In this section, right (left) zero semigroups are charachterized in terms of SI-ideals of S. A semigroup S is called *right (left) zero* if $xy = y$ $(xy = x)$ for all $x, y \in S$.

Proposition 5 *For a semigroup* S*, the following conditions are equivalent:*

1) *The set of all idempotent elements of* S *forms a left (right) zero subsemigroup of* S*.*

2) *For every* SI-*left (right) ideal* f_S *of* S, $f_S(e) = f_S(f)$ *for all idempotent elements* e *and* f *of* S.

Proof. First assume that the set I_S of all idempotent elements of S is a left zero subsemigroup of S. Let $e, f \in I_S$ and f_S be an SI-left ideal of S. Then, since

$$ef = e \text{ and } fe = f$$

$$f_S(e) = f_S(ef) \supseteq f_S(f) = f_S(fe) \supseteq f_S(e)$$

and so

$$f_S(e) = f_S(f).$$

Thus, (1) implies (2).

Conversely, assume that (2) holds. Since S is regular, it is obvious that $I_S \neq \emptyset$. Moreover, the soft characteristic function $\mathcal{S}_{L[f]}$ of the left ideal $L[f]$ of S is an SI-left ideal of S. Thus, by assumptio,n

$$\mathcal{S}_{L[f]}(e) = \mathcal{S}_{L[f]}(f) = U$$

and so $e \in L[f] = Sf$. (Here note that, if S is a regular semigroup, $L[a] = Sa$ for every $a \in S$ ([17]). Thus, for some $x \in S$,

$$e = xf = x(ff) = (xf)f = ef$$

This means that I_S is a left zero semigroup. Thus (2) implies (1). The case when S is right zero, the proof can be seen similarly. This completes the proof. □

Corollary 1 *For an idempotent semigroup* S, *the following conditions are equivalent:*

1) S *is left (right) zero.*

2) *For every* SI-*left (right) ideal* f_S *of* S, $f_S(e) = f_S(f)$ *for all elements* $e, f \in S$.

Proposition 6 *Let* S *be a group. Then, every* SI-*bi-ideal of* S *is a constant function.*

Proof. Let S be a group with identity e and f_S be any SI-bi-ideal of S and a be any element of S. Then,

$$f_S(a) = f_S(eae) \supseteq f_S(e) \cap f_S(e) = f_S(e) = f_S(ee) = f_S((aa^{-1})(a^{-1}a)) = f_S(a(a^{-1}a^{-1})a) \supseteq f_S(a) \cap f_S(a) = f_S(a)$$

and so $f_S(e) = f_S(a)$. This implies that f_S is a constant function. □

Proposition 7 *For a regular semigroup* S*, the following conditions are equivalent:*

1) S *is a group.*

2) *For every* SI*-bi-ideal* f_S *of* S*,* $f_S(e) = f_S(f)$ *for all idempotent elements* $e, f \in S$.

Proof. Assume that (1) holds. Let f_S be any SI-bi-ideal of S. Then, it follows from Proposition 6 that f_S is a constant function. This implies that

$$f_S(e) = f_S(f)$$

for all idempotent elements $e, f \in S$. Thus (1) implies (2).

Conversely, assume that (2) holds. Let e and f be any idempotent elements of S. As is well-known, if S is a regular semigroup, B[x], the principal ideal of S generated by $x \in S$ is $B[x] = xSx$ ([17]). Moreover, since the soft characteristic function $\mathcal{S}_{B[f]}$ of the bi-ideal $B[f]$ of S is an SI-bi-ideal of S and since $f \in B[f]$,

$$\mathcal{S}_{B[f]}(e) = \mathcal{S}_{B[f]}(f) = U$$

and so $e \in B[f] = fsf$, which means that $e = fxf$ for some $x \in S$. One can similarly obtain that $f = eye$ for some $y \in S$. Thus,

$$e = fxf = fx(ff) = (fxf)f = ef = e(eye) = (ee)ye = eye = f$$

Since S is regular, $I_S \neq \emptyset$ and S contains exactly one idempotent. Thus, it follows from ([17], p.33) that S is a group. Thus (2) implies (1). This completes the proof. □

6 Right (left) simple semigroups

In this section, soft simple semigroup is defined and the relation of soft simple semigroup with simple semigroup is given. A semigroup S is called *left (right) simple* if it contains no proper left (right) ideal of S and is called *simple* if it contains no proper ideal.

Definition 13 *A semigroup* S *is called soft left (right) simple if every* SI*-left (right) ideal of* S *is a constant function and is called soft simple if every* SI*-ideal of* S *is a constant function.*

Theorem 14 *For a semigroup S, the following conditions are equivalent:*

1) S *is left (right) simple.*

2) S *is soft left (right) simple.*

Proof. First assume that S is left simple. Let f_S be any SI-left ideal of S and a and b be any element of S. Then, it follows from ([17], p. 6) that there exist elements $x, y \in S$ such that $b = xa$ and $a = yb$. Hence, since S is an SI-left ideal of S,

$$f_S(a) = f_S(yb) \supseteq f_S(b) = f_S(xa) \supseteq f_S(a)$$

and so $f_S(a) = f_S(b)$. Since a and b be any elements of S, this means that f_S is a constant function. Thus, it is obtained that S is soft left simple and (1) implies (2).

Conversely, assume that (2) holds. Let A be any left ideal of S. Then, $\mathcal{S}_A$ is an SI-left ideal of S. By assumption, $\mathcal{S}_A$ is a constant function. Let x be any element of S. Then, since $A \neq \emptyset$,

$$\mathcal{S}_A(x) = U$$

and so $x \in A$. This implies that $S \subseteq A$, and so $S = A$. Hence, S is left simple and (2) implies (1). In the case, when S is soft right simple, the proof follows similarly. □

Theorem 15 *For a semigroup S, the following conditions are equivalent:*

1) S *is simple.*

2) S *is soft simple.*

As is well-known, a semigroup S is a group if it is left and right simple. From this, the following theorem:

Proposition 8 *For a semigroup S, the following conditions are equivalent:*

1) S *is a group.*

2) S *is both soft left and soft right simple.*

Proposition 9 *Let S be a left simple semigroup. Then, every SI-bi-ideal of S is an SI-right ideal of S.*

Proof. Let f_S be an SI-bi-ideal of S and a and b be any elements of S. Then, since S is left simple, there exists an element x in S such that

$$b = xa.$$

Then, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(a(xa)) = f_S(a) \cap f_S(a) = f_S(a)$$

which means that f_S is an SI-right ideal of S. This completes the proof. □

7 Semilattices of left (right) simple semigroups

In this section, a semigroup that is a semilattice of left (right) simple semigroups is characterized by SI-ideals. A semigroup S is a *semilattice of left simple semigroups* if it is the set-theoretical union of the family of left simple semigroups S_i $(i \in M)$ such that,

$$S = \bigcup_{i \in M} S_i$$

such that the products S_iS_j and S_jS_i are both contained in the same S_k $(k \in M)$.

