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New characterization of symmetric groups of prime degree

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Abstract. In this paper, will show that a symmetric group of prime degree $p \geq 5$ is recognizable by its order and a special conjugacy class size of $(p-1)!$.

1 Introduction

There are a few finite groups that are determined up to isomorphism solely by their order, such as $\mathbb{Z}_2$ or $\mathbb{Z}_{15}$. Still other finite groups are determined by their order together with other data, such as the number of elements of each order, the structure of the prime graph, the number of order components, the number of Sylow p-subgroups for each prime p, etc. In this paper, we investigate the possibility of characterizing Sym_p by simple conditions when p is prime number.

Our result states that: if p is a prime number, then the groups Sym_p are determined up to isomorphism by their order and a special conjugacy class size of $(p-1)!$. In fact, the main theorem of our paper is as follows.

Theorem A *Let* G *be a group. Then* $G \cong \mathrm{Sym}_p$ *if and only if* $|G| = p!$ *and* G *has a special conjugacy class size of* $(p-1)!$*, where* $p \geq 5$ *is a prime number.*

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For related results, Chen et al. in [14] shows that the projective special linear groups $L_2(p)$ recognizable by their order and a special conjugacy class size, where p is a prime number. As a consequence of their result, they showed that Thompson's conjecture is valid for $L_2(p)$.

Put $N(G) = \{n : G \text{ has a conjugacy class of size } n\}$. By Thompson's conjecture if L is a finite non-abelian simple group, G is a finite group with a trivial center, and $N(G) = N(L)$, then $L \cong G$.

Similar characterizations have been found in [9], [13], [2], and [3] for the groups: sporadic simple groups, simple K_3-groups (a finite simple group is called a simple K_n-group if its order is divisible by exactly n distinct primes), ${}^2D_n(2)$, ${}^2D_{n+1}(2)$ and alternating group of degree p, $p+1$, $p+2$, where p is a prime number.

The *prime graph* of a finite group G, denoted by $\Gamma(G)$, is the graph whose vertices are the prime divisors of G and where prime p is defined to be adjacent to prime q $(\neq p)$ if and only if G contains an element of order pq.

We denote by $\pi(G)$ the set of prime divisors of $|G|$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

We can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. The numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd component of G. Write $OC(G)$ for the set $\{m_1, m_2, \ldots, m_{t(G)}\}$ of order components of G and $T(G)$ for the set of connected components of G.

Williams and Kondrat'ev (see [11, 4, 5]), proved a series of important results on prime graphs. We apply these results to prove Theorem A.

Suppose G and S are two finite groups satisfying $t(\Gamma(S)) \geq 2$, $N(G) = N(S)$, and $Z(G) = 1$. Then it is proved in [7, Lemma 1.4] that $|G| = |S|$. Therefore, if $N(G) = N(\mathrm{Sym}_p)$, and $Z(G) = 1$, then $|G| = |\mathrm{Sym}_p|$. Now, Theorem A follows that $G \cong \mathrm{Sym}_p$. Hence, Thompson's conjecture is valid under a weak condition for the symmetric groups of prime degree.

According to the classification theorem of finite simple groups and [4, 5, 11, 12], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-3 in [1].

We say $p^k \parallel m$ if $p^k \mid m$ and $p^{k+1} \nmid m$. The other notation and terminology in this paper are standard, and the reader is referred to [10] if necessary.

2 Preliminary results

Definition 1 *A group* G *is a finite Frobenius group, if it contains a subgroup* H *such that* $1 \neq H \neq G$ *and* $H \cap H^g = 1$ $H \cap H^g = 1$ *for all* $g \in G - H$. *A subgroup with these properties is called a Frobenius complement of* G. *The Frobenius kernel of* G*, with respect to* H*, is defined by* $K = (G - \bigcup\limits_{g \in G} H^g) \bigcup \{1\}$.

Lemma 1 [6, Theorem 10.3.1] Let G be a Frobenius group with Frobenius kernel H and Frobenius complement K. Then $|K| \equiv 1 \pmod{|H|}$.

Lemma 2 [8, Lemma 8] Let G be a finite group with $t(G) \geq 2$ and N a normal subgroup of G. If N is a π_i-group for some prime graph component of G, and $\mu_1, \mu_2, \ldots, \mu_r$ are some of order components of G but not a π_i-number then $\mu_1 \mu_2 \ldots \mu_r$ is a divisor of $|N| - 1$.

Now we state the following lemma which is proved in [5, Lemma 6], with some differences and classify the simple groups of Lie type with prime odd order component by θ function, which is introduced later.

Lemma 3 *If* L *is a simple group of Lie type and has prime odd order component* $p \geq 17$ *and* $\pi(L)$ *has at most* $\theta(L)$ *prime numbers* t*, where* $\frac{p+1}{2} < t < p$. *Then* $\theta(L) \leq 3$.

Throughout the proof of the above Lemma, we can divide simple groups of Lie type, L, with prime odd order component $p \geq 17$, into the following cases.

(1) $\theta(L) = 0$ if L is isomorphic to $A_{p'-1}(q)$, $A_{p'}(q)$, where $q-1 \mid p'+1$, $A_2(2)$, ${}^2A_{p'-1}(q)$, ${}^2A_{p'}(q)$, where $q+1 \mid p'+1$, ${}^2A_3(2)$, $B_n(q)$, where $n = 2^{m'}$ and q is odd, $B_{p'}(3)$, $C_n(q)$, where $n = 2^{m'}$ or $(n, q) = (p', 3)$, $D_{p'+1}(3)$, $D_{p'}(q)$, for $q = 3$, 5, ${}^2D_n(q)$, for $(n, q) = (2^{m'}, q)$, $(p', 3)$, where $5 \leq p' \neq 2^{m'} + 1$ or $(2^{m'} + 1, 3)$, where $5 \leq p' \neq 2^{m'} + 1$, $G_2(q)$, where $q \equiv \epsilon \pmod 3$ for $\epsilon = \pm 1$, ${}^3D_4(q)$, $E_6(q)$ or ${}^2E_6(q)$;

(2) $\theta(L) = 1$ if L is isomorphic to one of the simple groups $A_1(q)$, where $2 \mid q$, $A_2(4)$, ${}^2A_5(2)$, $C_{p'}(2)$, $D_n(2)$, where $n = p'$ or $p' + 1$, ${}^2D_n(2)$, where $(n, q) = (2^{m'} + 1, 2)$ or $(p' = 2^{m'} + 1, 3)$, where $m' \geq 2$, $E_7(2)$, $E_7(3)$, $F_4(q)$, ${}^2F_4(q)$, where $q = 2^{2n+1} > 2$, or $G_2(q)$, where $3 \mid q$;

(3) $\theta(L) = 2$ if L is isomorphic to the simple groups $A_1(q)$, where $q \equiv \epsilon \pmod 4$ for $\epsilon = \pm 1$, ${}^2B_2(q)$, where $q = 2^{2m'+1} > 2$, or ${}^2G_2(q)$, where $q = 3^{2m'+1} > 3$;

(4) $\theta(L) = 3$ if L is isomorphic to the simple groups $E_8(q)$ or ${}^2E_6(2)$.

Lemma 4 [5, Lemma 1] If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2} < p_i < n$. Here

$$\begin{aligned}
s(n) &= 6 \text{ for } n \geq 48; \\
s(n) &= 5 \text{ for } 42 \leq n \leq 47; \\
s(n) &= 4 \text{ for } 38 \leq n \leq 41; \\
s(n) &= 3 \text{ for } 18 \leq n \leq 37; \\
s(n) &= 2 \text{ for } 14 \leq n \leq 17; \\
s(n) &= 1 \text{ for } 6 \leq n \leq 13.
\end{aligned}$$

In particular, for every natural number $n > 6$, there exists a prime p such that $\frac{n+1}{2} < p < n$, and for every natural number $n > 3$, there exists an odd prime number p such that $n - p < p < n$.

Definition 2 *A group G is called a 2-Frobenius group, if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.*

Lemma 5 [11, Theorem A] Let G be a finite group with more than one prime graph components. Then either G is a Frobenius or a 2-Frobenius group, or G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that such that H and G/K are π_1-groups, K/H is a nonabelian simple group with $\pi_i \subseteq \pi(K)$ for every $i > 1$, and H is a nilpotent group, especially, $K/H \trianglelefteq G/H \trianglelefteq \mathrm{Aut}(K/H)$.

3 Proofs

Proof of the Theorem A. Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

By hypothesis, there exists an element x of order p in G such that $C_G(x) = \langle x \rangle$ and $C_G(x)$ is a Sylow p-subgroup of G. By the Sylow's theorem, we have that $C_G(y) = \langle y \rangle$ for any element y in G of order p. So, $\{p\}$ is a prime graph component of G and $t(G) \geq 2$. In addition, p is the maximal prime divisor of $|G|$ and an odd order component of G. The proof of the Theorem A follows from the following Lemmas.

Lemma 6 *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group and H is a nilpotent group.*

Proof. First, we prove that G is neither Frobenius nor 2-Frobenius. Suppose that G is a Frobenius group with a Frobenius kernel H and a Frobenius complement K. Then $|K| \mid |H| - 1$, by Lemma 2. If $p \in \pi(H)$, then by Lemma 1, $|H| = p$ and $|K| = (p-1)!$. It follows that $(p-1)! \mid p-1$, a contradiction. If $p \in \pi(K)$, then $|K| = p$ and $|H| = (p-1)!$ by Lemma 1, and so $p \mid (p-1)! - 1$. By Wilson's theorem in number theory, we get a contradiction. Hence, G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \{p\} = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, and $|G/K| \mid (p-1)$. Then we have that K/H is of order p. By Lemma 4 and $|G| = p!$, we deduce that there exists $r \in \pi(G)$ such that $\frac{p-1}{2} < r < p$. So, $r \nmid p-1$ and hence, $|G/K| \mid (p-1)$ implies that $r \in \pi(H)$.

Let R be an r-Sylow subgroup of H. Thus $|G| = p!$ shows that R is a cyclic subgroup of order r. On the other hand, the same reasoning as before shows that $R \rtimes P$ is a Frobenius group, where P is a p-Sylow subgroup of K, so Lemma 1 forces $p \mid r-1$ and hence, $p+1 < r$, which is a contradiction. Therefore, G is not a 2-Frobenius group either. Now Lemma 5 imply Lemma 6. $\square$

If K/H has an element of order rq where r and q are primes, then G has also such element. Hence by definition of order components, an odd order component of G must be an odd order component of K/H. Note that $t(K/H) \geq 2$.

Lemma 7 (a) *If* $t \in \pi(H)$*, then* $t \leq \frac{p+1}{2}$*;*

(b) K/H *can not be isomorphic to a sporadic simple group. Moreover, if* K/H *is isomorphic to an alternating simple group, then we must have* $G \cong \mathrm{Sym}_p$*.*

Proof. (a) If t divides $|H|$ where $\frac{p+1}{2} < t < p$, then since H is nilpotent subgroup of G and the order of T, the Sylow t-subgroup of H, is equal to t. By Lemma 2, we must have $p \mid t-1$, which is impossible. Thus $|H|$ is not divisible by the primes t with $\frac{p+1}{2} < t < p$.

(b) We note that if $H \neq 1$, by nilpotency of H, we may assume that H is a t-group for $t \in \pi_1(G)$.

If $K/H \cong J_4$, then $p = 43$. Since $19 \in \pi(G) \backslash \pi(\mathrm{Aut}(J_4))$, then $19 \in \pi(H)$. By Lemma 2, $43 \mid 19^i - 1$ for $i = 1$ or 2, which is impossible.

If $K/H \cong M_{22}$, then $p = 11$. Since $5^2 \mid |G|$ and $5 \parallel |\mathrm{Aut}(M_{22})|$, $5 \in \pi(H)$. So by Lemma 2, we get a contradiction.

If $K/H \cong J_2$, then $p = 7$, but $2^4 \parallel |G|$ and $2^7 \parallel |K/H|$, which is impossible. If K/H is isomorphic to other sporadic simple groups we can view a contradiction similarly.

Now let K/H be isomorphic to an alternating group. By Tables 1 and 2 in [1], K/H must be isomorphic to Alt_n, where $n = p$, $p+1$ or $p+2$. If $n > p$, then $n! > p!$ and it contradicts $|K/H| \mid |G|$. Hence $K/H \cong \mathrm{Alt}_p$. Since $\mathrm{Aut}(K/H) \simeq \mathrm{Sym}_p$, G/H is isomorphic to Alt_p or Sym_p. If $G/H \simeq \mathrm{Sym}_p$, then $H = 1$. Therefore, $G \cong \mathrm{Sym}_p$

If $G/H \simeq \mathrm{Alt}_p$, then $H \cong Z_2$, by Lemma 2, we get a contradiction. □

Lemma 8 *If* $t \in \pi(G/H)$ *and* $\frac{p+1}{2} < t < p$, *then* $t \in \pi(K/H)$.

Proof. It follows from Lemma 7(a), and the proof of Lemma 6(d) in [5]. □

Lemma 9 K/H *can not be isomorphic to a simple group of Lie type.*

Proof.. By Lemmas 8 and 4, we must have $17 \leq p \leq 37$ and $\theta(K/H) \geq 2$. Therefore K/H is isomorphic to one of the following simple groups.

(1) $L_2(q)$, where $q \equiv \epsilon \pmod 4$ for $\epsilon = \pm 1$;

(2) ${}^2B_2(q)$, where $q = 2^{2m'+1} > 2$;

(3) ${}^2G_2(q)$, where $q = 3^{2m'+1} > 3$;

(4) $E_8(q)$ or ${}^2E_6(2)$.

Since one of the odd order components of K/H is equal to p, by Tables 2 and 3 in [1], we must have:

(1) $K/H \cong L_2(17)$, for $p = 17$;

(2) $K/H \cong L_2(19)$, ${}^2G_2(27)$ or ${}^2E_6(2)$, for $p = 19$;

(3) $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29 or 31;

(4) $K/H \cong L_2(37)$, ${}^2G_2(27)$ or ${}^2E_6(2)$, for $p = 37$.

