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Fredholm type integral equation with special functions

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Abstract. Recently Chaurasia and Gill [7], Chaurasia and Kumar [8] have solved the one-dimensional integral equation of Fredholm type involving the product of special functions. We solve an integral equation involving the product of a class of multivariable polynomials, the multivariable H-function defined by Srivastava and Panda [29, 30] and the multivariable I-function defined by Prasad [21] by the application of fractional calculus theory. The results obtained here are general in nature and capable of yielding a large number of results (known and new) scattered in the literature.

1 Introduction and preliminaries

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of any arbitrary real or complex order. The widely investigated subject of fractional calculus has gained importance and popularity during the past four decades or so, chiefly due to its demonstrated applications in

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numerous seemingly diverse fields of science and engineering including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics (see, for details, [6, 5, 15, 26, 31] and for recent works, see also [18, 19]). Under various fractional calculus operators, the computations of image formulas for special functions of one or more variables are important from the point of view the solution of differential and integral equations (see, [1, 2, 3, 13, 12, 15, 22, 23, 24, 25]). In this paper, we use the fractional calculus, more precisely the Weyl fractional operator to resolve the one-dimensional integral equation of Fredholm type involving the product of special functions. The integral equations occur in many fields of physics, mechanics and applied mathematics. In the last several years a large number of Fredholm type integral equations involving various polynomials or special functions as kernels have been studied by many authors notably Chaurasia et al. [7, 8], Buchman [4], Higgins [10], Love [16, 17], Prabhakar and Kashyap [20] and others. In the present paper, we obtain the solutions of following Fredholm integral equation.

$$\int_0^\infty y^{-\alpha}\, S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left[u_1\left(\frac{x}{y}\right)^t,\cdots,u_s\left(\frac{x}{y}\right)^t\right]\times H^{0,\lambda':(\alpha',\beta');\cdots;(\alpha^{(r)},\beta^{(r)})}_{A',C':(M',N');\cdots;(M^{(r)},N^{(r)})}$$
$$\left[\begin{array}{c} u_1(y)^p \\ \cdot \\ \cdot \\ u_r(y)^p \end{array}\ \middle|\ \begin{array}{c} [(g_j);\gamma',\cdots,\gamma^{(r)}]_{1,A'}:(q',\eta')_{1,M'};\cdots;(q^{(r)},\eta^{(r)})_{1,M^{(r)}} \\ [(f_j);\xi',\cdots,\xi^{(r)}]_{1,C'}:(p',\epsilon')_{1,N'};\cdots;(p^{(r)},\epsilon^{(r)})_{1,N^{(r)}} \end{array}\right]$$
$$\times\ I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \tag{1}$$
$$\left[\begin{array}{c} z_1\left(\frac{x}{y}\right)^q \\ \cdot \\ \cdot \\ z_r\left(\frac{x}{y}\right)^q \end{array}\ \middle|\ \begin{array}{c} (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots;(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}: \\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots;(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}: \end{array}\right.$$
$$\left.\begin{array}{c} (a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ (b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right]f(y)dy=g(x)\quad(0<x<\infty)$$

The multivariable I-function defined by Prasad [21] is an extension of the multivariable H-function defined by Srivastava and Panda [29, 30]. It is defined in term of multiple Mellin-Barnes type integral:

$$
\begin{aligned}
& I\left(z_1, \cdots, z_r\right)=I_{p_2, q_2, p_3, q_3 ; \cdots ; p_r, q_r: p^{(1)}, q^{(1)} ; \cdots ; p^{(r)}, q^{(r)}}^{0, n_2 ; 0, n_3 ; \cdots ; 0, n_r: m^{(1)}, n^{(1)} ; \cdots ; m^{(r)}, n^{(r)}} \\
& \left[\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right| \begin{array}{l} \left(a_{2j} ; \alpha_{2j}^{\prime}, \alpha_{2j}^{\prime\prime}\right)_{1, p_2} ; \cdots ; \\ \left(b_{2j} ; \beta_{2j}^{\prime}, \beta_{2j}^{\prime\prime}\right)_{1, q_2} ; \cdots ; \end{array} \\
& \left.\begin{array}{l} \left(a_{rj} ; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)}\right)_{1, p_r}:\left(a_j^{(1)}, \alpha_j^{(1)}\right)_{1, p^{(1)}} ; \cdots ;\left(a_j^{(r)}, \alpha_j^{(r)}\right)_{1, p^{(r)}} \\ \left(b_{rj} ; \beta_{rj}^{(1)}, \cdots, \beta_{rj}^{(r)}\right)_{1, q_r}:\left(b_j^{(1)}, \beta_j^{(1)}\right)_{1, q^{(1)}} ; \cdots ;\left(b_j^{(r)}, \beta_j^{(r)}\right)_{1, q^{(r)}} \end{array}\right] \\
& =\frac{1}{(2 \pi \omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(s_1, \cdots, s_r\right) \prod_{i=1}^r \theta_i\left(t_i\right) z_i^{t_i} d t_1 \cdots d t_r,
\end{aligned} \tag{2}
$$

the existence and convergence conditions for the defined integral (2), see Prasad [21].

The condition for absolute convergence of multiple Mellin-Barnes type contour (1) can be obtained by extension of the corresponding conditions for multivariable H-function:

$$|\arg z_i|<\frac{1}{2} \Omega_i \pi$$

where

$$
\begin{aligned}
\Omega_i= & \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}} \beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} \\
& +\left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right)+\cdots+\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) \\
& -\left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)}+\sum_{k=1}^{q_3} \beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right),
\end{aligned} \tag{3}
$$

where $i=1, \cdots, r$. The complex numbers z_i are not zero. We establish the asymptotic expansion in the following convenient form:

$$
\begin{aligned}
& I\left(z_1, \cdots, z_r\right)=O \quad \left(|z_1|^{\alpha_1}, \cdots,|z_r|^{\alpha_r}\right), \max \left(|z_1|, \cdots,|z_r|\right) \rightarrow 0 \\
& I\left(z_1, \cdots, z_r\right)=O \quad \left(|z_1|^{\beta_1}, \cdots,|z_r|^{\beta_r}\right), \min \left(|z_1|, \cdots,|z_r|\right) \rightarrow \infty
\end{aligned}
$$

where $k = 1, \cdots, r$; $\alpha'_k = \min\left[\Re\left(\frac{b_j^{(k)}}{\beta_j^{(k)}}\right)\right], j = 1, \cdots, m^{(k)}$;
and $\beta'_k = \max\left[\Re\left(\frac{a_j^{(k)}-1}{\alpha_j^{(k)}}\right)\right], j = 1, \cdots, n^{(k)}$.
The generalized class of multivariable polynomials defined by Srivastava [28], is given as

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1, \cdots, y_s] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!} A[N_1, K_1; \cdots; N_s, K_s]\, y_1^{K_1} \cdots y_s^{K_s}, \tag{4}$$

where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_s, K_s]$ are arbitrary constants, real or complex.

The generalized polynomials of one variable defined by Srivastava [27], is given in the following manner:

$$S_N^M(x) = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A[N, K]\, x^K, \tag{5}$$

where the coefficients$A[N, K]$ are arbitrary constants real or complex.
The series representation of the multivariable H-function defined by Srivastava and Panda [29, 30] is given by Chaurasia and Olka [9] as

$$H[u_1, \cdots, u_r] = H^{0,\lambda':(\alpha',\beta');\cdots;(\alpha^{(r)},\beta^{(r)})}_{A',C':(M',N');\cdots;(M^{(r)},N^{(r)})} \left[\begin{array}{c|c} u_1 & [(g_j);\gamma',\cdots,\gamma^{(r)}]_{1,A'} : (q^{(1)},\eta^{(1)})_{1,M^{(1)}};\cdots;(q^{(r)},\eta^{(r)})_{1,M^{(r)}} \\ \cdot & \\ \cdot & [(f_j);\xi',\cdots,\xi^{(r)}]_{1,C'} : (p^{(1)},\epsilon^{(1)})_{1,N^{(1)}};\cdots;(p^{(r)},\epsilon^{(r)})_{1,N^{(r)}} \\ u_r & \end{array} \right] \tag{6}$$
$$= \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n'_i=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i\, u_i^{U_i} (-)^{\sum_{i=1}^r n'_i}}{\prod_{i=1}^r \epsilon_{m_i}^i\, n'_i!},$$

where

$$\phi = \frac{\prod_{j=1}^{\lambda'} \Gamma\left(1 - g_j + \sum_{i=1}^r \gamma_j^{(i)} U_i\right)}{\prod_{j=\lambda'+1}^{A} \Gamma\left(g_j - \sum_{i=1}^r \gamma_j^{(i)} U_i\right) \prod_{j=1}^{C'} \Gamma\left(1 - f_j + \sum_{i=1}^r \xi_j^{(i)} U_i\right)}, \tag{7}$$

and

$$\phi_i = \frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}} \Gamma\left(p_j^{(i)} - \epsilon_j^{(i)} U_i\right) \prod_{j=1}^{\beta^{(i)}} \Gamma\left(1 - q_j^{(i)} + \eta_j^{(i)} U_i\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}} \Gamma\left(1 - p_j^{(i)} + \epsilon_j^{(i)} U_i\right) \prod_{j=\beta^{(i)}+1}^{M^{(i)}} \Gamma\left(q_j^{(i)} - \eta_j^{(i)} U_i\right)}, \tag{8}$$

for $i = 1, \cdots, r$ and $U_i = \frac{p_{m_i}^{(i)} + n_i'}{\epsilon_{m_i}^{(i)}}$, $\quad i = 1, \cdots, r$, which is valid under the following conditions:

$$\epsilon_{m_i}^{(i)} \left[p_j^{(i)} + p_i\right] \neq \epsilon_j^{(i)} \left[p_{m_i} + n_i\right],$$

for $j = m_i,\ m_i = 1, \cdots, \alpha^{(i)};\ \ p_i, n_i' = 0, 1, 2, \cdots ; u_i \neq 0$

$$\Sigma_i = \sum_{j=1}^{A'} \gamma_j^{(i)} - \sum_{j=1}^{C'} \xi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{D^{(i)}} \epsilon_j^{(i)} < 0, \quad \forall i \in \{1, \cdots, r\}. \tag{9}$$

Let $\mathfrak{I}$ denote the space of all functions f which are defined on $\mathbb{R}^+$ and satisfy
(i) $f \in C^\infty(\mathbb{R}^+)$
(ii) $\lim_{x\to\infty}[x^\gamma f^r(x)] = 0$ for all non-negative integersγ and r.
(iii) $f(x) = 0(1)$ as $x \to 0$.
For correspondence to the space of good functions defined on the whole real line $(-\infty, \infty)$.

The Riemann-Liouville fractional integral (of order μ) is defined by

$$\begin{aligned} D^{-\mu}\{f(x)\} &= {}_0D_x^{-\mu}\{f(x)\} \\ &= \frac{1}{\Gamma(\mu)} \int_0^x (x-\omega)^{\mu-1}\ f(\omega)\, d\omega \qquad (\Re(\mu) > 0, f \in \mathfrak{I}), \end{aligned} \tag{10}$$

where $D^\mu\{f(x)\} = \phi(x)$ is understood to mean that ϕ is a locally integrable solution of $f(x) = D^{-\mu}\{\phi(x)\}$, implying that D^μ is the inverse of the fractional operator $D^{-\mu}$.

The Weyl fractional (of order h) is defined by Laurent [14] as following:

$$W^{-h}\{f(x)\} = {}_xD_\infty^{-h}\{f(x)\} = \frac{1}{\Gamma(h)} \int_x^\infty (\xi - x)^{h-1}\ f(\xi) d\xi, \tag{11}$$

where $\Re(h) > 0$ and $f \in \mathfrak{I}$.

2 Solution of the Fredholm integral equation (1)

For convenience, we shall use these following notations:

$$U_r = p_2, q_2; p_3, q_3; \cdots ; p_{r-1}, q_{r-1}; \quad V_r = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}, \tag{12}$$

$$W_r = \left(p^{(1)}, q^{(1)}\right); \cdots ; \left(p^{(r)}, q^{(r)}\right); \quad X_r = \left(m^{(1)}, n^{(1)}\right); \cdots ; \left(m^{(r)}, n^{(r)}\right), \tag{13}$$

$$A = \left(a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}\right)_{1,p_2}; \cdots ; \left(a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots , \alpha_{(r-1)k}^{(r-1)}\right)_{1,p_{r-1}}, \tag{14}$$

$$\mathbb{A} = \left(a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots , \alpha_{rk}^{(r)}\right)_{1,p_r} : \left(a_k^{(1)}, \alpha_k^{(1)}\right)_{1,p^{(1)}}; \cdots ; \left(a_k^{(r)}, \alpha_k^{(r)}\right)_{1,p^{(r)}}, \tag{15}$$

$$B = \left(b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}\right)_{1,q_2}; \cdots ; \left(b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots , \beta_{(r-1)k}^{(r-1)}\right)_{1,q_{r-1}}, \tag{16}$$

$$\mathbb{B} = \left(b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots , \beta_{rk}^{(r)}\right)_{1,q_r} : \left(b_k^{(1)}, \beta_k^{(1)}\right)_{1,q^{(1)}}; \cdots ; \left(b_k^{(r)}, \beta_k^{(r)}\right)_{1,q^{(r)}}, \tag{17}$$

$$a_s = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_s)_{M_s K_s}}{K_s!} A\left[N_1, K_1; \cdots ; N_s, K_s\right], \tag{18}$$

$$U = (\alpha', \beta'); \cdots ; \left(\alpha^{(r)}, \beta^{(r)}\right); \; V = (M', N'); \cdots ; \left(M^{(r)}, N^{(r)}\right), \tag{19}$$

$$C = \left[(g_j); \gamma', \cdots , \gamma^{(r)}\right]_{1,A'} : \left(q^{(1)}, \eta^{(1)}\right)_{1,M^{(1)}}; \cdots ; \left(q^{(r)}, \eta^{(r)}\right)_{1,M^{(r)}}, \tag{20}$$

$$D = \left[(f_j); \xi', \cdots , \xi^{(r)}\right]_{1,C'} : \left(p^{(1)}, \epsilon^{(1)}\right)_{1,N^{(1)}}; \cdots ; \left(p^{(r)}, \epsilon^{(r)}\right)_{1,N^{(r)}}. \tag{21}$$

We have the following formula:

Lemma 1

$$W^{\beta-\alpha}\left[y^{-\alpha} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{y}\right)^t, \cdots , u_s\left(\frac{x}{y}\right)^t\right] H_{A',C';V}^{0,\lambda';U}\left(\begin{array}{c|c} v_1\left(\frac{x}{y}\right)^p & C \\ \vdots & \vdots \\ v_r\left(\frac{x}{y}\right)^p & D \end{array}\right)\right.$$

$$\left.\times\; I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c|c} z_1\left(\frac{x}{y}\right)^q & A; \mathbb{A} \\ \vdots & \vdots \\ z_r\left(\frac{x}{y}\right)^q & B; \mathbb{B} \end{array}\right)\right]$$

$$= y^{-\beta} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^{r} \phi_i v_i^{U_i} (-)^{\sum_{i=1}^{r} n_i'}}{\prod_{i=1}^{r} \epsilon_{m_i}^{i} n_i'!}$$

$$a_s u_1^{K_1} \cdots u_s^{K_s} \left(\frac{x}{y}\right)^{t\sum_{i=1}^{s} K_i + p\sum_{i=1}^{r} U_i} \times I^{V_r;0,n_r+1;X_r}_{U_r:p_r+1,q_r+1;W_r} \tag{22}$$

$$\left(\begin{matrix} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{matrix} \;\middle|\; \begin{matrix} A; (1-\beta - t\sum_{i=1}^{s} K_i - p\sum_{i=1}^{r} U_i; q, \cdots, q), \mathbb{A} \\ B; (1-\alpha - t\sum_{i=1}^{s} K_i - p\sum_{i=1}^{r} U_i; q, \cdots, q), \mathbb{B} \end{matrix} \right),$$

where ϕ_1, ϕ_i *and* a_s *are defined respectively by* (7), (8) *and* (18). *Also, provided that*

(a) $\Re(\alpha) > \Re(\beta)$,

(b) $\Re\left[\beta + p\sum_{i=1}^{r} \min_{1\le j\le M^{(i)}} \left(\frac{p_j^{(i)}}{\epsilon_j^{(i)}}\right) + q\sum_{i=1}^{r} \min_{1\le j\le m^{(i)}} \left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > 0$,

(c) $|\arg z_i| < \frac{1}{2}\Omega_i\pi$, *where* Ω_i *is defined by* (4),

(d) *which is valid under the following conditions:* $\epsilon_{m_i}^{(i)}[p_j^{(i)} + p_i] \neq \epsilon_j^{(i)}[p_{m_i} + n_i]$ *for* $j = m_i, m_i = 1 \cdots, \alpha^{(i)}$; $p_i, n_i' = 0, 1, 2, \cdots; v_i \neq 0$,

$$\Sigma_i = \sum_{j=1}^{A'} \gamma_j^{(i)} - \sum_{j=1}^{C'} \xi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{D^{(i)}} \epsilon_j^{(i)} < 0, \quad \forall i \in \{1, \cdots, r\}$$

Proof. To prove the lemma first we use the definition of Weyl fractional integral given by (11), express the class of multivariable polynomials $S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[.]$ in series with the help of (4), the multivariable H-function in series with the help of (6) and the multivariable I-function defined by Prasad [21] in Mellin-Barnes type contour integral. Now we interchange the order of summation and integrations (which is permissible under the conditions stated), we evaluate the t-integral and reinterpreting the resulting Mellin-Barnes contour integral as terms of the multivariable I-function, we obtain the desired result. □

Theorem 1

$$\int_0^{\infty} y^{-\beta} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^{r} \phi_i v_i^{U_i} (-)^{\sum_{i=1}^{r} n_i'}}{\prod_{i=1}^{r} \epsilon_{m_i}^{i} n_i'!}$$

$$a_s \left(\frac{x}{y}\right)^{t\sum_{i=1}^{s} K_i + p\sum_{i=1}^{r} U_i} u_1^{K_1} \cdots u_s^{K_s} \times I^{V_r;0,n_r+1;X_r}_{U_r:p_r+1,q_r+1;W_r}$$

$$\left(\begin{array}{c} z_1\left(\frac{x}{y}\right)^q \\ \vdots \\ z_r\left(\frac{x}{y}\right)^q \end{array}\middle|\begin{array}{c} A;\left(1-\beta-t\sum_{i=1}^{s}K_i-p\sum_{i=1}^{r}U_i;q,\cdots,q\right),\mathbb{A} \\ B;\left(1-\alpha-t\sum_{i=1}^{s}K_i-p\sum_{i=1}^{r}U_i;q,\cdots,q\right),\mathbb{B} \end{array}\right) f(y)\,dy$$

$$= \int_0^\infty y^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{y}\right)^t,\cdots,u_s\left(\frac{x}{y}\right)^t\right] H_{A',C';V}^{0,\lambda';u}\left(\begin{array}{c} v_1\left(\frac{x}{y}\right)^p \\ \vdots \\ v_r\left(\frac{x}{y}\right)^p \end{array}\middle|\begin{array}{c} C \\ \vdots \\ D \end{array}\right)$$

$$\times\, I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c} z_1\left(\frac{x}{y}\right)^q \\ \vdots \\ z_r\left(\frac{x}{y}\right)^q \end{array}\middle|\begin{array}{c} A,\mathbb{A} \\ \vdots \\ B,\mathbb{B} \end{array}\right) D^{\beta-\alpha}\left[f(x)\right]\,dy, \tag{23}$$

under the same conditions and notations that (22).

Proof. Let E denote the first member of the equation (23). Then using the 1 and applying (11), we have

$$E = \int_0^\infty \frac{f(y)}{\Gamma(\alpha-\beta)}\,(\xi-y)^{\alpha-\beta-1}\,\xi^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{\xi}\right)^t,\cdots,u_s\left(\frac{x}{\xi}\right)^t\right]$$

$$\times\, H\left[v_1\left(\frac{x}{\xi}\right)^p,\cdots,v_r\left(\frac{x}{\xi}\right)^p\right] I\left[z_1\left(\frac{x}{\xi}\right)^q,\cdots,z_r\left(\frac{x}{\xi}\right)^q\right] d\xi\,dy$$

$$= \int_0^\infty \xi^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{\xi}\right)^t,\cdots,u_s\left(\frac{x}{\xi}\right)^t\right] H\left[v_1\left(\frac{x}{\xi}\right)^p,\cdots,v_r\left(\frac{x}{\xi}\right)^p\right]$$

$$\times\, I\left[z_1\left(\frac{x}{\xi}\right)^q,\cdots,z_r\left(\frac{x}{\xi}\right)^q\right]\left\{\int_0^\xi \frac{(\xi-y)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\, f(y)\,dy\right\} d\xi. \tag{24}$$

□

The change the order of integration is assumed to be permissible just as in the proof of the lemma 1. Now by applying to definition (11) and (24), it gives

$$E = \int_0^\infty \xi^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{\xi}\right)^t,\cdots,u_s\left(\frac{x}{\xi}\right)^t\right] H\left[v_1\left(\frac{x}{\xi}\right)^p,\cdots,v_r\left(\frac{x}{\xi}\right)^p\right]$$

$$\times\, I\left[z_1\left(\frac{x}{\xi}\right)^q,\cdots,z_r\left(\frac{x}{\xi}\right)^q\right]\left\{D^{\beta-\alpha}\{f(y)\}\,dy\right\} d\xi. \tag{25}$$

We obtain the desired result where $f \in \mathfrak{I}$ and$x > 0$, under the same conditions that (22).

3 Particular cases

We obtain the similar formula with the class of polynomials of one variable defined by Srivastava [27]. We have

Corollary 1

$$\int_0^\infty y^{-\beta} \sum_{K=0}^{[N/M]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i v_i^{U_i} (-)^{\sum_{i=1}^r n_i'}}{\prod_{i=1}^r \epsilon^i_{m_i} n_i'!} \frac{(-N)_{MK}}{K!}$$

$$A[N,K] \left(\frac{x}{y}\right)^{tK+p\sum_{i=1}^r U_i} u^K \times I^{V_r;0,n_r+1;X_r}_{U_r:p_r+1,q_r+1;W_r}$$

$$\left(\begin{matrix} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{matrix} \left| \begin{matrix} A; \left(1-\beta-tK-p\sum_{i=1}^r U_i; q, \cdots, q\right), \mathbb{A} \\ B; \left(1-\alpha-tK-p\sum_{i=1}^r U_i; q, \cdots, q\right), \mathbb{B} \end{matrix} \right. \right) f(y)\, dy \tag{26}$$

$$= \int_0^\infty y^{-\alpha} S_N^M \left(u \left(\frac{x}{y}\right)^t\right) H^{0,\lambda';U}_{A',C';V} \left(\begin{matrix} v_1 \left(\frac{x}{y}\right)^p \\ \vdots \\ v_r \left(\frac{x}{y}\right)^p \end{matrix} \left| \begin{matrix} C \\ \vdots \\ D \end{matrix} \right. \right)$$

$$I^{V_r;0,n_r;X_r}_{U_r:p_r,q_r;W_r} \left(\begin{matrix} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{matrix} \left| \begin{matrix} A; \mathbb{A} \\ \vdots \\ B; \mathbb{B} \end{matrix} \right. \right) \times D^{\beta-\alpha}\left[f(x)\right] dy,$$

under the same conditions and notations that (22).

If the multivariable I-function reduces to the multivariable H-function defined by Srivastava and Panda [29, 30], we obtain the following result:

Corollary 2

$$\int_0^\infty y^{-\beta} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i v_i^{U_i} (-)^{\sum_{i=1}^r n_i'}}{\prod_{i=1}^r \epsilon^i_{m_i} n_i'!}$$

$$a_s \left(\frac{x}{y}\right)^{t\sum_{i=1}^s K_i + p\sum_{i=1}^r U_i} u_1^{K_1} \cdots u_s^{K_s} \times H^{0,n_r+1;X_r}_{p_r+1,q_r+1;W_r}$$

$$\left(\begin{matrix} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{matrix} \left| \begin{matrix} \left(1-\beta-t\sum_{i=1}^s K_i-p\sum_{i=1}^r U_i; q, \cdots, q\right), \mathbb{A} \\ \left(1-\alpha-t\sum_{i=1}^s K_i-p\sum_{i=1}^r U_i; q, \cdots, q\right), \mathbb{B} \end{matrix} \right. \right) f(y)\, dy$$

$$= \int_0^\infty y^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{y}\right)^t,\cdots,u_s\left(\frac{x}{y}\right)^t\right] H_{A',C';V}^{0,\lambda';U}\left(\begin{array}{c|c} v_1\left(\frac{x}{y}\right)^p & C \\ \vdots & \vdots \\ v_r\left(\frac{x}{y}\right)^p & D \end{array}\right)$$

$$\times\, H_{p_r,q_r;W_r}^{0,n_r;X_r}\left(\begin{array}{c|c} z_1\left(\frac{x}{y}\right)^q & \mathbb{A} \\ \vdots & \vdots \\ z_r\left(\frac{x}{y}\right)^q & \mathbb{B} \end{array}\right) D^{\beta-\alpha}\left[f(x)\right]\, dy, \tag{27}$$

under the same conditions and notations that (22) *with* $U_r = V_r = A = B = 0$.

4 Conclusion

The equation (1) is of general character. By suitably specializing the various parameters of the multivariable I-function, the multivariable H-function and the class of multivariable polynomials, our results can be reduce to a large number of integral equations involving various polynomials or special functions of one and several variables occur in many fields of physics, mechanics and applied mathematics.

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Maia type fixed point theorems for Ćirić-Prešić operators

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Abstract. The main aim of this paper is to obtain Maia type fixed point results for Ćirić-Prešić contraction condition, following Ćirić L. B. and Prešić S. B. result proved in [Ćirić L. B.; Prešić S. B. On Prešić type generalization of the Banach contraction mapping principle, Acta Math. Univ. Comenian. (N.S.), 2007, v 76, no. 2, 143–147] and Luong N. V. and Thuan N. X. result in [Luong, N. V., Thuan, N. X., Some fixed point theorems of Prešić-Ćirić type, Acta Univ. Apulensis Math. Inform., No. 30, (2012), 237–249]. We unified these theorems with Maia's fixed point theorem proved in [Maia, Maria Grazia. Un'osservazione sulle contrazioni metriche. (Italian) Rend. Sem. Mat. Univ. Padova 40 1968 139–143] and the obtained results are proved is the present paper. An example is also provided.

1 Introduction and preliminaries

Prešić S. B. [11] extended the famous Banach contraction principle [2] to the case of product spaces in 1965. Recently, in 2007, Ćirić and Prešić [10], generalized the Prešić's theorem introducing Ćirić-Prešić contraction condition. Other important Prešić fixed point theorem generalizations and some related results can be found in Păcurar's papers [7], [8].

The following result was given by M. G. Maia [4] in 1968 and is also a generalization of Banach contraction mapping principle for sets endowed with two

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comparable metrics. Maia type fixed point results for singlevalued or multi-valued operators have been studied in [9], [12], [13], [14].

Theorem 1 [14], [4]

Let X *be a nonempty set,* d *and* ρ *two metrics on* X *and* $f : X \to X$ *an operator. We suppose that:*

(i) $d(x, y) \le \rho(x, y)$ *for all* $x, y \in X$;

(ii) (X, d) *is a complete metric space;*

(iii) $f : (X, d) \to (X, d)$ *is continuous;*

(iv) $f : (X, \rho) \to (X, \rho)$ *is an* α*-contraction.*

Then:

(a) $F_f = \{x^*\}$;

(b) $f^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty$, *for all* $x \in X$;

(c) $f^n(x) \xrightarrow{\rho} x^*$ *as* $n \to \infty$, *for all* $x \in X$;

(d) $\rho(x, x^*) \le \frac{1}{1-\alpha}\rho(x, f(x))$, *for each* $x \in X$.

In 2007, Ćirić L. B. and Prešić S. B. generalized the Prešić's theorem introducing Ćirić-Prešić contraction condition. Their fixed point result can be stated as follows:

Theorem 2 [10] *Let* (X, d) *be a complete metric space,* k *a positive integer and* $T : X^k \to X$ *a mapping satisfying the following contractive type condition:*

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \le \lambda \max\{d(x_i, x_{i+1}) : 1 \le i \le k\}, \quad (1)$$

where $\lambda \in (0, 1)$ *is constant and* $x_1, \ldots, x_{k+1} \in X$.

Then there exists a point $x^* \in X$ *such that* $T(x^*, \ldots, x^*) = x^*$. *Moreover, if* $x_1, x_2, x_3, \ldots, x_{k+1}$ *are arbitrary points in X and for* $n \in \mathbb{N}$,

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}),$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ *is convergent and*

$$\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n).$$

If, in addition, we suppose that on diagonal $\Delta \subset X^k$,

$$d(T(u, \ldots, u), T(v, \ldots, v)) < d(u, v) \quad (2)$$

holds for all $u, v \in X$, *with* $u \ne v$, *then* x^* *is the unique fixed point of* T *in* X *with* $T(x^*, \ldots, x^*) = x^*$.

Remark 1 [10] *Theorem 2 is a generalization of Prešić fixed point theorem (see [11]), as the Prešić's contraction condition implies the conditions 1 and 2.*

$$\begin{aligned}
&d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \le \\
&\le \alpha_1 d(x_1, x_2) + \alpha_2 d(x_2, x_3) + \cdots + \alpha_k d(x_k, x_{k+1}) \le \\
&\le (\alpha_1 + \alpha_2 + \cdots + \alpha_k) \max\{d(x_1, x_2), d(x_2, x_3), \dots, d(x_k, x_{k+1})\} \le \\
&\le \lambda \max\{d(x_i, x_{i+1}) : 1 \le i \le k\}
\end{aligned}$$

and

$$\begin{aligned}
&d(T(u, u, \dots, u), T(v, v, \dots, v)) \le \\
&\le d(T(u, u, \dots, u), T(u, \dots, u, v)) + d(T(u, \dots, u, v), T(u, \dots, u, v, v)) + \\
&\quad + \cdots + d(T(u, v, \dots, v), T(v, v, \dots, v)) \le \\
&\le \alpha_k d(u, v) + \alpha_{k-1} d(u, v) + \cdots + \alpha_1 d(u, v) = \\
&= (\alpha_k + \alpha_{k-1} + \cdots + \alpha_1) d(u, v) < d(u, v).
\end{aligned}$$

Following the above result, the next lemma is a generalization of the Prešić's lemma in [11].

Lemma 1 *Let $k \in \mathbb{N}, k \neq 0$ and $\lambda \in (0, 1)$. If $\{\Delta_n\}_{n \ge 1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \le \lambda \max\{\Delta_n, \Delta_{n+1}, \dots, \Delta_{n+k-1}\}, n \ge 1, \tag{3}$$

then there exist $L > 0$ and $\theta \in (0, 1)$ such that

$$\Delta_n \le L \cdot \theta^n, \text{ for all } n \ge 1. \tag{4}$$

Proof. Similarly with the proof of the result [10], we have:

Let $\Delta_1, \Delta_2, \dots, \Delta_k$ be k positive elements of the sequence $\{\Delta_n\}_{n \ge 1}$ satisfying (3).

Denoting $L = \max\{\Delta_1, \Delta_2, \dots, \Delta_k\}$, we obtain

$$\Delta_k \le \lambda \max\{\Delta_1, \Delta_2, \dots, \Delta_k\} = \lambda L$$

We assume that (4) holds for $n, n+1, \dots, n+k-1$ and we prove that it takes place for $n + k$.

$$\Delta_i \le L\theta^i, \; i = n, n+1, \dots, n+k-1$$

where $\theta = \lambda^{\frac{1}{k}}$, $L = \max\{\frac{\Delta_1}{\theta}, \frac{\Delta_2}{\theta^2}, \dots, \frac{\Delta_k}{\theta^k}\}$.

$$\begin{aligned}\Delta_{n+k} &\le \lambda \max\{\Delta_n, \Delta_{n+1}, \dots, \Delta_{n+k-1}\} \\ &\le \lambda \max\{L\theta^n, L\theta^{n+1}, \dots, L\theta^{n+k-1}\} \\ &= L\lambda \max\{\theta^n, \theta^{n+1}, \dots, \theta^{n+k-1}\}.\end{aligned}$$

As $\theta \in (0,1)$, $\theta^{n+1} < \theta^n$, we have

$$\begin{aligned}&\Delta_{n+k} \le L\lambda\theta^n \ (0 < \theta < 1) \\ &\Delta_{n+k} \le L\theta^{n+k}.\end{aligned}$$

□

Remark 2 [12] *For any operator* $f : X^k \to X$, k *a positive integer, we can define its associate operator* $F : X \to X$ *by*

$$F(x) = f(x, \dots, x), \ x \in X.$$

$x \in X$ *is a fixed point of* $f : X^k \to X$ *if and only if* x *is a fixed point of its associate operator* F.

Remark 3 *Particular cases:*
1. From Maia's fixed point theorem when $d \equiv \rho$, *we get Banach's fixed point theorem.*
2. A Maia type fixed point theorem for Prešić-Kannan operators has been obtained by Balazs M. [1].

Starting from these results, the aim of this paper is to extend Theorem 2 and Theorem 2.2, Theorem 2.5 from [5], to the case of a set endowed with two comparable metrics.

2 The main results

Theorem 3 *Let* X *be a nonempty set,* d *and* ρ *two metrics on* X, k *a positive integer,* $\lambda \in (0,1)$ *a constant and* $f : X^k \to X$ *a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \lambda \max\{\rho(x_i, x_{i+1}) : 1 \le i \le k\} \quad (5)$$

for any $x_1, x_2, \dots, x_{k+1} \in X$.

We suppose that:
(i) $d(x,y) \le \rho(x,y)$ *for all* $x, y \in X$;
(ii) (X, d) *is a complete metric space;*
(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous;*
(iv) *on diagonal* $\Delta \subset X^k$

$$d(f(x,x,\dots,x), f(y,y,\dots,y)) < d(x,y) \tag{6}$$

holds for all $x, y \in X$*, with* $x \neq y$.
Then:
(a) f *has a unique fixed point* x^**,* $F_f = \{x^*\}$*,* $f(x^*, x^*, \dots, x^*) = x^*$;
(b) *the sequence* $\{x_n\}_{n\ge 1}$ *with* $x_1, x_2, \dots, x_k \in X$*, and*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n \ge 1$$

converges to x^* *w.r.t.* d.

Proof. Let $\{x_n\}_{n\ge 1}$, $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n \ge 1$, with $x_1, x_2, \dots, x_k$ arbitrary elements in X.

$$\begin{aligned}\rho(x_{n+k}, x_{n+k+1}) &= \rho(f(x_n, x_{n+1}, \dots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \dots, x_{n+k})) \\ &\le \lambda \max\{\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2}), \dots, \rho(x_{n+k-1}, x_{n+k})\}\end{aligned}$$

Denoting $\rho(x_n, x_{n+1}) = \Delta_n$ we have

$$\Delta_{n+k} \le \lambda \max\{\Delta_n, \Delta_{n+1}, \dots, \Delta_{n+k-1}\},\ n \ge 1$$

The conditions in Lemma 1 are fulfilled and there exist $L > 0$ and $\theta \in (0,1)$ such that

$$\begin{aligned}\Delta_n &\le L\theta^n,\ n \ge 1 \\ \rho(x_{n+k}, x_{n+k+1}) &\le \lambda \max\{L\theta^n, L\theta^{n+1}, \dots, L\theta^{n+k-1}\} \\ &\le \lambda L\theta^n\end{aligned}$$

From [10], $\lambda = \theta^k$, so $\rho(x_{n+k}, x_{n+k+1}) \le L\theta^{n+k}$.
For $n, p \in \mathbb{N}^*$ with $p > n$, we have

$$\begin{aligned}\rho(x_n, x_{n+p}) &= \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \dots + \rho(x_{n+p-1}, x_{n+p}) \le \\ &\le L\theta^n + L\theta^{n+1} + \dots + L\theta^{n+k-1} = \\ &= L\theta^n(1 + \theta + \dots + \theta^{p-1}) = \\ &= L\theta^n \frac{1-\theta^p}{1-\theta},\ n \ge 1,\ p \ge 1.\end{aligned}$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X,ρ). From (i) it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in the complete metric space (X,d), so $\{x_n\}_{n\geq 1}$ is also convergent: there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_{n+k},x^*) = 0$.

By the continuity of f and the associate operator $F : X \to X$, $F(x) = f(x,x,\dots,x)$, for any $x \in X$, we have:

$$\begin{aligned} d(F(x^*),x^*) &= d(f(x^*,\dots,x^*),x^*) = \\ &= d(f(\lim_{n\to\infty} x_n,\dots,\lim_{n\to\infty} x_{n+k-1}),x^*) = \\ &= \lim_{n\to\infty}(d(f(x_n,\dots,x_{n+k-1})),x^*) = \lim_{n\to\infty} d(x_{n+k},x^*) = 0. \end{aligned}$$

Therefore $x^* = f(x^*,\dots,x^*) = F(x^*)$ is a fixed point of f.

We suppose there exists another fixed point of f, $y^* = f(y^*,\dots,y^*)$,

$$d(x^*,y^*) = d(f(x^*,\dots,x^*),f(y^*,\dots,y^*))$$

from (iv) we have

$$d(x^*,y^*) < d(x^*,y^*)$$

which is a contradiction. The uniqueness of the fixed point is proved. □

Remark 4 *We have the following important particular cases of Theorem 3:*
1. If $k = 1$, by Theorem 3 we get Maia fixed point theorem.
2. If $d = \rho$, by Theorem 3 we get Ćirić and Prešić fixed point theorem [10].

Following the results in [3], we extend them to the case of a set endowed with two comparable metrics.

Remark 5 [3]

Let Φ denote all functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying
(i) φ *is continuous and non-decreasing;*
(ii) $\sum_{i=1}^{\infty} \varphi^i(t) < \infty$, *for all* $t \in (0,\infty)$.

Lemma 2 [5] *Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is non-decreasing. Then for every $t > 0$, $\lim_{n\to\infty} \varphi^n(t) = 0$, implies $\varphi(t) < t$.*

Remark 6 [3]

Property (ii) from Remark 5 implies $\lim_{n\to\infty} \varphi^n(t) = 0$ for every $t > 0$. Therefore, by Lemma 2, if $\varphi \in \Phi$ then $\varphi(t) < t$, for every $t > 0$.

Theorem 4 *Let* X *be a nonempty set,* d *and* ρ *two metrics on* X*,* k *a positive integer,* $\varphi \in \Phi$ *and* $f : X^k \to X$ *a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \varphi(\max\{\rho(x_i, x_{i+1}) : 1 \le i \le k\}) \quad (7)$$

for any $x_1, x_2, \dots, x_{k+1} \in X$.

We suppose that:

(i) $d(x, y) \le \rho(x, y)$ *for all* $x, y \in X$;

(ii) (X, d) *is a complete metric space;*

(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous;*

(iv) *on diagonal* $\Delta \subset X^k$

$$d(f(x, x, \dots, x), f(y, y, \dots, y)) < d(x, y) \quad (8)$$

holds for all $x, y \in X$*, with* $x \ne y$.

Then:

(a) f *has a unique fixed point* x^**,* $F_f = \{x^*\}$*,* $f(x^*, x^*, \dots, x^*) = x^*$;

(b) *the sequence* $\{x_n\}_{n\ge1}$ *with* $x_1, x_2, \dots, x_k \in X$*, and*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), \ n \ge 1$$

converges to x^* *w.r.t.* d.

Proof. Let $\{x_n\}_{n\ge1}$, $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1})$, $n \ge 1$, with $x_1, x_2, \dots, x_k \in X$. For simplicity, we set

$$\theta = \max\{\rho(x_1, x_2), \rho(x_2, x_3), \dots, \rho(x_k, x_{k+1})\}.$$

If $x_1 = x_2 = \dots = x_{k+1} = x^*$, then x^* is a fixed point of f, therefore we assume they are not all equal, i.e., $\theta > 0$.

We have

$$\begin{aligned}
\rho(x_{k+1}, x_{k+2}) &= \rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \\
&\le \varphi(\max\{\rho(x_1, x_2), \rho(x_2, x_3), \dots, \rho(x_k, x_{k+1}\}) \le \\
&\le \varphi(\theta) < \theta. \\
\rho(x_{k+2}, x_{k+3}) &= \rho(f(x_2, x_3, \dots, x_{k+1}), f(x_3, x_4, \dots, x_{k+2})) \le \\
&\le \varphi(\max\{\rho(x_2, x_3), \rho(x_3, x_4), \dots, \rho(x_{k+1}, x_{k+2}\}) \le \\
&\le \varphi(\max\{\theta, \varphi(\theta)\}) = \varphi(\theta) < \theta. \\
&\dots \\
\rho(x_2k, x_{2k+1}) &= \rho(f(x_k, x_{k+1}, \dots, x_{2k-1}), f(x_{k+1}, x_{k+2}, \dots, x_{2k})) \le \\
&\le \varphi(\max\{\rho(x_k, x_{k+1}), \rho(x_{k+1}, x_{k+2}), \dots, \rho(x_{2k-1}, x_{2k}\}) \le \\
&\le \varphi(\max\{\theta, \varphi(\theta), \dots, \varphi(\theta)\}) = \varphi(\theta) < \theta.
\end{aligned}$$

$$\begin{aligned}\rho(x_{2k+1}, x_{2k+2}) &= \rho(f(x_{k+1}, x_{k+2}, \ldots, x_{2k}), f(x_{k+2}, x_{k+3}, \ldots, x_{2k+1})) \leq \\ &\leq \varphi(\max\{\rho(x_{k+1}, x_{k+2}), \rho(x_{k+2}, x_{k+3}), \ldots, \rho(x_{2k}, x_{2k+1}\}) \leq \\ &\leq \varphi(\max\{\varphi(\theta), \varphi(\theta), \ldots, \varphi(\theta)\}) = \varphi^2(\theta) < \varphi(\theta).\end{aligned}$$

By induction, we get

$$\rho(x_{nk+1}, x_{nk+2}) \leq \varphi^n(\theta), \ n \geq 1$$

or

$$\rho(x_{n+1}, x_{n+2}) \leq \varphi^{\left[\frac{n}{k}\right]}(\theta), \ n \geq k.$$

By property (ii) from Remark 5, we have

$$\lim_{n\to\infty} \rho(x_{n+1}, x_{n+2}) = 0 \tag{9}$$

For $n, p \in \mathbb{N}$, $n > k$, we have

$$\begin{aligned}\rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+p-1}, x_{n+p}) \leq \\ &\leq \varphi^{\left[\frac{n-1}{k}\right]}(\theta) + \varphi^{\left[\frac{n}{k}\right]}(\theta) + \cdots + \varphi^{\left[\frac{n+p-2}{k}\right]}(\theta)\end{aligned} \tag{10}$$

Denoting $l = \left[\frac{n-1}{k}\right]$ and $m = \left[\frac{n+p-2}{k}\right]$, $l \leq m$.