Theorem 16 [17, 34] *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left simple semigroups.*

2) *S is left regular and every left ideal of S is two-sided.*

3) *S is left regular and $AB = BA$ for any left ideals A and B of S.*

Theorem 17 [39] *For a left regular semigroup S, the following conditions are equivalent:*

1) *Every left ideal of S is a two-sided ideal of S.*

2) *Every SI-left ideal of S is an SI-ideal of S.*

Theorem 18 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left simple semigroups.*

2) *S is left regular and every* SI*-left ideal of* S *is an* SI*-ideal of* S*.*

3) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for every* SI*-left ideals of* S*.*

4) *The set of all* SI*-left ideals of* S *is a semilattice under the soft int-product.*

5) *The set of all left ideals of* S *is a semilattice under the multiplication of subsets.*

Proof. The equivalence of (1) and (2) follows from Theorem 16 and Theorem 17. Assume that (2) holds. Let f_S and g_S be any SI-left ideals of S and a be any element of S. Then, since S is left regular, there exists an element $x \in S$ such that $a = xa^2$. By assumption, f_S is also an SI-right ideal of S. So,

$$\begin{aligned}(f_S \circ g_S)(a) &= \bigcup_{a=yz} (f_S(y) \cap g_S(z)) \\ &\supseteq (f_S(xa) \cap g_S(a)) \\ &\supseteq (f_S(a) \cap g_S(a)) \\ &= (f_S \tilde{\cap} g_S)(a)\end{aligned}$$

Thus, $f_S \circ g_S \tilde{\supseteq} f_S \tilde{\cap} g_S$. On the other hand, by assumption, g_S is SI-right ideal of S, and so

$$\begin{aligned}(f_S \circ g_S)(a) &= \bigcup_{a=yz} (f_S(y) \cap g_S(z)) \\ &\subseteq (f_S(yz) \cap g_S(yz)) \\ &= f_S(a) \cap g_S(a) \\ &= (f_S \tilde{\cap} g_S)(a)\end{aligned}$$

Thus, $f_S \circ g_S \tilde{\subseteq} f_S \tilde{\cap} g_S$. Thus, $f_S \circ g_S = f_S \tilde{\cap} g_S$ and so (2) implies (3).

(3) implies (4) is clear. Assume that (4) holds. Let A and B be any left ideals of S and a be any element of BA. Since the soft characteristic function $\mathcal{S}_A$ and $\mathcal{S}_B$ are SI-left ideals of S,

$$\mathcal{S}_{AB}(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = (\mathcal{S}_B \circ \mathcal{S}_A)(a) = \mathcal{S}_{BA}(a) = U$$

which implies that $a \in AB$. Thus, $BA \subseteq AB$. Similarly, $AB \subseteq BA$. Thus, $AB = BA$.

In order to see that any left ideal A of S is idempotent, let a be any element of A. Since $\mathcal{S}_A$ is an SI-left ideal of S,

$$\mathcal{S}_{A^2}(a) = (\mathcal{S}_A \circ \mathcal{S}_A)(a) = \mathcal{S}_A = U$$

and so $a \in A^2$. Thus, $A \subseteq A^2$ and so $A = A^2$. Therefore (4) implies (5).

Finally, assume that (5) holds. Let A be any left ideal of S and a be any element of S. Then, since S itself is a left ideal, by assumption

$$AS = SA \subseteq A$$

Thus, A is a right ideal of S, and so A is a two-sided ideal of S.

Let a be any element of S. Then, since the left ideal $L[a]$ of S is idempotent by assumption and since $a \in L[a]$,

$$a \in L[a]L[a] = (\{a\} \cup Sa)(\{a\} \cup Sa) = \{a^2\} \cup aSa \cup Sa^2 \cup SaSa \subseteq \\ \{a^2\} \cup (aS)aSa \cup Sa^2 \cup SaSa \subseteq \{a^2\} \cup SaSa \cup Sa^2 \subseteq \{a^2\} \cup Sa^2$$

which implies that S is left-regular. Thus, it follows by Theorem 16-(2) that S is a semilattice of left simple groups. That is to say (5) implies (1). This completes the proof. □

The left-right dual of Theorem 18 reads as follows:

Theorem 19 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of right simple semigroups.*

2) S *is right regular and every* SI*-right ideal of* S *is an* SI*-ideal of* S.

3) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for every* SI*-right ideals of* S.

4) *The set of all* SI*-right ideals of* S *is a semilattice under the soft int-product.*

5) *The set of all right ideals of* S *is a semilattice under the multiplication of subsets.*

Theorem 20 [39] *For a semigroup S, the following conditions are equivalent:*

1) S *is left regular.*

2) *For every* SI*-left ideal* f_S *of* S, $f_S(a) = f_S(a^2)$ *for all* $a \in S$.

Theorem 21 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of left simple semigroups.*

2) *For every* SI*-left ideal* f_S *of* S, $f_S(a) = f_S(a^2)$ *and* $f_S(ab) = f_S(ba)$ *for all* $a, b \in S$.

Proof. Assume that S is a semilattice of left simple semigroups. Let f_S be any SI-left ideal of S. Then, by Theorem 16-(2), S is left regular and f_S is an SI-ideal of S. Let a be any element of S. Thus, by Theorem 20,

$$f_S(ab) = f_S((ab)^2) = f_S(a(ba)b) \supseteq f_S(ba).$$

Similarly, $f_S(ba) \supseteq f_S(ab)$. Hence,

$$f_S(ab) = f_S(ba).$$

Thus, (1) implies (2).

Conversely, assume that (2) holds. Let f_S be any SI-ideal of S. Since $f_S(a) = f_S(a^2)$ for all $a \in S$, it follows from Theorem 20 that S is left regular. Let A and B be any left ideal of S and ab be any element of AB. Since the soft characteristic function $\mathcal{S}_{L[ba]}$ of the the left ideal $L[ba]$ is an SI-left ideal of S and since $ba \in L[ba]$,

$$\mathcal{S}_{L[ba]}(ab) = \mathcal{S}_{L[ba]}(ba) = U$$

This implies that

$$ab \in L[ba] = \{ba\} \cup Sba \subseteq BA \cup SBA \subseteq BA$$

and so $AB \subseteq BA$. Similarly, it can be seen that the converse inclusion holds. Thus,

$$AB = BA$$

Then, it follows by Theorem 16-(3) that S is a semilattice of left simple semigroups. Therefore (3) implies (1). This completes the proof. □

The right dual of Theorem 21 is as following:

Theorem 22 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of right simple semigroups.*

2) *For every SI-right ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

8 A semilattice of left (right) groups

In this section, a semigroup that is a semilattice of left (right) simple groups is characterized by SI-ideals. An element a of S is said to be *left (right) cancellable* if, for any $x, y \in S$ $ax = ay$ $(xa = ya)$ implies $x = y$. A semigroup S is called *left (right) cancellative* if every element of S is left (right) cancellative. A semigroup S is called a *left group* if it is left simple and right cancellable ([17]), that is, for all $a \in S$, there exists a unique element $x \in S$ such that $xa^2 = a$ ([33]). Dually, a semigroup S is called a *right group* if it is right simple and left cancellable.

