Acta Universitatis Sapientiae

Mathematica

Volume 9, Number 2, 2017

Sapientia Hungarian University of Transylvania
Scientia Publishing House

Contents

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 275–282

DOI: 10.1515/ausm-2017-0020

Para-mixed linear spaces

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Abstract. We consider the paracomplex version of the notion of mixed linear spaces introduced by M. Jurchescu in [4] by replacing the complex unit i with the paracomplex unit j, $j^2 = 1$. The linear algebra of these spaces is studied with a special view towards their morphisms.

Introduction

It is well-known that up to isomorphisms there are three 2-dimensional real algebras: $\mathbb{C} = \mathbb{R}[X]/(x^2+1)$, $\mathbb{A} = \mathbb{R}[X]/(x^2-1)$, $\mathbb{D} = \mathbb{R}[X]/(x^2)$. The theory of the first algebra is richer than the other two, a fact corresponding to the field property of $\mathbb{C}$. Similar to the complex case, the paracomplex algebra $\mathbb{A}$ has the basis $\{1, j\}$ with $j^2 = 1$; therefore the elements of $\mathbb{A}$ are $z = x + jy$ with x and y real numbers. For historical details about the paracomplex algebra please see the survey [3].

Similar to the linear complex geometry there exists a paracomplex version as follows: let V be a real linear space. A *paracomplex structure* on V is an involution $J : V \to V$, $J^2 = 1_V$, such that the eigenspaces $V_\pm := \ker(1_V \pm J)$ have the same dimension. The pair (V, J) is then called *a paracomplex linear space*. If the hypothesis regarding the eigenspaces is dropped then we obtain the notion of *almost paracomplex structure*. An $\mathbb{A}$-linear map between the (almost) paracomplex linear spaces (V, J) and (V', J') is a linear map $T : V \to V'$

2010 Mathematics Subject Classification: 15A03, 15A04
Key words and phrases: paracomplex structure, para-mixed linear space, morphism

satisfying $T \circ J = J' \circ T$. Just like complex vector spaces are vector spaces over the field $\mathbb{C}$, the almost paracomplex linear spaces are free modules over $\mathbb{A}$.

In a series of papers ([4]-[6]) M. Jurchescu defined and uses the notions of *mixed linear space* and *mixed manifold* having as (local) model the direct product $\mathbb{R}^m \times \mathbb{C}^n$. The mixed manifolds as differentiable families of complex spaces are useful at the cross-road of complex analysis and complex geometry, for example regarding the smoothly parameterized Čech cohomology of complex manifolds, [1]. Here, following his ideas and restricted at the linear level we define the concept of *para-mixed linear space* by replacing $\mathbb{C}$ with $\mathbb{A}$. A lot of properties of algebraic nature are similar to these frameworks and we hope to use further the notions considered now.

The paper is structured in two sections. The first one is devoted to the general theory of para-mixed linear spaces including the adapted linear maps between them and also their subspaces. A special attention is dedicated to the finite-dimensional case. The second section treats special morphisms between para-mixed linear spaces and the notion of *paracomplexification* of such spaces.

1 Para-mixed linear spaces and subspaces

Definition 1 *A (almost) para-mixed linear space is a triple* (E, E_p, P) *where* E *is a real linear space and* E_p *is a linear subspace of* E *endowed with an (almost) paracomplex structure* P*. Then* E_p *is called the paracomplex part of* E *while the quotient real linear space* $E_r = E/E_p$ *is the real part of* E*. The para-mixed space is pure real if* $E_p = 0$ *(i.e.* $E_r = E$*) and pure paracomplex if* $E_r = 0$ *(i.e.* $E_p = E$*). In the following we place always in the "almost" case and for simplicity we will drop this epithet.*

Example 1 i) *The fundamental example is* $E = E_f = \mathbb{R}^m \times \mathbb{A}^n$ *with* $E_p = \mathbb{A}^n$ *and* $E_r = \mathbb{R}^m$*. Hence, a para-mixed linear space can be thought as a (trivial) vector bundle over* E_r *with paracomplex fibres* E_p*; also, para-mixed linear spaces can be though as linear families of paracomplex spaces. Let us point out that vertical bundles endowed with paracomplex structures are recently studied in [2] and the geometry of polynomial sub-endomorphisms on a pair of distributions for a given manifold are studied in [7].*

ii) *The paracomplex linear spaces will be considered as pure paracomplex para-mixed linear spaces while the real linear spaces will be considered as pure real para-mixed linear spaces. A para-mixed linear space* E *is simultaneous pure real and pure paracomplex if and only if* $E = \{0\}$*.*

Definition 2 *A linear map* $T \in L_{\mathbb{R}}(E, F) := L(E, F)$ *between two para-mixed linear spaces is called a morphism if* $T(E_p) \subseteq F_p$ *and the induced map* $T_p = T|_{E_p} : E_p \to F_p$ *is an* $\mathbb{A}$*-linear map.* T *is called an antimorphism if* $T(E_p) \subseteq F_p$ *and* T_p *is an* $\mathbb{A}$*-antilinear map:* $T(x + jy) = Tx - jTy$. T_p *is the paracomplex part of* T *while* $T_r = T|_{E_r} : E_r \to F_r$ *is the real part of* T*. Denotes by* $\mathrm{Hom}(E, F)$ *the set of all morphisms from* E *to* F *and* $T \in \mathrm{Hom}(E, F)$ *is called isomorphism if both* T_p *and* T_r *are bijective maps.*

Remark 1 i) *If* E *is a pure real para-mixed linear space and* $T \in L(E, F)$ *then* T *is both morphism and antimorphism. If* F *is a pure real para-mixed linear space and* $T \in L(E, F)$ *then* T *is a morphism if and only if* $T_p = 0$*. If* E *is a general para-mixed linear space then* $(1_{E_r}, P) \in \mathrm{Hom}(E, E)$ *by considering the decomposition* $E = E_r \otimes E_p$ *(see also the Corollary 1 below).*

ii) *The class of para-mixed linear spaces with their morphisms defines a category which contains the category of real linear spaces as well as the category of paracomplex linear spaces. The consideration of the paracomplex part (for spaces and morphisms) is a functor from the category of para-mixed linear spaces to the category of paracomplex linear spaces and similar for the consideration of the real part (for spaces and morphisms).*

iii) *Fix* E *and* F *two para-mixed linear spaces and* $T_1, T_2 \in L(E, F)$ *two (anti)morphisms. Consider also two real numbers* α, β*. It follows that* $\alpha T_1 + \beta T_2$ *is also an (anti)morphism with* $(\alpha T_1 + \beta T_2)_i = (\alpha T_1)_i + (\beta T_2)_i$ *for* $i \in \{p, r\}$*.*

iv) *The linear map* $(x, y) \in \mathbb{R}^2 \to z = x + jy \in \mathbb{A}$ *is a bijective morphism which is not an isomorphism.*

v) $\mathrm{Hom}(E, F)$ *is a para-mixed linear space with the paracomplex part* $\mathrm{Hom}(E, F)_p = \{T \in \mathrm{Hom}(E, F); T_r = 0\}$ *and real part* $\mathrm{Hom}(E, F)_r = \mathrm{Hom}(E, F_r) \simeq \mathrm{Hom}(E_r, F_r)$*. The corresponding paracomplex structure is the map* $T \in \mathrm{Hom}(E, F) \to T \circ (1_{E_r}, P) \in \mathrm{Hom}(E, F)$*.*

vi) *Fix* $T \in \mathrm{Hom}(E, F)$ *and let* H *be another para-mixed linear space. Let* $T_* : \mathrm{Hom}(H, E) \to \mathrm{Hom}(H, F)$ *be the composition with* T *at left and* $T^* : \mathrm{Hom}(F, H) \to \mathrm{Hom}(E, H)$ *be the composition with* T *at right. Then* T_* *and* T^* *are morphisms with respect to the para-mixed structure from v).*

A first structural result is provided by:

Proposition 1 *In the category of para-mixed linear spaces a given para-mixed linear space* E *is isomorphic with the direct product* $E_r \times E_p$*.*

Proof. We have the canonical maps: $i : E_p \to E$ and $\pi : E \to E_r$. There exists the maps $q : E \to E_p$ and $\rho : E_r \to E$ such that: $q \circ i = 1_{E_p}$, $i \circ q + \rho \circ \pi = 1_E$. It follows that these maps i, π, q, ρ are morphisms of para-mixed linear spaces and $(\rho, i) : E_r \times E_p \to E$ is an isomorphism. $\square$

It follows directly:

Corollary 1 *Let* $E = E_r \times E_p$ *and* $F = F_r \times F_p$ *be para-mixed linear spaces and* $T \in L(E, F)$*. Then* $T \in Hom(E, F)$ *if and only if it has the expression:*

$$T = \begin{pmatrix} T_2 & 0 \\ \alpha & T_1 \end{pmatrix} \tag{1}$$

where $T_1 : E_p \to F_p$ *is a* $\mathbb{A}$*-linear map while* $T_2 : E_r \to F_r$ *and* $\alpha : E_r \to F_p$ *are real linear maps. In this decomposition,* T_1 *is the paracomplex part of* T *and* T_2 *is the real part of* T*.*

A characterization of isomorphisms is provided by:

Proposition 2 *Let* E*,* F *and* T *as above. Then the following statements are equivalent:*
i) T *is an isomorphism,*
ii) T*,* T_1 *and* T_2 *are all bijective,*
iii) *two of the maps* T*,* T_1 *and* T_2 *are bijective.*

A special study can be performed in finite-dimension:

Definition 3 *Let* E *be a para-mixed linear space and* $m, n \in \mathbb{N}$*. We say that* E *is of* (m, n)*-type if* E_r *is a real linear space of dimension* m *and* E_p *is a paracomplex space of dimension* n*. A frame on* E *is a set of vectors* $B = \{e_1, ..., e_m, e_{m+1}, ..., e_{m+n}\}$ *with* $\{e_{m+1}, ..., e_{m+n}\}$ *a basis in* E_p *and* $\{[e_1], ..., [e_m]\}$ *a basis in* E_r *where* $[e]$ *is the class of* $e \in E$ *considered in* E_r*.*

A characterization of this notion is:

Proposition 3 *Fix* E *a para-mixed linear space of* (m, n)*-type and* $B = \{e_1, ..., e_{m+n}\} \subset E$ *with* $\{e_{m+1}, ..., e_{m+n}\} \subset E_p$*. Then* B *is a frame on* E *if and only if the map:*

$$T : (x^1, ..., x^{m+n}) \in E_f = \mathbb{R}^m \times \mathbb{A}^n \to x = x^i e_i \in E \tag{2}$$

belongs to $Hom(E_f, E)$*.*

Definition 4 *Let* E *be para-mixed linear space and* W *a subspace of* E*. Then* W *is called para-mixed subspace of* E *if* $W \cap E_p$ *is a paracomplex subspace of* E_p *which means that* $x \in W \cap E_p$ *implies* $jx \in E_p$*.*

A para-mixed subspace W will be considered itself as a para-mixed linear space with $W_p = W \cap E$. Hence, the inclusion map $i : W \to E$ is a morphism with i_p and i_r injective maps.

Example 2 i) *For* $T \in \mathrm{Hom}(E, F)$ *the kernel* $\ker T = T^{-1}(0_F)$ *is a para-mixed subspace of* E.

ii) *The intersection of an arbitrary family of para-mixed subspaces is again a para-mixed subspace.*

iii) *Let* E *be a pure real para-mixed linear space and* $W \subset E$ *a (real) subspace. Then* W *is a para-mixed subspace. A similar property holds for the paracomplex case.*

2 Monomorphisms and epimorphisms

Definition 5 *Let* $T \in \mathrm{Hom}(E, F)$.
i) T *is called monomorphism if there exists a para-mixed linear space* G *and* $R \in \mathrm{Hom}(G, F)$ *such that the map* $(T, R) : E \times G \to F$ *is an isomorphism.*

ii) T *is called epimorphism if there exists a para-mixed linear space* G *and* $R \in \mathrm{Hom}(E, G)$ *such that the map* $(R, T) : E \to F \times G$ *is an isomorphism.*

A characterization of these types of morphisms is given by:

Proposition 4 *Let* $T \in \mathrm{Hom}(E, F)$ *be given.*
I) *The following statements are equivalent:*
a) T *is a monomorphism,*
b) T *and* T_r *are injective maps,*
c) T_p *and* T_r *are injective maps,*
d) T *have an inverse morphism at left.*
II) *Also, the following statements are equivalent:*
e) T *is an epimorphism,*
f) T *and* T_p *are surjective maps,*
g) T_p *and* T_r *are surjective maps,*
h) T *have an inverse morphism at right.*

Proof. a)⇒b). From hypothesis the maps (T, R) and $(T, R)_r : E_r \times G_r \to F_r$ is bijective and then T, T_r are injective. b)⇒c). It is obvious.
c)⇒d). Consider the decomposition (1) of T. Since T_p is injective it follows the existence of R_1 an $\mathbb{A}$-linear map which is inverse at left. Similar, from T_r being

injective it result the existence of R_2 a $\mathbb{R}$-linear map which is inverse at left. The map $R : F \to E$ given by:

$$R = \begin{pmatrix} R_r & 0 \\ -R_p \alpha R_r & R_p \end{pmatrix}$$

is a morphism from F to E with $R \circ T = Id_E$.

d)⇒a). Let $R : F \to E$ be the inverse at left of T and consider $G = \ker R$ together with the inclusion $i : G \to F$. From $R \circ (1_F - T \circ R) = 0$ it results the existence of $w \in Hom(F, G)$ such that $T \circ R + i \circ w = 1_F$. Then $w \circ i = 1_G$ and $w \circ T = 0$. Let $A = (T, i) : E \times G \to F$ and $B = (R, w) : F \to E \times G$. With the equations above it follows that A and B are isomorphisms with $B = A^{-1}$.

The equivalences from II are analogous. □

Corollary 2 *Fix* $T \in Hom(E, F)$*. Then* T *is a monomorphism if and only if* $T(E)$ *is a para-mixed subspace of* F *and the induced map* $T' : E \to T(E)$ *is an isomorphism. Also,* T *is an epimorphism if and only if the co-induced map* $T'' : E/\ker T \to F$ *is an isomorphism.*

Proof. Suppose that T is a monomorphism. Since T_r is injective it results that $T(E) \cap F_p = T_p(E_p)$ and so, $T(E)$ is a para-mixed subspace in F. It follows also that $T' \in Hom(E, T(E))$ and its paracomplex part T'_p is surjective. From T=injective we get that T' and T'_p are bijective maps and then T' is an isomorphism. Similar arguments hold for the second part. □

Let us remark that an injective $T \in Hom(E, F)$ is not a-priori a monomorphism and the example is provided by the inclusion $\mathbb{R} \to \mathbb{A}$. In order to include this class we consider:

Definition 6 $T \in Hom(E, F)$ *is called cartesian monomorphism if it is injective and for every para-mixed linear space* G *and every map* $\alpha : G \to E$ *we have that* $T \circ \alpha$ *is a morphism if and only if* α *is a morphism.*

This notion is useful for another concept:

Definition 7 *A paracomplexification of the para-mixed linear space* E *is a pair* (E^p, ρ) *with* E^p *a paracomplex linear space and* $\rho \in Hom(E, E^p)$ *injective and satisfying* $\rho(E) + j\rho(E) = E^p$ *and* $\rho(E) \cap j\rho(E) = \rho(E_p)$.

A characterization of this notion is given by:

Theorem 1 i) *Every para-mixed linear space* E *have a paracomplexification.*

ii) *The morphism* ρ *is a cartesian monomorphism.*

iii) *A pair* (E^p, ρ) *is a paracomplexification of* E *if and only if* E^p *is a paracomplex linear space and the map* $\rho^* : \mathrm{Hom}(E^p, F) \to \mathrm{Hom}(E, F)$ *given in Remark 1*(vi) *is bijective for any paracomplex linear space* F.

Proof. i) Let $E = E_r \times E_p$ the canonical decomposition of E and consider the space:

$$E^p = (E_r \otimes_{\mathbb{R}} \mathbb{A}) \times E_p.$$

It results that E^p is a paracomplex linear space. One define the map $\rho : E \to E^p$ by $\rho(x_2, x_1) = (x_2 \otimes 1, x_1)$ and a straightforward computation gives that (E^p, ρ) is a paracomplexification of E.

ii) Let G be a para-mixed linear space and $\alpha : G \to E$ such that $\rho \circ \alpha \in \mathrm{Hom}(G, E^p)$. Fix $z \in G_p$; then:

$$\rho \circ \alpha(z) = j(\rho \circ \alpha)(jz) \in \rho(E) \cap j\rho(E) = \rho(E_p)$$

and the injectivity of ρ yields that $\alpha(z) \in E_p$. Also, $\rho \circ \alpha(jz) = j(\rho \circ \alpha(z)) = \rho(j\alpha(z))$ and again the injectivity of ρ gives $\alpha(jz) = j\alpha(z)$. These facts together with the $\mathbb{R}$-linearity means that $\alpha \in \mathrm{Hom}(G, E)$.

iii) Fix (E^p, ρ) a paracomplexification of E, F a paracomplex linear space and $u \in \mathrm{Hom}(E, F)$. Define then $\nu : E^p \to F$:

$$\nu(\rho(x) + j\rho(y)) := u(x) + ju(y)$$

for all $x, y \in E$. Since $\rho(x) + j\rho(y) = \rho(x') + j\rho(y')$ if and only if $y' - y \in E_p$ and $x - x' = j(y' - y)$ it results that ν is well defined. It follows that ν is a $\mathbb{A}$-linear map and $\nu \circ \rho = u$. We get also the uniqueness of ν with these two properties. □

Example 3 i) *Let* E *be a real linear space endowed with the paracomplex structure* J. *Consider then* $E^p = (E, J) \oplus (E, -J)$ *and the diagonal map* $\rho : E \to E^p$. *Then* (E^p, ρ) *is a paracomplexification of* E. *Indeed, every vector* $(e_1, e_2) \in E^p$ *has a decomposition* $(e_1, e_2) = (x + Jy, x - Jy)$ *with* $x, y \in E$. *More precisely,* $x = \frac{1}{2}(e_1 + e_2)$ *and* $y = \frac{1}{2}(Je_1 - Je_2)$.

Let now F *be a paracomplex linear space and* $u : E \to F$ *a* $\mathbb{R}$*-linear map. The unique* $\mathbb{A}$*-linear map* $\nu : E^p \to F$ *satisfying* $\nu \circ \rho = u$ *is:*

$$\nu(e_1, e_2) = \frac{1}{2}\left(u(e_1) - ju(Je_1)\right) + \frac{1}{2}\left(u(e_2) + ju(Je_2)\right).$$

ii) *Let* E, F *be para-mixed linear spaces and fix* $\rho : \mathsf{F} \to \mathsf{F}^{p}$ *a paracomplexification of* F. *Then* $\rho_* : \mathsf{Hom}(\mathsf{E},\mathsf{F}) \to \mathsf{Hom}(\mathsf{E},\mathsf{F}^{p})$ *given in Remark 1*(vi) *is a cartesian monomorphism. In general,* $(\mathsf{Hom}(\mathsf{E},\mathsf{F}^{p}),\rho_*)$ *is not a paracomplexification of* $\mathsf{Hom}(\mathsf{E},\mathsf{F})$ *since* $\mathsf{Hom}(\mathsf{E},\mathsf{F}^{p})$ *is a paracomplex linear space satisfying* $\rho_*(\mathsf{Hom}(\mathsf{E},\mathsf{F}^{p})) \cap j\rho_*(\mathsf{Hom}(\mathsf{E},\mathsf{F}^{p}))$ *but generally it do not satisfies* $\rho_*(\mathsf{Hom}(\mathsf{E},\mathsf{F}^{p})) + j\rho_*(\mathsf{Hom}(\mathsf{E},\mathsf{F}^{p})) = (\rho_*(\mathsf{Hom}(\mathsf{E},\mathsf{F}^{p})))^{p}$.

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Received: February 16, 2017

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 283–290

DOI: 10.1515/ausm-2017-0021

The sharp version of a strongly starlikeness condition

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Abstract. In this paper we give the best form of a strongly starlikeness condition. Some consequences of this result are deduced. The basic tool of the research is the method of differential subordinations.

1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane. Let $\mathcal{A}$ be the class of analytic functions f, which are defined on the unit disk $\mathbb{U}$ and have the properties $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$, consisting of functions for which the domain $f(\mathbb{U})$ is starlike with respect to 0 is denoted by S^*. An analytic characterization of S^* is given by

$$S^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U} \right\}.$$

In connection with the starlike functions has been introduced the following class

$$SS^*(\alpha) = \left\{ f \in \mathcal{A} : \left|\arg \frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2},\ \alpha \in (0,1],\ z \in \mathbb{U} \right\},$$

2010 Mathematics Subject Classification: 11A25
Key words and phrases: starlike functions, strongly starlike functions, subordination

which is the class of strongly starlike functions of order α.
Another subclass of $\mathcal{A}$ we deal with is the following

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < b, \quad z \in \mathbb{U} \right\}, \tag{1}$$

where $b > 0$.
The authors of [3] proved the following result:

Theorem 1 *If the function* f *belongs to the class* $\mathcal{G}_{b(\beta)}$ *with*

$$b(\beta) = \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}},$$

where $0 < \beta \leq 1$*, then* $f \in SS^*(\beta)$.

Let $-1 \leq B < A \leq 1$. The class $S^*(A, B)$ is defined by the equality

$$S^*(A,B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \; z \in \mathbb{U} \right\}.$$

An other result regarding the class $\mathcal{G}_b$ is the following theorem published in [4].

Theorem 2 *Assume that* $-1 \leq B < A \leq 1$ *and* $b(1 + |A|)^2 \leq |A - B|$*. If* $f \in \mathcal{G}_b$*, then* $f \in S^*(A, B)$.

The aim of this paper is to prove the sharp version of Theorem 1, and an improvement of Theorem 2.
In our work we need the following results.

2 Preliminaries

Let f and g be analytic functions in $\mathbb{U}$. The function f is said to be subordinate to g, written $f \prec g$, if there is a function w analytic in $\mathbb{U}$, with $w(0) = 0$, $|w(z)| < 1$, $z \in \mathbb{U}$ and $f(z) = g(w(z))$, $z \in \mathbb{U}$. Recall that if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Lemma 1 [1] *Let* $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$ *be analytic in* $\mathbb{U}$ *with* $p(z) \not\equiv a$, $n \geq 1$ *and let* $q : \mathbb{U} \to \mathbb{C}$ *be an analytic and univalent function with* $q(0) = a$. *If* p *is not subordinate to* q, *then there are two points* $z_0 \in \mathbb{U}$, $|z_0| = r_0$ *and* $\zeta_0 \in \partial\mathbb{U}$ *and a real number* $m \in [n, \infty)$, *so that* q *is defined in* ζ_0, $p(\mathbb{U}(0, r_0)) \subset q(\mathbb{U})$, *and:*

(i) $p(z_0) = q(\zeta_0)$,
(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$,
(iii) $\mathrm{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m\,\mathrm{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right)$.

We note that $z_0 p'(z_0)$ *is the outward normal to the curve* $p(\partial\mathbb{U}(0, r_0))$ *at the point* $p(z_0)$, *while* $\partial\mathbb{U}(0, r_0)$ *denotes the border of the disc* $\mathbb{U}(0, r_0)$.

A basic result we need in our research is the following:

Lemma 2 *If* $f \in \mathcal{A}$, $b \in [0, 1)$, *and* $p(z) = \dfrac{zf'(z)}{f(z)}$, *then the inequality*

$$\left|\frac{zp'(z)}{p^2(z)}\right| < b, \quad z \in \mathbb{U}, \tag{2}$$

implies that

$$p(z) \prec \frac{1}{1 - bz}.$$

The result is sharp.

Proof. If the subordination $p(z) \prec q(z) = \dfrac{1}{1 - bz}$ does not holds, then there are two points $z_0 \in \mathbb{U}$, $|z_0| = r_0 < 1$ and $\zeta_0 \in \partial\mathbb{U}$ and a real number $m \in [1, \infty)$, so that q is defined in ζ_0, $p(\mathbb{U}(0, r_0)) \subset q(\mathbb{U})$, and:

$$p(z_0) = q(\zeta_0) = \frac{1}{1 - b\zeta_0}$$

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = m\frac{b\zeta_0}{(1 - b\zeta_0)^2}.$$

Thus we get

$$\frac{z_0 p'(z_0)}{p^2(z_0)} = mb\zeta_0. \tag{3}$$

Since $|mb\zeta_0| \geq b$, it follows that the equality (3) contradicts (2), and the proof is done. □

3 Main results

The following theorem is the sharp version of Theorem 1.

Theorem 3 *If* $\alpha \in (0,1)$, *and* $f \in \mathcal{G}_{b(\alpha)}$, *where* $b(\alpha) = \sin\left(\alpha\frac{\pi}{2}\right)$, *then* $f \in SS^*(\alpha)$. *The result is sharp.*

Proof. If we denote $p(z) = \dfrac{zg'(z)}{g(z)}$, then the condition $f \in \mathcal{G}_{b(\alpha)}$ becomes

$$\left|\frac{zp'(z)}{p^2(z)}\right| < b(\alpha), \quad z \in \mathbb{U}, \tag{4}$$

and according to Lemma 2 we get

$$p(z) \prec q(z) = \frac{1}{1-b(\alpha)z}.$$

The domain $D = q\big(\mathbb{U}\big)$ is symmetric with respect to the real axis and the boundary of D is the curve

$$\Gamma = \begin{cases} x(\theta) = \operatorname{Re}\frac{1}{1-b(\alpha)e^{i\theta}} = \frac{1-b(\alpha)\cos\theta}{1+b^2(\alpha)-2b(\alpha)\cos\theta}, \\ y(\theta) = \operatorname{Im}\frac{1}{1-b(\alpha)e^{i\theta}} = \frac{b(\alpha)\sin\theta}{1+b^2(\alpha)-2b(\alpha)\cos\theta}, \end{cases} \quad \theta \in [-\pi, \pi].$$

The subordination $p(z) \prec q(z)$ implies that $|\arg(p(z))| \le \arctan(M)$, where M is the slope of the tangent line to the curve Γ trough the origin.
The equation of the tangent line is

$$\frac{x - x(\theta)}{x'(\theta)} = \frac{y - y(\theta)}{y'(\theta)}.$$

This tangent line crosses the origin if and only if

$$\frac{x(\theta)}{x'(\theta)} = \frac{y(\theta)}{y'(\theta)},$$

and this equation is equivalent to

$$2b(\alpha)\cos^2\theta - (3b^2(\alpha)+1)\cos\theta + b(\alpha)(b^2(\alpha)+1) = 0.$$

After a short calculation we get $\cos\theta = b(\alpha)$ and this implies

$$M = \frac{y'(\theta)}{x'(\theta)} = \frac{y(\theta)}{x(\theta)} = \frac{b(\alpha)\sin\theta}{1-b(\alpha)\cos\theta} = \frac{b(\alpha)}{\sqrt{1-b^2(\alpha)}}.$$

Finally if we put $b(\alpha) = \sin\left(\alpha\frac{\pi}{2}\right)$, then it follows that $|\arg(p(z))|$ $< \arctan(M) = \arctan\frac{b(\alpha)}{\sqrt{1-b^2(\alpha)}} = \alpha\frac{\pi}{2},\ z \in \mathbb{U}$.
Thus we have proved the implication

$$\left|\frac{zp'(z)}{p^2(z)}\right| < \sin\left(\alpha\frac{\pi}{2}\right) \Rightarrow \quad |\arg(p(z))| < \arctan(M) = \alpha\frac{\pi}{2},$$

and the proof is done. □

Putting $\alpha = 1$ in Theorem 3, we get the following starlikeness condition, which is the sharp version of Corollary 1 from [3].

Corollary 1 *If* $f \in \mathcal{A}$ *and*

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < 1, \quad z \in \mathbb{U},$$

then $f \in S^*$.

For $\alpha = \frac{1}{2}$, we get the sharp version of Corollary 2 from [3].

Corollary 2 *If* $f \in \mathcal{A}$ *and*

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < \frac{\sqrt{2}}{2}, \quad z \in \mathbb{U},$$

then $f \in SS^*\left(\frac{1}{2}\right)$.

Theorem 4 *If* $f \in \mathcal{G}_b$ *and* $b(1 + A - B + |B|) < A - B$, *then* $f \in S^*(A, B)$.

