Optimization of Youla-parametrized Controllers

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Abstract: The so-called Youla-parametrized controllers provide a special way to design the best regulators for open-loop stable plants. The authors formerly extended this parametrization for two-degree of freedom (*TDOF*) control systems. The control error in a generic *TDOF* control system has three major parts: design-, realizability- and modeling-loss. This new decomposition opens useful ways for practical optimization and the paper investigates the optimality of the second term.

Keywords:Optimal performance, sensitivity function, error decomposition, YOULA parametrization

1. Introduction

Control system optimization is usually based on the error signal or the error transfer function of the closed-loop. The last one is called sensitivity function (*SF*), so any such optimization procedure is strongly connected to the sensitivity or the robustness of control systems. One widely applied possibility to optimize a proper norm formulated for the closed-loop *SF* is to consider the criterion as a function of the loop-parameters (design, regulator, constraints, etc.) and to solve the strongly nonlinear constrained mathematical programming problem. The existing advanced software tools (MATLAB™, MATHEMATICA™, etc.) are capable to solve such complex tasks, however give relatively little understanding for the influence of the different factors. These methods do not analyze the internal properties of the control error and the different contributing parts of the sensitivity.

This is why such decomposition of the original problem is preferred and proposed here, where the separate tasks are easy to be understood and well scaled in the design parameters and factors. The introduced new decomposition helps to analyze the reachable minimum of the different components, so it is possible to see the theoretical limits of the optimization of control systems. The different components of the decomposition are connected to different well-formulated subtasks, which can be solved separately and help to form logical iterative procedures.

2. Decomposition of the Sensitivity Function in *TDOF* Systems

Assume that the plant to be controlled is factorable as

$$P = P_+ P_- = \frac{\mathrm{B}}{\mathrm{A}} = \frac{\mathrm{B}_+ \mathrm{B}_-}{\mathrm{A}} \tag{1}$$

where $P_+ = \mathrm{B}_+ / \mathrm{A}$ means the *inverse stable* (*IS*) and $P_- = \mathrm{B}_-$ the *inverse unstable* (*IU*) factors, respectively.

In a practical case only the model $\hat{P}$ of the process is known. Assume that the model $\hat{P}$, is similarly factorable as the process in (1)

$$\hat{P} = \hat{P}_+ \hat{P}_- = \frac{\hat{\mathrm{B}}}{\hat{\mathrm{A}}} = \frac{\hat{\mathrm{B}}_+ \hat{\mathrm{B}}_-}{\hat{\mathrm{A}}} \tag{2}$$

where $\hat{P}_+ = \hat{\mathrm{B}}_+ / \hat{\mathrm{A}}$ means the *IS*, $P_- = \hat{\mathrm{B}}_-$ the *IU* (or any invariant) process factors, respectively. Introduce the additive

$$\Delta = P - \hat{P} \tag{3}$$

and relative model errors

$$\ell = \frac{\Delta}{\hat{P}} = \frac{P - \hat{P}}{\hat{P}} \tag{4}$$

The complementary sensitivity function (CSF) of a one-degree of freedom (ODOF) control system is

$$T = \frac{CP}{1+CP} = \hat{T}\frac{1+\ell}{1+\hat{T}\ell} \quad ; \quad \hat{T} = \frac{\hat{C}P}{1+\hat{C}P} \tag{5}$$

where $\hat{T}$ is the *CSF* of the model based *ODOF* system. For a two-degree of freedom (*TDOF*) control system [2] it is reasonable to request the design goals by two stable and usually strictly proper transfer functions R_{r} and R_{n}, that are partly capable to place desired poles in the tracking and the regulatory transfer functions, furthermore they are usually referred as reference signal and output disturbance predictors. They can even be called as reference models, so reasonably $R_{\mathrm{r}}(\omega = 0) = 1$ and $R_{\mathrm{n}}(\omega = 0) = 1$ unity gains are selected.

An important new observation is that the *SF* ($S = 1 - T$) can be decomposed into additive components according to different principles:

$$S = \underbrace{\left(1 - R_{\rm n}\right)}_{S_{\rm des}} + \overbrace{\underbrace{\left(R_{\rm n} - \hat{T}\right)}_{S_{\rm real}} - \underbrace{\left(T - \hat{T}\right)}_{S_{\rm id}}}^{S_{\rm perf}} = S_{\rm des} + S_{\rm real} + S_{\rm id} = \underbrace{\left(1 - R_{\rm n}\right)}_{S_{\rm des}} + \underbrace{\left(R_{\rm n} - T\right)}_{S_{\rm perf}} = S_{\rm des} + S_{\rm perf} =$$
$$= \underbrace{\left(1 - \hat{T}\right)}_{S_{\rm cont}} + S_{\rm id} = \underbrace{\left(1 - \hat{T}\right)}_{S_{\rm des} + S_{\rm real}} + S_{\rm id} = S_{\rm cont} + S_{\rm id} \tag{6}$$

Here $S_{\rm des} = \left(1 - R_{\rm n}\right)$ is the design, $S_{\rm real} = \left(R_{\rm n} - \hat{T}\right)$ is the realizability, $S_{\rm id} = -\left(T - \hat{T}\right) = \hat{T} - T$ is the modeling (or identification) degradation, respectively. Furthermore $S_{\rm cont} = \left(1 - \hat{T}\right)$ and $S_{\rm perf} = \left(P_{\rm n} - T\right)$ are the overall control and performance degradations, respectively. The *SF* depends on the model-based *SF* ($\hat{S} = 1 - \hat{T}$) as

$$S = \frac{1}{1 + CP} = \hat{S}\frac{1}{1 + \hat{T}\ell} = \hat{S} + S_{\rm id} \quad ; \quad \hat{S} = \frac{1}{1 + C\hat{P}} \tag{7}$$

The term $S_{\rm id}$ can be further simplified

$$S_{\rm id} = S - \hat{S} = \hat{T} - T = -\frac{\hat{T}\hat{S}\ell}{1 + \hat{T}\ell} = -\hat{T}S\ell\Big|_{\ell \to 0} \approx -\hat{T}\hat{S}\ell \tag{8}$$

It is easy to see that $\left|\hat{T}\hat{S}\right|$ has its maximum at the cross over frequency $\omega_{\rm c}$, which means that the model minimizing $S_{\rm id}$ is the most accurate around this medium frequency range. (Note that the accuracy of the estimated model at a given frequency is inverse proportional to the weight in the modeling error at that frequency.) The realizability and identification degradations can be called as systematic ($S_{\rm syst}$) and random ($S_{\rm rand}$) components, too.

In a general case the overall *CSF* of a *TDOF* control system has the form $T_{\rm r} = FT$ and a similar decomposition can be introduced for the tracking error function $S_{\rm r} = 1 - T_{\rm r}$ as for S in (6):

$$S_{\rm r} = \left(1 - R_{\rm r}\right) + \left(R_{\rm r} - \hat{T}_{\rm r}\right) - \left(T_{\rm r} - \hat{T}_{\rm r}\right) = S_{\rm des}^{\rm r} + S_{\rm real}^{\rm r} + S_{\rm id}^{\rm r} \tag{9}$$

The overall transfer function of the *TDOF* system is

$$T_{\mathrm{r}} = \hat{T}_{\mathrm{r}} \frac{1+\ell}{1+\hat{T}\ell} \qquad (10)$$

The term $S_{\mathrm{id}}^{\mathrm{r}}$ can be further simplified

$$S_{\mathrm{id}}^{\mathrm{r}} = \hat{T}_{\mathrm{r}} - T_{\mathrm{r}} = -\frac{\hat{T}_{\mathrm{r}}\hat{S}\ell}{1+\hat{T}\ell} = -\hat{T}_{\mathrm{r}} S\ell\Big|_{\ell\to 0} \approx -\hat{T}_{\mathrm{r}}\hat{S}\ell \qquad (11)$$

In an ideal control system it is required to follow the transients given by R_{r} and R_{n} (more exactly $(1-R_{\mathrm{n}})$), i.e., the ideal overall transfer characteristics of the *TDOF* control system would be

$$y^{\mathrm{o}} = R_{\mathrm{r}} y_{\mathrm{r}} - (1-R_{\mathrm{n}}) y_{\mathrm{n}} = y_{\mathrm{r}}^{\mathrm{o}} + y_{\mathrm{n}}^{\mathrm{o}} \qquad (12)$$

while a practical, realizable control can provide only

$$y = T_{\mathrm{r}} y_{\mathrm{r}} - S y_{\mathrm{n}} = T_{\mathrm{r}} y_{\mathrm{r}} - (1-T) y_{\mathrm{n}} \quad ; \quad \hat{y} = \hat{T}_{\mathrm{r}} y_{\mathrm{r}} - \hat{S} y_{\mathrm{n}} = \hat{T}_{\mathrm{r}} y_{\mathrm{r}} - (1-\hat{T}) y_{\mathrm{n}} \qquad (13)$$

for the true (y) and model-based ($\hat{y}$) closed-loop control output signals. Here y_{r}, y and y_{n} are the reference, process output and disturbance (or output noise) signals, respectively.

Express the deviation between the ideal (y^{o}) and the realizable best (y) closed-loop output signals as

$$\Delta y = y^{\mathrm{o}} - y = (R_{\mathrm{r}} - T_{\mathrm{r}}) y_{\mathrm{r}} - (R_{\mathrm{n}} - T) w = S_{\mathrm{perf}}^{\mathrm{r}} y_{\mathrm{r}} - S_{\mathrm{perf}}^{\mathrm{n}} y_{\mathrm{n}} \qquad (14)$$

where $S_{\mathrm{perf}}^{\mathrm{r}}$ is the performance degradation for tracking and $S_{\mathrm{perf}}^{\mathrm{w}} = S_{\mathrm{perf}}$ is the performance degradation for the disturbance rejection (or control) behaviors, respectively. Similar equation can be obtained for the deviation between the ideal (y^{o}) and the model based ($\hat{y}$) closed-loop outputs

$$\Delta\hat{y} = y^{\mathrm{o}} - \hat{y} = (R_{\mathrm{r}} - \hat{T}_{\mathrm{r}}) y_{\mathrm{r}} - (R_{\mathrm{n}} - \hat{T}) y_{\mathrm{n}} = S_{\mathrm{real}}^{\mathrm{r}} y_{\mathrm{r}} - S_{\mathrm{real}}^{\mathrm{n}} y_{\mathrm{n}} \qquad (15)$$

where $S_{\mathrm{real}}^{\mathrm{r}}$ is the realizability degradation for tracking and $S_{\mathrm{real}}^{\mathrm{w}} = S_{\mathrm{real}}$ is the realizability degradation for the disturbance rejection (control) behaviors, respectively. So

$$\Delta y = \Delta\hat{y} - \left(S_{\mathrm{id}}^{\mathrm{r}} y_{\mathrm{r}} - S_{\mathrm{id}}^{\mathrm{n}} y_{\mathrm{n}}\right) \tag{16}$$

It is important to note that the term S_{real} (and $S_{\mathrm{real}}^{\mathrm{r}}$) can be made zero for *IS* processes only, however, for *IU* plants the reachable minimal value of S_{real} (and $S_{\mathrm{real}}^{\mathrm{r}}$) always depends on the invariant factors and never becomes zero.

3. Control Error Decomposition in Youla-parametrized Systems

If the applied regulator design is based on the *Youla-parametrization* (*YP*) [6], [7] then the realizable best all stabilizing and the model based regulators are

$$C = \frac{Q}{1 - QP} \quad ; \quad \hat{C} = \frac{Q}{1 - Q\hat{P}} \tag{17}$$

where the "parameter" Q ranges over all proper ($Q(\omega = \infty)$ is finite), stable transfer functions. The *CSF*'s of the true and model-based *ODOF* control systems are

$$T = \frac{\hat{C}P}{1 + \hat{C}P} = \frac{Q\hat{P}(1+\ell)}{1 + Q\hat{P}\ell} \quad ; \quad \hat{T} = \frac{\hat{C}\hat{P}}{1 + \hat{C}\hat{P}} = Q\hat{P} \tag{18}$$

Only in case of *YP* one can also compute the realizable best *CSF*

$$T_* = \frac{CP}{1 + CP} = QP = Q\hat{P}(1+\ell) = \hat{T}(1+\ell) \tag{19}$$

The *SF* of the model based and true closed-loops are now

$$\hat{S} = \frac{1}{1 + \hat{C}\hat{P}} = 1 - Q\hat{P} \tag{20}$$

and

$$S = \frac{1}{1 + \hat{C}P} = \frac{1 - Q\hat{P}}{1 + Q\hat{P}\ell} = \frac{\hat{S}}{1 + \hat{T}\ell} \tag{21}$$

The realizable best *SF*, corresponding to T_* is

$$S_* = \frac{1}{1+CP} = 1 - QP = 1 - Q\hat{P}\left(1+\ell\right) = \hat{S} - \hat{T}\ell \tag{22}$$

The decomposition of the *SF* is

$$S = \left(1-R_\mathrm{n}\right) + \left(R_\mathrm{n} - \hat{T}\right) - \left(T - \hat{T}\right) = S_\mathrm{des} + S_\mathrm{real} + S_\mathrm{id} = \left(1-R_\mathrm{n}\right) + \left(R_\mathrm{n} - Q\hat{P}\right) - \frac{Q\hat{P}\left(1-Q\hat{P}\right)}{1+Q\hat{P}\ell}\ell \tag{23}$$

where the identification degradation is

$$S_\mathrm{id} = -\left.\frac{Q\hat{P}\left(1-Q\hat{P}\right)}{1+Q\hat{P}\ell}\ell\right|_{\ell\approx 0} \approx -Q\hat{P}\left(1-Q\hat{P}\right)\ell \tag{24}$$

It is interesting to note that for the realizable best case the decomposition of $S_* = 1 - T_*$ results in

$$S_* = 1 - QP = S_\mathrm{des} + S_\mathrm{real} + S^*_\mathrm{id} = \left(1-R_\mathrm{n}\right) + \left(R_\mathrm{n} - Q\hat{P}\right) - QM\ell =$$
$$= S_\mathrm{des} + S^*_\mathrm{perf} = \left(1-R_\mathrm{n}\right) + \left(R_\mathrm{n} - QS\right) \tag{25}$$

where

$$S^*_\mathrm{id} = -Q\hat{P}\ell = \frac{1}{S}S_\mathrm{id} \tag{26}$$

This last expression is different from the form (8), because at the optimal point, when $\hat{P} = P$, the *Y-parametrized* closed-loop virtually opens, therefore the weighting by $\hat{S}$ is missing here.