From inequality 10, we have

$$\begin{aligned}\rho(x_n, x_{n+p}) \leq &\underbrace{\varphi^l(\theta) + \varphi^l(\theta) + \cdots + \varphi^l(\theta)}_{k \text{ times}} + \\ &+ \underbrace{\varphi^{l+1}(\theta) + \varphi^{l+1}(\theta) + \cdots + \varphi^{l+1}(\theta)}_{k \text{ times}} + \\ &+ \cdots + \\ &+ \underbrace{\varphi^m(\theta) + \varphi^m(\theta) + \cdots + \varphi^m(\theta)}_{k \text{ times}}\end{aligned}$$

and that is

$$\rho(x_n, x_{n+p}) \leq k \sum_{i=l}^{m} \varphi^i(\theta). \tag{11}$$

By property (ii)

$$\lim_{l\to\infty} \sum_{i=l}^{m} \varphi^i(\theta) = 0.$$

So we have that $\rho(x_n, x_{n+p}) \to 0$, when $n \to \infty$. The sequence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, ρ). From (i) it follows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in the complete metric space (X, d), so $\{x_n\}_{n\geq 1}$ is also convergent: there exists $x^* \in X$ such that $d(x_{n+k}, x^*) = 0$.

By the continuity of f and the associate operator $F : X \to X$, $F(x) = f(x, x, \dots, x)$, for any $x \in X$, we have

$$\begin{aligned} d(F(x^*), x^*) &= d(f(x^*, \dots, x^*), x^*) = \\ &= d(f(\lim_{n\to\infty} x_n, \dots, \lim_{n\to\infty} x_{n+k-1}), x^*) = \\ &= \lim_{n\to\infty} (d(f(x_n, \dots, x_{n+k-1})), x^*) = \lim_{n\to\infty} d(x_{n+k}, x^*) = 0. \end{aligned}$$

Therefore $x^* = f(x^*, \dots, x^*) = F(x^*)$ is a fixed point of f.

We suppose there exists another fixed point of f, $y^* = f(y^*, \dots, y^*)$, $x^* \neq y^*$,

$$d(x^*, y^*) = d(f(x^*, \dots, x^*), f(y^*, \dots, y^*))$$

from (iv) we have

$$d(x^*, y^*) = d(f(x^*, \dots, x^*), f(y^*, \dots, y^*)) < d(x^*, y^*)$$

which is a contradiction. The uniqueness of the fixed point is proved. □

Remark 7 *We have the following particular cases of Theorem 4:*

1. If $\varphi(t) = \lambda t$, for all $t \in [0, \infty)$ and $\lambda \in (0, 1)$, by Theorem 4 we get Theorem 3.

2. If $d = \rho$, by Theorem 4 we get Theorem 2.2 in [3].

The next theorem is an extension of Theorem 4 to monotone nondecreasing mappings in ordered metric spaces. First we recall some useful notions [3]:

Let $(X, \preceq)$ be a partially ordered set and we consider the following partial order on X^k

for $x, y \in X^k$, $x = (x_1, x_2, \dots, x_k)$, $y = (y_1, y_2, \dots, y_k)$

$$x \sqsubseteq y \Leftrightarrow x_1 \preceq y_1, x_2 \preceq y_2, \dots, x_k \preceq y_k.$$

Definition 1 [3] *Let $(X, \preceq)$ be a partially ordered set and $f : X^k \to X$ a mapping.*

f is said to be monotone non-decreasing if for all $x, y \in X^k$,

$$x \sqsubseteq y \Rightarrow f(x) \preceq f(y),$$

where $x = (x_1, x_2, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_k)$.

Theorem 5 *Let* X *be a nonempty set,* $(X, \preceq)$ *a partially ordered set,* d *and* ρ *two metrics on* X*,* k *a positive integer,* $\varphi \in \Phi$ *and* $f : X^k \to X$ *a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \varphi(\max\{\rho(x_i, x_{i+1}) : 1 \leq i \leq k\}) \quad (12)$$

for any $x_1, x_2, \ldots, x_{k+1} \in X$ *and* $x_1 \preceq x_2 \preceq \cdots \preceq x_{k+1}$.

We suppose that:

(i) $d(x, y) \leq \rho(x, y)$ *for all* $x, y \in X$;

(ii) (X, d) *is a complete metric space;*

(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous*

or

X *has the property: if* $\{x_n\}_{n\geq 1}$ *is a monotone non-decreasing sequence,* $x_n \to x$ *then* $x_n \preceq x$*, for any* $n \geq 1$*;*

(iv) *there exists* k *elements* $x_1, x_2, \ldots, x_k \in X$ *such that*

$$x_1 \preceq x_2 \preceq \cdots \preceq x_k \text{ and } x_k \preceq f(x_1, x_2, \ldots, x_k);$$

(v) *on diagonal* $\Delta \subset X^k$

$$d(f(x, x, \ldots, x), f(y, y, \ldots, y)) < d(x, y) \quad (13)$$

holds for all $x, y \in X$*, with* $x \neq y$.

Then:

(a) f *has a unique fixed point* x^**,* $F_f = \{x^*\}$*,* $f(x^*, x^*, \ldots, x^*) = x^*$;

(b) *the sequence* $\{x_n\}_{n\geq 1}$ *with* $x_1, x_2, \ldots, x_k \in X$*, and*

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}),\ n \geq 1,\ x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \ldots$$

converges to x^* *w.r.t.* d.

Proof. From (iv), if we denote $x_{k+1} = f(x_1, x_2, \ldots, x_k) \succeq x_k$, $x_{k+2} = f(x_2, x_3, \ldots, x_{k+1}) \succeq x_{k+1}$ and so on, we obtain the sequence $\{x_n\}_{n\geq 1}$,

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}),\ n \geq 1,\ x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \ldots$$

The alternative assumption (iii) is usual in fixed point theory in ordered metric spaces. The first paper that first considered this assumption is due to Nieto, Juan J.; Rodríguez-López, Rosana. Existence and uniqueness results for fuzzy differential equations subject to boundary value conditions. Mathematical models in engineering, biology and medicine, 264–273, AIP Conf. Proc., 1124, Amer. Inst. Phys., Melville, NY, 2009.

For the next part of the proof, see the proof of Theorem 4. □

Corollary 1 *Let* X *be a nonempty set,* $(X, \preceq)$ *a partially ordered set,* d *and* ρ *two metrics on* X*,* k *a positive integer,* $\lambda \in (0,1)$ *a constant and* $f : X^k \to X$ *a mapping satisfying the following condition:*

$$\rho(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \lambda \max\{\rho(x_i, x_{i+1}) : 1 \leq i \leq k\} \tag{14}$$

for any $x_1, x_2, \dots, x_{k+1} \in X$ *and* $x_1 \preceq x_2 \preceq \dots \preceq x_{k+1}$.

We suppose that:

(i) $d(x,y) \leq \rho(x,y)$ *for all* $x, y \in X$;

(ii) (X, d) *is a complete metric space;*

(iii) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous*

or

X *has the property: if* $\{x_n\}_{n\geq 1}$ *is a monotone non-decreasing sequence,* $x_n \to x$ *then* $x_n \preceq x$*, for any* $n \geq 1$*;*

(iv) *there exists* k *elements* $x_1, x_2, \dots, x_k \in X$ *such that*

$$x_1 \preceq x_2 \preceq \dots \preceq x_k \text{ and } x_k \preceq f(x_1, x_2, \dots, x_k);$$

(v) *on diagonal* $\Delta \subset X^k$

$$d(f(x, x, \dots, x), f(y, y, \dots, y)) < d(x, y) \tag{15}$$

holds for all $x, y \in X$*, with* $x \neq y$.

Then:

(a) f *has a unique fixed point* x^**,* $F_f = \{x^*\}$*,* $f(x^*, x^*, \dots, x^*) = x^*$*;*

(b) *the sequence* $\{x_n\}_{n\geq 1}$ *with* $x_1, x_2, \dots, x_k \in X$*, and*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), \ n \geq 1, \ x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq \dots$$

converges to x^* *w.r.t.* d.

Remark 8 *We have the following particular cases of Theorem 5:*

1. If $\varphi(t) = \lambda t$*, for all* $t \in [0, \infty)$ *and* $\lambda \in (0,1)$*, by Theorem 5 we get Theorem 3 for ordered metric space, see Corollary 1.*

2. If $d = \rho$*, by Theorem 5 we get Theorem 2.5 in* [3].

The following example, adapted after Example 1 in [10], illustrates the result in this paper.

Example 1 *Let* d *be the euclidean distance and* ρ *be the sum-distance, metrics on* $X = [0,1] \cup [2,3]$*. For* $k = 2$*, let* $f : X^2 \to X$ *be a mapping defined by*

$$f(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{4}; \ (x_1, x_2) \in [0,1] \times [0,1] \\ \frac{x_1+x_2+4}{4}; \ (x_1, x_2) \in [2,3] \times [2,3] \\ \frac{x_1+x_2-2}{4}; \ (x_1, x_2) \in [0,1] \times [2,3] \text{ or } (x, y) \in [2,3] \times [0,1]. \end{cases}$$

satisfying the condition 14.

$$d(f(x_1,x_2),f(x_2,x_3)) = \sqrt{(f(x_1,x_2)-f(x_2,x_3))^2} = |f(x_1,x_2)-f(x_2,x_3)|$$
$$= \rho(f(x_1,x_2),f(x_2,x_3))$$

$f:(X^2,d)\to(X,d)$ *is continuous. Hence, the conditions* $(i)-(iv)$ *from Theorem 3 are satisfied.*

Let $\{x_n\}_{n\geq 1}$*, defined by* $x_{n+2}=f(x_n,x_{n+1})$.

For $n=1$*, we have* $x_3=f(x_1,x_2)$.

Then,

for any $x_1,x_2\in[0,1]$ *we have* $f(x_1,x_2)=x_3\in[0,1]$*, and*

for any $x_1,x_2\in[2,3]$ *we have* $f(x_1,x_2)=x_3\in[2,3]$.

For $x_1,x_2\in[0,1]$ *or* $x_1,x_2\in[2,3]$ *we have*

$$\rho(f(x_1,x_2),f(x_2,x_3)) = |\frac{x_1+x_2}{4}-\frac{x_2+x_3}{4}| = |\frac{x_1-x_2}{4}+\frac{x_2-x_3}{4}| \leq$$
$$\leq |\frac{x_1-x_2}{4}|+|\frac{x_2-x_3}{4}| \leq \frac{1}{4}\cdot\max\{\rho(x_1,x_2),\rho(x_2,x_3)\}.$$

For $(x_1,x_2)\in[0,1]\times[2,3]$ *or* $(x_1,x_2)\in[2,3]\times[0,1]$ *we have* $f(x_1,x_2)=x_3\in[0,1]$.

Therefore,

if $x_2\in[2,3]$*, then*

$$\rho(f(x_1,x_2),f(x_2,x_3)) = |\frac{x_1+x_2}{4}-\frac{x_2+x_3}{4}| \leq \frac{1}{4}\cdot\max\{\rho(x_1,x_2),\rho(x_2,x_3)\}.$$

if $x_2\in[0,1]$*, then*

$$\rho(f(x_1,x_2),f(x_2,x_3)) = |\frac{x_1+x_2-2}{4}-\frac{x_2+x_3}{4}| = |\frac{x_1-x_2}{4}-\frac{1}{2}+\frac{x_2-x_3}{4}| \leq$$
$$\leq |\frac{x_1-x_2}{4}-\frac{1}{2}|+|\frac{x_2-x_3}{4}| < |\frac{x_1-x_2}{4}|+|\frac{x_2-x_3}{4}| \leq$$
$$\leq \frac{1}{4}\cdot\max\{\rho(x_1,x_2),\rho(x_2,x_3)\}$$

So f *is a Ciric-Prešić operator, with* $\lambda=\frac{1}{4}\in(0,1)$.

Since $\lambda=\frac{1}{4}\in(0,1)$*, it follows that* $\{x_n\}_{n\geq 1}$ *is a Cauchy sequence in* (X,ρ). *From* (i) *we have that* $\{x_n\}_{n\geq 1}$ *is a Cauchy sequence in* (X,d)*, which in the complete metric space* (X,d)*, is also convergent. So there exists* $x^*\in[0,1]\cup[2,3]$ *such that*

$$\lim_{n\to\infty} x_n = x^*,\ \ x^* = f(x^*,x^*).$$

$$d(x_3, 0) = d(f(x_1, x_2), f(0,0)) = \sqrt{(f(x_1,x_2) - f(0,0))^2} = |f(x_1,x_2) - f(0,0)| = 0$$

$$d(x_3, 2) = d(f(x_1, x_2), f(2,2)) = \sqrt{(f(x_1,x_2) - f(2,2))^2} = |f(x_1,x_2) - f(2,2)| = 0$$

From the continuity of f *in* (X, d)*, we have*

$$\lim_{n\to\infty} f(x_1, x_2) = f(0,0),$$

and

$$\lim_{n\to\infty} f(x_1, x_2) = f(2,2),$$

so $f(0,0) = 0$ *and* $f(2,2) = 2$*,* $F_f = \{0; 2\}$.

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On generalized Laguerre matrix polynomials

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Abstract. The main object of the present paper is to introduce and study the generalized Laguerre matrix polynomials for a matrix that satisfies an appropriate spectral property. We prove that these matrix polynomials are characterized by the generalized hypergeometric matrix function. An explicit representation, integral expression and some recurrence relations in particular the three terms recurrence relation are obtained here. Moreover, these matrix polynomials appear as solution of a differential equation.

1 Introduction

Laguerre, Hermite, Gegenbauer and Chebyshev matrix polynomials sequences have appeared in connection with the study of matrix differential equations [8, 7, ?, 4]. In [13], the Laguerre and Hermite matrix polynomials were introduced as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. The Laguerre matrix polynomials were introduced and studied in [11, ?, ?, ?]. In [?], it is shown that these matrix polynomials are orthogonal with respect to a non-diagonal

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Sobolev-Laguerre matrix polynomials matrix moment functional. Recently, the numerical inversion of Laplace transforms using Laguerre matrix polynomials has been given in [?]. A generalized form of the Gegenbauer matrix polynomials is presented in [2]. Moreover, two generalizations of the Hermite matrix polynomials have been given in [1, ?].

The main aim of this paper is to consider a new generalization of the Laguerre matrix polynomials. The structure of this paper is the following. After a section introducing the notation and preliminary results, we characterize, in Section 3, the definition of the generalized Laguerre matrix polynomials and an explicit representation and integral expression are given. Finally, Section 4 deals with some recurrence relations in particular the three terms recurrence relation for these matrix polynomials. Furthermore, we prove that the generalized Laguerre matrix polynomials satisfy a matrix differential equation.

2 Preliminaries

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. We say that a matrix A is a positive stable if Re$(\mu)>0$ for every eigenvalue $\mu\in\sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus [5, p. 558], it follows that $f(A)g(A)=g(A)f(A)$. The reciprocal gamma function denoted by $\Gamma^{-1}(z)=1/\Gamma(z)$ is an entire function of the complex variable z. Then, for any matrix A in $\mathbb{C}^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Furthermore, if

$$A+nI \text{ is invertible for every integer } n\geq 0, \tag{1}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and it follows that [6, p. 253]

$$(A)_n=A(A+I)...(A+(n-1)I);\ n\geq 1, \tag{2}$$

with $(A)_0=I$.

For any non-negative integers m and n, from (2), one easily obtains

$$(A)_{n+m}=(A)_n(A+nI)_m, \tag{3}$$

and

$$(A)_{mn}=m^{mn}\prod_{s=1}^{m}\left(\frac{1}{m}(A+(s-1)I)\right)_n. \tag{4}$$

Let P and Q be commuting matrices in $\mathbb{C}^{N\times N}$ such that for all integer $n \geq 0$ one satisfies the condition

$$P + nI, \quad Q + nI, \quad \text{and} \quad P + Q + nI \quad \text{are invertible.} \tag{5}$$

Then by [10, Theorem 2] one gets

$$B(P, Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P + Q), \tag{6}$$

where the gamma matrix function, $\Gamma(A)$, and the beta matrix function, $B(P, Q)$, are defined respectively [9] by

$$\Gamma(A) = \int_0^\infty \exp(-t)t^{A-I}dt, \tag{7}$$

and

$$B(P, Q) = \int_0^1 t^{P-I}(1 - t)^{Q-I}dt. \tag{8}$$

In view of (7), we have [10, p. 206]

$$(A)_n = \Gamma(A + nI)\Gamma^{-1}(A); \quad n \geq 0. \tag{9}$$

If λ is a complex number with $\mathrm{Re}(\lambda) > 0$ and A is a matrix in $\mathbb{C}^{N\times N}$ with $A + nI$ invertible for every integer $n \geq 1$, then the n-th Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ is defined by [8, p. 58]

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k\lambda^k}{k!(n-k)!}(A + I)_n[(A + I)_k]^{-1}x^k, \tag{10}$$

and the generating function of these matrix polynomials is given [8] by

$$G(x, t, \lambda, A) = (1 - t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1 - t}\right) = \sum_{n\geq 0} L_n^{(A,\lambda)}(x)t^n. \tag{11}$$

According to [8], Laguerre matrix polynomials satisfy the three-term recurrence relation

$$\begin{aligned}(n + 1)L_{n+1}^{(A,\lambda)}(x) + \left[\lambda xI - (A + (2n + 1)I)\right]L_n^{(A,\lambda)}(x)&\\ + (A + nI)L_{n-1}^{(A,\lambda)}(x) = \theta; \quad n \geq 0,&\end{aligned} \tag{12}$$

with $L_{-1}^{(A,\lambda)}(x) = \theta$ and $L_0^{(A,\lambda)}(x) = I$ where θ is the zero matrix in $\mathbb{C}^{N\times N}$.

Definition 1 [2] *Let p and q be two non-negative integers. The generalized hypergeometric matrix function is defined in the form:*

$$
\begin{aligned}
&{}_pF_q(A_1,\dots,A_p;B_1\dots,B_q;z)\\
&\qquad = \sum_{n\geq 0}(A_1)_n\dots(A_p)_n[(B_1)_n]^{-1}\dots[(B_q)_n]^{-1}\frac{z^n}{n!},
\end{aligned}
\tag{13}
$$

where A_i and B_j are matrices in $\mathbb{C}^{N\times N}$ such that the matrices $B_j; 1\leq j\leq q$ satisfy the condition (1).

With $p=1$ and $q=0$ in (13), one gets the following relation due to [11, p. 213]

$$
(1-z)^{-A} = \sum_{n\geq 0}\frac{1}{n!}(A)_n z^n, \quad |z|<1. \tag{14}
$$

The following lemma provides results about double matrix series. The proof are analogous to the corresponding for the scalar case c.f [**?**, p. 56] and [**?**, p. 101].

Lemma 1 [2, 3, **?**] *If $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n\geq 0$ and $k\geq 0$, then it follows that:*

$$
\sum_{n\geq 0}\sum_{k\geq 0}A(k,n) = \sum_{n\geq 0}\sum_{k=0}^{n}A(k,n-k), \tag{15}
$$

$$
\sum_{n\geq 0}\sum_{k=0}^{\lfloor n/m\rfloor}A(k,n) = \sum_{n\geq 0}\sum_{k\geq 0}A(k,n+mk), \tag{16}
$$

and

$$
\sum_{n\geq 0}\sum_{k\geq 0}B(k,n) = \sum_{n\geq 0}\sum_{k=0}^{\lfloor n/m\rfloor}B(k,n-mk) \quad ;n>m, \tag{17}
$$

where $\lfloor a\rfloor$ is the standard floor function which maps a real number a to its next smallest integer.

It is obviously desirable, by (2), to have the following:

$$
\begin{aligned}
\frac{1}{(n-mk)!}I &= \frac{(-1)^{mk}}{n!}(-nI)_{mk}\\
&= \frac{(-1)^{mk}}{n!}m^{mk}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}I\right)_k; \quad 0\leq mk\leq n.
\end{aligned}
\tag{18}
$$

3 Definition of generalized Laguerre matrix polynomials

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1) and let λ be a complex number with $\mathrm{Re}(\lambda) > 0$. For a positive integer m, we can define the generalized Laguerre matrix polynomials [**GLMPs**] by

$$F(x,t,\lambda,A) = (1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right) = \sum_{n=0}^{\infty} L_{n,m}^{(A,\lambda)}(x)t^n. \tag{19}$$

By (14) one gets

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\lambda^k}{k!n!} x^{mk}(A+I+mkI)_n t^{n+mk} = \sum_{n=0}^{\infty} L_{n,m}^{(A,\lambda)}(x)t^n,$$

which by using (17) and (3) and equating the coefficients of t^n, yields an explicit representation for the **GLMPs** in the form:

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-1)^k\lambda^k}{k!(n-mk)!}(A+I)_n[(A+I)_{mk}]^{-1}x^{mk}. \tag{20}$$

It should be observed that when $m = n$, the explicit representation (20) becomes

$$L_{n,n}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!} - \lambda x^n I.$$

If $m > n$, then from (20) one gets

$$L_{n,m}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!}.$$

Moreover, it is evident that

$$L_{n,m}^{(A,\lambda)}(0) = \frac{(A+I)_n}{n!} \quad \text{and} \quad L_{n,m}^{(A,\lambda)}(x) = L_{n,m}^{(A,1)}(\lambda^{\frac{1}{m}}x).$$

Note that the expression (20) coincides with (10) for the case $m = 1$.

In view of (4) and (18), we can rewrite the formula (20) in the form

$$L_{n,m}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!} \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-1)^{(m+1)k}\lambda^k}{k!} x^{mk} \prod_{p=1}^{m} \left(\frac{p-n-1}{m}I\right)_k \tag{21}$$

$$\times\left[\prod_{s=1}^{m}\left(\frac{1}{m}(A+sI)\right)_k\right]^{-1}.$$

Therefore, in view of (13), the hypergeometric matrix representation of **GLMPs** is given in the form:

$$L_{n,m}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!}\,{}_mF_m\left(\frac{-n}{m}I,\cdots,\frac{(-n+m-1)}{m}I;\frac{A+I}{m},\cdots,\frac{A+mI}{m};(-1)^{m+1}\lambda x^m\right). \tag{22}$$

We give a generating matrix function of **GLMPs**. This result is contained in the following.

Theorem 1 *Let* A *be a matrix in* $\mathbb{C}^{N\times N}$ *satisfying* (1) *and let* λ *be a complex number with* $\mathrm{Re}(\lambda)>0$. *Then*

$$\sum_{n\geq 0}[(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)t^n = e^t\,{}_0F_m\left(-;\frac{A+I}{m},\cdots,\frac{A+mI}{m};-\lambda\left(\frac{xt}{m}\right)^m\right). \tag{23}$$

Proof. By virtue of (20) and applying (16), we have

$$\sum_{n\geq 0}[(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)t^n = \left(\sum_{n\geq 0}\frac{t^n}{n!}\right)\left(\sum_{k\geq 0}\frac{(-1)^k\lambda^k}{k!}[(A+I)_{mk}]^{-1}x^{mk}t^{mk}\right),$$

which, by using (4) and (13), reduces to (23). □

It is clear that

$$e^t\,{}_0F_m\left(-;\frac{A+I}{m},\cdots,\frac{A+mI}{m};-\lambda\left(\frac{xyt}{m}\right)^m\right) = e^{(1-y)t}e^{ty}\,{}_0F_m\left(-;\frac{A+I}{m},\cdots,\frac{A+mI}{m};-\lambda\left(\frac{xyt}{m}\right)^m\right).$$

Thus, by using (23) and applying (15), it follows that

$$\sum_{n\geq 0}[(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(xy)t^n = \sum_{n\geq 0}\sum_{k=0}^{n}\frac{(1-y)^{n-k}y^k}{(n-k)!}[(A+I)_k]^{-1}L_{k,m}^{(A,\lambda)}(x)t^n.$$

By equating the coefficients of t^n, in the last series, one gets

$$L_{n,m}^{(A,\lambda)}(xy) = (A+I)_n\sum_{k=0}^{n}\frac{(1-y)^{n-k}y^k}{(n-k)!}[(A+I)_k]^{-1}L_{k,m}^{(A,\lambda)}(x).$$

Let B be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1). From (3), (4), (14) and (16) and taking into account (20) we have

$$\begin{aligned}\sum_{n\geq 0}&(B)_n[(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)t^n\\ &=(1-t)^{-B}\sum_{n\geq 0}\frac{(-\lambda)^n}{n!}(B)_{mn}[(A+I)_{mn}]^{-1}\left(\frac{xt}{1-t}\right)^{mn}.\end{aligned}\tag{24}$$

By using (4) and (13), the equation (24) gives the following generating function of **GLMPs**:

$$\begin{aligned}&\sum_{n\geq 0}(B)_n[(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)t^n=(1-t)^{-B}\\ &{}_mF_m\left(\frac{B}{m},\cdots,\frac{B+(m-1)I}{m};\frac{A+I}{m},\cdots,\frac{A+mI}{m};-\lambda\left(\frac{xt}{1-t}\right)^m\right).\end{aligned}\tag{25}$$

Clearly, (25) reduces to (19) when $B=A+I$.

We now proceed to give an integral expression of **GLMPs**. For this purpose, we state the following result.

Theorem 2 *Let A and B be positive stable matrices in $\mathbb{C}^{N\times N}$ such that $AB=BA$. Then*

$$\begin{aligned}L_{n,m}^{(A+B,\lambda)}(x)&=\Gamma(A+B+(n+1)I)\Gamma^{-1}(B)\Gamma^{-1}(A+(n+1)I)\\ &\quad\times\int_0^1 t^A(1-t)^{B-I}L_{n,m}^{(A,\lambda)}(xt)dt.\end{aligned}\tag{26}$$

Proof. According to (8) and (20), we can write

$$\begin{aligned}\Psi&=\int_0^1 t^A(1-t)^{B-I}L_{n,m}^{(A,\lambda)}(xt)dt\\ &=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k}{k!(n-mk)!}(A+I)_n[(A+I)_{mk}]^{-1}x^{mk}B(A+(mk+1)I,B),\end{aligned}\tag{27}$$

and since the summation in the right-hand side of the above equality is finite, then the series and the integral can be permuted. Hence by (6) and (9) it follows that

$$\begin{aligned}\Psi &= \Gamma\big(A+(n+1)I\big)\Gamma(B)\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k x^{mk}}{k!(n-mk)!}\Gamma^{-1}\big(A+B+(mk+1)I\big)\\ &= \Gamma\big(A+(n+1)I\big)\Gamma(B)\Gamma^{-1}\big(A+B+(n+1)I\big)\\ &\qquad \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k x^{mk}}{k!(n-mk)!}(A+B+I)_n[(A+B+I)_{mk}]^{-1}.\end{aligned}\tag{28}$$

From (20), (27) and (28), the expression (26) holds. □

We conclude this section giving an integral form of **GLMPs**.

Theorem 3 *For* **GLMPs** *the following holds*

$$\begin{aligned}\int_0^\infty x^A L_{n,m}^{(A,\lambda)}(x)e^{-x}dx &= \frac{\Gamma(A+(n+1)I)}{n!}\\ &\quad {}_mF_0\left(\frac{-n}{m},\cdots,\frac{-n+m-1}{m};-;(-1)^{m+1}\lambda m^m\right).\end{aligned}\tag{29}$$

Proof. From (7), (9) and (20), it follows that

$$\begin{aligned}\int_0^\infty x^A L_{n,m}^{(A,\lambda)}(x)e^{-x}dx &= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k}{k!(n-mk)!}\\ &\qquad (A+I)_n[(A+I)_{mk}]^{-1}\Gamma(A+(mk+1)I)\\ &= \Gamma(A+(n+1)I)\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k}{k!(n-mk)!}.\end{aligned}$$

Using (18) and taking into account (13) we arrive at (29). □

4 Recurrence relations

In addition to the three terms recurrence relation, some differential recurrence relations of **GLMPs** are obtained here.

Theorem 4 *The generalized Laguerre matrix polynomials satisfy the following relations:*

$$\begin{aligned}\sum_{r=0}^{2m}\binom{2m}{r}(-1)^r(n+1-r)L_{n+1-r,m}^{(A,\lambda)}(x) &= (A+I)\sum_{r=0}^{2m-1}\binom{2m-1}{r}(-1)^r L_{n-r,m}^{(A,\lambda)}(x)\\ -m\lambda x^m\sum_{r=0}^{m}\binom{m}{r}(-1)^r L_{n-r-m+1,m}^{(A,\lambda)}(x) &- m\lambda x^m\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^r L_{n-r-m,m}^{(A,\lambda)}(x),\end{aligned} \tag{30}$$

and

$$\sum_{r=0}^{m}\binom{m}{r}(-1)^r DL_{n-r,m}^{(A,\lambda)}(x) = -\lambda m x^{m-1}L_{n-m,m}^{(A,\lambda)}(x). \tag{31}$$

Proof. Differentiating (19) with respect to t yields

$$\begin{aligned}(1-t)^{2m}\sum_{n\geq 1} nL_{n,m}^{(A,\lambda)}(x)t^{n-1} &= (A+I)(1-t)^{2m-1}\sum_{n\geq 0}L_{n,m}^{(A,\lambda)}(x)t^n\\ -\lambda m x^m t^{m-1}(1-t)^m\sum_{n\geq 0}L_{n,m}^{(A,\lambda)}(x)t^n &- \lambda x^m t^m(1-t)^{m-1}\sum_{n\geq 0}L_{n,m}^{(A,\lambda)}(x)t^n.\end{aligned}$$

With the help of the binomial theorem, it follows that

$$\begin{aligned}&\sum_{n\geq 0}\sum_{r=0}^{2m}\binom{2m}{r}(-1)^r(n+1)L_{n+1,m}^{(A,\lambda)}(x)t^{n+r}\\ &= (A+I)\sum_{n\geq 0}\sum_{r=0}^{2m-1}\binom{2m-1}{r}(-1)^r L_{n,m}^{(A,\lambda)}(x)t^{n+r}\\ &\quad -\lambda x^m\Bigg[m\sum_{n\geq m-1}\sum_{r=0}^{m}\binom{m}{r}(-1)^r L_{n-m+1,m}^{(A,\lambda)}(x)t^{n+r}\\ &\quad +\sum_{n\geq m}\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^r L_{n-m,m}^{(A,\lambda)}(x)t^{n+r}\Bigg].\end{aligned}$$

Hence, by equating the coefficients of t^n, equation (30) holds.

Now, by differentiating (19) with respect to x one gets

$$\sum_{n\geq 0}DL_{n,m}^{(A,\lambda)}(x)t^n = \frac{-\lambda m x^{m-1}t^m}{(1-t)^m}(1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right).$$

Thus, it follows that

$$\sum_{n\geq m}\sum_{r=0}^{m}\binom{m}{r}(-1)^r DL_{n-r,m}^{(A,\lambda)}(x)t^n = -\lambda m x^{m-1}\sum_{n\geq m} L_{n-m,m}^{(A,\lambda)}(x)t^n,$$

which, by equating the coefficients of t^n, gives (31). □

It is worthy to mention that (30) reduces to (12) for $m = 1$. Also, for the case $m = 1$, the expression (31) gives the result for the Laguerre matrix polynomials in the form

$$DL_n^{(A,\lambda)}(x) = DL_{n-1}^{(A,\lambda)}(x) - \lambda L_{n-1}^{(A,\lambda)}(x).$$

Differentiating (19) with respect to x again we obtains

$$\begin{aligned}\sum_{n\geq 0} DL_{n,m}^{(A,\lambda)}(x)t^n &= -\lambda m x^{m-1} t^m (1-t)^{-(A+(m+1)I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right)\\ &= -\lambda m x^{m-1}\sum_{n\geq m} L_{n-m,m}^{(A+mI,\lambda)}(x)t^n.\end{aligned}$$

Hence, by equating the coefficients of t^n, we readily obtain

$$DL_{n,m}^{(A,\lambda)}(x) = -\lambda m x^{m-1} L_{n-m,m}^{(A+mI,\lambda)}(x). \tag{32}$$

It may be noted that the formula (32) reduces to the result of [12, p. 16] for Laguerre matrix polynomials, when $m = 1$, in the form

$$DL_n^{(A,\lambda)}(x) = -\lambda L_{n-1}^{(A+I,\lambda)}(x).$$

Using the fact that

$$(1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right) = (1-t)^m (1-t)^{-(A+(m+1)I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right),$$

and (19), one gets

$$\sum_{n\geq 0} L_{n,m}^{(A,\lambda)}(x)t^n = \sum_{n\geq r}\sum_{r=0}^{m}\binom{m}{r}(-1)^r L_{n-r,m}^{(A+mI,\lambda)}(x)t^n.$$

Hence, we obtain that

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{r=0}^{m}\binom{m}{r}(-1)^r L_{n-r,m}^{(A+mI,\lambda)}(x).$$

Let B be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1) with $AB = BA$. Note that

$$(1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right) = (1-t)^{-(A-B)}(1-t)^{-(B+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right).$$

Using (14), (15) and (19) , it follows that

$$\sum_{n\geq 0} L_{n,m}^{(A,\lambda)}(x)t^n = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(A-B)_k}{k!} L_{n-k,m}^{(B,\lambda)}(x)t^n.$$

Identifying the coefficients of t^n, in the last series, gives

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(A-B)_k}{k!} L_{n-k,m}^{(B,\lambda)}(x). \tag{33}$$

By reversing the order of summation in (33), we obtain that

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(A-B)_{n-k}}{(n-k)!} L_{k,m}^{(B,\lambda)}(x). \tag{34}$$

And finally, we prove the following result.

Theorem 5 *The* **GLMPs** *is a solution of the following differential equation*

$$\begin{aligned}\Bigg[\Theta\prod_{s=1}^{m}\left(\frac{1}{m}(\Theta-1)I+\frac{1}{m}(A+sI)\right)+(-1)^m\lambda m x^m \\ \times\prod_{p=1}^{m}\left(\frac{1}{m}\Theta+\frac{p-n-1}{m}\right)I\Bigg]L_{n,m}^{(A,\lambda)}(x)=\theta,\end{aligned} \tag{35}$$

where $\Theta = x\frac{d}{dx}$.

Proof. It is clear that $\frac{1}{m}\Theta x^{mk} = kx^{mk}$. According to (22) we can write

$$\begin{aligned}W &= {}_mF_m\left(-\frac{n}{m}I,\ldots,-\frac{n-m+1}{m}I;\frac{A+I}{m},\ldots,\frac{A+mI}{m};(-1)^{m+1}\lambda x^m\right)\\ &= \sum_{k=0}^{\lfloor n/m\rfloor}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}\right)_k\left[\prod_{s=1}^{m}(\frac{1}{m}(A+sI)_k\right]^{-1}(-1)^{(m+1)k}\lambda^k\frac{x^{mk}}{k!}.\end{aligned}$$

It follows after replacing k by $k+1$ and using (3) that

$$\begin{aligned}
&\frac{1}{m}\Theta\prod_{s=1}^{m}\left(\frac{1}{m}(\Theta-1)I+\frac{1}{m}(A+sI)\right)W\\
&=\sum_{k=0}^{\lfloor n/m\rfloor}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}\right)_{k+1}\left[\prod_{s=1}^{m}\left(\frac{1}{m}(A+sI)\right)_{k}\right]^{-1}(-1)^{(m+1)(k+1)}\lambda^{k+1}\frac{x^{m(k+1)}}{k!}\\
&=(-1)^{m+1}\lambda x^{m}\sum_{k=0}^{\lfloor n/m\rfloor}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}\right)_{k+1}\left[\prod_{s=1}^{m}\left(\frac{1}{m}(A+sI)\right)_{k}\right]^{-1}(-1)^{(m+1)k}\lambda^{k}\frac{x^{mk}}{k!}\\
&=(-1)^{m+1}\lambda x^{m}\prod_{p=1}^{m}\left(\frac{1}{m}\Theta+\frac{p-n-1}{m}\right)W.
\end{aligned}$$

Therefore, W is a solution of the following differential equation

$$\left[\frac{1}{m}\Theta\prod_{s=1}^{m}\left(\frac{1}{m}(\Theta-1)I+\frac{1}{m}(A+sI)\right)+(-1)^{m}\lambda x^{m}\prod_{p=1}^{m}\left(\frac{1}{m}\Theta+\frac{p-n-1}{m}\right)\right]W=\theta.$$

Since $W=n![(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)$, then (35) follows immediately. $\square$

It is worth noticing that taking $m=1$ in (35) gives the following [8]

$$\left[xI\frac{d^2}{dx^2}+(A+(1-\lambda x)I)\frac{d}{dx}+\lambda nI\right]L_{n}^{(A,\lambda)}(x)=\theta.$$

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On a quadratic type functional equation on locally compact abelian groups

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Abstract. Let $(G,+)$ be a locally compact abelian Hausdorff group, $\mathcal{K}$ is a finite automorphism group of G, $\kappa = card\mathcal{K}$ and let μ be a regular compactly supported complex-valued Borel measure on G such that $\mu(G) = \frac{1}{\kappa}$. We find the continuous solutions $f, g : G \to \mathbb{C}$ of the functional equation

$$\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G f(x+k\cdot y+\lambda\cdot s)d\mu(s) = g(y)+\kappa f(x),\ x,y\in G,$$

in terms of k-additive mappings. This equations provides a common generalization of many functional equations (quadratic, Jensen's, Cauchy equations).

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1 Introduction

Throughout this paper, Let $(G,+)$ be a locally compact abelian Hausdroff group, $\mathcal{K}$ be a finite automorphism group of G, $\kappa := card\mathcal{K}$, μ be a regular compactly supported complex-valued Borel measure on G such that $\mu(G) = \int_G d\mu(t) = \frac{1}{\kappa}$, $M_C(G)$ be the space of all regular compactly supported complexe-valued Borel measures on G. By $C(G)$ we mean the algebra of all continuous functions from G into $\mathbb{C}$. All terminology in this paper concerning harmonic analysis according to the monograph in [4].

The following generalization of Cauchy and quadratic functional equation is

$$\sum_{n=0}^{N-1} f(x+w^n y) = Nf(x) + g(y)\ ,x,y \in \mathbb{C}, \tag{1}$$

where $N \in \{2,3,...\}$ and w is a primitive N^{th} root of unity, $f,g : \mathbb{C} \to \mathbb{C}$ are continuous, was solved by Stetkær [7]. Łukasik [6] derived an explicit formula solutions of the following functional equation

$$\sum_{k\in\mathcal{K}} f(x+k\cdot y) = \kappa f(x) + h(y)\ ,x,y \in G, \tag{2}$$

where $\kappa := card\mathcal{K}$. such that $g(y) - g(0) = h(y)$.

The purpose of this paper is to derive an explicit solutions of the following integral-functional equation

$$\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}} \int_G f(x+k\cdot y+\lambda\cdot s)d\mu(s) = g(y) + \kappa f(x)\ ,x,y \in G. \tag{3}$$

This equation is a generalization of (1) and (2). In fact, Eq. (2) results from (3) by taking $\mu = \frac{1}{\kappa}\delta_a$, where δ_a denotes the Dirac measure concentrated at a. Furthermore, using our main result Theorem 2, we find the solutions of the following functional equations:

A/

$$f(x+y+a) + f(x+\sigma(y)+a) = 2f(x) + g(y)\ ,x,y \in G,$$

when we take $\mu = \frac{1}{2}\delta_a, \mathcal{K} = \{I,\sigma\}$ where σ is an automorphism of the abelian group G such that $\sigma^2 = id_G, f,g : G \to \mathbb{C}$ where investigated by Stetkær [7].

B/

$$\sum_{k=1}^{n} f(x+y+a_k) = nf(x) + ng(y)\ ,x,y \in G,$$

when we take $\mu = \frac{\sum_{k=1}^{n} \delta_{a_k}}{n}, \mathcal{K} = \{I\}$ and $a_1, \dots, a_n \in G$.
C/

$$f(x+y+a) = f(x) + g(y)\ , x, y \in G$$

when we take $\mu = \delta_a, \mathcal{K} = \{I\}$.
Furthermore, we find the continuous solutions of some functional equations by using our main result.

2 Notations and preliminary results

In this section, we need to introduce some notions and notations which we will need in the sequel.

A function $A : G \to \mathbb{C}$ is said to be additive provided if $A(x+y) = A(x) + A(y)$ for all $x, y \in G$. In this case, it is easily seen that $A(rx) = rA(x)$ for all $x \in G$ and all $r \in \mathbb{Z}$.

Let $k \in \mathbb{N}$ and $A_k : G^k \to \mathbb{C}$ be a function, then we say that A_k is k-additive if it is additive in each variable. In addition, we say that A is symmetric if

$$A(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}) = A(x_1, x_2, \dots, x_k)$$

whenever $x_1, x_2, \dots, x_k \in G$ and σ is a permutation of $\{1, 2, \dots, k\}$.

Let $A_k : G^k \to \mathbb{C}$ be symmetric and k-additive and let $A_k(x) = A(x, x, \dots, x)$ for $x \in G$ and note that $A_k(rx) = r^k A_k(x)$ whenever $x \in G$ and $r \in \mathbb{Z}$.

In this way a function $A_k : G^k \longrightarrow \mathbb{C}$ which satisfies for all $r \in \mathbb{Z}$ and $x \in G$, $A_k(rx) = r^k A_k$ will be called a $\mathbb{Z}$-homogeneous form of degree k, (assuming $A_k \neq 0$).

A function $p : G \to \mathbb{C}$ is called a generalized polynomial of degree at most $N \in \mathbb{N}$ if there exist $a_0 \in G$ and $\mathbb{Z}$-homogeneous forms $A_k : G \to \mathbb{C}$ (for $1 \leq k \leq N$) of degree k, such that

$$p(x) = a_0 + \sum_{k=1}^{N} A_k(x), \quad x \in G.$$

The following theorem were proved by Łukasik in [6].

Theorem 1 *Let $(G, +)$ be an abelian topological group. The functions $f, h \in G \to \mathbb{C}$ satisfy the functional equation (2) if and only if there exist symmetric $k-$additive mappings $A_k : G^k \to \mathbb{C}$, $k \in \{1, ..., \kappa\}$ and $A_0, B_0 \in \mathbb{C}$ such that*

$$f(x) = A_0 + A_1(x) + \ldots + A_\kappa(x, \dots, x),\ x \in G,$$

$$h(x) = B_0 + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_\kappa(\lambda \cdot x, \ldots, \lambda \cdot x), \; x \in G,$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \; x, y \in G, 1 \leq i \leq j-1, 2 \leq j \leq \kappa.$$

3 Main results

In this section, we describe the solutions of Eq.(3).

