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Topological properties for a perturbed first order sweeping process

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Abstract. In this paper, we consider a perturbed sweeping process for a class of subsmooth moving sets. The perturbation is general and takes the form of a sum of a single-valued mapping and a set-valued mapping. In the first result, we study some topological proprieties of the attainable set, the set-valued mapping considered here is upper semi-continuous with convex values. In the second result, we treat the autonomous problem under assumptions that do not require the convexity of the values and that weaken the assumption on the upper semi-continuity. Then, we deduce a solution of the time optimality problem.

1 Introduction

The attainable sets plays an important role in control theory; many problems of optimization, dynamics, planning procedures in mathematical economy and game theory can be stated and solved in terms of attainable sets. The perturbed state-dependent sweeping process is an evolution differential inclusion governed by the normal cone to a mobile set depending on both time and state variables, of the following form:

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + F(t,u(t)), \text{ a.e } t \in [T_0,T]; \\ x(t) \in C(t,u(t)), \ \forall t \in [T_0,T]; \quad u(T_0) = u_0 \in C(T_0,u_0), \end{cases}$$

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where $N_{C(t,u(t))}(u(t))$ is the normal cone to $C(t,u(t))$ at $u(t)$ and F is a set-valued or single-valued mapping playing the role of a perturbation to the problem, that is an external force applied on the system. This type of problems was initiated by J. J. Moreau in the 1970's and extensively studied by himself when the sets $C(t)$ are assumed to be convex and $F \equiv \{0\}$ (see [24, 25, 26, 27]). The original motivation is to model quasistatic evolution in elastoplasticity, friction dynamics, granular material, contact dynamics. However, many applications of the sweeping processes can be also found nowadays in nonsmooth mechanics, convex optimization, modeling of crowd motion, mathematical economics, dynamic networks, switched electrical circuits, etc, see for example [2, 15, 16, 19, 22] and the references therein. Existence (and possibly uniqueness) of solutions of such systems and their classical variants subjected to perturbation forces, state-dependent, second order sweeping processes, etc, have been studied fruitfully in the literature see for example [1, 3, 7, 8, 9, 10, 11, 14, 17, 21, 22, 28, 29, 30, 31] and the references therein.

In [12], a generalization of convexity has been defined, that is the almost convexity of sets, the authors have shown the existence of solution to the upper semi-continuous differential inclusions $\dot{x}(t) \in F(x(t))$, $x(0) = a$. This almost convexity condition has been used successfully by [3, 4, 5] to study the perturbed first order Moreau's sweeping process, the right-hand side contains a set-valued perturbation with almost convex values.

In this work, we extend the results in [3] in many direction. At first, we study in finite dimensional space, the existence of solution and the compactness of the attainable sets for the problem

$$(\mathcal{SP}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + F\big(t,u(t)\big) + f\big(t,u(t)\big), & \text{a.e. } t \in [T_0,T]; \\ u(t) \in C\big(t,u(t)\big), \ \forall t \in [T_0,T]; \quad u(T_0) = u_0 \in C(T_0,u_0), \end{cases}$$

when F is a set-valued mapping with nonempty closed convex values, upper semi-continuous and the element of minimum norm satisfies a linear growth condition, f is a continuous single-valued mapping and the moving sets $C(t,x)$ are equi-uniformlt-subsmooth. It is important to emphasize that this class of sets, introduced by D. Aussel, A. Daniilidis and L. Thibault in [6], is an extension of convexity and prox-regularity of a set. In this way, the result concerning existence of solution of the first order differential inclusion is more general. Second, we define a larger class contains set-valued mappings with almost convex values and their translated, then we study the existence of solution to the autonomous problem

$$(\mathcal{ASP}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(u(t)\right)}\big(u(t)\big) + F\big(u(t)\big) + f\big(u(t)\big), \text{ a.e. } t \in [T_0, T]; \\ u(t) \in C\big(u(t)\big), \ \forall t \in [T_0, T]; \quad u(T_0) = u_0 \in C(u_0), \end{cases}$$

under the weaker assumption on the upper semi-continuity and the almost convexity of the values of F. We mention that C, F and f are assumed time independent for purely technical reasons. As will be shown, our almost convexity does not imply that the set of solutions to $(\mathcal{ASP})$ is compact in the space of continuous functions with uniform convergence, as happens in the case of the assumption of convexity, but only that the sections of this set of solutions are compact. As an application, we consider the autonomous control system

$$(\mathcal{ASP}_{\mathcal{O}}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(u(t)\right)}\big(u(t)\big) + h\big(u(t), z(t)\big) + f\big(u(t)\big), \text{ a.e. } t \in [T_0, T]; \\ z(t) \in U(u(t)), \ u(t) \in C\big(u(t)\big), \ \forall t \in [T_0, T]; \quad u(T_0) = u_0 \in C(u_0), \end{cases}$$

controlled by parameters $z(t) \in U(u(t))$, where $U : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping with compact values that is upper semi-continuous on $\mathbb{R}^n$. Under the almost convexity assumption on the sets

$$F\big(u(t)\big) = h\big(u(t), U(u(t))\big) = \{h\big(u(t), z(t)\big)\}_{z(t) \in U(u(t))}$$

and $F\big(u(t)\big) + f\big(u(t)\big)$ the solutions of the control problem $(\mathcal{ASP}_{\mathcal{O}})$ are solutions to the $(\mathcal{ASP})$, in which the controls do not appear explicitly, we say that F is parameterized by elements of U. The equivalence between a control system and the corresponding differential inclusion is the central idea used to prove the existence of solution to the minimum time problem for $(\mathcal{ASP}_{\mathcal{O}})$.
This paper is organized as follows: in the first section, we introduce preliminaries and background. In the second, we study the existence of solution to the problem $(\mathcal{SP})$ and some topological proprieties of the attainable set when the perturbation is convex. In the last section, we prove the existence of solution for a differential inclusion $(\mathcal{ASP})$ with almost convex perturbation and we deduce a solution of the time optimality problem.

2 Preliminaries and background

Throughout this paper $\mathbf{R}^n$ is the n-dimensional Euclidean space, $\mathcal{I} = [T_0, T]$ $(T > T_0 \geq 0)$ an interval of $\mathbf{R}$, $\mathbf{B}$ is the closed unit ball centered at the origin of $\mathbf{R}^n$ and $\mathbf{B}(a, \eta)$ the open ball of center a and radius $\eta > 0$. We denote

by $\mathcal{C}_{\mathbf{R}^n}(\mathcal{I})$ the Banach space of all continuous maps from $\mathcal{I}$ into $\mathbf{R}^n$ endowed with the sup-norm, $L^1_{\mathbf{R}^n}(\mathcal{I})$ stands for the space of all Lebesgue integrable $\mathbf{R}^n$-valued mappings defined on $\mathcal{I}$. A map $u : \mathcal{I} \to \mathbf{R}^n$ is absolutely continuous if there is a mapping $g \in L^1_{\mathbf{R}^n}(\mathcal{I})$ such that $u(t) = u(T_0) + \int_{T_0}^t g(s)\, ds$, for all $t \in \mathcal{I}$. For a nonempty closed subset K of $\mathbf{R}^n$, $co(K)$ $\big($resp. $\overline{co}(K)\big)$ stands for the convex $\big($resp. closed convex$\big)$ hull of K, which can be characterized by $\overline{co}(K) = \big\{x \in \mathbf{R}^n, \forall x' \in \mathbf{R}^n, \langle x', x\rangle \leq \delta^*(x', K)\big\}$ where $\delta^*(x', K) = \sup\limits_{y\in K}\langle x', y\rangle$ is the support function of K at $x' \in \mathbf{R}^n$. We denote by $d_K(\cdot)$ the usual distance function associated with K, i.e., $d_K(x) = \inf\limits_{y\in K}\|x - y\|$, $\mathrm{Proj}_K(x) = \{y \in K : d_K(x) = \|x - y\|\}$ the projection set of x into K and by $m(K) = \mathrm{Proj}_K(0)$ the element of K with minimal norm, it is unique whenever K is a closed convex. If F is a measurable set-valued mapping, with nonempty closed convex values, then F admits a measurable selection with minimal norm $m(F(x)) = \mathrm{Proj}_{F(x)}(0)$.
We will need the concept of Clarke subdifferential and normal cone. For a locally Lipschitzian function $\varphi : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$, the Clarke subdifferential $\partial\varphi(x)$ of φ at x is the nonempty convex compact subset of $\mathbf{R}^n$, given by (see[13])

$$\partial\varphi(x) = \{\xi \in \mathbf{R}^n : \langle \xi, v\rangle \leq \varphi^o(x, v), \ \text{ for all } v \in \mathbf{R}^n\},$$

where $\varphi^o(x, v) = \limsup\limits_{\substack{y\to x\\ t\downarrow 0}} \dfrac{\varphi(y + tv) - \varphi(y)}{t}$ is the generalized directional derivative of φ at x in the direction v. The Clarke normal cone $N_K(x)$ at $x \in K$ is defined from T^C_K by polarity, that is,

$$N_K(x) = \{\xi \in \mathbf{R}^n : \langle \xi, v\rangle \leq 0, \ \text{for all } v \in T^C_K(x)\},$$

where $T^C_K(x)$ is the Clarke tangent cone at $x \in K$ given by $T^C_K(x) = \big\{v \in \mathbf{R}^n : d^o_K(x, v) = 0\big\}$.

The concept of Fréchet subdifferential will be needed. A vector $v \in \mathbf{R}^n$ is a Fréchet subdifferential $\partial^F\varphi(x)$ of φ at x (see[23]) provided that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle v, y - x\rangle \leq \varphi(y) - \varphi(x) + \varepsilon\|y - x\|, \ \text{ for all } y \in \mathbf{B}(x, \delta).$$

We always have the inclusion $\partial^F\varphi(x) \subset \partial\,\varphi(x)$, for all $x \in K$. The Fréchet normal cone at $x \in K$ is given by $N^F_K(x) = \partial^F\psi_K(x)$, where ψ_K is the indicator function of K, that is, $\psi_K(x) = 0$ if $x \in K$ and $\psi_K(x) = +\infty$ otherwise. So we have the inclusion $N^F_K(x) \subset N_K(x)$, for all $x \in K$. On the other hand, the

Fréchet normal cone is also related (see[23]) to the Fréchet subdifferential of the distance function, since for all $x \in K$

$$\partial^F d_K(x) = N_K^F(x) \cap \mathbf{B}. \tag{1}$$

An important property is that, whenever $y \in \mathrm{Proj}_K(x)$, one has

$$x - y \in N_K^F(y) \qquad \text{hence also} \qquad x - y \in N_K(y). \tag{2}$$

Now, we introduce a class of subsmooth sets introduced in [6].

Definition 1 *Let K be a closed subset of $\mathbf{R}^n$. The set K is called subsmooth at $x_0 \in K$, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x_1, x_2 \in \mathbf{B}(x_0, \delta) \cap K$ and $\xi_i \in N_K(x_i) \cap \mathbf{B}$ ($i \in \{1, 2\}$), on has*

$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|. \tag{3}$$

The set K is subsmooth, if it subsmooth at each point of K. We say that K is uniformly subsmooth, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (3) holds for all $x_1, x_2 \in K$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_K(x_i) \cap \mathbf{B}$ ($i \in \{1, 2\}$).

The following subdifferential regularity of the distance function remains true for subsmooth sets (see [6]).

Proposition 1 *Let K be a closed set of $\mathbf{R}^n$. If K is subsmooth at $x \in K$, then*

$$N_K(x) = N_K^F(x) \qquad \text{and} \qquad \partial d_K(x) = \partial^F d_K(x). \tag{4}$$

The concept of equi-uniformly subsmoothness will also be helpful.

Definition 2 *Let $(K(q))_{q \in Q}$ be a family of closed sets of $\mathbf{R}^n$ with parameter $q \in Q$. This family is called equi-uniformly subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $q \in Q$, the inequality (3) holds for all $x_1, x_2 \in K(q)$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_{K(q)}(x_i) \cap \mathbf{B}$.*

The next proposition provides partial upper semi-continuity property. For the proof, we refer the reader to [21].

Proposition 2 *Let $\{K(t, x) : (t, x) \in \mathcal{I} \times \mathbf{R}^n\}$ be a family of nonempty closed sets of $\mathbf{R}^n$, which is equi-uniformly-subsmooth and let a real $\eta \geq 0$. Assume that there exist a real constants $L_1 \geq 0$, $L_2 \in [0, 1[$ such that, for any $x_1, x_2, y \in \mathbf{R}^n$ and $t, s \in \mathcal{I}$*

$$|d_{K(t,x_1)}(y) - d_{K(s,x_2)}(y)| \leq L_1 |t - s| + L_2 \|x_1 - x_2\|. \tag{5}$$

Then, the following assertions hold:

(i) *for all* $(t, x, y) \in \mathrm{Gph}\, K$, *we have* $\eta \partial d_{K(t,x)}(y) \subset \eta \mathbf{B}$;

(ii) *for any sequence* $\left(t_n, x_n\right)_n$ *in* $\mathcal{I} \times \mathbf{R}^n$ *converging to* (t, x), *any* $(y_n)_n$ *converging to* $y \in K(t, x)$ *with* $y_n \in K(t_n, x_n)$ *and any* $\xi \in \mathbf{R}^n$, *we have*

$$\limsup_{n \to +\infty} \delta^*\Big(\xi, \eta \partial d_{K(t_n, x_n)}(y_n)\Big) \leq \delta^*\Big(\xi, \eta \partial d_{K(t,x)}(y)\Big).$$

In the next, we give the definition of the almost convex sets and attainable sets.

Definition 3 [12] *For a vector space* X, *a set* $D \subset X$ *is called almost convex if for every* $\xi \in \mathrm{co}(D)$ *there exist* λ_1 *and* λ_2, $0 \leq \lambda_1 \leq 1 \leq \lambda_2$ *such that* $\lambda_1 \xi \in D$ *and* $\lambda_2 \xi \in D$.

Any convex set is almost convex since $D = \mathrm{co}(D)$. If Q is a convex set not containing the origin, $D = \partial Q$ is almost convex, and if the convex set Q contains the origin, one take $D = \{0\} \cup \partial Q$. The origin plays a particular role in the definition of almost convexity. It ensues that the class of almost convex sets is not stable under translation, for example the set for example the set $K = \{0, 1\}$ is almost convex, while $K - \frac{1}{2} = \{-\frac{1}{2}, \frac{1}{2}\}$ is not.

Definition 4 *The attainable set of any problem at time* $\tau \in \mathcal{I}$ *is defined by*

$$\mathcal{R}_{u_0}(\tau) = \{x \in \mathbf{R}^n : x = u(\tau) \text{ such that } u(\cdot) \in \mathcal{S}_\tau(u_0)\},$$

where $\mathcal{S}_\tau(u_0)$ *is the set of the trajectories of our problem on the interval* $[T_0, t]$.

We will also need the following result, which is a discrete version of Gronwall's Lemma.

Lemma 1 *Let* $\alpha > 0$, (a_n) *and* (b_n) *two nonnegative sequence such that*

$$a_n \leq \alpha + \sum_{k=0}^{n-1} b_k a_k, \quad \text{for all } n \in \mathbf{N}.$$

Then, for every $n \in \mathbf{N}^*$, *we have*

$$a_n \leq \alpha \exp\left(\sum_{k=0}^{n-1} b_k\right).$$

3 Convex case

In this section, we study the existence of solution and some topological properties of the attainable set for the sweeping process $(\mathcal{SP})$ when F is an upper semi-continuous set-valued mapping with nonempty closed convex values unnecessarily bounded.

Theorem 1 *Let* $C : \mathcal{I} \times \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ *be a set-valued mapping with nonempty closed values satisfying:*

$(\mathcal{A}_1^C)$ *for all* $(t, x) \in \mathcal{I} \times \mathbf{R}^n$*, the sets* $C(t, x)$ *are equi-uniformly subsmooth;*

$(\mathcal{A}_2^C)$ *there are two constants* $L_1 \geq 0$*,* $L_2 \in [0, 1[$ *such that, for all* $t, s \in \mathcal{I}$ *and any* $x, u, v \in \mathbf{R}^n$ *on has*

$$\left|d_{C(t,u)}(x) - d_{C(s,v)}(x)\right| \leq L_1|t - s| + L_2\left\|u - v\right\|.$$

Let $F : \mathcal{I} \times \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ *be a set-valued mapping with nonempty closed convex values, upper semi-continuous such that:*

$(\mathcal{A}^F)$ *for some real* $\alpha \geq 0$*,* $d_{F(t,x)}(0) \leq \alpha\big(1 + \|x\|\big)$*, for all* $(t, x) \in \mathcal{I} \times \mathbf{R}^n$.

And $f : \mathcal{I} \times \mathbf{R}^n \to \mathbf{R}^n$ *be a continuous mapping such that:*

$(\mathcal{A}^f)$ *for some real* $\beta > 0$*,* $\left\|f(t, x)\right\| \leq \beta(1 + \|x\|)$*, for all* $(t, x) \in \mathcal{I} \times \mathbf{R}^n$.

Then, for any $u_0 \in C(0, u_0)$

(1) *the problem* $(\mathcal{SP})$ *admits a Lipschitz solution;*

(2) *for* $\tau \in \mathcal{I}$ *fixed, the attainable set* $\mathcal{R}_{u_0}(\tau)$ *is compact;*

(3) *the set-valued mapping* $\mathcal{R}_{u_0}(\cdot)$ *is upper semi-continuous.*

Proof. (1) The existence of solution: for each $(t, x) \in \mathcal{I} \times \mathbf{R}^n$, we put $m(t, x) = \mathrm{Proj}_{F(t,x)}(0)$ the element of minimal norm of F and $h(t, x) = m(t, x) + f(t, x)$. It follows that, $\|h(t, x)\| \leq \gamma\big(1 + \|x\|\big)$ with $\gamma = \alpha + \beta$. For each $n \in \mathbf{N}^*$, we consider a partition of $\mathcal{I}$ by $\mathcal{I}_i^n = [t_i^n, t_{i+1}^n[,\ t_i^n = T_0 + i\mu_n, \mu_n = \dfrac{T - T_0}{n},\ i \in \{0, 1, \cdots, n-1\}$ and $\mathcal{I}_0^n = \{t_0^n\} = \{T_0\}$.
Step 1. We define inductively the sequence $(x_i^n)_{0 \leq i \leq n}$ in $\mathbf{R}^n$. Putting $x_0^n = u_0 \in C(t_0^n, x_0^n)$ and for each $i \in \{1, 2, \cdots, n-1\}$ the following inclusions is well defined

$$x_{i+1}^n \in C\big(t_{i+1}^n, x_i^n\big), \tag{6}$$

$$x_i^n + \mu_n h(t_i^n, x_i^n) - x_{i+1}^n \in N_{C\left(t_{i+1}^n, x_i^n\right)}(x_{i+1}^n). \tag{7}$$

Indeed, for $i = 0$ and since $C(t_1^n, x_0^n)$ has closed values, we can take

$$x_1^n \in \mathrm{Proj}_{C(t_1^n, x_0^n)}\left(x_0^n + \mu_n h(t_0^n, x_0^n)\right),$$

clearly

$$x_1^n \in C\left(t_1^n, x_0^n\right). \tag{8}$$

Then, by (2), we obtain

$$x_0^n + \mu_n h(t_0^n, x_0^n) - x_1^n \in N_{C\left(t_1^n, x_0^n\right)}(x_1^n).$$

Using $(\mathcal{A}_2^C)$ and (8), we get

$$\begin{aligned}
\|x_1^n - x_0^n\| &\leq d_{C\left(t_1^n, x_0^n\right)}\left(x_0^n + \mu_n h(t_0^n, x_0^n)\right) + \mu_n \|h(t_0^n, x_0^n)\| \\
&\leq \left| d_{C\left(t_1^n, x_0^n\right)}(x_0^n) - d_{C\left(t_0^n, x_0^n\right)}(x_0^n) \right| + 2\mu_n \|h(t_0^n, x_0^n)\| \\
&\leq L_1 \mu_n + 2\gamma \mu_n (1 + \|x_0^n\|).
\end{aligned}$$

Assume that, for $i \in \{0, 1, \cdots, n-1\}$ the points $x_1^n, x_2^n, \cdots, x_i^n$ have been constructed satisfying (6) and (7). Since $C(t_{i+1}^n, x_i^n)$ is closed, we can take

$$x_{i+1}^n \in \mathrm{Proj}_{C\left(t_{i+1}^n, x_i^n\right)}\left(x_i^n + \mu_n h(t_i^n, x_i^n)\right),$$

and

$$x_{i+1}^n \in C\left(t_{i+1}^n, x_i^n\right).$$

Using the characterization of the normal cone in terms of projection operator, we can write a.e. $t \in \mathcal{I}$

$$x_i^n + \mu_n h(t_i^n, x_i^n) - x_{i+1}^n \in N_{C\left(t_{i+1}^n, x_i^n\right)}(x_{i+1}^n).$$

By $(\mathcal{A}_2^C)$ and (6), we get

$$\begin{aligned}
\|x_{i+1}^n - x_i^n\| &\leq d_{C\left(t_{i+1}^n, x_i^n\right)}\left(x_i^n + \mu_n h(t_i^n, x_i^n)\right) + \|\mu_n h(t_i^n, x_i^n)\| \\
&\leq \left| d_{C\left(t_{i+1}^n, x_i^n\right)}(x_i^n) - d_{C\left(t_i^n, x_{i-1}^n\right)}(x_i^n) \right| + 2\mu_n \|h(t_i^n, x_i^n)\| \\
&\leq L_1 \mu_n + L_2 \|x_i^n - x_{i-1}^n\| + 2\gamma \mu_n (1 + \|x_i^n\|).
\end{aligned}$$

By induction, we find for $i \in \{0, 1, \cdots, n-1\}$,

$$\|x_{i+1}^n - x_i^n\| \leq (L_1 + 2\gamma)\mu_n \sum_{k=0}^{i} L_2^k + 2\gamma\mu_n \sum_{k=0}^{i} L_2^{i-k}\|x_k^n\|,$$

since $L_2 \in [0, 1[$, we get

$$\|x_{i+1}^n - x_i^n\| \leq \frac{L_1 + 2\gamma}{1 - L_2}\mu_n + 2\gamma\mu_n \sum_{k=0}^{i} L_2^{i-k}\|x_k^n\|. \tag{9}$$

Furthermore, we have

$$\begin{aligned}
\|x_i^n - x_0^n\| &\leq \|x_i^n - x_{i-1}^n\| + \|x_{i-1}^n - x_{i-2}^n\| + \cdots + \|x_1^n - x_0^n\| \\
&\leq \frac{L_1 + 2\gamma}{1 - L_2}\mu_n + 2\gamma\mu_n \sum_{k=0}^{i-1} L_2^{i-k}\|x_k^n\| + \frac{L_1 + 2\gamma}{1 - L_2}\mu_n \\
&+ 2\gamma\mu_n \sum_{k=0}^{i-2} L_2^{i-k}\|x_k^n\| + \cdots + \mu_n(L_1 + 2\gamma) + 2\gamma\mu_n\|x_0^n\| \\
&\leq \frac{L_1 + 2\gamma}{1 - L_2}\mu_n(i-1) + 2\gamma\mu_n\|x_0^n\| \sum_{k=0}^{i-1} L_2^k + 2\gamma\mu_n\|x_1^n\| \sum_{k=0}^{i-1} L_2^k \\
&+ 2\gamma\mu_n\|x_2^n\| \sum_{k=0}^{i-1} L_2^k + \cdots + 2\gamma\mu_n\|x_{i-1}^n\| \sum_{k=0}^{i-1} L_2^k \\
&\leq T\frac{L_1 + 2\gamma}{1 - L_2} + \frac{2\gamma T}{1 - L_2} \sum_{k=0}^{i-1} \|x_k^n\|.
\end{aligned}$$

Then,

$$\|x_i^n\| \leq \|x_0^n\| + T\frac{L_1 + 2\gamma}{1 - L_2} + \frac{2\gamma T}{1 - L_2} \sum_{k=0}^{i-1} \|x_k^n\|.$$

By Lemma 1 and for all $i \in \{0, 1, \cdots, n-1\}$, we can write

$$\|x_i^n\| \leq \left(\|x_0^n\| + T\frac{L_1 + 2\gamma}{1 - L_2}\right) \exp\left(\frac{2\gamma T}{1 - L_2}\right) = \eta. \tag{10}$$

Using relations (9) and (10), we get

$$\|x_{i+1}^n - x_i^n\| \leq \frac{L_1 + 2\gamma}{1 - L_2}\,\mu_n + 2\gamma\,\mu_n \sum_{k=0}^{i} L_2^{i-k}\eta.$$

Since $L_2 \in [0,1[$, we obtain

$$\|x_{i+1}^n - x_i^n\| \leq \frac{1}{1-L_2}\,\mu_n\big(L_1 + 2\gamma + 2\gamma\eta\big). \tag{11}$$

Step 2. Construction of sequence $\big(u_n(\cdot)\big)_{n\geq 0}$.
For any $t \in \mathcal{I}_i^n$ with $i \in \{0,1,\cdots,n-1\}$ and for every $n \geq 1$, we define

$$u_n(t) = x_i^n + \big(t - t_i^n\big)\frac{x_{i+1}^n - x_i^n}{\mu_n}. \tag{12}$$

Observe that $u_n(t_i^n) = x_i^n$, and

$$\dot{u}_n(t) = \frac{x_{i+1}^n - x_i^n}{\mu_n}. \tag{13}$$

By (6) and (7) we can write

$$u_n(t_{i+1}^n) \in C\big(t_{i+1}^n, u_n(t_i^n)\big) \tag{14}$$

$$\dot{u}_n(t) \in -N_{C\left(t_{i+1}^n, u_n(t_i^n)\right)}\big(u_n(t_{i+1}^n)\big) + h\big(t_i^n, u_n(t_i^n)\big), \text{ a.e. } t \in \mathcal{I}_i^n. \tag{15}$$

Relations (11) and (13) imply that

$$\|\dot{u}_n(t)\| \leq \frac{1}{1-L_2}\big(L_1 + 2\gamma + 2\gamma\eta\big) = \Delta. \tag{16}$$

Now let us defined the step functions from $\mathcal{I}$ to $\mathcal{I}$ by

$$\theta_n(t) = \begin{cases} t_i^n & \text{if } t \in \mathcal{I}_i^n, \\ t_{n-1}^n & \text{if } t = T_0. \end{cases} \tag{17}$$

$$\rho_n(t) = \begin{cases} t_{i+1}^n & \text{if } t \in \mathcal{I}_i^n, \\ T & \text{if } t = T. \end{cases} \tag{18}$$

Observe that, for all $t \in \mathcal{I}$,

$$\lim_{n\to+\infty}\big|\theta_n(t) - t\big| = \lim_{n\to+\infty}\big|\rho_n(t) - t\big| = 0. \tag{19}$$

Combining (14), (15), (17) and (18), it results

$$u_n\big(\rho_n(t)\big) \in C\big(\rho_n(t), u_n(\theta_n(t))\big), \quad \text{for all } t \in \mathcal{I}, \tag{20}$$

$$\dot{u}_n(t) \in -N_{C\left(\rho_n(t), u_n(\theta_n(t))\right)}\big(u_n(\rho_n(t))\big) + h\big(\theta_n(t), u_n(\theta_n(t))\big), \text{ a.e. } t \in \mathcal{I}, \tag{21}$$

Furthermore, for all $t \in \mathcal{I}$, we have

$$\left\|h\big(\theta_n(t), u_n(\theta_n(t))\big)\right\| \leq \gamma(1+\eta) = \Theta, \tag{22}$$

and

$$\left\|m\big(\theta_n(t), u_n(\theta_n(t))\big)\right\| \leq \alpha(1+\eta) \tag{23}$$

with $m\big(\theta_n(\cdot), u_n(\theta_n(\cdot))\big) = \mathrm{Proj}_{F(\theta_n(\cdot), u_n(\theta_n(\cdot)))}(0)$.
Step 3. The convergence of the sequences.
By relation (12) and (16) we have for all $t \in \mathcal{I}$,

$$\left\|u_n(\rho_n(t))\right\| - \left\|u_n(t)\right\| \leq \left\|u_n(\rho_n(t)) - u_n(t)\right\| \leq \|\dot{u}_n(s)\| \big(\rho_n(t) - t\big) \leq \Delta\ \big(\rho_n(t) - t\big),$$

then

$$\lim_{n \to +\infty} \left\|u_n(\rho_n(t)) - u_n(t)\right\| = 0. \tag{24}$$

In the same way

$$\lim_{n \to +\infty} \left\|u_n(\theta_n(t)) - u_n(t)\right\| = 0. \tag{25}$$

So, $(u_n(t))_{n\geq 1}$ is relatively compact for all $t \in \mathcal{I}$, on the other hand $(u_n(\cdot))_{n\geq 1}$ is equi-continuous according to (16). Using Ascoli-Arzelà's theorem, $(u_n(\cdot))_{n\geq 1}$ is relatively compact in $\mathcal{C}_{\mathbf{R}^n}(\mathcal{I})$, so we can extract a subsequence of $(u_n(\cdot))_{n\geq 1}$ (that we do not relabel) which converges uniformly to some mapping $u(\cdot) \in \mathcal{C}_{\mathbf{R}^n}(\mathcal{I})$ and $(\dot{u}_n(\cdot))_{n\geq 1}$ converges weakly in $L^1_{\mathbf{R}^n}(\mathcal{I})$ to a mapping y with $\|y(t)\| \leq \Delta$. Fixing $t \in \mathcal{I}$ and taking any $\xi \in \mathbf{R}^n$, the above weak convergence in $L^1_{\mathbf{R}^n}(\mathcal{I})$ yields

$$\lim_{n \to +\infty} \int_{T_0}^{T} \Big\langle \chi_{\mathcal{I}}(s)\ \xi, \dot{u}_n(s) \Big\rangle\ ds = \int_{T_0}^{T} \Big\langle \chi_{\mathcal{I}}(s)\ \xi, y(s) \Big\rangle\ ds$$

or equivalently

$$\lim_{n \to +\infty} \Big\langle \xi, u_0 + \int_{T_0}^{t} \dot{u}_n(s)\ ds \Big\rangle = \Big\langle \xi, u_0 + \int_{T_0}^{t} y(s)\, ds \Big\rangle.$$

Then, $\lim\limits_{n \to +\infty} \int_{T_0}^{t} \dot{u}_n(s)\ ds = \int_{T_0}^{t} y(s)\ ds$. Since $u_n(\cdot)$ is an of absolutely continuous mapping, we get

$$\lim_{n \to +\infty} \big(u_n(t) - u_0\big) = \lim_{n \to +\infty} \int_{T_0}^{t} \dot{u}_n(s)\ ds = \int_{T_0}^{t} y(s)\, ds.$$

Then $u(t) = u_0 + \int_{T_0}^{t} y(s)\, ds$ and $y = \dot{u}$.
Let set $\Big(m\big(\theta_n(\cdot), u_n(\theta_n(\cdot))\big)\Big)_n = \big(p_n(\cdot)\big)_n$, for all $n \geq 0$, by (23) we get

$\|p_n(t)\| \le \alpha(1+\eta)$, which means that (p_n) is integrably bounded, so, by extracting a subsequence, not relabeled, we may assume that (p_n) weakly converges in $L^1_{\mathbf{R}^n}(\mathcal{I})$ to some mapping $p \in L^1_{\mathbf{R}^n}(\mathcal{I})$, with $\|p(t)\| \le \alpha(1+\eta)$ for all $t \in \mathcal{I}$.
Let put $\Big(f\big(\theta_n(\cdot), u_n(\theta_n(\cdot))\big)\Big)_n = \big(q_n(\cdot)\big)_n$, according to the continuity of f, (19) and (25) we get that $(q_n(\cdot))$ converges to $q(\cdot)$ and for all $t \in \mathcal{I}$, $\|q(t)\| \le \beta(1+\eta)$.
Step 4. We prove that the mapping u is a solution of $(\mathcal{SP})$. Fix any $t \in \mathcal{I}$, by $(\mathcal{A}_2^C)$ and (20), we have

$$\begin{aligned} & d_{C(t,u(t))}\big(u_n(t)\big) \le \|u_n(t) - u_n(\rho_n(t))\| + d_{C\big(t,u(t)\big)}\big(u_n(\rho_n(t))\big) \\ & \le \|u_n(t) - u_n(\rho_n(t))\| + \Big| d_{C\big(t,u(t)\big)}\big(u_n(\rho_n(t))\big) - d_{C\big(\rho_n(t), u_n(\theta_n(t))\big)}\big(u_n(\rho_n(t))\big)\Big| \\ & \le \|u_n(\rho_n(t)) - u_n(t)\| + L_1|t - \rho_n(t)| + L_2\big\|u(t) - u_n(\theta_n(t))\big\|. \end{aligned}$$

Using (19), (24), (25), and by passing to the limit in the preceding inequality, thanks to the closedness of $C(t, u(t))$, we get

$$u(t) \in C\big(t, u(t)\big), \quad \text{for all } t \in \mathcal{I}.$$

Furthermore, by (16) and (22), we have

$$\big\| -\dot{u}_n(t) + p_n(t) + q_n(t)\big\| \le \Delta + \Theta = \Upsilon. \tag{26}$$

Then, (21) and (26) yield that

$$-\dot{u}_n(t) + p_n(t) + q_n(t) \in N_{C\big(\rho_n(t), u_n(\theta_n(t))\big)}\big(u_n(\rho_n(t))\big) \cap \Upsilon \mathbf{B},$$

from relation (1) and Proposition 1, we get

$$-\dot{u}_n(t) + p_n(t) + q_n(t) \in \Upsilon \partial d_{C\big(\rho_n(t), u_n(\theta_n(t))\big)}\big(u_n(\rho_n(t))\big), \text{ a.e. } t \in \mathcal{I} \tag{27}$$

and

$$p_n(t) \in F\big(\theta_n(t), u_n(\theta_n(t))\big). \tag{28}$$

Since $\big(-\dot{u}_n + p_n + q_n, p_n\big)$ weakly converges in $L^1_{\mathbf{R}^n \times \mathbf{R}^n}(\mathcal{I})$ to $\big(-\dot{u} + p + q, p\big)$, by Mazur's Lemma, there exists a sequence $\big(\omega_n, \zeta_n\big)_{n \ge 1}$ with

$$\omega_n \in \text{co}\{-\dot{u}_k + p_k + q_k\} \quad \text{and} \quad \zeta_n \in \text{co}\{p_k, \ k \ge n\}, \ n \ge 0$$

such that $\big(\omega_n, \zeta_n\big)_{n \ge 1}$ converges strongly in $L^1_{\mathbf{R}^n \times \mathbf{R}^n}(\mathcal{I})$ to $\big(-\dot{u} + p + q, p\big)$. By extracting a subsequence if necessary, we suppose that $\big(\omega_n, \zeta_n\big)_{n \ge 1}$ converges a.e. to $\big(-\dot{u}+p+q,p\big)$. Then, there is a Lebesgue negligible set $\mathcal{S}\subset\mathcal{I}$ such that, for every $t\in\mathcal{I}\backslash\mathcal{S}$, on one hand $\big(\omega_n,\zeta_n\big)_{n\geq 1}$ converges strongly to $\big(-\dot{u}+p+q,p\big)$ and on the other hand the inclusions (27) and (28) hold true for every integer n as well as the inclusions

$$-\dot{u}(t)+p(t)+q(t)\in\bigcap_{n\geq 0}\overline{\{\omega_k(t),\ k\geq n\}}\subset\bigcap_{n\geq 0}\overline{co}\{-\dot{u}_k(t)+p_k(t)+q_k(t),\ k\geq n\},\tag{29}$$

and

$$p(t)\in\bigcap_{n\geq 0}\overline{\{p_k(t),\ \ k\geq n\}}\subset\bigcap_{n\geq 0}\overline{co}\{p_k(t),\ \ k\geq n\},\tag{30}$$

Fix any $t\in\mathcal{I}\backslash\mathcal{S}$ and $z\in\mathbf{R}^n$ the relations (27) and (29) gives

$$\langle z,-\dot{u}(t)+p(t)+q(t)\rangle\leq\limsup_{n\to+\infty}\delta^*\Big(z,\Upsilon\partial d_{C\left(\rho_n(t),u_n(\theta_n(t))\right)}\big(u_n(\rho_n(t))\big)\Big).$$

By Proposition 2, we get

$$\langle z,-\dot{u}(t)+p(t)+q(t)\rangle\leq\delta^*\Big(z,\Upsilon\partial d_{C\left(t,u(t)\right)}\big(u(t)\big)\Big).$$

Since $\Upsilon\partial d_{C\left(t,u(t)\right)}\big(u(t)\big)$ is closed convex values, we obtain

$$-\dot{u}(t)+p(t)+q(t)\in\Upsilon\partial d_{C(t,u(t))}\big(u(t)\big)\subset N_{C\left(t,u(t)\right)}\big(u(t)\big).\tag{31}$$

Furthermore, according to (28), (30) and the upper semi-continuous of F, we have

$$\langle z,p(t)\rangle\leq\limsup_{n\to+\infty}\delta^*\Big(z,F(\theta_n(t),u_n(\theta_n(t))\Big)\leq\delta^*\Big(z,F(t,u(t))\Big).$$

Since F has closed convex values, we conclude that $p(t)\in F\big(t,u(t)\big)$ for all $t\in\mathcal{I}\backslash\mathcal{S}$. By (31)

$$\dot{u}(t)\in-N_{C\left(t,u(t)\right)}\big(u(t)\big)+F\big(t,u(t)\big)+f\big(t,u(t)\big),\ \ \text{a.e. }\ t\in\mathcal{I}.$$

2) It suffice to show that the solution set

$$S_\tau(u_0)=\{u\in\mathcal{C}_{\mathbf{R}^n}([T_0,\tau]):u\text{ is a Lipschitz solution of }(\mathcal{SP})\}$$

is compact for $\tau\in\mathcal{I}$. By part 1, we have $S_\tau(u_0)\neq\emptyset$. Let $(u_n)_n$ be a sequence in $S_\tau(u_0)$. Then, for each $n\in\mathbf{N}$, u_n is a Lipschitz solution of $(\mathcal{SP})$ with

$$\big\|\dot{u}_n(\tilde{t})\big\|\leq\Delta,\ \ \text{a.e. }\ \tilde{t}\in[T_0,\tau],\tag{32}$$

and

$$\left\|u_n(\tilde{t})\right\| \leq \left\|u_0\right\| + \int_{T_0}^{\tilde{t}} \left\|\dot{u}_n(s)\right\| \, ds \leq \left\|u_0\right\| + \Delta(\tilde{t} - T_0).$$

Then, $(u_n(\tilde{t}))_n$ is relatively compact in $\mathbf{R}^n$, in addition, it is equi-continuous according to (32). By Arzelà-Ascoli theorem $\left(u_n\right)_n$ is relatively compact in $\mathcal{C}_{\mathbf{R}^n}([T_0, \tau])$, so, we can extract a subsequence of $(u_n)_n$ (that we do not relabel) which converges uniformly to some mapping u on $[T_0, \tau]$. By the inequality (32), $(\dot{u}_n)_n$ converges in $L^1_{\mathbf{R}^n}([T_0, \tau])$ to mapping $\dot{u}(\cdot) \in L^1_{\mathbf{R}^n}([T_0, \tau])$ with $\left\|\dot{u}(\tilde{t})\right\| \leq \Delta$ a.e. $\tilde{t} \in [T_0, \tau]$. For the rest of the demonstration we can follow the proof of the part 1 to get

$$\dot{u}(\tilde{t}) \in -N_{C\left(\tilde{t}, u(\tilde{t})\right)}\left(u(\tilde{t})\right) + F\left(\tilde{t}, u(\tilde{t})\right) + f\left(\tilde{t}, u(\tilde{t})\right), \text{ a.e. } \tilde{t} \in [T_0, \tau].$$

Then, $S_\tau(u_0)$ is compact.

3) Now we show the upper semi-continuity of the set-valued mapping $\mathcal{R}_{u_0}(\cdot)$ on $\mathcal{I}$. Consider the graph of $\mathcal{R}_{u_0}(\cdot)$ defined by

$$\mathsf{Gph}(\mathcal{R}_{u_0}) = \{(\tau, x) \in \mathcal{I} \times \mathbf{R}^n : x \in \mathcal{R}_{u_0}(\tau)\}.$$

Let $(\tau_n, x_n) \in \mathsf{Gph}(\mathcal{R}_{u_0})$ converges to (τ, x), then, for all $n \geq 0$ there exists a Lipschitz mapping $(u_n(\cdot)) \in S_\tau(u_0)$ such that $u_n(\tau_n) = x_n \in \mathcal{R}_{u_0}(\tau_n)$, by the compactness of $S_\tau(u_0)$ we can extract a subsequence of $(u_n(\cdot))_n$ (that we do not relabel) which converges uniformly to the Lipschitz mapping $u(\cdot) \in S_\tau(u_0)$, and we have

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} u_n(\tau_n) = u(\tau),$$

so $x \in \mathcal{R}_{u_0}(\tau)$. We deduce that $\mathsf{Gph}(\mathcal{R}_{u_0})$ is closed, then $\mathcal{R}_{u_0}(\cdot)$ is upper semi-continuous. □

4 Almost convex case

In this section we study the existence of solution and a property of the attainable set to the perturbed sweeping process $(\mathcal{ASP})$, when we weaken the condition of convexity and upper semi-continuity. Then we present an existence result of the minimum time of the problem $(\mathcal{ASP}_{\mathcal{O}})$. We begin by the following preliminary lemma.

Lemma 2 *Let* $G : \mathbf{R}^n \rightharpoonup \mathbf{R}^n$ *be a measurable set valued mapping with nonempty compact and almost convex values. Then, there exist two integrable functions* $\xi_1(\cdot)$ *and* $\xi_2(\cdot)$ *defined on* $\mathcal{I}$*, satisfying* $0 \leq \xi_1(t) \leq 1 \leq \xi_2(t)$ *and for* $t \in \mathcal{I}$

$$\xi_1(t)\, m\big(u(t)\big) \in G\big(u(t)\big) \quad \textit{and} \quad \xi_2(t)\, m\big(u(t)\big) \in G\big(u(t)\big). \tag{33}$$

Proof. By the almost convexity of the values of G there exist two nonempty set-valued mappings $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ such that

$$\Omega_1(t) = \big\{\xi_1 \in [0,1] \ : \ \xi_1\, m\big(u(t)\big) \in G\big(u(t)\big)\big\},$$

and

$$\Omega_2(t) = \big\{\xi_2 \in [1,+\infty[\ : \ \xi_2\, m\big(u(t)\big) \in G\big(u(t)\big)\big\}.$$

Let show that $\Omega_1(\cdot)$ is measurable. Consider its graph

$$\mathrm{Gph}(\Omega_1) = \{(t,\xi_1) \in \mathcal{I} \times [0,1] : \xi_1\, m\big(u(t)\big) \in G\big(u(t)\big)\},$$

then,

$$\mathrm{Gph}(\Omega_1) = \{(t,\xi_1) \in \mathcal{I} \times [0,1] : d_{G\left(u(t)\right)}\big(\xi_1\, m\big(u(t)\big)\big) = 0\}$$

$$= \sigma^{-1}\big(\{0\}\big) \cap \Big(\mathcal{I} \times [0,1]\Big)$$

where $\sigma : (t,\xi_1) \mapsto d_{G\left(u(t)\right)}\big(\xi_1\, m\big(u(t)\big)\big)$ is measurable. Then $\mathrm{Gph}(\Omega_1)$ is measurable. It follows that Ω_1 is measurable on $\mathcal{I}$, then there exists a measurable selection $\xi_1(\cdot)$ defined on $\mathcal{I}$. The proof that $\Omega_2(\cdot)$ is measurable is similar, since $G(u(t))$ is bounded, and the same reasoning as in the previous point can be applied . Then, there exists measurable selection $\xi_2(\cdot)$ defined on $\mathcal{I}$. □

Consider the following assumptions:

Assumption 1: Let $C : \mathbf{R}^n \rightharpoonup \mathbf{R}^n$ be a set-valued mapping with nonempty closed values satisfying:

($\mathcal{H}_1^C$) for all $x \in \mathbf{R}^n$, the sets $C(x)$ are equi-uniformly subsmooth;

($\mathcal{H}_2^C$) there is a constant $L_2 \in [0,1[$ and for any $x,u,v \in \mathbf{R}^n$ on has

$$\big|d_{C(u)}(x) - d_{C(v)}(x)\big| \le L_2\big\|u - v\big\|.$$

Assumption 2: Let $F : \mathbf{R}^n \rightharpoonup \mathbf{R}^n$ be a measurable set valued mapping with nonempty compact and almost convex values such that:

1. ($\mathcal{H}_1^F$) the set-valued mapping $\mathrm{co}(F(\cdot))$ is upper semi-continuous on $\mathbf{R}^n$;
2. ($\mathcal{H}_2^F$) for some real $\alpha > 0$, $d_{\mathrm{co}(F(x))}(0) \le \alpha\big(1 + \|x\|\big)$ for all $x \in \mathbf{R}^n$.

Assumption 3: Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous mapping such that, for some real $\beta \geq 0$,

$$\|f(x)\| \leq \beta(1 + \|x\|), \ \forall x \in \mathbf{R}^n.$$

Let

$$X = \{F : \mathbf{R}^n \rightharpoonup \mathbf{R}^n : F \text{ satisfies Assumption 1}\},$$

$$Y = \{f \in \mathcal{C}_{\mathbf{R}^n}(\mathbf{R}^n) : f \text{ satisfies Assumption 2}\}.$$

Since the class of almost convex sets is not stable under translation, we will define a larger class

$$Z = \{F \in X, \exists\ f \in Y : \ F + f \text{ has almost convex values}\}$$

which contains the set-valued mappings with almost convex values and their translated.

Theorem 2 *Assume that the Assumption 1 holds and let $F \in Z$. Then, for every $u_0 \in C(u_0)$,*

1. *the problem $(\mathcal{ASP})$ admits a solution;*
2. *for all $\tau \in \mathcal{I}$, the attainable set of the problem $(\mathcal{ASP})$ at τ, $\mathcal{R}_{u_0}(\tau)$ coincides with $\mathcal{R}^{co}_{u_0}(\tau)$, the attainable set at τ of the convexified problem.*

Proof. 1) (a) Let $[\alpha, \beta] \subset \mathcal{I}$ be an interval, and assume that, there exist two integrable functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ such that $0 \leq \xi_1(t) \leq 1 \leq \xi_2(t)$ for all $t \in [\alpha, \beta]$. In addition, assume that $\xi_1(\cdot) > 0$ a.e., using the same technique as in the proof in [5] and [12], there exist two measurable subsets of $[\alpha, \beta]$, having characteristic functions χ_1 and χ_2 such that $\chi_1 + \chi_2 = \chi_{[\alpha,\beta]}$ and an absolutely continuous function $y = y(t)$ on $[\alpha, \beta]$, such that

$$\dot{y}(t) = \frac{1}{\xi_1(t)}\chi_1(t) + \frac{1}{\xi_2(t)}\chi_2(t) \ \text{ and } \ y(\beta) - y(\alpha) = \beta - \alpha.$$

(b) By Theorem 1 there exists a Lipschitz solution $x : \mathcal{I} \to \mathbf{R}^n$ of the convexified problem

$$(\mathcal{ASP}_{co}) \quad \begin{cases} \dot{u}(t) \in -N_{C(u(t))}\big(u(t)\big) + co\big(F(u(t))\big) + f\big(u(t)\big), \ \text{ a.e. } \ t \in \mathcal{I}; \\ u(t) \in C\big(u(t)\big), \quad \forall t \in \mathcal{I}; \quad u(T_0) = u_0 \in C(u_0). \end{cases}$$

Let set $m_T\big(x(\tau)\big) = \mathrm{Proj}_{co(F(x(\tau)))+f(x(\tau)}(0)$ and consider the closed set

$$\mathcal{A} = \{\tau \in \mathcal{I} : \ m_T\big(x(\tau)\big) = 0\}.$$

Case 1: $\mathcal{A}$ is empty. In this case $\xi_1(\tau) > 0$, so, we can apply the part (a) to the interval $\mathcal{I}$. Set $y(\tau) = T_0 + \int_{T_0}^{\tau} \dot{y}(s)\, ds$ is increasing and we have $y(T_0) = T_0$ and $y(T) = T$, so, y defined from $\mathcal{I}$ into itself. Let $\vartheta : \mathcal{I} \to \mathcal{I}$ be its inverse, then $\vartheta(T_0) = T_0$, $\vartheta(T) = T$, $1 = \dot{y}\big(\vartheta(\tau)\big)\, \dot{\vartheta}(\tau)$ and

$$\dot{\vartheta}(\tau) = \xi_1\big(\vartheta(\tau)\big)\, \chi_1(\tau) + \xi_2\big(\vartheta(\tau)\big)\, \chi_2(\tau).$$

Define the map $\tilde{x} : \mathcal{I} \to \mathbf{R}^n$, as $\tilde{x}(\tau) = x\big(\vartheta(\tau)\big)$ for all $\tau \in \mathcal{I}$, then we have

$$\frac{d}{d\tau}\tilde{x}(\tau) = \dot{\vartheta}(\tau)\, \frac{d}{d\tau} x\big(\vartheta(\tau)\big) \in \dot{\vartheta}(\tau)\big(- N_{C\left(x(\vartheta(\tau))\right)}\big(x(\vartheta(\tau))\big) + m_T\big(x(\vartheta(\tau))\big),$$

using the property of the normal cone and the definition of the set Z. we get, far all $\tau \in \mathcal{I}$

$$\begin{aligned} \frac{d}{d\tau}\tilde{x}(\tau) &\in -N_{C\left(x(\vartheta(\tau))\right)}\big(x(\vartheta(\tau))\big) + F\big(x(\vartheta(\tau))\big) + f\big(x(\vartheta(\tau))\big) \\ &= -N_{C\left(\tilde{x}(\tau)\right)}\big(\tilde{x}(\tau)\big) + F\big(\tilde{x}(\tau)\big) + f\big(\tilde{x}(\tau)\big). \end{aligned}$$

Case 2: $\mathcal{A}$ is non-empty. Let $c = \sup\{\tau,\ \tau \in \mathcal{A}\}$, so that $c \in \mathcal{A}$ because $\mathcal{A}$ is closed relative to $\mathcal{I}$. The complement of $\mathcal{A}$ is open relative to $\mathcal{I}$, it consists of at most countably many overlapping open intervals $]\alpha_i, \beta_i[$, with the possible exception of one of the form $[c, \beta_{i_i}[$. For each i, apply part (a) to the interval $]\alpha_i, \beta_i[$, to infer the existence of two measurable subsets of $]\alpha_i, \beta_i[$ with characteristic functions $\chi_1^i(\cdot)$ and $\chi_2^i(\cdot)$ such that $\chi_1^i(\cdot) + \chi_2^i(\cdot) = \chi_{]\alpha_i, \beta_i[}(\cdot)$. Setting, $\dot{y}(\tau) = \dfrac{1}{\xi_1(\tau)}\, \chi_1^i(\tau) + \dfrac{1}{\xi_2(\tau)}\, \chi_2^i(\tau)$, we obtain $\int_{\alpha_i}^{\beta_i} \dot{y}(\tau) d\tau = \beta_i - \alpha_i$.
On $[T_0, c]$, set

$$\dot{y}(\tau) = \frac{1}{\xi_2(\tau)}\, \chi_{\mathcal{A}}(\tau) + \sum_i \big(\frac{1}{\xi_1(\tau)}\, \chi_1^i(\tau) + \frac{1}{\xi_2(\tau)}\, \chi_2^i(\tau)\big),$$

where the sum is over all intervals contained in $[T_0, c]$, in addition to that $\xi_2(\tau) \geq 1$ and $\int_{T_0}^{c} \dot{y}(\tau) d\tau = \kappa \leq c - T_0$. Setting $y(\tau) = T_0 + \int_{T_0}^{\tau} \dot{y}(\tau) d\tau$, we obtain that $y(\cdot)$ is an invertible map from $[T_0, c]$ to $[T_0, \kappa]$. Define $\vartheta = \vartheta(\tau)$ from $[T_0, \kappa]$ to $[T_0, c]$ to be the inverse of $y(\cdot)$, then extend $\vartheta(\cdot)$ as an absolutely continuous map $\tilde{\vartheta}(\cdot)$ on $[T_0, c]$. Setting $\dot{\tilde{\vartheta}}(\tau) = 0$ for all $\tau \in]\kappa, c]$. We prove the mapping $\tilde{x}(\tau) = x(\tilde{\vartheta}(\tau))$ is a solution of the problem $(\mathcal{ASP})$ on $[T_0, c]$ satisfying $\tilde{x}(c) = x(c)$.
For $\tau \in [T_0, \kappa]$, we get $\tilde{\vartheta}(\tau) = \vartheta(\tau)$ it is invertible and

$$\dot{\vartheta}(\tau) = \xi_2(\vartheta(\tau))\, \chi_A(\vartheta(\tau)) + \sum_i \big(\xi_1(\vartheta(\tau))\chi_1^i(\vartheta(\tau)) + \xi_2(\vartheta(\tau))\chi_2^i(\vartheta(\tau))\big).$$

As $\frac{d}{d\tau}\tilde{x}(\tau) = \dot{\vartheta}(\tau)\,\frac{d}{d\tau}x(\vartheta(\tau))$, we have

$$\frac{d}{d\tau}\tilde{x}(\tau) = \dot{\vartheta}(\tau)\Big(-N_{C\left(x(\vartheta(\tau))\right)}\big(x(\vartheta(\tau))\big) + m_T\big(x(\tau)\big)\Big). \tag{34}$$

Using (33) and the properties of the normal cone we get

$$\begin{aligned}\frac{d}{d\tau}\tilde{x}(\tau) &\in -N_{C\left(x(\vartheta(\tau))\right)}\big(x(\vartheta(\tau))\big) + F\big(x(\vartheta(\tau))\big) + f\big(x(\vartheta(\tau))\big)\\ &\in -N_{C\left(\tilde{x}(\tau)\right)}\big(\tilde{x}(\tau)\big) + F\big(\tilde{x}(\tau)\big) + f\big(\tilde{x}(\tau)\big).\end{aligned}$$

For $\tau \in]\kappa, c]$, we get $\vartheta(\kappa) = c$ and $\dot{\tilde{\vartheta}}(\tau) = 0$, then we have $\tilde{\vartheta}(\tau) = \tilde{\vartheta}(\kappa) = \vartheta(\kappa)$, so $\tilde{x}(\tau) = x(\tilde{\vartheta}(\tau)) = x(\tilde{\vartheta}(\kappa)) = \tilde{x}(\kappa)$, then $\tilde{x}$ is constant on $]\kappa, c]$ and we have $\frac{d}{d\tau}\tilde{x}(\tau) = 0 \in co(F(\tilde{x}(\tau)) + f(\tilde{x}(\tau))$, in addition $0 \in N_{C\left(\tilde{x}(\tau)\right)}\big(\tilde{x}(\tau)\big)$, we conclude that for all $\tau \in]\kappa, c]$

$$\frac{d}{d\tau}\tilde{x}(\tau) = 0 \in -N_{C\left(\tilde{x}(\tau)\right)}\big(\tilde{x}(\tau)\big) + F\big(\tilde{x}(\tau)\big) + f\big(\tilde{x}(\tau)\big).$$

On $]c, T]$, $\mathcal{A}$ is empty and $\xi_1(\tau) > 0$, then we can repeat the arguments of the part (a). We conclude, That $\tilde{x}$ is a solution of the problem $(\mathcal{ASP})$.
2) For all $\tau \in \mathcal{I}$, $\mathcal{R}_{u_0}(\tau) \subset \mathcal{R}^{co}_{u_0}(\tau)$. It is enough to prove the converse inclusion. Let $u(\tau) \in \mathcal{R}^{co}_{u_0}(\tau)$, so, $u(t)$ is a Lipschitz solution of $(\mathcal{ASP}_{co})$ on $[T_0, \tau]$. Then the proof of Theorem 2 can be repeated on $[T_0, \upsilon]$ and we find a solution $\tilde{u}(\cdot) : [T_0, \upsilon] \to \mathbf{R}^n$ of the problem $(\mathcal{ASP})$ such that $\tilde{u}(\tau) = u(\tau) \in \mathcal{R}_{u_0}(t)$. Then $\mathcal{R}^{co}_{u_0}(t) \subset \mathcal{R}_{u_0}(t)$. Hence we get the needed coincidence. □

The following corollary to Theorem 2, to be compared with Theorem 1 of Filippov [20], shows that, in the case of autonomous control systems, for the existence of a time optimal solution, Filippov's assumption that the set $h(x, U(x))$ is convex can be replaced by the weaker assumption that the same set is almost convex.

Corollary 1 *Assume that Assumption 1 holds. Let* $U : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a set-valued mapping with compact valued that is upper semi-continuous on* $\mathbb{R}^n$ *and* $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping satisfying the following assumption*

$(\mathcal{H}^h)$ *there is a nonnegative constant* α, *such that* $\|h(x, y)\| \le \alpha(1 + \|x\|)$, *for all* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$;

We associate with these data the set-valued mapping $F : \mathbf{R}^n \rightharpoonup \mathbf{R}^n$ *defined by*

$$F(x) = \{h\big(x, z\big)\}_{z \in U(x)}, \quad \textit{for all } x \in \mathbb{R}^n.$$

Assume that $F \in Z$, *where*

$$Z = \{F \in X, \exists\ f \in Y : \ F + f \textit{ has almost convex values}\}.$$

Let u_0 *and* ζ *be given in* $\mathbf{R}^n$ *such that* $u_0 \in C(u_0)$ *and for some* $\bar{t} \in [T_0, T]$, $\zeta \in \mathcal{R}_{u_0}(\bar{t})$. *Then, the problem of reaching* ζ *from* u_0 *in a minimum time admits a solution.*

Proof. Consider

$$\mathcal{M} = \{t \in [T_0, \bar{t}] \ : \ \zeta \in \mathcal{R}_{u_0}(t)\}.$$

By hypothesis $\mathcal{M} \neq \emptyset$. We put $\tau = \inf \mathcal{M}$, then, there exists a decreasing sequence (τ_n) in $[T_0, \bar{t}]$ converges to τ, and a mapping $u_n(\cdot)$ solution of

$$\begin{cases} \dot{u}(t) \in -N_{C\left(u(t)\right)}\big(u(t)\big) + F\big(u(t)\big) + f\big(u(t)\big) & \text{a.e. } t \in [T_0, \tau_n]; \\ u(t) \in C\big(u(t)\big), \ \forall t \in [T_0, \tau_n]; \quad u(T_0) = u_0 \in C(u_0). \end{cases}$$

such that for all $n \geq 1$, $u_n(\tau_n) = \zeta$. Also, for all $n \geq 1$, $u_n(\cdot)$ is solution of

$$\begin{cases} \dot{u}(t) \in -N_{C\left(u(t)\right)}\big(u(t)\big) + co(F)\big(u(t)\big) + f\big(u(t)\big) & \text{a.e. } t \in [T_0, \tau_n]; \\ u(t) \in C\big(u(t)\big), \ \forall t \in [T_0, \tau_n]; \quad u(T_0) = u_0 \in C(u_0). \end{cases}$$

Let $w_n(t) = u_n(t)$ for $t \in [0, \tau]$ and $n \geq 1$, $w_n(\cdot) \in S_\tau(u_0)$, by the proof of theorem 2 this set is compact, then by extracting a subsequence if necessary we may conclude that $(w_n(\cdot)$ converges uniformly to $w(\cdot) \in S_\tau(u_0)$. On the other hand, we have $\zeta = u_n(\tau_n) \in \mathcal{R}^{co}_{u_0}(\tau_n)$, by Theorem 2 again, the multifunction $\mathcal{R}^{co}_{u_0}(\cdot)$ is upper semi-continuous with nonempty compact values, so we get $\limsup_{n \to \infty} \mathcal{R}^{co}_{u_0}(\tau_n) = \mathcal{R}^{co}_{u_0}(\tau)$. Then, $\zeta \in \mathcal{R}^{co}_{u_0}(\tau) = \mathcal{R}_{u_0}(\tau)$. Consequently, w is the solution of the problem $(\mathcal{ASP}_{\mathcal{O}})$ that reaches ζ in the minimum time, and τ is the value of the minimum time. □

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References

[1] S. Adly, F. Nacry and L. Thibault, Discontinuous sweeping process with prox-regular sets, *ESAIM Control Optim. Calc. Var.*, **23** (2017), 1293–1329.

[2] K. Addi, B. Brogliato and D. Goeleven, Aqualitative mathematical analysis of a class of linear variational inequalities via semi-complementarity problems, *Appl. Electron. Math. Program.*, **126** (2011), 31–67.

[3] D. Affane, M. Aissous and M. F. Yarou, Existence results for sweeping process with almost convex perturbation, *Bull. Math. Soc. Sci. Math. Roumanie*, **2** (2018), 119–134.

[4] D. Affane, M. Aissous and M. F. Yarou, Almost mixed semi-continuous perturbation of Moreau's sweeping process, *Evol. Equ. Control Theory* , **1** (2020), 27–38.

[5] D. Affane and D. Azzam-Laouir, Almost convex valued perturbation to time optimal control sweeping processes, *ESAIM Control Optim. Calc. Var*, **23** (2017), 1–12.

[6] D. Aussel, A Danilis and L. Thibault, Subsmooth sets: functional characterizations and related concepts, *Trans. Amer. Math. Soc.* **357** (2004), 1275–1301.

[7] H. Benabdellah, Existence of solutions to the nonconvex sweeping process, *J. Differential Equations*, **164** (2000), 286–295.

[8] M. Bounkhel and M. F. Yarou, Existence results for first and second order nonconvex sweeping processs with delay, *Port. Math.,* **61** (2004), 207–230.

[9] C. Castaing, A. G. Ibrahim and M. F. Yarou , Some contributions to nonconvex sweeping process, *J. Nonlinear Convex Anal.*, **10** (2009), 1–20.

[10] N. Chemetov and M. D. P. Monteiro Marques, Non-convex Quasi-variational Differential Inclusions, *Set-Valued Var. Anal.*, **5** (2007), 209–221.

[11] K. Chraibi, Resolution du problème de rafle et application a un problème de frottement, *Topol. Methods Nonlinear Anal.*, **18** (2001), 89–102.

[12] A. Cellina and A. Ornelas, Existence of solution to differential inclusion and the time optimal control problems in the autonomous case, *SIAM J. Control Optim.*, **42** (2003), 260–265.

[13] F. H. Clarke, Y. S. Ledyaev, R. J. Stern and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.

[14] G. Colombo and V. V. Goncharov, The sweeping processes without convexity, *Set-Valued Anal.*, **18** (1999), 357–374.

[15] G. Colombo, R. Henrion, N. D Hoang and B. Sh. Mordukhovich, Discrete approximations of a controlled sweeping process, *Set-Valued Var. Anal.*, **23** (2015), 69–86.

[16] G. Colombo, R. Henrion, N. D Hoang and B. Sh. Mordukhovich, Optimal control of the sweeping process over polyhedral controlled sets, *J. Differ. Equ.*, **260** (2016), 3397–3447.

[17] G. Colombo and L. Thibault, Prox-regular sets and applications, in Handbook of Nonconvex Analysis and Applications, *International Press-Somerville*, MA, (2010), 99–182.

[18] S. Di Marino, F. Santambrogio and B. Maury, Measure sweeping processes, *J. Conv. An.*, **23** (2015), 567–601.

[19] D. Goeleven, *Complementarity and Variational Inequalities in Electronics, Mathematical Analysis and its Applications*, Academic Press, London, 2017.

[20] A. F. Filippov, On certain questions in the theory of optimal control, *Vestnik. Univ., Ser. Mat. Mech.*, **2** (1959), 25–32; Translated in *SIAM J. Control*, **1** (1962), 76–84.

[21] T. Haddad, J. Noel and L. Thibault, Perturbation sweeping process with a subsmooth set depending on the state,*Linear Nonlinear Anal.*, **2** (2016), 155–174.

[22] M. Kunze and M. D. P. Monteiro Marques, *An introduction to Moreau's sweeping process, in B. Brogliato, (ed.) Impacts in Mechanical Systems. Analysis and Modelling*, Springer, Berlin, 2000 1–60.

[23] B. Sh. Mordukhovich and Y. Shao, Nonsmooth sequential analysis in asplund space, *Trans. Amer. Math. Soc.*, **4** (1996), 1235–1279.

[24] J. J. Moreau, Rafle par un convexe variable I, *Sem. Anal. Convexe Montpellier*, exposé No. **15** (1971).

[25] J. J. Moreau, Rafle par un convexe variable II, *Sem. Anal. Convexe Montpellier*, exposé No. **3** (1972).

[26] J. J. Moreau, Evolution problem associated with amoving convex set in a Hilbert Space, *J. Differ. Equ.*, **26** (1977), 347–374.

[27] J. J. Moreau, *Unilateral contact and dry friction in finite freedom dynamics*, Nonsmooth Mechanics and Applications 302 in CISM, Courses and Lectures, Springer Verlag, 1988.

[28] V. Recupero, A continuity method for sweeping processes, *J. Differ. Equ.*, **251** (2011), 2125–2142 .

[29] V. Recupero, BV continuous sweeping processes, *J. Differ. Equ.*, **259** (2015), 4253–4272.

[30] V. Recupero, Sweeping processes and rate independence, *J. Convex Anal.*, **23** (2016), 921–946.

[31] V. Recupero, F. Santambrogio, Sweeping processes with prescribed behavior on jumps, *Ann. Mat. Pura Appl.*, **197** (2018), 1311–1332.

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Frames associated with shift invariant spaces on positive half line

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Abstract. In this paper, we introduce the notion of Walsh shift-invariant space and present a unified approach to the study of shift-invariant systems to be frames in $L^2(\mathbb{R}^+)$. We obtain a necessary condition and three sufficient conditions under which the Walsh shift-invariant systems constitute frames for $L^2(\mathbb{R}^+)$. Furthermore, we discuss applications of our main results to obtain some known conclusions about the Gabor frames and wavelet frames on positive half line.

1 Introduction

Shift-invariant spaces play an important role in modern analysis for the past two decades because of their rich underlying theory and their applications

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in time frequency analysis, approximation theory, numerical analysis, digital signal and image processing and so on. A shift-invariant space serves as a universal model for sampling problem as it includes a large class of functions whether bandlimited or not by appropriately choosing a generator. The concept of shift-invariant subspace of $L^2(\mathbb{R})$ was introduced by Helson [15]. In fact, he introduced range functions and used this notion to completely characterize shift invariant spaces. Later on, a considerable amount of research has been conducted using this framework in order to describe and characterize frames and bases of these spaces. For example, de Boor et al.[7] gave the general structure of these spaces in $L^2(\mathbb{R}^n)$ using the machinery of fiberization based on range functions. This has been further developed in the work of Ron and Shen [25] with the introduction of the technique of Gramians and dual Gramians. Bownik [8] gave a characterization of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ following an idea from Helson's book [15]. The invariance properties of shift-invariant spaces under non-integer translations were completely characterized by Aldroubi et al.[4] and they showed that the principal shift-invariant spaces generated by a compactly supported function is not invariant under such translations. In [23], authors constructed p-frames for the weighted shift-invariant spaces and investigated their frame properties under some mild technical conditions on the frame generators. On the other side, the study of shift-invariant spaces and frames have been extended to locally compact Abelian groups in [9], nilpotent Lie groups in [11] and non-abelian compact groups in [24]. The results of Aldroubi et al.[4] were further generalized to the context of LCA groups by Anastasio et al.[5]. They provide necessary and sufficient conditions for an H-invariant space to be M-invariant space by means of range functions, where H is a countable uniform lattice in G and M is any closed subgroup of G containing H. Shift-invariant spaces for local fields were first introduced and investigated by Ahmadi et al.[2]. More precisely, they studied shift-invariant spaces of $L^2(G)$, where G is a locally compact abelian group, or in general a local field, with a compact open subgroup. The general results in Euclidean spaces to characterize tight frame generators for the shift-invariant subspaces was studied by Labate in [19]. Some applications of this general result are then obtained, among which are the characterization of tight wavelet frames and tight Gabor frames [20, 21]. In his recent paper, Behera [6] showed that every closed shift-invariant subspace of $L^2(\mathbb{R}^+)$ is generated by the Λ-translates of a countable number of functions, where K is the local field of positive characteristic and Λ is the associated translation set.

In the framework of mathematical analysis and linear algebra, redundant representations are obtained by analysing vectors with respect to an overcomplete system.Then the obtained vectors are interpreted using the frame theory as introduced by Duffin and Schaeffer [12] and recently studied at depth, see [10] and the compressive list of references therein. Most commonly used coherent/structured frames are wavelet, Gabor, and wave-packet frames which are a mixture type of wavelet and Gabor frames [10]. Frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, oversampling, and irregular sampling ploy a role. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine. Recall that a countable collection $\{f_k : k \in \mathbb{Z}\}$ in an infinite-dimensional separable Hilbert space $\mathcal{H}$ is called a *frame* if there exist positive constants A and B such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \tag{1}$$

holds for every $f \in \mathcal{H}$ and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. If we only require the second inequality to hold in (1), then $\{f_k : k \in \mathbb{Z}\}$ is called a *Bessel collection*. A frame is tight if $A = B$ in (1) and if $A = B = 1$ it is called a *Parseval frame* or a *normalized tight frame*.

During last two decades there is a substantial body of work that has been concerned with the wavelet and Gabor frames on positive half line. Kozyrev [16] found a compactly supported p-adic wavelet basis for $L^2(\mathbb{Q}_p)$ which is an analog of the Haar basis. It turns out that these wavelets are eigenfunctions of some p-adic pseudodifferential operators in [18]. Such property used to solve p-adic pseudodifferential equations which are needed for some physical problems. Khrennikov et al. [17] developed a method to find explicitly the solution for a wide class of evolutionary linear pseudo-differential equations. Farkov [13] indicated several differences between the constructed wavelets in Walsh analysis and the classical wavelets, and characterized all compactly supported refinable functions on the Vilenkin group G_p with $p \geq 2$. Manchanda et al. [22] introduced the vector-valued wavelet packets and obtained their properties and orthogonality formulas. Albeverio et al. [3] presented a complete characterization of scaling functions generating an p-MRA, suggested a method for constructing sets of wavelet functions, and proved that any set of wavelet functions generates a p-adic wavelet frame. Shah [27] constructed Gabor frame on positive half line and obtain necessary and sufficient conditions for Gabor frames in $L^2(\mathbb{R}^+)$. More Recently, Zhang [28] characterize the shift-invariant

Bessel sequences, frame sequences and Riesz sequences in $L^2(\mathbb{R}^+)$ and give a characterization of dual wavelet frames using Walsh-Fourier transform.

Motivated and inspired by the above work, we introduce the notion of Walsh shift-invariant spaces and establish some necessary and sufficient conditions under which shift-invariant systems become frames in $L^2(\mathbb{R}^+)$. Furthermore, we use these results to give some necessary conditions and sufficient conditions for Gabor frames and wavelet frames on positive half line.

The paper is structured as follows. In Section 2, we give a brief introduction to Walsh-Fourier analysis including the definition of shift-invariant spaces on half line. In Section 3, we obtain a necessary condition for the shift-invariant system to be a frame for $L^2(\mathbb{R}^+)$. In Section 4, we establish sufficient conditions for shift-invariant systems to be frames. Sections 5 and 6 discusses applications of the our main results to Gabor frames and wavelet frames, respectively on positive half line.

2 Preliminaries and shift-invariant spaces on positive half line

As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x] (\operatorname{mod} p), \qquad x_{-j} = [p^{1-j} x] (\operatorname{mod} p), \tag{2}$$

where $x_j, x_{-j} \in \{0, 1, \dots, p-1\}$. Clearly, x_j and x_{-j} are the digits in the p-expansion of x:

$$x = \sum_{j<0} x_{-j} p^{-j-1} + \sum_{j>0} x_j p^{-j}.$$

Moreover, the first sum on the right is always finite. Besides,

$$[x] = \sum_{j<0} x_{-j} p^{-j-1}, \qquad \{x\} = \sum_{j>0} x_j p^{-j},$$

where $[x]$ and $\{x\}$ are, respectively, the integral and fractional parts of x.

Consider on $\mathbb{R}^+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = x_j + y_j (\mathrm{mod}\, p)$, $j \in \mathbb{Z} \setminus \{0\}$, where $\zeta_j \in \{0, 1, \ldots, p-1\}$ and x_j, y_j are calculated by (2). Clearly, $[x \oplus y] = [x] \oplus [y]$ and $\{x \oplus y\} = \{x\} \oplus \{y\}$. As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

Let $\varepsilon_p = \exp(2\pi i/p)$, we define a function $r_0(x)$ on $[0, 1)$ by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \\ \varepsilon_p^\ell, & \text{if } x \in \left[\ell p^{-1}, (\ell+1)p^{-1}\right), \quad \ell = 1, 2, \ldots, p-1. \end{cases}$$

The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1) = r_0(x), \forall\ x \in \mathbb{R}^+$. Then, the system of *generalized Walsh functions* $\{w_m(x) : m \in \mathbb{Z}^+\}$ on $[0, 1)$ is defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \ldots, p-1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of generalized Walsh functions. For $x, y \in \mathbb{R}^+$, let

$$\chi(x, y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{3}$$

where x_j, y_j are given by equation (2).

We observe that

$$\chi\left(x, \frac{m}{p^n}\right) = \chi\left(\frac{x}{p^n}, m\right) = w_m\left(\frac{x}{p^n}\right), \qquad \forall\ x \in [0, p^n),\ m, n \in \mathbb{Z}^+,$$

and

$$\chi(x \oplus y, z) = \chi(x, z)\, \chi(y, z), \quad \chi(x \ominus y, z) = \chi(x, z)\, \overline{\chi(y, z)},$$

where $x, y, z \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational. It is well known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1)$ (See [14, 26]).

The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\, \overline{\chi(x, \xi)}\, dx, \tag{4}$$

where $\chi(x,\xi)$ is given by (3). The Walsh-Fourier operator $\mathcal{F} : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $\mathcal{F}f = \widehat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [14, 26]). In particular, if $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, then $\widehat{f} \in L^2(\mathbb{R}^+)$ and

$$\left\|\widehat{f}\right\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}. \tag{5}$$

Moreover, if $f \in L^2[0,1)$, then we can define the Walsh-Fourier coefficients of f as

$$\widehat{f}(n) = \int_0^1 f(x)\overline{w_n(x)}\,dx. \tag{6}$$

The series $\sum_{n\in\mathbb{Z}^+} \widehat{f}(n) w_n(x)$ is called the *Walsh-Fourier series* of f. Therefore, from the standard L^2-theory, we conclude that the Walsh-Fourier series of f converges to f in $L^2[0,1)$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_0^1 \left|f(x)\right|^2 dx = \sum_{n\in\mathbb{Z}^+} \left|\widehat{f}(n)\right|^2. \tag{7}$$

By p-adic interval $I \subset \mathbb{R}^+$ of range n, we mean intervals of the form

$$I = I_n^k = \left[kp^{-n}, (k+1)p^{-n}\right), \quad k \in \mathbb{Z}^+.$$

The p-adic topology is generated by the collection of p-adic intervals and each p-adic interval is both open and closed under the p-adic topology (see [14]). The family $\left\{[0, p^{-j}) : j \in \mathbb{Z}\right\}$ forms a fundamental system of the p-adic topology on $\mathbb{R}^+$. Therefore, the generalized Walsh functions $w_j(x), 0 \le j \le p^n - 1$, assume constant values on each p-adic interval I_n^k and hence continuous on these intervals. Thus, $w_j(x) = 1$ for $x \in I_n^0$.

Let $\mathcal{E}_n(\mathbb{R}^+)$ be the space of p-adic entire functions of order n, that is, the set of all functions which are constant on all p-adic intervals of range n. Thus, for every $f \in \mathcal{E}_n(\mathbb{R}^+)$, we have

$$f(x) = \sum_{k\in\mathbb{Z}^+} f(p^{-n}k)\chi_{I_n^k}(x), \quad x \in \mathbb{R}^+. \tag{8}$$

Clearly each Walsh function of order up to p^{n-1} belongs to $\mathcal{E}_n(\mathbb{R}^+)$. The set $\mathcal{E}(\mathbb{R}^+)$ of p-adic entire functions on $\mathbb{R}^+$ is the union of all the spaces $\mathcal{E}_n(\mathbb{R}^+)$. It is clear that $\mathcal{E}(\mathbb{R}^+)$ is dense in $L^p(\mathbb{R}^+), 1 \le p < \infty$ and each function in $\mathcal{E}(\mathbb{R}^+)$ is of compact support. Thus, we consider the following set of functions

$$\mathcal{E}^0(\mathbb{R}^+) = \left\{f \in \mathcal{E}(\mathbb{R}^+) : \ \widehat{f} \in L^\infty(\mathbb{R}^+) \text{ and } \operatorname{supp} f \subset \mathbb{R}^+ \setminus \{0\}\right\}. \tag{9}$$

For any $A \in \mathbb{R}^+$ and $m, n \in \mathbb{Z}^+$, we define the following operators on $L^2(\mathbb{R}^+)$ as:

$$T_n f(x) = f\big(x \ominus n\big);\; E_m f(x) = \chi_m\big(x\big) f(x);\; D_A f(x) = \sqrt{A} f\,(Ax)\,.$$

Then for any $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, the following results can easily be verified:

$$\mathcal{F}\big\{T_n f(x)\big\} = E_{-n}\mathcal{F}\big\{f(x)\big\};\; \mathcal{F}\big\{E_m f(x)\big\} = T_m \mathcal{F}\big\{f(x)\big\};$$
$$\mathcal{F}\big\{D_{A^j} f(x)\big\} = D_{A^{-j}}\mathcal{F}\big\{f(x)\big\}.$$

Definition 1 *A closed subspace S of $L^2(\mathbb{R}^+)$ is called a Walsh shift invariant space if $T_k\varphi_\alpha(x) \in S$, for every $\varphi_\alpha \in S, k \in \mathbb{Z}^+$, and $\alpha \in \Lambda$, where T_k is the translation operator and Λ is a countable indexing set.*

A closed shift-invariant subspace S of $L^2(\mathbb{R}^+)$ is said to be generated by $\Psi \subset L^2(\mathbb{R}^+)$ if $S = \overline{span}\,\big\{T_k\psi_\alpha(x) := \psi_\alpha\big(x \ominus k\big) : k \in \mathbb{Z}^+, \psi_\alpha \in \Psi\big\}$. The cardinality of a smallest generating set Ψ for S is called the length of S which is denoted by $|S|$. If $|S| =$ finite, then S is called a finite Walsh shift-invariant space (FSI) and if $|S| = 1$, then S is called a principal Walsh shift-invariant space (PSI). Moreover, the spectrum of a Walsh shift-invariant space is defined to be

$$\sigma(S) = \Big\{\xi \in [0,1) : \widehat{S}(\xi) \neq \{0\}\Big\}, \tag{10}$$

where $\widehat{S}(\xi) = \big\{\widehat{\varphi}_\alpha\big(\xi \oplus k\big) \in l^2(\mathbb{Z}^+) : \varphi_\alpha \in S, k \in \mathbb{Z}^+, \alpha \in \Lambda\big\}\,.$

It is easy to verify that the system

$$\Gamma = \Big\{T_k\varphi_\alpha(x) =: \varphi_\alpha\big(x \ominus k\big) : k \in \mathbb{Z}^+, \alpha \in \Lambda\Big\}, \tag{11}$$

is a Walsh shift-invariant system with respect to lattice $\mathbb{Z}^+$, where $\varphi_\alpha(x) \in L^2(\mathbb{R}^+)$.

Definition 2 *The Walsh shift-invariant system Γ defined by (11) is called shift-invariant frame if there exist constants C and D with $0 < C \leq D < \infty$ such that*

$$C\big\|\varphi\big\|_2^2 \leq \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+}\big|\big\langle \varphi, T_n\varphi_\alpha\big\rangle\big|^2 \leq D\big\|\varphi\big\|_2^2, \tag{12}$$

holds for every $\varphi \in L^2(\mathbb{R}^+)$. The largest constant A and the smallest constant B satisfying (12) are called the optimal lower and upper frame bounds, respectively. A frame is a tight frame if A and B are chosen so that $A = B$ and is a Parseval frame if $A = B = 1$.

Since the set $\mathcal{E}(\mathbb{R}^+)$ is dense in $L^2(\mathbb{R}^+)$ and is closed under the Fourier transform, the set $\mathcal{E}^0(\mathbb{R}^+)$ defined by (9) is also dense in $L^2(\mathbb{R}^+)$. Therefore, it is sufficient to prove that the Walsh shift-invariant system Γ given by (11) is a frame for $L^2(\mathbb{R}^+)$ if the inequalities in (12) holds for all $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$.

3 Necessary condition for Walsh shift invariant space to be frame for $L^2(\mathbb{R}^+)$

In this section, we shall study the necessary condition for the Walsh shift-invariant system Γ defined by (11) to be frame for $L^2(\mathbb{R}^+)$.

Theorem 1 *If the Walsh shift-invariant system* Γ *defined by* (11) *is a frame for* $L^2(\mathbb{R}^+)$ *with bounds* A *and* B*, then*

$$A \leq G_\Gamma(\xi) \leq B, \quad \text{a.e. } \xi \in \mathbb{R}^+ \tag{13}$$

where $G_\Gamma(\xi) = \sum_{\alpha\in\Lambda} \left|\widehat{\varphi}_\alpha(\xi)\right|^2$.

Proof. Since the Walsh shift-invariant system Γ is a frame for $L^2(\mathbb{R}^+)$ with bounds A and B, we have

$$A\|\varphi\|_2^2 \leq \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+} \left|\langle \varphi, T_n\varphi_\alpha\rangle\right|^2 \leq B\|\varphi\|_2^2, \quad \text{for all } \varphi \in L^2(\mathbb{R}^+). \tag{14}$$

Notice that for all $\varphi \in L^2(\mathbb{R}^+)$ and $m \in \mathbb{Z}^+$. By substituting φ by $T_m\varphi$, equation (14) can be rewritten as

$$A\|\varphi\|_2^2 \leq \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+} \left|\langle T_m\varphi, T_n\varphi_\alpha\rangle\right|^2 \leq B\|\varphi\|_2^2.$$

Or, equivalently

$$A\|\varphi\|_2^2 \leq \int_{[0,1)} \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+} \left|\langle T_m\varphi, T_n\varphi_\alpha\rangle\right|^2 dx \leq B\|\varphi\|_2^2, \quad \text{for all } \varphi \in L^2(\mathbb{R}^+). \tag{15}$$

Since $\mathbb{R}^+ = \bigcup_{n\in\mathbb{Z}^+} \left(\mathbb{T} \oplus n\right)$ is a disjoint union,where $\mathbb{T} = [0, 1)$. Therefore, it follows from the Plancherel theorem that

$$\begin{aligned}\int_{\mathbb{T}} \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+} \left|\left\langle T_m\varphi, T_n\varphi_\alpha\right\rangle\right|^2 dx &= \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+}\int_{\mathbb{T}} \left|\left\langle \varphi, T_{n\ominus m}\varphi_\alpha\right\rangle\right|^2 dx \\ &= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+} \left|\left\langle \varphi, T_m\varphi_\alpha\right\rangle\right|^2 dx \\ &= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+} \left|\left\langle \widehat{\varphi}, (T_m\varphi_\alpha)^\wedge\right\rangle\right|^2 dx \\ &= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+} |\langle \widehat{\varphi}, E_{\ominus m}\widehat{\varphi}_\alpha\rangle|^2 dx \\ &= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+} \left|\int_{\mathbb{R}^+} \widehat{\varphi}(\xi)\overline{\widehat{\varphi}_\alpha(\xi)}\, w_m(\xi)\, d\xi\right|^2 dx.\end{aligned} \tag{16}$$

Clearly, for all $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$, we have $\widehat{\varphi}(\xi)\overline{\widehat{\varphi}_\alpha(\xi)} \in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$. Therefore, it follows from (16) and the Plancherel theorem that for all $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$,

$$\begin{aligned}\int_{\mathbb{T}} \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+} \left|\left\langle T_m\varphi, T_n\varphi_\alpha\right\rangle\right|^2 dx &= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+} \left|\left(\widehat{\varphi}\overline{\widehat{\varphi}_\alpha}\right)^\vee(\xi)\right|^2 dx \\ &= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+} \left|\widehat{\varphi}(\xi)\overline{\widehat{\varphi}_\alpha}(\xi)\right|^2 d\xi \\ &= \int_{\mathbb{R}^+} \left|\widehat{\varphi}(\xi)\right|^2 \sum_{\alpha\in\Lambda}\left|\widehat{\varphi}_\alpha(\xi)\right|^2 d\xi.\end{aligned} \tag{17}$$

Combining (15) and (17), we observe that for all $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$, we have

$$A\|\varphi\|_2^2 \le \int_{\mathbb{R}^+} \left|\widehat{\varphi}(\xi)\right|^2 \sum_{\alpha\in\Lambda}\left|\widehat{\varphi}_\alpha(\xi)\right|^2 d\xi \le B\|\varphi\|_2^2. \tag{18}$$

Making appropriate choices of $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$ in (18), we obtain

$$A \le G_\Gamma(\xi) \le B, \quad \text{a.e. } \xi \in \mathbb{R}^+.$$

Thus the proof of Theorem 1 is complete. □

4 Sufficient conditions for Walsh shift invariant space to be frame for $L^2(\mathbb{R}^+)$

In this section, we derive three sufficient conditions for the Walsh shift-invariant system Γ to be a frame for $L^2(\mathbb{R}^+)$.

In order to prove our results, we need the following lemma.