The decomposition of the tracking error function for the *YP* is

$$S_\mathrm{r} = 1 - T_\mathrm{r} = \left(1-R_\mathrm{r}\right) + \left(R_\mathrm{r} - Q_\mathrm{r}\hat{P}\right) - \left(T_\mathrm{r} - \hat{T}_\mathrm{r}\right) = S^\mathrm{r}_\mathrm{des} + S^\mathrm{r}_\mathrm{real} + S^\mathrm{r}_\mathrm{id} \tag{27}$$

where

$$S^\mathrm{r}_\mathrm{id} = -\left.\frac{Q_\mathrm{r}\hat{P}\left(1-Q\hat{P}\right)}{1+Q\hat{P}\ell}\ell\right|_{\ell\approx 0} \approx -Q_\mathrm{r}\hat{P}\left(1-Q\hat{P}\right)\ell \tag{28}$$

4. A *GTDOF* Controller for Stable Linear Plants

In many practical cases the plant to be controlled is stable, and a *TDOF* control system is required because of the high performance double tracking and regulatory requirements [2], [4]. An ideal solution for this task is the *generic two-degree of freedom* (*GTDOF*) scheme introduced in [6]. This framework and topology is based on the *YP* [6] providing stabilizing regulators for open-loop stable plants.

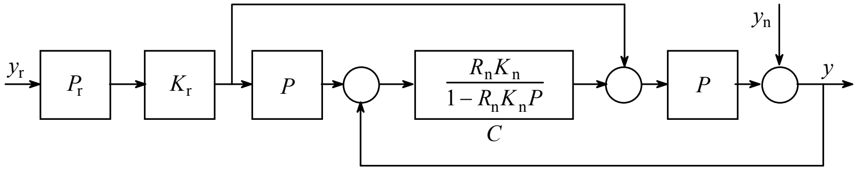

Figure 1. The generic TDOF (GTDOF) control system

A *GTDOF* control system is shown in Fig. 1. The realizable best regulator of the *GTDOF* scheme can be given by an explicit form for $\ell = 0$

$$C_* = \frac{Q_*}{1-Q_* P} = \frac{R_n K_n}{1-R_n K_n P} = \frac{R_n G_n P_+^{-1}}{1-R_n G_n P_-} \tag{29}$$

where

$$Q_* = Q_n^* = R_n K_n = R_n G_n P_+^{-1} \tag{30}$$

is the associated optimal *Y-parameter* furthermore

$$Q_r^* = R_r K_r = R_r G_r P_+^{-1} \quad ; \quad K_n = G_n P_+^{-1} \quad ; \quad K_r = G_r P_+^{-1} \tag{31}$$

It is interesting to see how the transfer characteristics of the closed-loop look like:

$$y = R_r K_r P y_r + \left(1 - R_n K_n P\right) y_n = T_r y_r + S y_n = y_* = R_r G_r P_- y_r + \left(1 - R_n G_n P_-\right) y_n = y_t + y_d \tag{32}$$

where y_t is the tracking (servo) and y_d is the regulating (disturbance rejection or control) independent behavior of the closed-loop response, respectively.
So the invariant factor P_- cannot be eliminated, consequently the ideal design goals R_r and R_n are biased by $G_r P_-$ and $G_n P_-$. We cannot reach the ideal tracking $y_r^o = P_r y_r$ and regulatory $y_n^o = \left(1 - R_n\right) y_n$ behaviors (see (12)), because of the invariant factor (mainly zeros) in the *IU* factor P_-. (In a general case the time delay should also be considered here as an invariant factor.) The realizable

best transients, corresponding to (13) and (32), is given by $R_r G_r P_-$ and $\left(1-R_n G_n P_-\right)$ respectively, where G_r and G_n can optimally attenuate the influence of P_-. (Unfortunately P_- does not depend on the control design. This factor is a basic behavior of the process, so it can be considerably changed only via certain technological changes.) The unity gain of R_n ensures integral action in the regulator, which is maintained if and only if the applied optimization provides $G_n P_-(\omega=0)=1$, or including R_n the condition is $R_n G_n P_-(1)=1$.

The model based version of the *YP* regulator $\hat{C}=C\left(\hat{P}\right)$ in the *GTDOF* scheme means that P is substituted by $\hat{P}$ in equations (29)-(31).

The decomposition of the *SF* in the true *GTDOF* control system by (23) is

$$S=S_{\mathrm{des}}+S_{\mathrm{real}}+S_{\mathrm{id}}=\left(1-R_n\right)+R_n\left(1-G_n\hat{P}_-\right)-\frac{R_n G_n\hat{P}_-\left(1-R_n G_n\hat{P}_-\right)}{1+Q\hat{P}\ell}\ell \tag{33}$$

5. Optimization of the *SF* in H_2, L_2 Norm Spaces

Investigate the direct optimization of the *SF* in H_2 norm space first for the *GTDOF* control system. It is easy to check from (32) that the *SF* for *YP* regulator when $\ell=0$ (exact model matching case) is

$$S=1-R_n G_n P_-=S_n \tag{34}$$

and for the tracking sensitivity

$$S_r=1-R_r G_r P_- \tag{35}$$

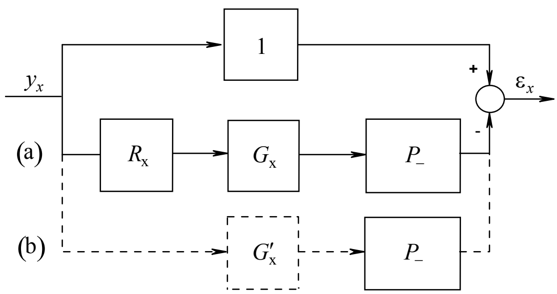

Figure 2. Reduced form of H_2, L_2 *optimality of the SF for the GTDOF control* ($Y_x=\mathcal{L}\left(y_x\right)=s^{-j}$)

In the sequel a more general formulation of the second norm as performance measure

$$J_x^2 = \left\| Y_x \left(1 - R_x G_x P_-\right)\right\|_2 = \left\| \frac{1}{s^j}\left(1 - R_x G_x P_-\right)\right\|_2 = \begin{cases} H_2 \text{ norm}; j = 0 \\ L_2 \text{ norm}; j \geq 1 \end{cases} \tag{36}$$

will be applied. Here a form $Y_x = s^{-j}$ was used for the excitation. This norm is the H_2 system (or operator) norm for $j = 0$ and is the L_2 signal norm for $j \geq 1$. One must know that the L_2 norm is usually formulated for real functions and not for complex functions as here. So the optimality of the *SF* for a *GTDOF* system can be ensured by minimizing the $\left\| Y_x \left(1 - R_x G_x P_-\right)\right\|_2$ type cost functions, see Fig.2a. (Here the subscript x depends on the tracking or the control problem.) Because the regulator cancels the *IS* factor of the process, this is the reduced or residual form of the optimal scheme, so the task to be solved is

$$G_x = \arg\left\{\min_{G_x}\left[J_x\right]\right\} = \arg\left\{\min_{G_x}\left[\left\|Y_x S_x\right\|_2\right]\right\} = \arg\left\{\min_{G_x}\left[\left\|Y_x\left(1 - R_x G_x P_-\right)\right\|_2\right]\right\} \tag{37}$$

It is also important to note that the "generalized energy norm" formulated by (36) is bounded for $j = 0$ and for $j \geq 1$, too because the low-pass Y_x is multiplied by a high-pass control error term $\left(1 - R_x G_x P_-\right)$, which is satisfied for closed-loop system with type number higher than zero.

Note that the above minimization in H_2-norm space is equivalent to the minimum mean square error (*MSE*) or minimum variance (*MV*) problem (the classical Wiener paradigm of optimal stochastic systems), if the external excitation y_x is a white noise sequence, i.e. $j = 0$ [5].

Assume a unity gain

$$R_x = B_x / A_x \tag{38}$$

IS reference model with coprime B_x and A_x, thus

$$J_x^2 = \left\| \frac{1}{s^j}\left(1 - \frac{B_x}{A_x} G_x B_-\right)\right\|_2 = \left\|\frac{B_-}{B_-^-}\right\|_2 \left\| \frac{B_-^-}{B_- s^j} - \frac{B_x G_x B_-^-}{A_x s^k}\right\|_2 \tag{39}$$

where $B_- = P_-$ is usually selected monic and unity gain, i.e., $B_-(0) = 1$ with the form of $B_-(s) = \prod_{i=1}^{m_-}\left(1 - s/z_i\right)$ for m_- number unstable zeros. B_-^- contains the special z_i^* zeros, which are the unstable zeros of B_- mirrored on the imaginary axis: $\mathrm{Re}\left\{z_i^*\right\} = -\mathrm{Re}\left\{z_i\right\}$ and $\mathrm{Im}\left\{z_i^*\right\} = \mathrm{Im}\left\{z_i\right\}$.

Thus B_-^- is stable now so $\left\| B_- / B_-^- \right\|_2$ is a stable norm-preserving inner function. Taking the usual decomposition

$$\frac{B_-^-}{B_- s^j} = \frac{R}{B_-} + \frac{K}{s^j} = \frac{R s^j + K B_-}{B_- s^j} \tag{40}$$

where in the right hand side R / B_- is the non-causal and K / s^j is the causal part consequently (38) becomes

$$J_x^2 = \left\| \frac{R}{B_-} + \frac{K}{s^j} - \frac{B_x G_x B_-^-}{A_x s^j} \right\|_2 \tag{41}$$

Following the orthogonality principle [8] of the classical Wiener design the H_2, L_2 optimal G_x can be obtained making the sum of the causal parts zero, i.e.

$$\frac{K}{s^j} - \frac{B_x G_x B_-^-}{A_x s^j} = 0 \tag{42}$$

The H_2, L_2 optimal embedded filter G_x is obtained as

$$G_x = \frac{A_x K}{B_x B_-^-} \tag{43}$$

R and K can be obtained from the special Diophantine equation (see 40)

$$B_-^- = R s^j + K B_- \tag{44}$$

It is interesting to observe that using the obtained H_2, L_2 optimal G_x, the original $\left\| Y_x \left(1 - R_x G_x P_-\right) \right\|_2$ form changes to $\left\| Y_x \left(1 - G'_x P_-\right) \right\|_2$, where

$$G'_x = K / B_-^- \tag{45}$$

because the optimization cancels all stable factors. This optimal embedded filter corresponds to the scheme shown in Fig.2b, which is the classical Nehari problem [3], [8]. The minimum of the cost function is

$$J_x = \left\| R / B_-^- \right\|_2 \tag{46}$$

The H_2 optimal regulator using (29) and (43) is

$$C_* = \frac{\mathrm{KA}}{\mathrm{B}_+\left(\mathrm{B}_-^- - \mathrm{KB}_-\right)} \tag{47}$$

Investigating the product

$$G_\mathrm{x}\mathrm{B}_- = \frac{\mathrm{A}_\mathrm{x}\mathrm{KB}_-}{\mathrm{B}_\mathrm{x}\mathrm{B}_-^-} = \frac{\mathrm{A}_\mathrm{x}}{\mathrm{B}_\mathrm{x}}\left(1 - \frac{\mathrm{K}}{\mathrm{B}_-^-}s^j\right) \tag{48}$$

where (44) was used. It is easy to see that $G_\mathrm{x}\mathrm{B}_-\big|_{s\to 0} = 1$, providing <u>integrating regulator, can be obtained if, and only if</u> $j \geq 1$, i.e. only for L_2 optimality. This is an important result: <u>the original formulation of the</u> H_2 <u>optimality of</u> $\|S\|_2$ <u>using the operator norm of the *SF* cannot provide an integrating regulator</u>. This is why the $Y_\mathrm{x}(s) = s^{-j}$ excitation was introduced and the extended norm $J_\mathrm{x}^2 = \|\varepsilon_\mathrm{x}\|_2$ was used in the optimization, see (36).

This result is not surprising, because the sensitivity function is usually a high-pass filter, however, the applicability of the Parseval theorem to calculate the H_2 norm directly for an error transfer function requires a minimal pole access one. This requirement is ensured by $j \geq 1$.

Example 1.
Assume a first order reference model $R_\mathrm{x} = 1/(1+sT_\mathrm{w})$ and be $\mathrm{B}_- = 1 - sT$, so $\mathrm{B}_-^- = 1 + sT$. The solution of the Diophantine equation (44) for $j = 0$ gives $\mathrm{R} = 2$ and $\mathrm{K} = -1$. The H_2 optimal filter is now

$$G_\mathrm{x} = -\frac{1+sT_\mathrm{w}}{1+sT} \tag{49}$$

and the H_2 optimal regulator is

$$C_* = \frac{\mathrm{A}(-1)}{\mathrm{B}_+\left[(1+sT)-(-1)(1-sT)\right]} = \frac{\mathrm{A}}{2\mathrm{B}_+} \tag{50}$$

which is not an integrating one.

Solving the problem for $j = 1$ gives $\mathrm{R} = 2T$ and $\mathrm{K} = 1$. The L_2 optimal filter is now

$$G_{\mathrm{x}} = \frac{1+sT_{\mathrm{w}}}{1+sT} \tag{51}$$

and the L_2 optimal regulator is

$$C_{*} = \frac{\mathrm{A}\left(+1\right)}{\mathrm{B}_{+}\left[\left(1+sT\right)-\left(+1\right)\left(1-sT\right)\right]} = \frac{\mathrm{A}}{2\mathrm{B}_{+}Ts} \tag{52}$$

which is an integrating one.

An important conclusion that the optimization of the *SF* in H_2 and L_2 norm spaces does not depend on the R_{x} reference model only on the invariant process factor B_{-} and j. This observation induced the need to investigate such criteria, which depend on the design goal, too.

6. The H_2, L_2 Optimal Realizability Loss

The optimization of the *GTDOF* control system can also be formulated by the *SF* decomposition introduced in section 2 and 3. Corresponding cost functions for the tracking and control properties can always be constructed by using the triangle inequality

$$\begin{aligned} J_{\mathrm{tracking}} &\le J_{\mathrm{des}}^{\mathrm{r}} + J_{\mathrm{real}}^{\mathrm{r}} + J_{\mathrm{id}}^{\mathrm{r}} = \left\| S_{\mathrm{des}}^{\mathrm{r}} \right\|_2 + \left\| S_{\mathrm{real}}^{\mathrm{r}} \right\|_2 + \left\| S_{\mathrm{id}}^{\mathrm{r}} \right\|_2 \\ J_{\mathrm{control}} &\le J_{\mathrm{des}}^{\mathrm{n}} + J_{\mathrm{real}}^{\mathrm{n}} + J_{\mathrm{id}}^{\mathrm{n}} = \left\| S_{\mathrm{des}} \right\|_2 + \left\| S_{\mathrm{real}} \right\|_2 + \left\| S_{\mathrm{id}} \right\|_2 \end{aligned} \tag{53}$$

The possible minimization of the second terms will be discussed in the sequel.