Theorem 2 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group and let* $\mu \in M_C(G)$ *such that* $\mu(G) = \frac{1}{\kappa}$. *The functions* $f, g \in C(G)$ *satisfy the functional equation (3) if and only if there exist symmetric* $k-$*additive mappings* $A_k \in C(G)$, $k \in \{1, ..., \kappa\}$ *such that*

$$f(x) = f(0) + P(x) = f(0) + A_1(x) + \ldots + A_\kappa(x, \ldots, x), x \in G,$$

$$g(x) = g(0) + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_k(\lambda \cdot x, \ldots, \lambda \cdot x), x \in G,$$

where

$$g(0) = \kappa \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s),$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \; x, y \in G, 1 \leq i \leq j-1, 2 \leq j \leq \kappa.$$

Proof. Assume that the functions $f, g \in C(G)$ satisfy the functional equation (3). It is easy to check that if we put $y = 0$ in (3), we get

$$\sum_{k \in \mathcal{K}} \sum_{\lambda \in \mathcal{K}} \int_G f(x + \lambda \cdot s) d\mu(s) = g(0) + \kappa f(x).$$

Therefore,

$$\kappa \sum_{\lambda \in \mathcal{K}} \int_G f(x + \lambda \cdot s) d\mu(s) = g(0) + \kappa f(x), \tag{4}$$

for all $x \in G$. Replacing x by $x + k \cdot y$ in (4), we obtain

$$\sum_{\lambda \in \mathcal{K}} \int_G f(x + k \cdot y + \lambda \cdot s) d\mu(s) = \frac{1}{\kappa} g(0) + f(x + k \cdot y),$$

from which we infer that

$$\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G f(x+k\cdot y+\lambda\cdot s)d\mu(s)=\sum_{k\in\mathcal{K}} f(x+k\cdot y)+g(0). \tag{5}$$

By using (3) and (5), we observe that

$$\sum_{k\in\mathcal{K}} f(x+k\cdot y)=g(y)+\kappa f(x)-g(0). \tag{6}$$

Putting $g(y)-g(0)=h(y)$ in (6), we obtain

$$\sum_{k\in\mathcal{K}} f(x+k\cdot y)=\kappa f(x)+h(y),$$

which means that the functions $f,h\in C(G)$ satisfy the equation (2). According to Theorem (1), there exist symmetric $k-$additive mappings $A_k\in C(G)$, $k\in\{1,....,\kappa\}$ such that

$$\begin{aligned} f(x)&=f(0)+P(x)=f(0)+A_1(x)+\ldots+A_\kappa(x,\ldots,x), x\in G,\\ g(x)&=g(0)+\sum_{\lambda\in\mathcal{K}}A_1(\lambda\cdot x)+\ldots+\sum_{\lambda\in\mathcal{K}}A_k(\lambda\cdot x,\ldots,\lambda\cdot x), x\in G, \end{aligned}$$

We compute the left hand side of (3) to be

$$\begin{aligned} &\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G f(x+k\cdot y+\lambda\cdot s)d\mu(s)\\ &\qquad=\kappa f(0)+\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G P(x+k\cdot y+\lambda\cdot s)d\mu(s)\\ &\qquad=\kappa f(0)+\kappa P(x)+g(y)\\ &\qquad=\kappa f(0)+\kappa P(x)+g(0)+\sum_{\lambda\in\mathcal{K}}P(\lambda\cdot y). \end{aligned}$$

In put $x=y=0$, we get

$$\kappa\sum_{\lambda\in\mathcal{K}}\int_G P(\lambda\cdot s)d\mu(s)=2\kappa P(0)+g(0)$$

Hence

$$g(0)=\kappa\sum_{\lambda\in\mathcal{K}}\int_G P(\lambda\cdot s)d\mu(s)$$

Conversely, assume that there exist symmetric k-additive mappings $A_k \in C(G)$, $k \in \{1, ..., \kappa\}$ such that

$$f(x) = f(0) + P(x) = f(0) + A_1(x) + \ldots + A_\kappa(x, \ldots, x), x \in G,$$

where

$$P(x) = \sum_{k=1}^{\kappa} A_k(x),$$

$$g(x) = g(0) + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_k(\lambda \cdot x, \ldots, \lambda \cdot x), x \in G,$$

where

$$g(0) = \kappa \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s),$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \;\; x, y \in G, 1 \leq i \leq j-1, 2 \leq j \leq \kappa.$$

Then $f, g \in C(G)$ and a small computation shows that

$$\sum_{k \in \mathcal{K}} P(x + k \cdot y) = \kappa P(x) + \sum_{k \in \mathcal{K}} P(k \cdot y), \;\; P(0) = 0. \tag{7}$$

using (7) at the second equality (3) is given as follows

$$\begin{aligned}
&\sum_{k \in \mathcal{K}} \sum_{\lambda \in \mathcal{K}} \int_G f(x + k \cdot y + \lambda \cdot s) d\mu(s) \\
&\quad = \sum_{k \in \mathcal{K}} \int_G \sum_{\lambda \in \mathcal{K}} \left[f(0) + P(x + k \cdot y + \lambda \cdot s) d\mu(s) \right] \\
&\quad = \kappa f(0) + \sum_{k \in \mathcal{K}} P(x + k \cdot y) + \sum_{k \in \mathcal{K}} \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s) \\
&\quad = \kappa f(0) + \kappa P(x) + \sum_{k \in \mathcal{K}} P(k \cdot y) + \kappa \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s) \\
&\quad = \kappa f(x) + g(y).
\end{aligned}$$

Hence, the functions $f, g \in C(G)$ satisfy the functional equation (3). $\square$

4 Applications

Corollary 1 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group. The function* $f \in C(G)$ *satisfies the functional equation*

$$\sum_{k \in \mathcal{K}} f(x + k \cdot y) = \kappa f(x)$$

if and only if there exist symmetric $k-$*additive mappings* $A_k \in C(G^k), k \in \{0, ..., \kappa - 1\}$ *such that*

$$f(x) = f(0) + A_1(x) + \ldots + A_{\kappa-1}(x, \ldots, x), \; x \in G.$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \; x, y \in G, 1 \le i \le j, 2 \le j \le \kappa - 1.$$

Proof. By putting $\mu = \frac{\delta_0}{\kappa}$ and $g = 0$ in Theorem 2 and from [[6], Theorem 5] we get the desired result. $\square$

Corollary 2 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group, and choose an arbitrarily element* $a \in G$. *The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$f(x + y + a) = f(x) + g(y)$$

if and only if there exists a mapping $A \in C(G)$, *such that*

$$f(x) = f(0) + A(x), \; x \in G$$

$$g(x) = g(0) + A(x), \; x \in G.$$

Proof. By similar the method, we put $\mu = \delta_a$, $\mathcal{K} = \{I\}$ in Theorem 2 and by a simple calculation we get $g(0) = P(a)$. Hence we get the desired result. $\square$

Corollary 3 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group, and choose an arbitrarily element* $a \in G$. *The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$f(x + y + a) + f(x + \sigma(y) + a) = 2f(x) + g(y)$$

if and only if there exists a symmetric bi-additive mapping $A_k \in C(G^k), k \in \{1, 2\}$ *such that*

$$f(x) = f(0) + A_1(x) + A_2(x, x), \ x \in G$$

$$g(x) = g(0) + A_1(x) + A_1(\sigma(x)) + A_2(x, x) + A_2(\sigma(x), \sigma(x)), \ x \in G$$

$$A_2(x, y) + A_2(x, \sigma(y)) = 0, \ g(0) = 2P(a).$$

Proof. Using Theorem 2, by a simple calculation we get the desired result by putting $\mu = \frac{\delta_a}{2}$, $\mathcal{K} = \{I, \sigma\}$.

□

Corollary 4 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group,* $\mu \in M_C(G)$ *and choose arbitrarily elements* $a_1, \ldots, a_n \in G$.

The functions $f, g \in C(G)$ *satisfy the functional equation*

$$\sum_{k=1}^{n} f(x + y + a_k) = nf(x) + ng(y)$$

if and only if there exists a symmetric additive mapping $A \in C(G)$ *such that*

$$f(x) = f(0) + A(x), \ x \in G,$$

$$g(x) = g(0) + A(x), \ x \in G,$$

$$g(0) = \frac{1}{n} \sum_{k=1}^{n} P(a_k)$$

Proof. It easy to prove the result by tacking $\mu = \frac{\sum_{k=1}^{n} \delta_{a_k}}{n}$ and $\mathcal{K} = \{I\}$ in Theorem 2. □

Corollary 5 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group, let* $\mu \in M_C(G)$ *and choose arbitrarily element* $a \in G$. *The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$\sum_{k \in \mathcal{K}} f(x + k \cdot y + a) = \kappa f(x) + g(y).$$

if and only if there exists a symmetric k-additive mapping $A \in C(G)$ *such that*

$$f(x) = f(0) + A_1(x) + \ldots + A_\kappa(x, \ldots, x), \ x \in G,$$

$$g(x) = g(0) + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_\kappa(\lambda \cdot x, \ldots, \lambda \cdot x), x \in G$$

$$g(0) = \kappa P(a)$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \ \ x, y \in G, 1 \le i \le j, 2 \le j \le \kappa - 1.$$

Proof. By similar method, we get the result by putting $\mu = \frac{\delta_a}{\kappa}$ in Theorem 2. $\square$

Corollary 6 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group, and let* $\mu \in M_C(G)$. *The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$\int_G f(x + y + t) d\mu(t) = f(x) + g(y)$$

if and only if there exists a symmetric additive mapping $A : G \to \mathbb{C}$ $A \in C(G)$ *such that*

$$f(x) = f(0) + A(x) \ , x \in G,$$

$$g(x) = g(0) + A(x) \ , x \in G,$$

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Fixed point iterations for Prešić-Kannan nonexpansive mappings in product convex metric spaces

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Abstract. We introduce Prešić-Kannan nonexpansive mappings on the product spaces and show that they have a unique fixed point in uniformly convex metric spaces. Moreover, we approximate this fixed point by Mann iterations. Our results are new in the literature and are valid in Hilbert spaces, CAT(0) spaces and Banach spaces simultaneously.

1 Introduction and preliminaries

Let (X, d) be a metric space and $F : X \to X$ be a mapping. Then F is called nonexpansive if

$$d(Fx, Fy) \leq d(x, y) \tag{1}$$

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for all $x, y \in X$.

F is said to be Kannan nonexpansive provided that

$$d(Fx, Fy) \leq \frac{1}{2}\left[d(x, Fx) + d(y, Fy)\right] \tag{2}$$

for all $x, y \in X$.

Nonexpansive mappings are always continuous but Kannan nonexpansive mappings are discontinuous in general, see [10]. Therefore, conditions (1) and (2) are independent, that is, there exists a nonexpansive mapping which is not Kannan nonexpansive and a Kannan nonexpansive which is not nonexpansive. So, we cannot compare both the mappings directly. It is well known that nonexpansive and Kannan nonexpansive mappings on a nonempty, compact and convex subset C of a Banach space X, have a fixed point [13]. For a weakly compact and convex subset C of a Banach space X, the existence of fixed points for nonexpansive and Kannan nonexpansive mappings cannot be obtained. This fact was studied by Alspach[1] who showed that there is a weakly compact and convex subset C of $L^1[0,1]$ such that $F : C \to C$ is a nonexpansive mapping without a fixed point.

In 1973, Kannan [9] proved that if X is a reflexive Banach space and for any convex F-invariant subset H of C – nonempty bounded, closed and convex subset of X – which has more than one point and

$$\sup_{y \in H} \|y - Fy\| < \operatorname{diam}(H),$$

then the Kannan nonexpansive mapping $F : C \to C$ has a fixed point. After that Soardi [24] proved the existence of a fixed point for the Kannan nonexpansive mapping $F : C \to C$ by using the notion of normal structure and obtained a similar result to Kirk's fixed point theorem for Kannan nonexpansive mappings.

For two related but distinct concepts of nonexpansive bivariate mappings, see [3].

A convex structure [25] in a metric space X is a mapping $W : X^2 \times I \to X$ satisfying:

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha) d(u, y)$$

for all $u, x, y \in X$ and $\alpha \in I = [0, 1]$. In general, W is not continuous but in this paper we shall assume that W is continuous. Hadamard manifolds [5] and geodesic spaces [4] are the nonlinear examples of a convex metric space, while Hilbert spaces and Banach spaces are the linear ones.

A convex metric space X is uniformly convex [23] if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r(1-\delta(\varepsilon)) < r$ for all $r > 0$ and $x, y, z \in X$ with $d(z,x) \leq r$, $d(z,y) \leq r$ and $d(x,y) \geq r\varepsilon$.

Uniformly convex Banach spaces and $CAT(0)$ spaces are nice examples of a uniformly convex metric space[11].

A convex metric space X satisfy Property(C) if

$$W(x,y,\lambda) = W(y,x,1-\lambda)$$

and Property(H) if

$$d\left(W(x,y,\lambda), W(x,z,\lambda)\right) \leq (1-\lambda)\, d(y,z)$$

for all $x, y, z \in X$ and $\lambda \in (0,1)$.

It is shown in [19] that Property(C) and (H) together imply continuity of W and Property(C) holds in uniformly convex metric spaces.

Recently, Fukhar et. al [7] has proved the following result:

Theorem 1 *Let C be a nonempty, closed, convex and bounded subset of a complete and uniformly convex metric space X satisfying Property (H). If $F : C \to C$ is a continuous mapping satisfying*

$$d(Fx, Fy) \leq a_1 d(x,y) + a_2 d(Fx, x) + a_3 d(Fy, y) + a_4 d(Fx, y) + a_5 d(Fy, x)$$

for all $x, y \in C$, where $a_i \geq 0$ and $\sum_{i=1}^{5} a_i \leq 1$, then F has a fixed point in C.

An interesting generalization of Banach contraction principle has been obtained by Prešić [20]:

Theorem 2 *Let (X,d) be a complete metric space, k a positive integer, $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k \in \mathbb{R}_+, \sum_{i=1}^{k} \alpha_i = \alpha < 1$ and $f : X^k \to X$ a mapping satisfying*

$$d\left(f(x_0, x_1, ..., x_{k-1}), f(x_1, x_2, ..., x_k)\right) \leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i)$$

for all $x_0, x_1, ..., x_k \in X$. Then f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*, x^*, ..., x^*) = x^*$ and the sequence defined by

$$x_{n+1} = f(x_{n-k+1}, ..., x_n), \quad n = k-1, k, k+1, ...$$

converges to x^* for any $x_0, x_1, ..., x_{k-1} \in X$.

Some generalizations of Theorem 2 have been obtained in [6, 21] (see also [15, 16, 17, 18]).

Recall the definition of a Prešić nonexpansive mapping, first introduced in [2].

A mapping $f : X^k \to X$ is a Prešić nonexpansive if

$$d(f(x_0, x_1, ..., x_{k-1}), f(x_1, x_2, ..., x_k)) \leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i)$$

for all $x_0, x_1, ..., x_k \in X$, where k is a positive integer, and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k \in \mathbb{R}_+, \alpha_1 + \alpha_2 \leq 1$.

Since in the definition of Prešić nonexpansive mapping, the constant $\alpha \leq 1$, therefore this class of mappings includes the class of Prešić contractions appearing in Theorem 2. In the case $k = 1$, the Prešić nonexpansiveness condition reduces to Banach contractive condition if $\alpha < 1$ and to the nonexpansiveness condition if $\alpha = 1$.

A fixed point result about Prešić nonexpansive mappings in a nonlinear domain, namely, $CAT(0)$ spaces has been obtained in [8] in the form of the following result:

Theorem 3 *Let* C *be a bounded, closed and convex subset of a complete* $CAT(0)$ *space* X, k *a positive integer, and let* $f : X^k \to X$ *be a Prešić nonexpansive mapping. Then* f *has a fixed point, that is, there exists* $x^* \in X$ *such that* $f(x^*, x^*, ..., x^*) = x^*$.

We note that for $k = 1$, Theorem 3 becomes Kirk fixed point theorem in [12].

These and similar facts have motivated us to study the generalization of Kannan nonexpansive mappings in product metric spaces and product convex metric spaces.

A mapping $f : X^k \to X$ is a Prešić Kannan nonexpansive if

$$d(f(x_0, x_1, ..., x_{k-1}), f(x_1, x_2, ..., x_k)) \leq \frac{1}{k(k+1)} \sum_{i=0}^{k} d(x_i, f(x_i, x_i, ..., x_i))$$

for all $x_0, x_1, ..., x_k \in X$.

For $k = 1$, it reduces to the classical Kannan nonexpansive mapping.

Example 1 *Let* $I = [0, 1]$ *be the unit interval with the usual Euclidean norm and let* $f : I^2 \to I$ *be given by*

$$f(x,y)=\begin{cases}\frac{1}{6} \text{ if } x<\frac{3}{4}, y\in I\\ \frac{1}{15} \text{ if } x\geq\frac{3}{4}, y\in I.\end{cases}$$

Then f is a Prešić Kannan nonexpansive but not Prešić nonexpansive.

Proof. In this particular case, Prešić kannan nonexpansive condition becomes:

$$|f(x_0,x_1)-f(x_1,x_2)|\leq\frac{1}{6}\left(\begin{array}{c}|x_0-f(x_0,x_0)|+|x_1-f(x_1,x_1)|\\ +|x_2-f(x_2,x_2)|\end{array}\right) \tag{3}$$

for any $x_0,x_1,x_2\in I$.

The way of defining f, we write $I^2=\cup_{i=1}^4 I_i$ where

$$\begin{aligned} I_1 &= \left\{(x,y):0\leq x,y\leq\frac{3}{4}\right\}, I_2=\left\{(x,y):\frac{3}{4}\leq x\leq 1, 0\leq y\leq\frac{3}{4}\right\}\\ I_3 &= \left\{(x,y):\frac{3}{4}\leq x,y\leq 1\right\}, I_4=\left\{(x,y):0\leq x\leq\frac{3}{4},\frac{3}{4}\leq y\leq 1\right\}.\end{aligned}$$

Now we discuss five cases.

Case I: $(x_0,x_1)\in I_1$or $(x_0,x_1)\in I_3$ and $x_2\in I$.Then $f(x_0,x_1)=f(x_1,x_2)$.

Consequently, inequality(3) holds for any x_0,x_1,x_2 in the specified domain of f.

Case II: $(x_0,x_1)\in I_2, x_2<\frac{3}{4}$. Then $f(x_0,x_0)=\frac{1}{15}=f(x_0,x_1), f(x_1,x_1)=f(x_1,x_2)=f(x_2,x_2)=\frac{1}{6}$.

Thus the condition(3) becomes:

$$\frac{1}{10}\leq\frac{1}{6}\left(\left|x_0-\frac{1}{15}\right|+\left|x_1-\frac{1}{6}\right|+\left|x_2-\frac{1}{6}\right|\right),$$

but $\frac{3}{4}\leq x_0\leq 1$ implies $\frac{41}{60}\leq x_0-\frac{1}{15}\leq\frac{14}{15}$ implies $\left|x_0-\frac{1}{15}\right|\geq\frac{41}{60}; 0\leq x_1\leq\frac{3}{4}$ implies $-\frac{1}{6}\leq x_1-\frac{1}{6}\leq\frac{7}{12}$ implies $\left|x_1-\frac{1}{6}\right|\geq\frac{1}{6}; 0\leq x_2<\frac{3}{4}$ implies $-\frac{1}{6}\leq x_1-\frac{1}{6}<\frac{7}{12}$ implies $\left|x_1-\frac{1}{6}\right|\geq\frac{1}{6}$.

It follows that $\frac{1}{10}\leq\frac{1}{6}\left(\frac{41}{60}+\frac{2}{3}\right)\leq\frac{1}{6}\left(\left|x_0-\frac{1}{15}\right|+\left|x_1-\frac{1}{6}\right|+\left|x_2-\frac{1}{6}\right|\right)$ holds.

Case III: $(x_0,x_1)\in I_2, x_2\geq\frac{3}{4}$. Then $f(x_0,x_0)=f(x_0,x_1)=f(x_2,x_2)=\frac{1}{15}, f(x_1,x_1)=f(x_1,x_2)=\frac{1}{6}$.

Thus the condition(3) becomes:

$$\frac{1}{10}\leq\frac{1}{6}\left(\left|x_0-\frac{1}{15}\right|+\left|x_1-\frac{1}{6}\right|+\left|x_2-\frac{1}{15}\right|\right).$$

But $\left|x_0-\frac{1}{15}\right|\geq\frac{41}{60}; \left|x_1-\frac{1}{6}\right|\geq\frac{1}{6}; \frac{3}{4}\leq x_2\leq 1$ implies $\frac{41}{60}\leq x_1-\frac{1}{15}<\frac{14}{15}$ implies $\left|x_1-\frac{1}{15}\right|\geq\frac{41}{60}$.

It follows that $\frac{1}{10} \le \frac{1}{6}\left(\frac{41}{30} + \frac{1}{6}\right) \le \frac{1}{6}\left(\left|x_0 - \frac{1}{15}\right| + \left|x_1 - \frac{1}{6}\right| + \left|x_2 - \frac{1}{6}\right|\right)$ holds.

Case IV: $(x_0, x_1) \in I_4, x_2 < \frac{3}{4}$ and Case V: $(x_0, x_1) \in I_4, x_2 \ge \frac{3}{4}$ follow similarly.

Now we show that f is not Prešić nonexpansive. For the above defined f, the Prešić nonexpansive condition becomes:

$$|f(x_0, x_1) - f(x_1, x_2)| \le \alpha_1 |x_0 - x_1| + \alpha_1 |x_1 - x_2| \tag{4}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}_+, \alpha_1 + \alpha_2 \le 1$.

Set $x_0 = \frac{3}{4}, x_1 = \frac{7}{10} = x_2$, $f(x_0, x_1) = \frac{1}{15}, f(x_1, x_2) = \frac{1}{6}$ in (4), we get

$$\frac{1}{10} \le \alpha_1 \frac{1}{20} \Longleftrightarrow 2 \le \alpha_1 \le 1,$$

a contradiction. □

2 Main results

Kannan [9] proved the following result:

Theorem 4 *Let* X *be a compact metric space and let* $F : X \to X$ *be a continuous Kannan nonexpansive mapping satisfying the property that for every closed subset* C *of* X *which contains more than one point in* C *and* F *maps* C *into itself. If there exists* $x \in C$ *such that* $d(x, F(x)) < \sup_{y \in C} d(y, F(y))$, *then* F *has a unique fixed point in* X.

We extend Theorem 4 for Prešić-Kannan nonexpansive mappings on the product of compact metric spaces.

Theorem 5 *Let* (X, d) *be a compact metric space and let* $f : X^k \to X$ *be a continuous Prešić Kannan nonexpansive mapping satisfying the property that for every closed subset* C *of* X *which contains more than one point in* C *and* $f\left(C^k\right) \subseteq C$, *there exists* $x \in C$ *such that* $d(x, f(x, x, ..., x)) < \sup_{y \in C} d(y, f(y, y, ..., y))$. *Then* f *has a unique fixed point* x^* *in* X, *that is,* $f(x^*, x^*, ..., x^*) = x^*$.

Proof. Define $F : C \to C$ by

$$F(z) = f(z, z, ..., z), \quad z \in C.$$

For any $x, y \in C$, we have

$$\begin{aligned}
& d(F(x), F(y)) = d(f(x, x, ..., x), f(y, y, ..., y)) \\
& \quad \leq d(f(x, x, ..., x), f(x, ..., x, y)) + d(f(x, ..., x, y), f(x, ..., x, y, y)) \\
& \quad + ... + d(f(x, y, ..., y), f(y, y, ..., y)) \\
& \quad \leq \frac{1}{k(k+1)} \left[\begin{array}{c} \underbrace{d(x, f(x, x, ..., x)) + ... + d(x, f(x, x, ..., x))}\text{(k times)} \\ +d(y, f(y, y, ..., y)) \end{array} \right] \\
& \quad + \frac{1}{k(k+1)} \left[\begin{array}{c} \underbrace{d(x, f(x, x, ..., x)) + ... + d(x, f(x, x, ..., x))}\text{(k-1 times)} \\ +\underbrace{d(y, f(y, y, ..., y)) + d(y, f(y, y, ..., y))}\text{(2 times)} \end{array} \right] \\
& \quad + ... \\
& \quad + \frac{1}{k(k+1)} \left[\begin{array}{c} d(x, f(x, x, ..., x)) \\ +\underbrace{d(y, f(y, y, ..., y)) + ... + d(y, f(y, y, ..., y))}\text{(k-1 times)} \end{array} \right] \\
& \quad = \frac{1}{k(k+1)} \left[\begin{array}{c} d(x, F(x))(k + ... + 2 + 1) \\ +d(y, F(y))(1 + 2 + ... + k) \end{array} \right] \\
& \quad = \frac{1}{k(k+1)} . \frac{k(k+1)}{2} [d(x, F(x)) + d(y, F(y))] \\
& \quad = \frac{1}{2} [d(x, F(x)) + d(y, F(y))] .
\end{aligned}$$

This shows that F is a Kannan nonexpansive mapping and the conclusion follows immediately from Theorem 4. □

Theorem 6 *If, in addition to the hypotheses of Theorem 5, we have*

$$d(f(x_1, x_2, ..., x_k), x^*) \leq \sum_{i=1}^{k} \alpha_i d(x_i, x^*)$$

for all $x_1, x_2, ..., x_k \in X$ *and* $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k \in \mathbb{R}_+$ *with* $x_i \neq x^*$ *(a unique fixed point of* f*) and* $\sum_{i=1}^{k} \alpha_i < 1$*. Then, for every* $x \in X$, $f^n(x, x, ..., x) \to x^*$.

Proof. Define $F : X \to X$ by

$$F(z) = f(z, z, ..., z), \quad z \in C.$$

For any $x \in X$ such that $x \neq x^* = F(x^*)$, we have

$$\begin{aligned} d(F(x), x^*) &= d(f(x, x, ..., x), x^*) \\ &\leq d(f(x, x, ..., x), f(x, ..., x, x^*)) \\ &\quad + d(f(x, ..., x, x^*), f(x, ..., x, x^*, x^*)) \\ &\quad + ... + d(f(x, x^*, ..., x^*), x^*) \\ &\leq \alpha_k d(x, x^*) + \alpha_{k-1} d(x, x^*) + ... + \alpha_1 d(x, x^*) \\ &= \sum_{i=1}^{k} \alpha_i d(x, x^*) \\ &= \alpha d(x, x^*) \\ &< d(x, x^*). \end{aligned}$$

That is,

$$d(F(x), x^*) < d(x, x^*). \tag{5}$$

Since X is compact, there exists a subsequence $\{F^{n_i}(x)\}$ of $\{F^n(x)\}$ such that $F^{n_i}(x) \to z \in X$.

Note that

$$\begin{aligned} d(F^n(x), x^*) &= d(F^n(x), F(x^*)) \\ &\leq \frac{1}{2}\left\{d\left(F^{n-1}(x), F^n(x)\right) + d(x^*, F(x^*))\right\} \\ &= \frac{1}{2} d\left(F^{n-1}(x), F^n(x)\right) \\ &\leq \frac{1}{2}\left\{d\left(F^{n-1}(x), x^*\right) + d(F^n(x), x^*)\right\}. \end{aligned}$$

That is,

$$d(F^n(x), x^*) \leq d\left(F^{n-1}(x), x^*\right).$$

Thus $\{d(F^n(x), x^*)\}$ is a nonincreasing sequence and is therefore convergent. Since $F^{n_i}(x) \to z$, we have $\{d(F^{n_i}(x), x^*)\} \to d(z, x^*)$ and

$$d\left(F^{n_i+1}(x), x^*\right) \leq d\left(F^{n_i+1}(x), F(z)\right) + d(F(z), x^*).$$

We claim that $z = x^*$. If not, then by the continuity of F and $F^{n_i}(x) \to z$, we have

$$\begin{aligned} d(z, x^*) &= \lim_{i\to\infty} d(F^{n_i}(x), x^*) = \lim_{n\to\infty} d(F^n(x), x^*) \\ &= \lim_{i\to\infty} d\left(F^{n_i+1}(x), x^*\right) \leq d(F(z), x^*), \end{aligned}$$

a contradiction to (5).

Hence

$$\lim_{n\to\infty} d\left(F^{n}\left(x\right),x^{*}\right)=\lim_{i\to\infty} d\left(F^{n_i}\left(x\right),x^{*}\right)=\lim_{i\to\infty} d\left(F^{n_i}\left(x\right),z\right)=0.$$

This completes the proof. $\square$

Next we give our convergence result for a Prešić-Kannan nonexpansive mapping on the product of convex metric spaces.

Theorem 7 *Let* X *be a convex metric space with continuous convex structure* W *and* C *be a nonempty bounded closed and convex subset of* X*. Let* $f: C^k \to C$ *be a continuous Prešić Kannan nonexpansive mapping with a fixed point* x^**, that is,* $f\left(x^*, x^*, ..., x^*\right) = x^*$*. Set* $F_\lambda\left(x\right) = W\left(x, f\left(x, x, ..., x\right), \lambda\right)$ *for some* $\lambda \in (0,1)$ *and let* $d\left(F_\lambda\left(x\right), x^*\right) < d\left(x, x^*\right)$ *for* $x \neq x^*$*. Generate* $\{x_n\}$ *by:* $x_1 \in C, x_{n+1} = F_\lambda\left(x_n\right) \supset F_\lambda\left(x_{n_i}\right) = x_{n_i+1} \to z$*. Then* $z = x^*$ *and* $x_n \to x^*$*.*

Proof. Note that x^* is a fixed point of F_λ as $F_\lambda\left(x^*\right) = W\left(x^*, f\left(x^*, x^*, ..., x^*\right), \lambda\right) = W\left(x^*, x^*, \lambda\right) = x^*$.

Define $F: C \to C$ by

$$F\left(z\right) = f\left(z, z, ..., z\right) \quad \text{for all } z \in C.$$

As calculated in Theorem 5, F is a Kannan nonexpansive mapping.

Since

$$\begin{aligned} d\left(F\left(x_n\right), x^*\right) &= d\left(F\left(x_n\right), F\left(x^*\right)\right) \\ &\leq \frac{1}{2}\left\{d\left(x_n, F\left(x_n\right)\right) + d\left(x^*, F\left(x^*\right)\right)\right\} \\ &= \frac{1}{2} d\left(x_n, F\left(x_n\right)\right) \\ &\leq \frac{1}{2}\left\{d\left(x_n, x^*\right) + d\left(F\left(x_n\right), x^*\right)\right\}, \end{aligned}$$

we get

$$d\left(F\left(x_n\right), x^*\right) \leq d\left(x_n, x^*\right).$$

Therefore

$$\begin{aligned} d\left(x_{n+1}, x^*\right) &= d\left(F_\lambda\left(x_n\right), x^*\right) \\ &= d\left(W\left(x_n, F\left(x_n\right), \lambda\right), x^*\right) \\ &\leq \lambda d\left(x_n, x^*\right) + (1-\lambda)\, d\left(F\left(x_n\right), x^*\right) \\ &\leq \lambda d\left(x_n, x^*\right) + (1-\lambda)\, d\left(x_n, x^*\right) \\ &= d\left(x_n, x^*\right). \end{aligned}$$

This gives that $\{d(x_n, x^*)\}$ is a nonincreasing sequence and is therefore convergent.

Also

$$d(x_{n_i+1}, x^*) \leq d(x_{n_i+1}, F_\lambda(z)) + d(F_\lambda(z), x^*).$$

Since F_λ is continuous and $x_{n_i+1} \to z$,

$$\lim_{i\to\infty} d(x_{n_i+1}, x^*) \leq d(F_\lambda(z), x^*).$$

Hence

$$d(z, x^*) = \lim_{i\to\infty} d(x_{n_i+1}, x^*) \leq d(F_\lambda(z), x^*).$$

By the given fact that $d(F_\lambda(x), x^*) < d(x, x^*)$ for $x \neq x^*$, we conclude that $z = x^*$. This completes the proof. □

Next, we approximate fixed point of a Prešić-Kannan nonexpansive mapping by using Mann iterations [14] in the product of uniformly convex metric spaces. The following lemma is crucial to prove our next theorem.

Lemma 1 [22] *Let X be a uniformly convex metric space satisfying Property(H). Then for $\varepsilon > 0$ and $r > 0$, there exists $\alpha(\varepsilon) > 0$ such that*

$$d(W(x, y, c), z) \leq r(1 - 2\min\{c, 1-c\}\alpha(\varepsilon))$$

for all $x, y, z \in X, d(x, z) \leq r, d(y, z) \leq r, d(x, y) \geq r\varepsilon$ and $c \in [0, 1]$.

Theorem 8 *Let C be a nonempty, bounded, closed and convex subset of a complete uniformly convex metric space X satisfying Property(H). If $\{c_n\}$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} c_n(1 - c_n) > 0$ and f is a continuous Prešić Kannan nonexpansive mapping from C^k onto a compact subset of C, then the sequence $\{x_n\}$ defined by:*

$$x_1 \in C, x_{n+1} = W(x_n, f(x_n, x_n, ..., x_n), c_n),$$

converges to a unique fixed point x^* of f.

Proof. As shown in Theorem 5, the mapping $F : C \to C$ defined by

$$F(z) = f(z, z, ..., z) \quad \text{for all } z \in C$$

is Kannan nonexpansive. Further choosing $\alpha_1 = \alpha_4 = \alpha_5 = 0, \alpha_2 = \frac{1}{2} = \alpha_3$ in Theorem 1, we see that F has a unique fixed point $x^* \in C$. As calculated in Theorem 7, T satisfy the inequality

$$d(Fx_n, x^*) \leq d(x_n, x^*).$$

Therefore

$$\begin{aligned} d(x_{n+1}, x^*) &= d\left(W\left(x_n, F\left(x_n\right), c_n\right), x^*\right) \\ &\leq c_n d(x_n, x^*) + c_n d(F\left(x_n\right), x^*) \\ &= c_n d(x_n, x^*) + c_n d(F\left(x_n\right), x^*) \\ &\leq d(x_n, x^*). \end{aligned}$$

This gives that $\{d\left(x_n, x^*\right)\}$ is a nonincreasing sequence and is therefore convergent. Let $\lim_{n\to\infty} d(x_n, x^*) = c$ (say). Since $\{d\left(x_n, Fx_n\right)\}$ is bounded, therefore $\inf_{n\geq 1} d\left(x_n, Fx_n\right)$ exists. We claim that $\inf_{n\geq 1} d\left(x_n, Fx_n\right) = 0$.Assume that $\inf_{n\geq 1} d\left(x_n, Tx_n\right) = \sigma > 0$. Then

$$d\left(x_n, Tx_n\right) = d(x_n, x^*).\frac{\sigma}{d(x_n, x^*)} \geq d(x_n, x^*).\frac{\sigma}{d(x_1, x^*)}.$$

Hence by Lemma 1, there exists $\alpha\left(\frac{\sigma}{d(x_1,x^*)}\right) > 0$ such that

$$\begin{aligned} d(x_{n+1}, x^*) &= d\left(W\left(x_n, T\left(x_n\right), c_n\right), x^*\right) \\ &\leq d(x_n, x^*)\left(1 - 2\min\{c_n, 1 - c_n\}\,\alpha\left(\frac{\sigma}{d(x_1, x^*)}\right)\right). \end{aligned}$$

That is

$$2c_n\left(1 - c_n\right)\alpha\left(\frac{\sigma}{d(x_1, x^*)}\right) \leq d(x_{n+1}, x^*) - d(x_n, x^*). \tag{6}$$

By $\liminf_{n\to\infty}$ on both sides in (6), we get that

$$\alpha\left(\frac{\sigma}{d(x_1, x^*)}\right)\liminf_{n\to\infty} c_n\left(1 - c_n\right) = 0.$$

But $\liminf_{n\to\infty} c_n\left(1 - c_n\right) > 0$ implies that $\alpha\left(\frac{\sigma}{d(x_1,x^*)}\right) = 0$, a contradiction. Therefore $\inf_{n\geq 1} d\left(x_n, Tx_n\right) = 0$. Then there exists a subsequence $\{x_{n_i}\}$ such that $\lim_{i\to\infty} d\left(x_{ni}, F\left(x_{n_i}\right)\right) = 0$. Since F maps C into a compact subset of C, this implies that there exists a subsequence $\left\{x_{n_{i_j}}\right\}$ of $\{x_{n_i}\}$ such that $\left\{F\big(x_{n_{i_j}}\big)\right\}$ converges to a point u in C as $j \to \infty$. Since $\lim_{j\to\infty} d\big(x_{n_{i_j}}, F\big(x_{n_{i_j}}\big)\big) = 0$ we obtain that $\lim_{j\to\infty} x_{n_{i_j}} = u = \lim_{j\to\infty} F\big(x_{n_{i_j}}\big)$. But F is continuous, so $u = F\big(u\big)$. Since

$$d\left(u, x^*\right) = d\left(F\left(u\right), F\left(x^*\right)\right) \leq \frac{1}{2}\{d\left(u, F\left(u\right)\right) + d\left(x^*, F\left(x^*\right)\right)\} = 0,$$

$\lim_{j\to\infty} x_{n_{i_j}} = u, \lim_{n\to\infty} d(x_n, x^*)$ exists, therefore $x_n \to x^*$.

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Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers

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Abstract. In this paper, we introduce and investigate new subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers. Furthermore, we find estimates of first two coefficients of functions in these classes. Also, we determine Fekete-Szegö inequalities for these function classes.

1 Introduction

Let $\mathbb{U} = \{z : |z| < 1\}$ denote the unit disc on the complex plane. The class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

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in the open unit disc $\mathbb{U}$ with normalization $f(0) = f'(0) - 1 = 0$ is denoted by $\mathcal{A}$ and the class $\mathcal{S} \subset \mathcal{A}$ is the class which consists of univalent functions in $\mathbb{U}$.

The Koebe one quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f \in \mathcal{A}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } \ f(f^{-1}(w)) = w\left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{2}$$

One can see a short history and examples of functions in the class Σ in [12]. Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [1, 2, 8, 12, 13, 14]).

An analytic function f is subordinate to an analytic function F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, provided there is an analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F(\omega(z))$. It follows from Schwarz Lemma that

$$f(z) \prec F(z) \quad \Longleftrightarrow f(0) = F(0) \text{ and } \ f(\mathbb{U}) \subset F(\mathbb{U}) \ , z \in \mathbb{U}$$

(for details see [3], [7]). We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Recently,in [11], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition:

Definition 1 *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

It should be observed $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^*$.

Later, Dziok et al. in [4] and [5] defined and introduced the class $\mathcal{KSL}$ and $\mathcal{SLM}_\alpha$ of convex and α−convex functions related to a shell-like curve connected with Fibonacci numbers, respectively. These classes can be given in the following definitions.

Definition 2 *The function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{KSL}$ *of convex shell-like functions if it satisfies the condition that*

$$1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

Definition 3 *The function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{SLM}_\alpha$, $(0 \le \alpha \le 1)$ *if it satisfies the condition that*

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

The class $\mathcal{SLM}_\alpha$ is related to the class $\mathcal{KSL}$ only through the function $\tilde{p}$ and $\mathcal{SLM}_\alpha \neq \mathcal{KSL}$ for all $\alpha \neq 1$. It is easy to see that $\mathcal{KSL} = \mathcal{SLM}_1$.

Besides, let's define the class $\mathcal{SLG}_\gamma$ of so-called gamma-starlike functions related to a shell-like curve connected with Fibonacci numbers as follows.

Definition 4 *The function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{SLG}_\gamma$, $(\gamma \ge 0)$, *if it satisfies the condition that*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$\left(10x - \sqrt{5}\right)y^2 = \left(\sqrt{5} - 2x\right)\left(\sqrt{5}x - 1\right)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n \tau + u_{n-1}.$$

In [10], taking $\tau z = t$, Raina and Sokół showed that

$$\begin{aligned}
\tilde{p}(z) &= \frac{1+\tau^2 z^2}{1 - \tau z - \tau^2 z^2} = \left(t + \frac{1}{t}\right)\frac{t}{1-t-t^2} \\
&= \frac{1}{\sqrt{5}}\left(t + \frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t} - \frac{1}{1-\tau t}\right) \\
&= \left(t + \frac{1}{t}\right)\sum_{n=1}^{\infty}\frac{(1-\tau)^n - \tau^n}{\sqrt{5}} t^n \\
&= \left(t + \frac{1}{t}\right)\sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty}(u_{n-1} + u_{n+1})\tau^n z^n,
\end{aligned} \tag{3}$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \tau = \frac{1-\sqrt{5}}{2} \quad (n = 1, 2, \ldots). \tag{4}$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \cdots$.

And they got

$$\begin{aligned}\tilde{p}(z) &= 1+\sum_{n=1}^{\infty}\tilde{p}_n z^n = 1+(u_0+u_2)\tau z+(u_1+u_3)\tau^2 z^2\\ &\quad +\sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau^n z^n\\ &= 1+\tau z+3\tau^2 z^2+4\tau^3 z^3+7\tau^4 z^4+11\tau^5 z^5+\cdots.\end{aligned} \tag{5}$$

Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0)=1$ and $\mathrm{Re}\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1 [5] *The function* $\tilde{p}(z)=\frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$ *belongs to the class* $\mathcal{P}(\beta)$ *with* $\beta=\sqrt{5}/10 \approx 0.2236$.

Now we give the following lemma which will use in proving.

Lemma 1 [9] *Let* $p \in \mathcal{P}$ *with* $p(z)=1+c_1 z+c_2 z^2+\cdots$, *then*

$$|c_n| \le 2, \qquad \text{for} \qquad n \ge 1. \tag{6}$$

In this present work, we introduce two subclasses of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for these function classes. Also, we give bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$ for each subclass.

2 Bi-univalent function class $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$

In this section, we introduce a new subclass of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class by subordination.