Proof. Let $q, h : \mathbb{U} \to \mathbb{C}$ be the functions defined by

$$q(z) = \frac{1}{1-bz}, \quad h(z) = \frac{1+Az}{1+Bz}.$$

According to Lemma 2 we have $p(z) = \frac{zf'(z)}{f(z)} \prec q(z)$ which is equivalent to

$$p(\mathbb{U}) \subset q(\mathbb{U}). \tag{5}$$

We will prove that $q(\mathbb{U}) \subset h(\mathbb{U})$. A simple calculation shows that the domains $q(\mathbb{U})$ and $h(\mathbb{U})$ are convex.
The border of the domain $q(\mathbb{U})$ is the curve

$$\Gamma: \quad q(e^{i\theta}) = \frac{1}{1-be^{i\theta}}, \quad \theta \in [0, 2\pi],$$

and the border of $h(\mathbb{U})$ is the curve

$$\Delta: \quad h(e^{i\eta}) = \frac{1+Ae^{i\eta}}{1+Be^{i\eta}}, \quad \eta \in [0, 2\pi].$$

The inequality $b(1+A-B+|B|) < A-B$ is equivalent to $\frac{b}{1-b} < \frac{A-B}{1+|B|}$.
This inequality implies

$$|q(e^{i\theta})-1| = \frac{b}{|1-be^{i\theta}|} \le \frac{b}{1-b} < \frac{A-B}{1+|B|} \le \frac{A-B}{|1+Be^{i\eta}|} = |h(e^{i\eta})-1|.$$

Thus we get

$$|q(e^{i\theta})-1| < |h(e^{i\eta})-1|, \quad \text{for every } \theta, \eta \in [0, 2\pi]. \tag{6}$$

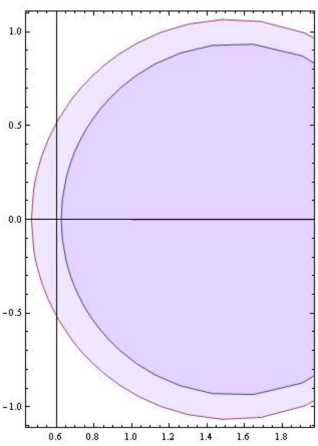
Since $1 \in q(\mathbb{U})$ and $1 \in h(\mathbb{U})$, the inequality (6) implies that the curve Γ is inside the curve Δ.
This means that

$$q(\mathbb{U}) \subset h(\mathbb{U}). \tag{7}$$

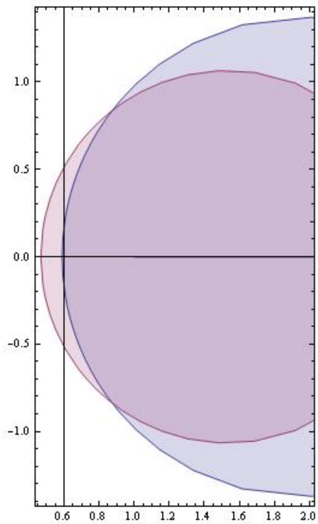
For example if we consider

$$q(z) = \frac{1}{1-0.6z} \text{ and } h(z) = \frac{1+0.3z}{1-0.5z}$$

and the inequality $b(1+A-B+|B|) < A-B$ is satisfied for $b = 0.6$, $A = 0.3$ and $B = -0.5$ then we obtain the following graphics:



which shows that $q(\mathbb{U}) \subset h(\mathbb{U})$. For $b = 0.7$, $A = 0.3$ and $B = -0.5$ the inequality $b(1 + A - B + |B|) < A - B$ is not satisfied and consequently we obtain the following image:



which shows that $q(\mathbb{U}) \not\subset h(\mathbb{U})$. Finally (5) and (7) implies $p(\mathbb{U}) \subset h(\mathbb{U})$ and since h is univalent we infer $\dfrac{zf'(z)}{f(z)} = p(z) \prec h(z), \quad z \in \mathbb{U}$.
This subordination is equivalent to $f \in S^*(A, B)$. $\square$

If $0 \leq B < A \leq 1$, then we get the following corollary, which improvs the result of Theorem 2.

Corollary 3 *Let* $0 \leq B < A \leq 1$ *and* $b \in (0, +\infty)$ *such that* $b(1+A) \leq 1+B$. *If* $f \in \mathcal{G}_b$, *then* $f \in S^*(A, B)$.

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Received: January 8, 2017

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 291–302

DOI: 10.1515/ausm-2017-0022

A result regarding monotonicity of the Gamma function

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Abstract. In this paper we analyze the monotony of the function $\frac{\ln\Gamma(x)}{\ln(x^2+\tau)-\ln(x+\tau)}$, for $\tau > 0$. Such functions have been used from different authors to obtain inequalities concerning the gamma function.

1 Introduction

In [8] the author proved the following double inequality:

$$\frac{x^2+1}{x+1} \le \Gamma(x+1) \le \frac{x^2+2}{x+2},\ x \in [0,1]. \tag{1}$$

In [12] the authors improved this inequality proving that

$$\left(\frac{x^2+1}{x+1}\right)^{2(1-\gamma)} \le \Gamma(x+1) \le \left(\frac{x^2+1}{x+1}\right)^{\gamma},\ x \in [0,1]. \tag{2}$$

2010 Mathematics Subject Classification: 33B15
Key words and phrases: Gamma function, monotonic function

Other improvements of (1) can be found in [9], [10] and [11]. The inequality (2) is equivalent to

$$2(1-\gamma) > \frac{\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)} > \gamma, \quad x \in (0,1).$$

The authors of [12] proved inequality (2) using the monotony of the function

$$g:(0,\infty)\to\mathbb{R}, \quad g(x)=\frac{\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)}.$$

In connection with this function they formulated the following conjecture: if $\tau>0$, then the mapping $u_\tau:(0,\infty)\to\mathbb{R}$ defined by

$$u_\tau(x)=\begin{cases}\frac{\ln\Gamma(x)}{\ln(x^2+\tau)-\ln(x+\tau)}, & x\neq 1\\ -(1+\tau)\gamma, & x=1\end{cases} \tag{3}$$

is strictly increasing. This conjecture was confirmed for $\tau\in(0,1)$ in [6]. We found a counterexample regarding this conjecture: if $\tau=1000$, then

$$\begin{aligned} u_\tau(11) &= \frac{\ln\Gamma(11)}{\ln\frac{1121}{1011}} = \frac{\ln 3628800}{\ln\frac{1121}{1011}} < \frac{\ln 24^5}{\ln\frac{1121}{1011}} = \frac{\ln 24}{\ln\left(\frac{1121}{1011}\right)^{\frac{1}{5}}} = \frac{\ln\Gamma(5)}{\ln\left(\frac{1121}{1011}\right)^{\frac{1}{5}}} \\ &< \frac{\ln\Gamma(5)}{\ln\left(\frac{1025}{1005}\right)} = u_\tau(5). \end{aligned}$$

Numerical results suggest that there is a value $\tau_0\in(212,213)$ such that if $\tau\in(0,\tau_0)$ then u_τ is strictly increasing. We will prove a partial result regarding this question.

Theorem 1 *The function u_τ is strictly increasing on the interval $(0,\infty)$ for all τ, $0<\tau\le 25$.*

2 Preliminaries

In order to prove our main results we need the following lemmas.

Lemma 1 [3] *Let $h,k:[a,b]\to\mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further let $k'(x)\neq 0$, $x\in(a,b)$. If h'/k' is strictly increasing (resp. decreasing) on (a,b), then the functions*

$$x\longmapsto\frac{h(x)-h(a)}{k(x)-k(a)} \qquad x\longmapsto\frac{h(x)-h(b)}{k(x)-k(b)}$$

are also strictly increasing (resp. decreasing) on (a,b).

Lemma 2 *If* $\tau > 1$, *then the function* $u_\tau : (0, \infty) \to \mathbb{R}$ *defined by*

$$u_\tau(x) = \begin{cases} \frac{\ln \Gamma(x)}{\ln(x^2+\tau) - \ln(x+\tau)}, & x \neq 1 \\ -(1+\tau)\gamma, & x = 1 \end{cases}$$

is strictly increasing on the interval $(0, x_1)$, *where* x_1 *is the positive root of the equation* $x^2 + 2\tau x - \tau = 0$.

Proof. According to [4] we have $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}$. It is easily seen that $\frac{1}{2} > x_1 = \frac{\tau}{\tau + \sqrt{\tau^2+\tau}} > \frac{1}{4}$. If $x \in (0, x_1)$, then $\frac{\tau - 2\tau x - x^2}{(x^2+\tau)(x+\tau)} > 0$, $\frac{1}{x} + \gamma - \sum_{n=1}^{\infty} \frac{x}{n(n+x)} > 0$, $\Gamma(x) > 1$, and this implies

$$u_\tau'(x) = \frac{\left(\frac{1}{x} + \gamma - \sum\limits_{n=1}^{\infty} \frac{x}{n(n+x)}\right) \ln \frac{x+\tau}{x^2+\tau} + \frac{\tau - 2\tau x - x^2}{(x^2+\tau)(x+\tau)} \ln \Gamma(x)}{\ln^2\left(\frac{x^2+\tau}{x+\tau}\right)} > 0.$$

Thus u_τ is strictly increasing on the interval $(0, x_1)$. $\square$

Lemma 3 *The unique positive root of the equation* $\psi(x) = -\frac{1}{x} - \gamma + \sum\limits_{n=1}^{\infty} \frac{x}{(n+x)n} = 0$ *is* $x_2 = 1.4616\ldots$. *If* $\tau > 1$, *then the function*

$$v : (x_1, \infty) \to \mathbb{R}, \quad v(x) = \frac{-\frac{1}{x} - \gamma + \sum\limits_{n=1}^{\infty} \frac{x}{(n+x)n}}{\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}}, \tag{4}$$

is strictly increasing on the interval (x_1, x_2), *where* x_1 *is defined in Lemma 2.*

Proof. We have $v'(x) = \frac{A(x)}{\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right)^2}$, where

$$\begin{aligned} A(x) = &\left(\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) \\ &+ \left(\frac{1}{x} + \gamma - \sum_{n=1}^{\infty} \frac{x}{n(n+x)}\right)\left(\frac{-2x^2+2\tau}{(x^2+\tau)^2} + \frac{1}{(x+\tau)^2}\right). \end{aligned} \tag{5}$$

Since $\frac{1}{3} < x_1$, and the following inequalities hold

$$\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} > \frac{1}{x^2} + \frac{\gamma}{x} - \sum_{n=1}^{\infty} \frac{1}{n(n+x)} > 0, \ x \in (\frac{1}{3}, x_2),$$

$$\text{and} \quad \frac{2x}{x^2+\tau} - \frac{1}{x+\tau} = \frac{x^2 + 2\tau x - \tau}{(x+\tau)(x^2+\tau)} > 0, \quad x \in (x_1, x_2),$$

it follows that

$$\begin{aligned}
A(x) &> \left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{2x}{x^2+\tau}-\frac{1}{x+\tau}\right)\\
&+\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{-2x^3+2\tau x}{(x^2+\tau)^2}+\frac{x}{(x+\tau)^2}\right)\\
&=\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{2x^3+2\tau x}{(x^2+\tau)^2}-\frac{x+\tau}{(x+\tau)^2}+\frac{-2x^3+2\tau x}{(x^2+\tau)^2}+\frac{x}{(x+\tau)^2}\right)\\
&=\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{4\tau x}{(x^2+\tau)^2}-\frac{\tau}{(x+\tau)^2}\right)\\
&=\tau\left(\frac{1}{x^2}+\frac{\gamma}{x}-\sum_{n=1}^{\infty}\frac{1}{n(n+x)}\right)\left(\frac{x^3(4-x)+6\tau x^2+\tau^2(4x-1)}{(x^2+\tau)^2(x+\tau)^2}\right)\\
&> 0,\ x\in(x_1,x_2),
\end{aligned}$$

and we get $\nu'(x)>0,\quad x\in(x_1,x_2)$. Thus ν is a strictly increasing function on the interval (x_1,x_2). □

Lemma 4 *Suppose* $\tau>1$. *The equation* $\psi(x)=\psi'(x)$ *has a unique positive root* $x_3=2.2324\ldots$. *The function* $\nu:(x_1,\infty)\to\mathbb{R}$ *defined by (4) is strictly increasing on the interval* (x_2,x_3).

Proof. We will prove this lemma in two steps. We have $x_2<\frac{3}{2}$.
In the first step we discuss the case $(x_2,\frac{3}{2})$.
According to the mean value theorem for every $x\in(x_2,\frac{3}{2})$ there are the values $c_x,d_x\in(x_2,x)$ such that $\psi(x)=\psi(x)-\psi(x_2)=\psi'(c_x)(x-x_2)$ and $\psi'(x_2)-\psi'(x)=-\psi''(d_x)(x-x_2)$. These two equalities imply

$$\psi(x)=\psi(x)-\psi(x_2)=\psi'(c_x)(x-x_2)<\psi'(x_2)\left(\frac{3}{2}-x_2\right)<\frac{4}{100}\psi'(x_2)$$

and

$$\begin{aligned}
\psi'(x_2)-\psi'(x)=-\psi''(d_x)(x-x_2)&=2(x-x_2)\left(\sum_{n=0}^{\infty}\frac{1}{(n+d_x)^3}\right)\\
&<\frac{8}{100}\psi'\left(\frac{3}{2}\right)\le\frac{8}{100}\psi'(x).
\end{aligned}$$

□

Thus we get $0 < \psi(x) < \frac{4}{100}(1 + \frac{8}{100})\psi'(x) < \frac{1}{12}\psi'(x), \ x \in (x_2, \frac{3}{2})$ and consequently

$$A(x) > \psi(x)\left(\frac{24x}{x^2+\tau} - \frac{12}{x+\tau} + \frac{2x^2-2\tau}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right)$$
$$= \psi(x)\frac{B(x)}{(x^2+\tau)^2(x+\tau)^2},$$

where $B(x) = 12x^5 + (1+36\tau)x^4 + 24\tau^2x^3 + 4\tau x^2(x-1) + \tau^2 x(26x-16) + 24x\tau^3 - 14\tau^3 - \tau^2$, and A is defined by (5). It is easily seen that if $x \in (x_2, \frac{3}{2})$, then $B(x) > 0$, and consequently $\nu'(x) > 0$, for $x \in (x_2, \frac{3}{2})$.
In the second step suppose $x \in (\frac{3}{2}, x_3)$. We have in this case $0 < \psi(x) \le \psi'(x)$, where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. A short calculation leads to

$$A(x) > \psi(x)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau} + \frac{2x^2-2\tau}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) = \psi(x)\frac{C(x)}{(x^2+\tau)^2(x+\tau)^2},$$

where $C(x) = x^5 + (1+3\tau)x^4 + 4\tau x^2(x-1) + \tau^2 x(4x-5) + \tau^2(2x^3-1) + \tau^3(2x-3) > 0, \ x \in (\frac{3}{2}, x_3)$. Consequently we obtain $\nu'(x) > 0, \ x \in (\frac{3}{2}, x_3)$, and the proof is completed.

Lemma 5 *If* $x \in [2,3)$, *then*

$$\frac{6}{7}(\ln x - \frac{7}{25}) > \ln\Gamma(x), \tag{6}$$

and

$$-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty}\frac{x}{n(n+x)} > \ln x - \frac{7}{25}. \tag{7}$$

If $\tau = 25$, *then*

$$\ln\frac{x^2+\tau}{x+\tau} \ge \frac{6}{7}\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right), \quad x \in [2.23, 3]. \tag{8}$$

Proof. Let $\nu_5 : [2,3] \to \mathbb{R}$ be the function defined by $\nu_5(x) = \frac{6}{7}(\ln x - \frac{7}{25}) - \ln\Gamma(x)$. We have

$$\nu_5'(x) = \frac{6}{7x} - \psi(x) = \frac{13}{7x} + \gamma - \sum_{n=1}^{\infty}\frac{x}{n(n+x)},$$

and

$$v_5''(x) = -\frac{6}{7x^2} - \psi'(x) = -\frac{13}{7x^2} - \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} < 0, \quad x \in [2,3].$$

The monotony of v_5' and the inequalities $v_5'(2) > 0, \ \ v_5'(3) < 0$ implies that the equation $v_5'(x) = 0$ has exactly one root $x_1 \in (2,3)$ and $v_5'(x) > 0, \quad x \in (2, x_1)$, and $v_5'(x) < 0, \quad x \in (x_1, 3)$.
The monotony of v_5 implies

$$v_5(x) \geq \min\{v_5(2), v_5(3)\} > 0, \quad x \in (2,3),$$

and thus the inequality (6) holds.
In order to prove (7), we define the function $v_6 : [2,3] \to \mathbb{R}$,

$$v_6(x) = \psi(x) - \ln x + \frac{7}{25} = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} - \ln x + \frac{7}{25}.$$

We have $v_6'(x) = -\frac{1}{x} + \psi'(x) = -\frac{1}{x} + \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} > 0, \ x \in [2,3]$, and consequently

$$v_6(x) \geq v_6(2) > 0, \ x \in [2,3].$$

Thus the inequality (7) holds.
The third inequality can be proved as follows.
Let $v_7 : [2.23, \infty) \to \mathbb{R}$ be the function defined by $v_7(x) = \ln \frac{x^2+\tau}{x+\tau} - \frac{6}{7}\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right)$. We have $v_7'(x) = \frac{D(x)}{(x^2+\tau)^2(x+\tau)^2}$, where $\alpha = \frac{6}{7}$ and $D(x) = x^5 + (3\tau+3\alpha)x^4 + (2\tau^2 + 4\alpha\tau)x^3 + 2(1+\alpha)\tau^2 x^2 + (2\tau^3 - (4\alpha+1)\tau^2)x - (2\alpha+1)\tau^3 + \alpha\tau^2$. A suitable alignment in the numerator of v_7' shows that $v_7'(x) > 0, \quad x \in [2.23, 3]$. Thus we get

$$v_7(x) \geq v_7(2.23) > 0, \quad x \in [2.23, 3],$$

and the inequality (8) follows. □

Lemma 6 *If* $x \in [3, \infty)$, *then*

$$(x-2)(\ln x - \frac{1}{4}) > \ln \Gamma(x). \tag{9}$$

If $x \in [3, \infty)$, *then*

$$-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} > \ln x - \frac{1}{4}, \quad x \in (3, \infty). \tag{10}$$

If $x \in [3, \infty)$, *and* $\tau = 25$, *then*

$$\ln \frac{x^2+\tau}{x+\tau} \geq (x-2)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right), \quad x \in (3, \infty). \tag{11}$$

Proof. In order to prove inequality (9) we define the function $v_8 : [3, \infty) \to \mathbb{R}$ by $v_8(x) = (x-2)(\ln x - \frac{1}{4}) - \ln \Gamma(x)$. We have

$$v_8'(x) = \ln x - \frac{1}{4} + \frac{x-1}{x} + \gamma - \sum_{n=1}^{\infty} \frac{x}{n(n+x)},$$

and

$$v_8''(x) = \frac{1}{x} + \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}.$$

It is easily seen that

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)^2} < \sum_{n=1}^{\infty} \frac{1}{(n+x)(n-1+x)} = \frac{1}{x}, \ x \in [3, \infty).$$

Thus we have $v_8''(x) > 0, \ x \in [3, \infty),$ consequently v_8' is strictly increasing and

$$v_8'(x) > v_8'(3) = \ln 3 + \gamma - 1 - \frac{5}{12} > 0, \ x \in (3, \infty).$$

This means that v_8 is strictly increasing too and

$$v_8(x) > v_8(3) = \ln 3 - \frac{1}{4} - \ln 2 > 0, \quad x \in (3, \infty).$$

The inequality (10) can be proved as follows. Let the function $v_9 : [3, \infty) \to \mathbb{R}$ be defined by $v_9(x) = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} - \ln x + \frac{1}{4}$. We have

$$v_9'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} - \frac{1}{x}.$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{(n+x)^2} > \sum_{n=0}^{\infty} \frac{1}{(n+x)(n+1+x)} = \frac{1}{x}, \ x \in [3, \infty),$$

it follows that $v_9'(x) > 0, \ x \in [3, \infty),$ consequently v_9 is strictly increasing and

$$v_9(x) > v_9(3) = 1 + \frac{3}{4} - \gamma - \ln 3 > 0, \ x \in (3, \infty).$$

$\square$

Finally, in order to prove (11), we define the function $\nu_{10} : [3,\infty) \to \mathbb{R}$ by $\nu_{10}(x) = \ln \frac{x^2+\tau}{x+\tau} - (x-2)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right)$, where $\tau = 25$. We have

$$\begin{aligned} \nu'_{10}(x) &= (x-2)\left(\frac{2x^2-2\tau}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \\ &= (x-2)\frac{x^4+4\tau x^3+2(\tau^2-2\tau)x^2-4\tau^2 x-2\tau^3-\tau^2}{(x^2+\tau)^2(x+\tau)^2} \\ &= (x-2)\frac{x^4+100x^3+1150x^2-2500x-31875}{(x^2+\tau)^2(x+\tau)^2}. \end{aligned}$$

The Descartes rule of signs implies that the equation $x^4+100x^3+1150x^2-2500x-31875 = 0$ has no more than one positive root, thus it is easily seen that the equation $\nu'_{10}(x) = 0$ has exactly one root $x_0 = 5.13\ldots$. This means that ν_{10} is stictly decreasing on the interval $[3, x_0]$ and strictly increasing on $[x_0, \infty)$. Consequently $\min_{x\in[3,\infty)} \nu_{10}(x) = \nu_{10}(x_0) = 0.01\ldots > 0$, and this implies

$$\nu_{10}(x) > 0, \quad \text{for} \quad \text{all } x \in [3,\infty).$$

3 Proof of the main result

In this section we shall prove the main theorems.

Theorem 2 *Let the function $g_{\alpha,\beta} : (0,\infty) \to \mathbb{R}$ be defined by*

$$g_{\alpha,\beta}(x) = \begin{cases} \frac{\ln(x^2+\alpha)-\ln(x+\alpha)}{\ln(x^2+\beta)-\ln(x+\beta)}, & x \in (0,1)\cup(1,\infty) \\ \frac{1+\beta}{1+\alpha}, & x = 1. \end{cases} \tag{12}$$

If $\alpha > \beta > 0$, then the mapping $g_{\alpha,\beta}$ is strictly increasing on the interval $(0,\infty)$.

Proof. We will prove the theorem in two steps. Let $x_1 = \frac{\beta}{\beta+\sqrt{\beta^2+\beta}}$ be the positive root of the equation $x^2+2\beta x-\beta = 0$, and let $x_2 = \frac{\alpha}{\alpha+\sqrt{\alpha^2+\alpha}}$ be the positive root of $x^2+2\alpha x-\alpha = 0$.

In the first step let $x \in (0,1)$. Since $\left(\frac{x^2+2\alpha x-\alpha}{x^2+2\beta x-\beta}\right)' = \frac{2(\alpha-\beta)(x-x^2)}{(x^2+2\beta x-\beta)^2} > 0$, $x \in (0,x_1)\cup(x_2,1)$, it follows that the function $h : (0,\infty) \to \mathbb{R}$ defined by

$$h(x) = \frac{\big(\ln(x^2+\alpha)-\ln(x+\alpha)\big)'}{\big(\ln(x^2+\beta)-\ln(x+\beta)\big)'} = \frac{x^2+\beta}{x^2+\alpha}\cdot\frac{x+\beta}{x+\alpha}\cdot\frac{x^2+2\alpha x-\alpha}{x^2+2\beta x-\beta},$$

is strictly increasing on the intervals $(0, x_1)$ and $(x_2, 1)$, (because h is a product of positive strictly increasing functions). Now Lemma 1 implies that $g_{\alpha,\beta}$ is strictly increasing on $(0, x_1)$ and $(x_2, 1)$ too. On the other hand

$$g'_{\alpha,\beta}(x) = \frac{D(x)}{(\ln(x^2+\beta) - \ln(x+\beta))^2},$$

where

$$D(x) = \frac{x^2+2\alpha x-\alpha}{(x^2+\alpha)(x+\alpha)} \ln\frac{x^2+\beta}{x+\beta} - \frac{x^2+2\beta x-\beta}{(x^2+\beta)(x+\beta)} \ln\frac{x^2+\alpha}{x+\alpha}.$$

Since $\frac{x^2+2\alpha x-\alpha}{(x^2+\alpha)(x+\alpha)} \ln\frac{x^2+\beta}{x+\beta} > 0,\ x \in (x_1, x_2)$, and $\frac{x^2+2\beta x-\beta}{(x^2+\beta)(x+\beta)} \ln\frac{x^2+\alpha}{x+\alpha} < 0,\ x \in (x_1, x_2)$, it follows that $D(x) > 0,\ x \in (x_1, x_2)$, and consequently $g'(x) > 0,\ x \in (x_1, x_2)$.

We have deduced that $g_{\alpha,\beta}$ is a strictly increasing function on the intervals $(0, x_1)$, (x_1, x_2), and $(x_2, 1)$. The continuity of $g_{\alpha,\beta}$ implies that this function is strictly increasing on $(0, 1)$.

In the second step we prove that $g_{\alpha,\beta}$ is strictly increasing on $(1, \infty)$. We will prove that

$$D(x) > 0, \quad x \in (1, \infty). \tag{13}$$

Let $k : (0, \infty) \to \mathbb{R}$ be the function defined by $k(\tau) = \frac{\ln(x^2+\tau)-\ln(x+\tau)}{\frac{2x}{x^2+\tau}-\frac{1}{x+\tau}}$. The following equivalence chain holds

$$g'_{\alpha,\beta}(x) > 0 \quad \Leftrightarrow D(x) > 0 \quad \Leftrightarrow k(\beta) > k(\alpha), \tag{14}$$

providing that $x \in (1, \infty)$, and $\alpha > \beta > 0$.

Consequently in order to prove that $g_{\alpha,\beta}$ is strictly increasing we have to show that if $x \in (1, \infty)$ is a fixed number, then k is strictly decreasing on $(0, \infty)$. We have

$$k'(\tau) = \frac{E(\tau)}{(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau})^2},$$

$$E(\tau) = \left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) + \left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \ln\frac{x^2+\tau}{x+\tau}.$$

It is easily seen that if $\tau \in (0, \infty)$ and $x \in (1, \infty)$, then $\frac{1}{x^2+\tau} - \frac{1}{x+\tau} < 0$, $\frac{2x}{x^2+\tau} - \frac{1}{x+\tau} > 0, \quad \ln\frac{x^2+\tau}{x+\tau} > 0$.

This second case has two sub-cases.

First suppose that $\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2} \leq 0$, for some $x \in (1, \infty)$, $\tau \in (0, \infty)$. In this case we have $\left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) < 0$ and $\left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \ln \frac{x^2+\tau}{x+\tau} \leq 0$. Thus it follows $E(\tau) < 0$, and so we get $k'(\tau) < 0$, and we are done.
Now we suppose $\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2} > 0$.
In this case we use the well-known inequality $t-1 \geq \ln t$, $\quad t \in (0, \infty)$. Putting $t = \frac{x^2+\tau}{x+\tau}$ we get $\ln \frac{x^2+\tau}{x+\tau} \leq \frac{x^2-x}{x+\tau}$, for every $x \in (1, \infty)$, $\tau \in (0, \infty)$, and it follows that

$$
\begin{aligned}
E(\tau) &= \left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) + \left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \\
\ln \frac{x^2+\tau}{x+\tau} &\leq \left(\frac{1}{x^2+\tau} - \frac{1}{x+\tau}\right)\left(\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}\right) \\
&\quad + \left(\frac{2x}{(x^2+\tau)^2} - \frac{1}{(x+\tau)^2}\right) \cdot \frac{x^2-x}{x+\tau} \\
&= \frac{(x-x^2)(x^2+2\tau x-\tau)}{(x^2+\tau)^2(x+\tau)^2} + \frac{[2x(x+\tau)^2-(x^2+\tau)^2](x^2-x)}{(x^2+\tau)^2(x+\tau)^3} \\
&= \frac{(x-x^2)(x^4-x^3+\tau x^2-\tau x)}{(x^2+\tau)^2(x+\tau)^3} < 0.
\end{aligned}
$$

Consequently, provided that x is fixed, $x \in (1, \infty)$, the inequality $k'(\tau) < 0$ holds for every $\tau \in (0, \infty)$. According to (14) it follows $g'_{\alpha,\beta}(x) > 0$, $x \in (1, \infty)$, and the proof is finished. □

Theorem 3 *If* $\tau = 25$, *then the mapping* u_τ *is strictly increasing on the interval* $(0, \infty)$, *where* u_τ *is defined by* (3).

Proof. Provided that $\tau = 25$, Lemma 2 implies that the function u_τ is strictly increasing on the interval $(0, x_1)$, where x_1 is the positive root of the equation $x^2 + 2\tau x - \tau = 0$.
Let $x_2 = 1.4616\ldots$ be the positive root of the equation $\psi(x) = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{(n+x)n} = 0$. If $\tau = 25$, then Lemma 3 implies that the function

$$
v : (x_1, \infty) \to \mathbb{R}, \quad v(x) = \frac{-\frac{1}{x} - \gamma + \sum\limits_{n=1}^{\infty} \frac{x}{(n+x)n}}{\frac{2x}{x^2+\tau} - \frac{1}{x+\tau}},
$$

is strictly increasing on the interval (x_1, x_2).
Let $x_3 = 2.2324\ldots$ be the positive root of the equation $\psi(x) = \psi'(x)$. Since

Lemma 4 implies that ν is strictly increasing on the interval (x_2, x_3), it follows that ν is strictly increasing on (x_1, x_3). Now this result and Lemma 1 imply that the mapping u_τ is also strictly increasing on the intervals $(x_1, 1)$ and $(1, x_3)$.
Further we will prove that u_τ is strictly increasing on $(x_3, 3)$. We observe that if $x \in (x_3, 3)$ and $\tau = 25$ we can multiply the inequalities (6), (7), (8) and it follows that

$$\left(-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}\right) \ln \frac{x^2+\tau}{x+\tau} > \frac{x^2+2\tau x-\tau}{(x^2+\tau)(x+\tau)} \ln \Gamma(x), \tag{15}$$

and consequently we obtain

$$u_\tau'(x) = \frac{\left(-\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}\right) \ln \frac{x^2+\tau}{x+\tau} - \frac{x^2+2\tau x-\tau}{(x^2+\tau)(x+\tau)} \ln \Gamma(x)}{\ln^2 \left(\frac{x^2+\tau}{x+\tau}\right)} > 0,\ x \in (x_3, 3).$$

Summarizing, if $\tau = 25$, then we have proved that the function u_τ is strictly increasing on the intervals $(0, x_1)$, (x_1, x_3), $(x_3, 3)$. The continuity of u_τ implies that u_τ is strictly increasing on the interval $(0, 3)$.
We will prove in the followings that if $\tau = 25$, then u_τ is strictly increasing on $(3, \infty)$.
It is easily seen that multiplying the inequalities (9), (10), and (11) the inequality (15) follows in case $\tau = 25$ and $x \in (3, \infty)$. Thus we have $u_{25}'(x) > 0$, $x \in (3, \infty)$, and so u_{25} is strictly increasing on $(3, \infty)$. The continuity of u_{25} implies that this function is strictly increasing on $(0, \infty)$. □

Proof of Theorem 1.: From the equality

$$u_\tau(x) = u_{25}(x) \cdot g_{25,\tau}(x),$$

and from the results of Theorem 2. and Theorem 3. we infer that u_τ is strictly increasing on the interval $(0, \infty)$ in case of every given $\tau \in (0, 25]$. □

Other interesting results regarding the Γ function can be found in [1], [2], [5], [6] and [7].