Lemma 1 *Suppose that Walsh shift-invariant system* Γ *is defined by* (11). *If* $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$ *and* $\operatorname{ess\,sup}\left\{\sum_{\alpha\in\Lambda}\left|\widehat{\varphi}_\alpha(\xi)\right|^2 : \xi \in \mathbb{R}^+\right\} < \infty$, *then*

$$\sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+}\left|\langle \varphi, T_n\varphi_\alpha\rangle\right|^2 = \int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2 \sum_{\alpha\in\Lambda} |\widehat{\varphi}_\alpha(\xi)|^2\, d\xi + R(\varphi), \tag{19}$$

where

$$R(\varphi) = \sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+} \widehat{\varphi}(\xi)\overline{\widehat{\varphi}_\alpha(\xi)}\widehat{\varphi}(\xi\oplus m)\overline{\widehat{\varphi}_\alpha(\xi\oplus m)}\, d\xi. \tag{20}$$

Furthermore, the iterated series in (20) *is absolutely convergent.*

Proof. By Parseval formula, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^+}\left|\langle \varphi, T_n\varphi_\alpha\rangle\right|^2 &= \sum_{n\in\mathbb{Z}^+}\left|\left\langle \widehat{\varphi}, (T_n\varphi_\alpha)^\wedge\right\rangle\right|^2 \\
&= \sum_{n\in\mathbb{Z}^+}\left|\int_{\mathbb{R}^+} \widehat{\varphi}(\xi)\overline{\widehat{\varphi}_\alpha(\xi)}\, w_n(\xi)\, d\xi\right|^2 \\
&= \sum_{n\in\mathbb{Z}^+}\int_{\mathbb{R}^+}\left\{\sum_{m\in\mathbb{Z}^+}\int_{[0,1)} \widehat{\varphi}(\xi\oplus m)\overline{\widehat{\varphi}_\alpha(\xi\oplus m)}\, w_n(\xi\oplus m)\, d\xi\right\} \\
&\qquad\qquad \times\ \overline{\widehat{\varphi}(\xi)}\widehat{\varphi}_\alpha(\xi)\, \overline{w}_n(\xi)\, d\xi.
\end{aligned}$$

Notice that $\widehat{\varphi}$ has compact support and $w_{nm} \equiv 1$ for all $m, n \in \mathbb{Z}^+$. Therefore, by the convergence theorem of Fourier series on $\mathbb{T}$, we obtain

$$\sum_{n\in\mathbb{Z}^+}\left|\langle \varphi, T_n\varphi_\alpha\rangle\right|^2 = \int_{\mathbb{R}^+} \overline{\widehat{\varphi}(\xi)}\widehat{\varphi}_\alpha(\xi)\left\{\sum_{m\in\mathbb{Z}^+} \widehat{\varphi}(\xi\oplus m)\overline{\widehat{\varphi}_\alpha(\xi\oplus m)}\right\} d\xi. \tag{21}$$

We claim that

$$\sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+}\left|\langle \varphi, T_n\varphi_\alpha\rangle\right|^2 = \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2\, |\widehat{\varphi}_\alpha(\xi)|^2\, d\xi + R(\varphi), \tag{22}$$

hold for all $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$. In fact, by (21), we have

$$\begin{aligned}
&\sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+}\left|\left\langle\varphi, T_n\varphi_\alpha\right\rangle\right|^2 \\
&= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+}\left\{|\widehat{\varphi}(\xi)|^2\,|\widehat{\varphi}_\alpha(\xi)|^2 + \overline{\widehat{\varphi}(\xi)}\widehat{\varphi}_\alpha(\xi)\sum_{m\in\mathbb{Z}^+}\widehat{\varphi}(\xi\oplus m)\overline{\widehat{\varphi}_\alpha(\xi\oplus m)}\right\} \\
&= \sum_{\alpha\in\Lambda}\int_{\mathbb{R}^+}|\widehat{\varphi}(\xi)|^2\,|\widehat{\varphi}_\alpha(\xi)|^2\,d\xi + R(\varphi).
\end{aligned}$$

This is just (22). Finally, by the condition that $\operatorname{ess\,sup}\left\{\sum_{\alpha\in\Lambda}\left|\widehat{\varphi}_\alpha(\xi)\right|^2 : \xi\in\mathbb{R}^+\right\} < \infty$, and by invoking Levi Lemma, we have

$$\sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+}\left|\left\langle\varphi, T_n\varphi_\alpha\right\rangle\right|^2 = \int_{\mathbb{R}^+}|\widehat{\varphi}(\xi)|^2\sum_{\alpha\in\Lambda}|\widehat{\varphi}_\alpha(\xi)|^2\,d\xi + R(\varphi).$$

We now proceed to prove that the iterated series in (20) is absolutely convergent. Note that

$$\left|\overline{\widehat{\varphi}_\alpha(\xi)}\widehat{\varphi}_\alpha(\xi\oplus m)\right| \leq \frac{1}{2}\left[|\widehat{\varphi}_\alpha(\xi)|^2 + \left|\widehat{\varphi}_\alpha(\xi\oplus m)\right|^2\right].$$

We have

$$\begin{aligned}
\left|R(\varphi)\right| &= \left|\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\widehat{\varphi}(\xi)\overline{\widehat{\varphi}_\alpha(\xi)}\widehat{\varphi}(\xi\oplus m)\overline{\widehat{\varphi}_\alpha(\xi\oplus m)}\,d\xi\right| \\
&\leq \sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\left|\widehat{\varphi}(\xi)\overline{\widehat{\varphi}_\alpha(\xi)}\widehat{\varphi}(\xi\oplus m)\overline{\widehat{\varphi}_\alpha(\xi\oplus m)}\,d\xi\right| \\
&\leq \frac{1}{2}\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\left|\widehat{\varphi}(\xi)\widehat{\varphi}(\xi\oplus m)\right|\,|\widehat{\varphi}_\alpha(\xi)|^2\,d\xi \\
&\qquad\qquad + \frac{1}{2}\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\left|\widehat{\varphi}(\xi)\widehat{\varphi}(\xi\oplus m)\right|\,\left|\widehat{\varphi}_\alpha(\xi\oplus m)\right|^2\,d\xi.
\end{aligned}$$

Hence it suffices to verify that the series

$$\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\left|\widehat{\varphi}(\xi)\widehat{\varphi}(\xi\oplus m)\right|\,|\widehat{\varphi}_\alpha(\xi)|^2\,d\xi$$

is convergent. In fact,

$$\begin{aligned}
&\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\left|\widehat{\varphi}(\xi)\widehat{\varphi}(\xi\oplus m)\right|\,|\widehat{\varphi}_\alpha(\xi)|^2\,d\xi \\
&\qquad\qquad \leq \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^+}\sum_{\alpha\in\Lambda}|\widehat{\varphi}_\alpha(\xi)|^2\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\left|\widehat{\varphi}(\xi)\widehat{\varphi}(\xi\oplus m)\right|\,d\xi.
\end{aligned} \tag{23}$$

Since $m \in \mathbb{N}$ and $\varphi \in \mathcal{E}^0(\mathbb{R}^+)$, hence, only finite terms of the iterated series in (23) are non-zero. Consequently, (23) becomes

$$\sum_{\alpha\in\Lambda}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\left|\widehat{\varphi}(\xi)\widehat{\varphi}\big(\xi\oplus m\big)\right|\left|\widehat{\varphi}_\alpha(\xi)\right|^2 d\xi \\ \leq \text{ess sup}_{\xi\in\mathbb{R}^+}\sum_{\alpha\in\Lambda}\left|\widehat{\varphi}_\alpha(\xi)\right|^2 C\left\|\widehat{\varphi}\right\|^2, \tag{24}$$

where C is a constant. Using the assumption $\operatorname{ess\,sup}_{\xi\in\mathbb{R}^+}\sum_{\alpha\in\Lambda}\left|\widehat{f}_\alpha(\xi)\right|^2 < \infty$ and equation (24), it follows that the series (20) is absolutely convergent for all $f \in \mathcal{E}^0(\mathbb{R}^+)$. The proof of the lemma is completed. □

To establish the first sufficient condition of shift-invariant frame for $L^2(\mathbb{R}^+)$, we put

$$\underline{G}_\Gamma = \operatorname{ess\,inf}\left\{G_\Gamma(\xi) : \xi \in \mathbb{R}^+\right\}, \quad \overline{G}_\Gamma = \operatorname{ess\,sup}\left\{G_\Gamma(\xi) : \xi \in \mathbb{R}^+\right\},$$

$$\Theta_f\big(m\big) = \operatorname{ess\,sup}\left\{\sum_{\alpha\in\Lambda}\left|\widehat{\varphi}_\alpha(\xi)\widehat{\varphi}_\alpha\big(\xi\oplus m\big)\right| : \xi \in \mathbb{R}^+\right\},$$

where $G_\Gamma(\xi)$ is the same as the one in (13).

Theorem 2 *Suppose* $\left\{\varphi_\alpha(x) : \alpha \in \Lambda\right\} \subset L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ *such that*

$$C_1 = \underline{G}_\Gamma - \sum_{m\in\mathbb{N}}\left[\Theta_\varphi\big(m\big)\cdot\Theta_\varphi\big(\ominus m\big)\right]^{1/2} > 0, \tag{25}$$

$$D_1 = \overline{G}_\Gamma + \sum_{m\in\mathbb{N}}\left[\Theta_\varphi\big(m\big)\cdot\Theta_\varphi\big(\ominus m\big)\right]^{1/2} < +\infty. \tag{26}$$

Then $\left\{T_n\varphi_\alpha(x) : n \in \mathbb{Z}^+, \alpha \in \Lambda\right\}$ *is a frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_1 *and* D_1.

Proof. By Lemma 1 and (26), equation (19) holds, where

$$R(\varphi) = \sum_{m\in\mathbb{N}}\int_{\mathbb{R}^+}\widehat{\varphi}(\xi)\overline{\widehat{\varphi}\big(\xi\oplus m\big)}\sum_{\alpha\in\Lambda}\overline{\widehat{\varphi}_\alpha(\xi)}\widehat{\varphi}_\alpha\big(\xi\oplus m\big)\, d\xi. \tag{27}$$

Using the Cauchy–Schwarz inequality, the change of variables $\eta = \xi \oplus m$

and (27), we obtain

$$
\begin{aligned}
\left|R(\varphi)\right| &\leq \sum_{m\in\mathbb{N}} \left\{ \int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2 \left| \sum_{\alpha\in\Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| d\xi \right\}^{1/2} \\
&\qquad\qquad \times \left\{ \int_{\mathbb{R}^+} \left|\widehat{\varphi}(\xi \oplus m)\right|^2 \left| \sum_{\alpha\in\Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| d\xi \right\}^{1/2} \\
&= \sum_{m\in\mathbb{N}} \left\{ \int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2 \left| \sum_{\alpha\in\Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| d\xi \right\}^{1/2} \\
&\qquad\qquad \times \left\{ \int_{\mathbb{R}^+} |\widehat{\varphi}(\eta)|^2 \left| \sum_{\alpha\in\Lambda} \overline{\widehat{\varphi}_\alpha(\eta \ominus m)} \widehat{\varphi}_\alpha(\eta) \right| d\eta \right\}^{1/2} \\
&\leq \int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2 \, d\xi \sum_{m\in\mathbb{N}} \left[\Theta_\varphi(m) \cdot \Theta_\varphi(\ominus m) \right]^{1/2} \\
&\leq \|\varphi\|_2^2 \sum_{m\in\mathbb{N}} \left[\Theta_\varphi(m) \cdot \Theta_\varphi(\ominus m) \right]^{1/2}.
\end{aligned} \tag{28}
$$

Consequently, it follows from equations (19), (25), (26) and (28) that

$$
C_1\|\varphi\|_2^2 \leq \sum_{\alpha\in\Lambda} \sum_{n\in\mathbb{Z}^+} \left|\langle \varphi, T_n\varphi_\alpha \rangle\right|^2 \leq D_1\|\varphi\|_2^2, \quad \text{for all } \varphi \in \mathcal{E}^0(\mathbb{R}^+).
$$

The proof of Theorem 2 is complete. ☐

Remark 1 *It is easy to see that*

$$
\sum_{m\in\mathbb{N}} \Theta_\varphi(m) = \sum_{m\in\mathbb{N}} \Theta_\varphi(\ominus m).
$$

Set $\Theta_\varphi = \sum_{m\in\mathbb{N}} \Theta_\varphi(m)$. *Then by* (27) *and the Cauchy–Schwarz inequality, we have*

$$
\left|R(\varphi)\right| \leq \sum_{m\in\mathbb{N}} \left[\Theta_f(m) \cdot \Theta_f(\ominus m) \right]^{1/2} \|f\|_2^2 \leq \Theta_f \|f\|_2^2.
$$

As a consequence, the following second sufficient condition is obtained.

Theorem 3 *Suppose* $\{\varphi_\alpha(x) : \alpha \in \Lambda\} \subset L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ *such that*

$$C_2 = \underline{G}_\Gamma - \Theta_f > 0, \tag{29}$$

$$D_2 = \overline{G}_\Gamma + \Theta_f < +\infty. \tag{30}$$

Then $\{T_n\varphi_\alpha(x) : n \in \mathbb{Z}^+, \alpha \in \Lambda\}$ *is a frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_2 *and* D_2*.*

The proof follows in the similar lines to that of Theorem 2.

Using a different estimation technique, we are in a position to position to provide the third sufficient condition for the Walsh shift-invariant system $\mathcal{W}(\varphi, \alpha, k)$ to be frame of $L^2(\mathbb{R}^+)$ as follows:

Theorem 4 *Suppose* $\{\varphi_\alpha(x) : \alpha \in \Lambda\} \subset L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ *such that*

$$C_3 = \operatorname{ess}\inf_{\xi \in \mathbb{R}^+} \left[\sum_{\alpha \in \Lambda} |\widehat{\varphi}_\alpha(\xi)|^2 - \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| \right] > 0, \tag{31}$$

$$D_3 = \operatorname{ess}\sup_{\xi \in \mathbb{R}^+} \left[\sum_{m \in \mathbb{Z}^+} \left| \sum_{\alpha \in \Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| \right] < +\infty. \tag{32}$$

Then $\{T_n\varphi_\alpha(x) : n \in \mathbb{Z}^+, \alpha \in \Lambda\}$ *is a frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_3 *and* D_3*.*

Proof. We estimate $R(\varphi)$ in (27) by another technique. Using the Cauchy–Schwarz inequality, the change of variables $\eta = \xi \oplus m$ and the Levi Lemma, we have

$$\begin{aligned}
\left|R(\varphi)\right| &\le \sum_{m \in \mathbb{N}} \left\{ \int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2 \left| \sum_{\alpha \in \Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| d\xi \right\}^{1/2} \\
&\qquad \times \left\{ \int_{\mathbb{R}^+} \left|\widehat{\varphi}(\xi \oplus m)\right|^2 \left| \sum_{\alpha \in \Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| d\xi \right\}^{1/2} \\
&\le \left\{ \int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2 \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| d\xi \right\}^{1/2} \\
&\qquad \times \left\{ \int_{\mathbb{R}^+} |\widehat{\varphi}(\eta)|^2 \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\widehat{\varphi}_\alpha(\eta)} \widehat{\varphi}_\alpha(\eta \ominus m) \right| d\eta \right\}^{1/2} \\
&= \int_{\mathbb{R}^+} |\widehat{\varphi}(\xi)|^2 \sum_{m \in \mathbb{N}} \left| \sum_{\alpha \in \Lambda} \overline{\widehat{\varphi}_\alpha(\xi)} \widehat{\varphi}_\alpha(\xi \oplus m) \right| d\xi.
\end{aligned} \tag{33}$$

Consequently, it follows from equations (19), (31), (32) and (33) that

$$C_3\|\varphi\|_2^2 \le \sum_{\alpha\in\Lambda}\sum_{n\in\mathbb{Z}^+}\left|\langle\varphi, T_n\varphi_\alpha\rangle\right|^2 \le D_3\|\varphi\|_2^2, \quad \text{for all } \varphi\in\mathcal{E}^0(\mathbb{R}^+).$$

The proof of Theorem 4 is complete. □

5 Walsh shift-invariant systems as Gabor frames on positive half line

In this section, we apply Theorems 1, 2 and 4 to the Gabor systems and obtain some results on Gabor frames on local fields of positive characteristic.

Gabor systems are the collection of functions

$$\mathcal{G}(g,m,n) = \left\{M_m T_n g(x) =: w_m(x) g\big(x\ominus n\big) : m,n\in\mathbb{Z}^+\right\} \tag{34}$$

which are built by the action of modulations and translations of a single $g \in L^2(\mathbb{R}^+)$. If we interchange the role of the translation and modulation operators, we have the system

$$\mathcal{G}'(g,m,n) = \left\{T_n M_m g(x) =: w_m(x) g\big(x\ominus n\big) : m,n\in\mathbb{Z}^+\right\}. \tag{35}$$

It is immediate to see that the system $\mathcal{G}(g,m,n)$ is a frame of $L^2(\mathbb{R}^+)$ if and only if $\mathcal{G}'(g,m.n)$ is a frame of $L^2(\mathbb{R}^+)$, and the frame bounds are the same in the two cases. It is evident that Gabor system (35) is shift-invariant. So, the main results in Sections 3 and 4 can apply directly to the Gabor systems.

Setting $\Lambda = \{m : m\in\mathbb{Z}^+\}$, and for all $\alpha\in\Lambda$, we take $\varphi_\alpha = E_m g(x)$. Then the system $\Gamma(\varphi,\alpha,m)$ given by (11) reduces to the Gabor system $\mathcal{G}(g,m,n)$ defined by (34). Notice that for all $\alpha\in\Lambda$,

$$\widehat{\varphi}_\alpha(\xi) = \widehat{g}\big(\xi\ominus m\big).$$

Therefore, for all $n\in\mathbb{Z}^+$, we have

$$\sum_{\alpha\in\Lambda}\overline{\widehat{\varphi}_\alpha(\xi)}\,\widehat{\varphi}_\alpha\big(\xi\oplus n\big) = \sum_{m\in\mathbb{Z}^+}\overline{\widehat{g}\big(\xi\ominus m\big)}\,\widehat{g}\big(\xi\ominus m\oplus n\big).$$

Using Theorems 1, 2 and 4, we obtain

Theorem 5 *If the Gabor system* $\mathcal{G}(g,m,n)$ *defined by* (34) *is a frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_4 *and* D_4*, then*

$$C_4 \leq \sum_{m\in\mathbb{Z}^+} \left|\widehat{g}(\xi \ominus m)\right|^2 \leq D_4, \quad a.e.\, \xi \ \in \mathbb{R}^+. \tag{36}$$

Theorem 6 *Suppose* $g \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ *such that*

$$C_5 = \text{ess} \inf_{\xi\in\mathbb{R}^+} \sum_{m\in\mathbb{Z}^+} \left|\widehat{g}(\xi \ominus m)\right|^2 - \sum_{n\in\mathbb{N}} \Big[\Theta_g(n)\cdot\Theta_g(\ominus n)\Big]^{1/2} > 0, \tag{37}$$

$$D_5 = \text{ess} \sup_{\xi\in\mathbb{R}^+} \sum_{m\in\mathbb{Z}^+} \left|\widehat{g}(\xi \ominus m)\right|^2 + \sum_{n\in\mathbb{N}} \Big[\Theta_g(n)\cdot\Theta_g(\ominus n)\Big]^{1/2} < +\infty, \tag{38}$$

then $\left\{M_m T_n g(x) : m,n \in \mathbb{Z}^+\right\}$ *is a Gabor frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_5 *and* D_5*, where*

$$\Theta_g(n) = \text{ess}\sup \left\{ \sum_{m\in\mathbb{Z}^+} \left|\widehat{g}(\xi\ominus m)\widehat{g}(\xi\ominus m\oplus n)\right| : \xi\in\mathbb{R}^+ \right\}.$$

Theorem 7 *Suppose* $g \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ *such that*

$$C_6 = \text{ess} \inf_{\xi\in\mathbb{R}^+} \left\{ \sum_{m\in\mathbb{Z}^+} \left|\widehat{g}(\xi\ominus m)\right|^2 - \sum_{n\in\mathbb{N}} \left| \sum_{m\in\mathbb{Z}^+} \overline{\widehat{g}(\xi\ominus m)}\widehat{g}(\xi\ominus m\oplus n) \right| \right\} > 0, \tag{39}$$

$$D_6 = \text{ess} \sup_{\xi\in\mathbb{R}^+} \left\{ \sum_{n\in\mathbb{Z}^+} \left| \sum_{m\in\mathbb{Z}^+} \overline{\widehat{g}(\xi\ominus m)}\widehat{g}(\xi\ominus m\oplus n) \right| \right\} < +\infty. \tag{40}$$

Then $\left\{M_m T_n g(x) : m,n \in \mathbb{Z}^+\right\}$ *is a Gabor frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_6 *and* D_6*.*

Remark 2 *Since*

$$\Big\langle \varphi, T_n M_m g \Big\rangle = \Big\langle \varphi^\vee, (T_n M_m g)^\vee \Big\rangle = \Big\langle \varphi^\vee, T_{\ominus m} M_n g^\vee \Big\rangle$$

by the Plancherel Theorem, similarly to the case in the frequency domain, we able to present similar results in the time domain. They were omitted.

6 Walsh shift-invariant systems as wavelet frames on positive half line

In this section, we apply Theorems 1, 2 and 4 to the wavelet systems and obtain some results on wavelet frames on positive half line.

For a given $\psi \in L^2(\mathbb{R}^+)$, define the wavelet system

$$\mathcal{W}(\psi, j, k) = \left\{\psi_{j,k}(x) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\right\} \tag{41}$$

where $\psi_{j,k}(x) = p^{j/2}\psi\big(p^j x \ominus k\big)$. In general, the wavelet system $\mathcal{W}(\psi, j, k)$ is not shift-invariant, and thus the main results in Sections 3 and 4 do not apply directly to a wavelet system. But we can use a quasi-wavelet system to investigate the wavelet system. The *quasi-wavelet system* generated by $\psi \in L^2(\mathbb{R}^+)$ is defined by

$$\widetilde{\mathcal{W}}\big(\widetilde{\psi}, j, k\big) = \left\{\widetilde{\psi}_{j,k}(x) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\right\} \tag{42}$$

where

$$\widetilde{\psi}_{j,k}(x) = \begin{cases} D_{A^j} T_k \psi(x) = A^{j/2}\psi\big(A^j x \ominus k\big), & j \geq 0, k \in \mathbb{Z}^+, \\ A^{j/2} T_k D_{A^j}\psi(x) = A^j \psi\Big(A^j\big(x \ominus k\big)\Big), & j < 0, k \in \mathbb{Z}^+. \end{cases} \tag{43}$$

It is easy to see that the quasi-wavelet system is shift-invariant. There is some sort of equivalence between wavelet and quasi-wavelet systems. Indeed, Abdullah [1] proved in full generality the following result on positive half line.

Theorem 8 *Let* $\psi \in L^2(\mathbb{R}^+)$*. Then*

(a) $\mathcal{W}(\psi, j, k)$ *is a Bessel family if and only if* $\widetilde{\mathcal{W}}\big(\widetilde{\psi}, j, k\big)$ *is a Bessel family. Furthermore, their exact upper bounds are equal.*

(b) $\mathcal{W}(\psi, j, k)$ *is a frame for* $L^2(\mathbb{R}^+)$ *if and only if* $\widetilde{\mathcal{W}}\big(\widetilde{\psi}, j, k\big)$ *is a frame for* $L^2(\mathbb{R}^+)$*. Furthermore, their lower and upper exact bounds are equal.*

For $j \in \mathbb{Z}^+$*, let* $\mathcal{N}_j$ *denotes a full collection of coset representatives of* $\mathbb{Z}^+/A^j\mathbb{Z}^+$*, i.e.,*

$$\mathcal{N}_j = \begin{cases} \left\{0, 1, 2, \ldots, A^j - 1\right\}, & j \geq 0 \\ \{0\}, & j < 0. \end{cases} \tag{44}$$

Then, $\mathbb{Z}^+ = \bigcup_{n\in\mathcal{N}_j} \big(n \oplus A^j\mathbb{Z}^+\big)$, and for any distinct $n_1, n_2 \in \mathcal{N}_j$, we have $\big(n_1 \oplus A^j\mathbb{Z}^+\big) \cap \big(n_2 \oplus A^j\mathbb{Z}^+\big) = \emptyset$. Thus, every non-negative integer k can uniquely be written as $k = rA^{-j} \oplus s$, where $r \in \mathbb{Z}^+, s \in \mathcal{N}_j$.

We now show that the quasi-wavelet system $\widetilde{\mathcal{F}}(\widetilde{\psi}, j, k)$ given by (42) is invariant under translations by $k, k \in \mathbb{Z}^+$. In fact

$$T_k\widetilde{\psi}_{j,n}(x) = \widetilde{\psi}_{j,0}(x \ominus k) = A^j\psi\big(A^j(x \ominus k)\big) = \widetilde{\psi}_{j,k}, \quad \text{if } j < 0,$$

and for $j \geq 0, n \in \mathcal{N}_j$, we have

$$\begin{aligned} T_k\widetilde{\psi}_{j,n}(x) &= \widetilde{\psi}_{j,n}(x \ominus k) \\ &= \psi_{j,d}(x \ominus k) \\ &= A^{j/2}\psi\Big(A^j(x \ominus k) \ominus n\Big) \\ &= A^{j/2}\psi\Big(A^j x \ominus (A^j k \oplus n)\Big) \\ &= \psi_{j,kA^j\oplus n}(x). \end{aligned}$$

Therefore, the quasi-wavelet system can also be represented as Suppose that $\Lambda = \{(j, n) : j \in \mathbb{Z}, n \in \mathcal{N}_j\}$. Then, for all $\alpha \in \Lambda$, we set

$$\widetilde{\mathcal{W}}(\widetilde{\psi}, j, k) = \Big\{T_k\widetilde{\psi}_{j,n}(x) : j \in \mathbb{Z}, k \in \mathbb{Z}^+, n \in \mathcal{N}_j\Big\}. \tag{45}$$

Suppose that $\Lambda = \{(j, n) : j \in \mathbb{Z}, n \in \mathcal{N}_j\}$. Then, for all $\alpha \in \Lambda$, we set

$$\varphi_\alpha(x) = \begin{cases} A^{j/2}\psi(A^j x \ominus n), & \text{if } j \geq 0, \\ A^j\psi\Big(A^j(x \ominus n)\Big), & \text{if } j < 0. \end{cases} \tag{46}$$

Therefore, one can easily see that $\varphi_\alpha \in L^2(\mathbb{R}^+)$ and consequently, the system $\{T_k\varphi_\alpha : k \in \mathbb{Z}^+, \alpha \in \Lambda\}$ is the quasi-wavelet system $\widetilde{\mathcal{F}}(\widetilde{\psi}, j, k)$. Moreover, the Fourier transform of (46) yields

$$\widehat{\varphi}_\alpha(\xi) = \begin{cases} A^{-j/2}\widehat{\psi}(A^j\xi)\,\overline{w_n(A^j\xi)}, & \text{if } j \geq 0, \\ \widehat{\psi}(A^j\xi)\,\overline{w_n(A^j\xi)}, & \text{if } j < 0. \end{cases} \tag{47}$$

Thus, for all $m \in \mathbb{Z}^+$, we have

$$\begin{aligned} &\sum_{\alpha\in\Lambda} |\widehat{\varphi}_\alpha(\xi)|\,\big|\widehat{\varphi}_\alpha(\xi \oplus m)\big| \\ &\quad = \sum_{j<0}\Big|\widehat{\psi}(A^j\xi)\Big|\,\Big|\widehat{\psi}\Big(A^j(\xi \oplus m)\Big)\Big| + \sum_{j\geq 0}\sum_{n\in\mathcal{N}_j} A^{-j/2}\Big|\widehat{\psi}(A^j\xi)\Big|\,\Big|\widehat{\psi}\Big(A^j(\xi \oplus m)\Big)\Big| \end{aligned}$$

$$= \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(A^j\xi)\right| \left|\widehat{\psi}\Big(A^j(\xi\oplus m)\Big)\right|.$$

As a consequence of Theorems 1, 2, 4 and Theorem 8, a necessary condition and two sufficient conditions for wavelet frames on positive half line.

Theorem 9 *If the quasi-wavelet system* $\left\{\widetilde{\psi}_{j,k} : j\in\mathbb{Z}, k\in\mathbb{Z}^+\right\}$ *defined by* (42) *is a frame for* $L^2(\mathbb{R}^+)$ *with bounds* A_7 *and* B_7*, then*

$$C_7 \le \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(A^j\xi)\right|^2 \le D_7, \quad a.e.\ \xi\ \in \mathbb{R}^+. \tag{48}$$

Theorem 10 *Let* $\psi \in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$*. If*

$$C_8 = \operatorname*{ess\,inf}_{\xi\in\mathbb{R}^+} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(A^j\xi)\right|^2 - \sum_{n\in\mathbb{N}} \Big[\Theta_\psi(n)\cdot\Theta_\psi(\ominus n)\Big]^{1/2} > 0, \tag{49}$$

$$D_8 = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^+} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(A^j\xi)\right|^2 + \sum_{n\in\mathbb{N}} \Big[\Theta_\psi(n)\cdot\Theta_\psi(\ominus n)\Big]^{1/2} < +\infty. \tag{50}$$

Then $\left\{\widetilde{\psi}_{j,k}(x) : j\in\mathbb{Z}, k\in\mathbb{Z}^+\right\}$ *is a wavelet frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_8 *and* D_8*, where*

$$\Theta_\psi(n) = \operatorname{ess\,sup}\left\{\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(A^j\xi)\widehat{\psi}\Big(A^j(\xi\oplus n)\Big)\right| : \xi\in\mathbb{R}^+\right\}.$$

Theorem 11 *Let* $\psi \in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$*. If*

$$C_9 = \operatorname*{ess\,inf}_{\xi\in\mathbb{R}^+} \left\{\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(A^j\xi)\right|^2 - \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{N}} \left|\widehat{\psi}(A^j\xi)\widehat{\psi}\Big(A^j(\xi\oplus n)\Big)\right|\right\} > 0, \tag{51}$$

$$D_9 = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^+} \left\{\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^+} \left|\widehat{\psi}(A^j\xi)\widehat{\psi}\Big(A^j(\xi\oplus n)\Big)\right|\right\} < +\infty. \tag{52}$$

Then $\left\{\widetilde{\psi}_{j,k}(x) : j\in\mathbb{Z}, k\in\mathbb{Z}^+\right\}$ *is a wavelet frame for* $L^2(\mathbb{R}^+)$ *with bounds* C_9 *and* D_9*.*

References

[1] Abdullah, Affine and quasi-sffine frames on positive half line, *J. Math. Ext.*, **10** (3) (2016), 47–61.

[2] A. Ahmadi, A. A. Hemmat and R. R. Tousi, Shift-invariant spaces for local fields, *Int. J. Wavelets Multiresolut. Inf. Process.*, **9** (3) (2011), 417–426.

[3] S. Albeverio, S. Evdokimov, M. Skopina, p-adic multiresolution analysis and wavelet frames, *J. Fourier Anal. Appl.*, **16** (2010), 693–714.

[4] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson and U. Molter, Invariance of a shift-invariant space, *J. Fourier Anal. Appl.*, **16** (2010), 60–75.

[5] M. Anastasio, C. Cabrelli and V. Paternostro, Extra invariance of shift-invariant spaces on LCA groups, *J. Math. Anal. Appl.*, **370** (2010), 530–537.

[6] B. Behera, Shift-invariant subspaces and wavelets on local fields, *Acta Math. Hungar.*, **148** (2016), 157–173.

[7] C. de-Boor, R.A. De-Vore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$, *J. Funct. Anal.*, **119** (1994) 37–78.

[8] M. Bownik, The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$, *J. Funct. Anal.*, **177** (2000), 282–309.

[9] C. Cabrelli and V. Paternostro, Shift-invariant spaces on LCA groups, *J. Funct. Anal.*, **258** (2010), 2034–2059.

[10] O. Christensen, *An Introduction to Frames and Riesz Bases*, Second Edition, Birkhäuser, Boston, 2016.

[11] B. Currey, A. Mayeli and V. Oussa, Characterization of shift-invariant spaces on a class of nilpotent Lie groups with applications, *J. Fourier Anal. Appl.*, **20** (2014), 384–400.

[12] R. J. Duffin and A. C. Shaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, **72** (1952), 341–366.

[13] Y. A. Farkov, Constructions of MRA-based wavelets and frames in Walsh analysis, *Poincare J. Anal. Appl.*, **2** (2015), 13–36.

[14] B. I. Golubov, A. V. Efimov and V. A. Skvortsov (1991), Walsh Series and Transforms: Theory and Applications, (Kluwer, Dordrecht).

[15] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, London, 1964.

[16] S. V. Kozyrev, Wavelet analysis as a p-adic spectral analysis, *Izv. Akad. Nauk, Ser. Mat.*, **66** (2002), 149–158.

[17] A. Yu. Khrennikov and V. M. Shelkovich, An infinite family of p-adic non-Haar wavelet bases and pseudo-differential operators, p*-Adic Numb. Ultrametr. Anal. Appl.*, **3** (2009) 204–216.

[18] S. V. Kozyrev, p-adic pseudodifferential operators and p-adic wavelets, *Theor. Math. Phys.*, **138** (2004), 1–42.

[19] D. Labate, A unified characterization of reproducing systems generated by a finite family, J. Geom. Anal., **12** (2002), 469–491.

[20] D. Li and T. Qian, Sufficient conditions for shift-invariant systems to be frames in $L^2(\mathbb{R}^n)$, *Acta Math. Sinica, English Series.*, **29** (8) (2013), 1629–1636.

[21] D. Li G. Wu and X. Yang, Unified conditions for wavelet frames, *Georgian Math. J.*, **18** (2011), 761–776.

[22] P. Manchanda and V. Sharma, Construction of vector valued wavelet packets on $\mathbb{R}^+$ using Walsh-Fourier transform. *Indian J. Pure Appl. Math.*, **45** (2014), 539–553.

[23] S. Pilipović and S. Simić, Construction of frames for shift-invariant spaces, *J. Funct. Spaces Appl.*, (2013) Article ID. 163814, 7 pages.

[24] R. Radha and N.S. Kumar, Shift-invariant subspaces on compact groups, *Bull. Sci. Math.*, **137** (2013), 485–497.

[25] A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$, *Canad. J. Math.*, **47** (1995), 1051–1094.

[26] F. Schipp, W. R. Wade and P. Simon (1990), Walsh Series: An Introduction to Dyadic Harmonic Analysis, (Adam Hilger, Bristol and New York).

[27] F. A. Shah, Gabor frames on a half-line, *J. Contemp. Math. Anal.*, **47** (5) (2012), 251–260.

[28] Y. Zhang, Walsh Shift-Invariant Sequences and p-adic Nonhomogeneous Dual Wavelet Frames in $L^2(\mathbb{R}^+)$, *Results Math.*, **74** 111 (2019), 26 pp.

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CLT for single functional index quantile regression under dependence structure

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Abstract. In this paper, we investigate the asymptotic properties of a nonparametric conditional quantile estimation in the single functional index model for dependent functional data and censored at random responses are observed. First of all, we establish asymptotic properties for a conditional distribution estimator from which we derive an central limit theorem (CLT) of the conditional quantile estimator. Simulation study is also presented to illustrate the validity and finite sample performance of the considered estimator. Finally, the estimation of the functional index via the pseudo-maximum likelihood method is discussed, but not tackled.

1 Introduction

Multivariate regression analysis is a powerful statistical tool in biomedical research and many fields of life (Muharisa *et al.* [27]) with numerous applications.

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While linear regression can be used to model the expected value (ie, mean) of a continuous outcome given the covariates in the model, quantile regression can be used to compare the entire distribution of a continuous response or a specific quantile of the response between groups. Despite the regression function is of interest, other statistics such as quantile and mode regression might be important from a theoretical and a practical point of view. Quantile regression is a common way to describe the dependence structure between a response variable Y and some covariate X. Unlike the regression function that relies only on the central tendency of the data, the conditional quantile function allows the analyst to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. Moreover, it is well known that conditional quantiles can give a good description of the data (see, Chaudhuri *et al.* [9]), such as robustness to heavy-tailed error distributions and outliers to ordinary mean-based regression. As a particular case, note that the conditional median is useful for asymmetric distributions.

Quantile regression(QR) is one of the major statistical tools and is gradually developing into a comprehensive strategy for completing the regression prediction. It is emerging as a popular statistical approach, which complements the estimation of conditional mean models. While the latter only focuses on one aspect of the conditional distribution of the dependent variable, the mean, quantile regression provides more detailed insights by modeling conditional quantiles. Her can therefore detect whether the partial effect of a regressor on the conditional quantiles is the same for all quantiles or differs across quantiles, and can provide evidence for a statistical relationship between two variables even if the mean regression model does not. In many fields of applications like quantitative finance, econometrics, marketing and also in medical and biological sciences, QR is a fundamental element for data analysis, modeling and inference. An application in finance is the analysis of conditional Value-at-Risk, moreover, her is the development of statistical tools used to explain the relationship between response and predictor variables (see Yanuar *et al.* [37]). The quantile method is a technique of dividing a group of data into several parts after the data is sorted from the smallest to the largest Yanuar *et al.* [36]. QR enjoys some very appealing features. Apart from enabling some very exible patterns of partial effects, quantile regressions are also interesting because they satisfy some equivariance and robustness principles.

The advantage of the QR methodology is that it allows for understanding relationships between variables outside of the conditional mean of the response; it is useful for understanding an outcome at its various quantiles and comparing groups or levels of an exposure on those quantiles. QR is a common way to describe the dependence structure between a response variable Y and some covariate X. Unlike the regression function (which is defined as the conditional mean) that relies only on the central tendency of the data, the conditional quantile function allows the analysts to estimate the functional dependence between variables for all portions of the conditional distribution of the response variable. Moreover, quantiles are well known for their robustness to heavy-tailed error distributions and outliers which allow to consider them as a useful alternative to the regression function Chaouch and Khardani [8].

Moreover, it is a statistical technique intended to estimate, and conduct inference about, conditional quantile functions. Just as classical linear regression methods based on minimizing sums of squared residuals enable one to estimate models for conditional mean functions, quantile regression methods offer a mechanism for estimating models for the conditional median function, and the full range of other conditional quantile functions. By supplementing the estimation of conditional mean functions with techniques for estimating an entire family of conditional quantile functions, quantile regression is capable of providing a more complete statistical analysis of the stochastic relationships among random variables.

For example, QR has been used in a broad range of application settings. Reference growth curves for children's height and weight have a long history in pediatric medicine; quantile regression methods may be used to estimate upper and lower quantile reference curves as a function of age, sex, and other covariates without imposing stringent parametric assumptions on the relationships among these curves. In ecology, theory often suggests how observable covariates affect limiting sustainable population sizes, and quantile regression has been used to directly estimate models for upper quantiles of the conditional distribution rather than inferring such relationships from models based on conditional central tendency. In survival analysis, and event history analysis more generally, there is often also a desire to focus attention on particular segments of the conditional distribution, for example survival prospects of the oldest-old, without the imposition of global distributional assumptions.

In recent years, estimating conditional quantiles has received increasing interest in the literature, for both independent and dependent data; Samanta [31] established a nonparametric estimation of conditional quantiles, Wang and Zhao [35] presented a kernel estimator for conditional t-quantiles for mixing samples and established its strong uniform convergence. Ferraty *et al.* [15] studied the estimation of a conditional quantiles for functional dependent data with application to the climatic El Ninö phenomenon. Ezzahrioui & Elias Ould-Saïd [14] considered the estimation of the conditional quantile function when the covariates take values in some abstract function space, the almost complete convergence and the asymptotic normality of the kernel estimator of the conditional quantile under the α-mixing assumption were established. Ferraty *et al.* [15] introduced a nonparametric estimator of the conditional quantile defined as the inverse of the conditional cumulative distribution function (df) when data are dependent.

In life time data analysis, nonparametrically estimated conditional survival curves (such as the conditional Kaplan-Meier estimate) are useful for assessing the influence of risk factors, predicting survival probabilities, and checking goodness-of-fit of various survival regression models. It is well known that in medical studies the observation on the survival time of a patient is often incomplete due to right censoring. Classical examples of the causes of this type of censoring are that the patient was alive at the termination of the study, that the patient withdrew alive during the study, or that the patient died from other causes than those under study. The censored quantile regression model is derived from the censored model. This method is used to overcome problems in modeling censored data as well as to overcome the assumptions of linear models that are not met, in this linear models Sarmada and Yanuar [32] have compared the results of the analysis of the quantile regression method with the censored quantile regression method for censored data. In the context of censored data, Gannoun *et al.* [17] introduced a local linear (LL) estimator of the quantile regression and established its almost sure consistency (without rate) as well as its asymptotic normality in the independent and identically distributed (i.i.d.) case. El Ghouch and Van Keilegom [13] considered the LL estimation of the quantile regression and its first derivative under an α-mixing assumption and studied their asymptotic properties. Ould-Saïd [28] constructed a kernel estimator of the conditional quantile under an i.i.d. censorship model and established its strong uniform convergence rate. Under an α-mixing assumption, Liang and Alvarez [21] established the strong uniform convergence (with rate) of the conditional quantile function as well as its asymptotic distribution.

The single index model is a natural extension of the linear regression model for applications in which linearity does not hold. This last approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. In the past few recent years, the single functional index models have received much attention, and it has been studied extensively in both statistical and econometric literatures. Interesting to this methods, many authors worked on this sort of problems, see for instance Aït-Saidi *et al.* [1, 2]. Attaoui *et al.* [3] investigated the kernel estimator of the conditional density of a scalar response variable Y, given a Hilbertian random variable X when the observations are from a single functional index model. Ling *et al.* [24] reconsidered the kernel estimator of the conditional density when the scalar response variable Y and the Hilbertian random variable X also come from the single functional index model. The asymptotic results such as pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimation with rates in the setting of the α mixing functional data are also obtained, which extend the i.i.d. case in Attaoui *et al.* [3] to the dependence setting. Ling & Xu [23] investigated the estimation of conditional density function based on the single-index model for functional time series data. Under α-mixing condition, the asymptotic normality of the conditional density estimator and the conditional mode estimator where obtained. Attaoui [4] studied a nonparametric estimation of the conditional density of a scalar response variable given a random variable taking values in separable Hilbert space when the variables satisfy the strong mixing dependency, based on the single-index structure.

Inspired by all the papers above, our work in this paper aims to contribute to the research on functional nonparametric regression model, by giving an alternative estimation of QR estimation in the single functional index model with randomly right-censored data under α-mixing conditions whose definition is given below.

Recall that a process $(X_i, Y_i)_{i\geq 1}$ is called α-mixing or strongly mixing (see Lin and Lu [22]) for more details and examples, if

$$\sup_k \sup_{A\in\mathcal{F}_1^k} \sup_{B\in\mathcal{F}_{n+k}^{\infty}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| = \alpha(n) \to 0 \quad \text{as} \quad n \to \infty,$$

where $\mathcal{F}_j^k$ denotes the σ-field generated by the random variables $\{(X_i,\ Y_i),\ j \leq i \leq k\}$. The process $\{(X_i,\ Y_i),\ i \geq 1\}$ is said to be arithmetically α mixing with order $a > 0$, if $\exists C > 0,\ \alpha(n) \leq Cn^{-a}$.

The strong-mixing condition is reasonably weak and has many practical applications (see, e.g., Cai [6], Doukhan [11], Dedecker *et al.* [10] Ch. 1, for more details). In particular, Masry and Tøjstheim [25] proved that, both ARCH processes and nonlinear additive autoregressive models with exogenous variables, which are particularly popular in finance and econometrics, are stationary and α-mixing.

This article is organized as follows: In Section 2, we describe our model and construct precisely the QR estimator based on the functional stationary data under censorship model. In Section 3, we build up asymptotic theorems for our model. Section 4 illustrates those asymptotic properties through some simulated. Finally, the proofs of the main results are postponed to Section 5.

2 Notations and estimators of the semi-parametric framework

2.1 The model

Let (X, T) be a pair of random variables where T is a real-valued random variable and X takes its values in a separable Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|$ generated by an inner product $<\cdot,\cdot>$. Let C be a censoring variable with common continuous distribution function G. The continuity of G allows to use the convergence results for the Kaplan and Meier estimator of G. (see [19]).

From now on we suppose that (X, T) and C are independent. It is plausible whenever the censoring is independent of the characteristics of the patients under study. In the right censorship model, the pair (T, C) is not directly observed and the corresponding available information is given by $Y = \min(T, C)$ and $\delta = \mathbf{1}_{\{T \leq C\}}$, where $\mathbf{1}_A$ is the indicator function of the set A.

Such censorship models have been amply studied in the Literature for real or multi-dimensional random variables, and in nonparametric frameworks the kernel techniques are particularly used (see Tanner and Wong [33], Padgett [29], Lecoutre and Ould-Saïd [20] and Van Keilegom and Veraverbeke [34], for a necessarily non-exhaustive sample of literature in this area).

Furthermore, let $(X_i, T_i)_{1\leq i\leq n}$ be the statistical sample of pairs which are identically distributed like (X, T), but not necessarily independent, $(C_i)_{1\leq i\leq n}$ is a sequence of i.i.d. random variables which is independent of $(X_i, T_i)_{1\leq i\leq n}$. Therefore, we assume that the sample $\{(X_i, \delta_i, Y_i), i = 1, \ldots, n\}$ is at our disposal. Moreover, we consider $d_\theta(\cdot,\cdot)$ a semi-metric associated with the single index $\theta \in \mathcal{H}$ defined by $d_\theta(x_1, x_2) := | < x_1 - x_2, \theta > |$, for x_1 and x_2 in $\mathcal{H}$.

For a fixed x in $\mathcal{H}$, the conditional cumulative distribution function (*cond-cdf*) of Y given $<\theta, X> = <\theta, x>$, is defined as follows:

$$\forall t \in \mathbb{R},\ F(\theta, t, x) := \mathbb{P}(Y \leq t | < X, \theta > = < x, \theta >).$$

Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of Y given $<\theta, X>$. Now, let $\zeta_\theta(\gamma, x)$ be the γth-conditional quantile of the distribution of Y given $<\theta, X> = <\theta, x>$. Formally, $\zeta_\theta(\gamma, x)$ is defined as:

$$\zeta_\theta(\gamma, x) := \inf\{t \in \mathbb{R} :\ F(\theta, t, x) \geq \gamma\}, \qquad \forall \gamma \in (0, 1).$$

In order to simplify our framework and to focus on the main interest of our paper (the functional feature of $< \theta, X >$), we assume that $F(\theta, \cdot, x)$ is strictly increasing and continuous in a neighborhood of $\zeta_\theta(\gamma, x)$. This is insuring that the conditional quantile $\zeta_\theta(\gamma, x)$ is uniquely defined by:

$$\zeta_\theta(\gamma, x) \;=\; F^{-1}(\theta, \gamma, x) \;\; \text{equivalently} \;\; \widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x)\;, x) = \gamma. \tag{1}$$

Next, in all what follows, we assume only smoothness restrictions for the *cond-cdf* $F(\theta, \cdot, x)$ through nonparametric modeling. Assume also $(X_i, T_i)_{i\in\mathbb{N}}$ is an α-mixing sequence, which is one among the most general mixing structures.

2.2 The estimators

The kernel estimator $F_n(\theta, \cdot, x)$ of $F(\theta, \cdot, x)$ is presented as follows:

$$F_n(\theta, t, x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - T_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{2}$$

where K is a kernel function, H a cumulative distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) a sequence of positive real numbers. Note that using similar ideas, Roussas [30] introduced some related estimates but in the special case when X is real, while Samanta [31] produced previous asymptotic study.

As a by-product of (1) and (2), it is easy to derive an estimator $\zeta_{\theta,n}(\gamma, x)$ of $\zeta_\theta(\gamma, x)$:

$$\zeta_{\theta,n}(\gamma, x) \;=\; F_n^{-1}(\theta, \gamma, x). \tag{3}$$

Such an estimator is unique as soon as H is an increasing continuous function. Such an approach has been largely used in the case where the variable X is of finite dimension (see *e.g* Whang and Zhao [35], Cai [7], Zhou and Liang [38] or Gannoun *et al.* [17]).

The objective of this section is to adapt these ideas under functional random variable X, and build a kernel type estimator of the conditional distribution $F(\theta, \cdot, X)$ adapted for censored samples. In the censoring case, based on the observed sample $(X_i, \delta_i, Y_i)_{i=1,\ldots,n}$ we define the following "pseudo-estimator" of $F(\theta, \cdot, X)$ which is used as an intermediate estimator, thus we can reformulate the expression (2) as follows:

$$\widetilde{F}(\theta,t,x)=\frac{\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K\left(h_K^{-1}(<x-X_i,\theta>)\right)H\left(h_H^{-1}(t-Y_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}(<x-X_i,\theta>)\right)}. \tag{4}$$

In practice $\bar{G}(\cdot)=1-G(\cdot)$ is unknown, hence it is impossible to use the estimator (6). Then, we replace $\bar{G}(\cdot)$ by its Kaplan and Meier [19] estimate $\bar{G}_n(\cdot)$ given by

$$\bar{G}_n(t)=1-G_n(t)=\begin{cases}\prod_{i=1}^{n}\left(1-\frac{1-\delta_{(i)}}{n-i+1}\right)^{1_{\{Y_{(i)}\le t\}}}, & \text{if } t<Y_{(n)};\\ 0, & \text{if } t\ge Y_{(n)}.\end{cases} \tag{5}$$

where $Y_{(1)}<Y_{(2)}<\ldots<Y_{(n)}$ are the order statistics of Y_i and $\delta_{(i)}$ is the concomitant of $Y_{(i)}$. Therefore, a full estimator of the conditional distribution function $F(\theta,\cdot,x)$ is defined as:

$$\widehat{F}(\theta,t,x)=\frac{\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K\left(h_K^{-1}(<x-X_i,\theta>)\right)H\left(h_H^{-1}(t-Y_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}(<x-X_i,\theta>)\right)}, \tag{6}$$

which is rewritten also as:

$$\widehat{F}(\theta,t,x)=\frac{\widehat{F}_N(\theta,t,x)}{\widehat{F}_D(\theta,x)}. \tag{7}$$

Consequently, a natural estimator of $\zeta_\theta(\gamma,x)$ is given by

$$\begin{aligned}\widehat{\zeta}_\theta(\gamma,x) &= \widehat{F}^{-1}(\theta,\gamma,x)\\ &= \inf\{t\in\mathbb{R}:\ \widehat{F}(\theta,t,x)\ge\gamma\},\end{aligned} \tag{8}$$

which satisfies

$$\widehat{F}(\theta,\widehat{\zeta}_\theta(\gamma,x)\ ,x)=\gamma. \tag{9}$$

3 Assumptions and results

3.1 Assumptions on the functional variable

Let N_x be a fixed neighborhood of x and let $B(x, h)$ be the ball of center x and radius h, namely $B_\theta(x, h) = \{f \in \mathcal{H}/0 < | < x - f, \theta > | < h\}$. Assume that, $(C_i)_{i\geq 1}$ and $(T_i)_{i\geq 1}$ are independent and we assume that $\tau_G := \sup\{t : G(t) < 1\}$ and let τ be a positive real number such that $\tau < \tau_G$.

let's consider the following hypotheses:

(H1) $\forall h > 0,\ \mathbb{P}\left(X \in B_\theta(x, h)\right) = \phi_{\theta,x}(h) > 0,$

(H2) $(X_i, Y_i)_{i\in\mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify:

$$\exists a > 0,\ \exists c > 0 :\ \forall n \in \mathbb{N},\ \alpha(n) \leq cn^{-a}.$$

(H3) $0 < \sup_{i\neq j} \mathbb{P}\left((X_i, X_j) \in B_\theta(x, h) \times B_\theta(x, h)\right) = \mathcal{O}\left(\frac{(\phi_{\theta,x}(h_K))^{(a+1)/a}}{n^{1/a}}\right).$

3.2 The nonparametric model

As usually in nonparametric estimation, we suppose that the cond-cdf $F(\theta, \cdot, x)$ verifies some smoothness constraints. Let b_1 and b_2 be two positive numbers; such that:

(H4) $\forall(x_1, x_2) \in N_x \times N_x$, $\forall(t_1, t_2) \in \mathcal{S}^2_{\mathbb{R}}$,

(i) $|F(\theta, t_1, x_1) - F(\theta, t_2, x_2)| \leq C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + |t_1 - t_2|^{b_2}\right),$

(ii) $\int_{\mathbb{R}} tf(\theta, t, x)dt < \infty$ for all $\theta, x \in \mathcal{H}$.

To this end, we need some assumptions concerning the kernel estimator $\widehat{F}(\theta, \cdot, x)$:

(H5) $\forall(t_1, t_2) \in \mathbb{R}^2,\ |H(t_1) - H(t_2)| \leq C|t_1 - t_2|$ with $\int H^{(1)}(t)dt = 1$, $\int H^2(t)dt < \infty$ and $\int |t|^{b_2} H^{(1)}(t)dt < \infty.$

(H6) K is a positive bounded function with support $[0, 1]$.

(H7) The df of the censored random variable, G has bounded first derivative G'.

(H8) For all $u \in [0,1]$, $\lim_{h \to 0} \frac{\phi_{\theta,x}(uh)}{\phi_{\theta,x}(h)} = \lim_{h \to 0} \xi_h^{\theta,x}(u) = \xi_0^{\theta,x}(u)$.

(H9) The bandwidth h_H satisfies,

(i) $nh_H^2 \phi_{\theta,x}^2(h_K) \longrightarrow \infty$, and $\frac{nh_H^3 \phi_{\theta,x}(h_K)}{\log^2 n} \longrightarrow \infty$ as $n \to \infty$.

(ii) $nh_H^2 \phi_{\theta,x}^3(h_K) \longrightarrow 0$, as $n \to \infty$.

(H10) There exist sequences of integers (u_n) and (v_n) increasing to infinity such that $(u_n + v_n) \le n$, satisfying

(i) $v_n = o((n\phi_{\theta,x}(h_K))^{1/2})$ and $\left(\frac{n}{\phi_{\theta,x}(h_K)}\right)^{1/2} \alpha(v_n) \to 0$ as $n \to 0$,

(ii) $q_n v_n = o((n\phi_{\theta,x}(h_K))^{1/2})$ and $q_n \left(\frac{n}{\phi_{\theta,x}(h_K)}\right)^{1/2} \alpha(v_n) \to 0$ as $n \to 0$

where q_n is the largest integer such that $q_n(u_n + v_n) \le n$.

3.3 Comments of the assumptions

(H1) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X, while (H3) concerns the behavior of the joint distribution of the pairs (X_i, X_j). Indeed, this hypothesis is equivalent to assume that, for n large enough

$$\sup_{i \neq j} \frac{\mathbb{P}\left((X_i, X_j) \in B_\theta(x,h) \times B_\theta(x,h)\right)}{\mathbb{P}\left(X \in B_\theta(x,h)\right)} \le C \left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}.$$

This is one way to control the local asymptotic ratio between the joint distribution and its margin. Remark that the upper bound increases with a. In other words, more the dependence is strong, more restrictive is (H3). The hypothesis (H2) specifies the asymptotic behavior of the α-mixing coefficients. Let's note that (H4) is used for the prove of the the almost complete convergence of $\widehat{\zeta}_\theta(\gamma, x)$. Assumptions (H5), (H6) and (H7) are classical in nonparametric estimation. To establish the asymptotic normality, dealing with strong mixing random variables (under (H2)), we use the well-known sectioning device introduced by Doob [12] in (H10).

This part of paper is devoted to the main result, the asymptotic normality of $\widehat{F}(\theta, t, x)$ and $\widehat{\zeta}_\theta(\gamma, x)$.

Theorem 1 *Under Assumptions (H1)-(H10), we have*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{\sigma^2(\theta,t,x)}\right)^{1/2}\left(\widehat{F}(\theta,t,x)-F(\theta,t,x)\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,1), \tag{10}$$

where $\sigma^2(\theta,t,x)=\dfrac{a_2(\theta,x)}{(a_1(\theta,x))^2}F(\theta,t,x)\left(\dfrac{1}{\bar{G}(t}-F(\theta,t,x)\right)$

and

$$\frac{\phi_{\theta,x}(h_K)\mathbb{E}K_1^2(x,\theta)}{\mathbb{E}^2K_1(x,\theta)}=:\frac{a_2(\theta,x)}{(a_1(\theta,x))^2}.$$

Theorem 2 *If the Assumptions (H1)-(H10) are satisfied, and et* γ *is the unique order of the quantile such that* $\gamma=F(\theta,\zeta_\theta(\gamma,x),x)=F_n(\theta,\widehat{\zeta}_\theta(\gamma,x),x)$*,*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{\Sigma^2(\theta,\zeta_\theta(\gamma,x),x)}\right)^{1/2}(\zeta_{\theta,n}(\gamma,x)-\zeta_\theta(\gamma,x))\xrightarrow{\mathcal{D}}\mathcal{N}(0,1), \tag{11}$$

where $\Sigma(\theta,\zeta_\theta(\gamma,x),x)=\dfrac{\sigma(\theta,\zeta_\theta(\gamma,x),x)}{f(\theta,\zeta_\theta(\gamma,x),x)}$.

As one can see, the asymptotic variance $\Sigma(\theta,\zeta_\theta(\gamma,x),x)$ depends on some unknown functions $f(\theta,\zeta_\theta(\gamma,x),x)$ and $\phi_{\theta,x}(h_K)$ and other theoretical quantities $F(\theta,\zeta_\theta(\gamma,x),x)$, $\overline{G}(\cdot)$ and $\zeta_\theta(\gamma,x)$ that have to be estimated in practice. Therefore, $\overline{G}(\cdot)$, $F(\theta,t,x)$ and $\zeta_\theta(\gamma,x)$ should be replaced, respectively, by the Kaplan-Meier's estimator $\overline{G}_n(\cdot)$, the kernel-type estimator of the joint distribution $\widehat{f}(\theta,\zeta_\theta(\gamma,x),x)$ and $\zeta_{\theta,n}(\gamma,x)$ the conditional quantile estimator given by equation (8). Moreover, using the decomposition given by assumption (H1), one can estimate $\phi_{\theta,x}(z)$ by $F_{x,n}(z)=1/n\sum_{i=1}^n \mathbf{1}_{\{X_i\in B_\theta(x,z)\}}$.

The corollary below allows one to obtain a confidence interval in practice since all quantities are known.

3.4 Confidence intervals

Now based on the quantities estimation, we easily get a plug-in estimator $\widehat{\Sigma}(\theta,\zeta_{\theta,n}(\gamma,x),x)$ of $\Sigma(\theta,\zeta_\theta(\gamma,x),x)$. The Theorem (2) can be now used to provide the $100(1-\gamma)\%$ confidence bands for $\zeta_\theta(\gamma,x)$ which is given, for $x\in\mathcal{H}$, by

$$\left[\zeta_{\theta,n}(\gamma,x)-c_{\gamma/2}\frac{\widehat{\Sigma}(\theta,\zeta_{\theta,n}(\gamma,x),x)}{\sqrt{nF_{x,n}(h_K)}},\zeta_{\theta,n}(\gamma,x)+c_{\gamma/2}\frac{\widehat{\Sigma}(\theta,\zeta_{\theta,n}(\gamma,x),x)}{\sqrt{nF_{x,n}(h_K)}}\right]$$

where $c_{\gamma/2}$ is the upper $\gamma/2$ quantile of the distribution of $\mathcal{N}(0,1)$.

4 Finite sample performance

This section considers simulated as well as real data studies to assess the finite-sample performance of the proposed estimator and compare it to its competitor. More precisely, we are interested in comparing the conditional quantile estimator based on single functional index model (SFIM) to the kernel-type conditional quantile estimator (NP) introduced in Chaouch and Khardani [8] when the data is dependent and the response variable is subject to a random right-censorship phenomena. Throughout the simulation part, the n i.i.d. random variables $(C_i)_i$ (censured variables) are simulated through the exponential distribution $\mathcal{E}(1.5)$. Similarly, in the real data applications, the censored variables are simulated according to the aforementioned exponential law.

The single functional index $\theta \in \mathcal{H}$ is usually unknown and has to be estimated in practice. This topic was discussed in single functional regression model literature and an estimation approaches based on cross-validation or maximum-likelihood methods were discussed, for instance, in Aït Saidi *et al.* [2] and the references therein. Another alternative, which will be adopted in this section, consists in selecting $\theta(t)$ among the eigenfunctions of the covariance operator $\mathbb{E}[(X' - \mathbb{E}(X')) < X', . >_{\mathcal{H}}]$, where $X(t)$ is, for instance, a diffusion-type process defined on a real interval $[a, b]$ and $X'(t)$ its first derivative (see, for instance, Attaoui and Ling [5]). Given a training sample $\mathcal{L}$, the covariance operator can be estimated by its empirical version $\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X'_i - \mathbb{E}X')\,^t(X'_i - \mathbb{E}X')$. Consequently, one can obtain a discretized version of the eigenfunctions $\theta_i(t)$ by applying the principle component analysis method. Let $\theta^\star$ be the first eigenfunction corresponding to the highest eigenvalue of the empirical covariance operator, which will replace θ in the simulation steps to calculate the estimator of the conditional distribution as well as the conditional quantiles.

4.1 Simulation study

We generate n copies, say $(X_i, \delta_i, Y_i)_{i=1,\dots,n}$, of (X, δ, Y), where X and Y are simulated according to the following functional regression model.

$$T_i = R(X_i) + \epsilon_i,\ i = 1, \dots, n,$$

where ϵ_i is the error assumed to be generated according to an autoregressive model defined as:

$$\epsilon_i = 1/\sqrt{2}\epsilon_{i-1} + \eta_i,\ i = 1, \dots, n,$$

where $(\eta_i)_i$ a sequence of i.i.d. random variables normally distributed with a variance equal to 0.1. The functional covariate X is assumed to be a diffusion process defined on $[0, 1]$ and generated by the following equation:

$$X(t) = A(2 - \cos(\pi t W)) + (1 - A)\cos(\pi t W),\ t \in [0, 1],$$

where $W \rightsquigarrow \mathcal{N}(0, 1)$ and $A \rightsquigarrow$ Bernoulli$(1/2)$.

Figure 1 depicts a sample of 100 realizations of the functional random variable X sampled in 100 equidistant points over the interval $[0, 1]$.

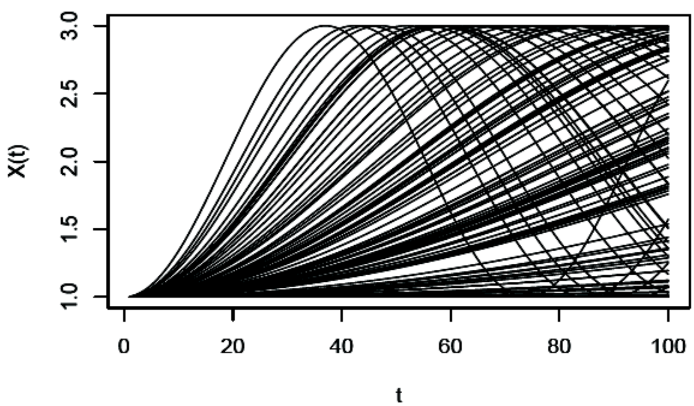

Figure 1: A sample of 100 curves $\{X_i(t),\ t \in [0, 1]\}_{i=1,\ldots,100}$

On the other side, a nonlinear functional regression, defined as follows, is considered

$$R(X) = \frac{1}{4}\int_0^1 \left(X'(t)\right)^2 dt,$$

the computation of our estimator is based on the observed data $(X_i, \delta_i, Y_i)_{i=1}^n$, where $Y_i = \min(T_i, C_i)$, $\delta_i = \mathbf{1}_{\{T_i \le C_i\}}$.

To assess the accuracy of the proposed estimator, we split the generated data into a training $(\mathcal{L})$ and a testing $(\mathcal{J})$ subsamples. The training subsample is used to estimate the single functional index and to select the smoothing parameters h_k and h_H. Whereas the testing subsample is used to assess and compare the single functional index based estimator of the conditional quantile, namely $\widehat{\zeta}_\theta(\gamma, \cdot)$, to the kernel-type conditional quantile estimator, say $\widehat{\zeta}(\gamma, \cdot)$, which is introduced in Chaouch and Khardani [8] as follows:

$$\widehat{\zeta}(\gamma, x) = \inf\left\{y \in \mathbb{R},\ \widehat{F}^x(y) \ge \gamma\right\},$$

where

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^n \frac{\delta_i}{\bar{G}_n(Y_i)} K\left(h_K^{-1} d(x, X_i)\right) H\left(h_H^{-1}(y - Y_i)\right)}{\sum_{i=1}^n K\left(h^{-1} d(x, X_i)\right)}, \quad \forall y \in \mathbb{R}.$$

Figure 2 displays the first three eigenfunctions calculated from the estimated covariance operator using the data in the training subsample.

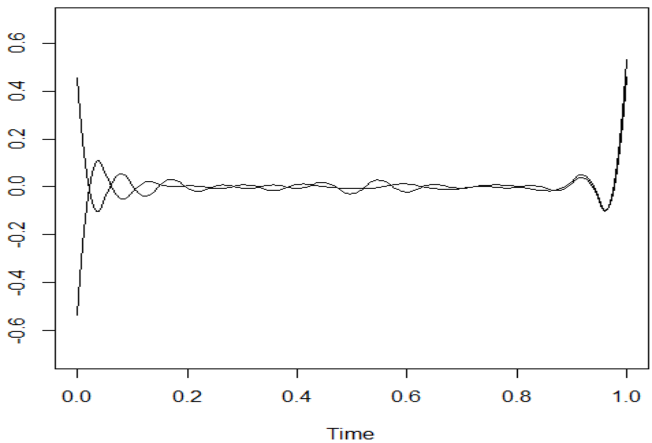

Figure 2: The first three eigenfunctions (respectively, continuous, dashed and dotted lines) representing $\theta_i(t), i = 1, 2$

Given an $X = x$, we can observe that the random variable T has a normal distribution with mean equal to $R(x)$ and standard deviation equal to 0.2. Therefore, the conditional median is equal to $R(x)$. A 500 Monte-Carlo simulations are performed in order to assess the estimation accuracy of $R(x)$ using the conditional median estimation by the single functional index approach and by the nonparametric approach. The simulations were performed for two sample sizes $n = 100, 500$ and for two Censorship Rates $CR = 60\%, 30\%$. Furthermore, some tuning parameters have to be specified. The kernel $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{2}\left(1 - u^2\right) \mathbf{1}_{[0,1]}$ and the cumulative distribution function $H(u) = \int_{-\infty}^{u} \frac{3}{4}\left(1 - z^2\right) \mathbf{1}_{[-1,1]}(z)\, dz$. As shown in Figure 1 the covariate is a smooth process and the regression function $R(\cdot)$ is defined as the integral of the derivative of the functional random variable X. Consequently, according to Ferraty and Vieu [16], the appropriate choice of the semi-metric is the L_2 distance between the first derivatives of the curves. In this section, we assume that $h := h_K = h_H$, is selected using a cross-validation method based on the k-nearest neighbors as described in Ferraty and Vieu [16], p. 102.

We consider the absolute error (AE) as a measure of accuracy of the estimators:

$$AE_{k,\theta} = |\widehat{\zeta}_\theta(0.5, x) - R(x)| \quad \text{and} \quad AE_k = |\widehat{\zeta}(0.5, x) - R(x)|, \quad k = 1, \ldots, 500,$$

where $\widehat{\zeta}_\theta(0.5, x)$ and $\widehat{\zeta}(0.5, x)$ are, respectively, the estimators of the conditional median using the single functional index model and the nonparametric approach. Table 1 shows that the SFIM estimator performs better that the NP one in estimating $R(x)$. Higher is the sample size and lower is the censorship rate better will be the accuracy of the SFIM compared to the NP one. Moreover, even when CR=60% and $n = 100$, the SFIM estimator is still performing better than the NP one.

Table 1: First, second and third quartile of the Absolute errors ($AE_{k,\theta}$ and AE_k, $k = 1, \ldots, 500$) obtained for CR=60% and CR=30%(between parentheses).

	n=100		n=500	
	NP	SFIM	NP	SFIM
1st quartile of AE	0.709	0.69	0.62	0.53
	(0.29)	(0.212)	(0.136)	(0.097)
Median of AE	0.955	0.93	0.95	0.75
	(0.557)	(0.573)	(0.584)	(0.346)
3rd quartile of AE	1.085	1.08	1.07	0.92
	(0.73)	(0.76)	(0.718)	(0.624)

The next phase of this simulation study consists in comparing the accuracy of the SFIM and the NP approaches in terms of prediction. For this purpose a sample of 550 observations was simulated according to the previous functional regression model defined above. A subsample of size 500 is considered for training and the remaining 50 observations are used for prediction assessment. The purpose consists in predicting the response variable Y_i in the test sample using the conditional median which is estimated either by SFIM or NP approach. An overall assessment of the predictions is performed using the median square error, where the square error (SE) is defined as follows: $SE_{j,\theta} := (Y_j - \widehat{\zeta}_\theta(0.5, x))$ and $SE_j := (Y_j - \widehat{\zeta}(0.5, x)), j = 1, \ldots, 50$. Two censorship rates are considered here: $CR = 45\%$ and $CR = 2\%$.

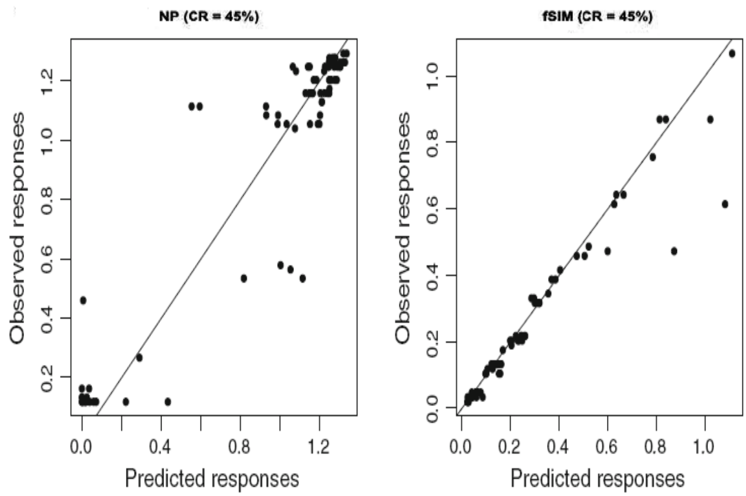

Figure 3: Prediction of $(Y_j)_{j=1,\ldots,50}$ in the test subsample when $CR = 45\%$.

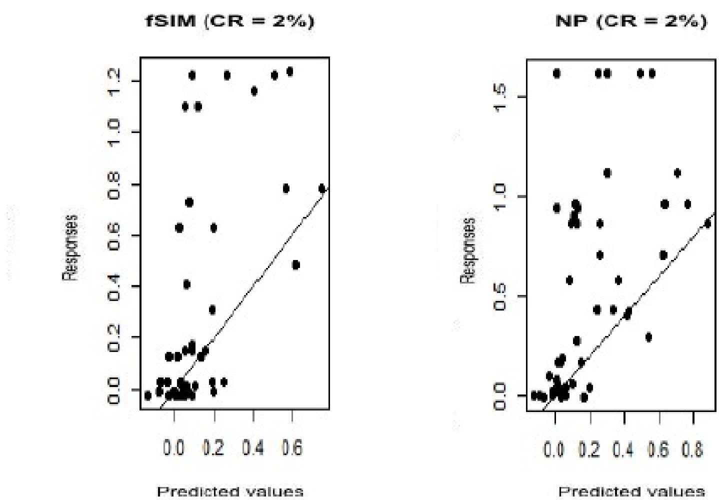

Figure 4: Prediction of $(Y_j)_{j=1,\ldots,50}$ in the test subsample when $CR = 2\%$.

Figures 3 and 4 show that the SFIM estimator performs better than the NP estimator in predicting the response variable in the testing subsample. The accuracy increases when the censorship rate decreases. Indeed when $CR =$

45%, the median square error is equal to 0.011 using the SFIM approach and 0.055 for the NP one. whereas, when CR = 2%, the median square error is equal to 0.008 for the SFIM and 0.012 for the NP approach.

5 Proofs

In order to prove our results, we introduce some further notations. Let First we consider the following decomposition

$$\begin{aligned}
\widehat{F}(\theta,t,x)-F(\theta,t,x) &= \frac{\widehat{F}_N(\theta,t,x)}{\widehat{F}_D(\theta,x)} - \frac{a_1(\theta,x)F(\theta,t,x)}{a_1(\theta,x)} \\
&= \frac{1}{\widehat{F}_D(\theta,x)}\left(\widehat{F}_N(\theta,t,x)-\mathbb{E}\widehat{F}_N(\theta,t,x)\right) \\
&- \frac{1}{\widehat{F}_D(\theta,x)}\left(a_1(\theta,x)F(\theta,t,x)-\mathbb{E}\widehat{F}_N(\theta,t,x)\right) \\
&+ \frac{F(\theta,t,x)}{\widehat{F}_D(\theta,x)}\left(a_1(\theta,x)-\mathbb{E}\left[\widehat{F}_D(\theta,x)\right]\right) \\
&- \frac{F(\theta,t,x)}{\widehat{F}_D(\theta,x)}\left(\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)\right) \\
&= \frac{1}{\widehat{F}_D(\theta,x)}A_n(\theta,t,x)+B_n(\theta,t,x)
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
A_n(\theta,t,x) &= \frac{1}{n\mathbb{E}K_1(x,\theta)}\sum_{i=1}^{n}\left\{\left(\frac{\delta_i}{\bar{G}_n}H_i(t)-F(\theta,t,x)\right)K_i(\theta,x)\right. \\
&\qquad \left. -\mathbb{E}\left[\left(\frac{\delta_i}{\bar{G}_n}H_i(y)-F(\theta,t,x)\right)K_i(\theta,x)\right]\right\} \\
&= \frac{1}{n\mathbb{E}K_1(x,\theta)}\sum_{i=1}^{n}N_i(\theta,t,x).
\end{aligned}$$

It follows that,

$$\begin{aligned}
n\phi_{\theta,x}(h_K)Var\left(A_n(\theta,t,x)\right) &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(x,\theta)}Var(N_1) \\
&+ \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}^{n}\sum Cov(N_i,N_j)
\end{aligned} \tag{13}$$

$$= V_n(\theta,t,x) + \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}^{n}\sum \mathrm{Cov}(N_i,N_j).$$

Lemma 1 *Under hypotheses (H1)-(H3) and (H6)-(H8) as $n \to \infty$ we have*

$$n\phi_{\theta,x}(h_K)\mathrm{Var}\left(A_n(\theta,t,x)\right) \longrightarrow V(\theta,t,x)$$

where $V(\theta,t,x) = \dfrac{a_2(\theta,x)}{(a_1(\theta,x))^2}F(\theta,t,x)\left(\dfrac{1}{\bar{G}(t}-F(\theta,t,x)\right)$.

Lemma 2 *Under hypotheses (H1)-(H3), (H6) and (H8)-(H10), as $n \to \infty$ we have*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{V(\theta,t,x)}\right)^{1/2} A_n(\theta,t,x) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

Lemma 3 *Under Assumptions (H1)-(H3) and (H6)-(H9)as $n \to \infty$ we have*

$$\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta,t,x) \longrightarrow 0 \quad \textit{in Probabilty.}$$

Next, Making use of Proposition 3.2 for $l = 1$ and Theorem 3.1, in *Kadiri et al.* [18] we get the following corollary.

Corollary 1 *Under hypotheses of Lemma 3, as $n \to \infty$ we have*

$$\frac{(n\phi_{\theta,x}(h_K))^{1/2}B_n(\theta,t,x)}{\widehat{f}\left(\theta,\zeta^*_{\theta,n}(\gamma,x),x\right)} \longrightarrow 0 \quad \textit{in Probabilty.}$$

Proof. [Proof of Theorem 1]
To prove Theorem 1, it suffices to use (12). Applying Lemmas Lemma 1 and Lemma 3, we get the result. □

Proof. [Proof of Theorem 2]
For Theorem 2, making use of (12), we have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}\left(\zeta_\theta(\gamma,x)-\zeta_{\theta,n}(\gamma,x)\right) &= \sqrt{n\phi_{\theta,x}(h_K)}\frac{F_n(\theta,\zeta_\theta(\gamma,x),x)}{F'_n(\theta,\zeta^*_{\theta,n}(\gamma,x),x)} \\ &\quad -\sqrt{n\phi_{\theta,x}(h_K)}\frac{F(\theta,\zeta_\theta(\gamma,x),x)}{F'_n(\theta,\zeta^*_{\theta,n}(\gamma,x),x)}\end{aligned}$$

$$= \frac{\sqrt{n\phi_{\theta,x}(h_K)}A_n(\theta,t,x)}{F'_n(\theta,\zeta^*_{\theta,n}(\gamma,x),x)} - \frac{\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta,t,x)}{F'_n(\theta,\zeta^*_{\theta,n}(\gamma,x),x)}.$$

Then using Theorem 1, Corollary 1 and Lemma 3 we obtain the result. □

Proof. [Proof of Lemma 1]

$$\begin{aligned} V_n(\theta,t,x) &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(\theta,x)}\mathbb{E}\left[K_1^2(\theta,x)\left(\frac{\delta_1}{\bar{G}(Y_1)}H_1(t)-F(\theta,t,x)\right)^2\right] \\ &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(\theta,x)}\mathbb{E}\left[K_1^2(\theta,x)\mathbb{E}\left(\left(\frac{\delta_1H_1(t)}{\bar{G}(Y_1)}-F(\theta,t,x)\right)^2|<\theta,X_1>\right)\right]. \end{aligned} \tag{14}$$

Using the definition of conditional variance, we have

$$\mathbb{E}\left[\left(\frac{\delta_1}{\bar{G}(Y_1)}H(h_H^{-1}(t-Y_1))-F(\theta,t,x)\right)^2|<\theta,X_1>\right] = J_{1n}+J_{2n}$$

where $J_{1n} = Var\left(\frac{\delta_1}{\bar{G}(Y_1)}H(h_H^{-1}(t-Y_1))|<\theta,X_1>\right)$,

$J_{2n} = \left[\mathbb{E}\left(\frac{\delta_1}{\bar{G}(Y_1)}H(h_H^{-1}(t-Y_1))|<\theta,X_1>\right)-F(\theta,t,x)\right]^2$

Concerning J_{1n},

$$\begin{aligned} J_{1n} &= \mathbb{E}\left[\frac{\delta_1}{\bar{G}^2(Y_1)}H^2\left(\frac{t-Y_1}{h_H}\right)|<\theta,x>\right] \\ &\quad -\left(\mathbb{E}\left[\frac{\delta_1}{\bar{G}(Y_1)}H\left(\frac{t-Y_1}{h_H}\right)|<\theta,X_1>\right]\right)^2 \\ &= \mathcal{J}_1+\mathcal{J}_2. \end{aligned}$$

As for $\mathcal{J}_1$, by the property of double conditional expectation, we get that,

$$\begin{aligned} \mathcal{J}_1 &= \mathbb{E}\left\{\mathbb{E}\left[\frac{\delta_1}{\bar{G}^2(Y_1)}H^2\left(\frac{t-Y_1}{h_H}\right)|<\theta,X_1>,T_1\right]\right\} \\ &= \mathbb{E}\left\{\frac{\delta_1}{\bar{G}^2(T_1)}H^2\left(\frac{t-T_1}{h_H}\right)\mathbb{E}\left[\mathbf{1}_{T_1\le C_1}|T_1\right]|<\theta,X_1>\right\} \\ &= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)}H^2\left(\frac{t-T_1}{h_H}\right)|<\theta,X_1>\right) \end{aligned} \tag{15}$$

$$= \int_{\mathbb{R}} \frac{1}{\bar{G}(v)} H^2\left(\frac{t-v}{h_H}\right) dF(\theta, v, X_1)$$
$$= \int_{\mathbb{R}} \frac{1}{\bar{G}(t-uh_H)} H^2(u) dF(\theta, t-uh_H, X_1).$$

By the first order Taylor's expansion of the function $\bar{G}^{-1}(\cdot)$ around zero, one gets

$$\begin{aligned} \mathcal{J}_1 &= \int_{\mathbb{R}} \frac{1}{\bar{G}(t)} H^2(u) dF(\theta, t-uh_H, X_1) \\ &+ \frac{h_H^2}{\bar{G}^2(t)} \int_{\mathbb{R}} uH(u)\bar{G}^{(1)}(t^*) f(\theta, t-uh_H, X_1) du + o(1) \end{aligned}$$

where t^* is between t and $t-uh_H$

Under hypothesis (H7) and using hypothesis (H3)-(ii), we get

$$\mathcal{J}_1' = \frac{h_H^2}{\bar{G}^2(t)} \int_{\mathbb{R}} uH^2(t)\bar{G}^{(1)}(t^*) f(\theta, t-uh_H, X_1) du = \mathcal{O}(h_H^2).$$

Indeed

$$\mathcal{J}_1' \leq h_H^2 \left(\sup_{u\in\mathbb{R}} |G'(u)|/\bar{G}^2(t)\right) \int_{\mathbb{R}} uf(\theta, t-uh_H, x) du.$$

On the other hand, by integrating by part and under assumption (H3)-(i), we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{H^2(u)}{\bar{G}(t)} dF(\theta, t-uh_H, X_1) &= \frac{1}{\bar{G}(t)} \int_{\mathbb{R}} 2H(u)H'(u)F(\theta, t-uh_H, X_1) du \\ &- \frac{1}{\bar{G}(t)} \int_{\mathbb{R}} 2H(u)H'(u)F(\theta, t, x) du \\ &+ \frac{1}{\bar{G}(t)} \int_{\mathbb{R}} 2H(u)H'(u)F(\theta, t, x) du. \end{aligned}$$

Clearly we have

$$\int_{\mathbb{R}} 2H(u)H'(u)F(\theta, t, x) du = \left[H^2(u)F(\theta, t, x)\right]_{-\infty}^{+\infty} = F(\theta, t, x) \qquad (16)$$

thus

$$\int_{\mathbb{R}} \frac{1}{\bar{G}(t)} H^2(u) dF(\theta, t-uh_H, X_1) = \frac{F(\theta, t, x)}{\bar{G}(t)} + \mathcal{O}(h_K^{\beta_1} + h_H^{\beta_2}). \qquad (17)$$

As for J_{2n}, by (H2), (H4) and (H5), and using Lemma 3.2 in Kadiri *et al.* [18] we obtain that

$$J_{2n} \longrightarrow 0, \quad \text{as } n \to \infty.$$

- Concerning $\mathcal{J}_2$

$$\begin{aligned}
\mathcal{J}_2' &= \mathbb{E}\left[\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) | < \theta, X_1 >\right] \\
&= \mathbb{E}\left(\mathbb{E}\left[\frac{\delta_1}{\bar{G}(Y_1)} H_1(t) | < \theta, X_1 >, T_1\right]\right) \\
&= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)} H\left(\frac{t-T_1}{h_H}\right) \mathbb{E}\left[\mathbf{1}_{T_1 \leq C_1} | T_1\right] | < \theta, X_1 >\right) \\
&= \mathbb{E}\left(H\left(\frac{t-T_1}{h_H}\right) | < \theta, X_1 >\right) \\
&= \int H\left(\frac{t-v}{h_H}\right) f(\theta, t, X_1) dv.
\end{aligned}$$

Moreover, we have by integration by parts and changing variables

$$\mathcal{J}_2' = F(\theta, t, x) \int H'(u) du + \int H'(u) \left(F(\theta, t - u h_H, x) - F(\theta, t, x)\right) du$$

the last equality is due to the fact that H' is a probability density.

Thus we have:

$$\mathcal{J}_2' = F(\theta, t, x) + \mathcal{O}\left(h_K^{\beta_1} + h_H^{\beta_2}\right). \tag{18}$$

Finally by hypothesis (H5) we get $\mathcal{J}_2 \underset{n\to\infty}{\longrightarrow} F^2(\theta, t, x)$.

Meanwhile, by (H1), (H4), (H6) and (H8), it follows that:

$$\frac{\phi_{\theta,x}(h_K)\mathbb{E}K_1^2(\theta, x)}{\mathbb{E}^2 K_1(\theta, x)} \underset{n\to\infty}{\longrightarrow} \frac{a_2(\theta, x)}{(a_1(\theta, x))^2}.$$

which leads to combining equations (14)-(18)

$$V_n(\theta, t, x) \underset{n\to\infty}{\longrightarrow} \frac{a_2(\theta, x)}{(a_1(\theta, x))^2} F(\theta, t, x) \left(\frac{1}{\bar{G}(t)} - F(\theta, t, x)\right). \tag{19}$$

Secondly, by the boundness of H and conditioning on $(< \theta, X_i >, < \theta, X_j >)$, we have

$$\mathbb{E}\left(|N_i N_j|\right) = \mathbb{E}\left[(\Omega_i)(\Omega_j) K_i(\theta, x) K_j(\theta, x)\right]$$

$$\begin{aligned}
&= \mathbb{E}\Big(\mathbb{E}\Big[(\Omega_i)(\Omega_j) \mid <\theta, X_i>, <\theta, X_j>\Big] K_i(\theta,x)K_j(\theta,x)\Big)\\
&\leq \left(1+\frac{1}{\bar{G}(\tau_F)}\right)^2 \mathbb{E}(K_i(\theta,x)K_j(\theta,x))\\
&\leq C\mathbb{P}\left((X_i,X_j)\in B_\theta(x,h)\times B_\theta(x,h)\right)\\
&\leq C\left(\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}\phi_{\theta,x}(h_K)\right)
\end{aligned}$$

where $\Omega_i = \dfrac{\delta_i}{\bar{G}_i} H_i(t) - F(\theta,t,x)$.

Then, taking

$$\begin{aligned}
\frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)} \sum_{|i-j|>0}^{n}\sum Cov(N_i,N_j) &= \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)} \sum_{0<|i-j|\leq m_n}^{n} Cov(N_i,N_j)\\
&\quad + \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)} \sum_{|i-j|>m_n}^{n} Cov(N_i,N_j)\\
&= K_{1n}+K_{2n}.
\end{aligned}$$

Therefore

$$K_{1n} \leq C\, m_n \left\{\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/a}\right\}, \quad \forall i \neq j.$$

Now choose $m_n = \left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{-1/a}$, we get $K_{1n} = o(1)$.

For K_{2n}: since the variable $(\Delta_i)_{1\leq i\leq n}$ is bounded (i.e, $\|\Delta_i\|_\infty < \infty$, we can use the Davydov-Rio's inequality. So, we have for all $i \neq j$,

$$|Cov(\Delta_i,\Delta_j)| \leq C\alpha(|i-j|).$$

By the fact, $\displaystyle\sum_{k\geq m_n+1} k^{-a} \leq \int_{m_n}^{\infty} v^{-a}dv = \frac{m_n^{-a+1}}{a-1}$, we get by applying (H1),

$$K_{2n} \leq \sum_{|i-j|\geq m_n+1} |i-j|^{-a} \leq \frac{n m_n^{-a+1}}{a-1}$$

with the same choice of m_n, we get $K_{2n} = o(1)$.