Firstly, let $p(z)=1+p_1 z+p_2 z^2+\cdots$, and $p \prec \tilde{p}$. Then there exists an analytic function u such that $|u(z)|<1$ in $\mathbb{U}$ and $p(z)=\tilde{p}(u(z))$. Therefore, the function

$$h(z)=\frac{1+u(z)}{1-u(z)}=1+c_1 z+c_2 z^2+\ldots \tag{7}$$

is in the class $\mathcal{P}(0)$. It follows that

$$u(z)=\frac{c_1 z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1 c_2+\frac{c_1^3}{4}\right)\frac{z^3}{2}+\cdots \tag{8}$$

and

$$\begin{aligned}\tilde{p}(u(z)) &= 1 + \tilde{p}_1 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\} \\ &\quad + \tilde{p}_2 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^2 \\ &\quad + \tilde{p}_3 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^3 + \cdots \qquad (9) \\ &= 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left\{ \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) \tilde{p}_1 + \frac{c_1^2}{4} \tilde{p}_2 \right\} z^2 \\ &\quad + \left\{ \frac{1}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} c_1 \left(c_2 - \frac{c_1^2}{2} \right) \tilde{p}_2 + \frac{c_1^3}{8} \tilde{p}_3 \right\} z^3 + \cdots .\end{aligned}$$

And similarly, there exists an analytic function ν such that $|\nu(w)| < 1$ in $\mathbb{U}$ and $p(w) = \tilde{p}(\nu(w))$. Therefore, the function

$$k(w) = \frac{1 + \nu(w)}{1 - \nu(w)} = 1 + d_1 w + d_2 w^2 + \ldots \qquad (10)$$

is in the class $\mathcal{P}(0)$. It follows that

$$\nu(w) = \frac{d_1 w}{2} + \left(d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left(d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots \qquad (11)$$

and

$$\begin{aligned}\tilde{p}(\nu(w)) &= 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left\{ \frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) \tilde{p}_1 + \frac{d_1^2}{4} \tilde{p}_2 \right\} w^2 \\ &\quad + \left\{ \frac{1}{2} \left(d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} d_1 \left(d_2 - \frac{d_1^2}{2} \right) \tilde{p}_2 + \frac{d_1^3}{8} \tilde{p}_3 \right\} w^3 + \cdots .\end{aligned} \qquad (12)$$

Definition 5 *For $0 \le \alpha \le 1$, a function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ if the following subordination hold:*

$$\alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) + (1 - \alpha) \left(\frac{z f'(z)}{f(z)} \right) \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \qquad (13)$$

and

$$\alpha \left(1 + \frac{w g''(w)}{g'(w)} \right) + (1 - \alpha) \left(\frac{w g'(w)}{g(w)} \right) \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2} \qquad (14)$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2).

Specializing the parameter $\alpha = 0$ and $\alpha = 1$ we have the following, respectively:

Definition 6 *A function* $f \in \Sigma$ *of the form* (1) *is said to be in the class* $\mathcal{SL}_\Sigma(\tilde{p}(z))$ *if the following subordination hold:*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{15}$$

and

$$\frac{wg'(w)}{g(w)} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \tag{16}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ *where* $z, w \in \mathbb{U}$ *and* g *is given by* (2).

Definition 7 *A function* $f \in \Sigma$ *of the form* (1) *is said to be in the class* $\mathcal{KL}_\Sigma(\tilde{p}(z))$ *if the following subordination hold:*

$$1+\frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{17}$$

and

$$1+\frac{wg''(w)}{g'(w)} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \tag{18}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ *where* $z, w \in \mathbb{U}$ *and* g *is given by* (2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases.

Theorem 2 *Let* f *given by* (1) *be in the class* $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$. *Then*

$$|a_2| \le \frac{|\tau|}{\sqrt{(1+\alpha)^2-(1+\alpha)(2+3\alpha)\tau}} \tag{19}$$

and

$$|a_3| \le \frac{|\tau|\left[(1+\alpha)^2-(3\alpha^2+9\alpha+4)\tau\right]}{2(1+2\alpha)(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]}. \tag{20}$$

Proof. Let $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ and $g = f^{-1}$. Considering (13) and (14), we have

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) = \tilde{p}(u(z)) \tag{21}$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right)=\tilde{p}(v(w)) \tag{22}$$

where $\tau=(1-\sqrt{5})/2\approx -0.618$ where $z,w\in\mathbb{U}$ and g is given by (2). Since

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)=1+(1+\alpha)a_2z+(2(1+2\alpha)a_3 \\ -(1+3\alpha)a_2^2)z^2+\cdots$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right)=1-(1+\alpha)a_2w+((3+5\alpha)a_2^2 \\ -2(1+2\alpha)a_3)w^2+\cdots$$

Thus we have

$$\begin{aligned}&1+(1+\alpha)a_2z+(2(1+2\alpha)a_3-(1+3\alpha)a_2^2)z^2+\Delta\Delta\Delta\\ &=1+\frac{\tilde{p}_1c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right]z^2\\ &\quad+\left[\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right]z^3+\cdots.\end{aligned} \tag{23}$$

and

$$\begin{aligned}&1-(1+\alpha)a_2w+((3+5\alpha)a_2^2-2(1+2\alpha)a_3)w^2+\Delta\Delta\Delta,\\ &=1+\frac{\tilde{p}_1d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right]w^2\\ &+\left[\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right]w^3+\cdots.\end{aligned} \tag{24}$$

It follows from (23) and (24) that

$$(1+\alpha)a_2=\frac{c_1\tau}{2}, \tag{25}$$

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{c_1^2}{4}3\tau^2, \tag{26}$$

and

$$-(1+\alpha)a_2=\frac{d_1\tau}{2}, \tag{27}$$

$$(3+5\alpha)a_2^2-2(1+2\alpha)a_3=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{d_1^2}{4}3\tau^2. \tag{28}$$

From (25) and (27), we have

$$c_1 = -d_1, \tag{29}$$

and

$$2a_2^2 = \frac{(c_1^2 + d_1^2)}{4(1+\alpha)^2}\tau^2. \tag{30}$$

Now, by summing (26) and (28), we obtain

$$2(1+\alpha)a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2. \tag{31}$$

By putting (30) in (31), we have

$$2(1+\alpha)\left[(-2-3\alpha)\tau + (1+\alpha)\right] a_2^2 = \frac{1}{2}(c_2 + d_2)\tau^2. \tag{32}$$

Therefore, using Lemma (1) we obtain

$$|a_2| \le \frac{|\tau|}{\sqrt{(1+\alpha)^2 - (1+\alpha)(2+3\alpha)\tau}}. \tag{33}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (26) and (28). So, we find

$$4(1+2\alpha)a_3 - 4(1+2\alpha)a_2^2 = \frac{1}{2}\left(c_2 - d_2\right)\tau. \tag{34}$$

Hence, we get

$$4(1+2\alpha)|a_3| \le 2|\tau| + 4(1+2\alpha)|a_2|^2. \tag{35}$$

Then, in view of (33), we obtain

$$|a_3| \le \frac{|\tau|\left[(1+\alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau\right]}{2(1+2\alpha)(1+\alpha)\left[(1+\alpha) - (2+3\alpha)\tau\right]}. \tag{36}$$

□

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, respectively.

Corollary 1 *Let* f *given by* (1) *be in the class* $\mathcal{SL}_\Sigma(\tilde{p}(z))$. *Then*

$$|a_2| \le \frac{|\tau|}{\sqrt{1-2\tau}} \tag{37}$$

and

$$|a_3| \le \frac{|\tau|(1-4\tau)}{2(1-2\tau)}. \tag{38}$$

Corollary 2 *Let* f *given by* (1) *be in the class* $\mathcal{KSL}_\Sigma(\tilde{p}(z))$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-10\tau}} \tag{39}$$

and

$$|a_3| \leq \frac{|\tau|(1-4\tau)}{3(2-5\tau)}. \tag{40}$$

3 Bi-univalent function class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$

In this section, we define a new class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ of $\gamma-$ bi-starlike functions associated with Shell-like domain.

Definition 8 For $\gamma \geq 0$, we let a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$, if the following conditions are satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{41}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau z-\tau^2w^2}, \tag{42}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2).

Remark 1 Taking $\gamma = 1$, we get $\mathcal{SLG}_{1,\Sigma}(\tilde{p}(z)) \equiv \mathcal{SL}_\Sigma(\tilde{p}(z))$ the class as given in Definition 5 satisfying the conditions given in (15) and (16).

Remark 2 Taking $\gamma = 0$, we get $\mathcal{SLG}_{0,\Sigma}(\tilde{p}(z)) \equiv \mathcal{KL}_\Sigma(\tilde{p}(z))$the class as given in Definition 6 satisfying the conditions given in (17) and (18).

Theorem 3 Let f given by (1) be in the class $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$. Then

$$|a_2| \leq \frac{\sqrt{2}|\tau|}{\sqrt{2(2-\gamma)^2-(5\gamma^2-21\gamma+20)\tau}}$$

and

$$|a_3| \leq \frac{|\tau|\left[2(2-\gamma)^2-(5\gamma^2-29\gamma+32)\tau\right]}{2(3-2\gamma)\left[2(2-\gamma)^2-(5\gamma^2-21\gamma+20)\tau\right]}.$$

Proof. Let $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ and $g = f^{-1}$ given by (2) Considering (41) and (42), we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}(u(z)) \tag{43}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \prec \tilde{p}(v(w)) \tag{44}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (2). Since,

$$\begin{aligned}\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} &= 1+(2-\gamma)a_2 z \\ +\left(2(3-2\gamma)a_3+\frac{1}{2}[(\gamma-2)^2-3(4-3\gamma)]a_2^2\right)z^2+\cdots &\prec \tilde{p}(u(z))\end{aligned} \tag{45}$$

and

$$\begin{aligned}\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} &= 1-(2-\gamma)a_2 w \\ +\left([8(1-\gamma)+\frac{1}{2}\gamma(\gamma+5)]a_2^2-2(3-2\gamma)a_3\right)w^2+\cdots &\prec \tilde{p}(v(w)).\end{aligned} \tag{46}$$

Equating the coefficients in(45) and (46), with (9) and (12) respectively we get,

$$(2-\gamma)a_2 = \frac{c_1\tau}{2}, \tag{47}$$

$$2(3-2\gamma)a_3+\frac{1}{2}[(\gamma-2)^2-3(4-3\gamma)]a_2^2 = \frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{c_1^2}{4}3\tau^2, \tag{48}$$

and

$$-(2-\gamma)a_2 = \frac{d_1\tau}{2}, \tag{49}$$

$$-2(3-2\gamma)a_3+[8(1-\gamma)+\frac{1}{2}\gamma(\gamma+5)]a_2^2 = \frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{d_1^2}{4}3\tau^2. \tag{50}$$

From (47) and (49), we have

$$a_2 = \frac{c_1\tau}{2(2-\gamma)} = -\frac{d_1\tau}{2(2-\gamma)}, \tag{51}$$

which implies

$$c_1 = -d_1 \tag{52}$$

and

$$a_2^2 = \frac{(c_1^2 + d_1^2)\tau^2}{8(2-\gamma)^2}. \tag{53}$$

Now, by summing (48) and (50), we obtain

$$(\gamma^2 - 3\gamma + 4)a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2. \tag{54}$$

Proceeding similarly as in the earlier proof of Theorem 2, using Lemma (1) we obtain

$$|a_2| \le \frac{\sqrt{2}|\tau|}{\sqrt{2(2-\gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau}}. \tag{55}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (48) and (50). So, we find

$$4(3-2\gamma)\,a_3 - 4(3-2\gamma)\,a_2^2 = \frac{1}{2}\,(c_2 - d_2)\,\tau. \tag{56}$$

Hence, we get

$$4(1+2\gamma)|a_3| \le 2|\tau| + 4(1+2\gamma)|a_2|^2. \tag{57}$$

Then, in view of (55), we obtain

$$|a_3| \le \frac{|\tau|\left[2(2-\gamma)^2 - (5\gamma^2 - 29\gamma + 32)\tau\right]}{2(3-2\gamma)\left[2(2-\gamma)^2 - (5\gamma^2 - 21\gamma + 20)\tau\right]}. \tag{58}$$

□

Remark 3 *By taking $\gamma = 1$ and $\gamma = 0$ in the above theorem, we have the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, as stated in Corollary 1 and Corollary 2 respectively.*

4 Fekete-Szegö inequalities for the function classes $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ and $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$

Fekete and Szegö [6] introduced the generalized functional $|a_3 - \mu a_2^2|$, where μ is some real number. Due to Zaprawa [15], in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$.

Theorem 4 *Let* f *given by (1) be in the class* $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & |\mu - 1| \le \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \\ \frac{|1-\mu|\tau^2}{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}, & |\mu - 1| \ge \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \end{cases}.$$

Proof. From (32) and (34)we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{\tau^2(c_2+d_2)}{4(1+\alpha)\,[(1+\alpha)-(2+3\alpha)\tau]} + \frac{\tau(c_2-d_2)}{8(1+2\alpha)} \\ &= \left(\frac{(1-\mu)\tau^2}{4(1+\alpha)\,[(1+\alpha)-(2+3\alpha)\tau]} + \frac{\tau}{8(1+2\alpha)}\right)c_2 \qquad (59) \\ &\quad + \left(\frac{(1-\mu)\tau^2}{4(1+\alpha)\,[(1+\alpha)-(2+3\alpha)\tau]} - \frac{\tau}{8(1+2\alpha)}\right)d_2. \end{aligned}$$

So we have

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{|\tau|}{8(1+2\alpha)}\right)c_2 + \left(h(\mu) - \frac{|\tau|}{8(1+2\alpha)}\right)d_2 \qquad (60)$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4(1+\alpha)\,[(1+\alpha)-(2+3\alpha)\tau]}. \qquad (61)$$

Then, by taking modulus of (60), we conclude that

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & 0 \le |h(\mu)| \le \frac{|\tau|}{8(1+2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \ge \frac{|\tau|}{8(1+2\alpha)} \end{cases}.$$

Taking $\mu = 1$, we have the following corollary.

Corollary 3 *If* $f \in \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$, *then*

$$|a_3 - a_2^2| \le \frac{|\tau|}{2(1+2\alpha)}. \qquad (62)$$

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the above theorem, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, respectively.

Corollary 4 *Let* f *given by (1) be in the class* $\mathcal{SL}_\Sigma(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2}, & |\mu - 1| \leq \frac{1-2\tau}{2|\tau|} \\ \frac{|1-\mu|\tau^2}{1-2\tau}, & |\mu - 1| \geq \frac{1-2\tau}{2|\tau|} \end{cases}.$$

Corollary 5 *Let* f *given by (1) be in the class* $\mathcal{KSL}_\Sigma(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{6}, & |\mu - 1| \leq \frac{2-5\tau}{3|\tau|} \\ \frac{|1-\mu|\tau^2}{2(2-5\tau)}, & |\mu - 1| \geq \frac{2-5\tau}{3|\tau|} \end{cases}.$$

In the following theorem, we find the Fekete-Szegö functional for $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$.

Theorem 5 *Let* f *given by (1) be in the class* $\mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(3-2\gamma)}, & |\mu - 1| \leq \frac{2(2-\gamma)^2+(-5\gamma^2+21\gamma-20)\tau}{4(3-2\gamma)|\tau|} \\ \frac{2|1-\mu|\tau^2}{2(2-\gamma)^2+[(-5\gamma^2+21\gamma-20)\tau]}, & |\mu - 1| \geq \frac{2(2-\gamma)^2+(-5\gamma^2+21\gamma-20)\tau}{4(3-2\gamma)|\tau|} \end{cases}.$$

Taking $\mu = 1$, we have the following corollary.

Corollary 6 *If* $f \in \mathcal{SLG}_{\gamma,\Sigma}(\tilde{p}(z))$, *then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{2(3-2\gamma)}. \tag{63}$$

By taking $\gamma = 1$ and $\gamma = 0$ in the above theorem, we have the Fekete-Szegö inequality for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, as stated in Corollary 4 and Corollary 5, respectively.

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A note on the paper "Contraction mappings in b-metric spaces" by Czerwik

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Abstract. In this paper we correct an inaccuracy that appears in the proof of Theorem 1. in Czerwik [1].

1 Introduction

It is well known that on certain function spaces the natural "norm" does not satisfy the triangle inequality (for example, on $L_p(\mathbb{R}^n)$ when $p \in (0, 1)$ the functional that is usually called the "p-norm" has this property). In view of this observation, Czerwik in [1] proposed a generalisation of metric spaces by relaxing the triangle inequality in a way that allows the extension of fixed point theory to cover also these badly behaved function spaces. The resulting notion of b-metric spaces created a new direction in which fixed point theory could be developed and Czerwik was the first to generalise Banach's fixed point theorem for this case. Since then many authors contributed to this development and nowadays the field occupies a considerable position in fixed point theory.

The aim of our paper is to correct an inaccuracy that appears in the proof of Czerwik's theorem mentioned above. This occurs in the proof of Theorem 1 in [1] (and also in the essentially same proof of Theorem 12.2 in [2]) at the

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step of proving that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence: the definition of $(x_k)_{k\in\mathbb{N}}$ depends on the choice of ε. However, in our proof we show that this inaccuracy can be corrected, and thus the conclusion of Czerwik's theorem remains true.

2 Main result

In this section we recall the notion of b-metric spaces, the statement of Theorem 1 in [1] and we present our corrected proof.

Definition 1 *We say that* (X, d) is a b-metric space with constant $s \geq 1$ *if* $d\colon X \times X \to [0, \infty)$ *satisfies the following conditions for every* $x, y, z \in X$*:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$*;*

(iii) $d(x, y) \leq s\big(d(x, z) + d(z, y)\big)$.

Theorem 1 *Let* (X, d) *be a complete* b*-metric space with constant* $s \geq 1$ *and suppose that* $T\colon X \to X$ *satisfies*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{1}$$

for all $x, y \in X$*, where* $\varphi\colon [0, \infty) \to [0, \infty)$ *is increasing and*

$$\lim_{n\to\infty} \varphi^n(t) = 0$$

for each $t \geq 0$*. Then* T *has a unique fixed point* $x^* \in X$ *and* $\lim_{n\to\infty} T^n(x) = x^*$ *for each* $x \in X$*.*

Proof. Let $x \in X$ and define $x_k = T^k x$ for every $k \in \mathbb{N}$. Since for every $m, n \in \mathbb{N}$ we can apply mn times (1) to get

$$d(T^n x_{mn}, x_{mn}) \leq \varphi^{mn}(d(x_n, x_0)),$$

it follows that

$$\lim_{m\to\infty} d(T^n x_{mn}, x_{mn}) = 0, \quad \forall n \in \mathbb{N}. \tag{2}$$

In the next step for any $\varepsilon > 0$ we construct an $\widetilde{n}$ and $\widetilde{m}$ such that

$$T^{\widetilde{n}}(B(x_{\widetilde{m}\widetilde{n}}, \varepsilon)) \subseteq B(x_{\widetilde{m}\widetilde{n}}, \varepsilon), \tag{3}$$

where $B(x_{\widetilde{m}\widetilde{n}}, \varepsilon) = \{u \in X \mid d(u, x_{\widetilde{m}\widetilde{n}}) < \varepsilon\}$. First we choose $\widetilde{n} \in \mathbb{N}$ such that $\varphi^{\widetilde{n}}(\varepsilon) < \frac{\varepsilon}{2s}$. By (2), for ε and $\widetilde{n}$ we can choose an $\widetilde{m} \in \mathbb{N}$ such that

$$d(T^{\widetilde{n}} x_{m\widetilde{n}}, x_{m\widetilde{n}}) < \frac{\varepsilon}{2s}, \quad \forall\, m \geq \widetilde{m}.$$

The inclusion given in (3) holds for these indices since for any $u \in B(x_{\widetilde{m}\widetilde{n}}, \varepsilon)$ both inequalities

$$d(T^{\widetilde{n}} u, T^{\widetilde{n}} x_{\widetilde{m}\widetilde{n}}) \leq \varphi^{\widetilde{n}}(d(u, x_{\widetilde{m}\widetilde{n}})) \leq \varphi^{\widetilde{n}}(\varepsilon) < \frac{\varepsilon}{2s}$$

and

$$d(T^{\widetilde{n}} x_{\widetilde{m}\widetilde{n}}, x_{\widetilde{m}\widetilde{n}}) < \frac{\varepsilon}{2s}$$

are satisfied, hence we can use the relaxed triangle inequality to obtain

$$\begin{aligned} d(T^{\widetilde{n}} u, x_{\widetilde{m}\widetilde{n}}) &\leq s\big(d(T^{\widetilde{n}} u, T^{\widetilde{n}} x_{\widetilde{m}\widetilde{n}}) + d(T^{\widetilde{n}} x_{\widetilde{m}\widetilde{n}}, x_{\widetilde{m}\widetilde{n}})\big) \\ &< s\left(\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s}\right) = \varepsilon. \end{aligned}$$

As a consequence of (3), we can conclude

$$d(x_{m\widetilde{n}}, x_{\widetilde{m}\widetilde{n}}) < \varepsilon, \tag{4}$$

for all $m \geq \widetilde{m}$.

We observe that (2) also implies that there exists an $m_0 \in \mathbb{N}$ such that

$$d(x_{m\widetilde{n}}, x_{m\widetilde{n}+p}) < \varepsilon, \quad \forall\, m \geq m_0, \forall\, p \in \{0, 1, \ldots, \widetilde{n} - 1\}. \tag{5}$$

Indeed, since (2) holds for $n = 1$, there exists $k_0 \in \mathbb{N}$ such that

$$d(x_{k+1}, x_k) < \frac{\varepsilon}{\widetilde{n} s^{\widetilde{n}}},$$

for all $k \geq k_0$. Let m_0 be such that $m_0 \widetilde{n} > k_0$. We can apply $p - 1$ times the relaxed triangle inequality to obtain

$$d(x_{m\widetilde{n}}, x_{m\widetilde{n}+p}) \leq \sum_{i=0}^{p} s^{i+1} d(x_{m\widetilde{n}+i}, x_{m\widetilde{n}+i+1}) \leq \sum_{i=0}^{\widetilde{n}-1} s^{\widetilde{n}} \frac{\varepsilon}{\widetilde{n} s^{\widetilde{n}}} = \varepsilon,$$

for all $m > m_0$ and $p \in \{0, ..., \widetilde{n} - 1\}$.

We have prepared all the necessary technicalities to prove that $(x_k)_{k \in \mathbb{N}}$ is a Cauchy sequence. For any $\varepsilon > 0$ first construct $\widetilde{n}, \widetilde{m}$ such that (3) holds and then m_0 such that (5) is also satisfied. Let $\overline{m} = \max\{\widetilde{m}, m_0\}$ and $k_1, k_2 \in \mathbb{N}$ such that $k_1, k_2 \geq \overline{m}\widetilde{n}$. We can write $k_1 = m_1\widetilde{n} + p_1$, $k_2 = m_2\widetilde{n} + p_2$, where $p_1, p_2 \in \{0, ..., \widetilde{n}-1\}$ and $m_1, m_2 \geq \overline{m}$. The construction of these indices implies

$$\begin{aligned} d(x_{k_1}, x_{m_1\widetilde{n}}) < \varepsilon \text{ and } d(x_{m_2\widetilde{n}}, x_{k_2}) < \varepsilon, &\quad \text{by (5)};\\ d(x_{m_1\widetilde{n}}, x_{\widetilde{m}\widetilde{n}}) < \varepsilon \text{ and } d(x_{\widetilde{m}\widetilde{n}}, x_{m_2\widetilde{n}}) < \varepsilon, &\quad \text{by (4)}. \end{aligned}$$

Therefore we can use the relaxed triangle inequality to obtain

$$\begin{aligned} d(x_{k_1}, x_{k_2}) &\leq s d(x_{k_1}, x_{m_1\widetilde{n}}) + s^2 d(x_{m_1\widetilde{n}}, x_{\widetilde{m}\widetilde{n}})\\ &\qquad + s^3 d(x_{\widetilde{m}\widetilde{n}}, x_{m_2\widetilde{n}}) + s^3 d(x_{m_2\widetilde{n}}, x_{k_2})\\ &\leq s\varepsilon + s^2\varepsilon + s^3\varepsilon + s^3\varepsilon\\ &\leq 4s^3\varepsilon. \end{aligned}$$

Thus we proved that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete, there exists an $x^* \in X$ such that $x_k \to x^*$. Then

$$\begin{aligned} s^{-1} d(x^*, Tx^*) &\leq \liminf_{k\to\infty} d(x_{k+1}, Tx^*)\\ &\leq \limsup_{k\to\infty} d(x_{k+1}, Tx^*)\\ &= \limsup_{k\to\infty} d(Tx_k, Tx^*)\\ &\leq \limsup_{k\to\infty} d(x_k, x^*) = 0, \end{aligned}$$

hence $Tx^* = x^*$.

It remains to prove that T has no other fixed point besides of x^*. Suppose that y^* is also a fixed point of T. Hence

$$0 \leq d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(d(x^*, y^*)),$$

and since φ is increasing we have

$$0 \leq d(x^*, y^*) \leq \varphi^n(d(x^*, y^*)), \quad \forall n \in \mathbb{N}.$$

If we let $n \to \infty$, the last inequality implies $d(x^*, y^*) = 0$.

□

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On generalized nonlinear Euler-Bernoulli Beam type equations

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Abstract. This paper is devoted to the study of a nonlinear Euler-Bernoulli Beam type equation involving both left and right Caputo fractional derivatives. Differently from the approaches of the other papers where they established the existence of solution for the linear Euler-Bernoulli Beam type equation numerically, we use the lower and upper solutions method with some new results on the monotonicity of the right Caputo derivative. Furthermore, we give the explicit expression of the upper and lower solutions. A numerical example is given to illustrate the obtained results.

1 Introduction

Fractional differential equations containing a composition of left and right fractional derivatives occur in the fractional theoretical mechanics and may arise naturally in the study of variational problems such as the fractional Euler-Lagrange equations, see [1–5, 8, 12]. The presence of both left and right fractional derivatives in the nonlinear differential equation poses many complications when trying to apply the existing methods, for this reason, most

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studies focus on the linear cases and use numerical analysis, we refer the reader to [1, 5, 12].

In [12] the authors discussed a linear fractional differential equation involving the right Caputo derivative and the left Riemann-Liouville fractional derivative and describing the height of granular material decreasing over time in a silo:

$$^{C}D^{\alpha}_{L^-} D^{\alpha}_{0^+} u(t) + bu(t) = 0, 0 \le t \le L, 0 < \alpha \le 1.$$

Recently in [5], the author solved analytically and numerically a linear fractional Euler–Bernoulli beam equation containing left and right fractional Caputo derivatives:

$$\begin{aligned} ^{C}D^{\alpha}_{L^-}{}^{C}D^{\alpha}_{0^+} u(t) &= f(t), 0 \le t \le L, 1 < \alpha \le 2 \\ u(0) &= u'(0) = u(L) = u'(L) = 0, \end{aligned}$$

that is derived by using a variational approach.

In [8] the authors proved existence of solutions for a nonlinear fractional oscillator equation containing both left Riemann–Liouville and right Caputo fractional derivatives

$$\begin{aligned} -^{C}D^{\alpha}_{1^-} D^{\beta}_{0^+} u(t) + \omega^2 u(t) &= f(t, u(t)), \\ 0 &\le t \le 1, \omega \in \mathbb{R}, \omega \neq 0, 0 < \alpha, \beta < 1 \\ u(0) &= 0, D^{\beta}_{0^+} u(1) = 0. \end{aligned}$$

The main tools for this study was the upper and lower solutions method.

Nonlinear fractional differential equations has been studied by different methods such fixed point theorems, upper and lower solutions method, successive approximations,... see [1–10, 13, 14].

In this paper we focus on a nonlinear Euler-Bernoulli Beam equation involving both left and right Caputo fractional derivatives

$$^{C}D^{\alpha}_{1^-}{}^{C}D^{\beta}_{0^+} u(t) = f(t, u(t)), 0 \le t \le 1, \tag{1}$$

with the boundary conditions

$$u(0) = u'(0) = u(1) = D^{\beta+1}_{0^+} u(1) = 0, \tag{2}$$

Where $1 < \alpha, \beta < 2$, $^{C}D^{\alpha}_{1^-}$ and $^{C}D^{\beta}_{0^+}$ denote respectively the right and the left sides Caputo derivatives and $D^{\beta+1}_{0^+}$ denotes the left Riemann–Liouville fractional derivative. Denote by (P1) the problem (1)-(2).

Note that the presence of both left and right fractional derivatives leads to great difficulties. To solve problem (P1) we apply the lower and upper solutions method and a new result on the monotonicity for the right Caputo derivative. To succeed with such approach, we transform the problem (P1) to a right Caputo fractional boundary value problem of lower order. After constructing the explicit expressions of the lower and upper solutions, we define a sequence of modified problems that we solve by Schauder fixed point theorem. An example is presented to illustrate the main theorem.

2 Preliminaries

In this section, we recall necessary definitions of fractional operators and their properties [11, 15, 16].

The left and right Caputo derivatives of order $n-1 < p < n$ are respectively defined as

$$\begin{aligned} {}^{C}D_{0^+}^{p} g(t) &= \left(I_{0^+}^{1-p} D^{n} g\right)(t), \\ {}^{C}D_{1^-}^{p} g(t) &= -\left(I_{1^-}^{1-p} D^{n} g\right)(t), \end{aligned}$$

and the Riemann -Liouville fractional derivative is defined as

$$D_{0^+}^{p} g(t) = D^{n}\left(I_{0^+}^{1-p} g\right)(t),$$

where D^n is the the classical derivative operator of order n and the operators $I_{0^+}^{p}$ and $I_{1^-}^{p}$ are respectively the left and right fractional Riemann–Liouville integrals of order $p > 0$ defined by

$$\begin{aligned} I_{0^+}^{p} g(t) &= \frac{1}{\Gamma(p)} \int_0^t \frac{g(s)}{(t-s)^{1-p}} ds, \\ I_{1^-}^{p} g(t) &= \frac{1}{\Gamma(p)} \int_t^1 \frac{g(s)}{(s-t)^{1-p}} ds. \end{aligned}$$

The composition rules of the fractional operators (for $n-1 < p < n$) are:

1- $I_{0^+}^{p}\,{}^{C}D_{0^+}^{p} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} t^{k}$.

2- $I_{1^-}^{p}\,{}^{C}D_{1^-}^{p} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{(-1)^{k} g^{(k)}(1)}{k!} (1-t)^{k}$.

3- ${}^{C}D_{1^-}^{p} D g(t) = -{}^{C}D_{1^-}^{p+1} g(t)$.

4- $D D_{0^+}^{p} g(t) = D_{0^+}^{p+1} g(t)$.

Next we give some results on the Caputo derivative of monotone functions.

Theorem 1 [8] *Assume that* $g \in C^1[0,1]$ *is such that* ${}^C D_{1-}^{\gamma} g(t) \geq 0$ *for all* $t \in [0,1]$ *and all* $\gamma \in (p,1)$ *with some* $p \in (0,1)$. *Then* g *is monotone decreasing. Similarly, if* ${}^C D_{1-}^{\gamma} g(t) \leq 0$ *for all* t *and* γ *mentioned above, then* g *is monotone increasing.*

3 Main results

To reduce the problem (P1) to an equivalent right Caputo fractional boundary value problem of lower order, we use the following Lemma

Lemma 1 *If a function* g *satisfies* $g(0) = g'(0)$, *then we have*

$${}^C D_{1^-}^{\alpha} {}^C D_{0^+}^{\beta} g(t) = -{}^C D_{1^-}^{\alpha-1} D_{0^+}^{\beta+1} g(t).$$

where $D_{0^+}^{\beta+1}$ *denotes the Riemann -Liouville fractional derivative.*

Proof. The proof is based on the composition rules of the fractional operators. □

From Lemma 2, equation (1) can be written as

$$-{}^C D_{1^-}^{\alpha-1} D_{0^+}^{\beta+1} u(t) = f(t, u(t)), 0 \leq t \leq 1.$$

Denote by (P2) the auxiliary problem:

$$(P2) \begin{cases} D_{0^+}^{\beta+1} u(t) = v(t), 0 \leq t \leq 1 \\ u(0) = u'(0) = u(1) = 0. \end{cases}$$

In the next lemma, we give the solution of (P2).

Lemma 2 *If* $1 < \beta < 2$, *then problem (P2) has a unique solution given by*

$$u(t) = I_{0^+}^{\beta+1} v(t) - t^{\beta} I_{0^+}^{\beta+1} v(1).$$

Let E denotes the Banach space $C([0,1], \mathbb{R})$ equipped with the uniform norm $\|u\| = \max_{t \in [0,1]} |u(t)|$. Define the operator T on E by

$$Tv(t) = I_{0^+}^{\beta+1} v(t) - t^{\beta} I_{0^+}^{\beta+1} v(1), t \in [0,1],$$

thus

$$u(t) = Tv(t), t \in [0,1].$$

From the boundary condition $D_{0^+}^{\beta+1}u(1) = 0$, we show that the problem (P1) is equivalent to the following right Caputo boundary value problem of order $0 < \alpha - 1 < 1$:

$$(P3) \begin{cases} -{}^{C}D_{1^-}^{\alpha-1}v(t) = f(t, Tv(t)), 0 \leq t \leq 1 \\ v(1) = 0. \end{cases}$$

Let us make the following hypotheses:

$(H1)$ There exists a constant $A \geq 0$ such that

$$f(t,x) \geq \frac{-A}{\Gamma(2-\alpha)}(1-t)^{1-\gamma},$$

for $0 \leq t \leq 1$, for all $\gamma \in [\alpha-1, 1)$ and $\frac{-A(\beta+1)}{\Gamma(3+\beta)} \leq x \leq 0$.

$(H2)$ There exists a constant $B \leq 0$ such that $A \geq |B|$ and

$$f(t,x) \leq \frac{-B}{\Gamma(2-\alpha)}(1-t)^{1-\gamma},$$

for $0 \leq t \leq 1$, for all $\gamma \in [\alpha-1, 1)$ and $0 \leq x \leq \frac{-B(\beta+1)}{\Gamma(3+\beta)}$.

To use Theorem 1, we have to adapt the definition of the lower and upper solutions for problem (P1) as follows:

Definition 1 *The functions $\underline{\sigma}$, $\overline{\sigma} \in AC^4[0,1]$ are called lower and upper solutions of problem (P1) respectively, if*

a) $-{}^{C}D_{1^-}^{\gamma}D_{0^+}^{\beta+1}\underline{\sigma}(t) \geq f(t, \underline{\sigma}(t))$,*for all* $t \in [0,1]$ *and for all* $\gamma \in [\alpha-1, 1)$ *and*

$\underline{\sigma}(0) \geq 0$, $\underline{\sigma}'(0) \geq 0$, $\underline{\sigma}(1) \geq 0$ *and* $D_{0^+}^{\beta+1}\underline{\sigma}(1) \geq 0$.

b) $-{}^{C}D_{1^-}^{\gamma}D_{0^+}^{\beta+1}\overline{\sigma}(t) \leq f(t, \overline{\sigma}(t))$,*for all* $t \in [0,1]$ *and for all* $\gamma \in [\alpha-1, 1)$ *and*

$\overline{\sigma}(0) \leq 0$, $\overline{\sigma}'(0) \leq 0$, $\overline{\sigma}(1) \leq 0$ *and* $D_{0^+}^{\beta+1}\overline{\sigma}(1) \leq 0$.

Where

$$AC^4[0,1] = \left\{u \in C^3[0,1], u^{(3)} \textit{ absolutely continuous function on } [0,1]\right\}.$$

Functions $\underline{\sigma}$ and $\overline{\sigma}$ are lower and upper solutions in reverse order if $\underline{\sigma}(t) \geq \overline{\sigma}(t), 0 \leq t \leq 1$.

Lemma 3 *Under the hypotheses (H1) and (H2), the problem (P1) has a lower and an upper solutions.*

Proof. Define $\varphi(t) = A(1-t)$, then we get

$$\begin{aligned} 0 &\geq T\varphi(t) = I_{0^+}^{\beta+1}\varphi(t) - t^{\beta} I_{0^+}^{\beta+1}\varphi(1) \\ &= \frac{At^{\beta}}{\Gamma(3+\beta)}\left(-t^2 + t(\beta+2) - (\beta+1)\right) \geq \frac{-A(\beta+1)}{\Gamma(3+\beta)}. \end{aligned}$$

By computation we obtain for $\gamma \in [\alpha-1, 1)$

$$^{C}D_{1-}^{\gamma}\varphi(t) = \frac{A}{\Gamma(2-\gamma)}(1-t)^{1-\gamma},$$

Now, we show that $\overline{\sigma}(t) = T\varphi(t)$ is an upper solution of problem (P1). By the help of hypothesis (H1), we have for all $t \in [0,1]$ and for all $\gamma \in [\alpha-1, 1)$

$$\begin{aligned} -{}^{C}D_{1-}^{\gamma} D_{0^+}^{\beta+1}\overline{\sigma}(t) &= -{}^{C}D_{1-}^{\gamma}\varphi(t) = \frac{-A}{\Gamma(2-\gamma)}(1-t)^{1-\gamma} \\ &\leq f(t, T\varphi(t)) = f(t, \overline{\sigma}(t)) \end{aligned}$$

in addition $\overline{\sigma}(0) \leq 0$, $\overline{\sigma}'(0) \leq 0, \overline{\sigma}(1) \leq 0$ and $D_{0^+}^{\beta+1}\overline{\sigma}(1) \leq 0$, consequently $\overline{\sigma}(t) = T\varphi(t)$ is an upper solution of problem (P1).

Similarly, setting $\psi(t) = B(1-t)$ and taking hypothesis (H2) into account, we show that $\underline{\sigma}(t) = T\psi(t)$ is a lower solution of problem (P1). Finally we write the explicit expressions of the upper and lower solutions as

$$\begin{aligned} \overline{\sigma}(t) &= T\varphi(t) = \frac{At^{\beta}}{\Gamma(3+\beta)}\left(-t^2 + t(\beta+2) - (\beta+1)\right) \leq 0, \\ \underline{\sigma}(t) &= T\psi(t) = \frac{Bt^{\beta}}{\Gamma(3+\beta)}\left(-t^2 + t(\beta+2) - (\beta+1)\right) \geq 0, \end{aligned}$$

and then $\underline{\sigma}$ and $\overline{\sigma}$ are lower and upper solutions in reverse order, i.e

$$\overline{\sigma}(t) \leq \underline{\sigma}(t), 0 \leq t \leq 1.$$

The proof is completed. □

Let us introduce the following sequence of modified problems $\left((P4)_{\gamma}\right)$, for $\gamma \in [\alpha-1, 1)$:

$$\left((P4)_{\gamma}\right)\begin{cases} -{}^{C}D_{1-}^{\gamma}\nu(t) = F\nu(t), 0 \leq t \leq 1 \\ \nu(1) = 0, \end{cases}$$

where the operator $F : E \to E$, is defined by

$$F\nu(t) = f(t, T(\min(\varphi, \max(\nu, \psi)))), 0 \leq t \leq 1.$$

The relation between the solution of the sequence of modified problem $\left((\mathrm{P4})_\gamma\right)$ and the solution of problem (P1) is given in the following lemma:

Lemma 4 *If ν is a solution of problem $\left((\mathrm{P4})_{\alpha-1}\right)$, then $u = T\nu$ is a solution of problem (P1) satisfying*

$$\overline{\sigma}(t) \leq u(t) \leq \underline{\sigma}(t), 0 \leq t \leq 1. \tag{3}$$

Proof. Let ν_γ be a solution of problem $\left((\mathrm{P4})_\gamma\right)$ for $\gamma \in [\alpha - 1, 1)$, we shall prove that

$$\psi(t) \leq \nu_\gamma(t) \leq \varphi(t), 0 \leq t \leq 1. \tag{4}$$

For this purpose, set $\epsilon(t) = \nu_\gamma(t) - \varphi(t)$. It's clear that $\epsilon(1) = 0$. Suppose the contrary, i.e. there exists $t_1 \in (0, 1]$ such that $\epsilon(t_1) > 0$, since ϵ is continuous, then there exist $a \in [0, t_1]$ and $b \in (t_1, 1]$ such that $\epsilon(b) = 0$ and $\epsilon(t) \geq 0$, for all $t \in [a, b]$. By taking the right Caputo derivative of ϵ, it yields

$$\begin{aligned} {}^C D_{1-}^{\gamma}\epsilon(t) &= {}^C D_{1-}^{\gamma}\nu_\gamma(t) - {}^C D_{1-}^{\gamma}\varphi(t) \\ &= -f(t, T(\min[\varphi, (\max(\nu_\gamma, \psi))])) - {}^C D_{1-}^{\gamma}\varphi(t) \\ &= -f(t, T\varphi(t)) - {}^C D_{1-}^{\gamma}\varphi(t). \end{aligned}$$

Taking in to account that $\overline{\sigma} = T\varphi(t)$ is an upper solution, we conclude that ${}^C D_{1-}^{\gamma}\epsilon(t) \leq 0$, $t \in [a, b]$, therefore, ϵ is increasing on $[a, b]$ by Theorem 1. Since $\epsilon(b) = 0$, then $\epsilon(t) \leq 0$, for all $t \in [a, b]$ and thus $\nu_\gamma(t) \leq \varphi(t)$, $t \in [a, b]$ that leads to a contradiction. Proceeding by the same way, we prove that $\psi(t) \leq \nu_\gamma(t)$, $t \in [0, 1]$.

Now, let ν be a solution of problem $((\mathrm{P4})_{\alpha-1})$, then thanks to inequalities (4) we have

$$-{}^C D_{1-}^{\alpha-1}\nu(t) = F\nu(t) = f(t, T\nu(t)),$$

hence ν is a solution of (P3) and consequently $u = T\nu$ is a solution of (P1). Let us rewrite the operator T as

$$T\nu(t) = \frac{1}{\Gamma(1+\beta)} \int_0^1 G(t, s)\nu(s)\, ds$$

where the Green function G given by

$$G(t, s) = \begin{cases} (t-s)^\beta - t^\beta (1-s)^\beta, s \leq t \\ -t^\beta (1-s)^\beta, s \geq t \end{cases}$$

is negative for $0 \leq s, t \leq 1$, consequently, by applying the operator T to (4) it yields

$$T\varphi(t) \leq Tv(t) \leq T\psi(t), 0 \leq t \leq 1,$$

thus (3) holds. This achieves the proof. □

Now we are ready to prove our main results for problem (P1):

Theorem 2 *Under the hypotheses (H1) and (H2), the problem (P1) has at least one solution u satisfying*

$$\overline{\sigma}(t) \leq u(t) \leq \underline{\sigma}(t), 0 \leq t \leq 1.$$

Proof. Define the operator R on E, by

$$\begin{aligned} Rv(t) &= -I_{1^-}^{\alpha-1}Fv(t) \\ &= -I_{1^-}^{\alpha-1}f(t, T(\min(\varphi, \max(v, \psi)))), 0 \leq t \leq 1. \end{aligned}$$

Let us remark that if R has a fixed point v then $u = Tv$ is a solution of (P1). Set

$$\Omega = \{v \in C[0,1], \|v\| \leq \frac{M}{\Gamma(\alpha)}\}.$$

where

$$M = \max\{|f(t,x)|, \overline{\sigma}(t) \leq x \leq \underline{\sigma}(t), 0 \leq t \leq 1\}.$$

Let us prove that $R(\Omega)$ is uniformly bounded, equicontinuous and $R(\Omega) \subset \Omega$. Let $v \in \Omega$, then $\overline{\sigma}(t) \leq T(\min(\varphi, \max(v, \psi)))(t) \leq \underline{\sigma}(t)$ we get

$$\begin{aligned} |Rv(t)| &\leq I_{1^-}^{\alpha-1}|f(t, T(\min(\varphi, \max(v, \psi)))(t))| \\ &= \frac{1}{\Gamma(\alpha-1)} \int_t^1 \frac{|f(s, T(\min(\varphi, \max(v, \psi)))(s))|}{(s-t)^{2-\alpha}} ds \\ &\leq \frac{M}{\Gamma(\alpha)}, \end{aligned}$$

therefore $R(\Omega)$ is uniformly bounded and $R(\Omega) \subset \Omega$. Let $0 \leq t_1 < t_2 \leq 1$, for simplicity we denote $g(t) = f(t, T(\min(\varphi, \max(v, \psi)))(t))$, we have

$$\begin{aligned} |Rv(t_1) - Rv(t_2)| &\leq \left| I_{1^-}^{\alpha-1}g(t_1) - I_{1^-}^{\alpha-1}g(t_2) \right| \\ &\leq \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (s-t_1)^{\alpha-2} |g(s)| ds + \\ &\quad \frac{1}{\Gamma(\alpha-1)} \int_{t_2}^{1} \left((s-t_1)^{\alpha-2} - (s-t_2)^{\alpha-2} \right) |g(s)| ds \\ &\leq \frac{M}{\Gamma(\alpha)} \left((1-t_1)^{\alpha-1} - (1-t_2)^{\alpha-1} \right) \to 0, t_1 \to t_2, \end{aligned}$$

hence, $R(\Omega)$ is equicontinuous. By Arzela-Ascoli Theorem we conclude that R is completely continuous. Finally an application of Schauder fixed point theorem implies that R has a fixed point $v \in \Omega$, and so $u = Tv$ is a solution of (P1) satisfying from Lemma 7, $\underline{\sigma}(t) \leq u(t) \leq \overline{\sigma}(t)$, $0 \leq t \leq 1$. The proof is completed. □

Next, we present an example to illustrate the obtained results.