Let $p = 17$. Then $K/H \cong L_2(17)$. Since $13 \notin \pi(L_2(17))$, we get a contradiction by Lemma 8.

Let $p = 19$. If $K/H \cong L_2(19)$, then since $17 \notin \pi(L_2(19))$, we get a contradiction by Lemma 8. Since $37 \mid |{}^2G_2(27)|$, $K/H \ncong {}^2G_2(27)$. For the case $K/H \cong {}^2E_6(2)$, we view a contradiction by $2^{36} \mid |{}^2E_6(2)|$.

Let $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29, 31, or 37. If $p = 37$, then since $23 \notin \pi(L_2(37))$, we get a contradiction by Lemma 8. Similarly, we view a contradiction when $p = 23$, 29, or 31.

Let $p = 37$. If $K/H \cong L_2(37)$, then since $23 \notin \pi(L_2(37))$, we get a contradiction. Since $23 \notin \pi({}^2G_2(27))$, $K/H \ncong {}^2G_2(27)$. If $K/H \cong {}^2E_6(2)$, we view a contradiction by $2^{36} \mid |{}^2E_6(2)|$. □

Now Lemmas 7(b) and 9 imply Theorem A.

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Properties of nearly ω-continuous multifunctions

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Abstract. Erdal Ekici has introduced and studied nearly continuous multifunctions in [5]. The purpose of the present paper is to introduce and study upper and lower nearly ω-continuous multifunctions as a weaker form of upper and lower nearly continuous multifunctions. Basic characterizations, several properties of upper and lower nearly ω-continuous multifunctions are investigated.

1 Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with

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such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. Several characterizations and properties of ω-closed sets were provided in [1], [2], [7] and [8]. Recently, Zorlutuna [14] introduced and studied the concept of ω-continuous multifunctions in topological spaces. In this paper, we introduce and study a new class of multifunction called near ω-continuous multifunctions in topological spaces.

2 Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [8]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [8] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ, denoted by τ_ω. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. We set $\omega O(X, x) = \{A : A \in \tau_\omega$ and $x \in A\}$ the neighborhood system at x in τ_ω. A point x of X is called a θ-cluster [12] point of $S \subset X$ if $\mathrm{Cl}(U) \cap S \neq \varnothing$ for every open subset of X containing x. The set of all θ-cluster points of S is called the θ-closure of S and is denoted by $\mathrm{Cl}_\theta(S)$. A subset S is said to be θ-closed if and only if $S = \mathrm{Cl}_\theta(S)$. The complement of a θ-closed set is said to be a θ-open set. The θ-interior [12] of A is defined as $\mathrm{Int}_\theta(A) = \{x \in X : \mathrm{Cl}(U) \subset A$ for some open set U containing $x\}$. By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A)$ $= \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F^+(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

Definition 1 [14] *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is said to be*

1. *upper* ω*-continuous if* $F^+(V) \in \omega O(X)$ *for each open set* V *of* Y*,*
2. *lower* ω*-continuous if* $F^-(V) \in \omega O(X)$ *for each open set* V *of* Y*.*

Definition 2 [4] *A subset* A *of a topological space* (X, τ) *is said to be* N*-closed if every cover of* A *by regular open sets of* X *has a finite subcover.*

Definition 3 [5] *A function* $F : (X, \tau) \to (Y, \sigma)$ *is said to be:*

1. *upper nearly continuous at a point* $x \in X$ *if for each open set* V *containing* $F(x)$ *and having* N*-closed complement, there exists an open set* U *containing* x *such that* $F(U) \subset V$*.*
2. *lower nearly continuous at a point* $x \in X$ *if for each open set* V *of* Y *meeting* $F(x)$ *and having* N*-closed complement, there exists an open set* U *of* X *containing* x *such that* $F(u) \cap V \neq \emptyset$ *for each* $u \in U$*.*
3. *upper (resp. lower) nearly continuous on* X *if it has this property at every point of* X*.*

3 Upper (Lower) nearly ω-continuous multifunctions

Definition 4 *A function* $F : (X, \tau) \to (Y, \sigma)$ *is said to be:*

1. *upper nearly* ω*-continuous at a point* $x \in X$ *if for each open set* V *containing* $F(x)$ *and having* N*-closed complement, there exists an* ω*-open set* U *containing* x *such that* $F(U) \subset V$*.*
2. *lower nearly* ω*-continuous at a point* $x \in X$ *if for each open set* V *of* Y *meeting* $F(x)$ *and having* N*-closed complement, there exists an* ω*-open set* U *of* X *containing* x *such that* $F(u) \cap V \neq \emptyset$ *for each* $u \in U$*.*
3. *upper (resp. lower) nearly* ω*-continuous on* X *if it has this property at every point of* X*.*

Example 1 *Let* $X = Y = \{a, b, c, d\}$*,* $\tau_X = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ *and* $\sigma_Y = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, b, c\}\}$*. Consider the multifunction* $F : (X, \tau_X) \to (Y, \sigma_Y)$ *defined as follows:* $F(a) = \{c\}, F(b) = \{a, b\}, F(c) = \{d\}, F(d) = \{a, b\}$*. It is easy to see that:* F *is upper (resp. lower) nearly* ω*-continuous on* X*.*

Example 2 *Let $\mathfrak{R}$ be the set of real numbers with the discrete topology τ_d. Consider the multifunction $F : (\mathfrak{R}, \tau_d) \to (\mathfrak{R}, \sigma_d)$ defined as follows: $F(x) = \{x\}$ for all $x \in \mathfrak{R}$. It is easy to see that: F is upper (resp. lower) nearly ω-continuous on X.*

It is clear that every upper (resp. lower) nearly continuous multifunction is upper (resp. lower) nearly ω-continuous multifunction, but the converse is not true in general as shown in the following example.

Example 3 *In the Example 1, F is upper (resp. lower) nearly ω-continuous on X but is not upper (resp. lower) nearly continuous on X*

Theorem 1 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is upper nearly ω-continuous.*
2. *$F^+(V)$ is ω-open for each open set V of Y having N-closed complement.*
3. *$F^-(K)$ is ω-closed for every N-closed and closed subset K of Y.*
4. *$\omega\,\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y having N-closed closure.*
5. *$F^+(\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^+(B))$ for every subset B of Y such that $Y \setminus \mathrm{Int}(B)$ is N-closed.*

Proof. (1)⇒(2): Let $x \in F^+(V)$ and V be any open set of Y having N-closed complement. From (1), there exists an ω-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \underset{x \in F^+(V)}{\cup} U_x$. Since any union of ω-open sets is ω-open, $F^+(V)$ is ω-open in (X, τ).
(2)⇒(3): Let K be any N-closed and closed set of Y. Then by (2), $F^+(Y \setminus K) = X \setminus F^-(K)$ is an ω-open set. Then it is obtained that $F^-(K)$ is an ω-closed set.
(3)⇒(4): Let B be any subset of Y having N-closed closure. By (3), we have $F^-(B) \subset F^-(\mathrm{Cl}(B)) = \omega\,\mathrm{Cl}(F^-(\mathrm{Cl}(B)))$. Hence $\omega\,\mathrm{Cl}(F^-(B)) \subset \omega\,\mathrm{Cl}(F^-(\mathrm{Cl}(B))) = F^-(\mathrm{Cl}(B))$.
(4)⇒(5): Let B be a subset of Y such that $Y \setminus \mathrm{Int}(B)$ is N-closed. Then by (4), we have $X \setminus \omega\,\mathrm{Int}(F^+(B)) = \omega\,\mathrm{Cl}(X \setminus F^+(B)) = \omega\,\mathrm{Cl}(F^-(Y \setminus B)) \subset F^-(\mathrm{Cl}(Y \setminus B)) = F^-(Y \setminus \mathrm{Int}(B)) = X \setminus F^+(\mathrm{Int}(B))$. Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^+(B))$.
(5)⇒(1): Let $x \in X$ and V be any open set of Y containing $F(x)$ and having N-closed complement. Then by (5), $x \in F^+(V) = F^+(\mathrm{Int}(V)) \subset \omega\,\mathrm{Int}(F^+(V))$.

There exists an ω-open set U containing x such that $U \subset F^+(V)$; hence $F(U) \subset V$. This shows that F is upper nearly ω-continuous. □

Theorem 2 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is lower nearly ω-continuous.*
2. *$F^-(V)$ is ω-open for each open set V of Y having N-closed complement.*
3. *$F^+(K)$ is ω-closed for every N-closed and closed set K of Y.*
4. *$\omega\,\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y having N-closed closure.*
5. *$F^-(\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^-(B))$ for every subset B of Y such that $Y \backslash \mathrm{Int}(B)$ is N-closed.*

Proof. The proof is similar to that of Theorem 1. □

Corollary 1 *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper nearly ω-continuous (resp. lower nearly ω-continuous) if $F^-(K)$ is ω-closed (resp. $F^+(K)$ is ω-closed) for every N-closed set K of Y.*

Proof. Let G be any open set of Y having N-closed complement. Then $Y \backslash G$ is N-closed. By the hypothesis, $X \backslash F^+(G) = F^-(Y \backslash G) = \omega\,\mathrm{Int}(F^-(Y \backslash G)) = \omega\,\mathrm{Cl}(X \backslash F^+(G)) = X \backslash \omega\,\mathrm{Int}(F^+(G))$ and hence, $F^+(G) = \omega\,\mathrm{Int}(F^+(G))$. It follows from Theorem 1, that F is upper nearly ω-continuous. The proof of lower nearly ω-continuity is entirely similar. □

Theorem 3 *Let (Y, σ) be a regular space. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *F is upper nearly ω-continuous;*
2. *$F^-(\mathrm{Cl}_\theta(B))$ is an ω-closed set in X for every subset B of Y such that $\mathrm{Cl}_\theta(B)$ is N-closed;*
3. *$F^-(K)$ is an ω-closed set in X for every θ-closed and N-closed set K of Y;*
4. *$F^+(V)$ is an ω-open set in X for every θ-open set V of Y having N-closed complement.*

Proof. (1)⇒(2): Let B be any subset of Y such that $\mathrm{Cl}_\theta(V)$ is N-closed. Then $\mathrm{Cl}_\theta(B)$ is closed and N-closed. By Theorem 1, $F^-(\mathrm{Cl}_\theta(B))$ is an ω-closed set in X.
(2)⇒(3): Let K be any N-closed and θ-closed set of Y. Then $K = \mathrm{Cl}_\theta(K)$ is N-closed. By (2), it follows that $F^-(K)$ is an ω-closed set in X
(3)⇒(4): Let V be any θ-open set of Y having N-closed complement. Then $Y\backslash V$ is θ-closed and N-closed and by (3), $F^-(Y\backslash V) = \omega\,\mathrm{Cl}(F^-(Y\backslash V))$. Therefore, $X\backslash F^+(V) = \omega\,\mathrm{Cl}(X\backslash F^+(V)) = X\backslash \omega\,\mathrm{Int}(F^+(V))$. Then $F^+(V)$ is an ω-open set in X.
(4)⇒(1): Let V be any open set of Y having N-closed complement. Since Y is regular, V is a θ-open set in Y having N-closed complement and by (4), we have $F^+(V)$ is an ω-open set in X. By Theorem 1, F is upper nearly ω-continuous. □

Theorem 4 *Let (Y, σ) be a regular space. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *F is lower nearly ω-continuous;*
2. *$F^+(\mathrm{Cl}_\theta(B))$ is an ω-closed set in X for every subset B of Y such that $\mathrm{Cl}_\theta(B)$ is N-closed;*
3. *$F^+(K)$ is an ω-closed set in X for every θ-closed and N-closed set K of Y;*
4. *$F^-(V)$ is an ω-open set in X for every θ-open set V of Y having N-closed complement.*

Proof. The proof is similar to that of Theorem 3. □

Definition 5 *A subset A of a topological space (X, τ) is said to be:*

(i) *α-regular [9] if for each $a \in A$ and any open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$;*

(ii) *α-paracompact [13] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.*

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the multifunction $\mathrm{Cl}\,F : (X, \tau) \to (Y, \sigma)$ is defined as follows $(\mathrm{Cl}\,F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\omega\,\mathrm{Cl}\,F$.

Lemma 1 [14] *If* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α-*paracompact and* α-*regular for each* $x \in X$, *then for each open set* V *of* Y, $(\mathrm{Cl}\, F)^+(V) = (\omega\, \mathrm{Cl}\, F)^+(V) = F^+(V)$.

Theorem 5 *Let* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α-*regular and* α-*paracompact for each* $x \in X$. *Then* F *is upper nearly* ω-*continuous if and only if* $G : (X, \tau) \to (Y, \sigma)$ *is upper nearly* ω-*continuous, where* G *denotes* $\mathrm{Cl}\, F$ *or* $\omega\, \mathrm{Cl}\, F$.

Proof. Suppose that F is upper nearly ω-continuous multifunction. Let V be any open set of Y having N-closed complement. Then by Lemma 1 and Theorem 1, we have $G^+(V) = F^+(V) = \omega\, \mathrm{Int}(F^+(V)) = \omega\, \mathrm{Int}(G^+(V))$. This shows that G is upper nearly ω-continuous. Conversely, suppose that G is upper nearly ω-continuous. Let V be any open set of Y having N-closed complement. Then by Lemma 1 and Theorem 1, we have $F^+(V) = G^+(V) = \omega\, \mathrm{Int}(G^+(V)) = \omega\, \mathrm{Int}(F^+(V))$. By Theorem 1, F is upper nearly ω-continuous. $\square$

Lemma 2 [14] *If* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α-*paracompact* α-*regular for each* $x \in X$, *then for each open set* V *of* Y, $(\mathrm{Cl}\, F)^-(V) = (\omega\, \mathrm{Cl}\, F)^-(V) = F^-(V)$.

Theorem 6 *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is lower nearly* ω-*continuous if and only if* $G : (X, \tau) \to (Y, \sigma)$ *is lower nearly* ω-*continuous, where* G *denotes* $\mathrm{Cl}\, F$ *or* $\omega\, \mathrm{Cl}\, F$.