Theorem 23 [33] *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left groups.*

2) *S is regular and $aS \subseteq Sa$ for every $a \in S$.*

Theorem 24 *Let S be a semigroup that is a semilattice of left groups. Then, every SI-(generalized) bi-ideal of S is an SI-right ideal of S.*

Proof. Let f_S be any SI-bi-ideal of S, and a and b any elements of S. Then, it follows from Theorem 23 that there exist elements $x, y \in S$ such that

$$a = axa \text{ and } ab = ya.$$

Thus,

$$ab = (axa)b = (ax)(ab) = (ax)(ya) = a(xy)a.$$

Since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(a(xy)a) \supseteq f_S(a) \cap f_S(a) = f_S(a).$$

Hence, f_S is an SI-right ideal of S. □

Corollary 2 *Let S be a semigroup that is a semilattice of left groups. Then, every SI-left ideal of S is an SI-right ideal of S, that is to say, S is soft left duo.*

Theorem 25 *Let S be a semigroup that is a semilattice of left groups. Then, every SI-interior ideal of S is an SI-left ideal of S.*

Proof. Let f_S be any SI-interior ideal of S, and a and b any elements of S. Then, it follows from Theorem 23 that there exist element $z \in S$ such that

$$b = bzb.$$

Thus,

$$ab = (axa)b = (ax)(ab) = (ax)(ya) = a(xy)a.$$

Since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(a(bzb)) = f_S((a)b(zb)) \supseteq f_S(b).$$

Hence, f_S is an SI-left ideal of S. □

Theorem 26 [40] *For a semigroup* S *the following conditions are equivalent:*

1) S *is regular.*

2) $f_S \tilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI*-quasi-ideal* f_S *of* S *and* SI*-ideal* g_S *of* S *over* U*.*

Theorem 27 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of left groups.*

2) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *and* SI*-left ideal* g_S *of* S*.*

3) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *and* SI*-ideal* g_S *of* S*.*

4) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *and* SI*-interior ideal* g_S *of* S*.*

5) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-left ideal* g_S *of* S*.*

6) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-ideal* g_S *of* S*.*

7) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-interior ideal* g_S *of* S*.*

8) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-left ideal* g_S *of* S*.*

9) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *and* SI*-left ideal* g_S *of* S*.*

10) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *and* SI*-ideal* g_S *of* S*.*

11) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *and* SI*-interior ideal* g_S *of* S*.*

12) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-one-sided ideal* f_S *and* SI*-ideal* g_S *of* S*.*

13) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-one-sided ideal* f_S *and* SI*-interior ideal* g_S *of* S*.*

14) *is regular left duo.*

Proof. First assume that (1) holds. Let f_S and g_S be any SI-generalize bi-ideal of S and SI-interior ideal of S, respectively and a be any element of S. Then, since S is regular by Theorem 23, there exists an element $x \in S$ such that

$$a = axa (= axaxa).$$

Since g_S is an SI-interior ideal of S, $g_S((x)a(xa)) \supseteq g_S(a)$. Thus,

$$\begin{aligned}(f_S \circ g_S)(a) &= \bigcup_{a=pq} (f_S(p) \cap g_S(q)) \\ &\supseteq f_S(a) \cap g_S((x)a(xa)) \\ &\supseteq f_S(a) \cap g_S(a) \\ &= (f_S \tilde{\cap} g_S)(a)\end{aligned}$$

and so $f_S \circ g_S \tilde{\supseteq} f_S \tilde{\cap} g_S$. Moreover, it follows by Theorem 24 that f_S is an SI-right ideal of S. Thus,

$$\begin{aligned}(f_S \circ g_S)(a) &= \bigcup_{a=pq} (f_S(p) \cap g_S(q)) \\ &\subseteq \bigcup_{a=pq} (f_S(pq) \cap g_S(pq)) \\ &= \bigcup_{a=pq} (f_S(a) \cap g_S(a)) \\ &= f_S(a) \cap g_S(a) \\ &= (f_S \tilde{\cap} g_S)(a)\end{aligned}$$

and so $f_S \circ g_S \tilde{\subseteq} f_S \tilde{\cap} g_S$. Therefore, $f_S \circ g_S = f_S \tilde{\cap} g_S$ and that (1) implies (10). It is clear that (10) implies (9), (9) implies (8), (8) implies (5), (5) implies (2), (10) implies (7), (7) implies (6), (6) implies (5), (5) implies (2), (7) implies (4), (4) implies (3), (3) implies (2) and (4) implies (12), (12) implies (11).

Assume that (2) holds. Then, it follows by Theorem 26 that S is regular. Let Q be any quasi-ideal of S. Then, the soft characteristic function $\mathcal{S}_Q$ is an SI-quasi-ideal of S. Since $\tilde{\mathbb{S}}$ itself is an SI-left ideal of S and so by assumption,

$$\mathcal{S}_Q = \mathcal{S}_Q \tilde{\cap} \tilde{\mathbb{S}} = \mathcal{S}_Q \circ \tilde{\mathbb{S}}.$$

Thus, $\mathcal{S}_Q$ is an SI-right ideal of S, and so Q is a right ideal of S. Thus, any quasi-ideal of S is a right ideal of S. Let $a \in S$. Then, the quasi-ideal Sa is a right ideal of S. Since S is regular,

$$aS \subseteq (aSa)S = ((aS)a)S \subseteq (Sa)S \subseteq Sa.$$

Thus, $aS \subseteq Sa$ and since S is regular, S is a semilattice of left groups by Theorem 23. Thus, (2) implies (1).

Assume that (11) holds. Let f_S and g_S be any SI-right ideal and any SI-left ideal of S, respectively. Then, since $\widetilde{\mathbb{S}}$ itself is an SI-ideal of S and so by assumption,

$$g_S = g_S \widetilde{\cap} \widetilde{\mathbb{S}} = g_S \circ \widetilde{\mathbb{S}}$$

Thus, g_S is an SI-right ideal of S, that is, g_S is an SI-ideal of S. Thus, by assumption, $f_S \circ g_S = f_S \widetilde{\cap} g_S$ for every SI-right ideal f_S of S over U and SI-left ideal g_S of S over U. It follows by Theorem 3 that S is regular. As is proved in (2) implies (1), $aS \subseteq Sa$. Thus, S is a semilattice of left groups, so (11) implies (1).

Assume that (1) holds. Then, it follows by Theorem 23 that S is regular. Moreover, it follows by Corollary 2 that S is soft left duo and so by Theorem 8, S is left duo. Thus (1) implies (13).

Conversely assume that (13) holds. Then, it follows by Theorem 8 that S is left duo, that is, every left ideal of S is a right ideal of S. In order to prove that S is semilattice of left groups, by Theorem 23, it suffices to show that $aS \subseteq Sa$ for all $a \in S$. As is proved in (2) implies (1), $aS \subseteq Sa$. Thus, S is a semilattice of left groups, so (13) implies (1). This completes the proof. □

The left-right dual of Theorem 27 is as following:

Theorem 28 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of right groups.*

2) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

3) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*ideal* g_S *of* S.

4) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*interior ideal* g_S *of* S.

5) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

6) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*ideal* g_S *of* S.

7) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*interior ideal* g_S *of* S.

8) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

9) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

10) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*ideal* g_S *of* S.

11) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*interior ideal* g_S *of* S.

12) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*one-sided ideal* f_S *and* SI-*ideal* g_S *of* S.

13) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*one-sided ideal* f_S *and* SI-*interior ideal* g_S *of* S.

14) S *is regular right duo.*

Theorem 29 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of left groups.*

2) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*left ideal* g_S *of* S.

3) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI-*bi-ideal* f_S *and* SI-*left ideal* g_S *of* S.

4) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*left ideal* g_S *of* S.

Proof. First assume that (1) holds. Let f_S and g_S be any SI-generalized bi-ideal of S. Then,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$$

On the other hand, since the SI-left ideal g_S is an SI-bi-ideal of S,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} (\widetilde{\mathbb{S}} \circ g_S) \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} g_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} g_S$$

Therefore,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S.$$

Let a be any element of S. Then, it follows by Theorem 23 that there exist elements $x, y \in S$ such that $a = axa$ and $ax = ya$. Hence,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Thus,

$$(f_S \circ g_S \circ f_S)(a) \quad = \quad [(f_S \circ g_S) \circ f_S](a)$$

$$\begin{aligned}
&= \bigcup_{a=pq} [(f_S \circ g_S)(p) \circ f_S(q)] \\
&\supseteq (f_S \circ g_S)(ax) \cap f_S(a) \\
&= \{ \bigcup_{ax=pq} (f_S(p) \cap g_S(q)) \cap f_S(a) \\
&\supseteq (f_S(axa) \cap g_S(xya)) \cap f_S(a) \\
&\supseteq (f_S(a) \cap g_S(a)) \cap f_S(a) \\
&= (f_S \widetilde{\cap} g_S)(a)
\end{aligned}$$

and so, $f_S \circ g_S \circ f_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. Thus, $f_S \circ g_S \circ f_S = f_S \widetilde{\cap} g_S$ and (1) implies (4). It is clear that (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let f_S be any SI-quasi ideal of S. Then, $\widetilde{\mathbb{S}}$ is an SI-left ideal of S and so by assumption,

$$f_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S \circ \widetilde{\mathbb{S}} \circ f_S$$

Thus,it follows by Theorem 4 that S is regular. On the other hand, let g_S be any SI-left ideal of S. then, by assumption,

$$g_S = \widetilde{\mathbb{S}} \widetilde{\cap} g_S = \widetilde{\mathbb{S}} \circ g_S \circ \widetilde{\mathbb{S}}$$

Thus, g_S is an SI-interior ideal of S. Since S is regular, g_S is an SI-ideal of S by Theorem 5. Therefore, every SI-left ideal of S is an ideal of S. It follows by Theorem 6 that every SI-left ideal of S is an SI-ideal of S. Let $a \in S$. Since S is regular, the left ideal Sa is an ideal of S. Thus,

$$aS \subseteq (aSa)S \subseteq a((Sa)S) \subseteq a(Sa) = (aS)a \subseteq Sa.$$

Thus, $aS \subseteq Sa$ and since S is regular, S is a semilattice of left groups by Theorem 23. Thus (2) implies (1). □

The left-right dual of Theorem 29 is as following:

Theorem 30 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of right groups.*

2) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every SI-quasi-ideal f_S and SI-right ideal g_S of S.*

3) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every SI-bi-ideal f_S and SI-right ideal g_S of S.*

4) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every SI-generalized bi-ideal f_S and SI-right ideal g_S of S.*

Theorem 31 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of left groups.*

2) $f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ *for every* SI*-quasi-ideal* f_S *and* SI*-left ideal* g_S *of* S*.*

3) $f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-left ideal* g_S *of* S*.*

4) $f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *and* SI*-left ideal* g_S *of* S*.*

Proof. First assume that (1) holds. Let f_S and g_S be any SI-generalized bi-ideal and SI-left ideal of S, respectively. Then,

$$f_S \circ \tilde{\mathbb{S}} \circ g_S = f_S \circ (\tilde{\mathbb{S}} \circ g_S) \tilde{\subseteq} f_S \circ g_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ g_S \tilde{\subseteq} g_S.$$

Moreover, by Theorem 24 that f_S is an SI-right ideal of S. Thus,

$$f_S \circ \tilde{\mathbb{S}} \circ g_S = (f_S \circ \tilde{\mathbb{S}}) \circ g_S \tilde{\subseteq} f_S \circ g_S \tilde{\subseteq} f_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} f_S.$$

Thus, $f_S \circ \tilde{\mathbb{S}} \circ g_S \tilde{\subseteq} f_S \tilde{\cap} g_S$.

Let a be any element of S. Then, it follows by Theorem 23 that there exist elements $x, y \in S$ such that $a = axa$ and $ax = ya$. Hence,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Thus,

$$\begin{aligned}
(f_S \circ \tilde{\mathbb{S}} \circ g_S)(a) &= [(f_S \circ \tilde{\mathbb{S}}) \circ g_S](a) \\
&= [\bigcup_{a=pq} (f_S \circ \tilde{\mathbb{S}})(p)] \circ g_S(q) \\
&\supseteq (f_S \circ \tilde{\mathbb{S}})(ax) \cap g_S(a) \\
&= \{\bigcup_{ax=pq} (f_S(p) \cap \tilde{\mathbb{S}}(q))\} \cap g_S(a) \\
&\supseteq (f_S(axa) \cap \tilde{\mathbb{S}}(aya)) \cap g_S(a) \\
&= (f_S(a) \cap U) \cap g_S(a) \\
&\supseteq f_S(a) \cap g_S(a) \\
&= (f_S \tilde{\cap} g_S)(a)
\end{aligned}$$

and so, $f_S \circ \tilde{\mathbb{S}} \circ g_S \tilde{\supseteq} f_S \tilde{\cap} g_S$. And so, $f_S \circ \tilde{\mathbb{S}} \circ g_S = f_S \tilde{\cap} g_S$. Thus, (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let f_S and g_S be any SI-quasi-ideal and SI-left ideal of S, respectively. Then, by assumption,

$$f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S = f_S \circ (\tilde{\mathbb{S}} \circ g_S) \tilde{\subseteq} f_S \tilde{\circ} g_S.$$

Hence, it follows by Theorem 3 that S is regular. Let g_S be any SI-left ideal of S. Then, since g_S is an SI-quasi-ideal of S and since $\tilde{\mathbb{S}}$ itself is an SI-left ideal of S,

$$g_S = g_S \tilde{\cap} \tilde{\mathbb{S}} = g_S \circ \tilde{\mathbb{S}} \circ \tilde{\mathbb{S}}.$$

Let L be any left ideal of S and $a \in L$. Then, the soft characteristic function $\mathcal{S}_L$ is an SI-left ideal of S. Thus,

$$\mathcal{S}_{LSS}(a) = (\mathcal{S}_L \circ \mathcal{S}_S \circ \mathcal{S}_S)(a) = \mathcal{S}_L(a) = U$$

which means that $a \in LSS$. Thus, $L \subseteq LSS$. Moreover, let $a \in LSS$. Then,

$$\mathcal{S}_L(a) = (\mathcal{S}_L \circ \mathcal{S}_S \circ \mathcal{S}_S)(a) = \mathcal{S}_{LSS}(a) = U$$

and so $a \in L$. Thus, $LSS \subseteq L$, and so $LSS = L$. Since Sa is a left ideal of S, $(Sa)SS = Sa$ and so,

$$aS \subseteq (aSa)S = a(Sa)S = a((Sa)SS)S \subseteq a((Sa)SS) \subseteq a(Sa) = (aS)a \subseteq Sa.$$

It follows by Theorem 23 that S is a semilattice of left groups and so (2) implies (1). □