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Received: October 15, 2017

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 303–323

DOI: 10.1515/ausm-2017-0023

Extensional quotient coalgebras

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Abstract. Given an endofunctor F of an arbitrary category, any maximal element of the lattice of congruence relations on an F-coalgebra (A, a) is called a coatomic congruence relation on (A, a). Besides, a coatomic congruence relation K is said to be factor split if the canonical homomorphism $\nu : A_K \to A_{\nabla_A}$ splits, where ∇_A is the largest congruence relation on (A, a). Assuming that F is a covarietor which preserves regular monos, we prove under suitable assumptions on the underlying category that, every quotient coalgebra can be made extensional by taking the regular quotient of an F-coalgebra with respect to a coatomic and not factor split congruence relation or its largest congruence relation.

1 Introduction

The study of coalgebras developed by J. J. M. M. Rutten [15] concerns the particular case of Set-endofunctors. The author develops the theory of universal coalgebras with the assumption that the functors preserve weak pullbacks. This property can see bisimulation equivalences corresponding notions as of congruence relations in universal algebras. In the same context, the largest bisimulation on any coalgebra is again the largest congruence on this coalgebra.

Many theoretical computer science structures, including automata, transition systems, object oriented systems and lazy data types can be modeled with a type functor preserving weak pullbacks. However there are viable examples

2010 Mathematics Subject Classification: 68Q85
Key words and phrases: bisimulation, congruence relation

of coalgebras (topological spaces, for instance) whose type functors do not obey such a restriction.

Certainly, the major advantage of coalgebras is that the theory can naturally deal with nondeterminism and undefinedness, concepts which are hard, or even impossible, to treat algebraically.

A universal algebra is called simple if it does not have any nontrivial congruence relation. The notion of simple coalgebra is obtained by applying the same definition. In other words, the largest congruence relation on a simple coalgebra is its diagonal. An extensional coalgebra is a coalgebra on which the largest bisimulation is its diagonal. Assuming the type functor preserves weak pullbacks, every extensional coalgebra is simple (see [4]).

A quotient algebra also called a factor algebra, is obtained by partionning the elements of an algebra into equivalence classes given by a congruence relation, that is an equivalence relation compatible with all the operators of the algebra. This is equivalent to consider the quotient of an algebra with respect to a congruence relation. The quotient algebra A/θ is simple if and only if θ is a maximal congruence on A or θ is the largest congruence relation on A (see [3]).

The purpose of this paper is to give a characterization theorem for extensional quotient coalgebras of an endofunctor, given an arbitrary underlying category. To this end, let F denote an endofunctor of an arbitrary category. Any maximal element of the lattice of congruence relations on an F-coalgebra (A, a) is called a coatomic congruence relation on (A, a). Besides, a coatomic congruence relation K is said to be factor split if the canonical homomorphism $\nu : A_K \to A_{\nabla_A}$ splits, where ∇_A is the largest congruence relation on (A, a). Suppose that the underlying category is regularly well powered, cocomplete, exact and equipped with epi-(regular mono) factorizations. If more, F is a covarietor which preserves regular monos then, every quotient coalgebra can be made extensional by taking the regular quotient of an F-coalgebra with respect to a coatomic and not factor split congruence relation or its largest congruence relation.

2 Basic notions

We recall here some definitions and usual properties for the following sections.

2.1 Factorization systems

They will be often used throughout this paper.

A *factorization system* (F.S) for a category $\mathcal{C}$ consists of a pair $(\mathcal{E},\mathcal{M})$ of classes of morphisms in $\mathcal{C}$ such that:

FS1. $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms of $\mathcal{C}$ and are closed under composition.

FS2. Every morphism f of $\mathcal{C}$ can be factored as $f = m \circ e$ for some morphisms $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

FS3. For all commutative squares

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "u"'] & \bullet \arrow[d, "v"] \arrow[dl, dashed, "w"] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a unique arrow w making both triangles commute.

2.2 Subobjects

Unions of regular subobjects are revisited (for more details, see [2]). Their existence allows one to construct pullbacks.

Recall that a regular mono is a morphism in some category which occurs as the equalizer of some parallel pair of morphisms. The dual concept is that of regular epi.

Let A be an object of a category $\mathcal{C}$. Denote by $\mathcal{M}_A$ the class of all regular monos of codomain A. Any member $f : B \to A$ of $\mathcal{M}_A$ is written (B, f). The relation $\leq_A$ defined on $\mathcal{M}_A$ by $(B, f) \leq_A (C, g)$ iff there is $h : B \to C$ such that $f = g \circ h$ is a preorder. This preorder induces an equivalence relation $\sim_A$ in $\mathcal{M}_A$, where $(B, f) \sim_A (C, g)$ iff $(B, f) \leq_A (C, g)$ and $(C, g) \leq_A (B, f)$. Also, the preorder $\leq_A$ in $\mathcal{M}_A$ induces an order, again denoted $\leq_A$, in the quotient class $\overline{\mathcal{M}}_A = \mathcal{M}_A / \sim_A$; more precisely $[(B, f)] \leq_A [(C, g)]$ iff $(B, f) \leq_A (C, g)$. A member of an equivalence class is called a *regular subobject* of A.

Definition 1 *A category $\mathcal{C}$ is said to be regularly well powered, if for each A in $\mathcal{C}$, $\overline{\mathcal{M}}_A$ is a set.*

An equivalence class $[(B, f)]$ will be also denoted by its representative f or simply by the domain B; and in this case one also says that f or B is a regular subobject of A.

Definition 2 *A regular image of a morphism* $f : A \to C$ *is a regular mono* $m : B \rightarrowtail C$ *through which* f *factors, which is minimal in the sense that, if* f *factors through any other regular mono* $B' \rightarrowtail C$*, then* B *is a regular subobject of* B'*.*

Suppose that $\mathcal{C}$ is a regularly well powered category admitting coproducts and epi-(regular mono) factorizations. The regular image of a cospan $(f_\alpha : A_\alpha \longrightarrow A)_\alpha$ in $\mathcal{C}$ is the smallest regular subobject E of A through which each f_α factors; that is, there exists a regular mono $m : E \to A$ and an epi sink $(g_\alpha : A_\alpha \longrightarrow E)_\alpha$ such that $(f_\alpha) = m \circ (g_\alpha)$. It is constructed in two steps as follows:

- By the universal property of coproducts, consider the unique morphism $f : \coprod_\alpha A_\alpha \to A$ such that $(f_\alpha) = f \circ (\mu_\alpha)$, where $(\mu_\alpha)_\alpha$ is the cospan of structural injections.
- Consider the epi-(regular mono) factorization of f: $\coprod_\alpha A_\alpha \xrightarrow{e} E \overset{m}{\rightarrowtail} A$.

Hence, the collection of morphisms $(e \circ \mu_\alpha : A_\alpha \longrightarrow E)_\alpha$ is an epi sink given that e is an epimorphism and $(\mu_\alpha)_\alpha$ is an epi sink. Particularly, the regular image or *union* of a cospan $(m_\alpha : S_\alpha \rightarrowtail A)_\alpha$ of regular subobjects in $\mathcal{C}$ is their supremum in the ordered set $(\overline{\mathcal{M}}_A, \leq_A)$. It will be denoted $\bigcup_{\alpha\in\lambda} Im(m_\alpha)$.

Definition 3 *In a category with binary products, a binary relation from* A *to* B *is a regular subobject of* $A \times B$*. This is represented by a regular mono* $m : R \rightarrowtail A \times B$ *or equivalently, by a pair of arrows*

$$\begin{tikzcd} & A \\ R \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & \\ & B \end{tikzcd}$$

with the property that the induced arrow $\langle r_1, r_2\rangle : R \to A \times B$ *is a regular mono. Also,* r_1 *and* r_2 *form a mono source because* $r_1 = p_1 \circ \langle r_1, r_2\rangle$ *and* $r_2 = p_2 \circ \langle r_1, r_2\rangle$ *with* p_1 *and* p_2 *which form a mono source as structural morphisms of the product of* A *and* B*. A relation from* A *to* A *is called a relation on* A*.*

Binary relations are ordered (as regular subobjects of $A \times A$) and can be composed. The relational composition is defined by applying the standard pullback construction as in the category of sets: given a binary relation R

(represented by $r_1 : R \to A$ and $r_2 : R \to B$) in a finitely complete category with epi-(regular mono) factorizations, form the pullback of r_1 and r_2.

$$
\begin{tikzcd}
R \times_A R \arrow[r, "t_1"] \arrow[d, "t_2"'] & R \arrow[r, "r_1"] \arrow[d, "r_2"] & A \\
R \arrow[r, "r_1"'] \arrow[d, "r_2"'] & A & \\
A & &
\end{tikzcd}
$$

Factorize $\langle r_1 \circ t_1, r_2 \circ t_2 \rangle : R \times_A R \to A \times A$ as an epimorphism followed by a regular mono, then the latter represents the composite $R \circ R$. R is said to be *transitive* if $R \circ R$ is smaller than R. The relation R is called *reflexive* if the diagonal map $\langle 1_A, 1_A \rangle : A \to A \times A$ factors through it and, *symmetric* if there is an arrow $\tau : R \to R$ such that $r_1 \circ \tau = r_2$ and $r_2 \circ \tau = r_1$. We say that R is an *equivalence relation* if it is reflexive, symmetric and transitive.

Pullbacks are constructed in the presence of unions of regular subobjects as follows.

Proposition 1 *Suppose that $\mathcal{C}$ is a regularly well powered category with coproducts, finite products and admitting epi-(regular mono) factorizations. Then it has pullbacks.*

Proof. Consider a cospan $(A \xrightarrow{f_1} C \xleftarrow{f_2} B)$ in $\mathcal{C}$.

$$
\begin{tikzcd}
A \arrow[dr, "f_1"] & & & & A \arrow[dr, "f_1"] & \\
& C & & R \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & & C \\
B \arrow[ur, "f_2"'] & & & & B \arrow[ur, "f_2"'] &
\end{tikzcd}
$$

Let us denote by $\mathrm{Rel}(A, B)$ the class of all binary relations R from A to B such that $f_1 \circ r_1 = f_2 \circ r_2$. This class is nonempty as we are going to show. Let 0 be the initial objest of $\mathcal{C}$ (the coproduct in $\mathcal{C}$ over the empty index set). The canonical arrow $! : 0 \to A \times B$ factorizes through a regular subobject $0'$ of $A \times B$, which is a member of $\mathrm{Rel}(A, B)$. Since the category $\mathcal{C}$ is regularly well powered, the class $\mathrm{Rel}(A, B)$ is a set. Let R denote again its supremum (the union); this supremum exists since $\mathcal{C}$ has coproducts. Denote by $u \colon R \rightarrowtail A \times B$ the regular mono making R a binary relation.

Consider a span $(Q, (g_i)_{i=1,2})$ such that $f_1 \circ g_1 = f_2 \circ g_2$. By the universal property of products, there is a unique arrow $g : Q \longrightarrow A \times B$ such that

$p_1 \circ g = g_1$ and $p_2 \circ g = g_2$; p_1 and p_2 being the structural morphisms of the product of A and B. Factorize g as an epimorphism followed by a regular mono:

$$\begin{tikzcd} Q \arrow[rr, "g"] \arrow[dr, "e_g"'] & & A \times B \\ & W \arrow[ur, "m_g"'] & \end{tikzcd}$$

Then $(W, (h_i)_{i=1,2})$, with $h_i = p_i \circ m_g$, is a binary relation from A to B such that $f_1 \circ h_1 = f_2 \circ h_2$. Hence, there is an arrow $s \colon W \to R$ such that $m_g = u \circ s$. As a result, $g_i = p_i \circ g = p_i \circ m_g \circ e_g = p_i \circ u \circ s \circ g = r_i \circ s \circ e_g$ with $r_i = p_i \circ u$; $i = 1, 2$. This implies that for any arrow $j : Q \to R$ such that $r_1 \circ j = g_1$ and $r_2 \circ j = g_2$, we have $r_1 \circ j = r_1 \circ (s \circ e_g)$ and $r_2 \circ j = r_2 \circ (s \circ e_g)$. Thereafter $s \circ e_g = j$ since the pair (r_1, r_2) is a mono source. Consequently, $s \circ e_g$ is the unique arrow from Q to R such that $r_1 \circ (s \circ e_g) = g_1$ and $r_2 \circ (s \circ e_g) = g_2$. This proves that R together with arrows $r_1 = p_1 \circ u$ and $r_2 = p_2 \circ u$ is the pullback of the cospan $(A \xrightarrow{f_1} C \xleftarrow{f_2} B)$. $\square$

Under Proposition 1, the category $\mathcal{C}$ is finitely complete; this is because it has finite products and pullbacks (see [14]).

2.3 Exact sequences

Set, the category of sets and mappings has *exact sequences*; this means that every equivalence relation is a kernel pair of its coequalizer. In other words, there is a ono-to-one correspondence between equivalence relations and regular quotients.

Replacing Set by a finitely complete category $\mathcal{C}$ with coequalizers, an exact sequence in $\mathcal{C}$ is a diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "e"] & B \end{tikzcd}$$

where R is the kernel pair of e and e is the coequalizer of the parallel pair (r_1, r_2). The category $\mathcal{C}$ is said to have exact sequences if every equivalence relation in $\mathcal{C}$ is the kernel pair of its coequalizer. Every topos has exact sequences (see [7]).

A category $\mathcal{C}$ will be called *regular* if every finite diagram has a limit, if every parallel pair of morphisms has a coequalizer and if regular epis are stable under pullbacks. A regular category with exact sequences is called *exact*.

2.4 Kleisli categories

Only monads on Set will be considered.

A *monad* on Set consists of a Set-endofunctor T together with

• a *unit* natural transformation $\eta : \mathtt{id} \Rightarrow \mathsf{T}$; that is, a function $\eta_X : X \to \mathsf{T}X$ for each set X satisfying a suitable naturality condition; and

• a *multiplication* natural transformation $\mu : \mathsf{T}^2 \Rightarrow \mathsf{T}$, consisting of functions $\mu_X : \mathsf{T}^2X \to \mathsf{T}X$ with X ranging over sets.

The unit and multiplication are required to satisfy the following compatibility conditions.

$$\begin{tikzcd} \mathsf{T}X \arrow[r, "\eta_{\mathsf{T}X}"] \arrow[dr, "\mathtt{id}"'] & \mathsf{T}^2X \arrow[d, "\mu_X"] & \mathsf{T}X \arrow[l, "\mathsf{T}\eta_X"'] \arrow[dl, "\mathtt{id}"] \\ & \mathsf{T}X & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{T}^3X \arrow[r, "\mathsf{T}\mu_X"] \arrow[d, "\mu_{\mathsf{T}X}"'] & \mathsf{T}^2X \arrow[d, "\mu_X"] \\ \mathsf{T}^2X \arrow[r, "\mu_X"'] & \mathsf{T}X \end{tikzcd}$$

The powerset functor $\mathcal{P}$ is a monad with a unit given by singletons and a multiplication given by unions. Every adjunction gives rise to a monad (see [10]).

Given any monad T, its *Kleisli category* $\mathcal{K}\ell(\mathsf{T})$ is defined as follows. Its objects are the objects of the base category, hence sets in our consideration. An arrow $X \to Y$ in $\mathcal{K}\ell(\mathsf{T})$ is the same thing as an arrow $X \to \mathsf{T}Y$. Identities and composition of arrows are defined using the unit and the multiplication of T. Moreover, there is a canonical adjunction $\mathsf{J} \dashv \mathsf{H}$, where the functor $\mathsf{J} : \mathsf{Set} \to \mathcal{K}\ell(\mathsf{T})$ carries a mapping $f : X \to Y$ to $\eta_Y \circ f : X \to \mathsf{T}Y$ in $\mathcal{K}\ell(\mathsf{T})$ (see [10]). For instance, the Kleisli category $\mathcal{K}\ell(\mathcal{P})$ of the powerset monad is up to isomorphism the category **Rel** of sets and binary relations (see [5]).

A functor $\bar{\mathsf{F}} : \mathcal{K}\ell(\mathsf{T}) \to \mathcal{K}\ell(\mathsf{T})$ is said to be a *lifting* of a Set-endofunctor F if the following diagram commutes.

$$\begin{tikzcd} \mathcal{K}\ell(\mathsf{T}) \arrow[r, "\bar{\mathsf{F}}"] & \mathcal{K}\ell(\mathsf{T}) \\ \mathsf{Set} \arrow[u, "\mathsf{J}"] \arrow[r, "\mathsf{F}"'] & \mathsf{Set} \arrow[u, "\mathsf{J}"'] \end{tikzcd}$$

A lifting $\bar{\mathsf{F}}$ of a Set-endofunctor F is in bijective with a *distributive law* $\lambda : \mathsf{F}\mathsf{T} \Rightarrow \mathsf{T}\mathsf{F}$ (see [12]).

3 Coalgebras of an endofunctor

Let F be an endofunctor of a category $\mathcal{C}$. An F-*coalgebra* or a *coalgebra of type* F is a pair (A, α) consisting of an object A in $\mathcal{C}$ together with a $\mathcal{C}$-morphism

$\mathsf{a} : \mathsf{A} \to \mathsf{FA}$. A is called the *carrier* or the *underlying object* and the arrow a the *coalgebra structure* of (A, a).

Given F-coalgebras (A, a) and (B, b), the arrow $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ in $\mathcal{C}$ is called an F-*morphism*, if the following diagram commutes

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\mathsf{a}"] \arrow[d, "\mathsf{f}"'] & \mathsf{FA} \arrow[d, "\mathsf{Ff}"] \\ \mathsf{B} \arrow[r, "\mathsf{b}"'] & \mathsf{FB} \end{tikzcd}$$

It is straightforward to check that the F-morphisms are stable under composition. We write $\mathcal{C}_\mathsf{F}$ the category of F-coalgebras and their homomorphisms.

Throughout all that follows, unless otherwise stated,

- $\mathcal{C}$ is a regularly well powered category equipped with epi-(regular mono) factorizations and admitting products;
- F denotes an endofunctor of $\mathcal{C}$.

3.1 Congruences

Definition 4 *Let* (A, a) *and* (B, b) *be* F-*coalgebras. A binary relation* K *from* A *to* B *is a precongruence if for every cospan* $(\mathsf{A} \xrightarrow{\mathsf{i}} \mathsf{Z} \xleftarrow{\mathsf{j}} \mathsf{B})$,

if $\begin{tikzcd} & \mathsf{A} \arrow[dr, "\mathsf{i}"] & \\ \mathsf{K} \arrow[ur] \arrow[dr] & & \mathsf{Z} \\ & \mathsf{B} \arrow[ur, "\mathsf{j}"'] & \end{tikzcd}$ *commutes then so does* $\begin{tikzcd} & \mathsf{A} \arrow[r, "\mathsf{a}"] & \mathsf{FA} \arrow[dr, "\mathsf{Fi}"] & \\ \mathsf{K} \arrow[ur] \arrow[dr] & & & \mathsf{FZ} \\ & \mathsf{B} \arrow[r, "\mathsf{b}"'] & \mathsf{FB} \arrow[ur, "\mathsf{Fj}"'] & \end{tikzcd}$

A congruence relation is a precongruence which is an equivalence relation.

Consider a Set-endofunctor F that preserves weak pullbacks. There exists a distributive law $\lambda : \mathsf{F}\mathcal{P} \Rightarrow \mathcal{P}\mathsf{F}$ given by

$$\lambda_\mathsf{X}(\mathsf{u}) = \{\nu \in \mathsf{FX} : (\nu, \mathsf{u}) \in \mathsf{Rel}_\mathsf{F}(\epsilon_\mathsf{X})\}$$

where $\mathsf{u} \in \mathsf{F}\mathcal{P}\mathsf{X}$ and $\mathsf{Rel}_\mathsf{F}(\epsilon_\mathsf{X}) \subseteq \mathsf{FX} \times \mathsf{F}\mathcal{P}\mathsf{X}$ is the F-relation lifting of the membership relation ϵ_X (see [5]). The functor $\bar{\mathsf{F}} : \mathbf{Rel} \to \mathbf{Rel}$ induced by this distributive law carries and arrow $\mathsf{R} : \mathsf{X} \to \mathsf{Y}$ in $\mathcal{K}\mathsf{l}(\mathcal{P})$ which is a binary relation from X to Y to its F-relation lifting $\mathsf{Rel}_\mathsf{F}(\mathsf{R})$. That is, $\bar{\mathsf{F}}\mathsf{R} = \mathsf{Rel}_\mathsf{F}(\mathsf{R}) : \mathsf{FX} \to \mathsf{FY}$ in $\mathcal{K}\mathsf{l}(\mathcal{P}) \cong \mathbf{Rel}$.

Given $\bar{\mathsf{F}}$-coalgebras (A, a) and (B, a). Let $K : A \to B$ be an $\bar{\mathsf{F}}$-morphism. The following diagram commutes as $\bar{\mathsf{F}}$ and F coincide on objects.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "K"'] & FA \arrow[d, "\bar{F}K"] \\ B \arrow[r, "b"'] & FB \end{tikzcd}$$

Also, for every cospan $(A \xrightarrow{i} Z \xleftarrow{j} B)$, if $j \circ K = i$ then $\bar{F}(j) \circ \bar{F}K = \bar{F}(i)$; hence $\bar{F}(j) \circ b \circ K = \bar{F}(j) \circ \bar{F}K \circ a = \bar{F}(i) \circ a$. This results the commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[dd, "K"'] & FA \arrow[dr, "\bar{F}i"] & \\ & & FZ \\ B \arrow[r, "b"'] & FB \arrow[ur, "\bar{F}j"'] & \end{tikzcd}$$

Then K is a precongruence. Consequently, any $\bar{\mathsf{F}}$-morphism is a precongruence.

Proposition 2 *Assume the category $\mathcal{C}$ has colimits. Congruence relations on an F-coalgebra (A, a) form a sup-complete lattice denoted* ***Con***(A, a)*. The supremum is given by*

$$\bigvee_{\alpha\in\Lambda} K_\alpha = [\cup\{\mathrm{Im}(m_\alpha : K_\alpha \to A \times A); \alpha \in \Lambda\}]^*$$

the smallest congruence relation greater than the union of all m_α.

Proof. Let $(m_\alpha : K_\alpha \to A \times A)_{\alpha\in\Lambda}$ be a nonempty family of congruences on an F-coalgebra (A, a) with projections k_1^α and k_2^α given $\alpha \in \Lambda$. Since the category $\mathcal{C}$ is regularly well powered, this family of regular subobjects of $A\times A$ is a set. Its supremum K exists therefore in $\mathcal{C}$. This is equivalent to consider a regular mono $m : K \to A\times A$ and an epi sink $(e_\alpha)_\alpha$ such that $(m_\alpha) = m\circ(e_\alpha)$. Furthermore, the category $\mathcal{C}$ has pullbacks under Proposition 1. Given $(A \xrightarrow{u} B \xleftarrow{v} A)$ the pushout of the projections k_1 and k_2 of K. Denote by $\mathsf{Pb}(u, v)$ the pullback of u and v. There is a unique arrow $s : K \to \mathsf{Pb}(u, v)$ such that $w_1 \circ s = k_1$ and $w_2 \circ s = k_2$; w_1 and w_2 being the structural morphisms of $\mathsf{Pb}(u, v)$. Consider a cospan $(A \xrightarrow{i} Z \xleftarrow{j} A)$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dr, "i"] & \\ \mathsf{Pb}(u,v) \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & Z \\ & A \arrow[ur, "j"'] & \end{tikzcd}$$

Then $i \circ k_1 = i \circ w_1 \circ s = j \circ w_2 \circ s = j \circ k_2$. By the universal property of pushouts, there is a unique arrow $w : B \to Z$ such that $w \circ u = i$ and $w \circ v = j$. For each $\alpha \in \Lambda$, $i \circ k_1^\alpha = j \circ k_2^\alpha$; this follows from the fact that $k_1^\alpha = k_1 \circ e_\alpha$ and $k_2^\alpha = k_2 \circ e_\alpha$. Hence $F(i) \circ a \circ k_1 \circ e_\alpha = F(j) \circ a \circ k_2 \circ e_\alpha$, because K_α is a precongruence. The equality $F(i) \circ a \circ k_1 = F(j) \circ a \circ k_2$ due to the collection $(e_\alpha)_\alpha$ is an epi sink. Particularly, the equality $F(u) \circ a \circ k_1 = F(v) \circ a \circ k_2$ holds. There is therefore a unique arrow $b : B \to FB$ turning u and v into F-morphisms. So, we have $F(i) \circ a \circ w_1 = F(w \circ u) \circ a \circ w_1 = F(w) \circ F(u) \circ a \circ w_1 = F(w) \circ b \circ u \circ w_1 = F(w) \circ b \circ v \circ w_2 = F(w) \circ F(v) \circ a \circ w_2 = F(w \circ v) \circ a \circ w_2 = F(j) \circ a \circ w_2$. This proves that the following diagram commutes.

$$\begin{tikzcd}
 & A \arrow[r, "a"] & FA \arrow[dr, "Fi"] & \\
Pb(u,v) \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & & FZ \\
 & A \arrow[r, "a"'] & FA \arrow[ur, "Fj"'] &
\end{tikzcd}$$

Thus $Pb(u,v)$ is a precongruence. Besides $u = v$ given that K is reflexive. That is, u is the coequalizer of the two projections k_1 and k_2. Consequently, $Pb(u,v)$ is an equivalence relation as the kernel pair of a regular mono. Hence $Pb(u,v)$ is a congruence relation on (A,a). It is easy to check that this is in fact the supremum of the family $(m_\alpha : K_\alpha \to A \times A)_{\alpha\in\Lambda}$.

The supremum of a family of congruences on an F-coalgebra (A,a) indexed over the empty set is $\Delta_A = \ker(1_A)$. It is the smallest congruence on (A,a). □

Write ∇_A to denote the largest congruence relation on (A,a).

Proposition 3 *Suppose that the category $\mathcal{C}$ is exact with colimits. For every* F*-coalgebra* (B,b)*, there is at most one* F*-morphism* $\varphi : (B,b) \to A_{\nabla_A}$.

Proof. By Proposition 1, the category $\mathcal{C}$ has pullbacks. Let us prove that there is at most one F-morphism with codomain A_{∇_A}. Assume there are two different F-morphisms $\varphi_1, \varphi_2 : (B,b) \to A_{\nabla_A}$. Let $\psi : A_{\nabla_A} \to C$ be their coequalizer. Then $\ker(\psi \circ \pi_{\nabla_A})$ is an equivalence relation on A, where $\pi_{\nabla_A} : A \to A_{\nabla_A}$ is the coequalizer of ∇_A. In addition, $\ker(\psi \circ \pi_{\nabla_A})$ is a precongruence. Indeed, consider a cospan $(A \xrightarrow{i} Z \xleftarrow{j} A)$ such that the following diagram commutes; t_1 and t_2 being the projections of $\ker(\psi \circ \pi_{\nabla_A})$.

$$\begin{tikzcd}
 & A \arrow[dr, "i"] & \\
\ker(\psi \circ \pi_{\nabla_A}) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & Z \\
 & A \arrow[ur, "j"'] &
\end{tikzcd}$$

Then $i = j$ given that $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ is a reflexive relation on A. Also, $\psi \circ \pi_{\nabla_A}$ is the coequalizer of $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ as the category $\mathcal{C}$ is regular. By the universal property of coequalizers, there is a unique arrow $u : C \to Z$ such that $u \circ (\psi \circ \pi_{\nabla_A}) = i$. Furthermore, π_{∇_A} is an F-morphism; this follows from the fact that ∇_A is a congruence relation. Hence $\psi \circ \pi_{\nabla_A}$ is an F-morphism. This implies that $F(i) \circ a \circ t_1 = F(j) \circ a \circ t_2$; that is, the following diagram commutes.

$$\begin{tikzcd}
 & A \arrow[r, "a"] & FA \arrow[dr, "Fi"] & \\
\mathsf{ker}(\psi \circ \pi_{\nabla_A}) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & & FZ \\
 & A \arrow[r, "a"'] & FA \arrow[ur, "Fj"'] &
\end{tikzcd}$$

So, $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ is a congruence relation on (A, a). Under condition that the category $\mathcal{C}$ has exact sequences, ∇_A is the kernel pair of π_{∇_A}. Consequently, ∇_A is properly smaller than $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ because φ_1 and φ_2 are different. This contradicts the fact that ∇_A is the largest congruence relation on (A, a). $\square$

Any maximal element of the lattice of congruence relations on (A, a) is called a *coatomic* congruence relation on (A, a).

3.2 Bisimulations

In the coalgebraic context, there are four notions of bisimulation that generalize the standard notion of bisimulation for labelled transition systems (i.e., coalgebras of the Set-endofunctor $\mathcal{P}(L \times (-))$), due to Milner [11] and Park [13]. Further, the four notions are related under certain conditions (see [16]). The definition we adopt here is a simplification of the bisimulation of Hermida and Jacobs [6].