Finally by

$$\frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}\sum^{n}Cov(N_i,N_j)=o(1), \tag{20}$$

this complete the proof of lemma. □

Proof. [Proof of Lemma 2]

We will establish the asymptotic normality of $A_n(\theta,t,x)$ suitably normalized. We have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}A_n(\theta,t,x) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n}N_i(\theta,t,x)\\ &= \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\sqrt{n}\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n}N_i(\theta,t,x)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Xi_i(\theta,t,x)=\frac{1}{\sqrt{n}}S_n.\end{aligned}$$

Now we can write, $\Xi_i=\dfrac{\sqrt{\phi_{\theta,x}(h_K)}}{\mathbb{E}K_1(\theta,x)}N_i$, we have

$$Var(\Xi_i)=\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(\theta,x)}Var(N_i)=V_n(\theta,t,x).$$

Note that by (19), we have $Var(\Xi_i)\longrightarrow V(\theta,t,x)$ as n goes to infinity and by (20), we have

$$\sum_{|i-j|>0}|Cov(\Xi_i,\Xi_j)|=\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(x,\theta)}\sum_{|i-j|>0}^{n}|Cov(N_i,N_j)|=o(n), \tag{21}$$

Obviously, we have

$$\sqrt{\frac{n\phi_{\theta,x}(h_K)}{V(\theta,t,x)}}\left(A_n(\theta,t,x)\right)=\left(nV(\theta,t,x)\right)^{-1/2}S_n.$$

Thus, the asymptotic normality of $(nV(\theta,t,x))^{-1/2}S_n$, is sufficient to show the proof of this Lemma. This last is shown by the blocking method, where the random variables Ξ_i are grouped into blocks of different sizes defined.

We consider the classical big- and small-block decomposition. We split the set $\{1, 2, \dots, n\}$ into $2k_n + 1$ subsets with large blocks of size u_n and small blocks of size v_n and put

$$k_n := \left[\frac{n}{u_n + v_n}\right].$$

Now by Assumption (H10)-(ii) allows us to define the large block size by

$$u_n =: \left[\left(\frac{n\phi_{\theta,x}(h_K)}{q_n}\right)^{1/2}\right].$$

Using Assumption (H10) and simple algebra allows us to prove that

$$\frac{v_n}{u_n} \to 0, \quad \frac{u_n}{n} \to 0, \quad \frac{u_n}{\sqrt{n\phi_{\theta,x}(h_K)}} \to 0, \quad \text{and} \quad \frac{n}{u_n}\alpha(v_n) \to 0. \tag{22}$$

Now, let Υ_j, Υ_j' and Υ_j'' be defined as follows:

$$\Upsilon_j(\theta, t, x) = \Upsilon_j = \sum_{i=j(u+v)+1}^{j(u+v)+u} \Xi_i(\theta, t, x), \quad 0 \le j \le k-1,$$

$$\Upsilon_j'(\theta, t, x) = \Upsilon_j' = \sum_{i=j(u+v)+u+1}^{(j+1)(u+v)} \Xi_i(\theta, t, x), \quad 0 \le j \le k-1,$$

$$\Upsilon_j''(\theta, t, x) = \Upsilon_j'' = \sum_{i=k(u+v)+1}^{n} \Xi_i(\theta, t, x), \quad 0 \le j \le k-1.$$

Clearly, we can write

$$\begin{aligned} S_n(\theta, t, x) = S_n &= \sum_{j=1}^{k-1} \Upsilon_j + \sum_{j=1}^{k-1} \Upsilon_j' + \Upsilon_k'' \\ &=: \Psi_n(\theta, t, x) + \Psi_n'(\theta, t, x) + \Psi_n''(\theta, t, x) \\ &=: \Psi_n + \Psi_n' + \Psi_n''. \end{aligned}$$

We prove that

$$\text{(i) } \frac{1}{n}\mathbb{E}(\Psi_n')^2 \longrightarrow 0, \quad \text{(ii) } \frac{1}{n}\mathbb{E}(\Psi_n'')^2 \longrightarrow 0, \tag{23}$$

$$\left|\mathbb{E}\left\{\exp\left(izn^{-1/2}\Psi_n\right)\right\} - \prod_{j=0}^{k-1}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Upsilon_j\right)\right\}\right| \longrightarrow 0, \tag{24}$$

$$\frac{1}{n}\sum_{j=0}^{k-1}\mathbb{E}\left(\Upsilon_j^2\right) \longrightarrow V(\theta,t,x), \tag{25}$$

$$\frac{1}{n}\sum_{j=0}^{k-1}\mathbb{E}\left(\Upsilon_j^2\mathbf{1}_{\{|\Upsilon_j|>\varepsilon\sqrt{nV(\theta,t,x)}\}}\right) \longrightarrow 0 \tag{26}$$

for every $\varepsilon > 0$.

Expression (23) show that the terms Ψ_n' and Ψ_n'' are asymptotically negligible, while Equations (24) and (25) show that the Υ_j are asymptotically independent, verifying that the sum of their variances tends to $V(\theta,t,x)$. Expression (26) is the Lindeberg-Feller's condition for a sum of independent terms. Asymptotic normality of S_n is a consequence of Equations (23)-(26).

- **Proof of (23)** Because $\mathbb{E}(\Xi_j) = 0, \ \forall j$, we have that

$$\mathbb{E}(\Psi_n')^2 = Var\left(\sum_{j=1}^{k-1}\Upsilon_j'\right) = \sum_{j=1}^{k-1}Var\left(\Upsilon_j'\right) + \sum_{|i-j|>0}^{k-1}Cov\left(\Upsilon_i',\Upsilon_j'\right) := \Pi_1 + \Pi_2.$$

By the second-order stationarity and (21) we get

$$\begin{aligned} Var\left(\Upsilon_j'\right) &= Var\left(\sum_{i=j(u_n+v_n)+u_n+1}^{(j+1)(u_n+v_n)}\Xi_i(\theta,t,x)\right) \\ &= v_n Var(\Xi_1(x)) + \sum_{|i-j|>0}^{v_n}Cov\left(\Xi_i(\theta,t,x),\Xi_j(\theta,t,x)\right) \\ &= v_n Var(\Xi_1(x)) + o(v_n). \end{aligned}$$

Then

$$\begin{aligned} \frac{\Pi_1}{n} &= \frac{kv_n}{n}Var(\Xi_1(\theta,t,x)) + \frac{k}{n}o(v_n) \\ &\leq \frac{kv_n}{n}\left\{\frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(x)}Var\left(\Xi_1(x)\right)\right\} + \frac{k}{n}o(v_n) \end{aligned}$$

$$\leq \frac{k v_n}{n}\left\{\frac{1}{\phi_{\theta,x}(h_K)}\mathrm{Var}\left(\Xi_1(x)\right)\right\}+\frac{k}{n}o(v_n).$$

Simple algebra gives us

$$\frac{k v_n}{n} \cong \left(\frac{n}{u_n+v_n}\right)\frac{v_n}{n} \cong \frac{v_n}{u_n+v_n} \cong \frac{v_n}{u_n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Using Equation (20) we have

$$\lim_{n\to\infty}\frac{\Pi_1}{n}=0. \tag{27}$$

Now, let us turn to Π_2/n. We have

$$\begin{aligned}\frac{\Pi_2}{n} &= \frac{1}{n}\sum_{|i-j|>0}^{k-1}\mathrm{Cov}\left(\Upsilon_i(x),\Upsilon_j(x)\right)\\ &= \frac{1}{n}\sum_{|i-j|>0}^{k-1}\sum_{l_1=1}^{v_n}\sum_{l_2=1}^{v_n}\mathrm{Cov}\left(\Xi_{m_j+l_1},\Xi_{m_j+l_2}\right),\end{aligned}$$

with $m_i = i(u_n+v_n)+u_n+1$. As $i \neq j$, we have $|m_i - m_j + l_1 - l_2| \geq u_n$. It follows that

$$\frac{\Pi_2}{n} \leq \frac{1}{n}\sum_{|i-j|\geq u_n}^{n}\mathrm{Cov}\left(\Xi_i(x),\Xi_j(x)\right)=o(1),$$

then

$$\lim_{n\to\infty}\frac{\Pi_2}{n}=0. \tag{28}$$

By Equations (27) and (28) we get Part(i) of the Equation(23).

We turn to (ii), we have

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left(\Psi_n''\right)^2 &= \frac{1}{n}\mathrm{Var}\left(\Upsilon_k''\right)\\ &= \frac{\vartheta_n}{n}\mathrm{Var}\left(\Xi_1(x)\right)+\frac{1}{n}\sum_{|i-j|>0}^{\vartheta_n}\mathrm{Cov}\left(\Xi_i(x),\Xi_j(x)\right),\end{aligned}$$

where $\vartheta_n = n - k_n(u_n+v_n)$; by the definition of k_n, we have $\vartheta_n \leq u_n + v_n$.

Then

$$\frac{1}{n}\mathbb{E}\left(\Psi_n''\right)^2 \le \frac{u_n+v_n}{n}\mathrm{Var}\left(\Xi_1(x)\right)+\frac{1}{n}\sum_{|i-j|>0}^{\vartheta_n}\mathrm{Cov}\left(\Xi_i(x),\Xi_j(x)\right)$$

and by the definition of u_n and v_n we achieve the proof of (ii) of Equation (23).

- **Proof of (24)** We make use of Volkonskii and Rozanov's lemma (see the appendix in Masry [26]) and the fact that the process $(X_i, X_j)$is strong mixing.

  Note that Υ_a is $\mathcal{F}_{i_a}^{j_a}$-mesurable with $i_a = a(u_n+v_n)+1$ and $j_a = a(u_n+v_n)+u_n$; hence, with $V_j = \exp\left(izn^{-1/2}\Psi_n\right)$ we have

$$\begin{aligned}\left|\mathbb{E}\{V_j\}-\prod_{j=0}^{k-1}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Upsilon_j\right)\right\}\right| &\le 16k_n\alpha(v_n+1)\\ &\cong \frac{n}{v_n}\alpha(v_n+1)\end{aligned}$$

  which goes to zero by the last part of Equation (22). Now we establish Equation (25).

- **Proof of (25)** Note that $\mathrm{Var}(\Psi_n)\longrightarrow V(\theta,t,x)$ by equation (23) (by the definition of the Ξ_i). Then because

$$\mathbb{E}\left(\Psi_n\right)^2=\mathrm{Var}\left(\Psi_n\right)=\sum_{j=0}^{k-1}\mathrm{Var}\left(\Upsilon_j\right)+\sum_{i=0}^{k-1}\sum_{\substack{j=1\\ i\neq j}}^{k-1}\mathrm{Cov}\left(\Upsilon_i,\Upsilon_j\right),$$

  all we have to prove is that the double sum of covariances in the last equation tends to zero. Using the same arguments as those previously used for Π_2 in the proof of first term of Equation (23) we obtain by replacing v_n by u_n we get

$$\frac{1}{n}\sum_{j=1}^{k-1}\mathbb{E}\left(\Upsilon_j^2\right)=\frac{ku_n}{n}\mathrm{Var}\left(\Xi_1\right)+o(1).$$

  As $\mathrm{Var}\left(\Xi_1\right)\longrightarrow V(\theta,t,x)$ and $\dfrac{ku_n}{n}\longrightarrow 1$, we get the result.

  Finally, we prove Equation (26).

- **Proof of (26)** Recall that

$$\Upsilon_j = \sum_{i=j(u_n+v_n)+1}^{j(u_n+v_n)+u_n} \Xi_i.$$

Finally for establish (26) it suffices to show for n large enough that the set $\{|\Upsilon_j| > \varepsilon\sqrt{nV(\theta,t,x)}\}$ is empty .

Making use Assumptions (H3) and (H5), we have

$$\left|\Xi_i\right| \leq C\left(\phi_{\theta,x}(h_K)\right)^{-1/2}$$

therefore

$$\left|\Upsilon_j\right| \leq Cu_n\left(\phi_{\theta,x}(h_K)\right)^{-1/2},$$

which goes to zero as n goes to infinity by Equation (22).

Since $|H_i(t) - F(\theta,t,x)| \leq 1$, then

$$\begin{aligned}\left|\Upsilon_j\right| &\leq \frac{u_n N_j}{\sqrt{\phi_{\theta,x}(h_K)}}\\ &\leq \frac{Cu_n}{\sqrt{\phi_{\theta,x}(h_K)}}.\end{aligned}$$

Thus

$$\frac{1}{\sqrt{n}}\left|\Upsilon_j\right| \leq \frac{Cu_n}{\sqrt{n\phi_{\theta,x}(h_K)}}.$$

Then for n large enough, the set $\left\{|\Upsilon_j| > \varepsilon\left(nV(\theta,t,x)\right)^{-1/2}\right\}$ becomes empty, this completes the proof and therefore that of the asymptotic normality of $(nV(\theta,t,x))^{-1/2}S_n$ and the Lemma 2.

□

Proof. [Proof of Lemma 3]

We have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta,t,x) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta,x)}\Big\{\mathbb{E}\widehat{F}_N(\theta,t,x) - a_1(\theta,x)F(\theta,t,x)\\ &\quad +F(\theta,t,x)\left(a_1(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\right)\Big\}.\end{aligned}$$

Firstly, observed that the results below as $n \to \infty$

$$\frac{1}{\phi_{\theta,x}(h_K)}\mathbb{E}\left[K^l\left(\frac{< x - X_i, \theta >}{h_K}\right)\right] \longrightarrow a_l(\theta, x), \quad \text{for } l = 1, 2, \tag{29}$$

$$\mathbb{E}\left[\widehat{F}_D(\theta, x)\right] \longrightarrow a_1(\theta, x), \tag{30}$$

and

$$\mathbb{E}\left[\widehat{F}_N(\theta, t, x)\right] \longrightarrow a_1(\theta, x)F(\theta, t, x), \tag{31}$$

can be proved in the same way as in Ezzahrioui and Ould-Saïd [14] corresponding to their Lemmas 5.1 and 5.2, and then their proofs are omitted.

Secondly, on the one hand, making use of (29), (30) and (31), we have as $n \to \infty$

$$\left\{\mathbb{E}\widehat{F}_N(\theta, t, x) - a_1(\theta, x)F(\theta, t, x) + F(\theta, t, x)\left(a_1(\theta, x) - \mathbb{E}\widehat{F}_D(\theta, x)\right)\right\} \longrightarrow 0.$$

On other hand,

$$\frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta, x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widetilde{F}'(\theta, t, x)}{\widehat{F}_D(\theta, x)\widetilde{F}'(\theta, t, x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widetilde{F}'(\theta, t, x)}{\widetilde{F}'_N(\theta, t, x)}. \tag{32}$$

Then using Proposition 3.2 in Kadiri *et al.* [18], it suffices to show that $\frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widetilde{F}'_N(\theta,t,x)}$ tends to zero as n goes to infinity.

Indeed

$$\widetilde{F}'_N(\theta, t, x) = \frac{1}{nh_H\mathbb{E}K_1(\theta, x)}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K\left(\frac{< x - X_i, \theta >}{h_K}\right)H'\left(\frac{t - Y_i}{h_H}\right).$$

Because $K(\cdot)H'(\cdot)$ is continuous with support on $[0, 1]$ then by (H5)-(ii) and (H6) $\exists\ m = \inf_{[0,1]} K(t)H'(t)$ it follows that

$$\widetilde{F}'_N(\theta, t, x) \geq \frac{m}{h_H\phi_{\theta,x}(h_K)}$$

which gives

$$\frac{n\phi_{\theta,x}(h_K)}{\widetilde{F}'_N(\theta, t, x)} \leq \frac{\sqrt{nh_H^2\phi_{\theta,x}(h_K)^3}}{m}.$$

Finally, using (H10), completes the proof of Lemma 3.

□

References

[1] A. Aït Saidi, F. Ferraty and R. Kassa, Single functional index model for a time series, *Revue Roumaine de Mathématique Pures Appliquées*, **50** (2002), 321–330.

[2] A. Aït Saidi, F. Ferraty, R. Kassa and P. Vieu, Cross-validated estimation in the single functional index model, *Statistics*, **42(6)** (2008), 475–494.

[3] S. Attaouti, A. Laksaci and E. Ould-Saïd, A note on the conditional density estimate in the single functional index model, *Statistics and Probability Letters*, **81(1)** (2011), 45–53.

[4] S. Attaoui, On the Nonparametric Conditional Density and Mode Estimates in the Single Functional Index Model with Strongly Mixing Data, *Sankhyã: The Indian Journal of Statistics*, **76(A)** (2014), 356–378.

[5] S. Attaoui and N. Ling, Asymptotic Results of a Nonparametric Conditional Cumulative Distribution Estimator in the single functional index Modeling for Time Series Data with Applications, *Metrika*, **79** (2016), 485–511.

[6] Z. Cai, Estimating a distribution function for censored time series data, *J. Multivariate Anal.*, **78** (2001), 299–318.

[7] Z. Cai, Regression quantiles for time series, *Econometric Theory*, 18 (2002), 169–192.

[8] M. Chaouch and S. Khardani, Randomly Censored Quantile Regression Estimation using Functional Stationary Ergodic Data, *Journal of Nonparametric Statistics*, 27 (2015), 65–87.

[9] P. Chaudhuri, K. Doksum and A. Samarov, On average derivative quantile regression, **Ann. Statist.**, **25** (1997),715–744.

[10] J. Dedecker, P. Doukhan, G. Lang, J.R. Leon, S. Louhichi and C. Prieur, *Weak Dependence: With Examples and Applications*, Lecture Notes in Statistics, **190**. New York: Springer-Verlag, 2007.

[11] P. Doukhan, *Mixing: Properties and Examples*, Lecture Notes in Statistics, **85**. New York: Springer-Verlag, 1994.

[12] J. L. Doob, *Stochastic Processes*, New York: Wiley, 1953.

[13] A. El Ghouch and I. Van Keilegom, Local Linear Quantile Regression with Dependent Censored Data, *Statistica Sinica*, **19** (2009), 1621–1640.

[14] M. Ezzahrioui and E. Ould-Saïd, Asymptotic results of a nonparametric conditional quantile estimator for functional time series, *Comm. Statist. Theory and Methods.*, **37(16-17)** (2008), 2735–2759.

[15] F. Ferraty, A. Rabhi and P. Vieu, Conditional quantiles for functional dependent data with application to the climatic ElNinõ phenomenon, *Sankhyã B, Special Issue on Quantile Regression and Related Methods*, **67(2)** (2005), 378–399.

[16] F. Ferraty and P. Vieu, *Nonparametric Functional Data Analysis: Theory and Practice*, Springer Series in Statistics, Springer, New York, 2006.

[17] A. Gannoun, J. Saracco and K. Yu, Nonparametric prediction by conditional median and quantiles, *J. Statist. Plann. Inference.*, **117** (2003), 207–223.

[18] N. Kadiri, A. Rabhi and A. Bouchentouf, Strong uniform consistency rates of conditional quantile estimation in the single functional index model under random censorship, *Journal Dependence Modeling*, **6(1)** (2018), 197–227.

[19] E. Kaplan and P. Meier, Nonparametric Estimation from Incomplete Observations, *Journal of the American Statistical Association*, **53** (1958), 457–481.

[20] J-P. Lecoutre and E. Ould-Saïd, Hazard rate estimation for strong mixing and censored processes, *J. Nonparametr. Stat.* **5** (1995), 83–89.

[21] H.Y. Liang and J. Ua-lvarez, Asymptotic Properties of Conditional Quantile Estimator for Cen- sored Dependent Observations, *Annals of the Institute of Statistical Mathematics*, **63** (2011), 267–289.

[22] Z. Lin and C. Lu, *Limit theory of mixing dependent random variables*, Mathematics and its applications, Sciences Press, Kluwer Academic Publishers, Beijing, 1996.

[23] N. Ling and Q. Xu, Asymptotic normality of conditional density estimation in the single index model for functional time series data, *Statistics and Probability Letters*, **82** (2012), 2235–2243.

[24] N. Ling, Z. Li and W. Yang, Conditional density estimation in the single functional index model for α-mixing functional data, *Communications in Statistics: Theory and Methods*, **43(3)** (2014), 441–454.

[25] E. Masry and D. Tøjstheim, Nonparametric estimation and identification of nonlinear time series, *Econometric Theor.*, **11** (1995), 258–289.

[26] E. Masry, Nonparametric Regression Estimation for Dependent Functional Data: Asymptotic Normality, *Stochastic Processes and their Applications*, 115 (2005), 155–177.

[27] C. Muharisa, F. Yanuar and D. Devianto, Simulation Study The Using of Bayesian Quantile Regression in Non-normal Error, *Cauchy: Jurnal Matematika Murni dan Aplikasi*, **5(3)** (2018), 121–126.

[28] E. Ould-Saïd, A Strong Uniform Convergence Rate of Kernel Conditional Quantile Estima- tor under Random Censorship, *Statistics and Probability Letters*, **76** (2006), 579–586.

[29] W-J. Padgett, *Nonparametric Estimation Of Density And Hazard Rate Functions When Samples Are Censored*, In P.R. Krishnaiah and C.R. Rao (Eds). Handbook of Statist. **7**, pp. 313–331. Elsevier/North-Holland. Amsterdam. Science Publishers, 1988.

[30] G. Roussas, Nonparametric estimation of the transition distribution function of a Markov process, *Ann. Statist.*, **40** (1969), 1386–1400.

[31] M. Samanta, Nonparametric estimation of conditional quantiles, *Statist. Probab. Lett.*, **7** (1989), 407–412.

[32] S. Sarmada and F. Yanuar, Quantile Regression Approach to Model Censored Data, *Science and Technology Indonesia*, **5(3)** (2020), 79–84.

[33] M. Tanner and W-H. Wong, The estimation of the hazard function from randomly censored data by the kernel methods, *Ann. Statist.*, **11** (1983), 989–993.

[34] I. Van-Keilegom and N. Veraverbeke, Hazard rate estimation in nonparametric regression with censored data, *Ann. Inst. Statist. Math.*, **53** (2001), 730–745.

[35] H. Wang and Y. Zhao, A kernel estimator for conditional $\mathbf{t}$-quantiles for mixing samples and its strong uniform convergence, (in chinese), *Math. Appl. (Wuhan).*, **12** (1999), 123–127.

[36] F. Yanuar, H. Laila and D. Devianto, The Simulation Study to Test the Performance of Quantile Regression Method With Heteroscedastic Error Variance, *Cauchy: Jurnal Matematika Murni dan Aplikasi*, **5(1)** (2017), 36–41.

[37] F. Yanuar, H. Yozza, F. Firdawati, I. Rahmi, and A. Zetra, Applying Bootstrap Quantile Regression for The Construction of a Low Birth Weight Model, *Makara Journal of Health Research*, **23(2)** (2019), 90–95.

[38] Y. Zhou and H. Liang, Asymptotic properties for L_1 norm kernel estimator of conditional median under dependence, *J. Nonparametr. Stat.*, **15** (2003), 205–219.

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Tripotent elements in quaternion rings over $\mathbb{Z}_p$

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Abstract. In this paper, we discuss tripotent[1] elements in the finite ring $\mathbb{H}/\mathbb{Z}_p$. We provide examples and establish conditions for tripotency. We follow similar methods used in [3] for idempotent elements in $\mathbb{H}/\mathbb{Z}_p$.

1 Introduction

Quaternions, denoted by $\mathbb{H}$, were first discovered by William. R. Hamilton in 1843 as an extension of complex numbers into four dimensions [9]. Namely, a quaternion is of the form $x = a_0 + a_1 i + a_2 j + a_3 k$, where a_i are reals and i, j, k are such that $i^2 = j^2 = k^2 = ijk = -1$. Algebraically speaking, $\mathbb{H}$ forms a division algebra (skew field) over R of dimension 4 ([9], p.195–196). A study of the finite ring[2] $\mathbb{H}/\mathbb{Z}_p$, where p is a prime number, looking into its structure and some of its properties, was done in [2]. A more detailed description of the structure $\mathbb{H}/\mathbb{Z}_p$ was given by Miguel and Serodio in [6]. Among others, they found the number of zero-divisors, the number of idempotent elements, and provided an interesting description of the zero-divisor graph. In particularly, they showed that the number of idempotent elements in $\mathbb{H}/\mathbb{Z}_p$ is p^2+p+2, for p odd prime. As discussed in [3], the only scalar idempotents in $\mathbb{H}/\mathbb{Z}_p$ are $a_0 = 0, 1$. Unlike that, as we will see below, there are scalar tripotents ($a_0 \neq 0, 1$) in

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$\mathbb{H}/\mathbb{Z}_p$. Yet, in both cases, there are no non-zero scalar multiple of the imaginary units (i.e. $x = bi$). Unlike also to the idempotent case, there are also pure imaginary tripotents (i.e. $x = a_1i+a_2j+a_3k$). There are also tripotent elements which are not idempotent. In the sections that follow, we give examples of tripotent elements in $\mathbb{H}/\mathbb{Z}_p$ and provide conditions for tripotency in $\mathbb{H}/\mathbb{Z}_p$.

2 Tripotent elements in $\mathbb{H}/\mathbb{Z}_p$

A quaternion x of the form $x = a_0+a_1i+a_2j+a_3k$ is said to be *tripotent* if $x^3 = x$. For the case of $\mathbb{H}/\mathbb{R}$ (i.e. $a_0, a_1, a_2, a_3 \in \mathbb{R}$), the only tripotent elements are $x = -1, x = 0$ and $x = 1$. However, for the case $\mathbb{H}/\mathbb{Z}_p$ (i.e. $a_0, a_1, a_2, a_3 \in \mathbb{Z}_p$), where p is a prime number, there are other possible tripotents other than, say, the obvious ones.

First notice the following: Take, for example, $p = 5$. If $a_0 \neq 0, a_1 = a_2 = a_3 = 0$, i.e. $\mathbb{H}/\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, a scalar tripotent is 4. For $\mathbb{H}/\mathbb{Z}_7$ is 6, $\mathbb{H}/\mathbb{Z}_{11}$ is 10, etc. In other words, for $\mathbb{H}/\mathbb{Z}_p$ the only scalar tripotent is $p–1$. This is not hard to show as $(p–1)^3 = (-1)^3 = -1 = p–1$. Furthermore, there are no tripotents that are non-zero scalar multiple of the imaginary units (say $x = bi$) because $x^3 = (bi)^3 = -bi = -x$ $(\neq x)$.

Furthermore, the existence of non-trivial tripotents is guaranteed as follows: As discussed in [2], [3] and [6], $\mathbb{H}/\mathbb{Z}_p$, which is not a division ring, has non-trivial idempotents[3]. But, it is not hard to show that idempotency implies tripotency due to the fact that in any ring $x^2 = x \Rightarrow x^3 = x$. (actually $x^2 = x \Rightarrow x^n = x$, for $n > 0$). Nevertheless, the converse is not true. For example, in $\mathbb{H}/\mathbb{Z}_5$, $3 + i$ is idempotent and hence also tripotent, but $2 + i$ is tripotent but not idempotent. (see also Example 1 and Remark 1).

The following propositions discuss the cases in which a non-scalar quaternion $x \in \mathbb{H}/\mathbb{Z}_p$, where p is a prime number, is tripotent.

Proposition 1 *Let* $x \in \mathbb{H}/\mathbb{Z}_p$ *be a quaternion of the form* $x = a_0 + a_1i$, *where* $a_0, a_1 \neq 0$. *Then,* x *is tripotent if and only if* $a_0^2 = \frac{1-p}{4}$ *and* $a_1^2 = \frac{p-1}{4}$, *where* p *prime number and* $p \neq 2, 3$.

Proof. Let $x = a_0 + a_1i$. Then:

$$x^3 = x \Rightarrow (a_0 + a_1i)^3 = a_0 + a_1i \Rightarrow a_0^3 - 3a_0a_1^2 + (3a_0^2a_1 - a_1^3)i = a_0 + a_1i$$

From the above we have the following two equations:

$$a_0^3 - 3a_0a_1^2 = a_0$$

$$3a_0^2a_1 - a_1^3 = a_1$$

These can be simplified into the following:

$$a_0^2 - 3a_1^2 = 1 \tag{1}$$
$$3a_0^2 - a_1^2 = 1 \tag{2}$$

One can solve for a_0^2 and a_1^2 as follows:

$$a_0^2 - 3a_1^2 = 3a_0^2 - a_1^2 \Rightarrow a_0^2 - 3a_0^2 = 3a_1^2 - a_1^2 \Rightarrow -2a_0^2 = 2a_1^2 \Rightarrow -a_0^2 = a_1^2 \tag{3}$$

Substituting for a_0^2 in (1) and solving for a_1^2 , we get:

$$-a_1^2 - 3a_1^2 = 1 \Rightarrow -4a_1^2 = 1 \Rightarrow a_1^2 = \frac{-1}{4}. \text{ Since } p = 0\,(\mathrm{mod}\,p)\ ,\ a_1^2 = \frac{p-1}{4}.$$

Solving for a_0^2, equation (3) gives: $a_0^2 = -\left(\frac{p-1}{4}\right) = \frac{1-p}{4}$.

To see if the quantities $\frac{p-1}{4}$ and $\frac{1-p}{4}$ are squares $\mathrm{mod}\,p$, we calculate the *Legendre Symbol*[4] for $\left(\frac{\frac{p-1}{4}}{p}\right)$ and $\left(\frac{\frac{1-p}{4}}{p}\right)$ respectively. The first gives:

$$\begin{aligned}\left(\frac{\frac{p-1}{4}}{p}\right) &= \left(\frac{p-1}{p}\right)\left(\frac{\frac{1}{4}}{p}\right) = \left(\frac{p-1}{p}\right)\cdot 1 = (p-1)^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} \\ &= \begin{cases} 1 & \text{if} \quad p \equiv 1 \pmod 4 \\ -1 & \text{if} \quad p \equiv 3 \pmod 4. \end{cases}\end{aligned}$$

Hence, there are *no* tripotents of the form $a_0 + a_1 i$, if $p \equiv 3 \pmod 4$. Elements of the form $a_0 + a_1 i$ are tripotent if $p \equiv 1 \pmod 4$ and, in that case, $a_0^2 = \frac{1-p}{4}$ and $a_1^2 = \frac{p-1}{4}$.

For the converse, it is not hard to show that given $a_0^2 = \frac{1-p}{4}$ and $a_1^2 = \frac{p-1}{4}$, we have that:

$$\begin{aligned} x^3 = (a_0 + a_1 i)^3 &= a_0^3 - 3a_0a_1^2 + (3a_0^2a_1 - a_1^3)i \\ &= a_0(a_0^2 - 3a_1^2) + a_1(3a_0^2 - a_1^2)i \\ &= a_0\left(\frac{1-p}{4} - 3\frac{p-1}{4}\right) + a_1\left(3\frac{1-p}{4} - \frac{p-1}{4}\right)i \\ &= a_0(1-p) + a_1(1-p)i \\ &= a_0 + a_1 i, \quad \text{as } p = 0 \pmod p \\ &= x \end{aligned}$$

Hence, x is tripotent. □

Example 1 *Let* $p = 13$. *Then, we have* $a_0^2 = \frac{1-13}{4} = \frac{-12}{4} = -3 = 10 \pmod{13}$ *and* $a_1^2 = \frac{13-1}{4} = \frac{12}{4} = 3$. *There are many values for* a_0 *and* a_1. *One pair of these possible values is* $a_0 = 6$ *and* $a_1 = 4$, *because* $6^2 = 36 = 10 \pmod{13}$ *and* $4^2 = 16 = 3 \pmod{13}$. *Therefore* $x = 6 + 4i$ *is a tripotent in* $\mathbb{H}/\mathbb{Z}_{13}$. *Notice also that* $x = 6 + 4i$ *is not an idempotent in* $\mathbb{H}/\mathbb{Z}_{13}$.

Remark 1 *As we have seen already above, there are tripotents which are also idempotents. As we explained already, idempotency implies tripotency, hence the tripotents which are also idempotents satisfy also the conditions of idempotency given in [3]. Namely,* $a_0 = \frac{p+1}{2}$ *and* $a_1^2 + a_2^2 + a_3^2 = \frac{p^2-1}{4}$. *Tripotents which are not idempotends, that is 'proper' tridempotents, do not satisfy these additional conditions. It is not hard to see that the conditions for idempotency imply the conditions for tripotency that we provide here, but not vice versa. (see more in Par. 3 for a general condition on when a tripotent is also idempotent). Notice also that in [3] it was shown that there are no pure imaginary idempotents of the form* $x = a_1 i + a_2 j + a_3 k$. *Yet, as Proposition 2 below shows, there are tripotents of that form. Hence, all pure imaginary elements are 'proper' tripotents.*

Proposition 2 *Let* $x \in \mathbb{H}/\mathbb{Z}_p$ *be a pure imaginary element of the form* $x = a_1 i + a_2 j + a_3 k$, *where at least two of* a_1, a_2, a_3 *are non-zero. Then,* x *is tripotent if and only if* $a_1^2 + a_2^2 + a_3^2 = p - 1$.

Proof. Let $x = a_1 i + a_2 j + a_3 k$. Then:

$$x^3 = x \Rightarrow (a_1 i + a_2 j + a_3 k)^3 = a_1 i + a_2 j + a_3 k$$

Expanding the above, we get:

$$a_1(-a_1^2 - a_2^2 - a_3^2)i + a_2(-a_1^2 - a_2^2 - a_3^2)j + a_3(-a_1^2 - a_2^2 - a_3^2)k = a_1 i + a_2 j + a_3 k$$

Hence, we obtain the following three equations:

$$a_1(-a_1^2 - a_2^2 - a_3^2) = a_1 \quad (4)$$
$$a_2(-a_1^2 - a_2^2 - a_3^2) = a_2 \quad (5)$$
$$a_3(-a_1^2 - a_2^2 - a_3^2) = a_3. \quad (6)$$

From the above three equations we get:

$$a_1 = 0 \quad \text{or} \quad -a_1^2 - a_2^2 - a_3^2 = 1$$

$$a_2 = 0 \quad \text{or} \quad -a_1^2 - a_2^2 - a_3^2 = 1$$

$$a_3 = 0 \quad \text{or} \quad -a_1^2 - a_2^2 - a_3^2 = 1.$$

From the equation $-a_1^2 - a_2^2 - a_3^2 = 1$, we have $a_1^2 + a_2^2 + a_3^2 = -1$. This can be written also as $a_1^2 + a_2^2 + a_3^2 = p - 1$, as $p \bmod p = 0$.

For the converse, given that $a_1^2 + a_2^2 + a_3^2 = p - 1$, it is not hard to see that: $x^3 = (a_1 i + a_2 j + a_3 k)^3 = a_1(-a_1^2 - a_2^2 - a_3^2)i + a_2(-a_1^2 - a_2^2 - a_3^2)j + a_3(-a_1^2 - a_2^2 - a_3^2)k = a_1(1-p)i + a_2(1-p)j + a_3(1-p)k = a_1 i + a_2 j + a_3 k = x$, as $p \bmod p = 0$. Hence, x is tripotent. □

Example 2 *Let* $p = 5$*. Then, we have* $a_1^2 + a_2^2 + a_3^2 = 5 - 1 = 4$*. We can have different combinations of numbers from* $\mathbb{Z}_5$ *that satisfy the above equation. One such combinations is* $a_1 = 3$*,* $a_2 = 4$ *and* $a_3 = 2$ *(i.e.* $3^2 + 4^2 + 2^2 = 29 = 4$ mod 5). *Hence,* $x = 3i + 4j + 2k$ *is a tripotent in* $\mathbb{H}/\mathbb{Z}_5$.

Theorem 1 *Let* $x \in \mathbb{H}/\mathbb{Z}_p$*, where p is prime and* $p \neq 2, 3$*, be an element of the from* $x = a_0 + a_1 i + a_2 j + a_3 k$*, where* $a_0 \neq 0$ *and at least one of* a_1, a_2, a_3 *is non-zero. Then, x is tripotent if and only if* $a_0^2 = \frac{1-p}{4}$ *and* $a_1^2 + a_2^2 + a_3^2 = \frac{p-1}{4}$.

Proof. Let $x = a_0 + a_1 i + a_2 j + a_3 k$. Then:

$$x^3 = x \Rightarrow (a_0 + a_1 i + a_2 j + a_3 k)^3 = a_0 + a_1 i + a_2 j + a_3 k.$$

After the multiplications, we get:

$$a_0\big(a_0^2 - 3(a_1^2 + a_2^2 + a_3^2)\big) + a_1\big(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)\big)i + a_2\big(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)\big)j + a_3\big(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)\big)k = a_0 + a_1 i + a_2 j + a_3 k.$$

Hence, we obtain the following four equations by equating the corresponding coefficients:

$$a_0\big(a_0^2 - 3(a_1^2 + a_2^2 + a_3^2)\big) = a_0 \tag{7}$$
$$a_1\big(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)\big) = a_1 \tag{8}$$
$$a_2\big(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)\big) = a_2 \tag{9}$$
$$a_3\big(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)\big) = a_3. \tag{10}$$

From the above four equations we get the following:

$$a_0 = 0 \quad \text{or} \quad a_0^2 - 3(a_1^2 + a_2^2 + a_3^2) = 1$$

$$\begin{aligned} a_1 = 0 \quad &\text{or} \quad 3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right) = 1 \\ a_2 = 0 \quad &\text{or} \quad 3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right) = 1 \\ a_3 = 0 \quad &\text{or} \quad 3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right) = 1. \end{aligned}$$

From the first, since $a_0 \neq 0$, we have $a_0^2 - 3\left(a_1^2 + a_2^2 + a_3^2\right) = 1$. In addition, from the last three we have $3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right) = 1$. Let $a_1^2 + a_2^2 + a_3^2 = \lambda$. Then, we have the following two equations:

$$a_0^2 - 3\lambda = 1 \tag{11}$$

$$3a_0^2 - \lambda = 1. \tag{12}$$

Combining the equations, we get:

$$a_0^2 - 3\lambda = 3a_0^2 - \lambda \Rightarrow -2a_0^2 = 2\lambda \Rightarrow a_0^2 = -\lambda.$$

Substituting a_0^2 for $-\lambda$ in (11), we get $\lambda = -\frac{1}{4} = \frac{p-1}{4}$, because $p \bmod p = 0$. Hence, $a_1^2 + a_2^2 + a_3^2 = \frac{p-1}{4}$. And, since $a_0^2 = -\lambda$, we get $a_0^2 = \frac{1-p}{4}$.

For the converse, given that $a_0^2 = \frac{1-p}{4}$ and $a_1^2 + a_2^2 + a_3^2 = \frac{p-1}{4}$, it is not hard to see that:

$$\begin{aligned} x^3 = \left(a_0 + a_1 i + a_2 j + a_3 k\right)^3 &= a_0\left(a_0^2 - 3\left(a_1^2 + a_2^2 + a_3^2\right)\right) + \\ &\quad a_1\left(3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right)i + \\ &\quad a_2\left(3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right)j + \\ &\quad a_3\left(3a_0^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right)k \\ &= a_0\left(\frac{1-p}{4} - 3\frac{p-1}{4}\right) + \\ &\quad a_1\left(3\frac{1-p}{4} - \frac{p-1}{4}\right)i + \\ &\quad a_2\left(3\frac{1-p}{4} - \frac{p-1}{4}\right)j + \\ &\quad a_3\left(3\frac{1-p}{4} - \frac{p-1}{4}\right)k \\ &= a_0(1-p) + a_1(1-p)i + a_2(1-p)j + a_3(1-p)k \\ &= a_0 + a_1 i + a_2 j + a_3 k \ , \text{as } p = 0 \mod p \\ &= x. \end{aligned}$$

Hence, x is tripotent. □

Example 3 *Let* $p = 7$. *Then,* $a_0^2 = \frac{1-7}{4} = \frac{-6}{4}$ *and* $a_1^2 + a_2^2 + a_3^2 = \frac{7-1}{4} = \frac{6}{4}$. *From the two equations we have* $4a_0^2 = -6 = 1 \pmod 7$ *and* $4(a_1^2 + a_2^2 + a_3^2) = 6$. *One possible solution is* $a_0 = 3$ *and* $a_1 = 2$, $a_2 = 2$, $a_3 = 2$. *This can be checked as follows:* $4(3^2) = 36 = 1 \pmod 7$ *and* $4(2^2 + 2^2 + 2^2) = 48 = 6 \pmod 7$. *Thus, the element* $x = 3 + 2i + 2j + 2k$ *is tripotent in* $\mathbb{H}/\mathbb{Z}_7$ *(but not idempotent). Another tripotent is* $x = 4 + 3i + j + 4k$, *which is also idempotent.*

Remark 2 *The equation* $a_1^2 + a_2^2 + a_3^2 = \frac{p-1}{4}$ *brings to mind the classic 'Sum of Three Squares Theorem' which was proved by Gauss in his Disquisitiones Arithmeticae in 1801.*[5] *As that theorem says, an integer n can be the sum of three squares if and only if* $n \neq 4^m(8k + 7), m, k, \geq 0$. *So, clearly, when* $n = 7$ *one does not have solutions to the equation* $a_1^2 + a_2^2 + a_3^2 = n$. *But, in our case (in this special 'modp' version), one does get solutions for* $p = 7$ *to the equation* $a_1^2 + a_2^2 + a_3^2 = \frac{p-1}{4}$, *as Example 3 above shows. More interestingly, we get solutions even if* $\frac{p-1}{4} = 4^m(8k+7), m, k, \geq 0$. *For example, for* $p = 113$, $\frac{p-1}{4} = \frac{113-1}{4} = 28 = 4^1(8.0+7)$, *but* $28 = 141 \pmod{113} = 4^2 + 5^2 + 10^2$. *And, given that* $a_0^2 = \frac{1-113}{4} = -28 = 85 \pmod{113}$, *the tripotent is* $56+4i+5j+10k$.

3 Connection to general rings and applications

There is a lot in the literature regarding tripotents, and k-potents in general, in more general rings R. It would be interesting to see if and how some of these results relate to the 'special', in a sense, ring $\mathbb{H}/\mathbb{Z}_p$.

In Zhou et al. [14] (Theorem 2.1), we are informed that in a commutative ring R every x is the sum of two idempotents if and only if $x^3 = x$. As $\mathbb{H}/\mathbb{Z}_p$ is not commutative, the above fails. For example, consider the idempotents $a = 3+i$ and $b = 3+j$ in $\mathbb{H}/\mathbb{Z}_5$. Then, $x = a+b = (3+i)+(3+j) = 6+i+j = 1 + i + j$, but x is not tripotent (because $1^2 \neq \frac{1-5}{4}$ and $1^2 + 1^2 \neq \frac{5-1}{4}$ from the Theorem 1 above). The above fails even when the idempotents commute. Take, for example, $a = b = 3 + i$ in $\mathbb{H}/\mathbb{Z}_5$.

Also, Mosic in [7] gives the relation between idempotent and tripotent elements in any associative ring R, generalizing the result on matrices by Baksalary and Trenkler [12]. Namely, for any $x \in R$, where 2, 3 are invertible, x is idempotent if and only if x is tripotent and $1-x$ is tripotent (or $1+x$ is invertible). Since $\mathbb{H}/\mathbb{Z}_p$ is associative, the result holds. As we have seen already in Par.2 above, idempotency implies tripotency. But for a tripotent to be idempotent it is also required that $1-x$ is tripotent. Take for example the tripotents in our Example 3 above. Namely, $x = 4+3i+j+4k$ and $x = 3+2i+2j+2k$ in

$\mathbb{H}/\mathbb{Z}_7$. The first is also an idempotent, but the second is not. It is not hard to check that directly or using the conditions for idempotency in [3]. Notice also that for the first case we have $1-x$ is tripotent (and $1+x$ is invertible as the $N(x)=2\neq 0$), where in the second case is $1-x$ is not tripotent (nor $1+x$ is invertible as the $N(x)=0$). More generally, in $\mathbb{H}/\mathbb{Z}_p$, one can see the conditions given by Mosic as follows: Theorem 1 says that if $x=a_0+a_1i+a_2j+a_3k$ is tripotent then $a_0^2=\frac{1-p}{4}$ and $a_1^2+a_2^2+a_3^2=\frac{p-1}{4}$. If $1-x$ is also a tripotent, then $(1-a_0)^2=\frac{1-p}{4}$ and $a_1^2+a_2^2+a_3^2=\frac{p-1}{4}$. Equating the corresponding first terms, one has $a_0^2=(1-a_0)^2 \Rightarrow 1-2a_0+a_0^2 \Rightarrow 2a_0=1 \Rightarrow a_0=\frac{1}{2}=\frac{p+1}{2}$, which is the first condition for idempotency in [3]. (the second condition in [3] is also true by simply noticing that $a_1^2+a_2^2+a_3^2=\frac{p-1}{4}=\frac{p^2-1}{4}$).

Finally, it is interesting to note any possible applications of idempotents, tripotent or more generally k-potent ring elements. Wu in [13] applies k-potent matrices in digital image encryption. A series of encryption key matrices is used, via matrix multiplications, to mask an image by altering the gray level of each pixel of the image. The original image then is transformed into a different image. k-potent matrices, and their 'variations', are used for the encryption key matrices. Wu defines them all via the equation: $A=\alpha I+\beta A$, where $\alpha\beta=0, \alpha,\beta\in\{-1,0,1\}$ and $k\geq 2$. (e.g. A is *periodic* with period $k-1$ if $A^k=A$ and k is the least positive integer as such, A is *skew-unipotent* if $A^k=-I$, etc).

4 Conclusion

In this paper, we talked about tripotent elements in $\mathbb{H}/\mathbb{Z}_p$. Unlike idempotents, there are scalar tripotents ($a_0\neq 0,1$) in $\mathbb{H}/\mathbb{Z}_p$. Yet, in both cases, there are no non-zero scalar multiple of the imaginary units (i.e. $x=bi$). Unlike also to the idempotent case, there are also pure imaginary tripotents (i.e. $x=a_1i+a_2j+a_3k$). There are also tripotent elements which are not idempotent. We provided examples of non-trivial tripotents and we established conditions for tripotency. The methodology we followed and the conditions we found were very similar as the one(s) in [3]. An interesting and possibly harder project is to look at the structure of $\mathbb{O}/\mathbb{Z}_p$, where $\mathbb{O}$ is the octonion division algebra, and discuss idempotent, tripotent and nilpotent elements in that finite ring.

Notes

1. Recall that x is *idempotent* if $x^2 = x$, and x is *tripotent* if $x^3 = x$. In general, x is *k-potent* if $x^k = x$, for some k.
2. $+$ and $\cdot$ on $\mathbb{H}$ are defined in ([5], p.124). As $p = 0\,(\mathrm{mod} p)$, on $\mathbb{H}/\mathbb{Z}_p$ they are defined as follows:

$$\begin{aligned}
x + y &= (a_0 + a_1 i + a_2 j + a_3 k) + (b_0 + b_1 i + b_2 j + b_3 k)\\
&= (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k\\
x \cdot y &= (a_0 + a_1 i + a_2 j + a_3 k) \cdot (b_0 + b_1 i + b_2 j + b_3 k)\\
&= a_0 b_0 + (p-1)a_1 b_1 + (p-1)a_2 b_2 + (p-1)a_3 b_3 +\\
&\quad (a_0 b_1 + a_1 b_0 + a_2 b_3 + (p-1)a_3 b_2)i +\\
&\quad (a_0 b_2 + (p-1)a_1 b_3 + a_2 b_0 + a_3 b_1)j +\\
&\quad (a_0 b_3 + a_1 b_2 + (p-1)a_2 b_1 + a_3 b_0)k.
\end{aligned}$$

3. In Herstein ([5], p.130), we have that: In a ring $\mathbb{F}$, if $x^2 = x$, for all x, then $\mathbb{F}$ is commutative. It is not hard to show that the converse is not true. (e.g. $\mathbb{F} = \mathbb{Z}_3$, 2 is not idempotent). Actually, a field $\mathbb{F}$ has only trivial idempotents. Hence, in $\mathbb{H}/\mathbb{Z}_p$ some elements are non-trivial idempotents and they were described in [3]. Interestingly, in Herstein ([5], p.136) we also have that: In a ring $\mathbb{F}$, if $x^3 = x$, for all x, then $\mathbb{F}$ is commutative. The latter is much harder to establish, but a solution (with an interesting story behind it) can be found in [4].
4. The *Legendre Symbol* $\left(\frac{a}{p}\right)$ is defined as follows:

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } \ a \ \text{ is a qudratic residue } \bmod p\\ -1 & \text{if } \ a \ \text{ is not a qudratic residue } \bmod p\\ 0 & \text{if } \ p/a.\end{cases}$$

5. For a proof see ([11], p.45). Also see [1] for a more elementary proof.

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References

[1] N. C. Ankeny, Sum of Three Squares, *Proc. Amer. Math. Soc.*, **8** (2) (1957), 316–319.

[2] M. Aristidou and A. Demetre, A Note on Quaternion Rings over $\mathbb{Z}_p$, *Int. J. Algebra*, **3** (15) (2009), 725–728.

[3] M. Aristidou and A. Demetre, Idempotent Elements in Quaternion Rings over $\mathbb{Z}_p$, *Int. J. Algebra*, **6** (5) (2012), 249–254.

[4] P. Damianou, Commutative Rings (in Greek), Proceedings of the 9^{th} Cyprus Conference of Mathematics Education and Science, Cyprus, 2007, 165–166.

[5] I. N. Herstein, *Topics in Algebra*, 2^{nd} ed., Wiley, NY, 1975.

[6] C. J. Miguel and R. Serodio, On the Structure of Quaternion Rings over $\mathbb{Z}_p$, *Int. J. Algebra*, **5** (27) (2011), 1313–1325.

[7] D. Mosic, Characterizations of k-potent Elements in Rings, *Ann. Mat. Pura Appl.*, **194** (4) (2015), 1157–1168

[8] R. S. Pierce, *Associative Algebras*, Springer, NY, 1982.

[9] R. Remmert, H. D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch and A.Prestel, *Numbers*, Springer, NY, 1991.

[10] R. D. Schafer, *An Introduction to Nonassociative Algebrass*, Academic Press, NY, 1966.

[11] J. P. Serre, *A Course in Arithmetic*, Springer, NY, 1973.

[12] G. Trenkler and O. M. Baksalary, On k-potent Matrices, *Electron. J. Linear Algebra*, **26** (2013), 446–470.

[13] Y. Wu, k-Potent Matrices - Construction and Applications in Digital Image Encryption, *Recent Advances in Applied Mathematics*, Proceedings of the 2010 American Conference on Applied Mathematics, USA, 2010, 455–460.

[14] Z. Ying, T. Kosan and Y. Zhou, Rings in Which Every Element is a Sum of Two Tripotents, *Canad. Math. Bull.*, **59** (3) (2016), 661–672.

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On the generalized Becker-Stark type inequalities

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Abstract. In this paper, we establish several generalized Becker-Stark type inequalities for the tangent function. We present unified proofs of many inequalities in the existing literature. Graphical illustrations of some obtained results are also presented.

1 Introduction

Becker and Stark [6] established the inequality

$$1-\frac{4x^2}{\pi^2}<\frac{x}{\tan x}<\frac{\pi^2}{8}-\frac{x^2}{2};\quad x\in(0,\pi/2). \tag{1}$$

The inequality (1) attracted many researchers and several of its variations and refinements have been established. We may refer to [8, 9, 10, 20, 21, 5, 11, 16,

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19, 3, 4], and the references therein for more details. Chen and Cheung [8] proved that the best possible constants for which the inequality

$$\left(1-\frac{4x^2}{\pi^2}\right)^{\beta} < \frac{x}{\tan x} < \left(1-\frac{4x^2}{\pi^2}\right)^{\alpha}; \quad x \in (0, \pi/2) \tag{2}$$

holds are $\alpha = \pi^2/12 \approx 0.8224$ and $\beta = 1$. The inequality (2) refines (1). Recently, Chen and Elezović [9] proved the following inequality:

$$\frac{\pi^2}{12} - \frac{2x^3}{3\pi} < \frac{x}{\tan x} < 1 - \frac{8x^3}{\pi^3}; \quad x \in (0, \pi/2). \tag{3}$$

Although the upper bound of (3) is sharper than the corresponding upper bound of (1), it is not sharper than the upper bound in (2).

The inequality

$$1 - \frac{4x^2}{\pi^2} < \frac{x}{\tan x} < 1 - \frac{x^2}{3}; \quad x \in (0, \pi/2) \tag{4}$$

was proved by Z.-H. Yang et. al. [19, (96)]. Before we proceed further, we would like to note that the right inequality in (4) is not good near the point $x = \pi/2-$ as well as that this inequality is not better than the right inequality in (2), as incorrectly stated in [19, Remark 17]. Strictly speaking, the following inequality, which appears as a part of the equation [19, (98)], is not true since the estimate

$$\left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\pi^2/24} < \frac{3}{3-x^2}, \quad x \in (0, \pi/2)$$

cannot be satisfied near the point $x = \pi/2-$. It is also known that

$$1 - \frac{2x}{\pi} < \frac{x}{\tan x} < \frac{\pi^2}{4} - \frac{\pi x}{2}; \quad x \in (0, \pi/2). \tag{5}$$

The left inequality of (5) is due to H.-F. Ge [12] and the right inequality of (5) is due to S. B. Stečkin [18].

Among all the inequalities (1)-(5), the inequality (2) is the best. In this paper, our aim is to obtain several generalized inequalities by studying the monotonicity of functions with one parameter. We will obtain or refine the above inequalities as particular cases of our results. We also aim to improve the lower bound of (2) in the interval $(0, \delta_*)$ where $\delta_* \approx 1.3407$ as well as the upper bound of (2) near the point $x = \pi/2^-$. Our new bounds may not be uniformly better than the ones in (2) but they certainly provide alternatives to the best bounds.

2 Preliminaries and lemmas

The following power series expansions involving Bernoulli numbers can be found in [13, 1.411]:

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}|B_{2k}|x^{2k-1} \; ; \quad |x| < \pi, \;\; x \neq 0 \tag{6}$$

and

$$\csc x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1}-1\right)}{(2k)!}|B_{2k}|x^{2k-1}; \quad |x| < \pi, \;\; x \neq 0, \tag{7}$$

where B_{2k} are the even indexed Bernoulli numbers. The expansion (7) can be rewritten as

$$\frac{x}{\sin x} = 1 + \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1}-1\right)}{(2k)!}|B_{2k}|x^{2k}; \quad |x| < \pi. \tag{8}$$

From (6), we obtain

$$\left(\frac{x}{\sin x}\right)^2 = -x^2(\cot x)' = 1 + \sum_{k=1}^{\infty} \frac{(2k-1)2^{2k}}{(2k)!}|B_{2k}|x^{2k}; \quad |x| < \pi, \;\; x \neq 0. \tag{9}$$

Also, with reference to [13, 1.518], we have

$$\ln(\tan x) = \ln x + \sum_{k=1}^{\infty} \frac{(2^{2k-1}-1)2^{2k}}{k(2k)!}|B_{2k}|x^{2k}; \quad 0 < x < \frac{\pi}{2}, \;\; x \neq 0. \tag{10}$$

In addition to the above formulas, we will also use the following lemmas in order to prove our main results. For Lemma 1, we refer to [2](see also [7, eqn (4.3), p. 42].

Lemma 1 *Let* $f_1(x)$ *and* $f_2(x)$ *be two real valued functions which are continuous on* $[a, b]$ *and derivable on* (a, b), *where* $-\infty < a < b < \infty$ *and* $g'(x) \neq 0$, *for all* $x \in (a, b)$. *Let,*

$$A(x) = \frac{f_1(x) - f_1(a)}{f_2(x) - f_2(a)}, \;\; x \in (a, b)$$

and

$$B(x) = \frac{f_1(x) - f_1(b)}{f_2(x) - f_2(b)}, \;\; x \in (a, b).$$

Then, we have

(i) $A(x)$ *and* $B(x)$ *are increasing on* (a, b) *if* $f_1'(x)/f_2'(x)$ *is increasing on* (a, b).

(ii) $A(x)$ *and* $B(x)$ *are decreasing on* (a, b) *if* $f_1'(x)/f_2'(x)$ *is decreasing on* (a, b).

The strictness of the monotonicity of $A(x)$ *and* $B(x)$ *depends on the strictness of monotonicity of* $f_1'(x)/f_2'(x)$.

The result below shows the relationship between two consecutive absolute Bernoulli numbers. It was established recently in [17].

Lemma 2 *For* $k \in \mathbb{N}$, *the Bernoulli numbers satisfy*

$$\frac{(2^{2k-1}-1)}{(2^{2k+1}-1)}\frac{(2k+1)(2k+2)}{\pi^2} < \frac{|B_{2k+2}|}{|B_{2k}|} < \frac{(2^{2k}-1)}{(2^{2k+2}-1)}\frac{(2k+1)(2k+2)}{\pi^2}.$$

Lemma 3 *Let* $A(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $B(x) = \sum_{k=0}^{\infty} b_k x^k$ *be convergent for* $|x| < R$, *where* a_k *and* b_k *are real numbers for* $k = 0, 1, 2, \cdots$ *such that* $b_k > 0$ *for* $k \geq 0$. *If the sequence* a_k/b_k *is strictly increasing (or decreasing), then the function* $A(x)/B(x)$ *is also strictly increasing (or decreasing) on* $(0, R)$.

For more details about Lemma 3, see, for instance, [14]. The following lemma can be found in [1].

Lemma 4 *For all integers* $k \in \mathbb{N}$, *we have*

$$\frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\alpha-2k}} < |B_{2k}| < \frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\beta-2k}}, \tag{11}$$

with the best constants $\alpha = 0$ *and* $\beta = 2 + (\ln(1 - 6/\pi^2))/\ln 2 \approx 0.6491$.

3 Main results

In this section, we will state and prove our main results. In the beginning, for any number $p \in \mathbb{R}$, we define

$$\phi_p(x) := \frac{\tan x - x}{x^p \tan x}, \quad x \in (0, \pi/2).$$

Then, the following result holds.

Theorem 1

I. $\phi_p(x)$ *is strictly increasing on* $(0, \pi/2)$ *if and only if* $p \leq 2$, *and*

II. $\phi_p(x)$ *is strictly decreasing on* $(0, \pi/2)$ *if and only if* $p \geq \pi^2/4 \approx 2.4674$.

Proof. By differentiation, we have

$$F_p(x) = \tan^2 x \cdot \phi_p'(x) = -px^{-(p+1)} \tan^2 x - (1-p)x^{-p} \tan x + x^{1-p} \sec^2 x.$$

Note that $\phi_p(x)$ is strictly increasing on $(0, \pi/2)$ if and only if $F_p(x) > 0$, $x \in (0, \pi/2)$, i.e.,

$$p < \frac{x^2 \sec^2 x - x \tan x}{\tan x(\tan x - x)} = \frac{\left(\frac{x}{\sin x}\right)^2 - x \cot x}{1 - x \cot x} =: f(x), \quad x \in (0, \pi/2).$$

From (6) and (9), we get

$$\begin{aligned} f(x) &= \frac{\sum_{k=1}^{\infty} \frac{2^{2k}(2k-1)}{(2k)!} |B_{2k}| x^{2k} + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} |B_{2k}| x^{2k}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} |B_{2k}| x^{2k}} \\ &= \frac{\sum_{k=1}^{\infty} \frac{2^{2k} 2k}{(2k)!} |B_{2k}| x^{2k}}{\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} |B_{2k}| x^{2k}} := \frac{\sum_{k=1}^{\infty} a_k x^{2k}}{\sum_{k=1}^{\infty} b_k x^{2k}}, \quad x \in (0, \pi/2). \end{aligned}$$

From this, we get $a_k/b_k = 2k$ $(k \in \mathbb{N})$. Since the sequence $\{a_k/b_k\}_{k=1}^{\infty}$ is strictly increasing, we conclude from Lemma 3 that the function $f(x)$ is strictly increasing on $(0, \pi/2)$. Hence, $\phi_p(x)$ is strictly increasing on $(0, \pi/2)$ if and only if $p \leq \inf\{f(x) : 0 < x < \pi/2\} = f(0^+) = 2$. Similarly, $\phi_p(x)$ is strictly decreasing on $(0, \pi/2)$ if and only if $F_p(x) < 0$, which is equivalent to saying that $p \geq \sup\{f(x) : 0 < x < \pi/2\} = f(\pi/2^-) = \pi^2/4$. □

Remark 1 *Suppose that* $p \in (2, \pi^2/4)$. *Since the function* $f(x)$ *is strictly increasing on* $(0, \pi/2)$, *we get from the above that there exists a unique point* $x_p \in (0, \pi/2)$ *such that* $f(x_p) = p$. *This implies* $f(x) < p$ *for* $x \in (0, x_p)$ *and* $f(x) > p$ *for* $x \in (x_p, \pi/2)$ *so that* $\phi_p(x)$ *is strictly decreasing on* $(0, x_p)$ *and strictly increasing on* $(x_p, \pi/2)$, *with* $\phi_p(x) \geq \phi_p(x_p)$ *for* $x \in (0, \pi/2)$.

Let $p \in (-\infty, 4] \setminus \{0\}$. Define now

$$\psi_p(x) := \frac{\ln\left(\frac{x}{\tan x}\right)}{\ln\left(1 - p\frac{x^2}{\pi^2}\right)}, \quad x \in (0, \pi/2).$$

Then, we have:

Theorem 2

I. $\psi_p(x)$ *is strictly decreasing on* $(0, \pi/2)$ *if and only if* $p < 0$, *and*

II. $\psi_p(x)$ *is strictly increasing on* $(0, \pi/2)$ *if and only if* $0 < p \leq 4$.

Proof. Set $\psi_1(x) := \ln(x/\tan x)$, $x \in (0, \pi/2)$ and $(\psi_2)_p(x) := \ln(1-(px^2/\pi^2))$, $x \in (0, \pi/2)$. Then $\psi_1(0^+) = 0 = (\psi_2)_p(0)$ and differentiation yields

$$\frac{\psi_1'(x)}{(\psi_2)_p'(x)} = \frac{1}{2p}\left(\pi^2 - px^2\right)\frac{x - \sin x \cos x}{x^2 \sin x \cos x} = \frac{1}{2p}(\psi_3)_p(x), \ \ x \in (0, \pi/2),$$

where, for every $x \in (0, \pi/2)$,

$$(\psi_3)_p(x) := \left(\pi^2 - px^2\right)\frac{x - \sin x \cos x}{x^2 \sin x \cos x} = \frac{(\pi^2 - px^2)}{x^2}\left(\frac{2x}{\sin 2x} - 1\right).$$

By (8), we get

$$\begin{aligned}
(\psi_3)_p(x) &= \frac{(\pi^2 - px^2)}{x^2}\sum_{k=1}^{\infty}\frac{2^{2k} - 2}{(2k)!}|B_{2k}|(2x)^{2k} \\
&= (\pi^2 - px^2)\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k} - 2)}{(2k)!}|B_{2k}|x^{2k-2} \\
&= \pi^2\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k} - 2)}{(2k)!}|B_{2k}|x^{2k-2} - p\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k} - 2)}{(2k)!}|B_{2k}|x^{2k} \\
&= \frac{2\pi^2}{3} + \sum_{k=1}^{\infty}\left(\frac{\pi^2 2^{2k+2}(2^{2k+2} - 2)}{(2k+2)!}|B_{2k+2}| - p\frac{2^{2k}(2^{2k} - 2)}{(2k)!}|B_{2k}|\right)x^{2k} \\
&= \frac{2\pi^2}{3} + \sum_{k=1}^{\infty} a_k x^{2k}, \quad x \in (0, \pi/2),
\end{aligned}$$

where

$$a_k := \frac{\pi^2 2^{2k+2}(2^{2k+2} - 2)}{(2k+2)!}|B_{2k+2}| - p\frac{2^{2k}(2^{2k} - 2)}{(2k)!}|B_{2k}| \ \ (k \in \mathbb{N}). \tag{12}$$

Case I. If $p < 0$, then $a_k > 0$ for $k \in \mathbb{N}$ and $(\psi_3)_p(x)$ is strictly increasing on $(0, \pi/2)$. Consequently, $\psi_p(x)$ is strictly decreasing on $(0, \pi/2)$ by Lemma 1.

Case II. If $p \in (0, 4]$, then $-p \geq -4$ and we have

$$a_k \geq \frac{\pi^2 2^{2k+2}(2^{2k+2}-2)}{(2k+2)!}|B_{2k+2}| - 4\frac{2^{2k}(2^{2k}-2)}{(2k)!}|B_{2k}| =: j(k), \quad k \in \mathbb{N}.$$

From Lemma 4, we calculate

$$\begin{aligned} a_k \geq j(k) &> \frac{2^{2k+3}}{\pi^{2k}}\left(\frac{2^{2k+2}-2}{2^{2k+2}-1} - \frac{2^{2k}-2}{2^{2k}-2^{\beta}}\right) \\ &= \frac{2^{2k+3}}{\pi^{2k}} \cdot \frac{2^{2k+2}(2-2^{\beta})-2^{2k}}{(2^{2k+2}-1)(2^{2k}-2^{\beta})} \\ &= \frac{2^{4k+3}}{\pi^{2k}} \cdot \frac{4(2-2^{\beta})-1}{(2^{2k+2}-1)(2^{2k}-2^{\beta})}. \end{aligned}$$

Since $4(2-2^{\beta}) \approx 1.7268$, we get $a_k > 0$ for $k \in \mathbb{N}$. This shows that $(\psi_3)_p(x)$ is strictly increasing on $(0, \pi/2)$. By Lemma 1, $\psi_p(x)$ is also strictly increasing on $(0, \pi/2)$.

□

If $p < 4$, then the function $\psi_p(x)$ cannot be defined for $x \geq \pi/2$. Suppose now that $p > 4$ and consider the function $\psi_p(x)$ defined for $x \in (0, \pi/\sqrt{p})$. Then, the following result holds.

Theorem 3 *The function* $\psi_p(x)$ *is strictly decreasing on* $(0, \pi/\sqrt{p})$ *if* $p \geq 84/15$.

Proof. Repeating verbatim the arguments used in the proof of Theorem 2, we get that

$$(\psi_3)_p(x) = \frac{2\pi^2}{3} + \sum_{k=1}^{\infty} a_k x^{2k}, \quad x \in (0, \pi/2),$$

where a_k ($k \in \mathbb{N}$) is given through (12). Let $c := 84/(15\pi^2)$. Then, we have

$$a_k \leq \pi^2\left(\frac{2^{2k+2}(2^{2k+2}-2)}{(2k+2)!}|B_{2k+2}| - c\frac{2^{2k}(2^{2k}-2)}{(2k)!}|B_{2k}|\right) =: l(k), \quad k \in \mathbb{N}.$$

Then $l(k) < 0$ if and only if

$$\frac{|B_{2k+2}|}{|B_{2k}|} < c\frac{2^{2k}(2^{2k}-2)}{(2k)!}\frac{(2k+2)!}{2^{2k+2}(2^{2k+2}-2)}$$

$$= \frac{c}{4}\frac{(2k+1)(2k+2)(2^{2k-1}-1)}{(2^{2k+1}-1)}.$$

From Lemma 2, we have

$$\frac{|B_{2k+2}|}{|B_{2k}|} < \frac{(2^{2k}-1)}{(2^{2k+2}-1)}\frac{(2k+1)(2k+2)}{\pi^2}.$$

Keeping in mind the arguments used in the proof of Theorem 2, it remains to be shown that

$$4(2^{2k}-1)(2^{2k+1}-1) < c\pi^2(2^{2k-1}-1)(2^{2k+2}-1),\ k \in \mathbb{N}.$$

After making a substitution $x = 4^k$ $(x \geq 4)$, it suffices to show that

$$2\frac{(x-1)(2x-1)}{(x-2)(4x-1)} \leq \frac{c}{4}\pi^2, \quad x \geq 4.$$

The equality holds for $x = 4$, while the strict inequality holds for $x > 4$ because the function

$$y = 2\frac{(x-1)(2x-1)}{(x-2)(4x-1)}, \quad x \geq 4$$

is strictly decreasing (https://www.desmos.com/calculator), as it can be easily approved. ☐

Next, we will show how our results give some known and other inequalities for $x/\tan x$. First of all, we can see by Theorem 1 that the function

$$\phi_2(x) = \frac{\tan x - x}{x^2 \tan x}$$

is strictly increasing on $(0, \pi/2)$. Hence,

$$\phi_2(0^+) = \frac{1}{3} < \phi_2(x) = \frac{\tan x - x}{x^2 \tan x} < \phi_2(\pi/2^-) = \frac{4}{\pi^2},$$

which gives the inequality (4). Similarly, $\phi_1(x)$ is strictly increasing on $(0, \pi/2)$ and thus with limits at extremities we obtain

$$1 - \frac{2x}{\pi} < \frac{x}{\tan x} < 1; \quad x \in (0, \pi/2). \tag{13}$$

This gives the left inequality of (5). Looking at the strictly decreasing function $\phi_3(x)$ on $(0, \pi/2)$ and the limit $\phi_3(\pi/2^-) = 8/\pi^3$, we get the right inequality of (3). Indeed, this inequality can be sharpened by considering $\phi_p(x)$ for $p = \pi^2/4$. Since $(\phi_p(x))_{p=\pi^2/4}$ is strictly decreasing on $(0, \pi/2)$, we obtain

$$(\phi_p(x))_{p=\pi^2/4} > \left(\phi_p(\pi/2^-)\right)_{p=\pi^2/4},$$

i.e.,

$$\frac{x}{\tan x} < 1 - \left(\frac{2x}{\pi}\right)^{\pi^2/4}; \quad x \in (0, \pi/2). \tag{14}$$

The inequality (14) is better than the right inequality of (2) near the point $x = \pi/2^-$. However, there is no strict comparison between the two.

Now it is easy to formulate the following

Corollary 1 *The exponents* 2 *and* $\pi^2/4$ *such that*

$$1 - \left(\frac{2x}{\pi}\right)^2 < \frac{x}{\tan x} < 1 - \left(\frac{2x}{\pi}\right)^{\pi^2/4}; \quad x \in (0, \pi/2) \tag{15}$$

are optimal.

Proof. Let

$$g(x) = \frac{\ln\left(1 - \frac{x}{\tan x}\right)}{\ln\left(\frac{2x}{\pi}\right)} = \frac{g_1(x)}{g_2(x)}.$$

Here $g_1(x)$ and $g_2(x)$ are such that $g_1(\pi/2^-) = 0 = g_2(\pi/2)$. Then

$$\frac{g_1'(x)}{g_2'(x)} = \frac{x^2 \sec^2 x - x \tan x}{\tan x \cdot (\tan x - x)} = f(x),$$

which is strictly increasing on $(0, \pi/2)$ as discussed in the proof of Theorem 1. Calculating the limits at extremities, we obtain the required. □

Several other inequalities can be established by using Theorem 1. We also have the following corollaries of Theorem 2.

Corollary 2 *If* $p \in (0, 4]$ *and* $x \in (0, \lambda)$, *where* $\lambda \in (0, \pi/2]$, *then the inequalities*

$$\left(1 - p\frac{x^2}{\pi^2}\right)^{\alpha} < \frac{x}{\tan x} < \left(1 - p\frac{x^2}{\pi^2}\right)^{\beta} \tag{16}$$

hold with the best possible constants $\alpha = \psi_p(\lambda^-)$ *and* $\beta = \pi^2/3p$.

Proof. From Theorem 2, $\psi_p(x)$ is strictly increasing on $(0,\lambda)$ for $p \in (0,4]$. So,

$$\psi_p(0^+) < \psi_p(x) < \psi(\lambda^-).$$

Since $\psi_p(0^+) = \pi^2/3p$, we get (16). □

Remark 2 *The inequality* (2) *can be deduced from Corollary 2, with* $p = 4$ *and* $\lambda = \pi/2$.

Corollary 3 *If* $a > 0$, *then the following inequality holds:*

$$\frac{x}{\tan x} < \left(\frac{\pi^2}{\pi^2 + ax^2}\right)^{\pi^2/3a}; \quad x \in (0, \pi/2). \tag{17}$$

Remark 3 *Graphically it is observed that the inequality* (17) *is in fact true for* $x \in (0, \pi)$.

We can use Theorem 3 to prove the following important corollary:

Corollary 4 *If* $x \in (0, \pi/\sqrt{p})$, *where* $p \geq 84/15$, *then the following inequality holds:*

$$\frac{x}{\tan x} \geq \left(1 - p\frac{x^2}{\pi^2}\right)^{\pi^2/3p}. \tag{18}$$

Furthermore, $\alpha = \pi^2/3p$ *is the optimal value for which* (18) *holds with a number* $p \geq 84/15$ *given in advance.*

Albeit not used henceforward, we will state and prove the following result:

Proposition 1 *Suppose that* $0 < p_1 < p_2$ *and* $x \in (0, \pi/\sqrt{p_2})$. *Then, we have*

$$\left(1 - p_2\frac{x^2}{\pi^2}\right)^{\pi^2/3p_2} < \left(1 - p_1\frac{x^2}{\pi^2}\right)^{\pi^2/3p_1}. \tag{19}$$

Proof. Let $0 < a < 1$. Then, the mapping $t \mapsto \ln(1 - at) - a\ln(1 - t)$, $0 \leq t < 1$ is strictly increasing because its first derivative is given by

$$t \mapsto a(1 - a)t(1 - t)^{-1}(1 - at)^{-1}, \quad t \in [0, 1).$$

Therefore, we have

$$\ln(1 - at) > a\ln(1 - t), \quad 0 < a < 1,\ 0 < t < 1. \tag{20}$$

Applying (20) with $a = p_1/p_2$ and $t = p_2x^2/\pi^2$, we get

$$\frac{\ln(1 - p_2x^2/\pi^2)}{p_2} < \frac{\ln(1 - p_1x^2/\pi^2)}{p_1}.$$

Multiplying both sides of the above inequality with $\pi^2/3$ and taking the exponents, we immediately get (19). □

Suppose now that $4 < p < 84/15$. We want to better explore the inequality (18) and the right part of the inequality (2) in this intermediate case. First of all, it is clear that there exists a sufficiently small real number $\epsilon_p > 0$ such that

$$\frac{x}{\tan x} > \left(1 - p\frac{x^2}{\pi^2}\right)^{\pi^2/3p}, \quad x \in \left((\pi/\sqrt{p}) - \epsilon_p, \pi/\sqrt{p}\right). \tag{21}$$

Set now

$$A := \left\{p > 4\,; \frac{x}{\tan x} > \left(1 - p\frac{x^2}{\pi^2}\right)^{\pi^2/3p} \text{ for all } x \in \left(0, \pi/\sqrt{p}\right)\right\}.$$

By Corollary 4, we have $[84/15, +\infty) \subseteq A$. On the other hand, Proposition yields that, if $p_0 > 4$ and $p_0 \notin A$, then $(4, p_0] \cap A = \emptyset$. Therefore, it is natural to ask: Can we calculate the set A intrinsically?

The answer is affirmative as the next result shows:

Theorem 4 *We have* $A = [7\pi^2/15, +\infty)$.

Proof. Define

$$h(x) := \ln\left(\frac{x}{\tan x}\right) - \frac{\pi^2}{3p}\ln\left(1 - p\frac{x^2}{\pi^2}\right), \quad x \in (0, \pi/\sqrt{p}).$$

Then $h(0+) = 0$ and

$$\begin{aligned} h'(x) &= \frac{3\pi^2 + x^2(2\pi^2 - 3p)}{3x(\pi^2 - px^2)} - \frac{2}{\sin(2x)} \\ &= \frac{[3\pi^2 + x^2(2\pi^2 - 3p)]\sin(2x) - 6x(\pi^2 - px^2)}{3x(\pi^2 - px^2)\sin(2x)}, \quad x \in (0, \pi/\sqrt{p}). \end{aligned}$$

Set $t := 2x \in (0, 2\pi/\sqrt{p})$ and

$$g(t) := \frac{\sin t}{t} - \frac{12\pi^2 - 3pt^2}{12\pi^2 + t^2(2\pi^2 - 3p)}, \quad t \in (0, 2\pi/\sqrt{p}).$$

Then, it can be easily seen that $h'(x) > 0$ if and only if $g(t) > 0$ if and only if $q(t) > 0$, where

$$q(t) := \sin(t) \cdot \Big[12\pi^2 + t^2(2\pi^2 - 3p)\Big] - t\Big[12\pi^2 - 3pt^2\Big], \quad t \in [0, 2\pi/\sqrt{p}).$$

Using https://www.symbolab.com/solver/partial-derivative-calculator, we get that $q^{(i)}(0) = 0$ for $i = 0, 1, 2, 3, 4$ as well as that

$$q^{(v)}(t) = t^2 \cos t \cdot (2\pi^2 - 3p) + 10t \sin t \cdot (2\pi^2 - 3p) + \cos t \cdot (60p - 28\pi^2),$$

for any $t \in [0, 2\pi/\sqrt{p})$. Since $2\pi^2 - 3p > 0$ for $p < 84/15$, we have that the assumption $p \geq 7\pi^2/15$ implies $q^{(v)}(0) \geq 0$ and $q^{(v)}(t) > 0$ for all $t \in (0, 2\pi/\sqrt{p})$. This simply implies $q(t) > 0$ for all $t \in (0, 2\pi/\sqrt{p})$ and therefore the function $h(x)$ is strictly increasing on $(0, \pi/\sqrt{p})$; therefore $h(x) > h(0+) = 0$ for all $x \in (0, \pi/\sqrt{p})$ and $[7\pi^2/15, +\infty) \subseteq A$. If $p < 7\pi^2/15$, then we have $q^{(i)}(0) = 0$ for $i = 0, 1, 2, 3, 4$ and $q^{(v)}(0) < 0$, so that $t = 0$ is a local maximum of function $q(t)$ (which can be extended to the even function defined on the whole real line) and therefore $q(t) < 0$ in a right neighborhood of point $t = 0$, which implies that $h'(x) < 0$ in a right neighborhood of point $x = 0$ and therefore $h(x) < 0$ in a right neighborhood of point $x = 0$; hence, $p \notin A$. Theorem 4 is proved. □

We now propose an alternative proof of Theorem 4 through the same baseline and the use of power series expansions.

Proof. [Alternative proof] Define

$$h(x) := \ln\Big(\frac{x}{\tan x}\Big) - \frac{\pi^2}{3p} \ln\Big(1 - p\frac{x^2}{\pi^2}\Big), \quad x \in (0, \pi/\sqrt{p}).$$

Then, by the power series expansion of the logarithmic function and (10), we have

$$h(x) = \frac{\pi^2}{3p} \sum_{k=1}^{\infty} \frac{1}{k} \frac{p^k}{\pi^{2k}} x^{2k} - \sum_{k=1}^{\infty} \frac{(2^{2k-1} - 1)2^{2k}}{k(2k)!} |B_{2k}| x^{2k}.$$

Since $|B_2| = 1/6$, after simplification, we get

$$h(x) = \sum_{k=2}^{\infty} c_{2k} x^{2k}, \tag{22}$$

where

$$c_{2k} := \frac{1}{k} \left[\frac{1}{3} \frac{p^{k-1}}{\pi^{2k-2}} - \frac{(2^{2k-1} - 1)2^{2k}}{(2k)!} |B_{2k}|\right].$$

Now, since $|B_4| = 1/30$, we have $c_4 = p/(6\pi^2) - 7/90$. So, if $p \geq 7\pi^2/15$, we obtain $c_4 \geq 0$. The rest of the proof consists in proving that $c_{2k} > 0$ for $k \geq 3$. It follows from Lemma 4 that, for any $k \in \mathbb{N}$,

$$|B_{2k}| < \frac{2^{2k-1}}{2^{2k-1}-1}\frac{2(2k)!}{(2\pi)^{2k}},$$

which implies that

$$\frac{(2^{2k-1}-1)2^{2k}}{(2k)!}|B_{2k}| < \frac{2^{4k}}{(2\pi)^{2k}} = \left(\frac{4}{\pi^2}\right)^k.$$

Therefore, if $p \geq 7\pi^2/15$, the following inequality holds:

$$c_{2k} > \frac{1}{k}\left[\frac{1}{3}\left(\frac{7}{15}\right)^{k-1} - \left(\frac{4}{\pi^2}\right)^k\right].$$

Now, remark that the inequality $(1/3)\,(7/15)^{k-1} > \left(4/\pi^2\right)^k$ is equivalent to $21/15 < (7\pi^2/60)^k$, which is true for $k \geq 3$ since $21/15 = 1.4$, $7\pi^2/60 \approx 1.151454 > 1$ and $(7\pi^2/60)^3 \approx 1.52665$. Thus, for $k \geq 3$, we have $c_{2k} > 0$. Now, if $p < 7\pi^2/15$, we have $c_4 < 0$. Owing to the expansion (22) and [15], there exists a $\delta > 0$ such that $h(x) < 0$ for $x \in (0, \delta)$. This ends the proof of Theorem 4. □

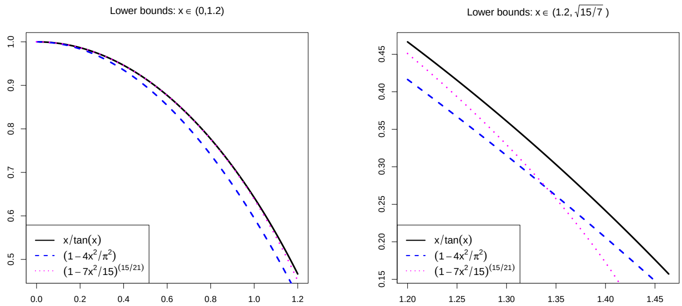
Now, Corollary 4 holds with $p \geq 7\pi^2/15$. For $p = 7\pi^2/15$, from Corollary 4, we get

$$\left(1 - \frac{7x^2}{15}\right)^{15/21} < \frac{x}{\tan x}; \quad x \in (0, \delta), \tag{23}$$

where $\delta = \sqrt{15/7} \approx 1.46385\cdots$.

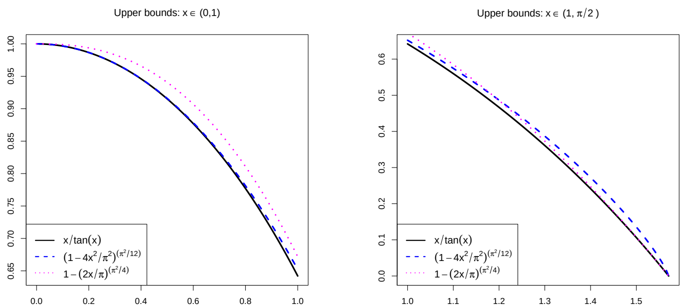
Now, let us compare graphically the bounds of $x/\tan x$ given in (2) with those obtained in (14) and in (23) in Figures 1 and 2, respectively. In each case, we distinguish two non-overlapping intervals of values for x to show some hierarchy for these bounds.

Based on Figure 1 and a numerical analysis, we see that, for $x \in (0, \delta_*)$ where $\delta_* \approx 1.3407$, the lower bound in (23) is stronger than the lower bound in (2). It is weaker for $x \in (\delta_*, \delta)$, where $\delta \approx 1.4638$. Also, based on Figure 2 and a numerical analysis, for $x \in (0, \mu)$, where $\mu \approx 1.1913$, the upper bound in (2) is stronger than the upper bound in (14). It is weaker for $x \in (\mu, \pi/2)$.



(a) (b)

Figure 1: Graphs of lower bounds of $x/\tan x$ in (2) and (23) for (a) $x \in (0, 1.2)$ and (b) $x \in (1.2, \sqrt{15/7})$.



(a) (b)

Figure 2: Graphs of upper bounds of $x/\tan x$ in (2) and (14) for (a) $x \in (0, 1)$ and (b) $x \in (1, \pi/2)$.

We conclude the paper by posing an open problem as follows:

Open Problem. Suppose that $p,\ \zeta > 0$. Then, determine the best possible constants $\alpha_{p,\zeta},\ \beta_{p,\zeta} \in \mathbb{R}$ such that the inequality

$$\left(1 - \frac{px^{\zeta}}{\pi^{\zeta}}\right)^{\beta_{p,\zeta}} < \frac{x}{\tan x} < \left(1 - \frac{px^{\zeta}}{\pi^{\zeta}}\right)^{\alpha_{p,\zeta}} ; \quad x \in (0, \pi/2) \cap (0, \pi/p^{1/\zeta})$$

holds.

Declarations

The authors declare no conflict of interest. The authors declare no funding. All the authors contributed equally to the findings.

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References

[1] H. Alzer, Sharp bounds for the Bernoulli numbers, *Archiv der Mathematik*, Vol. **74**, No. 3 (2000), 207–211.

[2] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal maps*, John Wiley and Sons, New York, 1997.

[3] Y. J. Bagul, On simple Jordan type inequalities, *Turkish J. Ineq.*, Vol. **3**, No. 1 (2019), 1–6.

[4] Y. J. Bagul, and C. Chesneau, New sharp bounds for tangent function, *Bulletin of the Allahabad Mathematical Society*, Vol. **34**, No. 2 (2019), 277–282.

[5] B. Banjac, M. Makragić and B. Malešević, Some notes on a method for proving inequalities by computer, *Results in Mathematics*, Vol. **69**, No. 1 (2016), 161–176.

[6] M. Becker, and E. L. Stark, On a hierarchy of quolynomial inequalities for $\tan x$, *Univ., Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. 602/633 (1978), 133–138.

[7] J. Cheeger, M. Gromov, M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, *J. Differential Geom.*, Vol. **17**, No. 1 (1982), 15–53.

[8] C.-P. Chen, and W.-S.Cheung, Sharp Cusa and Becker-Stark inequalities, *J. Inequal. Appl.*, 2011:136 (2011), Doi: https://doi.org/10.1186/1029-242X-2011-136.

[9] C.-P. Chen, and N. Elezović, Sharp Redheffer-type and Becker-Stark-type inequalities with an application, *Mathematical Inequalities and Applications*, Vol. **21**, No. 4 (2018), 1059–1078.

[10] C.-P. Chen, and J. Sándor, Sharp inequalities for trigonometric and hyperbolic functions, *Journal of Mathematical Inequalities*, Vol. **9**, No. 1 (2015), 203–217.

[11] L. Debnath, C. Mortici and L. Zhu, Refinements of Jordan-Stečkin and Becker-Stark inequalities, *Results. Math.* Vol. **67** (2015), 207–215.

[12] H.-F. Ge, New sharp bounds for the Bernoulli numbers and refinement of Becker-Stark inequalities, *J. Inequal. Appl.*, (2012), Article ID 137507, 7 pages.

[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products,* Elsevier, Seventh Edition, 2007.

[14] V. Heikkala, M. K. Vamanamurthy and M. Vuorinen, Generalized elliptic integrals, *Comput. Methods Funct. Theory*, Vol. **9**, No. 1 (2009), 75–109.

[15] B. Malešević and M. Makragić, Method for Proving Some Inequalities on Mixed Trigonometric Polynomial Functions, *J. Math. Inequal.*, Vol. **10**, No. 3 (2016), 849–876.

[16] Y. Nishizawa, Sharp Becker-Stark's type inequalities with power exponential functions, *J. Inequal. Appl.*, 2015, 402.

[17] F. Qi, A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers, *Journal of Computational and Applied Mathematics*, Vol. **351**, 1 May (2019), 1–5.

[18] S. B. Stečkin, Some remarks on trigonometric polynomials, *Uspekhi Matematicheskikh Nauk*, Vol. **10**, No. 1 (63) (1955), 159–166 (in Russian).

