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On weak (σ, δ)-rigid rings over Noetherian rings

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Abstract. Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers). Let σ be an endomorphism of R and δ a σ-derivation of R. We recall that a ring R is a weak (σ, δ)-rigid ring if $a(\sigma(a) + \delta(a)) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$ ($N(R)$ is the set of nilpotent elements of R). With this we prove that if R is a Noetherian integral domain which is also an algebra over $\mathbb{Q}$, σ an automorphism of R and δ a σ-derivation of R such that R is a weak (σ, δ)-rigid ring, then $N(R)$ is completely semiprime.

1 Introduction and preliminaries

Throughout this paper R will denote an associative ring with identity $1 \neq 0$, unless otherwise stated. The prime radical of a ring R denoted by $P(R)$ is the intersection of all prime ideals of R. The set of nilpotent elements of R is denoted by $N(R)$. The ring of integers, the field of rational numbers, the field

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of real numbers and the field of complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively, unless otherwise stated.

Krempa [4] introduced σ-rigid rings and proved that if σ is a rigid endomorphism of R, then it is a monomorphism preserving every minimal prime ideal and annihilator in R. Several properties of σ-rigid rings have been studied in [2, 3]. Weak σ-rigid rings were studied by Ouyang [7]. Bhat [1] gave a necessary and sufficient condition for a commutative Noetherian ring to be weak (σ, δ)-rigid ring.

In this article we investigate weak (σ, δ)-rigid rings over Noetherian rings.

Now let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. Recall that $\delta : R \to R$ an additive map such that

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R$$

is called a σ-derivation of R.

Example 1 *Let* F *be a field,* $R = F[x]$ *be the polynomial ring over* F*. Then* $\sigma : R \to R$ *defined as*

$$\sigma(f(x)) = f(-x) \textit{ is an automorphism.}$$

Define $\delta : R \to R$ *by*

$$\delta(f(x)) = f(x) - \sigma(f(x)).$$

Then δ *is a* σ*-derivation of* R*.*

1.1 σ-rigid ring

Recall that in Krempa [4], an endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies that $a = 0$, for all $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R.

Example 2 *Let* $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$*, where* F *is a field. Let* $\sigma : R \to R$ *be defined by*

$$\sigma\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \textit{ for } a, b, c \in F.$$

Then it can be seen that σ *is an endomorphism of* R*.*

Let $0 \neq a \in F$*. Then* $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \sigma\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

But $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence R *is not a* σ*-rigid ring.*

We recall that σ-rigid rings are reduced rings by Hong et. al. [3]. Recall that a ring R is reduced if it has no non-zero nilpotent elements. Observe that reduced rings are abelian.

1.2 Weak σ-rigid ring

Note that as in Ouyang [7], a ring R with an endomorphism σ is called a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.

Example 3 (Example 2.1 of [7]) *Let* σ *be an endomorphism of a ring* R*. Let*

$$A = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in R \right\}$$

be a subring of $T_3(R)$*, the ring of upper triangular matrices over* R*. Now* σ *can be extended to an endomorphism say* $\overline{\sigma}$ *of* A *by* $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$*. Then it can be seen that* A *is a weak* $\overline{\sigma}$*-rigid ring.*

Example 4 *Let* F *be a field and* $R = F(x)$*, the field of rational polynomials in one variable* x *over* F*. Then* $N(R) = \{0\}$*. Let* $\sigma : R \to R$ *be an endomorphism defined by*

$$\sigma(f(x)) = f(0).$$

Then R *is not a weak* σ*-rigid ring. For let* $f(x) = xa$*,* $f(0) = 0$ *and* $f(x)\sigma(f(x)) = xa.0 = 0 \in N(R)$*. But* $0 \neq f(x) \notin N(R)$*.*

Clearly the notion of a weak σ-rigid ring generalizes that of a σ-rigid ring. Also in [5], it has been shown that if R is a weak σ-rigid ring, then $N(R)$ is completely semi-prime where R is a Noetherian ring and σ an automorphism of R.The converse is not true.

Example 5 [6] *Let* F *be a field,* $R = F \times F$ *and* σ *an automorphism of* R *defined by*

$$\sigma((a, b)) = (b, a), \textit{ for } a, b \in F.$$

Then R *is a reduced ring and so* $N(R) = \{0\}$ *is completely semi-prime. But* R *is not a weak* σ*-rigid ring. Since* $(1, 0)\sigma((1, 0)) = (0, 0) \in N(R)$*, but* $(1, 0) \notin N(R)$*.*

1.3 Weak (σ, δ)-rigid rings

We generalize the above mentioned notions by involving σ and δ together as follows:

1.3.1 (σ, δ)-ring

Definition 1 (Definition 7 of [1]) *Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ, δ)-rigid ring if*

$$a(\sigma(a) + \delta(a)) \in P(R) \textit{ implies that } a \in P(R) \textit{ for } a \in R.$$

Definition 2 *Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ, δ)-rigid ring if*

$$a(\sigma(a) + \delta(a)) = 0 \textit{ implies that } a = 0 \textit{ for } a \in R.$$

Example 6 *Let $R = \mathbb{C}$ and $\sigma : R \to R$ be defined by*

$$\sigma(a + ib) = a - ib, \textit{ for all } a, b \in \mathbb{R}.$$

Then σ is an automorphism of R.
Define δ a σ-derivation of R as

$$\delta(z) = z - \sigma(z) \textit{ for } z \in R.$$

i.e., $\delta(a + ib) = a + ib - \sigma(a + ib) = a + ib - (a - ib) = 2ib$.

Let $A = a + ib$. Then $A[\sigma(A) + \delta(A)] = 0$ implies that

$$(a + ib)[\sigma(a + ib) + \delta(a + ib)] = 0$$

i.e. $(a + ib)[(a - ib) + 2ib] = 0$ or $(a + ib)(a + ib) = 0$ which implies that $a = 0, b = 0$. Therefore, $A = a + ib = 0$. Hence R is a (σ, δ)-rigid ring.

1.3.2 Weak (σ, δ)-rigid rings

Definition 3 *Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a weak (σ, δ)-rigid ring if $a(\sigma(a)+\delta(a)) \in N(R)$ implies and is implied by $a \in N(R)$ for $a \in R$.*

Example 7 *Let $R = \mathbb{Z}[\sqrt{2}]$. Then $\sigma : R \to R$ defined as*

$$\sigma(a + b\sqrt{2}) = (a - b\sqrt{2}) \textit{ for } a + b\sqrt{2} \in R$$

is an endomorphism of R.
For any $s \in R$. *Define* $\delta_s : R \to R$ *by*

$$\delta_s(a + b\sqrt{2}) = (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2}) \text{ for } a + b\sqrt{2} \in R.$$

Then δ_s *is a* σ*-derivation of* R. *Here* $N(R) = \{0\}$.
Further,

$$(a + b\sqrt{2})\{\sigma(a + b\sqrt{2}) + \delta_s(a + b\sqrt{2})\} \in N(R)$$

implies that

$$(a + b\sqrt{2})\{(a - b\sqrt{2}) + (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2})\} \in N(R)$$

or

$$(a + b\sqrt{2})\{a - b\sqrt{2} + as + bs\sqrt{2} - sa + sb\sqrt{2}\} \in N(R)$$

i.e.

$$(a + b\sqrt{2})\{a + (2s - 1)b\sqrt{2}\} \in N(R) = \{0\}$$

which gives $a = 0, b = 0$. *Hence* $a + b\sqrt{2} = 0 + 0\sqrt{2} \in N(R)$. *Thus* R *is a weak* (σ, δ)*-rigid ring.*

With this we prove the following:

Theorem A: Let R be Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. Then R a weak (σ, δ)-rigid ring implies that $N(R)$ is completely semi-prime.

The statement is proven in Theorem 1, to be found below.

2 Proof of the main result

We have the following before we prove the main result of this paper:

Recall that an ideal I of a ring R is called completely semi-prime if $a^2 \in I$ implies that $a \in I$ for $a \in R$.

Example 8 *Let* $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. *Then* $P_1 = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ *are prime ideals of* R.

It can be easily seen that P_1, P_2, P_3 *are completely semi-prime ideals.*

Proposition 1 *Let* R *be a ring,* σ *an automorphism of* R *and* δ *a* σ*-derivation of* R*. Then for* $u \neq 0$*,* $\sigma(u) + \delta(u) \neq 0$*.*

Proof. Let $0 \neq u \in R$, we show that $\sigma(u) + \delta(u) \neq 0$. Let for $0 \neq u$, $\sigma(u) + \delta(u) = 0$. This implies that

$$\delta(u) = -\sigma(u), \ \forall 0 \neq u \in R. \tag{1}$$

We know that for

$$0 \neq a, 0 \neq b \in R, \ \delta(ab) = \delta(a)\sigma(b) + a\delta(b).$$

By using equation 1, we have

$$\begin{aligned} -\sigma(ab) &= -\sigma(a)\sigma(b) + a(-\sigma(b)) \\ \Rightarrow \quad -\sigma(ab) &= -[\sigma(a) + a]\sigma(b) \\ \Rightarrow \quad \sigma(a)\sigma(b) &= -[\sigma(a) + a]\sigma(b) \\ \Rightarrow \quad \sigma(a) &= \sigma(a) + a \end{aligned}$$

Therefore, $a = 0$ which is not possible. Hence $\sigma(u) + \delta(u) \neq 0$. □

We now prove the main result of this paper in the form of the following Theorem:

Theorem 1 *Let* R *be a Noetherian integral domain which is also an algebra over* $\mathbb{Q}$*. Let* σ *be an automorphism of* R *and* δ *a* σ*-derivation of* R*. Then* R *a weak* (σ, δ)*-rigid ring implies that* $N(R)$ *is completely semi-prime.*

Proof. Let R be a weak (σ, δ)-rigid ring. Then we will show that $N(R)$ is completely semi-prime. Let $a \in R$. Since R is a weak (σ, δ)-rigid ring. Let

$$\{a(\sigma(a) + \delta(a))\}^2 \in N(R).$$

Then there exists a positive integer n such that $[a^2(\sigma(a) + \delta(a))^2]^n = 0$ which implies that $a^{2n}(\sigma(a) + \delta(a))^{2n} = 0$. But by Proposition (1), $\sigma(a) + \delta(a) \neq 0$. Hence $a^{2n} = 0$ which implies that $a^n = 0$, because R is an integral domain. Therefore, $a^n(\sigma(a) + \delta(a))^n = 0$ or $\{a(\sigma(a) + \delta(a))\}^n = 0$. Thus $a(\sigma(a) + \delta(a)) \in N(R)$. Hence $N(R)$ is completely semi-prime. □

The converse is not true.

Example 9 *Let* F *be a field. Let* $R = F \times F$ *and* σ *an automorphism of* R *defined by*

$$\sigma((a, b)) = (b, a) \text{ for } a, b \in F.$$

Then R *is a reduced ring and so* $N(R) = \{0\}$ *and therefore, it is completely semi-prime. Let* $r \in F$. *Define* $\delta_r : R \to R$ *by*

$$\delta_r((a, b)) = (a, b)r - r\sigma((a, b)) \text{ for } a, b \in F.$$

Then δ_r *is a* σ*-derivation of* R. *Also* R *is not a weak* (σ, δ)*-rigid ring. For take* $(1, -1) \in R$, $r = \frac{1}{2}$. *Then*

$$\begin{aligned} &\delta_r((1, -1)) = (1, -1)\tfrac{1}{2} - \tfrac{1}{2}\sigma((1, -1)) \\ &= (1, -1) \text{ and } (1, -1)[\sigma(1, -1) + \delta_r(1, -1)] \\ &= (1, -1)[(-1, 1) + (1, -1)] \\ &= (1, -1)(0, 0) = (0, 0) \in N(R). \end{aligned}$$

But $(1, -1) \notin N(R)$.

Corollary 1 *Let* R *be a commutative Noetherian, integral domain which is also an algebra over* $\mathbb{Q}$. *Let* σ *be an automorphism of* R *and* δ *a* σ*-derivation of* R. *Then* R *a weak* (σ, δ)*-rigid ring implies that* $N(R)$ *is completely semi-prime.*

Also we note that if R is a (σ, δ)-rigid ring then it is a weak (σ, δ)-rigid ring, but the converse need not be true as in the following example:

Example 10 *Let* σ *be an endomorphism of a ring* R *and* δ *a* σ*-derivation of* R. *Let* R *be a* (σ, δ)*-rigid ring. Then*

$$R_3 = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} : a, b, c, d \in R \right\}$$

is a subring of $T_3(R)$. *The endomorphism* σ *of* R *can be extended to the endomorphism* $\overline{\sigma} : R_3 \to R_3$ *defined by* $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ *and* δ *can be extended to* $\overline{\delta} : R_3 \to R_3$ *by* $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. *Let*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \left\{ \overline{\sigma} \left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \right) + \overline{\delta} \left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \right) \right\} \in N(R).$$

Then there is some positive integer n *such that*

$$\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \left\{ \begin{bmatrix} \sigma(a) & \sigma(b) & \sigma(c) \\ 0 & \sigma(a) & \sigma(d) \\ 0 & 0 & \sigma(a) \end{bmatrix} + \begin{bmatrix} \delta(a) & \delta(b) & \delta(c) \\ 0 & \delta(a) & \delta(d) \\ 0 & 0 & \delta(a) \end{bmatrix} \right\} \right)^n = 0$$

which implies that

$$\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}\left\{\begin{bmatrix} \sigma(a)+\delta(a) & \sigma(b)+\delta(b) & \sigma(c)+\delta(c) \\ 0 & \sigma(a)+\delta(a) & \sigma(d)+\delta(d) \\ 0 & 0 & \sigma(a)+\delta(a) \end{bmatrix}\right\}\right)^n = 0$$

or

$$\begin{bmatrix} a(\sigma(a)+\delta(a)) & a(\sigma(b)+\delta(b))+b(\sigma(a)+\delta(a)) & a(\sigma(c)+\delta(c))+b(\sigma(d)+\delta(d))+c(\sigma(a)+\delta(a)) \\ 0 & a(\sigma(a)+\delta(a)) & a(\sigma(d)+\delta(d))+d(\sigma(a)+\delta(a)) \\ 0 & 0 & a(\sigma(a)+\delta(a)) \end{bmatrix}^n$$

$= 0$, *which gives*

$$a(\sigma(a)+\delta(a)) \in N(R).$$

Since R *is reduced, we have*

$$a(\sigma(a)+\delta(a)) = 0$$

which implies that $a = 0$, *since* R *is a* (σ, δ)*-rigid ring. Hence*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \in N(R).$$

Conversely, assume that

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \in N(R).$$

Then there is some positive integer n *such that*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & * & * \\ 0 & a^n & * \\ 0 & 0 & a^n \end{bmatrix} = 0$$

which implies that $a = 0$, *because* R *is reduced (Here* $*$ *are non-zero terms involving summation of powers of some or all of a, b, c, d). So*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}\left\{\overline{\sigma}\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}\right) + \overline{\delta}\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}\right)\right\}$$

$$= \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 & \sigma(b) & \sigma(c) \\ 0 & 0 & \sigma(d) \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta(b) & \delta(c) \\ 0 & 0 & \delta(d) \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 & \sigma(b)+\delta(b) & \sigma(c)+\delta(c) \\ 0 & 0 & \sigma(d)+\delta(d) \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 0 & b(\sigma(d)+\delta(d)) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in N(R).$$

Therefore, R_3 *is a weak* $(\overline{\sigma}, \overline{\delta})$*-rigid ring. Also since* R_3 *is not reduced,* R_3 *is not a* $(\overline{\sigma}, \overline{\delta})$*-rigid ring.*

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Assessing the effect size of users' consciousness for computer networks vulnerability

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Abstract. In this paper the conditions and the findings of a simulation study is presented for assessing the effect size of users' consciousness to the computer network vulnerability in risky cyber attack situations at a certain business. First a simple model is set up to classify the groups of users according to their skills and awareness then probabilities are assigned to each class describing the likelihood of committing dangerous reactions in case of a cyber attack. To quantify the level of network vulnerability a metric developed in a former work is used. This metric shows the approximate probability of an infection at a given business with well specified parameters according to its location, the type of the attack, the protections used at the business etc. The findings mirror back the expected tendencies namely if the number of conscious user is on the

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rise the "relative improvement of the cyber security" is increasing. The tendencies in the change of this relative improvement are established, different graphs and curves are constructed to give an overall view for the influence of the different parameters. In addition to these general conclusions assessments are made for the magnitude and for the range of the relative cyber security improvement. An interesting findings that even in the case of small differences in skills making the users more conscious in their reactions can significantly enhance the level of cyber security at a business.

1 Introduction

Assessing the extent of vulnerability of a net of computers against outer cyber threats is of prime interest for both the IT experts sector and the businesses using computer networks. Most of the IT solutions concentrate on different hardware and software protections against the threats and little attention is paid on the effect of the users' behavior during their daily routine handling potentially risky situations. Opening potentially dangerous websites, clicking on links in emails from unknown source, downloading files to the computers are typical "user tricks" which may raise the level of risk of infections.

Making the users more conscious in their computer usage is an evidential tool for enhancing the cyber security of a business. However to give some measure for the effectiveness of these kind of efforts (trainings for employees, incentives, penalties, etc.) is essential for those offering these services and for the managements of the businesses as well.

In this paper the findings of computer simulation studies are presented where the users at a business are categorized according to their everyday computer usage. The three categories (Naive, Typical, and Conscious) encompasses the different types of user groups mainly reflecting their attitudes and behaviors in risky cyber threat situations.

To make the influence of the different groups sensible the $\mathfrak{p}_s$ metric for measuring cyber vulnerability developed in [4], [3], [2] and [1] is used. This $\mathfrak{p}_s$ metric is the probability that at least one cyber malware (virus) can successfully go through the IT protections and infects a certain net of computers.

The features of the given net and the prevailing threats at the location investigated are precisely characterized by different matrices describing the presence or absence of the different types of protections, the relative frequencies and the level of danger of different viruses and the level of danger of the different user tricks (Simple, Moderate, Complex).

This p_s metric holds the intrinsic characteristic of being monotonously increasing as the number of viruses, devices or users increases. In this study to ensure getting such a p_s metric which reflects the real differences of vulnerability attributable exclusively to the different categories of users' consciousness (awareness) and excludes the effect of the business size the number of users, devices, and threats are kept constant while the proportions of the different types of users are changed.

Besides these kept-constant parameters other input values like probabilities of occurrences of certain threats or probabilities of applying certain user tricks, etc. were also temporarily fixed in the analysis (at the values which are typical for the majority of businesses) but later the sensitivity of the p_s value against these varying input values is demonstrated.

The final goal was to illustrate the extent of changes in the value of p_s when the level of consciousness at a business increases due to some measures introduced by the management. The increase of consciousness is embodied in the increase of the proportion of Conscious users reevaluating some users' status from Typical or Naive to Conscious or in the increase of the proportion of Typical users reevaluating some users' status from Naive to Typical.

The direct relationship between the p_s value and some specific business financial indicator is not investigated here.

2 The model for users classification

According to their consciousness there are three distinguished class of users

- *Naive* users,
- *Typical* users,
- *Conscious* users.

As the names of the categories suggest Naive users are assumed to commit dangerous actions even in cases requiring very simple user tricks while Conscious users are victims only of threats requiring more sophisticated user tricks.

Let v be

$$\begin{bmatrix} c \\ t \\ n \end{bmatrix}$$

the vector where c (t, n reps.) is the number of the Conscious (Typical, Naive resp.) users. Let r be the number of users. Observe $c + t + n = r$.

There are three distinguished type of user tricks

- *Simple* user trick,
- *Moderate* user trick,
- *Complex* user trick.

The Simple (Complex resp.) user trick is the easiest (complicated resp.) trick for the threat to attack a given device.

Let P_{skills} be the

	Simple	Moderate	Complex
Conscious	$\beta_{C,1}$	$\beta_{C,2}$	$\beta_{C,3}$
Typical	$\beta_{T,1}$	$\beta_{T,2}$	$\beta_{T,3}$
Naive	$\beta_{N,1}$	$\beta_{N,2}$	$\beta_{N,3}$

3×3 matrix. The real number $\beta_{C,1}$ ($\beta_{C,2}$, $\beta_{C,3}$ resp.) is the probability that a Conscious user uses a Simple (Moderate, Complex resp.) user trick. The real number $\beta_{T,1}$ ($\beta_{T,2}$, $\beta_{T,3}$ resp.) is the probability that a Typical user uses a Simple (Moderate, Complex resp.) user trick. The real number $\beta_{N,1}$ ($\beta_{N,2}$, $\beta_{N,3}$ resp.) is the probability that a Naive user uses a Simple (Moderate, Complex resp.) user trick. It is assumed that $\beta_{i,1} < \beta_{i,2} < \beta_{i,3}$ and $\beta_{C,j} < \beta_{T,j} < \beta_{N,j}$ for $i = C, T, N$ and $j = 1, 2, 3$.

Combining ν and P_{skills} let $P_{user-usertrick}$ be the

	Simple	Moderate	Comlex
u_1	$\beta_{C,1}$	$\beta_{C,2}$	$\beta_{C,3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_c	$\beta_{C,1}$	$\beta_{C,2}$	$\beta_{C,3}$
u_{c+1}	$\beta_{T,1}$	$\beta_{T,2}$	$\beta_{T,3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{c+t}	$\beta_{T,1}$	$\beta_{T,2}$	$\beta_{T,3}$
u_{c+t+1}	$\beta_{N,1}$	$\beta_{N,2}$	$\beta_{N,3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{c+t+n}	$\beta_{N,1}$	$\beta_{N,2}$	$\beta_{N,3}$

$r \times 3$ matrix. Here $u_1, \ldots, u_c$ denote the users belonging to the group of Conscious users, $u_{c+1}, \ldots, u_{c+t}$ denote the users in the Typical group and $u_{c+t+1}, \ldots, u_{c+t+n}$ stand for the users in the Naive group.

In this paper this $P_{user-usertrick}$ matrix is the main tool to study the effect of users' behaviour at a business against different cyber threats. Changing the values of c, t, n within the fixed value of r ($r = c + t + n$) or changing the proportion (distribution) of probabilities in the rows/columns of this matrix enables us to study different situations and give an overview about the magnitude of the effect size of users' consciousness to the cyber vulnerability.

3 The p_s probability

In [2] the probability of infection p_s was introduced which is the probability that the investigated landscape will be infected by at least one malware. This can be calculated in the following form

$$p_s = 1 - \prod_{t=1,\dots,k;u=1,\dots,r;d=1,\dots,m} (1 - p_{user}(t,u) \cdot p_{device}(t,d) \cdot p_{prev}(t)) \qquad (1)$$

for any $u \in U$, $t \in T$ and $d \in D$, where U (T and D resp.) symbolizes the set of users (threats and devices resp.) at the landscape investigated. In (1) $p_{user}(t,u)$ is the probability that the threat t infects the landscape using at least one usertrick through the user u. In (1) $p_{device}(t,d)$ is the probability of a successful attempts of the threat t through any protection protecting the device d. In (1) $p_{prev}(t)$ is the probability that an attack is in the form of the threat t.

In [2] and [1] it was shown how these probabilities can be computed using several parameters describing the present state of the investigated landscape, the prevailing threats, the devices, the state of protections etc.

Since the number of these parameters influencing the value of p_s is numerous it is not easy to see general tendencies due to some selected specific parameters without keeping the others at constant values.

In this study where the aim was to get information about the effect of users' consciousness the parameter values referring to the other features of the investigated business have been fixed. Hence two types of parameters have been distinguished:

- kept-constant parameters and
- study parameters.

The kept-constant parameters are those which are not varied in the study at all. They can simply be regarded as those specifying the basic unit of comparison. For example the total number of users is fixed to 100 since this parameter can be regarded as the size of the users' sample taken from the population of all users at a given business.

The study parameters are being varied in the study. Since they are still rather numerous it was practical to temporarily select and fix them at some typical real world values (called "typical study values") and investigate the sensitivity of p_s to the deviation from these typical study values later.

4 The kept-constant parameters and their values in the study

The values of the kept-constant parameters:

- the number of malwares: $k = 10$,
- the number of users: $r = 100$,
- the number of devices: $m = 10$,
- the number of protections: $n = 10$,
- the number of user tricks: $i = 3$,
- the number of groups of user skills: $s = 3$,
- the probability that an attacker will use a particular threat or class of threats against the enterprise: $P_{prev} = \begin{bmatrix} 1/k, & 1/k, & \dots & ,1/k \end{bmatrix}$,
- the $Z_{device-elements}$ matrix which describes that in this situation each virus can work on each device:

$$Z_{device-elements} = \begin{array}{c|ccc} & 1 & \dots & k \\ \hline 1 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ m & 1 & \dots & 1 \end{array},$$

- the $Z_{device-prot-install}$ matrix which describes that in this situation each protection is installed on each device:

$$Z_{device-prot-install} = \begin{array}{c|ccc} & 1 & \dots & n \\ \hline 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ m & 0 & \dots & 0 \end{array},$$

- the P_{prot} matrix which describes that in this situation the probability of a successful attempt of a given threat through at a given protection:

$$P_{prot} = \begin{array}{c|ccc} & 1 & \dots & n \\ \hline 1 & 1/(nk) & \dots & 1/(nk) \\ \vdots & \vdots & \dots & \vdots \\ k & 1/(nk) & \dots & 1/(nk) \end{array}.$$

5 The study parameters and their "typical study values"

The "typical study values":

- The distribution of probabilities in the P_{skills} matrix:

$$P_{skills} = \begin{array}{l|ccc} & \text{Simple} & \text{Moderate} & \text{Complex} \\ \hline \text{Conscious} & p_0 & 3p_0 & 6p_0 \\ \text{Typical} & 3p_0 & 9p_0 & 18p_0 \\ \text{Naive} & 6p_0 & 18p_0 & 36p_0 \end{array}, \text{ where } p_0 = 0.0001.$$

- The distribution of probabilities in the $P_{usertrick}$ matrix:

$$P_{usertrick} = \begin{array}{c|ccc} & \text{Simple} & \text{Moderate} & \text{Complex} \\ \hline t_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ \vdots & \vdots & \vdots & \vdots \\ t_k & \alpha_1 & \alpha_2 & \alpha_3 \end{array}, \text{ where } \begin{array}{l} \alpha_1 = 0.6, \\ \alpha_2 = 0.3, \\ \alpha_3 = 0.1. \end{array}$$

Here $P_{usertrick}$ is a $k \times 3$ matrix. The real number α_1 (α_2, α_3 resp.) is the probability that a threat uses a Simple (Moderate, Complex resp.) user trick. The integer number k denotes the total number of threats involved in the study.

The fact that each row of the $P_{usertrick}$ matrix has the same values of α_1, α_2 and α_3 tacitly assumes that each virus behaves similarly that is all viruses involved in the analysis belong to the same group of viruses with respect to their user tricks required to activate them. Obviously it does not hold for all groups of viruses so later the effect of differently distributed α probabilities will be investigated.

6 The effect of users' consciousness for the p_s value in the case of "typical study values"

To illustrate the magnitude of the effect size of users' consciousness extreme situations have been analyzed where first all users were assumed to belong to one specific class of consciousness ("Original class") and gradually each user is trained to step up into a "higher" class of consciousness ("Improved class"). Three extreme versions of this "class change" are detailed:

- from Naive to Typical,
- from Naive to Conscious,
- from Typical to Conscious.

Various graphs have been constructed to visualize the effect size.

On the first kind of graphs the change of the absolute value of p_s probability is depicted as the function of the number of "reevaluated" users. This probability is denoted by $p(x)$ where x refers to the number of "reevaluated" users. (Sometimes they are called to "reeducated" users.) Accordingly $r - x$ refers to the number of users still in the "Original class". If $r = 100$, $p(0)$ refers to the case when all users are in their "Original class" while $p(100)$ indicates the situation when all users have been "reevaluated".

On the second kind of graphs the $\Delta(x)$ function, the relative change of the p_s probability is illustrated

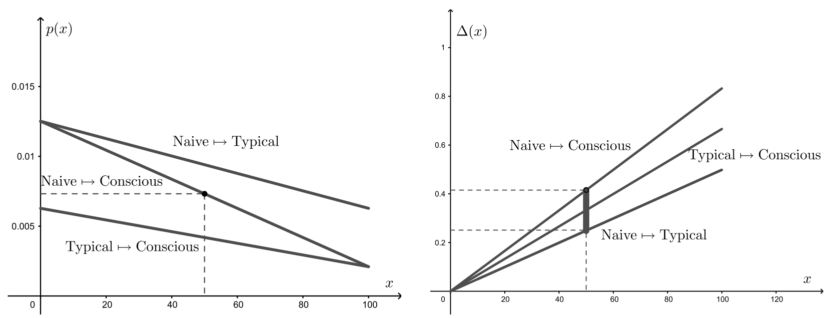
$$\Delta(x) = \frac{p(0) - p(x)}{p(0)}.$$

Since the absolute value of p_s is very much dependent on the different "class change" situations more practical to calculate the ratio of the probability change to the p_s value in the "Original class". This change can be interpreted as the "relative improvement of the defence".

From business point of view this relative improvement can be the basis of any management measures for the sake of improving cyber security through the enhancement of users' consciousness.

In Fig. 1(a) the $p(x)$ curves are shown for the three extreme "class change" situations. The changes of $p(x)$ are almost linear in all three cases, naturally the slope of the Naive $\mapsto$ Consious line is the steepest one.

On the Naive $\mapsto$ Consious curve in Fig. 1(a) it can be seen that the initial probability of the infection is $p(0) = 0.01252$ and if 50 Naive users are reeducated to Conscious users, then the probability of the infection is $p(50) = 0.00732$.



(a) The $p(x)$ curve (b) The $\Delta(x)$ curve

Figure 1: The "typical study values".

The $\Delta(x)$ curves can be constructed from the corresponding $p(x)$ curves hence for all three "class change" situations these curves also show almost perfect linearity. If 50 Naive users are reeducated to Conscious users, then the change of the defence is $p(0) - p(50) = 0.01252 - 0.00732 = 0.0052$ and the normalized change of defence with respect to the initial probability of the infection is $\Delta(50) = (p(0) - p(50))/p(0) = 0.41511$ which can be seen in Fig. 1(b).

These $\Delta(x)$ curves can be regarded as the most important findings of the simulation studies. Even for those who are not very familiar with the issues of cyber security the magnitude of the improvement can be convincing. Seeing the different extreme situations of "class change" one can find that significant improvement can be reached through the enhancement of the users' consciousness.

The range of this relative improvement for a specific x value can be assessed as the difference of the $\Delta(x)$ values for the Naive $\mapsto$ Conscious and the Naive $\mapsto$ Typical "class change" situations. Both the magnitude and the range of the improvement is monotonously increasing as the number of the "reevaluated" users is increasing. For example the relative improvement of the defence can vary from about 25 to 40 percent when half of the users' status has been changed. In Fig. 1(b) this range is indicated with a thick solid line.

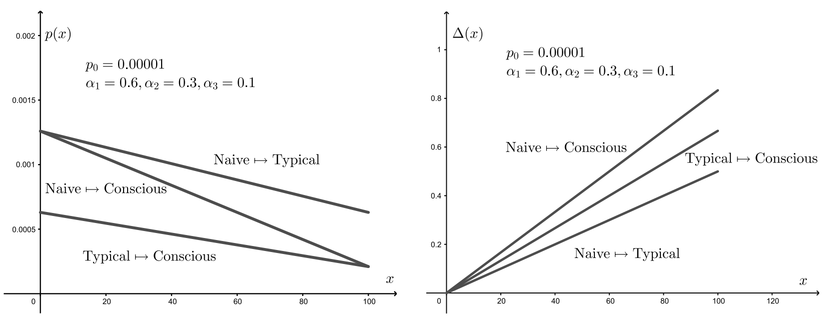
7 The sensitivity of the relative cyber security improvement to the deviations from the "typical study values"

In the remaining sections the sensitivity of the relative security improvement is investigated. As it was stated earlier the simulation studies were elaborated for those "typical study values" of the study parameters which are believed to be characteristic for real world average size businesses in every day cyber risk situations.

However it is worth to check whether slight or moderate deviations from these study values results in basically different conclusions or the findings are rather insensitive to these deviations.

7.1 Varying p_0

In the Fig. 2(a) - Fig. 4(b) the influence of the deviation from the $p_0 = 0.0001$ study value is shown. The range of the p_0 values goes from 0.00001 to 0.01.



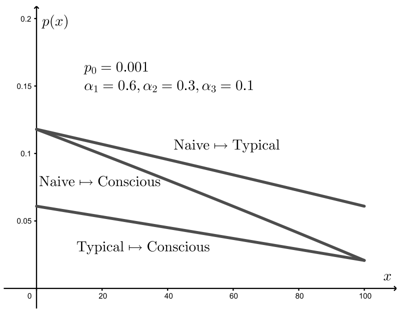
(a) The $p(x)$ curve

(b) The $\Delta(x)$ curve

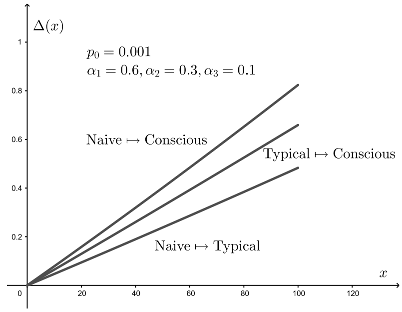
Figure 2: The value $p_0 = 0.00001$.

The magnitude of the $p(x)$ probabilities varies and the shape of the $p(x)$ and $\Delta(x)$ curves are slightly different from the almost linear "typical study values" curves.

With the increase of the p_0 probability the $\Delta(x)$ curves deviates from the linear relationships and tend to be more "exponential-like". At the same time the $\Delta(x)$ values for a specific x value are typically less then those for the typical $p_0 = 0.0001$ study value.

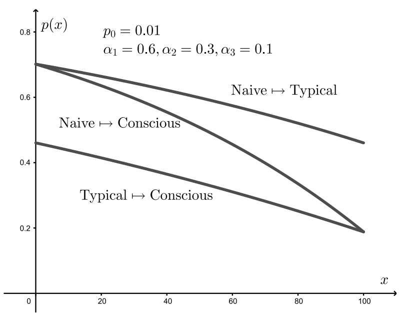


(a) The $p(x)$ curve

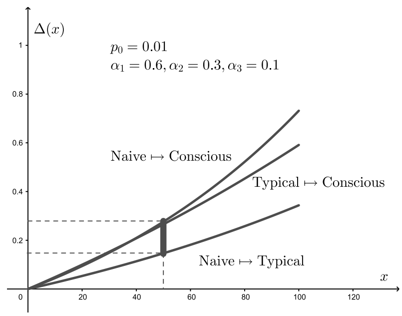


(b) The $\Delta(x)$ curve

Figure 3: The value $p_0 = 0.001$.



(a) The $p(x)$ curve



(b) The $\Delta(x)$ curve

Figure 4: The value $p_0 = 0.01$.

Besides these slight differences one can see that the range of the relative cyber security improvement still remain similar to the previously investigated cases. For example for the $p_0 = 0.01$ value the range of $\Delta(x)$ is somewhere between 15 and 30 percent if half of the users' status has been "reevaluated". In Fig. 4(b) this range is indicated with a thick solid line.

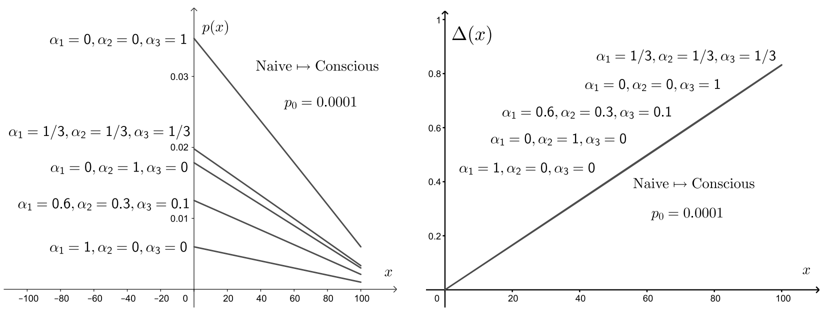
7.2 Varying α_1, α_2 and α_3

In the Fig. 5(a) - Fig. 5(b) the influence of the deviation from the $\alpha_1 = 0.6$, $\alpha_2 = 0.3$, $\alpha_3 = 0.1$ study values is shown.

The results are presented for the following sets of deviated α values:

- $\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 1,$
- $\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = 0,$
- $\alpha_1 = 1,\ \alpha_2 = 0,\ \alpha_3 = 0,$
- $\alpha_1 = 1/3,\ \alpha_2 = 1/3,\ \alpha_3 = 1/3.$

These sets of α values can be regarded as the representations of different virus groups requiring different user tricks to activate them.



(a) The $p(x)$ curve

(b) The $\Delta(x)$ curve

Figure 5: Varying α_1, α_2 and α_3.

The $\Delta(x)$ curves practically coincide with each other convincingly demonstrating the insensitivity of the $\Delta(x)$ function to the deviations from the typical α_1, α_2, α_3 study values.

7.3 Varying the ratio of the columns of P_{skills}

To check the sensitivity of the relative improvement to the ratio of the values in the columns of the P_{skills} matrix it is more convenient to rewrite the matrix into the form:

$$P_{\text{skills}} = \begin{array}{l|ccc} & \text{Simple} & \text{Moderate} & \text{Complex} \\ \hline \text{Conscious} & \gamma_1 p_0 & \gamma_2 p_0 & \gamma_3 p_0 \\ \text{Typical} & * & * & * \\ \text{Naive} & 6\gamma_1 p_0 & 6\gamma_2 p_0 & 6\gamma_3 p_0 \end{array}$$

Using these notations it is easy to express that the simulation studies covered the following sets of γ parameters:

- $\gamma_1 = 6$, $\gamma_2 = 3$, $\gamma_3 = 1$,
- $\gamma_1 = 1$, $\gamma_2 = 3$, $\gamma_3 = 6$,
- $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$.

Having investigated all three cases one can establish that the $\Delta(x)$ function is entirely insensitive to the different sets of γ values. The details are not shown here.

7.4 Varying the ratio of the rows of P_{skills}

To check the sensitivity of the relative improvement to the ratio of the values in the rows of the P_{skills} matrix it is more convenient to rewrite the matrix into the form:

$$P_{\text{skills}} = \begin{array}{l|ccc} & \text{Simple} & \text{Moderate} & \text{Complex} \\ \hline \text{Conscious} & p_0 & 3p_0 & 6p_0 \\ \text{Typical} & * & * & * \\ \text{Naive} & \delta p_0 & 3\delta p_0 & 6\delta p_0 \end{array}$$

Using these notations it is easy to express that the simulation studies covered the following set of δ values: $\delta = 1, 2, 4, 6, 12$. In Fig. 6 the $p(x)$ curves are shown for the "Naive-Conscious" "class change" situation for this set of δ values.

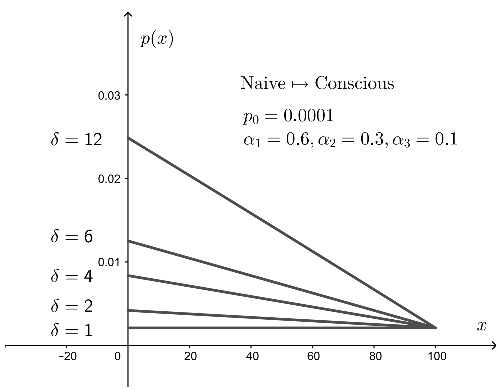
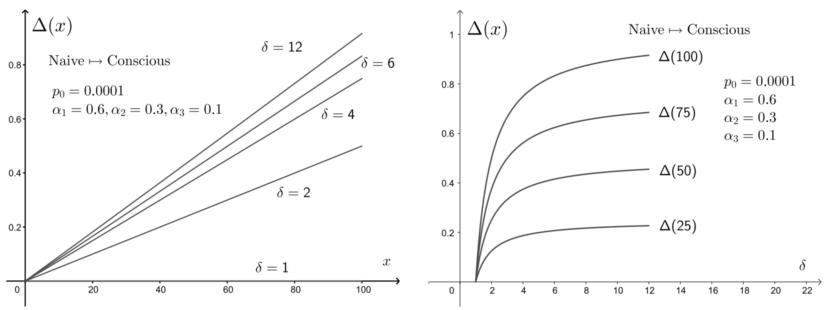


Figure 6: The $p(x)$ curve.

This sensitivity study when the effect of the varying δ value is being investigated is rather special. Since one specific δ value represents the differences between the Conscious and the Naive users in their skills (more precisely the ratio of the corresponding probabilities), it is straightforward that the effect of "reeducation" is larger if this difference is larger. If this difference is zero (the δ value is 1) the "reeducation" is obviously useless resulting in a $\Delta(x) = 0$ value.

Hence this sensitivity study is mainly for learning the form of the curves describing the relationship between the number of "reeducated" users and the relative cyber security improvement and also learning the relationship between the difference of skills and the relative cyber security improvement rather than simply establishing the fact of the existence of this sensitivity for the varying δ values.

In Fig. 7(a) the almost perfect linear association can be established between the number of "reeducated" users (x) and the relative cyber security improvement ($\Delta(x)$ values) for all δ values.



(a) The $\Delta(x)$ curve.

(b) The curves $\Delta(25)$, $\Delta(50)$, $\Delta(75)$, $\Delta(100)$

Figure 7: Varying δ.

In contrast the relationship between the difference in skills (δ) and the relative cyber security improvement ($\Delta(x)$) is far from linear. All curves for the different selected x values show steep increase in the relative improvement as the value of δ starts to depart from its initial value of 1 but later the slopes of the curves are becoming smaller and smaller seemingly tending to a limit value of $\Delta(x)$.

From practical point of view the steep starting phase of the curves can be of prime interest. It means that even in case of small differences between the user groups' skills it may be worth to take measures for enhancing the users awareness since significant increase may happen in the level of cyber security.

8 Conclusion

In this paper the conditions and the findings of a simulation study was presented for assessing the effect size of users' consciousness to the computer network vulnerability in risky cyber attack situations at a certain business.

First a simple model was set up to classify the groups of users according to their skills and awareness then probabilities were assigned to each class describing the likelihood of committing dangerous reactions in case of a cyber attack.

To quantify the level of danger a metric developed in a former work was used. This $\mathfrak{p}_s$ metric shows the approximate probability of an infection at a given business with well specified parameters according to its location, the type of the attack, the protections used at the business etc.

To be able to see the tendencies in vulnerability exclusively attributable to the users consciousness the set of the numerous parameters were grouped to kept-constant and study parameters.

First the "typical study values" of the study parameters were used in the simulations then the sensitivity of the findings was investigated to the deviations from these typical values.

On one hand the findings mirrored back the straightforward and expected tendencies namely either the number of "reeducated" user is increasing or the surmounted difference of the users' groups in their skills is increasing the "relative improvement of the cyber security" is increasing.

On the other hand the tendencies in the change of this relative improvement have been established, different graphs and curves have been constructed to give an overall view for the influence of the different parameters.

In addition to these general conclusions assessments were made for the magnitude and for the range of the relative cyber security improvement. Slight sensitivity was experienced to the departures from the typical study values. It was shown that even in the case of small differences in skills making the users more conscious in their reactions can significantly enhance the level of cyber security at a business.

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Trapezoid type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation

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Abstract. In this paper we establish some trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the g-*mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

1 Introduction

Let (a, b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). Following [18,

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p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ by

$$I_{a+,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{[g(x) - g(t)]^{1-\alpha}}, \quad a < x \le b$$

and

$$I_{b-,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{[g(t) - g(x)]^{1-\alpha}}, \quad a \le x < b.$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x - t)^{1-\alpha}}, \quad a < x \le b$$

and

$$J_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t) dt}{(t - x)^{1-\alpha}}, \quad a \le x < b,$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [18, p. 111]

$$H_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \le a < x \le b$$

and

$$H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \le a < x < b.$$

One can consider the function $g(t) = -t^{-1}$ and define the "*Harmonic fractional integrals*" by

$$R_{a+}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x - t)^{1-\alpha} t^{\alpha+1}}, \quad 0 \le a < x \le b$$

and

$$R_{b-}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t) dt}{(t - x)^{1-\alpha} t^{\alpha+1}}, \quad 0 \le a < x < b.$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the "β-*Exponential fractional integrals*"

$$E_{a+,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x \frac{\exp(\beta t) f(t) dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}}, \quad a < x \le b$$

and

$$E_{b-,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)}\int_x^b \frac{\exp(\beta t) f(t)\,dt}{[\exp(\beta t)-\exp(\beta x)]^{1-\alpha}},\ a \le x < b.$$

In the recent paper [14] we obtained the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1 *Let* $f : [a, b] \to \mathbb{C}$ *be a function of bounded variation on* $[a, b]$ *and* g *be a strictly increasing function on* (a, b), *having a continuous derivative* g' *on* (a, b). *For any* $x \in (a, b)$ *we have the inequalities*

$$\begin{aligned}
&\left| I_{a+,g}^{\alpha}f(x) + I_{b-,g}^{\alpha}f(x) - \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right) f(x)\right| \\
&\qquad\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\bigvee_t^x(f)\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)\bigvee_x^t(f)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_a^x(f) + [g(b)-g(x)]^{\alpha}\bigvee_x^b(f)\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}\begin{cases}
\left[\frac{1}{2}(g(b)-g(a)) + \left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\
\left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\text{with } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \\
\left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right],
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\left| I_{x-,g}^{\alpha}f(a) + I_{x+,g}^{\alpha}f(b) - \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right) f(x)\right| \\
&\qquad\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\bigvee_t^x(f)\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)\bigvee_x^t(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right] \\
&\qquad\le \frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_a^x(f) + [g(b)-g(x)]^{\alpha}\bigvee_x^b(f)\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}\begin{cases}
\left[\frac{1}{2}(g(b)-g(a)) + \left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\
\left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\text{with } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \\
\left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right].
\end{cases}
\end{aligned}$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the g-*mean of two numbers* $a, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

The following particular case for g-*mean* is of interest [14].

Corollary 1 *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
&\left|I^{\alpha}_{a+,g}f(M_g(a,b))+I^{\alpha}_{b-,g}f(M_g(a,b))-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)\bigvee_{M_g(a,b)}^t(f)\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right]\\
&\leq\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f);
\end{aligned}$$

and

$$\begin{aligned}
&\left|I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)\bigvee_x^t(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]\\
&\leq\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f).
\end{aligned}$$

Remark 1 *If we take in Theorem 1* $x = \frac{a+b}{2}$, *then we obtain similar mid-point inequalities, however the details are not presented here. Some applications for the Hadamard fractional integrals are also provided in* [14].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [16]-[27] and the references therein.

Motivated by the above results, in this paper we establish some trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the g-*mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

2 Some identities

We have:

Lemma 1 *Let* $f : [a, b] \to \mathbb{C}$ *be Lebesgue integrable on* $[a, b]$, g *be a strictly increasing function on* (a, b), *having a continuous derivative* g' *on* (a, b) *and* λ, μ *some complex parameters:*

(i) *For any* $x \in (a, b)$ *we have the representation*

$$I_{a+,g}^{\alpha}f(x) + I_{b-,g}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha+1)} \left(\lambda [g(x) - g(a)]^{\alpha} + \mu [g(b) - g(x)]^{\alpha}\right) + \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)[f(t) - \lambda]\,dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t) - \mu]\,dt}{[g(t) - g(x)]^{1-\alpha}}\right] \tag{1}$$

and

$$I_{x-,g}^{\alpha}f(a) + I_{x+,g}^{\alpha}f(b) = \frac{1}{\Gamma(\alpha+1)} \left(\lambda [g(x) - g(a)]^{\alpha} + \mu [g(b) - g(x)]^{\alpha}\right) + \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)[f(t) - \lambda]\,dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t) - \mu]\,dt}{[g(b) - g(t)]^{1-\alpha}}\right]. \tag{2}$$

(ii) *We have*

$$\frac{I_{b-,g}^{\alpha}f(a) + I_{a+,g}^{\alpha}f(b)}{2} = \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \frac{\lambda+\mu}{2} + \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t)[f(t) - \lambda]\,dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)[f(t) - \mu]\,dt}{[g(t) - g(a)]^{1-\alpha}}\right]. \tag{3}$$

Proof. (i) We observe that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,[f(t)-\lambda]\,dt}{[g(x)-g(t)]^{1-\alpha}} \\
&= I^{\alpha}_{a+,g}f(x) - \lambda\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,dt}{[g(x)-g(t)]^{1-\alpha}} \\
&= I^{\alpha}_{a+,g}f(x) - \frac{[g(x)-g(a)]^{\alpha}}{\alpha\Gamma(\alpha)}\lambda = I^{\alpha}_{a+,g}f(x) - \frac{[g(x)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\lambda
\end{aligned} \tag{4}$$

for $a < x \le b$ and, similarly,

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)\,[f(t)-\mu]\,dt}{[g(t)-g(x)]^{1-\alpha}} = I^{\alpha}_{b-,g}f(x) - \frac{[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}\mu \tag{5}$$

for $a \le x < b$.

If $x \in (a, b)$, then by adding the equalities (4) and (5) we get the representation (1).

By the definition of fractional integrals we have

$$I^{\alpha}_{x+,g}f(b) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)\,f(t)\,dt}{[g(b)-g(t)]^{1-\alpha}},\ \ a \le x < b$$

and

$$I^{\alpha}_{x-,g}f(a) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,f(t)\,dt}{[g(t)-g(a)]^{1-\alpha}},\ \ a < x \le b.$$

Then

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)\,[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I^{\alpha}_{x+,g}f(b) - \frac{[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{6}$$

for $a \le x < b$ and

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I^{\alpha}_{x-,g}f(a) - \frac{[g(x)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu \tag{7}$$

for $a < x \le b$.

If $x \in (a, b)$, then by adding the equalities (6) and (7) we get the representation (1).

If we take $x = b$ in (4) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b \frac{g'(t)\,[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I^{\alpha}_{a+,g}f(b) - \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{8}$$

while from $x = a$ in (5) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I^{\alpha}_{b-,g}f(a) - \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu. \tag{9}$$

If we add (8) with (9) and divide by 2 we get (3). □

Remark 2 *If we take in* (1) *and* (2) $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, *then we get*

$$\begin{aligned} &I^{\alpha}_{a+,g}f(M_g(a,b)) + I^{\alpha}_{b-,g}f(M_g(a,b)) \\ &\qquad = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{\lambda+\mu}{2}\right) \\ &+\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-\lambda]\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right] \end{aligned}$$

and

$$\begin{aligned} I^{\alpha}_{M_g(a,b)-,g}f(a) + I^{\alpha}_{M_g(a,b)+,g}f(b) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{\lambda+\mu}{2}\right) \\ &+ \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-\lambda]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]. \end{aligned}$$

The above lemma provides various identities of interest by taking particular values for the parameters λ and μ, out of which we give only a few:

Corollary 2 *With the assumptions of Lemma 1 we have:*

(i) *For any* $x \in (a,b)$,

$$\begin{aligned} I^{\alpha}_{a+,g}f(x) + I^{\alpha}_{b-,g}f(x) &= \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right)f(x) \\ &+ \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(x)]\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(x)]^{1-\alpha}}\right] \end{aligned} \tag{10}$$

and

$$I^{\alpha}_{x-,g}f(a) + I^{\alpha}_{x+,g}f(b) = \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right)f(x)$$

$$+\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]. \quad (11)$$

(ii) *For any* $x \in [a,b]$,

$$\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}f(x)$$
$$+\frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(a)]^{1-\alpha}}\right]. \quad (12)$$

The proof is obvious by taking $\lambda=\mu=f(x)$ in Lemma 1. These identities were obtained in [14]. If we take in (10)-(12) $x=M_g(a,b)=g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get the corresponding identities were obtained in [14].