The goal of this optimization step is to minimize the realizability loss $J_{\mathrm{real}}^{\mathrm{x}}$ using optimal embedded filters $G_{\mathrm{x}} = G_{\mathrm{x}}^{\mathrm{opt}}$ attenuating the influence of the invariant model factor $\hat{P}_{-}$

$$G_{\mathrm{x}}^{\mathrm{opt}} = \arg\left\{ \min_{G_{\mathrm{x}}} \left(J_{\mathrm{real}}^{\mathrm{x}} \right) \right\} = \arg\left\{ \min_{G_{\mathrm{x}}} \left\| Y_{\mathrm{x}} R_{\mathrm{x}} \left(1 - G_{\mathrm{x}} \hat{P}_{-} \right) \right\|_2 \right\} = \arg\left\{ \min_{G_{\mathrm{x}}} \left\| \frac{R_{\mathrm{x}}}{s^j} \left(1 - G_{\mathrm{x}} \hat{P}_{-} \right) \right\|_2 \right\}$$
(54)

This task corresponds to the model matching approach (see Fig 3) of control system design and different from the one given by (37). The optimal realizability degradation is considerably different for *IS* and *IU* processes. For the *IS* case $\hat{P}_{-} = 1$, so there is no optimization problem to be solved and the trivial

$G_x^{opt} = G_r = G_n = 1$ selection can be used. The realizability degradation is zero now.

The form (54) shows again the generalized formulation of the second norms similarly to (36). It is also important to note that the "generalized energy norm" formulated by (54) is bounded if the low-pass Y_x is multiplied by a high-pass model matching error term $R_x(1 - G_x P_-)$ as in (36).

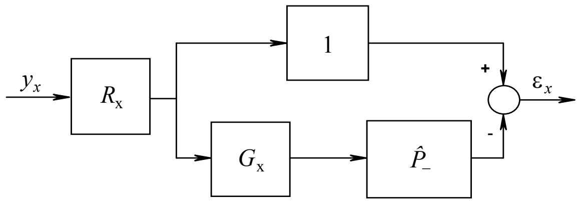

Figure 3. Reduced form of H_2, L_2 *optimality for* $\left\| S_{real}^x \right\|_2$

Assume the same reference model as in (39) and formulate the H_2, L_2 optimality for the cost function

$$J_{real}^x = \left\| Y_x R_x \left(1 - G_x \hat{B}_-\right) \right\|_2 = \left\| \frac{R_x}{s^j} \left(1 - G_x \hat{B}_-\right) \right\|_2 = \left\| \frac{\hat{B}_-}{\hat{B}_-^-} \right\|_2 \left\| \frac{\hat{B}_-^- B_x}{\hat{B}_- A_x s^j} - \frac{B_x G_x \hat{B}_-^-}{A_x s^j} \right\|_2 \quad (55)$$

where $\hat{B}_-^-$ has the same meaning as previously. Thus $\hat{B}_-^-$ is stable and $\left\| \hat{B}_- / \hat{B}_-^- \right\|_2$ is a stable norm-preserving inner function. Taking the usual decomposition

$$\frac{\hat{B}_-^- B_x}{\hat{B}_- A_x s^j} = \frac{R}{\hat{B}_-} + \frac{K}{A_x s^j} = \frac{R A_x s^j + K \hat{B}_-}{\hat{B}_- A_x s^j} \quad (56)$$

where in the right hand side $R / \hat{B}_-$ is the non-causal and $K / A_x s^j$ is the causal part consequently equation (55) becomes

$$J_x^2 = \left\| \frac{R}{\hat{B}_-} + \frac{K}{A_x s^j} - \frac{B_x G_x \hat{B}_-^-}{A_x s^j} \right\|_2 \quad (57)$$

Following the orthogonality principle the H_2, L_2 optimal G_x can be obtained making the causal part zero

$$\frac{\mathrm{K}}{\mathrm{A_x} s^j} - \frac{\mathrm{B_x} G_\mathrm{x} \hat{\mathrm{B}}_-^-}{\mathrm{A_x} s^j} = 0 \tag{58}$$

which results in

$$G_\mathrm{x} = \frac{\mathrm{K}}{\mathrm{B_x} \hat{\mathrm{B}}_-^-} \tag{59}$$

where R and K can be obtained from the special Diophantine equation

$$\hat{\mathrm{B}}_-^- \mathrm{B_x} = \mathrm{R}\,\mathrm{A_x} s^j + \mathrm{K}\hat{\mathrm{B}}_- \tag{60}$$

The minimum of the cost function formally is given by (46), where R is obtained from (60) now. The H_2, L_2 optimal regulator using (29) and (59) is

$$C_* = \frac{\mathrm{KA}}{\hat{\mathrm{B}}_+ \left(\mathrm{A_x} \hat{\mathrm{B}}_-^- - \mathrm{K}\hat{\mathrm{B}}_- \right)} \tag{61}$$

Let us investigate the product

$$G_\mathrm{x} \hat{\mathrm{B}}_- = \frac{\mathrm{K}\hat{\mathrm{B}}_-}{\mathrm{B_x}\hat{\mathrm{B}}_-^-} = 1 - \frac{\mathrm{A_x K}}{\mathrm{B_x}\hat{\mathrm{B}}_-^-} s^j \tag{62}$$

where (60) was used. It is easy to see that $G_\mathrm{x}\hat{\mathrm{B}}_-\Big|_{s\to 0} = 1$, providing <u>integrating regulator, can be obtained if, and only if</u> $j \ge 1$, <u>i.e. only for</u> L_2 <u>optimality</u>. <u>The original formulation of the</u> H_2 <u>optimality of</u> $\left\| S_\mathrm{real}^\mathrm{x} \right\|_2$ <u>using the classical operator norm</u> ($j = 0$) <u>cannot provide an integrating regulator</u>. This is why the $Y_\mathrm{x}(s) = s^{-j}$ excitation was introduced and the extended norm $J_\mathrm{x}^2 = \left\| \varepsilon_\mathrm{x} \right\|_2$ was used in the optimization, similarly to the previous section.

Example 2.
Solve the previous example with the H_2 optimal $\left\| S_\mathrm{real}^\mathrm{x} \right\|_2$ design principle. The solution of the Diophantine equation (60) for $j = 0$ gives

$$\mathrm{R} = \frac{2T}{T_\mathrm{w} + T} = \frac{2}{1+a} \quad ; \quad \mathrm{K} = \frac{T_\mathrm{w} - T}{T_\mathrm{w} + T} = \frac{a-1}{a+1} \quad ; \quad a = \frac{T_\mathrm{w}}{T} \tag{63}$$

The optimal filter is now

$$G_{\mathrm{x}} = \frac{T_{\mathrm{w}} - T}{T_{\mathrm{w}} + T}\frac{1}{1+sT} = \frac{a-1}{a+1}\frac{1}{1+sT} \tag{64}$$

It is easy to check that the gain of $G_{\mathrm{x}}\hat{\mathrm{B}}_{-}$ is not one now, so the corresponding regulator is not integrating. Solving the problem for $j=1$ gives

$$\mathrm{R} = \frac{2T^2}{T_{\mathrm{w}} + T} = \frac{2T}{1+a} = bT \ ; \ b = \frac{2T}{T_{\mathrm{w}} + T} = \frac{2}{1+a} \ ; \ \mathrm{K} = 1 + T_{\mathrm{g}}s = 1 + bT_{\mathrm{w}}s \ ; \ T_{\mathrm{g}} = bT_{\mathrm{w}} \tag{65}$$

and the obtained L_2 optimal filter is

$$G_{\mathrm{x}} = \frac{1+T_{\mathrm{g}}s}{1+sT} \tag{66}$$

It is easy to check that the gain of $G_{\mathrm{x}}\hat{\mathrm{B}}_{-}$ equals to one, so the corresponding regulator is an integrating one.

7. The H_∞, L_∞ Optimal Realizability Loss

The minimization of the realizability loss can be performed using the H_∞ norm, too. In this case the corresponding cost functions are

$$\begin{aligned} J_{\mathrm{tracking}} &\le J_{\mathrm{des}}^{\mathrm{r}} + J_{\mathrm{real}}^{\mathrm{r}} + J_{\mathrm{id}}^{\mathrm{r}} = \left\| S_{\mathrm{des}}^{\mathrm{r}} \right\|_\infty + \left\| S_{\mathrm{real}}^{\mathrm{r}} \right\|_\infty + \left\| S_{\mathrm{id}}^{\mathrm{r}} \right\|_\infty \\ J_{\mathrm{control}} &\le J_{\mathrm{des}}^{\mathrm{n}} + J_{\mathrm{real}}^{\mathrm{n}} + J_{\mathrm{id}}^{\mathrm{n}} = \left\| S_{\mathrm{des}}^{\mathrm{n}} \right\|_\infty + \left\| S_{\mathrm{real}}^{\mathrm{n}} \right\|_\infty + \left\| S_{\mathrm{id}}^{\mathrm{n}} \right\|_\infty \end{aligned} \tag{67}$$

where the optimization of the realizability loss requires the minimization of some $J_{\mathrm{real}}^{\mathrm{x}} = \left\| S_{\mathrm{real}}^{\mathrm{x}} \right\|_\infty \Big|_{\mathrm{x=r,n}}$ norms for tracking (r) and the disturbance rejection (n), respectively. These *SF* components have the general form of $\left\| S_{\mathrm{real}}^{\mathrm{x}} \right\| = \left\| R_{\mathrm{x}} \left(1 - G_{\mathrm{x}}\hat{P}_{-} \right) \right\|$, which can be simply rearranged equivalently to $\left\| S_{\mathrm{real}}^{\mathrm{x}} \right\| = \left\| R_{\mathrm{x}} - G'_{\mathrm{x}}\hat{P}_{-} \right\|$ as the Fig. 2 (a) and (b) show: consequently $G'_{\mathrm{x}} = R_{\mathrm{x}}G_{\mathrm{x}}$. Because the reference models have unity gains, it is enough to ensure the condition $G'_{\mathrm{x}}\hat{\mathrm{B}}_{-}\Big|_{s\to 0} = 1$ to the integral behavior of the regulator. If the excitation is not a Dirac impulse or a white noise it is reasonable to use the more general form of $\left\| S_{\mathrm{real}}^{\mathrm{r}} \right\| = \left\| Y_{\mathrm{x}} \left(R_{\mathrm{x}} - G'_{\mathrm{x}}\hat{P}_{-} \right) \right\|$, where $Y_{\mathrm{x}} = \mathcal{L}\left(y_{\mathrm{x}} \right) = s^{-j}$ is the Laplace transform of the well-known test signals. The goal of this optimization step is to minimize the realizability loss $J_{\text{real}}^{\text{x}}$ using the optimal embedded filters $G'_{\text{x}} = G_{\text{x}}^{\text{opt}}$ attenuating the influence of the invariant model factor $\hat{P}_{-}$ as

$$G_{\text{x}}^{\text{opt}} = \arg\left\{\min_{G'_{\text{x}}}\left(J_{\text{real}}^{\text{x}}\right)\right\} = \arg\left\{\min_{G'_{\text{x}}}\left\|\frac{1}{s^{j}}\left(R_{\text{x}} - G'_{\text{x}}\hat{P}_{-}\right)\right\|_{\infty}\right\} = \arg\left\{\min_{G'_{\text{x}}}\left[\sup_{\omega}\left|\frac{1}{s^{j}}\left(R_{\text{x}} - G'_{\text{x}}\hat{P}_{-}\right)\right|\right]\right\} \tag{68}$$

which is formulated in Fig 4. Here an s^{-j} form for the excitation Y_{x} was applied and R_{x} is a unity gain reference model. This formulation corresponds to the generalized approach introduced in (36), but for infinite norms, i.e.

$$J_{\text{x}}^{\infty} = \left\|Y_{\text{x}}\left(R_{\text{x}} - G'_{\text{x}}\hat{P}_{-}\right)\right\|_{\infty} = \left\|\frac{1}{s^{j}}\left(R_{\text{x}} - G'_{\text{x}}\hat{P}_{-}\right)\right\|_{\infty} = \begin{cases} \text{H}_{\infty} \text{ norm}; & j = 0 \\ \text{L}_{\infty} \text{ norm}; & j \geq 1 \end{cases} \tag{69}$$

This norm is the H_{∞} system (or operator) norm for $j = 0$ and is the L_{∞} signal norm for $j \geq 1$. One must know that the L_{∞} norm is usually formulated for real functions and not for complex functions as here. It is also important to note that this "generalized supremum norm" formulated by (69) is bounded for $j = 0$ and for $j \geq 1$ if the low-pass Y_{x} is multiplied by a high-pass control error term $\left(R_{\text{x}} - G'_{\text{x}}\hat{P}_{-}\right)$, which is satisfied for closed-loop system with type number higher than zero.

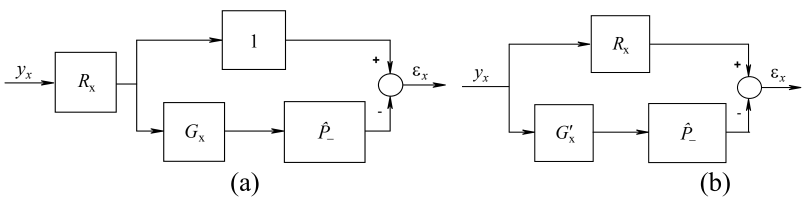

Figure 4. Reduced form of H_{∞} *optimality for* $\left\|S_{\text{real}}^{\text{x}}\right\|_{\infty}$

The optimal solution [8] lies in optimal interpolation theory and is known as the Nevanlinna-Pick problem. Assuming m_{-} number of unstable zeros in $\hat{P}_{-}$ the optimal W^{o} minimizing $\|W\|_{\infty}$ is an all-pass form

$$W^{\text{o}} = \begin{cases} \mu\dfrac{h^{\#}}{h} & \text{, if} \quad m_{-} \geq 1 \\ 0 & \text{, if} \quad m_{-} = 0 \end{cases} \tag{70}$$

where h is a Hurwitz polynomial of degree at most $\left(m_{-} - 1\right)$. The computation of $h^{\#}$ can be obtained by mirroring the zeros of h through the imaginary axis. The constant μ and the coefficients of h are real and are uniquely determined by the following - so-called - interpolation constraints

$$W(z_i) = \mu \frac{h^{\#}(z_i)}{h(z_i)} = \frac{1}{s^j}\left(R_x - G_x' \hat{P}_-\right)\Big|_{z_i} = \frac{R_x(z_i)}{z_i^j} = r_i \; ; \; i = 1, 2, \ldots, m_- \tag{71}$$

where $z_1, z_2, \ldots, z_{m_-}$ denote the distinct zeros of $\hat{P}_-$. (The multiplicity of zeros can be easily considered by additional differential interpolation constraints.) Because of the interpolation constraints (71) the m_- number of unknown parameters - $(m_- - 1)$ is in h and the m_--th is the μ itself - can be obtained from the m_- number of equations. It can also be observed that $\hat{P}_-$ is certainly the divisor of $R_x - s^j W$, thus

$$R_x - s^j W = \frac{B_x}{A_x} - \mu \frac{s^j h^{\#}}{h} = \frac{N}{D}\hat{P}_- = \frac{N}{D}\hat{B}_- \tag{72}$$

where the notation of $R_x = B_x / A_x$ and $\hat{P}_- = \hat{B}_-$ are used again and the form

$$G_x' = \frac{N}{D} \tag{73}$$

was introduced for the embedded filter. Comparing the two sides of (72) the polynomials D and N can be obtained by

$$D = A_x h \tag{74}$$

and

$$B_x h - N\hat{B}_- = A_x \mu s^j h^{\#} \quad \text{or} \quad N = \frac{B_x h - A_x \mu s^j h^{\#}}{\hat{B}_-} \tag{75}$$

where the last division is with no residue. The equation (71) can be rearranged into a "quasi" linear equation system in case of m_- disjunct real zeros z_i

$$\mu h^{\#}(z_i) = r_i h(z_i) \quad ; \quad i = 1, 2, \ldots, m_- \tag{76}$$

for the computation of h and μ. The minimum of the cost function is given by

$$\mu = \min\left\{\left\|W(j\omega)\right\|_\infty\right\} = \left\|W^o(j\omega)\right\|_\infty \tag{77}$$