Definition 5 *For any relation* R *from* A *to* B *in* $\mathcal{C}$*, we define the relation* $\bar{F}R$ *from* FA *to* FB *to be the regular image of the composite morphism* $FR \to F(A \times B) \to FA \times FB$.

A bisimulation between F*-coalgebras* (A, a) *and* (B, b) *is a binary relation* R *from* A *to* B *such that there is a morphism* $R \to \bar{F}R$ *making the following diagram commute.*

$$\begin{tikzcd}
A \arrow[d, "a"'] & R \arrow[l] \arrow[r] \arrow[d, dashed] & B \arrow[d, "b"] \\
FA & \bar{F}R \arrow[l] \arrow[r] & FB
\end{tikzcd}$$

A bisimulation on (A, a) *is a bisimulation between* (A, a) *and* (A, a)*. Any bisimulation on* (A, a) *which is an equivalence relation is called a bisimulation equivalence.*

Proposition 4 *Suppose that the category* $\mathcal{C}$ *has coproducts and the endofunctor* F *preserves regular monos. Then the union of any collection of bisimulations is a bisimulation.*

Proof. Given F-coalgebras (A, a) and (B, b). The class $(R_t)_{t\in T}$ of bisimulations between (A, a) and (B, b) is nonempty, since the category $\mathcal{C}$ has coproducts. Also, this class is a set because $\mathcal{C}$ is regularly well powered. Denote by $(\coprod_{t\in T} R_t, (\sigma_t)_{t\in T})$ the coproduct of R_t's. Each R_t is a regular subobject of $A \times B$ represented by a regular mono $m_t : R_t \to A \times B$. Let $u : \coprod_{t\in T} R_t \to A \times B$ be the unique arrow such that $u \circ \sigma_t = m_t$, for all $t \in T$. Denote by $\bar{F} \coprod_{t\in T} R_t$ the regular image of the composite morphism $F \coprod_{t\in T} R_t \to F(A \times B) \to FA \times FB$. The following diagram commutes.

$$\begin{tikzcd}
FR_t \arrow[r, "\bar{e}_t"] \arrow[d, "F(\sigma_t)"'] & \bar{F}R_t \arrow[ddl, dashed, "d"] \arrow[dd, "\bar{m}_t"] \\
F\coprod_{t\in T} R_t \arrow[d, "\bar{v}"'] & \\
\bar{F}\coprod_{t\in T} R_t \arrow[r, "\bar{u}"'] & FA \times FB
\end{tikzcd}$$

Under condition that the category $\mathcal{C}$ has epi-(regular mono) factorizations, there is a unique arrow $d : \bar{F}R_t \to \bar{F} \coprod_{t\in T} R_t$ making both triangles commute. By the universal property of coproducts, there is a unique arrow $\rho : \coprod_{t\in T} R_t \to \bar{F} \coprod_{t\in T} R_t$ such that $\rho \circ \sigma_t = d \circ r_t$, for all $t \in T$.

Factorize u as an epimorphism e followed by a regular mono $m : R \rightarrowtail A \times B$. Consider $m_1 : R_1 \rightarrowtail A$ and $m_2 : R_2 \rightarrowtail B$ the respective regular images of the morphisms $p_1 \circ u$ and $p_2 \circ u$; p_1 and p_2 being structural morphisms of the product of A and B. Then $(p_i \circ m) \circ e = m_i \circ e_i$; $i = 1, 2$. Hence, there is a unique arrow $w_i : R \to R_i$ such that $m_i \circ w_i = p_i \circ m$ and $w_i \circ e = e_i$. The morphisms w_1 and w_2 induce a unique arrow $\langle w_1, w_2\rangle : R \to R_1 \times R_2$ such that $v_1 \circ \langle w_1, w_2\rangle = w_1$ and $v_2 \circ \langle w_1, w_2\rangle = w_2$; v_1 and v_2 being the structural morphisms of the product of R_1 and R_2. Let $s : R_1 \times R_2 \to FA \times FB$ be the unique arrow such that $h_1 \circ s = a \circ m_1 \circ v_1$ and $h_2 \circ s = b \circ m_2 \circ v_2$; h_1 and h_2 being the structural morphisms of the product of FA and FB. In addition, consider the unique arrow $k : F(A \times B) \to FA \times FB$ such that $h_1 \circ k = Fp_1$

and $h_2 \circ k = Fp_2$. So, $\bar{u} \circ \bar{v} = k \circ F(u) = k \circ F(m) \circ F(e) = \bar{m} \circ \bar{e} \circ F(e)$. Because $\bar{v}$ is an epimorphism and $\bar{m}$ is a regular mono, there is a unique arrow $z : \bar{F}\coprod_{t\in T} R_t \to \bar{F}R$ such that $\bar{m} \circ z = \bar{u}$ and $z \circ \bar{v} = \bar{e} \circ F(e)$. Likewise, $F(m_i) \circ F(e_i) = F(m_i \circ e_i) = F(p_i \circ u) = F(p_i) \circ F(u) = h_i \circ k \circ F(u) = h_i \circ \bar{u} \circ \bar{v}$; $i = 1, 2$. Since the endofunctor F preserves regular monos, there is a unique arrow $c_i : \bar{F}\coprod_{t\in T} R_t \to FR_i$ such that $F(m_i) \circ c_i = h_i \circ \bar{u}$ and $c_i \circ \bar{v} = F(e_i)$; $i = 1, 2$. It follows that for all $t \in T$, $F(m_1) \circ c_1 \circ \rho \circ \sigma_t = h_1 \circ \bar{u} \circ \rho \circ \sigma_t = h_1 \circ \bar{u} \circ d \circ r_t = h_1 \circ \bar{m}_t \circ r_t = a \circ p_1 \circ m_t = a \circ p_1 \circ u \circ \sigma_t$ and $F(m_2) \circ c_2 \circ \rho \circ \sigma_t = h_2 \circ \bar{u} \circ \rho \circ \sigma_t = h_2 \circ \bar{u} \circ d \circ r_t = h_2 \circ \bar{m}_t \circ r_t = b \circ p_2 \circ m_t = b \circ p_2 \circ u \circ \sigma_t$. Whence $F(m_1) \circ c_1 \circ \rho = a \circ m_1 \circ e_1$ and $F(m_2) \circ c_2 \circ \rho = b \circ m_2 \circ e_2$ as the cospan $(\sigma_t)_{t\in T}$ is an epi sink.

These equalities are used to establish the following commutative diagram.

$$\begin{tikzcd}
\coprod_{t\in T} R_t \arrow[r, "e"] \arrow[d, "\rho"'] & R \arrow[d, "\langle w_1, w_2\rangle"] \arrow[ddl, dashed, "r"] \\
\bar{F}\coprod_{t\in T} R_t \arrow[d, "z"'] & R_1 \times R_2 \arrow[d, "s"] \\
\bar{F}R \arrow[r, "\bar{m}"'] & FA \times FB
\end{tikzcd}$$

Indeed, we have $h_1 \circ s \circ \langle w_1, w_2\rangle \circ e = a \circ m_1 \circ v_1 \circ \langle w_1, w_2\rangle \circ e = a \circ m_1 \circ w_1 \circ e = a \circ m_1 \circ e_1 = F(m_1) \circ c_1 \circ \rho = h_1 \circ \bar{u} \circ \rho = h_1 \circ \bar{m} \circ z \circ \rho$ and $h_2 \circ s \circ \langle w_1, w_2\rangle \circ e = b \circ m_2 \circ v_2 \circ \langle w_1, w_2\rangle \circ e = b \circ m_2 \circ w_2 \circ e = b \circ m_2 \circ e_2 = F(m_2) \circ c_2 \circ \rho = h_2 \circ \bar{u} \circ \rho = h_2 \circ \bar{m} \circ z \circ \rho$. Since the pair (h_1, h_2) is a mono source, the equality $s \circ \langle w_1, w_2\rangle \circ e = \bar{m} \circ z \circ \rho$ holds. Consequently, there is a unique arrow $r : R \to \bar{F}R$ making both triangles commute. Furthermore, we have that $(h_1 \circ \bar{m}) \circ r = h_1 \circ s \circ \langle w_1, w_2\rangle = a \circ m_1 \circ v_1 \circ \langle w_1, w_2\rangle = a \circ m_1 \circ w_1 = a \circ (p_1 \circ m)$ and $(h_2 \circ \bar{m}) \circ r = h_2 \circ s \circ \langle w_1, w_2\rangle = b \circ m_2 \circ v_2 \circ \langle w_1, w_2\rangle = b \circ m_2 \circ w_2 = b \circ (p_2 \circ m)$. Subsequently, R is a bisimulation as union of a collection of bisimulations. □

Any bisimulation equivalence is a congruence relation (see [16]). But the converse is not true (see [1]). Now, we are going to investigate the relationship between bisimulations and congruences.

The following fact is a generalization of the H. P. Gumm's result presented in [4].

Proposition 5 *Assume the category $\mathcal{C}$ has colimits and exact sequences. For every bisimulation R on an F-coalgebra (A, a) there is a smallest congruence relation $\langle R\rangle$ greater than R provided that R is reflexive.*

Proof. According to Proposition 1, the category $\mathcal{C}$ has pullbacks. Consider a bisimulation $(R, (r_i)_{i=1,2})$ on an F-coalgebra (A, a). Let $(A \xrightarrow{u} B \xleftarrow{v} A)$ be the pushout of r_1 and r_2. Denote by $Pb(u, v)$ the pullback of u and v. Then $Pb(u, v)$ is a precongruence. Indeed, given a cospan $(A \xrightarrow{i} Z \xleftarrow{j} A)$ such that the following diagram commutes; t_1 and t_2 being the projections of $Pb(u, v)$.

$$\begin{tikzcd}
 & A \arrow[dr, "i"] & \\
Pb(u,v) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & Z \\
 & A \arrow[ur, "j"'] &
\end{tikzcd}$$

By the universal property of pullbacks, there is a unique arrow $s : R \to Pb(u, v)$ such that $t_1 \circ s = r_1$ and $t_2 \circ s = r_2$. This implies that $i \circ r_1 = j \circ r_2$. Hence there is a unique arrow $w : B \to Z$ such that $w \circ u = i$ and $w \circ v = j$. In addition, $F(u) \circ a \circ r_1 = F(v) \circ a \circ r_2$ due to R is a precongruence as bisimulation (see [16]). Thus B is equipped with a coalgebra structure turning u and v into F-morphisms. For this reason, the equality $F(i) \circ a \circ t_1 = F(j) \circ a \circ t_2$ holds; that is, the following diagram commutes.

$$\begin{tikzcd}
 & A \arrow[r, "a"] & FA \arrow[dr, "Fi"] & \\
Pb(u,v) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & & FZ \\
 & A \arrow[r, "a"'] & FA \arrow[ur, "Fj"'] &
\end{tikzcd}$$

Also, $Pb(u, v)$ is an equivalence relation on A because R is a reflexive bisimulation. Finally, $Pb(u, v)$ is a congruence relation on (A, a) which is greater than R. Since the category $\mathcal{C}$ satisfies the exactness property, it is not hard to see that $Pb(u, v)$ is the smallest congruence relation with this property. □

Denote by **R-Bis**(A, a) the ordered set of reflexive bisimulations on (A, a). The Proposition 5 yields a functorial correspondence

$$\begin{array}{rccc} \Diamond_{(A,a)} : & \textbf{R-Bis}(A, a) & \longrightarrow & \textbf{Con}(A, a) \\ & R & \longmapsto & \langle R \rangle \end{array}$$

Otherwise, every congruence relation K on (A, a) is a reflexive relation on A as equivalence relation. Then the diagonal map $\langle 1_A, 1_A \rangle : A \to A \times A$ factors through K. But the diagonal map is a split mono and therefore a regular mono. Also, A is equipped with a bisimulation structure that comes from its coalgebra structure by epi-(regular mono) factorization of the composite morphism $FA \to F(A \times A) \to FA \times FA$. Hence A is a bisimulation on (A, a) smaller than K. If more, the endofunctor F preserves regular monos, there exists under Proposition 4, a largest bisimulation on (A, a) smaller than K, that we denote $\Box_{(A,a)}K$. Since A is a bisimulation on (A, a) smaller than K, the diagonal map factors through $\Box_{(A,a)}K$. So $\Box_{(A,a)}K$ is a reflexive bisimulation on (A, a). This defines a correspondence

$$\begin{array}{rccc} \Box_{(A,a)}: & \mathbf{Con}(A, a) & \longrightarrow & \mathbf{R\text{-}Bis}(A, a) \\ & K & \longmapsto & \Box_{(A,a)}K \end{array}$$

which extends to a functor.
Given a reflexive bisimulation R and a congruence relation K on (A, a), the following are equivalent:

(i) $\langle R \rangle$ is a regular subobject of K.

(ii) R is a regular subobject of $\Box_{(A,a)}K$.

Hence, assuming that F preserves regular monos, the functor $\Diamond_{(A,a)}$ is the left adjoint of the functor $\Box_{(A,a)}$.

Definition 6 *An endofunctor $F : \mathcal{C} \to \mathcal{C}$ is called a covarietor, provided that the forgetful functor $U_F : \mathcal{C}_F \to \mathcal{C}$ has a right adjoint.*

Given a topos $\mathcal{E}$ with a *natural number object* (see [7]). The endofunctor $M : \mathcal{E} \to \mathcal{E}$ that assigns to each object A in $\mathcal{E}$, the *free monoid* generated by A is a covarietor (see [8]).

The largest bisimulation on (A, a) which is denoted $\sim_A$ is a reflexive bisimulation.

Proposition 6 *Assume the category $\mathcal{C}$ has colimits with exact sequences and the endofunctor F is a covarietor which preserves regular monos. A nontrivial congruence relation K on (A, a) is coatomic or $K = \nabla_A$, provided that $\Box_{(A,a)}K = \sim_A$.*

Proof. Let K be a nontrivial congruence relation on (A, a), different from ∇_A and satisfying the condition $\Box_{(A,a)}K = \sim_A$. Suppose that there is a congruence relation L on (A, a), greater than K and different from ∇_A. By the universal property of coequalizers, there is a unique factorization $r : A_K \to A_L$ such that $\pi_L = r \circ \pi_K$, where π_K and π_L are respectively the coequalizers of K and L. Under

Proposition 1, the category $\mathcal{C}$ has pullbacks. Then the category $\mathcal{C}_F$ has also pullbacks, since the endofunctor F is a covarietor which preserves regular monos (see [9]). Since the category $\mathcal{C}$ has exact sequences, every equivalence relation in $\mathcal{C}$ is the kernel pair of its coequalizer. Furthermore, the coequalizer of any congruence relation is an F-morphism. Thereafter, the canonical arrow from the kernel pair of π_K in $\mathcal{C}_F$ to $A \times A$ factored through the largest bisimulation on (A, a) smaller than K. Likewise, the canonical arrow from the kernel pair of π_L in $\mathcal{C}_F$ to $A \times A$ factored through the largest bisimulation on (A, a) smaller than L. Besides, $\Box_{(A,a)} L$ is a regular subobject of $\Box_{(A,a)} K$, given that $\Box_{(A,a)} K = \sim_A$. As a consequence, there is a unique arrow $s : A_L \to A_K$ such that $\pi_K = s \circ \pi_L$. Then we get $\pi_K = (s \circ r) \circ \pi_K$; whence $s \circ r = 1_{A_K}$ since π_K is an epi. Thus r is an epi from the fact of the equality $\pi_L = r \circ \pi_K$, and a section; that is an iso. Hence K is coatomic.

On the other hand, $\sim_A$ is the largest bisimulation on (A, a) smaller than ∇_A. $\Box$

In general though $\langle \sim_A \rangle$ does not need to be the largest congruence on (A, a). For illustration, denote by $()^3_2 : \mathsf{Set} \to \mathsf{Set}$ the functor defined on objects as follows: for a set,

$$A^3_2 = \{(a_1, a_2, a_3) \in A^3 / \mid \{a_1, a_2, a_3\} \mid \leq 2\}$$

and for each mapping $f : A \longrightarrow B$,

$$f^3_2(a_1, a_2, a_3) = (f(a_1), f(a_2), f(a_3))$$

Consider the $()^3_2$-coalgebra (A, a) with $A = \{0, 1, 2\}$, $a(0) = (0, 0, 2)$, $a(1) = (1, 1, 2)$ and $a(2) = (1, 2, 2)$. Since the singleton $\{0\}$ can be provided with a $()^3_2$-coalgebra structure, the unique mapping $!_A : A \to \{0\}$ is a $()^3_2$-morphism. Its kernel pair is $A \times A$ and it is not a bisimulation on (A, a). This implies that $A \times A$ is the largest congruence on (A, a). However the largest bisimulation on (A, a) is the diagonal Δ_A. It is easy to check that $\langle \sim_A \rangle = \Delta_A$. Remark that $K = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$ is a coatomic congruence relation on (A, a).

4 Simple and extensional coalgebras

The largest bisimulation on a final coalgebra is its diagonal. Coalgebras which are not final but satisfy this condition are called extensional. They are said to satisfy the weaker condition of simplicity.

Definition 7 *An* F*-coalgebra* (S, s) *is extensional, if* Δ_S *is the largest bisimulation on* (S, s)*.*

The definition of extensionality reformulates the coinduction proof principle:

$$\frac{x \sim x'}{x = x'}$$

This means, in order to prove that two elements x and y are equal it is enough to prove that there exists a bisimulation R under which x is related to y; i.e., $(x, y) \in R$.

A coalgebra is called *simple* if it does not have any nontrivial congruence relation. Obviously every simple coalgebra is extensional, but the converse holds whenever the endofunctor preserves weak pullbacks.

Proposition 7 *For any* F*-coalgebra* (S, s) *the following are equivalent:*

(i) (S, s) *is extensional.*

(ii) *For every* F*-coalgebra* (A, a)*, there is at most one* F*-morphism* $\psi : (A, a) \to (S, s)$*.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that there are two different F-morphisms $\varphi_1, \varphi_2 : (A, a) \to (S, s)$. There is a unique arrow $\varphi : A \to S \times S$ such that $p_1 \circ \varphi = \varphi_1$ and $p_2 \circ \varphi = \varphi_2$, with p_1 and p_2 the structural morphisms of the product of S with itself. The arrow φ factorizes through a regular subobject R of $S \times S$ which is a nontrivial bisimulation on (S, s).

(ii) $\Longrightarrow$ (i). Suppose that Δ_S is not the largest bisimulation on (S, s). There is a bisimulation $(R, (r_i)_{i=1,2})$ on (S, s) with $r_1 \neq r_2$. $\square$

Recall the set $A = \{0, 1, 2\}$ together with the coalgebra structure $a : A \to A^3_2$ such that $a(0) = (0, 0, 2)$, $a(1) = (1, 1, 2)$ and $a(2) = (1, 2, 2)$, where $K = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$ is a coatomic congruence relation on (A, a). In particular, $K \neq \nabla_A = A \times A$, but the $()^3_2$-coalgebra on the quotient set $A_K = \{\bar{0}, \bar{2}\}$ is extensional; this is because the largest bisimulation on A_K is the diagonal Δ_{A_K} (see [4]).

Definition 8 *A coatomic congruence relation* K *on* (S, s) *is called factor split if the canonical homomorphism* $\nu : S_K \to S_{\nabla_S}$ *splits.*

$H = \{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1)\}$ is an equivalence relation on the set $A = \{0, 1, 2\}$. Let π_H denote the canonical projection of A onto $A_H = \{\bar{0}, \bar{1}\}$, the quotient set with respect to H. Provide A with the $()^3_2$-coalgebra structure α such that $\alpha(0) = (0, 0, 2)$, $\alpha(1) = (1, 1, 2)$ and $\alpha(2) = (1, 2, 2)$. Then π_H is a $()^3_2$-morphism given that A_H is equipped with the coalgebra structure $\alpha_H : A_H \to (A_H)^3_2$ defined as $\alpha_H(\bar{0}) = (\bar{0}, \bar{0}, \bar{1})$ and $\alpha_H(\bar{1}) = (\bar{1}, \bar{1}, \bar{1})$. Consequently, H is a congruence relation on (A, α). Also, H is coatomic as a maximal element of the lattice of congruence relations on (A, α). Since $\alpha_H(\bar{1}) = (\bar{1}, \bar{1}, \bar{1})$, the canonical homomorphism $\nu : A_H \to A_{\nabla_A} = \{0\}$ has a right-sided inverse. Hence, H is factor split.

For any coatomic and factor split congruence relation K on (S, s), denote by $\tau : S_{\nabla_S} \to S_K$ the right-sided inverse of the canonical homomorphism $\nu : S_K \to S_{\nabla_S}$. Then $\tau \circ \nu : S_K \to S_K$ and $1_{S_K} : S_K \to S_K$ are two different F-morphisms with codomain S_K. As a result, S_K is not extensional due to Proposition 7.

Lemma 1 *Suppose that the category $\mathcal{C}$ is exact with colimits. For any coatomic congruence relation K on an F-coalgebra (S, s), the quotient coalgebra S_K is extensional provided that K is not factor split.*

Proof. Given K a coatomic and not factor split congruence relation on (S, s). Suppose that the quotient coalgebra S_K is not extensional. Then the largest bisimulation on S_K is nontrivial. By Proposition 3, the canonical homomorphism ν from S_K to S_{∇_S} coequalizes its projections. Let $\varphi : S_K \to C$ denote the coequalizer of the projections of $\sim_{S_K}$, the largest bisimulation on (S, s). There is a unique arrow $t : C \to S_{\nabla_S}$ such that $t \circ \varphi = \nu$. Hence, $\ker(\varphi)$ is properly smaller than $\ker(\nu)$. Also, K is properly smaller than $\ker(\varphi \circ \pi_K)$ and $\ker(\pi_{\nabla_{S_K}} \circ \pi_K)$; π_K and $\pi_{\nabla_{S_K}}$ being respectively the coequalizer of the projections of K and the coequalizer of the projections of ∇_{S_K}, the largest congruence relation on S_K. But, $\ker(\varphi \circ \pi_K)$ and $\ker(\pi_{\nabla_{S_K}} \circ \pi_K)$ are congruence relations on (S, s). Consequently, $\ker(\varphi \circ \pi_K) = \nabla_A = \ker(\pi_{\nabla_{S_K}} \circ \pi_K)$ as K is coatomic. Besides, $\varphi \circ \pi_K$ and $\pi_{\nabla_{S_K}} \circ \pi_K$ are regular epis given that the category $\mathcal{C}$ is regular. This implies that $\varphi \circ \pi_K = \pi_{\nabla_{S_K}} \circ \pi_K$; that is, $\varphi = \pi_{\nabla_{S_K}}$ due to π_K is an epi. Then $\ker(\varphi) = \nabla_{S_K}$ because the category $\mathcal{C}$ has exact sequences. It follows that ∇_{S_K} is properly smaller than $\ker(\nu)$ which is a congruence relation on S_K. This is a contradiction. So, S_K is extensional. □

A quotient coalgebra can be made extensional by taking a regular quotient with respect to a coatomic and not factor split congruence relation or its largest congruence relation as the following states.

Proposition 8 *Assume the category $\mathcal{C}$ is exact with colimits and the endofunctor F is a covarietor which preserves regular monos. For every F-coalgebra*

(S, s) *and a congruence relation* K *on* (S, s)*, the quotient coalgebra* S_K *is extensional if and only if* K *is coatomic and not factor split or* $K = \nabla_S$*.*

Proof. Suppose that S_K is extensional. Let $(Q, (q_i)_{i=1,2})$ be a bisimulation on (S, s). There is a unique arrow $m : Q \to S \times S$ such that $p_1 \circ m = q_1$ and $p_2 \circ m = q_2$; p_1 and p_2 being the structural morphisms of the product of S with itself. Let π_K denote the coequalizer of K. The universal property of products yields a unique arrow $m_K : Q \to S_K \times S_K$ such that $\overline{p}_1 \circ m_K = \pi_K \circ p_1 \circ m$ and $\overline{p}_2 \circ m_K = \pi_K \circ p_2 \circ m$, with $\overline{p}_1$ and $\overline{p}_2$ the structural morphisms of the product of S_K with itself. The arrow m_K admits the epi-(regular mono) factorization

$$\begin{tikzcd} Q \arrow[rr, "m_K"] \arrow[dr, "u"'] & & S_K \times S_K \\ & R \arrow[ur, "\nu_K"'] & \end{tikzcd}$$

Let $r_K : S \times S \to S_K \times S_K$ be the unique arrow such that $\overline{p}_1 \circ r_K = \pi_K \circ p_1$ and $\overline{p}_2 \circ r_K = \pi_K \circ p_2$. Then $F(\nu_K) \circ F(u) = F(r_K) \circ F(m)$ due to $\nu_K \circ u = r_K \circ m$. Hence $h \circ F(\nu_K) \circ F(u) = h \circ F(r_K) \circ F(m)$, where $h : F(S_K \times S_K) \to FS_K \times FS_K$ is the unique arrow such that $\bar{t}_1 \circ h = F(\overline{p}_1)$ and $\bar{t}_2 \circ h = F(\overline{p}_2)$; $\bar{t}_1$ and $\bar{t}_2$ being the structural morphisms of the product of FS_K with itself. Furthermore, there is a unique arrow $\pi : FS \times FS \to FS_K \times FS_K$ such that $\bar{t}_1 \circ \pi = F(\pi_K) \circ t_1$ and $\bar{t}_2 \circ \pi = F(\pi_K) \circ t_2$, with t_1 and t_2 the structural morphisms of the product of FS with itself. Given $k : F(S \times S) \to FS \times FS$ the unique arrow such that $t_1 \circ k = F(p_1)$ and $t_2 \circ k = F(p_2)$, we have that $\bar{t}_i \circ \pi \circ k = F(\pi_K) \circ t_i \circ k = F(\pi_K) \circ F(p_i) = F(\overline{p}_i) \circ F(r_K) = \bar{t}_i \circ h \circ F(r_K)$; $i = 1, 2$. The equality $\pi \circ k = h \circ F(r_K)$ arises from the fact that the pair $(\bar{t}_1, \bar{t}_2)$ is a mono source. One deduces the following commutative diagram.

$$\begin{tikzcd} FQ \arrow[r, "\bar{e}"] \arrow[d, "F(u)"'] & \bar{F}Q \arrow[d, "\bar{m}"] \arrow[ddl, dashed, "w"] \\ FR \arrow[d, "\bar{u}"'] & FS \times FS \arrow[d, "\pi"] \\ \bar{F}R \arrow[r, "\bar{\nu}_K"'] & FS_K \times FS_K \end{tikzcd}$$

By the axiom FS3, there is a unique arrow $w : \bar{F}Q \to \bar{F}R$ making both triangles commute. In addition, there is a unique arrow $z : S_K \times S_K \to FS_K \times FS_K$ such that $\bar{t}_1 \circ z = s_K \circ \overline{p}_1$ and $\bar{t}_2 \circ z = s_K \circ \overline{p}_2$, where $s_K : S_K \to FS_K$ is the unique arrow turning π_K into an F-morphism. Denote by $q : Q \to \bar{F}Q$ the arrow such that $s \circ p_1 \circ m = t_1 \circ \bar{m} \circ q$ and $s \circ p_2 \circ m = t_2 \circ \bar{m} \circ q$. For $i = 1, 2$; the following holds:

$$\begin{aligned}
\bar{t}_i \circ z \circ \nu_K \circ u &= s_K \circ \overline{p}_i \circ \nu_K \circ u \\
&= s_K \circ \overline{p}_i \circ r_K \circ m \\
&= s_K \circ \pi_K \circ p_i \circ m \\
&= F(\pi_K) \circ s \circ p_i \circ m \\
&= F(\pi_K) \circ t_i \circ \bar{m} \circ q \\
&= \bar{t}_i \circ \pi \circ \bar{m} \circ q \\
&= \bar{t}_i \circ \bar{\nu}_K \circ w \circ q
\end{aligned}$$

Hence, $z \circ \nu_K \circ u = \bar{\nu}_K \circ w \circ q$ because the pair $(\bar{t}_1, \bar{t}_2)$ is a mono source. Since u is an epimorphism and $\bar{\nu}_K$ a regular mono, there is a unique arrow $r : R \longrightarrow \bar{F}R$ such that $\bar{\nu}_K \circ r = z \circ \nu_K$ and $r \circ u = w \circ q$. In fact, R is a bisimulation on S_K. Thus R is a regular subobject of Δ_{S_K} which is the largest bisimulation on S_K. This implies that $\pi_K \circ p_1 \circ m = \pi_K \circ p_2 \circ m$. Consequently Q is a regular subobject of K. Since Q is a bisimulation on (S, s), it is smaller than $\Box_{(S,s)}K$. Whence $\Box_{(S,s)}K$ is the largest bisimulation on (S, s). The Proposition 6 allows to conclude.

Conversely S_K is extensional arising from Propositions 3, 7 and Lemma 1.

□

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Received: August 1, 2015

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 324–335

DOI: 10.1515/ausm-2017-0024

Sesquilinear version of numerical range and numerical radius

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Abstract. In this paper by using the notion of sesquilinear form we introduce a new class of numerical range and numerical radius in normed space $\mathscr{V}$, also its various characterizations are given. We apply our results to get some inequalities.

1 Introduction and preliminaries

A related concept to our work is the notion of sesquilinear form. Sesquilinear forms and quadratic forms were studied extensively by various authors, who have developed a rich array of tools to study them; cf. [17, 19]. There is a considerable amount of literature devoting to the study of sesquilinear form. We refer to [1, 9, 22] for a recent survey and references therein.