Example 1 *Consider the problem (P1) with* $\alpha = \frac{5}{3}$, $\beta = \frac{3}{2}$ *and*

$$f(t,x) = \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x (1-t)^{\frac{1}{3}}, 0 \leq t \leq 1, x \in \mathbb{R}.$$

If we choose $A = 1$ *and* $B = -1$, *then Hypotheses (H1) and (H2) are satisfied, in fact for for*$\gamma \in [\frac{2}{3}, 1), 0 \leq t \leq 1, \frac{-5}{2\Gamma(\frac{9}{2})} \leq x \leq 0$, *we have*

$$\begin{aligned} f(t,x) &= \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x (1-t)^{\frac{1}{3}} = \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x (1-t)^{1-\frac{2}{3}} \\ &\geq \frac{-1}{\Gamma\left(\frac{1}{3}\right)} (1-t)^{1-\frac{2}{3}} \geq \frac{-1}{\Gamma\left(\frac{1}{3}\right)} (1-t)^{1-\gamma}. \end{aligned}$$

and if $\gamma \in [\frac{2}{3}, 1), 0 \leq t \leq 1, 0 \leq x \leq \frac{5}{2\Gamma(\frac{9}{2})}$, *it yields*

$$\begin{aligned} f(t,x) &= \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x (1-t)^{\frac{1}{3}} = \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x (1-t)^{1-\frac{2}{3}} \\ &\leq \frac{1}{\Gamma\left(\frac{1}{3}\right)} (1-t)^{1-\frac{2}{3}} \leq \frac{1}{\Gamma\left(\frac{1}{3}\right)} (1-t)^{1-\gamma}. \end{aligned}$$

The expressions of lower and upper solutions are

$$\begin{aligned} \overline{\sigma}(t) &= T\varphi(t) = \frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{9}{2}\right)} \left(-t^2 + \frac{7}{2}t - \frac{5}{2}\right) \leq 0, \\ \underline{\sigma}(t) &= T\psi(t) = \frac{-t^{\frac{3}{2}}}{\Gamma\left(\frac{9}{2}\right)} \left(-t^2 + \frac{7}{2}t - \frac{5}{2}\right) \geq 0. \end{aligned}$$

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Fundamental theorem of calculus under weaker forms of primitive

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Abstract. In this paper we will present abstract versions of fundamental theorem of calculus (FTC) in the setting of Kurzweil - Henstock integral for functions taking values in an infinite dimensional locally convex space. The result will also be dealt with weaker forms of primitives in a widespread setting of integration theories generalising Riemann integral.

1 Introduction and preliminaries

The (FTC) theorem is one of the celeberated results of classical analysis. The result establishes a relation between the notions of integral and derivative of a function. In its origional form FTC asserts that: if for the function $F : [a, b] \longrightarrow \mathbb{R}$, $F'(t)$ exists and $F'(t) = f(t)$ and if $f(t)$ if integrable then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Let us recall that a (tagged) partition of the interval $[a, b]$ is a finite set of non-overlapping subintervals $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$, where $a = x_0 < x_1 < \cdots <$

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$x_n = b$ and t_i's are the tags attached to each subinterval $[x_{i-1}, x_i]$. The norm or the mesh of the partition is define to be

$$|\mathcal{P}| = \max_{1 \le i \le n} (t_i - t_{i-1}).$$

Definition 1 A (bounded) function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be Riemann integrable if: $\exists\ x \in \mathbb{R}$ such that $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that for each (tagged) partition $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ of $[a, b]$ with $|P| < \delta$

$$|S(f, \mathcal{P}) - x| \le \epsilon,$$

where $S(f, \mathcal{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ is the Riemann sum of f corrsponding to the partition $\mathcal{P}$: The (unique) vector x, to be denoted by $\int_a^b f(t)dt$ shall be called the Riemann integral of f over $[a, b]$.

Theorem 1 *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is differentiable on* $[a, b]$ *and* $f'(t)$ *is (Riemann) integrable then*

$$\int_a^b f'(t)dt = f(b) - f(a).$$

In the preceeding therem the assumption of the integrability of the derivative $f'(t)$ is unavoidable. Below we give an explicate of FTC in the setting of Kurzweil – Henstock integral, where the integrability of the derivative comes for free.
Recalling that a gauge is a positive function $\delta : [a, b] \longrightarrow (0, \infty)$ and a partition $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is said to be δ-fine if $[x_{i-1}, x_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))\ \forall\ 1 \le i \le n$.

Definition 2 [2], [5] A function $f : [0, 1] \longrightarrow \mathbb{R}$ is said to be Kurzweil – Henstock integrable if there exists $x \in \mathbb{R}$ such that the following is true: for any $\epsilon > 0$, there exists a gauge $\delta(t) > 0$ on $[a, b]$ such that if $\{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is any δ-fine (tagged) partition of $[a, b]$ then

$$|S(f, \mathcal{P}) - x| \le \epsilon,$$

where x is the integral of f and $S(f, \mathcal{P}) = \sum\limits_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ is the Riemann sum, symbolically we write $f \in KH([0, 1])$.

The (KH–integral) integral is defined in almost the same way as Riemann integral through Riemann sums. The only difference is in defining the δ here it is assumed to be a positive function instead of a constant. The only technicality to be taken care of and the definition to make sense is that we must have a δ-fine partition for every gauge. Pierre Cousin [3] gives the existence of such a partition for every gauge $\delta(t)$ in the form of so called Cousin's lemma. Before proceeding further, we would like to show that the KH – integral sub- sumes Riemann integral properly through the famous Dirchlet's function $f : [0, 1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$$

We know that f is not Riemann integrable. Here we will show that f is KH – integrable. Let $\epsilon > 0$ be given and set

$$\delta(x) = \begin{cases} 1, & x \text{ is irrational} \\ \frac{\epsilon}{2^{i+1}}, & x = q_i, i \geq 1 \end{cases}$$

where q_i is the enumeration of rationals in $[0, 1]$. Now let $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a δ-fine partition of $[0, 1]$. If t_i is not rational, the term $f(t_i)(x_i - x_{i-1})$ in the Riemann sum of f with respect to $\mathcal{P}$ is 0. If t_i is rational and $t_i = q_j$ for some j, the term $f(t_i)(x_i - x_{i-1})$ in the Riemann sum is less than $2\delta(q_j) = \frac{\epsilon}{2^{j+1}}$. Thus we have

$$\left| \sum_{i=0}^{n-1} f(t_i)(x_i - x_{i-1}) \right| < 2 \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \epsilon$$

which shows that $f \in KH([0, 1])$. This is the most common example of a bounded function which is not Riemann integrable. But it turns out to be KH – integrable and furnishes a comparison between the two theories of integration.

Theorem 2 *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is differentiable on* $[a, b]$ *then* $f'(t)$ *is* (*Kurzweil – Henstock*) *integrable and*

$$\int_a^b f'(t)dt = f(b) - f(a).$$

A slight modification in the definition of δ makes an immense impact and if we take it to be a constant we get the Riemann integral. It is quite remarkable that the simple idea of replacing δ by a positive function $\delta(t)$ leads to a powerfull generalization of Riemann integral. The convergence theorems of the Lebesgue integral hold true in the setting of KH – integral and more importantly FTC holds in its full generality without the assumption of integrability of the derivative [1].

Definition 3 [2] Let $F, f : [a, b] \longrightarrow \mathbb{R}$, we say that:
(i) F is primitive of f on $[a, b]$ if $F'(x)$ exists and $F'(x) = f(x)$ for all $x \in [a, b]$.
(ii) F is a-primitive of f on $[a, b]$ if F is continuous, $F'(x)$ exists and $F'(x) = f(x)$ outside a null set $E \subset [a, b]$.
(iii) F is c-primitive of f on $[a, b]$ if F is continuous, $F'(x)$ exists and $F'(x) = f(x)$ outside a countable set $E \subset [a, b]$.
(iv) F is f-primitive of f on $[a, b]$ if F is continuous, $F'(x)$ exists and $F'(x) = f(x)$ outside a finite set $E \subset [a, b]$.

In the following example we show that the proof of Theorem 2 can be redesigned to permit one point of non-differentiability.

Example 1 Define $f : [0, 1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

f is not bounded on $[0, 1]$. If we take $F(x) = 2x$ for $x \in [0, 1]$ then F is continuous on $[0, 1]$ and $F'(x) = f(x)$ for all $x \in (0, 1]$ but $F'(0)$ does not exist. Hence F is an f-primitive of f on $[0, 1]$ with the exceptional set $E = \{0\}$. Now, if $t \in (0, 1]$ and $\epsilon > 0$ we can choose $\delta(t)$ in such a way that the conclusion of FTC holds true for F. To tackle with the point of exception 0 we choose $\delta(0) = \frac{\epsilon^2}{4}$ so that if $0 \leq v \leq \delta(0)$, then $F(v) - F(0) = 2\sqrt{v} \leq \epsilon$.
Now let $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a tagged partition of $[0, 1]$ that is δ-fine. If all of the tags belong to $(0, 1]$ the proof of Theorem 2 applies without any change. However, if the first tag $t_1 = 0$ then the first term in the Riemann sum $S(f, \mathcal{P})$ is equal to $f(0)(x_1 - x_0) = 0$. Also we have

$$|F(x_1) - F(x_0) - f(0)(x_1 - x_0)| = |F(x_1)| = 2\sqrt{x_1} \leq \epsilon.$$

We now apply the argument given in Theorem 2 to the remaining terms to obtain

$$\left| \sum_{i=2}^{n} F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \right| < \epsilon.$$

Therefore on adding these terms we have

$$|F(1) - F(0) - S(f, \mathcal{P})| \leq \epsilon + \epsilon = 2\epsilon$$

Since ϵ is arbitrary we conclude that $f \in KH([0, 1])$ and that

$$\int_0^1 f(t)dt = F(1) - F(0) = 2.$$

The argument of the above theorem can easily be carried over to any exceptional set of finitely many points and the conclusion of the theorem is sought for an f – primitive.
As a significant extension below we present a version of FTC where the conclusion holds true for a countably infinite (exceptional) set.

Theorem 3 *If* $f : [a, b] \longrightarrow \mathbb{R}$ *has a* c *– primitive* F *on* $[a, b]$ *then* $f \in KH[a, b]$ *and*

$$\int_a^b f(t)dt = F(b) - F(a)$$

In the preceeding theorem the conclusion holds true for a c – primitive that is if the exceptional set is taken to be a countably infinite set. We know that every countable set is a null set. So, it is natural to ask whether the gap between countable and the null set can be bridged. More precisely, can we replace above theorem by the assertion: if F is continuous function on $[a, b]$ and there exists a null set E such that $F'(x) = f(x)$ for all $x \in [a, b] - E$ then $f \in KH([0, 1])$ and

$$\int_a^b f(t)dt = F(b) - F(a).$$

On this account it becomes inevitable to discuss the the so-called Cantor-Lebesgue function on $[0, 1]$ the construction of the function is given as:
Define

$$\Lambda : [0, 1] \longrightarrow \mathbb{R}$$

by

$$\Lambda(x) = \lim_{n \longrightarrow \infty} \Lambda_n(x)$$

where $\Lambda_n(x)$ is taken to be $\frac{1}{2^n}$ on the left out intervals of $[0, 1]$ while constructing the Cantor set, $\Lambda_n(0) = 0$ and $\Lambda_n(1) = 1$.
It is easy to see that Λ is a continuous non-decreasing function and its derivative $\Lambda'(x) = 0$ for all points of $[0, 1]$ outside the Cantor set.
Now comming back to the question raised above we see that $\Lambda'(x)$ exists and $\Lambda'(x) = \Lambda(x)$ outside a (Cantor) null set. But

$$\int_0^1 \Lambda' = 0 \neq 1 = \Lambda(1) - \Lambda(0)$$

2 FTC – for functions taking values in a Frechet space

We begin this section by giving a formal definition of the Kurzweil – Henstock integral also known as gauge integral or generalized Riemann integral for functions taking values in a complete metrizable locally convex space known as Frechet space [7]. In this section X will denote a Frechet space, $p(X)$ a family of seminorms on X.

Definition 4 A function $f : [0,1] \longrightarrow X$ is said to be Kurzweil – Henstock integrable if there exists $x \in X$ for which the following is true: for any $\epsilon > 0$, and a seminorm $p \in p(X)$ there exists a gauge $\delta_{\epsilon,p} > 0$ on $[a,b]$ such that if $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is any $\delta_{\epsilon,p}$-fine (tagged) partition of $[a,b]$ then

$$p\left(S(f,\mathcal{P}) - x\right) \leq \epsilon,$$

where x is the integral of f and $S(f,\mathcal{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ is the Riemann sum, symbolically we write $f \in KH([0,1], X)$.

Lemma 1 *Let $F : [a,b] \to X$ be differentiable at a point $t \in [a,b]$, then given $\epsilon > 0$ there exists $\delta_{\epsilon,p}(t) > 0$ such that if $u, v \in [a,b]$ satisfy*

$$t - \delta_{\epsilon,p}(t) \leq u \leq t \leq v \leq t + \delta_{\epsilon,p}(t)$$

then

$$p\left(F(v) - F(u) - F'(t)(v-u)\right) \leq \epsilon(v-u)$$

Proof. By definition of the derivative at $t \in [0,1]$, we have, given $\epsilon > 0$ there exists $\delta_{\epsilon,p}(t) > 0$, such that

$$p\left(\frac{F(z) - F(t)}{z - t} - F'(t)\right) \leq \epsilon, \text{ for } |z-t| \leq \delta_{\epsilon,p}(t), z \in [a,b]$$

$$p\left(F(z) - F(t) - F'(t)(z-t)\right) \leq \epsilon|z-t| \text{ for all } z \in [a,b]$$

with

$$|z - t| \leq \delta_{\epsilon,p}(t).$$

In particular, if we pick $u \leq t$ and $v \geq t$ in this interval around t and note that $v - t \geq 0$ and $t - u \geq 0$, then we have

$$\begin{aligned} p\left(F(v) - F(u) - F'(t)(v-u)\right) &= p\Big(\left(F(v) - F(t) - F'(t)(v-t)\right) \\ &\quad -\left(F(u) - F(t) - F'(t)(t-u)\right)\Big) \\ &\leq p\left(F(v) - F(t) - F'(t)(v-u)\right) \\ &\quad + p\left(F(u) - F(t) - F'(t)(t-u)\right) \\ &\leq \epsilon(v-t) + \epsilon(t-u) \\ &= \epsilon(v-u) \end{aligned}$$

which implies,

$$p\left(F(v) - F(u) - F'(t)(v-u)\right) \leq \epsilon(v-u).$$

□

Now we will present the Frechet space analogue of FTC.

Theorem 4 *Let* X *be a Frechet space. If* $f : [a, b] \to X$ *has a primitive* F *i,e.,* $F : [a, b] \to X$ *is differentiable at every point of* $[a, b]$ *and* $F' = f$ *on* $[a, b]$ *then* $f \in KH([a, b], X)$ *and*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Proof. Since $F'(t)$ exists for every $t \in [a, b]$ and $F'(t) = f(t)$, given $\epsilon > 0$ there exists $\delta_{\epsilon,p}(t) > 0$ such that

$$p\left(\frac{F(z) - F(t)}{z - t} - f(t)\right) \leq \epsilon,$$

for

$$|z - t| \leq \delta_{\epsilon,p}(t), z \in [a, b]$$

which implies,

$$p\left(F(z) - F(t) - F'(t)(z-t)\right) \leq \epsilon|z - t| \text{ for all } z \in [a, b].$$

Therefore by Lemma 1, if $a \leq u \leq t \leq v \leq b$ and $0 < v - u \leq \delta_\epsilon(t)$, then

$$p(F(v) - F(u) - f(t)(v-u)) \leq \epsilon|v - u|.$$

If $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is a $\delta_{\epsilon,p}$-fine partition of $[a, b]$ then the telescoping sum $F(b) - F(a) = \sum_{i=1}^n \{F(x_i) - F(x_{i-1})\}$ satisfies the approximation

$$\begin{aligned} p\left(S(f, \mathcal{P}) - \big(F(b) - F(a)\big)\right) &= p\left(\sum_{i=1}^n (x_i - x_{i-1}) f(t_i) - \big(F(x_i) - F(x_{i-1})\big)\right) \\ &\leq \sum_{i=1}^n p\left((x_i - x_{i-1}) f(t_i) - \big(F(x_i) - F(x_{i-1})\big)\right) \\ &\leq \sum_{i=1}^n \epsilon (x_i - x_{i-1}) \\ &= \epsilon (b - a) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary letting $\epsilon \to 0$, we get $f \in HK([a, b], X)$ and

$$\int_a^b f(t) dt = F(b) - F(a).$$

□

Theorem 5 *If* $f : [a, b] \longrightarrow X$ *has a* c*-primitive* F *on* $[a, b]$ *the* $f \in KH([a, b], X)$ *and*

$$\int_a^b f(t) dt = F(b) - F(a).$$

Proof. Let $E = \{c_k\}_{k=1}^\infty$ be the exceptional set for the c-primitive. Since E is countable, it is a null set and without loss of generality we may suppose that $f(c_k) = 0$. We shall define a gauge $\delta_{\epsilon,p}$ on $[a, b]$. Given $\epsilon > 0$ if $t \in [a, b] - E$ we take $\delta_{\epsilon,p}$ as in Lemma 1. For $t \in E, t = c_k$ for some $k \in N$. Since F is continuous on $[a, b]$ we can choose $\delta_{\epsilon,p}(c_k) > 0$ such that

$$p\big(F(z) - F(c_k)\big) \leq \frac{\epsilon}{2^{k+2}} \ \forall \ z \in [a, b]$$

that satisfy

$$|z - c_k| \leq \delta_{\epsilon,p}(c_k).$$

Thus a gauge is defined on $[a, b]$.
Now let $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ be a $\delta_{\epsilon,p}$-fine partition of $[a, b]$. If none of the tags belong to E, then the proof given in the Theorem 4 applies without any change. However if $c_k \in E$ is the tag of some subinterval then,

$$p\big(F(x_i) - F(x_{i-1}) - f(c_k)(x_i - x_{i-1})\big)$$

$$
\begin{aligned}
&\leq p\big(F(x_i)-F(c_k)\big)+p\big(F(c_k)-F(x_{i-1})\big)+p\big(f(c_k)(x_i-x_{i-1})\big)\\
&\leq \frac{\epsilon}{2^{k+2}}+\frac{\epsilon}{2^{k+2}}\\
&= \frac{\epsilon}{2^{k+1}}
\end{aligned}
$$

Now each point of E can be the tag of at most two subintervals in $\mathcal{P}$ therefore for each $t_i \in E$ we have the following inequality satisfied

$$\sum_{t_i\in E} p\big(F(x_i)-F(x_{i-1})-f(t_i)(x_i-x_{i-1})\big) \leq \sum_{k=1}^{\infty}\frac{\epsilon}{2^k}=\epsilon.$$

Also for $t_i \notin E$, we have from Lemma 1

$$\sum_{t_i\notin E} p\big(F(x_i)-F(x_{i-1})-f(t_i)(x_i-x_{i-1})\big) \leq \epsilon\sum_{t_i\notin E}(x_i-x_{i-1}) \leq \epsilon(b-a).$$

Now $\mathcal{P}$ is $\delta_{\epsilon,p}$-fine, therefore we have

$$|F(b)-F(a)-S(f,\mathcal{P})| \leq \epsilon(b-a)$$

Letting $\epsilon \longrightarrow 0$, we conclude that $f \in KH([0,1],X)$ with integral $F(b)-F(a)$ which proves the theorem. $\square$

3 FTC – some interesting situations in vector integration

As pointed out in the Section 1 conclusion of the above theorem does not hold true even for a real valued function if the exceptional set E is taken to be a null set. But the problem has been dealt with and the conclusion sought, in the setting of Bochner integral by C. Volintiru [6] with the assumption that the Hausdorff measure of the image of E under F is 0.
Let (M,d) be a metric space and $A \subset M$. Let C_i be a covering of A with $\mathrm{diam}(C_i) \leq \delta\ \forall\ i$. Let $\mathcal{C}(A,\delta)$ be the collection of all such coverings of A. Now for $\alpha > 0$, define

$$h_\alpha^\delta(A)=\inf\left(\sum_i(\mathrm{diam} C_i)^\alpha : (C_i)\in\mathcal{C}(A,\delta)\right).$$

Then

$$h_\alpha(A)=\lim_{\delta\longrightarrow 0}h_\alpha^\delta(A)=\sup_{\delta>0}h_\alpha^\delta(A)$$

gives an outer measure on the power set of M which is countably additive on the σ-field of Borel subsets of M. This measure is known as Hausdorff measure.

Definition 5 A measurable function $f : \Omega \longrightarrow X$ (X a Banach space) is said to be Bochner integrable if there exists a sequence of simple function (f_n) such that

$$\lim_n \int_\Omega \|f_n - f\| d\mu = 0$$

In this case, $\int_E f d\mu$ is defined for each $E \in \sum$ by

$$\int_E f d\mu = \lim_n \int_E f_n d\mu$$

where $\int_E f_n$ is defined in the ususal way.

Theorem 6 *A measurable function $f : \Omega \longrightarrow X$ is Bochner integrable if and only if*

$$\int_\Omega \|f\| d\mu < \infty$$

Theorem 7 *If F is the a-primitive of the function $f : [a, b] \longrightarrow X$ such that $h_1\big(F(E)\big) = 0$ (E being the exceptional set). Then f is measurable and if we assume the integrability of f, then*

$$\int_a^b f(t) dt = F(b) - F(a)$$

Here we would like to pose following questions which appear to be open!

Problem 1 *Does the conclusion of FTC hold true in the setting of KH – integral for the exceptional set E to be a null set with the assumption that Hausdorff measure of $F(E)$ is taken to be zero.*

Problem 2 *Can we have an analogue of Theorem 7 in the setting of a more general class of integral (which subsumes KH–integral and Bochner integral) known as Pettis integral [4].*

There are two aspects of FTC to ponder upon. One about integrating the derivatives (what we have discussed) other differentiating the indefinite integrals. A lot of work has been done with regard to this aspect of FTC in the setting of KH–integral [2]. Below we state a result of paramount importance on diferentiating integrals and then conclude with a problem which appears to be open.

Theorem 8 *If* $f \in KH([a,b])$ *then any indefinite integral* F *is continuous on* $[a,b]$ *and an* a*-primitive of* f *that is* $F'(t) = f(t)\ \forall\ t \in [a,b] - E$*, where* E *is a null set.*

Problem 3 *If* $f : [a,b] \longrightarrow X$ *is Pettis integrable. Does the conclusion of above theorem hold true?*

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Some properties of analytic functions related with Booth lemniscate

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Abstract. The object of the present paper is to study of two certain subclass of analytic functions related with Booth lemniscate which we denote by $\mathcal{BS}(\alpha)$ and $\mathcal{BK}(\alpha)$. Some properties of these subclasses are considered.

1 Introduction

Let Δ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{A}$ be the class of normalized and analytic functions. Easily seen that any $f \in \mathcal{A}$ has the following form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (z \in \Delta). \tag{1}$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in Δ. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in Δ will be denote by $\mathcal{S}^*$ and the set of all functions $f \in \mathcal{A}$ that are convex univalent in

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Δ will be denote by $\mathcal{K}$. Analytically, the function $f \in \mathcal{A}$ is a starlike univalent function, if and only if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \Delta).$$

Also, $f \in \mathcal{A}$ belongs to the class $\mathcal{K}$, if and only if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \Delta).$$

For more details about this functions, the reader may refer to the book of Duren [2]. Define by $\mathfrak{B}$ the class of analytic functions $w(z)$ in Δ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \Delta)$. Let f and g be two functions in $\mathcal{A}$. Then we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a function $w \in \mathfrak{B}$ such that $f(z) = g(w(z))$ for all $z \in \Delta$. Furthermore, if the function g is univalent in Δ, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

Recently, the authors [10, 11], (see also [5]) have studied the function

$$F_\alpha(z) := \frac{z}{1 - \alpha z^2} = \sum_{n=1}^{\infty} \alpha^{n-1} z^{2n-1} \qquad (z \in \Delta,\ 0 \leq \alpha \leq 1). \tag{2}$$

We remark that the function $F_\alpha(z)$ is a starlike univalent function when $0 \leq \alpha < 1$. In addition $F_\alpha(\Delta) = D(\alpha)$ $(0 \leq \alpha < 1)$, where

$$D(\alpha) = \left\{x + iy \in \mathbb{C} : \ \left(x^2 + y^2\right)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha)^2} < 0\right\}$$

and

$$F_1(\Delta) = \mathbb{C} \backslash \{(-\infty, -i/2] \cup [i/2, \infty)\}.$$

For $f \in \mathcal{A}$ we denote by $\mathrm{Area}\, f(\Delta)$, the area of the multi-sheeted image of the disk $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ $(0 < r \leq 1)$ under f. Thus, in terms of the coefficients of f, $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ one gets with the help of the classical Parseval-Gutzmer formula (see [2]) the relation

$$\mathrm{Area}\, f(\Delta) = \int\!\!\int_{\Delta_r} |f'(z)|^2 dxdy = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \tag{3}$$

which is called the Dirichlet integral of f. Computing this area is known as the area problem for the functions of type f. Thus, a function has a finite Dirichlet integral exactly when its image has finite area (counting multiplicities). All polynomials and, more generally, all functions $f \in \mathcal{A}$ for which f' is bounded on Δ are Dirichlet finite. Now by (2), (3) and since $\sum_{n=1}^{\infty} nr^{2(n-1)} = 1/(1-r^2)^2$ we get:

Corollary 1 *Let* $0 \leq \alpha < 1$. *Then*

$$\mathrm{Area}\{F_\alpha(\Delta)\} = \frac{\pi}{(1-\alpha^2)^2}.$$

Let $\mathcal{BS}(\alpha)$ be the subclass of $\mathcal{A}$ which satisfy the condition

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec F_\alpha(z) \qquad (z \in \Delta). \tag{4}$$

The function class $\mathcal{BS}(\alpha)$ was studied extensively by Kargar et al. [5]. The function

$$\tilde{f}(z) = z\left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}, \tag{5}$$

is extremal function for several problems in the class $\mathcal{BS}(\alpha)$. We note that the image of the function $F_\alpha(z)$ $(0 \leq \alpha < 1)$ is the Booth lemniscate. We remark that a curve described by

$$\left(x^2+y^2\right)^2 - \left(n^4+2m^2\right)x^2 - \left(n^4-2m^2\right)y^2 = 0 \qquad (x,y) \neq (0,0),$$

(is a special case of Persian curve) was studied by Booth and is called the Booth lemniscate [1]. The Booth lemniscate is called elliptic if $n^4 > 2m^2$ while, for $n^4 < 2m^2$, it is termed hyperbolic. Thus it is clear that the curve

$$\left(x^2+y^2\right)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha)^2} = 0 \qquad (x,y) \neq (0,0),$$

is the Booth lemniscate of elliptic type. Thus the class $\mathcal{BS}(\alpha)$ is related to the Booth lemniscate.

In the present paper some properties of the class $\mathcal{BS}(\alpha)$ including, the order of strongly satarlikeness, upper and lower bound for $\mathfrak{Re}f(z)$, distortion and grow theorems and some sharp inequalities and logarithmic coefficients inequalities are considered. Also at the end, we introduce a certain subclass of convex functions.

2 Main results

Our first result is contained in the following. Further we recall that (see [12]) the function f is strongly starlike of order γ and type β in the disc Δ, if it satisfies the following inequality:

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} - \beta\right\}\right| < \frac{\pi\gamma}{2} \qquad (0 \le \beta \le 1, 0 < \gamma \le 1). \tag{6}$$

Theorem 1 *Let* $0 \le \alpha \le 1$ *and* $0 < \varphi < 2\pi$. *If* $f \in \mathcal{BS}(\alpha)$, *then* f *is strongly starlike function of order* $\gamma(\alpha, \varphi)$ *and type 1 where*

$$\gamma(\alpha, \varphi) := \frac{2}{\pi}\arctan\left(\frac{1+\alpha}{1-\alpha}|\tan\varphi|\right).$$

Proof. Let $z = re^{i\varphi}(r < 1)$ and $\varphi \in (0, 2\pi)$. Then we have

$$\begin{aligned} F_\alpha(re^{i\varphi}) &= \frac{re^{i\varphi}}{1 - \alpha r^2 e^{2i\varphi}} \cdot \frac{1 - \alpha r^2 e^{-2i\varphi}}{1 - \alpha r^2 e^{-2i\varphi}} \\ &= \frac{r(1-\alpha r^2)\cos\varphi + ir(1+\alpha r^2)\sin\varphi}{1 - 2\alpha r^2\cos 2\varphi + \alpha^2 r^4}. \end{aligned}$$

Hence

$$\left|\frac{\mathfrak{Im}\{F_\alpha(re^{i\varphi})\}}{\mathfrak{Re}\{F_\alpha(re^{i\varphi})\}}\right| = \left|\frac{(1+\alpha r^2)\sin\varphi}{(1-\alpha r^2)\cos\varphi}\right| = \frac{1+\alpha r^2}{1-\alpha r^2}|\tan\varphi| \qquad (\varphi \in (0, 2\pi)). \tag{7}$$

For such r the curve $F_\alpha(re^{i\varphi})$ is univalent in $\Delta_r = \{z : |z| < r\}$. Therefore

$$\left[\left(\frac{zf'(z)}{f(z)} - 1\right) \prec F_\alpha(z), \quad z \in \Delta_r\right] \Leftrightarrow \left[\left(\frac{zf'(z)}{f(z)} - 1\right) \in F_\alpha(\Delta_r), \quad z \in \Delta_r\right]. \tag{8}$$

Then by (7) and (8), we have

$$\begin{aligned} \left|\arg\left\{\frac{zf'(z)}{f(z)} - 1\right\}\right| &= \left|\arctan\frac{\mathfrak{Im}[(zf'(z)/f(z)) - 1]}{\mathfrak{Re}[(zf'(z)/f(z)) - 1]}\right| \\ &\le \left|\arctan\frac{\mathfrak{Im}(F_\alpha(re^{i\varphi}))}{\mathfrak{Re}(F_\alpha(re^{i\varphi}))}\right| \\ &< \arctan\left(\frac{1+\alpha r^2}{1-\alpha r^2}|\tan\varphi|\right), \end{aligned}$$

and letting $r \to 1^-$, the proof of the theorem is completed. □

In the sequel we define an analytic function $\mathcal{L}(z)$ by

$$\mathcal{L}(z) = \exp \int_0^z \frac{1+F_\alpha(t)}{t} dt \qquad (0 \le \alpha \le 3-2\sqrt{2}, t \neq 0), \tag{9}$$

where F_α is given by (2). Since the function F_α is convex univalent for $0 \le \alpha \le 3-2\sqrt{2}$, thus as result of (cf. [9]), the function $\mathcal{L}(z)$ is convex univalent function in Δ.

Theorem 2 *Let* $0 \le \alpha \le 3-2\sqrt{2}$. *If* $f \in \mathcal{BS}(\alpha)$, *then*

$$\mathcal{L}(-r) \le \Re\mathfrak{e}\{f(z)\} \le \mathcal{L}(r) \qquad (|z| = r < 1),$$

where $\mathcal{L}(.)$ *defined by* (9).

Proof. Suppose that $f \in \mathcal{BS}(\alpha)$. Then by Lindelöf's principle of subordination [4], we get

$$\begin{aligned}\inf_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\} \le \inf_{|z|\le r} \Re\mathfrak{e}\{f(z)\} &\le \sup_{|z|\le r} \Re\mathfrak{e}\{f(z)\} \\ &\le \sup_{|z|\le r} \Re\mathfrak{e}\{|f(z)|\} \le \sup_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\}.\end{aligned} \tag{10}$$

Because F_α is a convex univalent function for $0 \le \alpha \le 3-2\sqrt{2}$ and has real coefficients, hence $F_\alpha(\Delta)$ is a convex domain with respect to real axis. Moreover we have

$$\sup_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\} = \sup_{-r\le z\le r} \mathcal{L}(z) = \mathcal{L}(r)$$

and

$$\inf_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\} = \inf_{-r\le z\le r} \mathcal{L}(z) = \mathcal{L}(-r).$$

The proof of Theorem 2 is thus completed. □

Theorem 3 *Let* $f \in \mathcal{BS}(\alpha)$, $0 < \alpha \le 3-2\sqrt{2}$, $r_s(\alpha) = \frac{\sqrt{1+4\alpha}-1}{2\alpha} \le 0.8703$,

$$F_\alpha(r_s(\alpha)) = \max_{|z|=r_s(\alpha)<1} |F_\alpha(z)| \quad \text{and} \quad F_\alpha(-r_s(\alpha)) = \min_{|z|=r_s(\alpha)<1} |F_\alpha(z)|.$$

Then we have

$$\frac{1}{1+r_s^2(\alpha)}(F_\alpha(r_s(\alpha))-1) \le |f'(z)| \le \frac{1}{1-r_s^2(\alpha)}(F_\alpha(r_s(\alpha))+1) \tag{11}$$

and

$$\int_0^{r_s(\alpha)} \frac{F_\alpha(t)}{1+t^2} dt - \arctan r_s(\alpha) \le |f(z)| \le \frac{1}{2}\log\left(\frac{1+r_s(\alpha)}{1-r_s(\alpha)}\right) + \int_0^{r_s(\alpha)} \frac{F_\alpha(t)}{1-t^2} dt \tag{12}$$

Proof. Let $f \in \mathcal{BS}(\alpha)$. Then by definition of subordination we have

$$\frac{zf'(z)}{f(z)} = 1 + F_\alpha(w(z)), \tag{13}$$

where $w(z)$ is an analytic function $w(0) = 0$ and $|w(z)| < 1$. From [6, Corollary 2.1], if $f \in \mathcal{BS}(\alpha)$, then f is starlike univalent function in $|z| < r_s(\alpha)$, where $r_s(\alpha) = \frac{\sqrt{1+4\alpha}-1}{2\alpha}$. Thus if we define $q(z) : \Delta_{r_s(\alpha)} \to \mathbb{C}$ by the equation $q(z) := f(z)$, where $\Delta_{r_s(\alpha)} := \{z : |z| < r_s(\alpha)\}$, then $q(z)$ is starlike univalent function in $\Delta_{r_s(\alpha)}$ and therefore

$$\frac{r_s(\alpha)}{1 + r_s^2(\alpha)} \le |q(z)| \le \frac{r_s(\alpha)}{1 - r_s^2(\alpha)} \qquad (|z| = r_s(\alpha) < 1).$$

Now by (13), we have

$$zf'(z) = q(z)(F_\alpha(z) + 1) \qquad |z| = r_s(\alpha) < 1.$$

Since $w(\Delta_{r_s(\alpha)}) \subset \Delta_{r_s(\alpha)}$ and by the maximum principle for harmonic functions, we get

$$\begin{aligned} |f'(z)| &= \frac{|q(z)|}{|z|} |F_\alpha(w(z)) + 1| \\ &\le \frac{1}{1 - r_s^2(\alpha)} (|F_\alpha(w(z))| + 1) \\ &\le \frac{1}{1 - r_s^2(\alpha)} \left(\max_{|z|=r_s(\alpha)} |F_\alpha(w(z))| + 1 \right) \\ &\le \frac{1}{1 - r_s^2(\alpha)} (F_\alpha(r_s(\alpha)) + 1). \end{aligned}$$

With the same proof we obtain

$$|f'(z)| \ge \frac{1}{1 + r_s^2(\alpha)} (F_\alpha(r_s(\alpha)) - 1).$$

Since the function f is a univalent function, the inequality for $|f(z)|$ follows from the corresponding inequalities for $|f'(z)|$ by Privalov's Theorem [4, Theorem 7, p. 67]. $\square$

Theorem 4 *Let* $F_\alpha(z)$ *be given by* (2). *Then we have*

$$\frac{1}{1+\alpha} \le |F_\alpha(z)| \le \frac{1}{1-\alpha} \qquad (z \in \Delta - \{0\}, 0 < \alpha < 1). \tag{14}$$

Proof. It is sufficient that to consider $|F_\alpha(z)|$ on the boundary

$$\partial F_\alpha(\Delta) = \left\{ F_\alpha(e^{i\theta}) : \theta \in [0, 2\pi] \right\}.$$

A simple check gives us

$$x = \mathfrak{Re}\left\{ F_\alpha(e^{i\theta}) \right\} = \frac{(1-\alpha)\cos\theta}{1+\alpha^2 - 2\alpha\cos 2\theta} \tag{15}$$

and

$$y = \mathfrak{Im}\left\{ F_\alpha(e^{i\theta}) \right\} = \frac{(1+\alpha)\sin\theta}{1+\alpha^2 - 2\alpha\cos 2\theta}. \tag{16}$$

Therefore, we have

$$\left| F_\alpha(e^{i\theta}) \right|^2 = \frac{1}{1+\alpha^2 - 2\alpha\cos 2\theta} \tag{17}$$

$$= \frac{1}{1+\alpha^2 - 2\alpha(2t^2-1)} =: H(t) \qquad (t = \cos\theta). \tag{18}$$

Since $0 \leq t \leq 1$, it is easy to see that $H'(t) \leq 0$ when $-1 \leq t \leq 0$ and $H'(t) \geq 0$ if $0 \leq t \leq 1$. Thus

$$\frac{1}{(1+\alpha)^2} \leq H(t) \leq \frac{1}{(1-\alpha)^2} \qquad (-1 \leq t < 0)$$

and

$$\frac{1}{(1+\alpha)^2} \leq H(t) \leq \frac{1}{(1-\alpha)^2} \qquad (0 < t \leq 1).$$

This completes the proof. □

A simple consequence of Theorem 4 as follows.

Theorem 5 *If* $f \in \mathcal{BS}(\alpha)$ $(0 < \alpha < 1)$, *then*

$$\frac{1}{1+\alpha} \leq \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{1}{1-\alpha} \qquad (z \in \Delta).$$

The inequalities are sharp for the function $\tilde{f}$ *defined by* (5).

Proof. By definition of subordination, and by using of Theorem 4, the proof is obvious. For the sharpness of inequalities consider the function $\widetilde{f}$ which defined by (5). It is easy to see that

$$\left| \frac{z\widetilde{f}'(z)}{\widetilde{f}(z)} - 1 \right| = \left| \frac{z}{1-\alpha z^2} \right| = |F_\alpha(z)|$$

and concluding the proof. □

The logarithmic coefficients γ_n of $f(z)$ are defined by

$$\log\left\{\frac{f(z)}{z}\right\} = \sum_{n=1}^{\infty} 2\gamma_n z^n \qquad (z \in \Delta). \tag{19}$$

This coefficients play an important role for various estimates in the theory of univalent functions. For example, consider the Koebe function

$$k(z) = \frac{z}{(1-\mu z)^2} \qquad (\mu \in \mathbb{R}).$$

Easily seen that the above function $k(z)$ has logarithmic coefficients $\gamma_n(k) = \mu^n/n$ where $|\mu| = 1$ and $n \geq 1$. Also for $f \in \mathcal{S}$ we have

$$\gamma_1 = \frac{a_2}{2} \quad \text{and} \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right)$$

and the sharp estimates

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2}(1+2e^{-2}) \approx 0.635\ldots,$$

hold. Also, sharp inequalities are known for sums involving logarithmic coefficients. For instance, the logarithmic coefficients γ_n of every function $f \in \mathcal{S}$ satisfy the sharp inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6} \tag{20}$$

and the equality is attained for the Koebe function (see [3, Theorem 4]).

The following lemma will be useful for the next result.

Lemma 1 (see [5, Theorem 2.1]) *Let* $f \in \mathcal{A}$ *and* $0 \leq \alpha < 1$. *If* $f \in \mathcal{BS}(\alpha)$, *then*

$$\log\frac{f(z)}{z} \prec \int_0^z \frac{P_\alpha(t)-1}{t}\,dt \qquad (z \in \Delta), \tag{21}$$

where

$$P_\alpha(z) - 1 = \frac{2}{\pi(1-\alpha)} i \log\left(\frac{1-e^{\pi i(1-\alpha)^2}z}{1-z}\right) \qquad (z \in \Delta) \tag{22}$$

and

$$\widetilde{P}_\alpha(z) = \int_0^z \frac{P_\alpha(t)-1}{t}\,dt \qquad (z \in \Delta), \tag{23}$$

are convex univalent in Δ.