Corollary 3 *With the assumptions of Lemma 1 we have:*

$$I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)$$
$$=\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)\right)$$
$$+\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(a)]\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(t)-g(x)]^{1-\alpha}}\right] \quad (13)$$

and

$$I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)$$
$$=\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)\right)$$
$$+\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(a)]\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right], \quad (14)$$

for any $x \in (a,b)$

(ii) *We also have*

$$\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}$$
$$+\frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b \frac{g'(t)[f(t)-f(a)]\,dt}{[g(t)-g(a)]^{1-\alpha}}\right]. \quad (15)$$

The proof of (13) and (14) are obvious by taking $\lambda = f(a)$, $\mu = f(b)$ in Lemma 1. The proof of (15) follows by Lemma 1 on taking $\lambda = f(b)$ and $\mu = f(a)$.

Remark 3 *If we take in* (13) *and* (14) $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, *then we get*

$$\begin{aligned}
&I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) \\
&\qquad = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \left(\frac{f(a)+f(b)}{2}\right) \\
&+\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-f(a)]\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) \\
&\qquad = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \left(\frac{f(a)+f(b)}{2}\right) \\
&+\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-f(a)]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right].
\end{aligned}$$

3 Inequalities for bounded functions

Now, for ϕ, $\Phi \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions, see for instance [15]

$$\begin{aligned}
&\bar{U}_{[a,b]}(\phi,\Phi) \\
&:= \left\{f : [a,b] \to \mathbb{C} | \operatorname{Re}\left[(\Phi - f(t))\left(\overline{f(t)} - \overline{\phi}\right)\right] \geq 0 \text{ for almost every } t \in [a,b]\right\}
\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\phi,\Phi) := \left\{f : [a,b] \to \mathbb{C} \left| \left| f(t) - \frac{\phi+\Phi}{2}\right| \leq \frac{1}{2}|\Phi - \phi| \text{ for a.e. } t \in [a,b]\right.\right\}.$$

The following representation result may be stated.

Proposition 1 *For any* ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, *we have that* $\bar{U}_{[a,b]}(\phi,\Phi)$ *and* $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\phi,\Phi) = \bar{\Delta}_{[a,b]}(\phi,\Phi). \tag{16}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\phi + \Phi}{2} \right| \le \frac{1}{2} \left| \Phi - \phi \right|$$

if and only if

$$\operatorname{Re}\left[\left(\Phi - z \right) \left(\bar{z} - \phi \right) \right] \ge 0.$$

This follows by the equality

$$\frac{1}{4} \left| \Phi - \phi \right|^2 - \left| z - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re}\left[\left(\Phi - z \right) \left(\bar{z} - \phi \right) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (16) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 4 *For any $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]} \left(\phi, \Phi \right) = \{ f : [a, b] \to \mathbb{C} \mid \; & \left(\operatorname{Re} \Phi - \operatorname{Re} f\left(t \right) \right) \left(\operatorname{Re} f\left(t \right) - \operatorname{Re} \phi \right) \\ & + \left(\operatorname{Im} \Phi - \operatorname{Im} f\left(t \right) \right) \left(\operatorname{Im} f\left(t \right) - \operatorname{Im} \phi \right) \ge 0 \text{ for a.e. } t \in [a, b] \} . \end{aligned}$$

Now, if we assume that $\operatorname{Re}\left(\Phi\right) \ge \operatorname{Re}\left(\phi\right)$ and $\operatorname{Im}\left(\Phi\right) \ge \operatorname{Im}\left(\phi\right)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]} \left(\phi, \Phi \right) := \{ f : [a, b] \to \mathbb{C} \mid & \operatorname{Re}\left(\Phi \right) \ge \operatorname{Re} f\left(t \right) \ge \operatorname{Re}\left(\phi \right) \\ & \text{and } \operatorname{Im}\left(\Phi \right) \ge \operatorname{Im} f\left(t \right) \ge \operatorname{Im}\left(\phi \right) \text{ for a.e. } t \in [a, b] \} . \end{aligned}$$

One can easily observe that $\bar{S}_{[a,b]}\left(\phi, \Phi\right)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]} \left(\phi, \Phi \right) \subseteq \bar{U}_{[a,b]} \left(\phi, \Phi \right) .$$

We have:

Theorem 2 *Let $f : [a, b] \to \mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a, b]$, g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b) and ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$ such that $f \in \bar{\Delta}_{[a,b]}\left(\phi, \Phi\right)$.*

(i) *For any $x \in (a, b)$,*

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{\phi + \Phi}{2\Gamma\left(\alpha + 1\right)} \left(\left[g\left(x \right) - g\left(a \right) \right]^{\alpha} + \left[g\left(b \right) - g\left(x \right) \right]^{\alpha} \right) \right| \quad (17)$$

$$\leq \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right]$$

and

$$\left|I^{\alpha}_{x-,g}f(a)+I^{\alpha}_{x+,g}f(b)-\frac{\phi+\Phi}{2\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\right| \tag{18}$$
$$\leq \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right].$$

(ii) *We have*

$$\left|\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{\phi+\Phi}{2}\right| \tag{19}$$
$$\leq \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}|\Phi-\phi|[g(b)-g(a)]^{\alpha}.$$

Proof. Using the identity (1) for $\lambda=\mu=\frac{\phi+\Phi}{2}$, we have

$$\begin{aligned}
&I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)\\
&\qquad-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\frac{\phi+\Phi}{2}\\
&=\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\left[f(t)-\frac{\phi+\Phi}{2}\right]dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)\left[f(t)-\frac{\phi+\Phi}{2}\right]dt}{[g(t)-g(x)]^{1-\alpha}}\right]
\end{aligned} \tag{20}$$

for any $x\in(a,b)$.

Taking the modulus in (20), then we get

$$\begin{aligned}
&\left|I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\frac{\phi+\Phi}{2}\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\left|f(t)-\frac{\phi+\Phi}{2}\right|dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)\left|f(t)-\frac{\phi+\Phi}{2}\right|dt}{[g(t)-g(x)]^{1-\alpha}}\right]\\
&\leq\frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]\\
&=\frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right]
\end{aligned}$$

for any $x\in(a,b)$, which proves (17).

The inequality (18) follows in a similar manner from the identity (2).
The inequality (19) follows by (3) for $\lambda = \mu = \frac{\phi+\Phi}{2}$. □

Corollary 5 *With the assumptions of Theorem 2 we have*

$$\left| I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{\phi+\Phi}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} |\Phi-\phi| \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}$$

and

$$\left| I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{\phi+\Phi}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} |\Phi-\phi| \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}.$$

Remark 4 *If the function* $f : [a,b] \to \mathbb{R}$ *is measurable and there exists the constants* m, M *such that* $m \leq f(t) \leq M$ *for a.e.* $t \in [a,b]$, *then for any* $x \in (a,b)$ *we have by* (17) *and* (18) *that*

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{m+M}{2\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right) \right| \leq \frac{1}{2}(M-m) \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right]$$

and

$$\left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{m+M}{2\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right) \right| \leq \frac{1}{2}(M-m) \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right].$$

In particular,

$$\left| I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{m+M}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} (M-m) \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}$$

and

$$\left| I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{m+M}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} (M-m) \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}.$$

4 Trapezoid inequalities for functions of bounded variation

We have:

Theorem 3 *Let* $f : [a, b] \to \mathbb{C}$ *be a complex valued function of bounded variation on the real interval* $[a, b]$, *and* g *be a strictly increasing function on* (a, b), *having a continuous derivative* g' *on* (a, b). *Then we have the inequalities*

$$\begin{aligned}
&\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{[g(x) - g(a)]^{\alpha} f(a) + [g(b) - g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \bigvee_a^t (f)\, dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) \bigvee_t^b (f)\, dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[(g(x) - g(a))^{\alpha} \bigvee_a^x (f) + (g(b) - g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \left((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left((g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha} \right) \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \end{cases}
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{[g(x) - g(a)]^{\alpha} f(a) + [g(b) - g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \bigvee_a^t (f)\, dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) \bigvee_t^b (f)\, dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[(g(x) - g(a))^{\alpha} \bigvee_a^x (f) + (g(b) - g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \left((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left((g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha} \right) \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \end{cases}
\end{aligned} \tag{22}$$

for any $x \in (a, b)$

(ii) *We also have*

$$\begin{aligned}
&\left|\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right| \\
&\leq \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b \frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right] \\
&\leq \frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\bigvee_a^b(f).
\end{aligned} \tag{23}$$

Proof. Using the identity (13) and the properties of the modulus, we have

$$\begin{aligned}
&\left|I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)-\frac{\left[g(x)-g(a)\right]^{\alpha}f(a)+\left[g(b)-g(x)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left|f(t)-f(a)\right|dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b \frac{g'(t)\left|f(t)-f(b)\right|dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right]=:B(x)
\end{aligned}$$

for any $x \in (a,b)$.

Since f is of bounded variation on $[a,b]$, then we have

$$\left|f(t)-f(a)\right| \leq \bigvee_a^t(f) \leq \bigvee_a^x(f) \text{ for } a \leq t \leq x$$

and

$$\left|f(t)-f(b)\right| \leq \bigvee_t^b(f) \leq \bigvee_x^b(f) \text{ for } x \leq t \leq b.$$

Therefore

$$\begin{aligned}
B(x) &\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b \frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right] \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\bigvee_a^x(f)\int_a^x \frac{g'(t)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\bigvee_x^b(f)\int_x^b \frac{g'(t)\,dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right] \\
&= \frac{1}{\Gamma(\alpha)}\left[\frac{\left(g(x)-g(a)\right)^{\alpha}}{\alpha}\bigvee_a^x(f)+\frac{\left(g(b)-g(x)\right)^{\alpha}}{\alpha}\bigvee_x^b(f)\right] \\
&= \frac{1}{\Gamma(\alpha+1)}\left[\left(g(x)-g(a)\right)^{\alpha}\bigvee_a^x(f)+\left(g(b)-g(x)\right)^{\alpha}\bigvee_x^b(f)\right],
\end{aligned}$$

which proves the first two inequalities in (21).

The last part of (21) is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n \geq 0$

$$mc + nd \leq \begin{cases} \max\{m, n\}(c + d); \\ \\ (m^p + n^p)^{1/p}(c^q + d^q)^{1/q} \text{ with } p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

The inequality (22) follows in a similar way by utilising the equality (14).

From the equality (15) we have

$$\begin{aligned}
&\left| \frac{I_{b-,g}^{\alpha} f(a) + I_{a+,g}^{\alpha} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \frac{f(b) + f(a)}{2} \right| \\
&\leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t)|f(t) - f(b)|\,dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)|f(t) - f(a)|\,dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\
&\leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) \bigvee_t^b(f)\,dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) \bigvee_a^t(f)\,dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\
&\leq \frac{1}{2\Gamma(\alpha)} \left[\bigvee_a^b(f) \int_a^b \frac{g'(t)\,dt}{[g(b) - g(t)]^{1-\alpha}} + \bigvee_a^b(f) \int_a^b \frac{g'(t)\,dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\
&= \frac{1}{2\Gamma(\alpha)} \left[\bigvee_a^b(f) \frac{[g(b) - g(a)]^{\alpha}}{\alpha} + \bigvee_a^b(f) \frac{[g(b) - g(a)]^{\alpha}}{\alpha} \right] \\
&= \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \bigvee_a^b(f),
\end{aligned}$$

which proves (23). □

Corollary 6 *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
&\left| I_{a+,g}^{\alpha} f(M_g(a,b)) + I_{b-,g}^{\alpha} f(M_g(a,b)) - \frac{f(a) + f(b)}{2^{\alpha}\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) \bigvee_a^t(f)\,dt}{[g(M_g(a,b)) - g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) \bigvee_t^b(f)\,dt}{[g(t) - g(M_g(a,b))]^{1-\alpha}} \right] \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (g(b) - g(a))^{\alpha} \bigvee_a^b(f)
\end{aligned}$$

and

$$\left| I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \right|$$
$$\le \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) \bigvee_a^t (f)\, dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) \bigvee_t^b (f)\, dt}{[g(b)-g(t)]^{1-\alpha}} \right]$$
$$\le \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (g(b)-g(a))^{\alpha} \bigvee_a^b (f).$$

5 Inequalities for Hölder's continuous functions

We say that the function $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$ if

$$|f(t) - f(s)| \le H |t-s|^r \tag{24}$$

for any $t,\ s \in [a,b]$. If $r = 1$ and $H = L$ we call the function L-*Lipschitzian* on $[a,b]$.

Theorem 4 *Assume that* $f : [a,b] \to \mathbb{C}$ *is* r-H-*Hölder continuous on* $[a,b]$ *with* $r \in (0,1]$ *and* $H > 0$, *and* g *be a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b). *Then*

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{[g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right|$$
$$\le \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(t-a)^r\, dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)(b-t)^r\, dt}{[g(t)-g(x)]^{1-\alpha}} \right]$$
$$\le \frac{H}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} (x-a)^r + (g(b)-g(x))^{\alpha} (b-x)^r \right]$$
$$\le \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} [(x-a)^r + (b-x)^r]; \\ ((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p})^{1/p} ((x-a)^{rq} + (b-x)^{rq})^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ ((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha}) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \end{cases} \tag{25}$$

and

$$
\begin{aligned}
&\left| I^{\alpha}_{x-,g}f(a) + I^{\alpha}_{x+,g}f(b) - \frac{[g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
&\le \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(t-a)^r\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)(b-t)^r\,dt}{[g(b)-g(t)]^{1-\alpha}} \right] \\
&\le \frac{H}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha}(x-a)^r + (g(b)-g(x))^{\alpha}(b-x)^r \right] \\
&\le \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \left[(x-a)^r + (b-x)^r \right]; \\ \left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left((x-a)^{rq} + (b-x)^{rq} \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \end{cases}
\end{aligned} \tag{26}
$$

for any $x \in (a,b)$

(ii) *We also have*

$$
\begin{aligned}
&\left| \frac{I^{\alpha}_{b-,g}f(a) + I^{\alpha}_{a+,g}f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \frac{f(b)+f(a)}{2} \right| \\
&\le \frac{H}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t)(b-t)^r\,dt}{[g(b)-g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)(t-a)^r\,dt}{[g(t)-g(a)]^{1-\alpha}} \right] \\
&\le \frac{H}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} (b-a)^r .
\end{aligned} \tag{27}
$$

Proof. Using the identity (13) and the properties of the modulus, we have

$$
\begin{aligned}
&\left| I^{\alpha}_{a+,g}f(x) + I^{\alpha}_{b-,g}f(x) - \frac{[g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
&\le \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)|f(t)-f(a)|\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)|f(t)-f(b)|\,dt}{[g(t)-g(x)]^{1-\alpha}} \right] =: C(x)
\end{aligned}
$$

for any $x \in (a,b)$.

Since $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$, hence

$$
C(x) \le \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(t-a)^r\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)(b-t)^r\,dt}{[g(t)-g(x)]^{1-\alpha}} \right]
$$

$$\begin{aligned}
&\leq \frac{H}{\Gamma(\alpha)}\left[(x-a)^r\int_a^x \frac{g'(t)\,dt}{[g(x)-g(t)]^{1-\alpha}}+(b-x)^r\int_x^b \frac{g'(t)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]\\
&= \frac{H}{\Gamma(\alpha)}\left[(x-a)^r\frac{(g(x)-g(a))^\alpha}{\alpha}+(b-x)^r\frac{(g(b)-g(x))^\alpha}{\alpha}\right]\\
&= \frac{H}{\Gamma(\alpha+1)}\left[(x-a)^r(g(x)-g(a))^\alpha+(b-x)^r(g(b)-g(x))^\alpha\right],
\end{aligned}$$

for any $x\in(a,b)$, which proves the first two inequalities in (25). The rest is obvious.

The inequality (26) follows in a similar way by utilising the equality (14).

The inequality (27) follows by utilising the equality (15). □

Corollary 7 *With the assumptions of Theorem 4 we have*

$$\begin{aligned}
&\left|I^\alpha_{a+,g}f(M_g(a,b))+I^\alpha_{b-,g}f(M_g(a,b))-\frac{f(a)+f(b)}{2^\alpha\Gamma(\alpha+1)}[g(b)-g(a)]^\alpha\right|\\
&\leq \frac{H}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)(t-a)^r\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)(b-t)^r\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right]\\
&\leq \frac{H}{2^\alpha\Gamma(\alpha+1)}(g(b)-g(a))^\alpha\left[(M_g(a,b)-a)^r+(b-M_g(a,b))^r\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|I^\alpha_{M_g(a,b)-,g}f(a)+I^\alpha_{M_g(a,b)+,g}f(b)-\frac{f(a)+f(b)}{2^\alpha\Gamma(\alpha+1)}[g(b)-g(a)]^\alpha\right|\\
&\leq \frac{H}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)(t-a)^r\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)(b-t)^r\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]\\
&\leq \frac{H}{2^\alpha\Gamma(\alpha+1)}(g(b)-g(a))^\alpha\left[(M_g(a,b)-a)^r+(b-M_g(a,b))^r\right].
\end{aligned}$$

6 Applications for Hadamard fractional integrals

If we take $g(t)=\ln t$ and $0\leq a<x\leq b$, then by Theorem 3 for Hadamard fractional integrals H^α_{a+} and H^α_{b-} we have for $f:[a,b]\to\mathbb{C}$, a function of bounded variation on $[a,b]$ that

$$\left|H^\alpha_{a+}f(x)+H^\alpha_{b-}f(x)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^\alpha f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^\alpha f(b)}{\Gamma(\alpha+1)}\right|$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\bigvee_a^t(f)\,dt}{t}+\int_x^b \frac{\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\bigvee_t^b(f)\,dt}{t}\right]\\
&\leq \frac{1}{\Gamma(\alpha+1)}\left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}\bigvee_a^x(f)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}\bigvee_x^b(f)\right]\\
&\leq \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
&\left|H_{x-}^{\alpha}f(a)+H_{x+}^{\alpha}f(b)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|\\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\bigvee_a^t(f)\,dt}{t}+\int_x^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}\bigvee_t^b(f)\,dt}{t}\right]\\
&\leq \frac{1}{\Gamma(\alpha+1)}\left[\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}\bigvee_a^x(f)+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\bigvee_x^b(f)\right]\\
&\leq \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned}
\tag{29}
$$

for any $x\in(a,b)$

We also have

$$
\begin{aligned}
&\left|\frac{H_{b-}^{\alpha}f(a)+H_{a+}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right|\\
&\leq \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}\bigvee_t^b(f)\,dt}{t}+\int_a^b \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}g'(t)\bigvee_a^t(f)\,dt}{t}\right]\\
&\leq \frac{1}{\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\bigvee_a^b(f).
\end{aligned}
$$

If we take in (28) and (29) $x = G(a,b)$, then we get

$$\left| H^{\alpha}_{a+}f(G(a,b)) + H^{\alpha}_{b-}f(G(a,b)) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1} \bigvee_a^t (f)\, dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1} \bigvee_t^b (f)\, dt}{t}\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \bigvee_a^b (f)$$

and

$$\left| H^{\alpha}_{G(a,b)-}f(a) + H^{\alpha}_{G(a,b)+}f(b) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} \bigvee_a^t (f)\, dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} \bigvee_t^b (f)\, dt}{t}\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \bigvee_a^b (f).$$

Assume that $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$. If we take $g(t) = \ln t$ and $0 \leq a < x \leq b$ in Theorem 4, then we get

$$\left| H^{\alpha}_{a+}f(x) + H^{\alpha}_{b-}f(x) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} f(a) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right|$$
$$\leq \frac{H}{\Gamma(\alpha)}\left[\int_a^x \frac{\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}(t-a)^r\, dt}{t} + \int_x^b \frac{\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}(b-t)^r\, dt}{t}\right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)}\left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}(x-a)^r + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}(b-x)^r\right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha} \bigvee_a^b (f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases}$$

and

$$\begin{aligned}
&\left| H^{\alpha}_{x-}f(a)+H^{\alpha}_{x+}f(b)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \\
&\le \frac{H}{\Gamma(\alpha)}\left[\int_a^x \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}(t-a)^r\,dt}{t}+\int_x^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}(b-t)^r\,dt}{t}\right] \\
&\le \frac{H}{\Gamma(\alpha+1)}\left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}(x-a)^r+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}(b-x)^r\right] \\
&\le \frac{H}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}V_a^b(f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(V_a^x(f)\right)^q+\left(V_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}V_a^b(f)+\frac{1}{2}\left|V_a^x(f)-V_x^b(f)\right|\right]\end{cases}
\end{aligned} \tag{30}$$

for any $x\in(a,b)$.

We also have

$$\begin{aligned}
&\left|\frac{H^{\alpha}_{b-}f(a)+H^{\alpha}_{a+}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right| \\
&\le \frac{H}{2\Gamma(\alpha)}\left[\int_a^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}(b-t)^r\,dt}{t}+\int_a^b \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}(t-a)^r\,dt}{t}\right] \\
&\le \frac{H}{\Gamma(\alpha+1)}(b-a)^r\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}.
\end{aligned} \tag{31}$$

If we take in (30) and (31) $x=G(a,b)$, then we get

$$\begin{aligned}
&\left|H^{\alpha}_{a+}f(G(a,b))+H^{\alpha}_{b-}f(G(a,b))-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\right| \\
&\le \frac{H}{\Gamma(\alpha)}\left[\int_a^{G(a,b)}\frac{\left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1}(t-a)^r dt}{t}+\int_{G(a,b)}^b\frac{\left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1}(b-t)^r dt}{t}\right] \\
&\le \frac{1}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}(b-a)^r
\end{aligned}$$

and

$$\left|H^{\alpha}_{G(a,b)-}f(a)+H^{\alpha}_{G(a,b)+}f(b)-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}(t-a)^r\,dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}(b-t)^r\,dt}{t}\right]$$
$$\leq \frac{1}{2^\alpha\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^\alpha (b-a)^r.$$

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Composition iterates, Cauchy, translation, and Sincov inclusions

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Abstract. Improving and extending some ideas of Gottlob Frege from 1874 (on a generalization of the notion of the composition iterates of a function), we consider the composition iterates φ^n of a relation φ on X, defined by

$$\varphi^0 = \Delta_X, \qquad \varphi^n = \varphi \circ \varphi^{n-1} \quad \text{if} \quad n \in \mathbb{N}, \qquad \text{and} \qquad \varphi^\infty = \bigcup_{n=0}^{\infty} \varphi^n.$$

In particular, by using the relational inclusion $\varphi^n \circ \varphi^m \subseteq \varphi^{n+m}$ with $n, m \in \overline{\mathbb{N}}_0 = \{0\} \cup \mathbb{N} \cup \{\infty\}$, we show that the function α, defined by

$$\alpha(n) = \varphi^n \quad \text{for} \quad n \in \overline{\mathbb{N}}_0,$$

satisfies the Cauchy problem

$$\alpha(n) \circ \alpha(m) \subseteq \alpha(n+m), \qquad \alpha(0) = \Delta_X.$$

Moreover, the function f, defined by

$$f(n, A) = \alpha(n)[A] \qquad \text{for} \qquad n \in \overline{\mathbb{N}}_0 \quad \text{and} \quad A \subseteq X,$$

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satisfies the translation problem

$$f\big(n, f(m, A)\big) \subseteq f(n+m, A), \qquad f(0, A) = A.$$

Furthermore, the function F, defined by

$$F(A, B) = \big\{n \in \overline{\mathbb{N}}_0 : \quad A \subseteq f(n, B)\big\} \qquad \text{for} \qquad A, B \subseteq X,$$

satisfies the Sincov problem

$$F(A, B) + F(B, C) \subseteq F(A, C), \qquad 0 \in F(A, A).$$

Motivated by the above observations, we investigate a function F on the product set X^2 to the power groupoid $\mathcal{P}(U)$ of an additively written groupoid U which is supertriangular in the sense that

$$F(x, y) + F(y, z) \subseteq F(x, z)$$

for all $x, y, z \in X$. For this, we introduce the convenient notations

$$R(x, y) = F(y, x) \qquad \text{and} \qquad S(x, y) = F(x, y) + R(x, y),$$

and

$$\Phi(x) = F(x, x) \qquad \text{and} \qquad \Psi(x) = \bigcup_{y \in X} S(x, y).$$

Moreover, we gradually assume that U and F have some useful additional properties. For instance, U has a zero, U is a group, U is commutative, U is cancellative, or U has a suitable distance function; while F is nonpartial, F is symmetric, skew symmetric, or single-valued.

1 A few basic facts on relations

In [40], a subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation on X to itself is called a *relation on* X. More specially, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation on* X.

If F is a relation on X to Y, then by the above definitions we can also state that F is a relation on $X \cup Y$. However, for our present purposes, the latter view of the relation F would also be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images of* x *and* A *under* F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write xFy. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

The sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain and range of* F, respectively. If in particular $D_F = X$, then we say that F is a *relation of* X *to* Y, or that F is a *nonpartial relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

In particular, a function $\star$ of X to itself is called a *unary operation on* X, while a function $*$ of X^2 to X is called a *binary operation on* X. In this case, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then we can easily see that $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *inverse* F^{-1} can be defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$. Moreover, if G is a relation on Y to Z, then the *composition* $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$.

If F is a relation on X to Y, then a relation Φ of D_F to Y is called a *selection relation of* F if $\Phi \subseteq F$, i.e., $\Phi(x) \subseteq F(x)$ for all $x \in D_F$. By using the Axiom of Choice, it can be seen that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ of X to $\mathcal{P}(Y)$ and Φ of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi(x) = F(x)$ for all $x \in X$ and $\Phi(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y, while functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. They were briefly called *co-relations on* X *to* Y in [40].

In particular, a relation R on X can be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R can be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder* (*tolerance*) *relation*, and a symmetric (antisymmetric) preorder relation may be called an *equivalence* (*partial order*) *relation*.

For $A \subseteq X$, *Pervin's relation* $R_A = A^2 \cup A^c \times X$, with $A^c = X \setminus A$, is an important preorder on X. While, for a *pseudometric* d on X, *Weil's surrounding* $B_r = \{(x, y) \in X^2 : d(x, y) < r\}$, with $r > 0$, is an important tolerance on X.

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \big(A^c\big)^2$ is already an equivalence on X. And, more generally if $\mathcal{A}$ is a *partition of* X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is an equivalence on X which can, to some extent, be identified with $\mathcal{A}$.

2 A few basic facts on ordered sets and groupoids

If $\leq$ is a relation on X, then motivated by Birkhoff [5, p. 1] the ordered pair $X(\leq) = (X, \leq)$ is called a *goset* (generalized ordered set) [39]. In particular, it is called a *proset* (preordered set) if the relation $\leq$ is a preorder on X.

Quite similarly, a goset $X(\leq)$ is called a *poset* (partially ordered set) if the relation $\leq$ is a partial order on X. The importance of posets lies mainly in the fact that any family of sets forms a poset with set inclusion.

A function f of one goset $X(\leq)$ to another $Y(\leq)$ is called *increasing* if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in X$. The function f can now be briefly called *decreasing* if it is increasing as a function of $X(\leq)$ to the dual $Y(\geq)$.

An increasing function φ of the goset $X = X(\leq)$ to itself is called a *projection* (*involution*) *operation* on X if it is *idempotent* (*involutive*) in the sense that $\varphi \circ \varphi = \varphi$ $\big(\varphi \circ \varphi = \Delta_X\big)$. Note that $\varphi \circ \varphi = \Delta_X$ if and only if $\varphi^{-1} = \varphi$.

Moreover, a projection operation φ on a poset X is called a *closure operation on* X if it is *extensive* in the sense that $\Delta_X \leq \varphi$. That is, $x \leq \varphi(x)$ for all $x \in X$. The *interior operations* can again be most briefly defined by dualization.

If f is a function of one goset X to another Y and g is a function of Y to X such that, for any $x \in X$ and $y \in Y$, we have $f(x) \leq y$ if and only $x \leq g(y)$, then g is called a *Galois adjoint of* f [12, p. 155].

Hence, by taking $\varphi = g \circ f$, one can easily see that, for any $u, v \in X$, we have $f(u) \leq f(v)$ if and only if $u \leq \varphi(v)$. Moreover, if X and Y are prosets, then it can be shown that f is increasing, φ is a closure and $f = f \circ \varphi$ [39].

If $+$ is a binary operation on a set X, then the ordered pair $X(+) = (X, +)$ is called an *additive groupoid*. Recently, groupoids are usually called *magmas*, not to be confused with *Brandt groupoids* [6].

If X is a groupoid, then for any $A, B \subseteq X$ we may also naturally define $A + B = \{x + y : \ x \in A, \ y \in B\}$. Thus, by identifying singletons with their elements, X may be considered as a subgoupoid of its *power groupoid* $\mathcal{P}(X)$.

In a groupoid X, for any $n \in \mathbb{N}$ and $x \in X$ we may also naturally define $nx = x$ if $n = 1$, and $nx = (n-1)x + x$ if $n > 1$. Thus, for any $n \in \mathbb{N}$ and $A \subseteq X$, we may also naturally define $nA = \{nx : \ x \in A\}$.

If X is a *semigroup* (associative groupoid), then we have $(n+m)x = nx + mx$ and $(nm)x = n(mx)$ for all $n, m \in \mathbb{N}$ and $x \in X$. However, the equality $n(x+y) = nx + ny$ requires the elements $x, y \in X$ to be commuting [19].

If the groupoid X has a zero element 0, then we also naturally define $0x = 0$ for all $x \in X$. Moreover, if X is a group, then we also naturally define $(-n)x = n(-x)$ for all $n \in \mathbb{N}$ and $x \in X$. And thus also kA for all $k \in \mathbb{Z}$ and $A \subseteq X$.

Concerning the corresponding operations in $\mathcal{P}(X)$, we must be very careful.

Namely, in general, we only have $(n+m)A \subseteq nA + mA$ and $nA \subseteq \sum_{i=1}^{n} A$ for all $n, m \in \mathbb{N}$ and $A \subseteq X$. However, $\mathcal{P}(X)$ has a richer structure than X.

In particular, an element x of a groupoid X is called *left-cancellable* if $x+y = x+z$ implies $y = z$ for all $y, z \in X$. Moreover, the groupoid X is called *left-cancellative* if every element of X is left-cancellable.

"Right-cancellable" and *"right-cancellative"* are to be defined quite similarly. Moreover, for instance, the groupoid X is to be called *cancellative* if it is both left-cancellative and right-cancellative.

A semigroup X can be easily embedded in a *monoid* (semigroup with zero element), by adjoining an element 0 not in X, and defining $0+x = x+0 = x$ for all $x \in X$. Important monoids will be $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$.

3 The finite composition iterates of a relation

Notation 1 *In the sequel, we shall assume that X is a set, Δ is the identity function of X and φ is a relation on X.*

Note that the family $\mathcal{P}(X^2)$ of all relations on X forms a semigroup, with identity element Δ, with respect to the composition of relations. Therefore, we may naturally use the following

Definition 1 Define $\varphi^0 = \Delta$, and for any $n \in \mathbb{N}$

$$\varphi^n = \varphi \circ \varphi^{n-1}.$$

Remark 1 Thus, for each $n \in \mathbb{N}_0$, φ^n is also a relation on X which is called the nth composition iterate of φ.

Now, as a particular case of a more general theorem on monoids, we can state the following theorem whose direct proof is included here only for the reader's convenience.

Theorem 1 *For any $n, m \in \mathbb{N}_0$, we have*

$$\varphi^{n+m} = \varphi^n \circ \varphi^m.$$

Proof. For fixed $m \in \mathbb{N}_0$, we shall prove, by induction, that

$$\varphi^{m+n} = \varphi^n \circ \varphi^m$$

for all $n \in \mathbb{N}_0$. Hence, by the commutativity of the addition in $\mathbb{N}_0$, the assertion of the theorem follows.

By Definition 1, we evidently have $\varphi^{m+0} = \varphi^m = \Delta \circ \varphi^m = \varphi^0 \circ \varphi^m$. Therefore, the required equity is true for $n = 0$.

Let us suppose now that the required equality is true for some $n \in \mathbb{N}_0$. Then, by Definition 1, the above assumption, and the corresponding associativities, we have

$$\begin{aligned} \varphi^{m+(n+1)} &= \varphi^{(m+n)+1} = \varphi \circ \varphi^{m+n} \\ &= \varphi \circ (\varphi^n \circ \varphi^m) = (\varphi \circ \varphi^n) \circ \varphi^m = \varphi^{n+1} \circ \varphi^m. \end{aligned}$$

Therefore, the required equality is also true for $n + 1$. □

Remark 2 This theorem shows that the family $\{\varphi^n\}_{n=0}^{\infty}$ also forms a semigroup, with identity element Δ, with respect to composition.

By induction, we can also easily prove the less trivial part of the following

Theorem 2 *The following assertions are equivalent:*
(1) $\Delta \subseteq \varphi$; (2) $\varphi^n \subseteq \varphi^{n+1}$ *for all* $n \in \mathbb{N}_0$.

Remark 3 This theorem shows that the sequence $\left(\varphi^n\right)_{n=0}^{\infty}$ is increasing, with respect to set inclusion, if and only if the relation φ is reflexive on X.

Note that if in particular φ is reflexive on X and φ is a function, then we necessarily have $\varphi = \Delta$, and thus also $\varphi^n = \Delta$ for all $n \in \mathbb{N}_0$.

Therefore, in the important particular case when φ is a function of X to itself, Theorem 2 cannot have any significance.

4 The infinite composition iterate of a relation

In addition to Definition 1, we may also naturally use the following

Definition 2 Define

$$\varphi^{\infty} = \bigcup_{n=0}^{\infty} \varphi^n.$$

Remark 4 Moreover, the relations

$$\varliminf_{n\to\infty} \varphi^n = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} \varphi^k \qquad \text{and} \qquad \varlimsup_{n\to\infty} \varphi^n = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} \varphi^k$$

may also be naturally investigated.

Note that if in particular the sequence $\left(\varphi^n\right)_{n=0}^{\infty}$ is increasing with respect to set inclusion, then these relations coincide with φ^∞.

The relation φ^∞ is called the *preorder hull (closure) of* φ. Namely, we have

Theorem 3 *φ^∞ is the smallest preorder relation on X containing φ.*

Proof. By Definition 2, it is clear that $\Delta \subseteq \varphi^\infty$ and $\varphi \subseteq \varphi^\infty$. Thus, φ^∞ is reflexive and contains φ.

Moreover, if $(x, y) \in \varphi^\infty$ and $(y, z) \in \varphi^\infty$, then by Definition 2 there exist $m, n \in \mathbb{N}_0$ such that $(x, y) \in \varphi^m$ and $(y, z) \in \varphi^n$. Hence, by using Theorem 1, we can infer that $(x, z) \in \varphi^n \circ \varphi^m = \varphi^{n+m}$. Thus, by Definition 2, we also have $(x, z) \in \varphi^\infty$. Therefore, φ^∞ is also transitive.

On the other hand, if ψ is a relation on X such that $\varphi \subseteq \psi$, then we can note that $\varphi^n \subseteq \psi^n$ for all $n \in \mathbb{N}_0$, and thus by Definition 2 we have $\varphi^\infty \subseteq \psi^\infty$. Moreover, if ψ is reflexive, then $\psi^0 \subseteq \psi$. And, if ψ is transitive, then $\psi^n \subseteq \psi$ for all $n \in \mathbb{N}$. Therefore, if ψ is both reflexive and transitive, then by Definition 2 we have $\psi^\infty \subseteq \psi$, and thus also $\varphi^\infty \subseteq \psi$. □

Now, as an immediate consequence of this theorem, we can also state

Corollary 1 *The following assertions are equivalent:*
(1) $\varphi^\infty = \varphi$; (2) φ *is a preorder on* X.

Remark 5 From the above results, it is clear that ∞ is a closure operation on the poset $\mathcal{P}\left(X^2\right)$.

In general, it is not even finitely union preserving. However, it is compatible with the inversion of relations [18].

Moreover, in addition to Theorem 1, we can also easily prove the following

Theorem 4 *For any $n, m \in \overline{\mathbb{N}}_0$, we have*

$$\varphi^n \circ \varphi^m \subseteq \varphi^{n+m}.$$

Moreover, if φ is reflexive on X, then the corresponding equality is also true.

Proof. If in particular $n, m \in \mathbb{N}_0$, then by Theorem 1 the corresponding equality is true even if φ is not assumed to be reflexive.

Moreover, by using Definition 2 and Theorem 3, we can see that

$$\varphi^n \circ \varphi^\infty \subseteq \varphi^\infty \circ \varphi^\infty \subseteq \varphi^\infty = \varphi^{n+\infty}.$$

Furthermore, if φ is reflexive, then it is clear that we also have

$$\varphi^{\infty} = \Delta \circ \varphi^{\infty} \subseteq \varphi^{n} \circ \varphi^{\infty}.$$

Therefore, in this case, $\varphi^{n} \circ \varphi^{\infty} = \varphi^{\infty} = \varphi^{n+\infty}$ also holds. The case "$\infty + m$" can be treated quite similarly. $\square$

Remark 6 Now, in addition to Theorem 2, we can only state that $\varphi^{n} \subseteq \varphi^{\infty}$ for all $n \in \overline{\mathbb{N}}_0$.

However, by [30], we may naturally say that φ is n-*well-chained* if $\varphi^{n} = X^2$. And, φ is n-*connected* if $\varphi \cup \varphi^{-1}$ is n–well-chained.

Moreover, under the notation $\mathcal{T}_{\varphi} = \{A \subseteq X : \ \varphi[A] \subseteq A\}$ of [24], we have $\varphi^{\infty} = \bigcap_{A \in \mathcal{T}_{\varphi}} R_A$. And, φ^{∞} is the largest relation on X such that $\mathcal{T}_{\varphi^{\infty}} = \mathcal{T}_{\varphi}$.

5 From the composition iterates to a Cauchy inclusion

Now, extending an idea of Frege [15, 16], we may also naturally introduce

Definition 3 For any $n \in \overline{\mathbb{N}}_0$, define

$$\alpha(n) = \varphi^{n}.$$

Thus, α may be considered as a relation on $\overline{\mathbb{N}}_0$ to X^2, or as a function of $\overline{\mathbb{N}}_0$ to $\mathcal{P}(X^2)$, which can be proved to satisfy a Cauchy type inclusion.

First of all, by Theorem 1, we evidently have the following

Theorem 5 *For any* $n, m \in \mathbb{N}_0$, *we have*

$$\alpha(n+m) = \alpha(n) \circ \alpha(m).$$

Proof. By Definition 3 and Theorem 1, it is clear that

$$\alpha(n+m) = \varphi^{n+m} = \varphi^{n} \circ \varphi^{m} = \alpha(n) \circ \alpha(m).$$

$\square$

Remark 7 In addition to this theorem, it is also worth noticing that $\alpha(0) = \Delta$.

Moreover, by Theorem 2, we can also at once state the following

Theorem 6 *The following assertions are equivalent:*
(1) $\Delta \subseteq \varphi$; (2) $\alpha(n) \subseteq \alpha(n+1)$ *for all* $n \in \mathbb{N}_0$.

Remark 8 Thus, the restriction of the set-valued function α to $\mathbb{N}_0$ is increasing, with respect to set inclusion, if and only if the relation φ is reflexive on X.

By using Theorem 4 instead of Theorem 1, we can also easily establish

Theorem 7 *For any* $n, m \in \overline{\mathbb{N}}_0$ *we have*

$$\alpha(n) \circ \alpha(m) \subseteq \alpha(n+m).$$

Moreover, if φ *is reflexive on* X*, then the corresponding equality is also true.*

Remark 9 Now, in addition to Theorem 6, we can also state that $\alpha(n) \subseteq \alpha(\infty)$ for all $n \in \overline{\mathbb{N}}_0$.

Thus, in particular, the set-valued function α is increasing, with respect to set inclusion, if and only if the relation φ is reflexive on X.

6 From a Cauchy inclusion to a translation inclusion

Now, as an extension of our former observations, we may naturally start with

Notation 2 *Suppose that* U *is a additive groupoid and* α *is a relation on* U *to* X^2 *such that*

$$\alpha(u) \circ \alpha(v) \subseteq \alpha(u+v)$$

for all $u, v \in U$.

Thus, extending an idea of Frege [15, 16], we may also naturally introduce

Definition 4 For any $u \in U$ and $A \subseteq X$, define

$$f(u, A) = \alpha(u)[A].$$

Thus, f may be considered a relation on $U \times \mathcal{P}(X)$ to X, or as a function of $U \times \mathcal{P}(X)$ to $\mathcal{P}(X)$, which can be proved to satisfy a translation inclusion.

Theorem 8 *For any* $u, v \in U$ *and* $A \subseteq X$, *we have*

$$f\big(u, f(v, A)\big) \subseteq f(u+v, A).$$

Proof. By Definition 4 and the assumed superadditivity property of α, we have

$$\begin{aligned} f\big(u, f(v, A)\big) &= \alpha(u)\big[f(v, A)\big] = \alpha(u)\big[\alpha(v)[A]\big] \\ &= \big(\alpha(u) \circ \alpha(v)\big)[A] \subseteq \alpha(u+v)[A] = f(u+v, A). \end{aligned}$$

□

Remark 10 Thus, by identifying singleton with their elements, we may also write

$$f\big(u, f(v, x)\big) \subseteq f(u+v, x)$$

for all $u, v \in U$ and $x \in X$.

Now, to illustrate the appropriateness of Definition 4, we can also state

Example 1 *If in particular* α *is as in Definition 3, then by Definition 4 we have*

$$f(n, A) = \alpha(n)[A] = \varphi^n[A]$$

for all $n \in \overline{\mathbb{N}}_0$ *and* $A \subseteq X$*. Thus, in particular* $f(0, A) = A$ *for all* $A \subseteq X$*.*

7 From a translation inclusion to a Sincov inclusion

Now, as an extension of our former observations, we may also naturally start with the following

Notation 3 *Suppose that* U *is an additive groupoid,* X *is a goset and* f *is a function of* $U \times X$ *to* X *such that* f *is increasing in its second variable and*

$$f\big(u, f(v, x)\big) \leq f(u+v, x)$$

for all $u, v \in U$ *and* $x \in X$*.*

Thus, improving an idea of Frege [15, 16], we may also naturally introduce

Definition 5 For any $x, y \in X$, define

$$F(x, y) = \big\{u \in U : \quad x \leq f(u, y)\big\}.$$

Thus, F may be considered as a relation on X^2 to U, or as a function of X^2 to $\mathcal{P}(U)$, which can be proved to satisfy a Sincov type inclusion.

Theorem 9 *For any* $x, y, z \in X$*, we have*

$$F(x, y) + F(y, z) \subseteq F(x, z).$$

Proof. If

$$u \in F(x, y) \qquad \text{and} \qquad v \in F(y, z),$$

then by Definition 5 we get

$$x \leq f(u, y) \qquad \text{and} \qquad y \leq f(v, z).$$

Hence, by using the assumed increasingness and translation property of f, we can infer that

$$x \leq f(u, y) \leq f\big(u, f(v, z)\big) \leq f(u + v, z).$$

Therefore, by Definition 5, we have

$$u + v \in F(x, z).$$

Thus, the required inclusion is true. □

Now, to illustrate the appropriateness of Definition 5, we can also state

Example 2 *If* f *is as in Example 1, then by Definition 5 we have*

$$F(A, B) = \big\{n \in \overline{\mathbb{N}}_0 : \quad A \subseteq f(n, B)\big\} = \big\{n \in \overline{\mathbb{N}}_0 : \quad A \subseteq \varphi^n[B]\big\}$$

for all $A, B \subseteq X$*. Thus, in particular* $0 \in F(A, A)$ *for all* $A \subseteq X$*.*

Remark 11 By Aczél [1, pp. 223, 303 and 353], Sincov's functional equation and its generalizations have been investigated by a surprisingly great number of authors.

For some more recent investigations, see [4, 33, 38, 27, 34, 35, 7, 8, 3, 14, 13]. The most relevant ones are the set-valued considerations of Smajdor [38] and Augustová and Klapka [3].

Moreover, it is noteworthy that, by using the famous partial operation

$$(x, y) \bullet (y, z) = (x, z),$$

the above Sincov inclusion can be turned into a restricted Cauchy inclusion.

Therefore, some of the methods of the theory of superadditive functions and relations [28, 17, 29, 19] can certainly be applied to investigate the corresponding Sincov inequalities and inclusions.

8 Some immediate consequences of a Sincov inclusion

Now, motivated by our former observations, we may also naturally introduce the following notations and definitions.

Notation 4 *In what follows, we shall also assume that* X *is a set and* U *is an additive groupoid. Moreover, we shall suppose that* F *is a relation on* X^2 *to* U.

Definition 6 The relation F will be called *supertriangular* if

$$F(x, y) + F(y, z) \subseteq F(x, z)$$

for all $x, y, z \in X$.

Remark 12 Now, the relation F may also be naturally called *subtriangular* if the reverse inclusion holds. Moreover, F may be naturally called *triangular* if it is both subtriangular and supertriangular.

Subtriangular relations are certainly more important than the supertriangular ones. Namely, if a function d of X^2 to $[0, +\infty]$ satisfies the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$

for all $x, y, z \in X$, then the relation F, defined such that

$$F(x, y) = \big[0,\ d(x, y)\big] \qquad \Big(\ F(x, y) = \ [-d(x, y),\ d(x, y)]\ \Big)$$

for all $x, y \in X$, can, in general, be proved to be only subtriangular [2].

The $y = x$, $y = z$ and $z = x$ particular cases of the inclusion considered in Definition 6 strongly suggest the introduction of the following

Definition 7 For any $x, y \in X$, define

$$R(x, y) = F(y, x) \qquad \text{and} \qquad S(x, y) = F(x, y) + R(x, y).$$

Moreover, for any $x \in X$, define

$$\Phi(x) = F(x, x) \qquad \text{and} \qquad \Psi(x) = \bigcup_{y \in X} S(x, y).$$

Thus, R and S may be considered as relations on X^2 to U, and Φ and Ψ may be considered as relations on X to U.

Concerning these relations, we can easily prove the following

Theorem 10 *For any* $x, y \in X$ *we have*
(1) $\Phi(x) + \Phi(x) \subseteq \Psi(x)$;
(2) $R(x, x) = \Phi(x)$; (3) $S(x, x) = \Phi(x) + \Phi(x)$.

Proof. By Definition 7, we evidently have

$$R(x, x) = F(x, x) = \Phi(x),$$

and thus also

$$S(x, x) = F(x, x) + R(x, x) = \Phi(x) + \Phi(x).$$

Hence, by using the definition of Ψ, we can also easily note that

$$\Phi(x) + \Phi(x) = S(x, x) \subseteq \bigcup_{y \in X} S(x, y) = \Psi(x).$$

Therefore, assertions (2), (3) and (1) are true. □

Now, as a counterpart of [38, Lemma 1] of Wilhelmina Smajdor, we can also prove the following

Theorem 11 *If* F *is supertriangular, then for any* $x, y \in X$ *we have*
(1) $\Psi(x) \subseteq \Phi(x)$;
(2) $\Phi(x) + F(x, y) \subseteq F(x, y)$; (3) $F(x, y) + \Phi(y) \subseteq F(x, y)$.

Proof. By using Definition 7 and the corresponding particular cases of the inclusion considered in Definition 6, we can easily see that

$$\Phi(x) + F(x, y) = F(x, x) + F(x, y) \subseteq F(x, y)$$

and

$$F(x, y) + \Phi(y) = F(x, y) + F(y, y) \subseteq F(x, y).$$

Moreover,

$$S(x, y) = F(x, y) + R(x, y) = F(x, y) + F(y, x) \subseteq F(x, x) = \Phi(x),$$

and thus also

$$\Psi(x) = \bigcup_{y \in X} S(x, y) \subseteq \bigcup_{y \in X} \Phi(x) \subseteq \Phi(x).$$

Therefore, assertions (2), (3) and (1) are true even if only some consequences of the assumed inclusion property of F are supposed to hold. □

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 2 *If* F *is supertriangular, then for any* $x, y \in X$ *we have*

(1) $\Phi(x) + \Phi(x) \subseteq \Phi(x)$; (2) $\Psi(x) + \Psi(x) \subseteq \Psi(x)$;
(3) $\Psi(x) + \Phi(x) \subseteq \Psi(x)$; (4) $\Phi(x) + \Psi(x) \subseteq \Psi(x)$;
(5) $\Psi(x) + F(x, y) \subseteq F(x, y)$; (6) $F(x, y) + \Psi(y) \subseteq F(x, y)$.

Remark 13 By [8], in addition to Definition 6, the *separability equation*

$$F(x, y) + F(y, z) = F(x, z) + \Phi(y)$$

may also be naturally investigated.

Moreover, if in particular U is a group, then in addition to Definition 7, the *disymmetry relation* D *of* F, defined such that $D(x, y) = F(x, y) - R(x, y)$ for all $x, y \in X$, may also be naturally investigated.

9 The particular case when U has a zero element

Theorem 12 *If* F *is supertriangular,* U *has a one-sided zero element* 0 *and* $x \in X$ *is such that* $0 \in \Phi(x)$*, then*

(1) $\Phi(x) = \Psi(x)$; (2) $\Phi(x) = \Phi(x) + \Phi(x)$.

Proof. If 0 is a right zero element of U, then by using Theorems 10 and 11 we can see that

$$\Phi(x) = \Phi(x) + \{0\} \subseteq \Phi(x) + \Phi(x) \subseteq \Psi(x) \subseteq \Phi(x).$$

While, if 0 is a left zero element of U, then we can quite similarly see that

$$\Phi(x) = \{0\} + \Phi(x) \subseteq \Phi(x) + \Phi(x) \subseteq \Psi(x) \subseteq \Phi(x).$$

Therefore, in both cases, the required equalities are true. □

Remark 14 Note that if in particular F is as in Example 2, then $0 \in \Phi(A)$ holds for all $A \subseteq X$. Therefore, the above theorem can be applied.

Now, by using a somewhat more complicated argument, we can also prove

Theorem 13 *If* F *is supertriangular,* U *has a one-sided zero element* 0 *and* $x, y \in X$ *are such that*

$$0 \in F(x, y) \cap F(y, x),$$

then

(1) $\Phi(x) = \Psi(x) = F(x, y) = S(x, y)$; (2) $\Phi(x) = \Phi(x) + \Phi(y)$.

Proof. If 0 is a right zero element of U, then by using Theorem 11 we can see that

$$\begin{aligned}\Phi(x) &= \Phi(x) + \{0\} \subseteq \Phi(x) + F(x,y) \subseteq F(x,y) = F(x,y) + \{0\} \\ &\subseteq F(x,y) + F(y,x) = F(x,y) + R(x,y) = S(x,y) \subseteq \Psi(x) \subseteq \Phi(x).\end{aligned}$$

While, if 0 is a left zero element of U, then we can quite similarly obtain

$$\begin{aligned}\Phi(x) &= \{0\} + \Phi(x) \subseteq F(y,x) + \Phi(x) \subseteq F(y,x) = \{0\} + F(y,x) \\ &\subseteq F(x,y) + F(y,x) = F(x,y) + R(x,y) = S(x,y) \subseteq \Psi(x) \subseteq \Phi(x).\end{aligned}$$

Therefore, in both cases, assertion (1) is true.