Proof. By using Lemma 2, this shown similarly as in Theorem 1. $\square$

Remark 1 *It is well known that every upper (lower)* ω-*continuous multifunction is upper (lower) nearly* ω-*continuous, but the converse is not true in general as we can see in the following example.*

Example 4 *Let* $\mathfrak{R}$ *be the set of real numbers with the finite complement topology* τ_f *and the discrete topology* τ_d. *Consider the multifunction* $F : (\mathfrak{R}, \tau_f) \to (\mathfrak{R}, \tau_d)$, *defined by* $F(x) = \{x\}$. *Observe that* F *is an upper (lower) nearly* ω-*continuous multifunction in* $\mathfrak{R}$ *but* F *is not upper (lower)-continuous multifunction*

Now if we consider some additional condition, we can proof the converse.

Theorem 7 *Let* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* (Y, σ) *has a base of sets having* N-*closed complements. If* F *is lower nearly* ω-*continuous, then* F *is lower* ω-*continuous.*

Proof. Let V be any open set of Y. By the hypothesis, $V = \bigcup_{i \in I} V_i$, where V_i is an open set having N-closed complement for each $i \in I$. By Theorem 1, $F^-(V_i)$ is ω-open in X for each $i \in I$. Moreover, $F^-(V) = F^-(\cup\{V_i : i \in I\}) = \cup\{F^-(V_i) : i \in I\}$. Therefore, we have $F^-(V)$ is ω-open in X. Hence F is lower ω-continuous. □

Suppose that (X, τ), (Y, σ) and (Z, θ) are topological spaces. If $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the composite multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 8 *Let $F : (X, \tau) \to (Y, \sigma)$ and $G : (Y, \sigma) \to (Z, \theta)$ be multifunctions. If F is upper ω-continuous (resp. lower ω-continuous) and G is upper nearly continuous (resp. lower nearly continuous), then $G \circ F : (X, \tau) \to (Z, \theta)$ is upper nearly ω-continuous (resp. lower nearly ω-continuous).*

Proof. Let V be any open set of V having N-closed complement. Since G is upper nearly continuous (resp. lower nearly continuous), by Theorem 2 of [5], $F^+(V)$ (resp. $F^-(V)$) is an open set of y. Since F is upper ω-continuous (resp. lower ω-continuous), $(G \circ F)^+(V) = F^+(G^+(V)) = \omega\,\mathrm{Int}(F^+(G^+(V))) = \omega\,\mathrm{Int}((G \circ F)^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V)) = \omega\,\mathrm{Int}(F^-(G^-(V))) = \omega\,\mathrm{Int}((G \circ F)^-(V))$). By Theorem 1 (resp. Theorem 2), F is upper nearly ω-continuous (resp. lower nearly ω-continuous). □

Definition 6 *A topological space (Y, σ) is said to be N-normal [5] if for each disjoint closed sets K and H of Y, there exist open sets U and V having N-closed complement such that $K \subset U$, $H \subset V$ and $U \cap V = \emptyset$.*

Definition 7 *A topological space (X, τ) is said to be ω-T_2 [2] if for each distinct points $x, y \in X$, there exist ω-open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Theorem 9 *If $F : (X, \tau) \to (Y, \sigma)$ is an upper nearly ω-continuous multifunction satisfying the following conditions:*

1. *$F(x)$ is closed in Y for each $x \in X$,*
2. *$F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$,*
3. *(Y, σ) is an N-normal space,*

then (X, τ) is ω-T_2.

Proof. Let x and y be distinct points of X. Then, we have $F(x) \cap F(y) = \emptyset$. Since $F(x)$ and $F(y)$ are closed and Y is N-normal, there exist disjoint open sets U and V having N-closed complement such that $F(x) \subset U$ and $F(y) \subset V$. By Theorem 1, we obtain, an ω-open set $F^+(U)$ in X containing x and an ω-open set $F^+(V)$ in X containing y and $F^+(U) \cap F^+(V) = \emptyset$. This shows that X is ω-T_2. □

Theorem 10 *Let (X, τ) be a topological space. If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction F form (X, τ) into an N-normal space (Y, σ) satisfying the following conditions:*

1. *$F(x_1)$ and $F(x_2)$ are closed in Y,*

2. *F is upper nearly ω-continuous at x_1 and x_2, and*

3. *$F(x_1) \cap F(x_2) = \emptyset$,*

then (X, τ) is ω-T_2.

Proof. Let x_1 and x_2 be distinct points of X. Then, we have $F(x_1) \cap F(x_2) = \emptyset$. Since $F(x_1)$ and $F(x_2)$ are closed and Y is N-normal, there exist disjoint open sets V_1 and V_2 having N-closed complement such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Since F is upper nearly ω-continuous at x_1 and x_2, there exist U_1 and U_2 ω-open sets in X containing x_1 and x_2 respectively, such that $F(U_1) \subset V_1$ and $F(U_2) \subset V_2$. This implies that $U_1 \cap U_2 = \emptyset$. Hence (X, τ) is an ω-T_2-space. □

Theorem 11 *Let F and G be upper nearly ω-continuous and point closed multifunctions from a topological space X to a N-normal topological space Y. Then the set $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is ω-closed in X.*

Proof. Let $x \in X \backslash A$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions and Y is a N-normal space, it follows that there exists disjoint open sets U and V having N-closed complements containing $F(x)$ and $G(x)$, respectively. Since F and G are upper nearly ω-continuous, then the sets $F^+(U)$ and $G^+(V)$ are open and contain x. Let $H = F + (U) \cup G + (V)$. Then H is an ω-open set containing x and $H \backslash A = \emptyset$. Hence, A is ω-closed in X. □

Definition 8 *A topological space (X, τ) is said to be N-connected [6] if X cannot be written as the union of two disjoint nonempty open sets having N-closed complements.*

Definition 9 *A topological space* (X, τ) *is said to be* ω*-connected* [2] *if* X *cannot be written as the union of two disjoint nonempty* ω*-open sets.*

Theorem 12 *Let* (X, τ) *be a topological space. If* $F : (X, \tau) \to (Y, \sigma)$ *is an upper nearly* ω*-continuous or lower nearly* ω*-continuous surjective multifunction such tat* $F(x)$ *is connected for each* $x \in x$ *and* (X, τ) *is* ω*-connected, then* (Y, σ) *is* N*-connected.*

Proof. Suppose that (Y, σ) is not N-connected. There exist nonempty open sets U and V of Y having N-closed complement such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

1. $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,
2. $F^+(U) \cap F^+(V) = \emptyset$,
3. $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are ω-open sets in X.
(i) In case F is upper nearly ω-continuous by Theorem 1, $F^+(U)$ and $F^+(V)$ are ω-open sets in X.
(ii) In case F is lower nearly ω-continuous by Theorem 2, $F^+(V)$ is ω-closed set in X because U is clopen in (Y, σ), therefore, $F^+(V)$ is ω-open in X. Similarly $F^+(U)$ is ω-open in X. Therefore (X, τ) is not ω-connected. □

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, we define $D^+_{n\omega}(F)$ and $D^-_{n\omega}(F)$ as follows:
$D^+_{n\omega}(F) = \{x \in X : F \text{ is not upper nearly } \omega\text{-continuous at } x\}$.
$D^-_{n\omega}(F) = \{x \in X : F \text{ is not lower nearly } \omega\text{-continuous at } x\}$.

Theorem 13 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties hold:*

$$\begin{aligned} D^+_{n\omega} &= \bigcup_{G \in \sigma NC} \{F^+(G) \backslash \omega \operatorname{Int}(F^+(G))\} \\ &= \bigcup_{B \in iNC} \{F^+(\operatorname{Int}(B)) \backslash \omega \operatorname{Int}(F^+(B))\} \\ &= \bigcup_{B \in NC} \{\omega \operatorname{Cl}(F^-(B)) \backslash F^-(\operatorname{Cl}(B))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\omega \operatorname{Cl}(F^-(H)) \backslash F^-(H)\}, \text{where} \end{aligned}$$

σNC *is the family of all* σ*-open sets of* Y *having* N*-closed complement,*
iNC *is the family of all subsets* B *of* Y *such that* $Y \backslash \mathrm{Int}(B)$ *is* N*-closed,*
NC *is the family of all subsets* B *of* Y *having the* N*-closed closure,*
$\mathcal{F}$ *is the family of all closed and* N*-closed sets of* (Y, σ).

Proof. We shall only proof the first equality and the last equality since the proofs of other are similar to the first.
Let $x \in D^+_{n\omega}(F)$. Then, by Theorem 1, there exists an open set V of Y containing $F(x)$ and having N-closed complement such that $x \in \omega\,\mathrm{Int}(F^+(V))$. Therefore, $x \in F^+(V) \backslash \omega\,\mathrm{Int}(F^+(V)) \subset \bigcup\limits_{G \in \sigma NC} \{F^+(G) \backslash \omega\,\mathrm{Int}(F^+(G))\}$.
Conversely, let $x \in \bigcup\limits_{G \in \sigma NC} \{F^+(G) \backslash \omega\,\mathrm{Int}(F^+(G))\}$. Then there exists an open set V of Y having N-closed complement such that $x \in F^+(V) \backslash \omega\,\mathrm{Int}(F^+(V))$. By Theorem 1, $x \in D^+_{n\omega}(F)$. We prove the last equality.
$\bigcup\limits_{H \in \mathcal{F}} \{\omega\,\mathrm{Cl}(F^-(H)) \backslash F^-(H)\} \subset \bigcup\limits_{B \in NC} \{\omega\,\mathrm{Cl}(F^-(B)) \backslash F^-(\mathrm{Cl}(B))\} = D^+_{n\omega}(F)$.
Conversely, we have $D^+_{n\omega}(F) = \bigcup\limits_{B \in NC} \{\omega\,\mathrm{Cl}(F^-(B)) \cup \bigcup\limits_{H \in \mathcal{F}} \{\omega\,\mathrm{Cl}(F^-(H)) \backslash F^-(H)\}$. □

Theorem 14 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties hold:*

$$\begin{aligned} D^-_{n\omega} &= \bigcup_{G \in \sigma NC} \{F^-(G) \backslash \omega\,\mathrm{Int}(F^-(G))\} \\ &= \bigcup_{B \in iNC} \{F^-(\mathrm{Int}(B)) \backslash \omega\,\mathrm{Int}(F^-(B))\} \\ &= \bigcup_{B \in NC} \{\omega\,\mathrm{Cl}(F^+(B)) \backslash F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\omega\,\mathrm{Cl}(F^+(H)) \backslash F^+(H)\}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 13 □

Definition 10 *Let* (X, τ) *be a bitopological space and* A *be a subset of* X*. The* ω*-frontier of* A*,* $\omega Fr(A)$*, is defined by* $\omega Fr(A) = \omega\,\mathrm{Cl}(A) \cap \omega\,\mathrm{Cl}(X \backslash A) = \omega\,\mathrm{Cl}(A) \backslash \omega\,\mathrm{Int}(A)$.

Theorem 15 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*,* $D^+_n \omega(F)$ *(resp.* $D^-_{n\omega}(F)$*) is identical with the union of* ω*-frontiers of the upper (resp. lower) inverse images of* σ_i *open sets containing (resp. meeting)* $F(x)$ *and having* N*-closed complement.*

Proof. We shall prove the first case since the proof of the second is similar.
Let $x \in D^+_{n\omega}(F)$. Then, there exists an open set V of Y containing $F(x)$ and

having N-closed complement such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every open set U containing x. Then $x \in \omega \operatorname{Cl}(X \backslash F^+(V))$. On the other hand, since $x \in F^+(V) \subset \omega \operatorname{Cl}(F^+(V))$ and hence $x \in \omega \mathrm{Fr}(F^+(V))$. Conversely, suppose that F is upper nearly ω-continuous at $x \in X$. Then, for any open set V of Y containing $F(x)$ and having N-closed complement, there exists an ω-open set containing x such that $F(U) \subset V$; hence $x \in U \subset F^+(V)$. Therefore, we have $x \in U \subset \omega \operatorname{Int}(F^+(V))$. This contradicts to the fact that $x \in \omega \mathrm{Fr}(F^+(V))$. □

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Existence and uniqueness of solution for a class of nonlinear degenerate elliptic equation in weighted Sobolev spaces

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Abstract. In this work we are interested in the existence and uniqueness of solutions for the Navier problem associated to the degenerate nonlinear elliptic equations

$$\begin{aligned}\Delta(\nu(x)\,|\Delta u|^{r-2}\Delta u) &- \sum_{j=1}^{n} D_j\left[\omega_1(x)\mathcal{A}_j(x,u,\nabla u)\right]\\ &+ b(x,u,\nabla u)\,\omega_2(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x), \quad \text{in } \Omega\end{aligned}$$

in the setting of the Weighted Sobolev Spaces.