2010 Mathematics Subject Classification: 15A63, 15A03, 47A12, 47A30, 47A63, 47C10
Key words and phrases: sesquilinear form, numerical range, numerical radius, norm inequality

During the past decades, several definitions of the numerical range in various settings have been introduced by many mathematicians. For instance, Marcus and Wang [15] opened the concept of the rth permanent numerical range of operator A. Furthermore, Descloux in [3] defined the notion of the essential numerical range of an operator with respect to a coercive sesquilinear form. In 1977, Marvin [16] and in 1984, independently, Tsing [23] introduce and characterize a new version of numerical range in a space $\mathbb{C}^n$ equipped with a sesquilinear form. Li in [14], generalized the work of Tsing and explored fundamental properties and consequences of numerical range in the framework sesquilinear form. We also refer to another interesting paper by Fox [10] of this type.

The motivation of this paper is to introduce the notions of numerical range and numerical radius without the inner product structure. In fact, the result extends immediately to the case where the Hilbert space $\mathscr{H}$ and inner product $\langle\cdot,\cdot\rangle$, replaced by vector space $\mathscr{V}$ and sesquilinear form φ, respectively. For the sake of completeness, we reproduce the following definitions and preliminary results, which will be needed in the sequel.

A functional $\varphi : \mathscr{V}\times\mathscr{V} \to \mathbb{C}$ where $\mathscr{V}$ is complex vector space, is a sesquilinear form if satisfying the following two conditions:

(a) $\varphi(\alpha x_1 + \beta x_2, y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2, y)$,

(b) $\varphi(x, \alpha y_1 + \beta y_2) = \overline{\alpha}\varphi(x, y_1) + \overline{\beta}\varphi(x, y_2)$,

for any scalars α and β and any $x, x_1, x_2, y, y_1, y_2 \in \mathscr{V}$.

We now recall that, two typical examples of sesquilinear forms are as follows:

(I) Let A and B be operators on an inner product space $\mathscr{V}$. Then $\varphi_1(x, y) = \langle Ax, y\rangle$, $\varphi_2(x, y) = \langle x, By\rangle$, and $\varphi_3(x, y) = \langle Ax, By\rangle$ are sesquilinear forms on $\mathscr{V}$.

(II) Let f and g be two linear functionals on a vector space $\mathscr{V}$. Then $\varphi(x, y) = f(x)\overline{g(y)}$ is a sesquilinear form on $\mathscr{V}$.

A sesquilinear form φ on vector space $\mathscr{V}$ is called *symmetric* if $\varphi(x, y) = \overline{\varphi(y, x)}$, for all $x, y \in \mathscr{V}$. We say a sesquilinear form φ on vector space $\mathscr{V}$ is *positive* if $\varphi(x, x) \geq 0$, for all $x \in \mathscr{V}$. If $\mathscr{V}$ is a normed space, then φ is called *bounded* if $|\varphi(x, y)| \leq M\|x\|\|y\|$, for some $M > 0$ and all $x, y \in \mathscr{V}$.

It is worth to mention here that for a bounded sesquilinear form φ on $\mathscr{V}$ we have

$$|\varphi(x, y)| \leq \|\varphi\|\|x\|\|y\|,$$

for all $x, y \in \mathscr{V}$.

For each positive sesquilinear form φ on vector space $\mathscr{V}$, $\sqrt{\varphi(x,x)}$ is a semi norm; since satisfied the axioms of a norm exept that the implication $\sqrt{\varphi(x,x)} = 0 \Rightarrow x = 0$ may not hold; see [18, p. 52]. We notice that the norm of $\mathscr{V}$, will be denoted by $\|\cdot\|_{\varphi}$.

The operator A on the space $(\mathscr{V}, \|\cdot\|_{\varphi})$ is called *bounded* (in short $A \in \mathcal{B}(\mathscr{V})$) if

$$\|Ax\|_{\varphi} \leq M\|x\|_{\varphi},$$

for every $x \in \mathscr{V}$. The operator A in $\mathcal{B}(\mathscr{V})$ is called φ-*adjointable* if there exist $B \in \mathcal{B}(\mathscr{V})$ such that

$$\varphi(Ax, y) \leq \varphi(x, By)$$

for every $x, y \in \mathscr{V}$. In this case, B is φ-*adjoint* of A and it is denoted by A^*. If $A = A^*$, then A is called *self-adjoint* (for more information on related ideas and concepts we refer the reader to [21, p. 88-90]). Also, an operator A in $\mathcal{B}(\mathscr{V})$ is called φ-*positive* if it is self-adjoint and $\varphi(Ax, x) \geq 0$ for all $x \in \mathscr{V}$. The set of all φ-adjointable operators will denote by $\mathcal{L}(\mathscr{V})$.

In Section 2 we invoke some fundamental facts about the sesquilinear forms in vector space that are used throughout the paper. Some famous inequalities due to Kittaneh, Dragomir and Sándor are given. In Section 3 of this paper, we introduce and study the numerical range and numerical radius by using sesquilinear form φ in normed space $\mathscr{V}$, which we call them φ-*numerical range* and φ-*numerical radius*, respectively. Also some inequalities for φ-numerical radius are extended. For this purpose, we employ some classical inequalities for numerical radius in Hilbert space.

2 Some immediate results

We start our work by presenting some simple results. The following lemma is known as Polarization identity for sesquilinear forms; see [2, Theorem 4.3.7].

Lemma 1 *Let φ be a sesquilinear form on $\mathscr{V}$, then*

$$4\varphi(x,y) = \|x+y\|_{\varphi}^2 - \|x-y\|_{\varphi}^2 + i\,\|x+iy\|_{\varphi}^2 - i\,\|x-iy\|_{\varphi}^2. \tag{1}$$

The next lemma is known as the Cauchy-Schwarz inequality and follows from Lemma 1.

Lemma 2 *For any positive sesquilinear form φ on $\mathscr{V}$ we have*

$$|\varphi(x,y)| \leq \sqrt{\varphi(x,x)}\sqrt{\varphi(y,y)}.$$

Lemma 3 *The Schwarz inequality for φ-positive operators asserts that if A is a φ-positive operator in $\mathcal{L}(\mathscr{V})$, then*

$$|\varphi(Ax, y)|^2 \leq \varphi(Ax, x)\,\varphi(Ay, y), \tag{2}$$

for all x, y in $\mathscr{V}$.

The following lemma can be found in [13, Lemma 1].

Proposition 1 *Let A, B and C be operators in $\mathcal{L}(\mathscr{V})$, where A and B are φ-positive. Then $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a φ-positive operator in $\mathcal{L}(\mathscr{V} \oplus \mathscr{V})$ if and only if*

$$|\varphi(Cx, y)|^2 \leq \varphi(Ax, x)\,\varphi(By, y), \tag{3}$$

for all x, y in $\mathscr{V}$.

Proof. First assume that $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a φ-positive operator in $\mathcal{L}(\mathscr{V} \oplus \mathscr{V})$. Then by (2) we have

$$\left|\varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right)\right|^2 \leq \varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}\right)\varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right),$$

for all x, y in $\mathscr{V}$. A direct simplification of above inequality now yields (3). Conversely, assume that (2) holds, then for every x, y in $\mathscr{V}$,

$$\begin{aligned}
\varphi\left(\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) &= \varphi(Ax, x) + \varphi(C^*y, x) + \varphi(Cx, y) + \varphi(By, y) \\
&= \varphi(Ax, x) + \varphi(By, y) + 2\operatorname{Re}\varphi(Cx, y) \\
&\geq 2(\varphi(Ax, x))^{\frac{1}{2}}(\varphi(By, y))^{\frac{1}{2}} + 2\operatorname{Re}\varphi(Cx, y) \\
&\geq 2|\varphi(Cx, y)| + 2\operatorname{Re}\varphi(Cx, y) \\
&\geq 2|\varphi(Cx, y)| - 2|\varphi(Cx, y)| \\
&= 0.
\end{aligned}$$

This completes the proof of the theorem. $\square$

Remark 1 *If we put $C = AB$ in (3), then we obtain*

$$|\varphi(ABx, x)|^2 \leq \varphi\left(A^2x, x\right)\varphi\left(B^2y, y\right).$$

We will need the following definition to obtain our results. For more related details see [4, p. 1-5].

Definition 1 A functional $(\cdot,\cdot):\mathscr{V}\times\mathscr{V}\to\mathbb{C}$ is said to be a Hermitian form on linear space $\mathscr{V}$, if

(a) $(ax+by,z)=a(x,z)+b(y,z)$, for all $a,b\in\mathbb{C}$ and all $x,y,z\in\mathscr{V}$;

(b) $(x,y)=\overline{(y,x)}$, for all $x,y\in\mathscr{V}$.

Utilizing the Cauchy Schwarz inequality we can state the following result that will be useful in the sequel (see [7, Theorem 2]).

Lemma 4 *Let $(\mathscr{V},\varphi(\cdot,\cdot))$ be a complex vector space, then*

$$\begin{aligned}&\left(\|a\|_\varphi^2\|b\|_\varphi^2-|\varphi(a,b)|^2\right)\left(\|b\|_\varphi^2\|c\|_\varphi^2-|\varphi(b,c)|^2\right)\\&\geq\left|\varphi(a,c)\|b\|_\varphi^2-\varphi(a,b)\varphi(b,c)\right|^2.\end{aligned}\tag{4}$$

Proof. Let us consider the mapping $p_b:\mathscr{V}\times\mathscr{V}\to\mathbb{C}$, with $p_b(a,c)=\varphi(a,c)\|b\|_\varphi^2-\varphi(a,b)\varphi(b,c)$, for each $b\in\mathscr{V}\backslash\{0\}$. Obviously $p_b(\cdot,\cdot)$ is a non-negative Hermitian form and then writing Schwarz's inequality

$$|p_b(a,c)|^2\leq p_b(a,a)\,p_b(c,c),\ \ (a,c\in\mathscr{V})$$

we obtain the desired inequality (4). □

The following refinement of the Schwarz inequality holds (see [8, Theorem 4]):

Theorem 1 *Let $a,b\in\mathscr{V}$ and $e\in\mathscr{V}$ with $\|e\|_\varphi=1$, then*

$$\|a\|_\varphi\|b\|_\varphi\geq|\varphi(a,b)\varphi(a,e)\varphi(e,b)|+|\varphi(a,e)\varphi(e,b)|\geq|\varphi(a,b)|.\tag{5}$$

Proof. Applying the inequality (4), we can state that

$$\left(\|a\|_\varphi^2-|\varphi(a,e)|^2\right)\left(\|b\|_\varphi^2-|\varphi(b,e)|^2\right)\geq|\varphi(a,b)-\varphi(a,e)\varphi(e,b)|^2.\tag{6}$$

Utilizing the elementary inequality for real numbers

$$\left(m^2-n^2\right)\left(p^2-q^2\right)\leq(mp-nq)^2,$$

we can easily see that

$$(\|a\|_\varphi\|b\|_\varphi-|\varphi(a,e)\varphi(e,b)|)^2\geq\left(\|a\|_\varphi^2-|\varphi(a,e)|^2\right)\left(\left|\|b\|_\varphi^2-|\varphi(b,e)|^2\right|\right),\tag{7}$$

for any $a, b, e \in \mathscr{V}$ with $\|e\|_{\varphi} = 1$. Since, by the Schwarz's inequality

$$|\varphi(a, e)\, \varphi(e, b)| \leq \|a\|_{\varphi} \|b\|_{\varphi}.$$

Hence, by (6) and (7) we deduce the first part of (5). The second part of (5) is obvious. $\square$

If $\varphi(x, y) = 0$, x is said to be φ-*orthogonal* to y, and notation $x \perp^{\varphi} y$ is used. If $\varphi(x, x) = 0$ implies $x = 0$, then the relation $\perp^{\varphi}$ is symmetric. The notation $\mathscr{U} \perp^{\varphi} \mathscr{W}$ means that $x \perp^{\varphi} y$ when $x \in \mathscr{U}$ and $y \in \mathscr{W}$. Also $\mathscr{U}^{\perp}$ is the set of all $y \in \mathscr{V}$ that are orthogonal to every $x \in \mathscr{U}$. The following lemmas are known in the literature (see [21, p. 307-308]).

Lemma 5 *If* $x, y \in \mathscr{V}$, *and* $\varphi(x, x) = 0$ *implies* $x = 0$, *then*

$$\|y\|_{\varphi} \leq \|\lambda x + y\|_{\varphi} \quad (\lambda \in \mathbb{C}),$$

if and and only if $x \perp^{\varphi} y$.

Lemma 6 *Every non empty closed convex set* $\mathscr{U} \subset \mathscr{V}$ *contains a unique* x *of minimal norm.*

The next assertion is interesting on its own right.

Theorem 2 *If* $\mathscr{M}$ *is a closed subspace of* $\mathscr{V}$, *then*

$$\mathscr{V} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

3 φ-numerical range and φ-numerical radius

This section deals with the theory of sesquilinear forms, its generalizations and applications to numerical range and numerical radius of operators. The basic notions of numerical range and numerical radius can be found in [11]. Moreover, for a host of numerical radius inequalities, and for diverse applications of these inequalities, we refer to [6, 5, 20], and references therein. Before stating the results, we establish the notation some results from the literature.

Definition 2 The φ-*numerical range* of an operator A on vector space $\mathscr{V}$ is the subset of the complex numbers $\mathbb{C}$, given by

$$W_{\varphi}(A) = \{\varphi(Ax, x) : \; x \in \mathscr{V}, \|x\|_{\varphi} = 1\}.$$

Proposition 2 *The following properties of* $W_\varphi(A)$ *are immediate.*

(a) *If* φ *is symmetric then,* $W_\varphi(A^*) = \left\{\overline{\lambda} :\ \lambda \in W_\varphi(A)\right\}$.

(b) $W_\varphi(\alpha I + \beta A) = \alpha + \beta W_\varphi(A)$.

(c) $W_\varphi(U^* A U) = W_\varphi(A)$, *for any unitary operator* U.

Further, we list some basic properties of $W_\varphi(A)$:

Proposition 3 *Let* $A \in \mathcal{L}(\mathscr{V})$, φ *be a sesqulinear form on vector space* $\mathscr{V}$, *then*

(a) $W_\varphi(A)$ *is convex.*

(b) $\mathsf{Sp}(A) \subseteq \overline{W_\varphi(A)}$, *where* $\mathsf{Sp}(A)$ *denotes the spectrum of* A.

(c) *If* φ *is symmetric then,* A *is real if and only if* $W_\varphi(A)$ *is real.*

Definition 3 The φ-*numerical radius* of an operator A on $\mathscr{V}$ given by

$$\omega_\varphi(A) = \sup\{|\varphi(Ax, x)| :\ \|x\|_\varphi = 1\}.$$

Note that, if $\varphi(x, x) = 0$ implies $x = 0$ then $\omega_\varphi(\cdot)$ is a norm on the $\mathcal{L}(\mathscr{V})$ of all bounded linear operators $A : \mathscr{V} \to \mathscr{V}$, that is

(a) $\omega_\varphi(A) \geq 0$ for any $A \in \mathcal{L}(\mathscr{V})$ and $\omega_\varphi(A) = 0$ if and only if $A = 0$;

(b) $\omega_\varphi(\lambda A) = |\lambda|\, \omega_\varphi(A)$ for any $\lambda \in \mathbb{C}$ and $A \in \mathcal{L}(\mathscr{V})$;

(c) $\omega_\varphi(A + B) \leq \omega_\varphi(A) + \omega_\varphi(B)$ for any $A, B \in \mathcal{L}(\mathscr{V})$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

Proposition 4 *For each* $A \in \mathcal{L}(\mathscr{V})$

$$\omega_\varphi(A) \leq \|A\|_\varphi \leq 2\omega_\varphi(A), \tag{8}$$

where

$$\|A\|_\varphi = \sup\{|\varphi(Ax, y)| :\ \|x\|_\varphi = \|y\|_\varphi = 1\}.$$

We are now ready to construct our main results of this section.

Theorem 3 *Let* φ *be a symmetric sesquilinear form. Then* A *is self-adjoint if and only if* $W_\varphi(A)$ *is real.*

Proof. If A is self-adjoint, we have, for all $f \in \mathscr{V}$, $\varphi(Af, f) = \varphi(f, Af) = \overline{\varphi(Af, f)}$, and hence $W_\varphi(A)$ is real. Conversely, if $\varphi(Af, f)$ is real for all $f \in \mathscr{V}$, we have $\varphi(Af, f) = \varphi(f, Af) = 0 = \varphi((A - A^*) f, f)$. Thus the operator $A - A^*$ has only $\{0\}$ in its φ-numerical range. So $A - A^* = 0$ and $A = A^*$. $\square$

Theorem 4 *Let* $A \in \mathcal{L}(\mathscr{V})$. *If* $R(A) \perp^\varphi R(A^*)$, *then* $\omega_\varphi(A) = \frac{1}{2}\|A\|_\varphi$.

Proof. Let $x \in \mathscr{V}$, $\|x\|_\varphi = 1$. We can write $x = x_1 + x_2$, where $x_1 \in N(A)$, the null space of A, and $x_2 \in \overline{R(A^*)}$. Thus we have

$$\varphi(Ax, x) = \varphi(A(x_1 + x_2), x_1 + x_2) = \varphi(Ax_2, x_1).$$

Since $Ax_1 = 0$ and $\varphi(Ax_2, x_2) = \varphi(x_2, A^*x_2) = 0$. Thus

$$\begin{aligned} |\varphi(Ax, x)| &\leq \|A\|_\varphi \|x_1\| \|x_2\| \leq \frac{1}{2}\|A\|_\varphi (\|x_1\| + \|x_2\|) \\ &\qquad \text{(by the inequality } \|a\| \|b\| \leq \frac{1}{2}\left(\|a\|^2 + \|b\|^2\right)) \\ &= \frac{1}{2}\|A\|_\varphi \quad \text{(since } \|x_1\| + \|x_2\| = 1). \end{aligned}$$

Since x is arbitrary, we have

$$\omega_\varphi(A) \leq \frac{1}{2}\|A\|_\varphi \leq \omega_\varphi(A).$$

This completes the proof. $\square$

Our φ-numerical radius inequality for bounded operators can be stated as follows.

Theorem 5 *Let* $A, X \in \mathcal{L}(\mathscr{V})$, *then*

$$\omega_\varphi(AXA^*) \leq \|A\|_\varphi^2 \, \omega_\varphi(X). \tag{9}$$

Proof. Let $x \in \mathscr{V}$ be a unit vector. Then

$$\begin{aligned} |\varphi(AXA^*x, x)| &= |\varphi(XA^*x, A^*x)| \\ &\leq \|A^*x\|_\varphi^2 \, \omega_\varphi(x) \\ &\leq \|A^*\|_\varphi^2 \, \omega_\varphi(x) \\ &= \|A\|_\varphi^2 \, \omega_\varphi(x). \end{aligned}$$

Now the result follows immediately by taking supremum over all unit vectors in $\mathscr{V}$. $\square$

Remark 2 *Let* $A, X \in \mathcal{L}(\mathscr{V})$, *then*

$$\omega_\varphi(AXA^*) \le \|A\|_\varphi^2 \|X\|_\varphi. \tag{10}$$

Note that, by (8) *we can easily see that inequality* (9) *is sharper than inequality* (10).

The following result holds (see [12, Theorem 1], for the case of inner product):

Theorem 6 *Let* $A, B \in \mathcal{L}(\mathscr{V})$ *and* φ *is a bounded sesquilinear form, then*

$$\frac{1}{4}\|A^*A + AA^*\|_\varphi \le (\omega_\varphi(A))^2 \le \|A^*A + AA^*\|_\varphi.$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A. Then B and C are self-adjoint, and $A^*A + AA^* = 2\left(B^2 + C^2\right)$. Let x be any vector in $\mathscr{V}$. Then by the convexity of the function $f(t) = t^2$, we have

$$\begin{aligned}|\varphi(Ax, x)|^2 &= (\varphi(Bx, x))^2 + (\varphi(Cx, x))^2 \\ &\ge \frac{1}{2}(|\varphi(Bx, x)| + |\varphi(Cx, x)|)^2 \\ &\ge \frac{1}{2}|\varphi((B \pm C)x, x)|^2.\end{aligned}$$

Taking supremum over $x \in \mathscr{V}$ with $\|x\|_\varphi = 1$, produces

$$\frac{1}{2}\|B \pm C\|_\varphi^2 \le (\omega_\varphi(A))^2.$$

Since

$$\begin{aligned}2(\omega_\varphi(A))^2 &\ge \frac{1}{2}\left(\|B + C\|_\varphi^2 + \|B - C\|_\varphi^2\right) \\ &\ge \frac{1}{2}\left\|(B + C)^2 + (B - C)^2\right\|_\varphi \\ &= \left\|B^2 + C^2\right\|_\varphi \\ &= \frac{1}{2}\|A^*A + AA^*\|_\varphi,\end{aligned}$$

and hence

$$(\omega_\varphi(A))^2 \le \frac{1}{4}\|A^*A + AA^*\|_\varphi.$$

On the other hand

$$|\varphi(Ax, x)|^2 = (\varphi(Bx, x))^2 + (\varphi(Cx, x))^2 \le 2\left\|B^2 + C^2\right\|_\varphi.$$

Now by taking the supremum over $x \in \mathscr{V}$, with $\|x\|_{\varphi} = 1$ in the above inequality we infer that Theorem 6. $\square$

Now we state, another related φ-numerical radius inequality that has been given in [6, Theorem 36], for Hilbert space case.

Theorem 7 *Let* $A \in \mathcal{L}(\mathscr{V})$, *then*

$$\omega_{\varphi}^{2}(A) \leq \frac{1}{2}\left(\omega_{\varphi}\left(A^{2}\right) + \|A\|_{\varphi}^{2}\right). \tag{11}$$

Proof. By Theorem 1 observing that

$$|\varphi(a, b) - \varphi(a, e)\varphi(e, b)| \geq |\varphi(a, e)\varphi(e, b)| - |\varphi(a, b)|,$$

hence by first inequality in (5) we deduce

$$\frac{1}{2}\left(\|a\|_{\varphi}\|b\|_{\varphi} + |\varphi(a, b)|\right) \geq |\varphi(a, e)\varphi(e, b)|. \tag{12}$$

Choose in (12), $e = x$, $\|x\|_{\varphi} = 1$, $a = Ax$ and $b = A^{*}x$ to get

$$\frac{1}{2}\|Ax\|_{\varphi}\|A^{*}x\|_{\varphi} + \left|\varphi\left(A^{2}x, x\right)\right| \geq |\varphi(Ax, x)|^{2}, \tag{13}$$

for any $x \in \mathscr{V}$ with $\|x\|_{\varphi} = 1$. Taking the supremum in (13) over $x \in \mathscr{V}$ with $\|x\|_{\varphi} = 1$, we deduce the desired inequality (11). $\square$

Remark 3 *The concept of a sesquilinear form and quadratic form do not require the structure of an inner product space. They can be defined in any vector space. Something to notice about the definition of a sesquilinear form is the similarity it has with an inner product. In essence, a sesquilinear form is a generalization of an inner product. (Note that the inner product is a sesquilinear form but the converse is not true.)*

With regard to the point mentioned above, we can say that all of the inequalities which are obtained by Dragomir in [6] can be extended to vector space in a similar way. The details are left to the interested readers.

Acknowledgment

The authors would like to express our gratitude to the referee for the helpful comments.

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Received: December 8, 2016

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 336–347

DOI: 10.1515/ausm-2017-0025

Fibonacci words in hyperbolic Pascal triangles

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Abstract. The hyperbolic Pascal triangle $\mathcal{HPT}_{4,q}$ $(q \geq 5)$ is a new mathematical construction, which is a geometrical generalization of Pascal's arithmetical triangle. In the present study we show that a natural pattern of rows of $\mathcal{HPT}_{4,5}$ is almost the same as the sequence consisting of every second term of the well-known Fibonacci words. Further, we give a generalization of the Fibonacci words using the hyperbolic Pascal triangles. The geometrical properties of a $\mathcal{HPT}_{4,q}$ imply a graph structure between the finite Fibonacci words.

1 Introduction

The hyperbolic Pascal triangle $\mathcal{HPT}_{4,q}$ $(q \geq 5)$ is a new mathematical construction, which is a geometrical generalization of Pascal's arithmetical triangle [1]. In the present article we discuss the properties of the patterns of the rows of $\mathcal{HPT}_{4,q}$, which patterns give a new kind of generalizations of the well-known Fibonacci words. Our aim is to show the connection between the Fibonacci words and the hyperbolic Pascal triangles.

After a short introduction of the hyperbolic Pascal triangles and the finite Fibonacci words we define a new family of Fibonacci words and we present the relations between the hyperbolic Pascal triangles and the newly generalized Fibonacci words. Their connections will be illustrated by figures for better

2010 Mathematics Subject Classification: 05B30, 11B39
Key words and phrases: hyperbolic Pascal triangle, Fibonacci word

comprehension. As the hyperbolic Pascal triangles are based on the hyperbolic regular lattices, their geometrical properties provide a graph structure between the generalized finite Fibonacci words. The extension of this connection could provide a new family of binary words.

1.1 Hyperbolic Pascal triangles

In the hyperbolic plane there are infinite types of regular mosaics (or regular lattices), that are denoted by the Schläfli symbol $\{p, q\}$, where $(p-2)(q-2) > 4$. Each regular mosaic induces a so-called hyperbolic Pascal triangle, following and generalizing the connection between classical Pascal's triangle and the Euclidean regular square mosaic $\{4, 4\}$ (for more details see [1, 5, 6]).

The hyperbolic Pascal triangle $\mathcal{HPT}_{4,q}$ based on the mosaic $\{p, q\}$ can be depicted as a digraph, where the vertices and the edges are the vertices and the edges of a well-defined part of the lattice $\{p, q\}$, respectively. Further, the vertices possess a value each giving the number of the different shortest paths from the base vertex. Figure 1 illustrates the hyperbolic Pascal triangle when $\{p, q\} = \{4, 6\}$. Generally, for a $\{4, q\}$ configuration the base vertex has two edges, the leftmost and the rightmost vertices have three, the others have q edges. The square shaped cells surrounded by appropriate edges correspond to the regular squares in the mosaic. Apart from the winger elements, certain vertices (called "Type A" for convenience) have two ascendants and $q-2$ descendants, the others ("Type B") have one ascendant and $q-1$ descendants. In the figures of the present study we denote the type A vertices by red circles and the type B vertices by cyan diamonds, while the wingers by white diamonds. The vertices which are n-edge-long far from the base vertex are in row n.

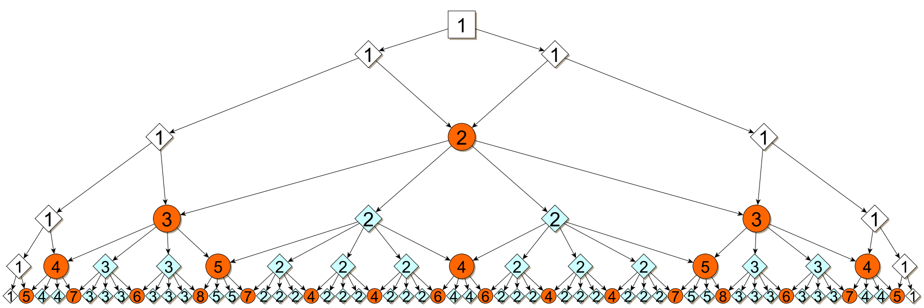

Figure 1: Hyperbolic Pascal triangle linked to $\{4, 6\}$ up to row 5

The general method of deriving the triangle is the following: going along the vertices of the j^{th} row, according to the type of the elements (winger, A, B), we draw the appropriate number of edges downwards (2, $q-2$, $q-1$, respectively). Neighbour edges of two neighbour vertices of the j^{th} row meet in the $(j+1)^{\text{th}}$ row, constructing a type A vertex. The other descendants of row j are type B in row $j+1$. Figure 2 also shows a growing algorithm of the different types except the leftmost items, that are always types B and A. (Compare Figure 2 with Figures 1 and 3.)

In the sequel, $\genfrac{\rangle}{\langle}{0pt}{}{n}{k}$ denotes the k^{th} element in row n, whose value is either the sum of the values of its two ascendants or the value of its unique ascendant. We note, that the hyperbolic Pascal triangle has the property of vertical symmetry.

In the following we generalize the Fibonacci word in a new (but not brand new) way and show that this generalization is the same as the patterns of nodes types A and B in rows of $\mathcal{HPT}_{4,q}$.