[19] Z.-H. Yang, Y.-L. Jiang, Y.-Q. Song and Y.-M. Chu, Sharp inequalities for trigonometric functions, *Abstract and Applied Analysis*, Vol. 2014, Article ID 601839 (2014), 18 pages.

[20] L. Zhu, Simple proofs of the Cusa-Huygens-type and Becker-Stark-type inequalities, *Journal of Mathematical Inequalities*, Vol. **7**, No. 4 (2013), 563–567.

[21] L. Zhu, and J.-K. Hua, Sharpening the Becker-Stark inequalities, *J. Inequal. Appl.*, Article ID 931275 (2010).

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Positive solution for singular third-order BVPs on the half line with first-order derivative dependence

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Abstract. In this paper, we investigate the existence of a positive solution to the third-order boundary value problem

$$\begin{cases} -u'''(t) + k^2 u'(t) = \phi(t) f(t, u(t), u'(t)), \; t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases}$$

where k is a positive constant, $\phi \in L^1(0, +\infty)$ is nonnegative and does vanish identically on $(0, +\infty)$ and the function $f : \mathbb{R}^+ \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}^+$ is continuous and may be singular at the space variable and at its derivative.

1 Introduction and main results

Boundary value problems for third-order differential equations arise in many branches of physics and engineering where, for physical considerations, the positivity of the solution is required. For instance, Danziger and Elemergreen (see [15], p. 133) have obtained the following third-order linear differential equations:

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$$\begin{aligned} &\alpha_3 u''' + \alpha_2 u'' + \alpha_1 u' + (1+k)\, u = kc, \ \ \theta < c \text{ and} \\ &\alpha_3 u''' + \alpha_2 u'' + \alpha_1 u' + u = 0, \ \ \theta > c. \end{aligned} \tag{1}$$

These equations describe the variation of thyroid hormone with time. Here $u = u(t)$ is the concentration of thyroid hormone at time t and $\alpha_3, \alpha_2, \alpha_2, k$ and c are constants.

A reduced version of the Hodgkin–Huxley model was proposed by Nagumo. He suggested the class of third-order differential equation

$$u''' - cu'' + f'(u)u' - \frac{b}{c}u = 0 \tag{2}$$

as a model exhibiting many of the features of the Hodgkin–Huxley equations, where f is a regular function. The Hodgkin–Huxley model is a system of nonlinear differential equations that approximates the electrical characteristics of excitable cells such as neurons and cardiac myocytes. Recall that the Hodgkin–Huxley model describes the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axom. The model has played a vital role in biophysics and neuronal modelling. For more details of Nagumo's equations, we refer to the paper by McKeen [22].

The Kuramoto–Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u^2 = 0$$

arises in a wide variety of physical phenomena. It was introduced to describe pattern formulation in reaction diffusion systems, and to model the instability of flame front propagation (see Y. Kuramoto and T. Yamada [18] and D. Michelson [23]). The travelling wave solutions of this partial differential equation (i.e. $u(x,t) = u(x - ct)$) solve the nonlinear third-order differential equation

$$\lambda u'''(x) + u'(x) + f(u) = 0, \tag{3}$$

where λ is a parameter depend on the constant c and f is an even function.

A three-layer beam is formed by parallel layers of different materials. For an equally loaded beam of this type, Krajcinovic in [17] proved that the deflection u is governed by the third order differential equation

$$-u''' + k^2 u' = a, \tag{4}$$

where k and a are physical parameters depending on the elasticity of the layers.

Study of existence of positive solutions for third-order bvps has received a great deal of attention and was the subject of many articles, see, for instance, [10, 11, 12, 13, 14, 21, 25, 27, 28, 29, 30, 31], for the case of finite intervals and [1, 2, 3, 4, 6, 7, 8, 9, 16, 19, 20, 24, 26] for the case posed on the half-line. Naturally, in such boundary value problems, the nonlinearity may have a singular dependence on time or on the space variable. This was the case in the papers [3, 6, 7, 8, 20, 21, 27, 28, 29], which motivated this work.

We are concerned in this paper by existence of a positive solution to the boundary value problem (bvp for short),

$$\begin{cases} -u'''(t) + k^2u'(t) = \phi(t) f(t, u(t), u'(t)), \ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{5}$$

where k is a positive constant, $\phi : (0, +\infty) \to \mathbb{R}^+$ is a measurable function, $f : \mathbb{R}^+ \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}^+$ is a continuous function and observe that the form of the differential equation in (5) is more general to those of (1)-(4). Here the constant k which may have a physical signification as in (4), will play an important role in finding a suitable framework for a fixed point formulation of bvp (5).

By positive solution to the bvp (5), we mean a function $u \in C^2(\mathbb{R}^+) \cap W^{3,1}(0, +\infty)$ such that $u > 0$ in $(0, +\infty)$ and $u(0) = u'(0) = \lim_{t\to+\infty} u'(t) = 0$, satisfying the differential equation in (5).

In all this paper, we let

$$\begin{aligned} &\gamma_1(t) = (e^{2kt} - 1)e^{-4kt}, \\ &\widetilde{\gamma}(t) = k^* e^{kt}\gamma_1(t) = k^*\left(1 - e^{-kt}\right)(1 + e^{-kt})e^{-kt}, \\ &\gamma(t) = \textstyle\int_0^t \widetilde{\gamma}(s)ds = \dfrac{k^*}{3k}\left(2 - 3e^{-kt} + e^{-3kt}\right) = \dfrac{k^*}{3k}\left(1 - e^{-kt}\right)^2(2 + e^{-kt}) \end{aligned}$$

where $k^* = \min(1, k)/2$ and we assume that the functions ϕ and f satisfy the following condition:

$$\begin{cases} \text{for all } R > 0 \text{ there exists a function } \Psi_R : (0, +\infty) \times (0, +\infty) \to (0, +\infty) \\ \text{such that } \Psi_R \text{ nonincreasing following its two variables,} \\ f(t, e^{kt}w, e^{kt}z) \le \Psi_R(w, z) \ \text{ for all } t, w, z \ge 0 \text{ with } |(w, z)| \le R, \\ \lim_{s\to+\infty} \phi(s)\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) = 0 \text{ and} \\ \int_0^{+\infty} \phi(s)\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) ds < \infty \text{ for all } r \in (0, R]. \end{cases} \tag{6}$$

Remark 1 *Notice that functions* m *in* $L^1(0,+\infty)$ *do not satisfy* $\lim_{t\to+\infty} m(t)=0$. *Indeed, the function*

$$m_0(t)=\begin{cases} 2n^4t-n(2n^4-1) & \text{if} \quad t\in\left[n-\frac{1}{2n^3},n\right] \\ -2n^4t+n(2n^4+1) & \text{if} \quad t\in\left[n,n+\frac{1}{2n^3}\right] \\ 0 & \text{if not} \end{cases}$$

is integrable since $\int_0^{+\infty} m_0(t)dt \le \sum_{n\ge1}\frac{1}{n^2} < \infty$, *and* $\lim_{n\to+\infty} m_0(n) = \lim_{n\to+\infty} n = +\infty$.

Hence, the condition $\int_0^{+\infty}\phi(s)\Psi_R\left(re^{-ks}\gamma(s),re^{-ks}\widetilde{\gamma}(s)\right)ds<\infty$ *in Hypothesis* (6) *does not imply that* $\lim_{s\to+\infty}\phi(s)\Psi_R\left(re^{-ks}\gamma(s),re^{-ks}\widetilde{\gamma}(s)\right)=0$.

Remark 2 *Observe that the case where the nonlinearity* f *satisfies the polynomial growth condition*

$$f(t,u,v)\le C\left(1+u^{\sigma}+v^{\mu}\right)$$

with $c,\sigma,\mu>0$, $\lim_{s\to+\infty}\phi(s)=0$ *and* $\int_0^{+\infty}\phi(s)ds<\infty$, *is a particular case where Condition* (6) *is satisfied.*

Remark 3 *Notice that if Hypothesis* (6) *holds then* $|\phi|_1=\int_0^{+\infty}\phi(s)ds<\infty$. *Indeed, for* $R=1$ *we have*

$$\infty>\int_0^{+\infty}\phi(s)\Psi_1\left(e^{-ks}\gamma(s),e^{-ks}\widetilde{\gamma}(s)\right)ds\ge\Psi_1\left(\gamma^+,\gamma^+\right)|\phi|_1,$$

where $\gamma^+=\max_{s>0}\left(e^{-ks}\left(\gamma(s)+\widetilde{\gamma}(s)\right)\right)$.

The statement of the main result needs to introduce the following notations. Let

$$f^0=\limsup_{|(w,z)|\to0}\left(\sup_{t\ge0}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),$$

$$f^\infty=\limsup_{|(w,z)|\to+\infty}\left(\sup_{t\ge0}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),$$

$$f_0(\theta)=\liminf_{|(w,z)|\to0}\left(\min_{t\in I_\theta}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),$$

$$f_\infty(\theta)=\liminf_{|(w,z)|\to+\infty}\left(\min_{t\in J_\theta}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),$$

where $|(w,z)|=|w|+|z|$, for $\theta>0$ $I_\theta=[0,\theta]$ and for $\theta>1$ $J_\theta=[1/\theta,\theta]$.

Let also,

$$\Gamma = (\Gamma_1 + \Gamma_2)^{-1},$$
$$\Theta_0(\theta) = (\Theta_{1,0}(\theta) + \Theta_{2,0}(\theta))^{-1} \text{ if } \theta > 0,$$
$$\Theta_\infty(\theta) = (\Theta_{1,\infty}(\theta) + \Theta_{2,\infty}(\theta))^{-1} \text{ if } \theta > 1,$$

where

$$\Gamma_1 = \sup_{t>0}\left(e^{-kt}\int_0^{+\infty} G(t,s)\phi(s)ds\right),$$
$$\Gamma_2 = \sup_{t>0}\left(e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\phi(s)ds\right),$$
$$\Theta_{1,0}(\theta) = \sup_{t>0}\left(e^{-kt}\int_0^{\theta} G(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),$$
$$\Theta_{2,0}(\theta) = \sup_{t>0}\left(e^{-kt}\int_0^{\theta} \widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),$$
$$\Theta_{1,\infty}(\theta) = \sup_{t>0}\left(e^{-kt}\int_{1/\theta}^{\theta} G(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),$$
$$\Theta_{2,\infty}(\theta) = \sup_{t>0}\left(e^{-kt}\int_{1/\theta}^{\theta} \widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),$$

and notice that Remark 3 guarantees that the constants Γ_1 and Γ_2 are finite.

Theorem 1 *Assume that Hypothesis (6) holds and one of the following conditions*

$$f^0 < \Gamma, \quad \Theta_\infty(\theta) < f_\infty(\theta) \text{ for some } \theta > 1 \tag{7}$$

$$f^\infty < \Gamma, \quad \Theta_0(\theta) < f_0(\theta) \text{ for some } \theta > 0 \tag{8}$$

is satisfied. Then the bvp (5) admits at least one positive solution.

Remark 4 *For the particular case where* $f(t,u,v) = \left(e^{-kt}(u+v)\right)^\sigma$ *with* $\sigma > 0$ *and* $\sigma \neq 1$*, we have* $f^0 = 0$ *and* $f_\infty(\theta) = +\infty$ *for all* $\theta > 0$ *if* $\sigma > 1$*, and* $f^\infty = 0$ *and* $f_0(\theta) = +\infty$ *for all* $\theta > 0$ *if* $\sigma < 1$*. Hence, Conditions (7) and (8) in Theorem 1 correspond to the superlinear case and the sublinear case of the nonlinearity* f*, respectively.*

2 Example

Consider the case of the bvp (5) where $\phi(t) = e^{-\alpha t}$, $\alpha > 0$ and

$$f(t, u, v) = A\left(\frac{u+v}{e^{kt}+u+v}\right)^p + B\left(\frac{u+v}{e^{kt}}\right)^q,$$

with $A, B > 0$, $p \leq 1$ and $q \geq 1$.

Thus, for all $t, w, z > 0$ we have

$$f(t, e^{kt}w, e^{kt}z) = A\left(\frac{w+z}{1+w+z}\right)^p + B\,(w+z)^q,$$

and if $|(w, z)| = w + z < R$, then

$$f(t, e^{kt}w, e^{kt}z) = A\left(\frac{w+z}{1+w+z}\right)^p + B\,(w+z)^q \leq \Psi_R\,(w, z)\,,$$

where

$$\Psi_R\,(w, z) = \begin{cases} AR^p + BR^q \text{ if } p \geq 0, \\ A\,(w+z)^p\,(1+R)^{-p} + BR^q \text{ if } p < 0. \end{cases}$$

Thus, if $p \geq 0$ then

$$\lim_{s\to+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma\,(s)\,, Re^{-ks}\widetilde{\gamma}\,(s)\right) = (AR^p + BR^q) \lim_{s\to+\infty} e^{-\alpha s} = 0,$$

$$\int_0^{+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma\,(s)\,, Re^{-ks}\widetilde{\gamma}\,(s)\right) ds = \frac{AR^p + BR^q}{\alpha} < \infty,$$

and if $p < 0$ then

$$\begin{aligned}\phi(s)\psi_R\left(Re^{-ks}\gamma\,(s)\,, Re^{-ks}\widetilde{\gamma}\,(s)\right) = BR^q e^{-\alpha s} + \\ A\,(1+R)^{-p}\,(k^*R)^p\, e^{-(\alpha+pk)s}\left(1-e^{-ks}\right)^p \rho(s),\end{aligned}$$

where

$$\rho(s) = \left(\frac{1}{3k}\left(1-e^{-ks}\right)\left(2+e^{-ks}\right) + e^{-ks}\left(1+e^{-ks}\right)\right)^p$$

satisfies

$$\left(\max\left(2, \frac{2}{3k}\right)\right)^p \leq \rho(s) \leq \left(\min\left(2, \frac{2}{3k}\right)\right)^p.$$

Therefore, we have

$$\lim_{s\to+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma\,(s)\,, Re^{-ks}\widetilde{\gamma}\,(s)\right) = 0 \text{ if and only if } \alpha > -pk$$

and

$$\int_0^{+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) ds < \infty \text{ if and only if } \alpha > -pk \text{ and } p > -1.$$

Straightforward computations lead to

$$f^{\infty} = f_{\infty}(\theta) = f_{\infty} = \begin{cases} +\infty & \text{if} \quad q > 1, \\ B & \text{if} \quad q = 1, \end{cases} \qquad \text{for all } \theta > 1$$

$$f^{0} = f_{0}(\theta) = f_{0} = \begin{cases} +\infty & \text{if} \quad p < 1, \\ A & \text{if} \quad p = 1 < q, \\ A + B & \text{if} \quad p = q = 1, \end{cases} \qquad \text{for all } \theta > 0.$$

We conclude from Theorem 1 and all the above calculations that this case of the bvp (5) admits a positive solution in each of the following situations:

1. $p = 1,\ q = 1,\ B < \Gamma$ and $A + B > \Theta_0(\theta)$ for some $\theta > 0$,
2. $p = 1,\ q > 1,$ and $A > \Theta_0(\theta)$ for some $\theta > 0$,
3. $p \in [0, 1),\ q = 1$ and $B < \Gamma$,
4. $p \in (-1, 0),\ q = 1,\ B < \Gamma$ and $\alpha > -pk$.

3 Abstract background

Let $(E, \|.\|)$ be a real Banach space. A nonempty closed convex subset C of E is said to be a cone in E if $C \cap (-C) = \{0_E\}$ and $tC \subset C$ for all $t \geq 0$.

Let Ω be a nonempty subset in E. A mapping $A : \Omega \to E$ is said to be compact if it is continuous and $A(\Omega)$ is relatively compact in E.

The main tool of this work is the following Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

Theorem 2 *Let P be a cone in E and let Ω_1, Ω_2 be bounded open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. If $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a compact mapping such that either*

1. $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*
2. $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$.

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

4 Fixed point formulation

In all this paper, we let

$$E = \{u \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} e^{-kt}u(t) = 0,\ \lim_{t\to+\infty} e^{-kt}u'(t) = 0\}.$$

Endowed with the norm $\|u\| = \|u\|_k + \|u'\|_k$ where $\|u\|_k = \sup_{t\geq 0}\left(e^{-kt}|u(t)|\right)$, E becomes a Banach space.

The following lemma is an adapted version for the case of the space E of Corduneanu's compactness criterion ([5], p. 62). It will be used in this work to prove that some operator is completely continuous.

Lemma 1 *A nonempty subset* M *of* E *is relatively compact if the following conditions hold:*

(a) M *is bounded in* E,

(b) *the sets* $\{u : u(t) = e^{-kt}x(t),\ x \in M\}$ *and* $\{u : u(t) = e^{-kt}x'(t),\ x \in M\}$ *are locally equicontinuous on* $[0, +\infty)$, *and*

(c) *the sets* $\left\{u : u(t) = e^{-kt}x(t),\ x \in M\right\}$ *and* $\left\{u : u(t) = e^{-kt}x'(t),\ x \in M\right\}$ *are equiconvergent at* $+\infty$.

In all this work, P denotes the cone in E defined by

$$P = \left\{u \in E :\ u'(t) \geq \widetilde{\gamma}(t)\|u\| \text{ and } u(t) \geq \gamma(t)\|u\| \text{ for all } t > 0\right\}.$$

Let $G, \widetilde{G} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be the functions defined by

$$G(t,s) = \frac{1}{k^2}\begin{cases} e^{-ks}(\cosh(kt) - 1) & \text{if} \quad t \leq s, \\ -e^{-kt}\sinh(ks) + \left(1 - e^{-ks}\right) & \text{if} \quad s \leq t, \end{cases}$$

$$\widetilde{G}(t,s) = \frac{\partial G}{\partial t}(t,s) = \frac{1}{k}\begin{cases} e^{-ks}\sinh(kt) & \text{if} \quad t \leq s, \\ e^{-kt}\sinh(ks) & \text{if} \quad s \leq t. \end{cases}$$

Lemma 2 *The functions* G *and* $\widetilde{G}$ *satisfy:*

(a) *For all* $t, s \in \mathbb{R}^+$ *we have* $G(t,s) \geq 0$ *and* $\widetilde{G}(t,s) \geq 0$.

(b) *The functions* G *and* $\widetilde{G}$ *are continuous and for all* $s \geq 0$, *we have*

$$G(0,s) = \widetilde{G}(0,s) = 0. \tag{9}$$

(c) *For all* $t, s \geq 0$, *we have*

$$G(t,s) \leq \frac{1}{k^2}(1-e^{-ks}) \leq \frac{1}{k^2}, \quad \widetilde{G}(t,s) \leq \widetilde{G}(s,s) \leq \frac{1}{2k}.$$

(d) *For all* $s, t, \tau \geq 0$, *we have*

$$\widetilde{G}(t,s)e^{-kt} \geq \gamma_1(t)\widetilde{G}(\tau,s)e^{-k\tau}.$$

(e) *For all* $t_2, t_1 \geq 0$, *we have*

$$\left|e^{-kt_2}G(t_2,s) - e^{-kt_1}G(t_1,s)\right| \leq \frac{3}{2k}|t_2 - t_1| \tag{10}$$

$$\left|e^{-kt_2}\widetilde{G}(t_2,s) - e^{-kt_1}\widetilde{G}(t_1,s)\right| \leq |t_2 - t_1| \tag{11}$$

Proof. Assertions (a), (b) and (c) are easy to prove, Assertion (d) is proved in [8]. Assertion (e) is obtained by the mean value theorem. □

Lemma 3 *Assume that Hypothesis* (6) *holds, then there exists a continuous operator* $T : P \setminus \{0\} \to P$ *such that for all* r, R *with* $0 < r < R$, $T(P \cap (\overline{B}(0,R) \setminus B(0,r)))$ *is relatively compact and fixed points of* T *are positive solutions to the bvp* (5).

Proof. The proof is divided into four steps.

Step 1. In this step we prove the existence of the operator T. To this aim let $u \in P \setminus \{0\}$. By means of Hypothesis (6) with $R = \|u\|$, for all $t > 0$ we have

$$\begin{aligned}\int_0^{+\infty} &G(t,s)\phi(s)\, f(s,u(s),u'(s))ds \\ &\leq \frac{1}{k^2}\int_0^{+\infty} \phi(s)\, f\left(s,u(s),u'(s)\right) ds \\ &= \frac{1}{k^2}\int_0^{+\infty} \phi(s)\, f\left(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right)\right) ds \\ &\leq \frac{1}{k^2}\int_0^{+\infty} \phi(s)\, \Psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) ds < \infty\end{aligned}$$

and

$$\int_0^{+\infty} \widetilde{G}(t,s)\phi(s)f\big(s,u(s),u'(s)\big)ds \leq \frac{1}{2k}\int_0^{+\infty} \phi(s)f\big(s,u(s),u'(s)\big)ds$$

$$\leq \frac{1}{2k}\int_0^{+\infty}\phi(s)\Psi_R\left(Re^{-ks}\gamma(s),Re^{-ks}\widetilde{\gamma}(s)\right)ds<\infty.$$

Thus, let v and w be the functions defined by

$$v(t)=\int_0^{+\infty}G(t,s)\phi\left(s\right)f(s,u(s),u'(s))ds$$
$$w(t)=\int_0^{+\infty}\widetilde{G}(t,s)\phi\left(s\right)f(s,u(s),u'(s))ds.$$

Since for all $t>0$,

$$\begin{aligned}v(t)=&-\frac{e^{-kt}}{k^2}\int_0^t\sinh(ks)\phi\left(s\right)f(s,u(s),u'(s))ds\\&+\frac{1}{k^2}\int_0^t(1-e^{-ks})\phi\left(s\right)f(s,u(s),u'(s))ds\\&+\frac{\cosh(kt)-1}{k^2}\int_0^t e^{-ks}\phi\left(s\right)f(s,u(s),u'(s))ds,\end{aligned}$$

we see that v is differentiable on $\mathbb{R}^+$ and for all $t\geq 0$,

$$\begin{aligned}v'(t)=&\frac{e^{-kt}}{k}\int_0^t\sinh(ks)\phi\left(s\right)f(s,u(s),u'(s))ds\\&+\frac{\sinh(kt)}{k}\int_t^{+\infty}e^{-ks}\phi\left(s\right)f(s,u(s),u'(s))ds\\=&\int_0^{+\infty}\widetilde{G}(t,s)\phi\left(s\right)f(s,u(s),u'(s))ds=w(t)\end{aligned}$$

with w continuous on $\mathbb{R}^+$.

At this stage we have proved that v belongs to $C^1(\mathbb{R}^+,\mathbb{R})$ and we need to prove that $v\in E$. Thus, we have to show that $\lim_{t\to+\infty}e^{-kt}v(t)=\lim_{t\to+\infty}e^{-kt}v'(t)=0$. Clearly for all $t>0$, $v(t),v'(t)>0$ and we have

$$\begin{aligned}e^{-kt}v(t)&=e^{-kt}\int_0^{+\infty}G(t,s)\phi\left(s\right)f\big(s,u(s),u'(s)\big)ds\\&\leq\frac{e^{-kt}}{k^2}\int_0^{+\infty}\phi\left(s\right)\Psi_R\left(Re^{-ks}\gamma(s),Re^{-ks}\widetilde{\gamma}(s)\right)ds\end{aligned}$$

and

$$e^{-kt}v'(t)=e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\phi\left(s\right)f\big(s,u(s),u'(s)\big)ds$$

$$\leq \frac{e^{-kt}}{2k} \int_0^{+\infty} \phi(s) \Psi_R \left(Re^{-ks} \gamma(s), Re^{-ks} \gamma(s) \right) ds.$$

The above two estimates prove that $\lim_{t \to +\infty} e^{-kt} v(t) = \lim_{t \to +\infty} e^{-kt} v'(t) = 0$.

Now for all $t, \tau > 0$, we have from Assertion (c) in Lemma 2

$$\begin{aligned} v'(t) &= e^{kt} \int_0^{+\infty} e^{-kt} \widetilde{G}(t,s) f\big(s, u(s), u'(s)\big) ds \\ &\geq e^{kt} \gamma_1(t) \int_0^{+\infty} e^{-k\tau} \widetilde{G}(\tau, s) f\big(s, u(s), u'(s)\big) ds \\ &= e^{kt} \gamma_1(t) e^{-k\tau} v'(\tau). \end{aligned}$$

Passing to the supremum on τ, we obtain

$$v'(t) \geq e^{kt} \gamma_1(t) \left\| v' \right\|_k \text{ for all } t > 0. \tag{12}$$

Since for all $t > 0$

$$v(t) = \int_0^t e^{k\xi} \left(e^{-k\xi} v'(\xi) \right) d\xi \leq \int_0^t e^{k\xi} d\xi \left\| v' \right\|_k \leq \frac{e^{kt}}{k} \left\| v' \right\|_k,$$

we have

$$\left\| v' \right\|_k \geq k \left\| v \right\|_k. \tag{13}$$

Therefore, (12) Combined with (13) leads to

$$v'(t) \geq k e^{kt} \gamma_1(t) \left\| v' \right\|_k \text{ for all } t > 0,$$

then to

$$v'(t) \geq \widetilde{\gamma}(t) \left\| v \right\| \text{ for all } t > 0. \tag{14}$$

Integrating (14), yields $v(t) \geq \gamma(t) \left\| v \right\|$ for all $t > 0$.

Thus, we have proved that $v \in P$ and the operator $T : P \setminus \{0\} \to P$ where for $u \in P \setminus \{0\}$

$$Tu(t) = \int_0^{+\infty} G(t,s) \phi(s) f\big(s, u(s), u'(s)\big) ds,$$

is well defined.

Step 2. In this step we prove that the operator T is continuous. Let (u_n) be a sequence in $P \setminus \{0\}$ such that $\lim_{n \to \infty} u_n = u_\infty$ in E with u_∞ in $P \setminus \{0\}$

and let $R > r > 0$ be such that $(u_n) \subset B(0,R) \setminus B(0,r)$. If Ψ_R is the function given by Hypothesis (6), then for all $n \geq 1$ we have

$$\|Tu_n - Tu_\infty\|_k = \sup_{t\geq 0} |Tu_n(t) - Tu_\infty(t)| \leq \frac{1}{k^2}\int_0^{+\infty} \phi(s)\left|f\big(s,u_n(s),u_n'(s)\big) - f\big(s,u_\infty(s),u_\infty'(s)\big)\right| ds$$

and

$$\|(Tu_n)' - (Tu_\infty)'\|_k = \sup_{t\geq 0} |(Tu_n)'(t) - (Tu_\infty)'(t)| \leq \frac{1}{2k}\int_0^{+\infty} \phi(s)\left|f\big(s,u_n(s),u_n'(s)\big) - f\big(s,u_\infty(s),u_\infty'(s)\big)\right| ds.$$

Because of

$$\left|f\big(s,u_n(s),u_n'(s)\big) - f\big(s,u_\infty(s),u_\infty'(s)\big)\right| \to 0, \text{ as } n \to +\infty$$

for all $s > 0$ and

$$\phi(s)\left|f\big(s,u_n(s),u_n'(s)\big) - f\big(s,u_\infty(s),u_\infty'(s)\big)\right| \leq 2\phi(s)\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\gamma(s)\right)$$

with $\int_0^{+\infty}\phi(s)\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\gamma(s)\right) ds < \infty$, the Lebesgue dominated convergence theorem guarantees that $\lim_{n\to\infty}\|Tu_n - Tu_\infty\| = 0$. Hence, we have proved that T is continuous.

Step 3. In this step, we prove that for $R > r > 0$, $T\left(P \cap \left(\overline{B}(0,R) \setminus B(0,r)\right)\right)$ is relatively compact. Set $\Omega = P \cap \left(\overline{B}(0,R) \setminus B(0,r)\right)$ and let $\Phi_{r,R}$ be defined by

$$\Phi_{r,R}(s) = \Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\gamma(s)\right)$$

where Ψ_R is the function given by Hypothesis (6). For all $u \in \Omega$, we have

$$\|Tu\| \leq \left(\frac{1}{k^2} + \frac{1}{2k}\right)\int_0^{+\infty}\phi(s)\,\Phi_{r,R}(s)ds < \infty,$$

proving that $T\Omega$ is bounded in E.

Now, let $t_1, t_2 \in [\eta, \xi]$, for all $u \in \Omega$, we have from (10) and (11) the estimates

$$|e^{-kt_1}Tu(t_1) - e^{-kt_2}Tu(t_2)| \leq \int_0^{+\infty} |e^{-kt_1}G(t_1,s) - e^{-kt_2}G(t_2,s)|\phi(s)\,\Phi_{r,R}(s)ds$$

$$\leq \frac{3}{2k}\left|t_2 - t_1\right| \int_0^{+\infty} \phi\left(s\right) \Phi_{r,R}(s) ds$$

and

$$\left|e^{-kt_1}(Tu)'(t_1) - e^{-kt_2}(Tu)'(t_2)\right| \leq \int_0^{+\infty} \left|e^{-kt_1}\widetilde{G}(t_1,s) - e^{-kt_2}\widetilde{G}(t_2,s)\right| \phi(s) \Phi_{r,R}(s) ds$$
$$\leq \left|t_2 - t_1\right| \int_0^{+\infty} \phi\left(s\right) \Phi_{r,R}(s) ds.$$

Proving the equicontinuity of $T\Omega$ on bounded intervals.

For all $u \in \Omega$ and $t > 0$, we have

$$\left|e^{-kt} Tu(t)\right| \leq \frac{e^{-kt}}{k^2} \int_0^{+\infty} \phi\left(s\right) \Phi_{r,R}(s) ds$$

and

$$\left|e^{-kt}\left(Tu\right)'(t)\right| \leq \frac{e^{-kt}}{k} \int_0^{+\infty} \phi\left(s\right) \Phi_{r,R}(s) ds.$$

Thus, the equiconvergence of $T\Omega$ follows from the fact that $\lim_{t\to\infty} e^{-kt} = 0$. In view of Lemma 1, $T\Omega$ is relatively compact in E.

Step 4. In this step we prove that fixed points of T are positive solutions to the bvp (5). Let $u \in P \setminus \{0\}$ be a fixed point of T, then for all $t > 0$ we have

$$u(t) = \int_0^{+\infty} G(t,s)\phi\left(s\right) f\big(s, u(s), u'(s)\big) ds \text{ and}$$
$$u'(t) = \int_0^{+\infty} \widetilde{G}(t,s)\phi\left(s\right) f\big(s, u(s), u'(s)\big) ds.$$

These with (9) lead to $u(0) = u'(0) = 0$.

Differentiating twice in

$$\begin{aligned} u'(t) &= \int_0^{+\infty} \widetilde{G}(t,s)\phi\left(s\right) f\big(s, u(s), u'(s)\big) ds \\ &= \frac{e^{-kt}}{k} \int_0^{t} \sinh(ks)\phi\left(s\right) f\big(s, u(s), u'(s)\big) ds \\ &\quad + \frac{\sinh(kt)}{k} \int_t^{+\infty} e^{-ks}\phi\left(s\right) f\big(s, u(s), u'(s)\big) ds, \end{aligned}$$

we see that $-u'''(t) + ku'(t) = \phi\left(t\right) f(t, u(t), u'(t))$ for all $t > 0$.

It remains to prove that $\lim_{t\to+\infty} u'(t) = 0$. We have

$$u'(t) = \frac{1}{ke^{kt}} \int_0^t \sinh(ks)\phi(s) f\big(s, u(s), u'(s)\big) ds + \frac{\sinh(kt)}{k} \int_t^{+\infty} e^{-ks}\phi(s) f\big(s, u(s), u'(s)\big) ds.$$

By means of Hypothesis (6) with $R = \|u\|$ and the L'Hopital's rule, we obtain

$$\begin{aligned} &\lim_{t\to+\infty} \frac{1}{ke^{kt}} \int_0^t \sinh(ks)\phi(s) f\big(s, u(s), u'(s)\big) ds \\ &\leq \lim_{t\to+\infty} \frac{1}{ke^{kt}} \int_0^t \sinh(ks)\phi(s) \Psi_R \left(Re^{-ks}\gamma(s), Re^{-ks}\gamma(s)\right) ds \\ &= \lim_{t\to+\infty} \frac{\sinh(kt)}{ke^{kt}} \phi(t) \Psi_R \left(Re^{-kt}\gamma(t), Re^{-kt}\gamma(t)\right) ds = 0. \end{aligned}$$

Also, we have

$$\begin{aligned} &\frac{\sinh(kt)}{k} \int_t^{+\infty} e^{-ks}\phi(s) f\big(s, u(s), u'(s)\big) ds \\ &\leq \frac{\sinh(kt)e^{-kt}}{k} \int_t^{+\infty} \phi(s) f\big(s, u(s), u'(s)\big) ds \to 0 \text{ as } t \to +\infty. \end{aligned}$$

The above calculations show that $\lim_{t\to+\infty} u'(t) = 0$, completing the proof of the lemma. □

5 Proof of Theorem 1

Step 1. Existence in the case where (7) holds

Let $\epsilon > 0$ be such that $(f^0 + \epsilon) < \Gamma$. For such a ϵ, there exists $R_1 > 0$ such that $f(t, e^{kt}w, e^{kt}z) \leq (f^0 + \epsilon)(w + z)$ for all w, z with $|(w, z)| \leq R_1$ and let $\Omega_1 = \{u \in E, \|u\| < R_1\}$.

Therefore, for all $u \in P \cap \partial\Omega_1$ and all $t > 0$, we have

$$\begin{aligned} e^{-kt}Tu(t) &= e^{-kt} \int_0^{+\infty} G(t, s)\phi(s) f\left(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right)\right) ds \\ &\leq \left(f^0 + \epsilon\right) e^{-kt} \int_0^{+\infty} G(t, s)\phi(s)e^{-ks}\left(u(s) + u'(s)\right) ds \\ &\leq \|u\| \left(f^0 + \epsilon\right) e^{-kt} \int_0^{+\infty} G(t, s)\phi(s) ds \leq \Gamma_1 \left(f^0 + \epsilon\right) \|u\|, \end{aligned}$$

leading to

$$\|Tu\|_k = \sup_{t>0}\left(e^{-kt}Tu(t)\right) \leq \left(f^0+\epsilon\right)\Gamma_1 \|u\|. \quad (15)$$

Similarly, we have

$$\begin{aligned} e^{-kt}(Tu)'(t) &= e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)f\Big(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right)\Big)ds \\ &\leq \left(f^0+\epsilon\right)e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)e^{-ks}\left(u(s)+u'(s)\right)ds \\ &\leq \|u\|\left(f^0+\epsilon\right)e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)ds \\ &\leq \left(f^0+\epsilon\right)\Gamma_2\|u\|, \end{aligned}$$

leading to

$$\left\|(Tu)'\right\|_k = \sup_{t>0}\left(e^{-kt}(Tu)'(t)\right) \leq \left(f^0+\epsilon\right)\Gamma_2\|u\|. \quad (16)$$

Summing (15) with (16), we obtain

$$\|Tu\| \leq \|u\|\left(f^0+\epsilon\right)\Gamma^{-1} \leq \|u\|.$$

Now, suppose that $f_\infty(\theta) > \Theta_\infty(\theta)$ for some $\theta > 1$ and let $\varepsilon > 0$ be such that
$(f_\infty(\theta)-\varepsilon) > \Theta_\infty(\theta)$. There exists $\widetilde{R}_2 > R_1$ such that $f(t, e^{kt}w, e^{kt}z) > (f_\infty(\theta)-\varepsilon)(w+z)$ for all $t \in J_\theta$ and all w, z with $|(w,z)| \geq \widetilde{R}_2$. Let $\gamma_\theta = \min\left\{\gamma(s)e^{-ks} : s \in J_\theta\right\}$, $R_2 = \widetilde{R}_2/\gamma_\theta$ and $\Omega_2 = \{u \in E : \|u\| < R_2\}$. For all $u \in P \cap \partial\Omega_2$, and all $t > 0$ we have

$$\begin{aligned} \|Tu\|_k \geq e^{-kt}Tu(t) &\geq e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)f\Big(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right)\Big)ds \\ &\geq (f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)ds \\ &\geq (f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)e^{-ks}u(s)ds \\ &\geq \|u\|(f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)e^{-ks}\gamma(s)\,ds \end{aligned}$$

and

$$\begin{aligned}\left\|(Tu)'\right\|_k &\geq e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)f\left(s,e^{ks}\left(e^{-ks}u(s)\right),e^{ks}\left(e^{-ks}u'(s)\right)\right)ds\\ &\geq (f_{\infty}(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)ds\\ &\geq (f_{\infty}(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)e^{-ks}u(s)ds\\ &\geq \|u\|(f_{\infty}(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\,ds.\end{aligned}$$

The above estimates lead to

$$\|Tu\|_k \geq (f_{\infty}(\theta)-\varepsilon)\Theta_{1,\infty}(\theta)\|u\|,$$
$$\left\|(Tu)'\right\|_k \geq (f_{\infty}(\theta)-\varepsilon)\Theta_{2,\infty}(\theta)\|u\|$$

then to

$$\|Tu\| \geq (f_{\infty}(\theta)-\varepsilon)\left(\Theta_{\infty}(\theta)\right)^{-1}\|u\| \geq \|u\|.$$

We deduce from Assertion 1 of Theorem 2, that T admits a fixed point $u \in P$ with
$R_1 \leq \|u\|_1 \leq R_2$ which is, by Lemma 3, a positive solution to Problem (5).

Step 2. Existence in the case where (8) holds

Suppose that $f_0(\theta) > \Theta_0(\theta)$ for some $\theta > 0$ and let $\varepsilon > 0$ be such that $(f_0(\theta)-\varepsilon) > \Theta_0(\theta)$. There exists R_1 such that $f(t,e^{kt}w,e^{kt}z) > (f_0(\theta)-\varepsilon)(w+z)$ for all w,z with $|(w,z)| \leq R_1$. Let $\Omega_1 = \{u \in E : \|u\| < R_1\}$, for all $u \in P \cap \partial\Omega_1$ and all $t > 0$, we have

$$\begin{aligned}\|Tu\|_k \geq e^{-kt}Tu(t) &\geq e^{-kt}\int_0^{\theta}G(t,s)\phi(s)f\left(s,e^{ks}\left(e^{-ks}u(s)\right),e^{ks}\left(e^{-ks}u'(s)\right)\right)ds\\ &\geq (f_0(\theta)-\varepsilon)e^{-kt}\int_0^{\theta}G(t,s)\phi(s)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)ds\\ &\geq (f_0(\theta)-\varepsilon)e^{-kt}\int_0^{\theta}G(t,s)\phi(s)e^{-ks}u(s)ds\\ &\geq \|u\|(f_0(\theta)-\varepsilon)e^{-kt}\int_0^{\theta}G(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\end{aligned}$$

and

$$\begin{aligned}\left\|(Tu)'\right\|_k &\geq e^{-kt}Tu(t) \geq e^{-kt}\int_0^\theta \widetilde{G}(t,s)\phi(s)f\Big(s,e^{ks}\Big(e^{-ks}u(s)\Big),e^{ks}\left(e^{-ks}u'(s)\right)\Big)ds\\ &\geq (f_0(\theta)-\varepsilon)e^{-kt}\int_0^\theta \widetilde{G}(t,s)\phi(s)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)ds\\ &\geq (f_0(\theta)-\varepsilon)e^{-kt}\int_0^\theta \widetilde{G}(t,s)\phi(s)e^{-ks}u(s)ds\\ &\geq \|u\|\,(f_0(\theta)-\varepsilon)e^{-kt}\int_0^\theta \widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\,ds.\end{aligned}$$

The above estimates lead to

$$\begin{aligned}\|Tu\|_k &\geq \big(f_0(\theta)-\varepsilon\big)\Theta_{1,0}(\theta)\,\|u\|\,,\\ \left\|(Tu)'\right\|_k &\geq \big(f_0(\theta)-\varepsilon\big)\Theta_{2,0}(\theta)\,\|u\|\end{aligned}$$

then to

$$\|Tu\| \geq \big(f_0(\theta)-\varepsilon\big)\left(\Theta_0(\theta)\right)^{-1}\|u\| \geq \|u\|\,.$$

Let $\epsilon > 0$ be such that $(f^\infty+\epsilon) < \Gamma$, there exists $R_\epsilon > 0$ such that

$$f(t,e^{kt}w,e^{kt}z) \quad \leq (f^\infty+\epsilon)(w+z)+\Psi_{R_\epsilon}(w,z)\,, \text{ for all } t,w,z>0,$$

where Ψ_{R_ϵ} is the functions given by Hypothesis (6) for $R=R_\epsilon$.

Let

$$\begin{aligned}&\Phi_\epsilon(t)=\Psi_{R_\epsilon}\left(R_\epsilon e^{-ks}\gamma(s),R_\epsilon e^{-ks}\widetilde{\gamma}(s)\right)\\ &\widetilde{R}_2=\frac{2\overline{\Psi_\epsilon}\Gamma}{\Gamma-(f^\infty+\epsilon)}\\ &\text{with } \overline{\Phi_\epsilon}=\sup_{t\geq 0}\left(e^{-kt}\int_0^{+\infty}G(t,s)\Phi_\epsilon(s)\,ds\right)\end{aligned}$$

and notice that $\Gamma^{-1}(f^\infty+\epsilon)R+2\overline{\Phi_\epsilon}\leq R$ for all $R\geq\widetilde{R}_2$.

Let $R_2>\max(R_1,\widetilde{R}_2,R_\epsilon)$ and $\Omega_2=\{u\in E,\ \|u\|<R_2\}$. For all $u\in P\cap\partial\Omega_2$ and all $t>0$, we have

$$\begin{aligned}e^{-kt}Tu(t) &= \int_0^{+\infty}G(t,s)\phi(s)f\left(s,e^{ks}\left(e^{-ks}u(s)\right),e^{ks}\left(e^{-ks}u'(s)\right)\right)ds\\ &\leq e^{-kt}\int_0^{+\infty}G(t,s)\phi(s)\Big((f^\infty+\epsilon)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)\\ &+\Psi_\epsilon\left(e^{-ks}u(s),e^{-ks}u'(s)\right)\Big)\,ds\end{aligned}$$

$$\leq (f^{\infty}+\epsilon)\,\|u\|\,e^{-kt}\int_0^{+\infty} G(t,s)\phi(s)ds + \overline{\Psi_\epsilon}$$
$$\leq (f^{\infty}+\epsilon)\,\|u\|\,\Gamma_1 + \overline{\Psi_\epsilon},$$

leading to

$$\|Tu\|_k \leq (f^{\infty}+\epsilon)\,\|u\|\,\Gamma_1 + \overline{\Psi_\epsilon}. \tag{17}$$

Similarly, we have

$$e^{-kt}\,(Tu)'(t) = \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)f\Big(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right)\Big)ds$$
$$\leq e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\phi(s)\left((f^{\infty}+\epsilon)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)\right.$$
$$\left.+\Psi_\epsilon\left(e^{-ks}u(s), e^{-ks}u'(s)\right)\right) ds$$
$$\leq (f^{\infty}+\epsilon)\,\|u\|\,e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\phi(s)ds + \overline{\Psi_\epsilon}$$
$$\leq (f^{\infty}+\epsilon)\,\|u\|\,\Gamma_2 + \overline{\Psi_\epsilon},$$

leading to

$$\left\|(Tu)'\right\|_k \leq (f^{\infty}+\epsilon)\,\Gamma_2\,\|u\| + \overline{\Psi_\epsilon}. \tag{18}$$

Summing (17) with (18), we obtain

$$\|Tu\| \leq (f^{\infty}+\epsilon)\,\Gamma^{-1}\,\|u\| + 2\overline{\Psi_\epsilon} \leq \|u\|\,.$$

We deduce from Assertion 2 of Theorem 2, that T admits a fixed point $u \in P$ with $R_1 \leq \|u\| \leq R_2$ which is, by Lemma 3, a positive solution to Problem (5).

Thus, the proof of Theorem 1 is complete.

6 Comments

1. Notice that the obtained positive solution in Theorem 1 is nondecreasing and bounded. Indeed, if $u \in P \setminus \{0\}$ is a fixed point of T with $\|u\| = R$, then for all $t > 0$

$$u'(t) = (Tu)'(t) = \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)f\big(s,u(s),u'(s)\big)ds > 0$$

and Hypothesis (6) leads to

$$u(t) = Tu(t) = \int_0^{+\infty} G(t,s)\phi(s)f\big(s,u(s),u'(s)\big)ds$$

$$\leq \int_0^{+\infty} G(t,s)\phi(s)\Psi_R\left(\left(e^{-ks}u(s)\right), \left(e^{-ks}u'(s)\right)\right)ds$$
$$\leq \frac{1}{k^2}\int_0^{+\infty} \phi(s)\Psi_R(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s))ds < \infty.$$

2. From the above comment arise the following question. Why we looked for solutions in the space E instead of looking for them in the natural space
$F = \left\{u \in C^1(\mathbb{R}^+) : \max\left(\sup_{t>0}|u(t)|, \sup_{t>0}|u'(t)|\right) < \infty\right\}$?
The answer is: There is no cone in F where we can realize the inequality $\|Tu\| \geq \|u\|$ in Theorem 2.

3. Notice that for $\theta > 1$, $\Gamma < \Theta_0(\theta) < \Theta_\infty(\theta)$ and let the interval $\mathcal{I} = (\Gamma, \Theta_\infty(\theta))$. In the particular case where the limits
$$f^0 = \lim_{|(w,z)|\to 0} \frac{f\left(t, e^{kt}w, e^{kt}z\right)}{w+z}, \qquad f^\infty = \lim_{|(w,z)|\to 0} \frac{f\left(t, e^{kt}w, e^{kt}z\right)}{w+z}$$
exist, then Theorem 1 claims that the bvp (5) admits a positive solution if f^0 and f^∞ are oppositely located relatively to the interval $\mathcal{I}$, that is the ratio $\left(f(t, e^{kt}w, e^{kt}z)/w+z\right)$ crosses the interval $\mathcal{I}$. Two questions arise from this observation; what happens if $\left(f(t, e^{kt}w, e^{kt}z)/w+z\right) > \Theta_\infty(\theta)$ or $\left(f(t, e^{kt}w, e^{kt}z)/w+z\right) < \Gamma$ for all $t, w, z > 0$?
The second question is: are the constants $\Gamma, \Theta_0(\theta), \Theta_\infty(\theta)$ the best ones? In an other manner, does exist two positive constants α and β with $\Gamma < \alpha < \beta < \Theta_0(\theta)$ such that if f^0 and f^∞ are oppositely located relatively to the interval (α, β), then the bvp (5) admits a positive solution?

4. Let $p > 1$ and consider the case where E is equipped with the norm $\|u\|_p = \sqrt[p]{\|u\|_k^p + \|u\|_k^p}$. In this case, under Hypothesis (6), we prove by the same arguments that the bvp (5) admits a positive solution if $f^0 < \Gamma_p < \Theta_\infty^p(\theta) < f_\infty(\theta)$ for some $\theta > 1$ or $f^\infty < \Gamma < \Theta_0^p(\theta) < f_0(\theta)$ for some $\theta > 0$, where
$$\Gamma_p = \left((\Gamma_1)^p + (\Gamma_2)^p\right)^{-1/p},$$
$$\Theta_0^p(\theta) = \left((\Theta_{1,0}(\theta))^p + (\Theta_{2,0}(\theta))^p\right)^{-1/p} \text{ for } \theta > 0,$$
$$\Theta_\infty^p(\theta) = \left((\Theta_{1,\infty}(\theta))^p + (\Theta_{2,\infty}(\theta))^p\right)^{-1/p} \text{ for } \theta > 0.$$
Noticing that $\Gamma_p > \Gamma$, $\Theta_0^p(\theta) > \Theta_0(\theta)$ and $\Theta_\infty^p(\theta) > \Theta_{p,\infty}(\theta)$ we understand that the problem posed in the above comment is a serious problem.

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References

[1] R. P. Agarwal and D. O'Regan, *Infinite interval problems for differential, difference and integral equations*, Kluwer Academic Publisher, Dordrecht, 2001.

[2] C. Bai and C. Li, Unbounded upper and lower solution method for third-order boundary value problem on the half-line, *Electron. J. Differential Equations*, 2009 No. 119 (2009), 1–12.

[3] Z. Benbaaziz and S. Djebali, On a singular multi-point third-order boundary value problem on the half-line, *Mathematica Bohemica*, (2019), DOI: 10.21136/MB.2019.0084-18.

[4] F. Bernis and L. A. Petelier,Two problems from draining flows involving third-order ordinary differential equations, *SIAM J. Math. Anal.*, **27** No. 2 (1996), 515–527.

[5] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[6] S. Djebali and O. Saifi, Singular ϕ-Laplacian third-order BVPs with derivative dependence, *Arch. Math.* (Brno), **52** (2016), 35–48.

[7] S. Djebali and O. Saifi, Upper and lower solution for ϕ-Laplacian third-order BVPs on the half-line, *Cubo A Mathematical Journal*, **16** No. 1 (2014), 105–116.

[8] S. Djebali and O. Saifi, Third order BVPs with ϕ-Laplacian operators on $[0, +\infty)$, *Afr. Diaspora J. Math.*, **16** No. 1 (2013), 1–17.

[9] S. Djebali and O. Saifi, Positive solutions for singular BVPs with sign changing and derivative depending nonlinearity on the half-line, *Acta Appl. Math.*, **110** (2010), 639–665.

[10] X. Feng, H. Feng and H. Tan, Existence and iteration of positive solutions for third-order Sturm-Liouville boundary value problem, *Appl. Math. Comput.*, **266** (2015), 634–641.

[11] Y. Feng, On the existence and multiplicity of positive periodic solutions of a nonlinear third-order equation, *Appl. Math. Lett.*, **22** (2009), 1220–1224.

[12] D. Fu and W. Ding, Existence of positive solutions for third-order boundary value problem with integral boundary conditions in Banach spaces, *Adv. Difference Equ.*, 2013, 2013:65.

[13] J. R. Graef, L. Kong and B. Yong, Positive solutions for third-order multipoint singular boundary value problems, *Czechoslovak Math. J.*, **60** No. 135 (2010), 173–182.

[14] Y. Guo, Y. Liu and Y. Liang, Positive solutions for the third-order boundary value problems with the second derivatives, *Bound. Value Probl.*, 2012, 2012:34.

[15] N. Finizio and G. Ladas, *Ordinary Differential Equations with Modern Applications,* Third Edition, Wadsworth Pub. Co., Belmont, 1988.

[16] S.[A. Iyase, On a third-order three point boundary value problem at resonance on the half-line, Arab. J. Math. (2019), 8:43–53.

[17] D. Krajcinovic, Sandwich Beam Analysis, *Appl. Mech.*, **1** No. 39, (1972), 773–778.

[18] Y. Kuramoto and T. Yamada, *Turbulent state in chemical reaction*, Progress of Theoretical Physics, 56 (1976), 679.

[19] H. Lian and J. Zhao, Existence of unbounded solution for a third-order boundary value problem on infinite intervals, *Discrete Dyn. Nat. Soc.*, Vol. 2012, Article ID 357697, 14 pages, doi:101155/2012/

[20] S. Liang and J. Zhang, Positive solutions for singular third-order boundary-value problem with dependence on the first order derivative on the half-line, *Acta. Appl. Math.*, **111** (2010), 27–43.

[21] Z. Liu, H. Chen and C. Liu, Positive solutions for singular third-order nonhomogeneous boundary value problems, *J. Appl. Math. Comput.*, **38** (2012), 161–172.

[22] H. P. McKean, Nagumo's equation, *Advances in Mathematics*, **4** (1970), 209–223.

[23] D. Michelson, Steady solutions of the Kuramoto-Sivashinsky equation, *Physica D*, **19** (1986), 89–111.

[24] P. K. Palamides and R. P. Agarwal, *An existence result for a singular third-order boundary value problem on* $[0, +\infty)$, *Appl. Math. Lett.*, **21** (2008), 1254–1259.

[25] H. Pang, W. Xie and L. Cao, Successive iteration and positive solutions for a third-order boundary value problem involving integral conditions, *Bound. Value Probl.*, (2015), 2015:39.

[26] H. Shi, M. Pei and L. Wang, Solvability of a third-order three point boundary value problem on a half-line, *Bull. Malays. Math. Sci. Soc.*, **38** No. 3 (2015), 909–926.

[27] Y. Sun, Triple positive solutions for a class of third-order p-Laplacian singular boundary value problems, *J. Appl. Math. Comput.*, **37** (2011), 587–599.

[28] Z. Wei, Some necessary and sufficient conditions for existence of positive solutions for third-order singular sublinear multi-point boundary value problems, *Acta Math. Sin.*, **34 B** (6) (2014), 1795–1810.

[29] Z. Wei, Some necessary and sufficient conditions for existence of positive solutions for third-order singular super-linear multi-point boundary value problems, *J. Appl. Math. Comput.*, **46** (2014), 407–422.

[30] Y. Wu and Z. Zhao, Positive solutions for a third-order boundary value problems with change of signs, *Appl. Math. Comput.*, **218** (2011), 2744–2749.

[31] J. Zhang, Z. Wei and W. Dong, The method of lower and upper solutions for third-order singular four-point boundary value problems, *J. Appl. Math. Comput.*, **36** (2011), 275–289.

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On a new one-parameter generalization of dual-complex Jacobsthal numbers

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Abstract. In this paper we define dual-complex numbers with generalized Jacobsthal coefficients. We introduce one-parameter generalization of dual-complex Jacobsthal numbers - dual-complex r-Jacobsthal numbers. We investigate some algebraic properties of introduced numbers, among others Binet type formula, Catalan, Cassini, d'Ocagne and Honsberger type identities. Moreover, we present the generating function, summation formula and matrix generator for these numbers. The results are generalization of the properties for the dual-complex Jacobsthal numbers.

1 Introduction

The Jacobsthal sequence $\{J_n\}$ is one of the special cases of sequences $\{a_n\}$ which are defined recurrently as a linear combination of the preceding k terms

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$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k} \quad \text{for } n \geq k, \tag{1}$$

where $k \geq 2$, b_i are integers, $i = 1, 2, \ldots, k$ and $a_0, a_1, \ldots, a_{k-1}$ are given numbers.

By recurrence (1) for $k = 2$ we get (among others) definitions of the well-known sequences:

$$\begin{array}{llll} F_n = F_{n-1} + F_{n-2}, & F_0 = 0, & F_1 = 1 & \text{(Fibonacci numbers)} \\ L_n = L_{n-1} + L_{n-2}, & L_0 = 2, & L_1 = 1 & \text{(Lucas numbers)} \\ J_n = J_{n-1} + 2J_{n-2}, & J_0 = 0, & J_1 = 1 & \text{(Jacobsthal numbers)} \\ P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0, & P_1 = 1 & \text{(Pell numbers).} \end{array}$$

Sequences defined by (1) are called sequences of the Fibonacci type. The first ten terms of the Jacobsthal sequence are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171$. This sequence is also given by formula $J_n = \frac{2^n - (-1)^n}{3}$, named as Binet type formula for the Jacobsthal numbers. Many authors have generalized the recurrence of the Jacobsthal sequence. In [4] a one-parameter generalization of the Jacobsthal numbers was introduced. We recall this generalization.

Let $n \geq 0$, $r \geq 0$ be integers. The nth r-Jacobsthal number $J(r, n)$ is defined by the following recurrence relation

$$J(r, n) = 2^r J(r, n-1) + (2^r + 4^r) J(r, n-2) \text{ for } n \geq 2 \tag{2}$$

with $J(r, 0) = 1$, $J(r, 1) = 1 + 2^{r+1}$.

For $r = 0$ we have $J(0, n) = J_{n+2}$. By (2) we obtain

$$\begin{aligned} J(r,0) &= 1 \\ J(r,1) &= 2 \cdot 2^r + 1 \\ J(r,2) &= 3 \cdot 4^r + 2 \cdot 2^r \\ J(r,3) &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r \\ J(r,4) &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r \\ J(r,5) &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r. \end{aligned}$$

In [4], it was proved that the r-Jacobsthal numbers can be used for counting of independent sets of special classes of graphs. We will recall some useful properties of the r-Jacobsthal numbers.

Theorem 1 [4] (*Binet type formula*) *Let $n \geq 0$, $r \geq 0$ be integers. Then the nth r-Jacobsthal number is given by*

$$J(r, n) = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} {\lambda_1}^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} {\lambda_2}^n,$$

where

$$\lambda_1 = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \lambda_2 = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Theorem 2 [4] *Let* $n \geq 1$, $r \geq 0$ *be integers. Then*

$$\sum_{l=0}^{n-1} J(r,l) = \frac{J(r,n) + (2^r + 4^r)J(r,n-1) - 2 - 2^r}{4^r + 2^{r+1} - 1}. \tag{3}$$

Theorem 3 [4] *(Cassini type identity) Let* $n \geq 1$, $r \geq 0$ *be integers. Then*

$$J(r,n+1)J(r,n-1) - J^2(r,n) = (-1)^n(2^r+1)^2(2^r+4^r)^{n-1}.$$

Proposition 1 [4] *Let* $n \geq 4$, $r \geq 0$ *be integers. Then*

$$J(r,n) = (3 \cdot 8^r + 2 \cdot 4^r)J(r,n-3) + (2 \cdot 16^r + 3 \cdot 8^r + 4^r)J(r,n-4).$$

Theorem 4 [4] *Let* n, m, r *be integers such that* $m \geq 2, n \geq 1$, $r \geq 0$. *Then*

$$J(r,m+n) = 2^r J(r,m-1)J(r,n) + (4^r + 8^r)J(r,m-2)J(r,n-1).$$

Theorem 5 [4] *The generating function of the sequence of* r*-Jacobsthal numbers has the following form*

$$f(x) = \frac{1 + (1+2^r)x}{1 - 2^r x - (2^r + 4^r)x^2}.$$

2 The dual-complex numbers and their properties

The set of dual numbers is defined in the following way

$$\mathbb{D} = \{d\colon d = u + v\varepsilon \,|\, u, v \in \mathbb{R},\ \varepsilon^2 = 0,\ \varepsilon \neq 0\}.$$

Dual numbers were introduced by Clifford ([5]). Dual-complex numbers are known generalization of complex and dual numbers. These numbers were introduced by Majernik [8]. The set of dual-complex numbers, denoted by $\mathbb{DC}$, is defined as follows

$$\mathbb{DC} = \{w\colon w = z_1 + \varepsilon z_2 \,|\, z_1, z_2 \in \mathbb{C}, \varepsilon^2 = 0,\ \varepsilon \neq 0\}.$$

If $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, then any dual-complex number can be written as

$$w = x_1 + ix_2 + \varepsilon y_1 + i\varepsilon y_2.$$

Let w_1, w_2 be any dual-complex numbers, $w_1 = z_1 + \varepsilon z_2$, $w_2 = z_3 + \varepsilon z_4$. Addition, subtraction and multiplication of them are defined by

$$w_1 \pm w_2 = (z_1 \pm z_3) + \varepsilon(z_2 \pm z_4),$$

$$w_1 \cdot w_2 = z_1 z_3 + \varepsilon(z_1 z_4 + z_2 z_3).$$

Table 1 presents multiplication scheme of dual-complex numbers.

$\cdot$	1	i	ε	$i\varepsilon$
1	1	i	ε	$i\varepsilon$
i	i	-1	$i\varepsilon$	$-\varepsilon$
ε	ε	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	$-\varepsilon$	0	0

Table 1.

Assuming that $\mathrm{Re}(w_2) \neq 0$, the division of two dual-complex numbers w_1, w_2 is given by

$$\frac{w_1}{w_2} = \frac{z_1 + \varepsilon z_2}{z_3 + \varepsilon z_4} = \frac{(z_1 + \varepsilon z_2)(z_3 - \varepsilon z_4)}{(z_3 + \varepsilon z_4)(z_3 - \varepsilon z_4)} = \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{z_3^2}.$$

The dual-complex numbers form a commutative ring with characteristics 0. Moreover, the multiplication of these numbers gives the dual-complex numbers the structure of 2-dimensional complex Clifford algebra and 4-dimensional real Clifford algebra.

Let $w = z_1 + \varepsilon z_2 = x_1 + ix_2 + \varepsilon y_1 + i\varepsilon y_2, z_2 \neq 0$. There are five different conjugations, denoted by w^*, of dual-complex number w:

$w^{*1} = (z_1)^* + \varepsilon(z_2)^* = (x_1 - ix_2) + \varepsilon(y_1 - iy_2)$ complex conjugation

$w^{*2} = z_1 - \varepsilon z_2 = (x_1 + ix_2) - \varepsilon(y_1 + iy_2)$ dual conjugation

$w^{*3} = (z_1)^* - \varepsilon(z_2)^* = (x_1 - ix_2) - \varepsilon(y_1 - iy_2)$ coupled conjugation

$w^{*4} = (z_1)^* \cdot \left(1 - \varepsilon \frac{z_2}{z_1}\right) = (x_1 - ix_2)\left(1 - \varepsilon \frac{y_1 + iy_2}{x_1 + ix_2}\right)$ dual $-$ complex conjugation

$w^{*5} = z_2 - \varepsilon z_1 = (y_1 + iy_2) - \varepsilon(x_1 + ix_2)$ anti $-$ dual conjugation.

Therefore, the norms of a dual-complex number w are defined as

$$N_w^{*1} = \|w \cdot w^{*1}\| = \sqrt{|z_1^2| + 2\varepsilon \mathrm{Re}(z_1(z_2)^*)}$$
$$N_w^{*2} = \|w \cdot w^{*2}\| = \sqrt{z_1^2}$$
$$N_w^{*3} = \|w \cdot w^{*3}\| = \sqrt{|z_1^2| - 2i\varepsilon \mathrm{Im}(z_1(z_2)^*)}$$
$$N_w^{*4} = \|w \cdot w^{*4}\| = \sqrt{|z_1^2|}$$
$$N_w^{*5} = \|w \cdot w^{*5}\| = \sqrt{z_1 z_2 + \varepsilon(z_2^2 - z_1^2)}.$$

In the literature there are a lot of studies about numbers of the Fibonacci type. Many authors investigated quaternions, split quaternions, hyperbolic numbers, dual-hyperbolic numbers and dual-complex numbers with Fibonacci, Lucas, Pell, Jacobsthal coefficients, see [1, 2, 7, 9, 10]. In [6] dual-complex Fibonacci and Lucas numbers were studied. In [3] many interesting properties of dual-complex k-Pell quaternions were given. In this paper we introduce dual-complex numbers with generalized Jacobsthal numbers coefficients. We use one-parameter generalization of the Jacobsthal numbers - r-Jacobsthal numbers.

3 The dual-complex r-Jacobsthal numbers

For nonnegative integers n and r the nth dual-complex r-Jacobsthal number $\mathbb{DC}J(r,n)$ is defined as

$$\mathbb{DC}J(r,n) = J(r,n) + iJ(r,n+1) + \varepsilon J(r,n+2) + i\varepsilon J(r,n+3), \quad (4)$$

where $J(r,n)$ is given by (2).

Note that for $r = 0$ we obtain $\mathbb{DC}J(0,n) = \mathbb{DC}J_{n+2}$, where $\mathbb{DC}J_n$ denotes the nth dual-complex Jacobsthal number.

Now we give five conjugations of dual-complex r-Jacobsthal numbers

1) complex conjugation

$$\mathbb{DC}J(r,n)^{*1} = J(r,n) - iJ(r,n+1) + \varepsilon J(r,n+2) - i\varepsilon J(r,n+3),$$

2) dual conjugation

$$\mathbb{DC}J(r,n)^{*2} = J(r,n) + iJ(r,n+1) - \varepsilon J(r,n+2) - i\varepsilon J(r,n+3),$$

3) coupled conjugation

$$\mathbb{DC}J(r,n)^{*3} = J(r,n) - iJ(r,n+1) - \varepsilon J(r,n+2) + i\varepsilon J(r,n+3),$$

4) dual-complex conjugation

$$\mathbb{DC}J(r,n)^{*4} = (J(r,n) - iJ(r,n+1))\left(1 - \varepsilon\frac{J(r,n+2) + iJ(r,n+3)}{J(r,n) + iJ(r,n+1)}\right),$$

5) anti-dual conjugation

$$\mathbb{DC}J(r,n)^{*5} = J(r,n+2) + iJ(r,n+3) - \varepsilon J(r,n) - i\varepsilon J(r,n+1).$$

By simple calculations we can give the following relations

$$\begin{aligned}
\mathbb{DC}J(r,n)\cdot\mathbb{DC}J(r,n)^{*1} &= J^2(r,n) + J^2(r,n+1) + 2\varepsilon[J(r,n)J(r,n+2)\\
&\quad + J(r,n+1)J(r,n+3)],\\
\mathbb{DC}J(r,n)\cdot\mathbb{DC}J(r,n)^{*2} &= J^2(r,n) - J^2(r,n+1) + 2iJ(r,n)J(r,n+1),\\
\mathbb{DC}J(r,n)\cdot\mathbb{DC}J(r,n)^{*3} &= J^2(r,n) + J^2(r,n+1) + 2i\varepsilon[J(r,n)J(r,n+3)\\
&\quad - J(r,n+1)J(r,n+2)],\\
\mathbb{DC}J(r,n)\cdot\mathbb{DC}J(r,n)^{*4} &= J^2(r,n) + J^2(r,n+1),\\
\mathbb{DC}J(r,n)\cdot\mathbb{DC}J(r,n)^{*5} &= J(r,n)J(r,n+2) - J(r,n+1)J(r,n+3)\\
&\quad + i[J(r,n)J(r,n+3) + J(r,n+1)J(r,n+2)]\\
&\quad + \varepsilon[-J^2(r,n) + J^2(r,n+1)\\
&\quad + J^2(r,n+2) - J^2(r,n+3)]\\
&\quad + 2i\varepsilon(J(r,n+2)J(r,n+3) - J(r,n)J(r,n+1)),\\
\mathbb{DC}J(r,n) + \mathbb{DC}J(r,n)^{*1} &= 2[J(r,n) + \varepsilon J(r,n+2)],\\
\mathbb{DC}J(r,n) + \mathbb{DC}J(r,n)^{*2} &= 2[J(r,n) + iJ(r,n+1)],\\
\mathbb{DC}J(r,n) + \mathbb{DC}J(r,n)^{*3} &= 2[J(r,n) + i\varepsilon J(r,n+3)],\\
\mathbb{DC}J(r,n) - \varepsilon\mathbb{DC}J(r,n)^{*5} &= J(r,n) + iJ(r,n+1),\\
\varepsilon\mathbb{DC}J(r,n) + \mathbb{DC}J(r,n)^{*5} &= J(r,n+2) + iJ(r,n+3).
\end{aligned}$$

Using the definition of the dual-complex r-Jacobsthal number we get the following recurrence relations.

Proposition 2 *Let* $n \geq 0$, $r \geq 0$ *be integers. Then*

$$\mathbb{DC}J(r,n+2) = 2^r\mathbb{DC}J(r,n+1) + (2^r + 4^r)\mathbb{DC}J(r,n)$$

with

$$\begin{aligned}
\mathbb{DC}J(r,0) = {} & 1 + i(2^{r+1} + 1) + \varepsilon(3\cdot 4^r + 2^{r+1}) + i\varepsilon(5\cdot 8^r + 5\cdot 4^r + 2^r),\\
\mathbb{DC}J(r,1) = {} & 2^{r+1} + 1 + i(3\cdot 4^r + 2^{r+1}) + \varepsilon(5\cdot 8^r + 5\cdot 4^r + 2^r)\\
& + i\varepsilon(8\cdot 16^r + 10\cdot 8^r + 3\cdot 4^r).
\end{aligned}$$

Proof. By formulas (4) and (2) we get

$$\begin{aligned}
&2^r\mathbb{DC}J(r,n+1)+(2^r+4^r)\mathbb{DC}J(r,n)\\
&\quad=2^r(J(r,n+1)+iJ(r,n+2)+\varepsilon J(r,n+3)+i\varepsilon J(r,n+4))\\
&\qquad+(2^r+4^r)(J(r,n)+iJ(r,n+1)+\varepsilon J(r,n+2)+i\varepsilon J(r,n+3))\\
&\quad=J(r,n+2)+iJ(r,n+3)+\varepsilon J(r,n+4)+i\varepsilon J(r,n+5)=\mathbb{DC}J(r,n+2).
\end{aligned}$$

□

Proposition 3 *Let* $n \geq 4$, $r \geq 0$ *be integers. Then*

$$\mathbb{DC}J(r,n)=(3\cdot 8^r+2\cdot 4^r)\mathbb{DC}J(r,n-3)+(2\cdot 16^r+3\cdot 8^r+4^r)\mathbb{DC}J(r,n-4).$$

Proof. Let $A=3\cdot 8^r+2\cdot 4^r$, $B=2\cdot 16^r+3\cdot 8^r+4^r$. By Proposition 1 we have

$$\begin{aligned}
\mathbb{DC}J(r,n)&=J(r,n)+iJ(r,n+1)+\varepsilon J(r,n+2)+i\varepsilon J(r,n+3)\\
&=A\cdot J(r,n-3)+B\cdot J(r,n-4)+i(A\cdot J(r,n-2)+B\cdot J(r,n-3))\\
&\quad+\varepsilon(A\cdot J(r,n-1)+B\cdot J(r,n-2))+i\varepsilon(A\cdot J(r,n)+B\cdot J(r,n-1))\\
&=A(J(r,n-3)+iJ(r,n-2)+\varepsilon J(r,n-1)+i\varepsilon J(r,n))\\
&\quad+B(J(r,n-4)+iJ(r,n-3)+\varepsilon J(r,n-2)+i\varepsilon J(r,n-1)).
\end{aligned}$$

Hence we get

$$\mathbb{DC}J(r,n)=A\cdot\mathbb{DC}J(r,n-3)+B\cdot\mathbb{DC}J(r,n-4).$$

□

Theorem 6 *Let* $n \geq 0$, $r \geq 0$ *be integers. Then*

$$\begin{aligned}
&\mathbb{DC}J(r,n)-i\mathbb{DC}J(r,n+1)-\varepsilon\mathbb{DC}J(r,n+2)+i\varepsilon\mathbb{DC}J(r,n+3)=\\
&=J(r,n)+J(r,n+2).
\end{aligned}$$

Proof. By simple calculations we get

$$\begin{aligned}
&\mathbb{DC}J(r,n)-i\mathbb{DC}J(r,n+1)-\varepsilon\mathbb{DC}J(r,n+2)+i\varepsilon\mathbb{DC}J(r,n+3)=\\
&\quad=J(r,n)+iJ(r,n+1)+\varepsilon J(r,n+2)+i\varepsilon J(r,n+3)\\
&\qquad-i\,(J(r,n+1)+iJ(r,n+2)+\varepsilon J(r,n+3)+i\varepsilon J(r,n+4))\\
&\qquad-\varepsilon\,(J(r,n+2)+iJ(r,n+3)+\varepsilon J(r,n+4)+i\varepsilon J(r,n+5))\\
&\qquad+i\varepsilon\,(J(r,n+3)+iJ(r,n+4)+\varepsilon J(r,n+5)+i\varepsilon J(r,n+6))\\
&\quad=J(r,n)+iJ(r,n+1)+\varepsilon J(r,n+2)+i\varepsilon J(r,n+3)\\
&\qquad-iJ(r,n+1)+J(r,n+2)-i\varepsilon J(r,n+3)+\varepsilon J(r,n+4)\\
&\qquad-\varepsilon J(r,n+2)-i\varepsilon J(r,n+3)+i\varepsilon J(r,n+3)-\varepsilon J(r,n+4)\\
&\quad=J(r,n)+J(r,n+2),
\end{aligned}$$

which ends the proof. □

In the next theorem we present the Binet type formula for the dual-complex r-Jacobsthal numbers.

Theorem 7 *Let $n \geq 0$, $r \geq 0$ be integers. Then the nth dual-complex r-Jacobsthal number is given by*

$$\mathbb{DC}J(r,n) = C_1\widehat{\lambda_1}\lambda_1^n + C_2\widehat{\lambda_2}\lambda_2^n, \tag{5}$$

where

$$\lambda_1 = 2^{r-1} + \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r}, \qquad \lambda_2 = 2^{r-1} - \frac{1}{2}\sqrt{4\cdot 2^r + 5\cdot 4^r},$$

$$\widehat{\lambda_1} = 1 + i\lambda_1 + \varepsilon\lambda_1^2 + i\varepsilon\lambda_1^3, \qquad \widehat{\lambda_2} = 1 + i\lambda_2 + \varepsilon\lambda_2^2 + i\varepsilon\lambda_2^3,$$

$$C_1 = \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} + 3\cdot 2^r + 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}, \qquad C_2 = \frac{\sqrt{4\cdot 2^r + 5\cdot 4^r} - 3\cdot 2^r - 2}{2\sqrt{4\cdot 2^r + 5\cdot 4^r}}.$$

Proof. By Theorem 1 we get

$$\begin{aligned}
\mathbb{DC}J(r,n) &= J(r,n) + iJ(r,n+1) + \varepsilon J(r,n+2) + i\varepsilon J(r,n+3)\\
&= C_1\lambda_1^n + C_2\lambda_2^n + i(C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1})\\
&\quad + \varepsilon(C_1\lambda_1^{n+2} + C_2\lambda_2^{n+2}) + i\varepsilon(C_1\lambda_1^{n+3} + C_2\lambda_2^{n+3})\\
&= C_1\lambda_1^n\left(1 + i\lambda_1 + \varepsilon\lambda_1^2 + i\varepsilon\lambda_1^3\right) + C_2\lambda_2^n\left(1 + i\lambda_2 + \varepsilon\lambda_2^2 + i\varepsilon\lambda_2^3\right)\\
&= C_1\widehat{\lambda_1}\lambda_1^n + C_2\widehat{\lambda_2}\lambda_2^n,
\end{aligned}$$

which ends the proof. □

Corollary 1 (*Binet type formula for dual-complex Jacobsthal numbers*) *Let $n \geq 0$ be an integer. Then*

$$\mathbb{DC}J_n = \frac{1}{3}\left[2^n(1 + 2i + 4\varepsilon + 8i\varepsilon) - (-1)^n(1 - i + \varepsilon - i\varepsilon)\right].$$

Proof. By formula (5), for $r = 0$ we obtain $\lambda_1 = 2$, $\lambda_2 = -1$, $C_1 = \frac{4}{3}$, $C_2 = -\frac{1}{3}$. Moreover,

$$\begin{aligned}
\mathbb{DC}J(0,n) &= \frac{4}{3}\cdot 2^n(1 + 2i + 4\varepsilon + 8i\varepsilon) - \frac{1}{3}(-1)^n(1 - i + \varepsilon - i\varepsilon)\\
&= \frac{1}{3}\cdot 2^{n+2}(1 + 2i + 4\varepsilon + 8i\varepsilon) - \frac{1}{3}(-1)^{n+2}(1 - i + \varepsilon - i\varepsilon)\\
&= \mathbb{DC}J_{n+2}.
\end{aligned}$$

□

4 Some identities involving the dual-complex r-Jacobsthal numbers

In this section we give some identities such as Catalan, Cassini and d'Ocagne type identities for the dual-complex r-Jacobsthal numbers. These identities can be proved by using formula (5). We will need the following lemma.

Lemma 1 *Let* $\widehat{\lambda_1} = 1 + i\lambda_1 + \varepsilon\lambda_1^2 + i\varepsilon\lambda_1^3$, $\widehat{\lambda_2} = 1 + i\lambda_2 + \varepsilon\lambda_2^2 + i\varepsilon\lambda_2^3$, *where*

$$\lambda_1 = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r},\ \ \lambda_2 = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Then

$$\begin{aligned} \widehat{\lambda_1}\widehat{\lambda_2} = \widehat{\lambda_2}\widehat{\lambda_1} \quad &= \quad 1 + 4^r + 2^r + 2^r i + (2^{r+1} + 5 \cdot 4^r + 5 \cdot 8^r + 3 \cdot 16^r)\varepsilon \\ &\quad +(3 \cdot 8^r + 2 \cdot 4^r)i\varepsilon. \end{aligned} \tag{6}$$

Proof. By simple calculations we get

$$\begin{aligned} \widehat{\lambda_1}\widehat{\lambda_2} \quad &= \quad 1 + i\lambda_2 + \varepsilon\lambda_2^2 + i\varepsilon\lambda_2^3 + i\lambda_1 - \lambda_1\lambda_2 + i\varepsilon\lambda_1\lambda_2^2 \\ &\quad -\varepsilon\lambda_1\lambda_2^3 + \varepsilon\lambda_1^2 + i\varepsilon\lambda_1^2\lambda_2 + i\varepsilon\lambda_1^3 - \varepsilon\lambda_1^3\lambda_2 \\ &= \quad 1 - \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)i + (\lambda_1^2 + \lambda_2^2)(1 - \lambda_1\lambda_2)\varepsilon \\ &\quad +(\lambda_1^3 + \lambda_2^3 + \lambda_1\lambda_2(\lambda_1 + \lambda_2))i\varepsilon. \end{aligned}$$

Using the equalities

$$\begin{aligned} \lambda_1\lambda_2 &= -(4^r + 2^r), \\ \lambda_1 + \lambda_2 &= 2^r, \\ \lambda_1^2 + \lambda_2^2 &= (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = 3 \cdot 4^r + 2^{r+1}, \\ \lambda_1^3 + \lambda_2^3 &= (\lambda_1 + \lambda_2)^3 - 3\lambda_1\lambda_2(\lambda_1 + \lambda_2) = 4 \cdot 8^r + 3 \cdot 4^r, \end{aligned}$$

we get the result. □

Theorem 8 (*Catalan type identity for dual-complex r-Jacobsthal numbers*) *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq m$. Then*

$$\begin{aligned} &(\mathbb{DC}J(r, n))^2 - \mathbb{DC}J(r, n - m) \cdot \mathbb{DC}J(r, n + m) = \\ &= -\frac{(-4^r - 2^r)^n (1 + 2^r)^2}{4 \cdot 2^r + 5 \cdot 4^r}\left(2 - \left(\frac{\lambda_1}{\lambda_2}\right)^m - \left(\frac{\lambda_2}{\lambda_1}\right)^m\right)\widehat{\lambda_1}\widehat{\lambda_2}, \end{aligned}$$

where $\widehat{\lambda_1}\widehat{\lambda_2}$ is given by (6).

Proof. By formula (5) we get

$$\begin{aligned}
&(\mathbb{DC}J(r,n))^2 - \mathbb{DC}J(r,n-m)\cdot \mathbb{DC}J(r,n+m)\\
&= (C_1\widehat{\lambda_1}\lambda_1^n + C_2\widehat{\lambda_2}\lambda_2^n)(C_1\widehat{\lambda_1}\lambda_1^n + C_2\widehat{\lambda_2}\lambda_2^n)\\
&\quad - (C_1\widehat{\lambda_1}\lambda_1^{n-m} + C_2\widehat{\lambda_2}\lambda_2^{n-m})(C_1\widehat{\lambda_1}\lambda_1^{n+m} + C_2\widehat{\lambda_2}\lambda_2^{n+m})\\
&= 2C_1C_2\widehat{\lambda_1}\widehat{\lambda_2}(\lambda_1\lambda_2)^n - C_1C_2\widehat{\lambda_1}\widehat{\lambda_2}\lambda_1^{n+m}\lambda_2^{n-m} - C_1C_2\widehat{\lambda_1}\widehat{\lambda_2}\lambda_1^{n-m}\lambda_2^{n+m}\\
&= C_1C_2\widehat{\lambda_1}\widehat{\lambda_2}(\lambda_1\lambda_2)^n\left(2-\left(\frac{\lambda_1}{\lambda_2}\right)^m-\left(\frac{\lambda_2}{\lambda_1}\right)^m\right).
\end{aligned}$$

Since $\lambda_1\lambda_2 = -(4^r+2^r)$ and $C_1C_2 = -\frac{(1+2^r)^2}{4\cdot 2^r+5\cdot 4^r}$, we have

$$\begin{aligned}
&(\mathbb{DC}J(r,n))^2 - \mathbb{DC}J(r,n-m)\cdot \mathbb{DC}J(r,n+m) =\\
&= C_1C_2(-4^r-2^r)^n\widehat{\lambda_1}\widehat{\lambda_2}\left(2-(\frac{\lambda_1}{\lambda_2})^m-\left(\frac{\lambda_2}{\lambda_1}\right)^m\right)\\
&= -\frac{(-4^r-2^r)^n(1+2^r)^2}{4\cdot 2^r+5\cdot 4^r}\left(2-(\frac{\lambda_1}{\lambda_2})^m-\left(\frac{\lambda_2}{\lambda_1}\right)^m\right)\widehat{\lambda_1}\widehat{\lambda_2},
\end{aligned}$$

which ends the proof. □

Corollary 2 (*Cassini type identity for dual-complex* r*-Jacobsthal numbers*) *Let* $n \geq 1$, $r \geq 0$ *be integers. Then*

$$(\mathbb{DC}J(r,n))^2 - \mathbb{DC}J(r,n-1)\cdot \mathbb{DC}J(r,n+1) = (-4^r-2^r)^{n-1}(1+2^r)^2\widehat{\lambda_1}\widehat{\lambda_2}.$$

In particular, by Theorem 8, we obtain the following formulas for the dual-complex Jacobsthal numbers.

Corollary 3 (*Catalan type identity for dual-complex Jacobsthal numbers*) *Let* $n \geq 0$, $m \geq 0$, *be integers such that* $n \geq m$. *Then*

$$(\mathbb{DC}J_n)^2 - \mathbb{DC}J_{n-m}\cdot \mathbb{DC}J_{n+m} = \frac{4}{9}(-2)^{n-m}\left((-2)^m-1)\right)^2(3+i+15\varepsilon+5i\varepsilon).$$

Corollary 4 (*Cassini type identity for dual-complex Jacobsthal numbers*) *Let* $n \geq 1$ *be an integer. Then*

$$(\mathbb{DC}J_n)^2 - \mathbb{DC}J_{n-1}\cdot \mathbb{DC}J_{n+1} = 4(-2)^{n-1}(3+i+15\varepsilon+5i\varepsilon).$$

Theorem 9 (*Vajda type identity for dual-complex* r*-Jacobsthal numbers*) *Let* $n \geq 0$, $m \geq 0$, $k \geq 0$, $r \geq 0$ *be integers such that* $n \geq k$. *Then*

$$\mathbb{DC}J(r, m+k) \cdot \mathbb{DC}J(r, n-k) - \mathbb{DC}J(r, m) \cdot \mathbb{DC}J(r, n) = \\ = -\frac{(1+2^r)^2(-4^r-2^r)^m}{4 \cdot 2^r + 5 \cdot 4^r} \left(\lambda_2^{n-m} \left[\left(\frac{\lambda_1}{\lambda_2} \right)^k - 1 \right] + \lambda_1^{n-m} \left[\left(\frac{\lambda_2}{\lambda_1} \right)^k - 1 \right] \right) \widehat{\lambda_1} \widehat{\lambda_2},$$

where $\widehat{\lambda_1}\widehat{\lambda_2}$ *is given by* (6).

Proof. By (5) we get

$$\begin{aligned} &\mathbb{DC}J(r, m+k) \cdot \mathbb{DC}J(r, n-k) - \mathbb{DC}J(r, m) \cdot \mathbb{DC}J(r, n) = \\ &= (C_1 \widehat{\lambda_1} \lambda_1^{m+k} + C_2 \widehat{\lambda_2} \lambda_2^{m+k})(C_1 \widehat{\lambda_1} \lambda_1^{n-k} + C_2 \widehat{\lambda_2} \lambda_2^{n-k}) \\ &\quad - (C_1 \widehat{\lambda_1} \lambda_1^m + C_2 \widehat{\lambda_2} \lambda_2^m)(C_1 \widehat{\lambda_1} \lambda_1^n + C_2 \widehat{\lambda_2} \lambda_2^n) \\ &= C_1 C_2 \widehat{\lambda_1} \widehat{\lambda_2} \left(\lambda_1^{m+k} \lambda_2^{n-k} + \lambda_1^{n-k} \lambda_2^{m+k} - \lambda_1^m \lambda_2^n - \lambda_1^n \lambda_2^m \right) \\ &= C_1 C_2 \widehat{\lambda_1} \widehat{\lambda_2} (\lambda_1 \lambda_2)^m \left(\lambda_2^{n-m} \left[\left(\frac{\lambda_1}{\lambda_2} \right)^k - 1 \right] + \lambda_1^{n-m} \left[\left(\frac{\lambda_2}{\lambda_1} \right)^k - 1 \right] \right) \\ &= -\frac{(1+2^r)^2(-4^r-2^r)^m}{4 \cdot 2^r + 5 \cdot 4^r} \left(\lambda_2^{n-m} \left[\left(\frac{\lambda_1}{\lambda_2} \right)^k - 1 \right] + \lambda_1^{n-m} \left[\left(\frac{\lambda_2}{\lambda_1} \right)^k - 1 \right] \right) \widehat{\lambda_1} \widehat{\lambda_2}. \end{aligned}$$

□

Theorem 10 (*Vajda type identity for dual-complex Jacobsthal numbers*) *Let* $n \geq 0$, $m \geq 0$, $k \geq 0$ *be integers such that* $n \geq k$. *Then*

$$\mathbb{DC}J_{m+k} \cdot \mathbb{DC}J_{n-k} - \mathbb{DC}J_m \cdot \mathbb{DC}J_n = \\ = -\frac{4}{9}(-2)^m \left((-1)^{n-m}[(-2)^k - 1] + 2^{n-m}[(-\frac{1}{2})^k - 1] \right) (3 + i + 15\varepsilon + 5i\varepsilon).$$

Theorem 11 (*d'Ocagne type identity for dual-complex* r*-Jacobsthal numbers*) *Let* $n \geq 0$, $m \geq 0$, $r \geq 0$ *be integers such that* $n \geq m$. *Then*

$$\mathbb{DC}J(r, n) \cdot \mathbb{DC}J(r, m+1) - \mathbb{DC}J(r, n+1) \cdot \mathbb{DC}J(r, m) = \\ = \frac{(1+2^r)^2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}}{4 \cdot 2^r + 5 \cdot 4^r} (-4^r - 2^r)^m \left(\lambda_1^{n-m} - \lambda_2^{n-m} \right) \widehat{\lambda_1} \widehat{\lambda_2},$$

where $\widehat{\lambda_1}\widehat{\lambda_2}$ *is given by* (6).