Now, assertion (2) can be easily derived from assertion (1), by noticing that

$$\Phi(x) = S(x,y) = F(x,y) + R(x,y) = F(x,y) + F(y,x) = \Phi(x) + \Phi(y).$$

□

From this theorem, it is clear that in particular we also have the following

Corollary 3 *If F is supertriangular and U has a one-sided zero element 0 such that $0 \in F(x,y)$ for all $x, y \in X$, then for any $x, y \in X$ we have*
(1) $\Phi(x) = \Psi(x) = F(x,y) = S(x,y)$; (2) $\Phi(x) = \Phi(x) + \Phi(y)$.

10 The particular case when U is a group

By using an argument of Frege [15, 16] and Sincov [36, 23], we can prove

Theorem 14 *If F is a nonpartial, triangular function and U is a group, then there exists a function ξ of X to U such that*

$$F(x,y) = \xi(x) - \xi(y)$$

for all $x, y \in X$.

Proof. By choosing $z \in X$, and defining

$$\xi(x) = F(x,z)$$

for all $x \in X$, we can see that

$$F(x,y) + \xi(y) = F(x,y) + F(y,z) = F(x,z) = \xi(x),$$

and thus $F(x,y) = \xi(x) - \xi(y)$ for all $x, y \in X$. □

Remark 15 If F is nonpartial and supertriangular and U is a group, then by using a similar argument we can only prove that

$$F(x,y) \subseteq \bigcap_{z\in X} \big(F(x,z) - F(y,z)\big)$$

for all $x,y \in X$.

Now, analogously to [38, Theorem 1] of Wilhelmina Smajdor, we can also prove

Theorem 15 *If* F *is nonpartial and supertriangular,* U *is a commutative group and* ϕ *is a triangular selection function of* F*, then*

$$F(x,y) = \phi(x,y) + \Phi(x)$$

for all $x,y \in X$.

Proof. Define

$$G(x,y) = -\phi(x,y) + F(x,y)$$

for all $x,y \in X$.

Then, because of $\phi(x,y) \in F(x,y)$, we evidently have

$$0 = -\phi(x,y) + \phi(x,y) \in -\phi(x,y) + F(x,y) = G(x,y)$$

for all $x,y \in X$. Moreover, by using the assumed triangularity properties of ϕ and F, we can easily see that

$$\begin{aligned} &G(x,y) + G(y,z) = -\phi(x,y) + F(x,y) - \phi(y,z) + F(y,z) = \\ &\quad -\big(\phi(x,y) + \phi(y,z)\big) + F(x,y) + F(y,z) \subseteq -\phi(x,z) + F(x,z) = G(x,z) \end{aligned}$$

for all $x,y,z \in X$.

Hence, by using Corollary 3 and the simple observation that

$$\phi(x,x) + \phi(x,x) = \phi(x,x),$$

and thus $\phi(x,x) = 0$ for all $x \in X$, we can already infer that

$$G(x,y) = G(x,x) = -\phi(x,x) + F(x,x) = \Phi(x),$$

and thus

$$-\phi(x,y) + F(x,y) = \Phi(x)$$

for all $x,y \in X$. Therefore, the required equality is also true. $\square$

Remark 16 It can be easily seen that a converse of Theorem 14 is also true. Therefore, if F is nonpartial and U is a group, then to find a triangular selection function ϕ of F, it is enough to find only a function ξ of X to U such that

$$\xi(x) - \xi(y) \in F(x, y)$$

for all $x, y \in X$.

11 The particular case when U is a commutative groupoid

Theorem 16 *If* F *is supertriangular and* U *is commutative, then* R *is also supertriangular.*

Proof. By Definitions 6 and 7 and the commutativity of U, we have

$$\begin{aligned} R(x, y) + R(y, z) &= F(y, x) + F(z, y) \\ &= F(z, y) + F(y, x) \subseteq F(z, x) = R(x, z) \end{aligned}$$

for all $x, y, z \in X$. □

Theorem 17 *If* U *is commutative, then for any* $x, y, z \in X$ *we have*
(1) $S(x, y) = S(y, x)$; (2) $S(x, y) \subseteq \Psi(x) \cap \Psi(y)$.

Proof. By Definition 7 and the commutativity of U, we have

$$\begin{aligned} S(x, y) &= F(x, y) + R(x, y) = R(y, x) + F(y, x) \\ &= F(y, x) + R(y, x) = S(y, x). \end{aligned}$$

Moreover, by the definition of Ψ, it is clear that $S(x, y) \subseteq \Psi(x)$. Hence, by using the above symmetry property of S, we can already infer that

$$S(x, y) = S(y, x) \subseteq \Psi(y),$$

and thus $S(x, y) \subseteq \Psi(x) \cap \Psi(y)$ also holds. □

Remark 17 Thus, if U is commutative, then S is already *pointwise symmetric* in the sense that $S(x, y) = S(y, x)$ for all $x, y \in X$.

Now, concerning the relation S, we can also prove the following

Theorem 18 *If* F *is supertriangular and* U *is a commutative semigroup, then* S *is also supertriangular.*

Proof. By using Definition 7, Theorem 16 and the commutativity and associativity of U, we can see that

$$\begin{aligned} S(x,y)+S(y,z) &= F(x,y)+R(x,y)+F(y,z)+R(y,z) \\ &= F(x,y)+F(y,z)+R(x,y)+R(y,z) \\ &\subseteq F(x,z)+R(x,z)=S(x,z) \end{aligned}$$

for all $x, y, z \in X$. □

12 The particular case when F is pointwise symmetric

In addition to Theorem 17, we can also prove the following

Theorem 19 *If* $x, y \in X$ *such that* $F(x,y)=F(y,x)$*, then*
(1) $R(x,y)=F(x,y)$;
(2) $S(x,y)=S(y,x)$;
(3) $S(x,y)=F(x,y)+F(x,y)$;
(4) $2F(x,y)\subseteq S(x,y)\subseteq \Psi(x)\cap\Psi(y)$.

Proof. By Definition 7 and the assumed symmetry property of F, we have

$$R(x,y)=F(y,x)=F(x,y),$$

and thus also

$$S(x,y)=F(x,y)+R(x,y)=F(x,y)+F(x,y).$$

Thus, assertions (1) and (3) are true.

Now, we can also easily see that

$$S(y,x)=F(y,x)+F(y,x)=F(x,y)+F(x,y)=S(x,y).$$

Therefore, assertion (2) is also true.

Hence, as in the proof of Theorem 17, we can already infer that

$$S(x,y)\subseteq \Psi(x)\cap\Psi(y).$$

Therefore, to complete the proof of assertion (4), it remains to note only that now

$$2F(x,y) \subseteq F(x,y) + F(x,y) = S(x,y)$$

is also true. □

Remark 18 Thus, not only the commutativity of U, but the pointwise symmetry of F also implies the pointwise symmetry of S.

By [8], in addition to the pointwise symmetry of F, one may also naturally investigate the case when F is only *weightable* in the sense that

$$w(x) + F(x,y) = R(x,y) + w(y)$$

for all $x, y \in X$ and some function (or relation) w on X to U.

However, it is now more important to note that, as an immediate consequence of our former results, we can also state

Corollary 4 *If* F *is supertriangular and* U *is commutative, then for any* $x, y \in X$ *we have*

$$2S(x,y) \subseteq S(x,y) + S(y,x) \subseteq S(x,x) \cap S(y,y).$$

Remark 19 Note that the latter corollary only needs the important consequence of the assumed inclusion property of F that $F(x,y) + F(y,x) \subseteq F(x,x)$ for all $x, y \in X$.

In Theorem 11, by using Definition 7, the latter property has been reformulated in the shorter form that $\Psi(x) \subseteq \Phi(x)$ for all $x \in X$. Now, this already implies that Ψ is a selection relation of Φ. Namely, if $x \in X$ such that $\Phi(x) \neq \emptyset$, then because of $\Phi(x) + \Phi(x) \subseteq \Psi(x)$, we also have $\Psi(x) \neq \emptyset$.

13 The particular case when U is a group and F is pointwise skew symmetric

Analogously to Theorem 19, we can also prove the following

Theorem 20 *If* U *is a group and* $x, y \in X$ *such that* $F(x,y) = -F(y,x)$*, then*

(1) $R(x,y) = -F(x,y)$; (2) $S(x,y) = -S(y,x)$;

(3) $S(x,y) = F(x,y) - F(x,y)$; (4) $S(x,y) \subseteq \Psi(x) \cap \big(-\Psi(y)\big)$.

Proof. To prove (4), note that now, in addition to $S(x,y) \subseteq \Psi(x)$, we also have

$$S(x,y) = -S(y,x) \subseteq -\Psi(y),$$

and thus $S(x,y) \subseteq \Psi(x) \cap \big(-\Psi(y)\big)$ also holds. $\square$

Remark 20 If in addition to the assumptions of this theorem $F(x,y) \neq \emptyset$ also holds, then from assertion (3) we can infer that $0 \in S(x,y)$.

Now, by using the corresponding definitions and Theorem 20, we can also prove

Theorem 21 *If U is a group and F is pointwise skew symmetric, then for any $x \in X$ we have*
(1) $\Phi(x) = -\Phi(x)$; (2) $\Psi(x) = -\Psi(x)$.

Proof. To prove (2), note that by Definition 7 and Theorem 20 we have

$$\Psi(x) = \bigcup_{y \in X} S(x,y) = \bigcup_{y \in X} \big(-S(x,y)\big) = -\bigcup_{y \in X} S(x,y) = -\Psi(x)$$

for all $x \in X$. $\square$

Remark 21 If in addition to the assumptions of this theorem, $\Phi(x) \neq \emptyset$ also holds, then from the inclusion

$$\Phi(x) - \Phi(x) = \Phi(x) + \Phi(x) \subseteq \Psi(x),$$

we can infer that $0 \in \Psi(x)$. Therefore, if in addition F is supertriangular, then because Theorem 11, we also have $0 \in \Phi(x)$.

Thus, by Theorem 12, we can also state the following

Theorem 22 *If U is a group and F is nonpartial, supertriangular and pointwise skew symmetric, then for any $x \in X$ we have*
(1) $\Phi(x) = \Psi(x)$; (2) $\Phi(x) = \Phi(x) + \Phi(x)$.

Now, by Theorems 20 and 21, we can also state the following

Theorem 23 *If U is a group and F is a nonpartial, pointwise skew symmetric function, then for any $x, y \in X$ we have*
(1) $S(x,y) = 0$; (2) $\Phi(x) = \Psi(x) = 0$.

The following example shows the three important consequences of the inclusion considered in Definition 6 do not imply, even in a very simple case, the validity of this inclusion itself.

Example 3 *If*

$$F(x, y) = \operatorname{sgn}(x - y)$$

for all $x, y \in \mathbb{R}$*, then* F *is a skew symmetric function of* $\mathbb{R}^2$ *to* $\mathbb{R}$ *such that, under the notation* $\Phi(x) = F(x, x)$*, for any* $x, y \in X$ *we have*

(1) $F(x, y) + F(y, x) = \Phi(x)$*;*

(2) $\Phi(x) + F(x, y) = F(x, y)$*;* (3) $F(x, y) + \Phi(y) = F(x, y)$*.*

However, F *is not either supertriangular nor subtriangular in both functional and relational sense.*

Namely, for instance, we have

$$F(2, 1) + F(1, 0) = 2 \qquad \textit{and} \qquad F(2, 0) = 1,$$

and

$$F(0, 1) + F(1, 2) = -2 \qquad \textit{and} \qquad F(0, 2) = -1.$$

14 The particular case when U is cancellative

Definition 8 In what follows, we shall denote by $\operatorname{lcan}(U)$ and $\operatorname{rcan}(U)$ the family of all left-cancellable and right-cancellable elements of the groupoid U, respectively.

Moreover, we shall also write $\operatorname{can}(U) = \operatorname{lcan}(U) \cap \operatorname{rcan}(U)$.

Remark 22 Thus, for any $u \in U$, we have $u \in \operatorname{lcan}(U)$ if and only if $u + v = u + w$ implies $u = w$ for all $v, w \in U$.

Moreover, for instance, we can state that U is left-cancellative if and only if $\operatorname{lcan}(U) = U$.

Lemma 1 *For any* $V, W \subseteq U$*,*

(1) $\operatorname{card}(V + W) \leq 1$ *and* $V \cap \operatorname{lcan}(U) \neq \emptyset$ *imply that* $\operatorname{card}(W) \leq 1$*;*

(2) $\operatorname{card}(V + W) \leq 1$ *and* $W \cap \operatorname{rcan}(U) \neq \emptyset$ *imply that* $\operatorname{card}(V) \leq 1$*.*

Proof. Assume that the conditions of (1) hold, $v \in V \cap \operatorname{lcan}(U)$ and $w_1, w_2 \in W$. Then, we have $v + w_1, v + w_2 \in V + W$. Hence, by using that $\operatorname{card}(V + W) \leq 1$, we can infer that $v + w_1 = v + w_2$. Moreover, since $v \in \operatorname{lcan}(U)$, we can also state that $w_1 = w_2$. Therefore, $\operatorname{card}(W) \leq 1$, and thus (1) also holds.

The proof of assertion (2) is quite similar. □

Now, by using this lemma, we can give some reasonable sufficient conditions in order that a suppertriangular relation should be a function.

Theorem 24 *If* F *is supertriangular and there exist* $x_0, y_0 \in X$ *such that*

(1) $\operatorname{card}\big(F(x_0, y_0)\big) \leq 1$*;*

(2) $F(x, y_0) \cap \operatorname{rcan}(U) \neq \emptyset$ *for all* $x \in X$*;*

(3) $F(x_0, y) \cap \operatorname{lcan}(U) \neq \emptyset$ *for all* $y \in X$*;*

then $\operatorname{card}\big(F(x, y)\big) \leq 1$ *for all* $x, y \in X$*, and thus* F *is a function.*

Proof. By the assumed inclusion property of F, we have

$$F(x_0, x) + F(x, y_0) \subseteq F(x_0, y_0)$$

for all $x \in X$. Hence, by using conditions (1) and (3) and Lemma 1, we can infer that

(a) $\operatorname{card}\big(F(x, y_0)\big) \leq 1$ for all $x \in X$.

Now, by the assumed inclusion property of F, we also have

$$F(x, y) + F(y, y_0) \subseteq F(x, y_0)$$

for all $x, y \in X$. Hence, by using assertion (a) condition (2) and Lemma 1, we can infer that

(b) $\operatorname{card}\big(F(x, y)\big) \leq 1$ for all $x, y \in X$.

Thus, the required assertion is true. □

From this theorem, by using Theorem 14, we can immediately derive

Corollary 5 *If* F *is nonpartial and supertriangular,* U *is a group and* $\operatorname{card}\big(F(x_0, y_0)\big) = 1$ *for some* $x_0, y_0 \in X$*, then there exists a function* ξ *of* X *to* U *such that*

$$F(x, y) = \xi(x) - \xi(y)$$

for all $x, y \in X$*.*

15 The particular case when U has a suitable distance function

Remark 23 A function d of X^2 to $[0, +\infty]$ is usually called a *distance function* on X.

Moreover, the extended real number

$$d(X) = \mathrm{diam}(X) = \sup\big\{ d(x, y) : \quad x, y \in X \big\}$$

is called the *diameter* of X.

Remark 24 Thus, we have $d(X) = -\infty$ if $X = \emptyset$, and $d(X) \geq 0$ if $X \neq \emptyset$. Moreover, if $X \neq \emptyset$, then $\mathrm{card}(X) = +\infty$ may also hold even if X is finite.

Definition 9 A distance function d on X will be called *admissible* if
(a) $d(X) < +\infty$;
(b) $d(x, y) = 0$ implies $x = y$ *for all* $x, y \in X$.
Moreover, the distance function d will be called *extremal* if
(c) for any $x, y \in X$ there exist $c \in \,]1, +\infty[$ and $z, w \in X$ such that

$$c d(x, y) \leq d(z, w).$$

Remark 25 If X is an additive groupoid, then to satisfy condition (c) we may naturally assume that for any $x, y \in X$, there exists $n \in \mathbb{N} \setminus \{1\}$ such that

$$n d(x, y) \leq d(nx, ny).$$

Namely, if X is a commutative abelian group and p is a function of U to $[0, +\infty]$ such that

$$n p(x) \leq p(nx)$$

for all $n \in \mathbb{N}$ and $x \in X$, then by defining

$$d(x, y) = p(-x + y)$$

for all $x, y \in X$, we have

$$\begin{aligned} n d(x, y) &= n p(-x + y) \leq p\big(n(-x + y)\big) \\ &= p\big(n(-x) + ny\big) = p(-nx + ny) = d(nx, ny) \end{aligned}$$

for all $n \in \mathbb{N}$ *and* $x, y \in U$.

The introduction of Definition 9 can only be motivated by the following

Lemma 2 *If there exists an extremal, admissible distance function* d *on* X, *then* $\mathrm{card}(X) \leq 1$.

Proof. If $X = \emptyset$, then the required assertion trivially holds. Therefore, we may assume that $X \neq \emptyset$, and thus $d(X) \neq -\infty$. Now, by condition (a), we can state that $d(X) \in \mathbb{R}$. Moreover, since d is nonnegative, we can now also note that $d(X) \geq 0$.

Thus, for every $\varepsilon > 0$, we have

$$d(X) - \varepsilon < d(X).$$

Therefore, by the definition of $d(X)$, there exist $x, y \in X$ such that $d(X) - \varepsilon < d(x, y)$, and thus

$$d(X) < d(x, y) + \varepsilon.$$

Moreover, by condition (c), there exist $c \in]1, +\infty[$ and and $z, w \in X$ such that

$$c d(xy) \leq d(z, w).$$

Combining the above two inequalities, we can see that

$$c d(x, y) < d(z, w) \leq d(X) < d(x, y) + \varepsilon,$$

and thus $(c-1) d(x, y) < \varepsilon$. Hence, by letting ε tend to zero, we can infer that $(c-1) d(x, y) \leq 0$. Therefore, since $c - 1 > 0$, we necessarily have $d(x, y) \leq 0$, and hence $d(x, y) \leq 0$ by the nonnegativity of d. Thus, we actually have

$$d(X) < d(x, x) + \varepsilon = \varepsilon.$$

Hence, by letting ε tend to zero, we can infer that $d(X) \leq 0$, and thus also $d(X) = 0$ by the nonnegativity of $d(X)$.

This, by condition (b), already implies that $\operatorname{card}(X) = 1$. Namely, if this is not the case, then by the assumption $X \neq \emptyset$, there exist $x, y \in X$ such that $x \neq y$. Hence, by condition (b) and the nonnegativity of d, we can infer that $d(x, y) > 0$, and thus also $d(X) > 0$ by the definition of $d(X)$. This contradiction proves that $\operatorname{card}(X) = 1$. □

Remark 26 From condition (c), by induction, we can infer that there exist sequences $(c_n)_{n=1}^{\infty}$ in $]1, +\infty[$ and $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in X such that

$$d(x, y) \prod_{i=0}^{n} c_i \leq d(x_n, y_n)$$

for all $n \in \mathbb{N}$. However, this fact cannot certainly be used to give a simpler proof for Lemma 2.

From Theorem 24, by using Lemma 2, we can immediately derive

Theorem 25 *If* F *is supertriangular and there exist* $x_0, y_0 \in X$*, such that*

(1) $F(x, y_0) \cap \operatorname{rcan}(U) \neq \emptyset$*; for all* $x \in X$*;*

(2) $F(x_0, y) \cap \operatorname{lcan}(U) \neq \emptyset$*; for all* $y \in X$*;*

(3) *there exists an extremal, admissible distance function on* $F(x_0, y_0)$*;*

then $\operatorname{card}\big(F(x, y)\big) \leq 1$ *for all* $x, y \in X$*, and thus* F *is a function.*

Proof. By assumption (3) and Lemma 2, we have $\operatorname{card}\big(F(x_0, y_0)\big) \leq 1$. Hence, by Theorem 24, we can see that the required assertion is also true. □

16 Contructions of supertriangular relations

Theorem 26 *If* V *is a subgroupoid of* U *and*

$$F(x, y) = V$$

for all $x, y \in X$*, then* F *is a supertriangular relation on* X *to* U*.*

Proof. We evidently have

$$F(x, y) + F(y, z) = V + V \subseteq V = F(x, z)$$

for all $x, y, z \in X$. □

Remark 27 Conversely, note that if F is a supertriangular relation on X^2 to U, then by Corollary 2 $\Phi(x) = F(x, x)$ is a subgroupoid of U for all $x \in X$.

Now, as a converse to Theorem 14, we can also easily prove the following

Theorem 27 *If* ξ *is a function of* X *to* U*,* U *is a group and*

$$F(x, y) = \xi(x) - \xi(y)$$

for all $x, y \in X$*, then* F *is a triangular function of* X^2 *to* U*.*

Proof. We evidently have

$$F(x, y) + F(y, z) = \xi(x) - \xi(y) + \xi(y) - \xi(z) = \xi(x) - \xi(z) = F(x, z)$$

for all $x, y, z \in X$. □

Remark 28 If ξ is only a relation of X to U, U is a group and $F(x, y) = \xi(x) - \xi(y)$ for all $x, y \in X$, then by using a similar argument we can only prove that F is a subtriangular relation of X^2 to U.

In addition to the above two theorems, it is also worth proving that the family of all supertriangular relations is closed under the usual pointwise operations.

Theorem 28 *If F is a supertriangular relation on X^2 to U and U is a commutative semigroup, then nF is also a supertriangular relation on X^2 to U for all $n \in \mathbb{N}$.*

Proof. If $n \in \mathbb{N}$, then by the corresponding definitions we have

$$\begin{aligned}\big(nF\big)(x, y) + \big(nF\big)(y, z) &= nF(x, y) + nF(y, z) \\ &= n\big(F(x, y) + F(y, z)\big) \subseteq nF(x, z) = \big(nF\big)(x, z)\end{aligned}$$

for all $x, y, z \in X$. □

Remark 29 If F is a supertriangular relation on X^2 to U and U has a zero element, then

$$\big(0F\big)(x, y) = \emptyset \quad \text{if} \quad F(x, y) = \emptyset \qquad \text{and} \qquad \big(0F\big)(x, y) = \{0\} \quad \text{if} \quad F(x, y) \neq \emptyset.$$

Therefore, $0F$ is a supertriangular function on X^2 to U.

Now, analogously to Theorem 28, we can also prove the following

Theorem 29 *If F is a supertriangular relation on X^2 to U and U is a commutative group, then kF is also a supertriangular relation on X^2 to U for all $k \in \mathbb{Z}$.*

Moreover, in addition to Theorems 28, we can also easily prove the following

Theorem 30 *If F and G are supertriangular relations on X^2 to U and U is a commutative semigroup, then $F + G$ is also a supertriangular relation on X^2 to U.*

Proof. By the corresponding definitions, it is clear that

$$\begin{aligned}&(F + G)(x, y) + (F + G)(y, z) = F(x, y) + G(x, y) + F(y, z) + G(y, z) \\ &\quad = F(x, y) + F(y, z) + G(x, y) + G(y, z) \subseteq F(x, z) + G(x, z) = (F + G)(x, z)\end{aligned}$$

for all $x, y, z \in X$. □

17 An application of the above results

Now, by using Theorems 26, 27 and 30, we can also easily establish

Theorem 31 *If* ξ *a function of* X *to* U, U *is a commutative group,* V *is a subgroupoid of* U *and*

$$F(x,y) = \xi(x) - \xi(y) + V$$

for all $x, y \in X$, *then* F *is a supertriangular relation on* X^2 *to* U *such that, under the notations of Definition 7, for any* $x, y \in X$ *we have:*

(1) $\Phi(x) = V$; (2) $S(x,y) = V + V$;

(3) $\Psi(x) = \emptyset$ *if* $X = \emptyset$ *and* $\Psi(x) = V + V$ *if* $X \neq \emptyset$.

Proof. From Theorems 26, 27 and 30, it is clear that F is supertriangular. Moreover, by the corresponding definitions, it is clear that

$$\Phi(x) = F(x,x) = \xi(x) - \xi(x) + V = V,$$

$$S(x,y) = F(x,y) + F(y,x) = \xi(x) - \xi(y) + V + \xi(y) - \xi(x) + V = V + V$$

and

$$\Psi(x) = \bigcup_{y \in X} S(x,y) = \bigcup_{y \in X} (V+V) = \begin{cases} \emptyset & \text{if} \quad X = \emptyset, \\ V+V & \text{if} \quad X \neq \emptyset. \end{cases}$$

□

Moreover, for an easy illustration of this theorem, we can also state

Example 4 *If* $r \geq 0$ *and*

$$F(x,y) = [x - y + r, \ +\infty[$$

for all $x, y \in \mathbb{R}$, *then* F *is a supertriangular relation of* $\mathbb{R}^2$ *to* $\mathbb{R}$ *such that, for any* $x, y \in X$, *we have:*

(1) $\Phi(x) = [r, +\infty[$; (2) $\Psi(x) = S(x,y) = [2r, +\infty[$.

To check this, note that, by taking $\xi = \Delta_{\mathbb{R}}$ *and* $V = [r, +\infty[$, *we have*

$$F(x,y) = [x - y + r, \ +\infty[\ = x - y + [r, +\infty[\ = \xi(x) - \xi(y) + V$$

for all $x, y \in X$. *Therefore, Theorem 31 can be applied.*

For instance, by assertion (2) of Theorem 31, we have

$$S(x,y) = V + V = [r, +\infty[\ + [r, +\infty[\ = [2r, +\infty[$$

for all $x, y \in X$.

Remark 30 Note that in the present particular case, for any $x, y \in \mathbb{R}$, we have:

(1) $0 \in \Phi(x) \iff r = 0$;
(2) $\Phi(x) = \Psi(x) \iff r = 0$;
(3) $x - y \in F(x, y) \iff r = 0$;
(4) $0 \in F(x, y) \iff r \leq y - x$;
(5) $0 \in F(x, y) \cap F(y, x) \iff r = 0, \quad x = y$.

To prove (5), note that by (4) we have

$$0 \in F(x, y) \cap F(y, x) \iff r \leq y - x,\ r \leq x - y \iff r \leq \min\{x - y,\ y - x\}.$$

Moreover, recall that $\min\{a, b\} = 2^{-1}\big(a + b - |a - b|\big)$ for all $a, b \in \mathbb{R}$, and thus in particular $\min\{x - y,\ y - x\} = -|x - y|$. Therefore,

$$r \leq \min\{x - y,\ y - x\} \iff r \leq -|x - y| \iff r = 0, \quad x = y.$$

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Direct and converse theorems for King operators

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Abstract. For the sequence of King operators, we establish a direct approximation theorem via the first order Ditzian-Totik modulus of smoothness, and a converse approximation theorem of Berens-Lorentz-type.

1 Introduction

Studying the connection between regular summability matrices and convergent positive linear operators, King [3] introduced an interesting Bernstein-type operator defined as follows:

$$(V_n f)(x) \equiv V_n(f;x) = \sum_{k=0}^{n} p_{n,k}(r_n(x)) f\left(\frac{k}{n}\right), \tag{1}$$

where $x \in [0,1]$, $f \in C[0,1]$, $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and

$$r_n(x) = \begin{cases} x^2, & \text{if} \quad n = 1 \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & \text{if} \quad n = 2, 3, \ldots \end{cases} \tag{2}$$

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For $r_n(x) = x$, $x \in [0,1]$, we recover from (1) the classical Bernstein operator:

$$(B_n f)(x) \equiv B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right).$$

It is known that

$$(B_n e_0)(x) = 1, \quad (B_n e_1)(x) = x \quad \text{and} \quad (B_n e_2)(x) = x^2 + \frac{x(1-x)}{n}, \tag{3}$$

where $e_j(x) = x^j$, $x \in [0,1]$ and $j \in \{0,1,2,\dots\}$. In contrast with (3), we have for V_n the relations (see [3, pp. 204-205]):

$$(V_n e_0)(x) = 1, \quad (V_n e_1)(x) = r_n(x) \quad \text{and} \quad (V_n e_2)(x) = x^2. \tag{4}$$

The goal of the paper is to obtain direct and converse approximation theorems for the operators given by (1)-(2). The direct result is established with the aid of the first order Ditzian-Totik modulus of smoothness defined by

$$\omega_\varphi^1(f;\delta) = \sup_{0<h\le\delta} \sup_{x\pm\frac{1}{2}h\varphi(x)\in[0,1]} \left| f(x+\frac{1}{2}h\varphi(x)) - f(x-\frac{1}{2}h\varphi(x)) \right|, \tag{5}$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. It is known [2, Theorem 2.1.1] that (5) is equivalent with the K-functional

$$K_{1,\varphi}(f;\delta) = \inf_{g\in W(\varphi)} \{\|f-g\| + \delta\|\varphi g'\|\}, \quad \delta > 0,$$

where $W(\varphi) = \{g \,|\, g \in A.C._{loc}[0,1], \|\varphi g'\| < \infty\}$, i.e. there exists $C_1 > 0$ such that

$$C_1^{-1}\omega_\varphi^1(f;\delta) \le K_{1,\varphi}(f;\delta) \le C_1\omega_\varphi^1(f;\delta). \tag{6}$$

Finally, a converse result of Berens-Lorentz-type is established for the operators V_n (see [1, p. 312, Lemma 5.2] and Lemma 3 below). Throughout this paper $C_1, C_2, \dots, C_{13}$ denote absolute positive constants.

2 Direct theorem

We have the following result for the functions defined by (2).

Lemma 1 *The functions* r_n, $n = 1,2,\dots,$ *satisfy the properties*

a) $0 \le r_n'(x) \le 2$ *for* $x \in [0,1]$ *and* $n = 1, 2, \ldots;$

b) $r_n(0) = 0$, $r_n(1) = 1$ *and* r_n *is strictly increasing function on* $[0,1]$ *for* $n = 1, 2, \ldots;$

c) $0 \le r_n(x) \le x \le 1$ *for* $x \in [0,1]$ *and* $n = 1, 2, \ldots;$

d) $0 \le x - r_n(x) \le \frac{2}{n}(1-x)$ *for* $x \in [0,1]$ *and* $n = 1, 2, \ldots;$

e) $x \le 2r_n(x)$ *for* $x \in \left[\frac{1}{n}, 1\right]$ *and* $n = 1, 2, \ldots$

Proof. a) Obviously $r_1'(x) = 2x$, $x \in [0,1]$. For $n \ge 2$, by simple computations, we obtain

$$r_n'(x) = \begin{cases} \lim\limits_{x \searrow 0} \dfrac{r_n(x) - r_n(0)}{x - 0} = 0, & \text{if} \quad x = 0 \\ \dfrac{\frac{n}{n-1}x}{\sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}}, & \text{if} \quad 0 < x \le 1. \end{cases}$$

Hence $0 \le r_n'(x) \le \dfrac{\frac{n}{n-1}x}{\sqrt{\frac{n}{n-1}}x} = \sqrt{\dfrac{n}{n-1}} \le \sqrt{2}$ for $x \in (0,1]$. Thus $0 \le r_n'(x) \le 2$ for $n = 1, 2, \ldots$ and $x \in [0,1]$.

b) It follows from (2) and a).

c) It follows from (2) by direct computations.

d) Obviously $0 \le x - r_1(x) = x(1-x) \le 2(1-x)$, $x \in [0,1]$. Using b) and c), we have $0 \le x - r_n(x) \le \frac{2}{n}(1-x)$ for $x = 0$ and $n \ge 2$, and

$$\begin{aligned} 0 \le x - r_n(x) &= x + \frac{1}{2(n-1)} - \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}} \\ &= \frac{\frac{x(1-x)}{n-1}}{x + \frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}} \\ &\le \frac{\frac{x(1-x)}{n-1}}{x} = \frac{1-x}{n-1} \le \frac{2}{n}(1-x) \end{aligned}$$

for $x \in (0,1]$ and $n \ge 2$.

e) For $n = 1$ the statement is obvious. For $n \ge 2$, we consider the function $h(x) = \frac{x}{r_n(x)}$, $x \in \left[\frac{1}{n}, 1\right]$. Then, by (2),

$$h'(x) = \frac{r_n(x) - x r_n'(x)}{r_n^2(x)} = r_n^{-2}(x) \left(r_n(x) + \frac{1}{2(n-1)} \right)^{-1/2}$$

$$\times \frac{1}{2(n-1)}\left[\frac{1}{2(n-1)}-\sqrt{\frac{n}{n-1}x^2+\frac{1}{4(n-1)^2}}\right]<0$$

for $x \in \left[\frac{1}{n}, 1\right]$. Hence

$$\begin{aligned} h(x) &\leq h\left(\frac{1}{n}\right)=\frac{\frac{1}{n}}{-\frac{1}{2(n-1)}+\sqrt{\frac{n}{n-1}\frac{1}{n^2}+\frac{1}{4(n-1)^2}}} \\ &= \frac{2(n-1)}{\sqrt{n}}\frac{1}{\sqrt{5n-4}-\sqrt{n}}=\frac{1}{2}\frac{\sqrt{5n-4}+\sqrt{n}}{\sqrt{n}} \leq 2 \end{aligned}$$

for $x \in \left[\frac{1}{n}, 1\right]$, which was to be proved. □

The operators V_n given by (1)-(2) are linear and positive. By Lemma 1, b), we have

$$(V_n f)(0)=f(0) \quad \text{and} \quad (V_n f)(1)=f(1) \tag{7}$$

for all $f \in C[0,1]$.

In the next theorem we establish the direct result.

Theorem 1 *There exists* $C_2>0$ *such that*

$$\|V_n f-f\| \leq C_2\, \omega_\varphi^1\left(f; \frac{1}{\sqrt{n}}\right) \tag{8}$$

for all $f \in C[0,1]$ *and* $n=1,2,\ldots$

Proof. Let $x \in (0,1)$ and $t \in [0,1]$. Taking into account [2, Lemma 9.6.1], we have

$$\left|\int_x^t \frac{du}{\varphi(u)}\right| \leq \varphi^{-1}(x)|t-x|^{1/2}\left|\int_x^t \frac{du}{|t-u|^{1/2}}\right|=2\varphi^{-1}(x)|t-x|. \tag{9}$$

Further, for $g \in W(\varphi)$, we have $g(t)=g(x)+\int_x^t g'(u)\,du$, $t \in [0,1]$ and $x \in (0,1)$. Hence, by (9), Hölder's inequality, (4) and Lemma 1, d), we get

$$\begin{aligned} |V_n(g;x)-g(x)| &= \left| V_n\left(\int_x^t g'(u)\,du; x\right)\right| \\ &\leq V_n\left(\left|\int_x^t |g'(u)|\,du\right|; x\right) \leq \|\varphi g'\| V_n\left(\left|\int_x^t \frac{du}{\varphi(u)}\,du\right|; x\right) \end{aligned}$$

$$
\begin{aligned}
&\leq 2\varphi^{-1}(x)\|\varphi g'\| V_n(|t-x|;x) \leq 2\varphi^{-1}(x)\|\varphi g'\| \left(V_n((t-x)^2;x)\right)^{1/2} \\
&= 2\varphi^{-1}(x)\|\varphi g'\| \left(V_n(e_2;x) - 2xV_n(e_1;x) + x^2 V_n(e_0;x)\right)^{1/2} \\
&= 2\varphi^{-1}(x)\|\varphi g'\|(x^2 - 2xr_n(x) + x^2)^{1/2} = 2\varphi^{-1}(x)\|\varphi g'\|(2x(x-r_n(x)))^{1/2} \\
&\leq 2\varphi^{-1}(x)\|\varphi g'\| \left(2x\frac{2}{n}(1-x)\right)^{1/2} = \frac{4}{\sqrt{n}}\|\varphi g'\|. \qquad (10)
\end{aligned}
$$

Due to (7), the estimation (10) is also valid for $x \in \{0, 1\}$.

On the other hand, by (4), we obtain $|(V_n f)(x)| \leq \sum_{k=0}^{n} p_{n,k}(r_n(x)) \left| f\left(\frac{k}{n}\right) \right| \leq \|f\| \sum_{k=0}^{n} p_{n,k}(r_n(x)) \leq \|f\|$, therefore

$$
\|V_n f\| \leq \|f\| \qquad (11)
$$

for all $f \in C[0, 1]$.

Now, in view of (4), (10) and (11), we find that

$$
\begin{aligned}
|V_n(f;x) - f(x)| &\leq |V_n(f-g;x)| + |V_n(g;x) - g(x)| + |g(x) - f(x)| \\
&\leq 2\|f-g\| + \frac{4}{\sqrt{n}}\|\varphi g'\| \leq 4\left\{\|f-g\| + \frac{1}{\sqrt{n}}\|\varphi g'\|\right\}.
\end{aligned}
$$

Taking the infimum on the right hand side over all $g \in W(\varphi)$, we obtain

$$
\|V_n f - f\| \leq 4\, K_{1,\varphi}\left(f; \frac{1}{\sqrt{n}}\right).
$$

Hence, by (6), we arrive at (8), which completes the proof. □

3 Converse theorem

We begin with the following remark.

Remark 1 *Due to (8), the condition* $\omega_\varphi^1(f;\delta) \leq C_3\delta^\alpha$, $\delta > 0$, $0 < \alpha < 1$ *implies that* $\|V_n f - f\| \leq C_4 n^{-\alpha/2}$, $n \geq 1$.

In what follows, we establish the converse result of the statement given in Remark 1. To achieve this we need some lemmas.

Lemma 2 *We have*

a) $\|\varphi(V_n f)'\| \le 8\sqrt{n}\|f\|$ *for* $f \in C[0,1]$ *and* $n = 1,2,\ldots;$

b) $\|\varphi(V_n g)'\| \le 32\|\varphi g'\|$ *for* $g \in W(\varphi)$ *and* $n = 1,2,\ldots$

Proof. a) Let $x \in (0,1)$. By [1, p. 305, (2.1)], we have for the derivatives of $p_{n,k}$ that

$$p'_{n,k}(x) = n\,[p_{n-1,k-1}(x) - p_{n-1,k}(x)] = \varphi^{-2}(x)(k-nx)p_{n,k}(x), \tag{12}$$

where $k = 1,2,\ldots,n$ and $p_{n-1,-1}(x) = p_{n-1,n}(x) = 0$. We distinguish two cases: $\underline{x \in (0, \frac{1}{n}]}$. By (1), (12), Lemma 1, a) and (4), we get

$$\begin{aligned}
|\varphi(x)(V_n f)'(x)| &= \varphi(x)r'_n(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))\, f\left(\frac{k}{n}\right)\right| \\
&= n\varphi(x)r'_n(x)\left|\sum_{k=0}^{n} [p_{n-1,k-1}(r_n(x)) - p_{n-1,k}(r_n(x))]\, f\left(\frac{k}{n}\right)\right| \\
&= n\varphi(x)r'_n(x)\left|\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x))\, \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right]\right| \\
&\le n\varphi(x)r'_n(x)\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| \\
&\le 2n\varphi(x)r'_n(x)\|f\|\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \le 4n\sqrt{x(1-x)}\|f\| \\
&\le 4\sqrt{n}\|f\|.
\end{aligned} \tag{13}$$

$\underline{x \in [\frac{1}{n}, 1)}$. Using (1), Lemma 1, a), (12), Hölder's inequality, (4) and Lemma 1, c), d), e), we get

$$\begin{aligned}
&|\varphi(x)(V_n f)'(x)| = \varphi(x)r'_n(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))\, f\left(\frac{k}{n}\right)\right| \\
&\le \quad 2\varphi(x)\|f\|\sum_{k=0}^{n} \varphi^{-2}(r_n(x))|k - nr_n(x)|p_{n,k}(r_n(x)) \\
&\le \quad 2\varphi(x)\varphi^{-2}(r_n(x))\|f\|\left(\sum_{k=0}^{n} (k-nr_n(x))^2 p_{n,k}(r_n(x))\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|\left(V_n(e_2;x)-2r_n(x)V_n(e_1;x)+r_n^2(x)V_n(e_0;x)\right)^{1/2}\\
&= 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|\left(x^2-2r_n^2(x)+r_n^2(x)\right)^{1/2}\\
&= 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|(x+r_n(x))^{1/2}(x-r_n(x))^{1/2}\\
&\le 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|\sqrt{2x}\sqrt{\frac{2}{n}(1-x)}=4\sqrt{n}\frac{\varphi^2(x)}{\varphi^2(r_n(x))}\|f\|\\
&= 4\sqrt{n}\frac{x}{r_n(x)}\frac{1-x}{1-r_n(x)}\|f\|\le 4\sqrt{n}\cdot 2\cdot 1\cdot\|f\|=8\sqrt{n}\|f\|. \qquad (14)
\end{aligned}
$$

Finally, by Lemma 1, a), we get $\varphi(0)(V_nf)'(0)=\varphi(0)\sum_{k=0}^{n}p_{n,k}(r_n(0))r_n'(0)f\left(\frac{k}{n}\right)$ $=0$ and $\varphi(1)(V_nf)'(1)=\varphi(1)\sum_{k=0}^{n}p_{n,k}(r_n(1))r_n'(1)f\left(\frac{k}{n}\right)=0$. Hence, due to (13) and (14), we obtain $\|\varphi(V_nf)'\|\le 8\sqrt{n}\|f\|$, which was to be proved.

b) The proof is similar to the above. Let $x\in\overline{\left(0,\frac{1}{n}\right]}$. Taking into account (1), (12), Lemma 1, a), (9), Hölder's inequality and (4), we get for $g\in W(\varphi)$ that

$$
\begin{aligned}
|(V_ng)'(x)| &= nr_n'(x)\left|\sum_{k=0}^{n}\left[p_{n-1,k-1}(r_n(x))-p_{n-1,k}(r_n(x))\right]g\left(\frac{k}{n}\right)\right|\\
&= nr_n'(x)\left|\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left[g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)\right]\right|\\
&\le 2n\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left|g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)\right|\\
&\le 2n\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|g\left(\frac{k+1}{n}\right)-g(x)\right|+\left|g\left(\frac{k}{n}\right)-g(x)\right|\right\}\\
&\le 2n\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|\int_x^{\frac{k+1}{n}}|g'(u)|\,du\right|+\left|\int_x^{\frac{k}{n}}|g'(u)|\,du\right|\right\}\\
&\le 2n\|\varphi g'\|\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|\int_x^{\frac{k+1}{n}}\frac{du}{\varphi(u)}\right|+\left|\int_x^{\frac{k}{n}}\frac{du}{\varphi(u)}\right|\right\}\\
&\le 4n\varphi^{-1}(x)\|\varphi g'\|\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|\frac{k+1}{n}-x\right|+\left|\frac{k}{n}-x\right|\right\}
\end{aligned}
$$

$$\leq \quad 4n\varphi^{-1}(x)\|\varphi g'\| \left\{ \left(\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k+1}{n} - x \right)^2 \right)^{1/2} \right.$$
$$\left. + \left(\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n} - x \right)^2 \right)^{1/2} \right\}. \tag{15}$$

Further, by (3), Lemma 1, c) and $x \in \left(0, \frac{1}{n}\right]$, we obtain

$$\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n} - x \right)^2 = \left(\frac{n-1}{n} \right)^2 \sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n-1} \right)^2$$
$$-2x\frac{n-1}{n} \sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \frac{k}{n-1} + x^2 \sum_{k=0}^{n-1} p_{n-1,k}(r_n(x))$$
$$= \left(\frac{n-1}{n} \right)^2 \left[r_n^2(x) + \frac{1}{n-1} r_n(x)(1 - r_n(x)) \right] - 2x\frac{n-1}{n} r_n(x) + x^2$$
$$= \frac{(n-1)(n-2)}{n^2} r_n^2(x) + \frac{n-1}{n} \left(\frac{1}{n} - 2x \right) r_n(x) + x^2$$
$$\leq \frac{(n-1)(n-2)}{n^2} r_n^2(x) + \frac{n-1}{n} \left(\frac{1}{n} + 2x \right) r_n(x) + x^2$$
$$\leq \frac{(n-1)(n-2)}{n^2} \frac{1}{n^2} + \frac{n-1}{n} \left(\frac{1}{n} + \frac{2}{n} \right) \frac{1}{n} + \frac{1}{n^2}$$
$$\leq \frac{1}{n^2} + \frac{3}{n^2} + \frac{1}{n^2} = \frac{5}{n^2}. \tag{16}$$

Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ with a and b real numbers, (16) and (3), we obtain

$$\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k+1}{n} - x \right)^2$$
$$\leq \quad 2\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n} - x \right)^2 + 2\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \frac{1}{n^2}$$
$$\leq \quad \frac{10}{n^2} + \frac{2}{n^2} = \frac{12}{n^2}. \tag{17}$$

Combining (15), (16) and (17), we get

$$|\varphi(x)(V_n g)'(x)| \leq 4n\|\varphi g'\| \left(\frac{\sqrt{12}}{n} + \frac{\sqrt{5}}{n} \right) \leq 23\|\varphi g'\|. \tag{18}$$

Let $x \in [\frac{1}{n}, 1)$. For $g \in W(\varphi)$, by (1), (4), (12), Lemma 1, a), (9), Hölder's inequality, (4) and Lemma 1, c), d), e), we find that

$$
\begin{aligned}
&|\varphi(x)(V_n g)'(x)| = \varphi(x)|(V_n g)'(x) - g(x)(V_n e_0)'(x)| \\
&= \varphi(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))r'_n(x)\left[g(\frac{k}{n}) - g(x)\right]\right| \\
&= \varphi(x)r'_n(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))\int_x^{\frac{k}{n}} g'(u)\,du\right| \\
&\leq 2\varphi(x)\left|\sum_{k=0}^{n} \varphi^{-2}(r_n(x))(k - nr_n(x))p_{n,k}(r_n(x))\int_x^{\frac{k}{n}} g'(u)\,du\right| \\
&\leq 2\varphi(x)\varphi^{-2}(r_n(x))\sum_{k=0}^{n} |k - nr_n(x)|p_{n,k}(r_n(x))\left|\int_x^{\frac{k}{n}} |g'(u)|\,du\right| \\
&\leq 4\varphi^{-2}(r_n(x))\|\varphi g'\|\sum_{k=0}^{n} |k - nr_n(x)|\left|\frac{k}{n} - x\right|p_{n,k}(r_n(x)) \\
&\leq 4n\varphi^{-2}(r_n(x))\|\varphi g'\|\left(\sum_{k=0}^{n}\left(\frac{k}{n} - r_n(x)\right)^2 p_{n,k}(r_n(x))\right)^{1/2} \\
&\quad\times\left(\sum_{k=0}^{n}\left(\frac{k}{n} - x\right)^2 p_{n,k}(r_n(x))\right)^{1/2} \\
&= 4n\varphi^{-2}(r_n(x))\|\varphi g'\|\left(V_n(e_2;x) - 2r_n(x)V_n(e_1;x) + r_n^2(x)V_n(e_0;x)\right)^{1/2} \\
&\quad\times\left(V_n(e_2;x) - 2xV_n(e_1;x) + x^2V_n(e_0;x)\right)^{1/2} \\
&= 4n\varphi^{-2}(r_n(x))\|\varphi g'\|\left(x^2 - r_n^2(x)\right)^{1/2}\left(2x^2 - 2xr_n(x)\right)^{1/2} \\
&= 4\sqrt{2}n\varphi^{-2}(r_n(x))\|\varphi g'\|\,(x + r_n(x))^{1/2}\sqrt{x}\,(x - r_n(x)) \\
&\leq 4\sqrt{2}n\varphi^{-2}(r_n(x))\|\varphi g'\|\sqrt{2x}\sqrt{x}\frac{2}{n}(1 - x) = 16\frac{\varphi^2(x)}{\varphi^2(r_n(x))}\|\varphi g'\| \\
&= 16\frac{x}{r_n(x)}\frac{1 - x}{1 - r_n(x)}\|\varphi g'\| \leq 32\|\varphi g'\|. \qquad (19)
\end{aligned}
$$

Finally, we have $\varphi(0)(V_n g)'(0) = 0 = \varphi(1)(V_n g)'(1)$. Hence, by (18) and (19), we obtain $\|\varphi(V_n g)'\| \leq 32\|\varphi g'\|$, which completes the proof. □

The next result is a weak-type version of the Berens-Lorentz lemma (see [1,

p. 312, Lemma 5.2]).

Lemma 3 *Let* $\phi : [0, a] \to [0, \infty)$ *be an increasing function with* $\phi(0) = 0$ *and* $0 < \alpha < 1$. *If* $0 < a \leq 1$, *then the inequalities*

$$\phi(a) \leq C_5 a^\alpha \tag{20}$$

and

$$\phi(x) \leq C_5 \left(y^\alpha + \frac{x}{y} \phi(y) \right), \quad 0 \leq x \leq y \leq a \tag{21}$$

imply for some $C_6 = C_6(\alpha) > 0$ *that*

$$\phi(x) \leq C_6 C_5 x^\alpha, \quad 0 \leq x \leq a. \tag{22}$$

Proof. Following the proof of Lemma 5.2 in [1, p. 312], it is easy to prove our result taking into account the slight modification on α. For completeness we give the proof.

For $0 < q < 1$, we define $x_k = q^k a$, $k = 0, 1, 2, \ldots$ If we take $C \geq 1$, then (20) implies (22) for $x = x_0$. We prove (22) for all $x = x_k$ by induction. Let $\phi(x_k) \leq C C_5 x_k^\alpha$, then, by (21),

$$\phi(x_{k+1}) \leq C_5 (x_k^\alpha + q\phi(x_k)) \leq C_5 (1 + qCC_5) x_k^\alpha \leq C_5 C x_{k+1}^\alpha,$$

provided $1 + qCC_5 \leq Cq^\alpha$. To achieve this, we first take q so small that $q^\alpha > C_5 q$, because $0 < \alpha < 1$, and then C sufficiently large. After this, for any $0 < x < a$, we select a k with $x_{k+1} \leq x \leq x_k$ and get with $C_6 := Cq^{-\alpha}$ the estimations $\phi(x) \leq \phi(x_k) \leq CC_5 x_k^\alpha \leq C_6 C_5 x^\alpha$. □

In the next theorem we establish the converse result. We set $C_{01}[0, 1] = \{f \in C[0, 1] : f(0) = f(1)\}$.

Theorem 2 *For* $f \in C_{01}[0, 1]$, $0 < \alpha < 1$ *and* V_n *defined by (1)-(2), the estimation*

$$\|V_n f - f\| \leq C_7 n^{-\alpha/2}, \quad n = 1, 2, \ldots \tag{23}$$

implies $\omega_\varphi^1(f; \delta) \leq C_8 \delta^\alpha$, $0 < \delta \leq 1$.