1 Introduction

In this work we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $X = W^{2,r}(\Omega,\nu) \cap W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (see Definition 4 and Definition 5) for the Navier problem

$$(P)\begin{cases} Lu(x) = f_0(x) - \displaystyle\sum_{j=1}^{n} D_j f_j(x), & \text{in } \Omega \\ u(x) = \Delta u(x) = 0, & \text{on } \partial\Omega \end{cases}$$

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where L is the partial differential operator

$$\begin{aligned} Lu(x) = \Delta(\nu(x)\,|\Delta u|^{r-2}\Delta u) - \sum_{j=1}^{n} D_j\left[\omega_1(x)\mathcal{A}_j(x,u(x),\nabla u(x))\right] \\ + \, b(x,u,\nabla u)\,\omega_2(x) \end{aligned}$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2 and ν are three weight functions, Δ is the Laplacian operator, $1 < p < \infty$ and the functions $\mathcal{A}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ $(j = 1, \ldots, n)$ and $b : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ satisfy the following assumptions:
(H1) The function $x \mapsto \mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$. The function $(\eta,\xi) \mapsto \mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.
(H2) there exists a constant $\theta_1 > 0$ such that

$$[\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\tilde{\eta},\tilde{\xi})].(\xi - \tilde{\xi}) \geq \theta_1\,|\xi - \tilde{\xi}|^p,$$

whenever $\xi, \tilde{\xi}\in\mathbb{R}^n$, $\xi\neq\tilde{\xi}$, $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi), \ldots, \mathcal{A}_n(x,\eta,\xi))$ (where a dot denote here the Euclidian scalar product in $\mathbb{R}^n$).
(H3) $\mathcal{A}(x,\eta,\xi).\,\xi \geq \lambda_1|\xi|^p + \Lambda_1|\eta|^p - g_1(x)|\eta| - g_2(x)|\xi|$, where λ_1 and Λ_1 are nonnegative constants, $g_1/\omega_2 \in L^{p\,'}(\Omega,\omega_2)$ and $g_2/\omega_1 \in L^{p\,'}(\Omega,\omega_1)$.
(H4) $|\mathcal{A}(x,\eta,\xi)| \leq K_1(x) + h_1(x)|\eta|^{p/p\,'} + h_2(x)|\xi|^{p/p\,'}$, where K_1, h_1 and h_2 are nonegative functions, with h_1 and $h_2\in L^\infty(\Omega)$, and $K_1\in L^{p\,'}(\Omega,\omega_1)$ (with $1/p + 1/p\,' = 1$).
(H5) The function $x \mapsto b(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$. The function $(\eta,\xi) \mapsto b(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.
(H6) there exists a constant $\theta_2 > 0$ such that

$$[b(x,\eta,\xi) - b(x,\tilde{\eta},\tilde{\xi})](\eta - \tilde{\eta}) \geq \theta_2\,|\eta - \tilde{\eta}|^p,$$

whenever $\eta, \tilde{\eta}\in\mathbb{R}$, $\eta\neq\tilde{\eta}$.
(H7) $b(x,\eta,\xi)\eta \geq \lambda_2|\xi|^p + \Lambda_2|\eta|^p - g_3(x)\,|\eta| - g_4(x)|\xi|$, where $\lambda_2\geq 0$ and $\Lambda_2 > 0$ are constants, $g_3/\omega_2\in L^{p\,'}(\Omega,\omega_2)$ and $g_4\omega_2/\omega_1\in L^{p\,'}(\Omega,\omega_1)$.
(H8) $|b(x,\eta,\xi)| \leq K_2(x) + h_3(x)|\eta|^{p/p\,'} + h_4(x)|\xi|^a$, where K_2, h_3 and h_4 are nonnegative functions, with $K_2 \in L^{p\,'}(\Omega,\omega_2)$, h_3 and $h_4 \in L^\infty(\Omega)$, and $a = (p-1)/q'$, where $1 < q < \infty$ $(1/q + 1/q\,' = 1)$.
(H9) $\lambda_1 + \lambda_2 > 0$.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1, 2, 4, 8, 13]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [10]). These classes have found many useful applications in harmonic analysis (see [12]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [9]). There are, in fact, many interesting examples of weights (see [8] for p-admissible weights).

In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), & \text{in} \quad \Omega \\ u(x) = 0, & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [7]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), & \text{in} \quad \Omega \\ u(x) = 0, & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [3]), where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [11] and the references therein), and the degenerated p-Laplacian has been studied in [4]. The problem with degenerated p-Laplacian and p-Biharmonic operators

$$\begin{cases} \Delta(\omega(x)|\Delta u|^{p-2}\Delta u) - \mathrm{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \mathrm{div}(G(x)), & \text{in} \quad \Omega \\ u(x) = \Delta u(x) = 0, \quad \text{in} \quad \partial\Omega \end{cases}$$

has been studied by the author in [2].

The following theorem will be proved in section 3.

Theorem 1 *Assume* (H1)-(H9). *If*
(i) $\nu \in A_r$ *and* $\omega_1, \omega_2 \in A_p$ $(1 < p, r, \infty)$, $\omega_1 \leq \omega_2$ *a.e.*, $\omega_2/\omega_1 \in L^q(\Omega, \omega_1)$ $(1 < q < \infty)$,
(ii) $f_0/\omega_2 \in L^{p'}(\Omega, \omega_2)$ *and* $f_j/\omega_1 \in L^{p'}(\Omega, \omega_1)$ $(j = 1, \ldots, n)$.
Then the problem (P) has a unique solution

$$u \in X = W^{2,r}(\Omega, \nu) \cap W_0^{1,p}(\Omega, \omega_1, \omega_2).$$

2 Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [6, 8, 12] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x;2\,r)) \leq C\,\mu(B(x;r))$ for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [8]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [12]).

If $\omega \in A_p$, then $\left(\frac{|E|}{|B|}\right)^p \leq C\frac{\mu(E)}{\mu(B)}$ whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [8]). Therefore, if $\mu(E) = 0$ then $|E| = 0$.

Definition 1 *Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that*

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{loc}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [13]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2 *Let $\Omega \subset \mathbb{R}^n$ be open, k be a nonnegative integer and $\omega \in A_p$ $(1 < p < \infty)$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$ for $1 \leq |\alpha| \leq k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by*

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p\,\omega(x)\,dx + \sum_{1\leq|\alpha|\leq k}\int_\Omega |D^\alpha u(x)|^p\,\omega(x)\,dx\right)^{1/p}. \quad (1)$$

We also define $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{k,p}(\Omega,\omega)}$.

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (1) (see Theorem 2.1.4 in [13]). The spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces.

It is evident that the weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (c_1 and c_2 positive constants), gives nothing new (the space $W_0^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $W_0^{k,p}(\Omega)$). Consequently, we shall be interested above in all such weight functions ω which either vanish in somewhere $\Omega \cup \partial\Omega$ or increase to infinity (or both).

Definition 3 *Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$, and let ω_1 and ω_2 be A_p-weights. We define the weighted Sobolev space $W^{1,p}(\Omega,\omega_1,\omega_2)$ as the set of functions $u \in L^p(\Omega,\omega_2)$ with weak derivatives $D_j u \in L^p(\Omega,\omega_1)$, $j = 1,\dots,n$. The norm of u in $W^{1,p}(\Omega,\omega_1,\omega_2)$ is given by*

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u(x)|^p \omega_2(x)\,dx + \sum_{j=1}^n \int_\Omega |D_j u(x)|^p \omega_1(x)\,dx\right)^{1/p}. \quad (2)$$

The space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2). The dual space of $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the space

$$\begin{aligned}&[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^* = W^{-1,p'}(\Omega,\omega_1,\omega_2)\\ &= \{T = f_0 - \operatorname{div} F : F = (f_1,\dots,f_n),\ \frac{f_0}{\omega_2} \in L^{p'}(\Omega,\omega_2),\ \frac{f_j}{\omega_1} \in L^{p'}(\Omega,\omega_1)\}.\end{aligned}$$

In this article we use the following results.

Theorem 2 *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$, μ-*a.e. on* Ω*;*
(ii) $|u_{m_k}(x)| \le \Phi(x)$, μ-*a.e. on* Ω*;*
(*where* $\mu(E) = \int_E \omega(x)\,dx$).

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [5]. □

Lemma 1 *Let $1 < p < \infty$.*

(a) *There exists a constant* α_p *such that*

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \leq \alpha_p \, |x-y|(|x|+|y|)^{p-2}, \ \forall \, x, y \in \mathbb{R}^n;$$

(b) *There exist two positive constants* β_p, γ_p *such that for every* $x, y \in \mathbb{R}^n$

$$\beta_p \, (|x|+|y|)^{p-2}|x-y|^2 \leq (|x|^{p-2}x - |y|^{p-2}y).(x-y) \leq \gamma_p \, (|x|+|y|)^{p-2}|x-y|^2.$$

Proof. See [3], Proposition 17.2 and Proposition 17.3. □

Lemma 2 *If* $\omega \in A_p$, *then* $\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega} \frac{\mu(E)}{\mu(B)}$, *whenever* B *is a ball in* $\mathbb{R}^n$ *and* E *is a measurable subset of B* (*where* $\mu(E) = \int_E \omega(x) \, dx$).

Proof. See Theorem 15.5 *Strong doubling of* A_p*-weights* in [8]. □

By Lemma 2, if $\mu(E) = 0$ then $|E| = 0$.

Definition 4 *We denote by* $X = W^{2,r}(\Omega, \nu) \cap W_0^{1,p}(\Omega, \omega_1, \omega_2)$ *with the norm*

$$\|u\|_X = \|u\|_{L^p(\Omega,\omega_2)} + \|\nabla u\|_{L^p(\Omega,\omega_1)} + \|\Delta u\|_{L^r(\Omega,\nu)}.$$

Definition 5 *We say that an element* $u \in X$ *is a* (*weak*) *solution of problem (P) if, for all* $\varphi \in X$,

$$\begin{aligned} &\int_\Omega |\Delta u|^{r-2} \, \Delta u \, \Delta \varphi \, \nu \, dx + \sum_{j=1}^n \int_\Omega \omega_1 \, \mathcal{A}_j(x, u(x), \nabla u(x)) D_j \varphi(x) dx \\ &+ \int_\Omega b(x, u, \nabla u) \varphi \, \omega_2 \, dx \\ &= \int_\Omega f_0(x) \varphi(x) dx + \sum_{j=1}^n \int_\Omega f_j(x) D_j \varphi(x) dx. \end{aligned}$$

3 Proof of Theorem 1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3 *Let* $A : X \to X^*$ *be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space* X. *Then for each* $T \in X^*$ *the equation* $Au = T$ *has a solution* $u \in X$.

Proof. See Theorem 26.A in [15]. $\square$

To prove the existence of solutions, we define $B, B_1, B_2, B_3 : X \times X \to \mathbb{R}$ and $T : X \to \mathbb{R}$ by

$$\begin{aligned}
B(u,\varphi) &= B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi),\\
B_1(u,\varphi) &= \sum_{j=1}^{n} \int_\Omega \omega_1\, \mathcal{A}_j(x,u,\nabla u) D_j\varphi\, dx = \int_\Omega \omega_1\, \mathcal{A}(x,u,\nabla u).\nabla\varphi\, dx,\\
B_2(u,\varphi) &= \int_\Omega b(x,u,\nabla u)\, \varphi\, \omega_2\, dx,\\
B_3(u,\varphi) &= \int_\Omega |\Delta u|^{r-2}\Delta u\, \Delta\varphi\, \nu\, dx,\\
T(\varphi) &= \int_\Omega f_0(x)\, \varphi(x)\, dx + \sum_{j=1}^{n} \int_\Omega f_j(x)\, D_j\varphi(x)\, dx.
\end{aligned}$$

Then $u \in X$ is a (weak) solution to problem (P) if for all $\varphi \in X$

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi) = T(\varphi).$$

Step 1. For $j = 1, \ldots, n$ we define the operator $F_j : X \to L^{p'}(\Omega, \omega_1)$ by

$$(F_j u)(x) = \mathcal{A}_j(x, u(x), \nabla u(x)).$$

We now show that operator F_j is bounded and continuous.
(i) Using (H4) and $\omega_1 \leq \omega_2$ we obtain

$$\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_j u(x)|^{p'}\omega_1\, dx = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\omega_1\, dx\\
&\leq \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)^{p'} \omega_1\, dx\\
&\leq C_p \int_\Omega \left[(K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega_1\right] dx \qquad (3)\\
&\leq C_p \left[\int_\Omega K_1^{p'}\, \omega_1\, dx + \int_\Omega h_1^{p'}|u|^p\, \omega_2\, dx\right.\\
&\qquad \left. + \int_\Omega h_2^{p'}|\nabla u|^p \omega_1\, dx\right],
\end{aligned}$$

where the constant C_p depends only on p. We have,

$$\int_\Omega h_1^{p'}|u|^p\, \omega_2\, dx \leq \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^p\, \omega_2\, dx \leq \|h_1\|^{p'}_{L^\infty(\Omega)}\, \|u\|^p_X,$$

and

$$\int_\Omega h_2^{p'} |\nabla u|^p \omega_1 \, dx \le \|h_2\|_{L^\infty(\Omega)}^{p'} \int_\Omega |\nabla u|^p \, \omega_1 \, dx \le \|h_2\|_{L^\infty(\Omega)}^{p'} \|u\|_X^p.$$

Therefore, in (3) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega_1)} \le C_p \left(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)}) \, \|u\|_X^{p/p'} \right),$$

and hence the boundedness.