After some straightforward manipulations one can obtain that

$$G'_{\mathrm{x}}\hat{\mathrm{B}}_{-} = \frac{\mathrm{B_x}h - \mathrm{A_x}\mu s^j h^{\#}}{\mathrm{A_x}h} = \frac{\mathrm{B_x}}{\mathrm{A_x}} - \frac{\mu h^{\#}}{h}s^j \qquad \text{or}$$

$$G_{\mathrm{x}}\hat{\mathrm{B}}_{-} = \frac{\mathrm{A_x}}{\mathrm{B_x}}\frac{\mathrm{B_x}h - \mathrm{A_x}\mu s^j h^{\#}}{\mathrm{A_x}h} = 1 - \frac{\mathrm{A_x}\mu h^{\#}}{\mathrm{B_x}}s^j \tag{78}$$

so it is easy to see that $G'_{\mathrm{x}}\hat{\mathrm{B}}_{-}\Big|_{s\to 0} = G_{\mathrm{x}}\hat{\mathrm{B}}_{-}\Big|_{s\to 0} = 1$, providing <u>integrating regulator, can be obtained if, and only if</u> $j \geq 1$, <u>i.e. only for</u> L_∞ <u>optimality</u>. <u>The original formulation of the</u> H_∞ <u>optimality of</u> $\left\|S^{\mathrm{x}}_{\mathrm{real}}\right\|_\infty$ <u>using the classical operator norm</u> ($j = 0$) <u>cannot provide an integrating regulator</u>.

Example 3.
Assume a first order reference model $R_{\mathrm{x}} = 1/(1+sT_{\mathrm{w}})$ and be $\hat{\mathrm{B}}_{-} = 1 - sT$. Because $m_{-} = 1$ the interpolation polynomial h is of zero order and the trivial $h = h^{\#} = 1$ constant can be selected. So μ can be easily obtained from the interpolation constraint (76): $\mu = R_{\mathrm{x}}(z_1 = 1/T) = T/(T_{\mathrm{w}} + T)$. The denominator polynomial of G'_{x} is $\mathrm{D} = 1 + sT_{\mathrm{w}}$ from (74). Apply $j = 0$ first. Because the right side of (75) is a first order polynomial therefore a scalar $\mathrm{N} = k$ can be used

$$1 - k(1 - sT) = (1 + sT_{\mathrm{w}})\mu \tag{79}$$

The solution gives $\mathrm{N} = k = T_{\mathrm{w}}/(T_{\mathrm{w}} + T)$ and μ is the same as obtained before. The H_∞ optimal filter is

$$G'_{\mathrm{x}} = \frac{\mathrm{N}}{\mathrm{D}} = \frac{\mathrm{N}}{\mathrm{A_x}h} = \frac{T_{\mathrm{w}}}{T_{\mathrm{w}} + T}\frac{1}{1 + sT_{\mathrm{w}}} \tag{80}$$

and it is easy to check that $G'_{\mathrm{x}}\hat{\mathrm{B}}_{-}\Big|_{s\to 0} \neq 1$, i.e. the optimal regulator is not integrating.

Example 4.
Apply $j = 1$ now, consequently (74) remains and only (79) will change to

$$1 - \mathrm{N}\,(1 - sT) = (1 + sT_{\mathrm{w}})\mu s^j \tag{81}$$

Searching a first order polynomial $\mathrm{N} = k(1 + s\tau)$, (81) will have the form

$$1-k(1+s\tau)(1-sT)=(1+sT_{\mathrm{w}})\mu s^{j} \tag{82}$$

Comparing the coefficients of the two sides the solution gives

$$k=1 \quad ; \quad \mu=\frac{T}{T_{\mathrm{w}}+T}T \quad ; \quad \tau=\frac{T_{\mathrm{w}}}{T_{\mathrm{w}}+T}T \tag{83}$$

The H_{∞} optimal filter is

$$G_{\mathrm{x}}'=\frac{\mathrm{N}}{\mathrm{D}}=\frac{\mathrm{N}}{\mathrm{A}_{\mathrm{x}}h}=\frac{1+s\tau}{1+sT_{\mathrm{w}}} \tag{84}$$

and it is easy to check that $G_{\mathrm{x}}'\hat{\mathrm{B}}_{-}\Big|_{s\to 0}=1$, i.e. the optimal regulator is integrating.

It is important to note that examples 3 and 4 are for a low order case ($m_{-}=1$), when $h=1$, just to demonstrate the computations. For higher order cases this optimality (the solution of Eqs. (71), (76)) requires to solve a nonlinear equations system contrary to examples 1 and 2, where the Diophantine equations are linear in the polynomial parameters. The original nonlinear task can be decomposed into a nonlinear (to determine μ and h) and a linear problem (to compute N by a simple polynomial division). Investigate first (71) and (76), which can also be written in a matrix form

$$\mathbf{r}=\mathbf{F}(\mu,r,\mathbf{z})\mathbf{h}(\mu,h) \tag{86}$$

Where

$$\mathbf{r}=\begin{bmatrix} r_1 & \dots & r_{m_-}\end{bmatrix}^{\mathrm{T}} \quad ; \quad \mathbf{z}=\begin{bmatrix} z_1 & \dots & z_{m_-}\end{bmatrix}^{\mathrm{T}} \tag{87}$$

$$\mathbf{h}(\mu,h)=\left[\mu,h_1,\dots,h_{m1}\right]^{\mathrm{T}} \quad ; \quad m_1=m_{-}-1 \tag{88}$$

$$\mathbf{F}(r,\mu,\mathbf{z})=\begin{bmatrix} 1 & f_{11} & \cdots & f_{1m_1} \\ 1 & f_{22} & \cdots & f_{2m_1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f_{m_-1} & \cdots & f_{m_-m_1} \end{bmatrix} \tag{89}$$

Here

$$f_{ik} = \sum_{k=1}^{m_1} \left[\mu \left(z_i^* \right)^k - r_i \left(z_i \right)^k \right] \quad ; \quad z_i^* = \text{sign}\left(h_i \right) z_i \tag{90}$$

The "quasi" linear term is used because the f_{ik} elements depend on μ, too. In such cases only iterative solutions can be formulated. One of the simplest methods is the relaxation type iterative algorithm

$$\mathbf{h}\left(\mu_{l+1}, h_{l+1} \right) = \left[\mathbf{F}\left(\mu_l, r, \mathbf{z} \right) \right]^{-1} \mathbf{r} \tag{91}$$

Having obtained the iterative solution for μ and h the polynomial N is computed from (75).

It is interesting to note that the order of the embedded filter G_x does not depend on j only on the invariant factor $\hat{B}_-$ (i.e., m_-) and the reference model R_x.

Example 5.

Consider a second order $B_- = \left(1 - sT_1 \right)\left(1 - sT_2 \right)$ *IU* process polynomial with $T_1 = 1$ and $T_2 = 2$. Select a unity gain first order reference model with $T_w = 0.5$. In this case $h\left(s \right) = 1 + h_1 s$ is first order because $m_- = 2$ and $m_1 = 1$. This is already a nonlinear problem as indicated above. The solution of the low order (85), however, still does not need the iterative algorithm, instead a second order polynomial equation system

$$\begin{aligned} &\mu^2 + \mu \frac{T_1 + T_2}{T_1 - T_2} \left(r_2 - r_1 \right) - r_1 r_2 = 0 \\ &h_1^2 + h_1 \frac{r_2 + r_1}{r_2 - r_1} \left(T_2 - T_1 \right) - T_1 T_2 = 0 \end{aligned} \tag{92}$$

can be formulated which has explicit analytical solution. Investigating the roots (both μ and h_1 must be positive) the following optimal solution is obtained for $j = 0$:

$$\mu = 0.9572 \quad \text{and} \quad h_1 = 0.1789 \tag{93}$$

Then N can be computed from the division (75)

$$N = 0.0428 \tag{94}$$

The L_∞ optimal filter is

$$G'_x = \frac{\mathrm{N}}{\mathrm{D}} = \frac{\mathrm{N}}{\mathrm{A_x} h} = \frac{0.0428}{(1+0.5s)(1+0.1789\mathrm{s})} \quad \text{and} \quad G_x = \frac{0.0428}{1+0.1789\mathrm{s}} \tag{95}$$

Applying the iterative algorithm (90) for $j=1$ the following numerical solutions are obtained

$$\mu = 3.1397 \quad \text{and} \quad h_1 = 0.6498 \tag{96}$$

The convergence of the iteration was very fast. Then N can be computed from the division in (75)

$$\mathrm{N} = 1 + 0.5s \tag{97}$$

and the L_∞ optimal filter is

$$G'_x = \frac{\mathrm{N}}{\mathrm{A_x} h} = \frac{1+0.5s}{(1+0.5s)(1+0.6498\mathrm{s})} = \frac{1}{1+0.6498\mathrm{s}} \quad \text{and} \quad G_x = \frac{1+0.5s}{1+0.6498\mathrm{s}} \tag{98}$$

8. Conclusions

The authors believe that the relatively easy and reasonably optimal solution of a generally very sophisticated control problem strongly depends on the proper decomposition of the original paradigm. These decompositions correspond to a natural control engineering practice, too, where the best reachable design goal and the way how to obtain it appear in a new iterative sequential procedure.

The paper investigated first the H_2, L_2 optimality of the *SF* in a *GTDOF* control system and the realizability loss in the decomposed form of the *SF*. The previous, classical optimality does not depend on the design goal, therefore the second one fits better to practical requirements. The optimization provides integrating regulator, iff a $Y_x(s) = s^{-j}$ form excitation is assumed with $j \geq 1$, which corresponds to L_2 optimality. <u>The classical H_2 optimality does not provide integrating regulator</u>.

Then the H_∞, L_∞ optimality of the realizability loss is investigated. The optimization provides also integrating regulator, iff a $Y_x(s) = s^{-j}$ form excitation is assumed with $j \geq 1$ which corresponds to L_∞ optimality. <u>The classical H_∞ optimality does not provide integrating regulator.</u>

Simple low order examples are also presented to demonstrate the computation of the optimal embedded filters for different cases and to help understanding the algorithms.

Finally an iterative method is also shown to solve the H_∞, L_∞ optimality of the realizability loss for higher order general cases.

The results can be easily applied for discrete time systems, too, where B_-^- contains the unstable zeros of B_- mirrored on the unit circle and $Y_x(z) = (z-1)^{-j}$. The major advantage of the application to discrete time system is, that the inclusion of the process time delay is relatively easy, because the transfer functions remain in the class of rational functions.

The obtained results are valid only for open-loop stable plants, where *Youla-parametrization* can be applied. This is the only parametrization, where the controller design uses explicit algebraic formulation, which is the basis for the above conclusions.

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Fuzzy Solution for Kano's Quality Model

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Abstract :For designing and developing products/services it is vital to know the relevancy of the performance generated by each technical attribute and how they can increase customer satisfaction. Improving the parameters of technical attributes requires financial resources, and the budgets are generally limited. Thus the optimum target is to achieve maximum customer satisfaction within given financial limits. Kano's quality model classifies the relationships between customer satisfaction and attribute-level performance and indicates that some of the attributes have a non-linear relationship to satisfaction, rather power-function should be used. For the customers' subjective evaluation these relationships are not deterministic and are uncertain. This paper proposes a method for fuzzy extension of Kano's model and presents numerical examples that can prove the efficiency of bacterial evolutionary algorithm in as well.

Key words: quality, Kano's model, fuzzy, bacterial evolutionary algorithm

1. Introduction

In the designing process of products/services, their technical attributes must be determined so that the maximum customer satisfaction can be achieved within acceptable and reasonable financial limits. Technical attributes have different effects on the satisfaction. Kano explored [5,6] that the features and characteristics of these relationships differ from the point of view of customers the utility functions are different as well. On the other hand customers requirements are not homogenous, they are changing in time and also differences can be detected even in the same market segment. Because of these differences in the mathematical model for Kano's quality assessment it is worth applying fuzzy numbers instead of crisp values. Results have been devoted to the relationship between technical attributes and customer requirements in correlation terms [3, 4], or represented the uncertainty of budgeting by fuzzy measures [11]. Assuming linear relationship [9] and [1] analyzed the issue. Reference [8] considered it as a linear problem by introducing the customer satisfaction coefficient. Reference [7] explored the asymmetric feature of the relationship between attribute-level performance and overall customer satisfaction and indicated indirectly that linear functions are not appropriate in each case. Application of fuzzy logic for ranking technical attributes is presented in [12].

In this paper the proposed fuzzy extension for Kano's model handles the customers assessment as a fuzzy number, the particular satisfaction functions are partly linear and partly non-linear and in a given market segment the overall satisfaction of all customers is the optimum criteria, so within budget limits set of technical attributes are to be determined that can maximize the overall satisfaction.

1. KANO's quality model

In his model [6] Kano distinguishes three types of product requirements which influence customer satisfaction in different ways when met (see Figure 1).

- Degressive or "must-be" requirements are basic criteria of a product. From a given point improving the technical attributes by unit results in minor increment in satisfaction, on the other hand not fulfilling the requirements induces dissatisfaction ("negative satisfaction"). See function *D* in Fig.1.
- One-dimensional requirements. Customer satisfaction is proportional to the level of fulfillment, the higher level of fulfillment, the higher the customer satisfaction, and vice versa. See function *L* in Fig.1.
- Progressive or attractive/excitement factors: fulfilling these requirement leads to more than proportional satisfaction. See function *P* in Fig.1.