We remark that an analytic function $P_{\mu,\beta} : \Delta \to \mathbb{C}$ by

$$P_{\mu,\beta}(z) = 1 + \frac{\beta-\mu}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\mu}{\beta-\mu}} z}{1-z}\right), \quad (\mu < 1 < \beta). \tag{24}$$

is a convex univalent function in Δ, and has the form:

$$P_{\mu,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{\beta-\mu}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\mu}{\beta-\mu}}\right), \qquad (n = 1, 2, \ldots). \tag{25}$$

The above function $P_{\mu,\beta}(z)$ was introduced by Kuroki and Owa [7] and they proved that $P_{\mu,\beta}$ maps Δ onto a convex domain

$$P_{\mu,\beta}(\Delta) = \{w \in \mathbb{C} : \mu < \mathfrak{Re}\{w\} < \beta\}, \tag{26}$$

conformally. Note that if we take $\mu = 1/(\alpha - 1)$ and $\beta = 1/(1-\alpha)$ in (24), then we have the function P_α which defined by (22). Now we have the following result about logarithmic coefficients.

Theorem 6 *Let $f \in \mathcal{A}$ belongs to the class $\mathcal{BS}(\alpha)$ and $0 < \alpha < 1$. Then the logarithmic coefficients of f satisfy the inequality*

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{(1-\alpha)^2}\left[\frac{\pi^2}{45} - \frac{1}{\pi^2}\left(\mathrm{Li}_4\left(e^{\pi(\alpha-2)i}\right) + \mathrm{Li}_4\left(e^{\pi(2-\alpha)i}\right)\right)\right], \tag{27}$$

where Li_4 is as following

$$\mathrm{Li}_4(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^4} = -\frac{1}{2}\int_0^1 \frac{\log^2(1/t)\log(1-tz)}{t} dt. \tag{28}$$

The inequality is sharp.

Proof. If $f \in \mathcal{BS}(\alpha)$, then by using Lemma 1 and with a simple calculation we get

$$\log\frac{f(z)}{z} \prec \sum_{n=1}^{\infty} \frac{2}{\pi n^2(1-\alpha)} i \left(1 - e^{\pi n(2-\alpha)i}\right) z^n \qquad (z \in \Delta). \tag{29}$$

Now, by putting (19) into the last relation we have

$$\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{1}{\pi n^2(1-\alpha)} i\left(1-e^{\pi n(2-\alpha)i}\right) z^n \qquad (z \in \Delta). \tag{30}$$

Again, by Rogosinski's theorem [2, 6.2], we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} |\gamma_n|^2 &\le \sum_{n=1}^{\infty} \left|\frac{1}{\pi n^2(1-\alpha)} i\left(1-e^{\pi n(2-\alpha)i}\right)\right|^2 \\ &= \frac{2}{\pi^2(1-\alpha)^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4} - \sum_{n=1}^{\infty}\frac{\cos\pi(2-\alpha)n}{n^4}\right)\end{aligned}$$

It is a simple exercise to verify that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \pi^4/90$ and

$$\sum_{n=1}^{\infty}\frac{\cos\pi(2-\alpha)n}{n^4} = \frac{1}{2}\left\{\mathrm{Li}_4\left(e^{-i(2-\alpha)\pi}\right) + \mathrm{Li}_4\left(e^{i(2-\alpha)\pi}\right)\right\}$$

and thus the desired inequality (27) follows. For the sharpness of the inequality, consider

$$F(z) = z\exp\widetilde{P}(z). \tag{31}$$

It is easy to see that the function $F(z)$ belongs to the class $\mathcal{BS}(\alpha)$. Also, a simple check gives us

$$\gamma_n(F(z)) = \frac{1}{\pi n^2(1-\alpha)} i\left(1-e^{\pi n(2-\alpha)i}\right).$$

Therefore the proof of this theorem is completed. □

Theorem 7 *Let* $f \in \mathcal{BS}(\alpha)$. *Then the logarithmic coefficients of* f *satisfy*

$$|\gamma_n| \le \frac{1}{2n} \quad (n \ge 1).$$

Proof. If $f \in \mathcal{BS}(\alpha)$, then by definition $\mathcal{BS}(\alpha)$, we have

$$\frac{zf'(z)}{f(z)} - 1 = z\left(\log\left\{\frac{f(z)}{z}\right\}\right)' \prec F_\alpha(z).$$

Thus

$$\sum_{n=1}^{\infty} 2n\gamma_n z^n \prec \sum_{n=1}^{\infty} \alpha^{n-1} z^{2n-1}.$$

Applying the Rogosinski theorem [8], we get the inequality $2n|\gamma_n| \le 1$. This completes the proof. □

3 The class $\mathcal{BK}(\alpha)$

In this section we introduce a new class. Our principal definition is the following.

Definition 1 *Let $0 \le \alpha < 1$ and F_α be defined by (2). Then $f \in \mathcal{A}$ belongs to the class $\mathcal{BK}(\alpha)$ if f satisfies the following:*

$$\frac{zf''(z)}{f'(z)} \prec F_\alpha(z) \qquad (z \in \Delta). \tag{32}$$

Remark 1 *By Alexander's lemma $f \in \mathcal{BK}(\alpha)$, if and only if $zf'(z) \in \mathcal{BS}(\alpha)$. Thus, if $f \in \mathcal{A}$ belongs to the class $\mathcal{BK}(\alpha)$, then*

$$\frac{\alpha}{\alpha-1} < \Re\mathfrak{e}\left\{1+\frac{zf''(z)}{f'(z)}\right\} < \frac{2-\alpha}{1-\alpha} \qquad (z \in \Delta).$$

The following theorem provides us a method of finding the members of the class $\mathcal{BK}(\alpha)$.

Theorem 8 *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{BK}(\alpha)$ if and only if there exists a analytic function q, $q(z) \prec F_\alpha(z)$ such that*

$$f(z) = \int_0^z \left(\exp \int_0^\zeta \frac{q(t)}{t}\right) d\zeta. \tag{33}$$

Proof. First, we let $f \in \mathcal{BK}(\alpha)$. Then from (32) and by definition of subordination there exists a function $\omega \in \mathfrak{B}$ such that

$$\frac{zf''(z)}{f'(z)} = F_\alpha(\omega(z)) \qquad (z \in \Delta). \tag{34}$$

Now we define $q(z) = F_\alpha(\omega(z))$ and so $q(z) \prec F_\alpha(z)$. The equation (34) readily gives

$$\{\log f'(z)\}' = \frac{q(z)}{z}$$

and moreover

$$f'(z) = \exp\left(\int_0^\zeta \frac{q(t)}{t} dt\right),$$

which upon integration yields (33). Conversely, by simple calculations we see that if f satisfies (33), then $f \in \mathcal{BK}(\alpha)$ and therefore we omit the details. $\square$

If we apply Theorem 8 with $q(z) = F_\alpha(z)$, then (33) with some easy calculations becomes

$$\hat{f}_\alpha(z) := z + \frac{z^2}{2} + \frac{1}{6}z^3 + \frac{1}{12}\left(\alpha + \frac{1}{2}\right)z^4 + \frac{1}{60}\left(4\alpha + \frac{1}{2}\right)z^5 + \cdots. \tag{35}$$

Theorem 9 *If a function $f(z)$ defined by (1) belongs to the class $\mathcal{BK}(\alpha)$, then*

$$|a_2| \le \frac{1}{2} \quad \text{and} \quad |a_3| \le \frac{1}{6}.$$

The equality occurs for $\hat{f}$ given in (35).

Proof. Assume that $f \in \mathcal{BK}(\alpha)$. Then from (32) we have

$$\frac{zf''(z)}{f'(z)} = \frac{\omega(z)}{1 - \alpha\omega^2(z)}, \tag{36}$$

where $\omega \in \mathfrak{B}$ and has the form $\omega(z) = b_1z + b_2z^2 + b_3z^3 + \cdots$. It is fairly well-known that if $|\omega(z)| = |b_1z + b_2z^2 + b_3z^3 + \cdots| < 1$ $(z \in \Delta)$, then for all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ we have $|b_k| \le 1$. Comparing the initial coefficients in (36) gives

$$2a_2 = b_1 \quad \text{and} \quad 6a_3 - 4a_2^2 = b_2. \tag{37}$$

Thus $|a_2| \le 1/2$ and $6a_3 = b_1^2 + b_2$. Since $|b_1|^2 + |b_2| \le 1$, therefore the assertion is obtained. □

Corollary 2 *It is well known that for $\omega(z) = b_1z + b_2z^2 + b_3z^3 + \cdots \in \mathfrak{B}$ for all $\mu \in \mathbb{C}$, we have $|b_2 - \mu b_1^2| \le \max\{1, |\mu|\}$. Therefore the Fekete-Szegö inequality i.e. estimates of $|a_3 - \mu a_2^2|$ for the class $\mathcal{BK}(\alpha)$ is equal to*

$$|a_3 - \mu a_2^2| \le \frac{1}{6}\max\left\{1, \left|\frac{3\mu}{2} - 1\right|\right\} \qquad (\mu \in \mathbb{C}).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

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On the sum of the Lah numbers and zeros of the Kummer confluent hypergeometric function

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Abstract. In the paper, the authors find the sum of the Lah numbers and make sure that the Kummer confluent hypergeometric function ${}_1F_1(n+1;2;z)$ has only $n-1$ real and negative zeros.

1 Notation and main results

In combinatorics, the Bell numbers, usually denoted by B_n for $n \in \{0\} \cup \mathbb{N}$, count the number of ways a set with n elements can be partitioned into disjoint

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and nonempty subsets. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930s. Every Bell number B_n can be generated by

$$e^{e^x-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}x^k$$

or, equivalently, by

$$e^{e^{-x}-1}=\sum_{k=0}^{\infty}(-1)^k B_k\frac{x^k}{k!}.$$

In combinatorics, the Stirling numbers arise in a variety of combinatorics problems. They are introduced in the eighteen century by James Stirling. There are two kinds of the Stirling numbers: the Stirling numbers of the first and second kinds. Every Stirling number of the second kind, usually denoted by $S(n,k)$, is the number of ways of partitioning a set of n elements into k nonempty subsets, can be computed by

$$S(n,k)=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n,$$

and can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},\quad k\in\{0\}\cup\mathbb{N}.$$

In combinatorics, the Lah numbers, discovered by Ivo Lah in 1955 and usually denoted by $L(n,k)$, count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets and have an explicit formula

$$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}.$$

The Lah numbers $L(n,k)$ can also be interpreted as coefficients expressing rising factorials $(x)_n$ in terms of falling factorials $\langle x\rangle_n$, where

$$(x)_n=\begin{cases}x(x+1)(x+2)\dots(x+n-1), & n\ge 1,\\ 1, & n=0\end{cases}$$

and

$$\langle x\rangle_n = \begin{cases} x(x-1)(x-2)\dots(x-n+1), & n \ge 1, \\ 1, & n = 0. \end{cases}$$

In combinatorics and the theory of polynomials, the partial Bell polynomials $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ can be defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n, \ell_i \in \{0\}\cup\mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}$$

and satisfy

$$B_{n,k}(1!, 2!, \dots, (n-k+1)!) = L(n,k). \tag{1}$$

The complete Bell polynomials $Y_n(x_1, x_2, \dots, x_n)$ are defined [3, p. 134] by

$$Y_n(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2}$$

and

$$Y_0(x_1, x_2, \dots, x_n) = 1. \tag{3}$$

In the theory of special functions, the generalized hypergeometric series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

is defined for complex numbers $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C}\setminus\{0, -1, -2, \dots\}$ and for positive integers $p, q \in \mathbb{N}$. The generalized hypergeometric series ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ converges absolutely for all $z \in \mathbb{C}$ if $p \le q$, for $|z| < 1$ if $p = q+1$, and for $|z| = 1$ if $p = q+1$ and $\Re[b_1 + \dots + b_q - (a_1 + \dots + a_p)] > 0$. Specially, the series

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$$

is called the Kummer confluent hypergeometric function and it is analytic for all $z \in \mathbb{C}$. See [4, pp. 3–5].

In [5] and [7], two explicit formulas for the Bell numbers B_n in terms of the Stirling numbers of the second kind $S(n,k)$ together with the Kummer confluent hypergeometric function ${}_1F_1(k+1; 2; 1)$ and the Lah numbers $L(n,k)$

respectively were established as follows. For $n \in \mathbb{N}$, the Bell numbers B_n can be expressed as

$$B_n = \frac{1}{e} \sum_{k=1}^{n} (-1)^{n-k} {}_1F_1(k+1;2;1) k! S(n,k) \tag{4}$$

and

$$B_n = \sum_{k=1}^{n} (-1)^{n-k} \left[\sum_{\ell=1}^{k} L(k,\ell) \right] S(n,k). \tag{5}$$

Comparing the formulas (4) with (5) motivates us to conjecture that

$$\frac{k!}{e} {}_1F_1(k+1;2;1) = \sum_{\ell=1}^{k} L(k,\ell), \quad k \in \mathbb{N}. \tag{6}$$

With the help of the famous software MATHEMATICA 9, we can verify that the equality (6) holds true for $1 \le k \le 9$ and they equal the following values respectively:

$$e, \quad \frac{3}{2}e, \quad \frac{13}{6}e, \quad \frac{73}{24}e, \quad \frac{167}{40}e, \quad \frac{4051}{720}e, \quad \frac{37633}{5040}e, \quad \frac{43817}{4480}e, \quad \frac{4596553}{362880}e.$$

This hints us that the above conjecture is true.

The aim of this paper is to prove a more general conclusions than the above conjecture. This general conclusion can be restated as the following theorems.

Theorem 1 *For $z \in \mathbb{C}$ and $n \in \mathbb{N}$, the formula*

$$\sum_{k=1}^{n} L(n,k) z^{k-1} = \frac{n!}{e^z} {}_1F_1(n+1;2;z) \tag{7}$$

is true. Specially, for $n \in \mathbb{N}$, the Lah number $L(n,k)$ and the complete Bell polynomials $Y_n(x_1, x_2, \dots, x_n)$ satisfy

$$\sum_{k=1}^{n} L(n,k) = \frac{n!}{e} {}_1F_1(n+1;2;1) \tag{8}$$

and

$$Y_n(1!, 2!, \dots, n!) = \frac{n!}{e} {}_1F_1(n+1;2;1) \tag{9}$$

respectively.

Theorem 2 *The Kummer confluent hypergeometric function* ${}_1F_1(n+1;2;z)$ *has only* $n-1$ *real and negative zeros.*

Remark 1 *The equations in* (4) *can be rewritten as*

$$\sum_{k=1}^{n}(-1)^{n-k}a_k S(n,k)=B_n,$$

where a_k *is sequence A000262 in the Online Encyclopedia of Integer Sequences. Such a sequence* a_k *has a nice combinatorial interpretation: it counts "the sets of lists, or the number of partitions of* $\{1,2\dots,k\}$ *into any number of lists, where a list means an ordered subset." This reveals the combinatorial interpretation of the special sequence* $k!{}_1F_1(k+1;2;1)$ *and the total sum* $\mathcal{L}_k=\sum_{\ell=1}^{k}L(k,\ell)$ *of the Lah numbers* $L(k,\ell)$*.*

2 Proofs of theorems

We now start out to prove Theorems 1 and 2.

Proof. [Proof of Theorem 1] It is easy to see that the equality (9) follows from substituting (1) into (8) and making use of (2) and (3). Hence, in what follows, we pay our attention to the proof of the formula (7).

In [6, p. 79, Theorem 2.1], we obtained

$$\sum_{k=1}^{n}L(n,k)x^k=\frac{e^{-x}}{x^n}\int_0^\infty I_1\big(2\sqrt{t}\,\big)t^{n-1/2}e^{-t/x}\,\mathrm{d}\,t \tag{10}$$

for $n\in\mathbb{N}$ and $x>0$, where the modified Bessel function of the first kind $I_\nu(z)$ can be defined by

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu} \tag{11}$$

for $\nu\in\mathbb{R}$ and $z\in\mathbb{C}$. See [1, p. 375, 9.6.10]. Substituting (11) for $\nu=1$

into (10) and straightforward computing arrive at

$$\begin{aligned}\sum_{k=1}^{n} L(n,k)x^k &= \frac{e^{-x}}{x^n}\int_0^\infty \sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}t^{n+k}e^{-t/x}\,\mathrm{d}\,t\\ &= \frac{e^{-x}}{x^n}\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\int_0^\infty t^{n+k}e^{-t/x}\,\mathrm{d}\,t\\ &= \frac{e^{-x}}{x^n}\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}x^{n+k+1}\Gamma(n+k+1)\\ &= e^{-x}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!(k+1)!}x^{k+1}\\ &= n!xe^{-x}\sum_{k=0}^{\infty}\frac{(n+1)_k}{(2)_k}\frac{x^k}{k!}\\ &= n!xe^{-x}{}_1F_1(n+1;2;x).\end{aligned}$$

Therefore, it follows that

$$\sum_{k=1}^{n} L(n,k)x^{k-1} = \frac{n!}{e^x}{}_1F_1(n+1;2;x) \tag{12}$$

for $x > 0$ and $n \in \mathbb{N}$.

Since the functions

$$\sum_{k=1}^{n} L(n,k)z^{k-1} \quad \text{and} \quad \frac{n!}{e^z}{}_1F_1(n+1;2;z)$$

are entire functions, that is, they are analytic on the whole complex plane $\mathbb{C}$, by the uniqueness theorem of analytic functions in the theory of complex functions, see [17, p. 210, Corollary], and by the formula (12), we easily derive the formula (7) for $z \in \mathbb{C}$ and $n \in \mathbb{N}$. The proof of Theorem 1 is complete. □

Proof. [Proof of Theorem 2] In [2, Lemma], the authors stated that if

$$P_{m,k}(x) = \sum_{n=1}^{m} L_k(m,n)x^n,$$

then the m roots of $P_{m,k}(x)$ are real, distinct, and non-positive for all $m \in \mathbb{N}$, where the associated Lah numbers $L_k(m,n)$ for $k > 0$ can be defined by

$$L_k(m,n) = \frac{m!}{n!}\sum_{r=1}^{n}(-1)^{n-r}\binom{n}{r}\binom{m+rk-1}{m}$$

and $L_k(m,n) = 0$ for $n > m$. Since $L_1(m,n) = L(m,n)$, see [2, p. 158, Eq. (4)], when $k = 1$, the polynomial $P_{m,k}(x)$ becomes

$$P_{m,1}(x) = \sum_{n=1}^{m} L(m,n)x^n.$$

The formula (7) implies that the integer polynomial $\frac{P_{m,1}(x)}{x}$ have the same zeros as the Kummer confluent hypergeometric function ${}_1F_1(n+1;2;z)$. Since the Kummer confluent hypergeometric function ${}_1F_1(n+1;2;z)$ has no positive zero, the zeros of $P_{m,1}(x)$ are non-positive, and then the Kummer confluent hypergeometric function ${}_1F_1(n+1;2;z)$ has only $n-1$ real and negative zeros. The proof of Theorem 2 is complete. $\square$

Remark 2 *The formula* (5) *has been generalized by R. B. Corcinoy, J. T. Malusay, J. A. Cillar, G. J. Rama, O. V. Silang, and I. M. Tacoloy in Philippines. There are more new results in [12] and [13, Section 5] for the Bell numbers* B_n*.*

Remark 3 *There are some new and closely related results published in [9, 10, 11, 14, 15, 16, 18] and references cited therein.*

Remark 4 *This paper is a revised version of the preprint [8].*

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An efficient numerical method for solving nonlinear Thomas-Fermi equation

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Abstract. In this paper, the nonlinear Thomas-Fermi equation for neutral atoms by using the fractional order of rational Chebyshev functions of the second kind (FRC2), $FU_n^\alpha(t, L)$, on an unbounded domain is solved, where L is an arbitrary parameter. Boyd (*Chebyshev and Fourier Spectral Methods, 2ed, 2000*) has presented a method for calculating the optimal approximate amount of L and we have used the same method for calculating the amount of L. With the aid of quasilinearization and FRC2 collocation methods, the equation is converted to a sequence of linear algebraic equations. An excellent approximation solution of $y(t)$, $y'(t)$, and $y'(0)$ is obtained.

1 Introduction

In this section, the introduction of numerical methods used for solving equations in unbounded domains is expressed. Furthermore, the mathematical model of Thomas-Fermi equation is introduced.

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1.1 The problems on unbounded domains

There are several numerical methods for solving differential equations on unbounded domains, such as:

1. *Finite difference method (FDM)*: One of the oldest and the simplest methods for solving differential equations is using the FDM approximations for derivatives. The FDMs are in a class of the discretization methods [2].

2. *Finite element method (FEM)*: One of the important methods used for solving the boundary value problems for partial differential equations is the finite element method [2].

3. *Meshfree methods*: Meshfree methods are those that do not require a connection between nodes of the simulation domain, i.e. a mesh, but are rather based on the interaction of each node with all its neighbors [3]. The use of Radial Basis Functions (RBFs) in meshless methods is very common in solving differential equations [4, 5]. This approach has recently received a great deal of attention from researchers [6, 7].

4. *Spectral methods*: Several approaches in Spectral methods have been proposed for solving the problems on unbounded domains:

   (a) Using functions such as Hermite, Sinc, Laguerre, and Bessel functions that are defined on the unbounded domains. This approach investigated by Parand et al. [8, 9], Funaro & Kavian [10], and Guo & Shen [11].

   (b) Mapping an unbounded equation to a bounded equation. Authors of [12, 13] have applied this approach in their works.

   (c) Replacing unbounded domains with $[-B, B]$ or $[0, B]$ by choosing B sufficiently large. This method is named domain truncation [14, 15].

   (d) Mapping the bounded basic functions to the unbounded basic functions. In this approach, the basic functions on a bounded domain convert to the functions on an unbounded domain. For example, Boyd [16] introduced a new spectral basis, called rational Chebyshev functions, on the unbounded domain by mapping on the Chebyshev polynomials, and also in Refs. [17, 18, 19]. There are three important mappings for this approach:

(A) Algebraic mapping: basic functions on a bounded domain $t \in [a, b]$ by using the transformation of $t = \frac{bx+aL}{x+L}$ convert to functions on an unbounded domain $x \in [0, \infty)$, where L is an arbitrary parameter [21].

(B) Exponential mapping: basic functions on a bounded domain $t \in [a, b]$ by using the transformation of $t = b + (a - b)e^{-\frac{x}{L}}$ convert to functions on an unbounded domain $x \in [0, \infty)$ [20].

(C) Logarithmic mapping: basic functions on a bounded domain $t \in [a, b]$ by using the transformation of $t = a+(b-a)\tanh(2\frac{x}{L})$ convert to functions on an unbounded domain $x \in [0, \infty)$.

In this paper, a Spectral method is introduced to solve unbounded problems by using the fractional order of rational Chebyshev orthogonal functions of the second kind.

1.2 The Thomas-Fermi equation

The Thomas-Fermi equation is an important nonlinear singular differential equation which is defined on semi-infinite domain [22, 23]:

$$\begin{aligned} \frac{d^2y(t)}{dt^2} - \frac{1}{\sqrt{t}}y^{\frac{3}{2}}(t) = 0, \qquad t \in [0, \infty), \\ y(0) = 1, \qquad y(\infty) = 0. \end{aligned} \tag{1}$$

The nonlinear Thomas-Fermi equation appears in the problem of determining the effective nuclear charge in heavy atoms, therefore, many great scholars were considered it, such as Fermi [24], Feynman (physics) [25], and Slater (chemistry) [26].

The initial slope $y'(0)$ is difficult for computing by any means and plays an important role in determining many properties of the physical of Thomas-Fermi atom [27]. It determines the energy of a neutral atom in Thomas-Fermi approximation:

$$E = \frac{6}{7}\left(\frac{4\pi}{3}\right)^{\frac{2}{3}} Z^{\frac{7}{3}}y'(0), \tag{2}$$

where Z is the nuclear charge.

For these reasons, the problem has been studied by many researchers and has been solved by different techniques where a number of them are listed in Table 1, in this table, the calculated value of $y'(0)$ by researchers is shown.

The rest of the paper is constructed as follows: the FRC2s and their properties are expressed in section 2. The methodology is explained in section 3. In section 4, results and discussions of the method are shown. Finally, a conclusion is provided.

2 Fractional order of rational Chebyshev functions of the second kind

In this section, the definition of the fractional order of rational Chebyshev functions of the second kind (FRC2s) and some theorems for them is provided.

2.1 The FRC2s definition

Using some transformations, some researchers have generalized the Chebyshev polynomials to semi-infinite or infinite domains, for example the rational Chebyshev functions on the semi-infinite domain [28], the rational Chebyshev functions on an infinite domain [1], and the generalized fractional order of the Chebyshev functions (GFCF) on finite interval $[0, \eta]$ [29, 30, 31] are introduced by using transformations $x = \frac{t-L}{t+L}$, $x = \frac{t}{\sqrt{t^2+L}}$, and $x = 1 - 2(\frac{t}{\eta})^\alpha$, respectively.

In the proposed work, by new transformation $x = \frac{t^\alpha - L}{t^\alpha + L}, L > 0$ on the Chebyshev polynomials of the second kind, the fractional order of rational Chebyshev functions of the second kind on domain $[0, \infty)$ is introduced, which is denoted by $FU_n^\alpha(t, L) = U_n(\frac{t^\alpha - L}{t^\alpha + L})$ where L is a numerical parameter.

The $FU_n^\alpha(t, L)$ can be calculated by using the following relation:

$$\begin{aligned}
&FU_0^\alpha(t, L) = 1, \qquad FU_1^\alpha(t, L) = 2\frac{t^\alpha - L}{t^\alpha + L}, \\
&FU_{n+1}^\alpha(t, L) = 2\frac{t^\alpha - L}{t^\alpha + L}\, FU_n^\alpha(t, L) - \, FU_{n-1}^\alpha(t, L), \quad n = 1, 2, \cdots, \qquad (3)
\end{aligned}$$

and we can also calculate:

$$FU_n^\alpha(t, L) = \sum_{k=0}^{n} \beta_{n,k}(t^\alpha + L)^{-k}, \qquad (4)$$

where

$$\beta_{n,k} = (-4L)^k \frac{(n + k + 1)!}{(n - k)!(2k + 1)!} \qquad \text{and} \qquad \beta_{0,k} = 1.$$

2.2 Approximation of functions

Any function of continuous and differentiable $y(t)$, $t \in [0, \infty)$, can be expanded as follows:

$$y(t) = \sum_{n=0}^{\infty} a_n \; FU_n^{\alpha}(t, L),$$

where the coefficients a_n can be obtained by:

$$a_n = \frac{8\alpha L^{\frac{3}{2}}}{\pi} \int_0^{\infty} FU_n^{\alpha}(t, L) \; y(t) \; w(t) dt, \qquad n = 0, 1, 2, \cdots .$$

In the numerical methods, we have to use first $(m+1)$-terms FRC2s and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m} a_n \; FU_n^{\alpha}(t, L). \tag{5}$$

Theorem 1 *The FRC2,* $FU_n^{\alpha}(t, L)$*, has precisely* n *real simple zeros on the interval* $(0, \infty)$ *in the form*

$$t_k = \left(L \frac{1 + \cos\left(\frac{k\pi}{n+1}\right)}{1 - \cos\left(\frac{k\pi}{n+1}\right)} \right)^{\frac{1}{\alpha}}, \qquad k = 1, 2, ..., n.$$

Proof. Chebyshev polynomial of the second kind $U_n(x)$ has n real simple zeros [1]:

$$x_k = \cos\left(\frac{k\pi}{n+1}\right), \qquad k = 1, 2, ..., n.$$

Therefore $U_n(x)$ can be written as

$$U_n(x) = (x - x_1)(x - x_2)...(x - x_n).$$

Using transformation $x = \frac{t^{\alpha} - L}{t^{\alpha} + L}$ yields to

$$FU_n^{\alpha}(t, L) = \left(\left(\frac{t^{\alpha} - L}{t^{\alpha} + L} \right) - x_1 \right) \left(\left(\frac{t^{\alpha} - L}{t^{\alpha} + L} \right) - x_2 \right) ... \left(\left(\frac{t^{\alpha} - L}{t^{\alpha} + L} \right) - x_n \right),$$

so, the real zeros of $FU_n^{\alpha}(t, L)$ are $t_k = \left(L \frac{1 + x_k}{1 - x_k} \right)^{\frac{1}{\alpha}}$. $\square$

Theorem 2 *The FRC2s are orthogonal on domain* $[0,\infty)$ *for all* $L > 0$ *with positive weight function* $w(t) = \frac{t^{\frac{3}{2}\alpha-1}}{(t^\alpha+L)^3}$ *as follows:*

$$\int_0^\infty FU_n^\alpha(t,L)\ FU_m^\alpha(t,L)\ w(t)\ dt = \frac{\pi}{8\alpha L^{\frac{3}{2}}}\delta_{mn}, \tag{6}$$

where δ_{mn} *is the Kronecker delta.*

Proof. The Chebyshev polynomials of the second kind $U_n(x)$ are orthogonal as [1]:

$$\int_{-1}^1 U_n(x)\ U_m(x)\ \sqrt{1-x^2}\ dx = \frac{\pi}{2}\delta_{mn}.$$

Now, by using transformation $x = \frac{t^\alpha - L}{t^\alpha + L}, L > 0$ on the integral, the theorem can be proved. □

3 The methodology

The quasi-linearization method (QLM) based on the Newton-Raphson method has introduced by Bellman and Kalaba [32, 33]. Some researchers have used this method in their works [34, 35, 36, 37].

Occasionally the linear ordinary differential equation that gets from the QLM at each iteration does not solve analytically. Hence we can use the Spectral methods to approximate the solution.

The QLM for Thomas-Fermi equation (1) is as follows:

$$\frac{d^2y_{n+1}}{dt^2} - \frac{3}{2\sqrt{t}}(y_n(t))^{1/2}y_{n+1}(t) = -\frac{1}{2\sqrt{t}}(y_n(t))^{3/2}, \tag{7}$$

$$y_{n+1}(0) = 1, \qquad y_{n+1}(\infty) = 0, \tag{8}$$

where $n = 0, 1, 2, 3, \cdots$.

The QLM iteration requires an initialization or "initial guess" $y_0(t)$. We assume that $y_0(t) \equiv 1$, i.e. the initial guess satisfies in the boundary condition at zero. Mandelzweig and Tabakin in Ref. [38] have shown that if the initial function is true in one of the conditions of (8) then the QLM is convergent.

Baker has shown that the solution of Eq. (1) is generated by the powers of $t^{\frac{1}{2}}$ as follows [39]:

$$\begin{aligned} y(t) = 1 + Bt + \frac{4}{3}t^{\frac{3}{2}} + \frac{2}{5}Bt^{\frac{5}{2}} + \frac{1}{3}t^3 + \frac{3}{70}B^2t^{\frac{7}{2}} + \frac{2}{15}Bt^4 \\ + \frac{4}{63}\left(\frac{2}{3} - \frac{1}{16}B^3\right)t^{\frac{9}{2}} + \cdots, \end{aligned} \tag{9}$$

for this reason, in Eq. (3), we assume that $\alpha = \frac{1}{2}$.

We apply the FRC2s collocation method to solve the linear ordinary differential equations at each iteration Eq. (7) with boundary conditions (8).

Approximation of functions $y_{n+1}(t)$ by using Eq. (5) is shown by $y_{m,n+1}(t)$. Now, for applying the collocation method, we construct the residual function for the Thomas-Fermi equation by substituting $y_{m,n+1}(t)$ for $y(t)$ in Eq. (1):

$$\mathrm{RES}_n^m(t) = \frac{d^2}{dt^2}\left(y_{m,n+1}(t)\right) - \frac{1}{\sqrt{t}}\left(y_{m,n+1}(t)\right)^{\frac{3}{2}}. \tag{10}$$

In this study, the roots of the FRC2s in the semi-infinite domain $[0, \infty)$ (Theorem 1) have been used as collocation points. Also, consider that all of the computations have been done by Maple 2015.

Boyd in Ref. [1] has provided the method of the experimental trial-and-error for calculating the approximation of the optimal value of L:

*“**The experimental trial-and-error method** (Optimizing infinite Interval Map Parameter) (Page 377 in Ref. [1]):*
Plot the coefficients a_i versus degree on a log-linear plot. If the graph abruptly flattens at some m, then this implies that L is TOO SMALL for the given m, and one should increase L until the flattening is postponed to $i = m$.”

It must be noted that the optimal value of L is dependent on m.

Fig. 1 presents the graph of the coefficients of $\log(|a_i|)$ for different values of L, $m = 200$ and $n = 50$, according to the above experimental trial-and-error method, the approximation optimal amount of L is about 21.

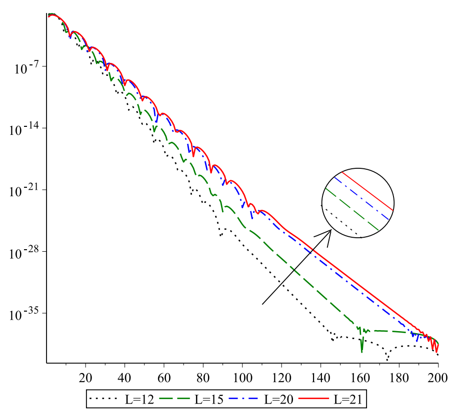


Figure 1: Graph of logarithm of coefficients $|a_i|$ with $m = 200$, $n = 50$, and different values of L, for calculating an approximation optimal value of L

Bellman & Kalaba [32] and Mandelzweig & Tabakin [38] proved the convergence of the QLM. Let $\delta y_{n+1}(t) \equiv y_{n+1}(t) - y_n(t)$, then it can show that $\| \delta y_{n+1} \| \leq k \| \delta y_n \|^2$ where k is a positive real constant [38]. Therefore, the convergence rate is of the order of 2, i.e. $O(h^2)$. We can also obtain for $(n+1)$-th iteration:

$$\| \delta y_{n+1} \| \leq (k \| \delta y_1 \|)^{2^n} / k. \tag{11}$$

Furthermore, it can be hoped that even if the initial guess is not appropriate, then after a while the solution converges [32].

4 Results and discussion

Calculating the amount of $y'(0)$ of Thomas-Fermi potential is very important for determining many physical properties of Thomas-Fermi atom.

Comparison of methods: Zaitsev et al. [40] showed that the Adams-Bashforth and Runge-Kutta methods to solve this equation on unbounded domains are ill-conditioned, hence, researchers have used the methods of numerical and semi-analytical for solving the equation, and some researchers can calculate very good solutions. For example, authors of [55, 57, 58, 59, 60, 61, 64, 68, 70] used the analytical methods for solving the equation and Amore et al. [68] were able to calculate the best solution using Pade-Hankel method, correct to 26 decimal places. Authors of [54, 56, 62, 63, 65, 66, 67] used the numerical methods for solving the equation and Parand & Delkhosh [73] were able to calculate the best solution using the combination of the quasilinearization method and the fractional order of rational Chebyshev collocation method, correct to 37 decimal places. In numerical methods, there is usually a numerical arbitrary parameter which selected by authors. Such as, in [54] the parameter is chosen 0.258497 to accuracy 10^{-6}, in [56] is chosen 0.93799968 to accuracy 10^{-8}, in [63] is chosen 0.62969503 to accuracy 10^{-6}, in [65] is chosen 0.0958885 to accuracy 10^{-7}, and in [67] is chosen 1.588071 to accuracy 10^{-7}. Here we choose $L = 21$ to accuracy 10^{-37}.

Table 1 presents some of the calculated values of $y'(0)$ of Thomas-Fermi potential by some researchers. It is clear that some researchers were able to calculate good solution and accuracy. The last three rows present the best solution obtained by the present method for different values of m.

Table 1: Comparison of the obtained values of $y'(0)$ by researchers, inaccurate digits are **bold**.

Author/Authors	Obtained value of $y'(0)$
Fermi (1928) [24]	-1.58
Baker (1930) [39]	-1.588**558**
Bush and Caldwell (1931) [41]	-1.58**9**
Miranda (1934) [42]	-1.5880**464**
Slater and Krutter (1935) [26]	-1.5880**8**
Feynman et al. (1949) [25]	-1.588**75**
Kobayashi et al. (1955) [43]	-1.58807**0972**
Mason (1964) [44]	-1.5880710
Laurenzi (1990) [45]	-1.588**588**
MacLeod (1992) [46]	-1.5880710226
Wazwaz (1999) [47]	-1.58807**6779**
Epele et al. (1999) [48]	-1.588**102**
Esposito (2002) [49]	-1.588
Liao (2003) [50]	-1.58**712**
Khan and Xu (2007) [51]	-1.58**6494973**
El-Nahhas (2008) [52]	-1.5**5167**
Yao (2008) [53]	-1.588**004950**
Parand and Shahini (2009) [54]	-1.58807**02966**
Marinca and Herianu (2011) [55]	-1.5880**659888**
Oulne (2011) [56]	-1.5880710**34**
Abbasbandy and Bervillier (2011) [57]	-1.58807102261137531271**9**
Fernandez (2011) [58]	-1.588071022611375313
Zhu et al. (2012) [59]	-1.58807**411**
Turkylmazoglu (2012) [60]	-1.5880**1**
Zhao et al. (2012) [61]	-1.5880710226
Boyd (2013) [62]	-1.5880710226113753127186845
Parand et al. (2013) [63]	-1.58807**0339**
Marinca and Ene (2014) [64]	-1.588071**9992**
Kilicman et al. (2014) [65]	-1.588071**347**
Jovanovic et al. (2014) [66]	-1.588071022**811**
Bayatbabolghani & Parand(2014)[67]	-1.588071
Amore et al. (2014) [68]	-1.58807102261137531271868450**8**
Fatoorehchi & Abolghasemi(2014)[69]	-1.58807**6818**
Liu and Zhu (2015) [70]	-1.58807**2**
Parand et al. (2016) [71]	-1.588071022611375312718684509
Parand et al. (2016) [72]	-1.588071022611375312718684509423
Parand and Delkhosh (2017) [73]	-1.5880710226113753127186845094239501095
Parand and Delkhosh (2017) [74]	-1.588071022611375312718684509
This paper [m=200]	-1.5880710226113753127186845094239501093
" " [m=100]	-1.5880710226113753127186845094239
" " [m=50]	-1.588071022611375312728

Table 2 presents the absolute errors in the calculation of $y'(0)$ for different values of m and the obtained results are compared with the best solution calculated in Ref. [73].

Table 2: Absolute errors of $y'(0)$ for different values of m and iterations

m	L_{opt}	10th Iter.	20th Iter.	30th Iter.	40th Iter.	50th Iter.
25	0.5	3.970e-08	3.939e-08	3.939e-08	3.939e-08	3.939e-08
75	5	6.667e-13	3.926e-18	5.878e-24	4.065e-30	4.646e-30
100	7	6.524e-13	5.656e-20	4.026e-24	8.314e-31	1.976e-33
175	19	6.524e-13	1.065e-25	1.349e-27	4.240e-31	1.908e-34
200	21	6.524e-13	3.477e-25	6.072e-29	9.237e-32	1.974e-37

Table 3: Obtained values of $y(t)$ by the present method for different values t

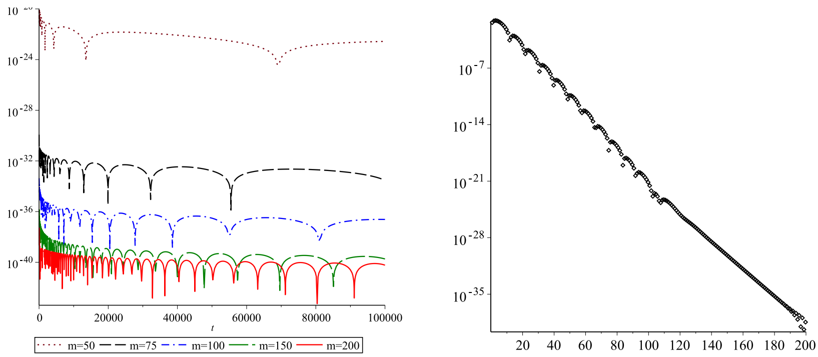
t	$y(t)$	t	$y(t)$	t	$y(t)$
0.25	0.7552014653133312	5	7.880777925136990e-2	125	5.423519678389911e-5
0.50	0.6069863833559799	6	5.942294925042258e-2	150	3.263396444625690e-5
0.75	0.5023468464123686	7	4.609781860449858e-2	175	2.115958647941346e-5
1.00	0.4240080520807056	8	3.658725526467680e-2	200	1.450180349694576e-5
1.25	0.3632014144595141	9	2.959093527054687e-2	300	4.548571953616680e-5
1.50	0.3147774637004581	10	2.431429298868086e-2	400	1.979732628112504e-5
1.75	0.2754513279960917	15	1.080535875582389e-2	500	1.034077168199939e-5
2.00	0.2430085071611195	20	5.784941191566940e-3	1000	1.351274773541057e-7
2.25	0.2158946265761301	25	3.473754416765632e-3	2000	1.733984751613821e-8
2.50	0.1929841234580007	50	6.322547829849047e-4	3000	5.189408334513832e-9
3.00	0.1566326732164958	75	2.182104320497469e-4	5000	1.130926706343084e-9
4.00	0.1084042569189077	100	1.002425681394073e-4	10000	1.42450045099559e-10

Tables 3 and 4 present the obtained results of $y(t)$ and $y'(t)$ by the present method for different values of t.

Table 4: Obtained values of $y'(t)$ by the present method for different values t

t	$y'(t)$	t	$y'(t)$	t	$y'(t)$
0.25	-0.7223069849102349	5	-2.356007495470051e-2	125	-1.202665391336449e-6
0.50	-0.4894116125745380	6	-1.586754953340707e-2	150	-6.091399478608917e-7
0.75	-0.3583068801675136	7	-1.114253181486708e-2	175	-3.410947673774533e-7
1.00	-0.2739890515933062	8	-8.088602969645474e-3	200	-2.057532316475268e-7
1.25	-0.2157941303007336	9	-6.033074714457392e-3	300	-4.365949618530290e-8
1.50	-0.1737387990139451	10	-4.602881871269254e-3	400	-1.436682305996181e-8
1.75	-0.1423209371968936	15	-1.515323082023606e-3	500	-6.034363442475256e-9
2.00	-0.1182431916254876	20	-6.472543327776920e-4	1000	-3.98801070822799e-10
2.25	-0.0994093212014470	25	-3.240429977697511e-4	2000	-2.57608536992070e-11
2.50	-0.0844261867988090	50	-3.249890204825881e-5	3000	-5.15300117644723e-12
3.00	-0.0624571308541209	75	-7.777974714283007e-6	5000	-6.75339712163883e-13
4.00	-0.0369437578241234	100	-2.739351068678330e-6	10000	-4.26161647550093e-14

Fig. 2 presents the graphs of the residual errors of RES_n^m of Eq. (10) with $m = 50, 75, 100, 150, 200$ and $n = 50$, and the logarithm of coefficients $|a_i|$ with $m = 200$ and $n = 50$, for showing the convergence of the method. It can see that the residual errors are very small value, about 10^{-39}.



(a) The residual errors (b) The logarithm of coefficients $|a_i|$

Figure 2: Graphs of the residual errors for different values of m and the logarithm of coefficients $|a_i|$, for showing the convergence of the method.