Proof. Using the Binet type formula (5) we get

$$\begin{aligned}
&\mathbb{DC}J(r,n)\cdot\mathbb{DC}J(r,m+1)-\mathbb{DC}J(r,n+1)\cdot\mathbb{DC}J(r,m)=\\
&=(C_1\widehat{\lambda_1}\lambda_1^n+C_2\widehat{\lambda_2}\lambda_2^n)(C_1\widehat{\lambda_1}\lambda_1^{m+1}+C_2\widehat{\lambda_2}\lambda_2^{m+1})\\
&\quad-(C_1\widehat{\lambda_1}\lambda_1^{n+1}+C_2\widehat{\lambda_2}\lambda_2^{n+1})(C_1\widehat{\lambda_1}\lambda_1^{m}+C_2\widehat{\lambda_2}\lambda_2^{m})\\
&=C_1C_2\widehat{\lambda_1}\widehat{\lambda_2}\left(\lambda_1^{m+1}\lambda_2^n+\lambda_1^n\lambda_2^{m+1}-\lambda_1^m\lambda_2^{n+1}-\lambda_1^{n+1}\lambda_2^m\right)\\
&=C_1C_2\widehat{\lambda_1}\widehat{\lambda_2}(\lambda_1\lambda_2)^m(\lambda_1\lambda_2^{n-m}+\lambda_1^{n-m}\lambda_2-\lambda_2^{n-m+1}-\lambda_1^{n-m+1})\\
&=C_1C_2(\lambda_2-\lambda_1)(\lambda_1\lambda_2)^m\widehat{\lambda_1}\widehat{\lambda_2}\left(\lambda_1^{n-m}-\lambda_2^{n-m}\right)\\
&=\frac{(1+2^r)^2\sqrt{4\cdot2^r+5\cdot4^r}}{4\cdot2^r+5\cdot4^r}(-4^r-2^r)^m\left(\lambda_1^{n-m}-\lambda_2^{n-m}\right)\widehat{\lambda_1}\widehat{\lambda_2}.
\end{aligned}$$

□

Corollary 5 (*d'Ocagne type identity for dual-complex Jacobsthal numbers*) *Let* $n\geq0$, $m\geq0$ *be integers such that* $n\geq m$. *Then*

$$\begin{aligned}
&\mathbb{DC}J_n\cdot\mathbb{DC}J_{m+1}-\mathbb{DC}J_{n+1}\cdot\mathbb{DC}J_m=\\
&\quad=\frac{4}{3}(-2)^m\left(2^{n-m}-(-1)^{n-m}\right)(3+i+15\varepsilon+5i\varepsilon).
\end{aligned}$$

Now we give a summation formula for the dual-complex r-Jacobsthal numbers.

Theorem 12 *Let* $n\geq1$, $r\geq0$ *be integers. Then*

$$\begin{aligned}
\sum_{l=0}^{n}\mathbb{DC}J(r,l)\;=\;&\frac{1}{4^r+2^{r+1}-1}[\mathbb{DC}J(r,n+1)+(2^r+4^r)\mathbb{DC}J(r,n)\\
&-(1+i+\varepsilon+i\varepsilon)(2+2^r)]\\
&-i-(2+2^{r+1})\varepsilon-(2^{r+2}+3\cdot4^r+2)i\varepsilon.
\end{aligned}$$

Proof. By formula (3) we have

$$\begin{aligned}
\sum_{l=0}^{n}\mathbb{DC}J(r,l)&=\sum_{l=0}^{n}(J(r,l)+iJ(r,l+1)+\varepsilon J(r,l+2)+i\varepsilon J(r,l+3))\\
&=\sum_{l=0}^{n}J(r,l)+\sum_{l=0}^{n}iJ(r,l+1)+\sum_{l=0}^{n}\varepsilon J(r,l+2)+\sum_{l=0}^{n}i\varepsilon J(r,l+3)\\
&=\frac{1}{4^r+2^{r+1}-1}[J(r,n+1)+(2^r+4^r)J(r,n)-2-2^r
\end{aligned}$$

$$+ i(J(r,n+2) + (2^r+4^r)J(r,n+1) - 2 - 2^r)$$
$$+ \varepsilon(J(r,n+3) + (2^r+4^r)J(r,n+2) - 2 - 2^r)$$
$$+ i\varepsilon(J(r,n+4) + (2^r+4^r)J(r,n+3) - 2 - 2^r)]$$
$$- iJ(r,0) - \varepsilon(J(r,0)+J(r,1)) - i\varepsilon(J(r,0)+J(r,1)+J(r,2)).$$

By simple calculations we get

$$\begin{aligned}\sum_{l=0}^{n} \mathbb{DC}J(r,l) = {} & \frac{1}{4^r+2^{r+1}-1}[J(r,n+1)+iJ(r,n+2) \\ & + \varepsilon J(r,n+3) + i\varepsilon J(r,n+4) \\ & + (2^r+4^r)(J(r,n)+iJ(r,n+1)+\varepsilon J(r,n+2)+i\varepsilon J(r,n+3)) \\ & - (2+2^r)(1+i+\varepsilon+i\varepsilon)] - i - (2^{r+1}+2)\varepsilon - (2^{r+2}+3\cdot 4^r+2)i\varepsilon \\ = {} & \frac{\mathbb{DC}J(r,n+1)+(2^r+4^r)\mathbb{DC}J(r,n)-(1+i+\varepsilon+i\varepsilon)(2+2^r)}{4^r+2^{r+1}-1} \\ & - i - (2+2^{r+1})\varepsilon - (2^{r+2}+3\cdot 4^r+2)i\varepsilon.\end{aligned}$$

□

In particular, we obtain the following formula for the dual-complex Jacobsthal numbers.

Corollary 6 *Let* $n \geq 1$ *be an integer. Then*

$$\sum_{l=0}^{n} \mathbb{DC}J_l = \frac{\mathbb{DC}J_{n+2} - \mathbb{DC}J_1}{2}.$$

Proof. By Theorem 12 for $r=0$ we have

$$\begin{aligned}\sum_{l=0}^{n} \mathbb{DC}J(0,l) &= \frac{\mathbb{DC}J(0,n+1)+2\mathbb{DC}J(0,n)-3(1+i+\varepsilon+i\varepsilon)}{2} \\ &\quad -(i+4\varepsilon+9i\varepsilon) \\ &= \frac{\mathbb{DC}J(0,n+2)-(3+5i+11\varepsilon+21i\varepsilon)}{2}.\end{aligned}$$

Using fact that $J(0,n) = J_{n+2}$ and $\mathbb{DC}J_0 = i+\varepsilon+3i\varepsilon$, $\mathbb{DC}J_1 = 1+i+3\varepsilon+5i\varepsilon$, we get

$$\begin{aligned}\sum_{l=0}^{n} \mathbb{DC}J_l &= \frac{\mathbb{DC}J_{n+2}-(3+5i+11\varepsilon+21i\varepsilon)}{2} + \mathbb{DC}J_0 + \mathbb{DC}J_1 \\ &= \frac{\mathbb{DC}J_{n+2}-(3+5i+11\varepsilon+21i\varepsilon)+2(1+2i+4\varepsilon+8i\varepsilon)}{2}. \\ &= \frac{\mathbb{DC}J_{n+2}-(1+i+3\varepsilon+5i\varepsilon)}{2} = \frac{\mathbb{DC}J_{n+2}-\mathbb{DC}J_1}{2},\end{aligned}$$

which ends the proof. □

The next theorem gives the Honsberger type identity for the dual-complex r-Jacobsthal numbers.

Theorem 13 *Let* $m \geq 2, n \geq 1$, $r \geq 0$ *be integers. Then*

$$\begin{aligned}
&2^r\mathbb{DC}J(r,m-1)\cdot\mathbb{DC}J(r,n)+(4^r+8^r)\mathbb{DC}J(r,m-2)\cdot\mathbb{DC}J(r,n-1)=\\
&\quad=2\mathbb{DC}J(r,m+n)-J(r,m+n)-J(r,m+n+2)\\
&\qquad-2\varepsilon J(r,m+n+4)+2i\varepsilon J(r,m+n+3).
\end{aligned}$$

Proof. By simple calculations we have

$$\begin{aligned}
&2^r\mathbb{DC}J(r,m-1)\cdot\mathbb{DC}J(r,n)=\\
&=2^r[J(r,m-1)J(r,n)+iJ(r,m-1)J(r,n+1)\\
&\quad+\varepsilon J(r,m-1)J(r,n+2)+i\varepsilon J(r,m-1)J(r,n+3)\\
&\quad+iJ(r,m)J(r,n)-J(r,m)J(r,n+1)+i\varepsilon J(r,m)J(r,n+2)\\
&\quad-\varepsilon J(r,m)J(r,n+3)+\varepsilon J(r,m+1)J(r,n)+i\varepsilon J(r,m+1)J(r,n+1)\\
&\quad+i\varepsilon J(r,m+2)J(r,n)-\varepsilon J(r,m+2)J(r,n+1)],\\
&(4^r+8^r)\mathbb{DC}J(r,m-2)\cdot\mathbb{DC}J(r,n-1)=\\
&=(4^r+8^r)[J(r,m-2)J(r,n-1)+iJ(r,m-2)J(r,n)\\
&\quad+\varepsilon J(r,m-2)J(r,n+1)+i\varepsilon J(r,m-2)J(r,n+2)\\
&\quad+iJ(r,m-1)J(r,n-1)-J(r,m-1)J(r,n)\\
&\quad+i\varepsilon J(r,m-1)J(r,n+1)-\varepsilon J(r,m-1)J(r,n+2)\\
&\quad+\varepsilon J(r,m)J(r,n-1)+i\varepsilon J(r,m)J(r,n)\\
&\quad+i\varepsilon J(r,m+1)J(r,n-1)-\varepsilon J(r,m+1)J(r,n)].
\end{aligned}$$

Hence

$$\begin{aligned}
&2^r\cdot\mathbb{DC}J(r,m-1)\cdot\mathbb{DC}J(r,n)+(4^r+8^r)\mathbb{DC}J(r,m-2)\cdot\mathbb{DC}J(r,n-1)=\\
&=2^r\,J(r,m-1)J(r,n)+(4^r+8^r)J(r,m-2)J(r,n-1)\\
&\quad+i[2^r\,J(r,m-1)J(r,n+1)+(4^r+8^r)J(r,m-2)J(r,n)\\
&\quad+2^r\,J(r,m)J(r,n)+(4^r+8^r)J(r,m-1)J(r,n-1)]\\
&\quad+\varepsilon[2^r\,J(r,m-1)J(r,n+2)+(4^r+8^r)J(r,m-2)J(r,n+1)\\
&\quad+2^r\,J(r,m+1)J(r,n)+(4^r+8^r)J(r,m)J(r,n-1)]\\
&\quad+i\varepsilon[2^r\,J(r,m-1)J(r,n+3)+(4^r+8^r)J(r,m-2)J(r,n+2)\\
&\quad+2^r\,J(r,m)J(r,n+2)+(4^r+8^r)J(r,m-1)J(r,n+1)]\\
&\quad-2^r\,J(r,m)J(r,n+1)-(4^r+8^r)J(r,m-1)J(r,n)\\
&\quad-\varepsilon[2^r\,J(r,m)J(r,n+3)+(4^r+8^r)J(r,m-1)J(r,n+2)\\
&\quad+2^r\,J(r,m+2)J(r,n+1)+(4^r+8^r)J(r,m+1)J(r,n)]\\
&\quad+i\varepsilon[2^r\,J(r,m+1)J(r,n+1)+(4^r+8^r)J(r,m)J(r,n)\\
&\quad+2^r\,J(r,m+2)J(r,n)+(4^r+8^r)J(r,m+1)J(r,n-1)].
\end{aligned}$$

Using Theorem 4, we get

$$\begin{aligned}&2^r\mathbb{DC}J(r,m-1)\cdot\mathbb{DC}J(r,n)+(4^r+8^r)\mathbb{DC}J(r,m-2)\cdot\mathbb{DC}J(r,n-1)=\\&=J(r,m+n)+2[iJ(r,m+n+1)+\varepsilon J(r,m+n+2)\\&\quad+i\varepsilon J(r,m+n+3)]-J(r,m+n+2)\\&\quad-2\varepsilon J(r,m+n+4)\varepsilon+2i\varepsilon J(r,m+n+3)\\&=2\mathbb{DC}J(r,m+n)-J(r,m+n)-J(r,m+n+2)\\&\quad-2\varepsilon J(r,m+n+4)+2i\varepsilon J(r,m+n+3).\end{aligned}$$

□

5 Generating functions and matrix generators

Now we give the generating function of the dual-complex r-Jacobsthal numbers.

Theorem 14 *The generating function of the dual-complex r-Jacobsthal numbers has the following form*

$$g(x)=\frac{\mathbb{DC}J(r,0)+(\mathbb{DC}J(r,1)-2^r\mathbb{DC}J(r,0))x}{1-2^rx-(2^r+4^r)x^2}.$$

Proof. Let

$$g(x)=\mathbb{DC}J(r,0)+\mathbb{DC}J(r,1)x+\mathbb{DC}J(r,2)x^2+\cdots+\mathbb{DC}J(r,n)x^n+\cdots$$

be the generating function of the dual-complex r-Jacobsthal numbers. Then

$$\begin{aligned}2^rxg(x)&=2^r\mathbb{DC}J(r,0)x+2^r\mathbb{DC}J(r,1)x^2+2^r\mathbb{DC}J(r,2)x^3\\&\quad+\cdots+2^r\mathbb{DC}J(r,n-1)x^n+\cdots\\(2^r+4^r)x^2g(x)&=(2^r+4^r)\mathbb{DC}J(r,0)x^2+(2^r+4^r)\mathbb{DC}J(r,1)x^3\\&\quad+(2^r+4^r)\mathbb{DC}J(r,2)x^4+\cdots\\&\quad+(2^r+4^r)\mathbb{DC}J(r,n-2)x^n+\cdots.\end{aligned}$$

By Proposition 2 we get

$$\begin{aligned}&g(x)-2^rxg(x)-(2^r+4^r)x^2g(x)\\&=\mathbb{DC}J(r,0)+(\mathbb{DC}J(r,1)-2^r\mathbb{DC}J(r,0))x\\&\quad+(\mathbb{DC}J(r,2)-2^r\mathbb{DC}J(r,1)-(2^r+4^r)\mathbb{DC}J(r,0))x^2+\cdots\\&=\mathbb{DC}J(r,0)+(\mathbb{DC}J(r,1)-2^r\mathbb{DC}J(r,0))x.\end{aligned}$$

Thus

$$g(x) = \frac{\mathbb{DC}J(r,0) + (\mathbb{DC}J(r,1) - 2^r\mathbb{DC}J(r,0))x}{1 - 2^r x - (2^r + 4^r)x^2}.$$

Moreover,

$$\begin{aligned}
\mathbb{DC}J(r,0) &= 1 + (2^{r+1}+1)i + (3\cdot 4^r + 2^{r+1})\varepsilon \\
&\quad +(5\cdot 8^r + 5\cdot 4^r + 2^r)i\varepsilon, \\
\mathbb{DC}J(r,1) - 2^r\mathbb{DC}J(r,0) &= 2^r + 1 + (4^r + 2^r)i + (2\cdot 8^r + 3\cdot 4^r + 2^r)\varepsilon \\
&\quad +(3\cdot 16^r + 5\cdot 8^r + 2\cdot 4^r)i\varepsilon.
\end{aligned}$$

□

Corollary 7 *The generating function of the dual-complex Jacobsthal sequence is*

$$g(x) = \frac{i + \varepsilon + 3i\varepsilon + (1 + 2\varepsilon + 2i\varepsilon)x}{1 - x - 2x^2}.$$

At the end we give the matrix representation of the dual-complex r-Jacobsthal numbers.

Theorem 15 *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$\begin{bmatrix} \mathbb{DC}J(r,n+1) & \mathbb{DC}J(r,n) \\ \mathbb{DC}J(r,n) & \mathbb{DC}J(r,n-1) \end{bmatrix} = \begin{bmatrix} \mathbb{DC}J(r,2) & \mathbb{DC}J(r,1) \\ \mathbb{DC}J(r,1) & \mathbb{DC}J(r,0) \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}^{n-1}. \tag{7}$$

Proof. (by induction on n) It is easy to check that for $n = 1$ the result holds. Assume that the formula (7) is true for $n \geq 1$. We will show that

$$\begin{bmatrix} \mathbb{DC}J(r,n+2) & \mathbb{DC}J(r,n+1) \\ \mathbb{DC}J(r,n+1) & \mathbb{DC}J(r,n) \end{bmatrix} = \begin{bmatrix} \mathbb{DC}J(r,2) & \mathbb{DC}J(r,1) \\ \mathbb{DC}J(r,1) & \mathbb{DC}J(r,0) \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}^{n}.$$

By induction's hypothesis and simple calculations we have

$$\begin{bmatrix} \mathbb{DC}J(r,2) & \mathbb{DC}J(r,1) \\ \mathbb{DC}J(r,1) & \mathbb{DC}J(r,0) \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{DC}J(r,n+1) & \mathbb{DC}J(r,n) \\ \mathbb{DC}J(r,n) & \mathbb{DC}J(r,n-1) \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2^r\mathbb{DC}J(r,n+1) + (2^r + 4^r)\mathbb{DC}J(r,n) & \mathbb{DC}J(r,n+1) \\ 2^r\mathbb{DC}J(r,n) + (2^r + 4^r)\mathbb{DC}J(r,n-1) & \mathbb{DC}J(r,n) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{DC}J(r, n+2) & \mathbb{DC}J(r, n+1) \\ \mathbb{DC}J(r, n+1) & \mathbb{DC}J(r, n) \end{bmatrix},$$

which ends the proof. □

Calculating the determinants in formula (7) we obtain the Cassini type identity for the dual-complex r-Jacobsthal numbers. We have

$$\begin{vmatrix} \mathbb{DC}J(r, n+1) & \mathbb{DC}J(r, n) \\ \mathbb{DC}J(r, n) & \mathbb{DC}J(r, n-1) \end{vmatrix} = \mathbb{DC}J(r, n+1) \cdot \mathbb{DC}J(r, n-1) - (\mathbb{DC}J(r, n))^2,$$

$$\begin{vmatrix} \mathbb{DC}J(r, 2) & \mathbb{DC}J(r, 1) \\ \mathbb{DC}J(r, 1) & \mathbb{DC}J(r, 0) \end{vmatrix} = \mathbb{DC}J(r, 2) \cdot \mathbb{DC}J(r, 0) - (\mathbb{DC}J(r, 1))^2.$$

$$\begin{vmatrix} 2^r & 1 \\ 2^r + 4^r & 0 \end{vmatrix}^{n-1} = (-(2^r + 4^r))^{n-1}.$$

Consequently,

$$\mathbb{DC}J(r, n+1) \cdot \mathbb{DC}J(r, n-1) - (\mathbb{DC}J(r, n))^2 = \\ = (-(2^r + 4^r))^{n-1} (\mathbb{DC}J(r, 2) \cdot \mathbb{DC}J(r, 0) - (\mathbb{DC}J(r, 1))^2).$$

Compliance with ethical standards

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[1] M. Akar, S. Yüce, S. Şahin, On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, *Journal of Computer Science & Computational Mathematics*, **8(1)** (2018) DOI: 10.20967/jcscm.2018.01.001.

[2] F. T. Aydin, Dual-complex k-Fibonacci numbers, *Chaos Solitons Fractals*, **115** (2018), 1–6.

[3] F. T. Aydin, Dual-complex k-Pell quaternions, *Notes on Number Theory and Discrete Mathematics*, vol. **25**, no. 3 (2019), 111–125.

[4] D. Bród, On a new Jacobsthal-type sequence, *Ars Combin.*, **150** (2020), 21–29.

[5] W. K. Clifford, Preliminary Sketch of Biquaternions, *Proc. Lond. Math. Soc.*, s1-4 (1873), 381–395.

[6] M. A. Güungör, A. Z. Azak, Investigation of Dual-Complex Fibonacci, Dual-Complex Lucas Numbers and Their Properties, *Adv. Appl. Clifford Algebr.*, **27** (2017), 3083–3096.

[7] A. F. Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, *Amer. Math. Monthly*, **70** (1963), 289–291.

[8] V. Majernik, Multicomponent number systems, *Acta Phys. Polon. A*, **90** (3) (1996), 491–498.

[9] A. Szynal-Liana, I. Włoch, A note on Jacobsthal quaternions, *Adv. Appl. Clifford Algebr.*, **26** (2016), 441–447.

[10] A. Szynal-Liana, I. Włoch, The Pell quaternions and the Pell octonions, *Adv. Appl. Clifford Algebr.*, **26** (2016), 435–440.

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Defining and investigating new soft ordered maps by using soft semi open sets

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Abstract. Here, we employ soft semi open sets to define new soft ordered maps, namely soft x-semi continuous, soft x-semi open, soft x-semi closed and soft x-semi homeomorphism maps, where x denotes the type of monotonicity. To show the relationships among them, we provide some illustrative examples. Then we give complete descriptions for each one of them. Also, we investigate "transmission" of these maps between soft and classical topological ordered spaces.

1 Introduction

In 1965, Nachbin [41] introduced new mathematical structure, namely topological ordered space. This structure consists of two independent concepts defined on a non-empty set X, one of them is a topological space (X, τ) and the other is a partially ordered set $(X, \preceq)$. McCartan [39] in 1968, studied separation axioms via topological ordered spaces. Kumar [35] defined the concepts of continuous and homeomorphism maps via topological ordered spaces. Recently, the authors of [1, 4, 6, 9, 10, 12, 23, 26, 28] have introduced and investigated many concepts via supra topological ordered spaces.

In 1999, Molotdsov [40] introduced the concept of soft sets for dealing with uncertainties and vagueness. Then, Maji et al. [38] put up the basis of soft

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set theory by defining some operations between two soft sets like soft subset and equality relations, and soft union and intersection. Shabir and Naz [45] initiated the idea of soft topological spaces and studied soft separation axioms. Later on, many researchers carried out several studies to discuss the topological notions on soft topologies (see, for example [2, 3, 7, 11, 15, 20, 22, 29, 30, 31, 36, 44]). Chen [24] and Mahanta and Das[37] displayed and probed the notions of soft semi open sets and soft semi separation axioms. Depend on soft semi open sets, some works were done (see, for example [5, 25, 33, 34]). At present, the notions of soft topological ordered spaces [16], supra soft topological ordered spaces [13] and soft ordered maps [13] were introduced and investigated.

This paper is organized as follows: In Section (2), we recall the previous definitions and results that we will need to prove our results. Section (3) gives other applications of soft semi open sets by defining some soft ordered maps, namely soft x-semi continuous, soft x-semi open, soft x-semi closed and soft x-semi homeomorphism maps for $x \in \{I, D, B\}$. These concepts are described and some examples are constructed to show the relationships among them. Also, we demonstrate the interrelationships between these soft maps and their counterparts of crisp ordered maps when the soft topology is extended. Section (4) concludes the paper.

2 Preliminaries

Let X and Ω be a universal set and a set of parameters, respectively. The power set of X is denoted by 2^X.

2.1 Soft sets

Definition 1 [40] *A notation G_Ω is said to be a soft set, terminologically, over X, if G is a map from Ω to 2^X. Usually, we write it as follows:*

$$G_\Omega = \{(\omega, G(\omega)) : \omega \in \Omega \text{ and } G(\omega) \in 2^X\}.$$

Through this article, $S(X_\Omega)$ denotes the family of soft sets on X with Ω.

Definition 2 [29, 45] *For $y \in X$ and G_Ω over X, we write that:*

1. $y \in G_\Omega$ (*resp.* $y \notin G_\Omega$) *if* $y \in G(\omega)$ (*resp.* $y \notin G(\omega)$) *for each* $\omega \in \Omega$.

2. $y \Subset G_\Omega$ (*resp.* $y \not\Subset G_\Omega$) *if* $y \in G(\omega)$ (*resp.* $y \notin G(\omega)$) *for some* $\omega \in \Omega$.

Definition 3 [38] *If* $G(\omega) = \emptyset$ *and* $F(\omega) = X$ *for each* $\omega \in \Omega$*, then* G_Ω *and* F_Ω *are respectively called null soft set and absolute soft set. They are respectively denoted by* $\widetilde{\emptyset}$ *and* $\widetilde{X}$*.*

Definition 4 [21] *The relative complement* G^c_Ω *of* G_Ω *is defined by* $G^c(\omega) = X \setminus G(\omega)$ *for each* $\omega \in \Omega$*.*

Definition 5 [46] *A soft mapping of* $S(X_\Omega,)$ *into* $S(Y_\Gamma)$*, denoted by* f_ϕ*, is a pair of mappings* $f : X \to Y$ *and* $\phi : \Omega \to \Gamma$ *such that the image of* $G_K \in S(X_\Omega,)$ *and pre-image of* $H_L \in S(Y_\Gamma)$ *are given by the following formulations:*

(i) $f_\phi(G_K) = (f_\phi(G))_\Gamma$ *is a soft subset of* $S(Y_\Gamma)$ *given by*

$$f_\phi(G)(\gamma) = \begin{cases} \bigcup_{a \in \phi^{-1}(\gamma) \bigcap K} f(G(a)) & : \quad \phi^{-1}(\gamma) \bigcap K \neq \emptyset \\ \emptyset & : \quad \phi^{-1}(\gamma) \bigcap K = \emptyset \end{cases}$$

for each $\gamma \in \Gamma$*.*

(ii) $f^{-1}_\phi(H_L) = (f^{-1}_\phi(H))_\Omega$ *is a soft subset of* $S(X_\Omega)$ *given by*

$$f^{-1}_\phi(H)(\omega) = \begin{cases} f^{-1}(H(\phi(\omega))) & : \quad \phi(\omega) \in L \\ \emptyset & : \quad \phi(\omega) \notin L \end{cases}$$

for each $\omega \in \Omega$*.*

Definition 6 [46] *If* f *and* ϕ *are injective (resp. surjective, bijective) maps, then* $f_\phi : S(X_\Omega) \to S(Y_\Gamma)$ *is said to be injective (resp. surjective, bijective).*

Proposition 1 [42] *Let* G_Ω *and* H_Γ *be soft subsets of* $S(X_\Omega)$ *and* $S(Y_\Gamma)$*, respectively. Then:*

(i) $G_\Omega \widetilde{\subseteq} f^{-1}_\phi f_\phi(G_\Omega)$*. If* f_ϕ *is injective, then* $G_\Omega = f^{-1}_\phi f_\phi(G_\Omega)$*.*

(ii) $f_\phi f^{-1}_\phi(H_\Gamma) \widetilde{\subseteq} H_\Gamma$*. If* f_ϕ *is surjective, then* $f_\phi f^{-1}_\phi(H_\Gamma) = H_\Gamma$*.*

Definition 7 [27], [42] *If there exist* $\omega \in \Omega$ *and* $x \in X$ *such that* $G(\omega) = \{x\}$ *and* $G(a) = \emptyset$ *for each* $a \in \Omega \setminus \{\omega\}$*, then* G_Ω *is called a soft point. Briefly, it is denoted by* P^x_ω*.*

If $x \in G(\omega)$, then $P^x_\omega \in G_\Omega$.

Definition 8 [16] *A triple* $(X, \Omega, \preceq)$ *is said to be a partially ordered soft set if* $(X, \preceq)$ *is a partially ordered set.*

$\preceq$ *is called linearly ordered if any pair of elements in the set of the relation are comparable, i.e., for each* $x, y \in X$ *either* $x \preceq y$ *or* $y \preceq x$*.*

Remark 1 *Through this paper, the notation* $\triangle$ *denotes a diagonal relation, i.e.* $\triangle = \{(x, x) : x \in X\}$*.*

Definition 9 [16] *An increasing soft operator* $i : (S(X_\Omega), \preceq) \to (S(X_\Omega), \preceq)$ *and a decreasing soft operator* $d : (S(X_\Omega), \preceq) \to (S(X_\Omega), \preceq)$ *are defined as follows: For each soft subset* G_Ω *of* $S(X_\Omega)$

1. $i(G_\Omega) = (iG)_\Omega$*, where a mapping* iG *of* Ω *into* 2^X *given by* $iG(\omega) = i(G(\omega)) = \{v \in X : y \preceq v$ *for some* $y \in G(\omega)\}$*.*

2. $d(G_\Omega) = (dG)_\Omega$*, where a mapping* dG *of* Ω *into* 2^X *given by* $dG(\omega) = d(G(\omega)) = \{v \in X : v \preceq y$ *for some* $y \in G(\omega)\}$*.*

Definition 10 [16] *A soft subset* G_Ω *of* $(X, \Omega, \preceq)$ *is said to be increasing (resp. decreasing) if* $G_\Omega = i(G_\Omega)$*(resp.* $G_\Omega = d(G_\Omega)$*).*

Theorem 1 [16] *If* $f_\phi : (S(X_\Omega), \preceq_1) \to (S(Y_\Gamma), \preceq_2)$ *is surjective and increasing (resp. decreasing), then the inverse image of each increasing (resp. decreasing) soft set is increasing (resp. decreasing).*

2.2 Soft topologies

Definition 11 [45] *A sub-collection* τ *of* $S(X_\Omega)$ *is called a soft topology on* X *provided that it is closed under finite soft intersection and arbitrary soft union.*

By a soft topological space we mean a triple (X, τ, Ω)*. Every member of* τ *is called soft open and its relative complement is called soft closed.*

Proposition 2 [45] *In* (X, τ, Ω)*, a class* $\tau_\gamma = \{G(\omega) : G_\Omega \in \tau\}$ *defines a classical topology on* X *for each* $\omega \in \Omega$*.*

Proposition 3 [42] *A class* $\tau^\star = \{G_\Omega : G(\omega) \in \tau_\gamma$ *for each* $\omega \in \Omega\}$ *defines a soft topology on* X *finer than* τ*.*

Henceforward, $\tau^\star$ is called an extended soft topology.

Definition 12 [24, 37] *A soft subset* H_Ω *of* (X, τ, Ω) *which satisfies* $H_\Omega \widetilde{\subseteq} cl(int(H_\Omega))$ *is said to be soft semi open. The relative complement of a soft semi open set is said to be soft semi closed.*

Definition 13 [24, 37, 45] *We associate a subset* H_Ω *of* (X, τ, Ω) *with the following four operators:*

(i) $\mathrm{int}(H_\Omega)$ (*resp.* $\mathrm{int}_s(H_\Omega)$) *is the largest soft open* (*resp. soft semi open*) *set contained in* H_Ω.

(ii) $\mathrm{cl}(H_\Omega)$ (*resp.* $\mathrm{cl}_s(H_\Omega)$) *is the smallest soft closed* (*resp. soft semi closed*) *set containing* H_Ω.

Definition 14 [24] $f_\phi : (X, \tau, A) \to (Y, \theta, B)$ *is said to be:*

(i) *soft semi continuous if* $f_\phi^{-1}(G_B)$ *is soft semi open for each* $G_B \in \theta$.

(ii) *soft semi open* (*resp. soft semi closed*) *if* $f_\phi(U_A)$ *is soft semi open* (*resp. soft semi closed*) *for each* $U_A(\mathrm{resp.}\ U_A^c) \in \tau$.

(iii) *a soft semi homeomorphism if it is bijective, soft semi continuous and soft semi open.*

Definition 15 [16] *We call a quadrable system* $(X, \tau, \Omega, \preceq)$ *a soft topological ordered space provided that* τ *is a soft topology and* $\preceq$ *is a partially ordered set on* X.

Henceforward, we use the two notations $(X, \tau, \Omega, \preceq_1)$ *and* $(Y, \theta, \Gamma, \preceq_2)$ *to denote soft topological ordered spaces.*

Definition 16 [17] *The composition of* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *and* $g_\lambda : (Y, \theta, \Gamma, \preceq_2) \to (Z, \upsilon, K, \preceq_3)$ *is a soft map* $f_\phi \circ g_\lambda : (X, \tau, \Omega, \preceq_1) \to (Z, \upsilon, K, \preceq_3)$ *and is given by* $(f_\phi \circ g_\lambda)(P_\omega^x) = f_\phi(g_\lambda(P_\omega^x))$.

Definition 17 [43] *A map* g *from* $(X, \tau, \preceq_1)$ *to* $(Y, \theta, \preceq_2)$ *is said to be:*

(i) *D* (*resp. I, B*) *-semi continuous if* $g^{-1}(G)$ *is D* (*resp. I, B*) *-semi open for each* $G \in \theta$.

(ii) *D* (*resp. I, B*) *-semi open if* $g(F)$ *is D* (*resp. I, B*) *-semi open for each* $F \in \tau$.

(iii) *D* (*resp. I, B*) *-semi closed if* $g(H)$ *is D* (*resp. I, B*) *-semi closed for each* $F^c \in \tau$.

(iv) *D* (*resp. I, B*) *-semi homeomorphism if it is bijective, D* (*resp. I, B*) *-semi continuous and D* (*resp. I, B*) *-semi open.*

3 New types of soft semi ordered maps

3.1 Soft $D(I, B)$-semi continuity

This subsection introduces the concepts of $D(I, B)$-semi continuity at soft point and ordinary point, where D, I and B denote "Decreasing", "Increasing" and "Balancing", respectively. We also give the equivalent terms for each one of these concepts at the ordinary points and provide some illustrative examples.

Definition 18 *A soft subset* H_Ω *of* $(X, \tau, \Omega, \preceq_1)$ *which is:*

(i) *soft semi open and increasing (resp. decreasing, balancing) is said to be SI (resp. SD, SB) -semi open.*

(ii) *soft semi closed and increasing (resp. decreasing, balancing) is said to be SI (resp. SD, SB) -semi closed.*

Definition 19 $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *is called:*

1. *SI (resp. SD, SB) -semi continuous at* $P^x_\omega \in \widetilde{X}$ *if for each soft open set* H_Γ *containing* $f_\phi(P^x_\omega)$*, there exists an SI (resp. SD, SB) -semi open set* G_Ω *containing* P^x_ω *such that* $f_\phi(G_\Omega) \widetilde{\subseteq} H_\Gamma$.
2. *SI (resp. SD, SB) -semi continuous at* $x \in X$ *if it is SI (resp. SD, SB) -semi continuous at each* P^x_ω.
3. *SI (resp. SD, SB) -semi continuous if it is SI (resp. SD, SB) -semi continuous at each* $x \in X$.

Theorem 2 $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *is SI (resp. SD, SB) -semi continuous iff the inverse image of each soft open set is SI (resp. SD, SB) -semi open.*

Proof. When f_ϕ is SD-semi continuous.
Necessity: Let $G_\Gamma \in \theta$. Without loss of generality, consider $f^{-1}(G_\Gamma) \neq \widetilde{\emptyset}$. By choosing $P^x_\omega \in X$ s.t. $P^x_\omega \in f^{-1}_\phi(G_\Gamma)$, we obtain $f_\phi(P^x_\omega) \in G_\Gamma$. Then there is an SD-semi open set H_Ω containing P^x_ω s.t. $f_\phi(H_\Omega) \widetilde{\subseteq} G_\Gamma$. Since P^x_ω is chosen arbitrary, then $f^{-1}_\phi(G_\Gamma) = \widetilde{\bigcup}_{P^x_\omega \in f^{-1}_\phi(G_\Gamma)} H_\Omega$; therefore, $f^{-1}_\phi(G_\Gamma)$ is an SD-semi open set.

Sufficiency: Let $G_\Gamma \in \theta$ such that $f_\phi(P^x_\omega) \in \theta$. Then $P^x_\omega \in f^{-1}_\phi(G_\Gamma)$. By hypothesis, $f^{-1}_\phi(G_\Gamma)$ is an SD-semi open set. Since $f_\phi(f^{-1}_\phi(G_\Gamma)) \widetilde{\subseteq} G_\Gamma$, then f_ϕ is an SD-semi continuous map at P^x_ω and since P^x_ω is selected randomly, then f_ϕ is an SD-semi continuous map. □

Remark 2 *It can be seen from Definition (19) the following.*

1. *Every SI (SD, SB) -semi continuous map is soft semi continuous.*
2. *Every SB-semi continuous map is SI (SD) -semi continuous.*

Examples given below manifest that the two results of the remark above are not reversible.

Example 1 *Let* $\Omega = \{\omega_1, \omega_2\}$ *be a parameters set and* $X = \{1, 2, 3, 4\}$ *be a universe set and consider* $\phi : \Omega \to \Omega$ *and* $f : X \to X$ *are two identity maps. Let* $\preceq = \triangle \bigcup \{(1, 3)\}$ *be a partial order relation on* X *and consider* $\tau = \{\widetilde{\emptyset}, \widetilde{X}, F_\Omega G_\Omega\}$ *and* $\theta = \{\widetilde{\emptyset}, \widetilde{Y}, H_\Omega L_\Omega\}$ *are two soft topologies on* X*, where* $F_\Omega = \{(\omega_1, \{1\}), (\omega_2, \{3, 4\})\}$*,* $G_\Omega = \{(\omega_1, \emptyset), (\omega_2, \{3\})\}$*,* $H_\Omega = \{(\omega_1, \{1\}), (\omega_2, \{2, 3\})\}$ *and* $L_\Omega = \{(\omega_1, \{1\}), (\omega_2, \{3\})\}$*. For a soft map* $f_\phi : (X, \tau, \Omega, \preceq) \to (X, \theta, \Omega, \preceq)$*, we note that* $f_\phi^{-1}(H_\Omega) = H_\Omega$ *and* $f_\phi^{-1}(L_\Omega) = L_\Omega$ *are soft semi open sets. So* f_ϕ *is a soft semi continuous map. But,* $f_\phi^{-1}(H_\Omega)$ *is neither an SD-semi open set nor an SI-semi open set. Hence* f_ϕ *is not SI (SD, SB)-semi continuous.*

Example 2 *By replacing a partial order relation (in the above example) by* $\preceq = \triangle \bigcup \{(2, 4)\}$(*resp.* $\preceq = \triangle \bigcup \{(4, 1)\}$)*, we obtain a soft map* f_ϕ *is SD-semi continuous* (*resp. SI-continuous*)*, but is not SB-semi continuous.*

Definition 20 *For any set* H_Ω *in* $(X, \tau, \Omega, \preceq)$*, we introduce the next operators:*

(i) H_Ω^{iso} (*resp.* $H_\Omega^{dso}, H_\Omega^{bso}$) *is the largest SI* (*resp. SD, SB*) *-semi open set contained in* H_Ω.

(ii) H_Ω^{iscl} (*resp.* $H_\Omega^{dscl}, H_\Omega^{bscl}$) *is the smallest SI* (*resp. SD, SB*) *-semi closed set containing* H_Ω.

Lemma 1 *The next properties are satisfied for a set* H_Ω *in* $(X, \tau, \Omega, \preceq)$.

(i) $(H_\Omega^{dscl})^c = (H_\Omega^c)^{iso}$.

(ii) $(H_\Omega^{iscl})^c = (H_\Omega^c)^{dso}$.

(iii) $(H_\Omega^{bscl})^c = (H_\Omega^c)^{bso}$.

Proof.

(i) $(H_\Omega^{dscl})^c = \{\widetilde{\bigcup} F_\Omega : F_\Omega$ is an SD-semi closed set containing $H_\Omega\}^c$
$= \widetilde{\bigcap}\{F_\Omega^c : F_\Omega^c$ is an SI-semi open set contained in $H_\Omega^c\} = (H_\Omega^c)^{iso}$.

By analogy with **(i)**, one can prove **(ii)** and **(iii)**. □

Theorem 3 *The next properties of* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *are equivalent:*

1. f_ϕ *is SI-semi continuous;*
2. $f_\phi^{-1}(L_\Gamma)$ *is an SD-semi closed subset of* $\widetilde{X}$ *for any soft closed set* L_Γ *in* $\widetilde{Y}$*;*
3. $(f_\phi^{-1}(M_\Gamma))^{dscl} \widetilde{\subseteq} f_\phi^{-1}(cl(M_\Gamma))$ *for every* $M_\Gamma \widetilde{\subseteq} \widetilde{Y}$*;*
4. $f_\phi(N_\Omega^{dscl}) \widetilde{\subseteq} cl(f_\phi(N_\Omega))$ *for every* $N_\Omega \widetilde{\subseteq} \widetilde{X}$*;*
5. $f_\phi^{-1}(int(M_\Gamma)) \widetilde{\subseteq} (f_\phi^{-1}(M_\Gamma))^{iso}$ *for every* $M_\Gamma \widetilde{\subseteq} \widetilde{Y}$.

Proof. $1 \Rightarrow 2$: Suppose L_Γ is a soft closed set in $\widetilde{Y}$. Then, $f_\phi^{-1}(L_\Gamma^c)$ is an SI-semi open set in $\widetilde{X}$. Now, $f_\phi^{-1}(L_\Gamma^c) = (f_\phi^{-1}(L_\Gamma))^c$; hence, $f_\phi^{-1}(L_\Gamma)$ is an SD-semi closed set.
$2 \Rightarrow 3$: It comes from 2 that $f_\phi^{-1}(cl(M_\Omega))$ is an SD-semi closed set in $\widetilde{X}$ for any $M_\Gamma \widetilde{\subseteq} \widetilde{Y}$. Therefore, $(f_\phi^{-1}(M_\Gamma))^{dscl} \widetilde{\subseteq} (f_\phi^{-1}(cl(M_\Gamma))^{dscl} = f_\phi^{-1}(cl(M_\Gamma))$.
$3 \Rightarrow 4$: We know that that $N_\Omega^{dscl} \widetilde{\subseteq} (f_\phi^{-1}(f_\phi(N_\Omega))^{dscl}$; according to 3 we have $(f_\phi^{-1}(f_\phi(N_\Omega))^{dscl} \widetilde{\subseteq} f_\phi^{-1}(cl(f_\phi(N_\Omega))$. Hence, $f_\phi(N_\Omega^{dscl}) \widetilde{\subseteq} cl(f_\phi(N_\Omega))$.
$4 \Rightarrow 5$: For any soft set M_Γ in $\widetilde{Y}$, we obtain from Lemma (1) that $f_\phi(\widetilde{X} - (f_\phi^{-1}(N_\Omega))^{iso}) = f_\phi(((f_\phi^{-1}(N_\Omega))^c)^{dscl})$. It follows from statement 4, that $f_\phi(((f_\phi^{-1}(N_\Omega))^c)^{dscl}) \widetilde{\subseteq} cl(f_\phi(f_\phi^{-1}(N_\Omega))^c) = cl(f_\phi(f_\phi^{-1}(N_\Omega^c))) \widetilde{\subseteq} cl(\widetilde{Y} - N_\Omega) = \widetilde{Y} - int(N_\Omega)$. Therefore $(\widetilde{X} - (f_\phi^{-1}(N_\Omega))^{iso}) \widetilde{\subseteq} f_\phi^{-1}(\widetilde{Y} - int(N_\Omega)) = \widetilde{X} - f_\phi^{-1}(int(N_\Omega))$. Thus $f_\phi^{-1}(int(N_\Omega)) \widetilde{\subseteq} (f_\phi^{-1}(N_\Omega))^{iso}$.
$5 \Rightarrow 1$: Consider M_Γ is a soft open set in $\widetilde{Y}$. Then $f_\phi^{-1}(M_\Gamma) = f_\phi^{-1}(int(M_\Gamma)) \widetilde{\subseteq} (f_\phi^{-1}(M_\Gamma))^{iso}$. So $(f_\phi^{-1}(M_\Gamma))^{iso} = f_\phi^{-1}(M_\Gamma)$ and this means that $f_\phi^{-1}(M_\Gamma)$ is an SI-semi open set in $\widetilde{X}$. □

Theorem 4 *The next properties of* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *are equivalent:*

1. f_ϕ *is SD-semi continuous (resp. SB-semi continuous);*
2. $f_\phi^{-1}(L_\Gamma)$ *is an SI-semi closed (resp. SB-semi closed) set in* $\widetilde{X}$ *for each soft closed set* L_Γ *in* $\widetilde{Y}$*;*
3. $(f_\phi^{-1}(M_\Gamma))^{iscl}\widetilde{\subseteq}f_\phi^{-1}(cl(M_\Gamma))$ *(resp.* $(f_\phi^{-1}(M_\Gamma))^{bscl}\widetilde{\subseteq}f_\phi^{-1}(cl(M_\Gamma)))$ *for every* $M_\Gamma\widetilde{\subseteq}\widetilde{Y}$*;*
4. $f_\phi(N_\Omega^{iscl})\widetilde{\subseteq}cl(f_\phi(N_\Omega))$ *(resp.* $f_\phi(N_\Omega^{bscl})\widetilde{\subseteq}cl(f_\phi(N_\Omega)))$ *for every* $N_\Omega\widetilde{\subseteq}\widetilde{X}$*;*
5. $f_\phi^{-1}(int(M_\Gamma))\widetilde{\subseteq}(f_\phi^{-1}(M_\Gamma))^{dso}$ *(resp.* $f_\phi^{-1}(int(M_\Gamma))\widetilde{\subseteq}(f_\phi^{-1}(M_\Gamma))^{bso}$*) for every* $M_\Gamma\widetilde{\subseteq}\widetilde{Y}$.

Proof. Similar to the proof of Theorem (3). □

Theorem 5 *Let a soft topology* $\tau^\star$ *be extended. Then* $g_\phi : (X, \tau^\star, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *is SI (resp. SD, SB) -semi continuous iff a crisp map* $g : (X, \tau^\star_\gamma, \preceq_1) \to (Y, \theta_{\phi(\omega)}, \preceq_2)$ *is I (resp. D, B) -semi continuous.*

Proof. $\Rightarrow$: Consider U is an open set in $(Y, \theta_{\phi(\omega)}, \preceq_2)$. Then there is a soft open set G_Γ in $(Y, \theta, \Gamma, \preceq_2)$ s.t. $G(\phi(\omega)) = U$. Since g_ϕ is an SI (resp. SD, SB) -semi continuous map, then $g_\phi^{-1}(G_\Gamma)$ is an SI (resp. SD, SB) -semi open set. From Definition (5), a soft set $g_\phi^{-1}(G_\Gamma) = (g_\phi^{-1}(G))_\Omega$ in $(X, \tau, \Omega, \preceq_1)$ is given by $g_\phi^{-1}(G)(\omega) = g^{-1}(G(\phi(\omega)))$ for any $\omega \in \Omega$. Now, $\tau^\star$ is extended; thus, a set $g^{-1}(G(\phi(\omega))) = g^{-1}(U)$ in $(X, \tau_\gamma, \preceq_1)$ is I (resp. D, B) -semi open. This proves that g is I (resp. D, B) -semi continuous.
$\Leftarrow$: Consider G_Γ is a soft open set in $(Y, \theta, \Gamma, \preceq_2)$. Then a soft set $g_\phi^{-1}(G_\Gamma) = (g_\phi^{-1}(G))_\Omega$ in $(X, \tau^\star, \Omega, \preceq_1)$ isgiven by $g_\phi^{-1}(G)(\omega) = g^{-1}(G(\phi(\omega)))$ for any $\omega \in \Omega$. Since amap g is I (resp. D, B) -semi continuous, a set $g^{-1}(G(\phi(\omega)))$ in $(X, \tau^\star_\gamma, \preceq_1)$ is I (resp. D, B) -semi open. Now, $\tau^\star$ is extended; thus, $g_\phi^{-1}(G_\Gamma)$ is an SI (resp. SD, SB) -semi open set in $(X, \tau^\star, \Omega, \preceq_1)$. This proves that a soft map g_ϕ is SI (resp. SD, SB) -semi continuous. □

Proposition 4 *Let* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *be SB-semi continuous and surjective. If* $\preceq_1$ *is linearly ordered, then* θ *is the indiscrete soft topology.*

3.2 Soft $I(D, B)$-semi open and soft $I(D, B)$-semi closed maps

In the following part, we present the notions of soft $I(D, B)$-semi open and soft $I(D, B)$-semi closed maps. Then, we elucidate the relationships among them with the help of examples. Finally, we characterize each one of these concepts and study some properties.

Definition 21 $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \tau, \Gamma, \preceq_2)$ *is called:*

(i) *SI* (*resp. SD, SB*) *-semi open if the image of any soft open set in* $\widetilde{X}$ *is an SI* (*resp. SD, SB*) *-semi open set in* $\widetilde{Y}$.

(ii) *SI* (*resp. SD, SB*) *-semi closed if the image of any soft closed set in* $\widetilde{X}$ *is an SI* (*resp. SD, SB*) *-semi closed set in* $\widetilde{Y}$.

Remark 3 *Note that:*

1. *an SI (SD, SB) -semi open map is soft semi open.*
2. *an SI (SD, SB) -semi closed map is soft semi closed.*
3. *an SB-semi open* (*resp. SB-semi closed*) *map is SI (SD) -semi open* (*resp. SI (SD) -semi closed*).

Examples given below manifest that the three results of the remark above are not reversible.

Example 3 *Let* Ω, X, $\phi : \Omega \to \Omega$, $f : X \to X$ *and* $\preceq$ *be the same as in Example* (1). *Consider* $\tau = \{\widetilde{\emptyset}, \widetilde{X}, F_\Omega\}$ *and* $\theta = \{\widetilde{\emptyset}, \widetilde{Y}, L_\Omega\}$ *are two soft topologies on* X, *where* $F_\Omega = \{(\omega_1, \{1\}), (\omega_2, \{3, 4\})\}$ *and* $L_\Omega = \{(\omega_1, \{1\}), (\omega_2, \{3\})\}$. *For a soft map* $f_\phi : (X, \tau, \Omega, \preceq) \to (X, \theta, \Omega, \preceq)$, *we note that* $f_\phi(F_\Omega) = F_\Omega$ *is a soft semi open set. So* f_ϕ *is a soft semi open map. On the other hand,* $f_\phi(F_\Omega)$ *is neither an SD-semi open set nor an SI-semi open set. Hence* f_ϕ *is not SI (SD, SB)-semi open. Also,* f_ϕ *is a soft semi closed map, but it is not SI (SD, SB)-semi closed.*

Example 4 *By replacing a partial order relation* (*given in the example above*) *by* $\preceq = \triangle \bigcup \{(2, 4)\}$, *we obtain* f_ϕ *is SI-semi open and SD-semi closed, but it is neither an SB-semi open map nor an SB-semi closed map. Also, if we replace only the partial order relation by* $\preceq = \triangle \bigcup \{(1, 2)\}$, *then the soft map* f_ϕ *is SD-open and SI-semi closed, but it is neither an SB-semi open map nor an SB-semi closed map.*

Theorem 6 *The next properties of* $f_\phi : (X, \theta, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *are equivalent:*

1. f_ϕ *is SI-semi open;*

2. $\mathrm{int}(f_{\phi}^{-1}(M_{\Gamma}))\widetilde{\subseteq}f_{\phi}^{-1}(M_{\Gamma}^{iso})$ *for any* $M_{\Gamma}\widetilde{\subseteq}\widetilde{Y}$;

3. $f_{\phi}(\mathrm{int}(N_{\Omega}))\widetilde{\subseteq}(f_{\phi}(N_{\Omega}))^{iso}$ *for any* $N_{\Omega}\widetilde{\subseteq}\widetilde{X}$.

Proof. $1 \Rightarrow 2$: It is clear that $\mathrm{int}(f_{\phi}^{-1}(M_{\Gamma}))$ is a soft open set in $\widetilde{X}$ for any soft set M_{Γ} in $\widetilde{Y}$. Now, $f_{\phi}(\mathrm{int}(f_{\phi}^{-1}(M_{\Gamma})))$ is an SI-semi open set in $\widetilde{Y}$. Since $f_{\phi}(\mathrm{int}(f_{\phi}^{-1}(M_{\Gamma})))\widetilde{\subseteq}f_{\phi}(f_{\phi}^{-1}(M_{\Gamma}))\widetilde{\subseteq}M_{\Gamma}$, then $\mathrm{int}(f_{\phi}^{-1}(M_{\Gamma}))\widetilde{\subseteq}f_{\phi}^{-1}(M_{\Gamma}^{iso})$.
$2 \Rightarrow 3$: Given a soft set N_{Ω} in $\widetilde{X}$, according to 2 $\mathrm{int}(f_{\phi}^{-1}(f_{\phi}(N_{\Omega})))\widetilde{\subseteq} f_{\phi}^{-1}((f_{\phi}(N_{\Omega}))^{iso})$. Since $\mathrm{int}(N_{\Omega})\widetilde{\subseteq}f_{\phi}^{-1}(f_{\phi}(\mathrm{int}(f_{\phi}^{-1}(f_{\phi}(N_{\Omega})))))\widetilde{\subseteq}f_{\phi}^{-1}((f_{\phi}(N_{\Omega}))^{iso})$, then $f_{\phi}(\mathrm{int}(N_{\Omega}))\widetilde{\subseteq}(f_{\phi}(N_{\Omega}))^{iso}$.
$3 \Rightarrow 1$: Let G_{Ω} be a soft open set in $\widetilde{X}$. Then $f_{\phi}(\mathrm{int}(G_{\Omega})) = f_{\phi}(G_{\Omega})\widetilde{\subseteq}(f_{\phi}(G_{\Omega}))^{iso}$. Hence, f_{ϕ} is an SI-semi open map. □

Following similar technique, the following two theorems are proved.

Theorem 7 *The following three properties of* $f_{\phi} : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *are equivalent:*

1. f_{ϕ} *is SD-semi open (resp. SB-semi open);*

2. $\mathrm{int}(f_{\phi}^{-1}(M_{\Gamma}))\widetilde{\subseteq}f_{\phi}^{-1}(M_{\Gamma}^{dso})$ (*resp.* $\mathrm{int}(f_{\phi}^{-1}(M_{\Gamma}))\widetilde{\subseteq}f_{\phi}^{-1}(M_{\Gamma}^{bso})$) *for every* $M_{\Gamma}\widetilde{\subseteq}\widetilde{Y}$;

3. $f_{\phi}(\mathrm{int}(N_{\Omega}))\widetilde{\subseteq}(f_{\phi}(N_{\Omega}))^{dso}$ (*resp.* $f_{\phi}(\mathrm{int}(N_{\Omega}))\widetilde{\subseteq}(f_{\phi}(N_{\Omega}))^{bso}$) *for every* $N_{\Omega}\widetilde{\subseteq}\widetilde{X}$.

Theorem 8 *The next statements hold for* $f_{\phi} : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$:

1. f_{ϕ} *is SI-semi closed iff* $(f_{\phi}(G_{\Omega}))^{iscl}\widetilde{\subseteq}f_{\phi}(\mathrm{cl}(G_{\Omega}))$ *for every* $G_{\Omega}\widetilde{\subseteq}\widetilde{X}$.

2. f_{ϕ} *is SD-semi closed iff* $(f_{\phi}(G_{\Omega}))^{dscl}\widetilde{\subseteq}f_{\phi}(\mathrm{cl}(G_{\Omega}))$ *for every* $G_{\Omega}\widetilde{\subseteq}\widetilde{X}$.

3. f_{ϕ} *is SB-semi closed iff* $(f_{\phi}(G_{\Omega}))^{bscl}\widetilde{\subseteq}f_{\phi}(\mathrm{cl}(G_{\Omega}))$ *for every* $G_{\Omega}\widetilde{\subseteq}\widetilde{X}$.

Proof. We only prove 1.
Necessity: Since f_{ϕ} is SI-semi closed, $f_{\phi}(\mathrm{cl}(G_{\Omega}))$ is an SI-semi closed set in $\widetilde{Y}$ and since $f_{\phi}(G_{\Omega})\widetilde{\subseteq}f_{\phi}(\mathrm{cl}(G_{\Omega}))$, $(f_{\phi}(G_{\Omega}))^{iscl}\widetilde{\subseteq}f_{\phi}(\mathrm{cl}(G_{\Omega}))$.
Sufficiency: Consider H_{Ω} is a soft closed set in $\widetilde{X}$. Then $f_{\phi}(H_{\Omega})\widetilde{\subseteq}(f_{\phi}(H_{\Omega}))^{iscl}\widetilde{\subseteq} f_{\phi}(\mathrm{cl}(H_{\Omega})) = f_{\phi}(H_{\Omega})$. Therefore, $f_{\phi}(H_{\Omega}) = (f_{\phi}(H_{\Omega}))^{iscl}$. This means that $f_{\phi}(H_{\Omega})$ is an SI-semi closed set. □

Theorem 9 *The next hold for a bijective soft map* $f_\phi : (X, \tau, \Omega, \preceq_1) \rightarrow (Y, \theta, \Gamma, \preceq_2)$*:*

(i) f_ϕ *is SI (resp. SD, SB) -semi open if and only if* f_ϕ *is SD (resp. SD, SB) -semi closed.*

(ii) f_ϕ *is SI (resp. SD, SB) -semi open if and only if* f_ϕ^{-1} *is SI (resp. SD, SB) -semi continuous.*

(iii) f_ϕ *is SD (resp. SI, SB) -semi closed if and only if* f_ϕ^{-1} *is SI (resp. SD, SB) -semi continuous.*

Proof. We prove the cases outside the parenthesis and the cases between parenthesis can be made similarly.

(i) Necessity: Let H_Ω be a soft closed set in $\widetilde{X}$ and consider f_ϕ is an SI-semi open map. Then H^c_Ω is soft open and $f_\phi(H^c_\Omega)$ is SI-semi open. Bijectiveness of f_ϕ leads to that $f_\phi(H^c_\Omega) = [f_\phi(H_\Omega)]^c$. This automatically implies that $f_\phi(H_\Omega)$ is SD-semi closed. Thus, f_ϕ is an SD-semi closed map. Following similar technique, the sufficient condition is proved.

(ii) Necessity: Let G_Ω be a soft open set in $\widetilde{X}$ and consider f_ϕ is an SI-semi open map. Then $f_\phi(G_\Omega)$ is SI-semi open. Bijectiveness of f_ϕ leads to that $f_\phi(G_\Omega) = (f_\phi^{-1})^{-1}(G_\Omega)$. This automatically implies that $(f_\phi^{-1})^{-1}(G_\Omega)$ is SI-semi open. Thus f_ϕ^{-1} is an SI-semi continuous map. Following similar technique, the sufficient condition is proved.

(iii) It follows from **(i)** and **(ii)**.

□

Theorem 10 *Let a soft topology* $\theta^\star$ *be extended and a map* ϕ *be injective. Then* $g_\phi : (X, \tau, \Omega, \preceq_1) \rightarrow (Y, \theta^\star, \Gamma, \preceq_2)$ *is SI (resp. SD, SB) -semi open iff a crisp map* $g : (X, \tau_\gamma, \preceq_1) \rightarrow (Y, \theta^\star_{\phi(\omega)}, \preceq_2)$ *is I (resp. D, B) -semi open.*

Proof. Let U be an open set in $(X, \tau_\gamma, \preceq_1)$ and $\phi(\omega) = f$. Then there is a soft open set G_Ω in $(X, \tau, \Omega, \preceq_1)$ s.t. $G(\omega) = U$. Since g_ϕ is an SI (resp. SD, SB) -semi open map, then $g_\phi(G_\Omega)$ is an SI (resp. SD, SB) -semi open set. Now, a soft set $g_\phi(G_\Omega) = (g_\phi(G))_\Gamma$ in $(Y, \theta, \Gamma, \preceq_2)$ is given by $g_\phi(G)(f) = \bigcup_{\omega \in \phi^{-1}(f)} g(G(\omega))$ for each $f \in \Gamma$. View of $\theta^\star$ is extended, a set $\bigcup_{\omega \in \phi^{-1}(f)} g(G(\omega)) = g(U)$ in $(Y, \theta_{\phi(\omega)}, \preceq_2)$ is I (resp. D, B) -semi open.

Hence a map g is I (resp. D, B) -semi open. Conversely, consider G_Ω is a soft open set in $(X, \tau, \Omega, \preceq_1)$. Then a soft set $g_\phi(G_\Omega) = (g_\phi(G))_\Gamma$ in $(Y, \theta^\star, \Gamma, \preceq_2)$ is given by $g_\phi(G)(f) = \bigcup_{\omega \in \phi^{-1}(f)} g(G(\omega))$ for each $f \in \Gamma$. Since a map g is I (resp. D, B) -semi open, a set $\bigcup_{\omega \in \phi^{-1}(f)} g(G(\omega))$ in $(Y, \theta^\star_{\phi(\omega)}, \preceq_2)$ is I (resp. D, B) -semi open. Now, $\theta^\star$ is an extended soft topology on Y, $g_\phi(G_\Omega)$ is an SI (resp. SD, SB) -semi open subset of $(Y, \theta^\star, \Gamma, \preceq_2)$. Hence a soft map g_ϕ is SI (resp. SD, SB) -semi open. □

Theorem 11 *Let a soft topology* $\theta^\star$ *be extended and a map* ϕ *is injective. Then* $g_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta^\star, \Gamma, \preceq_2)$ *is SI (resp. SD, SB) -semi closed iff a crisp map* $g : (X, \tau_\gamma, \preceq_1) \to (Y, \theta^\star_{\phi(\omega)}, \preceq_2)$ *is I (resp. D, B) -semi closed.*

Proposition 5 *Let* $x \in \{I, D, B\}$ *and consider* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *and* $g_\lambda : (Y, \theta, \Gamma, \preceq_2) \to (Z, \upsilon, K, \preceq_3)$ *are soft maps. Then:*

(i) *If* f_ϕ *is an Sx-semi continuous map and* g_λ *is a soft continuous map, then* $g_\lambda \circ f_\phi$ *is an Sx-continuous map.*

(ii) *If* f_ϕ *is a soft open (resp. soft closed) map and* g_λ *is an Sx-semi open (resp. Sx-semi closed) map, then* $g_\lambda \circ f_\phi$ *is an Sx-semi open (resp. Sx-semi closed) map.*

(iii) *If* $g_\lambda \circ f_\phi$ *is an Sx-open map and* f_ϕ *is surjective soft continuous, then* g_λ *is an Sx-open map.*

3.3 Soft $I(D, B)$-semi homeomorphism

We define and investigate in this subsection, the concepts of soft $I(D, B)$-semi homeomorphism maps. We discussed their main features and verify some findings related to them.

Definition 22 *A bijective soft map* $g_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *is called SI (resp. SD, SB) -semi homeomorphism if it is SI-semi continuous and SI-semi open (resp. SD-semi continuous and SD-semi open, SB-semi continuous and SB-semi open).*

Remark 4 *Note that:*

1. *an SI (SD, SB) -semi homeomorphism map is soft semi homeomorphism.*
2. *an SB-semi homeomorphism map is SI-semi homeomorphism or SD-semi homeomorphism.*

Examples given below manifest that the results of the remark above are not reversible.

Example 5 *Let* Ω, X, $\phi : \Omega \to \Omega$, $f : X \to X$ *and* $\preceq$ *be the same as in Example (1). Consider* $\tau = \{\widetilde{\emptyset}, \widetilde{X}, F_\Omega, L_\Omega\}$ *and* $\theta = \{\widetilde{\emptyset}, \widetilde{Y}, L_\Omega\}$ *are two soft topologies on* X*, where* $F_\Omega = \{(\omega_1, \{1\}), (\omega_2, \{3, 4\})\}$ *and* $L_\Omega = \{(\omega_1, \{1\}), (\omega_2, \{3\})\}$*. Then we find that* $f_\phi : (X, \tau, \Omega, \preceq) \to (X, \theta, \Omega, \preceq)$ *is a soft semi homeomorphism map, but it is neither an SD-semi homeomorphism map nor an SI-semi homeomorphism map. Hence* f_ϕ *is not SI (SD, SB)-semi homeomorphism.*

Example 6 *By replacing a partial order relation (given in example above) by* $\preceq = \triangle \bigcup \{(2, 4)\}$*, we find that a soft map* f_ϕ *is SI-semi homeomorphism, but it is not an SB-semi homeomorphism map. Also, replacing a partial order relation (given in example above) by* $\preceq = \triangle \bigcup \{(1, 2)\}$ *leads to that a soft map* f_ϕ *is SD-homeomorphism, but it is not an SB-semi homeomorphism map.*

Theorem 12 *Consider* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *is a bijective soft map and let* $(\gamma, \lambda) \in \{(Is, dscl), (Ds, iscl), (Bs, bscl)\}$*. Then* f_ϕ *is soft* γ*-homeomorphism if and only if* $(f_\phi(G_\Omega))^\lambda = f_\phi(cl(G_\Omega)) = cl(f_\phi(G_\Omega)) = f_\phi(G_\Omega^\lambda)$ *for every* $G_\Omega \widetilde{\subseteq} \widetilde{X}$*.*

Proof. We only prove the case of $(\gamma, \lambda) = (Is, dscl)$.
Necessity: The property f_ϕ is an SI-semi homeomorphism map implies that $f_\phi(G_\Omega^{dscl}) \widetilde{\subseteq} cl(f_\phi(G_\Omega))$ and $(f_\phi(G_\Omega))^{dscl} \widetilde{\subseteq} f_\phi(cl(G_\Omega))$ for every $G_\Omega \widetilde{\subseteq} \widetilde{X}$. So $f_\phi(cl(G_\Omega)) \widetilde{\subseteq} f_\phi(G_\Omega^{dscl}) \widetilde{\subseteq} cl(f_\phi(G_\Omega)) \widetilde{\subseteq} (f_\phi(G_\Omega))^{dscl}$ and $cl(f_\phi(G_\Omega)) \widetilde{\subseteq} (f_\phi(G_\Omega))^{dscl} \widetilde{\subseteq} f_\phi(cl(G_\Omega)) \widetilde{\subseteq} f_\phi(G_\Omega^{dscl})$. By the preceding two inclusion relations, we obtain the required equality relation.
Sufficiency: The equality relation $(f_\phi(G_\Omega))^{dscl} = f_\phi(cl(G_\Omega)) = cl(f_\phi(G_\Omega)) = f_\phi(G_\Omega^{dscl})$ implies that $f_\phi(G_\Omega^{dscl}) \widetilde{\subseteq} cl(f_\phi(G_\Omega))$ and $(f_\phi(G_\Omega))^{dscl} \widetilde{\subseteq} f_\phi(cl(G_\Omega))$. So f_ϕ is SI-semi continuous and SD-semi closed map. Hence the desired result is proved. □

Theorem 13 *If a bijective soft map* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *is SI-semi continuous (resp. SD-semi continuous, SB-semi continuous), Then the following three statements are equivalent:*

1. f_ϕ *is SI-semi homeomorphism (resp. SD-semi homeomorphism, SB-semi homeomorphism);*

2. f_ϕ^{-1} *is SI-semi continuous (resp. SD-semi continuous, SB-semi continuous);*

3. f_ϕ *is SD-semi closed (resp. SI-semi closed, SB-semi closed).*

Proof. $1 \Rightarrow 2$: Since f_ϕ is an SI-semi homeomorphism (resp. SD-semi homeomorphism, SB-semi homeomorphism) map, f_ϕ is SI-semi open (resp. SD-semi open, SB-semi open). It comes from item 2 of Theorem (9) that f_ϕ^{-1} is SI-semi continuous (resp. SD-semi continuous, SB-semi continuous).
$2 \Rightarrow 3$: It comes from item 3 of Theorem (9).
$3 \Rightarrow 1$: It suffices to prove that f_ϕ is an SI-semi open (resp. SD-semi open, SB-semi open) map. This comes from item 1 of Theorem (9). □

Theorem 14 *Let soft topologies* $\tau^\star$ *and* $\theta^\star$ *be extended on* X *and* Y*, respectively. Then a soft map* $g_\phi : (X, \tau^\star, \Omega, \preceq_1) \to (Y, \theta^\star, \Gamma, \preceq_2)$ *is SI (resp. SD, SB) -semi homeomorphism iff amap* $g : (X, \tau^\star_\gamma, \preceq_1) \to (Y, \theta^\star_{\phi(\omega)}, \preceq_2)$ *is I (resp. D, B) -semi homeomorphism.*

Proof. Directly from Theorem (5) and Theorem (10). □

Proposition 6 *Let the two soft topologies* τ *and* θ *on* X *and* Y*, respectively, do not belong to {soft discrete topology, soft indiscrete topology}. If a soft map* $f_\phi : (X, \tau, \Omega, \preceq_1) \to (Y, \theta, \Gamma, \preceq_2)$ *is SB-semi homeomorphism, then* $\preceq_1$ *and* $\preceq_2$ *is not linearly ordered.*

4 Conclusion

Study topological concepts in the ordered domain is an important issue because it helps to obtain some properties induced from the interaction between topology and algebra. Also, it helps to describe and solve some practical problems; see [8]

To this end, we [16] have formulated the concept of soft topological ordered spaces as an extension of the concept of soft topological spaces. Then we [17] have utilized monotone soft sets to define some soft ordered maps and investigated their main properties. In this work, we have used soft semi open sets to give the concepts of soft x-semi continuous, soft x-semi open, soft x-semi closed and soft x-semi homeomorphism maps for $x \in \{I, D, B\}$. We have given various characterizations for these concepts and have shown the relationships among them with the help of examples. It should be noted that results obtained herein and results obtained in [14] are independent of each other. Also, they are special case of results obtained in [32, 19] and are genuine generalizations of results obtained in [18].

References

[1] M. Abo-Elhamayel and T. M. Al-shami, Supra homeomorphism in supra topological ordered spaces, *Facta Universitatis, Series: Mathematics and Informatics*, **31** (5) (2016), 1091–1106.

[2] S. Al-Ghour, Soft ω^*-paracompactness in soft topological spaces, *International Journal of Fuzzy Logic and Intelligent Systems*, **21**(1) (2021), 57–65.

[3] S. Al-Ghour and W. Hamed, On two classes of soft sets in soft topological spaces, *Symmetry*, **12** (2020), 265; doi:10.3390/sym12020265

[4] T. M. Al-shami, Supra β-bicontinuous maps via topological ordered spaces, *Mathematical Sciences Letters*, **6**(3) (2017), 239–247.

[5] T. M. Al-shami, Corrigendum to "On soft topological space via semi-open and semi-closed soft sets, Kyungpook Mathematical Journal, 54 (2014) 221-236", *Kyungpook Mathematical Journal*, **58**(3) (2018), 583–588.

[6] T. M. Al-shami, On some maps in supra topological ordered spaces, *Journal of New Theory*, **20** (2018), 76–92.

[7] T. M. Al-shami, Soft somewhere dense sets on soft topological spaces, *Communications of the Korean Mathematical Society*, **33**(4) (2018), 1341–1356.

[8] T. M. Al-shami, Compactness on soft topological ordered spaces and its application on the information system, *Journal of Mathematics*, **Volume 2021**, Article ID 6699092, 12 pages.

[9] T. M. Al-shami, Sum of the spaces on ordered setting, *Moroccan Journal of Pure and Applied Analysis*, **6**(2) (2020), 255–265.

[10] T. M. Al-shami and M. Abo-Elhamayel, Novel class of ordered separation axioms using limit points, *Applied Mathematics & Information Sciences*, **14**(6)(2020), 1103–1111.

[11] T. M. Al-shami, I. Alshammari and B. A. Asaad, Soft maps via soft somewhere dense sets, *Filomat*, **34**(10)(2020), 3429–3440.

[12] T. M. Al-shami, M. A. Al Shumrani and C. Özel, Another form of supra ordered separation axioms, *Journal of Mathematical Extension*, **15**(1) (2021), 105–125.

[13] T. M. Al-shami and M. E. El-Shafei, On supra soft topological ordered spaces, *Arab Journal of Basic and Applied Sciences*, **26**(1) (2019), 433–445.

[14] T. M. Al-shami and M. E. El-Shafei, Some types of soft ordered maps via soft pre open sets, *Applied Mathematics & Information Sciences*, **13**(5) (2019), 707–715.

[15] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, Almost soft compact and approximately soft Lindelöf spaces, *Journal of Taibah University for Science*, **12**(5) (2018), 620–630.

[16] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, On soft topological ordered spaces, *Journal of King Saud University-Science*, **31**(4) (2019), 556–566.

[17] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, On soft ordered maps, *General letters in Mathematics*, **5**(3) (2018), 118–131.

[18] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, New types of soft ordered mappings via soft α-open sets, *Italian Journal of Pure and Applied Mathematics*, **42** (2019), 357–375.

[19] T. M. Al-shami, M. E. El-Shafei and B. A. Asaad, Other kinds of soft β mappings via soft topological ordered spaces, *European Journal of Pure and Applied Mathematics*, **12**(1) (2019), 176–193.

[20] T. M. Al-shami and L. D. R. Kočinac, The equivalence between the enriched and extended soft topologies, *Appl. Comput. Math.*, **18**(2) (2019), 149–162.

[21] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, *Computers and Mathematics with Applications*, **57** (2009), 1547–1553.

[22] A. Aygünoğlu and H. Aygün, Some notes on soft topological spaces, *Neural Computers and Applications*, **21** (2012), 113–119.

[23] B. A. Asaad, M. K. Tahat, T. M. Al-shami, Supra b maps via topological ordered spaces, *European Journal of Pure and Applied Mathematics*, **12** (3) (2019), 1231–1247.

[24] B. Chen, Soft semi-open sets and related properties in soft topological spaces, *Applied Mathematics & Information Sciences*, **7**(1) (2013), 287–294.

[25] B. Chen, Some local properties of soft semi-open sets, *Discrete Dynamics in Nature and Society*, **Volume 2013**, Article ID 298032, 6 pages.

[26] P. Das, Separation axioms in ordered spaces, *Soochow Journal of Mathematics*, **30**(4) (2004), 447–454.

[27] S. Das and S. K. Samanta, Soft metric, *Annals of Fuzzy Mathematics and Informatics*, **6**(1) (2013), 77–94.

[28] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, Strong separation axioms in supra topological ordered spaces, *Mathematical Sciences Letters*, **6**(3) (2017), 271–277.

[29] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, Partial soft separation axioms and soft compact spaces, *Filomat*, **32**(13) (2018), 4755–4771.

[30] M. E. El-Shafei and T. M. Al-shami, Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem, *Computational and Applied Mathematics*, **39**(3) (2020),138, https://doi.org/10.1007/s40314-020-01161-3

[31] M. E. El-Shafei and T. M. Al-shami, Some operators of a soft set and soft connected spaces using soft somewhere dense sets, *Journal of Interdisciplinary Mathematics*, (2021), https://doi.org/10.1080/09720502.2020.1842348

[32] M. E. El-Shafei and T. M. Al-shami, Some new types of soft b-ordered mappings, *International Journal of Advances in Mathematics*, **Volume 2019** (3) (2019), 1–14.

[33] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, γ-operation and decompositions of some forms of soft continuity in soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, **7**(2) (2014), 181–196.

[34] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi separation axioms and some types of soft functions, *Annals of Fuzzy Mathematics and Informatics*, **8**(2) (2014), 305–318.

[35] M. K. R. S. V. Kumar, Homeomorphism in topological ordered spaces, *Acta Ciencia Indica*, **XXVIII(M)**(1)(2012), 67–76.

[36] F. Lin, Soft connected spaces and soft paracompact spaces, International *Journal of Mathematical Science and Engineering*, **7**(2) (2013), 1–7.

[37] J. Mahanta and P. K. Das, On soft topological space via semi-open and semi-closed soft sets, *Kyungpook Mathematical Journal*, **54** (2014), 221–236.

[38] P. K. Maji, R. Biswas and R. Roy, Soft set theory, *Computers and Mathematics with Applications*, **45** (2003), 555–562.

[39] S. D. McCartan, Separation axioms for topological ordered spaces, *Mathematical Proceedings of the Cambridge Philosophical Society*, **64** (1986), 965–973.

[40] D. Molodtsov, Soft set theory-first results, *Computers and Mathematics with Applications*, **37** (1999), 19–31.

[41] L. Nachbin, *Topology and order*, D. Van Nostrand Inc. Princeton, New Jersey, (1965).

[42] S. Nazmul and S. K. Samanta, Neigbourhood properties of soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, **6**(1) (2013), 1–15.

[43] K. K. Rao and R. Chudamani, Semi-homeomorphism in topological ordered spaces, *International Journal of Mathematical Sciences, Technology and Humanities*, **39** (2012), 383–397.

[44] W. Rong, The countabilities of soft topological spaces, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, **6**(8) (2012), 952–955.

[45] M. Shabir and M. Naz, On soft topological spaces, *Computers and Mathematics with Applications*, **61** (2011), 1786–1799.

[46] I. Zorlutuna, M. Akdag, W. K. Min and S. K. Samanta, Remarks on soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, **2** (2012), 171–185.

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Generalizations of graded second submodules

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Abstract. Let G be a group with identity e. Let R be a graded ring, I a graded ideal of R and M be a G-graded R-module. Let $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function, where $S^{gr}(M)$ denote the set of all graded submodules of M. In this article, we introduce and study the concepts of graded ψ-second submodules and graded I-second submodules of a graded R-module which are generalizations of graded second submodules of M and investigate some properties of this class of graded modules.

1 Introduction

The study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction [13]. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded

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structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [1, 4, 10, 11, 12, 14]). Throughout this work, all graded rings are assumed to be commutative graded rings with identity, and all graded modules are unitary graded R-modules. We will denote the set of graded ideals of R by $S^{gr}(R)$ and the set of all graded submodules of M by $S^{gr}(M)$. Let G be a group with identity e and R be a ring. Then R is said to be a G-graded if $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are homogeneous of degree g. Consider $supp(R, G) = \{g \in G \mid R_g \neq 0\}$. For simplicity, we will denote the graded ring (R, G) by R. If $r \in R$, then r can be written as $\sum_{g\in G} r_g$, where r_g is the component r in R_g. Moreover, R_e is a subring of R and if R contains a unitary 1, then $1 \in R_e$. Furthermore, $h(R) = \bigcup_{g\in G} R_g$.

Let I be a left ideal of a graded ring R. Then I is said to be a graded ideal of R, if $I = \bigoplus_{g\in G}(I \cap R_g)$, i. e., for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. A proper graded ideal I of a graded ring R is said to be graded prime if whenever $r_gs_h \in I$ for some $r_g, s_h \in h(R)$, then $r_g \in I$ or $s_h \in I$. Graded primary (prime) ideals over commutative graded rings have been studied by [14].

Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M, G) = \{g \in G \mid M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i. e., for $m \in N$, $m = \sum_{g\in G} m_g$, where $m_g \in N$ for all $g \in G$. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$.

A proper graded submodule N of a graded R-module M is said to be graded prime, if $r_gm_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. A graded R-module M is called graded prime, if the zero graded submodule is graded prime in M. For more information about graded prime submodules over commutative graded rings see [3, 7, 9]. A graded R-module M is called graded finitely generated if $M = Rm_{g_1} + Rm_{g_2} + \cdots + Rm_{g_n}$ for some $m_{g_1}, \cdots, m_{g_n} \in h(M)$. Farshadifar and Ansari-Toroghy in [5, 6] introduced the concepts of I-second submodules of M and ψ-second submodules of M which are two generalizations of second submodules of M. In the first section of this paper, we introduce and study the notion of graded ψ-second submodules of a graded R-module M and we investigate some properties of such graded submodules. For example, in Theorem 7, we characterize graded ψ-second submodules of a graded R-module M. In the second section, we introduce the notion of graded I-second submodules of a graded R-module M and obtain some related results. For example, we prove when a graded submodule of a graded R-module is a graded I-second submodule.

2 Graded ψ-second submodules

In this section, we define and study graded ψ-second submodules of a graded module over a commutative graded ring.

The following Lemma is known, but we write it here for the sake of references.

Lemma 1 *Let M be a graded module over a graded ring R. Then the following hold:*

(i) *If I and J are graded ideals of R, then $I+J$ and $I\bigcap J$ are graded ideals of R.*

(ii) *If I is a graded ideal of R, N is a graded submodule of M, $r \in h(R)$ and $x \in h(M)$, then Rx, IN, rN and $(0:_M I)$ are graded submodules of M.*

(iii) *If N and K are graded submodules of M, then $N+K$ and $N\bigcap K$ are also graded submodules of M and $(N:_R M)$ is a graded ideal of R. Also, $Ann_R(M) = (0:_R M)$ is a graded ideal of R.*

(iv) *Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Definition 1 *Let M be a graded R-module and let $g \in G$.*

(a) *A non-zero submodule N_g of R_e- module M_g is said to be g-second submodule of M_g, if for each $r_e \in R_e$, either $r_eN_g = 0$ or $r_eN_g = N_g$.*

(b) *A non-zero graded submodule N of M is said to be a graded second submodule of M if for each $r_g \in h(R)$, either $r_gN = 0$ or $r_gN = N$.*

Definition 2 *Let M be a graded R-module and let $g \in G$. Let $\psi : S(M_g) \to S(M_g)\cup\{\emptyset\}$ be a function, where $S(M_g)$ is the set of all submodules of M_g. We say that a non-zero submodule N_g of R_e-module M_g is a g-ψ-second submodule, if $r_e \in R_e$, K a submodule of M_g, $r_eN_g \subseteq K$, and $r_e\psi(N_g) \nsubseteq K$, then $N_g \subseteq K$ or $r_eN_g = 0$.*

Definition 3 *Let* M *be a graded* R*-module,* $S^{gr}(M)$ *be the set of all graded submodules of* M*, and let* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be a function. We say that a non-zero graded submodule* N *of* M *is a graded* ψ*-second submodule of* M *if* $r_g \in h(R)$*,* K *a graded submodule of* M*,* $r_gN \subseteq K$*, and* $r_g\psi(N) \nsubseteq K$*, then* $N \subseteq K$ *or* $r_gN = 0$

We use the following functions $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$.

$$\begin{aligned} \psi_M(N) &= M, \ \forall N \in S^{gr}(M), \\ \psi_i(N) &= (N :_M \mathrm{Ann}_R^i(N)), \ \forall N \in S^{gr}(M), \ \forall i \in \mathbb{N}, \\ \psi_\sigma(N) &= \sum_{i=1}^{\infty} \psi_i(N), \ \forall N \in S^{gr}(M). \end{aligned}$$

Then it is clear that for any graded submodule and every positive integer n, we have the following implications:

$$\begin{aligned} \text{graded second} &\Rightarrow \text{graded } \psi_{n-1} - \text{second} \Rightarrow \text{graded } \psi_n - \text{second} \\ &\Rightarrow \text{graded } \psi_\sigma - \text{second} \end{aligned}$$

For functions $\psi, \theta : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$, we write $\psi \leq \theta$ if $\psi(N) \subseteq \theta(N)$ for each $N \in S^{gr}(M)$. So whenever $\psi \leq \theta$, any graded ψ-second submodule is graded θ-second.

Theorem 1 *Let* M *be a graded* R*-module and* N *be a graded submodule of* R*. Then the following statements are equivalent:*

(i) N *is a graded second submodule of* M*.*

(ii) $N \neq 0$ *and* $r_gN \subseteq K$*, where* $r_g \in h(R)$ *and* K *is a graded submodule of* M*, implies either* $r_gN = 0$ *or* $N \subseteq K$*.*

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (i) Let $r_g \in h(R)$ and $r_gN \neq 0$. Since $r_gN \subseteq r_gN$, so $N \subseteq r_gN$ by assumption. Therefore $r_gN = N$, as needed. □

Theorem 2 *Let* M *be a graded* R*-module,* N *a graded submodule of* M *and let* $g \in G$*. Let* $\psi : S(M_g) \to S(M_g) \cup \{\emptyset\}$ *be a function and* N_g *be a* g*-*ψ*-second submodule of* R_e*-module* M_g *such that* $\mathrm{Ann}_{R_e}(N_g)\psi(N_g) \nsubseteq N_g$*. Then* N_g *is a* g*-second submodule of* M_g*.*

Proof. Let $r_e \in R_e$ and K be a submodule of M_g such that $r_eN_g \subseteq K$. If $r_e\psi(N_g) \nsubseteq K$, then we are done because N_g is a g-ψ-second submodule of R_e-module M_g. Thus suppose that $r_e\psi(N_g) \subseteq K$. If $r_e\psi(N_g) \nsubseteq N_g$, then $r_e\psi(N_g) \nsubseteq N_g \cap K$. Since $r_eN_g \subseteq N_g \cap K$, then $N_g \subseteq N_g \cap K \subseteq K$ or $r_eN_g = 0$, as required. So let $r_e\psi(N_g) \subseteq N_g$. If $Ann_{R_e}(N_g)\psi(N_g) \nsubseteq K$, then $(r_e + Ann_{R_e}(N_g))\psi(N_g) \nsubseteq K$. Thus $(r_e + Ann_{R_e}(N_g))N_g \subseteq K$ implies that $N_g \subseteq K$ or $r_eN_g = (r_e+Ann_{R_e}(N_g))N_g = 0$, as needed. Hence let $Ann_{R_e}(N_g)\psi(N_g) \subseteq K$. Since $Ann_{R_e}(N_g)\psi(N_g) \nsubseteq N_g$, there exists $s_e \in Ann_{R_e}(N_g)$ such that $(s_e\psi(N_g) \nsubseteq N_g$. Thus $s_e\psi(N_g) \nsubseteq N_g \cap K$. Hence we have $(r_e + s_e)\psi(N_g) \nsubseteq N_g \cap K$. Therefore, $(r_e + s_e)N_g \subseteq N_g \cap K$ implies that $N_g \subseteq N_g \cap K \subseteq K$ or $(r_e + s_e)N_g = r_eN_g = 0$, as needed. □

Corollary 1 *Let M be a graded R-module, N a graded submodule of M and $g \in G$. Let $\psi : S(M_g) \to S(M_g) \cup \{\emptyset\}$ be a function and N_g be a g-ψ-second submodule of R_e-module M_g such that $(N_g :_{M_g} Ann^2_{R_e}(N_g)\psi(N_g) \subseteq \psi(N_g)$. Then N_g is a g-ψ_σ-second submodule of M_g.*

Proof. If N_g is a g-second submodule of M_g, then the result is clear. So assume that N_g is not g-second submodule of M_g. Then by Theorem 2, we have $Ann_{R_e}(N_g)\psi(N_g) \subseteq N_g$. Therefore, by assumption,

$$(N_g :_{M_g} Ann^2_{R_e}(N_g)) \subseteq \psi(N_g) \subseteq (N_g :_{M_g} Ann_{R_e}(N_g)).$$

We conclude that $\psi(N_g) = (N_g :_{M_g} Ann^2_{R_e}(N_g)) = (N_g :_{M_g} Ann_{R_e}(N_g))$, because $(N_g :_{M_g} Ann_{R_e}(N_g)) \subseteq (N_g :_{M_g} Ann^2_{R_e}(N_g))$. So we get

$$(N_g :_{M_g} Ann^3_{R_e}(N_g)) = (((N_g :_{M_g} Ann^2_{R_e}(N_g)) :_{M_g} Ann_{R_e}(\psi(N_g))) =$$

$$((N_g :_{M_g} Ann_{R_e}(N_g)) :_{M_g} Ann_{R_e}(N_g)) = (N_g :_{M_g} Ann^2_{R_e}(N_g)) = \psi(N_g).$$

By continuing, we get that $\psi(N_g) = (N_g :_{M_g} Ann^i_{R_e}(N_g))$ for all $i \geq 1$. Hence $\psi(N_g) = \psi_\sigma(N_g)$, as needed. □

Theorem 3 *Let M be a graded R-module and $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. Let N be a graded submodule of M such that for all graded ideals I and J of R, $(N :_M I) \subseteq (N :_M J)$ implies that $J \subseteq I$. If N is not a graded second submodule of M, then N is not a graded ψ_1-second submodule of M.*

Proof. Since N is not a graded second submodule of M, there exists $r_g \in h(R)$ and a graded submodule K of M such that $r_gN \neq 0$ and $N \nsubseteq K$, but $r_gN \subseteq K$

by Theorem 1. We have $N \nsubseteq N \cap K$ and $r_g N \subseteq N \cap K$. If $r_g(N :_M Ann_R(N)) \nsubseteq N \cap K$, then N is not a graded ψ_1-second submodule of M. Hence let $r_g(N :_M Ann_R(N)) \subseteq N \cap K$. Thus $r_g(N :_M Ann_R(N)) \subseteq N \cap K \subseteq N$. Therefore, $(N :_M Ann_R(N)) \nsubseteq (N :_M r_g)$ and so by assumption, $r_g \in Ann_R(N)$, which is a contradiction. □

Corollary 2 *Let M be a graded R-module and $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. Let N be a graded submodule of M such that for all graded ideals I and J of R, $(N :_M I) \subseteq (N :_M J)$ implies that $J \subseteq I$. Then N is a graded second submodule of M if and only if N is a graded ψ_1-second submodule of M.*

A graded R-module M is said to be a graded multiplication module if for every graded submodule N of M, there exists a graded ideal I of R such that $N = IM$. It is easy to see that M is a graded multiplication module if and only if $N = (N : M)M$ for each graded submodule N of M [8].