(ii) Let $u_m \to u$ in X as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega, \omega_1)$. We will apply the Lebesgue Dominated Theorem. If $u_m \to u$ in X, then $u_m \to u$ in $L^p(\Omega, \omega_2)$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega, \omega_1)$. Using Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and two functions $\Phi_1 \in L^p(\Omega, \omega_1)$ and $\Phi_2 \in L^p(\Omega, \omega_2)$ such that

$$\begin{aligned}
&u_{m_k}(x) \to u(x), \ \ \mu_2 - \text{a.e. in } \Omega, \\
&|u_{m_k}(x)| \le \Phi_2(x), \ \ \mu_2 - \text{a.e. in } \Omega, \\
&|\nabla u_{m_k}(x)| \to |\nabla u(x)|, \ \ \mu_1 - \text{a.e. in } \Omega, \\
&|\nabla u_{m_k}(x)| \le \Phi_1(x), \ \ \mu_1 - \text{a.e. in } \Omega.
\end{aligned}$$

where $\mu_i = \int_E \omega_i(x) \, dx$ $(i = 1, 2)$. Hence, using (H4) and $\omega_1 \le \omega_2$, we obtain

$$\begin{aligned}
&\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_1)}^{p'} = \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'} \omega_1 \, dx \\
&= \int_\Omega |\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{A}_j(x, u, \nabla u)|^{p'} \, \omega_1 \, dx \\
&\le C_p \int_\Omega \left(|\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x, u, \nabla u)|^{p'} \right) \omega_1 \, dx \\
&\le C_p \left[\int_\Omega \left(K_1 + h_1 |u_{m_k}|^{p/p'} + h_2 |\nabla u_{m_k}|^{p/p'} \right)^{p'} \omega_1 \, dx \right. \\
&\quad \left. + \int_\Omega \left(K_1 + h_1 |u|^{p/p'} + h_2 |\nabla u|^{p/p'} \right)^{p'} \omega_1 \, dx \right] \\
&\le 2\, C_p \int_\Omega \left(K_1 + h_1 \Phi_2^{p/p'} + h_2 \Phi_1^{p/p'} \right)^{p'} \omega_1 \, dx \\
&\le 2\, C_p \left[\int_\Omega K_1^{p'} \, \omega_1 \, dx + \int_\Omega h_1^{p'} \Phi_2^p \, \omega_1 \, dx + \int_\Omega h_2^{p'} \Phi_1^p \, \omega_1 \, dx \right]
\end{aligned}$$

$$\le 2\,C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)}+\|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_2^p\,\omega_2\,dx+\|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_1^p\,\omega_1\,dx\right]$$
$$\le 2\,C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)}+\|h_1\|^{p'}_{L^\infty(\Omega)}\,\|\Phi_2\|^p_{L^p(\Omega,\omega_2)}+\|h_2\|^{p'}_{L^\infty(\Omega)}\|\Phi_1\|^p_{L^p(\Omega,\omega_1)}\right].$$

By condition (H1), we have

$$F_j u_{m_k}(x)=\mathcal{A}_j(x,u_{m_k}(x),\nabla u_{m_k}(x))\to\mathcal{A}_j(x,u(x),\nabla u(x))=F_j u(x),$$

as $m_k\to+\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain $\|F_j u_{m_k}-F_j u\|_{L^{p'}(\Omega,\omega_1)}\to 0$, that is, $F_j u_{m_k}\to F_j u$ in $L^{p'}(\Omega,\omega_1)$. By the Convergence Principle in Banach spaces (see Proposition 10.13 in [14]), we have

$$F_j u_m\to F_j u \ \text{ in } \ L^{p'}(\Omega,\omega_1). \tag{4}$$

Step 2. Define the operator $G: X\to L^{r'}(\Omega,\nu)$, $(Gu)(x)=|\Delta u(x)|^{r-2}\Delta u(x)$. We also have that the operator G is continuous and bounded. In fact:
(i) We have

$$\begin{aligned}\|Gu\|^{r'}_{L^{r'}(\Omega,\nu)}&=\int_\Omega \big|\,|\Delta u|^{r-2}\,\Delta u\big|^{r'}\,\nu\,dx\\&=\int_\Omega |\Delta u|^{(r-2)\,r'}|\Delta u|^{r'}\,\nu\,dx=\int_\Omega |\Delta u|^r\,\nu\,dx\\&\le \|u\|^r_X.\end{aligned}$$

Hence, $\|Gu\|_{L^{r'}(\Omega,\nu)}\le\|u\|_X^{r/r'}$.
(ii) If $u_m\to u$ in X then $\Delta u_m\to\Delta u$ in $L^r(\Omega,\nu)$. By Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_3\in L^r(\Omega,\nu)$ such that

$$\begin{aligned}&\Delta u_{m_k}(x)\to\Delta u(x),\ \ \mu_3-a.e.\text{ in }\Omega\\&|\Delta u_{m_k}(x)|\le\Phi_3(x),\ \ \mu_3-a.e.\text{ in }\Omega,\end{aligned}$$

where $\mu_3(E)=\int_E \nu(x)\,dx$. Hence, using Lemma 1(a), we obtain, if $r\neq 2$

$$\begin{aligned}\|Gu_{m_k}-Gu\|^{r'}_{L^{r'}(\Omega,\nu)}&=\int_\Omega |\,Gu_{m_k}-Gu|^{r'}\,\nu\,dx\\&=\int_\Omega \Big|\,|\Delta u_{m_k}|^{r-2}\,\Delta u_{m_k}-|\,\Delta u|^{r-2}\,\Delta u\Big|^{r'}\,\nu\,dx\\&\le\int_\Omega\Big[\alpha_r\,|\Delta u_{m_k}-\Delta u|\,(\,|\Delta u_{m_k}|+|\Delta u|)^{(r-2)}\Big]^{r'}\,\nu\,dx\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_r^{r'} \int_\Omega |\Delta u_{m_k} - \Delta u|^{r'} (2\,\Phi_3)^{(r-2)\,r'}\, \nu\, dx \\
&\leq \alpha_r^{r'}\, 2^{(r-2)r'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^r\, \nu\, dx \right)^{r'/r} \times \left(\int_\Omega \Phi_3^{(r-2)\,r\,r'/(r-r')}\, \nu\, dx \right)^{(r-r')/r} \\
&\leq \alpha_r^{r'}\, 2^{(r-2)\,r'}\, \|u_{m_k} - u\|_X^{r'}\, \|\Phi\|_{L^r(\Omega,\nu)}^{r-r'},
\end{aligned}$$

since $(r-2)\,r\,r'/(r-r') = r$ if $r \neq 2$. If $r = 2$, we have

$$\|Gu_{m_k} - Gu\|^2_{L^2(\Omega,\nu)} = \int_\Omega |\Delta u_{m_k} - \Delta u|^2\, \nu\, dx \leq \|u_{m_k} - u\|_X^2.$$

Therefore (for $1 < r < \infty$), by the Lebesgue Dominated Convergence Theorem, we obtain $\|Gu_{m_k} - Gu\|_{L^r(\Omega,\nu)} \to 0$, that is, $Gu_{m_k} \to Gu$ in $L^{r'}(\Omega,\nu)$. By the Convergence Principle in Banach spaces (see Proposition 10.13 in [14]), we have

$$Gu_m \to Gu \text{ in } L^{r'}(\Omega,\nu). \tag{5}$$

Step 3. We define $H : X \to L^{p'}(\Omega,\omega_2)$ by $(Hu)(x) = b(x, u(x), \nabla u(x))$. We also have that the operator H is continuous and bounded. In fact,
(i) Using (H8) and $\alpha = (p-1)/q'$, we obtain

$$\begin{aligned}
\|Hu\|^{p'}_{L^{p'}(\Omega,\omega_2)} &= \int_\Omega |Hu|^{p'}\, \omega_2\, dx = \int_\Omega |b(x,u,\nabla u)|^{p'}\, \omega_2\, dx \\
&\leq \int_\Omega \left(K_2 + h_3|u|^{p/p'} + h_4|\nabla u|^{\alpha} \right)^{p'} \omega_2\, dx \\
&\leq C_p \int_\Omega \left[(K_2^{p'} + h_3^{p'}|u|^p + h_4^{p'}|\nabla u|^{\alpha\,p'})\omega_2 \right] dx \\
&= C_p \left[\int_\Omega K_2^{p'}\, \omega_2\, dx + \int_\Omega h_3^{p'}|u|^p\, \omega_2\, dx + \int_\Omega h_4^{p'}|\nabla u|^{\alpha\,p'} \omega_2\, dx. \right.
\end{aligned}$$

We have

$$\int_\Omega h_3^{p'}|u|^p \omega_2 \leq \|h_3\|^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^p\, \omega_2\, dx \leq \|h_3\|^{p'}_{L^\infty(\Omega)} \|u\|_X^p,$$

and

$$\begin{aligned}
\int_\Omega h_4^{p'} |\nabla u|^{a\,p'}\,\omega_2\,dx &\leq \|h_4\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^{p/q'}\,\omega_2\,dx \\
&= \|h_4\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^{p/q'}\,\frac{\omega_2}{\omega_1}\,\omega_1\,dx \\
&\leq \|h_4\|^{p'}_{L^\infty(\Omega)} \left(\int_\Omega |\nabla u|^p\,\omega_1\,dx\right)^{1/q'} \left(\int_\Omega \left(\frac{\omega_1}{\omega_2}\right)^q \omega_1\,dx\right)^{1/q} \\
&\leq \|h_4\|^{p'}_{L^\infty(\Omega)} \|u\|_X^{p/q'}\,\|\omega_2/\omega_1\|_{L^q(\Omega,\omega_1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Hu\|_{L^{p'}(\Omega,\omega_2)} \leq C_p \Big[&\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)} \|u\|_X^{p/p'} \\
&+ \|h_4\|_{L^\infty(\Omega)} \|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1} \|u\|_X^{(p-1)/q'} \Big].
\end{aligned}$$

(ii) By the same argument used in Step 1 (ii)(and condition (H5)), we obtain analogously, if $u_m \to u$ in X then

$$Hu_m \to Hu, \quad \text{in} \quad L^{p'}(\Omega,\omega_2). \tag{6}$$

Step 4. We also have

$$\begin{aligned}
|T(\varphi)| &\leq \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx \\
&= \int_\Omega \frac{|f_0|}{\omega_2}|\varphi|\omega_2\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_1}|D_j\varphi|\,\omega_1\,dx \\
&\leq \|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)} \|\varphi\|_{L^p(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \|D_j\varphi\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(\|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right) \|\varphi\|_X.
\end{aligned}$$

Moreover, using (H4), (H8) and the Hölder inequality, we also have

$$\begin{aligned}
|B(u,\varphi)| &\leq |B_1(u,\varphi)| + |B_2(u,\varphi)| + |B_3(u,\varphi)| \\
&\leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega_1\,dx + \int_\Omega |\Delta u|^{r-2}\,|\Delta u|\,|\Delta\varphi|\,\nu\,dx \\
&\quad + \int_\Omega |b(x,u,\nabla u)|\,|\varphi|\,\omega_2\,dx.
\end{aligned} \tag{7}$$

In (7) we have

$$
\begin{aligned}
&\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_1\,dx\\
&\leq \int_\Omega \Big(K_1 + h_1|u|^{p/p\,'} + h_2|\nabla u|^{p/p\,'}\Big)|\nabla\varphi|\,\omega_1\,dx\\
&\leq \|K_1\|_{L^{p\,'}(\Omega,\omega_1)}\|\nabla\varphi\|_{L^p(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p\,'}_{L^p(\Omega,\omega_2)}\|\nabla\varphi\|_{L^p(\Omega,\omega_1)}\\
&\quad + \|h_2\|_{L^\infty(\Omega)}\|\nabla u\|^{p/p\,'}_{L^p(\Omega,\omega_1)}\|\nabla\varphi\|_{L^p(\Omega,\omega_1)}\\
&\leq \Big(\|K_1\|_{L^{p\,'}(\Omega,\omega_1)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{p/p\,'}\Big)\|\varphi\|_X,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega |\,\Delta u|^{r-2}\,|\,\Delta u|\,|\,\Delta\,\varphi|\,\nu\,dx = \int_\Omega |\,\Delta u|^{r-1}\,|\,\Delta\,\varphi|\,\nu\,dx\\
&\leq \left(\int_\Omega |\,\Delta u|^r\,\nu\,dx\right)^{1/r\,'}\left(\int_\Omega |\,\Delta\varphi|^r\,\nu\,dx\right)^{1/r}\\
&\leq \|u\|_X^{r/r\,'}\|\varphi\|_X,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega |b(x,u,\nabla u)|\,|\varphi|\,\omega_2\,dx \leq \int_\Omega \Big(K_2 + h_3|u|^{p/p\,'} + h_4|\nabla u|^a\Big)|\varphi|\,\omega_2\,dx\\
&\leq \int_\Omega K_2\,|\varphi|\,\omega_2\,dx + \|h_3\|_{L^\infty(\Omega)}\int_\Omega |u|^{p/p\,'}|\varphi|\,\omega_2\,dx\\
&\quad + \|h_4\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^a|\varphi|\,\omega_2\,dx\\
&\leq \Big(\|K_2\|_{L^{p\,'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p\,'}\Big)\|\varphi\|_X\\
&\quad + \|h_4\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^{a\,p\,'}\omega_2\,dx\right)^{1/p\,'}\left(\int_\Omega |\varphi|^p\,\omega_2\,dx\right)^{1/p}\\
&\leq \Big(\|K_2\|_{L^{p\,'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p\,'}\Big)\|\varphi\|_X\\
&\quad + \|h_4\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^{p/q\,'}\frac{\omega_2}{\omega_1}\,\omega_1\,dx\right)^{1/p\,'}\|\varphi\|_X\\
&\leq \Big(\|K_2\|_{L^{p\,'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p\,'}\Big)\|\varphi\|_X
\end{aligned}
$$

$$+\|h_4\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^p\,\omega_1\,dx\right)^{1/(p'q')}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|\varphi\|_X$$

$$\leq \Big(\|K_2\|_{L^{p'}(\Omega,\omega_2)}+\|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p'}$$

$$+\|h_4\|_{L^\infty(\Omega)}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|u\|_X^{p/(q'p')}\Big)\|\varphi\|_X.$$

Therefore, in (7) we obtain, for all $u, \varphi \in X$

$$|B(u,\varphi)| \leq \Big[\|K_1\|_{L^{p'}(\Omega,\omega_1)}+\|K_2\|_{L^{p'}(\Omega,\omega_2)}$$

$$+\ (\|h_1\|_{L^\infty(\Omega)}+\|h_2\|_{L^\infty(\Omega)}+\|h_3\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}$$

$$+\|u\|_X^{r/r'}+\|h_4\|_{L^\infty(\Omega)}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|u\|_X^{p/(p'q')}\Big]\|\varphi\|_X.$$

Since $B(u,.)$ is linear, for each $u \in X$, there exists a linear and continuous operator $A : X \to X^*$ such that $\langle Au, \varphi\rangle = B(u,\varphi)$, for all $u,\ \varphi \in X$ (where $\langle f, x\rangle$ denotes the value of the linear functional f at the point x) and

$$\|Au\|_* \leq \|K_1\|_{L^{p'}(\Omega,\omega_1)}+\|K_2\|_{L^{p'}(\Omega,\omega_2)}$$

$$+(\|h_1\|_{L^\infty(\Omega)}+\|h_2\|_{L^\infty(\Omega)}+\|h_3\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}$$

$$+\|u\|^{r/r'}+\|h_4\|_{L^\infty(\Omega)}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|u\|_X^{p/(p'q')}.$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T,\ u \in X.$$