The left-right dual of Theorem 31 is as following:

Theorem 32 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of right groups.*

2) $f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ *for every SI-quasi-ideal f_S and SI-right ideal g_S of S.*

3) $f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ *for every SI-bi-ideal f_S and SI-right ideal g_S of S.*

4) $f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ *for every SI-generalized bi-ideal f_S and SI-right ideal g_S of S.*

Theorem 33 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left groups.*

2) $f_S \tilde{\cap} h_S \tilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every SI-quasi-ideal f_S, for every SI-ideal h_S and every SI-left ideal g_S of S.*

3) $f_S \tilde{\cap} h_S \tilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI*-bi-ideal* f_S*, for every* SI*-ideal* h_S *and every* SI*-left ideal* g_S *of* S*.*

4) $f_S \tilde{\cap} h_S \tilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S*, for every* SI*-ideal* h_S *and every* SI*-left ideal* g_S *of* S*.*

Proof. First assume that (1) holds. Let f_S be any SI-generalized bi-ideal of S, h_S be any SI-ideal of S and g_S be any SI-left ideal of S. Then,

$$f_S \circ h_S \circ g_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ (\tilde{\mathbb{S}} \circ g_S) \tilde{\subseteq} \tilde{\mathbb{S}} \circ g_S \tilde{\subseteq} g_S$$

and

$$f_S \circ h_S \circ g_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ h_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} h_S.$$

Moreover, by Theorem 24, since SI-generalized bi-ideal f_S of S is an SI-right ideal of S,

$$f_S \circ h_S \circ g_S \tilde{\subseteq} (f_S \circ \tilde{\mathbb{S}}) \circ \tilde{\mathbb{S}} \tilde{\subseteq} f_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} f_S.$$

Hence,

$$f_S \circ h_S \circ g_S \tilde{\subseteq} f_S \tilde{\cap} h_S \tilde{\cap} g_S.$$

Let $a \in S$. Then, by Theorem 23, $a = axa$ and $ax = ya$ for some $x, y \in S$. Then,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Hence,

$$\begin{aligned}
(f_S \circ h_S \circ g_S)(a) &= [(f_S \circ h_S) \circ g_S](a) \\
&= [\bigcup_{a=pq} (f_S \circ h_S)(p)] \circ g_S(q) \\
&\supseteq (f_S \circ h_S)(ax) \cap g_S(a) \\
&= \{\bigcup_{ax=pq} (f_S(p) \cap h_S(q))\} \cap g_S(a) \\
&\supseteq (f_S(axa) \cap h_S(xya)) \cap g_S(a) \\
&\supseteq (f_S(a) \cap h_S(a)) \cap g_S(a) \\
&= (f_S \tilde{\cap} h_S \tilde{\cap} g_S)(a)
\end{aligned}$$

and so, $f_S \circ h_S \circ g_S \tilde{\supseteq} f_S \tilde{\cap} h_S \tilde{\cap} g_S$. Thus, $f_S \circ h_S \circ g_S = f_S \tilde{\cap} h_S \tilde{\cap} g_S$ and (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Conversely, assume that (2) holds. Let f_S be any SI-quasi-ideal and g_S be any SI-left ideal of S. Then, since $\widetilde{\mathbb{S}}$ itself is an SI-ideal of S, by assumption that

$$f_S \widetilde{\cap} g_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} \widetilde{\cap} g_S = f_S \circ \widetilde{\mathbb{S}} \circ g_S = f_S \circ (\widetilde{\mathbb{S}} \circ g_S) \widetilde{\subseteq} f_S \circ g_S.$$

It follows by Theorem 3 that S is regular. As in the above Theorem, one can easily show that $aS \subseteq Sa$. Thus, S is a semilattice of left groups. Thus, (2) implies (1). This completes the proof. □

The left-right dual of Theorem 33 is as following:

Theorem 34 *For a semigroup* S, *the following conditions are equivalent:*

1) S *is a semilattice of right groups.*

2) $f_S \widetilde{\cap} h_S \widetilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*quasi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*right ideal* g_S *of* S.

3) $f_S \widetilde{\cap} h_S \widetilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*bi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*right ideal* g_S *of* S.

4) $f_S \widetilde{\cap} h_S \widetilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*right ideal* g_S *of* S.

9 A semilattice of groups

Let S be a semigroup. We shall say that S is a *semilattice of groups* if it is the set-theoretical union of a family of mutually disjoint subgroups G_i $(i \in M)$ such that, for any pair i, j in M, the products G_iG_j and G_jG_i are both contained in the same subgroups G_k $(k \in M)$. The following is due to [17, 28, 33].

Proposition 10 [17, 28, 33] *For a semigroup* S, *the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) S *is regular and* $aS = Sa$ *for all* $a \in S$.

3) $LR = L \cap R$ *for every left ideal* L *and every right ideal* R *of* S.

4) $LB = L \cap B$ *for every left ideal* L *and every bi-ideal* B *of* S.

5) $BR = B \cap R$ *for every bi-ideal* B *and every right ideal* R *of* S.

6) S *is regular and every one-sided ideal of* S *is two-sided.*

Proposition 11 *Let* S *be a semigroup that is a semilattice of groups. Then, every* SI-*(generalized) bi-ideal of* S *is an* SI-*ideal of* S.

Proof. Let f_S be any SI-bi-ideal of S and a and b be any elements of S. Then, it follows by Proposition 10 that

$$ab \in (aSa)S = (aS)(aS) = (aS)(Sa) = a(SS)a \subseteq aSa$$

Thus, there exists an element $x \in S$ such that $ab = axa$. Hence,

$$f_S(ab) = f_S(axa) \supseteq f_S(a) \cap f_S(a) = f_S(a).$$

Hence, f_S is an SI-right ideal of S. Similarly,

$$ab \in S(bSb) = (Sb)(Sb) = (bS)(Sb) = b(SS)b \subseteq bSb$$

Thus, there exists an element $x \in S$ such that $ab = bxb$. Hence,

$$f_S(ab) = f_S(bxb) \supseteq f_S(b) \cap f_S(a) = f_S(b).$$

Therefore, f_S is an SI-left ideal of S. That is to say, f_S is an SI-ideal of S. □

Proposition 12 [28] *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) *The set of all (generalized) bi-ideals of* S *is a semilattice under the multiplication of subsets.*

Now, see the characterization of a semigroup which s a semilattice of groups in terms of SI-ideals of semigroups.

Theorem 35 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*left ideal* f_S *and every* SI-*right ideal* g_S *of* S.

3) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*left ideal* f_S *and every* SI-*quasi ideal* g_S *of* S.

4) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*left ideal* f_S *and every* SI-*bi-ideal* g_S *of* S.

5) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*left ideal* f_S *and every* SI-*generalized bi-ideal* g_S *of* S.

6) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*quasi-ideal* f_S *and every* SI-*right ideal* g_S *of* S.

7) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*quasi-ideals* f_S *and* g_S *of* S.

8) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*quasi-ideal* f_S *and every* SI-*bi-ideal* g_S *of* S.

9) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*quasi-ideal* f_S *and every* SI-*generalized bi-ideal* g_S *of* S.

10) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*bi-ideal* f_S *and every* SI-*right ideal* g_S *of* S.

11) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*bi-ideal* f_S *and every* SI-*quasi-ideal* g_S *of* S.

12) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*bi-ideals* f_S *and* g_S *of* S.

13) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*bi-ideal* f_S *and every* SI-*generalized bi-ideal* g_S *of* S.

14) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*generalized bi-ideal* f_S *and every* SI-*right ideal* g_S *of* S.

15) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*generalized bi-ideal* f_S *and every* SI-*quasi-ideal* g_S *of* S.

16) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI-*generalized bi-ideal* f_S *and every* SI-*bi-ideal* g_S *of* S.

17) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*generalized bi-ideals* f_S *and* g_S *of* S.

18) S *is regular and every* SI-*one-sided ideal of* S *is an* SI-*ideal of* S.

19) *The set of all* SI-*quasi-ideals of* S *is a semilattice under the multiplication of soft int-product.*

20) *The set of all* SI-*bi-ideals of* S *is a semilattice under the multiplication of soft int-product.*

21) *The set of all* SI-*generalized-bi-ideals of* S *is a semilattice under the multiplication of soft int-product.*

Proof. First assume that (1) holds. In order to prove that (17) holds, let f_S and g_S be any SI-generalized bi-ideals of S. Then, it follows by Proposition 11 that f_S and g_S are SI-ideals of S. Since S is regular by Proposition 10, it follows from Theorem 3 that $f_S \circ g_S = f_S \widetilde{\cap} g_S$. Hence, it is obtained that (1) implies (17). It is clear that (17) implies (16), (16) implies (15), (15) implies (14), (14) implies (10), (10) implies (6), (6) implies (2), (17) implies (13), (13) implies (12), (12) implies (11), (11) implies (10), (13) implies (9), (9) implies (8), (8) implies (7), (7) implies (6) and (9) implies (5), (5) implies (4), (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let L and R be any left and right ideal of S, respectively. Then, the soft characteristic functions $\mathcal{S}_L$ and $\mathcal{S}_R$ are SI-left and SI-right ideal of S, respectively. Let a be any element of $L \cap R$. Then,

$$\mathcal{S}_{LR}(a) = (\mathcal{S}_L \circ \mathcal{S}_R)(a) = (\mathcal{S}_L \widetilde{\cap} \mathcal{S}_R)(a) = (\mathcal{S}_{L\cap R})(a) = U$$

and so $a \in LR$. Thus, $L \cap R \subseteq LR$.

Conversely, let a be any element of LR. Then,

$$(\mathcal{S}_{L\cap R})(a) = (\mathcal{S}_L \widetilde{\cap} \mathcal{S}_R)(a) = (\mathcal{S}_L \circ \mathcal{S}_R)(a) = \mathcal{S}_{LR}(a) = U,$$

and so $a \in L \cap R$. Thus, $LR \subseteq L \cap R$, hence $LR = L \cap R$. It follows by Proposition 10 that S is a semilattice of groups and so (2) implies (1).

Assume that (1) holds. Then, as shown above, (17) holds and (21) holds. It is obvious that (21) implies (20) and (20) implies (19). Assume that (19) holds. Then, since every SI-quasi-ideal of S is idempotent, it follows that S is regular ([40]. Let L and R be any left and right ideal of S, respectively. Then, since L and R are quasi-ideal of S, soft characteristic functions $\mathcal{S}_L$ and $\mathcal{S}_R$ are SI-quasi-ideal of S. Thus,

$$\mathcal{S}_{LR} = (\mathcal{S}_L \circ \mathcal{S}_R) = \mathcal{S}_R \circ \mathcal{S}_L) = \mathcal{S}_{RL}.$$

This implies that $LR = L \cap R$. Then, since S is regular,

$$R \cap L = RL = LR.$$

It follows by Proposition 12 that S is a semilattice of groups. Thus (19) implies (1).

Now assume that (2) holds. To see that (18) holds, let f_S be any SI-left ideal of S. Since $\widetilde{\mathbb{S}}$ is an SI-right ideal of S,

$$f_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S \circ \widetilde{\mathbb{S}}$$

Thus, f_S is an SI-right ideal of S. One can similarly show that every SI-right ideal of S is an SI-left ideal of S. As shown above, S is regular. Thus, (2) implies (18). Assume that (17) holds. In order to show that (1) holds, let A and B be any generalized bi-ideals of S and a be any element of AB. Then, the soft characteristic functions $\mathcal{S}_A$ and $\mathcal{S}_B$ are SI-generalized bi-ideals of S. Thus, by assumption,

$$(\mathcal{S}_B \circ \mathcal{S}_A)(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = \mathcal{S}_{AB}(a) = U$$

implying that $a \in BA$. Thus, $AB \subseteq BA$. It can be seen in a similar way that the converse inclusion holds. Thus, $AB = BA$. Now, let prove that any generalized bi-ideal of S is idempotent. Let B be any generalized bi-ideal of S and $a \in B$. Then, since the soft characteristic function $\mathcal{S}_B$ is an SI-generalized bi-ideal of S, by assumption

$$\mathcal{S}_{BB}(a) = (\mathcal{S}_B \circ \mathcal{S}_B)(a) = \mathcal{S}_B(a) = U$$

implying that $a \in BB$. Thus, $B \subseteq BB$. Similarly, one can show that $BB \subseteq B$. Hence, $B = BB$. This means that the set of all generalized bi-ideals of S is a semilattice under the multiplication of subsets. It follows by Proposition 12 that S is a semilattice of groups. Thus (2) implies (1). This completes the proof. □

Theorem 36 [39] *For a semigroup S the following conditions are equivalent:*

1) S *is completely regular.*

2) *Every bi-ideal of S is semiprime.*

3) *Every SI-bi-ideal of S is soft semiprime.*

4) $f_S(a) = f_S(a^2)$ *for every SI-bi-ideal f_S of S and for all $a \in S$.*

Theorem 37 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) *For every SI-quasi-ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

3) *For every SI-bi-ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

4) *For every* SI-*generalized bi-ideal* f_S *of* S, $f_S(a) = f_S(a^2)$ *and* $f_S(ab) = f_S(ba)$ *for all* $a, b \in S$.

Proof. First assume that (1) holds. Let f_S be any SI-generalized bi-ideal of S and a and b be any elements of S. Then, since S is regular by Proposition 10, there exists an element x in S such that $a = axa = axaxaxa$. Since $aS \subseteq Sa$ by Proposition 10, there exist elements $y, z \in S$ such that $xa = ya$ and $ax = za$. Thus,

$$a = axa = a(xaxaxa) = a(xa)x(ax)a = a(ya)x(za)a = a^2(yxz)a^2.$$

Hence, since f_S is an SI-generalized bi-ideal of S,

$$f_S(a) = f_S(a^2(yxz)a^2) \supseteq f_S(a^2) \cap f_S(a^2) = f_S(a^2) = f_S(a(axa)) = f_S(a(ax)a) \supseteq f_S(a) \cap f_S(a) = f_S(a)$$

and so $f_S(a) = f_S(a^2)$. Moreover, by Proposition 10,

$$(ab)^4 = a(ba)ba(ba)b \in (Sba)S(baS) = (baS)S(Sba).$$

Hence, there exists an element $u \in S$ such that $(ab)^4 = bauba$. Thus,

$$f_S(ab) = f_S((ab)^2) = f_S((ab)^4) = f_S((ba)u(ba)) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba).$$