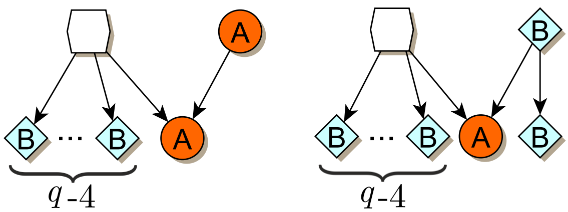

Figure 2: Growing method in Pascal triangles (except for the two leftmost items)

1.2 Fibonacci words

The most familiar and the most studied binary word in mathematics is the Fibonacci word. The finite Fibonacci words, f_i, are defined by the elements of the recurrence sequence $\{f_i\}_{i=0}^{\infty}$ over $\{0,1\}$ defined as follows

$$f_0 = 1, \qquad f_1 = 0, \qquad f_i = f_{i-1}f_{i-2}, \qquad (i \geq 2).$$

It is clear, that $|f_i| = F_{i+1}$, where F_i is the i-th Fibonacci number defined by the recurrence relation $F_i = F_{i-1} + F_{i-2}$ $(i \geq 2)$, with initial values $F_0 = 0$, $F_1 = 1$. The infinite Fibonacci word is $\mathbf{f} = \lim_{i\to\infty} f_i$. Table 1 shows the first few Fibonacci words. It is also well-known that the Fibonacci morphism (σ: $\{0,1\} \to \{0,1\}*$, $0 \to 01$, $1 \to 0$) acts between two consecutive finite Fibonacci words. For some newest properties (and further references) of Fibonacci words see [2, 3, 4, 7, 8].

$$\begin{array}{lcl} f_0 & = & 1 \\ f_1 & = & 0 \\ f_2 & = & 01 \\ f_3 & = & 010 \\ f_4 & = & 01001 \\ f_5 & = & 01001010 \\ f_6 & = & 0100101001001 \\ f_7 & = & 010010100100101001010 \end{array}$$

Table 1: The first Fibonacci words

2 $\{4, q\}$-Fibonacci words

There are some generalizations of Fibonacci words, one of them is the biperiodic Fibonacci word [2, 8]. For any two positive integers a and b, the biperiodic finite Fibonacci words sequence, say $\{\widehat{f}_i\}_{i=0}^{\infty}$, is defined recursively by

$$\widehat{f}_0 = 1, \ \widehat{f}_1 = 0, \ \widehat{f}_2 = 0^{a-1}1 = 00\ldots01,$$

and

$$\widehat{f}_i = \begin{cases} \widehat{f}_{i-1}^{a}\widehat{f}_{i-2}, & \text{if } i \text{ is even;} \\ \widehat{f}_{i-1}^{b}\widehat{f}_{i-2}, & \text{if } i \text{ is odd;} \end{cases} \quad (i \geq 3).$$

It has been proved [8], that if $i \geq 1$ then $|\widehat{f}_i| = F_i^{(a,b)}$, where for any two positive integers a and b, the biperiodic Fibonacci sequence $\{F_i^{(a,b)}\}_{i=0}^{\infty}$ is defined recursively by

$$F_0^{(a,b)} = 0, \ F_1^{(a,b)} = 1, \ F_i^{(a,b)} = \begin{cases} aF_{i-1}^{(a,b)} + F_{i-2}^{(a,b)}, & \text{if } i \text{ is even;} \\ bF_{i-1}^{(a,b)} + F_{i-2}^{(a,b)}, & \text{if } i \text{ is odd;} \end{cases} \quad (i \geq 2). \tag{1}$$

The first few terms are 0, 1, a, $ab+1$, a^2b+2a, $a^2b^2+3ab+1$, $a^3b^2+4a^2b+3a$, $a^3b^3 + 5a^2b^2 + 6ab + 1$. When $a = b = k$, this generalization gives the k-Fibonacci numbers and in the case $a = b = 1$, we recover the original Fibonacci numbers [2, 8].

Now let us define the finite $\{4,q\}$-Fibonacci words sequence $\{f_i^{[4,q]}\}_{i=0}^{\infty}$, shortly $\{f_i^{[q]}\}_{i=0}^{\infty}$, where $q \geq 5$, a new family of generalized Fibonacci words, and

$$f_0^{[q]} = 1,\ f_1^{[q]} = 0,\ f_i^{[q]} = \begin{cases} \left(f_{i-1}^{[q]}\right)^{q-4} f_{i-2}^{[q]}, & \text{if } i \text{ is even;} \\ f_{i-1}^{[q]} f_{i-2}^{[q]}, & \text{if } i \text{ is odd;} \end{cases} \quad (i \geq 2). \tag{2}$$

These new $\{4,q\}$-Fibonacci words are almost the same as the biperiodic Pascal words, $\widehat{f}_i$, if $a = 1$ and $b = q-4$. As the definitions for the second items vary, the odd and even situations are reversing. If $q = 5$, then $\{4,q\}$-Fibonacci words coincide with the classical Fibonacci words. (In Table 2 we list the first few $\{4,6\}$-Fibonacci words.) The infinite $\{4,q\}$-Fibonacci word is defined as $\mathbf{f}^{[q]} = \lim_{i\to\infty} f_i^{[q]}$ and $\mathbf{f} = \mathbf{f}^{[5]}$ (see Table 3).

$$\begin{aligned}
f_0^{[6]} &= 1 \\
f_1^{[6]} &= 0 \\
f_2^{[6]} &= 001 \\
f_3^{[6]} &= 0010 \\
f_4^{[6]} &= 00100010001 \\
f_5^{[6]} &= 001000100010010 \\
f_6^{[6]} &= 0010001000100100010001000100100010001
\end{aligned}$$

Table 2: The first few $\{4,6\}$-Fibonacci words

$$\begin{aligned}
\mathbf{f}^{[5]} &= 01001010010010100101001001010010010100101001001010010100\ldots \\
\mathbf{f}^{[6]} &= 00100010001001000100010001001000100010001001000100010010\ldots \\
\mathbf{f}^{[7]} &= 00010000100001000010001000010000100001000010001000010000\ldots \\
\mathbf{f}^{[8]} &= 00001000001000001000001000001000010000010000010000010000\ldots
\end{aligned}$$

Table 3: Some infinite $\{4,q\}$-Fibonacci words

In case of the extension of definition (2) to $q = 4$, the $f_{2k}^{[4]} = 1$, $f_{2k+1}^{[4]} = 1\ldots10$ (the number of 1's is k) for any $k \geq 1$ and there is no limit of $f_i^{[4]}$ if $i \to \infty$. Therefore, we investigate the $\{4,q\}$-Fibonacci words, when $q \geq 5$.

Let $\sigma^{[q]}$ be the $\{4,q\}$-Fibonacci morphism defined by

$$\{0,1\} \to \{0,1\}*, \quad 0 \to 0^{q-4}10,\ 1 \to 0^{q-4}1, \tag{3}$$

where $q \geq 5$.

Theorem 1 *The* $\{4, q\}$*-Fibonacci morphism,* $\sigma^{[q]}$*, acts between every second words of* $\{4, q\}$*-Fibonacci words, so that*

$$\sigma^{[q]}(f^{[q]}_{i-2}) = f^{[q]}_{i}, \qquad (i \geq 2). \tag{4}$$

Proof. We prove the assertion by induction on i. The statement is clearly true for $i = 2, 3$. Now we assume, that the result holds for any j, when $4 \leq j < i$. Let i be first even. Then

$$\begin{aligned} \sigma^{[q]}(f^{[q]}_{i-2}) &= \sigma^{[q]}\left(\left(f^{[q]}_{i-3}\right)^{q-4} f^{[q]}_{i-4}\right) = \left(\sigma^{[q]}(f^{[q]}_{i-3})\right)^{q-4} \sigma^{[q]}(f^{[q]}_{i-4}) \\ &= \left(f^{[q]}_{i-1}\right)^{q-4} f^{[q]}_{i-2} = f^{[q]}_{i} \end{aligned}$$

If i is odd the proof is similar, $\sigma^{[q]}(f^{[q]}_{i-2}) = \sigma^{[q]}\left(f^{[q]}_{i-3} f^{[q]}_{i-4}\right) = \cdots = f^{[q]}_{i}$. □

Remark 1 $\sigma^{[5]} = \sigma^2$ *and* $\sigma^2(f_i) = f_{i+2}$.

3 Connection between $\mathcal{HPT}_{4,q}$ and $\{4, q\}$-Fibonacci words

We consider again the hyperbolic Pascal triangle $\mathcal{HPT}_{4,q}$. Let us denote the left and right nodes '1' by type B (compare Figures 1 and 4). Let a_n and b_n be the number of vertices of type A and B in row n, respectively. Further let

$$s_n = a_n + b_n, \tag{5}$$

that gives the total number of the vertices in row $n \geq 0$. Then the ternary homogeneous recurrence relation

$$s_n = (q-1)s_{n-1} - (q-1)s_{n-2} + s_{n-3} \qquad (n \geq 4) \tag{6}$$

holds with initial values $s_0 = 1$, $s_1 = 2$, $s_2 = 3$, $s_3 = q$ (recall, that $q \geq 5$). For the explicit form see [1].

Lemma 1 *If* $n \geq 1$*, then*

$$s_n = u_n + 2, \tag{7}$$

where $u_1 = 0$*,* $u_2 = 1$ *and* $u_n = (q-2)u_{n-1} - u_{n-2}$*, if* $n \geq 3$.

Proof. Let $u_n = s_n - 2$, where $n \geq 1$. Then $u_1 = 0$, $u_2 = 1$ and $u_3 = s_3 - 2 = q - 2 = (q-2)u_2 - u_1$.

For general cases corresponding to $n \geq 4$, firstly, we have

$$\begin{aligned} u_n &= (q-1)s_{n-1} - (q-1)s_{n-2} + s_{n-3} - 2 \\ &= (q-1)(s_{n-1} - 2) - (q-1)(s_{n-2} - 2) + (s_{n-3} - 2) \\ &= (q-1)u_{n-1} - (q-1)u_{n-2} + u_{n-3}. \end{aligned}$$

This also means, that $\{s_n\}$ and $\{u_n\}$ have the same ternary recurrence relation (with different initial values).

Secondly, we show, that $\{u_n\}$ can be described by a binary recurrence relation too. (In contrast $\{s_n\}$ cannot.) Adding the equations $u_n = (q-2)u_{n-1} - u_{n-2}$ and $-u_{n-1} = -(q-2)u_{n-2} + u_{n-3}$, we obtain $u_n = (q-1)u_{n-1} - (q-1)u_{n-2} + u_{n-3}$. □

The first few terms of $\{u_i\}$ are 0, 1, $q-2$, $q^2 - 4q + 3$, $q^3 - 6q^2 + 10q - 4$, $q^4 - 8q^3 + 21q^2 - 20q + 5$.

Lemma 2 *Both of the sub-sequences consisting of every second term of* $\{F_i^{(a,b)}\}$ *satisfy the relation*

$$x_i = (ab+2)x_{i-2} - x_{i-4}, \qquad (i \geq 4). \tag{8}$$

Moreover, if $n \geq 2$ *then*

$$u_n = F_{2n-2}^{(1,q-4)}. \tag{9}$$

Proof. For the first few terms of $\{F_i^{(a,b)}\}$ the equation (8) is clearly true. We assume that for $i-1$ $(i \geq 6)$ equation (8) also holds. Then if i is even,

$$\begin{aligned} F_i^{(a,b)} &= aF_{i-1}^{(a,b)} + F_{i-2}^{(a,b)} \\ &= a\left((ab+2)F_{i-3}^{(a,b)} - F_{i-5}^{(a,b)}\right) + \left((ab+2)F_{i-4}^{(a,b)} - F_{i-6}^{(a,b)}\right) \\ &= (ab+2)\left(aF_{i-3}^{(a,b)} + F_{i-4}^{(a,b)}\right) - \left(aF_{i-5}^{(a,b)} + F_{i-6}^{(a,b)}\right) \\ &= (ab+2)F_{i-2}^{(a,b)} - F_{i-4}^{(a,b)}. \end{aligned}$$

If i is odd, the proof is the same. For the case $a = 1$ and $b = q-4$ we obtain the equation (9). □

Let $\{h_n^{[q]}\}_0^\infty$ be the sequence over $\{A, B\}$, where $h_n^{[q]}$ equals to the concatenations of the type of the vertices of row n in $\mathcal{HPT}_{4,q}$ from left to the right.

Further, we call the elements of this the $\{4, q\}$-hyperbolic Pascal words (shortly q-hyperbolic Pascal words). For example in the case of $q = 5$ (see Figure 3), we have

$$
\begin{aligned}
h_0^{[5]} &= \mathsf{B},\ h_1^{[5]} = \mathsf{BB},\ h_2^{[5]} = \mathsf{BAB},\ h_3^{[5]} = \mathsf{BABAB},\ h_4^{[5]} = \mathsf{BABABBABAB},\\
h_5^{[5]} &= \mathsf{BABABBABABBABBABABBABAB}.
\end{aligned}
$$

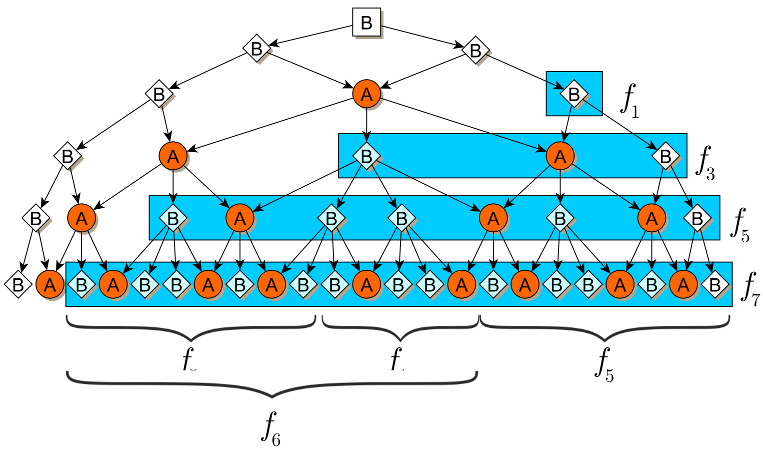

Figure 3: Pattern of $\mathcal{HPT}_{4,5}$ up to row 5 and some Fibonacci words

Let us consider the bijection

$$\phi : \{0, 1\} \to \{\mathsf{A}, \mathsf{B}\}, \qquad \phi(1) = \mathsf{A}, \qquad \phi(0) = \mathsf{B}. \tag{10}$$

Let the words u and v be over $\{0, 1\}$ and $\{\mathsf{A}, \mathsf{B}\}$, respectively. If $\phi(u) = v$, then we say that u is equivalent to v and we denote $u \equiv v$. For example from Figure 3 we have

$$
\begin{aligned}
f_1 = 0 \equiv \mathsf{B} = h_0^{[5]}, &\qquad 0f_1 = 00 \equiv \mathsf{BB} = h_1^{[5]},\\
f_3 = 01f_1 = 010 \equiv \mathsf{BAB} = h_2^{[5]}, &\qquad 01f_3 = 01010 \equiv \mathsf{BABAB} = h_3^{[5]}.
\end{aligned} \tag{11}
$$

Examining Figure 3 we can recognise that every second Fibonacci word is almost equivalent to the patterns of the rows in $\mathcal{HPT}_{4,5}$. (Compare the patterns of rows in Figure 4 and $f_{2n-3}^{[6]}$, $n = 2, 3, 4$.) The following theorem gives the exact relationship between $\mathcal{HPT}_{4,q}$ and $\{4, q\}$-Fibonacci words.

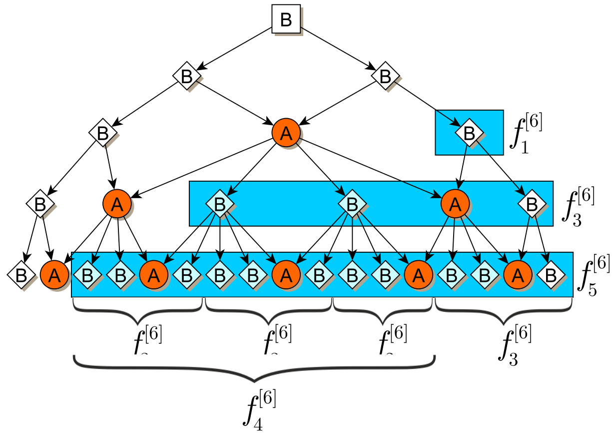

Figure 4: Pattern of $\mathcal{HPT}_{4,6}$ up to row 4 and some Fibonacci words

Theorem 2 *If* $n \geq 2$, *then*

$$01f_{2n-3}^{[q]} \equiv h_n^{[q]} \tag{12}$$

and

$$|f_{2n-3}^{[q]}| = F_{2n-2}^{(1,q-4)},$$

where $1 \equiv A$, $0 \equiv B$ *and* $|h_n^{[q]}| = s_n$.

Proof. If $n = 2$, then $01f_1^{[q]} = 010 \equiv BAB = h_2^{[q]}$. For higher values of n, examining the growing method of the hyperbolic Pascal triangles row by row based on Figure 2, we can recognise that except for the first two elements it can be described by the morphism

$$\lambda : \{A, B\} \to \{A, B\}* \quad \lambda(A) = (B)^{q-4}A, \quad \lambda(B) = (B)^{q-4}AB. \tag{13}$$

After comparing λ with the $\{4, q\}$-Fibonacci morphism $\sigma^{[q]}$ between every second $f_i^{[q]}$ according to Theorem 1, we can recognize that the growing methods (see Figure 2, (3) and (13)) are the same. This proves the equation (12), because the first two elements of all rows ($n \geq 2$) in $\mathcal{HPT}_{4,q}$ are B and A.

The second statement is a consequence of Lemma 2. □

4 Some properties of $\{4,q\}$-Fibonacci words

Presumably, the connection between the $\{4,q\}$-Fibonacci words and the hyperbolic Pascal pyramids can open new opportunities for examining the Fibonacci words. We show some properties of $\{4,q\}$-Fibonacci words in which we use these connections.

Let a binary word u be the concatenation of the words v and w, thus $u = vw$. If we delete w from the end of u, we get v. Let us denote it by $v = u \ominus w$. In words, the sign $\ominus$ acts so, that the word after the sign is deleted from the end of the word before the sign (if it is possible). For example $f_4 = f_5 \ominus f_3 = 01001\cancel{010}$, $f_6 = f_5 f_5 \ominus f_3 = 01001010 \cdot 01001\cancel{010}$ or $f_4^{[6]} = (f_3^{[6]})^3 \ominus f_5^{[6]} = 0010 \cdot 0010 \cdot 001\cancel{0}$.

Theorem 3 *All $\{4,q\}$-Fibonacci words with $(k \geq 2)$ can be given in terms of the previous two odd indexed ones, namely*

$$\begin{aligned} f_{2k}^{[q]} &= \left(f_{2k-1}^{[q]}\right)^{q-3} \ominus f_{2k-3}^{[q]}, \\ f_{2k+1}^{[q]} &= \left(\left(f_{2k-1}^{[q]}\right)^{q-3} \ominus f_{2k-3}^{[q]}\right) f_{2k-1}^{[q]}. \end{aligned}$$

Proof. Applying $f_{2k-1}^{[q]} = f_{2k-2}^{[q]} f_{2k-3}^{[q]}$ we can easily see that $f_{2k-2}^{[q]} = f_{2k-1}^{[q]} \ominus f_{2k-3}^{[q]}$. Furthermore, we can also see that $f_{2k}^{[q]} = \left(f_{2k-1}^{[q]}\right)^{q-4} f_{2k-2}^{[q]} = \left(f_{2k-1}^{[q]}\right)^{q-4} f_{2k-1}^{[q]} \ominus f_{2k-3}^{[q]} = \left(f_{2k-1}^{[q]}\right)^{q-3} \ominus f_{2k-3}^{[q]}$. The second equation is the corollary of the first one. □

If q tends to infinity, then the numbers of '0' in infinite $\{4,q\}$-Fibonacci words are relatively fast growing (see Table 3). Now let us derive these ratios.

Let $d_i^{[q]}$, $d_{i,0}^{[q]}$ and $d_{i,1}^{[q]}$ denote the numbers of all, '0' and '1' digits in the finite $\{4,q\}$-Fibonacci words, respectively. Then, let the limit $r_0^{[q]} = \lim_{i\to\infty}(d_i^{[q]}/d_{i,0}^{[q]})$ be the inverse density of '0' digits in the infinite $\{4,q\}$-Fibonacci word. Similarly, we denote the same density by $r_1^{[q]} = \lim_{i\to\infty}(d_i^{[q]}/d_{i,1}^{[q]})$ in the case of '1' digits.

Theorem 4 *The inverse density of '0' and '1' digits in the infinite $\{4,q\}$-Fibonacci words are*

$$\begin{aligned} r_0^{[q]} &= \frac{q-4+\sqrt{q(q-4)}}{2(q-4)}, \\ r_1^{[q]} &= \frac{q-2+\sqrt{q(q-4)}}{2}, \end{aligned}$$

where $q \geq 5$. *Moreover*

$$\lim_{q\to\infty} r_0^{[q]} = 1 \quad \textit{and} \quad \lim_{q\to\infty} r_1^{[q]} = \infty.$$

Proof. Firstly, let i be odd and large enough, so that $i = 2n-3$. As $01f_{2n-3}^{[q]} \equiv h_n^{[q]}$ from Theorem 2, we consider the ratio s_n/a_n from the hyperbolic Pascal triangle instead of the corresponding ratio $d_{2n-3}^{[q]}/d_{2n-3,0}^{[q]}$. Not only the sequence $\{s_n\}$ can be described by the ternary recurrence relation (6) but also the sequences $\{a_n\}$ and $\{b_n\}$ (more details in [1]). The solutions of the characteristic equations of their recurrence relations are positive real numbers. Moreover, it is well-known that the limit of s_n/a_n is the density of the coefficients of the largest solutions (all solutions are positive), i.e. $\alpha_s = -1/2+(q-2)\sqrt{q^2-4q}/(2q(q-4))$, $\alpha_a = (2-q)(1/2)+(q^2-4q+2)\sqrt{q^2-4q}/(2q(q-4))$ and $\alpha_b = (q-3)(1/2)+(1-q)\sqrt{q^2-4q}/(2q)$. Thus,

$$\lim_{n\to\infty} \frac{d_{2n-3}^{[q]}}{d_{2n-3,0}^{[q]}} = \lim_{n\to\infty} \frac{s_n}{b_n} = \lim_{n\to\infty} \frac{\alpha_s}{\alpha_b} = \frac{q-4+\sqrt{q^2-4q}}{2(q-4)},$$

$$\lim_{n\to\infty} \frac{d_{2n-3}^{[q]}}{d_{2n-3,1}^{[q]}} = \lim_{n\to\infty} \frac{s_n}{a_n} = \lim_{n\to\infty} \frac{\alpha_s}{\alpha_a} = \frac{q-2+\sqrt{q^2-4q}}{2}.$$

Secondly, let i be even. According to Theorem 3 all the even indexed $\{4,q\}$-Fibonacci words can be derived in terms of the previous two elements. We also obtain, that $d_{2k}^{[q]} = (q-3)d_{2k-1}^{[q]} - d_{2k-3}^{[q]}$, $d_{2k,0}^{[q]} = (q-3)d_{2k-1,0}^{[q]} - d_{2k-3,0}^{[q]}$ and $d_{2k,1}^{[q]} = (q-3)d_{2k-1,1}^{[q]} - d_{2k-3,1}^{[q]}$. From it we have

$$\lim_{n\to\infty} \frac{d_{2k}^{[q]}}{d_{2k,0}^{[q]}} = \lim_{n\to\infty} \frac{(q-3)d_{2k-1}^{[q]} - d_{2k-3}^{[q]}}{(q-3)d_{2k-1,0}^{[q]} - d_{2k-3,0}^{[q]}} = \lim_{n\to\infty} \frac{d_{2k-1}^{[q]}}{d_{2k-1,0}^{[q]}},$$

and the case for digits '1' is similar. For the limits of $r_0^{[q]}$ and $r_1^{[q]}$, the statement is obviously true. □

Naturally, if $q = 5$ the results of Theorem 4 give the known $r_0^{[5]} = \varphi$ and $r_1^{[5]} = 1+\varphi$ values, where φ is the golden ratio.

Finally, here are some properties, which can directly be obtained from the properties of $\mathcal{HPT}_{4,q}$:

- The words $01f_{2n-3}^{[q]}$ $(n \geq 2)$ are palindromes.

- The subword 11 never occurs in $\{4, q\}$-Fibonacci words.
- The subword $00\ldots0$ ($q-2$ digits 0) never occurs in words $f_i^{[q]}$.
- The last two digits of finite $\{4, q\}$-Fibonacci words are alternately 01 and 10.
- The infinite $\{4, q\}$-Fibonacci word has $n+1$ distinct subwords of length n, where $n \leq q-2$. In case $n = q-2$, they are $100\ldots01$ with $q-4$ digits 0 and the others are with only one digit 1, in case $n < q-2$ the subwords have at most one digit 1.

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Received: April 17, 2017

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 348–359

DOI: 10.1515/ausm-2017-0026

An explicit formula for derivative polynomials of the tangent function

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Abstract. In the paper, the authors derive an explicit formula for derivative polynomials of the tangent function, deduce an explicit formula for tangent numbers, pose an open problem about obtaining an alternative and explicit formula for derivative polynomials of the tangent function, and recommend some papers closely related to derivative polynomials of other elementary and applicable functions.

1 Introduction

It is not difficult to see that if f is a function whose derivative is a polynomial in f, that is, $f'(x) = P_1(f(x))$ for some polynomial P_1, then all the higher order derivatives of f are also polynomials in f, so we have a sequence of polynomials

2010 Mathematics Subject Classification: 26C99, 26A06, 26A09, 16A24, 33B10, 42A05
Key words and phrases: derivative polynomial; tangent function; explicit formula; tangent number; open problem

P_n defined by $f^{(n)}(x) = P_n(f(x))$ for $n \ge 0$. As usual, we call $P_n(u)$ the derivative polynomials of f. In fact, the polynomials P_n are determined by

$$P_0(u) = u, \quad P_{n+1}(u) = P_n'(u)P_1(u), \quad n \in \mathbb{N}.$$

For detailed information, please refer to [8, Section 2].

In 1945, Morley [10] observed that

$$\begin{gathered}(\tan x)' = 1 + \tan^2 x, \quad (\tan x)'' = 2\tan x + 2\tan^3 x,\\ (\tan x)''' = 2 + (2 + 2\cdot 3)\tan^2 x + 2\cdot 3\tan^4 x,\end{gathered} \tag{1}$$

a term $a_k \tan^k x$ in $(\tan x)^{(n)}$ gives $(\tan x)^{(n+1)}$ $ka_k \tan^{k-1} x + ka_k \tan^{k+1} x$, and then concluded that the coefficient of $\tan^{k-1} x$ in $(\tan x)^{(n+1)}$ is $(k-2)a_{k-2} + ka_k$, with $a_{k-2} = 0$ when $k \le 1$, and $a_k = 0$ when $k \ge n+2$.

In 1995, Hoffman [8, p. 25, (5)] obtained that the derivative polynomials P_n for the tangent function $\tan x$ defined by

$$\frac{\mathrm{d}^n(\tan x)}{\mathrm{d}\,x^n} = P_n(\tan x)$$

for $n \ge 0$ are polynomials of degree $n+1$ and satisfy the recurrence relation

$$P_{n+1}(u) = \sum_{k=0}^{n} \binom{n}{k} P_k(u)P_{n-k}(u) + \delta_{0n},$$

where

$$P_0(u) = u, \quad P_1(u) = 1 + u^2, \quad \text{and} \quad \delta_{ij} = \begin{cases} 0, & i \ne j; \\ 1, & i = j. \end{cases}$$

In [1, 9, 12, 26, 27, 32, 36], there are some explicit formulas and recurrence relations for the nth derivatives of trigonometric functions and other elementary functions. In [3, 4, 5, 20, 21, 26, 30, 33], there are some inequalities for trigonometric functions and other elementary functions. Specially, there are some explicit formulas and many other results on the nth derivative of the tangent function $\tan x$ in [11, 14].

Motivated by those results in [8, 10] and other references mentioned above, we are interested in the question: can one find explicit formulas for coefficients a_k of the derivative polynomials $P_n(u)$ for the tangent function $\tan x$?

The aim of this paper is to answer the above question. Our main results can be stated as the following theorem.

Theorem 1 *For $n \ge 0$, the derivative polynomials $P_n(u)$ of the tangent function $u = \tan x$ can be explicitly computed by*

$$P_n(u) = \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n+1-2k} u^{n+1-2k} \tag{2}$$

with

$$a_{2m-1,0} = (-1)^m \sum_{\ell=1}^{2m} (-1)^\ell 2^{2m-\ell} (\ell-1)! S(2m,\ell) \tag{3}$$

for $m \ge 1$ and

$$a_{n,n+1-2k} = (-1)^{k-1} \sum_{\ell=n+1-2k}^{n+1} (-1)^{n-\ell} 2^{n+1-\ell} (\ell-1)! \binom{\ell}{n+1-2k} S(n+1,\ell)$$

for $0 \le k \le \frac{1}{2}\left[n - \frac{1-(-1)^n}{2}\right]$, where $S(n,k)$ for $n \ge k \ge 1$ stand for the Stirling numbers of the second kind which can be generated by

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}, \quad k \in \mathbb{N}.$$

In Section 3 of this paper, we will pose an open problem about obtaining an alternative and explicit formula

$$a_{n,n-2m+1} = (n+1)! \sum_{\ell=0}^{m-1} (-1)^{m-1-\ell} b_{m,\ell} n^\ell, \quad n \ge 2, \quad 1 \le m \le \frac{1}{2}\left[n - \frac{1-(-1)^n}{2}\right] \tag{4}$$

for derivative polynomials $P_n(x)$ of the tangent function $\tan x$, where $b_{m,\ell}$ is a sequence to be determined.

In the final section of this paper, we give a consequence of Theorem 1 and recommend some papers closely related to derivative polynomials of other elementary and applicable functions.

2 Proof of Theorem 1

Now we start out to simply prove our Theorems 1 as follows.