Step 5. Using condition (H2), (H6) and Lemma 1(b), we have

$$\langle Au_1 - Au_2, u_1 - u_2\rangle = B(u_1, u_1-u_2) - B(u_2, u_1-u_2)$$

$$= \int_\Omega \omega_1\,\mathcal{A}(x,u_1,\nabla u_1).\nabla(u_1-u_2)\,dx + \int_\Omega |\Delta u_1|^{r-2}\,\Delta u_1\,\Delta(u_1-u_2)\,\nu\,dx$$

$$+\int_\Omega b(x,u_1,\nabla u_1)(u_1-u_2)\,\omega_2\,dx - \int_\Omega b(x,u_2,\nabla u_2)(u_1-u_2)\,\omega_2\,dx$$

$$-\int_\Omega \omega_1\,\mathcal{A}(x,u_2,\nabla u_2).\nabla(u_1-u_2)\,dx - \int_\Omega |\Delta u_2|^{r-2}\,\Delta u_2\,\Delta(u_1-u_2)\,\nu\,dx$$

$$= \int_\Omega \omega_1\left(\mathcal{A}(x,u_1,\nabla u_1) - \mathcal{A}(x,u_2,\nabla u_2)\right).\nabla(u_1-u_2)\,dx$$

$$
\begin{aligned}
&+\int_\Omega (|\Delta u_1|^{r-2}\Delta u_1 - |\Delta u_2|^{r-2}\Delta u_2)\,\Delta(u_1-u_2)\,\nu\,dx\\
&+\int_\Omega (b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2))(u_1-u_2)\,\omega_2\,dx\\
\geq{}& \theta_1\int_\Omega \omega_1\,|\nabla(u_1-u_2)|^p\,dx + \beta_r\int_\Omega (|\Delta u_1|+|\Delta u_2|)^{r-2}\,|\Delta u_1-\Delta u_2|^2\,\nu\,dx\\
&+\theta_2\int_\Omega |u_1-u_2|^p\omega_2\,dx\\
\geq{}& \theta_1\int_\Omega \omega_1\,|\nabla(u_1-u_2)|^p\,dx + \beta_r\int_\Omega (|\Delta u_1-\Delta u_2|)^{r-2}\,|\Delta u_1-\Delta u_2|^2\,\nu\,dx\\
&+\theta_2\int_\Omega |u_1-u_2|^p\omega_2\,dx\\
={}& \theta_1\int_\Omega \omega_1\,|\nabla(u_1-u_2)|^p\,dx + \beta_r\int_\Omega |\Delta u_1-\Delta u_2|^r\,\nu\,dx\\
&+\theta_2\int_\Omega |u_1-u_2|^p\omega_2\,dx\ \geq 0.
\end{aligned}
$$

Therefore, the operator A is monotone. Moreover, using (H3), (H7), (H9) and $\omega_1 \leq \omega_2$, we obtain

$$
\begin{aligned}
\langle Au,u\rangle ={}& B(u,u) = B_1(u,u)+B_2(u,u)+B_3(u,u)\\
={}& \int_\Omega \omega_1\,\mathcal{A}(x,u,\nabla u).\nabla u\,dx + \int_\Omega |\Delta u|^{r-2}\,\Delta u\,\Delta u\,\nu\,dx\\
&+\int_\Omega b(x,u,\nabla u)\,u\,\omega_2\,dx\\
\geq{}& \int_\Omega (\lambda_1|\nabla u|^p+\Lambda_1|u|^p-g_1|u|-g_2|\nabla u|)\,\omega_1\,dx + \int_\Omega |\Delta u|^r\,\nu\,dx\\
&+\int_\Omega (\lambda_2|\nabla u|^p+\Lambda_2|u|^p-g_3|u|-g_4|\nabla u|)\,\omega_2\,dx\\
\geq{}& (\lambda_1+\lambda_2)\int_\Omega |\nabla u|^p\,\omega_1\,dx + \int_\Omega |\Delta u|^r\,\nu\,dx + \Lambda_2\int_\Omega |u|^p\,\omega_2\,dx\\
&-\int_\Omega g_1|u|^p\,\omega_1\,dx - \int_\Omega g_2|\nabla u|^p\,\omega_1\,dx - \int_\Omega g_3|u|\omega_2\,dx - \int_\Omega g_4|\nabla u|\,\omega_2\,dx\\
\geq{}& \gamma\left(\|u\|^p_{L^p(\Omega,\omega_2)} + \|\nabla u\|^p_{L^p(\Omega,\omega_1)} + \|\Delta u\|^r_{L^r(\Omega,\nu)}\right) - \gamma_1\,\|u\|_X,
\end{aligned}
$$

where $\gamma = \min\{\lambda_1 + \lambda_2, \Lambda_2, 1\}$ and

$$\gamma_1 = \|g_1/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \|g_2/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \|g_3/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \|g_4\omega_2/\omega_1\|_{L^{p'}(\Omega,\omega_1)}.$$

Hence, since $1 < p, r < \infty$, we have

$$\frac{\langle Au, u\rangle}{\|u\|_X} \to +\infty, \text{ as } \|u\|_X \to +\infty,$$

that is, A is coercive (using that $\displaystyle\lim_{t+s+a\to\infty} \frac{t^p + s^p + a^r}{t+s+a} = \infty$, with $t > 0, s > 0$ and $a > 0$).

Step 6. We need to show that the operator A is continuous. Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}
&|B_1(u_m, \varphi) - B_1(u, \varphi)| \\
&\le \sum_{j=1}^{n} \int_\Omega |\mathcal{A}_j(x, u_m, \nabla u_m) - \mathcal{A}_j(x, u, \nabla u)||D_j\varphi|\, \omega_1\, dx \\
&= \sum_{j=1}^{n} \int_\Omega |F_j u_m - F_j u||D_j\varphi|\, \omega_1\, dx \\
&\le \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} \|D_j\varphi\|_{L^p(\Omega,\omega_1)} \\
&\le \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} \|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}
&|B_3(u_m, \varphi) - B_3(u, \varphi)| \\
&= \left| \int_\Omega |\Delta u_m|^{r-2} \Delta u_m\, \Delta\varphi\, \nu\, dx - \int_\Omega |\Delta u|^{r-2} \Delta u\, \Delta\varphi\, \nu\, dx \right| \\
&\le \int_\Omega \left| |\Delta u_m|^{r-2}\, \Delta u_m - |\Delta u|^{r-2} \Delta u \right| |\Delta\varphi|\, \nu\, dx \\
&= \int_\Omega |G u_m - G u|\, |\Delta\varphi|\, \nu\, dx \\
&\le \|G u_m - G u\|_{L^{r'}(\Omega,\nu)}\, \|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned} |B_2(u_m,\varphi)-B_2(u,\varphi)| &\leq \int_\Omega |b(x,u_m,\nabla u_m)-b(x,u,\nabla u)|\,|\varphi|\,\omega_2\,dx \\ &= \int_\Omega |Hu_m-Hu||\varphi|\,\omega_2\,dx \\ &\leq \|Hu_m-Hu\|_{L^{p'}(\Omega,\omega_2)}\|\varphi\|_X, \end{aligned}$$

for all $\varphi \in X$. Hence,

$$\begin{aligned} &|B(u_m,\varphi)-B(u,\varphi)| \\ &\leq |B_1(u_m,\varphi)-B_1(u,\varphi)|+|B_2(u_m,\varphi)-B_2(u,\varphi)|+|B_3(u_m,\varphi)-B_3(u,\varphi)| \\ &\leq \Bigg[\sum_{j=1}^n \|F_j u_m-F_j u\|_{L^{p'}(\Omega,\omega_1)}+\|Gu_m-Gu\|_{L^{r'}(\Omega,\nu)} \\ &\quad +\|Hu_m-Hu\|_{L^{p'}(\Omega,\omega_2)}\Bigg]\|\varphi\|_X. \end{aligned}$$

Then we obtain

$$\begin{aligned} \|Au_m-Au\|_* \leq \sum_{j=1}^n &\|F_j u_m-F_j u\|_{L^{p'}(\Omega,\omega_1)}+\|Gu_m-Gu\|_{L^{r'}(\Omega,\nu)} \\ &+\|Hu_m-Hu\|_{L^{p'}(\Omega,\omega_2)}. \end{aligned}$$

Therefore, using (4), (5) and (6) we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous (and this implies that A is hemicontinuous).

Therefore, by Theorem 3, the operator equation $Au=T$ has a solution $u\in X$ and it is a solution for problem (P).

Step 7. Let us now prove the uniqueness of the solution.

Suppose that $u_1,u_2\in X$ are two solutions of problem (P). Then

$$\begin{aligned} &\int_\Omega |\Delta u_i|^{r-2}\Delta u_i\,\Delta\varphi\,\nu\,dx+\int_\Omega \omega_1\,\mathcal{A}(x,u_i,\nabla u_i).\nabla\varphi\,dx \\ &\quad +\int_\Omega b(x,u_i,\nabla u_i)\,\varphi\,\omega_2\,dx \\ &= \int_\Omega f_0\,\varphi\,dx+\sum_{j=1}^n\int_\Omega f_j\,D_j\varphi\,dx, \end{aligned}$$

for all $\varphi \in X$, and $i = 1, 2$. Hence, we obtain

$$\int_\Omega \Big(|\Delta u_1|^{r-2}\Delta u_1 - |\Delta u_2|^{r-2}\Delta u_2 \Big) \Delta\varphi\, \nu\, dx + \int_\Omega \omega_1 \Big(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2) \Big).\nabla\varphi\, dx + \int_\Omega \Big(b(x, u_1, \nabla u_1) - b(x, u_2, \nabla u_2) \Big) \varphi\, \omega_2\, dx = 0.$$

In particular, for $\varphi = u_1 - u_2 \in X$ we have, by (H2), (H7) and Lemma 1(b),

$$\begin{aligned} 0 &= \int_\Omega \Big(|\Delta u_1|^{r-2}\Delta u_1 - |\Delta u_2|^{r-2}\Delta u_2 \Big)(\Delta u_1 - \Delta u_2)\, \nu\, dx \\ &\quad + \int_\Omega \omega_1 \Big(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2) \Big).(\nabla u_1 - \nabla u_2)\, dx \\ &\quad + \int_\Omega \Big(b(x, u_1, \nabla u_1) - b(x, u_2, \nabla u_2) \Big)(u - 1 - u_2)\, \omega_2\, dx \\ &\geq \beta_r \int_\Omega |\Delta u_1 - \Delta u_2|^r\, \nu\, dx + \theta_1 \int_\Omega |\nabla u_1 - \nabla u_2|^p\, \omega_1\, dx \\ &\quad + \theta_2 \int_\Omega |u_1 - u_2|^p\, \omega_2\, dx. \end{aligned}$$

Hence $\|u_1 - u_2\|_{L^p(\Omega,\omega_2)} = \|\nabla u_1 - \nabla u_2\|_{L^p(\Omega,\omega_1)} = \|\Delta u_1 - \Delta u_2\|_{L^r(\Omega,\nu)} = 0$. Since $u_1, u_2 \in X$, then $u_1 = u_2$ μ_2 a.e. Therefore, by Lemma 2, $u_1 = u_2$ a.e.

Example 1 *Let* $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. *Consider the weight functions* ω_1, ω_2 *and* ν, $\omega_1(x, y) = (x^2 + y^2)^{-1/4}$, $\omega_2(x, y) = (x^2 + y^2)^{-1/2}$ *and* $\nu(x, y) = (x^2 + y^2)^{-1/6}$ *(we have* $\omega_1, \omega_2 \in A_2$ $(p = 2)$ *and* $\nu \in A_3$ $(r = 3)$*), and the functions* $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ *and* $b : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} \mathcal{A}((x, y), \eta, \xi) &= h_2(x, y)\ \xi, \\ b((x, y), \eta, \xi) &= \eta\, (\cos^2(xy) + 1), \end{aligned}$$

where $h(x, y) = 2\, e^{(x^2+y^2)}$. *Let us consider the partial differential operator*

$$\begin{aligned} Lu(x, y) &= \Delta((x^2 + y^2)^{-1/6} |\Delta u|\, \Delta u) - \operatorname{div}((x^2 + y^2)^{-1/4}\, \mathcal{A}((x, y), u, \nabla u)) \\ &\quad + (x^2 + y^2)^{-1/2}\, b(x, u, \nabla u). \end{aligned}$$

Therefore, by Theorem 1, the problem

$$(P)\begin{cases} Lu(x) = \dfrac{\cos(xy)}{\sqrt{x^2+y^2}} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right), & \text{in } \Omega \\ u(x) = \Delta u(x) = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in X = W^{2,3}(\Omega, \nu) \cap W_0^{1,2}(\Omega, \omega_1, \omega_2)$.

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An extension of a variant of d'Alemberts functional equation on compact groups

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Abstract. All paper is related with the non-zero continuous solutions $f : G \to \mathbb{C}$ of the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G,$$

where σ, τ are continuous automorphism or continuous anti-automorphism defined on a compact group G and possibly non-abelian, such that $\sigma^2 = \tau^2 = id$. The solutions are given in terms of unitary characters of G.

1 Introduction

Let G be a compact group, let σ, τ be continuous automorphism or continuous anti-automorphism such that $\sigma^2 = \tau^2 = id$. We consider the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G, \tag{1}$$

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where $f : G \to \mathbb{C}$ is the function to determine. This equation, in the case where G is abelian, has been studied by many authors (see, e.g., Shin'ya [7, Corollary 3.12], and Stetkær [8, Theorem 14.9]). Eq. (1) is a generalization of the following variant of d'Alembert's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G, \tag{2}$$

which was introduced and solved on semi-groups by Stetkær in [9]. Some information, applications and numerous references concerning (2), d'Alembert's functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in \mathbb{R}, \tag{3}$$

and their further generalizations can be found e.g. in ([5, 3, 4, 1, 8, 9, 10, 11]).