Proof. The proof is based on Lemma 3 with $\phi(t) = \omega_\varphi^1(f; t)$, $t \in [0, 1]$. For $f \in C_{01}[0, 1]$, by Lemma 1, b), we have

$$\begin{aligned}
(V_1 f)(x) &= p_{1,0}(r_1(x)) f(0) + p_{1,1}(r_1(x)) f(1) = (1 - x^2) f(0) + x^2 f(1) \\
&= f(0) + x^2 (f(1) - f(0)) = f(0).
\end{aligned}$$

Therefore, by (5), $\omega_\varphi^1(f-V_1f;t)=\omega_\varphi^1(f;t)$, $t>0$. Hence, due to (5) and (23),

$$\omega_\varphi^1(f;1)=\omega_\varphi^1(f-V_1f;1)\le 2\|f-V_1f\|\le 2C_7. \tag{24}$$

Let $x\in[0,1]$ and $h>0$ such that $x\pm\frac{h}{2}\in[0,1]$, and let $\Delta_h^1 f(x)=f(x+\frac{h}{2})-f(x-\frac{h}{2})$. Then, by (23),

$$\begin{aligned}|\Delta_h^1 f(x)| &\le |\Delta_h^1(f-V_nf)(x)|+|\Delta_h^1(V_nf)(x)|\\ &\le 2\|f-V_nf\|+|\Delta_h^1(V_nf)(x)|\le 2C_7n^{-\alpha/2}+|\Delta_h^1(V_nf)(x)|. \end{aligned}\tag{25}$$

Using (6), we can choose $g=g_\delta\in A.C._{loc}[0,1]$ such that $\|f-g\|\le C_9\,\omega_\varphi^1(f;\delta)$ and $\|\varphi g'\|\le C_{10}\delta^{-1}\omega_\varphi^1(f;\delta)$. Hence, in view of Lemma 2,

$$\begin{aligned}|(V_nf)'(x)| &\le |(V_n(f-g))'(x)|+|(V_ng)'(x)|\\ &\le 8\sqrt{n}\varphi^{-1}(x)\|f-g\|+32\varphi^{-1}(x)\|\varphi g'\|\\ &\le 8\sqrt{n}C_9\varphi^{-1}(x)\omega_\varphi^1(f;\delta)+32C_{10}\delta^{-1}\varphi^{-1}(x)\omega_\varphi^1(f;\delta)\\ &\le C_{11}\varphi^{-1}(x)\left(\sqrt{n}+\frac{1}{\delta}\right)\omega_\varphi^1(f;\delta),\end{aligned}$$

where $C_{11}=8C_9+32C_{10}$. This implies that

$$|\Delta_h^1(V_nf)(x)|=\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}(V_nf)'(u)\,du\right|\le C_{11}\left(\sqrt{n}+\frac{1}{\delta}\right)\omega_\varphi^1(f;\delta)\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}\frac{du}{\varphi(u)}\right|.$$

Because of $x\pm\frac{h}{2}\in[0,1]$, we have $x\in(0,1)$. Using (9), we obtain

$$\begin{aligned}\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}\frac{du}{\varphi(u)}\right| &\le \left|\int_{x}^{x+\frac{h}{2}}\frac{du}{\varphi(u)}\right|+\left|\int_{x}^{x-\frac{h}{2}}\frac{du}{\varphi(u)}\right|\\ &\le 2\varphi^{-1}(x)\frac{h}{2}+2\varphi^{-1}(x)\frac{h}{2}=2\varphi^{-1}(x)h.\end{aligned}$$

Hence, by (25), we get

$$\begin{aligned}|\Delta_h^1 f(x)| &\le 2C_7n^{-\alpha/2}+C_{11}\left(\sqrt{n}+\frac{1}{\delta}\right)\omega_\varphi^1(f;\delta)2\varphi^{-1}(x)h\\ &\le C_{12}\left\{n^{-\alpha/2}+\left(h\sqrt{n}+\frac{h}{\delta}\right)\varphi^{-1}(x)\omega_\varphi^1(f;\delta)\right\}.\end{aligned}$$

Replacing h by $h\varphi(x)$ gives

$$|\Delta_{h\varphi(x)}^1 f(x)|\le C_{12}\left\{n^{-\alpha/2}+\left(h\sqrt{n}+\frac{h}{\delta}\right)\omega_\varphi^1(f;\delta)\right\}.$$

Now we choose $n \geq 1$ such that $\frac{1}{\sqrt{n}} \leq \delta \leq \frac{2}{\sqrt{n}}$, where $0 < \delta \leq 1$. Then we find that

$$|\Delta^1_{h\varphi(x)} f(x)| \leq C_{13} \left\{ \delta^\alpha + \frac{h}{\delta} \omega^1_\varphi(f;\delta) \right\}$$

for all x with $x \pm \frac{h}{2}\varphi(x) \in [0,1]$. Taking supremum over all h with $0 < h \leq t$, we obtain

$$\omega^1_\varphi(f;t) \leq C_{13} \left\{ \delta^\alpha + \frac{t}{\delta} \omega^1_\varphi(f;\delta) \right\}, \quad 0 < t \leq \delta. \tag{26}$$

Now (24) and (26) yield the assertion of our theorem by Lemma 3. □

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Fixed point theorem for new type of auxiliary functions

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Abstract. In this paper, we present some fixed point results satisfying generalized contractive condition with new auxiliary function in complete metric spaces. More precisely, the structure of the paper is the following. In the first section, we present some useful notions and results. The main aim of second section is to establish some new fixed point results in complete metric spaces. Finally, in the third section, we show the validity and superiority of our main results by suitable example. Also, as an application of our main result, some interesting corollaries have been included, which make our concepts and results effective. Our main result generalizes some well known existing results in the literature.

1 Introduction and preliminaries

The Banach contraction principle [10] is one of the revolutionary results of the fixed point theory, and it plays an imperative role to solve existence problems

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in many branches of nonlinear analysis. Inspired from the impact of this natural idea to functional analysis, a number of researchers have been extended and generalized this principle for different kinds of contractions in various spaces.

Let us denote:

$\Psi_1 = \{\psi_1 : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function such that $\psi_1(t) = 0$ if and only if $t = 0.\}$ (Altering distance function)

$\Psi_2 = \{\psi_2 : [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi_2(0) \geq 0$ and $\psi_2(t) > 0, t > 0.\}$ (Ultra-altering distance function)

$\Psi_3 = \{\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable function, summable on each compact subset of R^+, non-negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0.$ $\}$

In 2002, Branciari [3] introduced one of the genuine contraction, known as integral type contraction, as an analogue of Banach contraction principle [10].

Theorem 1 [3] *Let* (E, d) *be a complete metric space,* $k \in (0, 1)$*, and* $A : E \to E$ *is such that for each* $x, y \in E$

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \leq k \int_0^{d(x,y)} \varphi(t)dt, \tag{1}$$

where $\varphi \in \Psi_3$*. Then* A *has a unique fixed point of* $z \in E$*.*

Rhoades [5], in 2003, gave an extension of the result of Branciari [3] and proved following theorems.

Theorem 2 [5] *Let* (E, d) *be a complete metric space and* $A : E \to E$ *be a mapping such that,*

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \leq \beta \int_0^{M(x,y)} \varphi(t)dt,$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Ax), d(y,Ay), \frac{d(x,Ay) + d(y,Ax)}{2}\right\} \tag{2}$$

for all $x, y \in E$*,* $\beta \in [0, 1)$ *and* $\varphi \in \Psi_3$*. Then* A *has a unique fixed point* $z \in E$*.*

Theorem 3 [5] *Let us consider a complete metric space* (E, d) *and* $A : E \to E$ *is a mapping such that,*

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \leq \beta \int_0^{N(x,y)} \varphi(t)dt,$$

where

$$N(x,y) = \max\{d(x,y), d(x,Ax), d(y,Ay), d(x,Ay), d(y,Ax)\} \qquad (3)$$

for each $x, y \in E$, $\beta \in [0,1)$ *and* $\varphi \in \Psi_3$. *Then there exist a unique fixed point* $z \in E$ *such that* $Az = z$.

In 2010, Babu and Alemayehu [7] proved following theorem in complete metric spaces by using generalized $\phi-$ weak contraction.

Theorem 4 [7] *Let us consider a complete metric space* (E, d) *and* $A : E \to E$ *is such that for all* $x, y \in E$ *it satisfies*

$$d(Ax, Ay) \leq M(x,y) - \phi(M(x,y)),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Ax), d(y,Ay), \frac{d(x,Ay) + d(y,Ax)}{2}\right\}, \qquad (4)$$

and $\phi : [0,+\infty) \to [0,+\infty)$ *such that* $\phi(t) = 0$ *if and only if* $t = 0$. *Then there is a unique fixed point of* A *in* E.

In 2011, Samet and Yazidi [6] gave an extension of the result of Dass and Gupta [4] in the sense of Branciari integral type contraction, as follows

Theorem 5 [6] *Let* (E, d) *be a complete metric space and* A *be a self-map of* E *such that for each* $x, y \in E$,

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \leq \alpha \int_0^{M(x,y)} \varphi(t)dt + \beta \int_0^{d(x,y)} \varphi(t)dt$$

and

$$M(x,y) = \frac{d(y,Ay)[1 + d(x,Ax)]}{[1 + d(x,y)]},$$

where $\alpha, \beta > 0$ *are constants such that* $\alpha + \beta < 1$ *and* $\varphi \in \Psi_3$.
Then A *admits a unique fixed point* $z \in E$ *such that for each* $z \in E$, $A^n x \to z$ *as* $n \to \infty$.

In 2011, Gupta and Mani [14] proved a common fixed point theorem for two weakly compatible mappings using control functions ψ_1 and ψ_2 satisfying a contractive condition of integral type.

Theorem 6 *Let A and B be self compatible maps of a complete metric space (E, d) satisfying the following conditions:*

(i). $A(E) \subset B(E)$

(ii). $\psi_1\left(\int_0^{d(Ax,Ay)} \varphi(t)dt\right) \leq \psi_1\left(\int_0^{d(Bx,By)} \varphi(t)dt\right) - \psi_2\left(\int_0^{d(Bx,By)} \varphi(t)dt\right)$,

where $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2, \varphi \in \Psi_3$. Then there exist a unique common fixed point of A and B in E.

In 2013, Gupta and Mani [16] proved another generalization of the result of Branciari [3] using real valued function.

Theorem 7 [16] *Let A be a self map on complete metric space (E, d) such that for each $x, y \in E$*

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \leq \gamma\left(d\left(x,y\right)\right) \int_0^{m(x,y)} \varphi(t)dt$$

and

$$m\left(x,y\right) = \max\left\{\frac{d(x,Ax)d(y,Ay)}{d(x,y)}, d(x,y)\right\},$$

where $\varphi \in \Psi_3$ and $\gamma : R^+ \to [0,1)$ is a function with

$$\limsup_{\delta \to t} \gamma\left(\delta\right) < 1, \quad \forall \quad t > 0.$$

Then A has a unique fixed point in E.

Some other results in complete metric spaces satisfying integral type contractions are mentioned in [8, 9, 11, 12, 13, 17, 18]

In 2014-15, Ansari [1] introduced the notion of C -class function as a major generalization of Banach contraction principle. Currently this finding is one of the most attractive research topics in fixed point theory. Some other special cases of C-class functions can be found in [2]. Ansari [1] gave the following definitions and examples.

Definition 1 [1] *A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:*

1. $F(r,t) \leq r$ *for all* $r, t \in [0, \infty)$;

2. $F(r,t) = r$ *implies that either* $r = 0$ *or* $t = 0$;

Let us denote $\mathcal{C}$ *the family of* C*-class functions.*

Remark 1 *Clearly, for some* F *we have* $F(0,0) = 0$.

Example 1 [1] *The following functions* $F : [0,\infty)^2 \to \mathbb{R}$ *are elements of* $\mathcal{C}$*, for all* $r, t \in [0,\infty)$*:*

1. $F(r,t) = r - t$, $F(r,t) = r \Rightarrow t = 0$;

2. $F(r,t) = mr$, $0<m<1$, $F(r,t) = r \Rightarrow r = 0$;

3. $F(r,t) = \frac{r}{(1+t)^h}$; $h \in (0,\infty)$, $F(r,t) = r \Rightarrow r = 0$ *or* $t = 0$;

4. $F(r,t) = \log(t + a^r)/(1+t)$, $a > 1$, $F(r,t) = r \Rightarrow r = 0$ *or* $t = 0$;

5. $F(r,t) = \ln(1 + a^r)/2$, $a > e$, $F(r,t) = r \Rightarrow r = 0$;

6. $F(r,t) = (r + l)^{(1/(1+t)^p)} - l$, $l > 1, p \in (0,\infty)$, $F(r,t) = r \Rightarrow t = 0$;

7. $F(r,t) = r \log_{t+a} a$, $a > 1$, $F(r,t) = r \Rightarrow r = 0$ *or* $t = 0$;

8. $F(r,t) = r - (\frac{1+r}{2+r})(\frac{t}{1+t})$, $F(r,t) = r \Rightarrow t = 0$;

9. $F(r,t) = r\beta(r)$, $\beta : [0,\infty) \to [0,1)$, $F(r,t) = r \Rightarrow r = 0$;

10. $F(r,t) = r - \frac{t}{k+t}, F(r,t) = r \Rightarrow t = 0$;

11. $F(r,t) = r - \varphi(r), F(r,t) = r \Rightarrow r = 0$,*here* $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0 \Leftrightarrow t = 0$;

12. $F(r,t) = rh(r,t), F(r,t) = r \Rightarrow r = 0$,*here* $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is a continuous function such that* $h(r,t) < 1$ *for all* $t, s > 0$;

13. $F(r,t) = r - (\frac{2+t}{1+t})t$, $F(r,t) = r \Rightarrow t = 0$.

14. $F(r,t) = \sqrt[n]{\ln(1 + r^n)}$, $F(r,t) = r \Rightarrow r = 0$.

15. $F(r,t) = \phi(r), F(r,t) = r \Rightarrow r = 0$,*here* $\phi : [0,\infty) \to [0,\infty)$ *is a upper semi-continuous function such that* $\phi(0) = 0$, *and* $\phi(t) < t$ *for* $t > 0$,

16. $F(r,t) = \frac{r}{(1+r)^s}$*;* $s \in (0,\infty)$, $F(r,t) = r$ *implies* $r = 0$.

Remark 2 *We assume that is* F *increasing with respect to the first variable and decreasing with respect to the second variable.*

The aim of this contribution is to investigate some fixed point results using the concept of C -class function and control functions in the set up of complete metric spaces satisfying a generalized weak contraction. Our result mainly generalized the result of Rhoades [5] and Gupta and Mani [15].

2 Main result- fixed point results with auxiliary functions

The main result of this paper is the following theorem.

Theorem 8 *Let* (E,d) *be a complete metric space and* $A : E \to E$ *be a mapping such that for each* $x, y \in E$,

$$\psi_1\left(\int_0^{d(Ax,Ay)} \varphi(t)dt\right) \le F\left(\psi_1\left(\int_0^{M(x,y)} \varphi(t)dt\right), \psi_2\left(\int_0^{M(x,y)} \varphi(t)dt\right)\right), \tag{5}$$

where F *is a* C*-class function,* $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2, \varphi \in \Psi_3$ *and*

$$M(x,y) = \max\left\{d(x,y), d(x,Ax), d(y,Ay), \frac{d(x,Ay)+d(y,Ax)}{2}\right\}. \tag{6}$$

Then A *has a unique fixed point.*

Proof. Let $x_0 \in E$ be an arbitrary point. Choose a point x_1 in E such that $x_1 = Ax_0$. In general, choose x_{n+1} such that $x_{n+1} = Ax_n$ for $n = 0, 1, 2\cdots$. Suppose that $x_n \ne x_{n+1}$ for each integer $n > 1$, then from (5)

$$\psi_1\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right) \le F\left\{\begin{array}{c} \psi_1\left(\int_0^{M(x_{n-1},x_n)} \varphi(t)dt\right), \\ \psi_2\left(\int_0^{M(x_{n-1},x_n)} \varphi(t)dt\right) \end{array}\right\}, \tag{7}$$

where from (6),

$$M(x_{n-1},x_n) = \max\left\{\begin{array}{c} d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2} \end{array}\right\}$$

$$
\begin{aligned}
&= \max\left\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right), \frac{d\left(x_{n-1}, x_{n+1}\right)}{2}\right\} \\
&= \max\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\}.
\end{aligned} \tag{8}
$$

If $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$ for some n, then on combining equation (7) and (8), we get

$$
\psi_1\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right) \leq F\left\{\begin{array}{c} \psi_1\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right), \\ \psi_2\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right) \end{array}\right\}. \tag{9}
$$

Thus by definition of $F \in \mathcal{C}$, we get

$$
\text{either} \quad \psi_1\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right) = 0 \quad \text{or} \quad \psi_2\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right) = 0
$$

From definition of ψ_1 and ψ_2, it is possible only if

$$
\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0.
$$

This is a contradiction to our hypothesis . Thus $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, this implies

$$
\begin{aligned}
\psi_1\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right) &\leq F\left(\psi_1 \int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \psi_1 \int_0^{d(x_{n-1},x_n)} \varphi(t)dt\right) \\
&\leq \psi_1 \int_0^{d(x_{n-1},x_n)} \varphi(t)dt
\end{aligned}
$$

Since ψ_1 is continuous and non-decreasing, therefore

$$
\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \int_0^{d(x_{n-1},x_n)} \varphi(t)dt,
$$

thus $\left\{\int_0^{(d(x_n,x_{n+1})} \varphi(t)dt\right\}$ is monotone decreasing and lower bounded sequence. Therefore there exist $r \geq 0$ such that

$$
\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = r. \tag{10}
$$

Suppose that $r > 0$. Taking limit as $n \to \infty$ on both sides of eq. (9) and using eq. (10), we get

$$\psi_1(r) \leq F(\psi_1(r), \psi_2(r)),$$

implies from definition of $F \in \mathcal{C}$ that

$$\text{either} \quad \psi_1(r) = 0 \quad \text{or} \quad \psi_2(r) = 0.$$

Consequently, by definition of ψ_1 and ψ_2, we get $r = 0$.
Hence from eq. (10), we obtain

$$\lim_{n\to\infty} \int_0^{d(x_n, x_{n+1})} \varphi(t)dt = 0, \tag{11}$$

implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{12}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. Suppose it is not. Therefore for an $\epsilon > 0$, there exists two sub-sequences $\{x_{m(p)}\}$ and $\{x_{n(p)}\}$ of $\{x_n\}$ with $m(p) < n(p) < m(p+1)$ such that

$$d\left(x_{m(p)}, x_{n(p)}\right) \geq \epsilon, \qquad d\left(x_{m(p)}, x_{n(p)-1}\right) < \epsilon. \tag{13}$$

Consider

$$\begin{aligned} \psi_1\left(\int_0^{\epsilon} \varphi(t)dt\right) &\leq \psi_1\left(\int_0^{d\left(x_{m(p)}, x_{n(p)}\right)} \varphi(t)dt\right) \\ &\leq F\left\{\begin{array}{c} \psi_1\left(\int_0^{M\left(x_{m(p)-1}, x_{n(p)-1}\right)} \varphi(t)dt\right), \\ \psi_2\left(\int_0^{M\left(x_{m(p)-1}, x_{n(p)-1}\right)} \varphi(t)dt\right) \end{array}\right\}. \end{aligned} \tag{14}$$

Using (6)

$$\begin{aligned} M\left(x_{m(p)-1}, x_{n(p)-1}\right) &= \max\left\{\begin{array}{c} d\left(x_{m(p)-1}, x_{n(p)-1}\right), d\left(x_{m(p)-1}, x_{m(p)}\right), \\ d\left(x_{n(p)-1}, x_{n(p)}\right), \\ \frac{d\left(x_{m(p)-1}, x_{n(p)}\right) + d\left(x_{n(p)-1}, x_{m(p)}\right)}{2} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d\left(x_{m(p)-1}, x_{n(p)-1}\right), d\left(x_{m(p)-1}, x_{m(p)}\right), \\ d\left(x_{n(p)-1}, x_{n(p)}\right), z(m, n) \end{array}\right\}, \end{aligned} \tag{15}$$

where,

$$z(m,n) = \frac{d\left(x_{m(p)-1,x_{n(p)}}\right) + d\left(x_{n(p)-1,x_{m(p)}}\right)}{2}. \tag{16}$$

Thus

$$\begin{aligned}
&\int_0^{M\left(x_{m(p)-1,x_{n(p)-1}}\right)} \varphi(t)dt \\
&= \int_0^{\max\left\{d\left(x_{m(p)-1,x_{n(p)-1}}\right),d\left(x_{m(p)-1,x_{m(p)}}\right),d\left(x_{n(p)-1,x_{n(p)}}\right),z(m,n)\right\}} \varphi(t)dt \\
&= \max\left\{\begin{array}{c} \int_0^{d\left(x_{m(p)-1,x_{n(p)-1}}\right)} \varphi(t)dt, \int_0^{d\left(x_{m(p)-1,x_{m(p)}}\right)} \varphi(t)dt, \\ \int_0^{d\left(x_{n(p)-1,x_{n(p)}}\right)} \varphi(t)dt, \int_0^{z(m,n)} \varphi(t)dt \end{array}\right\}
\end{aligned} \tag{17}$$

Using (13) and triangle inequality, we get

$$\begin{aligned}
d\left(x_{m(p)-1}, x_{n(p)-1}\right) &\leq d\left(x_{m(p)-1}, x_{m(p)}\right) + d\left(x_{m(p)}, x_{n(p)-1}\right) \\
&< d\left(x_{m(p)-1}, x_{m(p)}\right) + \epsilon.
\end{aligned}$$

Therefore,

$$\lim_{p\to\infty} \int_0^{d\left(x_{m(p)-1,x_{n(p)-1}}\right)} \varphi(t)dt \leq \int_0^{\epsilon} \varphi(t)dt. \tag{18}$$

Also,

$$\begin{aligned}
z(m,n) &= \frac{d\left(x_{m(p)-1}, x_{n(p)}\right) + d\left(x_{n(p)-1}, x_{m(p)}\right)}{2} \\
&\leq \frac{d\left(x_{m(p)-1}, x_{m(p)}\right) + 2d\left(x_{m(p)}, x_{n(p)-1}\right) + d\left(x_{n(p)-1}, x_{n(p)}\right)}{2} \\
&\leq \frac{d\left(x_{m(p)-1}, x_{m(p)}\right) + d\left(x_{n(p)-1}, x_{n(p)}\right)}{2} + \epsilon.
\end{aligned}$$

Taking limit as $p \to \infty$ and using (12), we get

$$\lim_{p\to\infty} \int_0^{z(m,n)} \varphi(t)dt \leq \int_0^{\epsilon} \varphi(t)dt \tag{19}$$

Taking limit as $p \to \infty$ in equality (14) and using equations (15), (16), (17), (18) and (19) all together in (14), we get

$$\psi_1\left(\int_0^{\epsilon}\varphi(t)dt\right) \le F\left(\psi_1\left(\int_0^{\epsilon}\varphi(t)dt\right), \psi_2\left(\int_0^{\epsilon}\varphi(t)dt\right)\right).$$

Again from definition of $F \in \mathcal{C}$ we get,

$$\text{either} \quad \psi_1\left(\int_0^{\epsilon}\varphi(t)dt\right) = 0 \quad \text{or} \quad \psi_2\left(\int_0^{\epsilon}\varphi(t)dt\right) = 0.$$

It is possible only if, $\int_0^{\epsilon}\varphi(t)dt = 0$. This is a contradiction to our hypothesis. Therefore $\{x_n\}$ is a Cauchy sequence, call the limit α such that

$$\lim_{n\to\infty} Ax_{n-1} = \alpha. \tag{20}$$

Next we claim that α is the fixed point of map A.
That is $A\alpha = \alpha$, suppose it is not. Then $d(A\alpha, \alpha) > 0$.
Let $\delta = d(A\alpha, \alpha)$.
Consider,

$$\begin{aligned}\psi_1\left(\int_0^{\delta}\varphi(t)dt\right) &= \psi_1\left(\int_0^{d(A\alpha,\alpha)}\varphi(t)dt\right)\\ &\le F\left\{\begin{array}{c}\psi_1\left(\int_0^{M(\alpha,x_n)}\varphi(t)dt\right), \\ \psi_2\left(\int_0^{M(\alpha,x_n)}\varphi(t)dt\right)\end{array}\right\},\end{aligned} \tag{21}$$

where,

$$M(\alpha, x_n) = \max\left\{\begin{array}{c} d(\alpha, x_n), d(\alpha, A\alpha), d(x_n, x_{n+1}), \\ \frac{d(\alpha,x_{n+1})+d(x_n,A\alpha)}{2}\end{array}\right\}. \tag{22}$$

Since,

$$\lim_{n\to\infty} d(\alpha, x_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(\alpha, x_{n+1}) = 0. \tag{23}$$

Taking $\lim_{n\to\infty}$ in (21) and by using (20), (22), (23), we get

$$\psi_1\left(\int_0^{\delta}\varphi(t)dt\right) \le F\left\{\begin{array}{c}\psi_1\left(\int_0^{\max\left\{d(\alpha,A\alpha),\frac{d(\alpha,A\alpha)}{2}\right\}}\varphi(t)dt\right), \\ \psi_2\left(\int_0^{\max\left\{d(\alpha,A\alpha),\frac{d(\alpha,A\alpha)}{2}\right\}}\varphi(t)dt\right)\end{array}\right\}$$

$$\leq F\left\{\psi_1\left(\int_0^\delta \varphi(t)dt\right), \psi_2\left(\int_0^\delta \varphi(t)dt\right)\right\} \tag{24}$$

Thus we obtain,

$$\text{either} \quad \psi_1\left(\int_0^\delta \varphi(t)dt\right) = 0 \quad \text{or} \quad \psi_2\left(\int_0^\delta \varphi(t)dt\right) = 0$$

that is $\int_0^\delta \varphi(t)dt = 0$. Hence $\delta = 0$.
This implies $d(A\alpha, \alpha) = 0$. Therefore α is the fixed point of map A. Uniqueness of the fixed point can be easily obtain by using above inequality (21), (22), (24). This proves the main result. □

3 Applications and example

Next we give several corollaries, as a application of our main result, in the underlying spaces. Some of them are novel in literature

If we take $\psi_1(t) = t$ in Theorem 8, we get a new result.

Corollary 1 *Let* (E, d) *be a complete metric space and* $A : E \to E$ *be a mapping such that for each* $x, y \in E$,

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \leq F\left(\int_0^{M(x,y)} \varphi(t)dt, \psi_2\left(\int_0^{M(x,y)} \varphi(t)dt\right)\right)$$

for each $x, y \in E$, *where* $M(x, y)$ *is given in* (6), F *is a* C*-class function,* $\psi_2 \in \Psi_2, \varphi \in \Psi_3$.
Then A *has a unique fixed point.*

If we take $F(r, t) = \frac{r}{(1+t)^s}$ and assume $s = 1$ in Theorem 8, we find a very interesting novel result.

Corollary 2 *Let* (E, d) *be a complete metric space and* $A : E \to E$ *be a mapping such that for each* $x, y \in E$,

$$\psi_1\left(\int_0^{d(Ax,Ay)} \varphi(t)dt\right) \leq \frac{\psi_1\left(\int_0^{M(x,y)} \varphi(t)dt\right)}{1 + \psi_2\left(\int_0^{M(x,y)} \varphi(t)dt\right)},$$

where $M(x, y)$ *is given in* (6), $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2, \varphi \in \Psi_3$.
Then A *has a unique fixed point.*

If we take $F(r,t) = \lambda r$ for $0 < \lambda < 1$ and in Theorem 8, then we have following corollary.

Corollary 3 *Let* (E,d) *be a complete metric space and* $A : E \to E$ *be a mapping such that for each* $x, y \in E$*,*

$$\psi_1\left(\int_0^{d(Ax,Ay)} \varphi(t)dt\right) \le \lambda\psi_1\left(\int_0^{M(x,y)} \varphi(t)dt\right),$$

where $M(x,y)$ *is given in* (6)*,* $\psi_1 \in \Psi_1, \varphi \in \Psi_3$*.*
Then A *has a unique fixed point.*

If we assume that $\psi_1(t) = t$ in Corollary 3 then we obtain the result of Rhoades [5] (see Theorem 2 of [5]).

Corollary 4 *Let* (E,d) *be a complete metric space and* $A : E \to E$ *be a mapping such that for each* $x, y \in E$*,*

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \le \lambda \int_0^{M(x,y)} \varphi(t)dt,$$

where $M(x,y)$ *is given in* (6)*,* $\varphi \in \Psi_3$*.*
Then A *has a unique fixed point.*

If we take $F(r,t) = r - t$ in Theorem 8, then we obtain the result of Gupta and Mani [15].

Corollary 5 *Let* (E,d) *be a complete metric space and* $A : E \to E$ *be a mapping such that for each* $x, y \in E$*,*

$$\psi_1\left(\int_0^{d(Ax,Ay)} \varphi(t)dt\right) \le \psi_1\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \psi_2\left(\int_0^{M(x,y)} \varphi(t)dt\right)$$

where $M(x,y)$ *is given in* (6)*,* $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2, \varphi \in \Psi_3$*.*
Then A *has a unique fixed point.*

Remark 3 *It should be noted that in* [15]*, authors have considered an extra condition on* ψ_2*. But from above corollaries it is clear that we can deduce the same result without that extra assumption. Also, the result obtained in* [5] *and* [15] *are an element of* $\mathcal{C}$ *- class function as shown in Corollary 4 and Corollary 5. So the main result of this paper is more generalized than the other previously proved results.*

Now, we gave a non trivial example to justify the importance of our result.

Example 2 *Let* $E = N - \{\infty\}$ *and* d *is usual metric on* E. *Define a self maps* A *on* E *such that*

$$Ax = \frac{x}{3}, \quad \forall \quad x \in E.$$

Define a function $F : [0, \infty)^2 \to \mathbb{R}$ *as*

$$F(r, t) = mr, \quad \forall \;\; 0 < m = \frac{2}{3} < 1.$$

Then clearly, F *is a* C*-class function.*
Let us define $\psi_1, \varphi : [0, +\infty) \to [0, +\infty)$ *as*

$$\psi_1(t) = t, \quad \varphi(t) = \frac{t}{2}, \quad \forall \quad t \in [0, +\infty)$$

then for each $\epsilon > 0$, *clearly*

$$\int_0^{\epsilon} \varphi(t)dt = \frac{\epsilon^2}{4} > 0.$$

If $x = y$ *for all* $x, y \in E$, *then result holds trivially.*
So suppose that $x \neq y$ *for all* $x, y \in E$.
Since d *is usual metric for all* $x, y \in E$, *then on careful calculation, we get*

$$L.H.S. = \frac{|x - y|^2}{36},$$

and

$$R.H.S. = \frac{|x - y|^2}{3},$$

Then clearly, L.H.S $\leq$ *R.H.S for all* $x, y \in E$ *and, hence all conditions of Theorem 8 are verified.*
Clearly $0 \in E$ *is the unique fixed point of* A.

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Collatz conjecture revisited: an elementary generalization

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Abstract. Collatz conjecture states that iterating the map that takes even natural number n to $\frac{n}{2}$ and odd natural number n to $3n+1$, will eventually obtain 1. In this paper a new generalization of the Collatz conjecture is analyzed and some interesting results are obtained. Since Collatz conjecture can be seen as a particular case of the generalization introduced in this article, several more general conjectures are also presented.

1 Introduction

The Collatz conjecture remains today unsolved; as it has been for almost over 80 years. Although its statement is very simple and easy to understand, the nature of the problem makes extremelly to demonstrate or refuse. Articles such as [3] and [4] contain a huge amount of publications dealing with this problem and somehow trying to solve it.

Although the Collatz conjecture can be stated in several ways, in this paper we will use the following notation, i.e. modified Collatz function, that represents a slightly modification of the traditional formulation

$$C(n) = \begin{cases} \frac{3\cdot n+1}{2} & \text{if } n \equiv 1 (\text{mod } 2) \\ \\ \frac{n}{2} & \text{if } n \equiv 0 ((\text{mod } 2) \end{cases}$$

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By using the modified Collatz function, the Collatz conjecture can be stated in the following way:

Conjecture 1 *(Collatz) For every integer number* $n \in \mathbb{N}$*, there exists k such that* $C^{(k)}(n) = 1$*.*

In this sense, Terras [5] defined the $\mathtt{total\ stopping\ time}$ of an integer $n \in \mathbb{N}$, here we denote it by $\sigma_2(n)$, as the smallest integer k such that $C^{(k)}(n) = 1$ or $\sigma_2(n) = \infty$ if no such k exists. For example, if $n = 7$, then by successively applying C, we obtain the following sequence $7 \to 11 \to 17 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1$, and then $C^{11}(7) = 1$, so $\sigma_2(7) = 11$.

In this paper, we consider an elementary generalization of the Collatz problem. We also generalize the concept of $\mathtt{total\ stopping\ time}$ and, related with this new approach, we obtain several results for some classes of integer numbers that generalize some well-known results on this topic. In the last section of the paper, some calculations and new conjectures are introduced. These conjectures try to illustrate the idea that $\mathtt{traditional}$ Collatz problem is just a special case and that it can be seen as part of a more general point of view, what we have called of $\mathtt{Collatz\ numbers}$.

1.1 A more general $b-\mathtt{Collatz}$ function, C_b

For any natural number $b \geq 2$, we define the following $b - \mathtt{Collatz}$ function $C_b : \mathbb{N} \to \mathbb{N}$

$$C_b(n) = \begin{cases} \frac{(b+1)\cdot n+(b-x)}{b} & \text{if } n \equiv x (\text{mod } b),\ 1 \leq x \leq b-1 \\ \\ \frac{n}{b} & \text{if } n \equiv 0((\text{mod } b) \end{cases}$$

Clearly in the previous formula, if $b = 2$, we obtain the modified Collatz function stated in section [1]. In this sense, we defined the $b\text{-}\mathtt{total\ stopping\ time}$ of an integer $n \in \mathbb{N}$, denoted $\sigma_b(n)$, as the smallest integer k such that $C_b^{(k)}(n) = 1$ or $\sigma_b(n) = \infty$ if no such k exists. For example, if $b = 5$ and $n = 7$, then by successively applying C_5, we obtain the following sequence $7 \to 9 \to 11 \to 14 \to 17 \to 21 \to 26 \to 32 \to 39 \to 47 \to 57 \to 69 \to 83 \to 100 \to 20 \to 4 \to 5 \to 1$, so $C_5^{17}(7) = 1$ and finally $\sigma_5(7) = 17$

Theorem 1 *Let* $b, k, r, s \in \mathbb{N}$*, such that* $b \geq 2$*,* $k, r \geq 1$*,* $n = b^k \cdot r - s > 1$ *and* $1 \leq s \leq b-1$*. Then*

$$C_b^{(k)}(n) = (b+1)^k \cdot r - s.$$

Proof. We proceed by induction on t, $0 \le t \le k$. Case $t = 0$.

$$C_b^{(0)}(b^k \cdot r - s) = b^k \cdot r - s = ((b+1)^0 \cdot b^{k-0}) \cdot r - s.$$

So, the initial case holds. Now let us assume that

$$C_b^{(t)}(b^k \cdot r - s) = ((b+1)^t \cdot b^{k-t}) \cdot r - s,$$

for some $0 \le t < k$. Since $t < k$, then $((b+1)^t \cdot b^{k-t}) \cdot r - s \equiv -s \equiv b - s (\bmod b)$, so

$$\begin{aligned} C_b^{(t+1)}(b^k \cdot r - s) &= C_b(C_b^{(t)}(b^k \cdot r - s)) = C_b(((b+1)^t \cdot b^{k-t}) \cdot r - s) \\ &= \frac{(b+1) \cdot ((b+1)^t \cdot b^{k-t}) \cdot r - s) + s}{b} = \\ &= \frac{(b+1)^{(t+1)} \cdot b^{k-t} \cdot r - b \cdot s}{b} = \\ &= (b+1)^{(t+1)} \cdot b^{k-(t+1)} \cdot r - s. \end{aligned}$$

Thus,

$$C_b^{(t+1)}(b^k \cdot r - s) = (b+1)^{(t+1)} \cdot b^{k-(t+1)} \cdot r - s,$$

and the result follows. □

Corollary 1 *If* $b, k, r, s \in \mathbb{N}$*, such that* $b \ge 2$*,* $k, r \ge 1$*,* $n = b^k \cdot r - s > 1$ *and* $1 \le s \le b - 1$*, then*

$$\sigma_b(b^k \cdot r - s) = \sigma_b((b+1)^k \cdot r - s) + k.$$

Corollary 2 *If* $b \ge 2$*,* $k, r \ge 1$*,* $n = b^k - 1 > 1$*, then*

$$\sigma_b(b^k - 1) = \sigma_b((b+1)^k - 1) + k.$$

Corollary 3 *If* $b = 2$*,* $k, r \ge 1$*,* $n = 2^k \cdot r - 1$*, then*

$$\sigma_2(2^k \cdot r - 1) = \sigma_2(3^k \cdot r - 1) + k.$$

Theorem 2 *If* $b, k, r, t, s \in \mathbb{N}$*, such that* $b \ge 2$*,* $k > t \ge 1$*,* $r \ge 1$*, and* $1 \le s \le b - 1$*, then*

$$C_b^{(k)}(b^k \cdot r - b^t \cdot s) = (b+1)^{k-t} \cdot r - s.$$

Proof. Since $b^k \cdot r - b^t \cdot s$ is divisible by b^t, then $C_b^{(k)}(b^k \cdot r - b^t \cdot s) = C_b^{(k-t)}$ $(C_b^t(b^k \cdot r - b^t \cdot s)) = C_b^{(k-t)}(b^{k-t} \cdot r - s) = (b+1)^{k-t} \cdot r - s$, by Theorem 1. □

Corollary 4 *If* $b, k, r, t, s \in \mathbb{N}$*, such that* $b \geq 2$*,* $k > t \geq 1$*,* $r \geq 1$*, and* $1 \leq s \leq b - 1$*, then*

$$\sigma_b(b^k \cdot r - b^t \cdot s) = \sigma_b((b+1)^{k-t} \cdot r - s) + k.$$

Theorem 3 *Let* $b \geq 2$*,* $k > 2$ *and* $r \geq 1$*, then*
i) *If* $b = 2$ *then* $\sigma_b(b^k \cdot r - (2b-1)) = \sigma_b((b+1)^{k-2} \cdot r - 1) + k$.
ii) *If* $b > 2$ *then* $\sigma_b(b^k \cdot r - (2b-1)) = \sigma_b((b+1)^{k-1} \cdot r - 2) + k$.

Proof. Let $n = b^k \cdot r - (2b-1)$. Since $n \equiv 1 (\mathrm{mod}\ b)$, then

$$\begin{aligned} C_b^{(k)}(n) &= C_b^{(k-1)}(C_b(n)) = C_b^{(k-1)}(\frac{(b+1)\cdot(b^k \cdot r - 2b + 1)) + b - 1}{b}) \\ &= C_b^{(k-1)}((b+1)\cdot b^{k-1} \cdot r - 2b) = C_b^{(k-2)}(C_b((b+1)\cdot b^{k-1} \cdot r - 2b)) \\ &= C_b^{(k-2)}((b+1)\cdot b^{k-2} \cdot r - 2). \end{aligned}$$

If $b = 2$, then $C_b^{(k)}(n) = C_b^{(k-2)}(3 \cdot 2^{k-2} \cdot r - 2) = C_b^{(k-3)}(2^{k-3} \cdot 3 \cdot r - 1) = 3^{k-2} \cdot r - 1$. If $b > 2$, then by Theorem 1,

$$C_b^{(k)}(n) = C_b^{(k-2)}((b+1)\cdot b^{k-2} \cdot r - 2) = (b+1)^{k-1} \cdot r - 2.$$

□

Corollary 5 [6] *If* $k > 2$ *and* $r \geq 1$*, then* $\sigma_2(2^k \cdot r - 3) = \sigma_2(3^{k-2} \cdot r - 1) + k$.

Theorem 4 *Let* $b \geq 2$*,* $k > 2$ *and* $r, t \geq 1$*, then*
i) *If* $b = 2$*, then* $\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-2)} \cdot r - 1) + k$.
i) *If* $b > 2$*, then* $\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-1)} \cdot r - 2) + k$.

Proof. Let $n = b^k \cdot r - b^t \cdot (2b-1)$. Since $n \equiv 0 (\mathrm{mod}\ b^t)$, then

$$C_b^{(k)}(n) = C_b^{(k-t)}(\frac{n}{b^t}) = C_b^{k-t}(b^{k-t} \cdot r - (2b-1)).$$

By Theorem 3, if $b = 2$, then

$$C_b^{(k)}(n) = C_b^{k-t}(b^{k-t} \cdot r - (2b-1)) = (b+1)^{(k-t-2)} \cdot r - 1$$

and then

$$\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-2)} \cdot r - 1) + k.$$

In othen hand, if $b > 2$, then

$$C_b^{(k)}(n) = C_b^{k-t}(b^{k-t} \cdot r - (2b-1)) = (b+1)^{(k-t-1)} \cdot r - 2$$

and then $\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-1)} \cdot r - 2) + k$. □

Corollary 6 [6] *If* $k > 2$ *and* $r \geq 1$*, then* $\sigma_2(2^k \cdot r - 6) = \sigma_2(3^{k-3} \cdot r - 1) + k$.

Theorem 5 *Let* $b \geq 2$ *and* $k \equiv 0 (\bmod b)$*, then*
i) $\sigma_b(b^k - 1) = \sigma_b(b^{k-1} - 1) + 1$.
ii) $\sigma_b((b+1)^k - 1) = \sigma_b((b+1)^{k-1} - 1)$.

Proof. i) It enough to proof that

$$C_b^{(k+1)}(b^{k-1} - 1) = C_b^{(k+2)}(b^k - 1).$$

So, by Theorem 1, $C_b^{(k+1)}(b^{k-1}-1) = C_b^{(2)}((b+1)^{k-1}-1)$. Since $(b+1)^{k-1}-1 \equiv 0(\bmod b)$, then

$$C_b^{(2)}((b+1)^{k-1} - 1) = C_b(\frac{(b+1)^{k-1} - 1}{b}).$$

Finally, since $k \equiv 0(\bmod b)$ and $\frac{(b+1)^{k-1}-1}{b} \equiv b - 1(\bmod b)$, then

$$C_b^{(2)}((b+1)^{k-1} - 1) = C_b(\frac{(b+1)^{k-1} - 1}{b}) = \frac{(b+1)^k - 1}{b^2}.$$

In other hand, $C_b^{(k+2)}(b^k - 1) = C_b^{(2)}((b+1)^k - 1)$. So, since $k \equiv 0(\bmod b)$, then $(b+1)^k - 1 \equiv 0(\bmod b^2)$. Therefore,

$$C_b^{(k+2)}(b^k - 1) = C_b^{(2)}((b+1)^k - 1) = \frac{(b+1)^k - 1}{b^2},$$

and equality holds.
ii) From previous result i) and Corollary 4, we have

$$\begin{aligned}\sigma_b((b+1)^k - 1) &= \sigma_b(b^k - 1) - k = \sigma_b(b^{k-1} - 1) + 1 - k \\ &= \sigma_b((b+1)^{k-1} - 1) + (k-1) + 1 - k \\ &= \sigma_b((b+1)^{k-1} - 1).\end{aligned}$$

□

Corollary 7 [6] *If* k *is even and* $k > 2$, *then*

$$\sigma_2(2^k \cdot r - 1) = \sigma_2(2^{k-1} \cdot r - 1) + 1, \text{ and}$$
$$\sigma_2(3^k \cdot r - 1) = \sigma_2(3^{k-1} \cdot r - 1).$$

3. Some empirical results on C_b and open problems

The behaviour of C_b has been intensively studied during last years when $b = 2$. But, what can we state for C_b for other values of $b \geq 3$?. Let us see some examples for several values of b.

Case $b = 3$. Calculating $C_3(n)$ for some values of n

Iteration/n =	4	5	6	7	8	9	10	11	12
1	6	7	2	10	11	3	14	15	4
2	2	10	3	14	15	1	19	5	6
3	3	14	1	19	5	2	26	7	2
4	1	19	2	26	7	3	35	10	3
5	2	26	3	35	10	1	47	14	1
6	3	35	1	47	14	2	63	19	2
7	1	47	2	63	19	3	21	26	3
8	2	63	3	21	26	1	7	35	1
9	3	21	1	7	35	2	10	47	2
10	1	7	2	10	47	3	14	63	3
11	2	10	3	14	63	1	19	21	1
12	3	14	1	19	21	2	26	7	2
13	1	19	2	26	7	3	35	10	3
14	2	26	3	35	10	1	47	14	1
15	3	35	1	47	14	2	63	19	2
16	1	47	2	63	19	3	21	26	3
17	2	63	3	21	26	1	7	35	1
18	3	21	1	7	35	2	10	47	2
19	1	7	2	10	47	3	14	63	3

At first sight, there is no homogeneous behaviour for different values of n. In this case, for $n \in \{4, 6, 9, 12\}$ we found the cycle $2 \to 3 \to 1$ that repeats and contains the number 1.

In other hand, for $n \in \{5, 7, 8, 10, 11\}$, the behaviour is completely different than the previous one. In this case, the cycle $7 \to 10 \to 14 \to 19 \to 26 \to 35 \to 47 \to 63 \to 21$ repeats and does not contain the number 1. In fact, it seems that these are the only two cycles one can find when iterating C_3.

Case $b = 5$. Calculating $C_5(n)$ for some values of n

Iteration/n =	6	7	8	9	10	11	12	13	14
1	8	9	10	11	2	14	15	16	17
2	10	11	2	14	3	17	3	20	21
3	2	14	3	17	4	21	4	4	26
4	3	17	4	21	5	26	5	5	32
5	4	21	5	26	1	32	1	1	39
6	5	26	1	32	2	39	2	2	47
7	1	32	2	39	3	47	3	3	57
8	2	39	3	47	4	57	4	4	69
9	3	47	4	57	5	69	5	5	83
10	4	57	5	69	1	83	1	1	100
11	5	69	1	83	2	100	2	2	20
12	1	83	2	100	3	20	3	3	4
13	2	100	3	20	4	4	4	4	5
14	3	20	4	4	5	5	5	5	1
15	4	4	5	5	1	1	1	1	2
16	5	5	1	1	2	2	2	2	3
17	1	1	2	2	3	3	3	3	4
18	2	2	3	3	4	4	4	4	5
19	3	3	4	4	5	5	5	5	1
20	4	4	5	5	1	1	1	1	2
21	5	5	1	1	2	2	2	2	3

At first sight, there is an homogeneous behaviour for different values of n. Independently the value of n we select, we found the cycle $2 \to 3 \to 4 \to 5 \to 1$ that repeats and contains the number 1.

Case $b = 6$. Calculating $C_6(n)$ for some values of n

Iteration/ $n =$	7	8	9	10	11	12	13	14	15
1	9	10	11	12	13	2	16	17	18
2	11	12	13	2	16	3	19	20	3
3	13	2	16	3	19	4	23	24	4
4	16	3	19	4	23	5	27	4	5
5	19	4	23	5	27	6	32	5	6
6	23	5	27	6	32	1	38	6	1
7	27	6	32	1	38	2	45	1	2
8	32	1	38	2	45	3	53	2	3
9	38	2	45	3	53	4	62	3	4
10	45	3	53	4	62	5	73	4	5
11	53	4	62	5	73	6	86	5	6
12	62	5	73	6	86	1	101	6	1
13	73	6	86	1	101	2	118	1	2
14	86	1	101	2	118	3	138	2	3
15	101	2	118	3	138	4	23	3	4
16	118	3	138	4	23	5	27	4	5
17	138	4	23	5	27	6	32	5	6
18	23	5	27	6	32	1	38	6	1

By inspectioning this examples, it can be seen that there are two groups of numbers. First, numbers such as $n \in \{7, 9, 11, 13\}$ whose iterations contain the cycle $23 \to 27 \to 32 \to 38 \to 45 \to 53 \to 62 \to 73 \to 86 \to 101 \to 118 \to 138$. Secondly, numbers such as $n \in \{8, 10, 12, 14, 15\}$ whose iterations contain the cycle $2 \to 3 \to 4 \to 5 \to 6 \to 1$ that repeats and contains the number 1.

Case $b = 7$. Calculating $C_7(n)$ for some values of n

Iteration/ $n =$	8	9	10	11	12	13	14	15	16
1	10	11	12	13	14	15	2	18	19
2	12	13	14	15	2	18	3	21	22
3	14	15	2	18	3	21	4	3	26
4	2	18	3	21	4	3	5	4	30
5	3	21	4	3	5	4	6	5	35
6	4	3	5	4	6	5	7	6	5
7	5	4	6	5	7	6	1	7	6
8	6	5	7	6	1	7	2	1	7
9	7	6	1	7	2	1	3	2	1
10	1	7	2	1	3	2	4	3	2
11	2	1	3	2	4	3	5	4	3
12	3	2	4	3	5	4	6	5	4
13	4	3	5	4	6	5	7	6	5
14	5	4	6	5	7	6	1	7	6
15	6	5	7	6	1	7	2	1	7
16	7	6	1	7	2	1	3	2	1
17	1	7	2	1	3	2	4	3	2
18	2	1	3	2	4	3	5	4	3

By inspectioning this examples, it can be seen that in all cases, iterations contain the cycle $2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 1$ that repeats and contains the number 1.

Case $b = 9$. Calculating $C_9(n)$ for some values of n

Iteration/ $n =$	10	11	12	13	14	15	16	17	31	35
1	12	13	14	15	16	17	18	19	35	39
2	14	15	16	17	18	19	2	22	39	44
3	16	17	18	19	2	22	3	25	44	49
4	18	19	2	22	3	25	4	28	49	55
5	2	22	3	25	4	28	5	32	55	62
6	3	25	4	28	5	32	6	36	62	69
7	4	28	5	32	6	36	7	4	69	77
8	5	32	6	36	7	4	8	5	77	86
9	6	36	7	4	8	5	9	6	86	96
10	7	4	8	5	9	6	1	7	96	107
11	8	5	9	6	1	7	2	8	107	119
12	9	6	1	7	2	8	3	9	119	133
13	1	7	2	8	3	9	4	1	133	148
14	2	8	3	9	4	1	5	2	148	165
15	3	9	4	1	5	2	6	3	165	184
16	4	1	5	2	6	3	7	4	184	205
17	5	2	6	3	7	4	8	5	205	228
18	6	3	7	4	8	5	9	6	228	254
19	7	4	8	5	9	6	1	7	254	283
20	8	5	9	6	1	7	2	8	283	315
21	9	6	1	7	2	8	3	9	315	35
22	1	7	2	8	3	9	4	1	35	39
23	2	8	3	9	4	1	5	2	39	44
24	3	9	4	1	5	2	6	3	44	49
25	4	1	5	2	6	3	7	4	49	55

By inspectioning this examples, it can be seen that for $n \in \{10, 11, 12, 13, 14, 15, 16, 17\}$, iterations contain the cycle $2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 1$ that repeats and contains the number 1. In other hand, for $n \in \{31, 35\}$, for example, interations contain the cycle $35 \to 39 \to 44 \to 49 \to 55 \to 62 \to 69 \to 77 \to 86 \to 96 \to 107 \to 119 \to 133 \to 148 \to 165 \to 184 \to 205 \to 228 \to 254 \to 283 \to 315$.