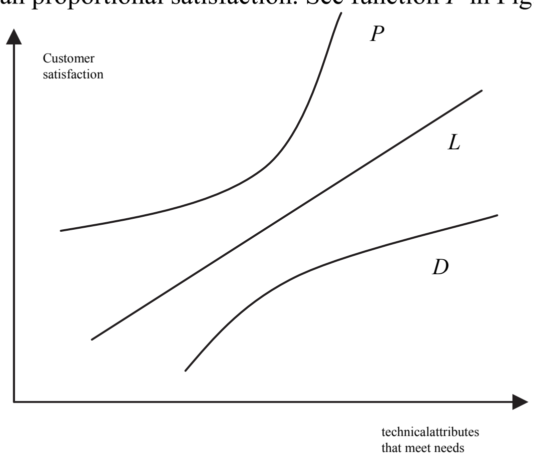

Figure 1. Kano's model of customer satisfaction

In this paper from now on we do not use the term "quality", because we are dealing only with technical attributes that result quality and have determined relationship to customer satisfaction. (Other cases are investigated in [3,5,8,9].)

2. Deterministic optimization of KANO's model

Questions of budgeting is not a key element of Kano's original model but we can reasonably assume that improving the level of technical attributes requires extra costs, so for each technical attribute a cost function can be set. The general target is to achieve the maximum economic result with the minimum use of resources, that is to maximize customer satisfaction with the minimum cost. The task can be mathematically formulated in two ways:

(A1) maximize overall satisfaction (*S*) not exceeding given cost limit (*C*)
or
(A2) achieve given overall satisfaction (*S*) with minimum cost (*C*).

(B) Increase customer satisfaction and decrease the cost at the same time, that is analogously to value analysis maximize *S/C*. This is the optimum point for the customer, since satisfaction fall versus unit cost is maximal.

Mathematical model

Let $$S_i(x_i) = b_i + a_i x_i^{\beta_i} \qquad i = 1,2 \ldots n \tag{1}$$

be the customer satisfaction generated by technical attribute x_i

$0 < x_i < < \infty$ is a real number variable
$a_i > 0$ is a real constant
$\beta_i > 0$ is a real constant
b_i is a constant such that $sg(b_i) = sg(\beta_i - 1)$

further $$C_i(x_i) = f_i + v_i\, x_i \qquad i = 1,2,\ldots n \tag{2}$$

be the cost of manufacturing technical attribute at level x_i , $f_i \geq 0$, $v_i \geq 0$, $x_i \geq 0$ are real constants

Let $$S = \sum_{i=1}^{n} (b_i + a_i x_i^{\beta_i}) \text{ be the overall satisfaction} \tag{3}$$

and $$C = \sum_{i=1}^{n} (f_i + v_i\, x_i) \text{ be the total cost.} \tag{4}$$

according to the "fixed costs - variable costs" methodology, and not considering the nonlinearity at 0 and 100% capacity usage (see Fig.2). Other shapes of cost functions do also exist, but in our case we follow the traditional solutions.

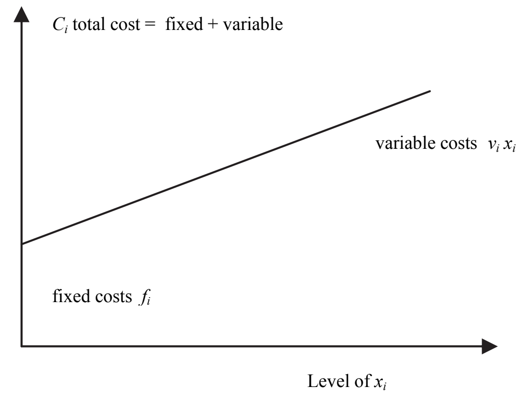

Figure 2. Cost of achieving different level of x technical attribute

Then the general formula for (A) is:

(A1) Let $\sum_{i=1}^{n} S_i(x_i) \rightarrow \max$, subject to $\sum_{i=1}^{n} C_i(x_i) \le C_o$, (5)

where C_0 is a given constant.

(A2) Let $\sum_{i=1}^{n} C_i(x_i) \rightarrow \min$, subject to $\sum_{i=1}^{n} S_i(x_i) \ge S_o$ (6)

where S_0 is a given constant.

(B) Let $\frac{\sum_{i=1}^{n} S_i(x_i)}{\sum_{i=1}^{n} C_i(x_i)} \rightarrow \max$, where $x_i < \infty$ (7)

In (A1) $x_i < \infty$ is automatically fulfilled but in (B) ∞/∞ type solutions might occur and must be eliminated.

Finding maximum and minimum values with limited x_i

In this paper we present a numerical example for (A1). For the approximation of the optimum the bacterial evolutionary algorithm [2, 10] is used. The original genetic algorithm is based on the process of evolution of biological organisms. It uses three operators: reproduction, crossover and mutation. The bacterial evolutionary algorithm is a more recent approach that gives an alternative to other evolutionary optimization algorithms, because it is simple and can achieve lower error than others within a short time. This method includes the bacterial mutation and the gene transfer operators. These operations were inspired by the microbial evolution phenomenon. The bacterial mutation operation optimizes the chromosome of one bacterium, the gene transfer operation allows to transfer information between the bacteria in the population. Each bacterium represents a solution of the original problem.

First the initial (random) bacteria population is created. The population consists of N_{ind} bacteria. The bacterial mutation is applied to each chromosome one by one. First N_{clones} copies (clones) are generated, then a certain part of the chromosome is randomly selected and the parameters of this selected part are randomly changed in each clone (mutation). Next all the clones and the original bacterium are evaluated and the best individual is selected. This individual transfers the mutated part into the other individuals. This process continues until all of the parts of the chromosome have been mutated and tested. At the end of this process the clones are eliminated. In the next step the gene transfer is applied, which allows the recombination of genetic information between two bacteria. First the population must be divided into two halves. The better bacteria are called superior half, the other bacteria are called inferior half. One bacterium is randomly chosen from the superior half this will be the source bacterium and another is randomly chosen from the inferior half, this will be the destination bacterium. A part from the source bacterium is chosen randomly and this part will overwrite the same part of the destination bacterium. This process is repeated for N_{inf} times per generation. When the maximum generation number is achieved the algorithm ends.

Parameters of the algorithm:

(N_{ind})	Number of bacteria in the population
(N_{clones})	Number of clones
(N_{inf})	Number of infections
(N_{gen})	Number of generations

Application for a numerical example

Find $x_1, x_2, \ldots, x_n$ that maximize S subject to $C \leq C_0$. Since function C_i is the cost of producing x_i, and an upper limit is given for $\sum C_i$, thus if we assume that only x_1 is produced and the other values are 0, then we obtain a $C_1(x_1) \leq C_0$ limit, and generally we can write:

$$0 \leq x_i \leq \frac{C_o - f_i}{v_i} \tag{8}$$

Let

$$\begin{aligned} S_1(x_1) &= 10 + 0.1\, x_1^{1.8} & C_1(x_1) &= 40 + 15\, x_1 \\ S_2(x_2) &= 5 + 0.15\, x_2^{1.8} & C_2(x_2) &= 20 + 20\, x_2 \\ S_3(x_3) &= 0 + 0.45\, x_3^{1} & C_3(x_3) &= 10 + 2\, x_3 \\ S_4(x_4) &= -5 + 0.15\, x_4^{0.5} & C_4(x_4) &= 20 + 0.2\, x_4 \\ S_5(x_5) &= -10 + 0.35\, x_5^{0.5} & C_5(x_5) &= 40 + 0.5\, x_5 \\ & & C_0 &= 1250 \end{aligned} \tag{9}$$

The result of the bacterial evolutionary algorithm is:

$$\begin{aligned} & x_1 = 0 \quad x_2 = 0 \quad x_3 = 558 \quad x_4 = 6 \quad x_5 = 5 \\ & S = 252 \\ & C = 1249.7 \end{aligned} \tag{10}$$

Results were obtained using the following algorithm parameters:

$$N_{gen}=1000 \quad N_{ind}=150 \quad N_{clones}=1000 \quad N_{inf}=30$$

As it is a stochastic algorithm the results of individual runs may differ, but they typically vary around central values. In about 3% of the runs the algorithm was not able to find the maximum value and got stuck at a local maximum, so it is adviceable doing several (e. g. at least three) runs.

3. Fuzzy extension of KANO's model

Classification of parameters in section 1. and the assignment to linear, degressive and progressive sets are based on classical characteristic functions of sets, that is a parameter is either linear or is not. This approach is not entirely in accordance with practical life even in case of one single customer, since assessment of the relevance of technical attributes is subjective and it can change in time as well. So the same technical attribute can be linear ("one-dimensional") in a case and can be progressive ("attractive") in another case. In such a situation an obvious solution can be the modification of exponent β subject to customers assessment that means the introduction of the time series $\beta(t)$, so:

$$S_i(x_i) = b_i + a_i x_i^{\beta_i(t)} \tag{11}$$

In this case forecasts for $\beta_i(t_1), \beta_i(t_2), \ldots.. \beta_i(t_k)$ must be calculated and the solution is the same as in 3.

The situation is significantly different when there are several customers at the same time (which is very often the case in practice), and some customers find a given technical attribute linear, some other assess it "must-be", and again some others possibly consider it as an attractive factor.

For example, let us assume that in a survey of 1000 it is found that technical attribute x_1 is considered as attractive by 700 customers, as linear by 200 ones and as "must-be" by 100 individual customers. In this case x_1 cannot be classified as a member of a crisp set of any type, it can be only determined with a certain degree of membership using a fuzzy set of technical attributes and constructing membership functions for x_1 , and similarly for each x_i . The following are examples for membership functions for x_1.

$\mu_P (x_1) = 0.7$ for progressive set
$\mu_L (x_1) = 0.2$ for linear set
$\mu_D (x_1) = 0.1$ for degressive set.

In our example the cost functions remain the same as they were, because customer assessment does not influence the actual resources required for producing the given level of technical attribute performance, so (2) can be used. However the overall satisfaction must be calculated subject to fuzzy approach.

4. Fuzzy optimization of KANO's model

The methodology of fuzzy solution depends on the approach applied for crisp (non-fuzzy) optimization. For the fuzzy extension there are two options.

In the first option we weight functions $S_i(x_i)$ in (5), (6) and (7) by their membership function values. Technical attributes x_i have satisfaction functions $S_{ij}(x_i)$ $(j = 1,2,3\ )$, so instead of (5) we can write:

$$\text{Let} \sum_{i=1}^{n}\sum_{j=1}^{3}\mu_{ij}(x_i)S_{ij}(x_i) \rightarrow \text{max, subject to } \sum_{i=1}^{n}C_i(x_i) \le C_o \qquad (12)$$

where $\mu_{ij}(x_i)$ $(j= 1,2,3$ *and* $i=1\ldots n)$ are the membership function values of the *i*th technical attribute for progressive, linear and degressive sets. We can rewrite (6) and (7) in the same way. Thus we back-up to the original crisp task, each satisfaction function is calculated by using three different exponents and weighted by membership function values. Disadvantage of this solution is that it can be applied only in the case when it is assumed that each progressive function has the same exponent and, analogously, no difference between the degressive exponents is allowed either. This strong restriction makes the practical use almost impossible.

An efficient solution in the practice is when the membership of each function is represented by the exponent β (option 2). The main features of each set are:

$0 < \beta < 1$ (degressive)
$\beta=1$ (linear)
$\beta > 1$ (progressive).

If β is considered as a fuzzy number then the features of each technical attribute are given by the shape of the membership function of β. (see Fig. 3.)

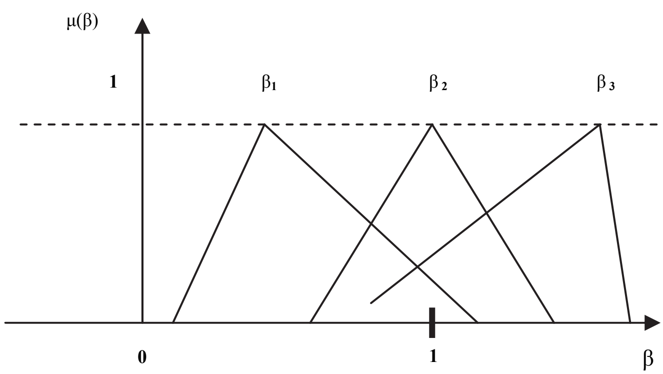

Figure 3. Exponent β as fuzzy number

The use of triangular fuzzy numbers is conform with the practical interpretation and can be easily handled by the proposed algorithm. The question is now how to interpret the fuzziness of exponents so that the original algorithm can be kept. An obvious solution is to weight the functions and sum up three weighted functions with weighted exponents:

- a "central" function, where $\beta_{i2} = \beta_i$
- a "left-side" function where exponent is calculated by using α-cuts

$$\beta_{i1} = \beta_i - (1-\alpha)(\beta_i - \beta_{iL}) \qquad (13)$$

- a "right-side" function, where the exponent is

$$\beta_{i3} = \beta_i + (1-\alpha)(\beta_{iR} - \beta_i) \qquad (14)$$

(see Fig.4). The weights of the functions are calculated in proportion to $\mu(\beta_i)$ membership function values based on similar considerations. Thus the function weights are:

$$1/(2\alpha + 1) \text{ for the "central" function} \qquad (15)$$

and

$$\alpha/(2\alpha + 1) \text{ for the "left-side" and "right-side" functions.} \qquad (16)$$

The calculation method for the exponents and the function weights is shown in Fig. 4.

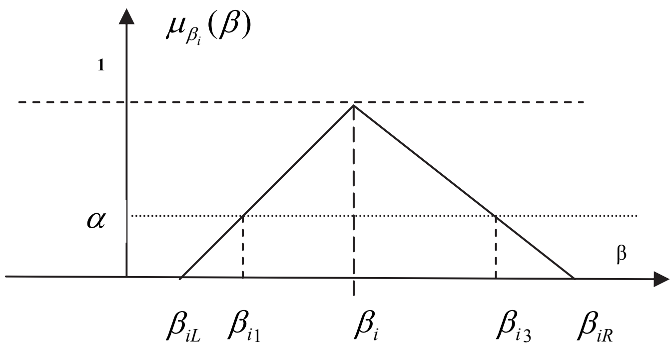

Figure 4. Weights and exponents based on α-cuts

Then instead (1) we can write:

$$S_i(x_i) = \frac{1}{2\alpha+1}(b_i + a_i x_i^{\beta_i}) + \frac{\alpha}{2\alpha+1}(b_i + a_i x_i^{\beta_i-(1-\alpha)(\beta_i-\beta_{iL})}) + \frac{\alpha}{2\alpha+1}(b_i + a_i x_i^{\beta_i+(1-\alpha)(\beta_{iR}-\beta_i)}) \qquad (17)$$

and rearranging (17) we obtain

$$S_i(x_i) = b_i + \frac{1}{2\alpha+1} a_i \, (x_i^{\beta_i} + \alpha x_i^{\beta_i-(1-\alpha)(\beta_i-\beta_{iL})} + \alpha x_i^{\beta_i+(1-\alpha)(\beta_{iR}-\beta_i)}) \qquad (18)$$

Equation (18) is simple enough for practical purposes and the applied bacterial evolutionary algorithm can be used with a minor modification.