5 Conclusion

In this paper, the combination of the methods of the quasilinearization and the FRC2s collocation is used for constructing an approximation of the solution of the nonlinear singular Thomas-Fermi equation on unbounded domain. The present method has several advantages. For example, for the first time, the fractional order of rational Chebyshev functions of the second kind (FRC2s) has been introduced as a new basic for Spectral methods. The fractional basis were used to solve an ordinary differential equation and this provides an insight into an important issue. The roots of the FRC2s are used on unbounded domain $[0, \infty)$ as collocation points for solving Thomas-Fermi equation and the problem does not convert to a bounded domain. Some researchers solved the equation by changing the variables in this equation [58, 62] or domain truncation [38] but we solved the problem without any changing on variables or domain in this equation. An approximate optimal value of L is calculated. The convergence of the obtained results is shown. The accurate solutions for $y(t)$, $y'(t)$ and $y'(0)$ by 200 collocation points are obtained. This article provided a good history of solving Thomas-Fermi equation by other researchers and the numerical methods to solve equations in unbounded domains.

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Totally geodesic property of the unit tangent sphere bundle with g-natural metrics

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Abstract. In this paper, we consider the tangent bundle of a Riemannian manifold (M, g) with g-natural metrics and among all of these metrics, we specify those with respect to which the unit tangent sphere bundle with induced g-natural metric is totally geodesic. Also, we equip the unit tangent sphere bundle T_1M with g-natural contact (paracontact) metric structures, and we show that such structures are totally geodesic K-contact (K-paracontact) submanifolds of TM, if and only if the base manifold (M, g) has positive (negative) constant sectional curvature. Moreover, we establish a condition for g-natural almost contact B-metric structures on T_1M such that these structures be totally geodesic submanifolds of TM.

1 Introduction

One of the classical research fields rising in both mathematics and physics is the notion of totally geodesic submanifold. This geometric motif has still remained a topic of debate in some various branches of physics such as string theory and cosmology as well as in differential geometry. In recent years, many

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mathematicians and physicists have placed this notion in the center of attention and have obtained some important results (see for example [9, 4]).

On the other hand the motif of lifted metric on the tangent bundle of Riemannian manifolds is widely considered as an interesting field by many mathematicians. This notion was first introduced by Sasaki and in recent years his works have generated strong motivation for other mathematicians to study and develop this concept on the tangent bundles of Riemannian manifolds. In [3], the authors introduced the notion of g-natural metrics on the tangent bundle of a Riemannian manifold (M, g). In the framework of g-natural metrics on the tangent bundle and tangent sphere bundle of a Riemannian manifold (M, g), Abbassi, et al. have made significant contributions (see for example [2, 3]).

The other fundamental motif in differential geometry of manifolds, given by Sasaki in [8], is the notion of the almost contact structure. As a counterpart of the almost contact metric structure, the notion of the almost contact B-metric structure has been an interesting research field for many geometrists in differential geometry of manifolds and geometric properties of such structures have been studied frequently (see for example [6]).

The aim of this paper is to specify all of g-natural metrics on the tangent bundle of a Riemannian manifold (M, g), such that with respect to them the unit tangent sphere bundle with induced g-natural metric is totally geodesic. The work is organized in the following way. In Section 2, we begin with a study on the concept of g-natural metrics on the tangent bundle and unit tangent sphere bundle of a Riemannian manifold (M, g) and we provide some necessary information about the mentioned spaces. We proceed in Section 3, to describe and study the totally geodesic property of the unit tangent sphere bundle and then we present the main theorem of the paper. In other words, we determine some conditions for the g-natural metric G on the tangent bundle TM, such that the unit tangent sphere bundle T_1M with the induced g-natural metric $\widetilde{G}$ from G is totally geodesic. In the next two sections, we equip the unit tangent sphere bundle T_1M with g-natural contact metric and paracontact metric structures, and we show that there is a direct correlation between sectional curvature of M and K-contact and K-paracontact totally geodesic property of T_1M. Also, we obtain a condition for a g-natural almost contact B-metric structure on T_1M such that this structure be totally geodesic submanifold of TM.

2 g-natural metric on sphere bundle

We provide some necessary information on g-natural metrics on the tangent bundle and unit tangent sphere bundle in this section.

2.1 g-natural metrics on the tangent bundle

We consider the $(n+1)$-dimensional Riemannian manifold (M, g) and denoting by ∇ its Levi-Civita connection, the tangent space $TM_{(x,u)}$ of the tangent bundle TM at a point (x, u) splits as

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical spaces with respect to ∇. The *horizontal lift* of $X \in M_x$ to $(x, u) \in TM$ is a unique vector $X^h \in \mathcal{H}_{(x,u)}$ such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. Moreover, for $X \in M_x$, the *vertical lift* of vector X is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Needless to say, 1-forms df on M are considered as functions on TM (i.e., $(df)(x, u) = uf$). The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$. As a result of this explanation, one can write each tangent vector $Z \in (TM)_{(x,u)}$ in the form $Z = X^h + Y^v$, where $X, Y \in M_x$, are uniquely determined vectors. Also, the geodesic flow vector field on TM is uniquely determined by $u^h_{(x,u)} = u^i(\frac{\partial}{\partial x^i})^h_{(x,u)}$, for any point $x \in M$ and $u \in TM_x$, with respect to the local coordinates $\{\frac{\partial}{\partial x^i}\}$ on M. In [2], the authors bring up a discussion on g-natural metrics on tangent bundle TM of a Riemannian manifold (M, g), including the following characterization.

Proposition 1 [2] *Let* (M, g) *be a Riemannian manifold and* G *be the g-natural metric on* TM. *Then there are six smooth functions* $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, *such that for every* $u, X, Y \in M_x$, *we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g(X, Y) + (\beta_1 + \beta_3)(r^2)g(X, u)g(Y, u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2)g(X, Y) + \beta_2(r^2)g(X, u)g(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2)g(X, Y) + \beta_1(r^2)g(X, u)g(Y, u), \end{cases} \quad (1)$$

where $r^2 = g(u, u)$.

As a prime example of Riemannian g-natural metrics on the tangent bundle, we express the Sasaki metric obtained from Proposition 1 with

$$\alpha_1(t) = 1, \qquad \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0.$$

2.2 g-natural metric on the unit tangent sphere bundle

Let (M, g) be a Riemannian manifold. The hyperspace

$$T_1M = \{(x, u) \in TM \mid g_x(u, u) = 1\},$$

in TM, is called the unit tangent sphere bundle over the Riemannian manifold (M, g). Denoting by $(T_1M)_{(x,u)}$, the tangent space of T_1M at a point $(x, u) \in T_1M$, we have

$$(T_1M)_{(x,u)} = \{X^h + Y^v | X \in M_x, Y \in \{u\}^\perp \subset M_x\}.$$

A g-natural metric on T_1M, is any metric $\widetilde{G}$, induced on T_1M by a g-natural metric G on TM. Using [5], we know that $\widetilde{G}$ is completely determined by the values of four real constants, namely

$$a = \alpha_1(1), \qquad b = \alpha_2(1), \qquad c = \alpha_3(1), \qquad d = \beta(1) = (\beta_1 + \beta_3)(1).$$

Let (M, g) be a $(2n + 1)$-dimensional Riemannian manifold. Considering an orthogonal basis $\{X_0 = u, X_1, \ldots, X_n\}$ on $x \in M$, we define $X_0^h = u^h$. The metric $\widetilde{G}$ on T_1M is completely determined by

$$\begin{cases} \widetilde{G}_{(x,u)}(X_i^h, X_j^h) = (a + c)g_x(X_i, X_j) + dg_x(X_i, u)g_x(X_j, u), \\ \widetilde{G}_{(x,u)}(X_i^h, Y_j^v) = bg_x(X_i, Y_j), \\ \widetilde{G}_{(x,u)}(Y_i^v, Y_j^v) = ag_x(Y_i, Y_j), \end{cases} \tag{2}$$

at any point $(x, u) \in T_1M$, for all $X_i, Y_j \in M_x$, with Y_j orthogonal to u [5].

Taking into account $\phi = a(a + c + d) - b^2$, using the Schmidt's orthogonalization process and some standard calculations, it can be shown that whenever $\phi \neq 0$, the following vector field on TM is normal to T_1M and is unitary at any point of T_1M for all $(x, u) \in TM$

$$N^G_{(x,u)} = \frac{1}{\sqrt{|(a + c + d)\phi|}}[-bu^h + (a + c + d)u^v].$$

Moreover, for a vector $X \in M_x$ at $(x, u) \in T_1M$, the *tangential lift* X^{t_G} with respect to G is defined as the tangential projection of the vertical lift of X to (x, u) with respect to N^G, in other words

$$X^{t_G} = X^v - \frac{\phi}{|\phi|}G_{(x,u)}(X^v, N^G_{(x,u)})N^G_{(x,u)} = X^v - \sqrt{\frac{|\phi|}{|a + c + d|}}g_x(X, u)N^G_{(x,u)}. \tag{3}$$

Also, if $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$. Assuming that $b = 0$, the tangential lift X^{t_G} and the classical tangential lift X^t defined for the case of the Sasaki metric coincide. In the most general case, we have

$$X^{t_G} = X^t + \frac{b}{a+c+d} g(X,u)u^h.$$

Remark 1 [5] *The tangential lift* u^{t_G} *to* $(x,u) \in T_1M$ *of the vector* u *is given by* $u^{t_G} = \frac{b}{a+c+d}u^h$, *that is,* u^{t_G} *is a horizontal vector. Therefore, the tangent space* $(T_1M)_{(x,u)}$ *of* T_1M *at* (x,u) *is spanned by vectors of the form* X^h *and* Y^{t_G} *as follows,*

$$(T_1M)_{(x,u)} = \{X^h + Y^{t_G} | X \in M_x, Y \in \{u\}^\perp \subset M_x\}, \tag{4}$$

hence, the operation of tangential lift from M_x *to a point* $(x,u) \in T_1M$ *will always be applied **only** to those vectors of* M_x *which are orthogonal to* u*.*

Taking into account Remark 1, the Riemannian metric $\widetilde{G}$ on T_1M, induced from G, is completely determined by the following identities.

$$\begin{cases} \widetilde{G}(X_1^h, X_2^h) = (a+c)g_x(X_1,X_2) + dg_x(X_1,u)g_x(X_2,u), \\ \widetilde{G}(X_1^h, Y_1^{t_G}) = bg_x(X_1,Y_1), \\ \widetilde{G}(Y_1^{t_G}, Y_2^{t_G}) = ag_x(Y_1,Y_2), \end{cases}$$

where $X_i, Y_i \in M_x$, for $i = 1,2$ with Y_i orthogonal to u. It should be noted that by the above equations, horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$, if and only if $b = 0$. Further details about g-natural metrics on the tangent bundle can be found in [5]. Here, we present the following propositions.

Proposition 2 [1] *Let* (M,g) *be a Riemannian manifold,* ∇ *its Levi-Civita connection and* R *its curvature tensor. Let* G *be the* g*-natural metric on* TM *given by* (1) *with* $a > 0$, $\alpha = a(a+c) - b^2 > 0$, *and* $\phi(t) = a(a+c+t\beta(t)) - b^2 > 0$, *for all* $t \in [0,\infty)$. *Then the Levi-Civita connection* $\overline{\nabla}$ *of* (TM,G) *is characterized by*

1.

$$(\overline{\nabla}_{X^h}Y^h)_{(x,u)} = \Big\{(\nabla_X Y)_x - \frac{ab}{2\alpha}[R(X_x,u)Y_x + R(Y_x,u)X_x]$$
$$+ \frac{b\beta(1)}{2\alpha}[g(X_x,u)Y_x + g(Y_x,u)X_x] + \frac{b}{\alpha\phi}[a^2\beta(1)g(R(X_x,u)Y_x,u)$$
$$+ (\alpha\beta'(1) - a\beta^2(1))g(X_x,u)g(Y_x,u)]u\Big\}^h + \Big\{\frac{b^2}{\alpha}R(X_x,u)Y_x$$

$$- \frac{a(a+c)}{2\alpha} R(X_x, Y_x)u - \frac{(a+c)\beta(1)}{2\alpha}[g(Y_x, u)X_x + g(X_x, u)Y_x]$$
$$+ \frac{1}{\alpha\phi}[-ab^2\beta(1)g(R(X_x, u)Y_x, u) + (-\alpha(a+c+\beta(1))\beta'(1)$$
$$+ b^2\beta^2(1))g(Y_x, u)g(X_x, u)]u\Big\}^v,$$

2.

$$(\overline{\nabla}_{X^h} Y^v)_{(x,u)} = \Big\{ - \frac{a^2}{2\alpha} R(Y_x, u)X_x + \frac{a\beta(1)}{2\alpha} g(X_x, u)Y_x$$
$$+ \frac{a}{2\alpha\phi}[a^2\beta(1)g(R(X_x, u)Y_x, u) + \alpha\beta(1)g(X_x, Y_x) + (2\alpha\beta'(1)$$
$$- a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^h + \Big\{(\nabla_X Y)_x + \frac{ab}{2\alpha} R(Y_x, u)X_x$$
$$- \frac{b\beta(1)}{2\alpha} g(X_x, u)Y_x + \frac{b}{2\alpha\phi}[-\alpha\beta(1)g(X_x, Y_x) - a^2\beta(1)g(R(X_x, u)Y_x, u)$$
$$- (2\alpha\beta'(1) - a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^v,$$

3.

$$(\overline{\nabla}_{X^v} Y^h)_{(x,u)} = \Big\{ - \frac{a^2}{2\alpha} R(X_x, u)Y_x + \frac{a\beta(1)}{2\alpha} g(Y_x, u)X_x$$
$$+ \frac{a}{2\alpha\phi}[a^2\beta(1)g(R(X_x, u)Y_x, u) + \alpha\beta(1)g(X_x, Y_x) + (2\alpha\beta'(1)$$
$$- a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^h + \Big\{\frac{ab}{2\alpha} R(X_x, u)Y_x - \frac{b\beta(1)}{2\alpha} g(Y_x, u)X_x$$
$$+ \frac{b}{2\alpha\phi}[-\alpha\beta(1)g(X_x, Y_x) - a^2\beta(1)g(R(X_x, u)Y_x, u) - (2\alpha\beta'(1)$$
$$- a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^v,$$

4.

$$(\overline{\nabla}_{X^v} Y^v)_{(x,u)} = 0,$$

for all vector fields X, Y *on* M *and* $(x, u) \in TM$, *where* $g_x(u, u) = 1$.

Proposition 3 [1] *At* $(x,u) \in T_1M$, *the Levi-Civita connection* $\widetilde{\nabla}$ *on* T_1M *is given by*

1.

$$
\begin{aligned}
(\widetilde{\nabla}_{X^h}Y^h)_{(x,u)} &= \Big\{(\nabla_X Y)_x - \frac{ab}{2\alpha}[R(X_x,u)Y_x + R(Y_x,u)X_x] + \frac{bd}{2\alpha}[g(X_x,u)Y_x \\
&+ g(Y_x,u)X_x] + \frac{b}{\alpha(a+c+d)}[(ad+b^2)g(R(X_x,u)Y_x,u) \\
&- d(a+c+d)g(X_x,u)g(Y_x,u)]u\Big\}^h + \Big\{\frac{b^2}{\alpha}R(X_x,u)Y_x \\
&- \frac{a(a+c)}{2\alpha}R(X_x,Y_x)u - \frac{(a+c)d}{2\alpha}[g(Y_x,u)X_x + g(X_x,u)Y_x] \\
&+ \frac{1}{\alpha}[-b^2 g(R(X_x,u)Y_x,u) + d(a+c)g(Y_x,u)g(X_x,u)]u\Big\}^{t_G},
\end{aligned}
$$

2.

$$
\begin{aligned}
(\widetilde{\nabla}_{X^h}Y^{t_G})_{(x,u)} &= \Big\{-\frac{a^2}{2\alpha}R(Y_x,u)X_x + \frac{ad}{2\alpha}g(X_x,u)Y_x \\
&+ \frac{1}{2\alpha(a+c+d)}[a(ad+b^2)g(R(X_x,u)Y_x,u) + \alpha d g(X_x,Y_x)]u\Big\}^h \\
&+ \Big\{(\nabla_X Y)_x + \frac{ab}{2\alpha}R(Y_x,u)X_x - \frac{bd}{2\alpha}g(X_x,u)Y_x - \frac{ab}{2\alpha}g(R(X_x,u)Y_x,u)u\Big\}^{t_G},
\end{aligned}
$$

3.

$$
\begin{aligned}
(\widetilde{\nabla}_{X^{t_G}}Y^h)_{(x,u)} &= \Big\{-\frac{a^2}{2\alpha}R(X_x,u)Y_x + \frac{ad}{2\alpha}g(Y_x,u)X_x \\
&+ \frac{1}{2\alpha(a+c+d)}[a(ad+b^2)g(R(X_x,u)Y_x,u) + \alpha d g(X_x,Y_x)]u\Big\}^h \\
&+ \Big\{\frac{ab}{2\alpha}R(X_x,u)Y_x - \frac{bd}{2\alpha}g(Y_x,u)X_x - \frac{ab}{2\alpha}g(R(X_x,u)Y_x,u)u\Big\}^{t_G},
\end{aligned}
$$

4.

$$
(\widetilde{\nabla}_{X^{t_G}}Y^{t_G})_{(x,u)} = 0,
$$

for all $(x,u) \in T_1M$ *and* X, Y *on* M *satisfying* (4).

3 Totally geodesic property of the sphere bundle

We consider a submanifold M of a (pseudo) Riemannian manifold $(\overline{M}, \overline{g})$. The (pseudo) Riemannian metric $\overline{g}$ induces a (pseudo) Riemannian metric g on the submanifold M. Then (M, g) is also called a (pseudo) Riemannian submanifold of $(\overline{M}, \overline{g})$. A submanifold M of a (pseudo) Riemannian manifold $(\overline{M}, \overline{g})$ is called totally geodesic if any geodesic on the submanifold M with its induced (pseudo) Riemannian metric g is also a geodesic on $(\overline{M}, \overline{g})$. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections on $(\overline{M}, \overline{g})$ and (M, g) respectively. The shape tensor or second fundamental form tensor II is a symmetric tensor field which can be defined as follows

$$\mathrm{II}(X, Y) = \overline{\nabla}_X Y - \nabla_X Y,$$

for all vector fields X, Y on M. The (pseudo) Riemannian submanifold M is totally geodesic provided its shape tensor vanishes, i.e. $\mathrm{II} = 0$ [7]. Here, we provide the main theorem of this paper.

Theorem 1 *The unit tangent sphere bundle* $(T_1M, \widetilde{G})$ *is a totally geodesic submanifold of* (TM, G) *if and only if* G *is a* g*-natural metric on* TM *with* $b = 0$ *and* $\beta'(1) = 0$.

Proof. First, notice that (4) yields that the tangent space of T_1M at (x, u) can be written as

$$(T_1M)_{(x,u)} = \mathrm{span}(u^h) \oplus \{X^h | X \perp u\} \oplus \{Y^v | Y \perp u\}. \tag{5}$$

Now, we compute the coefficients of the fundamental tensor II as follows. Taking into account Proposition 2 and Proposition 3 and (3) we get

$$\begin{aligned}
\mathrm{II}_{(x,u)}(X^h, Y^h) &= (\overline{\nabla}_{X^h} Y^h)_{(x,u)} - (\widetilde{\nabla}_{X^h} Y^h)_{(x,u)} \\
&= \left[\frac{ba^2\beta(1)}{\alpha\phi} - \frac{b(ad+b^2)}{\alpha(a+c+d)}\right] g(R(X_x, u)Y_x, u)u^h \\
&\quad + \left[-\frac{ab^2\beta(1)}{\alpha\phi} + \frac{b^2}{\alpha}\right] g(R(X_x, u)Y_x, u)u^v,
\end{aligned}$$

$$\begin{aligned}
\mathrm{II}_{(x,u)}(X^v, Y^h) = &\left[\frac{a^3\beta(1)}{2\alpha\phi} - \frac{a(ad+b^2)}{2\alpha(a+c+d)}\right] g(R(X_x, u)Y_x, u)u^h \\
&- \frac{b\alpha\beta(1)}{2\alpha\phi} g(X_x, Y_x)u^v + \left[-\frac{ba^2\beta(1)}{2\alpha\phi} + \frac{ab}{2\alpha}\right] g(R(X_x, u)Y_x, u)u^v,
\end{aligned}$$

$$\begin{aligned}\mathrm{II}_{(x,u)}(u^h,u^h) &= \left[\frac{b(\alpha\beta'(1)-a\beta^2(1))}{\alpha\phi}+\frac{bd}{\alpha}\right]u^h\\ &\quad+\left[\frac{(-\alpha(a+c+\beta(1))\beta'(1)+b^2\beta^2(1))}{\alpha\phi}\right]u^v,\end{aligned}$$

$$\mathrm{II}_{(x,u)}(X^v,Y^v)=\mathrm{II}_{(x,u)}(u^h,Y^h)=\mathrm{II}_{(x,u)}(X^v,u^h)=0,$$

for all X, Y satisfying (5) where $\alpha = a(a+c)-b^2$, $\phi = a(a+c+d)-b^2$ and $\beta(1) = d$. Therefore, the second fundamental form II vanishes if and only if the following system of equations

$$\begin{cases} \dfrac{ba^2\beta(1)}{\alpha\phi}=\dfrac{b(ad+b^2)}{\alpha(a+c+d)}, \quad \dfrac{ab^2\beta(1)}{\alpha\phi}=\dfrac{b^2}{\alpha}, \quad \dfrac{a^3\beta(1)}{2\alpha\phi}=\dfrac{a(ad+b^2)}{2\alpha(a+c+d)},\\[2ex] \dfrac{b\alpha\beta(1)}{2\alpha\phi}=0, \qquad \dfrac{ba^2\beta(1)}{2\alpha\phi}=\dfrac{ab}{2\alpha}, \qquad \dfrac{b(\alpha\beta'(1)-a\beta^2(1))}{\alpha\phi}=-\dfrac{bd}{\alpha},\\[2ex] \dfrac{-\alpha(a+c+\beta(1))\beta'(1)+b^2\beta^2(1)}{\alpha\phi}=0, \end{cases} \tag{6}$$

Satisfies. Standard calculations show that this system of equations satisfies if and only if $b = 0$ and $\beta'(1) = 0$. Hence, $(T_1M, \widetilde{G})$ is a totally geodesic submanifold of (TM, G) if and only if G is a g-natural metric on TM with $b = 0$ and $\beta'(1) = 0$. □

As immediate consequences of this theorem, we have the following corollaries.

Corollary 1 *The Sasaki metric obtained from (1) for*

$$\alpha_1(t)=1, \quad \alpha_2(t)=\alpha_3(t)=\beta_1(t)=\beta_2(t)=\beta_3(t)=0,$$

satisfies the conditions $b = 0$ and $\beta'(1) = (\beta_1+\beta_3)'(1) = 0$. Therefore, the unit tangent sphere bundle T_1M is a totally geodesic submanifold of (TM, g_S).

Corollary 2 *The Cheeger-Gromoll metric g_{CG}, as a classical example of g-natural metrics on the tangent bundle, is obtained for*

$$\alpha_1(t)=\beta_1(t)=-\beta_3(t)=\frac{1}{1+t}, \qquad \alpha_2(t)=\beta_2(t)=0, \qquad \alpha_3(t)=\frac{t}{1+t}.$$

So we have $b = \alpha_2(1) = 0$ and $\beta'(1) = (\beta_1+\beta_3)'(1) = 0$. Hence, T_1M with induced g-natural metric is a totally geodesic submanifold of (TM, g_{CG}).

Corollary 3 *Metrics of Cheeger-Gromoll type* $h_{m,r}$ *are obtained from* (1) *when*

$$\alpha_1(t) = \frac{1}{(1+t)^m}, \qquad \alpha_3(t) = 1 - \alpha_1(t),$$
$$\alpha_2(t) = \beta_2(t) = 0, \qquad \beta_1(t) = -\beta_3(t) = \frac{r}{(1+t)^m},$$

where $m \in \mathbb{R}$ *and* $r \geqslant 0$. *Obviously, these metrics satisfy* $b = 0$ *and* $\beta'(1) = 0$. *Hence, the unit tangent sphere bundle* T_1M *with induced* g*-natural metric is a totally geodesic submanifold of* $(TM, h_{m,r})$.

Corollary 4 *Kaluza-Klein metrics, are obtained from (1) for*

$$\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0.$$

Thus, Kaluza-Klein metrics satisfy $b = 0$ *and* $\beta'(1) = (\beta_1 + \beta_3)'(1) = 0$ *and therefore,* T_1M *with induced* g*-natural metric is a totally geodesic submanifold of* TM *with Kaluza-Klein metrics.*

4 g-natural contact and paracontact metric structures on tangent sphere bundle

In this section, we equip the unit tangent sphere bundle T_1M with g-natural contact metric and paracontact metric structures, and we show that there is a direct correlation between sectional curvature of M and K-contact and K-paracontact totally geodesic property of T_1M.

A $(2n+1)$-dimensional manifold M is called a contact manifold if it admits a global 1 form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M and a unique vector field ξ such that $\eta(\xi) = 1$ and $d\eta(\xi, .) = 0$. In addition, a Riemannian metric g is said to be an associated metric if there exists a tensor φ, of type $(1,1)$, such that

$$\eta = g(\xi, .), \qquad d\eta = g(\xi, .), \qquad \varphi^2 = -I + \eta \otimes \xi.$$

Moreover, a Riemannian metric g is said to be compatible with the contact structure if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M.

Now, (η, g, ξ, φ) is called a contact metric structure and $(M, \eta, g, \xi, \varphi)$ a contact metric manifold.

The following proposition determines the g-natural contact metric structure on unit tangent sphere bundle.

Proposition 4 [1] *Let* (M, g) *be a Riemannian manifold and* T_1M *be its unit tangent sphere bundle. Let* $\widetilde{G}$ *be a* g*-natural metric on* T_1M *given by* (2). *The set* $(\widetilde{G}, \eta, \varphi, \xi)$ *described by* (7)-(10) *is a family of contact metric structures over* T_1M.

$$\xi = u^h, \tag{7}$$

$$\eta(X^h) = g(X, u), \qquad \eta(X^{t_G}) = bg(X, u), \tag{8}$$

$$\begin{cases} \varphi(X^h) = \dfrac{1}{2\alpha}[-bX^h + (a+c)X^{t_G} + \dfrac{bd}{a+c+d}g(X,u)u^h], \\ \varphi(X^{t_G}) = \dfrac{1}{2\alpha}[-aX^h + bX^{t_G} + \dfrac{\phi}{a+c+d}g(X,u)u^h], \end{cases} \tag{9}$$

$$4\alpha = a + c + d = 1. \tag{10}$$

A K-contact manifold is a contact metric manifold $(M, g, \eta, \varphi, \xi)$ such that the characteristic vector field ξ is a Killing vector field with respect to g. We refer to [1] for more information on K-contact manifolds. Now, we provide the following statement.

Theorem 2 *Let* $\widetilde{G}$ *be a Riemannian* g*-natural metric on* T_1M *and* $(T_1M, \widetilde{G})$ *be a totally geodesic submanifold of* (TM, G). *The contact metric manifold* $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ *is K-contact if and only if the base manifold* (M, g) *has positive constant sectional curvature* $\frac{a+c}{a}$.

Proof. $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ is K-contact manifold if and only if the characteristic vector field ξ is a Killing vector field with respect to $\widetilde{G}$ and according to Theorem 2 in [1], ξ is Killing vector field if and only if $b = 0$ and (M, g) has constant sectional curvature $\frac{a+c}{a} > 0$. Using Theorem 1, the unit tangent sphere bundle $(T_1M, \widetilde{G})$ is a totally geodesic submanifold of (TM, G) if and only if $b = 0$ and $\beta'(1) = 0$. Consequently, totally geodesic submanifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$

of (TM, G) is K-contact if and only if the base manifold (M, g) has positive constant sectional curvature $\frac{a+c}{a} > 0$. □

Analogous to the contact cases, a $(2n + 1)$-dimensional manifold (M, g) is called a paracontact manifold if it admits a $(1, 1)$-tensor field φ, a vector field ξ and a 1-form η satisfying

$$\eta = g(\xi, .), \qquad d\eta = g(\xi, .), \qquad \varphi^2 = I - \eta \otimes \xi.$$

Also, a pseudo-Riemannian metric g is said to be compatible with the paracontact structure if

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all X, Y vector fields on M.

Now we report the following statement from [5].

Proposition 5 [5] *Let (M, g) be a Riemannian manifold and T_1M be its unit tangent sphere bundle. Let $\widetilde{G}$ be a g-natural metric on T_1M given by (2). The set $(\widetilde{G}, \eta, \varphi, \xi)$ described by (11)-(14) is a family of paracontact metric structures over T_1M.*

$$\xi = u^h, \tag{11}$$

$$\eta(X^h) = g(X, u), \qquad \eta(X^{t_G}) = bg(X, u), \tag{12}$$

$$\begin{cases} \varphi(X^h) = \dfrac{1}{2\alpha}[-bX^h + (a + c)X^{t_G} + \dfrac{bd}{a + c + d} g(X, u)u^h], \\[2ex] \varphi(X^{t_G}) = \dfrac{1}{2\alpha}[-aX^h + bX^{t_G} + \dfrac{\phi}{a + c + d} g(X, u)u^h], \end{cases} \tag{13}$$

$$-4\alpha = a + c + d = 1. \tag{14}$$

A paracontact metric structure (φ, g, η, ξ) is said to be K-paracontact if ξ is a Killing vector filed.

Remark 2 *According to [5], in order to construct a paracontact metric structure with an associated g-natural metric on the unit tangent sphere bundle T_1M, it requires to $a + c + d > 0$ and $\alpha < 0$. It deduces from $\alpha < 0$ that the induced g-natural metric $\widetilde{G}$ on T_1M is a non-degenerate pseudo-Riemannian metric. It can be shown that for $\alpha < 0$ and $\phi > 0$, Proposition 2 and Proposition 3 and consequently Theorem 1 remain true.*

Here, we have the following.

Theorem 3 *Let* $\widetilde{G}$ *be a pseudo-Riemannian* g*-natural metric on* T_1M *and* $(T_1M, \widetilde{G})$ *be a totally geodesic submanifold of* (TM, G)*. The paracontact metric manifold* $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ *is K-paracontact manifold if and only if the base manifold* (M, g) *has negative constant sectional curvature* $\frac{a+c}{a} < 0$*.*

Proof. The paracontact metric manifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ is K-paracontact if and only if ξ is a Killing vector field with respect to pseudo-Riemannian metric $\widetilde{G}$. It concludes from Theorem 3 of [5] that ξ is Killing vector field if and only if $b = 0$ and M has negative sectional curvature $\frac{a+c}{a}$. Moreover, by Theorem 1, $(T_1M, \widetilde{G})$ is a totally geodesic submanifold of (TM, G) if and only if $b = 0$ and $\beta'(1) = 0$. Consequently, totally geodesic submanifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ of (TM, G) is K-paracontact if and only if the base manifold (M, g) has negative constant sectional curvature $\frac{a+c}{a} < 0$. □

5 g-natural almost contact b-metric structures on unit tangent sphere bundle

In this section, we establish a condition for a g-natural almost contact B-metric structure on T_1M such that this structure be a totally geodesic submanifold of TM.

A $(2n+1)$-dimensional manifold M has an almost contact B-metric structure if it admits a tensor field φ of type $(1, 1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0,$$
$$\eta \circ \varphi = 0, \quad g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y).$$

Now we consider the unit tangent sphere bundle of a Riemannian manifold (M, g) with g-natural metric, and we equip it with an almost contact B-metric structure denoted briefly by $(T_1M, \varphi, \xi, \eta, \tilde{G})$, and also a basis $\{X^h, X^{t_G}, \xi\}$ such that $X^h, X^{t_G} \perp \xi$, with respect to $\tilde{G}$, where $\xi = u^h$. An almost contact structure on T_1M is defined by

$$\eta(X^h) = \eta(X^{t_G}) = 0, \;\; \eta(\xi) = 1, \;\; \varphi(X^h) = X^{t_G}, \quad \varphi(X^{t_G}) = -X^h, \quad \varphi(\xi) = 0.$$

In order to construct an almost contact B-metric structure with an associated g-natural metric on the unit tangent sphere bundle T_1M, it requires to $a +$

$c+d>0$ and $\alpha<0$. It deduces from $\alpha<0$ that the induced g-natural metric $\widetilde{G}$ on T_1M is a non-degenerate pseudo-Riemannian metric. It can be shown that for $\alpha<0$ and $\phi>0$, Proposition 2 and Proposition 3 and consequently Theorem 1 still remain true. Also, pseudo-Riemannian metric $\widetilde{G}$ must be of signature $(n,n+1)$ or $(n+1,1)$, therefore, it requires to $b=0$. Hence, the adapted g-natural metric on the unit tangent sphere bundle T_1M with almost contact B-metric structure is of following form

$$\begin{cases} \widetilde{G}_{(x,u)}(X_i^h,X_j^h)=(a+c)g_x(X_i,X_j)+dg_x(X_i,u)g_x(X_j,u),\\ \widetilde{G}_{(x,u)}(X_i^h,Y_j^{t_G})=0,\\ \widetilde{G}_{(x,u)}(Y_i^{t_G},Y_j^{t_G})=ag_x(Y_i,Y_j), \end{cases} \tag{15}$$

for all vector fields X,Y on M with $Y\perp u$. Also, we have following relations

$$\tilde{G}(\varphi X_i^h,\varphi X_j^h)=-\tilde{G}(X_i^h,X_j^h),\qquad \tilde{G}(\varphi X_i^{t_G},\varphi X_j^{t_G})=-\tilde{G}(X_i^{t_G},X_j^{t_G}),$$

which give that $\tilde{G}$ is a B-metric. As a result of these relations we have $a+c=-a$. Notice that using $b=0$ and $a+c=-a$, we conclude that $\tilde{G}$ is of signature $(n,n+1)$ or $(n+1,1)$. Now we provide the following statement.

Theorem 4 *The unit tangent sphere bundle* $(T_1M,\widetilde{G},\varphi,\eta,\xi)$ *equipped with a* g*-natural almost contact B-metric structure is a totally geodesic submanifold of* (TM,G) *if and only if* G *is a* g*-natural metric on* TM *with* $\beta'(1)=0$.

Proof. Taking into the account Theorem 1, the unit tangent sphere bundle $(T_1M,\widetilde{G})$ is a totally geodesic submanifold of (TM,G) if and only if G is a g-natural metric on TM with $b=0$ and $\beta'(1)=0$. Also, using (15) for a g-natural almost contact B-metric $\widetilde{G}$ we have $b=0$. Hence, $(T_1M,\widetilde{G},\varphi,\eta,\xi)$ is a totally geodesic submanifold of (TM,G) if and only if G is a g-natural metric on TM with $\beta'(1)=0$. □

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On the $\mathcal{L}$-duality of a Finsler space with exponential metric $\alpha e^{\beta/\alpha}$

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Abstract. The (α, β)-metrics are the most studied Finsler metrics in Finsler geometry with Randers, Kropina and Matsumoto metrics being the most explored metrics in modern Finsler geometry. The $\mathcal{L}$-dual of Randers, Kropina and Matsumoto space have been introduced in [3, 4, 5], also in recent the $\mathcal{L}$-dual of a Finsler space with special (α, β)-metric and generalized Matsumoto spaces have been introduced in [16, 17] . In this paper, we find the $\mathcal{L}$-dual of a Finsler space with an exponential metric $\alpha e^{\beta/\alpha}$, where α is Riemannian metric and β is a non-zero one form.

1 Introduction

The concept of $\mathcal{L}$-duality between Lagrange and Finsler spaces was introduced by R. Miron [8] in 1987. Since then it has been studied intensively by many Finsler geometers [3, 4, 5]. The $\mathcal{L}$-duals of a Finsler spaces with some special (α, β)-metrics have been obtained in [14, 15]. The concept of Finslerian and Lagrangian structures were introduced in the papers [9, 13] and the theory of higher order Lagrange and Hamilton spaces were discussed in [10, 11, 12]. Further, the geometry of higher order Finsler spaces have been studied in

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[1, 7, 11].
The importance of $\mathcal{L}$-duality is not limited to computing only the dual of some Finsler fundamental functions but many other geometrical problems have been solved by taking the $\mathcal{L}$-duals of Finsler spaces. In fact, duality has been used to solve the complex Zermelo nevigation problem of classifying Randers metrics of constant flag curvature [2] and it has been also used to study the geometry of a Cartan space [4]. In general, duality can be used to solve the geometrical problems of (α, β) metrics. Here, we study the $\mathcal{L}$-dual of the Finsler space associated with the exponential metric $\alpha e^{\beta/\alpha}$, where α is Riemannian metric and β is a non-zero one form.

2 The Legendre transformation

A Finsler space $F^n = (M, F(x, y))$ is said to have an (α, β)-metric if F is a positively homogeneous function of degree one in two variables α and β, where $\alpha^2 = a(y, y) = a_{ij}y^i y^j$, $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, α is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\widetilde{TM} = TM \setminus \{0\}$. A Finsler space with the fundamental function:

$$F(x, y) = \alpha(x, y) + \beta(x, y)$$

is called a *Randers space* [6].

A Finsler space having the fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)}$$

is called a *Kropina space* and one with

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)}$$

is called a *Matsumoto space.*

A Finsler space with the fundamental function:

$$F(x, y) = \alpha e^{\beta/\alpha} \tag{1}$$

is called a *Finsler space with exponential metric.*

Definition 1 *A Cartan space* C^n *is a pair* (M, H) *which consists of a real* n*-dimensional* C^∞*-manifold* M *and a Hamiltonian function* $H : T^*M \setminus \{0\} \to \mathfrak{R}$, *where* (T^*M, π^*, M) *is the cotangent bundle of* M *such that* $H(x,p)$ *has the following properties:*

1. *It is two homogeneous with respect to* $p_i \quad (i = 1, 2, \ldots, n)$.
2. *The tensor field* $g^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ *is nondegenerate.*

Let $C^n = (M, K)$ be an n-dimensional Cartan space having the fundamental function $K(x,p)$. We can also consider Cartan spaces having the metric functions of the following forms

$$K(x,p) = \sqrt{a^{ij}(x)p_i p_j} + b^i(x)p_i$$

or

$$K(x,p) = \frac{a^{ij}p_i p_j}{b^i(x)p_i}$$

and we will again call these spaces Randers and Kropina spaces respectively on the cotangent bundle T^*M.

Definition 2 *A regular Lagrangian* $L(x,y)$ *on a domain* $D \subset TM$ *is a real smooth function* $L : D \to R$ *and a regular Hamiltonian* $H(x,p)$ *on a domain* $D^* \subset T^*M$ *is a real smooth function* $H : D^* \to R$ *such that the matrices with entries*

$$\begin{aligned} g_{ab}(x,y) &= \dot{\partial}_a \dot{\partial}_b L(x,y) \quad \text{and} \\ g^{*ab}(x,p) &= \dot{\partial}^a \dot{\partial}^b H(x,p) \end{aligned}$$

are everywhere nondegenerate on D *and* D^* *respectively.*

Examples. (a) Every Finsler space $F^n = (M, F(x,y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$.

(b) Every Cartan space $C^n = (M, \bar{F}(x,p))$ is a Hamilton manifold with $H = \frac{1}{2}\bar{F}^2$. (Here $\bar{F}$ is positively 1-homogeneous in p_i and the tensor $\bar{g}^{ab} = \frac{1}{2}\dot{\partial}_a \dot{\partial}_b \bar{F}^2$ is nondegenerate).

(c) (M, L) and (M, H) with

$$L(x,y) = \frac{1}{2}a_{ij}(x)y^i y^j + b_i(x)y^i + c(x)$$

and

$$H(x,p) = \frac{1}{2}\bar{a}^{ij}(x)p_i p_j + \bar{b}^i(x)p_i + \bar{c}(x), \quad \text{where} \quad \bar{c} = b_i b^i - c,$$

are Lagrange and Hamilton manifolds respectively (Here $a_{ij}(x), \bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, b_i are components of covector field, $\bar{b}^i$ are the components of a vector field, C and $\bar{C}$ are the smooth functions on M).

Let $L(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. If $L \in F(D)$ is a differential map, we can consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$

$$\psi(x, y) = (x^i, \dot{\partial}_a L(x, y)),$$

which will be called the *Legendre transformation*.

It is easily seen that L is a regular Lagrangian if and only if ψ is a local diffeomorphism.

In the same manner if $H \in F(D^*)$ the fiber derivative is given locally by

$$\varphi(x, y) = (x^i, \dot{\partial}^a H(x, y)),$$

which is a local diffeomorphism if and only if H is regular.

Let us consider a regular Lagrangian L. Then ψ is a diffeomorphism between the open sets $U \subset D$ and $U^* \subset D^*$. We can define in this case the function:

$$H : U^* \to R, \ H(x, p) = p_a y^a - L(x, y), \tag{2}$$

where $y = (y^a)$ is the solution of the equations $p_a = \dot{\partial}_a L(x, y)$.

Also, if H is a regular Hamiltonian on M, ϕ is a diffeomorphism between same open sets $U^* \subset D^*$ and $U \subset D$, we can consider the function

$$L : U \to R, \ L(x, y) = p_a y^a - H(x, p), \tag{3}$$

where $y = (p_a)$ is the solution of the equations

$$y^a = \dot{\partial}^a H(x, p).$$

The Hamiltonian $H(x, p)$ given by (2) is the *Legendre transformation of the Lagrangian* L and the Lagrangian given by (3) is called the *Legendre transformation of the Hamiltonian* H.

If (M, K) is a Cartan space, then (M, H) is a Hamilton manifold [10, 13], where $H(x, p) = \frac{1}{2}K^2(x, p)$ is 2-homogenous on a domain of T^*M. So we get the following transformation of H on U:

$$L(x, y) = p_a y^a - H(x, p) = H(x, p). \tag{4}$$

Theorem 1 *The scalar field* $L(x, y)$ *given by* (4) *is a positively* 2*-homogeneous regular Lagrangian on* U.

Therefore, we get Finsler metric F *of* U*, so that*

$$L = \frac{1}{2}F^2.$$

Thus for the Cartan space (M, K) *we always can locally associate a Finsler space* (M, F) *which will be called the* $\mathcal{L}$*-dual of a Cartan space* $(M, C_{|U^*})$ *vice versa, we can associate, locally, a Cartan space to every Finsler space which will be called the* $\mathcal{L}$ *-dual of a Finsler space* $(M, F_{|U})$.