Similarly, $f_S(ba) \supseteq f_S(ab)$ and so $f_S(ab) = f_S(ba)$. Thus, (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Conversely, assume that (2) holds. Then, it follows by Theorem 36 that S is completely regular and so regular. Let a be any element of S. To see that $aS = Sa$, let ax be any element of aS. Since the soft characteristic function $\mathcal{S}_{B[xa]}$ of the principal bi-ideal $B[xa]$ is an SI-bi-ideal of S, by assumption,

$$\mathcal{S}_{B[xa]}(ax) = \mathcal{S}_{B[xa]}(xa) = U$$

and so $ax \in B[xa] = \{xa\} \cup (xa)^2 \cup (xa)S(xa)$. If $ax = xa$, then $ax = xa \in Sa$, and so $aS \subseteq Sa$. If $ax = (xa)^2$, then $ax = (xax)a \in Sa$. Hence, $aS \subseteq Sa$. If $ax \in (xa)S(xa)$, then

$$ax \in (xa)S(xa) = (xaSx)a \in Sa$$

and so $aS \subseteq Sa$. In any case, $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$. Thus, $aS = Sa$. Hence, it follows by Proposition 10 that S is a semilattice of groups. Thus, (2) implies (1). This completes the proof. $\square$

Theorem 38 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) $f_S \widetilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *of* S *and for all* SI*-ideal* g_S *of* S*.*

3) $f_S \widetilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *of* S *and for all* SI*-interior ideal* g_S *of* S*.*

4) $f_S \widetilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *of* S *and for all* SI*-ideal* g_S *of* S*.*

5) $f_S \widetilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *of* S *and for all* SI*-interior ideal* g_S *of* S*.*

6) $f_S \widetilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *of* S *and for all* SI*-ideal* g_S *of* S*.*

7) $f_S \widetilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *of* S *and for all* SI*-interior ideal* g_S *of* S*.*

Proof. First assume that (1) holds. Let f_S be any SI-generalized bi-ideal and g_S be any SI-interior ideal of S. It follows by Proposition 11 that f_S is an SI-ideal of S. Thus,

$$g_S \circ f_S \circ g_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S.$$

Moreover, $g_S \circ f_S \circ g_S \widetilde{\subseteq} g_S \circ (\widetilde{\mathbb{S}} \circ g_S) \widetilde{\subseteq} g_S \circ g_S \widetilde{\subseteq} g_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} g_S$. Therefore,

$$g_S \circ f_S \circ g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S.$$

Now, let a be any element of S. Since S is regular by Proposition 10, there exists an element $x \in S$ such that $a = axa$. Hence

$$\begin{aligned}
(g_S \circ f_S \circ g_S)(a) &= [(g_S \circ f_S) \circ g_S](a) \\
&= [\bigcup_{a=pq} (g_S \circ f_S)(p)] \circ g_S(q) \\
&\supseteq (g_S \circ f_S)(a) \cap g_S(xa) \\
&= \{\bigcup_{a=uv} (g_S(u) \cap f_S(v))\} \cap g_S(a) \\
&\supseteq (g_S(ax) \cap f_S(a)) \cap g_S(a) \\
&\supseteq f_S(a) \cap g_S(a)
\end{aligned}$$

$$= \quad (f_S \widetilde{\cap} g_S)(a)$$

and so, $g_S \circ f_S \circ g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. Thus, $g_S \circ f_S \circ g_S = f_S \widetilde{\cap} g_S$, so, (1) implies (7). It is clear that (7) implies (6), (6) implies (4), (4) implies (2) and (7) implies (5), (5) implies (3) and (3) implies (2).

Assume that (2) holds. Let Q and J be any quasi-ideal and ideal of S, respectively. Thus, the soft characteristic function $\mathcal{S}_Q$ and $\mathcal{S}_J$ are SI-quasi-ideal and SI-ideal of S, respectively. Hence, by assumption,

$$\mathcal{S}_{JQJ}(a) = (\mathcal{S}_J \circ \mathcal{S}_Q \circ \mathcal{S}_J)(a) = (\mathcal{S}_J \cap \mathcal{S}_Q)(a) = \mathcal{S}_{J\cap Q}(a) = U$$

which implies that $a \in JQJ$. Thus, $J \cap Q \subseteq JQJ$.

Now, let a be any element of JQJ. Then,

$$\mathcal{S}_{J\cap Q}(a) = (\mathcal{S}_J \cap \mathcal{S}_Q)(a) = (\mathcal{S}_J \circ \mathcal{S}_Q \circ \mathcal{S}_J)(a) = \mathcal{S}_{JQJ}(a) = U$$

which implies that $a \in J \cap Q$. Thus, $JQJ \subseteq J \cap Q$. Therefore, that $JQJ = J \cap Q$ for every quasi-ideal Q and ideal J of S, which implies that S is regular and (2) implies (1). This completes the proof. □

10 Soft normal semigroups

In this section, the concepts of soft normality in a semigroup is introduced. It is known that a semigroup S is called *normal* if $aS = Sa$ for all $a \in S$.

Definition 14 *An* SI-*quasi-ideal* f_S *of* S *is called* $Q-\mathsf{normal}$ *if* $f_S(ab) = f_S(ba)$ *for all* $a, b \in S$.

Definition 15 *An* SI-*bi-ideal* f_S *of* S *is called* $B-\mathsf{normal}$ *if* $f_S(ab) = f_S(ba)$ *for all* $a, b \in S$.

Definition 16 *A semigroup* S *is called soft* $B^\star-\mathsf{normal}$ *if every* SI-*bi ideal of* S *is* $B-\mathsf{normal}$.

Definition 17 *A semigroup* S *is called soft* $Q^\star-\mathsf{normal}$ *if every* SI-*quasi-ideal of* S *is* $Q-\mathsf{normal}$.

Theorem 39 *Any soft* $Q^\star-\mathsf{normal}$ *semigroup is normal.*

Proof. Let f_S be an SI-quasi-ideal of a soft $Q^\star-\mathsf{normal}$ semigroup of S. Let a be any element of S. To see that $aS = Sa$, let ax be any element of aS.

Since the soft characteristic function $\mathcal{S}_{Q[xa]}$ of the principal bi-ideal $Q[xa]$ is an SI-quasi-ideal of S, by assumption,

$$\mathcal{S}_{Q[xa]}(ax) = \mathcal{S}_{Q[xa]}(xa) = U$$

which implies that

$$ax \in Q[xa] = \{xa\} \cup (xaS \cap Sxa) \subseteq Sa$$

Thus, $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$ holds. Thus, $aS = Sa$ and S is normal. This completes the proof. □

The following theorem shows that the converse of Theorem 39 holds for a regular semigroup.

Theorem 40 *For a regular semigroup* S*, the following conditions are equivalent:*

1) S *is soft* $Q^{\star}-\mathrm{normal}$.