In [36, Theorem 2.1] and [36, Corollaries 2.1 and 2.2], it was obtained that

$$(\tan x)^{(n)} = (-i)^{n+1} \sum_{k=1}^{n+1} 2^{n+1-k}(k-1)!S(n+1,k)(i\tan x - 1)^k,$$

$$(\tan x)^{(n)} = (\tan x + i) \sum_{k=1}^{n} (2i)^{n-k}k!S(n,k)(\tan x - i)^k,$$

and

$$\begin{aligned}(\tan x)^{(n)} = \sum_{k=0}^{n+1} \Biggl[(-1)^{k+1} \cos\biggl(\frac{n+1+k}{2}\pi\biggr)\\ \times \sum_{\ell=\max\{1,k\}}^{n+1} (-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell)\binom{\ell}{k}\Biggr] \tan^k x. \quad (5)\end{aligned}$$

The identity (5) can be reformulated as

$$\begin{gathered}(\tan x)^{(n)} = -\cos\biggl(\frac{n+1}{2}\pi\biggr) \sum_{\ell=1}^{n+1}(-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell)\\ + \sum_{k=1}^{n+1}\Biggl[(-1)^{k+1} \cos\biggl(\frac{n+1+k}{2}\pi\biggr) \sum_{\ell=k}^{n+1}(-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell)\binom{\ell}{k}\Biggr] \tan^k x.\end{gathered}$$

Consequently, we arrives at

$$\begin{aligned}a_{2m-1,0} &= -\cos\biggl(\frac{2m}{2}\pi\biggr) \sum_{\ell=1}^{2m}(-1)^{2m-\ell-1}2^{2m-\ell}(\ell-1)!S(2m,\ell)\\ &= (-1)^m \sum_{\ell=1}^{2m}(-1)^{\ell}2^{2m-\ell}(\ell-1)!S(2m,\ell)\end{aligned}$$

for $m \ge 1$ and

$$\begin{aligned}a_{n,n+1-2m} &= (-1)^n \cos((n+1-m)\pi)\\ &\quad \sum_{\ell=n+1-2m}^{n+1} (-1)^{n-\ell}2^{n-\ell+1}(\ell-1)!S(n+1,\ell)\binom{\ell}{n+1-2m}\\ &= (-1)^{m-1} \sum_{\ell=n+1-2m}^{n+1} (-1)^{n-\ell}2^{n+1-\ell}(\ell-1)!S(n+1,\ell)\binom{\ell}{n+1-2m}\end{aligned}$$

for $0 \le m \le \frac{1}{2}\bigl[n - \frac{1-(-1)^n}{2}\bigr]$. The proof of Theorem 1 is thus complete.

3 An open problem

Now we would like to propose an open problem as follows.

The equation (2) means that

$$(\tan x)^{(n)} = \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1} \tan^{n-2k+1} x. \tag{6}$$

Differentiating with respect to x on both sides of (6) gives

$$\begin{aligned}
(\tan x)^{(n+1)} &= \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k} x\left(1+\tan^2 x\right) \\
&= \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k} x \\
&\quad + \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k+2} x \\
&= \sum_{k=1}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]+1} a_{n,n-2k+3}(n-2k+3)\tan^{n-2k+2} x \\
&\quad + \sum_{k=0}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} a_{n,n-2k+1}(n-2k+1)\tan^{n-2k+2} x \\
= \sum_{k=1}^{\frac{1}{2}\left[n+\frac{1-(-1)^n}{2}\right]} & [a_{n,n-2k+3}(n-2k+3) + a_{n,n-2k+1}(n-2k+1)]\tan^{n-2k+2} x \\
&+ a_{n,n+1}(n+1)\tan^{n+2} x + a_{n,\frac{1+(-1)^n}{2}}\frac{1+(-1)^n}{2}\tan^{\frac{(-1)^n-1}{2}} x.
\end{aligned}$$

Comparing this with

$$(\tan x)^{(n+1)} = \sum_{k=0}^{\frac{1}{2}\left[n+1+\frac{1+(-1)^n}{2}\right]} a_{n+1,n-2k+2}(\tan x)^{n-2k+2}$$

yields

$$a_{n+1,n+2} = a_{n,n+1}(n+1), \tag{7}$$

$$a_{n+1,\frac{1-(-1)^n}{2}} \tan^{\frac{1-(-1)^n}{2}} x = a_{n,\frac{1+(-1)^n}{2}} \frac{1+(-1)^n}{2} \tan^{\frac{(-1)^n-1}{2}} x, \tag{8}$$

and

$$a_{n+1,n-2k+2} = a_{n,n-2k+3}(n-2k+3) + a_{n,n-2k+1}(n-2k+1) \tag{9}$$

for $n \ge 1$ and $1 \le k \le \frac{1}{2}\left[n + \frac{1-(-1)^n}{2}\right]$.

The derivatives of the tangent function $\tan x$ in (1) means that $a_{0,1} = 1$, $a_{1,2} = 1$, $a_{2,3} = 2$, and $a_{3,4} = 2 \cdot 3$. Combining these values with (7) reveals that $a_{n,n+1} = n!$ for all $n \ge 0$.

The derivatives of the tangent function $\tan x$ in (1) also means that $a_{1,0} = 1$, $a_{2,1} = 2$, and $a_{3,0} = 2$. When $n = 2\ell$ for $\ell \ge 0$, the recurrence relation (8) becomes

$$a_{2\ell+1,0} = a_{2\ell,1}.$$

When $k = 1$, the recurrence relation (9) can be simplified as

$$a_{n+1,n} = a_{n,n+1}(n+1) + a_{n,n-1}(n-1) = a_{n,n-1}(n-1) + (n+1)!$$

for $n \ge 2$. From this recurrence relation, we acquire

$$a_{n,n-1} = \frac{1}{3}(n+1)!, \quad n \ge 2. \tag{10}$$

When $k = 2$, by (10), the recurrence relation (9) can be rearranged as

$$a_{n+1,n-2} = a_{n,n-1}(n-1) + a_{n,n-3}(n-3) = a_{n,n-3}(n-3) + (n-1)\frac{(n+1)!}{3}$$

for $n \ge 4$. Accordingly, it follows that

$$a_{n,n-3} = \frac{5n-8}{90}(n+1)!, \quad n \ge 4. \tag{11}$$

When $k = 3$, by (11), the recurrence relation (9) can be rewritten as

$$a_{n+1,n-4} = a_{n,n-3}(n-3) + a_{n,n-5}(n-5) = a_{n,n-5}(n-5) + (n-3)\frac{5n-8}{90}(n+1)!$$

for $n \ge 6$. Therefore, it follows that

$$a_{n,n-5} = \frac{35n^2 - 203n + 264}{5670}(n+1)!, \quad n \ge 6. \tag{12}$$

Similarly as above processing, we can procure that

$$a_{n,n-7}=\frac{175n^3-2205n^2+8654n-10272}{340200}(n+1)!,\quad n\ge 8, \tag{13}$$

$$a_{n,n-9}=\frac{385n^4-8470n^3+66539n^2-217910n+244704}{11226600}(n+1)!,\quad n\ge 10, \tag{14}$$

and the like. Accordingly, from (10), (11), (12), (13), and (14), we can conclude that

$$\begin{gathered}a_{n,n-2m+1}=(n+1)!\sum_{\ell=0}^{m-1}(-1)^{m-1-\ell}b_{m,\ell}n^{\ell},\\ n\ge 2,\quad 1\le m\le \frac{1}{2}\left[n-\frac{1-(-1)^n}{2}\right].\end{gathered} \tag{15}$$

Substituting this conclusion into (9) leads to

$$\begin{gathered}(n+2)!\sum_{\ell=0}^{k-1}(-1)^{k-1-\ell}b_{k,\ell}(n+1)^{\ell}=(n-2k+3)(n+1)!\sum_{\ell=0}^{k-2}(-1)^{k-2-\ell}b_{k-1,\ell}n^{\ell}\\ +(n-2k+1)(n+1)!\sum_{\ell=0}^{k-1}(-1)^{k-1-\ell}b_{k,\ell}n^{\ell},\\ \sum_{\ell=0}^{k-1}(-1)^{\ell+1}\left[(n+2)(n+1)^{\ell}-(n-2k+1)n^{\ell}\right]b_{k,\ell}\\ =(n-2k+3)\sum_{\ell=0}^{k-2}(-1)^{\ell}n^{\ell}b_{k-1,\ell},\end{gathered}$$

where $n\ge 4$ and $2\le k\le \frac{1}{2}\left[n-\frac{1-(-1)^n}{2}\right]$. Note that the sequence $b_{k,\ell}$ are independent of n.

To the best of our knowledge, we think that it is much difficult to explicitly determine the sequence $b_{m,\ell}$ in (15). Can one present a closed form for the sequence $b_{m,\ell}$ in (15)?

4 Remarks

Finally we comment on Theorem 1 and recommend some references closely related to derivative polynomials of other elementary and applicable functions.

Remark 1 *The expression* (3) *implies an explicit formula*

$$T_{2m-1} = (-1)^m \sum_{\ell=1}^{2m} (-1)^\ell 2^{2m-\ell} (\ell - 1)! S(2m, \ell), \quad m \geq 1$$

for tangent numbers T_{2m-1} *which can be generated by*

$$\tan x = \sum_{k=1}^{\infty} T_{2k-1} \frac{x^{2k-1}}{(2k-1)!}, \quad |x| < \frac{\pi}{2}.$$

For more information on tangent numbers T_{2m-1}*, please refer to* [1, 11, 14, 36] *and the closely related references therein.*

Remark 2 *It is worthwhile to recommending the paper* [2] *which was found on 3 March 2017 by the authors.*

Remark 3 *Except the above-mentioned literature, there are other papers such as* [6, 7, 13, 15, 16, 17, 18, 19, 22, 23, 24, 25, 28, 29, 31, 34, 35, 36, 37] *and the closely related references therein to discuss derivative polynomials of other elementary and applicable functions.*

Acknowledgements

The authors are grateful to the anonymous referees for their careful corrections to the original version of this paper.

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Received: February 8, 2017

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 360–373

DOI: 10.1515/ausm-2017-0027

A natural Galois connection between generalized norms and metrics

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Abstract. Having in mind a well-known connection between norms and metrics on vector spaces, for an additively written group X, we establish a natural Galois connection between functions of X to $\mathbb{R}$ and X^2 to $\mathbb{R}$.

1 Introduction

In this paper, for an additively written group X, we shall consider the sets

$$\mathcal{N} = \mathcal{N}(X) = \mathbb{R}^X \qquad \text{and} \qquad \mathcal{M} = \mathcal{M}(X) = \mathbb{R}^{X^2}.$$

to be equipped with the usual pointwise inequality of real-valued functions.

Moreover, having in mind a well-known connection between norms and metrics on vector spaces, for any $p \in \mathcal{N}$, $d \in \mathcal{M}$ and $x, y \in X$ we define

$$p_d(x) = d(0, x) \qquad \text{and} \qquad d_p(x, y) = p(-x + y).$$

Thus, it can be easily seen that, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}$,

(1) $d_p \leq d \implies p \leq p_d$, (2) $p \leq p_d \implies d_p \leq d_{p_d}$.

2010 Mathematics Subject Classification: O6A15, 20A99, 54E25
Key words and phrases: groups, preseminorms, invariant semimetrics, Galois connections

Moreover, if in particular

$$d(x, y) = |\varphi(x) - \varphi(y)|, \qquad \text{with} \qquad \varphi(x) = x/(1 + |x|),$$

for all $x, y \in \mathbb{R}$, then d is a metric on $\mathbb{R}$ such that $d_{p_d} \not\leq d$, despite that $p = p_{d_p}$ for all $p \in \mathcal{N}$.

Therefore, by defining

$$\mathcal{M}^{\wedge} = \mathcal{M}^{\wedge}(X) = \{\, d \in \mathcal{M}(X) : \quad d_{p_d} \leq d \,\},$$

we can note that the functions, defined by

$$f(p) = d_p \qquad \text{and} \qquad g(d) = p_d$$

for all $p \in \mathcal{N}$ and $d \in \mathcal{M}^{\wedge}$, establish an increasing Galois connection [21, 24] between the posets $\mathcal{N}$ and $\mathcal{M}^{\wedge}$ in the sense that, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}^{\wedge}$, we have

$$f(p) \leq d \iff p \leq g(d).$$

Some very particular Galois connections have also been investigated in Lambek [12] and our former papers [17, 18, 20, 3, 23, 25, 26]. However, to get a proper overview on Galois connections, the interested reader must consult most of the books [1, 2, 9, 7, 4, 5].

To feel the importance of our present Galois connection, note that if in particular $p \in \mathcal{N}$ is a preseminorm [16, 28] on X in the sense that

$$(1)\ \ p(0) \leq 0, \qquad (2)\ \ p(-x) \leq p(x), \qquad (3)\ \ p(x+y) \leq p(x) + p(y)$$

for all $x, y \in X$, then d_p is a left-invariant semimetric on X such that

$$d(p(x), p(y)) = |p(x) - p(y)| \leq d_p(x, y)$$

for all $x, y \in X$.

Conversely, if d is a left-invariant semimetric on X, then p_d is a preseminorm on X such that $d = d_{p_d}$. Therefore, preseminorms and left-invariant semimetrics are equivalent tools in a group. However, in contrast to the opinions of several authors, the former ones, being a function of only one variable, are certainly more convenient tools than the latter ones.

In this respect, it is also worth mentioning that if in particular d is the postman metric [22] on $\mathbb{C}$, i. e.,

$$d(x, y) = 0 \quad \text{if} \quad x = y \qquad \text{and} \qquad d(x, y) = |x| + |y| \quad \text{if} \quad x \neq y.$$

for any $x, y \in \mathbb{C}$, then d is a metric on $\mathbb{C}$ such that $d \in \mathcal{M}^{\wedge}(\mathbb{C})$, but $d \neq d_{p_d}$.

2 Generalized norms and metrics

Notation 1 In the sequel, we shall assume that X is an additively written group, and we shall write

$$\mathcal{N} = \mathcal{N}(X) = \mathbb{R}^{X} \qquad \text{and} \qquad \mathcal{M} = \mathcal{M}(X) = \mathbb{R}^{X^2}.$$

Moreover, on the sets $\mathcal{N}$ and $\mathcal{M}$ we shall consider the usual pointwise inequality of real-valued functions.

Remark 1 Thus, for instance, $\mathcal{N}$ is the space of all functions of X to $\mathbb{R}$. Moreover, since X^2 is also a group, we can note that $\mathcal{M}(X) = \mathcal{N}(X^2)$.

The members of the families $\mathcal{N}$ and $\mathcal{M}$ may be considered as certain generalized norms and metrics on X, respectively. They can be easily connected by the following

Definition 1 For any $d \in \mathcal{M}$, $p \in \mathcal{N}$ and $x, y \in X$, we define

$$p_d(x) = d(0, x) \qquad \text{and} \qquad d_p(x, y) = p(-x + y).$$

Remark 2 Moreover, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}$, we also define

$$f(p) = d_p \qquad \text{and} \qquad g(d) = p_d.$$

Thus, the functions f and g establish a natural connection between $\mathcal{N}$ and $\mathcal{M}$.

By Definition 1, we evidently have the following

Theorem 1 *For any* $p, q \in \mathcal{N}$ *and* $d, \rho \in \mathcal{M}$*,*

(1) $p \leq q \implies d_p \leq d_q$, (2) $d \leq \rho \implies p_d \leq p_\rho$.

Remark 3 Thus, by Remark 2, for any $p, q \in \mathcal{N}$ and $d, \rho \in \mathcal{M}$

(1) $p \leq q \implies f(p) \leq f(q)$, (2) $d \leq \rho \implies g(d) \leq g(\rho)$.

Therefore, the functions f and g are increasing.

Moreover, by using Definition 1, we can also easily prove the following

Theorem 2 *For any* $p \in \mathcal{N}$*, we have*

(1) $p = p_{d_p}$, (2) $d_p = d_{p_{d_p}}$.

Proof. For any $x \in X$, we have

$$p_{d_p}(x) = d_p(0, x) = p(-0 + x) = p(x).$$

Therefore, $d_{p_{d_p}} = p$, and thus (1) is true. Assertion (2) follows from (1). $\square$

Remark 4 By Theorem 2 and Remark 2, for any $p \in \mathcal{N}$ we have

(1) $p = g\big(f(p)\big)$, (2) $f(p) = f\Big(g\big(f(p)\big)\Big)$.

Hence, we at once see that f is injective and g maps the range of f onto $\mathcal{N}$. Moreover, $g \circ f$ and $f \circ g$ are the identity functions of $\mathcal{N}$ and $f[\mathcal{N}]$, respectively.

Now, as an immediate consequence of Theorems 1 and 2, we can also state

Theorem 3 *For any* $p \in \mathcal{N}$ *and* $d \in \mathcal{M}$,

(1) $d_p \leq d \implies p \leq p_d$, (2) $p \leq p_d \implies d_p \leq d_{p_d}$.

Proof. To prove (1), note that if $d_p \leq d$ holds, then by Theorem 1 we also have $p_{d_p} \leq p_d$. Moreover, by Theorem 2, we have $p_{d_p} = p$. Therefore, $p \leq p_d$ also holds. $\square$

Remark 5 By Theorem 3 and Remark 2, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}$

(1) $f(p) \leq d \implies p \leq g(d)$, (2) $p \leq g(d) \implies f(p) \leq f\big(g(d)\big)$.

3 Three important subfamilies of $\mathcal{M}$

Because of Theorem 3, we may naturally introduce the following

Definition 2 Define

$$\mathcal{M}^* = \mathcal{M}^*(X) = \big\{ d \in \mathcal{M}(X) : \quad d = d_{p_d} \big\},$$

$$\mathcal{M}^\wedge = \mathcal{M}^\wedge(X) = \big\{ d \in \mathcal{M}(X) : \quad d_{p_d} \leq d \big\},$$

$$\mathcal{M}^\vee = \mathcal{M}^\vee(X) = \big\{ d \in \mathcal{M}(X) : \quad d \leq d_{p_d} \big\}.$$

Remark 6 Thus, by Remark 2, we have

$$\mathcal{M}^* = \mathcal{M}^*(X) = \{ d \in \mathcal{M} : \quad d = f\big(g(d)\big)\},$$

$$\mathcal{M}^\wedge = \mathcal{M}^\wedge(X) = \{ d \in \mathcal{M} : \quad f\big(g(d)\big) \leq d \},$$

$$\mathcal{M}^\vee = \mathcal{M}^\vee(X) = \{ d \in \mathcal{M} : \quad d \leq f\big(g(d)\big)\}.$$

The importance of the family $\mathcal{M}^\wedge$ is already quite obvious from the following

Theorem 4 *For any* $d \in \mathcal{M}$, *the following assertions are equivalent:*

(1) $d \in \mathcal{M}^\wedge$, (2) $p \leq p_d \implies d_p \leq d$ *for all* $p \in \mathcal{N}$.

Proof. If $p \in \mathcal{N}$ and $p \leq p_d$, then by Theorem 1 we have $d_p \leq d_{p_d}$. Moreover, if in particular (1) holds, then by Definition 2 we have $d_{p_d} \leq d$. Therefore, if (1) holds, then $d_p \leq d$, and thus (2) also holds.

Conversely, if (2) holds, the from the trivial inequality $p_d \leq p_d$ we can already infer that $d_{p_d} \leq d$. Thus, by Definition 2, (1) also holds. □

Remark 7 By Theorem 4 and Remark 2, for any $d \in \mathcal{M}$ the following assertions are equivalent:

(1) $d \in \mathcal{M}^\wedge$, (2) $p \leq g(d) \implies f(p) \leq d$ for all $p \in \mathcal{N}$.

Now, as an immediate consequence of Theorems 3 and 4, we can also state

Theorem 5 *For any* $p \in \mathcal{N}$ *and* $d \in \mathcal{M}^\wedge$, *we have*

$$d_p \leq d \quad \iff \quad p \leq p_d.$$

Remark 8 Thus, by Remark 2, for any $p \in \mathcal{N}$ and $d \in \mathcal{M}^\wedge$ we have

$$f(p) \leq d \quad \iff \quad p \leq g(d).$$

This shows that the function f and the restriction of g to $\mathcal{M}^\wedge$ form an increasing Galois connection [19, 21, 24] between the posets $\mathcal{N}$ and $\mathcal{M}^\wedge$.

Thus, several consequences of Definition 1 can be immediately derived from the theory of Galois connections [1, 2, 9, 7, 4].

However, because of the simplicity of Definition 1, it seems now more convenient to apply some direct proofs.

For instance, by using Definition 2 and Theorem 2, we can easily prove

Theorem 6 *For any* $d \in \mathcal{M}$, *the following assertions are equivalent:*

(1) $d \in \mathcal{M}^*$, (2) $d = d_p$ *for some* $p \in \mathcal{N}$.

Proof. If (1) holds, then by Definition 2 we have $d = d_{p_d}$. Therefore, (2) also holds with $p = p_d$.

Moreover, by Theorem 2, we have $d_p = d_{p_{d_p}}$, and thus by Definition 2 $d_p \in \mathcal{M}^*$ for all $p \in \mathcal{N}$. Therefore, if (2) holds, then (1) also holds. □

Remark 9 From Theorem 6, by Remark 2, we can see that $\mathcal{M}^* = f[\mathcal{N}]$.

4 Some further characterizations of $\mathcal{M}^\wedge$ and $\mathcal{M}^*$

In addition to Theorem 4, we can also prove the following

Theorem 7 *For any* $d \in \mathcal{M}$, *the following assertions are equivalent:*

(1) $d \in \mathcal{M}^\wedge$,

(2) $d(0, y) \leq d(x, x+y)$ *for all* $x, y \in X$,

(3) $d(0, -x+y) \leq d(x, y)$ *for all* $x, y \in X$.

Proof. By Definition 2, (1) means only that $d_{p_d} \leq d$. That is,

$$d_{p_d}(x, y) \leq d(x, y)$$

for all $x\ y \in X$. Hence, by using that

$$d_{p_d}(x, y) = p_d(-x+y) = d(0, -x+y)$$

for all $x, y \in X$, we can see that (1) and (3) are equivalent.

Moreover, if (3) holds, then by writing $x+y$ in place of y, we can see that (2) also holds. While, if (2) holds, then by writing $-x+y$ in place of y we can see that (3) also holds. □

Analogously to Theorem 7, we can also prove the following

Theorem 8 *For any* $d \in \mathcal{M}$, *the following assertions are equivalent:*

(1) $d \in \mathcal{M}^\vee$,

(2) $d(x, x+y) \leq d(0, y)$ *for all* $x, y \in X$,

(3) $d(x, y) \leq d(0, -x+y)$ *for all* $x, y \in X$.

Now, by using that $\mathcal{M}^* = \mathcal{M}^{\wedge} \cap \mathcal{M}^{\vee}$, we can also prove the following

Theorem 9 *For any* $d \in \mathcal{M}$, *the following assertions are equivalent:*

(1) $d \in \mathcal{M}^*$,

(2) $d(0, y) = d(x, x+y)$ *for all* $x, y \in X$,

(3) $d(x, y) = d(0, -x+y)$ *for all* $x, y \in X$,

(4) $d(x, y) = d(z+x, z+y)$ *for all* $x, y, z \in X$,

(5) $d(x, y) \leq d(z+x, z+y)$ *for all* $x, y, z \in X$,

(6) $d(z+x, z+y) \leq d(x, y)$ *for all* $x, y, z \in X$.

Proof. By Theorems 7 and 8, it is clear that (1), (2) and (3) are equivalent. Moreover, if (4) holds, then by writing $-x$ in place of z we can see that (3) also holds.

While, if (3) holds, then we have

$$\begin{aligned} d(z+x, z+y) &= d\big(0, -(z+x)+z+y\big) \\ &= d\big(0, -x-z+z+y\big) = d(0, -x+y) = d(x, y) \end{aligned}$$

for all $x, y, z \in X$. Therefore, (4) also holds.

Now, since (4) trivially implies (5) and (6), it remains to show only that that (5) and (6) also imply (4). For this, note that if for instance (6) holds, then by writing $-z+x$ in place of x and $-z+y$ in place of y, we obtain

$$d(x, y) \leq d(-z+x, -z+y)$$

for all $x, y, z \in X$. Hence, by writing $-z$ in place of z, we can see that (5) also holds. Therefore, we actually have (4). $\square$

Remark 10 The above theorem shows that $\mathcal{M}^*$ is just the family of all left-invariant members of $\mathcal{M}$.

Moreover, by using Theorem 9, we can also prove the following

Theorem 10 *For a symmetric member* d *of* $\mathcal{M}$, *the following assertions are also equivalent:*

(1) $d \in \mathcal{M}^*$, (2) $d(x, 0) = d(z+x, z)$ *for all* $x, z \in X$.

Proof. If (1) holds, then from (4) in Theorem 9, by taking $y = 0$, we can at once see that (2) also holds even if d is not assumed to be symmetric.

While, if (2) holds, then by using the symmetry of d, we can see that

$$d(0, y) = d(y, 0) = d(x + y, x) = d(x, x + y)$$

for all $x, y \in X$. Therefore, by Theorem 9, assertion (1) also holds. □

5 Two illustrating particular metrics

Theorem 11 *Suppose that* X *is a normed space such that*

$$\| u + v \| < \| u \| + \| v \|$$

for some $u, v \in X$ *with* $u + v \neq 0$. *And, for any* $x, y \in X$, *define*

$$d(x, y) = 0 \quad \text{if} \quad x = y \qquad \text{and} \qquad d(x, y) = \| x \| + \| y \| \quad \text{if} \quad x \neq y .$$

Then, d *is a metric on* X *such that*

$$d \in \mathcal{M}^{\wedge}(X) \setminus \mathcal{M}^{\vee}(X).$$

Proof. To prove the latter statement, note that, for any $x, y \in X$ with $y \neq 0$, we have

$$d(0, y) = \| y \| = \| -x + x + y \| \leq \| x \| + \| x + y \| = d(x, x + y) .$$

Hence, since $d(0, 0) \leq d(x, x)$ trivially holds, by Theorem 7 we can see that $d \in \mathcal{M}^{\wedge}(X)$.

Moreover, note that for $x = -u$ and $y = u + v$ we have

$$d(0, y) = \| y \| = \| u + v \| < \| u \| + \| v \| = \| x \| + \| x + y \| = d(x, x + y) .$$

Therefore, by Theorem 8, $d \notin \mathcal{M}^{\vee}(X)$ also holds. □

Remark 11 To be more concrete, note that if for instance $X = \mathbb{R}^2$, and moreover $u = (1, 0)$ and $v = (0, 1)$, then

$$u + v = (1, 1) \neq (0, 0) \qquad \text{and} \qquad \| u + v \| = \sqrt{2} < 2 = \| u \| + \| v \| .$$

Theorem 12 *Suppose that* φ *is an injective function of a group* X *to a normed space* Y. *And, for any* $x, y \in X$, *define*

$$d(x, y) = \| \varphi(x) - \varphi(y) \| .$$

Then, d *is a metric on* X *such that*

(1) $d \in \mathcal{M}^{\wedge}(X)$ *if and only if* $\| \varphi(y) - \varphi(0) \| \leq \| \varphi(x+y) - \varphi(x) \|$ *for all* $x, y \in X$,

(2) $d \in \mathcal{M}^{\vee}(X)$ *if and only if* $\| \varphi(x+y) - \varphi(x) \| \leq \| \varphi(y) - \varphi(0) \|$ *for all* $x, y \in X$.

Proof. To prove (1), note that by Theorem 7 and the definition of d we have $d \in \mathcal{M}^{\wedge}(X)$ if and only if

$$d(0, y) \leq d(x, x+y), \qquad \text{i.e.,} \qquad \| \varphi(0) - \varphi(x) \| \leq \| \varphi(x) - \varphi(x+y) \|$$

for all $x, y \in X$. Therefore, (1) is true. □

Now, as an immediate consequence of this theorem, we can also state

Corollary 1 *Under the assumptions of Theorem 12, we have* $d \in \mathcal{M}^{*}(X)$ *if and only if* $\| \varphi(x+y) - \varphi(x) \| = \| \varphi(y) - \varphi(0) \|$ *for all* $x, y \in X$.

Remark 12 Note that in the above results, because of

$$d(x, y) = \| \varphi(x) - \varphi(y) \| = \| \varphi(x) - \varphi(0) - \big(\varphi(y) - \varphi(0) \big) \| ,$$

we may assume, without a genuine loss of generality, that $\varphi(0) = 0$.

Moreover, by using the notation

$$\Delta_y \varphi(x) = \varphi(x+y) - \varphi(x)$$

for all $x, y \in X$, the definition of d and the condition of Corollary 1 can be reformulated in the forms that

$$d(x, y) = \| \Delta_x \varphi(0) - \Delta_y \varphi(0) \| \qquad \text{and} \qquad \| \Delta_y \varphi(0) \| = \min_{x \in X} \| \Delta_y \varphi(x) \|$$

for all $x, y \in X$.

From Corollary 1, it is clear that in particular we also have

Corollary 2 *If in addition to the assumptions of Theorem 12, the function* φ *is additive, then* $d \in \mathcal{M}^*(X)$ *.*

Remark 13 In this respect, it is noteworthy that if φ is a function of a group X to a normed space Y such that

$$\|\varphi(x+y) - \varphi(x)\| \leq \|\varphi(y)\|$$

for all $x, y \in X$, then by writing $-u$ in place of x and $u+v$ in place of y we obtain

$$\|\varphi(v) - \varphi(-u)\| \leq \|\varphi(u+v)\|$$

for all $u, v \in X$.

Therefore, if in particular φ is odd, then we have

$$\|\varphi(u) + \varphi(v)\| \leq \|\varphi(u+v)\|$$

for all $u, v \in X$. (Note that the latter property already implies that $\varphi(0) = 0$ and φ is odd.)

Moreover, if in particular Y is an inner product space, then by a basic theorem of Maksa and Volkmann [14], we can state that φ is additive. (For some closely related results, see [6, 11, 15, 29, 30, 8, 27, 28].)

Concerning Theorem 12, it is also worth mentioning that Makai [13] proved that there exists a nowhere continuous additive function φ of $\mathbb{R}$ to itself such that $\varphi = \varphi^{-1}$. (For a more general result, see Kuzcma [10, p. 293].)