For a more general solution instead of the center, left-side and right-side functions approach we can consider any *m=2k+1* functions (*k=1,2,3, …*) in (17), so we can write the following (see Fig.5):

$$S_i(x_i) = \sum_{j=1}^{m} \frac{\mu(\beta_{ij})}{\sum_{j=1}^{m}\mu(\beta_{ij})}(b_i + a_i\ x_i^{\beta_{ij}}) = \frac{\sum_{j=1}^{m}\mu(\beta_{ij})\,(b_i + a_i\ x_i^{\beta_{ij}})}{\sum_{j=1}^{m}\mu(\beta_{ij})} \qquad (19)$$

which equals to (18) in case of *m=3*.

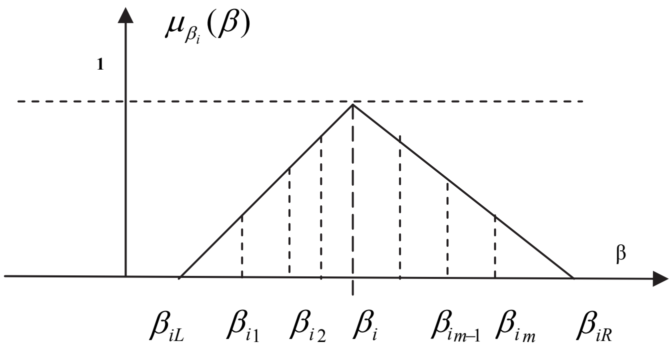

Figure 5. Exponents for m functions

Equation (19) can be transformed into the continuous case, then for any shape of membership function we can write:

$$S_i(x_i) = \frac{\int_0^\infty \mu(\beta_i)(b_i + a_i\ x_i^{\beta_i})\,d\beta_i}{\int_0^\infty \mu(\beta_i)\,d\beta_i} \qquad (20)$$

(Remark: Negative exponents are not allowed. Negative exponent of a technical attribute would mean that improving the level of technical attribute resulted in less satisfaction, which is obviously impossible in practice and thus without any interest in the model.)

Interpretation of fuzzy results

Obviously the overall satisfaction value obtained by using the fuzzy model is also a fuzzy number. Its α-cuts might be obtained from the respective results corresponding to the α-cuts of the exponent. Due to the concept we applied in this paper, the following definitions are proposed.

Let $S_i(x_i)$ be a triangular fuzzy number with membership function $\mu_S(x)$ defined on the set of real numbers by

$$\mu_S(x) = \begin{cases} \dfrac{x - S_L}{S_C - S_L} & \text{for } S_L \leq x \leq S_C \\ \dfrac{x - S_R}{S_C - S_R} & \text{for } S_C \leq x \leq S_R \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

where $[S_L , S_R]$ is the supporting interval, and

$$S_R = \max_{0 \langle \alpha \langle 1} S_i(x_i) \quad \text{and } S_L = \min_{0 \langle \alpha \langle 1} S_i(x_i) \tag{22}$$

5. Numerical example

For the fuzzy model we use the data of (9) with the following fuzzy exponents:

$\beta_1 = (1.7,\ 1.8,\ 2.2)$
$\beta_2 = (1.6,\ 1.8,\ 2.1)$
$\beta_3 = (0.7,\ 1.0,\ 1.2)$
$\beta_4 = (0.4,\ 0.5,\ 0.8)$
$\beta_5 = (0.4,\ 0.5,\ 0.7)$

where $\beta_i = (\beta_{iL} ,\ \beta_{i,}\ \beta_{iR})$ (see Fig.4). The results are shown in Table 1.

Table 1. Results of fuzzy approximation

α	$\sum S_i$	x_1	x_2	x_3	X_4	x_5
0.1	301	74.6	0	0	1.1	0.5
0.5	304	74.5	0	0	3	3
0.9	249	0	0	559	3	2.7
[a]1.0	252	0	0	558	6	5.5

Conclusions

The benefit of fuzzy extension can be measured by the advantage we obtain analyzing the outputs. Difference between overall satisfaction values is to be considered, but what is more important the structure of technical attribution has to be examined. The aim is to allocate the limited resources subject to the maximum profit requirement. Customer satisfaction is the key element of profitability and in the first step of achieving this satisfaction is based in the designing and resource allocating process. The customers' assessment of technical attributes is very uncertain especially at the beginning of product life-cycle so in Kano's model the exponents of satisfaction functions cannot be considered as deterministic values. We propose the fuzzy extension of the model in order to explore the possible alternative sets of technical attributes. In the numerical example the fuzzy solution is significantly different from the crisp (deterministic) version, not only in terms of total satisfaction but – what is more important – in terms of technical attribute levels. In case of $\alpha=0.9$ the x_i values are practically equal to the original solution, but at $\alpha=0.1$ the set of x_i-s transformed and attractive (progressive) technical attributes seem to be more important.

Acknowledgment

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Lifetime Engineering for Roads

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Abstract: Lifetime engineering is the concretisation of an innovative idea for solving the dilemma existing between in infrastructure as a long-term product and the short-term approach to its design, management and maintenance planning. Although lifetime engineering was originally developed for buildings and bridges, its principles can be readily utilised for roads. Sustainable road construction needs the assessment of Life Cycle Costs (LCC) of the structures, the encouragement of data collection for benchmarks, as well as public procurement and contract award incorporating LCC. The use of lifetime-oriented road management has become world-widely more and more widespread The possible high recycling rate in road construction is strived for. Several lifetime engineering elements (life cycle costing, pavement performance models, user cost calculation, internalisation of external road effects, evaluation of the actual effect of road maintenance to pavement performance etc.) already available in Hungarian road management are also introduced.

Keywords: lifetime engineering, road planning, decision making, maintenance, recycling, environmental impacts

1. Introduction

Our society is living through a period of great change, in which we can also see changes in the central goals and requirements of construction techniques. The challenge to the present generation is to lead rapid development of a global economy towards sustainability in relation to our entire society, economy, social welfare and ecology.

Buildings, and civil and industrial infrastructures are the longest lasting and most important products of our society. The economic value contained in buildings, and civil and industrial infrastructures are, to say the least, significant; and the safe, reliable and sound economic and ecological operation of these structures is greatly needed. In industrialised countries buildings and civil infrastructures represent about 80 per cent of national property. Construction plays a major role in the use of natural resources and in the development of the quality of the natural environment in our time. Consequently, building and civil engineering can make a major contribution to the sustainable development of society.
The sustainability of buildings and built environment can, in short, be defined as thinking in time spans of several generations. Sustainability includes social aspects (welfare, health, safety, comfort), economic aspects, functional aspects (usability for changing needs), technical aspects (serviceability, durability, reliability) and ecological aspects (consumption of natural resources such as energy, raw materials and water; air

water and soil pollution, waste production; and impact on biodiversity), all related over the entire life cycle of the built facilities. It could be claimed that a built facility can only be as good as its design. The technical definition for sustainable building can be: "Sustainable building is a technology and practice which meets the multiple requirements of the people and society in an optimal way during the life cycle on the built facility" [11].

Design is an important part of construction: translating the requirements of owners, users and society into performance requirements of the structural system; creating and optimising structural solution which fulfil those requirements, and finally, proving through analysis and dimensioning calculations, that these requirements are fulfilled.

Monetary costs are treated, as usual, by current value calculations. Environmental costs are the use of non-renewable natural resources (materials and energy), and the production of air, water or soil pollution. The consequences of air pollution are health problems, inconvenience for people, ozone depletion and global warming. These impacts dictate the environmental profiles of the structural and building service systems. The goal is to limit the environmental costs to permitted values and to minimise them. Integrated lifetime design is an important link in construction: translating the requirements of owners, users and society into performance requirements of the technical systems; creating and optimising technical solutions, which fulfil those requirements; and proving through analysis and dimensioning calculations that the performance requirements will be fulfilled over the entire design service life. The adoption of these new methods and processes will increase the need for renewed education and training of all those involved. This new model of integrated life cycle design, also called lifetime design, includes a framework for integrated structural life cycle design, a description of the design process and its phases, and special lifetime design methods with regard to different aspects discussed above.

Quality assurance has been widely systematised under the ISO 9000 standards. An environmental efficiency procedure is presented in the ISO 14000 standards. The impact of life cycle principles in construction is in the application of life cycle criteria in the quality assurance procedure. Multiple life cycle criteria are also applied during the selection of products, although most of the product specifications have already been produced at the design phase.

Integrated life cycle design supports and improved quality approach, which can be called life cycle quality. All its areas are treated over the life cycle of structures, and controlled in the design by technical performance parameters.

The life cycle performance of structures is highly dependent on maintenance. The first important instructions for life cycle maintenance are produced during the design stage. The structural system of a building or civil engineering facility needs a users' manual, just like a car or any other piece of equipment. The manual will be produced gradually during the design process in co-operation with those involved in design, manufacture and construction. The usual tasks of the structural designer are: compiling a list of maintenance task for the structural system, compiling and applying operational instructions, control and maintenance procedures and works, checking and co-ordinating the operational, control and maintenance instructions of product suppliers and contractors, preparing the relevant parts of the users' manual, and checking relevant parts of the final users' manual.

The active reduction of waste during construction, renovation and demolition is possible through the selective dismantling of structural systems, components and materials specifically for recycling. Selective dismantling includes detailed planning of the dismantling phases, and optimising the work sequences and logistics of the dismantling and selection process. The main goal is to separate the different types of materials and different types of components at the demolition phase in order to avoid multiple actions. The recyclability of the building materials and structural components depends on the degree and/or technical level of the desired reuse.

2. Lifetime engineering

Lifetime engineering is an innovative idea and a concretisation of this idea for solving the dilemma that currently exists between infrastructures as very long-term products and their short-term approach to design, management and maintenance planning [14]. The main elements of lifetime engineering are:

- lifetime investment planning and decision making,
- integrated lifetime design,
- integrated lifetime management and maintenance planning,
- modernisation, reuse, recycling and disposal,
- integrated lifetime environmental impact assessment and minimisation.

The integrated lifetime engineering methodology concerns the development and the use of technical performance parameters to guarantee that the structures meet throughout their whole life cycle the requirements coming from human conditions, economy, cultural, social and ecological considerations. Thus, using lifetime engineering, the human conditions (safety, health and comfort), the monetary (financial) economy and the economy of the nature (ecology) can be controlled and optimised tacking into account cultural and social needs.

For life cycle design, the actual analysis and design are expanded also to the levels of monetary economy and ecology. Life cycle expenses are calculated into present value or annual costs by discounting manufacture, construction, maintenance, repair, rehabilitation, reuse, recycling, disposal etc. expenses.

However, lifetime engineering was originally developed for buildings and bridges, its basic principles can be readily utilized for roads, as it will be shown subsequently.

3. Lifetime engineering development process

There is a clear need for a uniform approach for assessing, validating and operating infrastructures, buildings and industrial, facilities with the consideration of the generic requirements presented in Table 1 [10]

Table 1. Generic classified requirements of the structure

1. Human requirements	**2. Economic requirements**

• functionality in use
• safety
• health
• comfort
	• investment economy
	• construction economy
	• lifetime economy in:
	  o operation
	  o maintenance
	  o repair
	  o rehabilitation
	  o renewal
	  o demolition
	  o recovery and reuse
	  o disposal

3. Cultural requirements	**4. Ecological requirements**
• building traditions • life style • business culture • aesthetics • architectural styles and trends • imago	• raw materials economy • energy economy • environmental burdens economy • waste economy • biodiversity

Moving into lifetime technology means that all processes should be renewed. Furthermore, new methodologies and calculation methods are to be adopted from mathematics, physics, system engineering, environmental engineering etc. However, the need for strong systematic transparency and simplicity of design process has to be kept in mind in order to keep the multiple issues under control and to avoid excessive design activities. The adoption of new methods and process necessitates the renewal of the education and the training of all stakeholders.

Since the design for durability is an important element of lifetime engineering, its main principles will be briefly presented.

4. Design for durability

Durability design methods can be classified starting from most traditional and ending in most advanced methods as follows:

1. design based on structural detailing
2. reference factor method
3. limit sate durability design.

Durability design with structural detailing

Structural detailing for durability is a dominant practical method which is applied to all types of materials and structures. The principle is to specify structural design and details as well as materials so that both deterioration effects on structures, and the effects of environmental impacts on structures, can be eliminated or diminished. The first of these if typically dominant when designing structures, such as wooden buildings, which are sensitive to environmental effects. The second principle is appropriate for structures which can be designed to resist even stronger environmental impacts, such as concrete, coated steel or wooden structures.

The methods and details for durability detailing are presented in current norms and standards.

Reference factor method

The reference factor method aims to estimate the service life of a particular component or assembly in specific conditions. It is based on a reference service life – in essence the expected service life in the conditions that generally apply to that type of component or assembly – and a series of modifying factors that relate to the specific conditions of the case. The method uses modifying factors for each of the following:

- A quality of components
- B design level
- C work execution level
- D indoor environment
- E outdoor environment
- F in-use conditions
- G maintenance level.

Estimated service life of the component (ESLC):

$$ESLC = RSLC \times A \times B \times C \times D \times E \times F \times G$$

where RSLC is the reference service life of the component.

The reference factor method is always an additive method, because reference service life always has to be known. The reference factor method is most often needed because the environmental exposure (environmental load onto structure) usually varies over a wide scale. Many parametric methods including the values of parameters in different conditions already exist.

Limit sate durability design

Although this method is presented and applied in detail for concrete structures, similar methods can also be applied to steel, wooden and masonry structures. Deterioration processes, dictating environmental loads, degradation factors and degradation calculation models are different for different materials.

The simplest mathematical model for describing a 'faliure' event comprises a load variable S and a response variable R. In principle the variables S and R can be any quantity and be expressed in any units, the only requirement is that they are commensurable. Thus, for example, S can be a weathering effect and R can be the capability of the surface to resist the weathering effect without too much visual damage or loss of the concrete reinforcement cover.