2) S *is normal.*

Proof. It suffices to prove that (2) implies (1). Assume that (2) holds. Let f_S be any SI-quasi-ideal of S and a and b be any elements of S. Since S is regular and normal,

$$ab \in (aSa)(bSb) = (aS)(ab)(Sb) \subseteq (aS)(abSab)(Sb) = (aSa)b(Sa)(bSb) \subseteq (Sb)(Sa)S = (Sb)(aS)S = S(ba)SS = (ba)SSS \subseteq baS$$

This implies that there exists an element $x \in S$ such that $ab = bax$. Thus, since f_S is an SI-bi-ideal of S,

$$(f_S \circ \widetilde{\mathbb{S}})(ab) = \bigcup_{ab=pq} \{(f_S(p) \cap \widetilde{\mathbb{S}}(q)\} \supseteq f_S(ba) \cap \widetilde{\mathbb{S}}(x) = f_S(ba).$$

One can similarly show that

$$(\widetilde{\mathbb{S}} \circ f_S)(ab) \supseteq f_S(ba)$$

Since, f_S is an SI-quasi-ideal of S,

$$f_S(ab) \supseteq ((f_S \circ \widetilde{\mathbb{S}})\widetilde{\cap}(\widetilde{\mathbb{S}} \circ f_S))(ab) = (f_S \circ \widetilde{\mathbb{S}})(ab) \cap (\widetilde{\mathbb{S}} \circ f_S)(ab) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba)$$

Similarly, it can be proved that $f_S(ba) \supseteq f_S(ab)$. Thus, $f_S(ba) = f_S(ab)$, and so S is soft $Q^\star - \mathtt{normal}$ and that (2) implies (1). This completes the proof. □

Theorem 41 *Any soft* $B^\star - \mathtt{normal}$ *semigroup is normal.*

Proof. Let f_S be an SI-bi-ideal of a soft $B^\star - \mathtt{normal}$ semigroup of S. Let a be any element of S and ax be any element of aS. Since the soft characteristic function $\mathcal{S}_{B[xa]}$ of the principal bi-ideal $B[xa]$ is an SI-bi-ideal of S, by assumption,

$$\mathcal{S}_{B[xa]}(ax) = \mathcal{S}_{B[xa]}(xa) = U$$

which implies that

$$ax \in B[xa] = \{xa\} \cup \{xaxa\} \cup (xa)S(xa) \subseteq Sa$$

Thus, $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$ holds. Thus, $aS = Sa$ and S is normal. This completes the proof. □

The following theorem shows that the converse of Theorem 41 holds for a regular semigroup.

Theorem 42 *For a regular semigroup* S*, the following conditions are equivalent:*

1) S *is soft* $B^\star - \mathtt{normal}$*.*

2) S *is normal.*

Proof. It suffices to prove that (2) implies (1). Assume that (2) holds. Let f_S be any SI-bi-ideal of S and a and b be any elements of S. Since S is regular,

$$ab \in (aSa)(bSb) = (aS)(ab)(Sb) \subseteq (aS)(abSab)(Sb) = (aSa)b(Sa)(bSb) \subseteq (Sb)(aS)S = S(ba)SS = (ba)SSS \subseteq baS = (baSba)S = (baS)(Sba) = ba(SS)ba \subseteq baSba.$$

This implies that there exists an element $x \in S$ such that $a = baxba$. Thus, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S((ba)x(ba)) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba).$$

One can similarly show that $f_S(ba) \supseteq f_S(ab)$. Hence $f_S(ab) = f_S(ba)$ which implies that S is soft $B^\star - \mathtt{normal}$ and that (2) implies (1). This completes the proof. □

Proposition 13 *For an idempotent semigroup* S*, the following conditions are equivalent:*

1) S *is commutative.*

2) S *is soft* $Q^\star-\mathtt{normal}$*.*

3) S *is soft* $B^\star-\mathtt{normal}$*.*

Proof. (1) implies (3) and (3) implies (2) is obvious. Assume that (2) holds. Then, S is normal. Let $a, b \in S$. Then, $ab \in Sb = bS$. Thus, there exists an element x in S such that $ab = bx$. Similarly, $ba = yb$ for some $b \in S$. Hence, since S is idempotent,

$$ab = bx = (bb)x = b(bx) = b(ab) = (ba)b = (yb)b = yb = ba$$

which implies that S is commutative. Hence (2) implies (1). □

Definition 18 [33] *A semigroup* S *is called Archimedean if for all* $a, b \in S$*, there exists a positive integer* n *such that* $a^n \in SbS$*.*

Definition 19 [33] *A semigroup* S *is called weakly commutative if for all* $a, b \in S$*, there exists a positive integer* n *such that* $(ab)^n \in bSa$*.*

Proposition 14 [33] *Every weakly commutative semigroup is a semilattice of archimedean semigroups.*

Proposition 15 *Any soft* $B^\star-\mathtt{normal}$ *semigroup is a semilattice of Archimedean semigroups.*

Proof. Let S be any soft $B^\star-\mathtt{normal}$ semigroup. Let a and b be any elements of S, and f_S be any SI-bi-ideal of S. Since the soft characteristic function $\mathcal{S}_{B[ba]}$ of the principal bi-ideal $B[ba]$ is an SI-bi-ideal of S, by assumption,

$$\mathcal{S}_{B[ba]}(ab) = \mathcal{S}_{B[ba]}(ba) = U$$

and so

$$ab \in B[ba] = \{ba\} \cup \{baba\} \cup (baSba) \subseteq Sa$$

Thus, $(ab)^2 \in baSba \subseteq bSa$. Therefore, S is weakly commutative. Hence by Proposition 14, S is a semilattice of Archimedean semigroups. □

One can similarly prove the following proposition.

Proposition 16 *Any soft* $Q^\star-\mathtt{normal}$ *semigroup is a semilattice of Archimedean semigroups.*

Theorem 43 *For a completely regular semigroup* S*, the following conditions are equivalent:*

1) S *is soft* $Q^\star - \mathtt{normal}$.

2) S *is soft* $B^\star - \mathtt{normal}$.

3) *For each elements* a *and* b *of* S*, there exists a positive integer* n *such that* $(ab)^n \in baSba$.

Proof. It is obvious that (2) implies (1). Assume that (1) holds. Then, S is normal. Let a and b be any elements of S. Thus,

$$\begin{aligned}(ab)^3 &= ababab = a(ba)bab \subseteq (Sba)(baS) = (baS)(Sba) \\ &= (ba)SS(ba) \subseteq baSba\end{aligned}$$

which shows that (1) implies (3).

Conversely, assume that (3) holds. To see that (2) holds, l et f_S be any SI-bi-ideal of S and a and b be any elements of S. Then, by assumption, there exists a positive integer n such that $(ab)^n = baxba$. Since S is completely regular, for this positive integer, there exists an element $y \in S$ such that $ab = (ab)^n y (ab)^n$. Then, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S((ab)^n y (ab)^n) \supseteq f_S((ab)^n) \cap f_S((ab)^n) = f_S((ab)^n)) = \\ f_S(baxba) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba).$$

One can similarly show that $f_S(ba) \supseteq f_S(ab)$. Hence, $f_S(ab) = f_S(ba)$ which implies that f_S is soft $B^\star - \mathtt{normal}$. Thus, (3) implies (2). □

11 Conclusion

In this paper, certain classes of semigroups are characterized with regards to different soft intersection ideals of semigroups and soft normal semigroups are defined and the relation of this concept are studied with semigroups. Based on these results, some further work can be done on the properties of soft intersection semigroups and different classes of soft union ideals, which may be useful to characterize the classical semigroups.

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