A graded R-module M is said to be a graded comultiplication module if for every graded submodule N of M, there exists a graded ideal I of R such that $N = (0 :_M I)$ [2].

Definition 4 *Let R be a graded ring and $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$ be a function. A proper graded ideal P of R is called graded φ-prime, if for $a_g, b_h \in h(R)$, $a_g b_h \in P - \varphi(P)$, then $a_g \in P$ or $b_h \in P$.*

Definition 5 *Let M be a graded R-module and $\varphi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. A proper graded submodule N of M is said to be graded φ-prime, if for each $r_g \in h(R)$ and $m_g \in h(M)$, $r_g m_h \in N \setminus \varphi(N)$, then $m_h \in N$ or $r_g \in (N :_R M)$.*

Theorem 4 *Let M be a graded R-module, $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$, and $\theta : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be functions such that $\theta(P) = \varphi((P :_R M))M$. The following statements hold:*

(i) *If P is a graded θ-prime submodule of M such that $(\theta(P) :_R M) \subseteq \varphi((P :_R M))$, then $(P :_R M)$ is a graded φ-prime ideal of R.*

(ii) *If M is a graded multiplication R-module and $(P :_R M)$ is a graded φ-prime ideal of R, then P is a graded θ-prime submodule of M.*

Proof. (i) Let $a_g b_h \in (P :_R M) \setminus \varphi((P :_R M))$ for some $a_g, b_h \in h(R)$. If $a_g b_h M \subseteq \theta(P)$, then $a_g b_h \in \varphi((P :_R M))$, a contradiction. Thus $a_g b_h M \nsubseteq$

$\theta(P)$. Therefore, $a_gM \subseteq P$ or $b_hM \subseteq P$ because P is a graded θ-prime submodule of M. Thus $a_g \in (P :_R M)$ or $b_h \in (P :_R M)$, as needed.
(ii) Let $a_gm_h \in P \setminus \theta(P) = P \setminus \varphi((P :_R M))M$. Then $a_g((Rm_h :_R M))M \subseteq P$. If $a_g((P :_R M)) \subseteq \varphi((Rm_h :_R M))$, then $a_g((Rm_h :_R M))M \subseteq \varphi((P :_R M))M$. As M is a graded multiplication R-module, we have $a_gm_h \in Rm_h = (Rm_h :_R M)M$. Therefore, $a_gm_h \in \varphi((P :_R M))M$ which is a contradiction. Thus $a_g((Rm_h :_R M)) \nsubseteq \varphi((P :_R M))$ and so by assumption, $a_g \in (P :_R M)$ or $(Rm_h :_M M) \subseteq (P :_R M)$, as needed. □

Theorem 5 *Let M be a graded R-module, $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$, and $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be functions. Then the following hold:*

- (i) *If S is a graded ψ-second submodule of M such that $Ann_R(\psi(S)) \subseteq \varphi(Ann_R(S))$, then $Ann_R(S)$ is a graded φ-prime ideal of R.*
- (ii) *If M is a graded comultiplication R-module, S is a graded submodule of M such that $\psi(S) = (0 :_M \varphi(Ann_R(S)))$, and $Ann_R(S)$ is a graded φ-prime ideal of R, then S is a graded ψ-second submodule of M.*

Proof. (i) Let $a_gb_h \in Ann_R(S) \setminus \varphi(Ann_R(S))$ for some $a_g, b_h \in h(R)$. Then $a_gb_h\psi(S) \neq 0$ by assumption. If $a_g\psi(S) \subseteq (0 :_M b_h)$, then $a_gb_h\psi(S) = 0$, a contradiction. Thus $a_g\psi(S) \nsubseteq (0 :_M b_h)$. Therefore, $S \subseteq (0 :_M b_h)$ or $a_gS = 0$ because S is a graded ψ-second submodule of M. Hence $a_g \in Ann_R(S)$ or $b_h \in Ann_R(S)$, as required.
(ii) Let $a_g \in h(R)$ and K be a graded submodule of M such that $a_gS \subseteq K$ and $a_g\psi(S) \nsubseteq K$. As $a_gS \subseteq K$, we have $S \subseteq (K :_M a_g)$. It follows that

$$S \subseteq ((0 :_M Ann_R(K)) :_M a_g) = (0 :_M a_gAnn_R(K)).$$

This implies that $a_gAnn_R(K) \subseteq Ann_R((0 :_M a_gAnn_R(K))) \subseteq Ann_R(S)$. Hence $a_gAnn_R(K) \subseteq Ann_R(S)$. If $a_gAnn_R(K) \subseteq \varphi(Ann_R(S))$, then

$$\psi(S) = ((0 :_M \varphi(Ann_R(S)) = ((0 :_M Ann_R(K)) :_M a_g).$$

As M is a graded comultiplication R-module, we have $a_g\psi(S) \subseteq K$, a contradiction. Thus $a_gAnn_R(K) \nsubseteq \varphi(Ann_R(S))$ and so as $Ann_R(S)$ is a graded φ-prime ideal of R, we conclude that $a_gS = 0$ or

$$S = (0 :_M Ann_R(S)) \subseteq (0 :_M Ann_R(K)) = K$$

as needed. □

Example 1 *Let* $G = \mathbb{Z}_2$ *and* $R = \mathbb{Z}$ *be a* G*-graded ring with* $R_0 = \mathbb{Z}$ *and* $R_1 = \{0\}$. *Let* $M = \mathbb{Z} \times \mathbb{Z}$. *Then* M *is a* G*-graded* R*-module with* $M_0 = \mathbb{Z} \times \{0\}$ *and* $M_1 = \{0\} \times \mathbb{Z}$. *Consider the graded submodule* $S = (2\mathbb{Z} \times \{0\}) \oplus (\{0\} \times 2\mathbb{Z})$. *Clearly,* M *is not a graded comultiplication* R*-module. Suppose that* $\varphi : S^{gr}(R) \to S^{gr}(R) \cup \{\emptyset\}$ *and* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be functions such that* $\varphi(I) = I$ *for each graded ideal* I *of* R *and* $\psi(S) = M$. *Then* $Ann_R(S) = 0$ *is a graded* φ*-prime ideal of* R *and* $\psi(S) = M = (0 :_M \varphi(Ann_R(S)))$. *But since* $4S \subseteq (8\mathbb{Z} \times \{0\}) \oplus (\{0\} \times 8\mathbb{Z})$, $S \nsubseteq (8\mathbb{Z} \times \{0\}) \oplus (\{0\} \times 8\mathbb{Z})$, *and* $4S \neq 0$, *we have* S *is not a graded* ψ*-second submodule of* M.

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fractions $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \text{ and } g = (\deg s)^{-1}(\deg r)\}$. We write $h(S^{-1}R) = \bigcup_{g \in G} (S^{-1}R)_g$ [8].

Proposition 1 *Let* M *be a graded* R*-module,* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be a function and* N *be a graded* ψ*-second submodule of* M. *Then we have the following statements.*

(i) *If* K *is a graded submodule of* M *with* $K \subset N$ *and* $\psi_K : S^{gr}(M/K) \to S^{gr}(M/K) \cup \{\emptyset\}$ *be a function such that* $\psi_K(N/K) = \psi(N)/K$, *then* N/K *is a graded* ψ_K*-second submodule of* M/K.

(ii) *Let* N *be a graded finitely generated submodule of* M, S *be a multiplicatively closed subset of* R *with* $Ann_R(N) \cap S = \emptyset$, *and* $S^{-1}\psi : S^{gr}(S^{-1}M) \to S^{gr}(S^{-1}M) \cup \{\emptyset\}$ *be a function such that* $(S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N)$. *Then* $S^{-1}N$ *is a graded* $S^{-1}\psi$*-second submodule of* $S^{-1}M$.

Proof. (i) Since $K \subset N$, then $N/K \neq 0$. Let $r_g \in h(R)$, L/K be a graded submodule of M/K, $r_g(N/K) \subseteq L/K$ and $r_g\psi(N/K) \nsubseteq L/K$. We get $r_gN \subseteq L$ and $r_g\psi(N) \nsubseteq L$. Therefore, $r_gN = 0$ or $N \subseteq L$ since N is a graded ψ-second submodule of M. Hence $r_g(N/K) = 0$ or $N/K \subseteq L/K$, as needed.
(ii) Since N is graded finitely generated and $Ann_R(N) \cap S = \emptyset$, we get $S^{-1}(N) \neq 0$. Let $\frac{r}{s} \in h(S^{-1}R)$, $S^{-1}(K)$ be a graded submodule of $S^{-1}M$ and $\frac{r}{s}(S^{-1}\psi)(S^{-1}N) \nsubseteq S^{-1}K$. Thus we get $rN \subseteq K$ and $r\psi(N) \nsubseteq K$ $((S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N))$. Hence $N \subseteq K$ or $rN = 0$ since N is a graded ψ-second submodule of M. Therefore, $S^{-1}N \subseteq S^{-1}K$ or $\frac{r}{s}\psi(S^{-1}N) = 0$, and so $S^{-1}N$ is a graded $S^{-1}\psi$-second submodule of $S^{-1}M$. □

Let $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$ be two graded ring. The function $f : R \to S$ is called a graded homomorphism, if

(i) for any $a, b \in R$, $f(a+b) = f(a) + f(b)$,

(ii) for any $a, b \in R$, $f(ab) = f(a)f(b)$, and

(iii) $f(R_g) \subseteq S_g$ for any $g \in G$.

Proposition 2 *Let* M *and* M' *be graded* R*-modules and* $f : M \to M'$ *be a graded monomorphism. Let* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *and* $\psi' : S^{gr}(M') \to S^{gr}(M') \cup \{\emptyset\}$ *be functions such that* $\psi(f^{-1}(N')) = f^{-1}(\psi'(N'))$*, for each graded submodule* N' *of* M'*. If* N' *is a graded* ψ*-second submodule of* M' *such that* $N' \subseteq Im(f)$*, then* $f^{-1}(N')$ *is a graded* ψ*-second submodule of* M*.*

Proof. Since $N' \neq 0$ and $N' \subseteq Im(f)$, we have $f^{-1}(N') \neq 0$. Let $a_g \in h(R)$ and K be a graded submodule of M such that $a_g f^{-1}(N') \subseteq K$ and $a_g \psi(f^{-1}(N')) \nsubseteq K$. Then by assumptions, $a_g N' \subseteq f(K)$ and $a_g \psi'(N') \nsubseteq f(K)$. Thus $a_g N' = 0$ or $N' \subseteq f(K)$. Therefore, $a_g f^{-1}(N') = 0$ or $f^{-1}(N') \subseteq K$, as required. □

A proper graded submodule N of a graded R-module M is said to be $\mathsf{graded\ completely\ irreducible}$ if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of graded submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every graded submodule of M is an intersection of graded completely irreducible submodules of M.

Remark 1 *Let* N, K *be graded submodules of a graded* R*-module* M*. To prove* $N \subseteq K$*, it is enough to show that if* L *is a graded completely irreducible submodule of* M *such that* $K \subseteq L$*, then* $N \subseteq L$*.*

Proposition 3 *Let* M *be a graded* R*-module,* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be a function and let* N *be a graded* ψ_1*-second submodule of* M *. Then we have the following statements:*

(i) *If for* $a_g \in h(R)$, $a_g N \neq N$*, then* $(N :_M Ann_R(N)) \subseteq (N :_M a_g)$.

(ii) *If* J *is a graded ideal of* R *such that* $Ann_R(N) \subseteq J$ *and* $JN \neq N$*, then* $(N :_M Ann_R(N)) = (N :_M J)$.

Proof. (i) By Remark 1, there exists a graded completely irreducible submodule L of M such that $a_g N \subseteq L$ and $N \nsubseteq L$. If $a_g N = 0$, then we get $(N :_M Ann_R(N)) \subseteq (N :_M a_g)$. Hence let $a_g N \neq 0$. Since N is a graded ψ_1-second submodule of M, we have $a_g(N :_M Ann_R(N)) \subseteq L$. Now let H be a graded completely irreducible submodule of M such that $N \subseteq H$. Then $N \nsubseteq L \cap H$ and $a_g N \subseteq L \cap H$. Thus as N is a graded ψ_1-second submodule of M, we

have $a_g(N :_M Ann_R(N)) \subseteq L \cap H$. Hence $a_g(N :_M Ann_R(N)) \subseteq H$. Therefore, $a_g(N :_M Ann_R(N)) \subseteq N$ by Remark 1. Hence $(N :_M Ann_R(N)) \subseteq (N :_M a_g)$.
(ii) This follows from (i). □

Theorem 6 *Let* M *be a graded* R*-module,* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be a function and let* $g \in G$*. If* $(0 :_{M_g} a_e)$ *is a* g*-*ψ_1*-second submodule of* R_e*-module* M_g *such that* $(0 :_{M_g} a_e) \subseteq a_e(0 :_{M_g} a_e Ann_R(0 :_{M_g} a_e))$*, then* $(0 :_{M_g} a_e)$ *is a* g*-second submodule of* M_g*.*

Proof. Let $N = (0 :_{M_g} a_e)$ be a g-ψ_1-second submodule of M. Then $(0 :_{M_g} a_e) \neq 0$. Let $b_e \in R_e$ and K be a submodule of M_g such that $b_e(0 :_{M_g} a_e) \subseteq K$. If $b_e(N :_{M_g} Ann_{R_e}(N)) \not\subseteq K$, then $b_e(0 :_{M_g} a_e) = 0$ or $(0 :_{M_g} a_e) \subseteq K$ since $(0 :_{M_g} a_e)$ is a g-ψ_1-second submodule of M_g. So let $b_e(N :_{M_g} Ann_{R_e}(N)) \subseteq K$. Now we have $(a_e + b_e)(0 :_{M_g} a_e) \subseteq K$. If $(a_e + b_e)(N :_{M_g} Ann_{R_e}(N)) \not\subseteq K$, then as $(0 :_{M_g} a_e)$ is a g-ψ_1-second submodule of M_g, then $(a_e + b_e)(0 :_{M_g} a_e) = 0$ or $(0 :_{M_g} a_e) \subseteq K$ and we are done. Hence assume that $(a_e + b_e)(N :_{M_g} Ann_{R_e}(N)) \subseteq K$. Then $b_e(N :_{M_g} Ann_{R_e}(N)) \subseteq K$ gives that $a_e(N :_{M_g} Ann_{R_e}(N)) \subseteq K$. Therefore by assumption, $(0 :_{M_g} a_e) \subseteq K$ and the result follows from Theorem 1. □

Theorem 7 *Let* M *be a graded* R*-module,* $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ *be a functions, and* N *be a non-zero graded submodule of* M*. Then the following are equivalent:*

(i) N *is a graded* ψ*-second submodule of* M*;*

(ii) *For graded completely irreducible submodule* L *of* M *with* $N \not\subseteq L$*, we have* $(L :_R N) = Ann_R(N) \cup (L :_R \psi(N))$*;*

(iii) *For graded completely irreducible submodule* L *of* M *with* $N \not\subseteq L$*, we have* $(L :_R N) = Ann_R(N)$ *or* $(L :_R N) = (L :_R \psi(N))$*;*

(iv) *For any graded ideal* I *of* R *and any graded submodule* K *of* M*, if* $IN \subseteq K$ *and* $I\psi(N) \not\subseteq K$*, then* $IN = 0$ *or* $N \subseteq K$*.*

(v) *For each* $a_g \in h(R)$ *with* $a_g\psi(N) \not\subseteq a_gN$*, we have* $a_gN = N$ *or* $a_gN = 0$*.*

Proof. (i) ⇒ (ii) Let for a graded completely irreducible submodule L of M with $N \not\subseteq L$, we have $a_g \in (L :_R N) \setminus (L :_R \psi(N))$. Then $a_g\psi(N) \not\subseteq L$. Since N is a graded ψ-second submodule of M, we have $a_g \in Ann_R(N)$. As we may assume that $\psi(N) \subseteq N$, the other inclusion always holds.

(ii) ⇒ (iii) This follows from the fact that if a graded ideal is a union of two graded ideals, it is equal to one of them.
(iii) ⇒ (iv) Let I be a graded ideal of R and K be a graded submodule of M such that $IN \subseteq K$ and $I\psi(N) \nsubseteq K$. Suppose $I \nsubseteq Ann_R(N)$ and $N \nsubseteq K$. We show that $I\psi(N) \subseteq K$. Let $a \in I$ and L is a graded completely irreducible submodule of M with $K \subseteq L$. First, let $a \notin Ann_R(N)$. Then since $aN \subseteq L$, we have $(L :_R N) \neq Ann_R(N)$. Hence by assumption $(L :_R N) = (L :_R \psi(N))$. So $a\psi(N) \subseteq L$. Now let $a \in I \cap Ann_R(N)$. Let $b \in I \setminus Ann_R(N)$. Then $a + b \in I \setminus Ann_R(N)$. Hence by the first case, for each graded completely irreducible submodule L of M with $K \subseteq L$ we have $b\psi(N) \subseteq L$ and $(b + a)\psi(N) \subseteq L$. This gives that $a\psi(N) \subseteq L$. Thus in any case $a\psi(N) \subseteq L$. Thus $I\psi(N) \subseteq L$. Therefore, $a\psi(N) \subseteq K$ by Remark 1.
(iv) ⇒ (i) The proof is straightforward.
(i) ⇒ (v) Let $a_g \in h(R)$ such that $a_g\psi(N) \nsubseteq a_gN$. Then $a_gN \subseteq a_gN$ implies that $N \subseteq a_gN$ or $a_gN = 0$ by part (i). Thus $N = a_gN$ or $a_gN = 0$, as required. (v) ⇒ (i) Let $a_g \in h(R)$ and K be a graded submodule of M such that $a_gN \subseteq K$ and $a_g\psi(N) \nsubseteq K$. If $a_g\psi(N) \subseteq a_gN$, then $a_gN \subseteq K$ implies that $a_g\psi(N) \subseteq K$, a contradiction. Thus by part (v), $a_gN = N$ or $a_gN = 0$. Therefore, $N \subseteq K$ or $a_gN = 0$, as needed. □

Example 2 *Let N be a non-zero graded submodule of a graded R-module M and let $\psi : S^{gr}(M) \to S^{gr}(M) \cup \{\emptyset\}$ be a function. If $\psi(N) = N$, then N is a graded ψ-second submodule of M by Theorem 7 (v) ⇒ (i).*

Let R_1 and R_2 be two G-graded rings. Then $R = R_1 \times R_2$ becomes a G-graded ring with homogeneous elements $h(R) = \bigcup_{g\in G} R_g$, where $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a graded R_1-module and M_2 be a graded R_2-module. Then $M = M_1 \times M_2$ is a graded $R = R_1 \times R_2$-module.

Theorem 8 *Let $R = R_1 \times R_2$ be a graded ring and $M = M_1 \times M_2$ be a graded R-module where M_1 is a graded R_1-module and M_2 is a graded R_2-module. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. Then $S_1 \times 0$ is a graded $\psi^1 \times \psi^2$-second submodule of M, where S_1 is a graded ψ^1-second submodule of M_1 and $\psi^2(0) = 0$.*

Proof. Let $(r_g, r'_g) \in h(R)$ and $K_1 \times K_2$ be a graded submodule of M such that $(r_g, r'_g)(S_1 \times 0) \subseteq K_1 \times K_2$ and

$$(r_g, r'_g)((\psi^1 \times \psi^2)(S_1 \times 0)) = r_g\psi^1(S_1) \times r'_g\psi^2(0) = r_g\psi^1(S_1) \times 0 \nsubseteq K_1 \times K_2$$

Then $r_gS_1 \subseteq K_1$ and $r_g\psi^1(S_1) \nsubseteq K_1$. Hence $r_gS_1 = 0$ or $S_1 \subseteq K_1$ since S_1 is a graded ψ^1-second submodule of M_1. Therefore, $(r_g, r'_g)(S_1 \times 0) = 0 \times 0$ or $S_1 \times 0 \subseteq K_1 \times K_2$, as needed. □

3 Graded I-second submodules

Definition 6 *Let* R *be a graded ring,* M *be a graded* R*-module and* I *be a graded ideal of* R*.*
(a) *A proper graded ideal* P *of* R *is called graded* I*-prime, if* $a_gb_h \in P \setminus IP$*, then* $a_g \in P$ *or* $b_h \in P$*.*
(b) *A proper graded submodule* N *of* M *is called graded* I*-prime, if* $r_gm_h \in N \setminus IN$*, then* $m_h \in N$ *or* $r_g \in (N :_R M)$*.*

Theorem 9 *Let* I *be a graded ideal of a graded ring* R*. For a non-zero graded submodule* S *of a graded* R*-module* M *the following statements are equivalent:*

(i) *For each* $r_g \in h(R)$*, a submodule* K *of* M*,* $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$ *implies that* $S \subseteq K$ *or* $r_g \in Ann_R(S)$*;*

(ii) *For each* $r_g \notin (r_gS :_R (S :_M I))$*, we have* $r_gS = S$ *or* $r_gS = 0$*;*

(iii) $(K :_R S) = Ann_R(S \cup (K :_R (S :_M I))$*, for any submodule* K *of* M *with* $S \nsubseteq K$*;*

(iv) $(K :_R S) = Ann_R(S)$ *or* $(K :_R S) = (K :_R (S :_M I))$*, for any submodule* K *of* M *with* $S \nsubseteq K$*.*

Proof. (i) ⇒ (ii) Let $r_g \notin (r_gS :_R (S :_M I))$. Then as $r_gS \subseteq r_gS$, we have $S \subseteq r_gS$ or $r_gS = 0$ by part (i). Thus $r_gS = S$ or $r_gS = 0$.
(ii) ⇒ (i) Let $r_g \in h(R)$ and K be a graded submodule of M such that $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$. Then if $r_g \in (r_gS :_R (S :_M I))$, then $r_g \in (K :_R (S :_M I))$ which is a contradiction. Thus $r_g \notin (r_gS :_R (S :_M I))$. Now by part (ii), $r_gS = S$ or $r_gS = 0$. So $S \subseteq K$ or $r_gS = 0$, as needed.
(i) ⇒ (iii) Let $r_g \in (K :_R S)$ and $S \nsubseteq K$. If $r_g \notin (K :_R (S :_M I))$, then $r_g \in Ann_R(S)$ by part (i). Hence, $(K :_R S) \subseteq Ann_R(S)$. If $r_g \in (K :_R (S :_M I))$, then $(K :_R S) \subseteq (K :_R (S :_M I))$. Therefore, $(K :_R S) \subseteq Ann_R(S) \cup (K :_R (S :_M I))$. The other inclusion always holds.
(iii) ⇒ (iv) and (iv) ⇒ (i) are clear. □

Definition 7 *Let* I *be a graded ideal of* R. *We say that a non-zero graded submodule* S *of a graded* R*-module* M *is a graded* I*-second submodule of* M*, if satisfies the equivalent conditions of Theorem 9.*

Let I be a graded ideal of R. Clearly, every graded second submodule is a graded I-second submodule. But the converse is not true in general.

Example 3 (a) *If* $I = 0$*, then every graded module is a graded* I*-second submodule of itself but every graded module is not a graded second module. for example, let* $G = \mathbb{Z}_2$*,* $R = \mathbb{Z}$ *be a* G*-graded ring with* $R_0 = \mathbb{Z}$ *and* $R_1 = \{0\}$. *Then it is clear that the graded* R*-module* $M = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ *with* $M_0 = \mathbb{Z}$ *and* $M_1 = i\mathbb{Z}$ *is not a graded second module.*
(b) *Let* $G = \mathbb{Z}_2$*,* $R = \mathbb{Z}$ *and* $M = \mathbb{Z}_{12}[i] = \{\bar{a} + \bar{b}i \mid \bar{a}, \bar{b} \in \mathbb{Z}_{12}\}$. *Then* R *is a* G*-graded ring with* $R_0 = \mathbb{Z}$*,* $R_1 = \{0\}$ *and* M *is a graded* R*-module with* $M_0 = \mathbb{Z}_{12}$*,* $M_1 = i\mathbb{Z}_{12}$. *Consider the grade ideal* $I = 4\mathbb{Z} \oplus \{0\}$ *and the graded submodule* $S = \bar{3}\mathbb{Z} \oplus \{0\}$. *Thus* S *is a graded* I*-second submodule of* M*, but it is not a graded second submodule of* M.

Let I be a graded ideal of R and M be a graded R-module . If $I = R$, then every graded submodule is a graded I-second submodule. So in the rest of this paper we can assume that $I \neq R$.

Theorem 10 *Let* M *be a graded* R*-module and* I, J *be graded ideals of* R *such that* $I \subseteq J$. *If* S *is a graded* I*-second submodule of* M*, then* S *is a graded* J*-second submodule of* M.

Proof. The result follows from the fact that $I \subseteq J$ implies that $(r_g S :_R S) \setminus (r_g S :_R (S :_M J)) \subseteq (r_g S :_R S) \setminus (r_g S :_R (S :_M I))$, for each $r_g \in R$. □

Theorem 11 *Let* M *be a graded* R*-module and* $g \in G$. *If* I *is an ideal of* R_e *and* S *a* g*-*I*-second submodule of* R_e*-module* M_g *which is not* g*-second, then* $Ann_{R_e}(S)(S :_{M_g} I) \subseteq S$.

Proof. Assume on the contrary that $Ann_{R_e}(S)(S :_{M_g} I) \not\subseteq S$. We show that S is g-second. Let $rS \subseteq K$ for some $r \in R_e$ and a submodule K of M_g. If $r \notin (K :_{R_e} (S :_{M_g} I))$, then S is a g-I-second submodule implies that $S \subseteq K$ or $r \in Ann_{R_e}(S)$ as needed. So assume that $r \in (K :_{R_e} (S :_{M_g} I))$. First, suppose that $r(S :_{M_g} I) \not\subseteq S$. Then there exists a graded submodule L of M such that $S \subseteq L$ but $r_g(S :_{M_g} I) \not\subseteq L$. Then $r \in (K \cap L :_{R_e} S) \setminus (K \cap L :_{R_e} (S :_{M_g} I))$. So $S \subseteq K \cap L$ or $r_g \in Ann_{R_e}(S)$ and hence $S \subseteq K$ or $r \in Ann_{R_e}(S)$. So we can assume that $r(S :_{M_g} I) \subseteq S$. On the other hand, if $Ann_{R_e}(S)(S :_{M_g} I) \nsubseteq K$, then there exists $t \in Ann_{R_e}(S)$ such that $t \notin (K :_{R_e} (S :_{M_g} I))$. Then $t + r \in (K :_{R_e} S) \setminus (K :_{R_e} (S :_{M_g} I))$. Thus $S \subseteq K$ or $t + r \in Ann_{R_e}(S)$ and hence $S \subseteq K$ or $r \in Ann_{R_e}(S)$. So we can assume that $Ann_{R_e}(S)(S :_{M_g} I) \subseteq K$. Since $Ann_{R_e}(S)(S :_{M_g} I) \nsubseteq S$, there exists $t \in Ann_{R_e}(S)$, a submodule L of M such that $S \subseteq L$ and $t(S :_{M_g} I) \nsubseteq L$. Now we have $r + t \in (K \cap L :_{R_e} S) \setminus (K \cap L :_{R_e} (S :_{M_g} I))$. So S is a g- I-second submodule gives $S \subseteq K \cap L$ or $r + t \in Ann_{R_e}(S)$. Hence $S \subseteq K$ or $r \in Ann_{R_e}(S)$, as requested. □

Theorem 12 *Let* I *be a graded ideal of* R*,* M *a graded* R*-module and* S *be a graded submodule of* M*. Then we have the following.*

(i) *If* S *is a graded* I*-second submodule of* M *such that* $Ann_R((S :_M I)) \subseteq IAnn_R(S)$*, then* $Ann_R(S)$ *is a graded* I*-prime ideal of R.*

(ii) *If* M *is a graded comultiplication* R*-module and* $Ann_R(S)$ *is a graded* I*-prime ideal of* R*, then* S *is a graded* I*-second submodule of* M*.*

Proof. (i) Let $a_g b_h \in Ann_R(S) \setminus IAnn_R(S)$ for some $a_g, b_h \in h(R)$. Then $a_g S \subseteq (0 :_M b_h)$. As $a_g b_h \notin IAnn_R(S)$ and $Ann_R((S :_M I)) \subseteq IAnn_R(S)$, we have $a_g b_h \notin Ann_R((S :_M I))$. This implies that $a_g \notin ((0 :_M b_h) :_R (S :_M I))$. Thus $a_g \in Ann_R(S)$ or $S \subseteq (0 :_M b_h)$. Hence $a_g \in Ann_R(S)$ or $b_h \in Ann_R(S)$, as needed.
(ii) Let $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$ for some $r_g \in h(R)$ and graded submodule K of M. As M is a graded comultiplication R-module, there exists a graded ideal J of R such that $K = (0 :_M J)$. Thus $r_g J \subseteq Ann_R(S)$. Since $r_g \notin (K :_R (S :_M I))$, we have $J r_g (S :_M I) \neq 0$. This implies that $J r_g \nsubseteq Ann_R((S :_M I))$. Since always $IAnn_R(S) \subseteq Ann_R((S :_M I))$, we have $r_g J \nsubseteq IAnn_R(S)$. Thus by assumption, $r_g \in Ann_R(S)$ or $J \subseteq Ann_R(S)$ and so $S \subseteq (0 :_M J) = K$. □

Proposition 4 *Let* M *be a graded* R*-module and* I *a graded ideal of* R*. Let* N *be a graded* I*-second submodule of* M*. Then we have the following statements.*

(i) *If* K *is a graded submodule of* M *with* $K \subset N$*, then* N/K *is a graded* I*-second submodule of* M/K*.*

(ii) *Let* N *be a graded finitely generated submodule of* M*,* $S \subseteq h(R)$ *be a multiplicatively closed subset of* R *with* $Ann_R(N) \cap S = \emptyset$*. Then* $S^{-1}N$ *is a graded* $S^{-1}I$*-second submodule of* $S^{-1}M$*.*

Proof. (i) This follows from the fact that $r_g \notin (r_g(S/K) :_R (S/K :_{M/K} I))$ implies that $r_g \notin (r_gS :_R (S :_M I))$.
(ii) As $Ann_R(N) \cap S = \emptyset$ and N is graded finitely generated, $S^{-1}N \neq 0$. Now the claim follows from the fact that $r/s \notin ((r/s)S^{-1}N :_{S^{-1}} (S^{-1}N :_{S^{-1}M} S^{-1}I))$ implies that $r \notin (rN :_R (N :_M I))$. □

Proposition 5 *Let* I *be a graded ideal of* R*,* M *and* M' *be graded* R*-modules, and let* $f : M \to M'$ *be an R-monomorphism. If* N' *is a graded* I*-second submodule of* M' *such that* $N' \subseteq Im(f)$*, then* $f^{-1}(N')$ *is a graded* I*-second submodule of* M*.*

Proof. As $N' \neq 0$ and $N' \subseteq Im(f)$, we have $f^{-1}(N') \neq 0$. Let $r_g \notin (r_gf^{-1}(N') :_R (f^{-1}(N') :_M I))$; then one can see that $r_g \notin (r_gN' :_R (N' :_M I))$ using assumptions. Thus $r_gN' = 0$ or $r_gN' = N'$. This implies that $r_gf^{-1}(N') = 0$ or $r_gf^{-1}(N') = f^{-1}(N')$, as requested. □

Theorem 13 *Let* I *be a graded ideal of* R*,* M_1, M_2 *be graded* R*-modules, and let* N *be a graded submodule of* M_1*. Then* $N \oplus 0$ *is a graded* I*-second submodule of* $M_1 \oplus M_2$ *if and only if* N *is a graded* I*-second submodule of* M_1 *and for* $r_g \in (r_gN :_R (N :_{M_1} I))$*,* $r_gN \neq 0$*, and* $r_gN \neq N$*, we have* $r_g \in Ann_R((0 :_{M_2} I))$*.*

Proof. (⇒) Let $r_g \notin (r_gN :_R (N :_{M_1} I))$. Then $r_g \in (r_g(N \oplus 0) :_R (N \oplus 0 :_M I))$. Since $N \oplus 0$ is a graded I-second submodule, either $r_g(N \oplus 0) = N \oplus 0$ or $r_g(N \oplus 0) = 0 \oplus 0$. Thus either $r_gN = N$ or $r_gN = 0$, so N is graded I-second. Now, let $r_g \in (r_gN :_R (N :_{M_1} I))$, $r_gN \neq 0$, and $r_gN \neq N$. Assume on the contrary that $r_g \in Ann_R((0 :_{M_2} I))$. Then there exists $y_h \in M_2$ such that $Iy_h = 0$ and $r_gy_h \neq 0$. This implies that $r_g(0, y_h) \in r_g(N \oplus 0 :_M I) \setminus r_g(N \oplus 0)$. So since $N \oplus 0$ is a graded I-second submodule, either $r_g(N \oplus 0) = N \oplus 0$ or $r_g(N \oplus 0) = 0 \oplus 0$. Thus either $r_gN = N$ or $r_gN = 0$, which is a contradiction. Therefore, $r_g \in Ann_R((0 :_{M_2} I))$.
(⇐) Let $r_g \notin (r_g(N \oplus 0) :_R (N \oplus 0 :_M I))$. Then if $r_gN = N$ or $r_gN = 0$, the result is clear. So suppose that $r_gN \neq N$ and $r_gN \neq 0$. We show that $r_g(r_gN :_R (N :_{M_1} I))$ and this contradiction proves the result because N is a graded I-second submodule of M_1. Assume on the contrary that $r_g \in (r_gN :_R (N :_{M_1} I))$. Then by assumption, $r_g \in Ann_R((0 :_{M_2} I))$. This implies that if $(x_h, y_h) \in N \oplus (0 :_M I)$, then $r_g(x_h, y_h) \in r_g(N \oplus 0)$. Therefore, $r_g \in (r_g(N \oplus 0) :_R (N \oplus 0 :_M I))$, which is a desired contradiction. □

A non-zero graded R-module M is said to be graded secondary if for each $a_g \in h(R)$ the endomorphism of M given by multiplication by a_g is either surjective or nilpotent [4].

Corollary 3 *Let* I *and* P *be graded ideals of* R*,* M_1, M_2 *be graded* R*-modules, and let* N *be a graded submodule of* M_1*. Let* S_i *(*$1 \leq i \leq n$*) be graded* P*-secondary submodules of* M_1 *with* $\sum_{i=1}^{n} S_i = (N :_{M_1} I)$*. If* N *is a graded* I*-second submodule of* M_1 *and* $P \subseteq Ann_R((0 :_{M_2} I))$*, then* $N \oplus 0$ *is a graded* I*-second submodule of* $M_1 \oplus M_2$*.*

Proof. Let $r_g \in (r_g N :_R (N :_{M_1} I))$, $r_g N \neq 0$, and $r_g N \neq N$. Then we will prove that $r_g \in Ann_R((0 :_{M_2} I))$ and hence the result is obtained by Theorem 13. Assume on the contrary that $r_g \notin Ann_R((0 :_{M_2} I))$. Hence $r \notin P$. On the other hand, $r_g(\sum_{i=1}^{n} S_i) = r_g(N :_{M_1} I) \subseteq r_g N$. But $\sum_{i=1}^{n} S_i$ is a graded P-secondary submodule by [4], so either $r_g(\sum_{i=1}^{n} S_i) = \sum_{i=1}^{n} S_i$ or $r_g \in P$. This implies that $r_g N = N$ or $r_g \in P$, which is a contradiction. Thus $r_g \in Ann_R((0 :_{M_2} I))$. □

Theorem 14 *Let* I *be a graded ideal of* R *and* M *be a graded* R*-module. Then we have the following.*

(i) *If* $\bigcap_{n=1}^{\infty} I^n M = 0$ *and every proper graded submodule of* M *is graded* I*-prime, then every non-zero graded submodule of* M *is graded* I*-second.*

(ii) *If* $\sum_{n=1}^{\infty} (0 :_M I^n) = M$ *and every non-zero graded submodule of* M *is graded* I*-second, then every proper graded submodule of* M *is graded* I*-prime.*

Proof. (i) Let S be a non-zero graded submodule of M, $r_g \in (K :_R S) \setminus (K :_R (S :_M I))$ for some $r_g \in h(R)$ and a graded submodule K of M and $r_g S \neq 0$. If $r_g S \nsubseteq IK$, then as K is graded I-prime, we have $r_g M \subseteq K$ or $S \subseteq K$. If $r_g M \subseteq K$, then $r_g(S :_M I) \subseteq K$ which is a contradiction. So $S \subseteq K$ and we are done. Now suppose that $r_g S \subseteq IK$. As $r_g S \neq 0$ and $\bigcap_{n=1}^{\infty} I^n K = 0$, there exists a positive integer t such that $r_g S \nsubseteq I^t K$. Therefore, there is a positive integer h such that $r_g S \subseteq I^{h-1} K$ but $r_g S \nsubseteq I^h K$, where $2 \leq h \leq t$. Thus since $I^{h-1} K$ is graded I-prime, $S \subseteq I^{h-1} K$ or $r_g M \subseteq I^{h-1} K$. If $r_g M \subseteq I^{h-1} K$, then $r_g(S :_M I) \subseteq K$ which is a contradiction. So $S \subseteq I^{h-1} K$ as needed.
(ii) Let P be a proper graded submodule of M, $r_g K \subseteq P \setminus IP$ for some $r_g \in h(R)$ and a graded submodule K of M and $r_g M \nsubseteq P$. If $r_g(K :_M I) \nsubseteq P$, then as K is graded I-second, we have $r_g K = 0$ or $K \subseteq P$. If $r_g K = 0$, then $r_g K \subseteq IP$ which is a contradiction. So $K \subseteq P$ and we are done. Now suppose that $r_g(K :_M I) \subseteq P$.

As $r_g M \nsubseteq P$ and $\sum_{n=1}^{\infty}(K :_M I^n) = M$, there exists a positive integer t such that $r_g(K :_M I^t) \nsubseteq P$. Therefore, there is a positive integer h such that $r_g(K :_M I^{h-1}) \subseteq P$ but $r_g(K :_M I^h) \nsubseteq P$, where $2 \leq h \leq t$. Thus since $(K :_M I^{h-1})$ is graded I-second, $(K :_M I^{h-1}) \subseteq P$ or $r_g(K :_M I^{h-1}) = 0$. If $r_g(K :_M I^{h-1}) = 0$, then $0 = r_g K \subseteq IP$ which is a contradiction. So $K \subseteq (K :_M I^{h-1}) \subseteq P$, as needed. □

References

[1] R. Abu-Dawwas, M. Bataineh, Graded prime submodules over non-commutative rings, *Vietnam J. Math.*, **46(3)** (2018), 681–692.

[2] H. Ansari-Toroghy, F. Farshadifar, Graded comultiplication modules, *Chiang Mai J. Sci.*, **38(3)** (2011), 311–320.

[3] S. Ebrahimi Atani, On graded prime submodules, *Chiang Mai J. Sci.*, **33(1)** (2006), 3–7.

[4] S. Ebrahimi Atani, F. Farzalipour, On graded secondary modules, *Turk. J. Math.*, **31** (2007), 371–378.

[5] F. Farshadifar, H. Ansari-Toroghy, I-second submodules of a module, *Matematički Vesnik*, **72(1)** (2020), 58–65.

[6] F. Farshadifar, H. Ansari-Toroghy, ψ-second submodules of a module, *Beitr Algebra Geom*, **61** (2020), 571–578.

[7] F. Farzalipour, P. Ghiasvand, On the union of graded prime submodules, *Thai. J. of Math.*, **9(1)** (2011), 49–55.

[8] P. Ghiasvand, F. Farzalipour, Some properties of graded multiplication modules, *Far East J. Math. Sci.*, **34** (2009), 341–352.

[9] P. Ghiasvand, F. Farzalipour, On graded weak multiplication modules, *Tamkang J. of Math.*, **43(2)** (2012), 171–177.

[10] P. Ghiasvand, F. Farzalipour, On the graded primary radical of graded submodules, *Advances and Applications in Math. Sci.*, **10(1)** (2011), 1–7.

[11] N. Nastasescu, F. Van Oystaeyen, *Graded Rings Theory*, Mathematical Library 28, North Holand, Amsterdam, 1937.

[12] N. Nastasescu, F. Van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Mathematics, vol. 1836. Springer, Berlin, 2004.

[13] J. P. Serre, *Local Algebra*, Springer Verlag, 2000.

[14] M. Refaei, K. Alzobi, On graded primary ideals, *Turk. J. Math.*, **28(3)** (2004), 217–229.

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Sums and products of intervals in ordered groups and fields

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Abstract. We show that the sum of two intervals in an ordered dense Abelian group is also an interval such that the endpoints of the sum are equal to the sums of the endpoints. We prove analogous statements concerning to the product of two intervals.

1 Introduction

It is well known from elementary real analysis that if a, b, c, d are real numbers with $a < b$ and $c < d$, then

$$]a, b[+]c, d[=]a + c, b + d[, \tag{1}$$

moreover, if $0 \leqslant a < b$ and $0 \leqslant c < d$, then

$$]a, b[\cdot]c, d[=]ac, bd[. \tag{2}$$

The main purpose of this article is to show that equations (1), (2) remain valid in more general settings. Our references to ordered structures are [4], [8], [12], [13], [18].

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Now, we shall give a sort list of the necessary concepts and notations:

We say that $X = X(\leqslant)$ is a partially ordered set or a poset if X is a set and $\leqslant$ is a relation on X such that it is reflexive, symmetric and transitive.

A poset $X = X(\leqslant)$ is said to be ordered or a loset, if either $x \leqslant y$ or $y \leqslant x$ for all $x, y \in X$.

Let $X = X(*)$ be a groupoid in the sense that X is a nonempty set, $*$ is a binary operation on X. Then for any A, $B \subseteq X$ and $a \in X$ define

$$A * B := \{a * b \in X \mid a \in A, b \in B\},$$
$$a * B := \{a\} * B.$$

Let $X = X(\leqslant)$ be a poset and a ,$b \in X$ such that $a < b$, that is, $a \leqslant b$ but $a \neq b$. The open interval is defined by

$$]a, b[:= \{x \in X \mid a < x \text{ and } x < b\}.$$

The a and b are the endpoints of the interval $]a, b[$. Similarly, we can define $]a, b] := \{x \in X \mid a < x \leqslant b\}$, $[a, b[:= \{x \in X \mid a \leqslant x < b\}$, $[a, b] := \{x \in X \mid a \leqslant x \leqslant b\}$.

A poset $X = X(\leqslant)$ is said to be dense (in itself) if $]x, y[\neq \emptyset$ for all $x, y \in X$ with $x < y$.

An ordered group $\mathbb{G} = \mathbb{G}(+, \leqslant)$ is a group together with an order that is compatible with the group operation. A set of all positive elements of an ordered group $\mathbb{G}$ is denoted by $\mathbb{G}_+$, that is, $\mathbb{G}_+ := \{x \in \mathbb{G} \mid x > 0\}$.

An ordered group $\mathbb{G} = \mathbb{G}(+, \leqslant)$ is said to be Archimedean ordered if for all x, $y \in \mathbb{G}_+$ there exits a positive integer n such that $nx := x + \cdots + x > y$.

An ordered field $\mathbb{F} = \mathbb{F}(+, \cdot, \leqslant)$ is a field (the operation $\cdot$ is commutative) together with an order that is compatible with the field operations, in the sense, that if $x \leqslant y$, then $x + z \leqslant y + z$ for all $x, y, z \in \mathbb{F}$ and if $x \leqslant y$, then $xz \leqslant yz$ for all $x, y \in \mathbb{F}$ and $z \in \mathbb{F}_+ := \{x \in \mathbb{F} \mid x > 0\}$.

The foundations of the so-called interval arithmetic were laid by E. Moore, the first appearance of this topic was in 1959 [14], see also [15], [16] and [1]. Now, we shall show the Moore's formulas

$$\begin{aligned}
[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
[\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
[\underline{a}, \overline{a}] \cdot [\underline{b}, \overline{b}] &= [\min(\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{ab}), \max(\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{ab})], \\
[\underline{a}, \overline{a}]/[\underline{b}, \overline{b}] &= [\underline{a}, \overline{a}] \cdot [1/\overline{b}, 1/\underline{b}] \qquad 0 \notin [\underline{b}, \overline{b}]
\end{aligned} \tag{3}$$

for all $\underline{a}$, $\overline{a}$, $\underline{b}$, $\overline{b} \in \mathbb{R}$ with $\underline{a} \leqslant \overline{a}$ and $\underline{b} \leqslant \overline{b}$.

Due to [9] the results of Moore was extended to open ended unbounded intervals by R. J. Hanson (1968) [5], W. Kahan (1968) [10], E. Davis (1987) [2].

The famous Kohan-Novoa-Ratz arithmetic concerning to the division by an interval coutaining zero can be found in [11]:

$$
\mathbf{a}/\mathbf{b} = \begin{cases} \mathbf{a}\cdot[1/\overline{b}, 1/\underline{b}] \text{ for } 0 \notin \mathbf{b} \\ [-\infty, +\infty] \text{ for } 0 \in \mathbf{a} \text{ and } 0 \in \mathbf{b} \\ [\overline{a}/\underline{b}, +\infty] \text{ for } \overline{a} < 0 \text{ and } \underline{b} < \overline{b} = \underline{0} \\ [-\infty, \overline{a}/\overline{b}] \cup [\overline{a}/\underline{b}, +\infty] \text{ for } \overline{a} < 0 \text{ and } \underline{b} < 0 < \overline{b} \\ [-\infty, \overline{a}/\overline{b}] \text{ for } \overline{a} < 0 \text{ and } 0 = \underline{b} < \overline{b} \\ [-\infty, \underline{a}/\underline{b}] \text{ for } 0 < \underline{a} \text{ and } \underline{b} < \overline{b} = \underline{0} \\ [-\infty, \underline{a}/\underline{b}] \cup [\underline{a}/\underline{b}, +\infty] \text{ for } 0 < \underline{a} \text{ and } \underline{b} < 0 < \overline{b} \\ [\underline{a}/\overline{b}, +\infty] \text{ for } \underline{a} < 0 \text{ and } 0 = \underline{b} < \overline{b} \\ \emptyset \text{ for } 0 \notin \mathbf{a} \text{ and } 0 \in \mathbf{b} \end{cases} \qquad (4)
$$

for all closed ended bounded interval **a**, **b** of the real line.

An other way to extend the results of E. Moore that is to use the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with uppear additions of J. Moreau [17], that is,

$$
+\infty + (+\infty) = +\infty \quad \text{and} \quad +\infty + (-\infty) = +\infty.
$$

For example, in [3] can be found that

$$
\begin{aligned}
&]x+y, +\infty[\, =]x, \infty] +]y, +\infty[&& (x, y \in \overline{\mathbb{R}}),\\
&]x+y, +\infty] = [x, \infty] + [y, +\infty] && (x, y \in \overline{\mathbb{R}}),\\
&[-\infty, x+y[\, = [-\infty, x[+] - \infty, y] && (x, y \in \mathbb{R}\cup\{-\infty\} \text{ or } \mathbb{R}\cup\{+\infty\}),\\
&[-\infty, x+y] = [-\infty, x] + [-\infty, y] && (x, y \in \mathbb{R}\cup\{-\infty\} \text{ or } \mathbb{R}\cup\{+\infty\}).
\end{aligned}
$$

In [11] the author use intervals $X = [\underline{X}, \overline{X}]$ where $\underline{X}$ is the vector or matrix whose components are lower bounds of corresponding components of X, and $\overline{X}$ is the vector or matrix whose components are upper bounds of corresponding components of X.

In our former paper [6] we have investigated the sums and the products of intervals in ordered semigroups.

In our present paper we investigate the sums of open ended bounded intervals in ordered groups and the products of open ended buonded intervals in ordered fields. To obtained results will be used to extend additive and logarithmic functions [7].

Cases that arise when proving results for a product of two intervals **a** and **b** can be grouped according to the following criteria: the point 0 is an interior point neither of **a** nor of **b**; the point 0 is an interior point either of **a** or of **b**; the point 0 is an interior point both of **a** and **b**.

Finally, it is worth mentioning that if $X = X(\leqslant)$ is a loset, $]a, b[\, \subseteq X$ and $c \in \,]a, b[$, then

$$]a, b[\, = \,]a, c] \cup \,]c, b[\,. \tag{5}$$

If $X = X(\leqslant)$ is only a poset but is not a loset, then (5), in general, is not true. Thus, our applied arguments lose their validity on ordered structures in which the order is not linear.

2 Sum of intervals in ordered dense Abelian groups

In this section $\mathbb{G} = \mathbb{G}(+, \leqslant)$ is an ordered dense Abelian group, $\underline{a}$, $\overline{a}$, $\underline{b}$, $\overline{b}$ $\gamma \in \mathbb{G}$ with $\underline{a} < \overline{a}$ and $\underline{b} < \overline{b}$.

The following Proposition is trivial.

Proposition 1 $\gamma + \,]\underline{a}, \overline{a}[\, = \,]\gamma + \underline{a}, \gamma + \overline{a}[$, *and* $\gamma + \,]\underline{a}, \overline{a}] = \,]\gamma + \underline{a}, \gamma + \overline{a}]$.

Proposition 2 *If* α, $\beta \in \mathbb{G}_+$, *then* $]0, \alpha + \beta[\, \subseteq \,]0, \alpha[+]0, \beta[$.

Proof. Let $x \in \,]0, \alpha + \beta[$. Since $0 < \alpha < \alpha + \beta$ there are two cases:
1. Assume that $x \in \,]0, \alpha]$. Since $x > 0$ and $\beta > 0$ there exists an $y \in \mathbb{G}_+$ such that $y < x$ and $y < \beta$. Since $y < x < x+y$ thus we have that $0 < x-y < x \leqslant \alpha$ whence we obtain that

$$x = (x - y) + y \in \,]0, \alpha[+]0, \beta[\,.$$

2. Assume that $x \in \,]\alpha, \alpha + \beta[$. Then by Proposition 1. $x - \alpha \in \,]0, \beta[$ thus there exists an element $y \in \mathbb{G}_+$ such that $y < \beta - (x - \alpha)$ and $y < \alpha$. Since $y < \alpha < \alpha + y$ thus we have that $0 < \alpha - y < \alpha$. Since $y < \beta - (x - \alpha)$ thus we have that $0 < x - \alpha < y + (x - \alpha) = x + y - \alpha < \beta$. Thus

$$x = (\alpha - y) + (x + y - \alpha) \in \,]0, \alpha[+]0, \beta[\,.$$

□

The following Proposition is trivial.

Proposition 3 $]\underline{a},\overline{a}[+]\underline{b},\overline{b}[\subseteq]\underline{a}+\underline{b},\overline{a}+\overline{b}[$.

Theorem 1 $]a,\overline{a}[+]\underline{b},\overline{b}[=]\overline{a}+\underline{b},\overline{a}+\overline{b}[$.

Proof. By Proposition 3 it is enough to show that

$$]\underline{a}+\underline{b},\overline{a}+\overline{b}[\subseteq]\underline{a},\overline{a}[+]\underline{b},\overline{b}[\,.$$

By Proposition 1. and by Proposition 2. we have that

$$\begin{aligned}]\underline{a}+\underline{b},\overline{a}+\overline{b}[&= (\underline{a}+\underline{b})+]0,(\overline{a}-\underline{a})+(\overline{b}-\underline{b})[\subseteq \\ &(\underline{a}+]0,\overline{a}-\underline{a}[)+\big(\underline{b}+]0,\overline{b}-\underline{b}[\big)= \\ &]\underline{a},\overline{a}[+]\underline{b},\overline{b}[\end{aligned}$$

□

The following Theorem can be easily obtained by simply calculation.

Theorem 2 *If $\mathbb{G}=\mathbb{G}(+,\cdot,\leqslant)$ is an Archimedean ordered group, then the following assertions are equivalent:*

1. *$\mathbb{G}$ is dense.*
2. *$]\underline{a},\overline{a}[+]\underline{b},\overline{b}[=]\underline{a}+\underline{b},\overline{a}+\overline{b}[$ for all $\underline{a}$, $\overline{a}$, $\underline{b}$, $\overline{b}\in\mathbb{G}$ with $\underline{a}<\overline{a}$ and $\underline{b}<\overline{b}$.*
3. *$\mathbb{G}(+,\leqslant)$ is not isomorphic to the ordered group $\mathbb{Z}=\mathbb{Z}(+,\leqslant)$ (which is the group of all integers).*

3 The products of intervals in ordered fields

In this section $\mathbb{F}=\mathbb{F}(+,\cdot,\leqslant)$ is an ordered field, $\underline{a}$, $\overline{a}$, $\underline{b}$, $\overline{b}\in\mathbb{F}$ with $\underline{a}<\overline{a}$ and $\underline{b}<\overline{b}$. Define the intervals $\mathbf{a}$ and $\mathbf{b}$ by

$$\mathbf{a}:=]\underline{a},\overline{a}[\qquad\text{and}\qquad\mathbf{b}:=]\underline{b},\overline{b}[\,.$$

As a temporary device, use the notation for any open ended bounded interval $\mathbf{x}$ that

$$0<\mathbf{x},\ \text{ if } 0<x \text{ for all } x\in\mathbf{x},$$
$$\mathbf{x}<0,\ \text{ if } x<0 \text{ for all } x\in\mathbf{x}.$$

Proposition 4 *If α, β, $\gamma\in\mathbb{F}$ with $\alpha<\beta$, then*

$$\gamma\cdot]\alpha,\beta[=\begin{cases}]\gamma\beta,\gamma\alpha[, & \text{if } \gamma<0;\\ ]\gamma\alpha,\gamma\beta[, & \text{if } \gamma>0.\end{cases}$$

Proof. If $\gamma > 0$ and $x \in]\gamma\alpha, \gamma\beta[$, then $\gamma\alpha < x < \gamma\beta$ thus $\alpha < \frac{x}{\gamma} < \beta$ whence

$$x = \gamma \cdot \frac{x}{\gamma} \in \gamma \cdot]\alpha, \beta[\,.$$

The converse inclusion is trivial.

The case $\gamma < 0$ can be proved analogously. □

The following Proposition is trivial.

Proposition 5 *If* $0 < \mathbf{a}$ *and* $0 < \mathbf{b}$*, then* $\mathbf{a} \cdot \mathbf{b} \subseteq \left]\underline{a}\underline{b}, \overline{a}\overline{b}\right[$.

Proposition 6 *If* $0 < \mathbf{a}$ *and* $0 < \mathbf{b}$*, then* $\left]\underline{a}\underline{b}, \overline{a}\overline{b}\right[\subseteq \mathbf{a} \cdot \mathbf{b}$.

Proof. Let $x \in \left]\underline{a}\underline{b}, \overline{a}\overline{b}\right[$. There are two cases:
1. If either $\underline{a} = 0$ or $\underline{b} = 0$, say $\underline{a} = 0$, then there exists an $\varepsilon > 0$ such that

$$\varepsilon < \overline{b} - \frac{x}{\overline{a}}, \quad \text{and} \quad \varepsilon < \overline{b} - \underline{b}.$$

Since $0 < \varepsilon < \overline{b} - \frac{x}{\overline{a}}$ we have that $0 < \frac{x}{\overline{b}-\varepsilon} < \overline{a}$. Since $0 < \varepsilon < \overline{b} - \underline{b}$ we have that $\underline{b} < \overline{b} - \varepsilon < \overline{b}$. Thus we obtain that

$$x = \frac{x}{\overline{b} - \varepsilon} \cdot (\overline{b} - \varepsilon) \in]\underline{a}, \overline{a}[\cdot \left]\underline{b}, \overline{b}\right[\,.$$

2. If $\underline{a} \neq 0$ and $\underline{b} \neq 0$, then it is easy to see that $\underline{a}\underline{b} < \underline{a}\overline{b} < \overline{a}\overline{b}$ thus there is two sub-cases:
a. If $x \in \left]\underline{a}\underline{b}, \underline{a}\overline{b}\right]$, then there exists an $\varepsilon > 0$ such that

$$\varepsilon < \overline{a} - \underline{a} \quad \text{and} \quad \varepsilon < \frac{x}{\underline{b}} - \underline{a}.$$

Since $0 < \varepsilon < \overline{a} - \underline{a}$ thus we have that $\underline{a} < \underline{a} + \varepsilon < \overline{a}$ and since $0 < \varepsilon < \frac{x}{\underline{b}} - \underline{a}$ hence $\underline{b} < \frac{x}{\underline{a}+\varepsilon} < \overline{b}$. Thus we obtain that

$$x = (\underline{a} + \varepsilon) \cdot \frac{x}{\underline{a} + \varepsilon} \in]\underline{a}, \overline{a}[\cdot \left]\underline{b}, \overline{b}\right[\,.$$

b. If $x \in \left]\underline{a}\overline{b}, \overline{a}\overline{b}\right[$, then there exists an $\varepsilon > 0$ such that

$$\varepsilon < \overline{b} - \frac{x}{\overline{a}} \quad \text{and} \quad \varepsilon < \overline{b} - \underline{b}.$$

Since $0 < \varepsilon < \overline{b} - \frac{x}{\overline{a}}$ we have that $\underline{a} < \frac{x}{\overline{b}-\varepsilon} < \overline{a}$ and since $0 < \varepsilon < \overline{b} - \underline{b}$ we have that $\underline{b} < \overline{b} - \varepsilon < \overline{b}$. Thus we obtain that

$$x = \frac{x}{\overline{b} - \varepsilon} \cdot (\overline{b} - \varepsilon) \in \left]\underline{a}, \overline{a}\right[\cdot \left]\underline{b}, \overline{b}\right[.$$

□

As an immediate consequence of Propositions 4., 5. and 6. we can state that

Theorem 3 *If* $0 < \mathbf{a}$ *and* $0 < \mathbf{b}$, *then* $\mathbf{a} \cdot \mathbf{b} = \left]\underline{a}\underline{b}, \overline{a}\overline{b}\right[$.

First, we investigate the case, when the point 0 is an interior point neither of the interval $\mathbf{a}$ nor of the interval $\mathbf{b}$.

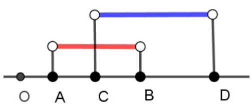
Theorem 4 *1. If* $0 < \mathbf{a}$ *and* $0 < \mathbf{b}$, *then* $\mathbf{a} \cdot \mathbf{b} = \left]\underline{a}\underline{b}, \overline{a}\overline{b}\right[$.

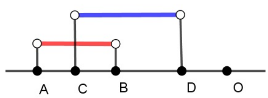
2. If $\mathbf{a} < 0$ *and* $\mathbf{b} < 0$, *then* $\mathbf{a} \cdot \mathbf{b} = \left]\overline{a}\overline{b}, \underline{a}\underline{b}\right[$.

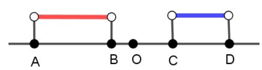
3. If $\mathbf{a} < 0$ *and* $0 < \mathbf{b}$, *then* $\mathbf{a} \cdot \mathbf{b} = \left]\underline{a}\overline{b}, \overline{a}\underline{b}\right[$.

4. If $\mathbf{b} < 0$ *and* $0 < \mathbf{a}$, *then* $\mathbf{a} \cdot \mathbf{b} = \left]\overline{a}\underline{b}, \underline{a}\overline{b}\right[$.

The following figures illustrate some cases of the Theorem.







Proof.
1. Is evident by Theorem 3.
2. By Proposition 4. and by assertion 1. we obtain that

$$\left]\underline{a}, \overline{a}\right[\cdot \left]\underline{b}, \overline{b}\right[= (-1)(-1)\left]-\overline{a}, -\underline{a}\right[\cdot \left]-\overline{b}, -\underline{b}\right[= \left]\overline{a}\overline{b}, \underline{a}\underline{b}\right[.$$

3. By Proposition 4. and by assertion 1. we obtain that

$$\left]\underline{a}, \overline{a}\right[\cdot \left]\underline{b}, \overline{b}\right[= (-1)\left]-\overline{a}, -\underline{a}\right[\cdot \left]\underline{b}, \overline{b}\right[= (-1)\left]-\overline{a}\underline{b}, -\underline{a}\overline{b}\right[= \left]\underline{a}\overline{b}, \overline{a}\underline{b}\right[.$$

4. Is an immediate consequence of assertion 3. □

Now we investigate the case, when the point 0 is an interior point either of the interval $\mathbf{a}$ or of the $\mathbf{b}$.

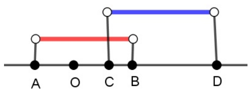
Theorem 5 *1. If* $0 \in \mathbf{a}$ *and* $0 < \mathbf{b}$*, then* $\mathbf{a} \cdot \mathbf{b} = \left]\underline{a}\underline{b}, \overline{a}\overline{b}\right[$.

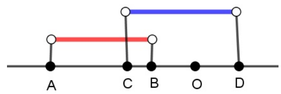
2. If $0 \in \mathbf{a}$ *and* $\mathbf{b} < 0$*, then* $\mathbf{a} \cdot \mathbf{b} =]\overline{a}\underline{b}, \underline{a}\underline{b}[$.

3. If $0 \in \mathbf{b}$ *and* $0 < \mathbf{a}$*, then* $\mathbf{a} \cdot \mathbf{b} = \left]\underline{a}\underline{b}, \overline{a}\overline{b}\right[$.

4. If $0 \in \mathbf{b}$ *and* $\mathbf{a} < 0$*, then* $\mathbf{a} \cdot \mathbf{b} = \left]\underline{a}\overline{b}, \underline{a}\underline{b}\right[$.

The following figures illustrate some cases of the Theorem.





Proof.
1. By assertions 2. and 1. of Theorem 4. we obtain that

$$\begin{aligned}]\underline{a}, \overline{a}[\cdot \left]\underline{b}, \overline{b}\right[= (]\underline{a}, 0[\cup \{0\} \cup]0, \overline{a}[) \cdot \left]\underline{b}, \overline{b}\right[= \\ ]\underline{a}, 0[\cdot \left]\underline{b}, \overline{b}\right[\cup \{0\} \cup]0, \overline{a}[\cdot \left]\underline{b}, \overline{b}\right[= \\ \left]\underline{a}\overline{b}, 0\right[\cup \{0\} \cup \left]0, \overline{a}\overline{b}\right[= \\ \left]\underline{a}\overline{b}, \overline{a}\overline{b}\right[. \end{aligned}$$

2. By assertions 2. and 4. of Theorem 4. we obtain that

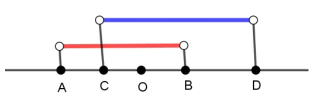
$$\begin{aligned}]\underline{a}, \overline{a}[\cdot \left]\underline{b}, \overline{b}\right[= (]\underline{a}, \overline{a}[\cdot \cup \{0\} \cup]0, \overline{a}[) \cdot \left]\underline{b}, \overline{b}\right[= \\ \left(]\underline{a}, 0[\cdot \left]\underline{b}, \overline{b}\right[\right) \cup \{0\} \cup \left(]0, \overline{a}[\cdot \left]\underline{b}, \overline{b}\right[\right) = \\ ]0, \underline{a}\underline{b}[\cup \{0\} \cup]\overline{a}\underline{b}, 0[= \\ ]\overline{a}\underline{b}, \underline{a}\underline{b}[. \end{aligned}$$

3. Can be obtained from assertion 1. by changing the roles of a and b.
4. Can be obtained from assertion 2. by changing the roles of a and b. □

Finally, we investigate the case, when the point 0 is an interior point both of $]\underline{a}, \overline{a}[$ and $\left]\underline{b}, \overline{b}\right[$.

Theorem 6 *If* $0 \in \mathbf{a} \cap \mathbf{b}$*, then* $\mathbf{a} \cdot \mathbf{b} = \left]\min\left\{\underline{a}\overline{b}, \overline{a}\underline{b}\right\}, \max\left\{\underline{a}\underline{b}, \overline{a}\overline{b}\right\}\right[$.

The following figure illustrates the Theorem.



Proof. By Theorem 5. we obtain that

$$\left]\underline{a},\overline{a}\right[\cdot\left]\underline{b},\overline{b}\right[=\left]\underline{a},\overline{a}\right[\cdot\left(\left]\underline{b},0\right[\cup\{0\}\cup\left]0,\overline{b}\right[\right)=$$
$$\left]\underline{a},\overline{a}\right[\cdot\left]\underline{b},0\right[\cup\{0\}\cup\left]\underline{a},\overline{a}\right[\cdot\left]0,\overline{b}\right[=$$
$$\left]\overline{a}\underline{b},\underline{a}\underline{b}\right[\cup\{0\}\cup\left]\underline{a}\overline{b},\overline{a}\overline{b}\right[=$$
$$\left]\min\left\{\underline{a}\overline{b},\overline{a}\underline{b}\right\},\max\left\{\underline{a}\underline{b},\overline{a}\overline{b}\right\}\right[.$$

□

Example 1 *Let $\mathbb{F}=\mathbb{Q}(\sqrt{2})$ equipped with the usuall field operations and order. Let $\mathbf{a}:=]-1-\sqrt{2},1+2\sqrt{2}]$, $\mathbf{b}:=[1-2\sqrt{2},2-\sqrt{2}]$. Calculate the product $\mathbf{a}\cdot\mathbf{b}$. Since*

$$\underline{a}\cdot\overline{b}=-\sqrt{2}=-1.4142\ldots \quad \text{and} \quad \boxed{\overline{a}\cdot\underline{b}}=-7$$
$$\boxed{\underline{a}\cdot\underline{b}}=3+2\sqrt{2}=4.4142\ldots \quad \text{and} \quad \overline{a}\cdot\overline{b}=-2+3\sqrt{2}=2.2426\ldots,$$

thus by Theorem 6. we obtain that

$$\mathbf{a}\cdot\mathbf{b}=\left[-7,3+2\sqrt{2}\right[.$$

Problem 1 *Check equation (4) for any ordered field $\mathbb{F}=\mathbb{F}(+,\cdot,\leqslant)$.*

References

[1] M. J. Cloud, R. B. Kearfott, R. E. Moore, Introduction to Interval Analysis, *Philadelphia: Society for Industrial and Applied Mathematics (SIAM).*

[2] E. Davis, Constraint propagation with interval labels . *Artifical Intelligence*, **32**(3) (1987) 281–331.

[3] Á. Figula, Á. Száz, Graphical relationships between the infimum and intersection convolutions. *Math. Pannonica*, **21/1** (2010) 23–35.

[4] L. Fuchs, Partially Ordered Algebraic Systems, *Dover Publications, Inc. Minesota, New York.* 1963.

[5] R. J. Hanson, Interval arithmetic as a closed arithmetic system on a computer. *Technical Memorandum 197, Jet Propulsion Laboratory, Section 314, California Institute of Technology, Pasadena, CA.*

[6] T. Glavosits, Zs. Karácsony, Sums and products of intervals in ordered semigroups, *An. St. Univ. Ovidius Constanta*, Vol. **29** (2) (2021), 187–198.

[7] T. Glavosits, Zs. Karácsony, Existence and unicity theorems for additive and logarithmic functional equations, accepted in *Communications in Mathematics*, 2021.

[8] T. Glavosits, Á. Száz, On the existence of nonnegativity domains of subsets of groups. *Demonstratio Math.*, **37** (2004), 505–516.

[9] L. Jaulin, M. Kieffer, O. Didrit, É. Walter, Applied Interval Analysis. *Springer, London*, (2001).

[10] W. Kahan, A more complete interval arithmetic, Lecture notes for a summer course. *University of Toronto, Canada*

[11] R. B. Kearfott, Rigorous Global Search: Continuous Problems, *Kluwer, Dordrecht*, (1996)

[12] F. W. Levi, Arithmetische Gesetze im Gebiete diskreter Gruppen, *Rend. Circ. Mat. Palermo*, **35** (1913), 225–236.

[13] P. Lorenzen, Abstrakte Begründung der multiplikativen Idelatheorie, *Math. Z.*, **45** (1939), 533–553.

[14] R. E. Moore, Automatic error analysis in digital computation. *Technical Report LMSD-48421 Lockheed Missiles and Space Co, Palo Alto, CA.*, (1959)

[15] R. E. Moore, Interval Arithmetic and Automatic Error Analysis in Digital Computing. *Department Analysis, Stanford University*, (1962)

[16] R. E. Moore, Interval Analysis. *Englewood Cliff, New Jersey, USA: Prentice-Hall.*, **35** (1966)

[17] J. J. Moreau, Inf-convolution, sous-additivité, convexité des fonctions numériques. *J. Math. Pures Appl.*, **49** (1970) 109–154.

[18] H. Simbireva, On the theory of partially ordered groups, *Mat. Sb.*, **20** (1947), 145–178. (in Russian)

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On f-rectifying curves in the Euclidean 4-space

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Abstract. A rectifying curve in the Euclidean 4-space $\mathbb{E}^4$ is defined as an arc length parametrized curve γ in $\mathbb{E}^4$ such that its position vector always lies in its rectifying space (i.e., the orthogonal complement ${N_\gamma}^\perp$ of its principal normal vector field N_γ) in $\mathbb{E}^4$. In this paper, we introduce the notion of an f-rectifying curve in $\mathbb{E}^4$ as a curve γ in $\mathbb{E}^4$ parametrized by its arc length s such that its f-position vector γ_f, defined by $\gamma_f(s) = \int f(s) d\gamma$ for all s, always lies in its rectifying space in $\mathbb{E}^4$, where f is a nowhere vanishing integrable function in parameter s of the curve γ. Also, we characterize and classify such curves in $\mathbb{E}^4$.

1 Introduction

Let $\mathbb{E}^3$ denote the Euclidean 3-space. Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit-speed curve (parametrized by arc length s) with at least four continuous derivatives. It is needless to mention that I denotes a non-trivial interval in $\mathbb{R}$, i.e., a connected set in $\mathbb{R}$ containing at least two points. For the curve γ in $\mathbb{E}^3$, we consider the Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$, where $T_\gamma = \gamma'$ is the unit tangent vector field of γ, N_γ is the unit principal normal vector field of γ obtained by

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normalizing the acceleration vector field T'_γ, $B_\gamma = T_\gamma \times N_\gamma$ is the unit binormal vector field of the curve γ so that the Frenet frame $\{T_\gamma, N_\gamma, B_\gamma\}$ is positive definite along γ having the same orientation as that of $\mathbb{E}^4$, and $\kappa_\gamma : I \longrightarrow \mathbb{R}$ is at least twice differentiable function with $\kappa_\gamma > 0$, known as the curvature of γ, and $\tau_\gamma : I \longrightarrow \mathbb{R}$ is a differentiable function, called the torsion of the curve γ. Then the Frenet formulae for the curve γ are given by ([1, 2])

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\gamma & 0 \\ -\kappa_\gamma & 0 & \tau_\gamma \\ 0 & -\tau_\gamma & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}.$$

The planes spanned by $\{T_\gamma, N_\gamma\}$, $\{N_\gamma, B_\gamma\}$ and $\{T_\gamma, B_\gamma\}$ are called the osculating plane, the normal plane and the rectifying plane of the curve γ, respectively (cf. [1, 2, 3]).

In the Euclidean 3-space $\mathbb{E}^3$, the notion of a rectifying curve was introduced by B.Y. Chen in [3] as a tortuous curve whose position vector always lies in the rectifying plane of the curve. That is, for a rectifying curve $\gamma : I \longrightarrow \mathbb{E}^3$, the position vector of γ can be expressed as

$$\gamma(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s), \quad s \in I,$$

for two unique smooth functions $\lambda, \mu : I \longrightarrow \mathbb{R}$.

Several characterizations and classification of the rectifying curves in $\mathbb{E}^3$ were studied in [3, 4, 5, 6]. Meanwhile, the notion of rectifying curves were extended to several sorts of Riemannian and pseudo-Riemannian spaces. As for example, many interesting characterizations and classification of rectifying curves in the higher dimensional Euclidean spaces were studied in [7, 8], and the same in Minkowski 3-space $\mathbb{E}^3_1$ were studied in [9, 10].

In [7], a rectifying curve in the Euclidean 4-space $\mathbb{E}^4$ was defined as a curve $\gamma : I \longrightarrow \mathbb{E}^4$ parametrized by its arc length s such that its position vector always lies in its rectifying space, i.e., in the orthogonal complement $N_\gamma{}^\perp$ of its principal normal vector field N_γ. In collateral to this, in this paper, we introduce the notion of an f-rectifying curve in $\mathbb{E}^4$ as a curve $\gamma : I \longrightarrow \mathbb{E}^4$ parametrized by its arc length s such that its f-position vector, denoted and defined by $\gamma_f(s) := \int f(s)\, d\gamma$ for all $s \in I$, always lies in its rectifying space. Here $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in arc length parameter s of the curve γ. In this regard, let us mention that non-null and null f-rectifying curves were investigated in Minkowski 3-space $\mathbb{E}^3_1$ [11, 12] and null f-rectifying curves were explored in Minkowski space-time $\mathbb{E}^4_1$ [13].

In the first section, we give requisite basic preliminaries and then introduce the notion of f-rectifying curves in $\mathbb{E}^4$. Thereafter, the second section is devoted to investigate some geometric characterizations of f-rectifying curves in $\mathbb{E}^4$. In the concluding section, we attempt for some classification of f-rectifying curves in terms of their f-position vectors in $\mathbb{E}^4$. Finally, we cite an example of an f-rectifying curve lying wholly in $\mathbb{E}^4$. This is how this paper is organised.

2 Preliminaries

The Euclidean 4-space $\mathbb{E}^4$ is the four dimensional real vector space $\mathbb{R}^4$ equipped with the standard inner product $\langle \cdot\,, \cdot \rangle$ defined by

$$\langle v, w \rangle := \sum_{i=1}^{4} v_i w_i$$

for all vectors $v = (v_1, v_2, v_3, v_4), w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$. As usual, the norm or length of a vector $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ is denoted and defined by

$$\|v\| := \sqrt{\langle v, v \rangle} = \sqrt{\sum_{i=1}^{4} v_i^2}\,.$$

Let $\gamma : J \longrightarrow \mathbb{E}^4$ be an arbitrary smooth curve in $\mathbb{E}^4$ parametrized by t and γ' stands for its velocity vector field. If we change the parameter t by arc length function $s = s(t)$ based at some $t_0 \in J$ given by $s(t) = \int_{t_0}^{t} \|\gamma'(u)\|\, du$ such that $\langle \gamma'(s), \gamma'(s) \rangle = 1$ for all possible s, then γ is a curve in $\mathbb{E}^4$ parametrized by arc length s or a unit-speed curve in $\mathbb{E}^4$. We may assume that γ is at least 4-times continuously differentiable. Now, let T_γ, N_γ, $B_{\gamma 1}$ and $B_{\gamma 2}$ denote the unit tangent vector field, the unit principal normal vector field, the unit first binormal vector field and the unit second binormal vector field of the curve γ in $\mathbb{E}^4$, respectively, so that for each $s \in I$, the set $\{T_\gamma(s), N_\gamma(s), B_{\gamma 1}(s), B_{\gamma 2}(s)\}$ forms an orthonormal basis for $\mathbb{E}^4$ at the point $\gamma(s)$. Also, let $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$ denote the first curvature, the second curvature and the third curvature of the curve γ in $\mathbb{E}^4$, respectively. Thus $\left\{T_\gamma, N_\gamma, B_{\gamma 1}, B_{\gamma 2}\right\}$ is the dynamic Frenet frame along the curve γ having the same orientation as that of $\mathbb{E}^4$. Then the Frenet formulae for the curve γ are given by ([14, 15])

$$\begin{pmatrix} T_\gamma' \\ N_\gamma' \\ B_{\gamma 1}' \\ B_{\gamma 2}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{\gamma 1} & 0 & 0 \\ -\kappa_{\gamma 1} & 0 & \kappa_{\gamma 2} & 0 \\ 0 & -\kappa_{\gamma 2} & 0 & \kappa_{\gamma 3} \\ 0 & 0 & -\kappa_{\gamma 3} & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_{\gamma 1} \\ B_{\gamma 2} \end{pmatrix}. \tag{1}$$

From the above formulae, it follows that $\kappa_{\gamma 3} \not\equiv 0$ if and only if the curve γ lies wholly in $\mathbb{E}^4$. This is equivalent to saying that $\kappa_{\gamma 3} \equiv 0$ if and only if the curve γ lies wholly in a hypersurface in $\mathbb{E}^4$ (cf. [14, 15]). We recall that the hypersurface in $\mathbb{E}^4$ defined by

$$\mathbb{S}^3(1) := \left\{ v \in \mathbb{E}^4 : \langle v, v \rangle = 1 \right\}$$

is called the unit-sphere with centre at the origin in $\mathbb{E}^4$. We also recall that the rectifying space of the curve γ is the orthogonal complement ${N_\gamma}^\perp$ of its principal normal vector field N_γ defined by

$${N_\gamma}^\perp := \left\{ v \in \mathbb{E}^4 : \langle v, N_\gamma \rangle = 0 \right\}.$$

Consequently, ${N_\gamma}^\perp$ at each point $\gamma(s)$ on γ is a three dimensional subspace of $\mathbb{E}^4$ spanned by $\left\{ T_\gamma(s), B_{\gamma 1}(s), B_{\gamma 2}(s) \right\}$.

3 f-rectifying curves in $\mathbb{E}^4$

As defined in [7], a unit-speed curve $\gamma : I \longrightarrow \mathbb{E}^4$ (parametrized by arc length function s) is a rectifying curve if and only if its position vector always lies in its rectifying space, i.e., if and only if its position vector can be expressed as

$$\gamma(s) = \lambda(s) T_\gamma(s) + \mu_1(s) B_{\gamma 1}(s) + \mu_2(s) B_{\gamma 2}(s)$$

for some differentiable functions $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ in parameter s, for each $s \in I$. Now, for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ in parameter s, the f-position vector of γ in $\mathbb{E}^4$ is denoted and defined by

$$\gamma_f(s) := \int f(s)\, d\gamma$$

for all $s \in I$. Here the integral sign is used in the sense that γ_f is an integral curve of the vector field fT_γ and after differentiating the previous equation we find $\gamma_f'(s) = f(s) T_\gamma(s)$ for all $s \in I$. Keeping in mind this notion of position vector of a curve in $\mathbb{E}^4$, we define an f-rectifying curve in $\mathbb{E}^4$ as follows:

Definition 1 *Let $\gamma : I \longrightarrow \mathbb{E}^4$ be a unit-speed curve (parametrized by arc length function s) with Frenet apparatus $\{T_\gamma, N_\gamma, B_{\gamma 1}, B_{\gamma 2}, \kappa_{\gamma 1}, \kappa_{\gamma 2}, \kappa_{\gamma 3}\}$. Also, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s with at least twice differentiable primitive function F. Then γ is called an f-rectifying* *curve in* $\mathbb{E}^4$ *if its* f*-position vector* γ_f *always lies in its rectifying space in* $\mathbb{E}^4$*, i.e., if its* f*-position vector* γ_f *in* $\mathbb{E}^4$ *can be expressed as*

$$\gamma_f(s) = \lambda(s)T_\gamma(s) + \mu_1(s)B_{\gamma 1}(s) + \mu_2(s)B_{\gamma 2}(s) \tag{2}$$

for all $s \in I$*, where* $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ *are three unique smooth functions in parameter* s*.*

4 Some geometric characterizations of f-rectifying curves in $\mathbb{E}^4$

In this section, we characterize unit-speed f-rectifying curves in $\mathbb{E}^4$ in terms of their curvatures and components of their f-position vectors. First, in the following theorem, we establish a necessary as well as sufficient condition for a unit-speed curve in $\mathbb{E}^4$ to be an f-rectifying curve.

Theorem 1 *Let* $\gamma : I \longrightarrow \mathbb{E}^4$ *be a unit-speed curve (parametrized by arc length* s*), having nowhere vanishing curvatures* $\kappa_{\gamma 1}$*,* $\kappa_{\gamma 2}$ *and* $\kappa_{\gamma 3}$*. Also, let* $f : I \longrightarrow \mathbb{R}$ *be a nowhere vanishing integrable function in parameter* s *with at least twice differentiable primitive function* F*. Then, up to isometries of* $\mathbb{E}^4$*,* γ *is congruent to an* f*-rectifying curve in* $\mathbb{E}^4$ *if and only if the following equation is satisfied:*

$$\frac{d}{ds}\left(\frac{\frac{d}{ds}\left(\frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)}F(s)\right)}{\kappa_{\gamma 3}(s)}\right) + \frac{\kappa_{\gamma 1}(s)\kappa_{\gamma 3}(s)}{\kappa_{\gamma 2}(s)}F(s) = 0 \tag{3}$$

for all $s \in I$.

Proof. Let us first assume that $\gamma : I \longrightarrow \mathbb{E}^4$ be an f-rectifying curve having nowhere vanishing curvatures $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$. Then for some differentiable functions $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ in parameter s, its f-position vector γ_f satisfies equation (2). Differentiating (2) and then applying (1), we obtain

$$\begin{aligned} f(s)T_\gamma(s) \;\; = \;\; & \lambda'(s)T_\gamma(s) + \big(\lambda(s)\kappa_{\gamma 1}(s) - \mu_1(s)\kappa_{\gamma 2}(s)\big)\, N_\gamma(s) \\ & + \big(\mu_1'(s) - \mu_2(s)\kappa_{\gamma 3}(s)\big)\, B_{\gamma 1}(s) \\ & + \big(\mu_2'(s) + \mu_1(s)\kappa_{\gamma 3}(s)\big)\, B_{\gamma 2}(s) \end{aligned} \tag{4}$$

for all $s \in I$. Equating the coefficients from both sides of (4), we get

$$\begin{cases} \lambda'(s) = f(s), \\ \lambda(s)\kappa_{\gamma 1}(s) - \mu_1(s)\kappa_{\gamma 2}(s) = 0, \\ \mu_1'(s) - \mu_2(s)\kappa_{\gamma 3}(s) = 0, \\ \mu_2'(s) + \mu_1(s)\kappa_{\gamma 3}(s) = 0 \end{cases} \tag{5}$$

for all $s \in I$. From first three equations of (5), after some steps, we find

$$\begin{cases} \lambda(s) = F(s), \\ \mu_1(s) = \dfrac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s), \\ \mu_2(s) = \dfrac{1}{\kappa_{\gamma 3}(s)} \dfrac{d}{ds}\left(\dfrac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s)\right) \end{cases} \tag{6}$$

for all $s \in I$. Substituting (6) in the fourth one of (5), we obtain

$$\frac{d}{ds}\left(\frac{\frac{d}{ds}\left(\frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s)\right)}{\kappa_{\gamma 3}(s)}\right) + \frac{\kappa_{\gamma 1}(s)\kappa_{\gamma 3}(s)}{\kappa_{\gamma 2}(s)} F(s) = 0$$

for all $s \in I$.

Conversely, we assume that $\gamma : I \longrightarrow \mathbb{E}^4$ is a unit-speed curve having nowhere vanishing curvatures $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$, and $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s with at least twice differentiable primitive function F such that the equation (3) is satisfied.

Let us define a vector field V along γ by

$$\begin{aligned} V(s) &= \gamma_f(s) - F(s)T_\gamma(s) - \frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s)\, B_{\gamma 1}(s) \\ &\quad - \frac{1}{\kappa_{\gamma 3}(s)} \frac{d}{ds}\left(\frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s)\right) B_{\gamma 2}(s) \end{aligned} \tag{7}$$

for all $s \in I$. Differentiating (7) and then applying (1) and (3), we find that $V'(s) = 0$ for all $s \in I$. This implies that V is constant along γ. Hence, up to isometries of $\mathbb{E}^4$, γ is congruent to an f-rectifying curve in $\mathbb{E}^4$. □

Remark 1 *For an f-rectifying curve in $\mathbb{E}^4$, if all of its curvature functions $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$ are non-zero constants, say, $\kappa_{\gamma 1}(s) = a_1 \neq 0$, $\kappa_{\gamma 2}(s) = a_2 \neq 0$ and $\kappa_{\gamma 3}(s) = a_3 \neq 0$ for all $s \in I$, then from (3), we obtain*

$$F''(s) + a_3^2 F(s) = 0. \tag{8}$$

If f is non-zero constant or linear, then from (8) we find $a_3 = 0$ which is a contradiction. Again, if f is non-linear, then from (8) we find a_3 is non-constant which is also a contradiction.

According to the above remark, we have the following theorem:

Theorem 2 *Let* $\gamma : I \longrightarrow \mathbb{E}^4$ *be a unit-speed curve having nowhere vanishing curvatures* $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ *and* $\kappa_{\gamma 3}$. *Then* γ *is not congruent to an* f*-rectifying curve for any choice of* f *if and only if all of its curvatures* $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ *and* $\kappa_{\gamma 3}$ *are constants.*

Now, it may happen that any two among the three nowhere vanishing curvatures $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$ are constants. Then, as some immediate consequences of Theorem 1, the following theorem provides some characterizations regarding the non-constant one.