If R and S are independent of time, the 'failure' event can be expressed as follows [10].

$$\{\text{failure}\} = \{R<S\} \quad (1)$$

The failure probability P_f is now defined as the probability of that 'failure':

$$P_f = P\{R<S\} \quad (2)$$

Either the resistance R or the load S or both can be time-dependent quantities. Thus the failure probability is also a time-dependent quantity. Considering $R(\tau)$ and $S(\tau)$ are instantaneous physical values of the resistance, and the load at the moment τ the failure probability in a lifetime t could be defined as:

$$P_f(t) = P\{R(\tau)<S(\tau)\} \text{ for all } \tau \leq t. \quad (3)$$

The determination of the function $P_f(t)$ according to Equation 3 is mathematically difficult. That is why R and S are considered to be stochastic quantities with time dependent or constant density distributions. By this means the failure probability can usually be defined as:

$$P_f(t) = P\{R(\tau)<S(t)\} \quad (4)$$

According to the Equation 4 the failure probability increases continuously with time.
At a given moment of time the probability of failure can be determined as the sum of products of two probabilities: 1) the probability that $R<S$, at $S = s$, and 2) the probability that $S = s$, extended for the whole range of S:

$$P_f(t) = P_f = \int P\ \{R<S \mid S = s\} P\{S=s\} \quad (5)$$

Considering continuous distributions the failure probability P_f at a certain moment of time can be determined using the convolution integral:

$$P_f = \int F_R(s) f_s(s) ds \quad (6)$$

where $F_R(s)$ is the distribution function of R,
$f_s(s)$ the probability density function of S, and
s the common quantity or measure of R and S.

The integral can be solved by approximative numerical methods.

5. Sustainable road construction

Sustainable development is a matter of satisfying the needs of present generations without compromising the ability of future generations to fulfil their own needs [13]. Sustainable development means sustainability not only ecologically (= environmentally) and economically but also socially and culturally.

SBIS (Sustainable Building Information System) has been established in Canada to provide users with non-commercial information about sustainable building around the world, and to point or link the user to more detailed sources of information elsewhere.

The World's Largest Life Cycle Assessment – LCA database is available in Japan quantifying the products' impact on the environment throughout its life cycle.

The series ISO 15686 "Building and constructed assets – Service life planning" offers new tools for the life cycle planning of buildings or other constructed assets included roads.

The final report of EC-project "Life Cycle Costs in Construction" [9] has the following recommendations:

- adoption of a common European methodology for assessing Life Cycle Costs (LCC) in construction, encouragement of data collection for benchmarks, supporting best practice and maintenance manuals,
- public procurement and contract award incorporating LCC,
- life cycle cost indicators displayed in buildings open to public,
- life cycle costing at the early design stage of a project,
- fiscal measures to encourage the use of LCC,
- development of guidance and fact sheets.

The ways in which built structures are procured and erected, used and operated, maintained, repaired, rehabilitated and finally demolished (and recycled, reused) constitute the complete cycle of sustainable construction activities. The use of materials, energy and water, and mobility should be minimised [13].

6. Life cycle of road structures

Several countries including Finland [7] have developed lifetime-oriented road management. Usually firstly significant contributions to the research and development of long-term pavement performance are made. The long-term performance models are naturally never complete (final). Secondly, the procurement methods go through profound changes. Typically, the road maintenance responsibility after investment is included in the contract. The daily maintenance contracts therefore cover larger geographic areas with longer contract periods. The contractors and consultants need to learn how to evaluate the life cycle and the life cycle costs of roads. Empirical knowledge, careful observation of existing structures and competence are the keys to success. The client has to clearly describe the targets and the desired quality levels. Besides, the client has also to be in charge of collecting preliminary data to ensure that tenders can be submitted without unnecessary risks.

The life cycle of pavement depends on the bearing capacity – and in a lot of countries the frost susceptibility – of pavement structure. Nowadays, these factors are usually satisfactorily taken into account for main roads. At the same time, for secondary roads the relevant threshold values allow greater variation. As a result, more maintenance measures are required during life cycle.
Typically, rutting type defects due to the deformation of unbound base or sub-base layers need to be repaired. However, unevenness is also often the cause of major maintenance. The selection of rehabilitation methods depends on the factors causing the actual defect. Environmental aspects are also more and more considered by the recycling of road structural material.

Construction expenses are generally rather high compared to life cycle maintenance costs. The former expenditure can be reduced by the possibly maximum rate of recycling, the minimisation of using materials from outside.

There is no generally accepted methodology to calculate the residual value for road structures at the end of investigation period. One of the possible ways is to consider the construction costs during the planning period as increasing (positive) factor and the deterioration (wear) of structures as decreasing (negative) factor. The remixing of wearing course slightly changes the actual residual value. The residual value of a road structure can be higher than the asset value at the beginning of the period.

The increasing use of performance-based specifications in road project tendering and contracting also paves the way for the application of the principles of lifetime engineering [8].

7. Some lifetime engineering elements available in Hungarian road management

Although it is evident that the lifetime engineering as a science has not been utilised, usually not even known by the Hungarian road engineers, several of its elements have already been worked out and applied in road (pavement, bridge etc.) management systems and in practice [4], as follows.

a.) Whole life (life cycle) costing is more and more used in the planning and design of major Hungarian road projects.
b.) Pavement performance models have been developed for the forecasting of future behaviour of various road pavement types as a function of time or traffic passed [5].

c.) The user costs (vehicle operating costs, time delay costs and accident costs) as a function of different pavement conditions levels were also estimated and utilised in the national economy level forecast of life cycle costs.

d.) Considerable effort has been made for the internalisation of such external road effects as air pollution, traffic noise and vibration.

e.) The actually effect of major road maintenance (rehabilitation) has also been evaluated using trial section monitoring information [1].

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Finite Element Mechanical Modeling Opportunities in Machine Design

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Abstract: In machine design one of the most important questions is, can a structure can work normally or does it fail at a given loading. This question can be answered in a theoretical way by using well founded mechanical models. The article gives a brief summary on the theoretical basis of mechanical modeling of machine parts, i.e. a review of the classical mechanical modeling opportunities for machine parts of both isotropic and orthotropic materials. Finite element (FE) modeling opportunities will also be demonstrated within some industrial problems.

Keywords: mechanics of materials, mechanical modeling, finite element (FE), industrial applications

1. Introduction

In machine design one of the main questions is to find the proper dimensions of certain machine parts so they do not fail. Solving this problem in a theoretical way one has to use failure criteria. Failure criteria are usually based on the knowledge of the strain and stress states of parts. Therefore the first task is to determine (at least approximately) the above states.

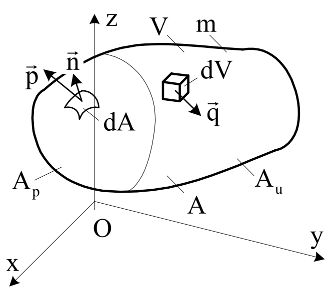


Figure 1. A general shaped machine part

Under normal working conditions machine parts usually have to remain in the elastic state. The strain and stress states of an elastic part or body can be determined by the governing equations of elasticity. These governing equations are a set of partial differential and algebraic equations that one can divide into the following groups [2]:
Stress equilibrium equations:

$$\mathbf{F}\cdot\nabla+\vec{q}=\vec{0}\,, \tag{1}$$

where $\mathbf{F}$ is the stress tensor, ∇ is the Hamilton operator, $\vec{q}$ is the body force vector per volume unit and $\cdot$ is the sign of scalar product.
Kinematical (strain-displacement) equations:

$$\mathbf{A}=\frac{1}{2}\left(\vec{u}\circ\nabla+\nabla\circ\vec{u}\right), \tag{2}$$

where $\mathbf{A}$ is the strain tensor, $\vec{u}$ is the displacement vector and $\circ$ is the sign of exterior (or outer) product.
Constitutive equations (material law)

$$\mathbf{F}=\frac{E}{1+\nu}\left(\mathbf{A}+\frac{1}{1-2\nu}A_I\mathbf{E}\right), \tag{3a}$$

where E is Young's modulus, ν is the Poisson's ratio, A_I is the first strain invariant and $\mathbf{E}$ is the unit tensor. The above (3a) Hooke's law is valid only in case of isotropic linear elasticity. The (3a) tensor equation can also be written in a matrix form:

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{12}\\ \gamma_{23}\\ \gamma_{13}\end{bmatrix}=\begin{bmatrix}\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{23}\\ \tau_{13}\end{bmatrix}, \tag{3b}$$

where ε and γ are normal and shear strain components of strain tensor $\mathbf{A}$ and σ and τ are normal and shear stress components of stress tensor $\mathbf{F}$.
In anisotropic linear elasticity the most frequent case is the orthotropic material behavior. An orthotropic material can be characterized by three Young's moduli E_1, E_2, E_3, by three of Young's moduli independent shear moduli G_{12}, G_{23}, G_{13} and by

Poisson's ratios ν_{12}, ν_{23}, ν_{13}, ν_{21}, ν_{32}, ν_{31}. But the Poisson's ratios depend on each other because of energetic reasons: $\frac{\nu_{12}}{E_2} = \frac{\nu_{21}}{E_1}$, $\frac{\nu_{13}}{E_3} = \frac{\nu_{31}}{E_1}$ and $\frac{\nu_{23}}{E_3} = \frac{\nu_{32}}{E_2}$, i.e. the material matrix is symmetric

Hooke's law for orthotropic material are as follows:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix}. \qquad (3c)$$

Kinematical boundary condition at A_u surface part of body:

$$\vec{u}\big|_{A_u} = \vec{u}_0, \qquad (4)$$

where $\vec{u}_0$ are given displacements.

Dynamical boundary conditions at A_p surface part of body:

$$\mathbf{F} \cdot \vec{n}\big|_{A_p} = \vec{p}_0, \ (5)$$

where $\vec{n}$ is the outer normal unit vector of A_p surface and $\vec{p}_0$ are given forces distributed on the A_p surface.

The equations (1)-(5) describes a boundary value problem of elasticity and provide theoretically the solution for the unknown $\vec{u}$, $\mathbf{A}$ and $\mathbf{F}$ fields.

In complex shaped parts and arbitrary loading and kinematical boundary conditions there is no exact solution that fulfills every equation. So one has to introduce simplifications and assumptions for the real problem i.e. has to model the real problem.

From the point of view of the investigation we retain in a modeling process the most important and essential properties of body or behavior of process and other properties or behavior will be neglected . In the field of mechanics of materials there are both classical and numerical opportunities for modeling an engineering problem.

2. Mechanical modeling

2.1. Classical models in mechanics of materials

In the classical models of mechanics of materials the set of governing equations of elasticity simplifies essentially by application of geometrical and kinematical / stress assumptions. For these simplified equations one can already find exact or well approximate solutions. The most important and common models of mechanics of materials are the following:

- *1D Axial structures (bars, rods, trusses, links etc.)*. The model of real structure is in this case a 1D (one dimensional) geometrical entity i.e. the centroid line of part. The quantities describing the mechanical behavior are bonded to this line. Solutions for axial structures problems are provided by different bending (Bernoulli, Timoshenko, etc.), torsion, etc. theories only in simple cases [2].
- *2D models: plane strain, plane stress and rotationally symmetric parts subjected to axial symmetric loading.* At plane strain problems the deformation occurs in a certain plane. Plane stress problems are in plane loaded plates. In axial symmetric problems the deformation occurs in the meridian plane of part. All three problems can be described by two basic variables that are functions of the same two space coordinates. For plane strain and plane stress problems the stress function method provides classical solutions for geometrically simple cases [2].
- *Plate and shell models.* Plates and shells are parts or bodies where it is possible to define a middle plane or middle surface. Every quantity describing the mechanical behavior is bonded to this plane/surface. For managing such structures different plate and shell bending theories (Kirchhoff-Love, Reissner-Mindlin, etc.) and shell membrane theories were developed. Using these theories it is possible to construct solutions for more simple cases such as circular and rectangular plates or cylindrical and spherical shells [8].

Using classical models and solutions of mechanics of materials for engineering problems one will usually encounter great, mainly mathematical difficulties. In axial structures it is very difficult to manage spatial structures with more hundreds or thousands of parts (bars, beams) with classical methods. In this area a special problem is the handling of the statically indetermination.

In 2D, plate and shell models one can find mathematically very complicated classical solutions that can manage only quite simple problems from an engineering point of view.

To overcome these difficulties the engineers usually use numerical methods to get approximate solutions for real engineering problems.

2.2. Finite element modeling tools

The finite element method (FEM) is one of the most efficient procedures for solving engineering problems in the field of mechanics of materials [1]. In this field a lot of reliable general purpose FEM codes are available for engineers that provide further modeling tools for solving real problems. By using FEM it is possible to manage both very complicated 3D real problems and every above classical problem of the mechanics of materials. The application of classical models results in a substantial decrease in the amount of computations. Beside this the FE method offers further modeling possibilities that also decrease the computational amount or leads to a more accurate approximation of a real problem. In the following the four most useful FEM modeling tools for machine design will be shown:

- *Symmetry conditions (consideration of an axis of symmetry in plane/2D case and a plane of symmetry in spatial/3D case).* Consideration of symmetry conditions is possible in case a structure has a symmetry axis or plane both in the sense of geometry and loading. In this case the half of the structure can be neglected and so the FEM mesh has to be set up for only half of the structure. This procedure reduces the number of unknown variables about on the half. On symmetry axis/plan it is necessary to constrain by kinematical boundary condition (4) that the nodes cannot leave the axis/plane. Naturally they can have arbitrary displacement along the axis or in the plane.
- *Sector symmetrical conditions.* A structure is sector symmetrical, if it has periodic parts in sense of geometry and loading. In such a structure the periodic parts can be assigned in several different ways (Figure 2.1). FEM allows modeling the structure by only one sector or periodic part of the whole body that causes a huge reduction in the number of unknown variables. The inner dashed lines/surfaces of a sector are constrained to have the same deformation. The fulfillment of this condition can be reached by signing node couples that have the same displacements.

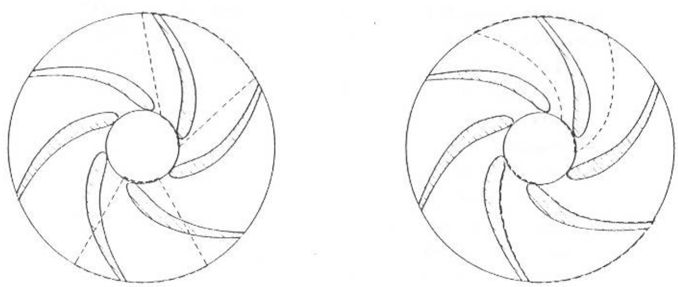


Figure 2. Sector symmetrical structure

- *Eccentric connections.* In machine structures it is a common solution to stiffen a thin wall by a beam of a given e.g. U cross section (Figure 3a). Often it can also happen that a wall thickness changes very quickly or by leaps and bounds (Figure 3b). In the first 3a case the centroid line does not coincide with the middle plane of wall and in the other 3b case the middle planes of plates of different thickness does not coincide with each other. FEM allows a rigid connection of these eccentric entities.