The purpose of the present paper is to solve the functional equation (1) in the case where G is a compact group and possibly non-abelian. Our approach uses the harmonic analysis and the representation theory on compact groups. We note that the idea of using Fourier analysis for solving (1) goes back to [2].

Throughout the rest of this paper, G is a compact group with identity element e. By solutions (resp. representations), we always mean continuous solutions (resp. continuous representations).

2 Preliminaries

In this section, we set up some notation and conventions and briefly review some fundamental facts in Fourier analysis which will be used later.

Let dx denotes the normalized Haar measure on G. Let $\hat{G}$ stand for the set of equivalence classes irreducible unitary representations of G. For $[\pi] \in \hat{G}$, the notation d_π denotes the dimension of the representation space of π and $\mathcal{E}_\pi = \text{span}\{\sqrt{\pi_{ij}} : i, j = 1, \dots, d_\pi\}$ the linear span of the matrix elements of π. For $f \in L^2(G)$, the Fourier transform of f is defined by

$$\hat{f}(\pi) = \int_G f(x)\pi(x)^{-1}dx \in M_{d_\pi}(\mathbb{C}) \text{ for all } \quad [\pi] \in \hat{G},$$

where $M_{d_\pi}(\mathbb{C})$ is the space of all $d_\pi \times d_\pi$ complex matrix.

As usual, the left and right regular representations of G in $L^2(G)$ are defined by

$$(L_y f)(x) = f(y^{-1}x) \quad \text{and} \quad (R_y f)(x) = f(yx),$$

respectively, where $f \in Ł^2(G)$ and $x, y \in G$.

The following properties will be useful later

$$\widehat{(L_y f)}(\pi) = \hat{f}(\pi)\pi(y)^{-1} \text{ and } \widehat{(R_y f)}(\pi) = \pi(y)\hat{f}(\pi)$$

for all $y \in G$, and $\pi \in \hat{G}$.

3 Main result

The following Lemmas will be used in the proof of Theorem 1.

Lemma 1 *Let* G *be a compact group and* π *be a unitary irreducible representation of* G*. Suppose every* $x \in G$*, there is* $c_x \in \mathbb{C}$ *such that*

$$\pi(\sigma(x)) + \pi(\tau(x)) = c_x I_{d_\pi}, \tag{4}$$

then $d_\pi = 1$.

Proof. Let $(\mathcal{H}; \langle\ ,\ \rangle)$ denote the complex Hilbert space on which the representation π acts. We will consider two cases, $\pi \circ \sigma \simeq \pi \circ \tau$ or not.

In the first case. From (4) we get that

$$\pi(\sigma(x))_{ij} + \pi(\tau(x))_{ij} = 0 \quad \text{for } i \neq j, \ \ 1 \leq i, j \leq d_\pi, \quad x \in G.$$

Since $\pi \circ \sigma \not\simeq \pi \circ \tau$ we have $\mathcal{E}_{\pi\circ\sigma} \perp \mathcal{E}_{\pi\circ\tau}$. Hence $(\pi \circ \sigma)_{ij} = 0$ for $i \neq j$, so $\pi \circ \sigma$ is a diagonal matrix. Since $\pi \circ \sigma$ is irreducible we have $d_\pi = 1$.

In the second case, i.e., $\pi \circ \sigma \simeq \pi \circ \tau$, there exists a unitary operator T on $\mathcal{H}$ such that

$$\pi \circ \sigma(x) = T^* \pi \circ \tau(x) T, \quad x \in G.$$

Since T is a unitary matrix, by the spectral theorem for normal operators applied to T, we infer that T is diagonalizable. Then $\mathcal{H}$ has an orthonormal basis $(e_1, e_2, \dots, e_{d_\pi})$ consisting of eigenvectors of T. We write $Te_i = \lambda_i e_i$ where $\lambda_i \in \mathbb{C}$ for $i = 1, 2, \dots, d_\pi$. Actually $|\lambda_i| = 1$, because T is unitary. For any $i = 1, 2, \dots, d_\pi$, we compute that

$$\begin{aligned}
(\pi \circ \sigma(x))_{ii} &= \langle \pi \circ \sigma(x) e_i, e_i \rangle = \langle (T)^* \pi(\tau(x)) e_i, e_i \rangle \\
&= \langle \pi(\tau(x)) Te_i, Te_i \rangle = \langle \lambda_i \pi(\tau(x)) e_i, \lambda_i e_i \rangle \\
&= \lambda_i \overline{\lambda_i} \langle \pi(\tau(x)) e_i, e_i \rangle = |\lambda_i|^2 (\pi \circ \tau(x))_{ii} = (\pi \circ \tau(x))_{ii},
\end{aligned}$$

for all $x \in G$. From (4), we infer that

$$2(\pi \circ \tau(x))_{ii} = 2f(x), \tag{5}$$

for all $i = 1, \ldots, d_\pi$ and $x \in G$. Then $d_\pi = 1$. Indeed, if $d_\pi > 1$, then (5) implies that $(\pi \circ \tau)_{ii} = (\pi \circ \tau)_{11}$ for all $i = 2 \ldots, d_\pi$. But if you use Schur's orthogonality relations which say $\frac{1}{d_\pi}(\pi \circ \tau)_{ii}$ is an orthonormal basis, we get a contradiction. Then $d_\pi = 1$. □

Lemma 2 *Let $f : G \to \mathbb{C}$ be a non-zero solution of (1). Then exists $[\pi] \in \widehat{G}$ such that $\widehat{f}(\pi)$ is invertible.*

Proof. Reformulate (3) to

$$2f(x)f = R_{\sigma(x)}f + L_{\tau(x^{-1})}f, \quad x \in G.$$

Taking the Fourier transform to the last equation and using the identities given in section 2, we have

$$\widehat{f}(\pi)\pi(\tau(x)) + \pi(\sigma(x))\widehat{f}(\pi) = 2f(x)\widehat{f}(\pi), \quad x \in G. \tag{6}$$

Since $f \not\equiv 0$, there exists $[\pi] \in \widehat{G}$ with $\widehat{f}(\pi) \neq 0$. Now, let ν be a vector in $\ker \widehat{f}(\pi)$. From (6), we infer that $\widehat{f}(\pi)\pi(\tau(x))\nu = 0$ for all $x \in G$, this implies that $\widehat{f}(\pi)\pi(x)\nu = 0$ for all $x \in G$. So $\pi(x) \ker \widehat{f}(\pi) \subset \ker \widehat{f}(\pi)$ for all $x \in G$. Since π is irreducible and $\widehat{f}(\pi) \neq 0$, we have $\ker \widehat{f}(\pi) = \{0\}$. This implies that $\widehat{f}(\pi)$ is bijective, thus invertible as a matrix. □

Lemma 3 *Let $f : G \to \mathbb{C}$ be a non-zero solution of (1). Then f is central.*

Proof. Using Lemma 2 and equality (6), we see that there exists $[\pi] \in \widehat{G}$ such that

$$\pi(\sigma(x)) + \widehat{f}(\pi)^{-1}\pi(\tau(x))\widehat{f}(\pi) = 2f(x)I_{d_\pi}, \quad x \in G. \tag{7}$$

Taking the trace on both sides of (7) we obtain that

$$tr(\pi(\sigma(x))) + tr(\pi(\tau(x))) = 2d_\pi f(x), \quad x \in G,$$

which abbreviates to

$$f(x) = \frac{1}{2d_\pi}(tr(\pi(\sigma(x))) + tr(\pi(\tau(x)))), \quad x \in G. \tag{8}$$

Each terms on the right hand side of (8) is a central function, because trace is a central function. Hence f is central. □

By help of the previous lemmas, we now describe the complete solution of (1) on an arbitrary compact group. It is clear that $f \equiv 0$ is a solution of (1), so in the following theorem we are only concerned with the non-zero solutions.

Theorem 1 *The non-zero solutions* $f : G \to \mathbb{C}$ *of* (1) *are the functions of the form* $f = (\chi + \chi \circ \sigma \circ \tau)/2$, *where* $\chi : G \to \mathbb{C}$ *is a character such that:*

1. $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$, *and*
2. χ *is* σ*-even and/or* τ*-even (i.e.,* $\chi \circ \sigma = \chi$ *and/or* $\chi \circ \tau = \chi$*).*

Proof. We have f is central. This implies that $\widehat{f}(\pi)$ is an intertwining operator for π. But π is irreducible, so $\widehat{f}(\pi) = \mu I_{d_\pi}$ for some $\mu \in \mathbb{C}$ by Schur's lemma.

Actually $\mu \neq 0$, because $\widehat{f}(\pi) \neq 0$. Now Eq. (7) coalesce into

$$\pi(\sigma(x)) + \pi(\tau(x)) = 2f(x) I_{d_\pi}, \qquad x \in G. \tag{9}$$

From $d_\pi = 1$, we see that π is a unitary character, say $\pi = \chi$, so

$$f = \frac{\chi \circ \sigma + \chi \circ \tau}{2}.$$

If $\chi \circ \sigma = \chi \circ \tau$, then letting $\chi := \chi \circ \sigma$ we have $f = \chi$. Substituting $f = \chi$ into (1) we get that $\chi \circ \sigma + \chi \circ \tau = 2\chi$. So $\chi = \chi \circ \sigma = \chi \circ \tau$. Then f has the desired form.

If $\chi \circ \sigma \neq \chi \circ \tau$, substituting $f = (\chi \circ \sigma + \chi \circ \tau)/2$ into (1) we find after a reduction that

$$\begin{aligned} &\chi \circ \sigma(x)[\chi(y) + \chi \circ \sigma \circ \tau(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] + \chi \circ \tau(x)[\chi \circ \tau \circ \sigma(y) \\ &+ \chi \circ \tau \circ \tau(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] = 0 \end{aligned}$$

for all $x, y \in G$. Since $\chi \circ \sigma \neq \chi \circ \tau$ we get from the theory of multiplicative functions (see for instance [9, Theorem 3.18]) that both terms are 0, so

$$\begin{cases} \chi \circ \sigma(x)[\chi(y) + \chi \circ \sigma \circ \tau(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] = 0 \\ \chi \circ \tau(x)[\chi \circ \tau \circ \sigma(y) + \chi(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] = 0 \end{cases} \tag{10}$$

for all $x, y \in G$. Since $\chi \circ \sigma \neq \chi \circ \tau$ at least one of $\chi \circ \sigma$ and $\chi \circ \tau$ is not zero. We have $\chi \circ \sigma \neq 0$ and $\chi \circ \tau \neq 0$. From (1), we have

$$\chi \circ \sigma + \chi \circ \tau = \chi + \chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma + \chi.$$

Using $\chi + \chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma + \chi$ and the fact that $\chi \circ \sigma \neq \chi \circ \tau$, we see that $\chi = \chi$ and $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$. Thus

$$\chi \circ \tau = \chi \circ \sigma \circ \tau \circ \sigma.$$

We now use $\chi \circ \sigma + \chi \circ \tau = \chi + \chi \circ \sigma \circ \tau$, we get that χ is σ-even or τ-even.

Finally, in view of these cases we deduce that f has the form stated in Theorem 1. □

Similarly to Theorem 1, we can get the solution of functional equation (1) when σ, τ are continuous anti-automorphism such that $\sigma^2 = \tau^2 = id$.

Theorem 2 *The non-zero solutions $f : G \to \mathbb{C}$ of (1) are the functions of the form $f = (\chi + \chi \circ \sigma \circ \tau)/2$, where $\chi : G \to \mathbb{C}$ is a character such that:*

1. *$\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$, and*

2. *χ is σ-even and/or τ-even (i.e., $\chi \circ \sigma = \chi$ and/or $\chi \circ \tau = \chi$).*

Proof. The proof is similar to the proof of Theorem 1. □

4 Some applications of the main result

As immediate consequences of Theorems 1 and 2, we have the following corollaries.

Corollary 1 *Let G be a compact group and σ be a continuous homomorphism or continuous anti-homomorphism such that $\sigma \circ \sigma = id$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = \chi$, where $\chi : G \to \mathbb{C}$ is a character such that χ is σ-even.

Proof. It suffices to take $\sigma(x) = \tau(x)$ for all $x \in G$ in Theorem 1 or in Theorem 2. □

Corollary 2 *Let G be a compact group and σ be a continuous homomorphism such that $\sigma \circ \sigma = id$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(x\sigma(y)) + f(yx) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \chi \circ \sigma)/2$, where $\chi : G \to \mathbb{C}$ is a character.

Proof. It suffices to take $\tau(x) = x$ for all $x \in G$ in Theorem 1. □

Corollary 3 *Let G be a compact group and τ be a continuous homomorphism such that $\tau \circ \tau = \mathrm{id}$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(xy) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \chi \circ \tau)/2$, where $\chi : G \to \mathbb{C}$ is a character.

Proof. It suffices to take $\sigma(x) = x$ for all $x \in G$ in Theorem 1. □

Corollary 4 *Let G be a compact group and σ be a continuous anti-homomorphism such that $\sigma \circ \sigma = \mathrm{id}$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(x\sigma(y)) + f(y^{-1}x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \overline{\chi \circ \sigma})/2$, where $\chi : G \to \mathbb{C}$ is a character such that χ is σ-even and/or $\overline{\chi} = \chi$.

Proof. It suffices to take $\tau(x) = x^{-1}$ for all $x \in G$ in Theorem 2. □

Corollary 5 *Let G be a compact group and τ be a continuous anti-homomorphism such that $\tau \circ \tau = \mathrm{id}$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(xy^{-1}) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \overline{\chi \circ \tau})/2$, where $\chi : G \to \mathbb{C}$ is a character such that χ is τ-even and/or $\overline{\chi} = \chi$.

Proof. It suffices to take $\sigma(x) = x^{-1}$ for all $x \in G$ in Theorem 2. □

Corollary 6 *The non-zero solutions $f : G \to \mathbb{C}$ of the functional equation*

$$f(xy) + f(yx) = 2f(x)f(y), \; x, y \in G,$$

are the functions of the form $f = \chi$, where $\chi : G \to \mathbb{C}$ is a unitary character.