3 The $\mathcal{L}$-dual of a Finsler space with exponential metric

In this case we put $\alpha^2 = y_i y^i$, $b^i = a^{ij}b_j$, $\beta = b_i y^i$, $\beta^* = b^i p_i$, $F^2 = y_i p^i$, $p^i = a^{ij}p_j$, $\alpha^{*2} = p_i p^i = a^{ij}p_i p_j$. we have $F = \alpha e^{\beta/\alpha}$ and

$$\begin{aligned}
p_i &= \frac{1}{2}\frac{\partial}{\partial y^i}F^2 = F\frac{\partial}{\partial y^i}F \\
&= F\left(\alpha_y^i e^{\beta/\alpha} + \alpha e^{\beta/\alpha}\frac{\alpha\beta_y^i - \beta\alpha_y^i}{\alpha^2}\right) \\
&= F\left(\frac{y_i}{\alpha}e^{\beta/\alpha} + \alpha e^{\beta/\alpha}\frac{\alpha b_i - \beta\dfrac{y_i}{\alpha}}{\alpha^2}\right) \\
&= F\left(\frac{y_i}{\alpha^2}F + F\frac{\alpha^2 b_i - \beta y_i}{\alpha^3}\right) \\
&= \frac{F^2}{\alpha^2}\left\{\left(1 - \frac{\beta}{\alpha}\right)y_i + \alpha b_i\right\}.
\end{aligned} \tag{5}$$

Contracting (5) with p^i and b^i respectively, we get

$$\begin{aligned}
\alpha^{*2} &= \frac{F^2}{\alpha^2}\left\{\left(1 - \frac{\beta}{\alpha}\right)y_i p^i + \alpha b_i p^i\right\} \\
&= \frac{F^2}{\alpha^2}\left\{\left(1 - \frac{\beta}{\alpha}\right)F^2 + \alpha\beta^*\right\}.
\end{aligned} \tag{6}$$

and

$$\beta^* = \frac{F^2}{\alpha^2}\left\{\left(1-\frac{\beta}{\alpha}\right)y_i b^i + \alpha b_i b^i\right\}$$
$$= \frac{F^2}{\alpha^2}\left\{\left(1-\frac{\beta}{\alpha}\right)\beta + \alpha b^2\right\}. \tag{7}$$

In [18], for a Finsler (α, β)-metric F on a Manifold M, one constructs a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ with $\|\beta\|_x < b_0, \forall x \in M$. The function ϕ satisfies $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, (|s| \leq b_0)$.
This mertic is a (α, β)-metric with $\phi = e^s$.

Using Shen's notation [18], put $s = \frac{\beta}{\alpha}$ and $\phi(s) = \frac{F}{\alpha} = e^s$ in (6) and (7), we get

$$\alpha^{*2} = \frac{F^2}{\alpha}\left\{\left(1-\frac{\beta}{\alpha}\right)\frac{F^2}{\alpha} + \beta^*\right\}$$
$$= Fe^s\{(1-s)Fe^s + \beta\} \tag{8}$$

and

$$\beta^* = Fe^s\left\{(1-s)s + b^2\right\} \tag{9}$$

Now, we have the following two theorems under two different cases:

Theorem 2 *Let* (M, F) *be a special Finsler space, where* F *is given by the equation* (1). *If* $b^2 = a_{ij}b^i b^j = 1$, *then the* $\mathcal{L}$*-dual of* (M, F) *is the space on* T^*M *having the fundamental function* $H(x, p)$ *given by the equations* (16).

Proof. From the equation (9), we get

$$F = \frac{\beta^*}{e^s\{(1-s)s+1\}} \tag{10}$$

and substituting F from the equation (10) in (8), we get

$$\alpha^{*2} = \frac{\beta^*}{\{(1-s)s+1\}}[(1-s)\frac{\beta^*}{\{(1-s)s+1\}} + \beta] \tag{11}$$

which implies that

$$(1+s-s^2)^2 - \delta(2-s^2) = 0$$
$$\text{or}\quad s^4 - 2s^3 + (-1+\delta)s^2 + 2s + 1 - 2\delta = 0, \tag{12}$$

where

$$\delta = \frac{\beta^{*2}}{\alpha^{*2}}.$$

Using Mathematica for solving the above equation (12), we get

$$s = (1 \pm \gamma_i)/2, \quad i = 1, 2 \tag{13}$$

where

$$\begin{aligned}
m_1 &= (1-\delta)/3, \\
m_2 &= 25 - 26\delta + \delta^2, \\
m_3 &= 125 - 195\delta + 69\delta^2 + \delta^3, \\
m_4 &= \delta\sqrt{25 - 48\delta + 21\delta^2 + 2\delta^3}, \\
m_5 &= m_3 + 3\sqrt{3}m_4{}^{1/3}, \\
m_6 &= \frac{m_2}{3m_5}, \\
m_7 &= \sqrt{2 - \delta - m_1 + m_7 + \frac{m_5}{3}}, \\
m_8 &= \sqrt{3 - \delta + m_1 - m_5 + m_6 + \frac{8\delta}{m_7}}, \\
\gamma_1 &= m_7 + m_8, \\
\text{and} \quad \gamma_2 &= m_7 - m_8.
\end{aligned}$$

From (10) and (13), we get

$$F = \frac{\beta^*}{e^{(1\pm\gamma_i)/2}\left\{1 + \frac{1\pm\gamma_i}{2} - (\frac{1\pm\gamma_i}{2})^2\right\}}. \tag{14}$$

As we know that $H(x, p) = \frac{1}{2}F^2$, therefore, by using the equation (14), we get

$$H(x, p) = \frac{\beta^{*2}}{e^{(1\pm\gamma_i)}\left\{1 + \frac{1\pm\gamma_i}{2} - (\frac{1\pm\gamma_i}{2})^2\right\}^2}, \tag{15}$$

putting $\beta^* = b^j p_j$, in equation (15), we get

$$H(x,p) = \frac{(b^j p_j)^2}{e^{(1\pm\gamma_i)}\left\{1 + \frac{1\pm\gamma_i}{2} - (\frac{1\pm\gamma_i}{2})^2\right\}^2}. \tag{16}$$

□

Theorem 3 *Let* (M, F) *be a special Finsler space, where* F *is given by the equation* (1)*. If* $b^2 = a_{ij}b^i b^j \neq 1$*, then the* $\mathcal{L}$*-dual of* (M, F) *is the space on* T^*M *having the fundamental function* $H(x,p)$ *given by the equations* (23)*.*

Proof. From (9), we get

$$F = \frac{\beta^*}{e^s\{(1-s)s + b^2\}}. \tag{17}$$

Substituting F from the equation (17) in (8), we get

$$\alpha^{*2} = \frac{\beta^*}{\{(1-s)s + b^2\}}[(1-s)\frac{\beta^*}{\{(1-s)s + b^2\}} + \beta] \tag{18}$$

which implies that

$$\begin{aligned} &(b^2 + s - s^2)^2 - \delta(1 + b^2 - s^2) = 0 \\ \text{or}\quad & s^4 - 2s^3 + (1 - 2b^2 + \delta)s^2 + 2b^2 s + b^4 - (1 + b^2)\delta = 0, \end{aligned} \tag{19}$$

where

$$\delta = \frac{\beta^{*2}}{\alpha^{*2}}.$$

Using Mathematica for solving the above equation (19), we get

$$s = (1 \pm \overline{\gamma}_i)/2, \quad i = 1, 2 \tag{20}$$

where

$$
\begin{aligned}
n_1 &= (1-2b^2+\delta)/3, \\
n_2 &= 1-10\delta+\delta^2+8(1-2\delta)b^2+16b^4, \\
n_3 &= 2(1-15\delta+39\delta^2+\delta^3)+(24-168\delta+60\delta^2)b^2 \\
&\quad +(96-192\delta)b^2+128b^6, \\
n_4 &= 432\delta^3(-1+7\delta+\delta^2)+432(1-19\delta+\delta^2+\delta^3)\delta^2b^2 \\
&\quad +432(8-28\delta-\delta^2)\delta^2b^4+6912\delta^2b^6, \\
n_5 &= 8(1-2b^2-3n_1), \\
n_6 &= n_1+\frac{2^{1/3}n_2}{3(n_3+\sqrt{n_4})}, \\
n_7 &= \sqrt{2b^2-\delta+n_7}, \\
n_8 &= 1+2b^2-\delta-n_7, \\
n_9 &= \frac{n_5}{4n_7}, \\
\overline{\gamma}_1 &= n_7+\sqrt{n_8-n_9} \\
\text{and}\quad \overline{\gamma}_2 &= n_7-\sqrt{n_8-n_9}.
\end{aligned}
$$

From (17) and (20), we get

$$
F=\frac{\beta^*}{e^{(1\pm\overline{\gamma}_i)/2}\left\{b_2+\dfrac{1\pm\overline{\gamma}_i}{2}-(\dfrac{1\pm\overline{\gamma}_i}{2})^2\right\}}. \tag{21}
$$

As we know that $H(x,p)=\frac{1}{2}F^2$, therefore by using (21), we get

$$
H(x,p)=\frac{\beta^{*2}}{e^{(1\pm\overline{\gamma}_i)}\left\{1+\dfrac{1\pm\overline{\gamma}_i}{2}-(\dfrac{1\pm\overline{\gamma}_i}{2})^2\right\}^2}, \tag{22}
$$

putting $\beta^*=b^jp_j$, in equation (22), we get

$$
H(x,p)=\frac{(b^jp_j)^2}{e^{(1\pm\overline{\gamma}_i)}\left\{1+\dfrac{1\pm\overline{\gamma}_i}{2}-(\dfrac{1\pm\overline{\gamma}_i}{2})^2\right\}^2}. \tag{23}
$$

□

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Certain classes of analytic functions defined by fractional q-calculus operator

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Abstract. In this paper, the concept of fractional q- calculus and generalized Al-Oboudi differential operator defining certain classes of analytic functions in the open disc are used. The results investigated for these classes of functions include the coefficient estimates, inclusion relations, extreme points and some more properties.

1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \tag{2}$$

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which are analytic in the open disc $\mathcal{U}$.

The q-shifted factorial is defined for α, $q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & n{=}0; \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & n \in \mathbb{N}, \end{cases} \tag{3}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha}; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \ (n > 0), \tag{4}$$

where the q-gamma functions [4, 5] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \ (0 < q < 1). \tag{5}$$

Note that, if $|q| < 1$, the q-shifted factorial (3) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{m=0}^{\infty} (1 - \alpha q^m).$$

Now recall the following q-analogue definitions given by Gasper and Rahman [4]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{6}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [4])

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \ (z \neq 0, \ q \neq 0). \tag{7}$$

$$\int_0^z f(t) d_q(t) = z(1-z) \sum_{m=0}^{\infty} q^m f(zq^m). \tag{8}$$

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^{\alpha}; q)_n}{(1-q)^n} = (\alpha)_n, \tag{9}$$

we observe that the q-shifted fractional (2) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

Now recall the definition of the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [7].

Definition 1 *(fractional q-integral operator). The fractional q-integral operator $I^{\delta}_{q,z}$ of a function $f(z)$ of order $\delta\,(\delta > 0)$ is defined by*

$$I^{\delta}_{q,z} = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z - tq)_{1-\delta} f(t) d_q t, \tag{10}$$

where $f(z)$ is a analytic in a simply connected region in the z-plane containing the origin. Here, the term $(z - tq)_{\delta-1}$ is a q-binomial function defined by

$$\begin{aligned}(z - tq)_{\delta-1} &= z^{\delta-1} \prod_{m=0}^{\infty} \left[\frac{1 - (\frac{tq}{z})q^m}{1 - (\frac{tq}{z})q^{\delta} + m - 1} \right] \\ &= z^{\delta}\, {}_1\phi_0 \left[q^{-\delta+1}; -; q, \frac{tq^{\delta}}{z} \right].\end{aligned} \tag{11}$$

According to Gasper and Rahman [4], the series ${}_1\phi_0[\delta; -; q, z]$ is single-valued when $|\arg(z)| < \pi$. Therefore, the function $(z - tq)_{\delta-1}$ in (11) is single-valued when $|\arg(\frac{-tq^{\delta}}{z}| < \pi$, $|tq^{\frac{\delta}{z}}| < 1$, and $|\arg(z)| < \pi$.

Definition 2 *(fractional q-derivative operator). The fractional q-derivative operator $D^{\delta}_{q,z}$ of a $f(z)$ of order $\delta (0 \leq \delta < 1)$ is defined by*

$$D^{\delta}_{q,z} f(z) = D_{q,z} I^{1-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(1 - \delta)} D_q \int_0^z (z - tq)_{-\delta} f(t) d_q t, \tag{12}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z - tq)_{-\delta}$ is removed as in Definition 1 above.

Definition 3 *(extended fractional q-derivative operator). Under the hypotheses of Definition 2, the fractional q-derivative for the function $f(z)$ of order δ is defined by*

$$D^{\delta}_{q,z} f(z) = D^{n}_{q,z} I^{n-\delta}_{q,z} f(z), \tag{13}$$

where, $n - 1 \leq \delta < n$, $n \in \mathbb{N}_0$.

The fractional q-defferintegral operator is defined by $\Omega^{\delta}_{q,z} f(z)$ for the function $f(z)$ of the form (1),

$$\Omega^{\delta}_{q} f(z) = \Gamma_q(2 - \delta) z^{\delta} D^{\delta}_{q,z} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_q(k + 1)\Gamma_q(2 - \delta)}{\Gamma_q(k + 1 - \delta)} a_k z^k, \tag{14}$$

where $D^{\delta}_{q,z}$ in (14), represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of $f(z)$ of order δ when $0 < \delta < 2$.

A linear multiplier fractional q-differintegral operator is defined as

$$\begin{aligned}
&\mathcal{D}_{q,\lambda}^{\delta,0} f(z) = f(z),\\
&\mathcal{D}_{q,\lambda}^{\delta,1} f(z) = (1-\lambda)\Omega_q^{\delta} f(z) + \lambda z D_q(\Omega_q^{\delta} f(z)),\\
&\mathcal{D}_{q,\lambda}^{\delta,2} f(z) = \mathcal{D}_{q,\lambda}^{\delta,1}(\mathcal{D}_{q,\lambda}^{\delta,1} f(z))\\
&\vdots
\end{aligned}$$

$$\mathcal{D}_{q,\lambda}^{\delta,n} f(z) = \mathcal{D}_{q,\lambda}^{\delta,1}(\mathcal{D}_{q,\lambda}^{\delta,n-1} f(z)). \tag{15}$$

We note that if $f \in \mathcal{A}$ is given by (1), then by (15), we have

$$\mathcal{D}_{q,\lambda}^{\delta,n} f(z) = z + \sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q) a_k z^k, \tag{16}$$

where

$$B(k,\ \delta,\ \lambda, n,\ q) = \left(\frac{\Gamma_q(2-\delta)\Gamma_q(k+1)}{\Gamma_q(k+1-\delta)}[([k]_q - 1)\lambda + 1]\right)^n. \tag{17}$$

It can be seen that, by specializing the parameters, the operator $\mathcal{D}_{q,\lambda}^{\delta,n}$ reduces tomany known and new integral and differential operators. In particular, when $\delta = 0$, and $q \to 1^-$ the operator $\mathcal{D}_{q,\lambda}^{\delta,n}$ reduces to the operator introduced by AL-Oboudi [1] and if $\delta = 0$, $\lambda = 1$ and $q \to 1^-$ and it reduces to the operator introduced by Sălăgean [9].

Now using above differential operator, we define the following subclass of $\mathcal{T}$.

Let $\mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the conditions

$$\Re\left\{\frac{z D_q(\mathcal{D}_{q,\lambda}^{\delta,n} f)}{\beta z D_q(\mathcal{D}_{q,\lambda}^{\delta,n} f) + (1-\beta)\mathcal{D}_{q,\lambda}^{\delta,n} f}\right\} > \alpha, \tag{18}$$

for some $\alpha,\ \beta\ (0 \le \alpha, \beta < 1)$, $\delta \le 2$, $\lambda > 0$ and $n \in \mathbf{N}_0$.

In particular, if $\delta = 0$, and $q \to 1^-$ we get the classes studied by Ravikumar, Dileep and Latha [8] and if $\delta = 0$, and $q \to 1^-$ and different parametric of values n we get the classes studied by Mostafa [6], Altintas and Owa [2].

2 Main results

Theorem 1 *A function* $f(z)$ *defined by* (2) *is in the class* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ *if and only if*

$$\sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)a_k[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]<1-\alpha, \tag{19}$$

where, $B(k,\ \delta,\ \lambda, n,\ q)$ *is defined in* (17)*,* $\alpha,\ \beta\quad (0\leq\alpha,\beta<1), \lambda>0$ *and* $n\in\mathbf{N}_0$.

Proof. Suppose $f\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$. Then

$$\Re\left\{\frac{zD_q(\mathcal{D}_{q,\lambda}^{\delta,n}f)}{\beta zD_q(\mathcal{D}_{q,\lambda}^{\delta,n}f)+(1-\beta)\mathcal{D}_{q,\lambda}^{\delta,n}f}\right\}>\alpha,$$

$$\Re\left\{\frac{z-\sum\limits_{k=2}^{\infty}B(k,\delta,\lambda,n,\ q)[k]_qa_kz^k}{\beta[z-\sum\limits_{k=2}^{\infty}B(k,\delta,\lambda,n,\ q)[k]_qa_kz^k]+(1-\beta)[z-\sum\limits_{k=2}^{\infty}B(k,\delta,\lambda,n,\ q)a_kz^k]}\right\}>\alpha,$$

$$\Re\left\{\frac{z-\sum\limits_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[k]_qa_kz^k}{z-\sum\limits_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_kz^k[\beta([k]_q-1)+1]}\right\}>\alpha.$$

Letting $z\to 1$, we get,

$$1-\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[k]_qa_k>\alpha\left\{1-\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_k[\beta([k]_q-1)+1]\right\}.$$

Equivalenty we have,

$$\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[k]_qa_k-\alpha\left\{\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_k[\beta([k]_q-1)+1]\right\}<(1-\alpha)$$

which implies

$$\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_k[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]<(1-\alpha).$$

Conversely, assume that (2.1) is be true. We have to show that (6) is satisfied or equivalently

$$\left|\left\{\frac{zD_q(\mathcal{D}^{\delta,n}_{q,\lambda}f)}{\beta zD_q(\mathcal{D}^{\delta,n}_{q,\lambda}f)+(1-\beta)\mathcal{D}^{\delta,n}_{q,\lambda}f}\right\}-1\right|<1-\alpha.$$

But

$$\left|\left\{\frac{z-\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)[k]_q a_k z^k}{z-\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k z^k[\beta([k]_q-1)+1]}\right\}-1\right|$$
$$=\left|\frac{\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k([k]_q-1)(\beta-1)z^k}{z-\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k[\beta([k]_q-1)+1]z^k}\right|$$
$$\leq\frac{\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k([k]_q-1)(\beta-1)|z^k|}{|z|-\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k[\beta([k]_q-1)+1]|z^k|}$$
$$\leq\frac{\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k([k]_q-1)(\beta-1)}{1-\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k[\beta([k]_q-1)+1]}.$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k([k]_q-1)(\beta-1)$$
$$\leq(1-\alpha)(1-\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k[\beta([k]_q-1)+1])$$

or

$$\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)a_k[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]<1-\alpha,$$

which is true by hypothesis. This completes the assertion of Theorem 1. □

Corollary 2 *If* $f \in \mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$, *then*

$$|a_k| \le \frac{1-\alpha}{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]}.$$

Theorem 3 *Let* $0 \le \alpha < 1,\quad 0 \le \beta_1 \le \beta_2 < 1,\quad n \in \mathbb{N}_0,$ *then* $\mathcal{T}_q^n(\alpha,\beta_1,\delta,\lambda) \subset \mathcal{T}_q^n(\alpha,\beta_2,\delta,\lambda)$.

Proof. For $f(z) \in \mathcal{T}_q^n(\alpha,\beta_2,\delta,\lambda)$. We have,

$$\begin{aligned}&\sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q) a_k[(1-\alpha\beta_2)[k]_q + \alpha\beta_2 - \alpha]\\ &\le \sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q) a_k[(1-\alpha\beta_1)[k]_q + \alpha\beta_1 - \alpha] < 1-\alpha.\end{aligned}$$

Hence $f(z) \in \mathcal{T}_q^n(\alpha,\beta_1,\delta,\lambda)$. □

Theorem 4 *Let* $f(z) \in \mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$. *Define* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{1-\alpha}{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]} z^k, \quad k = 2, 3, \cdots,$$

for some $\alpha,\ \beta\,(0 \le \beta < 1), n \in \mathbb{N}_0,\ \lambda > 0$ *and* $z \in \mathcal{U}$. *Then* $f(z) \in \mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ *if and only if* $f(z)$ *can be expressed as* $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ *where* $\mu_k \ge 0$ *and* $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. If $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ with $\sum_{k=1}^{\infty} \mu_k = 1,\quad \mu_k \ge 0$, then

$$\begin{aligned}&\sum_{k=2}^{\infty} \frac{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]\mu_k}{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]}(1-\alpha) \sum_{k=2}^{\infty} \mu_k(1-\alpha)\\ &= (1-\mu_1)(1-\alpha) \le (1-\alpha).\end{aligned}$$

Hence $f(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$.

Conversely, let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$, define

$$\mu_k = \frac{B(k, \delta, \lambda, n, q)\,[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]\,|a_k|}{(1-\alpha)}, \quad k = 2, 3, \cdots,$$

and define $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$. From Theorem (1), $\sum_{k=2}^{\infty} \mu_k \leq 1$ and hence $\mu_1 \geq 0$.

Since $\mu_k f_k(z) = \mu_k f(z) + a_k z^k$, $\sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} a_k z^k = f(z)$. □

Theorem 5 *The class $\mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$ is closed under convex linear combination.*

Proof. Let $f(z)$, $g(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$ and let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k.$$

For η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by $h(z) = (1-\eta) f(z) + \eta g(z), \quad z \in \mathcal{U}$ belongs to $\mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$. Now

$$h(z) = z - \sum_{k=2}^{\infty} [(1-\eta) a_k + \eta b_k] z^k.$$

Applying Theorem 1, to $f(z)$, $g(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} B(k, \delta, \lambda, n, q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]\,[(1-\eta) a_k + \eta b_k] \\
&= (1-\eta) \sum_{k=2}^{\infty} B(k, \delta, \lambda, n, q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha] a_k \\
&\quad + \eta \sum_{k=2}^{\infty} B(k, \delta, \lambda, n, q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha] b_k \\
&\leq (1-\eta)(1-\alpha) + \eta(1-\alpha) = (1-\alpha).
\end{aligned}$$

This implies that $h(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$. □

Corollary 6 *If* $f_1(z)$, $f_2(z)$ *are in* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ *then the function defined by* $g(z)=\frac{1}{2}[f_1(z)+f_2(z)]$ *is also in* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$.

Theorem 7 *Let for* $j=1,2,\cdots,k,\quad f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}z^k\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ *and* $0<\beta_j<1$ *such that* $\sum_{j=1}^{k}\beta_j=1$, *then the function* $F(z)$ *defined by* $F(z)=\sum_{j=1}^{k}\beta_j f_j(z)$ *is also in* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$.

Proof. For each $j\in\{1,2,3,\cdots,k\}$ we obtain

$$\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]|a_k|<(1-\alpha).$$

$$\begin{aligned}F(z)&=\sum_{j=1}^{k}\beta_j\left(z-\sum_{k=2}^{\infty}a_{k,j}z^k\right)=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{k}\beta_j a_{k,j}\right)z^k\\&\cdot\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]\left[\sum_{j=1}^{k}\beta_j a_{k,j}\right]\\&=\sum_{j=1}^{k}\beta_j\left[\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]\right]\\&<\sum_{j=1}^{k}\beta_j(1-\alpha)<(1-\alpha).\end{aligned}$$

Therefore $F(z)\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$. □

Bernardi Libera's integral operator is defined as

$$L_\gamma f(z)=\frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt,$$

which was studied by Bernardi in [3].

Theorem 8 *Let* $f(z)\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$. *The q-analogous Bernardi's integral operator defined by* $L_{q,\gamma}f(z)=\frac{[\gamma+1]_q}{z^\gamma}\int_0^z t^{\gamma-1}f(t)d_qt$ *then* $L_{q,\gamma}f(z)\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$.

Proof. We have

$$\begin{aligned}
L_{q,\gamma}f(z) &= \frac{[\gamma+1]_q}{z^\gamma} z(1-q)\sum_{j=0}^{\infty} q^j (zq^j)^{\gamma-1} f(zq^j) \\
&= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma} f(zq^j) \\
&= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma}\sum_{k=1}^{\infty} q^{jk} a_k z^k, \\
&= [\gamma+1]_q\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(1-q)q^{j(\gamma+k)} a_k z^k \\
&= z - \sum_{k=2}^{\infty}\frac{[\gamma+1]_q}{[\gamma+k]_q} a_k z^k.
\end{aligned}$$

Since $f \in \mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ and since $\dfrac{[\gamma+1]_q}{[\gamma+k]_q} < 1$, we have

$$\sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]\frac{[\gamma+1]_q}{[\gamma+k]_q} a_k < (1-\alpha). \qquad \square$$

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Cyclic flats and corners of the linking polynomial

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Abstract. Let $T(M;x,y) = \sum_{ij} T_{ij} x^i y^j$ denote the Tutte polynomial of the matroid M. If T_{ij} is a corner of $T(M;x,y)$, then T_{ij} counts the sets of corank i and nullity j and each such set is a cyclic flat of M. The main result of this article consists of extending the definition of cyclic flats to a pair of matroids and proving that the corners of the linking polynomial give the lower bound of the number of the cyclic flats of the matroid pair.

1 Introduction

Let A and B be two sets. We denote by $A \setminus B$ the set difference between A and B. We write $A \setminus e$ for $A \setminus \{e\}$. Similarly, we write $A \cup f$ instead of $A \cup \{f\}$. A *matroid* M defined on a finite nonempty set E consists of the set E and a collection $\mathcal{I}$ of subsets of E, satisfying the following axioms:

I1: $\emptyset \in \mathcal{I}$

I2: if $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$

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I3: if I_1 and $I_2 \in \mathcal{I}$ and, $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Elements of $\mathcal{I}$ are the *independent sets* of the matroid M. A *circuit* of M is a subset C that is not independent but $X \setminus e$ is independent for every $e \in X$. That is, a circuit is a minimal non-inependent set. A basis of M is a maximal independent set. The *dual* matroid of M, denoted by M^*, is the matroid whose bases are the complements of bases of M.

Let 2^E denotes the set of all the subsets of E and let $\mathcal{N}^+$ denotes the set of non-negative integers. The *rank function* of M, denoted by r, is a function from 2^E to $\mathcal{N}^+$, where, for $X \subseteq E$, $r(X)$ is the cardinality of the largest independent set I contained in X. The *dual* matroid of M, denoted by M^*, is the matroid whose bases are the complements of the bases of M.

For a matroid M defined on E, the Tutte polynomial of M, denoted by $T(M; x, y)$, is a two-variable polynomial defined as follows.

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where r is the rank function of M. This polynomial is much researched on since it encodes much information about the matroidal properties of combinatorial structures and is found useful in counting combinatorial invariants. For instance, let $G = (V, E)$ be a graph whose vertex set is V and edge set is E. Let M_G be the matroid defined on E, where the set of independent sets of M_G is the set of subsets $X \subseteq E$ that do not contain a closed path. It can be proved that for $A \subseteq E$, $r(A) = |V| - c(A)$, where $c(A)$ is the number of connected components of the graph (V, A). Thus, $T(M_G; 1-\lambda, 0)$ is proportional to the number of coloring of G using λ colors. See [3, 5, 6, 4, 17] for an extensive exposition to this topic.

Let M and N be two matroids defined on the set E with rank function r and s respectively. We call the pair (M, N) a *matroid pair*. The *dual matroid pair* is the pair (M^*, N^*) where M^* and N^* denote the dual matroids of M and N respectively. The *linking polynomial* of (M, N), denoted $Q(M, N; x, y, u, v)$ is defined in [24] as follows.

$$Q(M, N; x, y, u, v) = \sum_{X \subseteq E} x^{r(E)-r(X)} y^{|X|-r(X)} u^{s(E)-s(X)} v^{|X|-s(X)}.$$

The linking polynomial contains, as a specialisation the Tutte polynomial of a matroid and it also partially contains the Tutte invariant of *2-polymatroids* defined by Oxley and Whittle in [14, 15].

There is a *weak map* from a matroid M to another matroid N if every independent set in N is also independent in M, whereas a weak map is a *strong map* if every closed set of N is closed in M. A strong map is a *matroid perspective* if M and N are defined on the same set.

The linking polynomial is equivalent to the Tutte polynomial of a *matroid perspective*, the polynomial $T(P; x, y, z)$, defined and studied by the late Las Vergnas [9, 19, 20, 21, 22, 23] and in [10]. One of its most interesting evaluations is $T(P; 0, 0, 1)$. An *oriented matroid* is a matroid where an orientation is assigned to every element e. One of the simplest examples of oriented matroids is the cycle matroid of a graph G whose edges are oriented. If P is a perspective from an oriented matroid M to the oriented matroid N, then $T(P; 0, 0, 1)$ counts the number of subsets A such that A is acyclic in M and totally cyclic in N [22]. An obvious application is when there is a strong map from a cycle matroid of a graph G to a cycle matroid of a graph G' and one defines an orientation on the edges of G. This orientation is carried to the edges of G' in an obvious way. Then, $T(P; 0, 0, 1)$ counts the number of subsets A of edges, such that A is acyclic in G and totally cyclic in G'. This evaluation is paramount as it generalizes results on bounded regions of real hyperplane arrangements [25], non Radon partitions of real spaces [3]. More of such applications can be found in [19].

Moreover, the *bond matroid* of a graph G is the matroid whose independent sets are the subsets of edges of G that do not contain cutsets. Suppose that G and G^* are dually imbedded on a surface. Then there is a matroid perspective from the bond matroid of G^* to the cycle matroid of G. For the case of 4-valent graphs imbedded in the projective plane or a torus, Las Vergnas in [23] relates the Tutte polynomial of matroid perspective to Eulerian tours and cycles decompositions of G. This result sparks a renewed interest in the Tutte polynomial of matroid perspective because of its connection with the Bollobas-Riordan polynomial and Krushkal polynomial, which find many applications in the theory of graphs embedding on surfaces [8]. A generalization of matroid perspectives to a sequence of perspectives in [1] finds applications in electrical network theory. More applications of strong maps in engineering and in the theory of rigidity matroids can be seen in [2, 18]. Thanks to these many applications, the Tutte polynomial of matroid perspective deserves to be more studied algebraically and the Linking polynomial seems one of the best ways to investgate this algebraic structure. This paper looks at the corners of the linking polynomial and gives the lower bound of the number of the cyclic flats of the matroid pair.

For a matroid M defined on E, we write $M|X$ to denote the matroid M

restricted to the subset $X \subseteq E$. For a matroid M defined on E with rank function r, a subset X is a *flat* if for all $e \in E \setminus X$ we have $r(X \cup e) = r(X) + 1$.

A *cyclic flat* of M is a subset $X \subseteq E$ such that X is a flat of M and X is a union of circuits of M. In other words, X is a cyclic flat of M if X is a flat of M and there is no element $f \in X$ such that f is a coloop in $M|X$. A third equivalent definition is that X is a cyclic flat of a matroid M defined on E with rank function r if X is a flat of M and $r(X \setminus e) = r(X)$, for all $e \in X$.

2 Main results

We extend the above definitions to matroid pairs as follows. Let (M, N) be a matroid pair defined on E with rank funtions r and s for M and N respectively. We define a subset $X \subseteq E$ to be a *flat* of (M, N) if for all $f \in E \setminus X$

$$r(X \cup f) + s(X \cup f) \geq r(X) + s(X) + 1.$$

Further, a subset $X \subseteq E$ to be a *cyclic flat* of (M, N) if X is a flat of (M, N) and there is no element $e \in X$ such that e is a coloop in both $M|X$ and $N|X$. In other words, we say that X is a cyclic flat of the matroid pair (M, N) defined on E with rank functions r and s respectively if X is a flat of (M, N) and for all $e \in X$,

$$r(X \setminus e) + s(X \setminus e) > r(X) + s(X) - 2.$$

A classic result in Matroid Theory is as follows. X is a cyclic flat of a matroid M if and only if $E \setminus X$ is a cyclic flat of M^*. See [16] for an introduction to Matroid Theory. The next result extends this property to matroid pairs. If $P = (M, N)$ is a matroid pair, we denote by P^* the matroid pair (M^*, N^*).

Theorem 1 *Let P be a matroid pair. Then X is a cyclic flat of P if and only if $E \setminus X$ is a cyclic flat of P^*.*

Proof. First recall that if M is a matroid defined on E with rank function r, and r^* denotes the rank function of the dual matroid M^*, then for all $X \subseteq E$, we have

$$r^*(X) = |X| + r(E \setminus X) - r(E). \tag{1}$$

Suppose that X is a cyclic flat of P but $E \setminus X$ is not a cyclic flat of P^*. Then, either $E \setminus X$ is not a flat of (M^*, N^*) or $E \setminus X$ contains an element f which is a coloop in both $M^*|(E \setminus X)$ and $N^*|(E \setminus X)$.

Assume that $Y = E \setminus X$ is not a flat of (M^*, N^*). Then, for some element $g \in X$, we have

$$r^*(Y \cup g) + s^*(Y \cup g) < r^*(Y) + s^*(Y) + 1.$$

Therefore, by equation (1), we get

$$r(X \setminus g) + s(X \setminus g) < r(X) + s(X) - 1.$$

Equivalently,

$$r(X \setminus g) + s(X \setminus g) = r(X) + s(X) - 2.$$

Thus the element g is a coloop in both $M|X$ and $N|X$. Therefore X is not a cyclic flat of P, a contradiction.

If $Y = E \setminus X$ contains an element f which is coloop of Y in both M^* and N^* then $r^*(Y \setminus f) = r^*(Y) - 1$.

By equation (1), we get

$$|Y \setminus f| + r(E \setminus (Y \setminus f)) - r(E) = |Y| + r(E \setminus Y) - r(E) - 1.$$

Thus $r(X \cup f) = r(X)$. Similarly $s(X \cup f) = s(X)$.

Hence

$$r(X \cup f) + s(X \cup f) = r(X) + s(X) < r(X) + s(X) + 1.$$

Therefore, X is not a flat of P, a contradiction. To prove the converse one only needs to swap the roles of X and $E \setminus X$. □

In the sequel, we write $(ijkl) \leq (i'j'k'l')$ if $i \leq i'$ and $j \leq j'$ and $k \leq k'$ and $l \leq l'$, and we write $(ijkl) = (i'j'k'l')$ if $i = i'$ and $j = j'$ and $k = k'$ and $l = l'$. We say that $(ijkl)$ and $(i'j'k'l')$ are *incomparable* if some indices in $(ijkl)$ are strictly superior to the corresponding indices in $(i'j'k'l')$ and some other indices in $(ijkl)$ are strictly inferior to the corresponding indices in $(i'j'k'l')$. For all $X \subseteq E$ let $\mathrm{cor}_M(X)$ denote the integer $r(E) - r(X)$, $\mathrm{nul}_M(X)$ denote $|X| - r(X)$, $\mathrm{cor}_N(X)$ denote the integer $s(E) - s(X)$ and $\mathrm{nul}_N(X)$ denote $|X| - s(X)$. Let $\mathcal{E}_{ijkl}$ denote the family of subsets $X \subseteq E$ such that $\mathrm{cor}_M(X) = i$, $\mathrm{nul}_M(X) = j$, $\mathrm{cor}_N(X) = k$, $\mathrm{nul}_N(X) = l$.

The next result, proved in [11], is instrumental in the proof of Theorem 2.4.

Lemma 1 *Let (M, N) be a matroid pair defined on E. If $\mathcal{E}_{ijkl}$ is not empty then q_{ijkl} is positive and*

$$q_{ijkl} = |\mathcal{E}_{ijkl}| + \sum_{Y \notin \mathcal{E}_{ijkl}} \binom{\mathrm{cor}_M(Y)}{i}\binom{\mathrm{nul}_M(Y)}{j}\binom{\mathrm{cor}_N(Y)}{k}\binom{\mathrm{nul}_N(Y)}{l}$$

where the sum is only over the subsets Y such that $(i'j'k'l') > (ijkl)$.

A *corner* of $T(M;x,y)$ is a coefficient T_{ij} such that $T_{ij} > 0$ and there is no other positive coefficient $T_{i'j'}$ with $(i'j') > (ij)$. Corners of a Tutte polynomial $T(M;x,y)$ convey much information about the matroid M. In [5], Brylawski proved that if T_{ij} is a corner of $T(M;x,y)$, then T_{ij} counts the sets of corank i and nullity j and each such set is a cyclic flat of M. This result is strengthened in [13] as follows.

Theorem 2 [13, Theorem 4.11] *Suppose that* $T_{ij} > 0$ *for a matroid* M*. Then the following are equivalent.*
(i) T_{ij} *is a corner of* $T(M;x,y)$*.*
(ii) *Every set of corank* i *and nullity* j *is a cyclic flat.*
(iii) T_{ij} *counts the sets of corank* i *and nullity* j*.*

We extend Theorem 2 to matroid pairs as follows. A coefficient q_{ijkl} is called a *corner* in $Q(M,N;x,y,u,v)$ if
(i) $q_{ijkl} \neq 0$
(ii) $q_{i'j'k'l'} = 0$ for all $(i'j'k'l')$ such that $(i'j'k'l') > (ijkl)$.

Theorem 3 *Let* (M,N) *be a matroid pair defined on* E*. If* q_{ijkl} *is a corner of* $Q(M,N;x,y,u,v)$ *then every* $X \in \mathcal{E}_{ijkl}$ *is a cyclic flat of* (M,N)*.*

Proof. Suppose that q_{ijkl} is a corner of $Q(M,N)$ and $X \in \mathcal{E}_{ijkl}$. Suppose that X is not a cyclic flat of (M,N). Then, either X is not a flat of (M,N) or X contains an element e which is a coloop of X in both M and N.

Suppose that X is not a flat of (M,N). Then there is an element $e \in E \setminus X$ such that

$$r(X \cup e) + r(X \cup e) < r(X) + s(X) + 1.$$

Equivalently

$$r(X \cup e) + r(X \cup e) = r(X) + s(X).$$

Thus $r(X \cup e) = r(X)$ and $s(X \cup e) = s(X)$. Now, consider $X \cup e$.

$$\mathrm{cor}_M(X \cup e) = \mathrm{cor}_M(X), \quad \mathrm{cor}_N(X \cup e) = \mathrm{cor}_N(X)$$

$$\mathrm{nul}_M(X \cup e) = \mathrm{nul}_M(X) + 1, \quad \mathrm{nul}_N(X \cup e) = \mathrm{nul}_N(X) + 1.$$

Thus, if $X \in \mathcal{E}_{ijkl}$, then $\mathcal{E}_{ij'kl'}$ where $j' = j+1, l' = l+1$ is not empty. Hence by Lemma 1, $q_{i,j',k,l'} \neq 0$. Thus q_{ijkl} is not a corner as $(ij'kl') > (ijkl)$. Contradiction.

Suppose that X has an element e which is a coloop in both M and N. Consider the subset $X \setminus e$. Then,

$$\mathrm{cor}_M(X \setminus e) = \mathrm{cor}_M(X) + 1, \quad \mathrm{cor}_N(X \setminus e) = \mathrm{cor}_N(X) + 1$$

$$\mathrm{nul}_M(X \setminus e) = \mathrm{nul}_M(X), \quad \mathrm{nul}_N(X \setminus e) = \mathrm{nul}_N(X).$$

Thus, if $X \in \mathcal{E}_{ijkl}$, then $\mathcal{E}_{i'jk'l}$ where $i' = i+1, k' = k+1$ is not empty. Hence by Lemma 1, $q_{i',j,k',l} \neq 0$. Hence q_{ijkl} not a corner. □

Theorem 2, which is a strengthening of the result of Brylawski can not be generalised to matroid pairs. Indeed, Theorem 2 says, among other things, that if X is a cyclic flat of M of corank i and nullity j, then T_{ij} is not a corner if and only if there exists a subset Y such that $\mathrm{cor}(Y) = i$ and $\mathrm{nul}(Y) = j$ but Y is not a cyclic flat. But we have an example of a matroid pair (M, N) where all $X \in \mathcal{E}_{ijkl}$ are cyclic flats of (M, N) but q_{ijkl} is not a corner of $Q(M, N)$. Indeed, consider the matroid pair given in Figure 1.

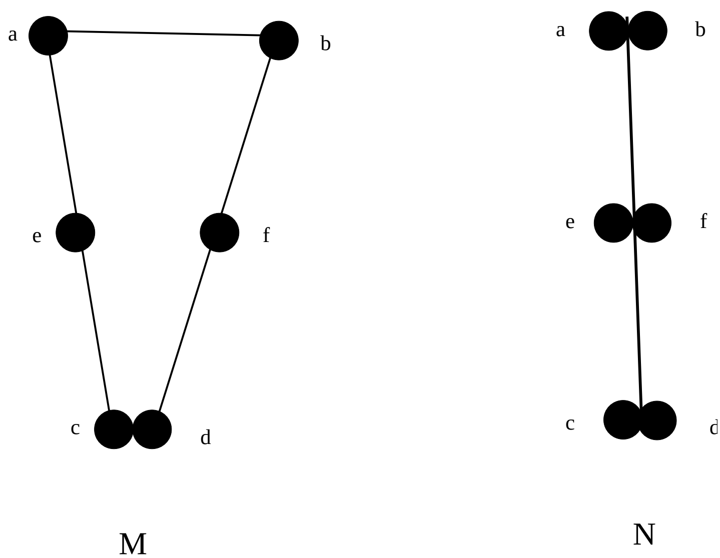

Figure 1: Example where $X \in \mathcal{E}_{ijkl}$ are cyclic flats of (M, N) but q_{ijkl} is not a corner of $Q(M, N)$.

The subset $\{a, b\} \in \mathcal{E}_{1011}$ is a cyclic flat of (M, N). The subset $\{c, d\} \in \mathcal{E}_{2111}$, thus $\mathcal{E}_{2111}$ is not empty. The subset $\{c, d\}$ is also a cyclic flat of (M, N). Since $(2111) > (1011)$, then q_{1011} is not a corner of $Q(M, N; x, y, u, v)$.

Suppose there is a subset $X \in \mathcal{E}_{1011}$ such that X is not a cyclic flat of (M, N). Since $\mathrm{cor}_M(X) = \mathrm{cor}_N(X) = \mathrm{nul}_N(X) = 1$ and $\mathrm{nul}_M(X) = 0$, such an X can be only $\{a, b\}$ or $\{e, f\}$. But they are both cyclic flats of N. Hence they are cyclic flats of (M, N). Contradiction. Therefore (M, N) does not contain such a subset X.

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