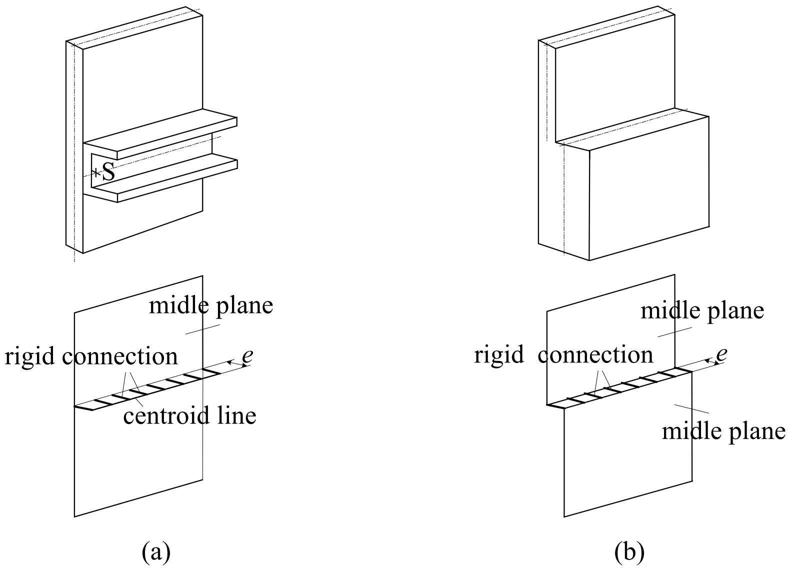


Figure 3. Sample eccentric connections

- *Continuous elastic support.* Different elastic machine parts are usually connected to each other by a surface contact. If one of the parts can be considered as a rigid body, then this connection can be modeled by a rigid support or a zero kinematical boundary condition (4), or by given distributed surface forces (5). But if both parts are elastic, the influence of the other part can be modeled using FEM with a continuous elastic support, that is using a continuously distributed spring system (Figure 4). Classical solutions for elastic supported parts exist only in a limited field of beam and plate problems. By using FEM the application of elastic supports can be extended for both other classical models and the general 3D case of problems of mechanics of materials.

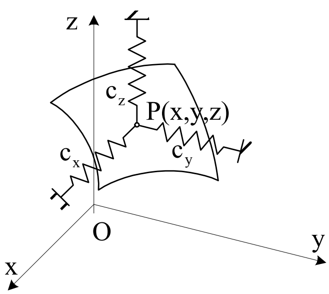
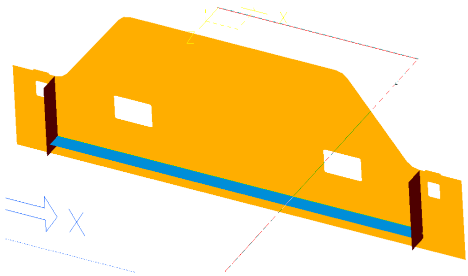


Figure 4. Continuous elastic support

3. Industrial applications

3.1. Machine parts of isotropic materials

3.1.1. Bending plate of a bending-off press

(a) (b)

Figure 5. Bending-off press and the model of its bending plate

From the point of view of metal forming the most important part of a bending-off press (Figure 5a) is the bending plate. This machine part is an in-plane loaded plate with holes and stiffening ribs. Because of the stiffeners this part can not be modeled as a simple plane stress problem but as a general shell problem. The reason is that the in-plane loaded plate obtains a bending influence from the stiffeners. The FEM model consists of general shell elements and it is advantageous to consider the axial symmetric behavior of the part (Figure 5b).

3.1.2. Steam condenser of a power station

The investigated steam condenser of a power station [4] is a huge complex steel construction with about 10x7x7 m dimensions. This structure is actually a thin walled housing stiffened by ribs and eccentric welded beams fastened to the side walls. Furthermore, internal channels, internal vertical plates and a very strong internal 3D-beam network are included in the structure (Figure 6).

When modeling a structure one had to apply almost every classical model of mechanics of materials: beams, bars, shells, etc. and further FEM based modeling tools such as symmetry condition and eccentric connections. Only by using above modeling tools can the number of unknown variables of FEM computation be kept under a considerable level.

The mechanical model (Figure 7) considers the double plane symmetry of structure. Figure 7a shows the parts of the condenser modeled by shell elements and Figure 7b demonstrates the internal beam network.

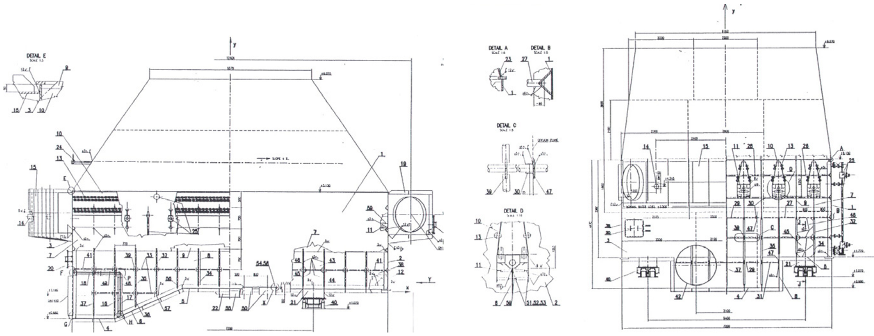
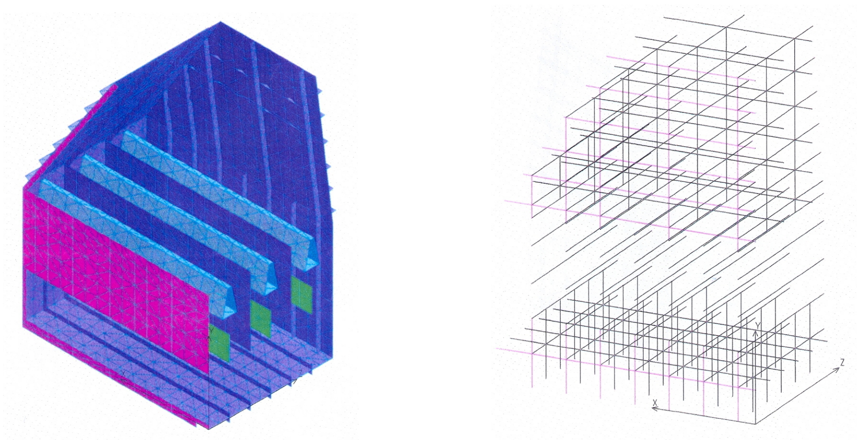


Figure 6. Draft of steam condenser

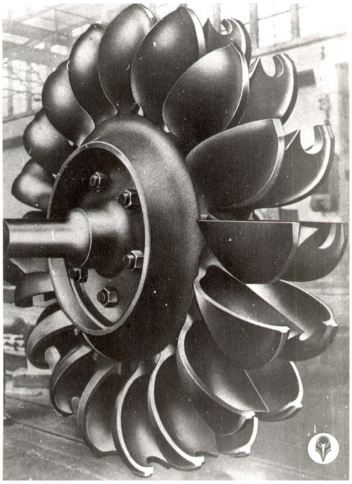


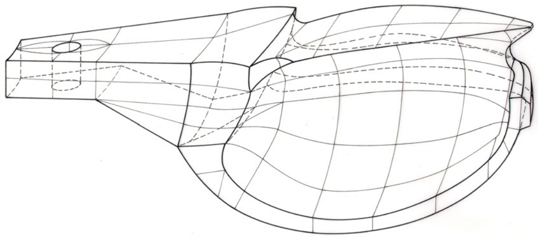
(a) (b)

Figure 7. FEM model of steam condenser

3.1.3. Impeller of a Pelton water turbine

The impeller of a Pelton water turbine (Figure 8a) is built up of periodic parts that consist of a bucket and the proper part of the running wheel. This very complex structure can be modeled well only with 3D finite elements [3]. Using the plane symmetry of the running wheel as well we can get the FEM computational model of the structure (Figure 8b). This model can be considered both a sector symmetric and a continuously elastic supported model. Computational results that origin from the above models are very close to each other.





(a) (b)

Figure 8. Impeller of water turbine and its FEM model

3.2. Machine parts of orthotropic materials

The weight reduction of machine parts is considered an important task in machine design. The usual way for achieving weight reduction is to replace metallic materials with fiber reinforced plastics (FRP) having similarly good stiffness and strength properties as metals. But the material behavior of FRP cannot be considered as an isotropic material.

FRP have inhomogeneous character as they are made of fibers having high strength and stiffness which are embedded in soft matrix material (e.g. in epoxy resin). Thus, the material properties of these materials essentially differ at certain points. The mechanical properties of FRP also depend on the direction of fibers, i.e. they are anisotropic. According to literature [7], [8] and to our own experience [5] the mechanical properties of these materials can be described in a macroscopic sense as homogeneous anisotropic materials. Macroscopic sense means that the material law is not valid in one point of the material, but in a certain area of the material which already contains the necessary number of fibers. In case of unidirectional (UD) fiber reinforcement (Figure 12a) and of plain weave fabrics (Figure 12b) the application of (12a) orthotropic material law delivers a good approximation for material behavior.

3.2.1 Preliminary gear study

As a preliminary analysis for production of composite gears we determined the mechanical behavior of two cylindrical rollers that were pressed to each other by force F (Figure 9). Both rollers consist of two parts: an inner metallic or in plane laminated core and an outer in tangential direction unidirectional (UD) reinfor-ced ring [6].

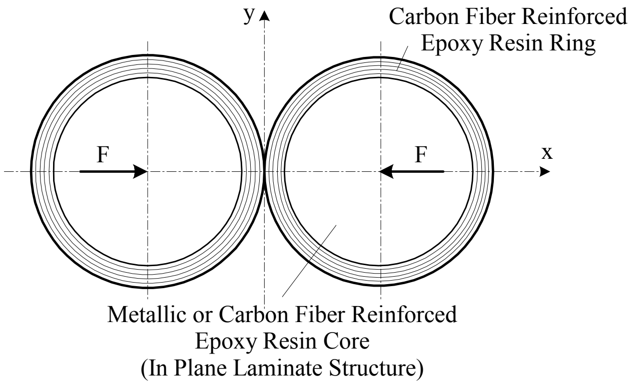
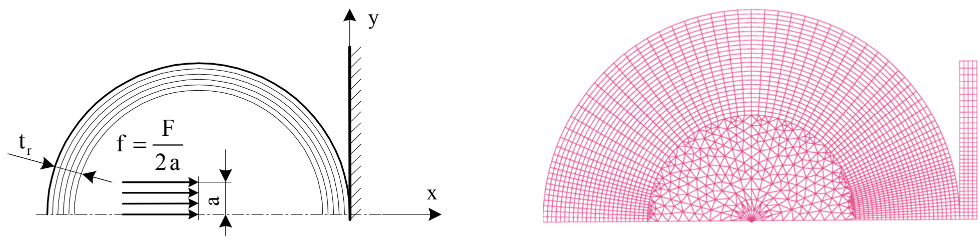


Figure 9. Contact problem of UD fiber reinforced rollers

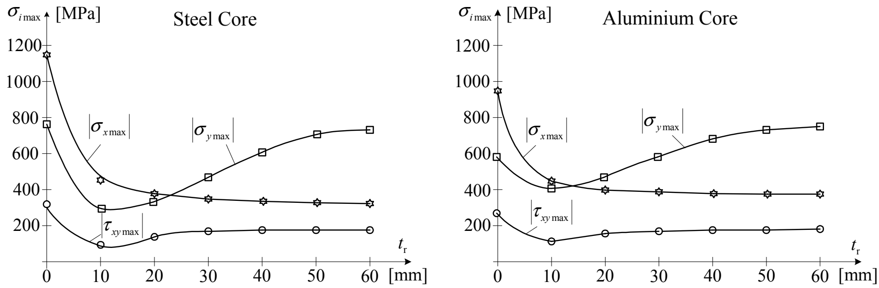


(a) (b)

Figure 10. The mechanical model and its FE mesh

Using double symmetry, the computational mechanical model can be obtained: a half cylindrical roller pressed against a rigid support plane with the load of F/2. In order to avoid stress concentration at the acting point of the F force, an equivalent distributed loading of f=F/(2a) was applied (Figure 10a).

The FE mesh must be much denser in the contact zone where high stress gradients are expected. In the quadrilateral elements applied to the FRP ring the principal material direction that are parallel and perpendicular respectively to the fiber direction were determined by the local nodal numbering of elements. The fulfillment of the symmetry conditions along the x axis was achieved by simple support at every node on the axis. For modeling the rigid plane support by a FE mesh stripe, a fixed hinge support is applied to the nodes of the roller's side line of the stripe.



(a)

(b)

Figure 11. Maximum stresses at metal cores

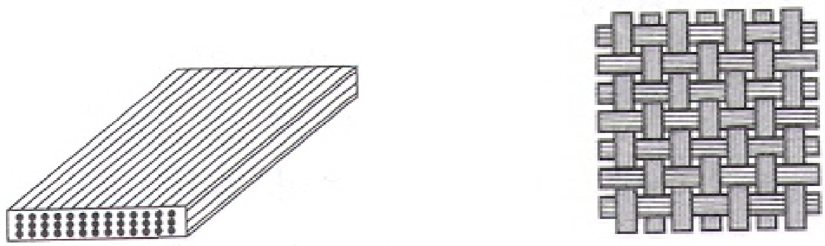
Figure 11 shows the maximum values of stress components as function of t_r radial thickness of the ring. For metal cores, an "optimal radial thickness area" of the UD reinforced ring was found in that every stress components has a low value.

3.2.2. Modeling of textile composites

In machine parts of composite materials the long fibers run in more layers. In addition, they run in different directions per layer. The layer in which fibers are running parallel to each other is called unidirectional (UD) layer (Figure 12a). It is also usual to apply layers with woven patterns, called fabric or textile composite materials (Figure 12b).

The main goal of the investigation was to prove that the applied measurement based linear orthotropic material modeling describes accurately enough the mechanical behavior of multilayered plain weave fabric composite parts from an engineering point of view [5].

The measurements and FEM computations were carried out for four layered parts made of fiber dominant FRP. The preimpregnated basic material had a matrix material volume proportion of 32%. The characteristic of the material was the following: 80% / 20% long / cross fiber orientation proportion, glass fiber reinforcement and epoxy matrix. The computation was performed by on one layer measured material properties, and two gauges were applied to the four layered specimen at the measurement (Figure 13).



(a) (b)

Figure 12. Unidirectional and plain weave layer

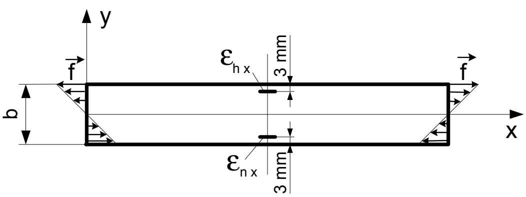


Figure 13. Location of gauges on the four layered bended part

FEM computational and gauges measured results showed a close agreement and so it succeeded to confirm the applicability of orthotropic material law.

Conclusions

Involving classical models of mechanics of materials the finite element method offers a lot of further modeling tools for solving engineering problems in the field of machine design. The paper summarizes the classical models and the most useful FE models of mechanics of materials and demonstrates by industrial application examples the wide range of applicability of modeling and that how it is possible to decrease the numerical computational amount by using FEM.

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