Theorem 3 *Let* $\gamma : I \longrightarrow \mathbb{E}^4$ *be a unit-speed curve (parametrized by arc length* s*), having nowhere vanishing curvatures* $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ *and* $\kappa_{\gamma 3}$. *Also, let* $f : I \longrightarrow \mathbb{R}$ *be a nowhere vanishing integrable function in parameter* s *with at least twice differentiable primitive function* F. *We have the following:*

(i) *If the first curvature* $\kappa_{\gamma 1}$ *and the second curvature* $\kappa_{\gamma 2}$ *are constants, then* γ *is congruent to an* f*-rectifying curve in* $\mathbb{E}^4$ *if and only if the third curvature* $\kappa_{\gamma 3}$ *satisfies the following differential equation:*

$$\kappa_{\gamma 3}(s)F''(s) - \kappa_{\gamma 3}'(s)F'(s) + \kappa_{\gamma 3}^3(s)F(s) = 0.$$

(ii) *If the first curvature* $\kappa_{\gamma 1}$ *and the third curvature* $\kappa_{\gamma 3}(= a)$ *are constants, then* γ *is congruent to an* f*-rectifying curve in* $\mathbb{E}^4$ *if and only if the second curvature* $\kappa_{\gamma 2}$ *satisfies the following differential equation:*

$$\frac{d^2}{ds^2}\left(\frac{F(s)}{\kappa_{\gamma 2}(s)}\right) + a^2 \frac{F(s)}{\kappa_{\gamma 2}(s)} = 0.$$

(iii) *If the second curvature* $\kappa_{\gamma 2}$ *and the third curvature* $\kappa_{\gamma 3}(= a)$ *are constants, then* γ *is congruent to an* f*-rectifying curve in* $\mathbb{E}^4$ *if and only if the first curvature* $\kappa_{\gamma 1}$ *satisfies the following differential equation:*

$$\frac{d^2}{ds^2}\left(\kappa_{\gamma 1}(s)F(s)\right) + a^2\kappa_{\gamma 1}(s)F(s) = 0.$$

Analogous characterizations can be derived as consequences of Theorem 1 when any one of $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ or $\kappa_{\gamma 3}$ is a constant.

Next, in the following theorem, we characterize unit-speed f-rectifying curves in $\mathbb{E}^4$ in terms of norm functions, tangential, normal, first and second binormal components of their f-position vectors.

Theorem 4 *Let* $\gamma : I \longrightarrow \mathbb{E}^4$ *be a unit-speed curve (parametrized by arc length* s*), having nowhere vanishing curvatures* κ_{γ_1}, κ_{γ_2} *and* κ_{γ_3}*. Also, let* $f : I \longrightarrow \mathbb{R}$ *be a nowhere vanishing integrable function in parameter* s *with at least twice differentiable primitive function* F*. If* γ *is an* f*-rectifying curve in* $\mathbb{E}^4$*, then the following statements are true:*

(i) *The norm function* $\rho = \|\gamma_f\|$ *is given by*

$$\rho(s) = \sqrt{F^2(s) + c^2}$$

for all $s \in I$*, where* c *is a non-zero constant.*

(ii) *The tangential component* $\langle \gamma_f, T_\gamma \rangle$ *of the* f*-position vector* γ_f *of* γ *is given by*

$$\langle \gamma_f(s), T_\gamma(s) \rangle = F(s)$$

for all $s \in I$.

(iii) *The normal component* $\gamma_f^{N_\gamma}$ *of the* f*-position vector* γ_f *of* γ *has constant length and the norm function* ρ *is non-constant.*

(iv) *The first binormal component* $\left\langle \gamma_f, B_{\gamma_1} \right\rangle$ *and the second binormal component* $\left\langle \gamma_f, B_{\gamma_2} \right\rangle$ *of the* f*-position vector* γ_f *of* γ *are respectively given by*

$$\left\langle \gamma_f(s), B_{\gamma_1}(s) \right\rangle = \frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s),$$
$$\left\langle \gamma_f(s), B_{\gamma_2}(s) \right\rangle = \frac{1}{\kappa_{\gamma_3}(s)} \frac{d}{ds} \left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s) \right)$$

for all $s \in I$.

Conversely, if $\gamma : I \longrightarrow \mathbb{E}^4$ *is a unit-speed curve (parametrized by arc length* s*), having nowhere vanishing curvatures* κ_{γ_1}, κ_{γ_2} *and* κ_{γ_3}*, and* $f : I \longrightarrow \mathbb{R}$ *is a nowhere vanishing integrable function in parameter* s *with at least twice differentiable primitive function* F *such that any one of the statements* (i), (ii), (iii) *or* (iv) *is true, then* γ *is an* f*-rectifying curve in* $\mathbb{E}^4$.

Proof. Let us first assume that $\gamma : I \longrightarrow \mathbb{E}^4$ is an f-rectifying curve having nowhere vanishing curvatures κ_{γ_1}, κ_{γ_2} and κ_{γ_3}. Then for some differentiable functions $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ in parameter s, the f-position vector γ_f of the curve γ in $\mathbb{E}^4$ satisfies equation (2) and from the proof of Theorem 1, we have (5) and (6). Now, from last two equations of (5), we easily find

$$\mu_1(s)\mu_1'(s) + \mu_2(s)\mu_2'(s) = 0$$

for all $s \in I$. Integrating previous equation, we obtain

$$\mu_1^2(s) + \mu_2^2(s) = c^2 \tag{9}$$

for all $s \in I$, where c is an arbitrary non-zero constant. We have the following:
(i) Using (2), (6) and (9), the norm function $\rho = \|\gamma_f\|$ is given by

$$\rho^2(s) = \|\gamma_f(s)\|^2 = \langle \gamma_f(s), \gamma_f(s)\rangle = F^2(s) + c^2,$$

i.e.,

$$\rho(s) = \sqrt{F^2(s) + c^2}$$

for all $s \in I$, where c is a non-zero constant.
(ii) Using (2) and (6), the tangential component $\langle \gamma_f, T_\gamma\rangle$ of γ_f is given by

$$\langle \gamma_f(s), T_\gamma(s)\rangle = \lambda(s) = F(s)$$

for all $s \in I$.
(iii) An f-position vector α_f of an arbitrary curve $\alpha : J \longrightarrow \mathbb{E}^4$ can be decomposed as

$$\alpha_f(t) = \nu(t)\, T_\gamma(t) + \alpha_f^{N_\gamma}(t), \quad t \in J,$$

for some differentiable function $\nu : I \longrightarrow \mathbb{R}$, where $\alpha_f^{N_\gamma}$ denotes the normal component of α_f. Here, γ is an f-rectifying curve in $\mathbb{E}^4$ and hence from (2), it is evident that the normal component $\gamma_f^{N_\gamma}$ of γ_f is given by

$$\gamma_f^{N_\gamma}(s) = \mu_1(s)B_{\gamma 1}(s) + \mu_2(s)B_{\gamma 2}(s)$$

for all $s \in I$. Therefore, we have

$$\left\|\gamma_f^{N_\gamma}(s)\right\| = \sqrt{\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s)\right\rangle} = \sqrt{\mu_1^2(s) + \mu_2^2(s)}$$

for all $s \in I$. Now, by using (9), we see that $\|\gamma_f^{N_\gamma}(s)\| = c$. This implies that $\gamma_f^{N_\gamma}$ has constant length. Furthermore, from statement (i), it follows that the norm function $\rho = \|\gamma_f\|$ is non-constant.

(iv) Using (2) and (6), the first binormal component $\langle \gamma_f, B_{\gamma 1} \rangle$ of γ_f is given by

$$\langle \gamma_f(s), B_{\gamma 1}(s) \rangle = \mu_1(s) = \frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s)$$

for all $s \in I$, and the second binormal component $\langle \gamma_f, B_{\gamma 2} \rangle$ of γ_f is given by

$$\langle \gamma_f(s), B_{\gamma 2}(s) \rangle = \mu_2(s) = \frac{1}{\kappa_{\gamma 3}(s)} \frac{d}{ds} \left(\frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s) \right)$$

for all $s \in I$.

Conversely, we assume that $\gamma : I \longrightarrow \mathbb{E}^4$ is a unit-speed curve having nowhere vanishing curvatures $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$, and $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s with at least twice differentiable primitive function F such that statement (i) or statement (ii) is true. For statement (i), we have

$$\langle \gamma_f(s), \gamma_f(s) \rangle = F^2(s) + c^2$$

for all $s \in I$, where c is a non-zero constant. On differentiation of last equation, we obtain

$$\langle \gamma_f(s), T_\gamma(s) \rangle = F(s) \tag{10}$$

for all $s \in I$. So in either case we have equation (10). Differentiating (10) and by using (1), we finally obtain

$$\langle \gamma_f(s), N_\gamma(s) \rangle = 0$$

for all $s \in I$. This asserts us that γ is an f-rectifying curve in $\mathbb{E}^4$.

Next, we assume that statement (iii) is true. Then $\|\gamma_f^{N_\gamma}\| = c$, say. Now, the normal component $\gamma_f^{N_\gamma}$ is given by

$$\gamma_f(s) = F(s)\, T_\gamma(s) + \gamma_f^{N_\gamma}(s)$$

for all $s \in I$. Therefore, we have

$$\langle \gamma_f(s), \gamma_f(s) \rangle = \langle \gamma_f(s), T_\gamma(s) \rangle^2 + c^2$$

for all $s \in I$. Differentiating previous equation and using (1), we obtain

$$\langle \gamma_f(s), N_\gamma(s) \rangle = 0$$

for all $s \in I$. This implies that γ is an f-rectifying curve in $\mathbb{E}^4$.

Finally, we assume that statement (iv) is true. Then we have

$$\left\langle \gamma_f(s), B_{\gamma 1}(s) \right\rangle = \frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s), \tag{11}$$

$$\left\langle \gamma_f(s), B_{\gamma 2}(s) \right\rangle = \frac{1}{\kappa_{\gamma 3}(s)} \frac{d}{ds} \left(\frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s) \right) \tag{12}$$

for all $s \in I$. Differentiating (11) and using (1), we obtain

$$-\kappa_{\gamma 2}(s) \left\langle \gamma_f(s), N_\gamma(s) \right\rangle + \kappa_{\gamma 3}(s) \left\langle \gamma_f(s), B_{\gamma 2}(s) \right\rangle = \frac{d}{ds} \left(\frac{\kappa_{\gamma 1}(s)}{\kappa_{\gamma 2}(s)} F(s) \right)$$

for all $s \in I$. From the equations (12) and last equation, we find

$$\left\langle \gamma_f(s), N_\gamma(s) \right\rangle = 0$$

for all $s \in I$. Consequently, γ is an f-rectifying curve in $\mathbb{E}^4$. □

5 Classification of f-rectifying curves in $\mathbb{E}^4$

In many papers, e.g., [3, 7, 8, 9], several interesting results were found primarily attempting towards the classification of the rectifying curves which are mostly based on their parametrizations. In this section we attempt for the same in the case of unit-speed f-rectifying curves in $\mathbb{E}^4$ but this classification is totally based on the parametrizations of their f-position vectors.

Theorem 5 *Let $\gamma : I \longrightarrow \mathbb{E}^4$ be a unit-speed curve (parametrized by arc length s) having nowhere vanishing curvatures $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$, and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s with at least twice differentiable primitive function F. Then γ is an f-rectifying curve in $\mathbb{E}^4$ if and only if, up to a parametrization, its f-position vector γ_f is given by*

$$\gamma_f(t) = \frac{c}{\cos\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)} \alpha(t) \tag{13}$$

for all $t \in J$, where c is a positive constant, $s_0 \in I$ and $\alpha : J \longrightarrow \mathbb{S}^3(1)$ is a unit-speed curve having $t : I \longrightarrow J$ as arc length function based at s_0.

Proof. Let us first assume that $\gamma : I \longrightarrow \mathbb{E}^4$ be an f-rectifying curve having nowhere vanishing curvatures $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\kappa_{\gamma 3}$. Then from the proof of Theorem 4, we have

$$\rho(s) = \sqrt{F^2(s) + c^2} \tag{14}$$

for all $s \in I$, where we may choose c as a positive real constant. Now, we define a new curve α in $\mathbb{E}^4$ by

$$\alpha(s) := \frac{1}{\rho(s)}\, \gamma_f(s) \tag{15}$$

for all $s \in I$. Then we find

$$\langle \alpha(s), \alpha(s) \rangle = 1 \tag{16}$$

for all $s \in I$. Therefore, α is a curve whose trace is lying wholly in the unit sphere $\mathbb{S}^3(1)$. Differentiating (16), we get

$$\big\langle \alpha(s), \alpha'(s) \big\rangle = 0, \tag{17}$$

for all $s \in I$. Now, from (14) and (15), we have

$$\gamma_f(s) = \alpha(s)\sqrt{F^2(s) + c^2} \tag{18}$$

for all $s \in I$. Differentiating (18), we find

$$f(s)T_\gamma(s) = \alpha'(s)\sqrt{F^2(s) + c^2} + \frac{\alpha(s)f(s)F(s)}{\sqrt{F^2(s) + c^2}} \tag{19}$$

for all $s \in I$. Using (16), (17) and (19), we obtain

$$\big\langle \alpha'(s), \alpha'(s) \big\rangle = \frac{c^2\, f^2(s)}{\left(F^2(s) + c^2\right)^2}$$

for all $s \in I$. Therefore, we get

$$\big\|\alpha'(s)\big\| = \sqrt{\langle \alpha'(s), \alpha'(s) \rangle} = \frac{c\, f(s)}{F^2(s) + c^2}$$

for all $s \in I$. Let $s_0 \in I$ and $t : I \longrightarrow J$ be arc length function of α based at s_0 given by

$$t = \int_{s_0}^{s} \big\|\alpha'(u)\big\|\, du.$$

Then we find

$$\begin{aligned} & t = \int_{s_0}^{s} \frac{c\, f(u)}{F^2(u) + c^2}\, du \\ \Longrightarrow \qquad & t = \arctan\left(\frac{F(s)}{c}\right) - \arctan\left(\frac{F(s_0)}{c}\right) \end{aligned}$$

$$\Longrightarrow \ F(s) = c \ \tan\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right).$$

Substituting previous equation in (18), we obtain the f-position vector of γ as follows:

$$\gamma_f(t) = \frac{c}{\cos\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)}\,\alpha(t)$$

for all $t \in J$.

Conversely, let γ be a curve in $\mathbb{E}^4$ such that its f-position vector γ_f is given by (13), where c is a positive constant, $s_0 \in I$ and $\alpha : J \longrightarrow \mathbb{S}^3(1)$ is a unit-speed curve having $t : I \longrightarrow J$ as arc length function based at s_0. Differentiating (13), we obtain

$$\gamma_f'(t) = \frac{c \sin\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)}{\cos^2\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)}\,\alpha(t) \ + \ \frac{c}{\cos\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)}\,\alpha'(t) \quad (20)$$

for all $t \in J$. Since α is a unit-speed curve in the unit-sphere $\mathbb{S}^3(1)$, we have $\langle \alpha'(t), \alpha'(t)\rangle = 1$, $\langle \alpha(t), \alpha(t)\rangle = 1$ and consequently $\langle \alpha(t), \alpha'(t)\rangle = 0$ for all $t \in J$. Therefore, from (13) and (20), we have

$$\begin{cases} \langle \gamma_f(t), \gamma_f(t)\rangle = \dfrac{c^2}{\cos^2\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)}, \\ \left\langle \gamma_f(t), \gamma_f'(t)\right\rangle = \dfrac{c^2 \ \sin\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)}{\cos^3\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)}, \\ \left\langle \gamma_f'(t), \gamma_f'(t)\right\rangle = \dfrac{c^2}{\cos^4\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)} \end{cases} \quad (21)$$

for all $t \in J$. We may reparametrize γ by

$$t = \arctan\left(\frac{F(s)}{c}\right) - \arctan\left(\frac{F(s_0)}{c}\right).$$

Then s becomes arc length parameter of γ and equations (21) reduce to

$$\begin{cases} \langle \gamma_f(s), \gamma_f(s) \rangle = \dfrac{c^2}{\cos^2\left(\arctan\left(\frac{F(s)}{c}\right)\right)}, \\ \left\langle \gamma_f(s), \gamma_f'(s) \right\rangle = \dfrac{c^2 \sin\left(\arctan\left(\frac{F(s)}{c}\right)\right)}{\cos^3\left(\arctan\left(\frac{F(s)}{c}\right)\right)}, \\ \left\langle \gamma_f'(s), \gamma_f'(s) \right\rangle = \dfrac{c^2}{\cos^4\left(\arctan\left(\frac{F(s)}{c}\right)\right)} \end{cases} \tag{22}$$

for all $s \in I$. Now, the normal component $\gamma_f^{N_\gamma}$ of γ_f is given by

$$\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s) \right\rangle = \langle \gamma_f(s), \gamma_f(s) \rangle - \frac{\langle \gamma_f(s), \gamma_f'(s) \rangle^2}{\left\langle \gamma_f'(s), \gamma_f'(s) \right\rangle}$$

for all $s \in I$. Then substituting (22) in last equation, we obtain

$$g\left(\gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s)\right) = \left\| \gamma_f^{N_\gamma}(s) \right\|^2 = c^2$$

for all $s \in I$. This implies that $\gamma_f^{N_\gamma}$ has a constant length. Also, the norm function ρ is given by

$$\rho(s) \;=\; \sqrt{g\left(\gamma_f(s), \gamma_f(s)\right)} \;=\; \frac{c}{\cos\left(\arctan\left(\frac{F(s)}{c}\right)\right)}$$

for all $s \in I$, and it is non-constant. Hence, by applying Theorem 4, we conclude that γ is an f-rectifying curve in $\mathbb{E}^4$. □

Finally, we cite an example of an f-rectifying curve lying wholly in $\mathbb{E}^4$.

Example 1 Let γ be a unit-speed curve (parametrized by arc length s) in $\mathbb{E}^4$. Let f be a nowhere vanishing integrable function in parameter s defined by

$$f(s) := \exp s.$$

Then its primitive function F is given by

$$F(s) = \exp s + c_1,$$

where c_1 is an arbitrary constant. We choose $c_1 = 0$ and substitute

$$F(s) = \tan\left(t + \arctan\left(\frac{F(0)}{1}\right)\right) = \tan\left(t + \frac{\pi}{4}\right),$$

i.e.,

$$s = \ln\left|\tan\left(t + \frac{\pi}{4}\right)\right|.$$

Now, let, up to a parametrization, the f-position vector γ_f of γ is given by

$$\gamma_f(t) = \frac{1}{\cos\left(t + \frac{\pi}{4}\right)}\,\alpha(t),$$

where α be a curve in $\mathbb{E}^4$ defined by

$$\alpha(t) := \frac{1}{\sqrt{2}}\left(\sin t, \cos t, \sin t, \cos t\right).$$

Evidently, we have $\langle \alpha(t), \alpha(t)\rangle = 1$ and $\langle \alpha'(t), \alpha'(t)\rangle = 1$ for all t. Therefore, α is a unit-speed curve in $\mathbb{S}^3(1)$ having t as arc length function based at 0. Consequently, γ is an f-rectifying curve and, up to a parametrization, it is given by

$$\gamma(t) = \frac{1}{2}\left(\ln\left|\frac{1+\sin 2t}{\cos 2t}\right|, \ln\left|\frac{1-\sin 2t}{\cos 2t}\right|, \ln\left|\frac{1+\sin 2t}{\cos 2t}\right|, \ln\left|\frac{1-\sin 2t}{\cos 2t}\right|\right).$$

Note: Examples of curves in $\mathbb{E}^4$ which are not f-rectifying for any choice of f are trivial and can be easily constructed by violating the condition stated in Theorem 1. For example, according to Theorem 2 (which is an immediate consequence of Theorem 1), curves in $\mathbb{E}^4$ having non-zero constant first, second and third curvatures are not f-rectifying.

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References

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Revised and Updated Second Edition, Dover Publications, Inc., Mineola, New York, 2016.

[2] A. Pressley, *Elementary Differential Geometry*, Second Edition, Springer-Verlag, London, 2010.

[3] B. Y. Chen, When does the position vector of a space curve always lie in its rectifying plane? *Amer. Math. Monthly*, **110** (2003), 147–152.

[4] B. Y. Chen, Rectifying curves and geodesics on a cone in the Euclidean 3-space, *Tamkang Jour. of Math.*, **48** (2017), 209–214.

[5] B. Y. Chen, F. Dillen, Rectifying curves as centrodes and extremal curves, *Bull. Inst. Math. Acad. Sinica*, **33** (2005), 77–90.

[6] S. Deshmukh, B. Y. Chen, S. Alshamari, On rectifying curves in Euclidean 3-space, *Turk. J. Math.*, **42** (2018), 609–620.

[7] K. Ilarslan, E. Nesovic, Some characterizations of rectifying curves in the euclidean space $\mathbf{E}^4$, *Turk. J. Math.*, **32**(1) (2008), 21–30.

[8] S. Cambie, W. Goemans, I. V. Bussche, Rectifying curves in the $\mathfrak{n}$-dimensional euclidean space, *Turk. J. Math.* **40**(1) (2016), 210–223.

[9] K. Ilarslan, E. Nešović, T. M. Petrović, Some characterization of rectifying curves in the Minkowski 3-Space, *Novi. Sad. Jour. Math.*, **33** (2003), 23–32.

[10] K. Ilarslan, E. Nešović, On rectifying curves as centrodes and extremal curves in the Minkowski 3-Space $\mathbb{E}^3_1$, *Novi. Sad. Jour. Math.*, **37** (2007), 53–64.

[11] Z. Iqbal, J. Sengupta, Non-null (spacelike or timelike) f-rectifying curves in the Minkowski 3-space $\mathbb{E}^3_1$, *Eurasian Bul. Math.*, **3** (1) (2020), 38–55.

[12] Z. Iqbal, J. Sengupta, Null (lightlike) f-rectifying curves in the Minkowski 3-space $\mathbb{E}^3_1$, *Fundam. J. Math. Appl.* **3** (1) (2020), 8–16.

[13] Z. Iqbal, J. Sengupta, Differential geometric aspects of lightlike f-rectifying curves in Minkowski space-time, *Diff. Geom. - Dyn. Syst.*, **22** (2020), 113–129.

[14] H. Gluck, Higher curvatures of curves in euclidean space, *Amer. Math. Mon.* **73**(7) (1966), 699–704.

[15] H. Gluck, Higher curvatures of curves in euclidean space, ii, *Amer. Math. Mon.* **74**(9) (1967), 1049–1056.

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Induced star-triangle factors of graphs

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Abstract. An induced star-triangle factor of a graph G is a spanning subgraph F of G such that each component of F is an induced subgraph on the vertex set of that component and each component of F is a star (here star means either $K_{1,n}$, $n \geq 2$ or K_2) or a triangle (cycle of length 3) in G. In this paper, we establish that every graph without isolated vertices admits an induced star-triangle factor in which any two leaves from different stars $K_{1,n}$ $(n \geq 2)$ are non-adjacent.

1 Introduction

A simple *graph* is denoted by $G(V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G)$ are respectively the vertex set and edge set of G.. The *order* and *size* of G are $|V(G)|$ and $|E(G)|$, respectively. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. A cycle of order n is denoted by C_n and a *triangle* is denoted by C_3. A complete bipartite graph $K_{1,n}$ is called a *star.* In $K_{1,n}$, the vertex of degree n is its *center* and all other vertices are *leaves.*

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A *matching* in a graph is a set of independent edges. That is, a subset M of the edge set E of G is a matching if no two edges of M have a common vertex. A matching M is said to be *maximal* if there is no matching N strictly containing M, that is, M is maximal if it cannot be enlarged. A matching M is said to be *maximum* if it has the largest possible cardinality, that is, M is maximum if there is no matching N such that $|N| > |M|$. A vertex v is said to be M-*saturated* (or saturated by M) if there is an edge $e \in M$ incident with v. A vertex which is not incident with any edge of M is said to be M-*unsaturated.* An M-*alternating path* in G is a path whose edges are alternately in $E(G) - M$ and M. That is, in an M-alternating path, the edges alternate between M-edges and non-M-edges. An M-alternating path whose end vertices are M-unsaturated is said to be an M-*augmenting path.*

For $S \subset V(G)$, the *induced graph* on S is a subgraph of G with vertex set S and the edge set consisting of all the edges of G which have both end vertices in S. An *induced star* of G is an induced subgraph of G which itself is a star.

For a set S of connected graphs, a spanning subgraph F of a graph G is called an S-*factor* of G if each component of F is isomorphic to an element of S. A spanning subgraph F of a graph G is a *star-cycle* factor of G if each component of F is a star or a cycle. A spanning subgraph S of a graph G will be called an *induced star-triangle factor* of G if each component of S is an induced star ($K_{1,n}$, $n \geq 2$, or K_2) or a triangle of G.

For a vertex subset S of $V(G)$, let $G[S]$ and $G - S$, respectively, denote the subgraph of G induced by S and $V(G) - S$. Further, let $iso(G)$ mean the number of isolated vertices in G and $Iso(G)$ be the set of isolated vertices of G. Clearly $|Iso(G)| = iso(G)$. For more definitions and notations, we refer to [7].

Tutte [8] characterized graphs having $\{K_2, C_n : n \geq 3\}$-factor. An elementary and short proof of Tutte's characterization can be seen in [1]. Las Vergnas [6] and Amahashi and Kano [2] showed that, for an integer $n \geq 2$, a graph has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $iso(G - S) \leq n|S|$ for all $S \subset V(G)$. Berge and Las Vergnas [3] showed the existence of $\{K_{1,n}, C_m : n \geq 1, m \geq 3\}$-factor in graphs. A short proof of this theorem can be seen in [4].

2 Main results

In [5], we established Boyer's conjecture on the dimension of sphere of influence of graphs having perfect matchings, by obtaining a factor of a given graph and then embedding that into a suitable finite dimensional Euclidean space. While working on the main conjecture, we encountered the following result, which we believe would of interest to a general reader.

Theorem 1 *Every graph without isolated vertices,*
admits an induced star-triangle factor in which any two leaves from different stars $K_{1,n}$ *(*$n \geq 2$*) are non adjacent.*

To prove the result, let G be any graph without isolated vertices.

Let $V(G)$ and $E(G)$, respectively, denote the vertex set and the edge set of G. Let M be the maximum matching in G, M' be the set of M-saturated vertices and I be the set of M-unsaturated vertices.

We adopt the following algorithm, which contains the gist of the proof of Theorem 1.

Algorithm 1

1. Let $M_1 = M$.
2. If $I \neq \emptyset$, then pick a vertex v from I, otherwise go to step 10.
3. Pick $u \in N(v)$ and call the edge uv as the *neighborhood edge* of v. (As v is not isolated, there exists an edge $uv \in E(G)$.) Then $u \in M'$. Otherwise $M \cup \{uv\}$ will be a larger matching than M, which is impossible.
4. Let $w \in M'$ such that $uw \in M$.
5. If S_u is not defined, define $S_u := \{w, v\}$, otherwise go to step 7.
6. Remove uw from M_1, go to step 8.
7. If S_u is defined then add v to S_u.
8. Set $J = I \setminus \{v\}$.
9. With $I = J$, go to step 2.
10. Stop.

At the end of this algorithm, we obtain a matching M_1, finitely many vertices $u_1, \dots, u_k$ and the corresponding sets $S_{u_1}, \dots, S_{u_k}$. Before we analyze these sets, let us consider an example to see how the algorithm works.

Example 1 *Consider a graph* G *on* 17 *vertices, given by Figure 1.*

Here

$$M = \{\{1,2\},\{7,8\},\{9,10\},\{13,14\},\{15,16\}\}$$

is a maximum matching and the corresponding set I *is given by* $\{3,4,5,6,11,12\}$. *Applying Algorithm 1, we obtain a factor of* G, *given by Figure 2.*

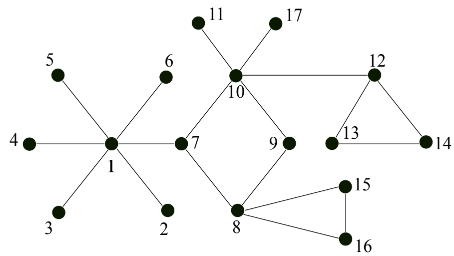

Figure 1: Graph G

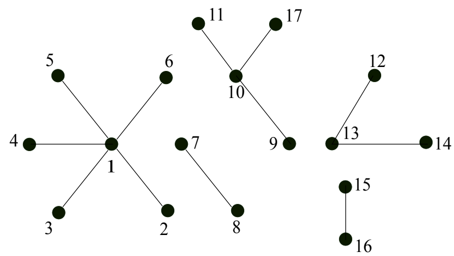

Figure 2: Output of Algorithm 1 on G

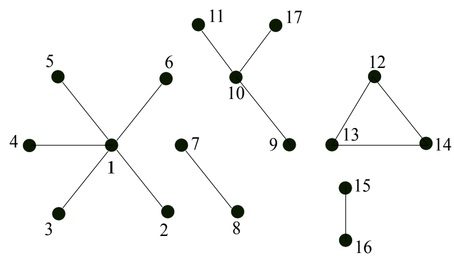

Figure 3: Star-triangle factor of G

After applying the procedure specified in the proof of Theorem 1 we will obtain the graph given by Figure 3, which is a required star-triangle factor of G. □

To prove Theorem 1, we need a series of lemmas. The first one is immediate.

Lemma 1 *1. Each* $v \in I$ *has exactly one neighborhood edge.*

2. *Each S_{u_i} has at least two vertices, exactly one vertex from M', and u_i has a matching edge with that vertex.*

Using this lemma, we obtain the following result.

Lemma 2 *For each $1 \leq i < j \leq k$, we have*

$$(\{u_i\} \cup S_{u_i}) \cap (\{u_j\} \cup S_{u_j}) = \emptyset.$$

Proof. It is enough to prove the result for $i = 1$ and $j = 2$. Assume that there exists some $x \in (\{u_1\} \cup S_{u_1}) \cap (\{u_2\} \cup S_{u_2})$.

If $x \in I$, then $x \in S_{u_1}$ and $x \in S_{u_2}$. Therefore, xu_1 and xu_2 are the neighborhood edges of x. By Lemma 1, x has only one neighborhood edge, a contradiction. Therefore $x \notin I$ and thus $x \in M'$.

If $x \in \{u_1, u_2\}$, without loss of generality, let $x = u_1$. Then $u_1 \in S_{u_2}$. By Lemma 1, S_{u_2} has only one vertex from M', and u_2 has a matching edge with that vertex. Therefore, u_1u_2 is a matching edge, that is, $u_1u_2 \in M$. This implies that $u_2 \in S_{u_1}$.

Also, by Lemma 1, we have $|S_{u_1}| \geq 2$ and $|S_{u_2}| \geq 2$. Choose $x_1 \in S_{u_1}$ and $x_2 \in S_{u_2}$ such that $\{x_1, x_2\} \cap \{u_1, u_2\} = \emptyset$. Then $\{x_1, x_2\} \subseteq I$ and thus $x_1 \neq x_2$.

Therefore, $x_1u_1u_2x_2$ is an augmenting path of M, which implies that M is not a maximum matching, a contradiction. Hence $x \notin \{u_1, u_2\}$.

Consequently $x \in M'$ such that $x \in S_{u_1}$ and $x \in S_{u_2}$. Again, Lemma 1 ensures that xu_1 and xu_2 are matching edges. Hence xu_1 and xu_2 are not independent edges, a contradiction. □

Lemma 3 *The residual set M_1 is a matching. Further, if M_1' is the set of vertices of M_1, then $V(G)$ can be partitioned as*

$$V(G) = \left(\dot\cup_{i=1}^{k}(\{u_i\}\dot\cup S_{u_i})\right)\dot\cup M_1'.$$

Proof. Since M_1 embeds inside the matching M, it is a matching in G.

Pick any $y \in V(G)$. Then, either $y \in M_1'$ or $y \notin M_1'$. If $y \notin M_1'$, then by our construction $y \in \{u_i\} \cup S_{u_i}$, for some i.

Therefore, $y \in \cup_{i=1}^{k}(\{u_i\} \cup S_{u_i})$.

Thence, $V(G) \subset \left(\cup_{i=1}^{k} (\{u_i\} \cup S_{u_i})\right) \cup M_1'$. The other inclusion is trivial.

To prove that the union is disjoint, let $x \in \{u\} \cup S_u$, for some $u \in V(G)$. Then, either $x \in I$ or $x \in M'$. If $x \in I$, then $x \notin M'$ and thus $x \notin M_1'$. If $x \in M'$, then either S_x is defined or $x \in S_{x'}$ where xx' is a matching edge removed from M_1. Therefore, $x \notin M_1'$ and thence

$$M_1' \cap (\cup_{S_u}(\{u\} \cup S_u)) = \emptyset.$$

This along with Lemma 2, establishes the result. □

Lemma 4 *If* $u \in \{u_1, \ldots, u_k\}$ *and if there are* $v, w \in S_u$ *such that* $vw \in E(G)$, *then*

$$S_u = \{w, v\}.$$

Proof. If possible, choose $v' \in S_u \setminus \{w, v\}$. By our construction, there exists some $v'' \in S_u$ such that $v''u \in M$. We have the following cases to consider.

1. If $v'' \notin \{w, v\}$, then by construction, S_u has exactly one vertex from M' and all other vertices from I. Therefore $vw \notin M$ and thus $\{vw\} \cup M$ is a matching in G, larger than M.

2. If $v'' = w$, then $vwuv'$ is an M-augmented path.

3. If $v'' = v$, then $wvuv'$ is an M-augmented path.

Therefore, in each case, the augmented paths contradict the choice of M as a maximum matching. This proves our assertion. □

Proof of Theorem 1: First, we make a small change in our notations from Algorithm 1.

For each $S_u = \{v_1, v_2\}$, if $v_1v_2 \in E(G)$, then destroy (remove) S_u means from now onwards this S_u does not exist. Instead, if such an S_u exists, we do the following.

If T is not defined, then define $T := \{\{u, v_1, v_2\}\}$, otherwise add $\{u, v_1, v_2\}$ to T.

Basically, we are separating out the class of triangles from stars. Thus, we have found mutually exclusive stars $\{u\} \cup S_u$, triangles and a matching M_1 in G covering all the vertices.

Now, we establish that the remaining sets $\{u\} \cup S_u$ are stars.

Claim 1. Each S_u is an independent set.

To see this, note that we first defined S_u as having one vertex from M' and other from I. Then we added some vertices from I to S_u. Therefore, each S_u has one vertex from M' and remaining vertices from I.

Let $\{v_1, v_2\} \subseteq S_u$. If $\{v_1, v_2\} \subseteq I$, then clearly $v_1v_2 \notin E(G)$. Otherwise, without loss of generality, assume that $v_1 \in M'$ and $v_2 \in I$.

If $|S_u| = 2$, then by our construction, we have
$v_1v_2 \notin E(G)$. If $|S_u| > 2$, then there exists some $v_3 \in S_u \setminus \{v_1, v_2\}$. Therefore, $v_3 \in I$ and $v_1u \in M$.

If $v_1v_2 \in E(G)$, then $v_2v_1uv_3$ is an M-augmenting path. Therefore, M is not a maximum matching, a contradiction. Hence, $v_1v_2 \notin E(G)$. This establishes claim 1.

So we obtain a matching M_1, finitely many induced stars and triangles, all of which span our given graph G. Note that the matching M_1 can also be treated as a finite collection of induced stars K_2. Consequently, we obtain an induced star-triangle factor of G.

To conclude our main result, we claim the following.

Claim 2. The set $\cup S_u$ is independent.

To see this, let $\{v_1, v_2\} \subseteq \cup S_u$. We have to prove that $v_1v_2 \notin E(G)$. If $\{v_1, v_2\} \subseteq S_u$, for some u, then this follows by Claim 1. Without loss of generality, it is enough to assume that $v_1 \in S_{u_1}$ and $v_2 \in S_{u_2}$.

We have the following cases to consider.

1. $\{v_1, v_2\} \subseteq M'$. To prove by contradiction, assume that $v_1v_2 \in E(G)$.

   By our construction, $|S_{u_1}| \geq 2$ and $|S_{u_2}| \geq 2$. Therefore, we can choose $x_1 \in S_{u_1}$ and $x_2 \in S_{u_2}$ such that $x_1 \neq v_1$ and $x_2 \neq v_2$. Then $\{x_1, x_2\} \subseteq I$ and $\{u_1v_1, u_2v_2\} \subseteq M$. Therefore, $x_1u_1v_1v_2u_2x_2$ is an M-augmenting path, concluding that M is not the maximum matching, a contradiction.

2. $\{v_1, v_2\} \subseteq I$. Clearly, $v_1v_2 \notin E(G)$, as I is an independent set.

3. $v_1 \in M'$ and $v_2 \in I$. (The other case $v_1 \in I$ and $v_2 \in M'$ is similar.) To prove by contradiction, assume that $v_1v_2 \in E(G)$.

   Since $|S_{u_1}| \geq 2$, there exists some $x_1 \in S_{u_1} \setminus \{v_1\}$. As S_u has only one vertex from M' and $v_1 \in M'$, we have $x_1 \in I$. Also, $u_1v_1 \in M$ ensures that $x_1u_1v_1v_2$ is an M-augmenting path. Thus, M is not the maximum matching, a contradiction.

Therefore, in every case $v_1v_2 \notin E(G)$. This establishes claim 2. Hence, Theorem 1 is proved. □

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References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs, *Lecture Notes in Math.*, **2031** Springer, 2011.

[2] A. Amahashi and M. Kano, On factors with given components, *Discrete Math.*, **42** (1983), 1–6.

[3] C. Berge and M. Las Vergnas, On the existence of subgraphs with degree constraints, *Nederl. Akad. Wetensch. Indag. Math.*, **40** (1978), 165–170.

[4] Y. Egawa, M. Kano and Z. Yan, Star-cycle factors of graphs, *Discussiones Mathematicae Graph Theory*, **34** (2014), 193–198.

[5] R. Kumar and S. P. Singh, SIG-dimension conjecture proved for graphs having a perfect matching, *Discrete Math. Algorithms Appl.*, **9,1** (2017), 1750008 (24 pages).

[6] M. Las Vergnas, An extension of Tutte's 1-factor theorem, *Discrete Math.*, **23** (1978), 241–255.

[7] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Hyderabad, India, 2012.

[8] W. T. Tutte, The 1-factors of oriented graphs, *Proc. Amer. Math. Soc.*, **4** (1953), 922–931.

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Generalized normal ruled surface of a curve in the Euclidean 3-space

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Abstract. In this study, we define the generalized normal ruled surface of a curve in the Euclidean 3-space E^3. We study the geometry of such surfaces by calculating the Gaussian and mean curvatures to determine when the surface is flat or minimal (equivalently, helicoid). We examine the conditions for the curves lying on this surface to be asymptotic curves, geodesics or lines of curvature. Finally, we obtain the Frenet vectors of generalized normal ruled surface and get some relations with helices and slant ruled surfaces and we give some examples for the obtained results.

1 Introduction

Ruled surfaces have an important role in many areas such as architecture, robotics, computer aided geometric design, physics, design problems in spatial mechanism, etc. In 1930, precontraint concrete has been discovered. Then, these surfaces have had an important role in architectural construction and used to construct the spiral stair-cases, roofs, water-towers and chimney-pieces. For instance, Eero Saarinen (1910–1961) used helicoid surface in staircase in General Motor Technical Center in Michigan. He also used ruled surfaces at Yale and M.I.T. buildings. Furthermore, Antonio Gaudí (1852-1926) designed the many pillars of the Sagrada Familia by using hyperbolic hyperboloids. Builder Felix Candela (1910-1997) has made extensive use of cylinders and

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the most familiar ruled surfaces [2]. Therefore, ruled surfaces have been the focus of study by many mathematicians and different kinds of such surfaces have been defined and studied. One of these kinds is rectifying developable of a curve which defined by Izumiya and Takeuchi as the envelope of the family of rectifying planes of a space curve. They have studied singularities of such surfaces and also given a local classification. They also defined and studied Darboux developable of a space curve whose singularities are given by the locus of the endpoints of modified Darboux vector of the curve [4, 5, 6, 7, 8].

Later, Soliman *et al* have made a different definition for the rectifying developable surface [13]. They have defined this surface as the surface whose generator line is unit Darboux vector of a space curve. They have obtained that this surface has pointwise 1-type Gauss map of the first kind with a base plane curve if and only if the base curve is a circle or straight line.

Recently, Önder and Kahraman defined general type of rectifying ruled surfaces as the surface whose rulings always lie on the rectifying plane of the base curve [12]. He has obtained many properties of these special ruled surfaces and showed that only the developable rectifying surfaces are the surfaces defined by Izumiya and Takeuchi.

Furthermore, in [11] Önder has defined some new types of ruled surfaces in 3-dimensional Euclidean space which are called slant ruled surfaces by using the "slant" concept in [4]. Later, Önder and Kaya have given some differential equation characterizations for slant ruled surfaces [10].

In this paper, we define a new type of ruled surfaces called generalized normal ruled surfaces in the Euclidean 3-space. We study their Gaussian and mean curvatures, investigate surface curves on generalized normal ruled surfaces and relations with other special ruled surfaces such as slant ruled surfaces.

2 Preliminaries

A surface F is called to be a ruled surface if it is drawn by the continuous movement of a straight line along a curve α. Such surfaces are parameterized as $F_{(\alpha,q)}(s,u) : I \times \mathbb{R} \to \mathbb{R}^3$, $\vec{F}_{(\alpha,q)}(s,u) = \vec{\alpha}(s) + u\vec{q}(s)$ where $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ is called the base curve and $q : \mathbb{R} \to \mathbb{R}^3 - \{0\}$ is called the ruling. If q is unit, the ruled surface $F_{(\alpha,q)}$ is cylindrical if and only if $\vec{q}\,' = 0$ and non-cylindrical otherwise, where the prime "$\,'$" shows derivative with respect to s. The curve $c = c(s)$ which satisfies the condition $\langle \vec{c}\,', \vec{q}\,' \rangle = 0$ is called the striction curve of ruled surface $F_{(\alpha,q)}$. The striction curve has an important geometric meaning such that if there exists a common perpendicular to two constructive rulings, then the foot of the common perpendicular on the main ruling is called a central point and striction curve is the locus of central points.

The Gauss map or the unit surface normal U of the ruled surface $F_{(\alpha,q)}$ is defined by

$$\vec{U}(s,u) = \frac{\frac{\partial \vec{F}_{(\alpha,q)}}{\partial s} \times \frac{\partial \vec{F}_{(\alpha,q)}}{\partial u}}{\| \frac{\partial \vec{F}_{(\alpha,q)}}{\partial s} \times \frac{\partial \vec{F}_{(\alpha,q)}}{\partial u} \|}. \tag{1}$$

If $\frac{\partial \vec{F}_{(\alpha,q)}}{\partial s} \times \frac{\partial \vec{F}_{(\alpha,q)}}{\partial u} = 0$ for some points (s_0, u_0), then such points are called singular points of the surface. Otherwise, they are called regular points. The ruled surface $F_{(\alpha,q)}$ is called to be developable if the surface normal does not change along a ruling $q = q_0$. Non-developable ruled surfaces are called skew surfaces. A ruled surface $F_{(\alpha,q)}$ is developable if and only if $\det(\vec{\alpha}', \vec{q}, \vec{q}\,') = 0$.

When $\|\vec{q}(s)\| = 1$, the vectors $\vec{h}(s) = \frac{\vec{q}\,'(s)}{\|\vec{q}\,'(s)\|}$ and $\vec{a}(s) = \vec{q}(s) \times \vec{h}(s)$ are called central normal and central tangent, respectively. The orthonormal frame $\{\vec{q}, \vec{h}, \vec{a}\}$ is called the Frenet frame of ruled surface $F_{(\alpha,q)}$ [9].

Definition 1 [11] *A ruled surface $F_{(\alpha,q)}$ is called a q-slant or a-slant (respectively, h-slant) ruled surface if its ruling q (respectively, central normal h) always makes a constant angle with a fixed direction.*

The first fundamental form I and second fundamental form II of ruled surface $F_{(\alpha,q)}$ are defined by

$$\begin{aligned} I &= E ds^2 + 2F ds du + G du^2 \\ II &= L ds^2 + 2M ds du + N du^2 \end{aligned} \tag{2}$$

where

$$\begin{aligned} E &= \left\langle \frac{\partial \vec{F}_{(\alpha,q)}}{\partial s}, \frac{\partial \vec{F}_{(\alpha,q)}}{\partial s} \right\rangle, \quad F = \left\langle \frac{\partial \vec{F}_{(\alpha,q)}}{\partial s}, \frac{\partial \vec{F}_{(\alpha,q)}}{\partial u} \right\rangle, \\ G &= \left\langle \frac{\partial \vec{F}_{(\alpha,q)}}{\partial u}, \frac{\partial \vec{F}_{(\alpha,q)}}{\partial u} \right\rangle \end{aligned} \tag{3}$$

and

$$L = \left\langle \frac{\partial^2 \vec{F}_{(\alpha,q)}}{\partial s^2}, \vec{U} \right\rangle, \quad M = \left\langle \frac{\partial^2 \vec{F}_{(\alpha,q)}}{\partial s \partial u}, \vec{U} \right\rangle, \quad N = \left\langle \frac{\partial^2 \vec{F}_{(\alpha,q)}}{\partial u^2}, \vec{U} \right\rangle. \tag{4}$$

The Gaussian curvature K and mean curvature H are defined by

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} \\ H &= \frac{EN - 2FM + GL}{2(EG - F^2)} \end{aligned} \tag{5}$$

respectively [1]. An arbitrary surface is called the flat surface if $K = 0$ and called minimal if $H = 0$ at all points of the surface.

Helicoid (or right helicoid) is a special kind of ruled surfaces which is generated by a line attached orthogonally to an axis such that the line moves along the axis and also rotates, both at constant speed.

The following theorem is known as Catalan theorem [3].

Theorem 1 *Among all ruled surfaces except planes only the helicoid and its fragments are minimal.*

Definition 2 *Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve and $\{\vec{T}, \vec{N}, \vec{B}\}$ be its Frenet frame. The ruled surfaces $F_{(\alpha,N)}$ and $F_{(\alpha,B)}$ are called principal normal surface and binormal surface of α, respectively, which are given by the parametrizations*

$$\vec{F}_{(\alpha,N)} = \vec{\alpha} \pm u\vec{N} \text{ and } \vec{F}_{(\alpha,B)} = \vec{\alpha} \pm u\vec{B}, \tag{6}$$

respectively.

3 Generalized normal ruled surfaces

In this section, we define generalized normal ruled surfaces of a curve in the Euclidean 3-space E^3.

Definition 3 *Let $\alpha(s) : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with arclength parameter s, Frenet frame $\{\vec{T}, \vec{N}, \vec{B}\}$, curvature κ and torsion τ. The ruled surface $F_{(\alpha,q_n)}(s,u) : I \times \mathbb{R} \to \mathbb{R}^3$ defined by*

$$\vec{F}_{(\alpha,q_n)}(s,u) = \vec{\alpha}(s) + u\vec{q}_n(s), \quad \vec{q}_n(s) = a_1(s)\vec{N}(s) + a_2(s)\vec{B}(s) \tag{7}$$

is called the generalized normal ruled surface (GNR-surface) of α where $a_1^2 + a_2^2 = 1$ and a_1, a_2 are smooth functions of arc-length parameter s.

From Definition 3, we can easily see that if $a_1 = 0$ and $a_2 = \pm 1$, then the GNR-surface $F_{(\alpha,q_n)}$ becomes binormal surface $F_{(\alpha,B)}$. Similarly, if $a_1 = \pm 1$ and $a_2 = 0$, then the GNR-surface $F_{(\alpha,q_n)}$ becomes principal normal surface $F_{(\alpha,N)}$.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $g : I \times \mathbb{R} \to \mathbb{R}$ be smooth functions defined by

$$\begin{aligned} f(s) &= a_1'(s)a_2(s) - a_1(s)a_2'(s) - \tau(s) \\ g(s,u) &= 1 - ua_1(s)\kappa(s) \end{aligned} \tag{8}$$

We will call f and g as the characterization functions of GNR-surface $F_{(\alpha,q_n)}$ and give the many properties of the surface with respect to f and g.

Theorem 2 *The surface $F_{(\alpha,q_n)}$ is not regular if and only if $f = g = 0$.*

Proof. From the partial derivatives of equation (7) we get

$$\begin{aligned} \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s} &= g\vec{T} + u(a_1' - a_2\tau)\vec{N} + u(a_1\tau + a_2')\vec{B} \\ \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial u} &= a_1\vec{N} + a_2\vec{B} \end{aligned} \tag{9}$$

Then, from the cross product of the last equations it follows

$$\frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s} \times \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial u} = uf\vec{T} - a_2g\vec{N} + a_1g\vec{B}$$

and we have that $f = g = 0$ if and only if $\frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s} \times \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial u} = 0$. □

This theorem allows to determinate the singular points of the GNR-surfaces as follows.

Proposition 1 (i) *If the surface has singular points, α is not a plane curve and not a line, for $a_1, a_2 \neq 0$, the locus of the singular points of the GNR-surface $F_{(\alpha,q_n)}$ is given by the curve γ defined by*

$$\vec{\gamma}(s) = \vec{\alpha}(s) + u(s)\vec{q}_n(s)$$

where

$$u(s) = \frac{1}{a_2\kappa \int \frac{\tau}{a_2^2} ds}.$$

(ii) *If α is a line, then the surface does not have singular points.*

(iii) *If the surface has singular points, α is a plane curve and not a line, then the locus of singular points of the GNR-surface $F_{(\alpha,q_n)}$ is given by the curve*

$$\vec{\gamma}(s) = \vec{\alpha}(s) + u(s)\vec{q}_n(s)$$

where

$$u(s) = \pm\frac{\sqrt{1 + C^2}}{C\kappa}.$$

and C is a real constant.

Proof. (i) From Theorem 2, for the singular points of $F_{(\alpha,q_n)}$ we have

$$a_1'(s)a_2(s) - a_1(s)a_2'(s) - \tau(s) = 0,$$
$$1 - ua_1(s)\kappa(s) = 0.$$

From the first equation of this system, we get $a_1 = a_2 \int \frac{\tau}{a_2^2} ds$ and from the second equation, we get $a_1 = \frac{1}{u\kappa}$. By eliminating a_1, we obtain the desired result.

(ii) In order to achieve singular points both f and g must be equal to zero. In the case of κ, g and κ cannot be zero at the same time since

$$g = 1 - ua_1(s)\kappa(s)$$

If we choose $g = \kappa = 0$, we obtain the contradiction $1 = 0$. Therefore, the surface does not have singular points if α is a line.

(iii) Since α is planar, we have $\tau = 0$. Then, from $f = 0$, for a real constant C, we get

$$\begin{aligned} a_1'(s)a_2(s) - a_1(s)a_2'(s) = 0 &\Rightarrow \frac{a_1'(s)}{a_1(s)} = \frac{a_2'(s)}{a_2(s)} \\ &\Rightarrow \ln|a_1(s)| = \ln|a_2(s)| + \ln|C|, C \in \mathbb{R} \\ &\Rightarrow a_1(s) = Ca_2(s). \end{aligned}$$

Since $a_1^2 + a_2^2 = 1$, it follows

$$a_1 = \pm\frac{C}{\sqrt{1+C^2}}, \qquad a_2 = \pm\frac{1}{\sqrt{1+C^2}}.$$

Therefore, in the case $\kappa \neq 0$ and $\tau = 0$, for the locus of the singular point of the GNR-surface $F_{(\alpha,q_n)}$ we get

$$\vec{\gamma}(s) = \vec{\alpha}(s) + u(s)\vec{q}_n(s)$$

where

$$u(s) = \pm\frac{\sqrt{1+C^2}}{C\kappa}.$$

□

Since we have $\vec{q}_n(s) = a_1(s)\vec{N}(s) + a_2(s)\vec{B}(s)$, by the derivation of the ruling with respect to s, it follows

$$\vec{q}_n' = -a_1\kappa\vec{T} + (a_1' - a_2\tau)\vec{N} + (a_1\tau + a_2')\vec{B}.$$

Then, the surface is cylindrical, i.e., $\vec{q}_n{}' = 0$ if and only if the following system hold

$$\begin{cases} a_1\kappa = 0, \\ a_1' - a_2\tau = 0, \\ a_1\tau + a_2' = 0. \end{cases}$$

This system is reduced to

$$\begin{cases} \kappa = 0, \\ a_1 = \text{constant}, \\ a_2 = \text{constant}. \end{cases} \tag{10}$$

when $a_1 \neq 0$. In this case, since $\kappa = 0$, we have that the tangent vector $\vec{T}$ is constant. Moreover, the principal normal vector $\vec{N}$ and binormal vector $\vec{B}$ are also constant vectors based on choice which are perpendicular to $\vec{T}$. Since the tangent of the base curve α is constant and the surface is cylindrical, the GNR-surface $F_{(\alpha,q_n)}$ becomes a plane. This gives the following corollary:

Corollary 1 *If $a_1 \neq 0$, among all GNR-surfaces $F_{(\alpha,q_n)}$ only the plane is cylindrical.*

In the case that $a_1 = 0$, the surface becomes binormal surface $F_{(\alpha,B)}$ and we get $\tau = 0$ which means that α is a planar curve. For this case we can give the following corollary

Corollary 2 *The binormal surface $F_{(\alpha,B)}$ is cylindrical if and only if $\tau = 0$, i.e., α is a planar curve.*

The striction parameter of the GNR-surface $F_{(\alpha,q_n)}$ can be achieved by

$$u(s) = -\frac{\langle \vec{\alpha}', \vec{q}_n{}' \rangle}{\langle \vec{q}_n{}', \vec{q}_n{}' \rangle} = \frac{a_1\kappa}{a_1^2\kappa^2 + (a_1' - a_2\tau)^2 + (a_1\tau + a_2')^2}. \tag{11}$$

If $a_1 = 0$, then we have $\vec{q}_n = \pm\vec{B}$. Thus, $\langle \vec{\alpha}', \vec{q}_n{}' \rangle = \langle \vec{T}, \mp\tau\vec{N} \rangle = 0$. Therefore, we have the following corollary:

Corollary 3 *The base curve α of surface $F_{(\alpha,q_n)}$ is its striction curve if and only if $F_{(\alpha,q_n)} = F_{(\alpha,B)}$ or α is a straight line.*

Proposition 2 *The GNR-surface $F_{(\alpha,q_n)}$ is developable if and only if $f = 0$.*

Proof. A ruled surface $F_{(\alpha,q)}$ is developable if and only if $\det(\vec{\alpha}', \vec{q}, \vec{q}\,') = 0$. Then, we get

$$\begin{aligned}\det(\vec{\alpha}', \vec{q}_n, \vec{q}_n') &= \det(\vec{T}, a_1\vec{N} + a_2\vec{B}, -a_1\kappa\vec{T} \\ &\quad + (a_1' - a_2\tau)\vec{N} + (a_1\tau + a_2')\vec{B}) \\ &= a_1a_2' - a_1'a_2 + \tau \\ &= -f\end{aligned} \tag{12}$$

From (12), it is clear that $\det(\vec{\alpha}', \vec{q}_n, \vec{q}_n') = 0$ if and only if $f = 0$. □

Considering Theorem 2 and Proposition 2, the following corollary is obtained.

Corollary 4 *A developable GNR-surface $F_{(\alpha,q_n)}$ is regular if and only if $g \neq 0$.*

The Gaussian map or the unit surface normal U of the GNR-surface $F_{(\alpha,q_n)}$ can easily be calculated from (1) as

$$\vec{U}(s,u) = \frac{1}{\sqrt{u^2f^2 + g^2}}\left(uf\vec{T} - a_2g\vec{N} + a_1g\vec{B}\right). \tag{13}$$

Considering base curve α on $F_{(\alpha,q_n)}$, the unit surface normal $\vec{U}$ can be restricted to α by taking $u = 0$, i.e.,

$$\vec{U}_\alpha = -a_2\vec{N} + a_1\vec{B}. \tag{14}$$

Then, we can give the followings:

Proposition 3 (i) *The base curve α is a geodesic if and only if α is a straight line or $F_{(\alpha,q_n)}=F_{(\alpha,B)}$.*

(ii) *The base curve α is an aymptotic curve if and only if α is a straight line or $F_{(\alpha,q_n)}=F_{(\alpha,N)}$.*

Proof. For the curve α to be a geodesic on $F_{(\alpha,q_n)}$, the directions of principal normal N of α and the surface normal $\vec{U}_\alpha$ along α must be identical. Then, α is a geodesic if and only if $\vec{U}_\alpha \times \vec{\alpha}'' = 0$. Using this equality, we get

$$\vec{U}_\alpha \times \vec{\alpha}'' = \left(-a_2\vec{N} + a_1\vec{B}\right) \times \kappa\vec{N} = -a_1\kappa\vec{T}.$$

which gives us (i).

Similarly, for the base curve α to be an asymptotic curve on $F_{(\alpha,q_n)}$, the principal normal N of α and surface normal $\vec{U}_\alpha$ along α must be perpendicular.

Then, α is an asymptotic curve if and only if $\langle \vec{U}_\alpha, \vec{\alpha}'' \rangle = 0$. Using this equality, we get

$$\langle \vec{U}_\alpha, \vec{\alpha}'' \rangle = \langle -a_2\vec{N} + a_1\vec{B}, \kappa\vec{N} \rangle = -a_2\kappa$$

which gives (ii). □

Proposition 4 *The base curve α is a line of curvature on the GNR-surface $F_{(\alpha,q_n)}$ if and only if the system*

$$\begin{cases} a_2' + a_1\tau = 0 \\ a_1' - a_2\tau = 0 \end{cases}$$

satisfies.

Proof. For the curve α to be a line of curvature on $F_{(\alpha,q_n)}$, the tangent vector $\vec{T}$ of the curve and the derivative of the surface normal along α must be in the same direction, i.e., $\vec{\alpha}' \times \vec{U}_\alpha' = 0$. Then, we get

$$\begin{aligned} \vec{\alpha}' \times \vec{U}_\alpha' &= \vec{T} \times \left(a_2\kappa\vec{T} - (a_2' + a_1\tau)\vec{N} + (a_1' - a_2\tau)\vec{B} \right) \\ &= -(a_1' - a_2\tau)\vec{N} - (a_2' + a_1\tau)\vec{B} \end{aligned}$$

Since the vectors $\vec{N}$ and $\vec{B}$ are linearly independent, we obtain that $\vec{\alpha}' \times \vec{U}_\alpha' = 0$ if and only if $a_2' + a_1\tau = 0$ and $a_1' - a_2\tau = 0$. □

Let now consider the Gaussian curvature K and the mean curvature H of GNR-surface $F_{(\alpha,q_n)}$. The fundamental coefficients of GNR-surface $F_{(\alpha,q_n)}$ are calculated from (3) and (4) as

$$\begin{aligned} E &= g^2 + u^2 \left[(a_1' - a_2\tau)^2 + (a_1\tau + a_2')^2 \right] \\ F &= 0 \\ G &= 1 \end{aligned} \tag{15}$$

and

$$\begin{aligned} L &= \frac{1}{\sqrt{u^2 f^2 + g^2}} \big[-uf \left(g' - u\kappa(a_1' - a_2\tau) \right) \\ &\qquad + g \left(-a_2\kappa g + a_2\tau(a_2' + a_1\tau) \right. \\ &\qquad \left. + u \left(-f' + a_1\tau(a_1' - a_2\tau) \right) \right) \big] \\ M &= \frac{-f}{\sqrt{u^2 f^2 + g^2}} \\ N &= 0 \end{aligned} \tag{16}$$

respectively. Then, from (5) the Gaussian curvature and mean curvature of GNR-surface $F_{(\alpha,q_n)}$ are

$$\begin{aligned} K &= -\frac{M^2}{E} = -\frac{f^2}{(u^2f^2+g^2)\left[g^2+u^2\left[(a_1'-a_2\tau)^2+(a_1\tau+a_2')^2\right]\right]} \\ H &= \frac{L}{2E} = \left[-uf\left(g'-u\kappa(a_1'-a_2\tau)\right)\right. \\ &\quad \left.+g\left(-a_2\kappa g+a_2\tau(a_1\tau+a_2')+u\left(-f'+a_1\tau(a_1'-a_2\tau)\right)\right)\right] \\ &\quad /\left\{2\sqrt{u^2f^2+g^2}\left[g^2+u^2\left[(a_1'-a_2\tau)^2+(a_1\tau+a_2')^2\right]\right]\right\} \end{aligned} \tag{17}$$

respectively. As we see from (17) and Proposition 2, a classical characterization for developable ruled surfaces stated as "a ruled surface is developable if and only if the Gaussian curvature vanishes" holds for GNR-surfaces.

From (17), we have the following corollary:

Corollary 5 *Between the Gaussian curvature* K *and the mean curvature* H *of a GNR-surface* $F_{(\alpha,q_n)}$ *the following relation exists*

$$KL + 2HM^2 = 0$$

or substituting L *and* M *from (16),*

$$\begin{aligned} \frac{K}{\sqrt{u^2f^2+g^2}} &\left[-uf\left(g'-u\kappa(a_1'-a_2\tau)\right)\right. \\ &+g\left(-a_2\kappa g+a_2\tau(a_1\tau+a_2')\right. \\ &\left.\left.+u\left(-f'+a_1\tau(a_1'-a_2\tau)\right)\right)\right] - \frac{2Hf^2}{u^2f^2+g^2} = 0. \end{aligned}$$

Proposition 5 *The GNR-surface* $F_{(\alpha,q_n)}$ *is minimal if and only if the equality*

$$\frac{f}{g} = \frac{-a_2\kappa g+a_2\tau(a_2'+a_1\tau)+u\left(-f'+a_1\tau(a_1'-a_2\tau)\right)}{u\left(g'-u\kappa(a_1'-a_2\tau)\right)} \tag{18}$$

satisfies.

Proof. The proof is clear from the second equality in (17). □

From Catalan theorem, Theorem 1, and Proposition 5 we obtain the following corollaries:

Corollary 6 *The GNR-surface* $F_{(\alpha,q_n)}$ *is a plane, a helicoid or its fragment if and only if* (18) *holds.*

Corollary 7 *Let the base curve* α *of the normal surface* $F_{(\alpha,q_n)}$ *be a straight line. Then,* $F_{(\alpha,q_n)}$ *is a minimal surface if and only if* f *is constant.*

Now, let $F_{(\alpha,q_n)}$ be developable. Then, $f = 0$ and we have $a_1'a_2 - a_1a_2' = \tau$. Since q_n is unit, $a_1a_1' + a_2a_2' = 0$. Now, it follows,

$$a_1\tau + a_2' = 0 \quad \text{and} \quad a_1' - a_2\tau = 0.$$

Now, from (18) we have that $a_2\kappa g^2 = 0$. From the last equations and Proposition 4, the followings are obtained:

Corollary 8 *The base curve* α *is always a line of curvature on the developable GNR-surface* $F_{(\alpha,q_n)}$.

Corollary 9 *A developable regular GNR-surface* $F_{(\alpha,q_n)}$ *is minimal (or equivalently a helicoid or a plane) if and only if* α *is a straight line or* $F_{(\alpha,q_n)} = F_{(\alpha,N)}$.

Since we assume $F_{(\alpha,q_n)}$ is developable, the partial derivatives given by (9) become

$$\begin{aligned} \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s} &= g\vec{T} \\ \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial u} &= a_1\vec{N} + a_2\vec{B} \end{aligned} \tag{19}$$

and the unit surface normal U becomes $\vec{U} = -a_2\vec{N} + a_1\vec{B}$. By considering the base $\left\{ \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s}, \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial u} \right\}$ of the tangent space $T_pF_{(\alpha,q_n)}$ at a point $p \in F_{(\alpha,q_n)}$, for a vector $v_p \in T_pF_{(\alpha,q_n)}$, the Weingarten map of $F_{(\alpha,q_n)}$ is given by $S_p = -D_p\nu : T_pF_{(\alpha,q_n)} \to T_{\nu_p}S^2$. Then, we have

$$S_p\left(\frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s}\right) = D_{\frac{\partial F_{(\alpha,q_n)}}{\partial s}}\vec{U} = -\frac{\partial \vec{U}}{\partial s} = -\frac{\partial}{\partial s}\left(-a_2\vec{N} + a_1\vec{B}\right) = -a_2\kappa\vec{T} = -\frac{a_2\kappa}{g}\frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s}$$

$$S_p\left(\frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s}\right) = D_{\frac{\partial F_{(\alpha,q_n)}}{\partial u}}\vec{U} = -\frac{\partial \vec{U}}{\partial u} = 0$$

Thus, at the non-singular points of the surface, the matrix form of the Weingarten map is given as

$$S_p = \begin{bmatrix} -\frac{a_2\kappa}{g} & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

From (20), it is easy to see that for a developable GNR-surface $F_{(\alpha,q_n)}$, the Gaussian curvature and mean curvature are

$$K = \det(S_p) = 0 \quad \text{and} \quad H = \frac{1}{2}\mathrm{tr}(S_p) = -\frac{a_2\kappa}{2g},$$

respectively. From the characteristic equation $\det(S_p - \lambda I) = 0$ of the matrix of Weingarten map, the principal curvatures of the developable GNR-surface $F_{(\alpha,q_n)}$ are $\lambda_1 = -\frac{a_2\kappa}{g}$, $\lambda_2 = 0$ and we have the following corollary:

Corollary 10 (i) *If $\kappa \neq 0$ and $a_2 \neq 0$, there are no umbilical points on the surface $F_{(\alpha,q_n)}$.*

(ii) *If $\kappa = 0$ or $a_2 = 0$, then we get $\lambda_1 = \lambda_2 = 0$ and all the points of the surface $F_{(\alpha,q_n)}$ are planar and the quadratic approach of the surface is a plane.*

(iii) *If $\kappa \neq 0$ and $a_2 \neq 0$, then $\lambda_1\lambda_2 = 0$, $\lambda_1 \neq 0$ and all points of the surface $F_{(\alpha,q_n)}$ are parabolic. Therefore, the quadratic approach of the surface is a parabolic cylinder.*

The principal directions $\vec{e}_1$, $\vec{e}_2$ of developable GNR-surface are defined by $S_p(\vec{e}_1) = \lambda_1 \vec{e}_1$, $S_p(\vec{e}_2) = \lambda_2 \vec{e}_2$ and calculated as

$$\begin{aligned} \vec{e}_1 &= g\vec{T} \\ \vec{e}_2 &= a_1\vec{N} + a_2\vec{B} \end{aligned} \tag{21}$$

respectively. Since a curve on a surface is a line of curvature if its tangent vector is a principal line, i.e., $S_p(\vec{T}) = m\vec{T}$, $m \in \mathbb{R}$, from the first equality, we have that the base curve α is always a line of curvature on developable GNR-surface $F_{(\alpha,q_n)}$, i.e., we have Corollary 3.13 again. Furthermore, for the parameter curves of the developable normal surface $F_{(\alpha,q_n)}$, we have the following corollary:

Corollary 11 *All parameter curves $F_{(\alpha,q_n)}(s,u_0)$ and $F_{(\alpha,q_n)}(s_0,u)$ of a developable GNR-surface $F_{(\alpha,q_n)}$ are lines of curvatures.*

Let now $\vec{v}_p$ be a unit tangent vector in the tangent space $T_pF_{(\alpha,q_n)}$ at a point p on developable GNR-surface $F_{(\alpha,q_n)}$. Then, $\vec{v}_p$ can be written as

$$\vec{v}_p = C(s,u)\frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s} + D(s,u)\frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial u} \tag{22}$$

where C, D are differentiable functions. Using the linearity of Weingarten map, we have

$$S_p(\vec{v}_p) = Ca_2\kappa\vec{T}$$

and substituting (19) in (22), we get

$$\vec{v}_p = Cg\vec{T} + D\vec{q}_n.$$

Since, the normal curvature in the direction of $\vec{v}_p$ is given by $k_n(\vec{v}_p) = \langle S_p(\vec{v}_p), \vec{v}_p\rangle$, we have

$$k_n(\vec{v}_p) = C^2 a_2\kappa g$$

If $C = 0$, $a_2 = 0$ or $\kappa = 0$, then $k_n(\vec{v}_p) = 0$ and we have the following theorem:

Theorem 3 (i) *If $\kappa \neq 0$, then a unit tangent vector $\vec{v}_p$ on the developable GNR-surface $F_{(\alpha,q_n)}$ is asymptotic if and only if $\vec{v}_p$ is a ruling i.e, $\vec{v}_p = \vec{q}_n$ or $a_2 = 0$.*

(ii) *If α is a straight line, then any tangent vector $\vec{v}_p$ is asymptotic.*

Let φ be a regular curve on the developable GNR-surface $F_{(\alpha,q_n)}$ given by the parametrization $\vec{\varphi}(t) = F_{(\alpha,q_n)}(s(t), u(t))$. Since the tangent vector of φ lies in the tangent space of $F_{(\alpha,q_n)}$, from (22) we get

$$\dot{\vec{\varphi}} = \frac{d\vec{\varphi}}{dt} = \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial s}\frac{ds}{dt} + \frac{\partial \vec{F}_{(\alpha,q_n)}}{\partial u}\frac{du}{dt}$$

which yields that

$$\frac{ds}{dt} = \dot{s} = C(s(t), u(t)) \quad \text{and} \quad \frac{du}{dt} = \dot{u} = D(s(t), u(t)).$$

From the derivative of $\vec{v}_p$ with respect to t, we obtain

$$\dot{\vec{v}}_p = \left(C^2 g_s + 2CDg_u + \dot{C}g\right)\vec{T} + \left(C^2 g\kappa + \dot{D}a_1\right)\vec{N} + \dot{D}a_2\vec{B}$$

where $g_s = \frac{\partial g}{\partial s}$ and $g_u = \frac{\partial g}{\partial u}$. Since the unit surface normal along φ is $\vec{U}_\varphi = -a_2\vec{N} + a_1\vec{B}$, then the geodesic curvature of the surface curve φ is given by

$$\begin{aligned}\kappa_g &= \langle \vec{v}_p, \dot{\vec{v}}_p \times \vec{U}_\varphi \rangle \\ &= Cg\left(\dot{D} - C^2 g g_u\right) - D\left(C^2 g_s + 2CDg_u + \dot{C}g\right).\end{aligned} \tag{23}$$

From (23), we have the following corollary:

Corollary 12 *A surface curve φ on the developable GNR-surface $F_{(\alpha,q_n)}$ is a geodesic if and only if*

$$Cg\left(\dot{D} - C^2 g g_u\right) - D\left(C^2 g_s + 2CDg_u + \dot{C}g\right) = 0.$$

The derivative of unit surface normal along φ with respect s is given by $\vec{U}'_\varphi = -S_p(\vec{v}_p) = -Ca_2\kappa\vec{T}$ and therefore the geodesic torsion of the surface curve φ is given by

$$\tau_g = \langle \vec{U}'_\varphi, \vec{U}_\varphi \times \vec{v}_p \rangle = CDa_2\kappa. \tag{24}$$

From the equation (24) we have the following corollary:

Corollary 13 (i) *If $\kappa = 0$, i.e., the base curve α is a straight line, then all surface curves are lines of curvature.*

(ii) *If $F_{(\alpha,q_n)} = F_{(\alpha,N)}$, then all surface curves are lines of curvature.*

(iii) *If* $C = 0$*,* $\kappa \neq 0$ *and* $a_2 \neq 0$*, then* $\vec{v}_p = \vec{q}_n$ *and only rulings are lines of curvature.*

(iv) *If* $D = 0$*,* $\kappa \neq 0$ *and* $a_2 \neq 0$*, then* $\vec{v}_p = \vec{T}$ *and only the base curve* α *is a line of curvature.*

Now, let us consider the Frenet frame of the GNR-surface $F_{(\alpha,q_n)}$. Since the vector $\vec{q}_n(s) = a_1(s)\vec{N}(s) + a_2(s)\vec{B}(s)$ is unit, we can take $a_1(s) = \cos(\theta(s))$ and $a_2(s) = \sin(\theta(s))$, i.e., $\vec{q}_n = \cos\theta\vec{N} + \sin\theta\vec{B}$ where θ is the angle function between $\vec{q}_n$ and $\vec{N}$. Then, the vectors of the Frenet frame $\{\vec{q}_n, \vec{h}, \vec{a}\}$ of the GNR-surface $F_{(\alpha,q_n)}$ are given by

$$\begin{aligned}\vec{q}_n &= \cos\theta\vec{N} + \sin\theta\vec{B}\\ \vec{h} &= \frac{1}{\sqrt{\kappa^2\cos^2\theta + z^2}}\left(-\kappa\cos\theta\vec{T} - z\sin\theta\vec{N} + z\cos\theta\vec{B}\right)\\ \vec{a} &= \frac{1}{\sqrt{\kappa^2\cos^2\theta + z^2}}\left(z\vec{T} - \kappa\cos\theta\sin\theta\vec{N} + \kappa\cos^2\theta\vec{B}\right)\end{aligned} \tag{25}$$

where $z = \theta' + \tau$. These equalities give the following proposition:

Proposition 6 *For a GNR-surface* $F_{(\alpha,q_n)}$*, the followings are equivalent.*

(i) *The angle function* θ *is given with the equality* $\theta = -\int \tau ds$*.*

(ii) *The central normal vector* $\vec{h}$ *of* $F_{(\alpha,q_n)}$ *coincides with the tangent vector* $\vec{T}$ *of the curve* α*.*

(iii) *The central tangent vector* $\vec{a}$ *of* $F_{(\alpha,q_n)}$ *lies in the normal plane of the curve* α*.*

Proof. If $\theta = -\int \tau ds$, then we have $z = 0$. Therefore, the proof is clear from (25). □

Corollary 14 *Let* $F_{(\alpha,q_n)}$ *be a GNR-surface with frame* $\{\vec{q}_n, \vec{h}, \vec{a}\}$ *and the angle function* θ *between* $\vec{q}_n$ *and* $\vec{N}$ *be given by* $\theta = -\int \tau ds$*. Then, the GNR-surface* $F_{(\alpha,q_n)}$ *is an* h*-slant ruled surface if and only if* α *is a general helix.*

4 Examples

In this section, we give some examples for the obtain results in the previous section.

Example 1 *Let take the z-axis as the base curve. Then,* $\vec{\alpha}_1(s) = (0, 0, s)$*. By choosing* $a_1(s) = \cos s$*,* $a_2(s) = \sin s$*, we have* $\vec{q}_n = (\cos s, \sin s, 0)$ *and GNR-surface* $F_{1(\alpha_1,q_n)}(s, u) = (u \cos s, u \sin s, s)$ *which is a right helicoid given in Figure 1. For this surface, it is obtained* $f = -1$ *and* $g = 1$ *which gives that* $F_{1(\alpha_1,q_n)}$ *is a regular and non-developable GNR-surface. The graph of the base curve* α_1 *is also given in Figure 1 colored red.*

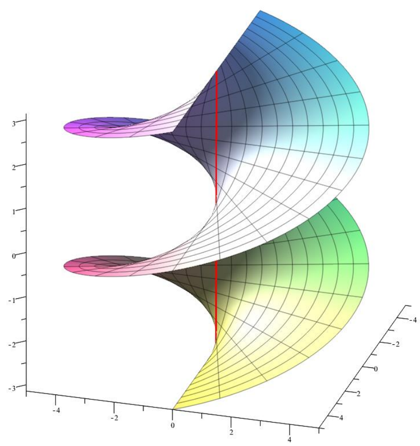

Figure 1: The GNR-surface $F_{1(\alpha_1,q_n)}$

Example 2 *Let the GNR-surface* $F_{2(\alpha_2,q_n)}$ *be given by the parametrization*

$$\vec{F}_{2(\alpha_2,q_n)}(s, u) = \vec{\alpha_2}(s) + u\vec{q}_n(s)$$

where $\alpha_2(s)$ *is a cylidrical helix given by the parametric form*

$$\vec{\alpha_2}(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$$

The Frenet vectors of α_2 *are*

$$\vec{T}(s) = \left(-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right),$$
$$\vec{N}(s) = \left(-\cos\left(\frac{s}{\sqrt{2}}\right), -\sin\left(\frac{s}{\sqrt{2}}\right), 0\right),$$
$$\vec{B}(s) = \left(\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right).$$

By choosing $a_1 = \sin\left(\frac{s}{2}\right)$, $a_2 = \cos\left(\frac{s}{2}\right)$, *we get*

$$\vec{q}_n(s) = \left(-\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) - \frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{2}\right)$$

and $f = 0$, $g = 1 - \frac{1}{2}u\sin\left(\frac{s}{2}\right)$. *For the function* g *to be zero, we get* $u = \frac{2}{\sin\left(\frac{s}{2}\right)}$. *Then, the locus of singular points of GNR-surface* $F_{2(\alpha_2, q_n)}$ *are given by the curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s)\gamma_3(s))$ *where*

$$\begin{aligned}
\gamma_1(s) &= \sqrt{2}\cot\left(\frac{s}{2}\right)\sin\left(\frac{s}{\sqrt{2}}\right) - \cos\left(\frac{s}{\sqrt{2}}\right), \\
\gamma_2(s) &= -\sqrt{2}\cot\left(\frac{s}{2}\right)\cos\left(\frac{s}{\sqrt{2}}\right) - \sin\left(\frac{s}{\sqrt{2}}\right), \\
\gamma_3(s) &= \frac{s}{\sqrt{2}} + \sqrt{2}\cot\left(\frac{s}{2}\right).
\end{aligned}$$

The graph of GNR-surface $F_{2(\alpha_2, q_n)}$ *is given in Figure 2. In the same figure, the graphs of base curve* α_2 *and locus of singular points of* $F_{2(\alpha_2, q_n)}$ *are also given by colored red and blue, respectively.*

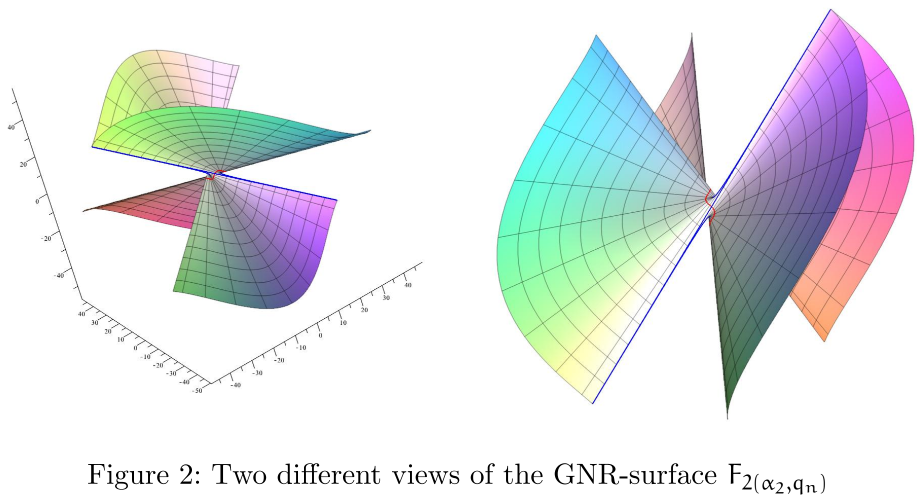

Figure 2: Two different views of the GNR-surface $F_{2(\alpha_2, q_n)}$

Example 3 *Let the GNR-surface* $F_{3(\beta, q_n)}$ *be given by the parametrization*

$$\vec{F}_{3(\beta, q_n)}(s, u) = \vec{\beta}(s) + u\vec{q}_n(s)$$

where $\vec{\beta}(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$ *is a special curve known as pedal curve with*

$$\beta_1(s) = \frac{3}{2}\cos\left(\frac{s}{2}\right) + \frac{1}{6}\cos\left(\frac{3s}{2}\right),$$
$$\beta_2(s) = \frac{3}{2}\sin\left(\frac{s}{2}\right) + \frac{1}{6}\sin\left(\frac{3s}{2}\right),$$
$$\beta_3(s) = \sqrt{3}\cos\left(\frac{s}{2}\right).$$

The Frenet vectors of β *are given by*

$$\vec{T}(s) = \left(-\frac{3}{4}\sin\left(\frac{s}{2}\right) - \frac{1}{4}\sin\left(\frac{3s}{2}\right), \cos^3\left(\frac{s}{2}\right), -\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right)\right),$$
$$\vec{N}(s) = \left(-\frac{\sqrt{3}}{2}\left(2\cos^2\left(\frac{s}{2}\right) - 1\right), -\sqrt{3}\cos\left(\frac{s}{2}\right)\sin\left(\frac{s}{2}\right), -\frac{1}{2}\right),$$
$$\vec{B}(s) = \left(\frac{1}{2}\cos\left(\frac{s}{2}\right)\left(2\cos^2\left(\frac{s}{2}\right) - 3\right), -\sin^3\left(\frac{s}{2}\right), \frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right)\right).$$

If we take $a_1(s) = \cos\left(\frac{s}{2}\right)$ *and* $a_2(s) = \sin\left(\frac{s}{2}\right)$, *we get* $\vec{q}_n(s) = (q_1(s), q_2(s), q_3(s))$ *where*

$$q_1(s) = \frac{1}{2}\cos\left(\frac{s}{2}\right)\left(2\sin\left(\frac{s}{2}\right)\cos^2\left(\frac{s}{2}\right) - 2\sqrt{3}\cos^2\left(\frac{s}{2}\right) - 3\sin\left(\frac{s}{2}\right) + \sqrt{3}\right),$$
$$q_2(s) = \sin\left(\frac{s}{2}\right)\left(\sin\left(\frac{s}{2}\right)\cos^2\left(\frac{s}{2}\right) - \sqrt{3}\cos^2\left(\frac{s}{2}\right) - \sin\left(\frac{s}{2}\right)\right),$$
$$q_3(s) = \frac{1}{2}\cos\left(\frac{s}{2}\right)\left(\sqrt{3}\sin\left(\frac{s}{2}\right) - 1\right),$$

and $f(s) = \frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right) - \frac{1}{2}$, $g(s,u) = 1 - \frac{\sqrt{3}}{2}u\cos^2\left(\frac{s}{2}\right)$. *Assuming* $k \in \mathbb{Z}$, *for the points*

$$(s_0, u_0) = \left(2\arcsin\left(\frac{1}{\sqrt{3}}\right) - 4k\pi, \frac{2}{\sqrt{3}\cos^2\left(\arcsin\left(\frac{1}{\sqrt{3}}\right) - 2k\pi\right)}\right),$$

and

$$(s_1, u_1) = \left(2\pi - 2\arcsin\left(\frac{1}{\sqrt{3}}\right) + 4k\pi, \frac{2}{\sqrt{3}\cos^2\left(\pi - \arcsin\left(\frac{1}{\sqrt{3}}\right) + 2k\pi\right)}\right),$$

we get $f = g = 0$ *and therefore these points are singular points on the GNR-surface* $F_{3(\beta,q_n)}$. *Therefore, except the points* (s_0, u_0) *and* (s_1, u_1), *the surface is regular. The graph of the GNR-surface* $F_{3(\beta,q_n)}$ *is given by Figure 3. The graphs of the base curve* β *and the images of singular points of the surface are also given in Figure 3 colored red and blue, respectively.*

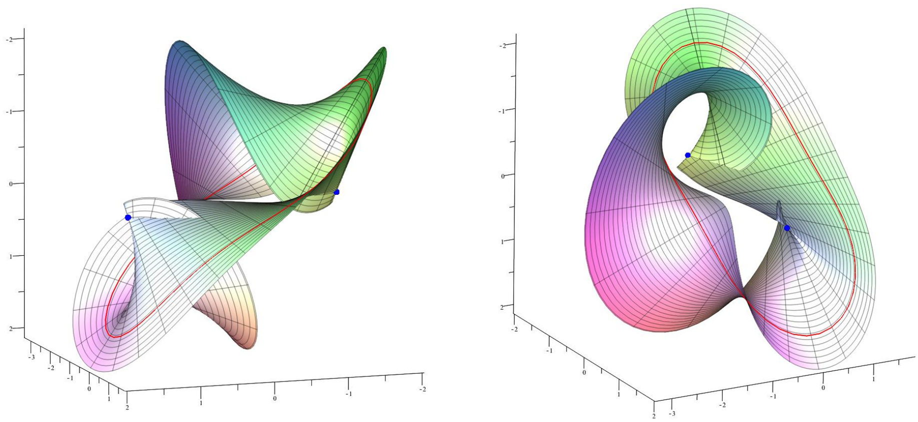

Figure 3: Two different views of the GNR-surface $F_{3(\beta,q_n)}$

Example 4 *Let the GNR-surface* $F_{4(\alpha_3,q_n)}$ *be given by the parametrization*

$$\vec{F}_{4(\alpha_3,q_n)}(s,u) = \vec{\alpha_3}(s) + u\vec{q}_n(s)$$

where

$$\vec{\alpha_3}(s) = \left(\sqrt{1+s^2}, s, \ln\left(s+\sqrt{1+s^2}\right)\right).$$

The Frenet vectors of α_3 *are*

$$\begin{aligned}
\vec{T} &= \left(\frac{\sqrt{2}s}{2\sqrt{s^2+1}}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2\sqrt{s^2+1}}\right), \\
\vec{N} &= \left(\frac{1}{\sqrt{s^2+1}}, 0, -\frac{s}{\sqrt{s^2+1}}\right), \\
\vec{B} &= \left(-\frac{\sqrt{2}s}{2\sqrt{s^2+1}}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2\sqrt{s^2+1}}\right).
\end{aligned}$$

By assuming $a_1(s) = s$ *and* $a_2(s) = \sqrt{1-s^2}$*, we get*

$$\vec{q}_n = \left(\frac{s}{\sqrt{s^2+1}} - \frac{\sqrt{2}}{2}\frac{s\sqrt{1-s^2}}{\sqrt{s^2+1}}, \frac{\sqrt{2}}{2}\sqrt{1-s^2}, -\frac{s^2}{\sqrt{s^2+1}} - \frac{\sqrt{2}}{2}\frac{\sqrt{1-s^2}}{s^2+1} \right),$$

and

$$f(s) = \frac{2s^2 - \sqrt{1-s^2} + 2}{2(s^2+1)\sqrt{1-s^2}}, \quad g(s,u) = \frac{2s^2 - u\sqrt{1-s^2} + 2}{2(s^2+1)}.$$

Then, the surface is regular and non-developable. The graph of GNR-surface $F_{4(\alpha_3, q_n)}$ *is given by Figure 4.*

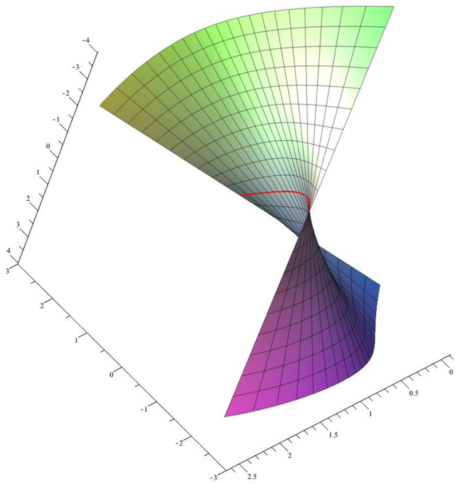

Figure 4: The GNR-surface $F_{4(\alpha_3, q_n)}$

5 Conclusions

In the Euclidean 3-space E^3, the principal normal surface and the binormal surface of a curve are the special ruled surfaces and they are well-known. Here, we have defined generalized normal ruled surface (GNR-surface) of a curve and showed that such surfaces are more general than the principal normal surface and binormal surface. We have given conditions when GNR-surfaces are minimal and investigated when the base curve is geodesic, asymptotic or a line of curvature on GNR-surface. By using a similar way of defining GNR-surfaces, other special ruled surfaces can be defined as well. In addition to this, classes of such ruled surfaces can be obtained. Therefore, new methods can be achieved for constructing new ruled surfaces related to a curve.