$\mathcal{C}ase$ $b = 10$. Calculating $C_{10}(n)$ for some values of n. We compute the fisrt 30 interations for esch value of n,

Iteration/n =	11	12	13	14	15	16	17	18	34	38
1	13	14	15	16	17	18	19	20	38	42
2	15	16	17	18	19	20	21	2	42	47
3	17	18	19	20	21	2	24	3	47	52
4	19	20	21	2	24	3	27	4	52	58
5	21	2	24	3	27	4	30	5	58	64
6	24	3	27	4	30	5	3	6	64	71
7	27	4	30	5	3	6	4	7	71	79
8	30	5	3	6	4	7	5	8	79	87
9	3	6	4	7	5	8	6	9	87	96
10	4	7	5	8	6	9	7	10	96	106
11	5	8	6	9	7	10	8	1	106	117
12	6	9	7	10	8	1	9	2	117	129
13	7	10	8	1	9	2	10	3	129	142
14	8	1	9	2	10	3	1	4	142	157
15	9	2	10	3	1	4	2	5	157	173
16	10	3	1	4	2	5	3	6	173	191
17	1	4	2	5	3	6	4	7	191	211
18	2	5	3	6	4	7	5	8	211	233
19	3	6	4	7	5	8	6	9	233	257
20	4	7	5	8	6	9	7	10	257	283
21	5	8	6	9	7	10	8	1	283	312
22	6	9	7	10	8	1	9	2	312	344
23	7	10	8	1	9	2	10	3	344	379
24	8	1	9	2	10	3	1	4	379	417
25	9	2	10	3	1	4	2	5	417	459
26	10	3	1	4	2	5	3	6	459	505
27	1	4	2	5	3	6	4	7	505	556
28	2	5	3	6	4	7	5	8	556	612
29	3	6	4	7	5	8	6	9	612	674
30	4	7	5	8	6	9	7	10	674	742

Then, we compute values for the rest of interations

Iteration/$n =$	11	12	13	14	15	16	17	18	34	38
31	5	8	6	9	7	10	8	1	742	817
32	6	9	7	10	8	1	9	2	817	899
33	7	10	8	1	9	2	10	3	899	989
34	8	1	9	2	10	3	1	4	989	1088
35	9	2	10	3	1	4	2	5	1088	1197
36	10	3	1	4	2	5	3	6	1197	1317
37	1	4	2	5	3	6	4	7	1317	1449
38	2	5	3	6	4	7	5	8	1449	1594
39	3	6	4	7	5	8	6	9	1594	1754
40	4	7	5	8	6	9	7	10	1754	1930
41	5	8	6	9	7	10	8	1	1930	193
42	6	9	7	10	8	1	9	2	193	213
43	7	10	8	1	9	2	10	3	213	235
44	8	1	9	2	10	3	1	4	235	259
45	9	2	10	3	1	4	2	5	259	285
46	10	3	1	4	2	5	3	6	285	314
47	1	4	2	5	3	6	4	7	314	346
48	2	5	3	6	4	7	5	8	346	381
49	3	6	4	7	5	8	6	9	381	420
50	4	7	5	8	6	9	7	10	420	42
51	5	8	6	9	7	10	8	1	42	47

In this case, for $n \in \{11, 12, 13, 14, 15, 16, 17, 18\}$ there is a common cycle $2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 1$ that repeats and contains the number 1. In other hand, for numbers such as $n \in \{34, 38\}$, one can find the large cycle $42 \to 47 \to 52 \to 58 \to 64 \to 71 \to 79 \to 87 \to 96 \to 106 \to 117 \to 129 \to 142 \to 157 \to 173 \to 191 \to 211 \to 233 \to 257 \to 283 \to 312 \to 344 \to 379 \to 417 \to 459 \to 505 \to 556 \to 612 \to 674 \to 742 \to 817 \to 899 \to 1088 \to 1197 \to 1317 \to 1449 \to 1594 \to 1754 \to 1930 \to 193 \to 213 \to 235 \to 259 \to 285 \to 314 \to 346 \to 381 \to 420$ that repeats and does not contain the number 1.

Case $b = 11$. Calculating $C_{11}(n)$ for some values of n

Iteration/n =	12	13	14	15	16	17	18	19	20	21
1	14	15	16	17	18	19	20	21	22	23
2	16	17	18	19	20	21	22	23	2	26
3	18	19	20	21	22	23	2	26	3	29
4	20	21	22	23	2	26	3	29	4	32
5	22	23	2	26	3	29	4	32	5	35
6	2	26	3	29	4	32	5	35	6	39
7	3	29	4	32	5	35	6	39	7	43
8	4	32	5	35	6	39	7	43	8	47
9	5	35	6	39	7	43	8	47	9	52
10	6	39	7	43	8	47	9	52	10	57
11	7	43	8	47	9	52	10	57	11	63
12	8	47	9	52	10	57	11	63	1	69
13	9	52	10	57	11	63	1	69	2	76
14	10	57	11	63	1	69	2	76	3	83
15	11	63	1	69	2	76	3	83	4	91
16	1	69	2	76	3	83	4	91	5	100
17	2	76	3	83	4	91	5	100	6	110
18	3	83	4	91	5	100	6	110	7	10
19	4	91	5	100	6	110	7	10	8	11
20	5	100	6	110	7	10	8	11	9	1
21	6	110	7	10	8	11	9	1	10	2
22	7	10	8	11	9	1	10	2	11	3
23	8	11	9	1	10	2	11	3	1	4
24	9	1	10	2	11	3	1	4	2	5
25	10	2	11	3	1	4	2	5	3	6
26	11	3	1	4	2	5	3	6	4	7
27	1	4	2	5	3	6	4	7	5	8
28	2	5	3	6	4	7	5	8	6	9
29	3	6	4	7	5	8	6	9	7	10
30	4	7	5	8	6	9	7	10	8	11
31	5	8	6	9	7	10	8	11	9	1
32	6	9	7	10	8	11	9	1	10	2
33	7	10	8	11	9	1	10	2	11	3
34	8	11	9	1	10	2	11	3	1	4
35	9	1	10	2	11	3	1	4	2	5

By inspectioning this examples, it can be seen that in all cases, iterations contain the cycle $2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 1$ that repeats and contains the number 1.

From previous empirical results, following definitions and conjectures can be stated

Definition 1 *For any* $b \geq 2$*, set* $\{2, 3, ..., b, 1\}$ *is called a* b*-trivial cycle.*

Definition 2 *Let* $b \geq 2$ *an integer number. Then, for any* $n \in \mathbb{N}$*, we define*

$$\mathtt{Iter}_b(n) = \{C_b^{(k)}(n) \mid k \geq 1\}.$$

Definition 3 *An integer number* $b \geq 2$ *is called Collatz number if for every* $n \in \mathbb{N}$*,* $\mathtt{Iter}_b(n)$ *contains the* b*-trivial cycle.*

Lemma 1 *Let* $b \geq 2$ *an integer number. Then, there exist numbers* $n \in \mathbb{N}$*, such that*
$i)$ $1 \in \mathtt{Iter}_b(n)$*, and*
$ii)$ $\{2, 3, ..., b, 1\} \subset \mathtt{Iter}_b(n)$*.*

Proof. Let $n = 2 \cdot b$, then it is easy to verify that $C_b(n) = 2$, $C_b^{(2)}(n) = 3$, ..., $C_b^{b-1}(n) = b$, $C_b^{(b)}(n) = 1$ and $C_b^{(b+1)}(n) = 2$. Thus, both previous statements can be easily checked. $\square$

If we analyze the list of positive integers numbers lower than 20, then we can see two completely different behaviour when iterating C_b function. In one hand, we can find values for b, such as $2, 5, 7, 8, ...$, in which interations of $C_b(n)$ on any integer number n, $\mathit{it\ seems}$, always end in 1, more concretely, interations contain the b-trivial cycle. And in other hand, values of b, such as $3, 4, 6, 9, 10, ...$, in which interations of $C_b(n)$ on any integer number n either end in 1 or in another non trivial cycle. Below you can find a table for different values of b, where one can find the lowest value of n, for which $\mathtt{Iter}_b(n)$ does not contain 1, but however, it contains a non trivial cycle.

b	n
3	5
4	11
6	7
9	31
10	34
11	588
12	767
15	49
16	35
17	19

Conjecture 2 *Let* $b \geq 2$ *an integer number. Then, one can find only following two possibilities*

Case i) *For all* $n \in \mathbb{N}$, $\mathtt{Iter}_b(n)$ *contains the* b*-trivial cycle, and then* b *is a Collatz number, or*

Case ii) *There are* $n \in \mathbb{N}$, *such that* $\mathtt{Iter}_b(n)$ *contains a common non-trivial cycle. There are also other values of* $n \in \mathbb{N}$, *for which* $\mathtt{Iter}_b(n)$ *contains the* b*-trivial cycle.*

Conjecture 3 *Numbers 2, 5, 7, 8, 13, 14, 18 and 19 are Collatz numbers.*

Conjecture 4 *There are infinite Collatz numbers.*

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Nonparametric estimation of trend function for stochastic differential equations driven by a bifractional Brownian motion

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Abstract. The main objective of this paper is to investigate the problem of estimating the trend function $S_t = S(x_t)$ for process satisfying stochastic differential equations of the type

$$dX_t = S(X_t)dt + \varepsilon dB_t^{H,K},\ X_0 = x_0,\ 0 \leq t \leq T,$$

where $\left\{B_t^{H,K}, t \geq 0\right\}$ is a bifractional Brownian motion with known parameters $H \in (0, 1)$, $K \in (0, 1]$ and $HK \in (1/2, 1)$. We estimate the unknown function $S(x_t)$ by a kernel estimator $\hat{S}_t$ and obtain the asymptotic properties as $\varepsilon \longrightarrow 0$. Finally, a numerical example is provided.

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1 Introduction

Fractional Brownian motion (fBm) is the most well-known and employed process with a long dependency-property for many real world applications including telecommunication, turbulence, finance, and so on. This process was introduced by Kolmogorov [5], then studied by many researchers including Mandelbrot and Van Ness [9] and Norros et al. [12].

The bifractional Brownian motion (bfBm) was introduced in Houdré and Villa [3], and further studied by Russo and Tudor [14] and Tudor and Xiao [16].

Nonparametric estimation of trend function for stochastic differential equations (SDEs) has caught the attention of different researchers. It was first investigated by Kutoyants [7] for the stochastic differential equation driven by a standard Brownian motion. After that, the problem was generalized by Mishra and Rao [10] for the stochastic differential equation driven by a fractional Brownian motion. Then, Mishra and Rao [11] presented nonparametric estimation of linear multiplier for fractional diffusion processes. Later, nonparametric inference for fractional diffusion were dealt by Saussereau [15]. Very recently, Prakasa Rao [13] investigated nonparametric estimation of trend function for SDEs driven by mixed fractional Brownian motion.

In this paper, we use the method developed by Kutoyants [7] to construct an estimate of the trend function S_t in a model described by stochastic differential equations driven by a bifractional Brownian motion. For this, let $\{X_t,\ 0 \le t \le T\}$ be the process governed by the following equation:

$$dX_t = S(X_t)dt + \varepsilon dB_t^{H,K},\ X_0 = x_0,\ 0 \le t \le T,$$

where $\varepsilon > 0$ and $B_t^{H,K}$ is a bifractional Brownian motion of parameters $H \in (0,1)$, $K \in (0,1]$, and $S(.)$ is an unknown function. In Kutoyants [7], the trend coefficient in a diffusion process was estimated from the process $\{X_t, 0 \le t \le T\}$. In this investigation, we use a similar approach and consider the estimate $\hat{S}_t$ of S_t as follows:

$$\hat{S}_t = \frac{1}{\phi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau,$$

where G is a bounded kernel with finite support with $\phi_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Under some hypotheses, we firstly prove the mean square consistency of the estimator. Then, we give a bound on the rate of convergence and prove the asymptotic normality of the estimator $\hat{S}_t$.

To the best of our knowledge, the problem of nonparametric estimation of trend function for stochastic differential equations driven by a bfBm has not been considered in the literature.

The rest of the paper is structured as follows. In Section 2, the basic properties of bifractional Brownian motion are stated. Section 3 is devoted to the preliminaries. Then, in Section 4, we give the main results; under some hypotheses, we establish the uniform consistency (Theorem 1), the rate of convergence (Theorem 2) as well as the asymptotic normality (Theorem 3) of the estimator. Further, in Section 5, a simulation example is carried out to illuminate our theoretical study. Section 6 is devoted to the technical proofs. Finally, we conclude the paper in Section 7.

2 Bifractional Brownian motion

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis satisfying the habitual hypotheses, i.e., a filtered probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\mathcal{F}_0$ contains every $\mathbb{P}$-null set.

Let $\{B_t^{H,K},\, t \geq 0\}$ be a normalized bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1]$, that is, a Gaussian process with continuous sample paths with $B_0^{H,K} = 0$ and the covariance:

$$R_{H,K}(t,s) = \mathbb{E}\left(B_t^{H,K} B_s^{H,K}\right) = \frac{1}{2^K}\left[(t^{2H} + s^{2H})^K + |s-t|^{2HK}\right], \quad t \geq 0, \ \ s \geq 0.$$

When $K = 1$, we retrieve the fractional Brownian motion while the case $K = 1$ and $H = 1/2$ corresponds to the standard Brownian motion.

The bfBm is an extension of the fBm which preserves many properties of the fBm, but not the stationarity of the increments. Russo and Tudor [14] showed that the bfBm $B^{H,K}$ behaves as a fBm of Hurst parameter HK.

According to Houdré and Villa [3] and Tudor and Xiao [16], the bfBm has the following properties:

1. $\mathbb{E}\left(B_t^{H,H}\right) = 0$ and $Var\left(B_t^{H,K}\right) = t^{2HK}$.

2. $B_t^{H,K}$ is said to be self-similar with index $HK \in (0,1)$, that is, for every constant $a > 0$,

$$\left\{B_{at}^{H,K},\, t \geq 0\right\} \triangleq \left\{a^{HK} B_t^{H,K},\, t \geq 0\right\}, \ \text{for each } \ a > 0, \tag{1}$$

   in the sense that the processes, on both sides of the equality sign, have the same finite dimensional distributions.

3. The process $B_t^{H,K}$ is not Markov and it is not a semi-martingale if $HK \neq 1/2$.

4. The trajectories of the process $B^{H,K}$ are Hölder continuous of order δ for any $\delta < HK$ and they are nowhere differentiable.

5. The bfBm $B^{H,K}$ is a quasi-helix in the sense of Kahane [4], for any $t, s \geq 0$ we have

$$2^{-K}(t-s)^{2HK} \leq \mathbb{E}\left[B_t^{H,K} - B_s^{H,K}\right]^2 \leq 2^{1-K}(t-s)^{2HK}.$$

The bfBm $B^{H,K}$ can be extended for $K \in (1,2)$ with $H \in (0,1)$ and $HK \in (0,1)$ (see Bardina and Es-Sebaiy [1] and Lifshits and Volkova [8]).

The stochastic calculus with respect to the bifractional Brownian motion has been recently developed by Kruk et al. [6]. More works on bifractional Brownian motion can be found in Tudor and Xiao [16], Es-sabaiy and Tudor [2], Yan et al. [17] and the references therein.

Fix a time interval $[0,T]$, we denote by $\mathcal{E}$ the set of step function on $[0,T]$. Let $\mathcal{H}_{B^{H,K}}$ be the canonical Hilbert space associated to the bfBm defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}_{B^{H,K}}} = R_{H,K}(t,s) = \int_0^T \int_0^T 1_{[0,t]}(u) 1_{[0,s]}(v) \frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} du dv,$$

where $R_{H,K}(t,s)$ is the covariance of $B_t^{H,K}$ and $B_s^{H,K}$. The application $\varphi \in \mathcal{E} \longrightarrow B^{H,K}(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{H,K}$ and it can be extended to $\mathcal{H}_{B^{H,K}}$. In this study, as $HK \in (1/2, 1)$ we will employ the subspace $|\mathcal{H}_{B^{H,K}}|$ of $\mathcal{H}_{B^{H,K}}$ which is defined as the set of measurable function φ on $[0,T]$ satisfying

$$\|\varphi\|_{|\mathcal{H}_{B^{H,K}}|} := \int_0^T \int_0^T |\varphi(u)| |\varphi(v)| \frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} du dv < \infty, \tag{2}$$

such that

$$\frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} = \alpha_{H,K} \left(t^{2H} + s^{2H}\right)^{K-2} (ts)^{2H-1} + \beta_{H,K} |t-s|^{2HK-2},$$

where

$$\alpha_{H,K} = 2^{-K+2} H^2 K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1} HK(2HK-1).$$

Note that, if $\varphi,\ \psi \in |\mathcal{H}_{B^{H,K}}|$, then their scalar product in $\mathcal{H}_{B^{H,K}}$ is given by

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{B^{H,K}}} = \int_0^T \int_0^T \varphi(u)\psi(v) \frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} du dv.$$

For $\varphi,\ \psi \in |\mathcal{H}_{B^{H,K}}|$, we have

$$\mathbb{E}\left(\int_0^T \varphi(u) dB_u^{H,K}\right) = 0,\ \ \mathbb{E}\left(\int_0^T \varphi(u) dB_u^{H,K} \int_0^T \psi(v) dB_v^{H,K}\right) = \langle \varphi, \psi \rangle_{\mathcal{H}_{B^{H,K}}}.$$

It is worth being pointed out that the canonical Hilbert space $\mathcal{H}_{B^{H,K}}$ associated with $B^{H,K}$ satisfies:

$$L^2([0,T]) \subset L^{1/HK}([0,T]) \subset |\mathcal{H}_{B^{H,K}}| \subset \mathcal{H}_{B^{H,K}}, \tag{3}$$

where $H \in (0,1)$, $K \in (0,1]$ and $HK \in (1/2,1)$.

3 Preliminaries

Let $\{X_t,\ 0 \leq t \leq T\}$ be a process governed by the following equation:

$$dX_t = S(X_t)dt + \varepsilon dB_t^{H,K},\ \ X_0 = x_0,\ 0 \leq t \leq T, \tag{4}$$

where $\varepsilon > 0$, $B_t^{H,K}$ a bifractional Brownian motion, and $S(.)$ is an unknown function. We suppose that x_t is a solution of the following equation

$$\frac{dx_t}{dt} = S(x_t),\ x_0,\ 0 \leq t \leq T. \tag{5}$$

We also suppose that the function $S : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(**A1**) There exists $L > 0$ such that

$$|S(x) - S(y)| \leq L\,|x-y|\,,\ \ x, y \in \mathbb{R}, \tag{6}$$

(**A2**) There exists $M > 0$ such that

$$|S(x)| \leq M(1+|x|),\ x \in \mathbb{R},$$

Then, the stochastic differential equation (4) has a unique solution $\{X_t,\ 0 \leq t \leq T\}$.

(**A3**) Assume that the function $S(x)$ is bounded by a constant C.

Since the function x_t satisfies (5), it follows that

$$|S(x_t) - S(x_s)| \leq L|x_t - x_s| = L\left|\int_s^t S(x_r)dr\right| \leq LC|t-s|, \ t, s \in [0, T].$$

Let us define $\Sigma_0(L)$ as the class of all functions $S(x)$ satisfying the assumption (A1) and uniformly bounded by the same constant C. Further, we denote by $\Sigma_k(L)$ the class of all function $S(x)$ which are uniformly bounded by the same constant C and which are k-times differentiable with respect to x satisfying the following condition

$$\left|S^k(x) - S^k(y)\right| \leq L\,|x-y|\,, \ x, y \in \mathbb{R}, \tag{7}$$

where $S^k(x)$ is the k-th derivative of $S(x)$.

Lemma 1 *Assume that hypothesis* (A1) *is verified. Let* X_t *and* x_t *be the solutions of the equations* (4) *and* (5) *respectively. Then, we have*

$$\sup_{0 \leq t \leq T} \mathbb{E}\,(X_t - x_t)^2 \leq e^{2LT}\varepsilon^2 T^{2HK}. \tag{8}$$

Proof of the Lemma 1

By (4) and (5), we have

$$X_t = x_0 + \int_0^t S(X_r)dr + \varepsilon B_t^{H,K},$$

and

$$x_t = x_0 + \int_0^t S(x_r)dr.$$

This implies

$$X_t - x_t = \int_0^t (S(X_r) - S(x_r))\,dr + \varepsilon B_t^{H,K}.$$

Thus

$$\begin{aligned} |X_t - x_t| &\leq \int_0^t |S(X_r) - S(x_r)|\,dr + \varepsilon|B_t^{H,K}| \\ &\leq L\int_0^t |X_r - x_r|\,dr + \varepsilon|B_t^{H,K}|. \end{aligned} \tag{9}$$

Putting $u_t = |X_t - x_t|$, we have

$$u_t \leq \int_0^t u_r dr + \varepsilon|B_t^{H,K}|.$$

By using Grönwall's inequality, we obtain

$$|X_t - x_t| \leq e^{Lt}\varepsilon \left|B_t^{H,K}\right|.$$

Then, since $\mathbb{E}\left(B_t^{H,K}\right)^2 = t^{2HK}$, we have

$$\mathbb{E}|X_t - x_t|^2 \leq e^{2Lt}\varepsilon^2 t^{2HK}.$$

Finally, we find

$$\sup_{0\leq t\leq T} \mathbb{E}\left(X_t - x_t\right)^2 < e^{2LT}\varepsilon^2 T^{2HK}.$$

4 Main results

The main goal of this work is to build an estimator of the trend function S_t in the model described by stochastic differential equation (4) using the method developed by Kutoyants [7]. Then, we study its asymptotic properties as $\varepsilon \longrightarrow 0$.

For all $t \in [0, T]$, the kernel estimator $\hat{S}_t$ of S_t is given by

$$\hat{S}_t = \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{10}$$

where $G(u)$ is a bounded function with finite support $[A, B]$ satisfying the following hypotheses:

(H1) $G(u) = 0$ for $u < A$ and $u > B$ and $\displaystyle\int_A^B G(u)du = 1$,

(H2) $\displaystyle\int_{-\infty}^{+\infty} G^2(u)du < \infty$,

(H3) $\displaystyle\int_{-\infty}^{+\infty} u^{2(k+1)}G^2(u)du < \infty$,

(H4) $\displaystyle\int_{-\infty}^{+\infty} |G(u)|^{\frac{1}{HK}}\, du < \infty$,

Further, we suppose that the normalizing function ϕ_ε satisfies:

(H5) $\phi_\varepsilon \longrightarrow 0$ and $\varepsilon^2\phi_\varepsilon^{-1} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

The following theorem gives the uniform convergence of the estimator $\hat{S}_t$.

Theorem 1 *Suppose that the assumptions* (A1)-(A3) *and* (H1)-(H5) *hold true. Further, suppose that the trend function* $S(x)$ *belongs to* $\Sigma_0(L)$*. Then, for any*

$0 < c \le d < T$ *and* $HK \in (1/2, 1)$*, the estimator* $\widehat{S}_t$ *is uniformly consistent, that is,*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{S(x) \in \Sigma_0(L)} \sup_{c \le t \le d} \mathbb{E}_S(|\widehat{S}_t - S(x_t)|^2) = 0. \tag{11}$$

The following additional assumptions are useful for the rest of the theoretical study. Assume that

(H6) $\displaystyle\int_{-\infty}^{+\infty} u^j G(u) du = 0$ for $j = 1, 2, ..., k$,

(H7) $\displaystyle\int_{-\infty}^{+\infty} u^{k+1} G(u) du < \infty$ and $\int_{-\infty}^{+\infty} u^{2(k+2)} G^2(u) du < \infty$.

The rate of convergence of the estimator $\widehat{S}_t$ is established in the following theorem.

Theorem 2 *Suppose that the function* $S(x) \in \Sigma_k(L)$*,* $HK \in (1/2, 1)$ *and* $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$*. Then, under the hypotheses* (A1)-(A3) *and* (H1)-(H7)*, we have*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{S(x) \in \Sigma_k(L)} \sup_{c \le t \le d} \mathbb{E}_S(|\widehat{S}_t - S(x_t)|^2) \varepsilon^{\frac{-2(k+1)}{k-HK+2}} < \infty. \tag{12}$$

Finally, the following theorem presents the asymptotic normality of the kernel type estimator $\widehat{S}_t$ of $S(x_t)$.

Theorem 3 *Suppose that the function* $S(x) \in \Sigma_{k+1}(L)$*,* $HK \in (1/2, 1)$ *and* $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$*. Then, under the hypotheses* (A1)-(A3) *and* (H1)-(H7)*, we have*

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}} \left(\widehat{S}_t - S(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(m, \sigma^2_{H,K}), \quad \textit{as } \varepsilon \longrightarrow 0,$$

where

$$m = \frac{S^{k+1}(x_t)}{(k+1)!} \int_{-\infty}^{+\infty} G(u) u^{k+1} du,$$

and

$$\begin{aligned}\sigma^2_{H,K} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G(u) G(v) \Big[& \alpha_{H,K} \left(u^{2H} + v^{2H}\right)^{K-2} (uv)^{2H-1} \\ & + \beta_{H,K} |u - v|^{2HK-2} \Big] du dv,\end{aligned}$$

with

$$\alpha_{H,K} = 2^{-K+2} H^2 K (K-1) \quad \textit{and} \quad \beta_{H,K} = 2^{-K+1} HK(2HK - 1).$$

5 Numerical example

The main objective of this part is to conduct a numerical study to illustrate our theoretical result. We compare our kernel estimator for stochastic differential equations driven by a bifractional Brownian motion to the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in Mishra and Prakasa Rao [10]. We compare numerically the variance $\sigma^2_{H,K}$ of our estimator to σ^2_H.

Consider a function G which satisfies hypotheses (H1)-(H7):

$$G(t) = \frac{15}{128}\left(63t^4 + 70t^2 + 15\right), \quad |t| \le 1.$$

– The variance of the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in Mishra and Prakasa Rao [10] is given as:

For all $H \in (1/2, 1)$,

$$\sigma^2_H = H(2H-1)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\,|u-v|^{2H-2}\,dudv,$$

– Using the result given in Theorem 3, the variance of our estimator is obtained as:

For all $H \in (0,1)$, $K \in (0,1]$ and $HK \in (1/2, 1)$, we have

$$\sigma^2_{H,K} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\Big[\alpha_{H,K}\left(u^{2H}+v^{2H}\right)^{K-2}(uv)^{2H-1} + \beta_{H,K}\,|u-v|^{2HK-2}\Big]\,dudv,$$

where

$$\alpha_{H,K} = 2^{-K+2}H^2K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1}HK(2HK-1).$$

Next, we compute the variances, the results are presented in the following Tables

Table 1: The variance values σ_H^2.

H	0.7	0.75	0.8	0.85	0.9	0.95
σ_H^2	1.1567	1.1900	1.1830	1.1506	1.1025	1.0452

Table 2: The variance values $\sigma_{H,K}^2$.

$K \setminus H$	0.7	0.75	0.8	0.85	0.9	0.95
0.75	0.6006	0.9458	1.1647	1.2965	1.3709	1.4091
0.8	0.8733	1.1230	1.2696	1.3462	1.3774	1.3801
0.85	1.0362	1.2107	1.3019	1.3376	1.3378	1.3159
0.9	1.1227	1.2382	1.2873	1.2930	1.2712	1.2326
0.95	1.1570	1.2264	1.2437	1.2274	1.1901	1.1402
1	1.1567	1.1900	1.1830	1.1506	1.1025	1.0452

From the obtained results in Tables 1 and 2, we clearly see that the variance of our estimator is less than that of the kernel estimator for stochastic differential equations driven by fractional Brownian motion. We can conclude that our kernel estimator for stochastic differential equations driven by a bifractional Brownian motion is better than that given in Mishra and Prakasa Rao [10].

6 Proof of Theorems

6.1 Proof of Theorem 1

From (4) and (10), we can see that

$$\begin{aligned}
\hat{S}_t - S(x_t) &= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right) dX_\tau - S(x_t) \\
&= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\left(S(X_\tau)d\tau + \varepsilon dB_\tau^{H,K}\right) - S(x_t) \\
&= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\left(S(X_\tau) - S(x_\tau)\right) d\tau \\
&\quad + \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right) S(x_\tau)d\tau - S(x_t) \\
&\quad + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right) dB_\tau^{H,K}.
\end{aligned}$$

Using the inequality $(\alpha+\beta+\gamma)^2 \leq 3\alpha^2+3\beta^2+3\gamma^2$, it yields

$$\begin{aligned}\mathbb{E}_S\left[\hat{S}_t - S(x_t)\right]^2 &\leq 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\left(S(X_\tau)-S(x_\tau)\right)d\tau\right]^2 \\ &\quad + 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\ &\quad + 3\mathbb{E}_S\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{H,K}\right]^2 \\ &\leq I_1+I_2+I_3.\end{aligned} \tag{13}$$

• **Concerning** I_1**.** Via inequalities (6) and (8) and hypotheses (H1)-(H2), we get

$$\begin{aligned} I_1 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\left(S(X_\tau)-S(x_\tau)\right)d\tau\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(X_{t+\phi_\varepsilon u})-S(x_{t+\phi_\varepsilon u})\right)du\right]^2 \\ &\leq 3(B-A)\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G^2(u)\left(S(X_{t+\phi_\varepsilon u})-S(x_{t+\phi_\varepsilon u})\right)^2 du\right] \\ &\leq 3(B-A)L^2\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G^2(u)\left(X_{t+\phi_\varepsilon u}-x_{t+\phi_\varepsilon u}\right)^2 du\right] \\ &\leq 3(B-A)L^2\left[\int_{-\infty}^{+\infty} G^2(u)\sup_{0\leq t+\phi_\varepsilon u\leq T}\mathbb{E}_S\left(X_{t+\phi_\varepsilon u}-x_{t+\phi_\varepsilon u}\right)^2 du\right] \\ &\leq 3(B-A)L^2e^{2LT}T^{2HK}\varepsilon^2 \\ &\leq C_1\varepsilon^2,\end{aligned} \tag{14}$$

where C_1 is a positive constant depending on T, L, H, K, and $(B-A)$.

• **Concerning** I_2**.** Let

$$\begin{aligned} I_2 &= 3\mathbb{E}_S\left[\tfrac{1}{\phi_\varepsilon}\int_0^T G\left(\tfrac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\,S(x_{t+\phi_\varepsilon u})du - S(x_t)\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(x_{t+\phi_\varepsilon u})-S(x_t)\right)du\right]^2 \end{aligned} \tag{15}$$

Next, by using hypotheses (A3) and (H3), we have

$$\begin{aligned} I_2 &\leq 3L^2 C_2^2 \mathbb{E}_S \left[\int_{-\infty}^{+\infty} G(u)(\phi_\varepsilon u)\, du \right]^2 \\ &\leq 3(B-A) L^2 C_2^2 \left[\int_{-\infty}^{+\infty} G^2(u) u^2 du \right] \phi_\varepsilon^2 \\ &\leq C_3 \phi_\varepsilon^2, \end{aligned}$$

where C_3 is a positive constant depending on L and $(B-A)$.

- **Concerning** I_3. Since $HK \in (1/2, 1)$, we have

$$\begin{aligned} I_3 &= 3\mathbb{E}_S \left[\frac{\varepsilon}{\phi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dB_\tau^{H,K} \right]^2 \\ &= 3\frac{\varepsilon^2}{\phi_\varepsilon^2} \mathbb{E}_S \left[\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dB_\tau^{H,K} \right]^2 \\ &\leq 3\frac{\varepsilon^2}{\phi_\varepsilon^2} \left[C(2, HK) \left(\int_0^T \left| G\left(\frac{\tau - t}{\phi_\varepsilon}\right) \right|^{\frac{1}{HK}} d\tau \right)^{2HK} \right] \\ &\leq C_4 \frac{\varepsilon^2}{\phi_\varepsilon^2} \left[\phi_\varepsilon^{2HK} \left(\int_{-\infty}^{+\infty} |G(u)|^{\frac{1}{HK}} du \right)^{2HK} \right] \\ &\leq C_5 \frac{\varepsilon^2}{\phi_\varepsilon} \phi_\varepsilon^{2HK-1} \quad \text{(using hypothesis (H4))}, \end{aligned} \tag{16}$$

where C_5 is a positive constant depending on H and K.

Combining (13)-(16), we have

$$\sup_{S(x)\in\Sigma_0(L)} \sup_{c\leq t\leq d} \mathbb{E}_S \left[\hat{S}_t - S(x_t)\right]^2 \leq C_6 \left(\varepsilon^2 + \phi_\varepsilon^2 + \frac{\varepsilon^2}{\phi_\varepsilon} \phi_\varepsilon^{2HK-1} \right).$$

Finally, under the assumption (H5), we obtain

$$\lim_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_0(L)} \sup_{c\leq t\leq d} \mathbb{E}_S \left[\hat{S}_t - S(x_t)\right]^2 = 0.$$

6.2 Proof of Theorem 2

Using the Taylor formula, we get

$$S(x_t) = S(x_{t_0}) + \sum_{j=1}^{k} S^j(x_{t_0}) \frac{(t - t_0)^j}{j!}$$

$$+\left(S^{k}(x_{t+\lambda(t-t_0)})-S^{k}(x_{t_0})\right)\frac{(t-t_0)^k}{k!},\ \lambda\in(0,1),$$

and

$$S\left(x_{t+\phi_\varepsilon u}\right)=S\left(x_t\right)+\sum_{j=1}^{k}S^{j}\left(x_t\right)\frac{(\phi_\varepsilon u)^j}{j!}$$
$$+\left(S^{k}(x_{t+\lambda(\phi_\varepsilon u)})-S^{k}(x_t)\right)\frac{(\phi_\varepsilon u)^k}{k!},\ \lambda\in(0,1).$$

Then, by substituting this expression in I_2, using inequality (7) and assumptions (H6)-(H7), we obtain

$$\begin{aligned}
I_2 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau-S(x_t)\right]^2\\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty}G\left(u\right)S(x_{t+\phi_\varepsilon u})du-S(x_t)\right]^2\\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty}G\left(u\right)\left(S(x_{t+\phi_\varepsilon u})-S(x_t)\right)du\right]^2\\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty}G(u)\left(\sum_{j=1}^{k}S^{j}(x_t)\frac{(\phi_\varepsilon u)^j}{j!}+\left(S^{k}(x_{t+\lambda(\phi_\varepsilon u)})-S^{k}(x_t)\right)\frac{(\phi_\varepsilon u)^k}{k!}\right)du\right]^2\\
&= 3\mathbb{E}_S\left[\frac{\phi_\varepsilon^k}{k!}\int_{-\infty}^{+\infty}G\left(u\right)u^k\left(S^{k}(x_{t+\lambda(\phi_\varepsilon u)})-S^{k}(x_t)\right)du\right]^2\ \ (\textit{by using (H6)})\\
&\le 3C_7^2L^2\left[\frac{\phi_\varepsilon^{k+1}}{k!}\int_{-\infty}^{+\infty}G\left(u\right)u^{k+1}du\right]^2\\
&\le 3C_7^2L^2\left(B-A\right)\frac{\phi_\varepsilon^{2(k+1)}}{(k!)^2}\left[\int_{-\infty}^{+\infty}G^2\left(u\right)u^{2(k+1)}du\right]\\
&\le C_8\phi_\varepsilon^{2(k+1)},
\end{aligned}\tag{17}$$

where C_8 is a positive constant depending on L and $(B-A)$.
Next, from (14), (16), and (17), we find

$$\sup_{S(x)\in\Sigma_k(L)}\sup_{c\le t\le d}\mathbb{E}_S\left|\hat{S}_t-S(x_t)\right|^2\le C_9\left(\varepsilon^2\phi_\varepsilon^{2HK-2}+\phi_\varepsilon^{2(k+1)}+\varepsilon^2\right).$$

Putting $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$, it yields

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_k(L)} \sup_{c\le t\le d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-2(k+1)}{k-HK+2}} < \infty.$$

This completes the proof of Theorem 2.

6.3 Proof of Theorem 3

From (4) and (10), we can see that

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}}\left(\hat{S}_t - S(x_t)\right) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(S(X_\tau) - S(x_\tau))\,d\tau\right.$$
$$\left. + \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right) S(x_\tau)d\tau - S(x_t) \ + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right) dB_\tau^{H,K}\right].$$

Therefore

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}}\left(\hat{S}_t - S(x_t)\right) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\left[\int_{-\infty}^{+\infty} G(u)\,(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u}))\,du\right.$$
$$\left. + \int_{-\infty}^{+\infty} G(u)\,(S(x_{t+\phi_\varepsilon u}) - S(x_t))\,du + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right) dS_\tau^H\right].$$

Thus

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}}\left(\hat{S}_t - S(x_t)\right) = r_1(t) + r_2(t) + \eta_\varepsilon(t).$$

Hence, by Slutsky's Theorem, it suffices to show the following three claims:

$$r_1(t) \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0 \text{ in probability.} \tag{18}$$

$$r_2(t) \longrightarrow m, \text{ as } \varepsilon \longrightarrow 0 \text{ in probability.} \tag{19}$$

and

$$\eta_\varepsilon(t) \longrightarrow \mathcal{N}(0, \sigma^2_{H,K}), \text{ as } \varepsilon \longrightarrow 0 \text{ in distribution.} \tag{20}$$

Proof of (18).
Let

$$r_1(t) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\int_{-\infty}^{+\infty} G(u)\,(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u}))\,du.$$

By applying the inequality (14), we have

$$0 \le \mathbb{E}\left[r_1^2(t)\right] \le \varepsilon^{\frac{-2(k+1)}{k-HK+2}} I_1 \le C_{10}\varepsilon^{\frac{2(1-HK)}{k-HK+2}}.$$

Therefore, using the Bienaymé-Tchebychev's inequality, as $\varepsilon \longrightarrow 0$, we obtain, for all $\alpha > 0$

$$P\left(|r_1(t)| > \alpha\right) \leq \frac{\mathbb{E}\left[r_1^2(t)\right]}{\alpha^2} \leq \frac{C_{10}\varepsilon^{\frac{2(1-HK)}{k-HK+2}}}{\alpha^2} \longrightarrow 0.$$

Proof of (19).

Let

$$r_2(t) = \varepsilon^{\frac{-(k+1)}{k-HK+2}} \int_{-\infty}^{+\infty} G\left(u\right)\left(S(x_{t+\phi_\varepsilon u}) - S(x_t)\right) du.$$

By taking any t, $u \in [0, T]$ and $b(x) \in \Sigma_{k+1}(L)$, via the Taylor expansion, we get

$$\begin{aligned} S\left(x_{t+\phi_\varepsilon u}\right) &= S\left(x_t\right) + \textstyle\sum_{j=1}^{k} S^j\left(x_t\right)\frac{(\phi_\varepsilon u)^j}{j!} + \frac{S^{k+1}(x_t)}{(k+1)!}\left(\phi_\varepsilon u\right)^{k+1} \\ &\quad + \left(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\right)\frac{(\phi_\varepsilon u)^{k+1}}{(k+1)!},\ \lambda \in (0,1), \end{aligned}$$

Making use of the conditions (H6), (H7), and choosing $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$, we obtain

$$\begin{aligned} \mathbb{E}\left[r_2(t) - m\right]^2 &= \mathbb{E}\left[\textstyle\int_{-\infty}^{+\infty} G\left(u\right)\left(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\right)\frac{(u)^{k+1}}{(k+1)!}du\right]^2 \\ &\leq C_{11}L^2C^2\left(\textstyle\int_{-\infty}^{+\infty} G(u)u^{k+2}\frac{\phi_\varepsilon}{(k+1)!}du\right)^2 \\ &\leq C_{12}\left(\textstyle\int_{-\infty}^{+\infty} G^2(u)u^{2(k+2)}du\right)\phi_\varepsilon^2 \\ &\leq C_{13}\phi_\varepsilon^2, \end{aligned}$$

where C_{13} is a positive constant which depends on L and k, and

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G\left(u\right)u^{k+1}du.$$

Therefore,

$$\mathbb{E}\left[r_2(t) - m\right]^2 \longrightarrow 0 \ \text{ as } \ \varepsilon \longrightarrow 0.$$

Then

$$r_2(t) \xrightarrow{\mathbb{P}} m.$$

Proof of (20).

Let

$$\eta_\varepsilon(t) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\varepsilon\phi_\varepsilon^{-1}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dB_\tau^{H,K}. \tag{21}$$

In fact, we have to evaluate the variance of (21). To this end, let

$$\mathbb{E}\left[\eta_\varepsilon(t)\right]^2 = \left(\varepsilon^{\frac{1-HK}{k-HK+2}}\phi_\varepsilon^{-1}\right)^2 \mathbb{E}\left(\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right) dB_\tau^{H,K}\right)^2.$$

Moreover, using equation (2), we have

$$\mathbb{E}\left[\eta_\varepsilon(t)\right]^2 = \left(\varepsilon^{\frac{1-HK}{k-HK+2}}\phi_\varepsilon^{-1}\right)^2 \left[\phi_\varepsilon^{2HK}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} du dv\right].$$

Then, by taking $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$, we get

$$\mathbb{E}\left[\eta_\varepsilon(t)\right]^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)\, G(v)\, \frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} du dv,$$

with

$$\frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} = \alpha_{H,K}\left(u^{2H}+v^{2H}\right)^{K-2}(uv)^{2H-1} + \beta_{H,K}\left|u-v\right|^{2HK-2},$$

where

$$\alpha_{H,K} = 2^{-K+2}H^2K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1}HK(2HK-1).$$

Finally, this last equation allows us to achieve the proof of Theorem 3.

7 Conclusion

This paper considered a nonparametric estimation of trend function for stochastic differential equations driven by a bifractional Brownian motion. We constructed an estimate of the trend function. Then, under some assumptions, we established the uniform consistency, the rate of convergence and the asymptotic normality of the proposed estimator. Further, an numerical example is provided. The present study has many applications in practical phenomena including telecommunications and economics.

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Orbital shadowing property on chain transitive sets for generic diffeomorphisms

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Abstract. Let $f : M \to M$ be a diffeomorphism on a closed smooth $n(\geq 2)$ dimensional manifold M. We show that C^1 generically, if a diffeomorphism f has the orbital shadowing property on locally maximal chain transitive sets which admits a dominated splitting then it is hyperbolic.

1 Introduction

Let M be a closed smooth $n(n \geq 2)$-dimensional Riemannian manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let Λ be a closed f-invariant set. We say that f has the *shadowing property* on Λ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. If $\Lambda = M$ then we say that f has the shadowing property. The shadowing property is very useful notion to investigate for hyperbolic structure. In fact, Robinson[22] and Sakai[24] proved that a diffeomorphism f has the C^1 robustly shadowing property if and only if it is structurally stable

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diffeomorphisms. Here, we say that f has the C^1 robustly shadowing property if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property. From the property, a general shadowing property was introduced by [20] which is called the orbital shadowing property. For the orbital shadowing property, many results published by the various view points (see [10, 13, 14, 15, 16, 17, 19]). We say that f has the *orbital shadowing property* on Λ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ pseudo orbit $\xi = \{x_i\}_{\in\mathbb{Z}} \subset \Lambda$ such that there is a point $y \in M$ such that

$$\mathrm{Orb}(y) \subset B_\epsilon(\xi) \text{ and } \xi \subset B_\epsilon(\mathrm{Orb}(y)).$$

If $\Lambda = M$ then we say that f has the orbital shadowing property. Let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then we say that f is Anosov.

Pilyugin *et al* [20] proved that if a diffeomorphism f has the C^1 robustly orbital shadowing property then it is structurally stable diffeomorphisms. Lee and Lee [10] proved that a volume preserving diffeomorphism f has the C^1 robustly orbital shadowing property then it is Anosov. Moreover, we can find similar results [12, 13, 14, 15]. We say that the set Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x. An invariant closed set $\mathcal{C}$ is called a *chain transitive* if for any $\delta > 0$ and $x, y \in \mathcal{C}$, there is δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n \geq 1) \subset \mathcal{C}$ such that $x_0 = x$ and $x_n = y$. It is clear that the transitive set Λ is the chain transitive set $\mathcal{C}$, but the converse is not true. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. For the relation between chain transitive sets and C^1 robustly shadowing theories, In [16], Lee proved that if a robustly chain transitive set with orbital shadowing then it is hyperbolic. We say that f has the C^1 stably shadowing property on Λ if there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has the shadowing property on $\Lambda_g(U)$, where $\Lambda_g(U)$ is the continuation of Λ. For $f \in \mathrm{Diff}(M)$, we say that a compact f-invariant set Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n.$$

In [18], Lee proved that if a diffeomorphism f has the C^1 stably shadowing property on chain transitive set $\mathcal{C}$ then it admits a dominated splitting. Sakai [23] proved that a diffeomorphism f has the C^1 stably shadowing property on chain transitive set $\mathcal{C}$ then it is hyperbolic.

We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff}(M)$; in this case $\mathcal{G}$ is dense in $\mathrm{Diff}(M)$. A property "P" is said to be *(C^1)-generic* if "P" holds for all diffeomorphisms which belong to some residual subset of $\mathrm{Diff}(M)$. For C^1 generic differeomorphism f, Abdenur and Díaz [3] suggested the problem : *if a C^1 generic diffeomorphism f has the shadowing property then is it hyperbolic?*

Unfortunately, this question still is open. For the problem, there are partial results [4, 9, 11]. Ahn *et al* [4] proved that for C^1 generic diffeomorphism f, if f has the shadowing property on a locally maximal homoclinic class then it is hyperbolic. Lee and Wen [11] proved that for C^1 generic diffeomorphism f, if f has the shadowing property on a locally maximal chain transitive set then it is hyperbolic. Very recently, Lee and Lee [9] proved that for C^1 generic diffeomorphism f, if f has the shadowing property on chain recurrence classes then it is hyperbolic. From the results, we study the orbital shadowing property for C^1 generic diffeomorphisms. The following is the main theorem of the paper.

Theorem A *For C^1 generic f, if f has the orbital shadowing property on a locally maximal $\mathcal{C}$ which admits a dominated splitting $E \oplus F$ then it is hyperbolic.*

2 Proof of Theorem A

Let M be as before, and let $f \in \mathrm{Diff}(M)$. A *periodic point* for f is a point $p \in M$ such that $f^{\pi(p)}(p) = p$, where $\pi(p)$ is the minimum period of p. Denote by $P(f)$ the set of all periodic points of f. Let p be a hyperbolic periodic point of f. A point $x \in M$ is called *chain recurrent* if for any $\delta > 0$, there is a finite δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n \geq 1)$ such that $x_0 = x$ and $x_n = x$. Denote by $\mathcal{CR}(f)$ the set of all chain recurrent points of f. We define a relation $\leftrightsquigarrow$ on $\mathcal{CR}(f)$ by $x \leftrightsquigarrow y$ if for any $\delta > 0$, there is a finite δ pseudo orbit $\{x_i\}_{i=0}^{n}$ such that $x_0 = x$ and $x_n = y$ and a finite δ pseudo orbit $\{w_i\}_{i=0}^{n}$ such that $w_0 = y$ and $w_n = x$. Then we know that the relation $\leftrightsquigarrow$ is an equivalence relation on $\mathcal{CR}(f)$. the equivalence classes are called the *chain recurrence classes* of f, denote by C_f. Note that if the class C_f has a hyperbolic periodic point p then we denote as $C(p, f)$.

It is well known that if p is a hyperbolic periodic point of f with period k

then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \quad \text{and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. The homoclinic class of a hyperbolic periodic point p is the closure of the transverse intersection of the $W^s(p)$ and $W^u(p)$, and it is denoted by $H(p, f)$. It is clear that $H(p, f)$ is compact, transitive and invariant sets. Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$ then index(p) = index(q), that is, $\dim W^s(p) = \dim W^s(q)$. By the Smale's transverse homoclinic point theorem, $H_f(p)$ coincides with the closure of the set of hyperbolic periodic points q of f such that $p \sim q$. Note that if p is a hyperbolic periodic point of f then there is a neighborhood U of p and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ there exists a unique hyperbolic periodic point p_g of g in U with the same period as p and index(p_g) = index(p). Such a point p_g is called the *continuation* of $p = p_f$. The following are results for C^1 generic diffeomorphisms (see [2]).

Lemma 1 *There is a residual set $\mathcal{G} \subset$ Diff(M) such that if $f \in \mathcal{G}$,*

(a) $H(p, f) = C(p, f)$*, for some hyperbolic periodic point* p (see [5]).

(b) *A compact f-invariant set $\mathcal{C}$ is chain transitive if and only if $\mathcal{C}$ is the Hausdorff limit of a sequence of periodic orbits of* f (see [8]).

(c) *A locally maximal transitive set Λ is a locally maximal $H(p, f)$ for some periodic point* $p \in \Lambda$ (see [1]).

(d) $H(p, f) = \overline{W^s(p)} \cap \overline{W^u(p)}$ (see [7]).

Remark 1 *Applying Pugh's closing lemma, we know that any transitive set Λ of a C^1-generic diffeomorphism f is the Hausdorff limit of a sequence of periodic orbits $\mathrm{Orb}_f(p_n)$ of f, that is, $\lim_{n\to\infty} \mathrm{Orb}_f(p_n) = \Lambda$. By Lemma 1* (b) *and* (c)*, a chain transitive set $\mathcal{C}$ is a transitive set Λ and so, a locally maximal chain transitive set $\mathcal{C} = H(p, f)$ for some periodic point p.*

Let Λ be a closed f-invariant set. We say that Λ is *Lyapunov stable* for f if for any open neighborhood U of Λ there is a neighborhood $V \subset U$ such that

$f^j(V) \subset U$ for all $j \in \mathbb{N}$. We say that the closed set is *bi-Lyapunov stable* if it is Lyapunov stable for f and f^{-1}. Potrie [21, Theorem 1.1] proved that C^1 generically, if a homoclinic class $H(p,f)$ is a bi-Lyapunov stable then it admits a dominated splitting.

A diffeomorphism f has a *heterodimensiional cycle* associated with the hyperbolic periodic points p and q of f if (i) the indice of the points p and q are different, and (ii) the stable manifold of p meets the unstable manifold of q and the same holds for the stable manifold of p and the unstable manifold of q (see [6]). We say that f has C^1 *robustly heterodimensional cycle* if f has a heterodiemnsional cycle associated with the hyperbolic periodic points p and q of f and there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has a heterodimensional cycle associated with the hyperbolic periodic points p_g and q_g, where p_g and q_g are the continuations of p and q for g.

Lemma 2 [6, Corollary 1.15] *There is a residual set* $\mathcal{T} \subset \mathrm{Diff}(M)$ *such that for any* $f \in \mathcal{T}$ *and every locally maximal chain recurrence class* C_f *of* f *there are two possibilities: either* C_f *is hyperbolic or it has a robustly heterodimensional cycle.*

Lemma 3 *Let* $C(p,f)$ *admits a dominated splitting* $E \oplus F$ *and let* $C(p,f)$ *be locally maximal. If a chain recurrence class* $C(p,f)$ *has heterodimensional cycle then* f *does not have the orbital shadowing property on* $C(p,f)$.

Proof. Suppose, by contradiction, that f has the orbital shadowing property on $C(p,f)$. Since $C(p,f)$ has heterodimensional cycle, there is q a hyperbolic periodic point in $C(p,f)$ such that $\mathrm{index}(p) \neq \mathrm{index}(q)$. Then we take $x, y \in C(p,f)$ such that $x \in W^s(p) \cap W^u(q)$ and $y \in W^u(p) \cap W^s(q)$. By assumption, x, y are not transverse intersection points. Since q is hyperbolic, $T_qM = E^s_q \oplus E^u_q$. Choose $\alpha > 0$ sufficiently small such that $W^s_{\alpha/4}(q) = \exp_q(E^s(\alpha/4))$ and $W^u_{\alpha/4}(q) = \exp_q(E^u(\alpha/4))$. Then we may assume that $y \in W^s_{\alpha/4}(q)$ and $x \in W^u_{\alpha/4}(q)$. Since $y \in W^u(p)$, there is $\eta > 0$ such that $y \in B_\eta(y) \cap W^u(p)$. Take a small arc $\mathcal{J}_y \subset B_\eta(y) \cap W^u(p)$ such that $T_y\mathcal{J}_y = T_yW^u(p)$. Since $C(p,f)$ admits a dominated splitting $E \oplus F$, we have $T_y\mathcal{J}_y = F_y = T_yW^u(p), F_q \subset E^u_q$ and $E^s_q \subset E_q$. Put $E^{u,1} = E_q \oplus E^u_q$ and $E^{u,2} = F_q$. Then $E^u_q = E^{u,1}_q \oplus E^{u,2}_q$, and $W^{u,1}_{\alpha/4}(q) = \exp_q(E^{u,1}_q(\alpha/4)), W^{u,2}_{\alpha/4}(q) = \exp_q(E^{u,2}_q(\alpha/4))$.