However, it is now more important to note that, by using Theorem 12, we can also prove the following

Theorem 13 *If* φ *is an injective function of* $\mathbb{R}$ *to a normed space* Y *such that*

$$\lim_{x \to -\infty} \varphi(x) = \alpha \qquad \text{and} \qquad \lim_{x \to +\infty} \varphi(x) = \beta$$

with $\alpha, \beta \in Y$ *such that* $\|\alpha - \varphi(0)\| < \|\alpha - \beta\|$*, and*

$$d(x, y) = \|\varphi(x) - \varphi(y)\|$$

for all $x, y \in \mathbb{R}$*, then* d *is a metric on* $\mathbb{R}$ *such that*

$$d \notin \mathcal{M}^{\wedge}(\mathbb{R}) \cup \mathcal{M}^{\vee}(\mathbb{R}).$$

Proof. To prove the latter statement, note that if $d \in \mathcal{M}^{\wedge}(\mathbb{R})$, then by Theorem 12 we have

$$\|\varphi(y) - \varphi(0)\| \leq \|\varphi(x+y) - \varphi(x)\|$$

for all $x, y \in \mathbb{R}$. Hence, by letting x tend to $+\infty$, we can infer that

$$\|\varphi(y) - \varphi(0)\| \leq 0,$$

and thus $\varphi(y) = \varphi(0)$ for all $y \in \mathbb{R}$. This contradiction proves that $d \notin \mathcal{M}^{\wedge}(\mathbb{R})$.

While, if $d \in \mathcal{M}^{\vee}(\mathbb{R})$, then by Theorem 12 we have

$$\|\varphi(x+y) - \varphi(x)\| \leq \|\varphi(y) - \varphi(0)\|$$

for all $x, y \in \mathbb{R}$. Hence, by letting y tend to $-\infty$, we can infer that

$$\|\alpha - \varphi(x)\| \leq \|\alpha - \varphi(0)\|,$$

for all $x \in \mathbb{R}$. Hence, by letting x tend to $+\infty$, we can infer that

$$\|\alpha - \beta\| \leq \|\alpha - \varphi(0)\|.$$

This contradiction proves that $d \notin \mathcal{M}^{\vee}(\mathbb{R})$. □

Remark 14 To be more concrete, note that if for instance

$$\varphi(x) = x/(1+|x|)$$

for all $x \in \mathbb{R}$, then φ is a strictly increasing function of $\mathbb{R}$ to itself such that $\varphi(0) = 0$,

$$\alpha = \lim_{x \to -\infty} \varphi(x) = -1 \qquad \text{and} \qquad \beta = \lim_{x \to +\infty} \varphi(x) = 1.$$

Therefore, $|\alpha - \varphi(0)| = 1 < 2 = |\alpha - \beta|$.

Acknowledgement

The work of the author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651.

Moreover, the author is indebted to Zsolt Páles for an observation leading to some shortenings of the proofs in Section 2.

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Received: February 2, 2017

Acta Univ. Sapientiae, Mathematica, 9, 2 (2017) 374–376

DOI: 10.1515/ausm-2017-0028

Does another Euclidean plane exist other than the parasphere?

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Abstract. It is shown that for the question of the title the answer is yes. We construct a plane in the hyperbolic space which is Euclidean.

Let us use the symbols on the Figure 9. of Bolyai's Appendix [1] and take a fourth plane (more precisely a half plane) which intersects the ABNM plane at an angle of $\frac{\pi}{2}$ and intersects the APM and the BND half planes.

According to the proof of Bolyai, the half planes APM resp. BND are intersecting each other if the angle between the APM and ABNM planes is $\frac{\pi}{2}$ and the angle between BND and ABNM (half) planes is arbitrarily less than $\frac{\pi}{2}$. Then it follows that the intersection lines of the fourth plane with the APM and BND half planes, respectively, are also intersecting each other. This means that in the fourth plane the Euclidean geometry is valid.

To describe the above construction in more detail, let us take a point R on the AM line and a point Z on the BN line such that the RZ line is perpendicular to BN.

2010 Mathematics Subject Classification: 51M09, 51M15
Key words and phrases: hyperbolic plane, Euclidean geometry

§9

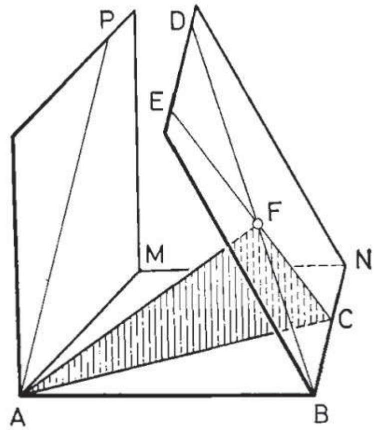

Figure 9

Take a plane containing the RZ line, which intersects the $ABNM$ plane at $\frac{\pi}{2}$ angle (δ plane). Take a half plane containing AM intersecting the $ABNM$ plane at $\frac{\pi}{2}$ angle (this will be denoted as AMP plane or β plane), and a half plane containing BN, which intersects the $ABNM$ plane at an angle less than $\frac{\pi}{2}$ (BND plane or α plane). Then, according to Bolyais proof, the two half planes (α and β half plane) intersect each other and the plane through RZ (δ plane) intersects the other two, and the intersection lines of δ plane with the β plane and α plane, respectively, also intersect each other while the α half planes dihedral angle with the AMBN plane less then $\frac{\pi}{2}$. (As Bolyai has proven, if the α half plane intersects the ABNM plane at an angle arbitrarily smaller than $\frac{\pi}{2}$, then the half planes α and β intersect each other.) Then it follows: in the δ plane the Euclidean geometry seems to be valid. The intersection line of the α and β half planes does not intersect the $ABNM$ plane, because the AM and BN lines are parallel.

Let us denote the intersection line of α and β half planes by K. The intersection lines of δ plane with the α and β half planes, respectively, both intersect K, because as Bolyai implicitly uses the statement: if there are two parallel lines in a plane and from one of the two we draw a perpendicular line in the plane, this latter line will intersect the other line. Also, K is parallel with AM and BN. If K would intersect AM or BN, then these two latter lines would also intersect each other.

Similar results were published recently by Miroslava Antic [2].
A very interesting related paper is published by Zoltán Győrfy [3].

Acknowledgements

It is a pleasure to express my thanks to my friend Éva Gramiss and to my son Ádám Székely for their constant help while writing this article, to my friend Dr. Zoltán Győrfy for discussing the content of this paper and to my friend Dr. Tamás Szabados for reading the final version of this paper and for his valuable advice. Special gratitude is due to Professor Tibor Weszely, the well-known expert of non-Euclidean geometry for reading an earlier version of this paper.

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Received: April 2, 2017

Acta Univ. Sapientiae, Mathematica, **9**, 2 (2017) 377–390

DOI: 10.1515/ausm-2017-0029

Uniqueness of polynomial and differential monomial

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Abstract. In this paper, we discuss the problem of meromorphic functions sharing small function and present one theorem which extend a result of K. S. Charak and Banarasi Lal [16].

1 Introduction and main results

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane.

Definition 1 *Let* $f(z)$ *and* $g(z)$ *be nonconstant meromorphic functions,* $a \in \mathbb{C} \cup \{\infty\}$*. We say that* f *and* g *share the value* a *CM if* $f - a$ *and* $g - a$ *have the same zeros with the same multiplicities.*

Definition 2 *We denote by* $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ *the counting function for zeros of* $f - a$ *with multiplicity* $\leq k$*, and by* $\overline{N}_{k)}\left(r, \frac{1}{(f-a)}\right)$ *the corresponding one for*

2010 Mathematics Subject Classification: 30D35
Key words and phrases: meromorphic function, sharing values, differential monomial, polynomial

which multiplicity is not counted. Let $N_{(k}\left(r,\frac{1}{(f-a)}\right)$ *be the counting function for zeros of* $f-a$ *with multiplicity at least* k *and* $\overline{N}_{(k}\left(r,\frac{1}{(f-a)}\right)$ *the corresponding one for which multiplicity is not counted. Set*

$$N_k\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\ldots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

Definition 3 *For two positive integers* n,p *we define* $\mu_p=\min\{n,p\}$ *and* $\mu_p^*=p+1-\mu_p$. *Then it is clear that*

$$N_p(r,0;f^n)\leq \mu_p N_{\mu_p^*}(r,0;f).$$

Definition 4 [17] *Let* z_0 *be a zero of* $f-a$ *of multiplicity* p *and a zero of* $g-a$ *of multiplicity* q. *We denote by* $\overline{N}_L(r,a;f)$ *the counting function of those* a*-points of* f *and* g *where* $p>q\geq 1$, *by* $N_E^{1)}(r,a;f)$ *the counting function of those* a*-points of* f *and* g *where* $p=q=1$ *and by* $\overline{N}_E^{(2}(r,a;f)$ *the counting function of those* a*-points of* f *and* g *where* $p=q\geq 2$, *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r,a;g), N_E^{1)}(r,a;g), \overline{N}_E^{(2}(r,a;g)$.

Definition 5 [18] *Let* k *be a non-negative integer or infinity. For* $a\in\mathbb{C}\cup\{\infty\}$ *we denote by* $E_k(a;f)$ *the set of all* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m\leq k$ *and* $k+1$ *times if* $m>k$. *If* $E_k(a;f)=E_k(a;g)$, *we say that* f,g *share the value* a *with weight* k.

The definition implies that if f,g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $n(>k)$, where m is not necessarily equal to n.

We write f,g share (a,k) to mean that f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for any integer p, $0\leq p<k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively.

With the notion of weighted sharing of values Lahiri-Sarkar [13] improved the result of Zhang [14]. In [15] Zhang extended the result of Lahiri-Sarkar [13] and replaced the concept of value sharing by small function sharing.

In 2008, Zhang and Lü [12] obtained the following result.

Theorem A *Let* k, n *be the positive integers,* f *be a non-constant meromorphic function, and* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$. *If* f^n *and* $f^{(k)}$ *share* a *IM and*

$$(2k+6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 2k + 12 - n,$$

or f^n *and* $f^{(k)}$ *share* a *CM and*

$$(k+3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > k + 6 - n,$$

then $f^n = f^{(k)}$.

In the same paper, T. Zhang and W. Lü asked the following question:

Question 1 What will happen if f^n and $(f^{(k)})^m$ share a meromorphic function $a(\not\equiv 0, \infty)$ satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$?

S. S. Bhoosnurmath and Kabbur [3] proved:

Theorem B *Let* f *be a non-constant meromorphic function and* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$. *Let* $P[f]$ *be a non-constant differential polynomial in* f. *If* f *and* $P[f]$ *share* a *IM and*

$$(2Q+6)\Theta(\infty, f) + (2 + 3\underline{d}(P))\delta(0, f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7,$$

or if f *and* $P[f]$ *share* a *CM and*

$$3\Theta(\infty, f) + (\underline{d}(P) + 1)\delta(0, f) > 4,$$

then $f \equiv P[f]$.

Banerjee and Majumder [2] considered the weighted sharing of f^n and $(f^m)^{(k)}$ and proved the following result:

Theorem C *Let* f *be a non-constant meromorphic function,* $k, n, m \in N$ *and* l *be a non negative integer. Suppose* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$ *such that* f^n *and* $(f^m)^{(k)}$ *share* (a, l). *If* $l \geq 2$ *and*

$$(k+3)\Theta(\infty, f) + (k+4)\Theta(0, f) > 2k + 7 - n,$$

or $l = 1$ *and*

$$\left(k+\frac{7}{2}\right)\Theta(\infty, f)+\left(k+\frac{9}{2}\right)\Theta(0, f) > 2k+8-n,$$

or $l = 0$ *and*

$$(2k+6)\Theta(\infty, f)+(2k+7)\Theta(0, f) > 4k+13-n,$$

then $f \equiv (f^m)^{(k)}$.

In 2015, Kuldeep S. Charak and Banarasi Lal [16] proved the following result:

Theorem D *Let* f *be a non-constant meromorphic function,* n *be a positive integer and* $a(\not\equiv 0, \infty)$ *be a meromorphic function satisfying* $T(r, a) = o(T(r, f))$ *as* $r \to \infty$. *Let* $P[f]$ *be a non-constant differential polynomial in* f. *Suppose* f^n *and* $P[f]$ *share* (a, l) *such that any one of the following holds*:
(i) *when* $l \geq 2$ *and*

$$(Q+3)\Theta(\infty, f)+2\Theta(0, f)+\overline{d}(P)\delta(0, f) > Q+5+2\overline{d}(P)-\underline{d}(P)-n,$$

(ii) *when* $l = 1$ *and*

$$\left(Q+\frac{7}{2}\right)\Theta(\infty, f)+\frac{5}{2}\Theta(0, f)+\overline{d}(P)\delta(0, f) > Q+6+2\overline{d}(P)-\underline{d}(P)-n,$$

(iii) *when* $l = 0$ *and*

$$(2Q+6)\Theta(\infty, f)+4\Theta(0, f)+2\overline{d}(P)\delta(0, f) > 2Q+10+4\overline{d}(P)-2\underline{d}(P)-n.$$

Then $f^n \equiv P[f]$.

Through the paper we shall assume the following notations. Let

$$\mathcal{P}(\omega) = a_{m+n}\omega^{m+n}+\ldots+a_n\omega^n+\ldots+a_0 = a_{n+m}\prod_{i=1}^{s}(\omega-\omega_{p_i})^{p_i}$$

where $a_j (j = 0, 1, 2, \ldots, n+m-1), a_{n+m} \neq 0$ and $\omega_{p_i} (i = 1, 2, \ldots, s)$ are distinct finite complex numbers and $2 \leq s \leq n+m$ and $p_1, p_2, \ldots, p_s, s \geq 2, n, m$ and k are all positive integers with $\sum_{i=1}^{s} p_i = n+m$. Also let $p > \max_{p \neq p_i, i=1,\ldots,r}\{p_i\}, r = s-1$, where s and r are two positive integers.

Let

$$P(\omega_1) = a_{n+m}\prod_{i=1}^{s-1}(\omega_1 + \omega_p - \omega_{p_i})^{p_i} = b_q\omega_1^q + b_{q-1}\omega_1^{q-1} + ... + b_0,$$

where $a_{n+m} = b_q, \omega_1 = \omega - \omega_p, q = n+m-p$. Therefore, $\mathcal{P}(\omega) = \omega_1^p P(\omega_1)$. Next we assume

$$P(\omega_1) = b_q \prod_{i=1}^{r}(\omega_1 - \alpha_i)^{p_i},$$

where $\alpha_i = \omega_{p_i} - \omega_p, (i = 1, 2, ..., r)$, be distinct zeros of $P(\omega_1)$.

In this paper we will prove one theorem which will improve and generalize Theorem D.

Theorem 1 *Let* $k(\geq 1)$,$n(\geq 1)$,$p(\geq 1)$ *and* $m(\geq 0)$ *be integers and* f *and* $f_1 = f - \omega_p$ *be two nonconstant meromorphic functions and* $M[f]$ *be a differential monomial of degree* d_M *and weight* Γ_M *and* k *is the highest derivative in* $M[f]$. *Let* $\mathcal{P}(z) = a_{m+n}z^{m+n} + ... + a_n z^n + ... + a_0$, $a_{m+n} \neq 0$, *be a polynomial in z of degree m+n such that* $\mathcal{P}(f) = f_1^p P(f_1)$. *Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $\mathcal{P}(f) - a$ *and* $M[f] - a$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$(3 + 2\lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + 2\mu_2 + 3 - p \qquad (1)$$

or $l = 1$ *and*

$$\left(\frac{7}{2} + 2\lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + \mu_2 + 4 + \frac{(m+n) - 3p}{2} \qquad (2)$$

or $l = 0$ *and*

$$(6 + 3\lambda)\Theta(\infty, f) + 2\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 3d_M\delta_{1+k}(0, f) > 3\Gamma_M + \mu_2 + 8 + 2(m+n) - 3p \qquad (3)$$

then $\mathcal{P}(f) \equiv M[f]$.

Following example shows that in Theorem 1 $a(z) \not\equiv 0$ is essential.

Example 1 *Let us take* $f(z) = e^{Lz}$ *where* $L \neq 0, \pm 1$ *and* $\mathcal{P}(f) = f^3, M[f] = f^{(2)}$. *Then* $\mathcal{P}(f)$ *and* $M[f]$ *share* $a = 0(\text{or}, \infty)$. *Here* $m = 0, p = n = 1, \omega_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 3$ *and* $\lambda = 2$. *Also* $\Theta(\infty; f) = 1 = \Theta(0; f)$ *and* $\delta_p(0; f) = 1, \forall q \in \mathbb{N}$. *Thus we see that the deficiency conditions stated in Theorem 1 are satisfied but* $\mathcal{P}(f) \not\equiv M[f]$.

The next example shows that the deficiency conditions stated in Theorem 1 are not necessary.

Example 2 *Let* $f(z) = C\cos z + D\sin z$, $CD \neq 0$. *Then* $\overline{N}(r, f) = S(r, f)$ *and*

$$\overline{N}(r, 0; f) = \overline{N}\left(r, \frac{C + iD}{C - iD}; e^{2iz}\right) \sim T(r, f).$$

Here $m = 0, p = n = 1, \omega_p = 0, d_M = 1, \mu_2 = 1, \Gamma_M = 4k + 1$ *and* $\lambda = 4k$. *Again* $\Theta(\infty, f) = 1$ *and* $\Theta(0, f) = \delta_p(0, f) = 0$. *Let* $m = 0$, *hence* $\mathcal{P}(f) = f$.

Therefore it is clear that $M[f] = f^{(4k)}$, *for* $k \in \mathbb{N}$ *and* $\mathcal{P}(f)$ *share* $a(z)$ *and the deficiency conditions in Theorem 1 are not satisfied, but* $\mathcal{P}(f) \equiv M$.

2 Lemmas

Lemma 1 [17] *For the differential monomial* $M[f]$,

$$N_p(r, 0; M[f]) \leq d_M N_{p+k}(r, 0; g) + \lambda \overline{N}(r, \infty, f) + S(r, f).$$

Lemma 2 [17] *Let* F *and* G *share* $(1, l)$. *Then*

$$\overline{N}_L(r, 1; F) \leq \frac{1}{l + 1}\overline{N}(r, \infty; F) + \frac{1}{l + 1}\overline{N}(r, 0; F) + S(r, F) \text{ if } l \geq 1,$$

and

$$\overline{N}_L(r, 1; F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F) \text{ if } l = 0.$$

Lemma 3 *Let* f *be a non-constant meromorphic function and* $a(z)$ *be a small function of* f. *Let us define* $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ *and* $G = \frac{M[f]}{a}$. *Then* $FG \not\equiv 1$.

Proof. *On contrary suppose* $FG \equiv 1$ *i.e*

$$f_1^p P(f_1) M[f] = a^2.$$

From above it is clear that the function f can't have any zero and poles. Therefore $\overline{N}(r, 0; f) = S(r, f) = \overline{N}(r, \infty; f)$. So by the First Fundamental Theorem and Lemma 1, we have

$$\begin{aligned}(m + n + d_M)T(r, f) &= T\left(r, \frac{a^2}{f_1^p P(f_1) f^{d_M}}\right) + S(r, f) \leq T\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f)\\ &\leq m\left(r, \frac{M[f]}{f^{d_M}}\right) + N\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f)\\ &\leq N\left(r, \frac{M[f]}{f^{d_M}}\right) + S(r, f)\end{aligned}$$

Then using Lemma 2 and from above inequality, we get

$$(m+n+d_M)T(r,f) \le d_M N(r,0;f) + \lambda \overline{N}(r,f) + S(r,f) \le S(r,f),$$

which is not possible.

Lemma 4 [17] *Let* f *be a non-constant meromorphic function and* $a(z)$ *be a small function of* f. *Let* $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ *and* $G = \frac{M[f]}{a}$ *such that* F *and* G *shares* $(1,\infty)$. *Then one of the following cases holds:*
1. $T(r) \le N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_L(r,\infty;F) + \overline{N}_L(r,\infty;G) + S(r)$,
2. $F \equiv G$,
3. $FG \equiv 1$.
where $T(r) = max\{T(r,F), T(r,G)\}$ *and* $S(r) = o(T(r))$, $r \in I$, I *is a set of infinite linear measure of* $r \in \{0,\infty\}$.

3 Proof of the Theorem

Proof.

Let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{M[f]}{a}$. Then $F-1 = \frac{f_1^p P(f_1)-a}{a}$ and $G-1 = \frac{M[f]-a}{a}$. Since $\mathcal{P}(f)$ and $M[f]$ share (a,l), it follows that F and G share $(1,l)$, except the zeros and poles of $a(z)$.

Define

$$\psi = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{4}$$

We consider the following cases:

Case 1. When $\psi \not\equiv 0$. Then from (4), we have $m(r,\psi) = S(r,f)$.
By the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F) + T(r,G) \le 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G}\right) \\ + \overline{N}\left(r,\frac{1}{G-1}\right) - N_0\left(r,\frac{1}{F'}\right) - N_0\left(r,\frac{1}{G'}\right) + S(r,f), \end{aligned} \tag{5}$$

where $N_0\left(r,\frac{1}{F'}\right)$ denotes the counting function of the zeros of F' which are not the zeros of $F(F-1)$ and $N_0\left(r,\frac{1}{G'}\right)$ denotes the counting function of the zeros of G' which are not the zeros of $G(G-1)$.

Subcase 1.1. When $l \geq 1$. Then from (4), we have,

$$\begin{aligned} N_E^{1)}\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{1}{\psi}\right) + S(r,f) \leq T(r,\psi) + S(r,f) = N(r,\psi) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) \\ &\quad + \overline{N}_L\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r,f), \end{aligned}$$

and so

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) = N_E^{1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) \\ &\quad + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) \\ &\quad + \overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r,f). \end{aligned} \tag{6}$$

For $l \geq 2$, we have

$$\begin{aligned} &2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\ &\quad \leq N\left(r, \frac{1}{G-1}\right) + S(r,f). \end{aligned}$$

Thus from (6), we obtain

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \leq \overline{N}(r,f) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) \\ &\quad + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r,f). \end{aligned} \tag{7}$$

Now from Lemma 1, (5) and (7) we obtain

$$\begin{aligned} T(r,F) &\le 3\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + S(r,f) \\ &\le 3\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + S(r,f) \\ &\le 3\overline{N}(r,f) + \mu_2 N_{\mu_2^*}(r,\omega_p;f) + 2d_M N_{1+k}\left(r,\frac{1}{f}\right) + 2\lambda\overline{N}(r,f) + S(r,f) \end{aligned}$$

$$\begin{aligned} (n+m)T(r,f) \le (3+2\lambda)\overline{N}(r,f) + \mu_2 N_{\mu_2^*}(r,\omega_p;f) + (m+n-p)T(r,f) \\ +2d_M N_{1+k}\left(r,\frac{1}{f}\right) + S(r,f) \end{aligned}$$

$$\begin{aligned} &\{(3+2\lambda)\Theta(\infty,f) + \mu_2\delta_{\mu_2^*}(r,\omega_p;f) + 2d_M\delta_{1+k}(0,f)\}T(r,f) \\ &\quad \le (3+2\lambda+2\mu_1+m+n-2p+2d_M)T(r,f) + S(r,f). \end{aligned}$$

$$\begin{aligned} &\{(3+2\lambda)\Theta(\infty,f) + \mu_2\delta_{\mu_2^*}(0,f) + 2d_M\delta_{1+k}(0,f) - \epsilon\}T(r,f) \\ &\qquad\qquad \le (2\Gamma_M + 3 + 2\lambda + 2\mu_2 - p)T(r,f) + S(r,f). \end{aligned}$$

which violates (1).
Next, consider the case when $l = 1$.
First note that

$$\overline{N}_L(r,\frac{1}{F-1}) \le \frac{1}{2}N(r,\frac{1}{F'}|F\neq 0) \le \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}(r,\frac{1}{F}), \tag{8}$$

when $N\left(r,\frac{1}{F'}|F\neq 0\right)$ denotes the zeros of F', that are not the zeros of F.
From (4) and (8), we have

$$\begin{aligned} 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) \\ \le N\left(r,\frac{1}{G-1}\right) + \overline{N}_L\left(r,\frac{1}{F-1}\right) + S(r,f) \\ \le N\left(r,\frac{1}{G-1}\right) + \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \end{aligned} \tag{9}$$

Thus, from (5) and (9) , we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&\le \overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\&+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+T(r,G)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned}\tag{10}$$

From Lemma 1, (5) and (10) we obtain

$$\begin{aligned}T(r,F)&\le 3\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\&\quad+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\&\le \frac{7}{2}\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\&\le \frac{7}{2}\overline{N}(r,f)+\mu_2N_{\mu_2^*}(r,\omega_p;f)+(m+n-p)T(r,f)+2d_MN_{1+k}\left(r,\frac{1}{f}\right)\\&\quad+2\lambda\overline{N}(r,f)+\frac{1}{2}\{\overline{N}(r,\omega_p;f)+(m+n-p)T(r,f)+S(r,f)\}\end{aligned}$$

$$\begin{aligned}(m+n)T(r,f)&\le\left(\frac{7}{2}+2\lambda\right)(1-\Theta(\infty,f))+\mu_2(1-\delta_{\mu_2^*}(\omega_p,f))+\frac{3}{2}(m+n-p)\\&+2d_M(1-\delta_{1+k}(0,f))+\frac{1}{2}(1-\Theta(\omega_p,f))+S(r,f).\end{aligned}$$

$$\begin{aligned}&\left\{\left(\frac{7}{2}+2\lambda\right)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(\omega_p,f))+2d_M\delta_{1+k}(0,f)+\frac{1}{2}\Theta(\omega_p;f)\right\}\\&\le\left(\frac{7}{2}+2\lambda+\mu_2+\frac{3}{2}(m+n-p)+2d_M+\frac{1}{2}-m-n+\epsilon\right)T(r,f)+S(r,f)\\&\le(2\Gamma_M+4+\mu_2+\frac{1}{2}m+\frac{1}{2}n-\frac{3}{2}p)T(r,f)+S(r,f)\end{aligned}$$

which violates (2).

Subcase 1.2. When $l=0$. Then, we have

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)=N_E^{1)}\left(r,\frac{1}{G-1}\right)+S(r,f),$$
$$\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)=\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+S(r,f),$$

and also from (4), we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\le N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)\\
&\quad+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\
&\le N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right)+S(r,f) \qquad (11)\\
&\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)\\
&+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)
\end{aligned}$$

From Lemma 2, (5) and (9), we obtain

$$\begin{aligned}
T(r,F)&\le 3\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&\quad+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)\\
&\le 6\overline{N}(r,f)+\mu_2 N_{\mu_2^*}(r,0;f)+(m+n-p)T(r,f)+3(d_M N_{1+k}(r,0;f)\\
&\quad+\lambda\overline{N}(r,f))+2\{\overline{N}(r,\omega_p;f)+(m+n-p)T(r,f)\}+S(r,f)
\end{aligned}$$

$$\begin{aligned}
(m+n)T(r,f)\le{}&(6+3\lambda)(1-\Theta(\infty,f))+\mu_2(1-\delta_{\mu_2^*}(r,f))+3(m+n-p)\\
&+3d_M(1-\delta_{1+k}(0,f))+2(1-\Theta(\omega_p,f))+S(r,f).
\end{aligned}$$

$$\begin{aligned}
&\{(6+3\lambda)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(r,f)+3d_M\delta_{1+k}(0,f)+2\Theta(\omega_p,f)-\epsilon\}T(r,f)\\
&\quad\le(6+3\lambda+\mu_2+3m+3n-3p+3d_M+2-m-n)T(r,f)+S(r,f)\\
&\quad\le(3\Gamma_M+\mu_2+2m+2n-3p+8-\epsilon)T(r,f)+S(r,f)
\end{aligned}$$

which violates (3).

Case 2. Let $H\equiv 0$.
On Integration we get

$$\frac{1}{G-1}\equiv\frac{A}{F-1}+B,$$

where $A(\neq 0), B$ are complex constants.
It is clear that F and G share $(1,\infty)$. Also by construction of F and G we see that F and G share $(\infty,0)$ also.

So using Lemma 1 and condition (2), we obtain

$$\begin{aligned}
&N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_L(r,\infty;F)\\
&\quad+\overline{N}_L(r,\infty;G)+S(r)\le 2\overline{N}(r,0;F)+2\overline{N}(r,0;G)+3\overline{N}(r,\infty;f)+S(r)\\
&\le 2(\overline{N}(r,\omega_p;f)+(m+n-p)T(r,f))+2(d_M N_{1+k}\left(r,\frac{1}{f}\right)+\lambda\overline{N}(r,f))\\
&\quad+3\overline{N}(r,f)+S(r)\le 2(1-\Theta(\omega_p,f))+2d_M(1-\delta_{1+k}(0,f))\\
&\quad+(3+2\lambda)(1-\Theta(\infty,f))+S(r)+(m+n-p)T(r,f)\\
&\le(3+2\lambda+2d_M+2+m+n-p)-(3+2\lambda+2d_M+2-p)T(r,f)+S(r)\\
&\le(m+n)T(r,f)+S(r)<T(r,F)+S(r).
\end{aligned}\tag{12}$$

Hence inequality (1) of Lemma 4 does not hold. Again in view of Lemma 3, we get $FG \not\equiv 1$. Therefore $F \equiv G$ i.e., $\mathcal{P}(f) \equiv M[f]$. □

Acknowledgement

The author (VH) is greatful to the University Grants Commission(UGC), New Delhi, India for supporting her research work by providing her with a Maulana Azad National Fellowship(MANF).

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