References

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey, 1976.

[2] M. Emmer, *Imagine Math Between Culture and Mathematics*, Springer, 2012 ed, 2012,

[3] A. T. Fomenko, A. A. Tuzhilin, E*lements of the geometry and topology of minimal surfaces in three dimensional space*, American Mathematical Society, Providence, Rhode Island, 2005.

[4] S. Izumiya, N. Takeuchi, New special curves and developable surfaces, *Turk. J. Math.* **28** (2004), 153–163.

[5] S. Izumiya, H. Katsumi, T. Yamasaki, The rectifying developable and the spherical Darboux image of a space curve, *Geometry and topology of caustics-Caustics '98- Banach Center Publications*, **50** (1999), 137–149.

[6] S. Izumiya, N. Takeuchi, Special curves and ruled surfaces, *Beitrage zur Algebra und Geometrie Contributions to Algebra and Geometry* **44**(1) (2003), 203–212.

[7] S. Izumiya, N. Takeuchi, Generic properties of helices and Bertrand curves. *Journal of Geometry*, **74** (2002), 97–109.

[8] S. Izumiya, N. Takeuchi, *Geometry of ruled surfaces*, Applicable Mathematics in the Golden Age (ed., J.C. Misra), Narosa Publishing House, New Delhi, 2003, 305–338.

[9] A. Karger, J. Novak, S*pace Kinematics and Lie Groups*, STNL Publishers of Technical Lit., Prague, Czechoslovakia, 1978.

[10] M. Önder, O. Kaya, Characterizations of Slant Ruled Surfaces in the Euclidean 3-space, *Caspian Journal of Mathematical Sciences*, **6**(1) (2017), 82–89.

[11] M. Önder, Slant ruled surfaces, *Transnational Journal of Pure and Applied Mathematics*, **1**(1) (2018), 63–82.

[12] M. Önder, T. Kahraman On rectifying ruled surfaces, *Kuwait J. Sci.*, **47**(4) (2020), 1–11.

[13] M. A. Soliman, H. N. Abd-Ellah, S. A. Hassan, S. Q. Saleh, Darboux Ruled Surfaces with Pointwise 1-Type Gauss Map, *Sohag J. Sci.*, **2**(1) (2017), 1–8.

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Some new inequalities via s-convex functions on time scales

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Abstract. In this paper, we prove some new integral inequalities for s-convex function on time scale. We give results for the case when time scale is $\mathbb{R}$ and when time scale is $\mathbb{N}$.

1 Introduction

The study of various types of integral inequalities for convex functions has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. Out of these inequalities Ostrowski inequality and Hermite-Hadamard inequality are the most common inequalities. These two inequalities have applications in numerical analysis, probability, optimization theory, stochastic, statistics, information and integral operator theory. Also these inequalities have various implementation in trapezoid, Simpson and quadrature rules, etc. The basic definitions of Ostrowski and Hermite-Hadamard inequalities are as follows.

The Ostrowski inequality [21] for a differetiable mapping Υ on the interior of an interval T with $|\,\Upsilon'(\mathsf{c})| \leq \mathsf{M}$, where Υ' implies first derivative of Υ, is

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defined as:

$$\left|\Upsilon(k) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon(c)dc\right| \le M(b_2 - b_1)\left[\frac{1}{4} + \frac{\left(k - \frac{b_1+b_2}{2}\right)^2}{(b_2 - b_1)^2}\right], \quad (1)$$

for $b_1 < b_2 \in \mathsf{T}$. This inequality gives an upper bound for the approximation of the integral average $-\frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon(c)dc$ by the value $\Upsilon(c)$ at point $c \in [b_1, b_2]$. The above inequality is then further generalized by researchers. For instance see [2, 6, 19]. On the other hand, for a convex function $\Upsilon : \mathsf{T} \to \mathbb{R}$ on an interval T, the Hermite-Hadamard inequality [10, 11] is defined as:

$$\Upsilon\left(\frac{b_1 + b_2}{2}\right) \le \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon(c)dc \le \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2}, \quad (2)$$

for all $b_1, b_2 \in \mathsf{T}$ with $b_1 < b_2$. The inequality (2) is the special case of Jensen inequality. For more generalizations and details see [9, 13, 14, 15, 16, 17, 18, 20].

During last few decades, the inequalities (1) and (2) have been proved on time scale, see [1, 3, 7, 8, 23] for more information. Of course the role of inequalities (1) and (2) on time scales are similar as in usual sense. Here we prove some Ostrowski and Hermite-Hadamard's type inequalities for s-convex functions on time scale. We also extend the results given in [22]. In [22], Tahir et. al. proved several useful identities for convex functions on time scales. By using some of these identities we find certain useful inequalities for s-convex functions. Our work has many mathematical applications and has flexibility to extend it for more useful results.

2 Preliminaries

A time scale is a nonepmty closed subset $\mathbb{T}$ of $\mathbb{R}$. Most common examples are $\mathbb{R}$ and $\mathbb{N}$.

The forward and the backward jump operators respectively, denoted by σ and ρ, are defined as:

$$\sigma(k) = \inf\{c \in \mathbb{T} : c > k\}, \quad \rho(k) = \sup\{c \in \mathbb{T} : c < k\},$$

for all $k \in \mathbb{T}$.

The number k is called right-scattered if $\sigma(k) > k$ and is called left scattered if $\rho(k) < k$. Moreover, k is called isolated if both the right-scattered and the left-scattered. Similarly, the number k is called right dense or left dense if

$\sigma(k) = k$ or $\rho(k) = k$, respectively. Furthermore, k is called dense if it is right dense and left dense simultaneously.

The mappings $\mu, \tau : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(k) := \sigma(k) - k, \qquad \tau(k) := k - \rho(k),$$

are known as forward and backward graininess functions, respectively.

A function $\Upsilon : \mathbb{T} \to \mathbb{R}$ is called rd-continuous C_{rd} if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$.

If $\Upsilon \in C_{rd}$ and $k_1 \in \mathbb{T}$, then we have

$$F(k) = \int_{k_1}^{k} \Upsilon(c)\Delta c, \quad k \in \mathbb{T}.$$

That is, for $\Upsilon \in C_{rd}$ implies $\int_{b_1}^{b_2} \Upsilon(c)\Delta c = F(b_1) - F(b_2)$, where $F^{\Delta} = \Upsilon$.

Theorem 1 ([4]) *Let* $b_1, b_2, b_3 \in \mathbb{T}$*,* $\Upsilon, \Upsilon_1, \Upsilon_2 \in C_{rd}$*,* $\varpi \in \mathbb{R}$ *and* σ *is forward jump operator, then*

(i). $\int_{b_1}^{b_2} (\Upsilon_1(c) + \Upsilon_2(c))\Delta c = \int_{b_1}^{b_2} \Upsilon_1(c)\Delta c + \int_{b_1}^{b_2} \Upsilon_2(c)\Delta c$*;*

(ii). $\int_{b_1}^{b_2} \varpi \Upsilon(c)\Delta c = \varpi \int_{b_1}^{b_2} \Upsilon(c)\Delta c$*;*

(iii). $\int_{b_2}^{b_1} \Upsilon(c)\Delta c = -\int_{b_1}^{b_2} \Upsilon(c)\Delta c$*;*

(iv). $\int_{b_1}^{b_2} \Upsilon(c)\Delta c = \int_{b_1}^{b_3} \Upsilon(c)\Delta c + \int_{b_3}^{b_2} \Upsilon(c)\Delta c$*;*

(v). $\int_{b_1}^{b_2} \Upsilon_1^{\sigma}(c)\, \Upsilon_2^{\Delta}(c)\Delta c = (\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^{\Delta}(c)\, \Upsilon_2(c)\Delta c$*;*

(vi). $\int_{b_1}^{b_2} \Upsilon_1(c)\, \Upsilon_2^{\Delta}(c)\Delta c = (\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^{\Delta}(c)\, \Upsilon_2^{\sigma}(c)\Delta c$*;*

(vii). $\int_{b_1}^{b_1} \Upsilon(c)\Delta c = 0$*;*

(viii). *If* $\Upsilon(c) \geq 0$ *for all* c*, then* $\int_{b_1}^{b_2} \Upsilon(c)\Delta c \geq 0$*;*

(ix). *If* $|\Upsilon_1(c)| \leq \Upsilon_2(c)$ *on* $[b_1, b_2]$*, then*

$$\left|\int_{b_1}^{b_2} \Upsilon_1(c)\Delta c\right| \leq \int_{b_1}^{b_2} \Upsilon_2(c)\Delta c.$$

From Theorem 1 (ix), for $\Upsilon_2(c) = |\Upsilon_1(c)|$ on $[b_1, b_2]$, we have

$$\left|\int_{b_1}^{b_2} \Upsilon(c)\Delta c\right| \leq \int_{b_1}^{b_2} |\Upsilon(c)|\Delta c.$$

Definition 1 ([12]) *Consider a time scale* $\mathbb{T}$ *and* $s \in (0,1]$. *A function* $\Upsilon : \mathsf{T} \subset \mathbb{T} \to \mathbb{R}_0$, *where* $\mathbb{R}_0 = [0, \infty)$, *is called* s*-convex function in second sense, if*

$$\Upsilon(tb_1 + (1-t)b_2) \leq t^s \Upsilon(b_1) + (1-t)^s \Upsilon(b_2), \tag{3}$$

for all $b_1, b_2 \in \mathsf{T}$ *and* $t \in [0,1]$.

3 Main results

First we prove the following identity.

Lemma 1 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $\Upsilon^\Delta \in C_{rd}$ *then following equality holds:*

$$\begin{aligned} &\frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c \\ &= \frac{1}{2(b_2 - b_1)}\left[\int_{b_1}^{b_2} (c - b_1)\Upsilon^\Delta(c)\Delta c - \int_{b_1}^{b_2} (b_2 - c)\Upsilon^\Delta(c)\Delta c\right]. \end{aligned} \tag{4}$$

Proof. By using the formula

$$\int_{b_1}^{b_2} \Upsilon_1(c)\Upsilon_2^\Delta(c)\Delta c(\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^\Delta(c)\Upsilon_2^\sigma(c)\Delta c,$$

with $\Upsilon_1(c) = \frac{c-b_1}{b_2-b_1}$, $\Upsilon_2(c) = \Upsilon(c)$ in first integral and $\Upsilon_1(c) = \frac{c-b_2}{b_1-b_2}$, $\Upsilon_2(c) = \Upsilon(c)$ in second integral , we have

$$\begin{aligned} &\int_{b_1}^{b_2} \frac{c - b_1}{b_2 - b_1}\Upsilon^\Delta(c)\Delta c - \int_{b_1}^{b_2} \frac{c - b_2}{b_1 - b_2}\Upsilon^\Delta(c)\Delta c \\ &= \left[\Upsilon(b_2) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c\right] - \left[-\Upsilon(b_1) + \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c\right] \quad (5) \\ &= \Upsilon(b_1) + \Upsilon(b_2) - \frac{2}{b_2 - b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c. \end{aligned}$$

Then by multiplying $\frac{1}{2}$ on both sides of equation (5), we get the required equality (4) (also see the proof of Lemma 3.1 in [5]). □

Corollary 1 *Let* $\mathbb{T} = \mathbb{R}$ *in Lemma 1, then we have*

$$\begin{aligned}&\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc \\ &= \frac{1}{2(b_2-b_1)}\left[\int_{b_1}^{b_2}(c-b_1)\,\Upsilon'(c)dc - \int_{b_1}^{b_2}(b_2-c)\,\Upsilon'(c)dc\right].\end{aligned} \tag{6}$$

Corollary 2 *Let* $\mathbb{T} = \mathbb{N}$ *in Lemma 1. Let* $b_1 = 0$, $b_2 = d$, $c = x$ *and* $\Upsilon(k) = c_k$*, then*

$$\frac{c_0 + c_d}{2} - \frac{1}{d}\sum_{x=0}^{d} c_x = \frac{1}{2d}\left[\sum_{x=0}^{d-1} x\Delta c_x - \sum_{x=0}^{d-1}(d-x)\Delta c_x\right]. \tag{7}$$

Corollary 3 *Under the assumptions of Lemma 1, we have*

$$\begin{aligned}&\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c \\ &= \frac{b_2-b_1}{2}\left[\int_0^1 t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t - \int_0^1 t\,\Upsilon^{\Delta}(tb_1+(1-t)b_2)\Delta t\right].\end{aligned} \tag{8}$$

Proof. In Lemma 1 using change of variable method, that is, by taking $t = \frac{c-b_1}{b_2-b_1}$, we find

$$\int_{b_1}^{b_2}\frac{c-b_1}{b_2-b_1}\,\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_0^1 t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t. \tag{9}$$

Similarly, by taking $t = \frac{c-b_2}{b_1-b_2}$, we get

$$\int_{b_1}^{b_2}\frac{c-b_2}{b_1-b_2}\,\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_0^1 t\,\Upsilon^{\Delta}(tb_1+(1-t)b_2)\Delta t. \tag{10}$$

Hence by using (9) and (10), we get the required equality (8). □

Theorem 2 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$*. Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o*, where* T^o *is the interior of* T*. If* $|\Upsilon^{\Delta}|$ *is* s*-convex then following inequality holds:*

$$\left|\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \leq \frac{b_2-b_1}{2}\lambda_1\left(|\Upsilon^{\Delta}(b_2)|+|\Upsilon^{\Delta}(b_1)|\right), \tag{11}$$

where

$$\lambda_1=\int_0^1 (t^{s+1}+t(1-t)^s)\Delta t.$$

Proof. Using Corollary 3, property of modulus and convexity of $|\Upsilon^{\Delta}|$, we find

$$\begin{aligned}
&\left|\frac{\Upsilon(b_2)+\Upsilon(b_1)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&\leq \frac{b_2-b_1}{2}\left[\int_0^1 t|\Upsilon^{\Delta}(tb_2+(1-t)b_1)|\Delta t \right.\\
&\quad \left. +\int_0^1 t|\Upsilon^{\Delta}(tb_1+(1-t)b_2)|\Delta t\right] \\
&\leq \frac{b_2-b_1}{2}\left[\int_0^1 t\{t^s|\Upsilon^{\Delta}(b_2)|+(1-t)^s|\Upsilon^{\Delta}(b_1)|\}\Delta t \right. \\
&\quad \left. +\int_0^1 t\{t^s|\Upsilon^{\Delta}(b_1)|+(1-t)^s|\Upsilon^{\Delta}(b_2)|\}\Delta t\right] \\
&= \frac{b_2-b_1}{2}\left[\left(|\Upsilon^{\Delta}(b_2)|+|\Upsilon^{\Delta}(b_1)|\right)\int_0^1 (t^{s+1}+t(1-t)^s)\Delta t\right] \\
&= \frac{b_2-b_1}{2}\lambda_1\left(|\Upsilon^{\Delta}(b_2)|+|\Upsilon^{\Delta}(b_1)|\right).
\end{aligned} \tag{12}$$

Hence the proof. □

Remark 1 *If* $\mathbb{T}=\mathbb{R}$, *then inequality* (11) *becomes:*

$$\left|\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc\right| \leq \frac{b_2-b_1}{2(s+1)}\left(|\Upsilon'(b_2)|+|\Upsilon'(b_1)|\right). \tag{13}$$

Theorem 3 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T}=[b_1,b_2]\subseteq\mathbb{T}$ *such that* $b_1<b_2\in\mathbb{T}$. *Let* $\Upsilon:\mathsf{T}\to\mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $|\Upsilon^{\Delta}|^q$ *is* s*-convex, for* $q>1$ *such that* $\frac{1}{r}+\frac{1}{q}=1$, *then following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Upsilon(b_2)+\Upsilon(b_1)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&\leq \frac{b_2-b_1}{2}\left(\int_0^1 t^r\Delta t\right)^{\frac{1}{r}} \\
&\times\left[\left(\int_0^1\left(t^s|\,\Upsilon^{\Delta}\,(b_2)|^q+(1-t)^s|\,\Upsilon^{\Delta}\,(b_1)|^q\right)\Delta t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(t^s|\,\Upsilon^{\Delta}\,(b_1)|^q+(1-t)^s|\,\Upsilon^{\Delta}\,(b_2)|^q\right)\Delta t\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{14}$$

Proof. Using Corollary 3, property of modulus, Holder's integral inequality and convexity of $|\,\Upsilon^{\Delta}\,|^q$, we find

$$\begin{aligned}
&\left|\frac{\Upsilon(b_2)+\Upsilon(b_1)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&=\frac{b_2-b_1}{2}\left|\int_0^1 t\,\Upsilon^{\Delta}\,(tb_2+(1-t)b_1)\Delta t-\int_0^1 t\,\Upsilon^{\Delta}\,(tb_1+(1-t)b_2)\Delta t\right| \\
&\leq\frac{b_2-b_1}{2}\left[\left|\int_0^1 t\,\Upsilon^{\Delta}\,(tb_2+(1-t)b_1)\Delta t\right|+\left|\int_0^1 t\,\Upsilon^{\Delta}\,(tb_1+(1-t)b_2)\Delta t\right|\right] \\
&\leq\frac{b_2-b_1}{2}\left(\int_0^1 t^r\Delta t\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|\Upsilon^{\Delta}(tb_2+(1-t)b_1)\right|\Delta t^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\Upsilon^{\Delta}(tb_1+(1-t)b_2)\right|^q\Delta t\right)^{\frac{1}{q}}\right] \\
&\leq\frac{b_2-b_1}{2}\left(\int_0^1 t^r\Delta t\right)^{\frac{1}{r}}\left[\left(\int_0^1\left(t^s|\,\Upsilon^{\Delta}\,(b_2)|^q+(1-t)^s|\,\Upsilon^{\Delta}\,(b_1)|^q\right)\Delta t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(t^s|\,\Upsilon^{\Delta}\,(b_1)|^q+(1-t)^s|\,\Upsilon^{\Delta}\,(b_2)|^q\right)\Delta t\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{15}$$

Hence the proof. □

Remark 2 *If* $\mathbb{T}=\mathbb{R}$, *then inequality* (14) *becomes*

$$\left|\frac{\Upsilon(b_1)+\Upsilon(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc\right|\leq\frac{b_2-b_1}{(r+1)^{\frac{1}{r}}}\left(\frac{|\,\Upsilon'\,(b_1)|^q+|\,\Upsilon'\,(b_2)|^q}{s+1}\right)^{\frac{1}{q}}.\tag{16}$$

Lemma 2 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $\Upsilon^\Delta \in C_{rd}$ *then following equality holds:*

$$\begin{aligned} &\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c \\ &= \int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\,\Upsilon^\Delta(c)\Delta c + \int_{\frac{b_1+b_2}{2}}^{b_2} \left(\frac{c-b_1}{b_2-b_1} - 1\right)\Upsilon^\Delta(c)\Delta c. \end{aligned} \tag{17}$$

Proof. By using the formula

$$\int_{b_1}^{b_2} \Upsilon_1(c)\,\Upsilon_2^\Delta(c)\Delta c = (\Upsilon_1\Upsilon_2)(b_2) - (\Upsilon_1\Upsilon_2)(b_1) - \int_{b_1}^{b_2} \Upsilon_1^\Delta(c)\,\Upsilon_2^\sigma(c)\Delta c,$$

with $\Upsilon_1(c) = \frac{c-b_1}{b_2-b_1}$, $\Upsilon_2(c) = \Upsilon(c)$ in first integral and $\Upsilon_1(c) = \frac{c-b_1}{b_2-b_1} - 1$, $\Upsilon_2(c) = \Upsilon(c)$ in second integral, we have

$$\begin{aligned} &\int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\,\Upsilon^\Delta(c)\Delta c + \int_{\frac{b_1+b_2}{2}}^{b_2} \frac{c-b_2}{b_2-b_1}\,\Upsilon^\Delta(c)\Delta c \\ &= \frac{1}{2}\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{\frac{b_1+b_2}{2}} \Upsilon^\sigma(c)\Delta c \\ &\quad + \frac{1}{2}\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{\frac{b_1+b_2}{2}}^{b_2} \Upsilon^\sigma(c)\Delta c \\ &= \Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c. \end{aligned} \tag{18}$$

Hence the proof. □

Corollary 4 *Let* $\mathbb{T} = \mathbb{R}$ *in Lemma 2, then*

$$\begin{aligned} &\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon(c)dc \\ &= \int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\,\Upsilon'(c)dc + \int_{\frac{b_1+b_2}{2}}^{b_2} \left(\frac{c-b_1}{b_2-b_1} - 1\right)\Upsilon'(c)dc. \end{aligned} \tag{19}$$

Corollary 5 *Let $\mathbb{T} = \mathbb{N}$ in Lemma 2. Let $b_1 = 0$, $b_2 = d$(with d is even), $c = x$ and $\Upsilon(k) = c_k$, then*

$$c_{\frac{d}{2}} - \frac{1}{d}\sum_{x=0}^{d} c_x = \frac{1}{d}\sum_{x=0}^{\frac{d}{2}-1} x\Delta c + \frac{1}{d}\sum_{x=\frac{d}{2}}^{d-1}(x-d)\Delta c. \tag{20}$$

Corollary 6 *Under the assumptions of Lemma 2, we have*

$$\begin{aligned}\Upsilon\left(\frac{b_1+b_2}{2}\right) &- \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\\ &= (b_2-b_1)\left[\int_0^{1/2} t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t\right.\\ &\left.+\int_{1/2}^{1}(t-1)\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t\right].\end{aligned} \tag{21}$$

Proof. In Lemma 2 using change of variable method, that is, by taking $t = \frac{c-b_1}{b_2-b_1}$, we find

$$\int_{b_1}^{\frac{b_1+b_2}{2}} \frac{c-b_1}{b_2-b_1}\,\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_0^{1/2} t\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t, \tag{22}$$

and

$$\int_{\frac{b_1+b_2}{2}}^{b_2}\left(\frac{c-b_1}{b_2-b_1}-1\right)\Upsilon^{\Delta}(c)\Delta c = (b_2-b_1)\int_{1/2}^{1}(t-1)\,\Upsilon^{\Delta}(tb_2+(1-t)b_1)\Delta t. \tag{23}$$

Hence by using (22) and (23), we get the required equality (21). □

Theorem 4 *Consider a time scale $\mathbb{T}$ and $\mathrm{T} = [b_1, b_2] \subseteq \mathbb{T}$ such that $b_1 < b_2 \in \mathbb{T}$. Let $\Upsilon : \mathrm{T} \to \mathbb{R}$ be a delta differentiable mapping on T^o, where T^o is the interior of T. If $|\Upsilon^{\Delta}|$ is s-convex then following inequality holds:*

$$\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \le (b_2-b_1)\left(H_1|\Upsilon^{\Delta}(b_2)| + H_2|\Upsilon^{\Delta}(b_1)|\right), \tag{24}$$

where

$$H_1 = \int_0^{\frac{1}{2}} t^{s+1}\Delta t + \int_{\frac{1}{2}}^{1} t^s(1-t)\Delta t, \text{ and } H_2 = \int_0^{\frac{1}{2}} t(1-t)^s\Delta t + \int_{\frac{1}{2}}^{1}(1-t)^{s+1}\Delta t.$$

Proof. Using Corollary 6, property of modulus and s-convexity of $|\Upsilon^{\Delta}|$, we find

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right| \\
&\leq (b_2-b_1)\left[\int_0^{1/2} t|\Upsilon^{\Delta}(tb_2+(1-t)b_1)|\Delta t\right. \\
&\quad\left.+\int_{1/2}^{1}|t-1||\Upsilon^{\Delta}(tb_2+(1-t)b_1)|\Delta t\right] \\
&\leq (b_2-b_1)\left[\int_0^{1/2} t\left(t^s|\Upsilon^{\Delta}(b_2)|+(1-t)^s|\Upsilon^{\Delta}(b_1)|\right)\Delta t\right. \\
&\quad\left.+\int_{1/2}^{1}(1-t)|\left(t^s|\Upsilon^{\Delta}(b_2)|+(1-t)^s|\Upsilon^{\Delta}(b_1)|\right)\Delta t\right] \\
&\leq (b_2-b_1)\left(H_1|\Upsilon^{\Delta}(b_2)|+H_2|\Upsilon^{\Delta}(b_1)|\right),
\end{aligned} \tag{25}$$

where

$$H_1=\int_0^{\frac{1}{2}} t^{s+1}\Delta t+\int_{\frac{1}{2}}^{1} t^s(1-t)\Delta t, \text{ and } H_2=\int_0^{\frac{1}{2}} t(1-t)^s\Delta t+\int_{\frac{1}{2}}^{1}(1-t)^{s+1}\Delta t.$$

Hence the proof is completed. □

Corollary 7 *If $\mathbb{T}=\mathbb{R}$ in Theorem 4, we get*

$$H_1=\int_0^{\frac{1}{2}} t^{s+1}dt+\int_{\frac{1}{2}}^{1} t^s(1-t)dt=\frac{1}{(s+1)(s+2)}\left[1-\frac{1}{2^{s+1}}\right],$$

and

$$H_2=\int_0^{\frac{1}{2}} t(1-t)^s dt+\int_{\frac{1}{2}}^{1}(1-t)^{s+1}dt=\frac{1}{(s+1)(s+2)}\left[1-\frac{1}{2^{s+1}}\right].$$

Hence inequality (24) becomes

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon(c)dc\right| \\
&\leq \frac{b_2-b_1}{(s+1)(s+2)}\left(1-\frac{1}{2^{s+1}}\right)\left(|\Upsilon'(b_1)|+|\Upsilon'(b_2)|\right).
\end{aligned} \tag{26}$$

Theorem 5 *Consider a time scale* $\mathbb{T}$ *and* $\mathsf{T} = [b_1, b_2] \subseteq \mathbb{T}$ *such that* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon : \mathsf{T} \to \mathbb{R}$ *be a delta differentiable mapping on* T^o, *where* T^o *is the interior of* T. *If* $|\Upsilon^\Delta|^q$ *is* s*-convex, for* $q > 1$ *such that* $\frac{1}{r} + \frac{1}{q} = 1$, *then following inequality holds:*

$$\begin{aligned} \left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^\sigma(c)\Delta c\right| &\leq (b_2-b_1)\left[\left(\int_0^{1/2} t^r \Delta t\right)^{\frac{1}{r}}\right. \\ &\times \left(|\Upsilon^\Delta(b_2)|^q \int_0^{1/2} t^s \Delta t + |\Upsilon^\Delta(b_1)|^q \int_0^{1/2}(1-t)^s\Delta t\right)^{\frac{1}{q}} \\ &+ \left(\int_{1/2}^1 (1-t)^r \Delta t\right)^{\frac{1}{r}} \\ &\left.\times \left(|\Upsilon^\Delta(b_2)|^q \int_{1/2}^1 t^s \Delta t + |\Upsilon^\Delta(b_1)|^q \int_{1/2}^1 (1-t)^s \Delta t\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{27}$$

Proof. Using Corollary 6, property of modulus, Holder's integral inequality and s-convexity of $|\Upsilon^\Delta|^q$, we find

$$\begin{aligned} &\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^\sigma(c)\Delta c\right| \\ &\leq (b_2-b_1)\left[\left|\int_0^{1/2} t\,\Upsilon^\Delta(tb_2+(1-t)b_1)\Delta t\right|\right. \\ &\quad \left.+\left|\int_{1/2}^1 (t-1)\,\Upsilon^\Delta(tb_2+(1-t)b_1)\Delta t\right|\right] \\ &\leq (b_2-b_1)\left[\left(\int_0^{1/2} t^r\Delta t\right)^{\frac{1}{r}}\left(\int_0^{1/2}|\Upsilon^\Delta(tb_2+(1-t)b_1)|^q\Delta t\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\int_{1/2}^1 |1-t|^r\Delta t\right)^{\frac{1}{r}}\left(\int_{1/2}^1|\Upsilon^\Delta(tb_2+(1-t)b_1)|^q\Delta t\right)^{\frac{1}{q}}\right] \\ &\leq (b_2-b_1)\left[\left(\int_0^{1/2} t^r\Delta t\right)^{\frac{1}{r}}\left(\int_0^{1/2}(t^s|\Upsilon^\Delta(b_2)|^q+(1-t)^s|\Upsilon^\Delta(b_1)|^q)\Delta t\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\int_{1/2}^1 (1-t)^r\Delta t\right)^{\frac{1}{r}}\left(\int_{1/2}^1(t^s|\Upsilon^\Delta(b_2)|^q+(1-t)^s|\Upsilon^\Delta(b_1)|^q)\Delta t\right)^{\frac{1}{q}}\right] \end{aligned} \tag{28}$$

$$= (b_2 - b_1)\left[\left(\int_0^{1/2} t^r \Delta t\right)^{\frac{1}{r}} \left(|\Upsilon^{\Delta}(b_2)|^q \int_0^{1/2} t^s \Delta t + |\Upsilon^{\Delta}(b_1)|^q \int_0^{1/2} (1-t)^s \Delta t\right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_{1/2}^{1} (1-t)^r \Delta t\right)^{\frac{1}{r}} \left(|\Upsilon^{\Delta}(b_2)|^q \int_{1/2}^{1} t^s \Delta t + |\Upsilon^{\Delta}(b_1)|^q \int_{1/2}^{1} (1-t)^s \Delta t\right)^{\frac{1}{q}} \right].$$

Hence the proof. □

Corollary 8 *If $\mathbb{T} = \mathbb{R}$ in Theorem 5, then we have*

$$\int_0^{\frac{1}{2}} t^r dt = \int_{\frac{1}{2}}^{1} (1-t)^r dt = \frac{1}{(r+1)2^{r+1}},$$
$$\int_0^{\frac{1}{2}} t^s dt = \int_{\frac{1}{2}}^{1} (1-t)^s dt = \frac{1}{(s+1)2^{s+1}},$$

and

$$\int_0^{\frac{1}{2}} (1-t)^s dt = \int_{\frac{1}{2}}^{1} t^s dt = \frac{1}{s+1} - \frac{1}{(s+1)2^{s+1}}.$$

Hence the inequality (27) becomes

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon(c)dc\right| \\
&\leq (b_2-b_1)\left(\frac{1}{2^{r+1}(r+1)}\right)^{\frac{1}{r}} \left[\left\{\frac{1}{2^{s+1}(s+1)}|\Upsilon'(b_2)|^q \right.\right. \\
&\quad \left. + \left(\frac{1}{s+1} - \frac{1}{2^{s+1}(s+1)}\right)|\Upsilon'(b_1)|^q\right\}^{\frac{1}{q}} \\
&\quad \left. + \left\{\frac{1}{2^{s+1}(s+1)}|\Upsilon'(b_1)|^q + \left(\frac{1}{s+1} - \frac{1}{2^{s+1}(s+1)}\right)|\Upsilon'(b_2)|^q\right\}^{\frac{1}{q}}\right].
\end{aligned} \tag{29}$$

Definition 2 **([5])** *Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by*

$$h_0(t, r) = 1 \text{ for all } r, t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t, r) = \int_r^t h_k(\tau, r)\Delta\tau$$

for all $r, t \in \mathbb{T}$.

For next result we need following lemma.

Lemma 3 ([22]) *Let* $\Upsilon : \mathbb{T} \to \mathbb{R}$ *be a differentaible mapping and* $b_1 < b_2 \in \mathbb{T}$. *Let* $\Upsilon^{\Delta} \in C_{rd}$ *then following holds:*

$$\begin{aligned}&\Upsilon\,(b_1)\{1-h_2(1,0)\}+\Upsilon(b_2)h_2(1,0)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\\&=\frac{b_2-b_1}{2}\int_0^1\int_0^1[\Upsilon^{\Delta}(tb_1+(1-t)b_2)-\Upsilon^{\Delta}(rb_1+(1-r)b_2)](r-t)\Delta t\Delta r.\end{aligned}\tag{30}$$

Theorem 6 *Let* $\Upsilon : \mathbb{T} \to \mathbb{R}$ *be a differentaible mapping and* $b_1 < b_2 \in \mathbb{T}$. *Let* $|\,\Upsilon^{\Delta}\,|$ *be* s*-convex function, then following inequality holds:*

$$\begin{aligned}&\left|\Upsilon\,(b_1)\{1-h_2(1,0)\}+\Upsilon(b_2)h_2(1,0)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right|\\&\leq\frac{b_2-b_1}{2}[A_1|\,\Upsilon^{\Delta}\,(b_1)|+A_2|\,\Upsilon^{\Delta}\,(b_2)|],\end{aligned}\tag{31}$$

where

$$A_1=\int_0^1\int_0^1(t^s+r^s)(r+t)\Delta t\Delta r,$$

$$A_2=\int_0^1\int_0^1((1-t)^s+(1-r)^s)(r+t)\Delta t\Delta r.$$

Proof. Using Lemma 3, modulus property and s-convexity of $|\,\Upsilon^{\Delta}\,|$, we have

$$\begin{aligned}&\left|\Upsilon\,(b_1)\{1-h_2(1,0)\}+\Upsilon(b_2)h_2(1,0)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\Upsilon^{\sigma}(c)\Delta c\right|\\&\leq\frac{b_2-b_1}{2}\int_0^1\int_0^1|\,\Upsilon^{\Delta}\,(tb_1+(1-t)b_2)-\Upsilon^{\Delta}(rb_1+(1-r)b_2)||r-t|\Delta t\Delta r\\&\leq\frac{b_2-b_1}{2}\int_0^1\int_0^1[|\,\Upsilon^{\Delta}\,(tb_1+(1-t)b_2)|+|\,\Upsilon^{\Delta}\,(rb_1+(1-r)b_2)|](r+t)\Delta t\Delta r\\&\leq\frac{b_2-b_1}{2}\int_0^1\int_0^1[(t^s|\,\Upsilon^{\Delta}(b_1)|+(1-t)^s|\,\Upsilon^{\Delta}\,(b_2)|)\\&\quad+(r^s|\,\Upsilon^{\Delta}\,(b_1)|+(1-r)^s|\,\Upsilon^{\Delta}\,(b_2)|)](r+t)\Delta t\Delta r\\&=\frac{b_2-b_1}{2}\int_0^1\int_0^1[(t^s+r^s)|\Upsilon^{\Delta}(b_1)|+((1-t)^s+(1-r)^s)|\Upsilon^{\Delta}(b_2)|](r+t)\Delta t\Delta r\\&=\frac{b_2-b_1}{2}[A_1|\,\Upsilon^{\Delta}\,(b_1)|+A_2|\,\Upsilon^{\Delta}\,(b_2)|],\end{aligned}\tag{32}$$

where

$$A_1 = \int_0^1 \int_0^1 (t^s + r^s)(r+t)\Delta t \Delta r,$$

$$A_2 = \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(r+t)\Delta t \Delta r.$$

Hence the proof. □

Corollary 9 *Let* $\mathbb{T} = \mathbb{R}$ *in Theorem 6, then we have* $\sigma(b) = b$ *and*

$$h_2(1,0) = \int_0^1 (\tau - 1) d\tau = \frac{1}{2}.$$

Also,

$$\begin{aligned} A_1 &= \int_0^1 \int_0^1 (t^s + r^s)(r+t) dt dr = \frac{3s+4}{(s+1)(s+2)}, \\ A_2 &= \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(r+t) dt dr = 2\beta(2, s+1) + \frac{1}{s+1}, \end{aligned} \tag{33}$$

and hence inequality (31) becomes,

$$\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Upsilon(c) dc \right|$$

$$\leq \frac{b_2 - b_1}{2} \left[\left(\frac{3s+4}{(s+1)(s+2)} \right) |\Upsilon'(b_1)| + \left(2\beta(2, s+1) + \frac{1}{s+1} \right) |\Upsilon'(b_2)| \right], \tag{34}$$

where β *is Beta function.*

Lemma 4 ([22]) *Let* $\Upsilon : \mathsf{T} \subseteq \mathbb{T} \to \mathbb{R}$ *be a delta differentaible mapping on* T^o *and* $b_1 < b_2 \in \mathsf{T}$. *Let* $\Upsilon^\Delta \in C_{rd}$ *then following equality holds:*

$$\begin{aligned} &\Upsilon\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Upsilon^\sigma(c) \Delta c \\ &= \frac{b_2 - b_1}{2} \int_0^1 \int_0^1 [\Upsilon^\Delta(tb_1 + (1-t)b_2) \\ &\quad - \Upsilon^\Delta(rb_1 + (1-r)b_2)](m(r) - m(t) \Delta t \Delta r, \end{aligned} \tag{35}$$

where

$$m(c) = \begin{cases} c, & c \in \left[0, \frac{1}{2}\right] \\ c-1, & c \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Theorem 7 *Let* $\Upsilon : \mathsf{T} \subseteq \mathbb{T} \to \mathbb{R}$ *be a delta differentaible mapping on* T^o *and* $b_1 < b_2 \in \mathsf{T}$*. Let* $|\,\Upsilon^\Delta\,|$ *be* s*-convex function, then following inequality holds:*

$$\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c\right| \leq \frac{b_2-b_1}{2}[B_1|\,\Upsilon^\Delta\,(b_1)| + B_2|\,\Upsilon^\Delta\,(b_2)|], \tag{36}$$

where

$$B_1 = \int_0^1\int_0^1 (t^s + r^s)(m(r)+m(t))\Delta t\Delta r,$$

$$B_2 = \int_0^1\int_0^1 ((1-t)^s + (1-r)^s)(m(r)+m(t))\Delta t\Delta r.$$

Proof. Using Lemma 4, modulus property and s-convexity of $|\,\Upsilon^\Delta\,|$, we have

$$\begin{aligned}
&\left|\Upsilon\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \Upsilon^\sigma(c)\Delta c\right| \\
&\leq \frac{b_2-b_1}{2}\int_0^1\int_0^1 |\,\Upsilon^\Delta\,(tb_1+(1-t)b_2) \\
&\quad - \Upsilon^\Delta(rb_1+(1-r)b_2)||m(r)-m(t)|\Delta t\Delta r \\
&\leq \frac{b_2-b_1}{2}\int_0^1\int_0^1 [|\,\Upsilon^\Delta\,(tb_1+(1-t)b_2)| \\
&\quad + |\,\Upsilon^\Delta\,(rb_1+(1-r)b_2)|](m(r)+m(t))\Delta t\Delta r \\
&\leq \frac{b_2-b_1}{2}\int_0^1\int_0^1 [(t^s|\,\Upsilon^\Delta\,(b_1)| + (1-t)^s|\,\Upsilon^\Delta\,(b_2)|) \\
&\quad + (r^s|\,\Upsilon^\Delta\,(p_1)| + (1-r)^s|\,\Upsilon^\Delta\,(p_2)|)](m(r)+m(t))\Delta t\Delta r \\
&= \frac{b_2-b_1}{2}\int_0^1\int_0^1 [(t^s+r^s)|\,\Upsilon^\Delta\,(b_1)| \\
&\quad + ((1-t)^s+(1-r)^s)|\,\Upsilon^\Delta\,(b_2)|](m(r)+m(t))\Delta t\Delta r \\
&= \frac{b_2-b_1}{2}[B_1|\,\Upsilon^\Delta\,(b_1)| + B_2|\,\Upsilon^\Delta\,(b_2)|],
\end{aligned} \tag{37}$$

where

$$B_1 = \int_0^1 \int_0^1 (t^s + r^s)(m(r) + m(t))\Delta t \Delta r,$$

$$B_2 = \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(m(r) + m(t))\Delta t \Delta r.$$

Hence the proof. □

Corollary 10 *Let* $\mathbb{T} = \mathbb{R}$ *in Theorem 7, then we have* $\sigma(b) = b$ *and*

$$B_1 = \int_0^1 \int_0^1 (t^s + r^s)(m(r) + m(t)dtdr = \frac{1}{s+1}\left[\frac{1}{2^s} - \frac{2}{s+2}\right], \tag{38}$$

$$\begin{aligned} B_2 &= \int_0^1 \int_0^1 ((1-t)^s + (1-r)^s)(m(r) + m(t)dtdr \\ &= 2\beta_{\frac{1}{2}}(2, s+1) - \frac{1}{s^{s+1}(s+2)}, \end{aligned} \tag{39}$$

and hence inequality (36) becomes,

$$\begin{aligned} &\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Upsilon(c)dc \right| \\ &\leq \frac{b_2 - b_1}{2}\left[\left(\frac{1}{s+1}\left[\frac{1}{2^s} - \frac{2}{s+2}\right]\right) |\Upsilon'(b_1)| \right. \\ &\quad \left. + \left(2\beta_{\frac{1}{2}}(2, s+1) - \frac{1}{s^{s+1}(s+2)}\right) |\Upsilon'(b_2)| \right], \end{aligned} \tag{40}$$

where β_u *is incomplete Beta function defined by*

$$\beta_u(b_1, b_2) = \int_0^u x^{b_1 - 1}(1-x)^{b_2 - 1}dx, \quad u \in (0, 1).$$

4 Conclusion

This research investigation includes some inequalities for s-convex function on time scales such as Hermite-Hadamard type inequalities. Some special cases are discussed, that is, when the time scale is $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$.

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References

[1] R. P. Agarwal, M. Bohner, A. Peterson, Inequalities on Time Scales: A Survey, *Math. Inequal. Appl.*, **4**(4) (2001), 537–557.

[2] M. Alomari, M. Darus, S. S. Dragomir, et al. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.*, **23** (2010), 1071–1076.

[3] M. R. S. Ammi, R. A. C. Ferreira, D. F. M. Torres, Diamond-α Jensen's inequality on time scales, *J. Inequal. Appl.*, Hindawi Publishing Corporation, **2008** (2008), 13 pages.

[4] M. Bohner, A. Peterson, Dynamic Equation on Time Scale: An Introduction with Application, Birkhauser, Boston, Mess, USA, 2001.

[5] M. Bohner, T. Matthews, Ostrowski inequalities on time scales, *J. Inequal. Pure Appl. Math.*, **9**(1) (2008).

[6] P. Cerone, S. S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, *Demonstratio Math.*, **37** (2004), 299–308.

[7] C. Dinu, Hermite-Hadamard inequality on time scale, *J. Inequal. Appl.*, Hindawi Publishing Corporation, **2008** (2018), p. 24.

[8] C. Dinu, Ostrowski type inequalities on time scales, *An. Univ. Craiova Ser. Mat. Inform.*, **34**(1) (2007), 43–58.

[9] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, **11**(5) (1998), 91–95.

[10] J. Hadamard, "Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann", *J. Math. Pures Appl.*, (1893), 171–215.

[11] Ch. Hermite, Sur deux limites d'une intégrale denie, Mathesis., **3** (1883), 82.

[12] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, A*equationes Math.* **48**(1) (1994), 100–111.

[13] N. Mehreen, M. Anwar, Integral inequalities for some convex functions via generalized fractional integrals, *J. Inequal. Appl.*, **2018** (2018), p. 208.

[14] N. Mehreen, M. Anwar, Hermite-Hadamard type inequalities via exponentially $\mathfrak{p}$-convex functions and exponentially $\mathfrak{s}$-convex functions in second sense with applications, *J. Inequal. Appl.*, **2019** (2019), p. 92.

[15] N. Mehreen, M. Anwar, Hermite-Hadamard type inequalities via exponentially $(\mathfrak{p}, \mathfrak{h})$-convex functions, *IEEE Access*, **8** (2020), 37589–37595.

[16] N. Mehreen, M. Anwar, On some Hermite-Hadamard type inequalities for $\mathfrak{tgs}$-convex functions via generalized fractional integrals, *Adv. Difference Equ.*, **2020** (2020), p. 6.

[17] N. Mehreen, M. Anwar, Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for $\mathfrak{p}$-convex functions via conformable fractional integrals, *J. Inequal. Appl.*, **2020** (2020), p. 107.

[18] N. Mehreen, M. Anwar, Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for $\mathfrak{p}$-convex functions via new fractional conformable integral operators, *J. Math. Compt. Sci.*, **19** (2019), 230–240.

[19] N. Mehreen, M. Anwar, Ostrowski type inequalities via some exponentially convex functions with applications, *AIMS Mathematics*, **5(2)** (2020), 1476–1483.

[20] N. Mehreen, M. Anwar, Some inequalities via ψ-Riemann—Liouville fractional integrals, *AIMS Mathematics*, **4(5)** (2019), 1403–1415.

[21] A. M. Ostrowski, Uber die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.

[22] S. F. Tahir, M. Mushtaq, M. Muddassar, A new interpretation of Hermite-Hadamard's inequalities by the way of time scales, *J. Comput. Anal. Appl.*, **26**(2) (2019), 223–233.

[23] H. Yaldiz, P. Agarwal, s-convex functions on discrete time domains, *Analysis*, (2017).

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Certain classes of bi-univalent functions associated with the Horadam polynomials

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Abstract. In this paper we consider two subclasses of bi-univalent functions defined by the Horadam polynomials. Further, we obtain coefficient estimates for the defined classes.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in Δ.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both the function f and its inverse f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1).

In 2010, Srivastava et al. [28] revived the study of bi-univalent functions by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were found in the very recent investigations (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30]) and including the references therein. The afore-cited all these papers on the subject were actually motivated by the work of Srivastava et al. [28]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

For analytic functions f and g in Δ, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)) \qquad (z \in \Delta).$$

This subordination will be denoted here by

$$f \prec g \qquad (z \in \Delta)$$

or, conventionally, by

$$f(z) \prec g(z) \qquad (z \in \Delta).$$

In particular, when g is univalent in Δ,

$$f \prec g \qquad (z \in \Delta) \;\Leftrightarrow\; f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The Horadam polynomials $h_n(x,\ a,\ b;\ p,\ q)$, or briefly $h_n(x)$ are given by the following recurrence relation (see [14, 15]):

$$h_1(x) = a, \quad h_2(x) = bx \quad \text{and} \quad h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \quad (n \geq 3) \tag{2}$$

for some real constants a, b, p and q.

The generating function of the Horadam polynomials $h_n(x)$ (see [15]) is given by

$$\Pi(x,\ z) := \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}\ . \tag{3}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is, $x \neq \Re(z)$.

Note that for particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ leads to various polynomials, among those, we list a few cases here (see, [14, 15] for more details):

1. For $a = b = p = q = 1$, we have the Fibonacci polynomials $F_n(x)$.
2. For $a = 2$ and $b = p = q = 1$, we obtain the Lucas polynomials $L_n(x)$.
3. For $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n(x)$.
4. For $a = b = p = 2$ and $q = 1$, we attain the Pell-Lucas polynomials $Q_n(x)$.
5. For $a = b = 1,\ p = 2$ and $q = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind
6. For $a = 1,\ b = p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $U_n(x)$ of the second kind.

Abirami et al. [1] considered bi- Mocanu - convex functions and bi-$\mu-$ starlike functions to discuss initial coefficient estimations of Taylor-Macularin series which is associated with Horadam polynomials, Abirami et al. [2] discussed coefficient estimates for the classes of $\lambda-$bi-pseudo-starlike and bi-Bazilevič functions using Horadam polynomial, Alamoush [3, 4] defined subclasses of bi-starlike and bi-convex functions involving the Poisson distribution series involving Horadam polynomials and a class of bi-univalent functions associated with Horadam polynomials respectively and obtained initial coefficient estimates, Altınkaya and Yalçın [7, 8] obtained coefficient estimates for Pascu-type bi-univalent functions and for the class of linear combinations of bi-univalent functions by means of (p, q)-Lucas polynomials respectively, Aouf et al. [10] discussed initial coefficient estimates for general class of pascu-type bi-univalent functions of complex order defined by $q-$Sălăgean operator and associated with Chebyshev polynomials, Awolere and Oladipo [11] found initial coefficients of bi-univalent functions defined by sigmoid functions involving pseudo-starlikeness associated with Chebyshev polynomials, Naeem et al. [18] considered a general class of bi-Bazilevič type functions associated with Faber polynomial to discuss n-th coefficients estimates, Magesh and Bulut [19] discussed Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, Orhan et al. [21] discussed initial estimates and Fekete-Szegö bounds for bi-Bazilevič functions related to shell-like curves, Sakar and Aydogan [23] obtained initial bounds for the class of generalized Sălăgean type bi-$\alpha-$ convex functions of complex order associated with the Horadam polynomials, Singh et al. [24] found coefficient estimates for bi-α-convex functions defined by generalized Sãlãgean operator related to shell-like curves connected with Fibonacci numbers, Srivastava et al. [25] introduced a technique by defining a new class bi-univalent functions associated with the Horadam polynomials to discuss the coefficient estimates, Srivastava et al. [27] gave a direction to study the Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Srivastava et al. [29] obtained general coefficient $|a_n|$ for a general class analytic and bi-univalent functions defined by using differential subordination and a certain fractional derivative operator associated with Faber polynomial, Wanas and Alina [30] discussed applications of Horadam polynomials on Bazilevič bi-univalent functions by means of subordination and found initial bounds. Motivated in these lines, estimates on initial coefficients of the Taylor-Maclaurin series expansion (1) and Fekete-Szegő inequalities for certain classes of bi-univalent functions defined by means of Horadam polynomials are obtained. The classes introduced in this paper are motivated by the corresponding classes investigated in [16, 20].

2 Coefficient estimates and Fekete-Szegő inequalities

A function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{G}^*_\Sigma(\alpha,\ x)$ for $0 \le \alpha \le 1$ and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)f'(z) \prec \Pi(x,\ z)+1-a$$

and for $g(w)=f^{-1}(w)$

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)g'(w) \prec \Pi(x,\ w)+1-a,$$

where the real constant a is as in (2).

Remark 1 *The classes $\mathcal{K}_\Sigma(x)$ and $\mathcal{H}_\Sigma(x)$ are defined by $\mathcal{G}^*_\Sigma(1,\ x) := \mathcal{K}_\Sigma(x)$ and introduced by [1] and $\mathcal{G}^*_\Sigma(0,\ x) := \mathcal{H}_\Sigma(x)$ introduced by [4] respectively.*

For functions in the class $\mathcal{G}^*_\Sigma(\alpha,\ x)$, the following coefficient estimates and Fekete-Szegő inequality are obtained.

Theorem 1 *Let $f(z)=z+\sum\limits_{n=2}^{\infty}a_n z^n$ be in the class $\mathcal{G}^*_\Sigma(\alpha,\ x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|(3-\alpha)\,b^2x^2-4\,\left(px^2b+qa\right)\right|}}, \qquad \textit{and} \qquad |a_3| \le \frac{|bx|}{3\,(\alpha+1)}+\frac{b^2x^2}{4}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3-\nu a_2^2\right| \le \begin{cases} \dfrac{|bx|}{3\alpha+3} & \textit{if } |\nu-1| \le \dfrac{\left|(3-\alpha)\,b^2x^2-4\,\left(px^2b+qa\right)\right|}{b^2x^2\,(3\alpha+3)} \\[2ex] \dfrac{|bx|^3\,|\nu-1|}{\left|(3-\alpha)\,b^2x^2-4\,\left(px^2b+qa\right)\right|} & \textit{if } |\nu-1| \ge \dfrac{\left|(3-\alpha)\,b^2x^2-4\,\left(px^2b+qa\right)\right|}{b^2x^2\,(3\alpha+3)}. \end{cases}$$

Proof. Let $f \in \mathcal{G}^*_\Sigma(\alpha,\ x)$ be given by the Taylor-Maclaurin expansion (1). Then, there are analytic functions r and s such that

$$r(0)=0; \quad s(0)=0, \quad |r(z)|<1 \quad \text{and} \quad |s(w)|<1 \quad (\forall\ z,\ w \in \Delta),$$

and we can write

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)f'(z) = \Pi(x,\ r(z))+1-a \tag{4}$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)g'(w)=\Pi(x,\ s(w))+1-a. \tag{5}$$

Equivalently,

$$\begin{aligned}\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)&+(1-\alpha)f'(z)\\ &=1+h_1(x)-a+h_2(x)r(z)+h_3(x)[r(z)]^2+\cdots\end{aligned} \tag{6}$$

and

$$\begin{aligned}\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)&+(1-\alpha)g'(w)\\ &=1+h_1(x)-a+h_2(x)s(w)+h_3(x)[s(w)]^2+\cdots.\end{aligned} \tag{7}$$

From (6) and (7) and in view of (3), we obtain

$$\begin{aligned}\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)&+(1-\alpha)f'(z)\\ &=1+h_2(x)r_1z+[h_2(x)r_2+h_3(x)r_1^2]z^2+\cdots\end{aligned} \tag{8}$$

and

$$\begin{aligned}\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)&+(1-\alpha)g'(w)\\ &=1+h_2(x)s_1w+[h_2(x)s_2+h_3(x)s_1^2]w^2+\cdots.\end{aligned} \tag{9}$$

If

$$r(z)=\sum_{n=1}^{\infty}r_nz^n \qquad \text{and} \qquad s(w)=\sum_{n=1}^{\infty}s_nw^n,$$

then it is well known that

$$|r_n|\le 1 \qquad \text{and} \qquad |s_n|\le 1 \qquad (n\in\mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (8) and (9), we have

$$2a_2=h_2(x)r_1 \tag{10}$$

$$3\,(\alpha+1)\,a_3-4a_2^2\alpha=h_2(x)r_2+h_3(x)r_1^2 \tag{11}$$

$$-\,2a_2=h_2(x)s_1 \tag{12}$$

and

$$2\,(\alpha+3)\,a_2^2 - 3\,(\alpha+1)\,a_3 = h_2(x)s_2 + h_3(x)s_1^2. \tag{13}$$

From (10) and (12), we can easily see that

$$r_1 = -s_1, \qquad \text{provided} \qquad h_2(x) = bx \neq 0 \tag{14}$$

and

$$\begin{aligned} 8\,a_2^2 &= (h_2\,(x))^2\left({r_1}^2 + {s_1}^2\right) \\ a_2^2 &= \frac{1}{8}\,(h_2\,(x))^2\left({r_1}^2 + {s_1}^2\right). \end{aligned} \tag{15}$$

If we add (11) to (13), we get

$$2\,a_2^2\,(3-\alpha) = (r_2 + s_2)\,h_2\,(x) + h_3\,(x)\left({r_1}^2 + {s_1}^2\right). \tag{16}$$

By substituting (15) in (16), we obtain

$$a_2^2 = \frac{(r_2+s_2)\,(h_2\,(x))^3}{2\,(3-\alpha)\,(h_2\,(x))^2 - 8\,h_3\,(x)} \tag{17}$$

and by taking $h_2(x) = bx$ and $h_3(x) = bpx^2 + qa$ in (17), it further yields

$$|a_2| \leq \frac{|bx|\,\sqrt{|bx|}}{\sqrt{|(3-\alpha)\,b^2x^2 - 4\,(px^2b + qa)|}}. \tag{18}$$

By subtracting (13) from (11) we get

$$6\,(\alpha+1)\left(a_3 - {a_2}^2\right) = (r_2 - s_2)\,h_2\,(x) + \left({r_1}^2 - {s_1}^2\right)h_3\,(x).$$

In view of (14) , we obtain

$$a_3 = \frac{(r_2 - s_2)\,h_2\,(x)}{6(\alpha+1)} + {a_2}^2. \tag{19}$$

Then in view of (15), (19) becomes

$$a_3 = \frac{(r_2 - s_2)\,h_2\,(x)}{6(\alpha+1)} + \frac{1}{8}\,(h_2\,(x))^2\left({r_1}^2 + {s_1}^2\right).$$

Applying (2), we deduce that

$$|a_3| \leq \frac{|bx|}{3\,(\alpha+1)} + \frac{b^2x^2}{4}.$$

From (19), for $\nu \in \mathbb{R}$, we write

$$a_3 - \nu a_2^2 = \frac{h_2(x)\,(r_2 - s_2)}{6\,(\alpha+1)} + (1-\nu)\,a_2^2. \tag{20}$$

By substituting (17) in (20), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{h_2(x)\,(r_2 - s_2)}{6\,(\alpha+1)} + \left(\frac{(1-\nu)\,(r_2+s_2)\,(h_2\,(x))^3}{2\,(3-\alpha)\,(h_2\,(x))^2 - 8\,h_3\,(x)}\right) \\ &= h_2(x)\left\{\left(\Lambda_1(\nu,\ x) + \frac{1}{6\,(\alpha+1)}\right) r_2 + \left(\Lambda_1(\nu,\ x) - \frac{1}{6\,(\alpha+1)}\right) s_2\right\}, \end{aligned} \tag{21}$$

where

$$\Lambda_1(\nu,\ x) = \frac{(1-\nu)\,[h_2(x)]^2}{2\,(3-\alpha)\,(h_2\,(x))^2 - 8\,h_3\,(x)}.$$

Hence, in view of (2) we conclude that

$$\left|a_3 - \nu a_2^2\right| \le \begin{cases} \dfrac{|h_2(x)|}{3(\alpha+1)} & ; 0 \le |\Lambda_1(\nu,\ x)| \le \dfrac{1}{6\,(\alpha+1)} \\[2ex] 2\,|h_2(x)|\,|\Lambda_1(\nu,\ x)| & ; |\Lambda_1(\nu,\ x)| \ge \dfrac{1}{6\,(\alpha+1)} \end{cases}$$

and in view of (2), it evidently completes the proof of Theorem 1. □

Taking $\alpha = 1$ in Theorem 1, we have following corollary.

Corollary 1 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{K}_\Sigma(x)$. *Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|2b^2x^2 - 4\,(px^2b + qa)|}}, \qquad \textit{and} \qquad |a_3| \le \frac{|bx|}{6} + \frac{b^2x^2}{4}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \le \begin{cases} \dfrac{|bx|}{6} & \textit{if } |\nu - 1| \le \dfrac{\left|b^2x^2 - 2\,(px^2b + qa)\right|}{3b^2x^2} \\[2ex] \dfrac{|bx|^3\,|\nu - 1|}{|2b^2x^2 - 4\,(px^2b + qa)|} & \textit{if } |\nu - 1| \ge \dfrac{\left|b^2x^2 - 2\,(px^2b + qa)\right|}{3b^2x^2}. \end{cases}$$

Taking $\alpha = 0$ in Theorem 1, we have following corollary.

Corollary 2 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{H}_\Sigma(x)$. *Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|3b^2x^2 - 4\ (px^2b + qa)|}}, \qquad \textit{and} \qquad |a_3| \leq \frac{|bx|}{3} + \frac{b^2x^2}{4}$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{3} & \textit{if } |\nu - 1| \leq \dfrac{\left|3b^2x^2 - 4\ \left(px^2b + qa\right)\right|}{3b^2x^2} \\ \dfrac{|bx|^3\,|\nu - 1|}{|3b^2x^2 - 4\ (px^2b + qa)|} & \textit{if } |\nu - 1| \geq \dfrac{\left|3b^2x^2 - 4\ \left(px^2b + qa\right)\right|}{3b^2x^2}. \end{cases}$$

Next, a function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{L}_\Sigma(x)$ and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} \prec \Pi(x,\ z) + 1 - a$$

and for $g(w) = f^{-1}(w)$

$$\frac{1 + \dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} \prec \Pi(x,\ w) + 1 - a,$$

where the real constant a is as in (2).

For functions in the class $\mathcal{L}_\Sigma(x)$, the following coefficient estimates and Fekete-Szegő inequality are obtained.

Theorem 2 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{L}_\Sigma(x)$. *Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|px^2b + qa|}}, \qquad \textit{and} \qquad |a_3| \leq \frac{|bx|}{4} + b^2x^2$$

and for $\nu \in \mathbb{R}$

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{4} & \text{if} \quad |\nu - 1| \leq \dfrac{\left|bpx^2 + aq\right|}{4b^2x^2} \\[2ex] \dfrac{|bx|^3\,|\nu - 1|}{|bpx^2 + aq|} & \text{if} \quad |\nu - 1| \geq \dfrac{\left|bpx^2 + aq\right|}{4b^2x^2}. \end{cases}$$

Proof. Let $f \in \mathcal{L}_\Sigma(x)$ be given by the Taylor-Maclaurin expansion (1). Then, there are analytic functions r and s such that

$$r(0) = 0; \quad s(0) = 0, \quad |r(z)| < 1 \quad \text{and} \quad |s(w)| < 1 \quad (\forall\, z,\ w \in \Delta),$$

and we can write

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = \Pi(x,\ r(z)) + 1 - a \tag{22}$$

and

$$\frac{1 + \dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} = \Pi(x,\ s(w)) + 1 - a. \tag{23}$$

Equivalently,

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = 1 + h_1(x) - a + h_2(x)r(z) + h_3(x)[r(z)]^2 + \cdots \tag{24}$$

and

$$\frac{1 + \dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} = 1 + h_1(x) - a + h_2(x)s(w) + h_3(x)[s(w)]^2 + \cdots. \tag{25}$$

From (24) and (25) and in view of (3), we obtain

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = 1 + h_2(x)r_1 z + [h_2(x)r_2 + h_3(x)r_1^2]z^2 + \cdots \tag{26}$$

and

$$\frac{1+\frac{wg''(w)}{g'(w)}}{\frac{wg'(w)}{g(w)}} = 1 + h_2(x)s_1w + [h_2(x)s_2 + h_3(x)s_1^2]w^2 + \cdots . \tag{27}$$

If

$$r(z) = \sum_{n=1}^{\infty} r_n z^n \qquad \text{and} \qquad s(w) = \sum_{n=1}^{\infty} s_n w^n,$$

then it is well known that

$$|r_n| \leq 1 \qquad \text{and} \qquad |s_n| \leq 1 \qquad (n \in \mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (26) and (27), we have

$$a_2 = h_2(x)r_1 \tag{28}$$

$$4\left(a_3 - a_2^2\right) = h_2(x)r_2 + h_3(x)r_1^2 \tag{29}$$

$$-a_2 = h_2(x)s_1 \tag{30}$$

and

$$4\left(a_2^2 - a_3\right) = h_2(x)s_2 + h_3(x)s_1^2. \tag{31}$$

From (28) and (30), we can easily see that

$$r_1 = -s_1, \qquad \text{provided} \qquad h_2(x) = bx \neq 0 \tag{32}$$

and

$$\begin{aligned} 2\,a_2^2 &= (h_2\,(x))^2 \left({r_1}^2 + {s_1}^2\right) \\ a_2^2 &= \frac{1}{2}\,(h_2\,(x))^2 \left({r_1}^2 + {s_1}^2\right) . \end{aligned} \tag{33}$$

If we add (29) to (31), we get

$$0 = (r_2 + s_2)\, h_2\,(x) + h_3\,(x) \left({r_1}^2 + {s_1}^2\right). \tag{34}$$

By substituting (33) in (34), we obtain

$$a_2^2 = -\frac{(r_2 + s_2)\,(h_2\,(x))^3}{2\,h_3\,(x)} \tag{35}$$

and by taking $h_2(x) = bx$ and $h_3(x) = bpx^2 + qa$ in (35), it further yields

$$|a_2| \quad \leq \quad \frac{|bx|\sqrt{|bx|}}{\sqrt{|px^2b + qa|}}. \tag{36}$$

By subtracting (31) from (29) we get

$$-8\left(a_2{}^2 - a_3\right) = (r_2 - s_2)\, h_2(x) + \left(r_1{}^2 - s_1{}^2\right) h_3(x)$$

In view of (32) , we obtain

$$a_3 = \frac{1}{8}(r_2 - s_2)\, h_2(x) + a_2{}^2. \tag{37}$$

Then in view of (33), (37) becomes

$$a_3 \quad = \quad \frac{1}{8}(r_2 - s_2)\, h_2(x) + \frac{1}{2}(h_2(x))^2 \left(r_1{}^2 + s_1{}^2\right).$$

Applying (2), we deduce that

$$|a_3| \quad \leq \quad \frac{|bx|}{4} + b^2x^2.$$

From (37), for $\nu \in \mathbb{R}$, we write

$$a_3 - \nu a_2^2 = \frac{1}{8} h_2(x)\,(r_2 - s_2) + (1 - \nu)\, a_2^2. \tag{38}$$

By substituting (35) in (38), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{1}{8} h_2(x)\,(r_2 - s_2) + \left(\frac{(\nu - 1)\,(r_2 + s_2)\,(h_2(x))^3}{2\, h_3(x)}\right) \\ &= h_2(x) \left\{ \left(\Lambda_2(\nu,\ x) + \frac{1}{8}\right) r_2 + \left(\Lambda_2(\nu,\ x) - \frac{1}{8}\right) s_2 \right\}, \end{aligned} \tag{39}$$

where

$$\Lambda_2(\nu,\ x) = \frac{(\nu - 1)\,(h_2(x))^2}{2\, h_3(x)}.$$

Hence, in view of (2) we conclude that

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|h_2(x)|}{4}; & 0 \leq |\Lambda_2(\nu,\ x)| \leq \dfrac{1}{8} \\[2ex] 2\,|h_2(x)|\,|\Lambda_2(\nu,\ x)|; & |\Lambda_2(\nu,\ x)| \geq \dfrac{1}{8} \end{cases}$$

and in view of (2), it evidently completes the proof of Theorem 2. □

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References

[1] C. Abirami, N. Magesh and J. Yamini, Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials, Abstr. Appl. Anal. **2020**, Art. ID 7391058, 8 pp.

[2] C. Abirami, N. Magesh, J. Yamini, and N. B. Gatti, Horadam polynomial coefficient estimates for the classes of $\lambda-$bi-pseudo-starlike and bi-Bazilevič functions, *J. Anal.*, (2020), 1–10.

[3] A. G. Alamoush, Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials, *Malaya J. Mat.*, **7** (2019), no. 4, 618–624.

[4] A. G. Alamoush, On a subclass of bi-univalent functions associated to Horadam polynomials, *Int. J. Open Problems Complex Anal.*, **12** (2020), no. 1, 58 – 65.

[5] I. Aldawish, T. Al-Hawary and B. A. Frasin, Subclasses of bi-Univalent functions defined by Frasin differential operator, *Mathematics*, **8** (2020), 783, 1–11.

[6] R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramanian, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.*, **25** (2012), no. 3, 344–351.

[7] Ş. Altınkaya and S. Yalçın, On the $(\mathfrak{p}, \mathfrak{q})$-Lucas polynomial coefficient bounds of the bi-univalent function class σ, *Bol. Soc. Mat. Mex.*, (3) **25** (2019), no. 3, 567–575.

[8] Ş. Altınkaya and S. Yalçın, $(\mathfrak{p}, \mathfrak{q})$-Lucas polynomials and their applications to bi-univalent functions, *Proyecciones*, **38** (2019), no. 5, 1093–1105.

[9] O. Al-Refai and M. Ali, General coefficient estimates for bi-univalent functions: a new approach, *Turk. J. Math.*, **44** (2020), 240– 251.

[10] M. K. Aouf, A. O. Mostafa, R. E. El. Morsy, Coefficient bounds for general class of bi-univalent functions of complex order associated with $\mathsf{q}-$Sălăgean operator and Chebyshev polynomials, *Electr. J. Math. Anal. Appl.*, **8** (2020), no. 2, 251–260.

[11] I. T. Awolere and A. T. Oladipo, Coefficients of bi-univalent functions involving pseudo-starlikeness associated with Chebyshev polynomials, *Khayyam J. Math.*, **5** (2019), no. 1, 140–149.

[12] M. Çağlar, E. Deniz, H. M. Srivastava, Second Hankel determinant for certain subclasses of bi-univalent functions, *Turk J. Math.*, **41**, (2017), 694–706.

[13] S. M. El-Deeb, T. Bulboaca and B. M. El-Matary, Maclaurin coefficient estimates of bi-Univalent functions connected with the $\mathsf{q}-$derivative, *Mathematics*, **8** (2020), 418, 1–14.

[14] A. F. Horadam, J. M. Mahon, Pell and Pell–Lucas polynomials, *Fibonacci Quart.*, **23**, (1985), 7–20.

[15] T. Hörçum, E. Gökçen Koçer, On some properties of Horadam polynomials, *Int Math Forum*, **4** (2009), 1243–1252.

[16] A. Y. Lashin, Coefficient estimates for two subclasses of analytic and bi-univalent functions, *Ukrainian Math. J.*, **70** (2019), no. 9, 1484–1492

[17] P. Long, H. Tang, W. Wang, Fekete-Szegö functional problems for certain subclasses of bi-univalent functions involving the Hohlov operator, *J. Math. Res. Appl.*, **40** (2020), no. 1, 1–12.

[18] M. Naeem, S. Khan, and F. M. Sakar, Faber polynomial coefficients estimates of bi-univalent functions, *Inter. J. Maps Math.*, **3** (2020), no. 1, 57–67.

[19] N. Magesh and S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, *Afr. Mat.*, **29** (2018), no. 1-2, 203–209.

[20] N. Magesh and J. Yamini, Fekete-Szegő problem and second Hankel determinant for a class of bi-univalent functions, *Tbilisi Math. J.*, **11** (2018), no. 1, 141–157.

[21] H. Orhan, N. Magesh and C. Abirami, Fekete-Szegö problem for bi-Bazilevič functions related to shell-like curves, *AIMS Mathematics*, **5** (2020), no. 5, 4412–4423.

[22] S. Porwal and S. Kumar, New subclasses of bi-univalent functions defined by multiplier transformation, *Stud. Univ. Babeş-Bolyai Math.*, **65** (2020), no. 1, 47–55.

[23] F. M. Sakar and S. M. Aydogan, Initial bounds for certain subclasses of generalized Sălăgean type bi-univalent functions associated with the Horadam polynomials, *J. Quality. Measur. Anal.*, **15** (2019), no. 1, 89–100.

[24] G. Singh, G. Singh and G. Singh, A subclass of bi-univalent functions defined by generalized Sãlãgean operator related to shell-like curves connected with Fibonacci numbers, *Int. J. Math. Math. Sci.*, **2019**, Art. ID 7628083, 1–7.

[25] H. M. Srivastava, Altınkaya and S. Yalçın, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, *Iran. J. Sci. Technol. Trans. A Sci.*, **43** (2019), no. 4, 1873–1879.

[26] H. M. Srivastava, F. M. Sakar and H. Özlem Güney, Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination, *Filomat*, **32** (2018), no. 4, 1313–1322.

[27] H. M. Srivastava, S. S. Eker, S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, *Bull. Iranian Math. Soc.*, **44** (2018), no. 1, 149–157.

[28] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (2010), no. 10, 1188–1192.

[29] H. M. Srivastava, A. Motamednezhad and E. A. Adegani, Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator, *Mathematics*, **8** (2020), 172, 1–12.

[30] A. K. Wanas and A. L. Alina, Applications of Horadam polynomials on Bazilevic bi-univalent function satisfying subordinate conditions, *IOP Conf. Series: J. Phy.*, **1294** (2019), 032003, 1–6.

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On sums of monotone functions over smooth numbers

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Abstract. In this article, we are going to look at the requirements regarding a monotone function $f \in \mathbb{R} \to \mathbb{R}_{\geq 0}$, and regarding the sets of natural numbers $(A_i)_{i=1}^{\infty} \subseteq \mathrm{dmn}(f)$, which requirements are sufficient for the asymptotic

$$\sum_{\substack{n \in A_N \\ P(n) \leq N^\theta}} f(n) \sim \rho(1/\theta) \sum_{n \in A_N} f(n)$$

to hold, where N is a positive integer, $\theta \in (0,1)$ is a constant, $P(n)$ denotes the largest prime factor of n, and ρ is the Dickman function.

1 Introduction

In his article [3], Croot gave a sufficient condition to express sums of non-negative functions over smooth natural numbers, using the Dickman function ρ. The result can be summarized as

$$\sum_{\substack{1 \leq n \leq N \\ P(n) \leq N^\theta}} f(n) \sim \rho(1/\theta) \sum_{1 \leq n \leq N} f(n) \tag{1}$$

where f is a non-negative function defined over $\mathbb{N}$, $\theta \in (0,1)$ is a constant, and $P(n)$ denotes the largest prime factor of n, with the convention that $P(1) = 1$.

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The Dickman function can be taken as the limit

$$\rho(1/\theta) = \lim_{N\to\infty} \frac{\Psi(N, N^\theta)}{N} \tag{2}$$

which limit exists if $\theta > 0$, see the article of Dickman [4]. Here $\Psi(x, y)$ is the count of y-smooth positive integers smaller than-, or equal to x. For a recollection about the behavior of the function ρ, and about smooth integers, see article [6], and chapter III.5 in [7].

The method of Croot is specialized for the problem tackled by him, and it is difficult to apply in more general situations. We are going to look at when we can say that the asymptotic equality (1) holds, based on properties of the examined function, which properties are easier to check.

Based on the properties of the function Ψ, it is easy to see that the idea works for functions $f(n) := c$, with any real constant c, as the equalities

$$\sum_{\substack{1\le n\le N\\ P(n)\le N^\theta}} c = c\Psi(N, N^\theta) = \frac{\Psi(N, N^\theta)}{N} \sum_{1\le n\le N} c$$

hold. We are expecting a similar result for more general functions. Concerning the basic properties of the examined functions, we expect them to be non-negative, monotone changing functions, which are not the constant zero function. As we are going to apply Abel's identity to handle certain sums, a heavier requirement arises, namely that the examined functions should be continuously differentiable.

A sufficient condition for (1) to hold is — informally — that f shouldn't change too fast. To introduce the concept in iterations, first we say that $f(x)$ should be in $o(x^\alpha) \cap \omega(x^{-\alpha})$ for every $\alpha > 0$, so f should be changing with at most the speed of the polylogarithmic functions or their reciprocals. As a second iteration, because we will bound the derivative of f, we will actually need a bit stronger requirement, namely that $f'(x)$ should be in $o(x^{\alpha-1})$ while $f(x) \in \omega(x^{-\alpha})$ for every $\alpha > 0$. (We need this, because differentiation doesn't preserve inequalities.) As a third, and final iteration, we can actually lighten these requirements a bit. Let

$$L_1 := \{f \in \mathbb{R} \to \mathbb{R} : \forall\alpha > 0, f'(x) \in \mathcal{O}(x^{\alpha-1}) \wedge f(x) \in \omega(x^{-\alpha})\}$$

and

$$L_2 := \{f \in \mathbb{R} \to \mathbb{R} : \forall\alpha > 0, f'(x) \in o(x^{\alpha-1}) \wedge f(x) \in \Omega(x^{-\alpha})\}$$

then let $L := L_1 \cup L_2$. We will show that $f \in L$ is a sufficient condition for (1) to hold. It's worth mentioning, that we cannot lighten both conditions at the same time. (Regarding the asymptotic notation, we refer to section 3.1 of [2], and to section 4.1.1 of [5]. Take note that we use these notations in the sense that they express a bound on the *absolute value* of the examined function.)

As a final generalization, instead of looking at the sum going from some initial positive value up until N, we will sum the examined function over some sets $(A_i)_{i=1}^{\infty} \subseteq \mathrm{dmn}(f)$. The only requirement concerning these sets is that they should be "dense" among the natural numbers, i. e. $|A_N| \sim N$ should hold.

Proposition 1 *Let* $\theta \in (0,1)$, $m \in \mathbb{N}$, *and let* $f : [m, +\infty) \to \mathbb{R}_{\geq 0}$ *be a monotone, continuously differentiable function which is in* L. *Take the sets* $(A_i)_{i=1}^{\infty} \subseteq \{m, \dots, N\}$, *where* $N > m$ *is an integer, which sets satisfy* $|A_N| \sim N$. *Then*

$$\sum_{\substack{n \in A_N \\ P(n) \leq N^{\theta}}} f(n) \sim \rho(1/\theta) \sum_{n \in A_N} f(n).$$

2 Proof of the proposition

First, we separately prove a lemma, which we are going to use after the application of Abel's identity, to bound the remaining integral term.

Lemma 1 *Let* $m \in \mathbb{N}$, *and let* $f : [m, +\infty) \to \mathbb{R}_{\geq 0}$ *be a monotone, continuously differentiable function which is in* L. *Then*

$$\frac{1}{f(x)} \int_m^x \lfloor t \rfloor |f'(t)| \, dt \in o(x).$$

Proof.

- Assume that $f \in L_1$, and take an arbitrary real $\alpha > 0$. Then because $f'(x) \in \mathcal{O}(x^{\alpha-1})$, there exists a real $c > 0$, and a real x_c, such that for every real $x > x_c$, we have that $|f'(x)| \leq cx^{\alpha-1}$ holds. So the inequality

$$\frac{1}{f(x)} \int_m^x \lfloor t \rfloor |f'(t)| \, dt < \frac{c}{f(x)} \int_m^x t^{\alpha} \, dt \tag{3}$$

holds when $x > \max(m, x_c)$. Because $f(x) \in \omega(x^{-\alpha})$, for every real $\varepsilon > 0$, there exists a real x_ε, such that for every real $x > x_\varepsilon$, we have that $|f(x)| > \varepsilon x^{-\alpha}$ holds. By this, when $x > \max(m, x_c, x_\varepsilon)$, the right hand side of inequality (3) is smaller than

$$\frac{c}{\varepsilon x^{-\alpha}} \int_m^x t^\alpha \, dt < \frac{c}{\varepsilon(\alpha+1)} x^{2\alpha+1} \to \frac{c}{\varepsilon} x$$

as α goes to zero. So for every real $\delta = c/\varepsilon > 0$, there exists a real $x_\delta = \max(m, x_c, x_\varepsilon)$, such that for every real $x > x_\delta$, we have the left hand side of (3) is smaller than δx.

- Assume that $f \in L_2$, and take an arbitrary real $\alpha > 0$. Then because $f(x) \in \Omega(x^{-\alpha})$, there exists a real $c > 0$, and a real x_c, such that for every real $x > x_c$, we have that $|f(x)| \geq c x^{-\alpha}$ holds. So the inequality

$$\frac{1}{f(x)} \int_m^x \lfloor t \rfloor |f'(t)| \, dt \leq \frac{1}{c x^{-\alpha}} \int_m^x \lfloor t \rfloor |f'(t)| \, dt \tag{4}$$

holds when $x > \max(m, x_c)$. Because $f'(x) \in o(x^{\alpha-1})$, for every real $\varepsilon > 0$, there exists a real x_ε, such that for every real $x > x_\varepsilon$, we have that $|f'(x)| < \varepsilon x^{\alpha-1}$ holds. By this, when $x > \max(m, x_c, x_\varepsilon)$, the right hand side of inequality (4) is smaller than

$$\frac{\varepsilon}{c x^{-\alpha}} \int_m^x t^\alpha \, dt < \frac{\varepsilon}{c(\alpha+1)} x^{2\alpha+1} \to \frac{\varepsilon}{c} x$$

as α goes to zero. So for every real $\delta = \varepsilon/c > 0$, there exists a real $x_\delta = \max(m, x_c, x_\varepsilon)$, such that for every real $x > x_\delta$, we have the left hand side of (4) is smaller than δx.

□

Now we are going to give an asymptotic for the sum of our examined function over the sets A_N by using Abel's identity.

Lemma 2 *Let* $m \in \mathbb{N}$*, and let* $f : [m, +\infty) \to \mathbb{R}_{\geq 0}$ *be a monotone, continuously differentiable function which is in* L*. Take the sets* $(A_i)_{i=1}^{\infty} \subseteq \{m, \ldots, N\}$*, where* $N > m$ *is an integer, which sets satisfy* $|A_N| \sim N$*. Then*

$$\sum_{n \in A_N} f(n) \sim N f(N).$$

Proof. First, we split the examined sum as

$$\sum_{n \in A_N} f(n) = \sum_{m \leq n \leq N} f(n) - \sum_{n \in \{m, \ldots, N\} \setminus A_N} f(n). \tag{5}$$

Because f has a continuous derivative on the interval $[m, +\infty)$, we can apply Abel's identity, see theorem 4.2 in section 4.3 of the book of Apostol [1], to get the equality

$$\sum_{m<n\leq N} f(n) = Nf(N) - mf(m) - \int_m^N \lfloor t \rfloor f'(t)\, dt. \tag{6}$$

- Assume that f is monotone increasing. Then

$$\sum_{n\in\{m,\ldots,N\}\setminus A_N} f(n) \leq f(N)(N - m + 1 - |A_N|).$$

  Using this inequality, and equality (6), we get that the left hand side of equality (5) is greater than-, or equal to

$$f(N)\left((1-m)\frac{f(m)}{f(N)} - \frac{1}{f(N)}\int_m^N \lfloor t \rfloor f'(t)\, dt + m - 1 + |A_N|\right)$$

  which, by lemma 1, is greater than-, or equal to $f(N)\left(|A_N| + o(N)\right)$. By this, we have that the limit

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \geq \lim_{N\to+\infty}\left(\frac{|A_N|}{N} + o_N(1)\right) = 1$$

  because $|A_N| \sim N$. Regarding the upper bound of the limit, we have

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \leq \lim_{N\to+\infty} \frac{f(N)\sum_{n\in A_N} 1}{Nf(N)} = \lim_{N\to+\infty} \frac{|A_N|}{N} = 1$$

  because f is monotone increasing, and $|A_N| \sim N$.

- Assume that f is monotone decreasing. Then

$$\sum_{n\in\{m,\ldots,N\}\setminus A_N} f(n) \geq f(N)(N - m + 1 - |A_N|).$$

  Using this inequality, and equality (6), we get that the left hand side of equality (5) is less than-, or equal to

$$f(N)\left((1-m)\frac{f(m)}{f(N)} + \frac{1}{f(N)}\int_m^N \lfloor t \rfloor |f'(t)|\, dt + m - 1 + |A_N|\right)$$

where we could switch the sign of the integral, because f is monotone decreasing, so f' is non-positive on $[m, N]$. By lemma 1, this is less than-, or equal to $f(N)(|A_N| + o(N))$. Based on this, using the same reasoning as in the case when f was monotone increasing, we can show that

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \leq 1$$

holds. Regarding the lower bound of the limit, we have

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \geq \lim_{N\to+\infty} \frac{f(N)\sum_{n\in A_N} 1}{Nf(N)} = \lim_{N\to+\infty} \frac{|A_N|}{N} = 1$$

because f is monotone decreasing, and $|A_N| \sim N$.

□

Proof. (Proposition 1) Fix a smoothness $\theta \in (0, 1)$, and assume that we have a function f, and sets A_N satisfying the requirements mentioned in the proposition. We will show that the limit

$$\lim_{N\to+\infty} \frac{\sum_{\substack{n\in A_N \\ P(n)\leq N^\theta}} f(n)}{\rho(1/\theta)\sum_{n\in A_N} f(n)} \tag{7}$$

is equal to one, separately when f is monotone increasing, and when f is monotone decreasing. Assuming that N is big enough, we can guarantee that A_N is not empty, thus the sums in the numerator and the denominator are not zero.

- Assume that f is monotone increasing. Then the limit (7) is less than-, or equal to

  $$\lim_{N\to+\infty} \frac{f(N)\Psi(N, N^\theta)}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

  because f is monotone increasing, and $A_N \subseteq \{m, \dots, N\}$. Using lemma 2, this is equal to

  $$\frac{1}{\rho(1/\theta)} \lim_{N\to+\infty} \frac{\Psi(N, N^\theta)}{N(1 + o_N(1))} = 1$$

  based on the limit (2). Regarding the lower bound of the limit, first we note that

  $$\sum_{\substack{n\in A_N \\ P(n)\leq N^\theta}} f(n) = \sum_{n\in A_n} f(n) - \sum_{\substack{n\in A_N \\ P(n)>N^\theta}} f(n) \tag{8}$$

where

$$\sum_{\substack{n\in A_N \\ P(n)>N^\theta}} f(n) \le f(N) \sum_{\substack{n\in A_N \\ P(n)>N^\theta}} 1 \le f(N)(N-\Psi(N,N^\theta))$$

because f is monotone increasing, and $A_N \subseteq \{m,\ldots,N\}$. By these, we have that the limit (7) is greater than-, or equal to

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n) - f(N)(N-\Psi(N,N^\theta))}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

where, by using lemma 2, we get

$$\frac{1}{\rho(1/\theta)}\left(1 - \lim_{N\to+\infty}\frac{1}{1+o_N(1)} + \lim_{N\to+\infty}\frac{\Psi(N,N^\theta)}{N(1+o_N(1))}\right) = 1$$

based on the limit (2).

- Assume that f is monotone decreasing. Because

$$\sum_{\substack{n\in A_N \\ P(n)\le N^\theta}} 1 = \sum_{\substack{1\le n\le N \\ P(n)\le N^\theta}} 1 - \sum_{\substack{n\in\{1,\ldots,N\}\setminus A_N \\ P(n)\le N^\theta}} 1 \ge \Psi(N,N^\theta) - N + |A_N|$$

we have that the limit (7) is greater than-, or equal to

$$\lim_{N\to+\infty} \frac{f(N)(\Psi(N,N^\theta) - N + |A_N|)}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

because f is monotone decreasing. Here, by using lemma 2, we get

$$\frac{1}{\rho(1/\theta)}\lim_{N\to+\infty}\left(\frac{\Psi(N,N^\theta)}{N(1+o_N(1))} - \frac{1}{1+o_N(1)} + \frac{|A_N|}{N(1+o_N(1))}\right) = 1$$

based on $|A_N| \sim N$, and on the limit (2). Regarding the upper bound of the limit, because

$$\sum_{\substack{n\in A_N \\ P(n)>N^\theta}} 1 = \sum_{\substack{1\le n\le N \\ P(n)>N^\theta}} 1 - \sum_{\substack{n\in\{1,\ldots,N\}\setminus A_N \\ P(n)>N^\theta}} 1 \ge |A_N| - \Psi(N,N^\theta)$$

we have that the limit (7) is less than-, or equal to

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n) - f(N)(|A_N| - \Psi(N,N^\theta))}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

based on equality (8). Here, by using lemma 2, we get

$$\frac{1}{\rho(1/\theta)}\left(1-\lim_{N\to+\infty}\frac{|A_N|}{N(1+o_N(1))}+\lim_{N\to+\infty}\frac{\Psi(N,N^{\theta})}{N(1+o_N(1))}\right)=1$$

based on $|A_N| \sim N$, and on the limit (2).

□

References

[1] T. M. Apostol, Introduction to Analytic Number Theory, *Springer-Verlag*, (1976).

[2] T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein, Introduction to Algorithms, *The MIT Press*, third edition, (2009).

[3] E. Croot, On a combinatorial method for counting smooth numbers in sets of integers, *J. Number Theory*, **126** (2007), 237–253.

[4] K. Dickman, On the frequency of numbers containing prime factors of a certain relative magnitude, *Ark. Mat. Astr. Fys.*, **22** (1930), 1–14.

[5] D. H. Greene, D. E. Knuth, Mathematics for the Analysis of Algorithms, *Birkhäuser*, third edition, (1990).

[6] A. Hildebrand, G. Tenenbaum, Integers without large prime factors, *J. Théor. Nr. Bordx* **5(2)** (1993), 411–484.

[7] G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory, *AMS*, third edition (2015).

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