Let $P^u : B_{\alpha/4}(q) \to E^u_q$ and $P^s : B_{\alpha/4}(q) \to E^s_q$ be the projections parallel to E^s_q and E^u_q, respectively. Then $P^u(f^n(\mathcal{J}_y)) \cap B_{\alpha/4}(q) \to W^{u,1}_{\alpha/4}(q)$ and $P^s(f^n(\mathcal{J}_y)) \cap B_{\alpha/4}(q) \to q$ as $n \to \infty$.

Take $\epsilon = \min\{\alpha/4, \eta, d(x, W^{u,1}_{\alpha/4}(p))/2\}$, and let $0 < \delta < \epsilon$ be the number of the orbital shadowing property. Since $y \in W^s(q) \cap W^u(p)$ and $x \in W^u(q) \cap W^s(p)$, there are $i_1 > 0$ and $i_2 > 0$ such that (i) $d(f^{i_1}(y), f^{-i_1}(x)) < \delta$ and $d(f^{-i_2}(y), f^{i_2}(x)) < \delta$, (ii) $\max\{d_H(P^s(f^{i_1+j}(\mathcal{J}_y), q), d_H(P^u(f^{i_1+j}(\mathcal{J}_y)) \cap B_{\alpha/4}(q), W^{u,1}_{\alpha/4}(q))\} < \epsilon$ for all $j \geq 0$, where d_H is the Hausdorff metric. Then we have a δ-pseudo orbit as follows:

$$\begin{aligned}\xi = \{&y, f(y), \dots, f^{i_1-1}(y), f^{-i_1}(x), \dots, f^{-1}(x),\\ &x, f(x), \dots, f^{i_2-1}(x), f^{-i_2}(y), \dots, f^{-1}(y), y\} \subset C(p, f).\end{aligned}$$

Since f has the orbital shadowing property on $C(p, f)$ and $C(p, f)$ is locally maximal, there is a point $w \in C(p, f)$ such that

$$Orb(w) \subset B_\epsilon(\xi) \text{ and } \xi \subset B_\epsilon(Orb(w)).$$

First, we assume that there is $k > 0$ such that $f^k(w) \in \mathcal{J}_y \setminus \{y\}$. Then if $f^k(w) \in P^u(f^n(\mathcal{J}_y))$ then since $P^u(f^n(\mathcal{J}_y)) \to W^{u,1}(q)(n \to \infty)$, $f^{k+n}(w) \to W^{u,1}(q)$ as $n \to \infty$. Thus there is $j > 0$ such that $d(f^{k+j}(w), f^j(y)) > 8\epsilon$ and so, $d(f^{k+j}(w), q) > 2\epsilon$ which is a contradiction. If $f^k(w) \notin P^u(\mathcal{J}_y)$ then by λ-lemma, $f^n(\mathcal{J}_y) \to W^u(q)$ as $n \to \infty$. Then there is $l > 0$ such that $d(f^{k+l}(w), q) > 4\epsilon$. Since $x \in W^u(q)$, there is $m > 0$ such that $d(f^{-m}(x), q) < \epsilon$ for some $m < i_2$. Then we know that $f^{k+l}(w) \notin B_\epsilon(\xi)$ which is a contradiction by the orbital shadowing property on $C(p, f)$. Then for all $i \in \mathbb{Z}$, $f^i(w) \notin \mathcal{J}_y \setminus \{y\}$.

We assume that there is $k > 0$ such that $f^k(w) = y$. Since $y \in W^s(q) \cap W^u(p)$ and $x \in W^u(q) \cap W^s(p)$, we know $Orb(x) \cap Orb(y) = \emptyset$. Then we have $\xi \not\subset B_\epsilon(Orb(w))$ which is a contradiction by the orbital shadowing property on $C(p, f)$. Thus we know $Orb(w) \cap \mathcal{J}_y = \emptyset$.

Finally, we assume that there is $k > 0$ such that $f^k(w) \in B_\eta(y) \setminus \mathcal{J}_y$. Then for all $z \in B_\eta(y) \setminus \mathcal{J}_y$, there is $k > 0$ such that $d(f^{-k}(x), f^k(z)) > 2\epsilon$ since $x \in W^u(q)$ and q is hyperbolic saddle. Then we have $\xi \not\subset B_\epsilon(Orb(w))$ which is a contradiction by the orbital shadowing property on $C(p, f)$. Consequently, if a locally maximal chain recurrence class $C(p, f)$ admits a dominated splitting and f has the orbital shadowing property on $C(p, f)$ then it does not the heterodimensional cycle. $\square$

Proof of Theorem A. Let $f \in \mathcal{G} \cap \mathcal{T}$ and let f has the orbital shadowing property on a locally maximal chain transitive set $\mathcal{C}$. Since $f \in \mathcal{G}$, by Remark 1 $\mathcal{C} = C(p, f)$ for some hyperbolic periodic point p. Since chain transitive set $\mathcal{C}$ admits a dominated splitting $E \oplus F$, and f has the orbital shadowing property on a locally maximal chain transitive $\mathcal{C}$, by Lemma 3, $\mathcal{C} = C(p, f)$ does not have the heterodimensional cycles. It is clear that a locally maximal $C(p, f)$ does not have the robustly hetrodimensional cycle. Thus by Lemma 2, a locally maximal chain transitive set $\mathcal{C}$ is hyperbolic. $\square$

In Abdenur *et al* [2] the authors proved that every locally maximal homoclinic class with a non-empty interior is the whole space.

Corollary 1 *Let* $f : M \to M$ *be a diffeomorphism with* $\dim M = 3$. *For* C^1 *generic* f, *if* f *has the orbital shadowing property on a locally maximal chain transitive set* $\mathcal{C}$ *which admits a dominated splitting* $E \oplus F$ *then it is Anosov.*

The following was proved in [21, Proposition 1.2] which means that a homoclinic class admits a codimension one dominated splitting then it has a non-empty interior.

Lemma 4 *There is a residual set* $\mathcal{R} \subset \mathrm{Diff}(M^3)$ *such that for any* $f \in \mathcal{R}$, *if a homoclinic class* H *admits a codimension one dominated splitting then it has non-empty interior.*

Proof of Corollary 1. Let $f \in \mathcal{G} \cap \mathcal{T} \cap \mathcal{R}$ and let f has the orbital shadowing property on a locally maximal chain transitive set $\mathcal{C}$ which admits a dominated splitting $E \oplus F$. By Remark 1, a locally maximal chain transitive $\mathcal{C} = H(p, f)$. Since $f \in \mathcal{R}$, by Lemma 4, a hoomoclinic class $H(p, f)$ has nonempty interior. Since $H(p, f)$ is locally maximal, by [2, Thereom 3], $H(p, f) = M$. Since f has the orbital shadowing property on a locally maximal chain transitive set $\mathcal{C}$ which admits a dominated splitting $E \oplus F$, by Theorem A, it is hyperbolic, and so it is Anosov. $\square$

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Restrained domination in signed graphs

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Abstract. A signed graph Σ is a graph with positive or negative signs attatched to each of its edges. A signed graph Σ is balanced if each of its cycles has an even number of negative edges. Restrained dominating set D in Σ is a restrained dominating set of its underlying graph where the subgraph induced by the edges across $\Sigma[D : V \setminus D]$ and within $V \setminus D$ is balanced. The set D having least cardinality is called minimum restrained dominating set and its cardinality is the restrained domination number of Σ denoted by $\gamma_r(\Sigma)$. The ability to communicate rapidly within the network is an important application of domination in social networks. The main aim of this paper is to initiate a study on restrained domination in the realm of different classes of signed graphs.

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1 Introduction

Graphs used in this article, unless otherwise mentioned will be undirected, simple and finite. For a graph $G = (V, E)$, the degree of a vertex v, denoted by $\mathrm{deg}(v)$ is the number of edges incident to the vertex v(loops counted twice incase of multigraph). The maximum degree of G is denoted by $\Delta(G)$ and the minimum degree of G is denoted by $\delta(G)$. If $\mathrm{deg}(v) = 1$, then v is called a pendant vertex. For all terminology and notation in graph theory, we refer the reader to the text-book by Harary [1]. The graphs with positive or negative signs attatched to each of its arcs are called signed graphs. Zaslavasky [2], formally defines a signed graph as $\Sigma = (G, \sigma)$, where G is the underlying unsigned graph consisting of $G = (V, E)$ and $\sigma : E \to \{+, -\}$ is the function assigning signs to the edges of the graph. The edges which receive $+(-)$ signs, are called positive(negative) edges of Σ.

A signed graph Σ is all-positive(all-negative) if all its edges are positive (negative). If it is an all-positive or all-negative, then it is said to be homogenous else heterogenous. Switching Σ with respect to a marking μ where $\mu : V \to \{+1, -1\}$ is the operation of negating every edge whose end vertices are of opposite signs. Σ is said to be balanced if each of its cycle has an even number of negative edges. Equivalently, a signed graph is balanced if it can be switched to an all-positive signed graph. For further details on theory of signed graphs, the reader is referred to [3, 2].

Domination in graph theory for unsigned graphs is one of the continuing research of the well-researched region. Detailed survey of the same can be found in the book by Haynes et al. [4]. In 2013, Acharya [5] introduced the theory of dominance for signed graphs as well as signed digraphs. A subset $D \subseteq V$ of vertices of $\Sigma = (G, \sigma)$ is a dominating set of Σ, if there exists a marking $\mu : V \to \{+1, -1\}$ of Σ such that every vertex not in D is adjacent to at least one vertex in D and $\sigma(uv) = \mu(u)\mu(v)$, $\forall\ u \in V \setminus D$. The minimum cardinality of a dominating set in Σ is called its domination number, denoted by $\gamma(\Sigma)$. Germina and Ashraf [6, 7] gave characterization for open domination and double domination in signed graphs. In 2015, Walikar et al. [8] introduced the concept of signed domination for signed graphs.

In a social network, if all individuals are connected to at least one such person who can be reached directly, an emergency message can easily be sent to all participants in the network, thus reducing delay time. Nevertheless, it is also important to examine positive and negative relationships between individuals when examining social network interactions. This situation can be modeled on what is known as the dominating set problem in signed graphs.

In this paper, we introduce the concept of restrained domination for signed graphs. In addition, we determine the best possible bounds on $\gamma_r(G)$ for certain classes of signed graphs.

2 Definitions and results

The concept of restrained domination in graphs was introduced by Domke et al. [9] in 1999. A set $D \subseteq V$ is a restrained dominating set of graph $G = (V, E)$, if every vertex in $V \setminus D$ is adjacent to a vertex in D as well as another vertex in $V \setminus D$. The restrained domination number of graph G denoted by $\gamma_r(G)$ is the smallest cardinality of a restrained dominating set of G. We will now define the concept of restrained domination for signed graphs and then find the best possible general bounds for some classes of signed graph. In this paper, we will be using the notation $\Sigma[D : V \setminus D]$ when $D \subseteq V$, to denote the subgraph of Σ induced by the edges of Σ with one end point in D and the other end point in $V \setminus D$. Induced subgraph in $V \setminus D$ is denoted by $\Sigma[V \setminus D]$.

Definition 1 *A subset $D \subseteq V$ of vertices of $\Sigma = (V, E, \sigma)$ is a restrained dominating set if there exists a marking $\mu : V \to \{+1, -1\}$ of Σ such that every vertex in $V \setminus D$ is adjacent to a vertex in D as well a vertex in $V \setminus D$ and for every vertex u in $V \setminus D$, $\sigma(uv) = \mu(u)\mu(v)$ $\forall$ $v \in D$ and $v \in V \setminus D$.*

The minimum cardinality of a restrained dominating set D is called restrained domination number of Σ denoted by $\gamma_r(\Sigma)$. Every restrained dominating set of a signed graph Σ of order n follows the inequality $1 \leq \gamma_r(\Sigma) \leq n$. As each vertex in $V \setminus D$ is adjacent to at least one other vertex in $V \setminus D$, the cardinality of the set $V \setminus D$ is always greater than or equal to two. Hence, $\gamma_r(\Sigma)$ can never be equal to $n - 1$. Before proceeding further with results on bounds for few classes of signed graphs, we state some important results used for obtaining these bounds.

Proposition 1 *Let G be any graph of order n with $\delta(G) = 1$. Then, every restrained dominating set of graph G must necessarily have all its pendant vertices.*

Switching $\Sigma = (G, \sigma)$ with respect to μ means forming the switched graph $\Sigma^\mu = (\Sigma, \sigma^\mu)$, whose underlying graph is the same but whose sign function is defined on an edge $e : vw$ by $\Sigma^\mu(e) = \mu(v)\sigma(e)\mu(w)$. In case of balanced signed graphs, when Σ is switched with respect to μ, we obtain an all-positive signed graph. Hence, we can state the following lemma:

Lemma 1 [2] *A signed graph Σ is balanced if and only if it can be switched to an all-positive signed graph.*

Let, D^r_Σ be the set of all restrained dominating sets of signed graphs and $D^r_{|\Sigma|}$ be the set of all restrained dominating sets of its underlying graphs. Then, for balanced signed graphs Σ, $D^r_\Sigma = D^r_{|\Sigma|}$. But, note that this equality does not hold true for all unbalanced signed graphs. For example, consider a 6-cycle graph $\Sigma = C_6$ with three negative edges, denoted as $C_6^{(3)}$. The underlying graph $|\Sigma| = C_6$ will have all independent vertices in its minimum restrained dominating set. Whereas, in Σ there exists no such independent vertices in restrained dominating set. This leads to the conclusion of following proposition:

Proposition 2 *The set D^r_Σ of all restrained dominating sets of a signed graph Σ is contained in the set $D^r_{|\Sigma|}$ of all restrained dominating sets of its underlying graph $|\Sigma|$.*

Proposition 3 *For any finite balanced signed graph, $\gamma_r(\Sigma) = \gamma_r(|\Sigma|)$.*

Clearly, from Lemma 1, we can conclude for balanced signed graphs $D^r_\Sigma = D^r_{|\Sigma|}$ and hence the result holds true.

Further, since $D^r_\Sigma \subseteq D^r_{|\Sigma|}$, we can conclude $\gamma_r(|\Sigma|) \leq \gamma_r(\Sigma)$. In the following results, we derive bounds for some classes of signed graphs.

Theorem 1 *If $P_n^{(r)}$ is a signed path with n vertices and r negative edges, $\gamma_r(P_n^{(r)}) = \gamma_r(P_n) = n - 2\lfloor(n-1)/3\rfloor$ for $n \geq 1$ and $0 < r \leq n-1$.*

Proof. In the case of restrained dominating set of P_n, it is proved in [9] that $\gamma_r(P_n) = n - 2\lfloor(n-1)/3\rfloor$. Since signed paths are trivially balanced, then by Proposition 3, the theorem follows. □

Theorem 2 *If $C_n^{(r)}$ is a signed cycle with $n \equiv 1, 2$ (mod 3) and $0 < r \leq n$, then $\gamma_r(C_n^{(r)}) = \gamma_r(C_n) = n - 2\lfloor n/3\rfloor$.*

Proof. Let Σ be a signed cycle $C_n^{(r)}$ with $n \equiv 1, 2 \pmod 3$ and $0 < r \leq n$. Restrained domination number as proved in [9] for C_n is $n - 2\lfloor n/3\rfloor$. We consider following two different cases to derive bounds for $\gamma_r(C_n)$:

Case 1 *$r \equiv 0(\text{mod } 2)$.*

Since, Σ has an even number of negative edges, therefore Σ is balanced. Thus by Proposition 3, $\gamma_r(C_n) = \gamma_r(\Sigma) = n - 2\lfloor n/3 \rfloor$.

Case 2 *$r \equiv 1$(mod 2).*

Let D be the minimum restrained dominating set of the underlying graph $|\Sigma|$. We need to check if D is a minimum restrained dominating set of Σ also. Let us suppose, D is restrained dominating set of Σ. Then, there exists at least one pair of adjacent vertices in D. Thus, $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ will always be acyclic, which is trivially balanced. Since, set D satisfies the property given in the definition, therefore it is a restrained dominating set of Σ. We know, $D^r_\Sigma \subseteq D^r_{|\Sigma|}$ by Proposition 2. We can thus conclude, D is minimum restrained dominating set of Σ and hence follows the theorem. $\square$

Theorem 3 *Let $C_n^{(r)}$ be a signed cycle with $n \equiv 0$ (mod 3) and $0 < r \leq n$, then*

$$\gamma_r(C_n^{(r)}) = \begin{cases} n - 2\lfloor n/3 \rfloor & \text{if } r \text{ is even} \\ n - 2\lfloor n/3 \rfloor + 2 & \text{if } r \text{ is odd} \end{cases}$$

Proof. Let Σ be a signed cycle $C_n^{(r)}$ with $n \equiv 0 \pmod 3$ and $0 < r \leq n$. Proceeding in a similar way as previous theorem, we form two cases for $\gamma_r(C_n)$:

Case 1 *$r \equiv 0$(mod 2)*

Since, Σ has an even number of negative edges, therefore Σ is balanced. Thus by Proposition 3, $\gamma_r(C_n) = \gamma_r(\Sigma) = n - 2\lfloor n/3 \rfloor$.

Case 2 *$r \equiv 1$(mod 2)*

Let D be the minimum restrained dominating set of the underlying graph $|\Sigma|$. We need to check, if D is restrained dominating set of Σ. In this case we observe that the set D has all vertices at a distance three from each other. Therefore, $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ will be a cycle with odd number of negative edges and hence not balanced. Thus by definition, D will not be a restrained dominating set of Σ. We now need to add more vertices to D. Suppose, a vertex $v_1 \in V \setminus D$ is added to the set D. The neighboring vertex of v_1 in $V \setminus D$, say v_2 then has no neighboring vertex in $V \setminus D$ and is not a restrained dominating set. Thus, we will need to add more vertices to D. Let us add $N(v_1) \in V \setminus D$ to the set D. Then, there exists only signed paths in $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$, which is trivially balanced . Since, we added two more vertices to the set D, therefore $\gamma_r(\Sigma) = \gamma_r(C_n) + 2 = n - 2\lfloor n/3 \rfloor + 2$. $\square$

Theorem 4 *If* $K^{(r)}_{1,n-1}$ *is a star signed graph with n vertices and r negative edges, then* $\gamma_r(K^{(r)}_{1,n-1}) = n$.

Proof. Let Σ be a star signed graph $K^{(r)}_{1,n-1}$ with r negative edges. Since, $\gamma_r(K_{1,n-1}) \leq \gamma_r(K^{(r)}_{1,n-1}) \leq n$ and $\gamma_r(K_{1,n-1}) = n$, the theorem holds true. $\square$

For complete signed graph K_n, $n \geq 5$, we can derive a general bounds as shown in Theorem 5 to obtain γ_r. But, $K^{(r)}_4$ doesnot satisfy this theorem. Hence, we state the following proposition.

Note that, paw graph is the graph obtained by joining a vertex of cycle graph C_3 to a singleton graph K_1 . In the following proposition, $P_2 \cup P_2$ is the union of two disconnected paths P_2.

Proposition 4 *Let* Σ *be a* $K^{(r)}_4$ *graph with r negative edges and r is even and let* $\langle I \rangle$ *be all-negative edge induced subgraph of* Σ*. Then,*

$$\gamma_r(\Sigma) = \begin{cases} 1, & \text{if } \langle I \rangle \cong C_4 \\ 2, & \text{if } \langle I \rangle \cong P_3 \text{ or } \langle I \rangle \text{ is a paw graph} \\ 4, & \text{if } \langle I \rangle \cong K^{(6)}_4 \text{ or } \langle I \rangle \cong P_2 \cup P_2. \end{cases}$$

Theorem 5 *If p is the order of the subgraph induced by negative edges of a complete signed graph* K_n *with n vertices, n*$\geq$ *5, then*

$$\gamma_r(K^{(r)}_n) = \begin{cases} p & \text{if } p < n-1 \\ n & \text{otherwise} \end{cases}$$

Proof. Let Σ be any complete signed graph having r negative edges and $n \geq 5$ and D be the minimum restrained dominating set of Σ. We need to show that all the vertices incident to any negative edge in Σ belongs to the set D. We prove this by contradiction. Suppose there exists at least one negative edge in Σ with end vertices say v_1 and v_2, such that either both or one end vertex is not in D. Then, the negative edge v_1v_2 will either be in $\Sigma[V \setminus D]$ or $\Sigma[D : V \setminus D]$. Now, there exists at least one C_3 in $\Sigma[V \setminus D]$ or $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ having odd number of negative edges, and thus Σ is not balanced. This implies, by Definition 1 that D is not a minimum restrained dominating set. D satisfies Definition 1 only when there does not exists any negative edge in $\Sigma[V \setminus D]$ or $\Sigma[D : V \setminus D]$, which contradicts our assumption. Therefore, $\gamma_r(K^{(r)}_n) = p$, for $p < n-1$. By Definition 1, $\gamma_r(K^{(r)}_n)$ can never be equal to $n-1$.

Hence, $\gamma_r(K_n^{(r)}) > n-1$ for $p > n-1$. But, we know $\gamma_r(K_n^{(r)}) \leq n$. Therefore, $\gamma_r(K_n^{(r)}) = n$. □

Restrained domination for complete bipartite signed graph Σ varies based on the number of negative edges in Σ and hence, for large graphs it is difficult to find exact restrained domination number. In the following theorems, we have generalized some of those cases for complete bipartite signed graphs $K_{m,m}^{(r)}$ with $2m$ vertices and concluded by giving the bounds on γ_r for any complete bipartite signed graph.

Theorem 6 *Let $K_{m,m}^{(r)}$ be a complete bipartite signed graph with $2m$ vertices and r negative edges and $\langle I \rangle$ denote the subgraph induced by all negative edges, then $\gamma_r(K_{m,m}^{(r)}) = 2$ in any one of the following conditions:*

1. *$\langle I \rangle \cong K_{1,m-1}$ or $\langle I \rangle \cong K_{1,m}$.*

2. *All the edges are negative, i.e. $\langle I \rangle \cong K_{m,m}^{(r)}$, where $r = m$*

Proof. Let Σ be a complete bipartite graph $K_{m,m}^{(r)}$ with $2m$ vertices and r negative edges and D be the restrained dominating set of Σ. We denote $\langle I \rangle$ for the subgraph induced by all negative edges of Σ.

Case 1 $\langle I \rangle \cong K_{1,m-1}$ *or* $\langle I \rangle \cong K_{1,m}$

Any induced cycle in a complete bipartite graph is always even. Also, for a signed graph to be balanced, every cycle in the graph must have an even number of negative edges. Moreover, degree of every vertex in cycle is always 2. Let u be the vertex to which all the negative edges are incident. All the induced cycles of Σ not including vertex u are all positive and hence satisfy the marking $\sigma(vw) = \mu(v)\mu(w)\ \forall\ v, w \neq u$. Thus, we need to check for the induced cycles in Σ containing the vertex u. In case of $\langle I \rangle \cong K_{1,m}$, Σ can be switched to all positive signed graph, and hence, by Proposition 3, $\gamma_r(K_{m,m}) = \gamma_r(K_{m,m}^{(r)}) = 2$ for $\langle I \rangle \cong K_{1,m}$. In case of $\langle I \rangle \cong K_{1,m-1}$, every cycle incuding vertex u will either have two negative edges incident to vertex u, which is always an even cycle or it will have 1 negative and 1 positive edge incident to u, which is an odd cycle. Hence, in case of odd cycle if we take the end vertices of the positive edge incident to vertex u in the set D, we get the desired result.

Case 2 $\langle I \rangle \cong K_{m,m}$.

This implies Σ is all negative and switching graph Σ, we obtain all positive $K_{m,m}$ graph. Thus, by Lemma 1, $\gamma_r(\Sigma) = 2$. □

Theorem 7 *Let Σ be a complete bipartite signed graph $K^{(r)}_{m,m}$ with $2m$ vertices and r negative independent edges with $r \leq m$. Then*

$$\gamma_r(\Sigma) = \begin{cases} 2r, & \text{if } r < m \\ 2(r-1), & \text{if } r = m. \end{cases}$$

Proof. Let Σ be $K^{(r)}_{m,m}$ with $2m$ vertices and r negative independent edges with $r \leq m$. Let D be the restrained dominating set of Σ. Since, all the negative edges in Σ are independent, therefore no two negative edges have at least one end point in common. In this case, there always exists at least one cycle in Σ containing odd number of negative edges and hence Σ is not balanced. Thus, to satisfy Definition 1 we will need to choose all the end points of the negative edges in the set D such that $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ is balanced.

Case 1 Suppose, there are r independent negative edges with $r < m$. Then, number of vertices in D will be twice the number of negative edges and hence $\gamma_r(\Sigma) = 2r$ for $r < m$.

Case 2 Now, suppose that the number of independent negative edges r is equal to m. Then, D must include all the vertices of Σ and hence, $\gamma_r(\Sigma)$ must be equal to $2r$. But, this is not the minimum restrained domination number and hence, we need to remove some vertices from the set D. Since, γ_r cannot be equal to $2m-1$, we will remove two vertices from set D. The set D is now minimum restrained dominating set. Thus, $\gamma_r(\Sigma) = 2(r-1)$.

□

Thus we can conclude with the following corollary:

Corollary 1 *Let Σ be any complete bipartite signed graph $K^{(r)}_{m,m}$ with $2m$ vertices and r negative edges, then $2 \leq \gamma_r(K^{(r)}_{m,m}) \leq 2(m-1)$.*

3 Conclusion

In this paper, we introduced the concept of restrained domination for signed graphs and determined the bounds on $\gamma_r(\Sigma)$ for certain classes of signed graphs. As for further work, it can be extended in finding bounds for other classes of derived signed graphs. Also, it would be interesting to study on the critical concepts of restrained domination in signed graphs. The above concepts is very much useful in fault tolerence analysis of communication networks, social networks and security systems.

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Graded Morita theory over a G-graded G-acted algebra

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Abstract. We develop a group graded Morita theory over a G-graded G-acted algebra, where G is a finite group.

1 Introduction

Let G be a finite group. In this article we develop a group graded Morita theory over a G-graded G-acted algebra, which is motivated by the problem to give a group graded Morita equivalences version of relations between character triples (see [11]) as in [9]. We will follow, in the development of graded Morita theory over a G-graded G-acted algebra, the treatment of Morita theory given by C. Faith in 1973 in [5]. Significant in this article is the already developed graded Morita theory. Graded Morita theory started in 1980 when R. Gordon and E. L. Green have characterized graded equivalences in the case of $G = \mathbb{Z}$, in [6]. Furthermore, in 1988 it was observed to work for arbitrary groups G by C. Menini and C. Năstăsescu, in [10]. We will make use of their results under the form given by A. del Río in 1991 in [4] and we will also use the graded Morita theory developed by P. Boisen in 1994 in [2].

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This paper is organized as follows: In Section §2, we introduce the general notations. In Section §3, we recall from [9] the definition of a G-graded G-acted algebra, we fix one to which we will further refer to by $\mathscr{C}$, we recall the definition of a G-graded algebra over said $\mathscr{C}$, and we recall the definition of a graded bimodule over $\mathscr{C}$. Moreover, we will give some new examples, useful for this article, for each notion. In Section §4, we construct the notion of a G-graded Morita context over $\mathscr{C}$ and we will give an appropriate example. In Section §5, we introduce the notions of graded functors over $\mathscr{C}$ and of graded Morita equivalences over $\mathscr{C}$ and finally we state and prove two Morita-type theorems using the said notions.

2 Notations and preliminaries

Throughout this article, we will consider a finite group G. We shall denote its identity by 1.

All rings in this paper are associative with identity $1 \neq 0$ and all modules are unital and finitely generated. We consider $\mathcal{O}$ to be a commutative ring.

Let A be a ring. We denote by A-Mod the category of all left A-modules. We shall usually write actions on the left, so in particular, by module we will usually mean a left module, unless otherwise stated. The notation ${}_A M$ (respectively, ${}_A M_{A'}$) will be used to emphasize that M is a left A-module (respectively, an (A, A')-bimodule).

Let $A = \bigoplus_{g\in G} A_g$ be a G-graded $\mathcal{O}$-algebra. We denote by A-Gr the category of all G-graded left A-modules. The forgetful functor from A-Gr to A-Mod will be denoted by U. For $M = \bigoplus_{g\in G} M_g \in A$-Gr and $g \in G$, the g-suspension of M is defined to be the G-graded A-module $M(g) = \oplus_{h\in G} M(g)_h$, where $M(g)_h = M_{gh}$. For any $g \in G$, $T_g^A : A\text{-Gr} \to A\text{-Gr}$ will denote (as in [4]) the g-suspension functor, i.e. $T_g^A(M) = M(g)$ for all $g \in G$. The stabilizer of M in G is, by definition [7, §2.2.1], the subgroup

$$G_M = \{g \in G \mid M \simeq M(g) \text{ as G-graded left } A\text{-modules}\}.$$

Let $M, N \in A$-Gr. We denote by $\mathrm{Hom}_A(M, N)$, the additive group of all A-linear homomorphisms from M to N. Because G is finite, E. C. Dade showed in [3, Corollary 3.10] that $\mathrm{Hom}_A(M, N)$ is G-graded. More precisely, if $g \in G$, the component of degree g (furthermore called the g-component) is defined as in [7, 1.2]:

$$\mathrm{Hom}_A(M, N)_g := \{f \in \mathrm{Hom}_A(M, N) \mid f(M_x) \subseteq N_{xg}, \text{ for all } x \in G\}.$$

We denote by id_X the identity map defined on a set X.

3 Graded bimodules over a G-graded G-acted algebra

We consider the notations given in Section §2. We recall the definition of a G-graded G-acted algebra and an example of a G-graded G-acted algebra as in [9]:

Definition 1 *An algebra $\mathscr{C}$ is a G-graded G-acted algebra if*

1. *$\mathscr{C}$ is G-graded, i.e. $\mathscr{C} = \oplus_{g\in G}\mathscr{C}_g$;*
2. *G acts on $\mathscr{C}$ (always on the left in this article);*
3. *$\forall h \in G$, $\forall c \in \mathscr{C}_h$ we have that ${}^g c \in \mathscr{C}_{{}^g h}$ for all $g \in G$.*

We denote the identity component (the 1-component) of $\mathscr{C}$ by $\mathscr{Z} := \mathscr{C}_1$, which is a G-acted algebra.

Let $A = \bigoplus_{g\in G} A_g$ be a strongly G-graded $\mathcal{O}$-algebra with identity component $B := A_1$. For the sake of simplicity, we assume that A is a crossed product (the generalization is not difficult, see for instance [7, §1.4.B.]). This means that we can choose invertible homogeneous elements u_g in the component A_g.

Example 1 *By Miyashita's theorem* [7, p.22]*, we know that the centralizer $C_A(B)$ is a G-graded G-acted $\mathcal{O}$-algebra: for all $h \in G$ we have that*

$$C_A(B)_h = \{a \in A_h \mid ab = ba, \forall b \in B\},$$

and the action is given by ${}^g c = u_g c u_g^{-1}$, $\forall g \in G$, $\forall c \in C_A(B)$. Note that this definition does not depend on the choice of the elements u_g and that $C_A(B)_1 = Z(B)$ (the center of B).

We recall the definition of a G-graded $\mathcal{O}$-algebra over a G-graded G-acted algebra $\mathscr{C}$ as in [9]:

Definition 2 *Let $\mathscr{C}$ be a G-graded G-acted $\mathcal{O}$-algebra. We say that A is a G-graded $\mathcal{O}$-algebra over $\mathscr{C}$ if there is a G-graded G-acted algebra homomorphism*

$$\zeta : \mathscr{C} \to C_A(B),$$

i.e. for any $h \in G$ and $c \in \mathscr{C}_h$, we have $\zeta(c) \in C_A(B)_h$, and for every $g \in G$, we have $\zeta({}^g c) = {}^g\zeta(c)$.

An important example of a G-graded $\mathcal{O}$-algebra over a G-graded G-acted algebra, is given by the following lemma:

Lemma 1 *Let P be a G-graded A-module. Assume that P is G-invariant. Let $A' = \mathrm{End}_A(P)^{op}$ be the set of all A-linear endomorphisms of P. Then A' is a G-graded $\mathcal{O}$-algebra over $C_A(B)$.*

Proof. By [3, Theorem 2.8], we have that there exists some $U \in B\text{−mod}$ such that P and $A \otimes_B U$ are isomorphic as G-graded left A-modules, henceforth for simplicity we will identify P as $A \otimes_B U$.

Because G is finite, E. C. Dade proved in [3, Corollary 3.10 and §4] that $A' = \mathrm{End}_A(P)^{\mathrm{op}}$ is a G-graded $\mathcal{O}$-algebra. Moreover, by [3, §4] we have that P is actually a G-graded (A, A')-bimodule.

Now, the assumption that P is G-invariant, according to [3, Corollary 5.14] and [7, §2.2.1], implies that $A' = \mathrm{End}_A(P)^{\mathrm{op}}$ is a crossed product and that P is isomorphic to its g-suspension, $P(g)$, for all $g \in G$. Henceforth, we can choose invertible homogeneous elements u'_g in the component A'_g, for all $g \in G$ such that

$$u'_g : P \to P(g).$$

By taking the truncation functor $(-)_1$ (more details are given in [3]) we obtain the isomorphism:

$$(u'_g)_1 : P_1 \to (P(g))_1,$$

where $P_1 = B \otimes_B U \simeq U$ and $(P(g))_1 = A_g \otimes_B U = u_g B \otimes_B U$. We fix arbitrary $a \in A$ and $u \in U$. We have:

$$u'_g(a \otimes_B u) = a u'_g(1_A \otimes_B u),$$

but $1_A \otimes_B u \in P_1$, henceforth:

$$u'_g(a \otimes_B u) = a(u'_g)_1(1_A \otimes_B u),$$

but there exists an unique $b \in B$ such that $(u'_g)_1(1_A \otimes_B u) = u_g b \otimes_B u = u_g \otimes_B bu$. Therefore, by defining $\varphi_g(u) := bu$, we obtain a map $\varphi_g : U \to U$, which is clearly well-defined. Moreover, we have:

$$u'_g(a \otimes_B u) = a u_g \otimes_B \varphi_g(u), \text{ for all } a \in A \text{ and } u \in U.$$

It is straightforward to prove that $\varphi_g : U \to U$ admits an inverse and that

$$u'^{-1}_g(a \otimes_B u) = a u_g^{-1} \otimes_B \varphi_g^{-1}(u), \text{ for all } a \in A \text{ and } u \in U.$$

We consider the G-graded algebra homomorphism from [8, Lemma 3.2.]:

$$\theta : C_A(B) \to A' = \mathrm{End}_A(P)^{\mathrm{op}}, \qquad \theta(c)(a \otimes u) = ac \otimes u,$$

where $c \in C_A(B)$, $a \in A$, and $u \in U$. First, we will prove that the image of θ is a subset of $C_{A'}(B')$. Indeed, consider $b' \in B'$ and $c \in C_A(B)$. We want:

$$\theta(c) \circ b' = b' \circ \theta(c).$$

Consider $a \otimes u \in A \otimes U = P$. We have:

$$(\theta(c) \circ b')(a \otimes u) = \theta(c)(b'(a \otimes u)) = a\theta(c)(b'(1_A \otimes u)),$$

because b' and $\theta(c)$ are A-linear. We fix $b'(1_A \otimes u) = a_0 \otimes u_0 \in A \otimes_B U$, but because $b' \in B' = A'_1 = \mathrm{End}_A(P)_1^{\mathrm{op}}$ we know that b' preserves the grading, so $1_A \in A_1$ implies that $a_o \in B$. Hence:

$$(\theta(c) \circ b')(a \otimes u) = a\theta(c)(a_0 \otimes u_0) = aa_0c \otimes u_0 = aca_0 \otimes u_0.$$

Following, we have that:

$$\begin{aligned} (b' \circ \theta(c))(a \otimes u) &= (b'(\theta(c)(a \otimes u)) = b'(ac \otimes u) \\ &= acb'(1_A \otimes u) = aca_0 \otimes u_0. \end{aligned}$$

Henceforth, the image of θ is a subset of $C_{A'}(B')$. Second, we prove that θ is G-acted, in the sense that:

$$\theta({}^g c) = {}^g(\theta(c)), \text{ for all } g \in G.$$

Indeed, we fix $g \in G$, and $a \otimes u \in A \otimes_B U = P$. We have:

$$\theta({}^g c)(a \otimes u) = \theta(u_g c u_g^{-1})(a \otimes u) = au_g c u_g^{-1} \otimes u$$

and

$$\begin{aligned} {}^g(\theta(c))(a \otimes u) &= (u'_g \cdot \theta(c) \cdot u_g'^{-1})(a \otimes u) = (u_g'^{-1} \circ \theta(c) \circ u'_g)(a \otimes u) \\ &= u_g'^{-1}(\theta(c)(u'_g(a \otimes u))) = u_g'^{-1}(\theta(c)(au_g \otimes \varphi_g(u))) \\ &= u_g'^{-1}(au_g c \otimes \varphi_g(u)) = au_g c u_g^{-1} \otimes \varphi_g^{-1}(\varphi_g(u)) \\ &= au_g c u_g^{-1} \otimes u. \end{aligned}$$

Finally, by taking $\zeta' : C_A(B) \to C_{A'}(B')$ to be the corestriction of θ to $C_{A'}(B')$, we obtain that A' is G-graded $\mathcal{O}$-algebra over $C_A(B)$, via the G-graded G-acted homomorphism ζ'. $\square$

Let $A' = \bigoplus_{g\in G} A'_g$ be another strongly G-graded $\mathcal{O}$-algebra with the identity component $B' := A'_1$. Again, we will consider that also A' is a crossed product, hence we will choose invertible homogeneous elements u'_g in the component A'_g, for all $g \in G$.

Now, we assume that A and A' are both strongly G-graded $\mathcal{O}$-algebras over a G-graded G-acted algebra $\mathscr{C}$, each endowed with G-graded G-acted algebra homomorphism $\zeta : \mathscr{C} \to C_A(B)$ and $\zeta' : \mathscr{C} \to C_{A'}(B')$ respectively.

We recall the definition of a G-graded bimodule over $\mathscr{C}$ as in [9]:

Definition 3 *We say that $\tilde{M}$ is a G-graded (A, A')-bimodule over $\mathscr{C}$ if:*

1. *$\tilde{M}$ is an (A, A')-bimodule;*
2. *$\tilde{M}$ has a decomposition $\tilde{M} = \bigoplus_{g\in G} \tilde{M}_g$ such that $A_g\tilde{M}_xA'_h \subseteq \tilde{M}_{gxh}$, for all $g, x, h \in G$;*
3. *$\tilde{m}_g \cdot c = {}^g c \cdot \tilde{m}_g$, for all $c \in \mathscr{C}, \tilde{m}_g \in \tilde{M}_g, g \in G$, where $c \cdot \tilde{m} = \zeta(c) \cdot \tilde{m}$ and $\tilde{m} \cdot c = \tilde{m} \cdot \zeta'(c)$, for all $c \in \mathscr{C}, \tilde{m} \in \tilde{M}$.*

Remark 1 *Condition 3. of Definition 3 can be rewritten* (see [9] for the proof) *as follows:*

3'. $m \cdot c = c \cdot m$, *for all* $c \in \mathscr{C}$, $m \in \tilde{M}_1$.

An example of a G-graded bimodule over a G-graded G-acted algebra is given by the following proposition:

Proposition 1 *Let $\mathscr{C}$ be a G-graded G-acted algebra and A a strongly G-graded $\mathcal{O}$-algebra over $\mathscr{C}$. Let P be a G-invariant G-graded A-module. Let $A' = \mathrm{End}_A(P)^{op}$. Then the following statements hold:*

1. *A' is a G-graded $\mathcal{O}$-algebra over $\mathscr{C}$;*
2. *P is a G-graded (A, A')-bimodule over $\mathscr{C}$.*

Proof. *1.* By Lemma 1, we know that A' is G-graded $\mathcal{O}$-algebra over $C_A(B)$ and let $\theta : C_A(B) \to C_{A'}(B')$ to be its G-graded G-acted structure homomorphism. Now, given that A is a strongly G-graded $\mathcal{O}$-algebra over $\mathscr{C}$, we have a G-graded G-acted algebra homomorphism $\zeta : \mathscr{C} \to C_A(B)$ and by taking $\zeta' : \mathscr{C} \to C_{A'}(B')$ to be the G-graded G-acted algebra homomorphism obtained by composing ζ with θ, we obtain that A' is also a G-graded $\mathcal{O}$-algebra over $\mathscr{C}$, with its structure given by ζ'. Hence, the first statement of this proposition was proved.

2. Without any loss in generality, we will identify P with $A \otimes_B U$, for some $U \in B-$mod. Following the proof of Lemma 1 we know that P is a G-graded (A, A')-bimodule. We now check that P is G-graded (A, A')-bimodule over $\mathscr{C}$. Indeed, we fix $g \in G$, $p_g = a_g \otimes u \in P_g$ and $c \in \mathscr{C}$. We have:

$$p_g \cdot c = (a_g \otimes u) \cdot c = (a_g \otimes u) \cdot \zeta'(c) = a_g \zeta(c) \otimes u,$$

but $a_g \in A_g$ so there exists a $b \in B$ such that $a_g = u_g b$, therefore:

$$\begin{aligned} p_g \cdot c &= u_g b \zeta(c) \otimes u = u_g \zeta(c) b \otimes u \\ &= u_g \zeta(c) u_g^{-1} u_g b \otimes u = {}^g\zeta(c) a_g \otimes u \\ &= \zeta({}^g c) a_g \otimes u = \zeta({}^g c)(a_g \otimes u) \\ &= \zeta({}^g c) \cdot p_g = {}^g c \cdot p_g. \end{aligned}$$

Henceforth, the last statement of this proposition has been proved. □

4 Graded Morita contexts over $\mathscr{C}$

We consider the notations given in Section §2. Let $\mathscr{C}$ be a G-graded G-acted algebra. We introduce the notion of a G-graded Morita context over $\mathscr{C}$, following the treatment given in [5, §12].

Definition 4 *Consider the following Morita context:*

$$(A, A', \tilde{M}, \tilde{M}', f, g).$$

We call it a G-graded Morita context over $\mathscr{C}$ if:

1. *A and A' are strongly G-graded $\mathcal{O}$-algebras over $\mathscr{C}$;*
2. *${}_A\tilde{M}_{A'}$ and ${}_{A'}\tilde{M}'_A$ are G-graded bimodules over $\mathscr{C}$;*
3. *$f : \tilde{M} \otimes_{A'} \tilde{M}' \to A$ and $g : \tilde{M}' \otimes_A \tilde{M} \to A'$ are G-graded bimodule homomorphisms such that by setting $f(\tilde{m} \otimes \tilde{m}') = \tilde{m}\tilde{m}'$ and $g(\tilde{m}' \otimes \tilde{m}) = \tilde{m}'\tilde{m}$, we have the associative laws:*

   $$(\tilde{m}\tilde{m}')\tilde{n} = \tilde{m}(\tilde{m}'\tilde{n}) \quad \text{and} \quad (\tilde{m}'\tilde{m})\tilde{n}' = \tilde{m}'(\tilde{m}\tilde{n}'),$$

   for all $\tilde{m}, \tilde{n} \in \tilde{M}$, $\tilde{m}', \tilde{n}' \in \tilde{M}'$.

If f and g are isomorphisms, then $(A, A', \tilde{M}, \tilde{M}', f, g)$ is called a surjective G-graded Morita context over $\mathscr{C}$.

As an example of a G-graded Morita context over $\mathscr{C}$, we have the following proposition which arises from [5, Proposition 12.6].

Proposition 2 *Let* A *be a strongly* G*-graded* $\mathcal{O}$*-algebra over* $\mathscr{C}$*, let* P *be a* G*-invariant* G*-graded* A*-module, let* $A' = \mathrm{End}_A(P)^{op}$ *and let* $P^* := \mathrm{Hom}_A(P, A)$ *be the* A*-dual of* P*. Then*

$$(A, A', P, P^*, (\cdot,\cdot), [\cdot,\cdot])$$

is a G*-graded Morita context over* $\mathscr{C}$*, where* $(\cdot,\cdot)$ *is a* G*-graded* (A, A)*-homomorphism, called the evaluation map, defined by:*

$$\begin{aligned}&(\cdot,\cdot) : P \otimes_{A'} P^* \to A,\\ &x \otimes \varphi \mapsto \varphi(x), \ \textit{for all } \varphi \in P^*, \ x \in P,\end{aligned}$$

and where $[\cdot,\cdot]$ *is a* G*-graded* (A', A')*-homomorphism defined by:*

$$\begin{aligned}&[\cdot,\cdot] : P^* \otimes_A P \to A',\\ &\varphi \otimes x \mapsto [\varphi, x], \ \textit{for all } \varphi \in P^*, \ x \in P,\end{aligned}$$

where for every $\varphi \in P^*$ *and* $x \in P$*,* $[\varphi, x]$ *is an element of* A' *such that*

$$y[\varphi, x] = \varphi(y) \cdot x, \ \textit{for all } y \in P.$$

Proof. For the sake of simplicity, we will assume that A is a crossed product as in Section §3. By Proposition 1, we have that A' is also a G-graded $\mathcal{O}$-algebra over $\mathscr{C}$ and that P is a G-graded (A, A')-bimodule over $\mathscr{C}$. Now, it is known that the A-dual of P, $P^* := \mathrm{Hom}_A(P, A)$ is a (A', A)-bimodule, where for each $\varphi \in P^*$ and for each $p \in P$, we have:

$$(a'\varphi a)(p) = (\varphi(pa'))a,$$

for all $a' \in A'$ and $a \in A$. By [7, §1.6.4.], we know that P^* is actually a G-graded (A', A)-bimodule, where for all $g \in G$, the g-component is defined as follows:

$$P^*_g = \{\varphi \in P^* \mid \varphi(P_x) \subseteq A_{xg}, \text{ for all } x \in G\}.$$

We prove that P^* is a G-graded (A', A)-bimodule over $\mathscr{C}$. Consider $g, h \in G$, $\varphi_g \in P^*_g$, $c \in C$ and $p_h \in P_h$. We have:

$$(\varphi_g c)(p_h) = (\varphi_g)(p_h)c.$$

Because $(\varphi_g)(p_h) \in A_{hg}$ we can choose a homogeneous element $u_{hg} \in A_{hg}$ and $b \in B$ such that $(\varphi_g)(p_h) = u_{hg}b$. Henceforth,

$$\begin{aligned}(\varphi_g c)(p_h) &= u_{hg}bc = u_{hg}cb = u_{hg}cu_{hg}^{-1}u_{hg}b = {}^{hg}c\,u_{hg}b\\ &= {}^{hg}c(\varphi_g)(p_h) = (\varphi_g)({}^{hg}c\,p_h)\\ &= (\varphi_g)(p_h\,{}^{g}c) = ({}^{g}c\,\varphi_g)(p_h),\end{aligned}$$

thus $\varphi_g c = {}^{g}c\,\varphi_g$, therefore P^* is a G-graded (A', A)-bimodule over $\mathscr{C}$. Next, following [5, §12], it is clear that $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are an (A, A)-homomorphism and an (A', A')-homomorphism, respectively. We now check if they are graded, as in the sense of [2, §3]: Indeed, consider $p_g \in P_g$ and $\varphi_h \in P^*_h$. We have:

$$(p_g, \varphi_h) = \varphi_h(p_g),$$

which is an element of A_{gh}, given the gradation of $P^* := \mathrm{Hom}_A(P, A)$. Also, for every $y_k \in P_k$, we have:

$$[\varphi_h, p_g](y_k) = \varphi_h(y_k)p_g,$$

which is an element of A_{khg}, given the gradation of P^* and of P, therefore $[\varphi_h, p_g]$ is an element of A_{hg}. Finally, we verify the associative law of the two homomorphism: Let $p, q \in P$ and $\varphi, \psi \in P^*$. We have:

$$(p, \varphi)q = \varphi(p)q \quad \text{and} \quad p[\varphi, q] = \varphi(p)q,$$

hence

$$(p, \varphi)q = p[\varphi, q];$$

and for all $y \in P$, we have:

$$([\varphi, p]\psi)(y) = \psi(y[\varphi, p]) = \psi(\varphi(y)p) = \varphi(y)\psi(p),$$

because $\varphi(y) \in A$ and ψ is A-linear, and also we have

$$(\varphi(p, \psi))(y) = (\varphi\psi(p))(y) = \varphi(y)\psi(p),$$

because $\psi(p) \in A$, hence

$$[\varphi, p]\psi = \varphi(p, \psi).$$

Therefore $(A, A', P, P^*, (\cdot,\cdot), [\cdot,\cdot])$ is a G-graded Morita context over $\mathscr{C}$. $\square$

If $(A, A', \tilde{M}, \tilde{M}', f, g)$ is a surjective G-graded Morita context over $\mathscr{C}$, then by Proposition 12.7 of [5], we have that A' is isomorphic to $\mathrm{End}_A(\tilde{M})^{\mathrm{op}}$ and we have a bimodule isomorphism between $\tilde{M}'$ and $\tilde{M}^* = \mathrm{Hom}_A(\tilde{M}, A)$. Henceforth, in this situation, the example given by Proposition 2 is essentially unique up to an isomorphism.

Given Corollary 12.8 of [5], the example given by Proposition 2 is a surjective G-graded Morita context over $\mathscr{C}$ if and only if ${}_AP$ is a progenerator.

5 Graded Morita theorems over $\mathscr{C}$

Consider the notations given in §2. Let $\mathscr{C}$ be a G-graded G-acted algebra. We denote by A and A' two strongly G-graded $\mathcal{O}$-algebras over $\mathscr{C}$ (with identity components $B := A_1$ and $B' := A'_1$), each endowed with G-graded G-acted algebra homomorphism $\zeta : \mathscr{C} \to C_A(B)$ and $\zeta' : \mathscr{C} \to C_{A'}(B')$ respectively. According to [4] we have the following definitions:

Definition 5 1. *We say that the functor $\tilde{\mathcal{F}} : A\text{-Gr} \to A'\text{-Gr}$ is G-graded if for every $g \in G$, $\tilde{\mathcal{F}}$ commutes with the g-suspension functor, i.e. $\tilde{\mathcal{F}} \circ T_g^A$ is naturally isomorphic to $T_g^{A'} \circ \tilde{\mathcal{F}}$;*

2. *We say that A and A' are G-graded Morita equivalent if there is a G-graded equivalence: $\tilde{\mathcal{F}} : A\text{-Gr} \to A'\text{-Gr}$.*

Assume that A and A' are G-graded Morita equivalent. Therefore, we can consider the G-graded functors:

$$\begin{tikzcd} A\text{-Gr} \arrow[r, shift left, "\tilde{\mathcal{F}}"] & A'\text{-Gr} \arrow[l, shift left, "\tilde{\mathcal{G}}"] \end{tikzcd}$$

which give a G-graded Morita equivalence between A and A'. By Gordon and Green's result [4, Corollary 10], this is equivalent to the existence of a Morita equivalence between A and A' given by the following functors:

$$\begin{tikzcd} A\text{-Mod} \arrow[r, shift left, "\mathcal{F}"] & A'\text{-Mod}; \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd}$$

such that the following diagram is commutative:

$$\begin{tikzcd} A\text{-Gr} \arrow[r, shift left, "\tilde{\mathcal{F}}"] \arrow[d, "U"'] & A'\text{-Gr} \arrow[l, shift left, "\tilde{\mathcal{G}}"] \arrow[d, "U'"] \\ A\text{-Mod} \arrow[r, shift left, "\mathcal{F}"] & A'\text{-Mod} \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd}$$

in the sense that:

$$U' \circ \tilde{F} = F \circ U, \qquad U \circ \tilde{G} = G \circ U',$$

where U' is the forgetful functor from A'-Gr to A'-Mod.

Lemma 2 *If* $\tilde{\mathsf{P}}$ *is a* G*-graded* A*-module, then* $\tilde{\mathsf{P}}$ *and* $\tilde{\mathcal{F}}(\tilde{\mathsf{P}})$ *have the same stabilizer in* G.

Proof. Let $g \in G_{\tilde{P}}$. We have $\tilde{P} \simeq \tilde{P}(g)$ as G-graded A-modules. Because $\tilde{\mathcal{F}}$ is a graded functor, we have that it commutes with the g-suspension functor. Thus $\tilde{\mathcal{F}}(\tilde{P}(g)) \simeq \tilde{\mathcal{F}}(\tilde{P})(g)$ in A'-Gr. Henceforth, $\tilde{\mathcal{F}}(\tilde{P}) \simeq \tilde{\mathcal{F}}(\tilde{P})(g)$ in A'-Gr, thus $g \in G_{\tilde{\mathcal{F}}(\tilde{P})}$. Hence $G_{\tilde{P}} \subseteq G_{\tilde{\mathcal{F}}(\tilde{P})}$. The converse, $G_{\tilde{\mathcal{F}}(\tilde{P})} \subseteq G_{\tilde{P}}$, is straightforward, thus $G_{\tilde{P}} = G_{\tilde{\mathcal{F}}(\tilde{P})}$. $\square$

Consider $\tilde{P}$ and $\tilde{Q}$ two G-graded A-modules. We have the following morphism:

$$\mathrm{Hom}_A(\tilde{P},\tilde{Q}) \xrightarrow{\quad\quad\tilde{\mathcal{F}}\quad\quad} \mathrm{Hom}_{A'}(\tilde{\mathcal{F}}(\tilde{P}),\tilde{\mathcal{F}}(\tilde{Q})). \qquad (*)$$

By following the proofs of Lemma 1 and Proposition 1, we have a G-graded homomorphism from $\mathscr{C}$ to $\mathrm{End}_A(\tilde{P})^{\mathrm{op}}$ (the composition between the structure homomorphism $\zeta : \mathscr{C} \to C_A(B)$ and the morphism $\theta : C_A(B) \to \mathrm{End}_A(\tilde{P})^{\mathrm{op}}$ from [8, Lemma 3.2.]) and that $\tilde{P}$ is a G-graded $(A, \mathrm{End}_A(\tilde{P})^{\mathrm{op}})$-bimodule. Then, by the restriction of scalars we obtain that $\tilde{P}$ is a right $\mathscr{C}$-module. Analogously $\tilde{Q}$, $\tilde{\mathcal{F}}(\tilde{P})$ and $\tilde{\mathcal{F}}(\tilde{Q})$ are also right $\mathscr{C}$-modules, thus $\mathrm{Hom}_A(\tilde{P},\tilde{Q})$ and $\mathrm{Hom}_A(\tilde{\mathcal{F}}(\tilde{P}),\tilde{\mathcal{F}}(\tilde{Q}))$ are G-graded $(\mathscr{C},\mathscr{C})$-bimodules. This allows us to state the following definition:

Definition 6
1. We say that the functor $\tilde{\mathcal{F}}$ is over $\mathscr{C}$ if the morphism $\tilde{\mathcal{F}}$ (see (∗)) is a morphism of G-graded $(\mathscr{C},\mathscr{C})$-bimodules;
2. We say that A and A' are G-graded Morita equivalent over $\mathscr{C}$ if there is a G-graded equivalence over $\mathscr{C}$: $\tilde{\mathcal{F}} : A\text{-Gr} \to A'\text{-Gr}$.

Theorem 1 (Graded Morita I over $\mathscr{C}$) *Let* $(A, A', \tilde{M}, \tilde{M}', f, g)$ *be a surjective* G*-graded Morita context over* $\mathscr{C}$*. Then the functors*

$$\tilde{M}' \otimes_A - : A\text{-Gr} \to A'\text{-Gr}$$
$$\tilde{M} \otimes_{A'} - : A'\text{-Gr} \to A\text{-Gr}$$

are inverse G*-graded equivalences over* $\mathscr{C}$.

Proof. Given [2, Theorem 3.2 (Graded Morita I) 6.] we already know that the pair of functors $\tilde{M}' \otimes_A -$ and $\tilde{M} \otimes_{A'} -$ are inverse G-graded equivalences. It remains to prove that they are also over $\mathscr{C}$. We will only prove that the functor $\tilde{M}' \otimes_A -$ is over $\mathscr{C}$ because the proof for the latter functor is similar. Consider $\tilde{P}$ and $\tilde{Q}$ two G-graded A-modules.

First, we will prove that the morphism

$$\tilde{M}' \otimes_A - : \mathrm{Hom}_A(\tilde{P}, \tilde{Q}) \to \mathrm{Hom}_{A'}(\tilde{M}' \otimes_A \tilde{P}, \tilde{M}' \otimes_A \tilde{Q}) \qquad (**)$$

is a $(\mathscr{C}, \mathscr{C})$-bimodule homomorphism. Indeed, consider $\varphi \in \mathrm{Hom}_A(\tilde{P}, \tilde{Q})$, then $\tilde{M}' \otimes_A \varphi \in \mathrm{Hom}_{A'}(\tilde{M}' \otimes_A \tilde{P}, \tilde{M}' \otimes_A \tilde{Q})$. Consider $c, c' \in \mathscr{C}$. We only need to prove that

$$\tilde{M}' \otimes_A (c\varphi c') = c(\tilde{M}' \otimes_A \varphi)c'.$$

Let $\tilde{m}' \in \tilde{M}'$ and $\tilde{p} \in \tilde{P}$. We have:

$$(\tilde{M}' \otimes_A (c\varphi c'))(\tilde{m}' \otimes \tilde{p}) = \tilde{m}' \otimes (c\varphi c')(\tilde{p}) = \tilde{m}' \otimes \varphi(\tilde{p}c)c'$$

and

$$\begin{aligned} (c(\tilde{M}' \otimes_A \varphi)c')(\tilde{m}' \otimes \tilde{p}) &= ((\tilde{M}' \otimes_A \varphi)((\tilde{m}' \otimes \tilde{p})c))c' \\ &= ((\tilde{M}' \otimes_A \varphi)(\tilde{m}' \otimes \tilde{p}c))c' \\ &= (\tilde{m}' \otimes \varphi(\tilde{p}c))c' \\ &= \tilde{m}' \otimes \varphi(\tilde{p}c)c'. \end{aligned}$$

Henceforth $\tilde{M}' \otimes_A (c\varphi c') = c(\tilde{M}' \otimes_A \varphi)c'$, thus the morphism $\tilde{M}' \otimes_A -$ (see (**)) is a $(\mathscr{C}, \mathscr{C})$-bimodule homomorphism.

Second, we will prove that the morphism $\tilde{M}' \otimes_A -$ is a G-graded $(\mathscr{C}, \mathscr{C})$-bimodule homomorphism, i.e. it is grade preserving. Consider $g \in G$ and $\varphi_g \in \mathrm{Hom}_A(\tilde{P}, \tilde{Q})_g$. We want $\tilde{M}' \otimes_A \varphi_g \in \mathrm{Hom}_{A'}(\tilde{M}' \otimes_A \tilde{P}, \tilde{M}' \otimes_A \tilde{Q})_g$, i.e. by [7, §1.2], if for some $h \in G$ and $\tilde{m}' \otimes \tilde{p} \in (\tilde{M}' \otimes_A \tilde{P})_h$, then we must have $(\tilde{M}' \otimes_A \varphi_g)(\tilde{m}' \otimes \tilde{p}) \in (\tilde{M}' \otimes_A \tilde{Q})_{hg}$. Beforehand, because $\tilde{m}' \otimes \tilde{p} \in (\tilde{M}' \otimes_A \tilde{P})_h$, by [7, §1.6.4], there exists some $x, y \in G$ with $h = xy$ such that $\tilde{m}' \in \tilde{M}'_x$ and $\tilde{p} \in \tilde{P}_y$. We have:

$$\begin{aligned} (\tilde{M}' \otimes_A \varphi_g)(\tilde{m}' \otimes \tilde{p}) &= \tilde{m}' \otimes \varphi_g(\tilde{p}) \in \tilde{M}'_x \otimes_A \varphi_g(\tilde{P}_y) \\ &\subseteq \tilde{M}'_x \otimes_A \tilde{Q}_{yg} \subseteq (\tilde{M}' \otimes_A \tilde{Q})_{xyg} \\ &= (\tilde{M}' \otimes_A \tilde{Q})_{hg}. \end{aligned}$$

Henceforth, the morphism $\tilde{M}' \otimes_A - : \mathrm{Hom}_A(\tilde{P}, \tilde{Q}) \to \mathrm{Hom}_{A'}(\tilde{M}' \otimes_A \tilde{P}, \tilde{M}' \otimes_A \tilde{Q})$ is a G-graded $(\mathscr{C}, \mathscr{C})$-bimodule homomorphism. $\square$

By Proposition 2 and the observations made in Section §4, the following corollary is straightforward.

Corollary 1 *Let P be a G-invariant G-graded A-module and $A' = \mathrm{End}_A(P)^{op}$. If ${}_AP$ is a progenerator, then $P \otimes_{A'} -$ is a G-graded Morita equivalence over $\mathscr{C}$ between A'-Gr and A-Gr, with $P^* \otimes_A -$ as its inverse.*

Theorem 2 (Graded Morita II over $\mathscr{C}$) *Assume that A and A' are G-graded Morita equivalent over $\mathscr{C}$ and let*

$$A\text{-Gr} \underset{\tilde{\mathcal{G}}}{\overset{\tilde{\mathcal{F}}}{\rightleftarrows}} A'\text{-Gr}$$

be inverse G-graded equivalences over $\mathscr{C}$. Then this equivalence is given by the following G-graded bimodules over $\mathscr{C}$: $P = \tilde{\mathcal{F}}(A)$ and $Q = \tilde{\mathcal{G}}(A')$. More exactly, P is a G-graded (A', A)-bimodule over $\mathscr{C}$, Q is a G-graded (A, A')-bimodule over $\mathscr{C}$ and the following natural equivalences of functors hold:

$$\tilde{\mathcal{F}} \simeq P \otimes_A - \quad \text{and} \quad \tilde{\mathcal{G}} \simeq Q \otimes_{A'} -.$$

Proof. By [4, Corollary 10 (Gordon-Green)], we know that $P = \tilde{\mathcal{F}}(A)$ is a G-graded (A', A)-bimodule, $Q = \tilde{\mathcal{G}}(A')$ is a G-graded (A, A')-bimodule and that that the following natural equivalences of functors hold: $\tilde{\mathcal{F}} \simeq P \otimes_A -$ and $\tilde{\mathcal{G}} \simeq Q \otimes_{A'} -$. Moreover, we have that ${}_A P$ is a progenerator.

It remains to prove that P and Q are G-graded bimodules over $\mathscr{C}$. We will only prove that P is G-graded bimodule over $\mathscr{C}$, because for Q the reasoning is similar.

By the hypothesis, A and A' are G-graded Morita equivalent over $\mathscr{C}$, hence $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ are over $\mathscr{C}$. Therefore the function:

$$\mathrm{Hom}_A({}_A A, {}_A A) \xrightarrow{\tilde{\mathcal{F}}} \mathrm{Hom}_{A'}({}_{A'} P, {}_{A'} P)$$

is an isomorphism of G-graded $(\mathscr{C}, \mathscr{C})$-bimodules, where if $f \in \mathrm{Hom}_A({}_A A, {}_A A)$, we have that $(c_1 f c_2)(a) = f(a c_1) c_2$, for all $a \in A$, $c_1, c_2 \in \mathscr{C}$. This means that the function

$$\alpha : A \to \mathrm{End}_{A'}(P)^{\mathrm{op}}, \quad \alpha(a) = \tilde{\mathcal{F}}(\rho(a)), \text{ for all } a \in A,$$

(where $\rho(c) : a \mapsto ac$, for all $a \in A$) is an isomorphism of G-graded $(\mathscr{C}, \mathscr{C})$-bimodules. Moreover, by the bimodule structure definition of P (see [1]), we have that $\alpha(a)(p) = pa$ for all $a \in A$ and for all $p \in P$.

It is clear that ${}_A A$ is G-invariant, hence by Lemma 2, P is also G-invariant. Henceforth, by Proposition 1, P is a G-graded $(A', \mathrm{End}_{A'}(P)^{\mathrm{op}})$-bimodule over $\mathscr{C}$. Consider the structural homomorphisms $\zeta : \mathscr{C} \to A$, $\zeta' : \mathscr{C} \to A'$ and $\zeta'' : \mathscr{C} \to \mathrm{End}_{A'}(P)^{\mathrm{op}}$, thus for all $a \in A$ and for all $c_1, c_2 \in \mathscr{C}$ we have:

$$\alpha(\zeta(c_1) a \zeta(c_2)) = \zeta''(c_1) \alpha(a) \zeta''(c_2).$$

By taking $a = 1_A$ and $c_2 = 1_{\mathscr{C}}$ we obtain $\alpha \circ \zeta = \zeta''$.

Let $g \in G$, $p_g \in P_g$ and $c \in \mathscr{C}$. We want $p_g \cdot c = {}^g c \cdot p_g$. We have:

$$p_g \cdot c = p_g \cdot \alpha(\zeta(c)) = p_g \cdot \zeta''(c) = {}^g\zeta'(c)p_g = {}^g c p_g.$$

Henceforth the statement is proved. □

6 Conclusion

We have developed a G-graded Morita theory over a G-graded G-acted algebra for the case of finite groups.

In Section §3, we recalled from [9] the notions of a G-graded G-acted algebra, of a G-graded algebra over a G-graded G-acted algebra and that of a G-graded bimodule over a G-graded G-acted algebra and we gave some useful examples for each notion.

In Section §4, we introduced the notion of a G-graded Morita context over a G-graded G-acted algebra and gave a standard example.

The main results are in Section §5, where a notion of graded functors over G-graded G-acted algebras and of graded Morita equivalences over G-graded G-acted algebras are introduced and two Morita-type theorems are proved using these notions: We proved that by taking a G-graded bimodule over a G-graded G-acted algebra we obtain a G-graded Morita equivalence over the said G-graded G-acted algebra and that by being given a G-graded Morita equivalence over a G-graded G-acted algebra, we obtain a G-graded bimodule over the said G-graded G-acted algebra, which induces the given G-graded Morita equivalence.

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Fixed points for a pair of weakly compatible mappings satisfying a new type of ϕ - implicit relation in $\mathcal{S}$ - metric spaces

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Abstract. The purpose of this paper is to introduce a new type of ϕ - implicit relation in $\mathcal{S}$ - metric spaces and to prove a general fixed point for a pair of weakly compatible mappings, which generalize Theorems 1, 2, 4 [23], Theorems 1-7 [13], Corollary 2.19 [13], Theorems 2.2, 2.4 [19], Theorems 3.2, 3.3, 3.4 [20] and other known results.

1 Introduction

Let X be a nonempty set and $f, g : X \to X$ two self mappings. A point $x \in X$ is said to be a coincidence point of f and g if $fx = gx = w$. The set of all coincidence points of f and g is denoted $\mathcal{C}(f, g)$ and w is said to be a point of coincidence of f and g.

In [8], Jungck defined f and g to be weakly compatible if $fgx = gfx$, for all $x \in \mathcal{C}(f, g)$.

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The notion of weakly compatible mappings is used to proof the existence of common fixed point for pairs of mappings.

A new class of generalized metric space, named D - metric space, is introduced in [5, 6]. In [11, 12], Mustafa and Sims proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced a new generalized metric spaces, named G - metric space. There exists a vast literature in the study of fixed points in G - metric spaces.

In [10], Mustafa initiated the study of fixed points for weakly compatible mappings in G - metric spaces.

Recently in [22], the authors introduced a new class of generalized metric space, named S - metric space. Quite recently in [7], the authors proved that the notions of G - metric spaces and S - metric space are independent.

Other results in the study of fixed points in S - metric space are obtained in [13, 19, 20, 21] and in other papers. Some results of fixed points for weakly compatible mappings in S - metric spaces are obtained in [23, 2].

In [14, 15], several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by implicit function.

The study of fixed point for mappings satisfying an implicit relation in G - metric spaces is initiated in [16, 17] and in other papers.

The notion of ϕ - maps is introduced in [9]. In [3], Altun and Turkoglu introduced a new class of implicit relation satisfying a ϕ - map.

A general fixed point theorem for mappings satisfying ϕ - implicit relations in G - metric spaces is obtained in [18].

The purpose of this paper is to introduce a new type of ϕ - implicit relation in S - metric spaces and to prove a general fixed point theorem for a pair of weakly compatible mappings in S - metric spaces, generalizing Theorems 1, 2, 4 [23], Theorems 1-7 [13], Corollary 2.19 [13], Theorems 2.2, 2.4 [19], Theorems 3.2, 3.3, 3.4 [20] and other known results.

2 Preliminaries

Definition 1 ([21, 22]) *A S - metric on a nonempty set X is a function $S : X^3 \to \mathbb{R}_+$ such that for all $x, y, z, a \in X$:*

$(S_1) : S(x, y, z) = 0$ *if and only if* $x = y = z$;

$(S_2) : S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called a S - metric space.

Example 1 *Let* $X = \mathbb{R}$ *and* $S(x, y, z) = |x - z| + |y - z|$. *Then,* $S(x, y, z)$ *is a* S *- metric on* $\mathbb{R}$ *and is named the usual* S *- metric on* X.

Lemma 1 **([4, 5])** *If* S *is a* S *- metric on a nonempty set* X*, then*

$$S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X.$$

Definition 2 **([22])** *Let* (X, S) *be a* S *- metric space. For* $r > 0$ *and* $x \in X$ *we define the open ball with center* x *and radius* r*, denoted* $B_S(x, r)$*, respectively closed ball, denoted* $\overline{B}_S(x, r)$*, the sets:*

$$B_S(x, r) = \{y \in X : S(x, x, y) < r\},$$

respectively,

$$\overline{B}_S(x, r) = \{y \in X : S(x, x, y) \leq r\}$$

The topology induced by S - metric on X is the topology determined by the base of all open balls in X.

Definition 3 **([22])** a) *A sequence* $\{x_n\}$ *in a* S *- metric space* (X, S) *is convergent to* x*, denoted* $x_n \to x$ *or* $\lim_{n\to\infty} x_n = x$*, if* $S(x_n, x_n, x) \to 0$ *as* $n \to \infty$*, that is, for* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *we have* $S(x_n, x_n, x) < \varepsilon$.

b) *A sequence* $\{x_n\}$ *in* (X, S) *is a Cauchy sequence if* $S(x_n, x_n, x_m) \to 0$ *as* $n, m \to \infty$*, that is, for* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $m, n \geq n_0$ *we have* $S(x_n, x_n, x_m) < \varepsilon$.

c) *A* S *- metric space* (X, S) *is complete if every Cauchy sequence is convergent.*

Example 2 (X, S) *by Example 1 is complete.*

Lemma 2 **([22])** *Let* (X, S) *be a* S *- metric space. If* $x_n \to x$ *and* $y_n \to y$*, then* $S(x_n, x_n, y_n) \to S(x, x, y)$.

Lemma 3 **([22])** *Let* (X, S) *be a* S *- metric space and* $x_n \to x$ *. Then* $\lim_{n\to\infty} x_n$ *is unique.*

Lemma 4 **([4])** *Let* (X, S) *be a* S *- metric space and* $\{x_n\}$ *be a sequence in* X *such that*

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ *is not a Cauchy sequence, then there exists an* $\varepsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers with* $n_k > m_k > k$ *such that*

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon,\ S\left(x_{m_{k-1}}, x_{m_{k-1}}, x_{n_k}\right) < \varepsilon$$

and

(i) $\lim_{n\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon$,

(ii) $\lim_{n\to\infty} S\left(x_{m_k}, x_{m_k}, x_{n_{k-1}}\right) = \varepsilon$,

(iii) $\lim_{n\to\infty} S\left(x_{m_{k-1}}, x_{m_{k-1}}, x_{n_k}\right) = \varepsilon$,

(iv) $\lim_{n\to\infty} S\left(x_{m_{k-1}}, x_{m_{k-1}}, x_{n_{k-1}}\right) = \varepsilon$.

Definition 4 ([9]) *Let* Φ *be the set of all functions such that* $\phi : [0, \infty) \to [0, \infty)$ *is a nondecreasing function satisfying* $\lim_{n\to\infty} \phi^n(t) = 0$ *for all* $t \in [0, \infty)$. *If* $\phi \in \Phi$, *then* ϕ *is called* ϕ *- mapping. Furthermore, if* $\phi \in \Phi$, *then:*

(i) $\phi(t) < t$ *for all* $t \in (0, \infty)$,

(ii) $\phi(0) = 0$.

The following theorems are recently published in [23].

Theorem 1 (Theorem 1 [23]) *Let* (X, S) *be a* S *- metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, gz) \leq \phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}) \tag{1}$$

for all $x, y, z \in X$.

If $f(X) \subset g(X)$ *and one of* $f(X)$ *or* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique point of coincidence.*

Moreover, if f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Theorem 2 (Theorem 2 [23]) *Let* (X, S) *be a* S *- metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq \max\{\phi(S(gx, gx, fx)), \phi(S(gy, gy, fy)), \phi(S(gz, gz, fz))\} \tag{2}$$

for all $x, y, z \in X$.

If $f(X) \subset g(X)$ *and one of* $f(X)$ *or* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Theorem 3 (Theorem 4 [23]) *Let* (X, S) *be a* S *- metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq k_1\phi(S(gx, gx, fx)) + k_2\phi(S(gy, gy, fy)) + k_3\phi(S(gz, gz, fz)) \tag{3}$$

for all $x, y, z \in X$, $k_1 + k_2 + k_3 < 1$.

If $f(X) \subset g(X)$ *and one of* $f(X)$ *or* $g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Remark 1 1) *Since* $\phi(t)$ *is nondecreasing, then*

$$\phi(\max\{t_2, t_3, t_4, t_5, t_6\}) = \max\{\phi(t_2), \phi(t_3), \phi(t_4), \phi(t_5), \phi(t_6)\}.$$

Hence, Theorem 2 is Theorem 1.

2) *By (3) we obtain*

$$\begin{aligned} S(fx, fy, fz) &\leq (k_1 + k_2 + k_3)\max\{\phi(S(gx, gx, fx)), \\ &\qquad \phi(S(gy, gy, fy)), \phi(S(gz, gz, fz))\} \\ &= (k_1 + k_2 + k_3)\phi(\max\{S(gx, gx, fx), S(gy, gy, fy), \\ &\qquad S(gz, gz, fz)\}) \\ &\leq \phi(\max\{\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}\}). \end{aligned}$$

Hence,

$$S(fx, fy, fz) \leq \phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}),$$

which is the inequality (1). Hence, Theorem 3 is a particular case of Theorem 1.

3) *In the proof of Theorem 1 is used* $x = y$*. Hence in Theorem 1 we have a new form of inequality (1):*

$$S(fx, fx, fy) \leq \phi(\max\{S(gx, gx, fx), g(fy, gy, fy)\}).$$

3 ϕ - implicit relations

Let $\mathcal{F}_\phi$ be the set of all lower semi - continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

(F_1) : F is nonincreasing in variable t_6,

(F_2) : There exists $\phi \in \mathcal{F}_\phi$ such that for all $u, v \geq 0$, $F(u, v, v, u, 0, 2u + v) \leq 0$ implies $u \leq \phi(v)$;

(F_3) : $F(t, t, 0, 0, t, t) > 0, \ \forall t > 0$.

In all the following examples, (F_1) is obviously.

Example 3 $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3, ..., t_6\}$, *where* $k \in \left[0, \frac{1}{3}\right)$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - k(u + 2v) \leq 0$. If $u > v$, then $u(1 - 3k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq 3kv$ and F satisfies (F_2) for $\phi(t) = 3kt$.

(F_3) : $F(t, t, 0, 0, t, t) = t(1 - k) > 0, \ \forall t > 0$.

Example 4 $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{3}\right\}$, *where* $k \in [0, 1)$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - k\max\left\{u, v, \frac{2u + v}{3}\right\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq kv$ and F satisfies (F_2) for $\phi(t) = kt$.

(F_3) : $F(t, t, 0, 0, t, t) = t(1 - k) > 0, \ \forall t > 0$.

Example 5 $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a, b, c, d, e \geq 0$ *and* $a + b + c + 3e < 1$ *and* $a + d + e < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - av - bv - cu - e(2u + v) \leq 0$. If $u > v$, then $u[1 - (a + b + c + 3e)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq (a + b + c + 3e)v$ and F satisfies (F_2) for $\phi(t) = (a + b + c + 3e)t$.

(F_3) : $F(t, t, 0, 0, t, t) = t[1 - (a + d + e)] > 0, \ \forall t > 0$.

Example 6 $F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, *where* $a, b, c, d \geq 0$, $a + b + c < 1$ *and* $a + d < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u^2 - u(av + bv + cu) \leq 0$. If $u > v$, then $u^2[1 - (a + b + c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq (a + b + c)v$ and F satisfies (F_2) for $\phi(t) = (a + b + c)t$.

(F_3) : $F(t, t, 0, 0, t, t) = t^2[1 - (a + d)] > 0, \ \forall t > 0$.

Example 7 $F(t_1, ..., t_6) = t_1^2 - at_2^2 - \frac{bt_5t_6}{1 + t_3^2 + t_4^2}$, *where* $a, b \geq 0$ *and* $a + b < 1$.

(F_2): Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u+v) = u^2 - av^2 \leq 0$, which implies $u \leq \sqrt{a}v$. Hence, F satisfies (F_2) for $\phi(t) = \sqrt{a}t$.

(F_3): $F(t, t, 0, 0, t, t) = t^2[1-(a+b)] > 0, \ \forall t > 0$.

In the following examples, if $\phi \in \Phi$, then F satisfy properties (F_1), (F_2), (F_3).

Example 8 $F(t_1, ..., t_6) = t_1 - \phi\left(\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{3}\right\}\right)$.

(F_2): Let $u, v \geq 0$ and

$$F(u, v, v, u, 0, 2u+v) = u - \phi\left(\max\left\{u, v, \frac{2u+v}{3}\right\}\right) \leq 0.$$

If $u > v$, then $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi(t) > 0, \ \forall t > 0$.

Example 9 $F(t_1, ..., t_6) = t_1 - \phi\left(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{3}\right\}\right)$.

(F_2): Let $u, v \geq 0$ and

$$F(u, v, v, u, 0, 2u+v) = u - \phi\left(\max\left\{u, \frac{u+v}{2}, \frac{2u+v}{3}\right\}\right) \leq 0.$$

If $u > v$, then $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi(t) > 0, \ \forall t > 0$.

Example 10 $F(t_1, ..., t_6) = t_1 - \phi(at_2 + b\max\{t_3, t_4\} + c\max\{t_5, t_6\})$, *where* $a, b, c \geq 0$ *and* $a + b + 3c < 1$.

(F_2): Let $u, v \geq 0$ and

$$F(u, v, v, u, 0, 2u+v) = u - \phi(av + b\max\{u, v\} + c(2u+v)) \leq 0.$$

If $u > v$, then $u - \phi((a+b+3c)u) \leq 0$, which implies $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$ and $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi(at + ct) \geq t - \phi((a+b+3c)t) \geq t - \phi(t) > 0, \ \forall t > 0$.

Example 11 $F(t_1, ..., t_6) = t_1 - \phi\left(a\sqrt{t_1 t_2} + b\sqrt{t_3 t_4} + c\sqrt{t_5 t_6}\right)$, *where* $a, b, c \geq 0$ *and* $a + b + c < 1$.

(F_2): Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u+v) = u - \phi\left(a\sqrt{uv} + b\sqrt{uv}\right) \leq 0$. If $u > v$, then $u \leq \phi((a+b)u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi((a+c)t) \geq t - \phi((a+b+c)t) \geq t - \phi(t) > 0, \ \forall t > 0$.

Example 12 $F(t_1, ..., t_6) = t_1 - \phi\left(at_2, \dfrac{b\sqrt{t_5 t_6}}{1+t_3+t_4}\right)$, *where* $a, b \geq 0$ *and* $a + b < 1$.

(F_2): Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u+v) = u - \phi(av) \leq 0$. If $u > v$, then $u - \phi(av) \leq 0$ implies $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi((a+b)t) \geq t - \phi(t) > 0, \ \forall t > 0$.

In the following examples, the proofs are similar to the proof of Example 12 and thus are omitted.

Example 13 $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4, t_5, t_6\}$, *where* $a, b \geq 0$ *and* $a + 3b < 1$.

If $F(u, v, v, u, 0, 2u+v) \leq 0$, then we have $u \leq \phi(v)$, where $\phi(t) = (a+3b)t$.

Example 14 $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - d\max\{t_5, t_6\}$, *where* $a, b, c, d \geq 0$ *and* $a + b + c + 3d < 1$.

If $F(u, v, v, u, 0, 2u+v) \leq 0$ then we have $u \leq \phi(v)$, where $\phi(t) = (a + b + c + 3d)t$.

Example 15 $F(t_1, ..., t_6) = t_1 - at_2 - d\max\{t_3, t_4\} - bt_5 - ct_6$, *where* $a, b, c, d \geq 0$, $a + 3c + d \geq 0$, $a + 3c + d < 1$ *and* $a + b + c < 1$.

If $F(u, v, v, u, 0, 2u+v) \leq 0$ then $u \leq \phi(v)$, where $\phi(t) = (a + 3c + d)t$.

Example 16 $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - et_4 - ct_5 - dt_6 - f\max\{t_2, t_3, ..., t_6\}$, *where* $a, b, c, d, e, f \geq 0$, $a + b + e + 3d + 3f < 1$ *and* $a + c + e + f < 1$.

If $F(u, v, v, u, 0, 2u+v) \leq 0$ then $u \leq \phi(v)$, where $\phi(t) = (a + b + e + 3d + 3f)t$.

Example 17 $F(t_1, ..., t_6) = t_1 - a(t_5 + t_6) - bt_2 - c\max\{t_3, t_4\}$, *where* $a, b, c \geq 0$ *and* $3a + b + c < 1$.

If $F(u, v, v, u, 0, 2u + v) \leq 0$ then $u \leq \phi(v)$, where $\phi(t) = (3a + b + c)\,t$.

Example 18 $F(t_1, ..., t_6) = t_1 - a(t_3 + t_4) - bt_2 - c\max\{t_5, t_6\}$, *where* $a, b, c \geq 0$ *and* $2a + b + 3c < 1$.

If $F(u, v, v, u, 0, 2u + v) \leq 0$ then $u \leq \phi(v)$, where $\phi(t) = (2a + b + 3c)\,t$.

Example 19 $F(t_1, ..., t_6) = t_1 - a\max\{t_4 + t_5, t_3 + t_6\} - bt_2$, *where* $a, b, c \geq 0$ *and* $4a + b < 1$.

If $F(u, v, v, u, 0, 2u + v) \leq 0$ then $u \leq \phi(v)$, where $\phi(t) = (4a + b)\,t$.

4 Main results

Lemma 5 ([1]) *Let* f *and* g *be weakly compatible self mappings of a nonempty set* X. *If* f *and* g *have a unique point of coincidence* $w = fx = gx$ *for some* $x \in X$, *then* w *is the unique common fixed point of* f *and* g.

Theorem 4 *Let* (X, S) *be a* S *- metric space and* $f, g : X \to X$ *such that*

$$F\begin{pmatrix} S(fx, fx, fy), S(gx, gx, gy), S(gx, gx, fx), \\ S(gy, gy, fy), S(gy, gy, fx), S(gx, gx, fy) \end{pmatrix} \leq 0 \tag{4}$$

for all $x, y \in X$ *and some* $F \in \mathcal{F}_\phi$.

If $f(X) \subset g(X)$ *(or* $g(X) \subset f(X)$*) and* $g(X)$ *(or* $f(X)$*) is a complete subspace of* (X, S), *then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process we define the sequence $\{x_n\}$ satisfying

$$fx_n = gx_{n+1} \text{ for } n \in \mathbb{N}.$$

Then, by (4) for $x = x_{n-1}$ and $y = x_n$ we have

$$F\begin{pmatrix} S(fx_{n-1}, fx_{n-1}, fx_n), S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), \\ S(gx_n, gx_n, fx_n), S(gx_n, gx_n, fx_{n-1}), S(gx_{n-1}, gx_{n-1}, fx_n) \end{pmatrix} \leq 0$$

$$F\begin{pmatrix} S(gx_n, gx_n, gx_{n+1}), S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, gx_n), \\ S(gx_n, gx_n, gx_{n+1}), 0, S(gx_{n-1}, gx_{n-1}, gx_{n+1}) \end{pmatrix} \leq 0 \tag{5}$$

By (S_2) and Lemma 1 we have

$$S\left(gx_{n-1},gx_{n-1},gx_{n+1}\right)\leq 2S\left(gx_n,gx_n,gx_{n+1}\right)+S\left(gx_{n-1},gx_{n-1},gx_n\right).$$

By (5) and (F_1) we obtain

$$F\left(\begin{array}{c} S\left(gx_n,gx_n,gx_{n+1}\right),S\left(gx_{n-1},gx_{n-1},gx_n\right),S\left(gx_{n-1},gx_{n-1},gx_n\right),\\ S\left(gx_n,gx_n,gx_{n+1}\right),0,2S\left(gx_n,gx_n,gx_{n+1}\right)+S\left(gx_{n-1},gx_{n-1},gx_n\right)\end{array}\right)\leq 0.$$

By (F_2) we obtain

$$S\left(gx_n,gx_n,gx_{n+1}\right)\leq\phi\left(S\left(gx_{n-1},gx_{n-1},gx_n\right)\right),\ \text{for } n=1,2,...$$

which implies

$$S\left(gx_n,gx_n,gx_{n+1}\right)\leq\phi^n\left(S\left(gx_0,gx_0,gx_1\right)\right).$$

Letting n tend to infinity we obtain

$$\lim_{n\to\infty}S\left(gx_n,gx_n,gx_{n+1}\right)=0.$$

We prove that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then, by Lemma 4, there exists an $\varepsilon>0$ and two sequences m_k and n_k with $n_k>m_k>k$ and $S\left(x_{m_k},x_{m_k},x_{n_k}\right)\geq\varepsilon$ and $S\left(x_{m_k-1},x_{m_k-1},x_{n_k}\right)<\varepsilon$ and satisfying the inequalities (i) - (iv) by Lemma 4.

By (4) for $x=x_{m_k-1}$ and $y=x_{n_k-1}$ we have

$$F\left(\begin{array}{c} S\left(fx_{m_k-1},fx_{m_k-1},fx_{n_k-1}\right),S\left(gx_{m_k-1},gx_{m_k-1},gx_{n_k-1}\right),\\ S\left(gx_{m_k-1},gx_{m_k-1},fx_{m_k-1}\right),S\left(gx_{n_k-1},gx_{n_k-1},fx_{n_k-1}\right),\\ S\left(gx_{n_k-1},gx_{n_k-1},fx_{m_k-1}\right),S\left(gx_{m_k-1},gx_{m_k-1},fx_{n_k-1}\right)\end{array}\right)\leq 0$$

$$F\left(\begin{array}{c} S\left(gx_{m_k},gx_{m_k},gx_{n_k}\right),S\left(gx_{m_k-1},gx_{m_k-1},gx_{n_k-1}\right),\\ S\left(gx_{m_k-1},gx_{m_k-1},gx_{m_k}\right),S\left(gx_{n_k-1},gx_{n_k-1},gx_{n_k}\right),\\ S\left(gx_{n_k-1},gx_{n_k-1},gx_{m_k}\right),S\left(gx_{m_k-1},gx_{m_k-1},gx_{n_k}\right)\end{array}\right)\leq 0. \tag{6}$$

By Lemma 1,

$$S\left(gx_{m_k-1},gx_{m_k-1},gx_{n_k}\right)=S\left(gx_{n_k},gx_{n_k},gx_{m_k-1}\right)$$

and

$$S\left(gx_{n_k-1},gx_{n_k-1},gx_{m_k}\right)=S\left(gx_{m_k},gx_{m_k},gx_{n_k-1}\right).$$

Letting n tend to infinity in (6) we obtain

$$F(\varepsilon, \varepsilon, 0, 0, \varepsilon, \varepsilon) \leq 0,$$

a contradiction of (F_3).

Hence, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, then $\{gx_n\}$ is convergent to a point $t \in g(X)$. Hence, there exists $p \in X$ such that $gp = t$ and $\lim_{n\to\infty} gx_n = gp$. We prove that $fp = gp$.

By (4) for $x = x_n$ and $y = p$ we have

$$F\left(\begin{array}{c} S(gx_n, gx_n, fp), S(gx_n, gx_n, gp), S(gx_n, gx_n, fx_n), \\ S(gp, gp, fp), S(gp, gp, fx_n), S(gx_n, gx_n, fp) \end{array}\right) \leq 0.$$

Letting n tend to infinity we obtain

$$F(S(gp, gp, fp), 0, 0, S(gp, gp, fp), 0, S(gp, gp, fp)) \leq 0.$$

By (F_1) we have

$$F(S(gp, gp, fp), 0, 0, S(gp, gp, fp), 0, 2S(gp, gp, fp)) \leq 0,$$

which implies $S(gp, gp, fp) = 0$. Hence $gp = fp = t$.

We prove that t is the unique point of coincidence of f and g. Suppose that there exists $z = fw = gw$. By (4) we obtain

$$F\left(\begin{array}{c} S(fp, fp, fw), S(gp, gp, gw), S(gp, gp, fp), \\ S(gw, gw, fw), S(gw, gw, fp), S(gp, gp, fw) \end{array}\right) \leq 0,$$

$$F(S(t, t, z), S(t, t, z), 0, 0, S(z, z, t), S(t, t, z)) \leq 0.$$

By Lemma 1 we have

$$F(S(t, t, z), S(t, t, z), 0, 0, S(t, t, z), S(t, t, z)) \leq 0,$$

a contradiction of (F_3) if $S(t, t, z) > 0$. Hence, $z = t$ and t is the unique point of coincidence of f and g.

Moreover, if f and g are weakly compatible, then by Lemma 5, f and g have a unique common fixed point t. □

If $\phi(t) = kt$, $k \in [0, 1)$, by Example 8 and Theorem 4 we obtain

Corollary 1 *Let* (X, S) *be a* S *- metric space and* $f, g : X \to X$ *such that*

$$S(fx, fx, fy) \leq k \max \left\{ \begin{array}{c} S(gx, gx, gy), S(gx, gx, fx), S(gy, gy, fy), \\ \dfrac{S(gy, gy, fx) + S(gx, gx, fy)}{3} \end{array} \right\} \quad (7)$$

where $k \in [0,1)$.

If $f(X) \subset g(X)$ *(or* $g(X) \subset f(X)$*) and* $g(X)$ *(or* $f(X)$*) is a complete subspace of* (X,S)*, then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Example 20 *Let* $X = \mathbb{R}$ *and* $S(x,y,z) = |x-z| + |y-z|$*. Then* $S(X)$ *is a complete* S *- metric space. Let* $fx = 2x-2$*,* $gx = 3x-4$*. Then* $f(X) = \mathbb{R}$*,* $g(X) = \mathbb{R}$ *and* $f(X) \subset g(X)$*. If* $fx = gx$*, then* $x = 2$ *which implies* $\mathcal{C}(f,g) = \{2\}$ *and* $fg2 = gf2 = 2$ *and* $x = 2$ *is the unique point of coincidence of* f *and* g *and* f *and* g *are weakly compatible. On the other hand,* $S(fx,fx,fy) = 4|x-z|$ *and* $S(gx,gx,gy) = 6|x-y|$*. Hence,* $S(fx,fx,fy) \leq kS(gx,gx,gy)$*, for* $k \in \left[\frac{2}{3},1\right)$*. This implies*

$$S(fx,fx,fy) \leq k\max\left\{\begin{array}{c} S(gx,gx,gy), S(gx,gx,fx), S(gy,gy,fy), \\ \dfrac{S(gy,gy,fx)+S(gx,gx,fy)}{3} \end{array}\right\}$$

for $k \in \left[\frac{2}{3},1\right)$*. By Corollary 1,* f *and* g *have a unique common fixed point* $x = 2$*.*

If $g(x) = x$, then by Theorem 4 we obtain

Theorem 5 *Let* (X,S) *be a complete* S *- metric space and* $f : X \to X$ *such that*

$$F(S(fx,fx,fy), S(x,x,y), S(x,x,fx), S(y,y,fy), S(y,y,fx), S(x,x,fy)) \leq 0,$$

for all $x,y \in X$ *and some* $F \in \mathcal{F}_\phi$*.*

Then f *has a unique fixed point.*

Corollary 2 *Let* (X,S) *be a complete* S *- metric space and* $f : X \to X$ *such that*

$$S(fx,fx,fy) \leq k\max\{S(x,x,y), S(x,x,fx), S(y,y,fy), S(x,x,fy), S(x,x,fy)\},$$

for all $x,y \in X$ *and* $k \in \left[0,\frac{1}{3}\right]$*. Then* f *has a unique fixed point.*

Proof. The proof follows by Theorem 5 and Example 4. □

Remark 2 1) *By Examples 13 - 19 and Theorem 4 we obtain Theorems 1-7* [13].
2) *By Example 4 and Theorem 4 we obtain Corollary 2.19* [13].
3) *By Example 5 and Theorem 4 we obtain Theorems 2.2, 2.4* [19] *and Theorems 3.2, 3.3, 3.4* [20].

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On Euler products with smaller than one exponents

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Abstract. Investigation has been made regarding the properties of the $\prod_{p\leq n}(1\pm 1/p^s)$ products over the prime numbers, where we fix the $s\in\mathbb{R}$ exponent, and let the $n\geq 2$ natural bound grow toward positive infinity. The nature of these products for the $s\geq 1$ case is known. We get approximations for the case when $s\in[1/2,1)$, furthermore different observations for the case when $s<1/2$.

1 Introduction

In this article, we will investigate the asymptotical properties of Euler products. More precisely, we are going to look at how does the

$$\prod_{p\leq n}\left(1\pm\frac{1}{p^s}\right) \tag{1}$$

products over the prime numbers behave asymptotically, when we fix the $s\in\mathbb{R}$ exponent, and let the $n\geq 2$ natural bound grow toward positive infinity.

Due to the connection with the Riemann zeta function and Dirichlet series, the "classical" Euler products were and are the subject of a thorough investigation. Some of the results concerning them can describe the nature of the products which we will examine, so we are going to shortly sum up the properties which can be already stated based on these results.

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First we concentrate on the negative case of expression (1). It is well known that

$$\lim_{n\to\infty}\prod_{p\leq n}\left(1-\frac{1}{p^s}\right)=\frac{1}{\zeta(s)}$$

when $s>1$, see [2] section 11.5. (This result dates back to Euler, who pointed out this connection in [9] by using simple algebraic manipulations on - what is now known as - the Dirichlet series form of the Riemann zeta function.) As when $s=1$, Mertens proved his infamous result in [12], stating that

$$\lim_{n\to\infty}\ln n\prod_{p\leq n}\left(1-\frac{1}{p}\right)=e^{-\gamma}$$

holds, which is usually referred to as Mertens' third theorem. (To obtain this result, one has to know how the $\zeta(s)$ Riemann zeta function behaves in the neighbourhood of its pole at $s=1$.) When $s=0$, then the product is zero, and when s is negative, then the product diverges.

Concerning the positive case of expression (1), one can use the

$$\left(1+\frac{1}{p^s}\right)\left(1-\frac{1}{p^s}\right)=1-\frac{1}{p^{2s}} \tag{2}$$

equation when $s\in\mathbb{R}$, to transform the results from the negative case. Because of this, we get that

$$\lim_{n\to\infty}\prod_{p\leq n}\left(1+\frac{1}{p^s}\right)=\frac{\zeta(s)}{\zeta(2s)}$$

when $s>1$, and also that

$$\lim_{n\to\infty}\frac{1}{\ln n}\prod_{p\leq n}\left(1+\frac{1}{p}\right)=\frac{6}{\pi^2}e^{\gamma}$$

holds when $s=1$. (These results are not new, see again section 11.5 in [2].) When s is zero, the product grows as $2^{\pi(n)}$, where π is the prime counting function.

What remains in both cases is when $s\in(0,1)$, and $s<0$ in the positive case of expression (1). We are going to concentrate on the positive case, because equation (2) can be applied to transform our results back to the negative case. We will rely on the following theorem, see theorem 2.7.1 from [3].

Theorem 1 *Let* f *be continuously differentiable on an open interval containing* $[2, \infty)$*, and let* $\pi(x) = \mathrm{Li}(x) + \epsilon(x)$*, where* $\mathrm{Li}(x) = \int_2^x dt/\ln(t)$ *is the offset logarithmic integral. Now if* $x \geq 2$*, then*

$$\sum_{p \leq x} f(p) = \int_2^x \frac{f(t)}{\ln(t)}\, dt + \epsilon(x) f(x) - \int_2^x \epsilon(t) f'(t)\, dt.$$

The precision which we can achieve while applying this theorem depends heavily on the applied ϵ error term. Most of the results in this area give absolute bound to the $\pi(x) - \mathrm{li}(x) = \varepsilon(x)$ error term, where

$$\mathrm{li}(x) = \lim_{h \to 0^+} \left(\int_0^{1-h} \frac{dt}{\ln t} + \int_{1+h}^x \frac{dt}{\ln t} \right)$$

is the logarithmic integral. Take note that the applied ϵ and the later ε differs; we are going to use the later ε throughout the paper. Because of this, we have to substitute the ϵ error term in theorem 1 as

$$\epsilon(x) = \pi(x) - \mathrm{Li}(x) = \pi(x) - \mathrm{li}(x) + \mathrm{li}(2) = \varepsilon(x) + \mathrm{li}(2)$$

later on. Take note that the value of $\mathrm{li}(2)$ is 1.04516378 approximately. Regarding the ε error term, Koch showed in [10], that if the Riemann hypothesis is true, then $\varepsilon(x) \in \mathcal{O}(\sqrt{x} \ln x)$. This has been made more precise by Schoenfeld in [14], showing that

$$|\varepsilon(x)| < \frac{\sqrt{x} \ln x}{8\pi} \tag{3}$$

holds for all $x \geq 2657$. As of now, this is the best possible error bound depending on the validity of the Riemann hypothesis. A weaker result from the same article of Koch states that

$$|\varepsilon(x)| < \mathcal{O}\left(x^{1/2+\sigma}\right)$$

for all $\sigma > 0$, if the Riemann hypothesis is true. Kotnik in his [11] article improves this by *conjecturing* that even

$$|\varepsilon(x)| < \sqrt{x} \tag{4}$$

holds for all $x \geq 2$ according to his investigations. We will use this later inequality in our calculations. Now we give our results for the $s \in [1/2, 1)$ case.

Proposition 1 *If the conjecture of Kotnik is true, then*

$$\lim_{n\to\infty} e^{-\mathrm{li}(n^{1-s})} \prod_{p\leq n}\left(1+\frac{1}{p^s}\right) = \varphi(s)e^{\mathcal{O}\left(\Gamma\left(1-\frac{1}{2s}\right)\right)} \tag{5}$$

holds when $s \in (1/2, 1)$*, where*

$$\varphi(s) = \frac{\sqrt{2s-1}}{(1-s)\sqrt{\ln 2}} \tag{6}$$

furthermore

$$\prod_{p\leq n}\left(1+\frac{1}{\sqrt{p}}\right) = e^{\mathrm{li}(\sqrt{n})+\mathcal{O}(\ln n)} \tag{7}$$

holds when $s = 1/2$*.*

In the case when Kotnik's conjecture would turn out to be false, one could fall back to using the result of Schoenfeld, see inequality (3), which would yield similar results, but with a more complex right hand side in equation (5), furthermore a much rougher asymptotic in place of equation (7). The plot of the $\varphi(s)$ function for $s \in (1/2, 1)$ can be seen on figure 1.

Take note that the gamma function on the right hand side of equation (5) goes to positive infinity as s approaches $1/2$ from the right, and $\varphi(s)$ goes to positive infinity as s approaches 1 from the left. When s is not near $1/2$ or 1, the right hand side of equation (5) is smaller than a constant depending on s, because next to the Γ function, we only have constant terms hidden behind the asymptotic, see the proof in section 2.

It is noted that to obtain the third theorem of Mertens, one has to know how the Riemann zeta function behaves in the neighbourhood of its pole. Here, the Riemann hypothesis is a much stronger assumption, which relies on the exact behaviour of the Riemann zeta function in the critical strip. (For more information about the critical strip, see for example [7].)

As for the cases when $s \in (0, 1/2)$ and when $s < 0$, we are going to get much rougher results. These will be more like observations, and we are going to give them in section 3, where we will state our remarks.

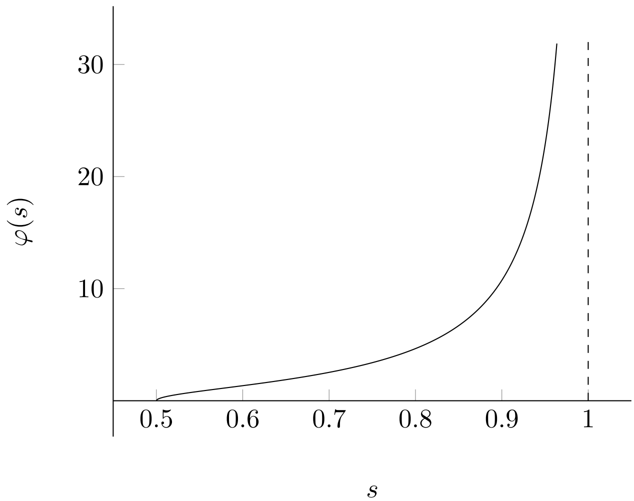

Figure 1: Plot of the $\varphi(s)$ function for $s \in (1/2, 1)$ from equation (6). Take note that $\varphi(s)$ approaches positive infinity as s approaches 1 from the left.

Our results have strong connection with the logarithmic integral, which is not so surprising due to the fact that we apply theorem 1 to obtain them. We know, see either chapter 5, equation 5.1.3 and 5.1.10 in [1], or article [6], that

$$\mathrm{li}(x) = \gamma + \ln\ln x + \sum_{k=1}^{\infty} \frac{\ln^k x}{k!k} \tag{8}$$

holds when $x > 1$. As one approaches $x = 1$ either from the left or the right, the $\mathrm{li}(x)$ logarithmic integral grows toward negative infinity. Using similar arguments as in [6], one can derive a formula, which is very similar to equation (8), in the case when $x \in (0, 1)$.

Lemma 1 *When* $x \in (0, 1)$, *then*

$$\mathrm{li}(x) = \gamma + \ln\ln\frac{1}{x} + \sum_{k=1}^{\infty} \frac{\ln^k x}{k!k}$$

holds.

Proof. Let $x \in (0, 1)$. By substituting t with e^{-u}, we get that

$$\mathrm{li}(x) = \int_0^x \frac{1}{\ln t}\,dt = -\int_{-\ln x}^{\infty} \frac{e^{-u}}{u}\,du = -\int_0^{\infty} \frac{e^{-u}}{u}\,du + \int_0^{-\ln x} \frac{e^{-u}}{u}\,du$$

equalities hold, because $-\ln x > 0$ as $x \in (0, 1)$. By splitting the first integral on the right hand side, we get

$$-\int_0^1 \frac{e^{-u}}{u}\,du - \int_1^\infty \frac{e^{-u}}{u}\,du + \int_0^{-\ln x} \frac{e^{-u}}{u}\,du + \int_{\ln x}^{-\ln x} \frac{1}{u}\,du \tag{9}$$

where one should observe that the last term which we have added is zero. We want to introduce γ into this expression, and because

$$\gamma = \int_0^1 \frac{1-e^{-u}}{u}\,du - \int_1^\infty \frac{e^{-u}}{u}\,du$$

holds, see page 103 of [5], we want to "cut down" $\int_0^1 1/u\,du$ from the last term in expression (9). Two scenarios can occur based on the value of x. If $x \in (0, e^{-1})$, then $-\ln x \geq 1$, so one can do the

$$\int_{\ln x}^{-\ln x} \frac{1}{u}\,du = \int_{\ln x}^{0} \frac{1}{u}\,du + \int_0^1 \frac{1}{u}\,du + \int_1^{-\ln x} \frac{1}{u}\,du$$

split, from which we get that expression (9) is equal to

$$\gamma + \ln\ln\frac{1}{x} + \int_0^{-\ln x} \frac{e^{-u}}{u}\,du + \int_{\ln x}^{0} \frac{1}{u}\,du$$

where interchanging the limits of the integration in the third term, and by applying the $u = -t$ substitution in the last term, one can get

$$\gamma + \ln\ln\frac{1}{x} + \int_{-\ln x}^{0} \frac{1-e^{-u}}{u}\,du$$

where the integral can be exchanged with the sum given in the lemma, as in article [6]. If $x \in (e^{-1}, 1)$, then $0 < -\ln x < 1$, so one can do the

$$\int_{\ln x}^{-\ln x} \frac{1}{u}\,du = \int_{\ln x}^{0} \frac{1}{u}\,du + \int_0^1 \frac{1}{u}\,du - \int_{-\ln x}^{1} \frac{1}{u}\,du$$

split, which we can transform back to the previous case by interchanging the limits of the integration in the last term. □

Take note that when $x \in (0, 1)$, then $\mathrm{li}(x)$ is smaller than zero and monotone decreases toward negative infinity, furthermore when $x \in (1, +\infty)$ then it monotone increases from negative infinity to positive infinity, see figure 2.

The reason why we have given the results in proposition 1 by using the logarithmic integral, and not dissecting it further is because it is hard to give a concise and also precise approximation for the logarithmic integral with elementary functions.

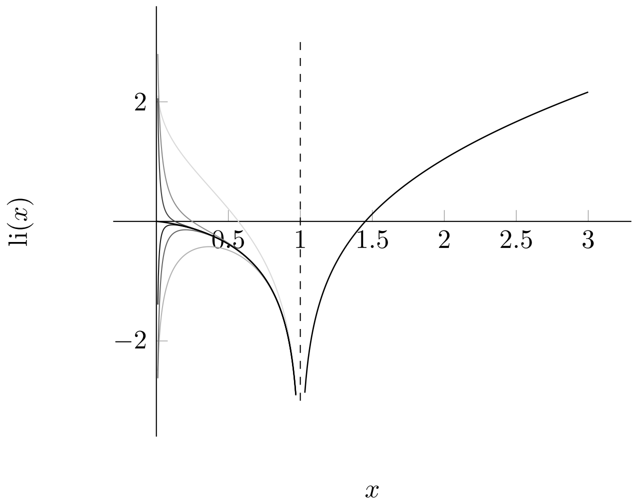

Figure 2: Plot of the logarithmic integral. The curves between 0 and 1 correspond to the truncated versions of the equation from lemma 1. (They get darker as we take more terms from the sum.)

2 Proof of the proposition

As it is noted, from now on we are going to concentrate on the positive case of expression (1), while $s \in [1/2, 1)$ holds.
Proof. Changing the product into summation in expression (1), we get

$$\exp\left(\ln \prod_{p\leq n}\left(1+\frac{1}{p^s}\right)\right) = \exp\left(\sum_{p\leq n}\ln\left(1+\frac{1}{p^s}\right)\right) \tag{10}$$

where we are going to apply theorem 1 on the argument of the exponential function on the right hand side. According theorem 1 and our observations after it, this sum is equal to

$$\int_2^n \frac{\ln\left(1+\frac{1}{t^s}\right)}{\ln t}\,dt + (\varepsilon(n)+\mathrm{li}(2))\ln\left(1+\frac{1}{n^s}\right) + s\int_2^n \frac{\varepsilon(t)+\mathrm{li}(2)}{t(t^s+1)}\,dt \tag{11}$$

because there exists an open interval containing $[2, \infty)$ on which we can continuously differentiate $\ln(1+1/t^s)$, furthermore

$$\frac{d}{dt}\ln\left(1+\frac{1}{t^s}\right) = -\frac{s}{t(t^s+1)}$$

holds in the said interval. In the following sections we will examine the terms of expression (11) piecewise, then sum our results in section 2.4.

2.1 First term

Concerning the first term in expression (11), we are going to use a series representation of the logarithmic function from [1], equation 4.1.24, which goes as

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \tag{12}$$

where $|x| \leq 1$ and $x \neq -1$. So

$$\int_2^n \frac{\ln\left(1+\frac{1}{t^s}\right)}{\ln t}\,dt = \int_2^n \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kt^{ks}\ln t}\,dt = \int_2^n \sum_{k=1}^{\infty} f_{k,s}(t)\,dt \tag{13}$$

because $|1/t^s| \leq 1$ when $t \in [2, n]$ and $s \in [1/2, 1]$. Now we are going to show that the order of the integration and the summation can be interchanged in equation (13). For every $k > 0$ and $s \in [1/2, 1]$, we have that $f_{k,s}$ is continuous on $[2, n]$, which means that it is measurable on $[2, n]$. What we have to show is that

$$\sum_{k=1}^{\infty} \int_2^n |f_{k,s}(t)|\,dt = \sum_{k=1}^{\infty} \int_2^n \frac{1}{kt^{ks}\ln t}\,dt < \frac{1}{\ln 2} \sum_{k=1}^{\infty} \int_2^n \frac{1}{t^{ks}}\,dt$$

converges. Because the integrand on the right hand side is a positive, measurable function on $[2, n]$ for every $k > 0$ and $s \in [1/2, 1]$, we can interchange the order of the summation and the integration, which – based on the sum of the geometric series – gives us

$$\frac{1}{\ln 2} \int_2^n \sum_{k=1}^{\infty} \frac{1}{t^{ks}}\,dt = \frac{1}{\ln 2} \int_2^n \frac{1}{t^s - 1}\,dt \leq \frac{1}{\ln 2} \int_2^n \frac{1}{\sqrt{t} - 1}\,dt < +\infty$$

because $s \in [1/2, 1]$, so one can interchange the order of summation and integration in equation (13) as

$$\int_2^n \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kt^{ks}\ln t}\,dt = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_2^n \frac{1}{t^{ks}\ln t}\,dt \tag{14}$$

where we have two cases during the evaluation of the integral inside the summation.

1. If $s = 1/m$ for some $m > 0$ integer, then the integral on the right side in equation (14) will be simply $\ln \ln n - \ln \ln 2$ when $k = m$.

2. Otherwise, when $s \in (1/2, 1)$ and there doesn't exist an m integer such that $s = 1/m$, then the integral can be treated as follows. Let us set $t = r^\lambda$, then we get that

$$\int_2^n \frac{1}{t^{ks} \ln t}\, dt = \int_{2^{1/\lambda}}^{n^{1/\lambda}} \frac{r^{\lambda - ks\lambda - 1}}{\ln r}\, dr$$

where we want $\lambda - ks\lambda - 1$ to be zero, so we have to set $\lambda = 1/(1 - ks)$, which gives us that

$$\int_{2^{1/\lambda}}^{n^{1/\lambda}} \frac{r^{\lambda - ks\lambda - 1}}{\ln r}\, dr = \int_{2^{1-ks}}^{n^{1-ks}} \frac{1}{\ln r}\, dr = \mathrm{li}(n^{1-ks}) - \mathrm{li}(2^{1-ks})$$

holds in this case.

Using these results, we get that when $s = 1/m$ for some $m > 0$ integer, then equation (14) is equal to

$$\frac{(-1)^{m+1}}{m} (\ln \ln n - \ln \ln 2) + \sum_{k \in \mathbb{N}^+ \setminus \{m\}} \frac{(-1)^{k+1}}{k} \Big(\mathrm{li}(n^{1-ks}) - \mathrm{li}(2^{1-ks})\Big) \tag{15}$$

otherwise when $s \in (1/2, 1)$ and there doesn't exist an m integer such that $s = 1/m$, then equation (14) is equal to the

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \Big(\mathrm{li}(n^{1-ks}) - \mathrm{li}(2^{1-ks})\Big) \tag{16}$$

sum. We are going to investigate these sums depending on the value of s separately in the cases when $s \in (1/2, 1)$, and when $s = 1/2$.

2.1.1 Above half

When $s \in (1/2, 1)$, then there is surely no such m integer that $s = 1/m$, so we are going to concentrate on expression (16) in this case. We will show that the sum can be actually split into two sums; one which contains only the $\mathrm{li}(n^{1-ks})$ terms, and another, which contains only the $\mathrm{li}(2^{1-ks})$ terms. Considering the members of the first sum, when $1 - ks < 0$, then

$$\lim_{n \to \infty} \mathrm{li}(n^{1-ks}) = 0$$

holds. Because $s \in (1/2, 1)$, this is true when $k > 1$. Regarding the members of the second sum, because the logarithmic integral is negative on $(0, 1)$, we have that

$$0 < \frac{-\mathrm{li}(2^{1-ks})}{k} = -\frac{1}{k} \lim_{h \to 0^+} \int_h^{2^{1-ks}} \frac{1}{\ln t}\,dt \tag{17}$$

holds for $k > 2$ and $s \in (1/2, 1)$. With these assumptions about k and s, we have that the $1/\ln t$ function is continuous on $[h, 2^{1-ks}]$ for every $h \in (0, 2^{1-ks})$, so we can apply the mean value theorem and get that the right hand side of equation (17) is smaller than

$$-\frac{1}{k} \lim_{h \to 0^+} \frac{2^{1-ks} - h}{\ln 2^{1-ks}} = \frac{2^{1-ks}}{k(ks-1)\ln 2} \tag{18}$$

because $|\ln t|$ increases as t approaches zero from the right. Take note that the right hand side of equation (18) decreases monotonically to zero as k approaches infinity when $s \in (1/2, 1)$. To simplify the discussion, we introduce the

$$\alpha_{i,s}(x) := \sum_{k=i}^{\infty} \frac{(-1)^{k+1}}{k} \mathrm{li}(x^{1-ks})$$

notation. We have arrived at the conclusion that $\alpha_{2,s}(n)$ converges to zero as n approaches positive infinity, and $\alpha_{3,s}(2)$ also converges based on the alternating series test, so the sum in expression (16) can be split, and it is equal to

$$\mathrm{li}(n^{1-s}) + \alpha_{2,s}(n) - \mathrm{li}(2^{1-s}) + \frac{1}{2}\mathrm{li}(2^{1-2s}) + \mathcal{O}(1)$$

when $s \in (1/2, 1)$. Now we are going to bound the third and fourth terms in this last expression. Using equation (8) for the third term, we have

$$\mathrm{li}(2^{1-s}) = \gamma + \ln\ln 2^{1-s} + \mathcal{O}(1)$$

furthermore using the result of lemma 1 for the fourth term, we get

$$\mathrm{li}(2^{1-2s}) = \gamma + \ln\ln \frac{1}{2^{1-2s}} + \mathcal{O}(1)$$

where the

$$\sum_{k=1}^{\infty} \frac{\ln^k x}{k!k} < \sum_{k=0}^{\infty} \frac{\ln^k x}{k!} = x$$

inequality was applied in both cases. Summing these results, we have that expression (16) is equal to

$$\mathrm{li}(n^{1-s}) + \alpha_{2,s}(n) + \ln \frac{\sqrt{2s-1}}{(1-s)\sqrt{\ln 2}} + \mathcal{O}(1) \tag{19}$$

in this case. Now we keep the $\alpha_{2,s}(n)$ term, because it will disappear when we will let n approach positive infinity at the end.

2.1.2 At half

As we lower s from one toward zero, the $s = 1/2$ is the first case where we have to use expression (15), which gives us

$$\mathrm{li}(\sqrt{n}) - \mathrm{li}(\sqrt{2}) - \frac{1}{2}(\ln\ln n - \ln\ln 2) + \sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{k} \left(\mathrm{li}(n^{1-k/2}) - \mathrm{li}(2^{1-k/2})\right)$$

where we can split the sum again, based on the arguments in section 2.1.1. Summing the constants, we get that expression (15) is equal to

$$\mathrm{li}(\sqrt{n}) + \alpha_{3,1/2}(n) - \frac{1}{2}\ln\ln n + \mathcal{O}(1) \tag{20}$$

when $s = 1/2$. Yet again, the $\alpha_{3,1/2}(n)$ term will disappear when we will approach positive infinity with n at the end.

2.2 Second term

As for the second term in expression (11), we are going to use the inequalities 4.1.33 from [1], which state that

$$\frac{x}{1+x} < \ln(1+x) < x$$

holds for every $x > -1$, $x \neq 0$, from which it follows that

$$\frac{1}{x^s+1} < \ln\left(1 + \frac{1}{x^s}\right) < \frac{1}{x^s}$$

holds for every $x \in (0, \infty)$ and $s \in \mathbb{R}$. Using the error term from inequality (4) in the second term of expression (11), we get that

$$\left|(\varepsilon(n) + \mathrm{li}(2)) \ln\left(1 + \frac{1}{n^s}\right)\right| \leq (|\varepsilon(n)| + \mathrm{li}(2)) \ln\left(1 + \frac{1}{n^s}\right)$$

$$< \frac{\sqrt{n} + \mathrm{li}(2)}{n^s} \tag{21}$$

so if $s > 1/2$, then the absolute value of the second term converges to zero as n goes to infinity. When $s = 1/2$, then its absolute value is smaller than

$$1 + \frac{\mathrm{li}(2)}{\sqrt{n}} \tag{22}$$

otherwise, its absolute value behaves asymptotically as

$$\mathcal{O}\left(n^{1/2-s}\right) \tag{23}$$

when $s < 1/2$.

2.3 Third term

For the third term in expression (11), we can assume without loss of generality that ε and also $|\varepsilon|$ is Riemann-integrable on $[2, n]$, so we can write

$$\left| s \int_2^n \frac{\varepsilon(t) + \mathrm{li}(2)}{t(t^s+1)}\, dt \right| \leq s \int_2^n \frac{|\varepsilon(t)| + \mathrm{li}(2)}{t(t^s+1)}\, dt < s \int_2^n \frac{\sqrt{t} + \mathrm{li}(2)}{t(t^s+1)}\, dt \tag{24}$$

where we have substituted the error term from inequality (4). This is equal to

$$\beta_s(n) := 2s \left[\sqrt{t} \cdot {}_2F_1\left(1, \frac{1}{2s}; 1 + \frac{1}{2s}; -t^s\right)\right]_2^n + \mathrm{li}(2) \left[\ln \frac{t^s}{t^s+1} \right]_2^n \tag{25}$$

because the first part of the integral on the right hand side can be transformed into the form of the ${}_2F_1$ Gauss hypergeometric function, and the second part can be decomposed into partial fractions. The transformation of the first part can be done as

$$\begin{aligned} \int \frac{1}{\sqrt{x}(x^s+1)}\, dx &= \int_0^x u^{-\frac{1}{2}} (1+u^s)^{-1}\, du + C \\ &= \frac{1}{s} \int_0^{x^s} r^{\frac{1}{2s}-1} (1+r)^{-1}\, dr + C \\ &= \frac{\sqrt{x}}{s} \int_0^1 t^{\frac{1}{2s}-1} (1+x^s t)^{-1}\, dt + C \end{aligned}$$

where first we switched the indefinite integral into a definite one, then we have applied the $u^s = r$ substitution and finally the $r = x^s t$ substitution. The Gauss hypergeometric function can be written in the

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a}\, dt$$

form, see equation 15.3.1 in [1]. Using this, we have that $z = -x^s$, $a = 1$ and $b = 1/2s$. Because $c - b - 1 = 0$ should hold, we get our missing c and with it, the first part of expression (25). As for the second part, we have

$$\mathrm{li}(2)\int_2^n \frac{s}{t(t^s+1)}\,dt = \mathrm{li}(2)\int_2^n \frac{s}{t} - \frac{st^{s-1}}{t^s+1}\,dt$$

where the second fraction is a logarithmic derivative. We are going to investigate the resulting expression (25) separately in the cases when $s \in (1/2, 1)$, and when $s = 1/2$.

2.3.1 Above half

First, we deal with the upper limit of the integration. Substituting n into expression (25), we get

$$2s\sqrt{n} \cdot {}_2F_1\left(1, \frac{1}{2s}; 1 + \frac{1}{2s}; -n^s\right) + \mathrm{li}(2)\ln\frac{n^s}{n^s+1} \tag{26}$$

which, by using the

$${}_2F_1\left(a, b; c; z\right) = (1-z)^{-b}{}_2F_1\left(b, c-a; c; \frac{z}{z-1}\right)$$

linear transformation formula, see equation 15.3.5 in [1], can be transformed into the

$$2s\frac{\sqrt{n}}{(n^s+1)^{1/2s}}{}_2F_1\left(\frac{1}{2s}, \frac{1}{2s}; 1 + \frac{1}{2s}; \frac{n^s}{n^s+1}\right) + \mathrm{li}(2)\ln\frac{n^s}{n^s+1}$$

form. Because $s \in (1/2, 1)$, we get that

$$\lim_{n\to+\infty}\frac{\sqrt{n}}{(n^s+1)^{1/2s}}{}_2F_1\left(\frac{1}{2s}, \frac{1}{2s}; 1 + \frac{1}{2s}; \frac{n^s}{n^s+1}\right) = {}_2F_1\left(\frac{1}{2s}, \frac{1}{2s}; 1 + \frac{1}{2s}; 1\right)$$

holds. When c is not zero or a negative integer, furthermore $\Re(c - a - b) > 0$ is true, then one can apply the

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

substitution, see equation 15.1.20 in [1]. Because $s \in (1/2, 1)$, the conditions are satisfied, and we can utilise the above mentioned formula to get

$$2s{}_2F_1\left(\frac{1}{2s}, \frac{1}{2s}; 1 + \frac{1}{2s}; 1\right) = 2s\Gamma\left(1 + \frac{1}{2s}\right)\Gamma\left(1 - \frac{1}{2s}\right) \tag{27}$$

which has an anomaly at $s = 1/2$. For the second term, observe that

$$\lim_{n\to+\infty} \ln \frac{n^s}{n^s+1} = 0$$

holds. Now we deal with the lower limit of the integral. By substituting 2 into expression (25) we get

$$2s\sqrt{2} \cdot {}_2F_1\left(1, \frac{1}{2s}; 1+\frac{1}{2s}; -2^s\right) + \mathrm{li}(2) \ln \frac{2^s}{2^s+1} \tag{28}$$

where the second term is a small constant, so we will concentrate on the value of the hypergeometric function. Using another

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

linear transformation formula, see equation 15.3.4 in [1], we get that the hypergeometric function in expression (28) is equal to

$$\frac{1}{2^s+1} {}_2F_1\left(1, 1; 1+\frac{1}{2s}; \frac{2^s}{2^s+1}\right)$$

where the last argument is in $(0, 1)$. We are going to give an upper bound for this expression. If $a \leq 1$ and $0 < b \leq c$, then

$${}_2F_1(-a, b; c; z)^{1/a} \geq \left[\left(1-\frac{b}{c}\right) + \frac{b}{c}(1-z)^a\right]^{1/a}$$

for all $z \in (0, 1)$, see [8] furthermore [13] and [4]. Because $s \in (1/2, 1)$, we can apply this inequality, which means that

$${}_2F_1\left(1, 1; 1+\frac{1}{2s}; \frac{2^s}{2^s+1}\right) \leq \left(1-\frac{1}{1+\frac{1}{2s}}\right) + \frac{1}{1+\frac{1}{2s}}\left(1-\frac{2^s}{2^s+1}\right)^{-1}$$

which is just a small constant when $s \in (1/2, 1)$. Joining our results so far, we get that $\beta_s(n)$ from expression (25) then converges to the right hand side of equation (27) plus some constant as n approaches positive infinity when $s \in (1/2, 1)$.

2.3.2 At half

One of the special elementary cases of the hypergeometric function is the

$$ {}_2F_1\left(1,1;2;z\right) = -z^{-1}\ln(1-z) $$

equality, see equation 15.1.3 in [1]. This means that when $s = 1/2$, then the first part of expression (25) is equal to

$$ \Big[\ln(\sqrt{t}+1)\Big]_2^n \in \mathcal{O}(\ln n) \tag{29} $$

and because we have tried to bound the absolute value of the third term from above, this partial result already spoils our pursuit of reaching a constant error term in this special case. Regarding the second part of expression (25), what was said in the previous section still holds, meaning that the second part converges to a small constant as n approaches infinty.

2.4 Summing the parts

Now we are going to sum our results. In the case when $s \in (1/2, 1)$, the first term of expression (11) is equal to expression (19), the absolute value of the second term is smaller than the right hand side of inequality (21) and the absolute value of the third term is smaller than expression (25). So expression (11) is equal to

$$ \mathrm{li}(n^{1-s}) + \ln\frac{\sqrt{2s-1}}{(1-s)\sqrt{\ln 2}} + \mathcal{O}(1) + \mathcal{O}\left(\alpha_{2,s}(n) + \frac{\sqrt{n}+\mathrm{li}(2)}{n^s} + \beta_s(n)\right) $$

in this case. Reintroducing this into equation (10), we get that the positive case of expression (1) is equal to

$$ \frac{\sqrt{2s-1}}{(1-s)\sqrt{\ln 2}}\exp\left(\mathrm{li}(n^{1-s}) + \mathcal{O}(1) + \mathcal{O}\left(\alpha_{2,s}(n) + \frac{\sqrt{n}+\mathrm{li}(2)}{n^s} + \beta_s(n)\right)\right) $$

where, after dividing with $\exp\left(\mathrm{li}(n^{1-s})\right)$ and taking the limit in n, we get the sought equality (5). As for the case when $s = 1/2$ the first term of expression (11) is equal to expression (20), the absolute value of the second term is smaller than expression (22), and the absolute value of the third term is smaller than expression (29). By these, expression (11) is equal to

$$ \mathrm{li}(\sqrt{n}) + \mathcal{O}(\ln n) $$

and by reintroducing this into equation (10), we get equality (7). □

3 Remarks

3.1 Below half

The problem in this region is that either when using expression (15) or expression (16) the result becomes more and more complicated as we approach zero with s from the right. This happens mainly because the $\mathrm{li}(n^{1-ks})$ terms only disappear when $k > 1/s$, but also because one has to pay close attention to the $\mathrm{li}(2^{1-ks})$ terms when 2^{1-ks} is close to one. (The $\mathrm{li}(n^{1-ks})$ terms behave more nicely, because they avoid the anomaly at one.) Furthermore, without taking the third term into consideration, the second term already contributes a rough asymptotical term, see expression (23).

3.2 The case of negative exponents

When the s exponent is negative, instead of expression (11), the sum in equation (10) is equal to

$$\int_2^n \frac{\ln\left(1+t^{|s|}\right)}{\ln t}\,dt + (\varepsilon(n)+\mathrm{li}(2))\ln\left(1+n^{|s|}\right) - |s|\int_2^n \frac{\varepsilon(t)+\mathrm{li}(2)}{t^{1-|s|}\left(1+t^{|s|}\right)}\,dt \quad (30)$$

because there exists an open interval containing $[2,\infty)$ on which we can continuously differentiate $\ln\left(1+t^{|s|}\right)$, furthermore

$$\frac{d}{dt}\ln\left(1+t^{|s|}\right) = |s|\frac{t^{|s|-1}}{1+t^{|s|}}$$

holds. We cannot apply equation (12) like in section 2.1. Instead, we will use the following simple estimations, which can be shown using the properties of the logarithmic function. For every $s > 0$ real number, there exists such $c_s > 1$ constant, that

$$\ln x^s \le \ln(1+x^s) \le c_s \ln x^s$$

holds for every $x > 1$ real number. Applying these estimations on the first term on expression (30), we get

$$|s|\int_2^n dt \le \int_2^n \frac{\ln\left(1+t^{|s|}\right)}{\ln t}\,dt \le c_s|s|\int_2^n dt$$

which shows us that the first term grows as $\Theta(n)$. Proceeding to the second term in expression (30), we have that its absolute value is smaller than

$$(|\varepsilon(n)|+\mathrm{li}(2))\ln\left(1+n^{|s|}\right) \le c_s|s|\left(\sqrt{n}+\mathrm{li}(2)\right)\ln n$$

which grows as $\mathcal{O}\left(\sqrt{n}\ln n\right)$. Still assuming that ε and also $|\varepsilon|$ is Riemann-integrable on $[2, n]$, the absolute value of the third term in expression (30) is smaller than or equal to

$$|s|\int_2^n \frac{|\varepsilon(t)| + \mathrm{li}(2)}{t^{1-|s|}\left(1 + t^{|s|}\right)}\,dt$$

which we can estimate from above by dropping the plus one from the denominator. This way, we get

$$|s|\int_2^n \frac{|\varepsilon(t)| + \mathrm{li}(2)}{t}\,dt < |s|\int_2^n \frac{\sqrt{t} + \mathrm{li}(2)}{t}\,dt = |s|\left[2\sqrt{t} + \mathrm{li}(2)\ln t\right]_2^n$$

which grows as $\mathcal{O}\left(\sqrt{n}\right)$. As it can be seen, when the s exponent is negative in equation (10), then the first term dominates so the positive case of the product in expression (1) grows as $\exp\left(\Theta(n)\right)$.

3.3 Remark about the strength of the method

Finally, we are going to look at how the theorem performs in the $s = 1$ case. Following the same path as in section 2, for the first term in expression (11) we should use expression (15), which is equal to

$$\ln\ln n - \ln\ln 2 + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k}\left(\mathrm{li}(n^{1-k}) - \mathrm{li}(2^{1-k})\right)$$

in this case. As in section 2.1.1, we can deduce that this expression is equal to

$$\ln\ln n + \alpha_{2,1}(n) + \mathcal{O}(1) \tag{31}$$

where the $\alpha_{2,1}(n)$ term disappears as we approach positive infinity with n. The absolute value of the second term in expression (11) is smaller than

$$\frac{\sqrt{n} + \mathrm{li}(2)}{n} \tag{32}$$

which vanishes as n tends to positive infinity, so what remains is the absolute value of the third term in the expression (11). Based on inequality (24) we get that its absolute value is smaller than

$$\beta(n) := \int_2^n \frac{\sqrt{t} + \mathrm{li}(2)}{t(t+1)}\,dt = 2\left[\arctan\sqrt{t}\right]_2^n + \left[\ln\frac{t}{t+1}\right]_2^n \tag{33}$$

which converges to a small constant when n approaches infinity. Using expression (31), expression (32), and finally expression (33), we get that equation (10) is equal to

$$\exp\left(\ln\ln n + \mathcal{O}(1) + \mathcal{O}\left(\alpha_{2,1}(n) + \frac{\sqrt{n} + \mathrm{li}(2)}{n} + \beta(n)\right)\right)$$

which in turn means that

$$\lim_{n\to\infty} \frac{1}{\ln n} \prod_{p\leq n} \left(1 + \frac{1}{p}\right) = e^{\mathcal{O}(1)}$$

holds.

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More on decomposition of generalized continuity

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Abstract. In this paper a new class of sets termed as ω_μ^*-open sets has been introduced and studied. Using these concept, a unified theory for decomposition of (μ, λ)-continuity has been given.

1 Introduction

For the last one decade or so, the notion of generalized topological spaces and several classes of generalized types of open sets are being studied by different mathematicians. Our aim here is to study the notion of decomposition of continuity by using the concept of generalized topology introduced by Á. Császár [2]. On the otherhand the notion of decompositions of continuity was first introduced by Tong [18, 19] by defining $\mathcal{A}$-sets and $\mathcal{B}$-sets. After then decompositions of continuity and some of its weak forms have been studied by Ganster and Reilly [7, 8], Yalvac [20], Hatir and Noiri [10, 11], Przemski [14], Noiri and Sayed [13], Dontchev and Przemski [5], Erguang and Pengfei [6] and many others. Decompositions of regular open sets and regular closed sets are given by using PS-regular sets in [9]. Since then the notion of decompositions of continuity is one of the most important area of research.

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We first recall some definitions given in [2]. Let X be a non-empty set and $\exp X$ denote the power set of X. We call a class $\mu \subsetneq \exp X$ a generalized topology (briefly, GT) [1, 2], if $\varnothing \in \mu$ and μ is closed under arbitrary unions. A set X, with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ). A GT μ is said to be a quasi topology (briefly QT) [3] if $M, M' \in \mu$ implies $M \cap M' \in \mu$. The pair (X, μ) is said to be a QTS if μ is a QT on X. For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. A GTS (X, μ) is called a μ-space [13] or a strong GTS [4] if $X \in \mu$. A subset A of a topological space (X, τ) is called ω-closed [12] if it contains all its condensation points. The complement of an ω-closed set is called an ω-open set. It is well known that a subset A of a space (X, τ) is ω-open if and only if for each $x \in A$, there exists $U \in \tau$ containing x such that $U \setminus A$ is countable.

The purpose of this paper is to introduce the decomposition theorem for the (μ, λ) continuous functions introduced in [1] which is a generalization of continuity and different weak forms of continuity. Throughout the paper, by (X, μ) and (Y, λ) we shall mean GTS unless otherwise stated.

2 ω_μ^*-open sets and its properties

Definition 1 *Let (X, μ) be a GTS. A subset A of X is called an ω_μ^*-open (ω_μ-open [15]) set if for each $x \in A$, there exists a μ-open set U containing x such that $U \setminus i_\mu(A)$ (resp. $U \setminus A$) is countable. The complement of an ω_μ^*-open (resp. ω_μ-open) set is known as an ω_μ^*-closed (resp. ω_μ-closed [15]) set.*

It follows from Definition 1 that every ω_μ^*-open set is an ω_μ-open set and every μ-open set is ω_μ^*-open set but the converses are false as shown in Example 3.

Remark 1 *Let μ be a GT on a topological space (X, τ). If $\tau \subsetneq \mu$, then the following relations hold:*

$$\begin{array}{ccccc} \omega\text{-open set} & \Leftarrow & \text{open set} & \Rightarrow & \mu\text{-open set} \\ \Downarrow & & & & \Downarrow \\ \omega_\mu\text{-open set} & & \Leftarrow & & \omega_\mu^*\text{-open set} \end{array}$$

Example 1 (a) *Let $X = \mathbb{R}$, τ be the usual topology on $\mathbb{R}$. Let $\mu = \{\varnothing, X, \mathbb{Q}\}$. Then μ is a GT on the topological space (X, τ). It is easy to see that $\mathbb{Q}$ is an ω_μ^*-open set but not an ω-open set.*

(b) *Let* $X = \mathbb{R}$ *and* μ *be the usual topology on* $\mathbb{R}$. *Then* μ *is a GT on* X. *It is easy to see that* $\mathbb{I}$, *the set of irrationals is an* ω_μ*-open set but not an* ω^*_μ*-open set.*

(c) *Let* $X = \mathbb{R}$ *and* $\mu = \{A \subseteq X : 0 \in A\} \cup \{\varnothing\}$. *Then* μ *is a GT on the set* X. *It is easy to see that* $\mathbb{I}$, *the set of irrationals is* ω_μ*-open but not* μ*-open.*

(d) *Let* $X = \mathbb{R}$, $\mu = \{A : A$ *is uncountable*$\} \cup \{\varnothing\}$ *and* $\tau = \{\varnothing, X, \mathbb{Q}\}$. *Then* μ *is a GT on the topological space* (X, τ). *It can be easily verified that* $\mathbb{Q}$ *is an* ω*-open set but not an* ω^*_μ*-open set.*

The family of all ω^*_μ-open sets of a GTS (X, μ) is denoted by $\omega^*_\mu(X)$ or simply by ω^*_μ.

Proposition 1 (a) *In a GTS* (X, μ), *the collection of all* ω^*_μ*-open sets forms a GT on* X.
(b) *If* (X, μ) *is a QTS, then the collection of all* ω^*_μ*-open sets forms a QT on* X.

Proof. (a) It is obvious that $\varnothing$ is an ω^*_μ-open set. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of ω^*_μ-open subsets of X. Then for each $x \in \cup\{A_\alpha : \alpha \in \Lambda\}$, $x \in A_\alpha$ for some $\alpha \in \Lambda$. Thus there exists $U \in \mu$ containing x such that $U \setminus i_\mu(A_\alpha)$ is countable. Now as $U \setminus i_\mu(\cup\{A_\alpha : \alpha \in \Lambda\}) \subseteq U \setminus i_\mu(A_\alpha)$, thus $U \setminus i_\mu(\cup\{A_\alpha : \alpha \in \Lambda\})$ is countable. Hence $\cup\{A_\alpha : \alpha \in \Lambda\}$ is an ω^*_μ-open set.

(b) It follows from (a) that (X, ω^*_μ) is a GTS. Let A and B be two ω^*_μ-open sets and $x \in A \cap B$. Then there exist μ-open sets U and V containing x such that $U \setminus i_\mu(A)$ and $V \setminus i_\mu(B)$ are countable. Then $U \cap V$ is a μ-open set containing x and $(U \cap V) \setminus i_\mu(A \cap B) = (U \cap V) \setminus i_\mu(A) \cap i_\mu(B) \subseteq [U \setminus i_\mu(A)] \cup [V \setminus i_\mu(B)]$. Thus $(U \cap V) \setminus i_\mu(A \cap B)$ is countable so that ω^*_μ is a QT on X. ☐

Theorem 1 *A subset* A *of a GTS* (X, μ) *is an* ω^*_μ*-open set if and only if for each* $x \in A$, *there exist* $U_x \in \mu$ *containing* x *and a countable subset* C *such that* $U_x \setminus C \subseteq i_\mu(A)$.

Proof. Let A be an ω^*_μ-open set in X and $x \in A$. Then there exists $U_x \in \mu$ containing x such that $U_x \setminus i_\mu(A)$ is countable. Let $C = U_x \setminus i_\mu(A) = U_x \cap (X \setminus i_\mu(A))$. Then $U_x \setminus C \subseteq i_\mu(A)$.

Conversely, let $x \in A$ and there exist $U_x \in \mu$ containing x and a countable subset C such that $U_x \setminus C \subseteq i_\mu(A)$. Then $U_x \setminus i_\mu(A) \subseteq C$ and hence $U_x \setminus i_\mu(A)$ is a countable set. Thus A is an ω^*_μ-open set in X. ☐

Theorem 2 *Let* (X, μ) *be a GTS and* $C \subseteq X$. *If* C *is* ω_μ^**-closed, then* $C \subseteq K \cup B$ *for some* μ*-closed set* K *and a countable subset* B.

Proof. If C be ω_μ^*-closed, then $X \setminus C$ is ω_μ^*-open and hence for each $x \in X \setminus C$, there exist $U \in \mu$ containing x and a countable subset B such that $U \setminus B \subseteq i_\mu(X \setminus C) = X \setminus c_\mu(C)$. Thus $c_\mu(C) \subseteq X \setminus (U \setminus B) = X \setminus (U \cap (X \setminus B)) = (X \setminus U) \cup B$. Let $K = X \setminus U$. Then K is μ-closed such that $C \subseteq K \cup B$. □

Proposition 2 *In a GTS* (X, μ), $\omega_\mu^* = \omega^*_{\omega_\mu^*}$, *where* ω_μ^* *denotes the family of* ω_μ^**-open sets of the GTS* (X, μ).

Proof. By Remark 1, we have $\omega_\mu^* \subseteq \omega^*_{\omega_\mu^*}$. Let $A \in \omega^*_{\omega_\mu^*}$. Then for each $x \in A$, there exist $U_x \in \omega_\mu^*$ with $x \in U_x$ and a countable set C_x such that $U_x \setminus C_x \subseteq i_\mu(A)$. Furthermore there exist a $V_x \in \mu$ with $x \in V_x$ and a countable set D_x such that $V_x \setminus D_x \subseteq i_\mu(U_x)$. Thus $V_x \setminus (C_x \cup D_x) = (V_x \setminus D_x) \setminus C_x \subseteq U_x \setminus C_x \subseteq i_\mu(A)$. Since $C_x \cup D_x$ is a countable set, we obtain $A \in \omega_\mu^*$. □

Remark 2 *If* (X, μ) *be a* μ*-space, then* (X, ω_μ^*) *is an* ω_μ^**-space.*

Definition 2 *A subset* A *of a GTS* (X, μ) *is called an* (i) (ω_μ, ω)*-set if* $i_{\omega_\mu^*}(A) = i_{\omega_\mu}(A)$.
(ii) (ω_μ, μ)*-set if* $i_{\omega_\mu^*}(A) = i_\mu(A)$.

Remark 3 *Every* ω_μ^**-open set is an* (ω_μ, ω)*-set and every* μ*-open set is an* (ω_μ, μ)*-set but the converses are usually not true.*

Example 2 (a) *Let* $X = \mathbb{R}$, $\mu = \{\varnothing, \mathbb{I}, X\}$. *Then* μ *is a GT on* X. *It can be checked that* $\mathbb{N}$ (= *the set of natural numbers*) *is not an* ω_μ^**-open but an* (ω_μ, ω)*-set.*

(b) *Let* $X = \mathbb{R}$, $\mu = \{\varnothing, (2, 3), X\}$. *Then* μ *is a GT on* X. *The set* $(1, \frac{3}{2})$ *is an* (ω_μ, μ)*-set but not a* μ*-open set.*

Theorem 3 *A subset* A *of a GTS* (X, μ) *is* ω_μ^**-open if and only if* A *is* ω_μ*-open and an* (ω_μ, ω)*-set.*

Proof. Since every ω_μ^*-open set is ω_μ-open (from definition) and an (ω_μ, ω)-set (by Remark 3), we have nothing to show.

Conversely, let A be an ω_μ-open and an (ω_μ, ω)-set. Then $A = i_{\omega_\mu}(A) = i_{\omega_\mu^*}(A)$. Thus A is an ω_μ^*-open set. □

Theorem 4 *A subset A of a GTS (X, μ) is μ-open if and only if A is ω_μ^*-open and an (ω_μ, μ)-set.*

Proof. One part follows from the fact that every μ-open set is ω_μ^*-open and (ω_μ, μ)-set.

Conversely, let A be an ω_μ^*-open set and an (ω_μ, μ)-set. Then $A = i_{\omega_\mu^*}(A) = i_\mu(A)$. Thus A is μ-open. □

Definition 3 *A GTS (X, μ) is called*
(i) *μ-locally countable if each $x \in X$ is contained in a countable μ-open set.*
(ii) *anti μ-locally countable if each non-empty μ-open subsets are uncountable.*

Theorem 5 *Let (X, μ) be a μ-locally countable space. Then*
(i) *for any subset A of X, A is ω_μ^*-open.*
(ii) *A is ω_μ^*-open if and only if A is ω_μ-open.*

Proof. (i) Let $A \subsetneqq X$ and $x \in A$. Then there is a countable μ-open set U containing x. Then $U \setminus i_\mu(A)$ is a countable set. Thus A is ω_μ^*-open.

(ii) One part follows from the fact that every ω_μ^*-open is ω_μ-open.

Conversely, let A be ω_μ-open. Then by (i), A is ω_μ^*-open. □

Remark 4 *Let (X, μ) be a countable GTS. Then for any subset A of X, A is ω_μ^*-open.*

Theorem 6 (i) *Let (X, μ) be a GTS and $A \subsetneqq X$. If (X, μ) is anti μ-locally countable, then so is (X, ω_μ^*).*
(ii) *Let (X, μ) be a QTS which is anti μ-locally countable. Then for any μ-open subset A of X, $c_\mu(A) = c_{\omega_\mu^*}(A)$.*

Proof. (i) Let A be an ω_μ^*-open set and $x \in A$. Then there exist $U_x \in \mu$ containing x and a countable subset C such that $U_x \setminus C \subsetneqq i_\mu(A)$. Thus $i_\mu(A)$ is uncountable and hence A is uncountable.

(ii) Let $x \in c_\mu(A)$ and G be an ω_μ^*-open set containing x. Then there exist $U_x \in \mu$ containing x and a countable subset C such that $U_x \setminus C \subsetneqq i_\mu(G)$. Then $(U_x \setminus C) \cap A \subsetneqq i_\mu(G) \cap A$ i.e., $(U_x \cap A) \setminus C \subsetneqq i_\mu(G) \cap A$. Since U_x is a μ-open set, $U_x \cap A \neq \varnothing$ and thus $U_x \cap A$ is a non-empty μ-open set. Hence by anti μ-locally countableness of (X, μ), it follows that $U_x \cap A$ is uncountable and hence $(U_x \cap A) \setminus C$ is also uncountable. Thus $i_\mu(G) \cap A$ is uncountable. Hence $G \cap A \neq \varnothing$. So $x \in c_{\omega_\mu^*}(A)$ i.e, $c_\mu(A) \subsetneqq c_{\omega_\mu^*}(A)$. The other part is obvious. □

Definition 4 *A μ-space (X,μ) is said to be μ-Lindelöf* [17] *if every cover of X by μ-open sets has a countable subcover.*

A subset A of a μ-space (X,μ) is said to be μ-Lindelöf relative to X [17] *if every cover of A by μ-open sets of X has a countable subcover.*

Theorem 7 *Let (X,μ) be a μ-Lindelöf GTS and A be a μ-closed, ω^*_μ-open subset of X. Then $A \setminus i_\mu(A)$ is countable.*

Proof. Clearly A is a μ-Lindelöf space (see Corollary 3.6 of [15]). For each $x \in A$, there exist $U_x \in \mu$ containing x and a countable subset C such that $U_x \setminus C \subseteqq i_\mu(A)$. Thus $\{U_x : x \in A\}$ is cover of A by μ-open subsets of X. Hence by μ-Lindelöfness of A, it has a countable subcover $\{U_n : n \in \mathbb{N}\}$. Since $A \setminus i_\mu(A) \subseteqq \cup\{U_n \setminus i_\mu(A) : n \in \mathbb{N}\}$, $A \setminus i_\mu(A)$ becomes countable. □

3 Decomposition of continuity by ω^*_μ-open sets

Definition 5 *A function $f : (X,\mu) \to (Y,\lambda)$ is said to be ω^*_μ-continuous (resp. ω_μ-continuous* [15]*, (μ,λ)-continuous* [1]*) if for every $x \in X$ and every λ-open set V of Y containing $f(x)$, there exists an ω^*_μ-open (resp. ω_μ-open, μ-open) set U containing x such that $f(U) \subseteqq V$.*

Definition 6 *A function $f : (X,\mu) \to (Y,\lambda)$ is said to be weakly ω^*_μ-continuous if for every $x \in X$ and every λ-open set V of Y containing $f(x)$, there exists an ω^*_μ-open set U containing x such that $f(U) \subseteqq c_\lambda(V)$.*

Theorem 8 *For a function $f : (X,\mu) \to (Y,\lambda)$ the following properties are equivalent:*
(i) *f is ω^*_μ-continuous;*
(ii) *$f : (X,\omega^*_\mu) \to (Y,\lambda)$ is (ω^*_μ,λ)-continuous;*
(iii) *$f^{-1}(V) \in \omega^*_\mu$ for every $V \in \lambda$.*

Proof. Obvious. □

Remark 5 *Let $f : (X,\mu) \to (Y,\lambda)$ be a function. Then the following relations hold: (μ,λ)-continuity $\Rightarrow$ ω^*_μ-continuity $\Rightarrow$ ω_μ-continuity $\Rightarrow$ weakly ω^*_μ-continuity.*

Example 3 (a) *Let $X = \{a,b,c\}$, $\mu = \{\varnothing,\{b\},\{a,c\},\{b,c\},X\}$ and $\lambda = \{\varnothing,\{a,c\}$, $\{a,b\}$,*

$X\}$. *Then* μ *and* λ *are two GT's on* X. *It can be verified that the identity mapping* $f : (X, \mu) \to (X, \lambda)$ *is* ω^*_μ*-continuous but not* (μ, λ)*-continuous.*

(b) *Let* $X = \mathbb{R}$, $\mu =$ *the usual topology on* $\mathbb{R}$, $Y = \{a, b, c, d\}$ *and* $\lambda = \{\varnothing, Y, \{a\}, \{a, b\}, \{c, d\}, \{a, b, c\}, Y\}$. *Consider the mapping* $f : (X, \mu) \to (Y, \lambda)$ *defined by*

$$f(x) = \begin{cases} a, & \text{if} \quad x \in \mathbb{I} \\ b, & \text{if} \quad x \notin \mathbb{I} \end{cases}$$

It can be checked that f *is* ω_μ*-continuous but not* ω^*_μ*-continuous.*

(c) *Let* $X = \mathbb{R}$ *be the set of real numbers,* $\mu = \{\varnothing, \mathbb{R}, \mathbb{I}\}$, $Y = \{a, b, c, d\}$ *and* $\lambda = \{\varnothing, Y, \{d\}, \{c, d\}, \{a, b, c\}\}$. *Consider the mapping* $f : (X, \mu) \to (Y, \lambda)$ *defined by*

$$f(x) = \begin{cases} a, & \text{if} \quad x \in \mathbb{I} \cup \{0\} \\ b, & \text{if} \quad x \notin \mathbb{I} \cup \{0\} \end{cases}$$

It can be verified that f *is* ω_μ*-continuous but not weakly* ω^*_μ*-continuous.*

Definition 7 *A function* $f : (X, \mu) \to (Y, \lambda)$ *is said to be* (ω^*_μ, ω)*-continuous (resp.* (ω^*_μ, μ)*-continuous) if for every* λ*-open set* A *of* Y, $f^{-1}(A)$ *is an* (ω_μ, ω)*-set (resp. an* (ω_μ, μ)*-set).*

Remark 6 *Every* (μ, λ)*-continuous function is* (ω^*_μ, μ)*-continuous and every* ω^*_μ*-continuous function is* (ω^*_μ, ω)*-continuous but the converses are not true.*

Example 4 (a) *Let* $X = \mathbb{R}$, $\mu = \{\varnothing, (2, 3), X\}$. *Then* μ *is a GT on* X. *Let* $B = (1, \frac{3}{2})$ *and* $\lambda = \{\varnothing, B, X \setminus B, X\}$. *Consider the mapping* $f : (X, \mu) \to (X, \lambda)$ *defined by*

$$f(x) = \begin{cases} \frac{5}{4}, & \text{if} \quad x \in (1, 2) \\ 4, & \text{if} \quad x \notin (1, 2) \end{cases}$$

It can be verified that f *is* (ω^*_μ, μ)*-continuous but not* (μ, λ)*-continuous.*

(b) *Let* $X = \mathbb{R}$, $\mu = \{\varnothing, \mathbb{I}, X\}$ *where* $\mathbb{I}$ *is the set of irrationals. Then* μ *is a GT on* X. *Consider the mapping* $f : (X, \mu) \to (X, \mu)$ *defined by*

$$f(x) = \begin{cases} \sqrt{2}, & \text{if} \quad x \in \mathbb{N} \\ 1, & \text{if} \quad x \notin \mathbb{N} \end{cases}$$

It can be verified that f *is* (ω^*_μ, ω)*-continuous but not* ω^*_μ*-continuous.*

Definition 8 *For any subset* A *of a GTS* (X, μ), *the* μ*-frontier* [16] *of* A *is denoted by* $\mathrm{Fr}_\mu(A)$ *and defined by* $\mathrm{Fr}_\mu(A) = c_\mu(A) \cap c_\mu(X \setminus A)$.

Definition 9 *A function* $f : (X, \mu) \to (Y, \lambda)$ *is said to be co-weakly* (ω^*_μ, λ)*-continuous if* $f^{-1}(Fr_\lambda(V))$ *is* ω^*_μ*-closed in* X *for every* λ*-open set* V *in* Y.

Theorem 9 *For a function* $f : (X, \mu) \to (Y, \lambda)$ *the following are equivalent:*
(i) f *is* ω^*_μ*-continuous.*
(ii) f *is* ω_μ*-continuous and* (ω_μ, ω)*-continuous.*

Proof. (i) $\Leftrightarrow$ (ii) Follows from Theorem 3. □

Theorem 10 *Let* (X, μ) *be a QTS. Then for a function* $f : (X, \mu) \to (Y, \lambda)$ *the following are equivalent:*
(i) f *is* ω^*_μ*-continuous.*
(ii) f *is co-weakly* (ω^*_μ, λ)*-continuous and weakly* ω^*_μ*-continuous.*

Proof. (i) $\Rightarrow$ (ii): Obvious.
(ii) $\Rightarrow$ (i): Let f be a co-weakly (ω^*_μ, λ)-continuous and weakly ω^*_μ-continuous function. Let $x \in X$ and V be a λ-open set of Y containing $f(x)$. As f is a weakly ω^*_μ-continuous function, there exists a ω^*_μ-open set U containing x such that $f(U) \subseteqq c_\lambda(V)$. Since $Fr_\lambda(V) = c_\lambda(V) \cap c_\lambda(X \setminus V) = c_\lambda(V) \setminus V$, we have $f(x) \notin Fr_\lambda(V)$. Since f is co-weakly (ω^*_μ, λ)-continuous, $x \in U \setminus f^{-1}(Fr_\lambda(V))$, which is ω^*_μ-open in X. Then for every $y \in f(U \setminus f^{-1}(Fr_\lambda(V)))$, $y = f(x_1)$ for a point $x_1 \in U \setminus f^{-1}(Fr_\lambda(V))$. Thus we have $f(x_1) = y \in f(U) \subseteqq c_\lambda(V)$ and $y \notin Fr_\lambda(V)$ with $f(x_1) \in V$. Thus $f(U \setminus f^{-1}(Fr_\lambda(V))) \subseteqq V$. Hence f is ω^*_μ-continuous. □

Theorem 11 *For a function* $f : (X, \mu) \to (Y, \lambda)$ *the following are equivalent:*
(i) f (μ, λ)*-continuous.*
(ii) f *is* ω^*_μ*-continuous and* (ω^*_μ, μ)*-continuous.*

Proof. Follows from Theorem 4. □

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