Proof. It suffices to take $\sigma(x) = \tau(x) = x$ for all $x \in G$ in Theorem 1. □

Corollary 7 *The non-zero solutions $f : G \to \mathbb{C}$ of the functional equation*

$$f(xy^{-1}) + f(y^{-1}x) = 2f(x)f(y), \; x, y \in G,$$

are the functions of the form $f = \chi$, where $\chi : G \to \mathbb{C}$ is a unitary character such that $\overline{\chi} = \chi$.

Proof. It suffices to take $\sigma(x) = \tau(x) = x^{-1}$ for all $x \in G$ in Theorem 2. □

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On the growth measures of entire and meromorphic functions focusing their generalized relative type and generalized relative weak type

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Abstract. In this paper we study some comparative growth properties of composite entire and meromorphic functions on the basis of their generalized relative order, generalized relative type and generalized relative weak type with respect to another entire function.

1 Introduction

Let f be an entire function defined in the finite complex plane $\mathbb{C}$. The maximum modulus function corresponding to entire f is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. If f is non-constant then it has the following property:

Property (A) [2] A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [2].

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For any two entire functions f and g, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. The order (lower order) of an entire function f which is generally used in computational purpose is defined in terms of the growth of f respect to the $\exp z$ function which is as follows:

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}\right).$$

When f is meromorphic, $M_f(r)$ cannot be defined as f is not analytic. In this case one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as maximum modulus function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)\left(\bar{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a)\log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a)\log r\right),$$

moreover we denote by $n_f(r,a)\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

And the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the proximity function of f is defined as follows:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \ \text{ for all } x \geqslant 0\,.$$

Also we may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r, a)$.

If f is entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Further, if f is non-constant entire then $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exist and is such that $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of the Nevanlinna's Characteristic functions of the meromorphic functions f and g. Moreover in case of meromorphic functions, the growth indicators such as order and lower order which are classical in complex analysis are defined in terms of their growths with respect to the $\exp z$ function as the following:

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

Bernal [1], [2] introduced the relative order between two entire functions to avoid comparing growth just with $\exp z$. Extending the notion of relative order as cited in the reference, Lahiri and Banerjee [9] introduced the definition of *relative order* of a meromorphic functions with respect to another entire function.

For entire and meromorphic functions, the notion of the growth indicators of its such as *generalized order*, *generalized type* and *generalized weak type* are classical in complex analysis and during the past decades, several researchers have already been continued their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the growth indicator such as *generalized order*, *generalized type* and *generalized weak type*. But at that time, the concept of *generalized relative order* and consequently *generalized relative type* and *generalized relative weak type* of entire and meromorphic function with respect to another entire function which have been discussed in the next section was mostly unknown to complex analysis and was not aware of the technical advantage given by such notion which gives an idea to avoid comparing growth just with exp function to calculate *generalized order*, *generalized type* and *generalized weak type* respectively. Therefore the growth of composite entire and meromorphic functions can be studied on the basis of their *generalized relative order, generalized* *relative type* and *generalized relative weak* which has been investigated in this paper.

2 Notation and preliminary remarks

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [8] and [12]. Hence we do not explain those in details. In the consequence we use the following notation:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, ... \text{ and } \log^{[0]} x = x.$$

Now we just recall some definitions which will be needed in the sequel.

Definition 1 *The* order ρ_f *and* lower order λ_f *of an entire function* f *are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \textit{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

When f *is meromorphic then*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \textit{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

In this connection Sato [10] define the *generalized order* $\rho_f^{[l]}$ (respectively, *generalized lower order* $\lambda_f^{[l]}$) of an entire function f which is defined as

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}\right)$$

where $l = 1, 2, 3 \ldots.$

For meromorphic f, the above definition reduces to

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r}\right)$$

for any $l \geq 1$.

These definitions extended the definitions of *order* ρ_f and *lower order* λ_f of an entire or meromorphic function f which are classical in complex analysis for integer $l = 2$ since these correspond to the particular case $\rho_f^{[2]} = \rho_f(2, 1) = \rho_f$ and $\lambda_f^{[2]} = \lambda_f(2, 1) = \lambda_f$.

Definition 2 *The* type σ_f *and* lower type $\overline{\sigma}_f$ *of an entire function* f *are defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}, \ 0 < \rho_f < \infty.$$

If f *is meromorphic then*

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Consequently the *generalized type* $\sigma_f^{[l]}$ and *generalized lower type* $\overline{\sigma}_f^{[l]}$ of an entire function f are defined as

$$\sigma_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f^{[l]}}} \text{ and } \overline{\sigma}_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f^{[l]}}}, \ 0 < \rho_f^{[l]} < \infty$$

where $l \geq 1$. If f is meromorphic then

$$\sigma_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-2]} T_f(r)}{r^{\rho_f^{[l]}}} \text{ and } \overline{\sigma}_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-2]} T_f(r)}{r^{\rho_f^{[l]}}}, \ 0 < \rho_f^{[l]} < \infty$$

where $l \geq 1$. Moreover, when $l = 2$ then $\sigma_f^{[2]}$ and $\overline{\sigma}_f^{[2]}$ are correspondingly denoted as σ_f and $\overline{\sigma}_f$ which are respectively known as *type* and *lower type* of entire or meromorphic f.

Datta and Jha [6] introduced the definition of *weak type* of an entire function of finite positive lower order in the following way:

Definition 3 [6] *The* weak type τ_f *and the growth indicator* τ_f *of an entire function* f *of finite positive lower order* λ_f *are defined by*

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty.$$

When f *is meromorphic then*

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty.$$

Similarly, extending the notion of *weak type* as introduced by Datta and Jha [6], one can define *generalized weak type* to determine the relative growth of two entire functions having same non zero finite *generalized lower order* in the following manner:

Definition 4 *The* generalized weak type $\tau_f^{[l]}$ *for* $l \geq 1$ *of an entire function* f *of finite positive* generalized lower order $\lambda_f^{[l]}$ *are defined by*

$$\tau_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f^{[l]}}},\ 0 < \lambda_f^{[l]} < \infty.$$

Also one may define the growth indicator $\overline{\tau}_f^{[l]}$ *of an entire function* f *in the following way:*

$$\overline{\tau}_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f^{[l]}}}\ ,\ 0 < \lambda_f^{[l]} < \infty.$$

When f *is meromorphic then*

$$\tau_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-2]} T_f(r)}{r^{\lambda_f^{[l]}}}\ \textit{and}\ \overline{\tau}_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-2]} T_f(r)}{r^{\lambda_f^{[l]}}},\ 0 < \lambda_f^{[l]} < \infty.$$

If an entire function g is non-constant then $M_g(r)$ and $T_g(r)$ are both strictly increasing and continuous function of r. Hence there exists inverse functions $M_g^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} M_g^{-1}(s) = \infty$ and $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ with $\lim_{s\to\infty} T_g^{-1}(s) = \infty$ respectively.

Bernal [1], [2] introduced the definition of *relative order* of af an entire function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu)\ \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [11] if $g(z) = \exp z$.

Similarly, one can define the *relative lower order* of an entire function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Extending this notion, Lahiri and Banerjee [9] introduced the definition of *relative order* of a meromorphic function f with respect to an entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu)\ \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [9] if $g(z) = \exp z$.

In the same way, one can define the *relative lower order* of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

Further, Banerjee and Jana [6] gave a more generalized concept of relative order of a meromorphic function with respect to an entire function in the following way:

Definition 5 [6] *If* $l \geq 1$ *is a positive integer, then the* l- th generalized relative order *of a meromorphic function* f *with respect to an entire function* g*, denoted by* $\rho_g^{[l]}(f)$ *is defined by*

$$\rho_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log r}.$$

Likewise one can define the generalized relative lower order *of a meromorphic function* f *with respect to an entire function* g *denoted by* $\lambda_g^{[l]}(f)$ *as*

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log r}.$$

In the case of meromorphic functions, it therefore seems reasonable to define suitably the *generalized relative type* and *generalized relative weak type* of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite *generalized relative order* or *generalized relative lower order* with respect to an entire function. Next we give such definitions of *generalized relative type* and *generalized relative weak type* of a meromorphic function f with respect to an entire function g which are as follows:

Definition 6 *The* generalized relative type $\sigma_g^{[l]}(f)$ *of a meromorphic function* f *with respect to an entire function* g *are defined as*

$$\sigma_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_g^{-1} T_f(r)}{r^{\rho_g^{[l]}(f)}}, \text{ where } 0 < \rho_g^{[[l]]}(f) < \infty.$$

Similarly, one can define the generalized lower relative type $\overline{\sigma}_g(f)$ *in the following way:*

$$\overline{\sigma}_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} T_g^{-1} T_f(r)}{r^{\rho_g^{[l]}(f)}}, \text{ where } 0 < \rho_g^{[l]}(f) < \infty.$$

Definition 7 *The* generalized relative weak type $\tau_g^{[l]}(f)$ *of a meromorphic function* f *with respect to an entire function* g *with finite positive relative lower order* $\lambda_g^{[l]}(f)$ *is defined by*

$$\tau_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} T_g^{-1} T_f(r)}{r^{\lambda_g^{[l]}(f)}}.$$

In a like manner, one can define the growth indicator $\overline{\tau}_g^{[l]}(f)$ *of a meromorphic function* f *with respect to an entire function* g *with finite positive relative lower order* $\lambda_g^{[l]}(f)$ *as*

$$\overline{\tau}_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_g^{-1} T_f(r)}{r^{\lambda_g^{[l]}(f)}}.$$

3 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [3] *Let* f *be meromorphic and* g *be entire then for all sufficiently large values of* r,

$$T_{f \circ g}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 2 [4] *Let* f *be meromorphic and* g *be entire and suppose that* $0 < \mu < \rho_g \leq \infty$. *Then for a sequence of values of* r *tending to infinity,*

$$T_{f \circ g}(r) \geq T_f(\exp(r^\mu)).$$

Lemma 3 [7] *Let* f *be an entire function which satisfy the Property (A),* $\beta > 0$, $\delta > 1$ *and* $\alpha > 2$. *Then*

$$\beta T_f(r) < T_f\left(\alpha r^\delta\right).$$

4 Main results

In this section we present the main results of the paper.

Theorem 1 *Let* f *be meromorphic,* g *and* h *be any two entire functions such that* $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, $\sigma_g < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\limsup \frac{\log^{[l]} T_h^{-1} T_{f \circ g}(r)}{\log^{[l]} T_h^{-1} T_f(\exp r^{\rho_g})} \leq \frac{\sigma_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(f)}.$$

Proof. Let us suppose that $\alpha > 2$.
Since $T_h^{-1}(r)$ is an increasing function r, it follows from Lemma 1, Lemma 3 and the inequality $T_g(r) \le \log M_g(r)$ {cf. [8]} that for all sufficiently large values of r we have

$$\begin{aligned} T_h^{-1}T_{fog}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\}T_f(M_g(r))\right] \\ \text{i.e., } T_h^{-1}T_{fog}(r) &\leqslant \alpha\left[T_h^{-1}T_f(M_g(r))\right]^{\delta} \\ \text{i.e., } \log^{[l]} T_h^{-1}T_{fog}(r) &\leqslant \log^{[l]} T_h^{-1}T_f(M_g(r)) + O(1) \end{aligned} \tag{1}$$

$$\begin{aligned} &\text{i.e., } \frac{\log^{[l]} T_h^{-1}T_{fog}(r)}{\log^{[l]} T_h^{-1}T_f(\exp r^{\rho_g})} \\ &\le \frac{\log^{[l]} T_h^{-1}T_f(M_g(r)) + O(1)}{\log^{[l]} T_h^{-1}T_f(\exp r^{\rho_g})} = \frac{\log^{[l]} T_h^{-1}T_f(M_g(r)) + O(1)}{\log M_g(r)} \cdot \\ &\qquad \frac{\log M_g(r)}{r^{\rho_g}} \cdot \frac{\log \exp r^{\rho_g}}{\log^{[l]} T_h^{-1}T_f(\exp r^{\rho_g})} \end{aligned} \tag{2}$$

$$\begin{aligned} &\text{i.e., } \limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1}T_{fog}(r)}{\log^{[l]} T_h^{-1}T_f(\exp r^{\rho_g})} \\ &\le \limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1}T_f(M_g(r)) + O(1)}{\log M_g(r)} \cdot \limsup_{r\to\infty} \frac{\log M_g(r)}{r^{\rho_g}} \cdot \\ &\qquad \limsup_{r\to\infty} \frac{\log \exp r^{\rho_g}}{\log^{[l]} T_h^{-1}T_f(\exp r^{\rho_g})} \end{aligned}$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1}T_{fog}(r)}{\log^{[l]} T_h^{-1}T_f(\exp r^{\rho_g})} \le \rho_h^{[l]}(f) \cdot \sigma_g \cdot \frac{1}{\lambda_h^{[l]}(f)} .$$

Thus the theorem is established. □

In the line of Theorem 1 the following theorem can be proved:

Theorem 2 *Let f be a meromorphic function, g and h be any two entire functions such that $\lambda_h^{[l]}(g) > 0$, $\rho_h^{[l]}(f) < \infty$, $\sigma_g < \infty$ and h satisfy the Property (A) where $l > 1$. Then*

$$\limsup \frac{\log^{[l]} T_h^{-1}T_{fog}(r)}{\log^{[l]} T_h^{-1}T_g(\exp r^{\rho_g})} \le \frac{\sigma_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(g)}.$$

Using the notion of lower type we may state the following two theorems without proof because it can be carried out in the line of Theorem 1 and Theorem 2 respectively.

Theorem 3 *Let* f *be meromorphic,* g *and* h *be any two entire functions such that* $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, $\overline{\sigma}_g < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf \frac{\log^{[l]} T_h^{-1} T_{f \circ g}(r)}{\log^{[l]} T_h^{-1} T_f(\exp r^{\rho_g})} \leq \frac{\overline{\sigma}_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(f)}.$$

Theorem 4 *Let* f *be a meromorphic function,* g *and* h *be any two entire functions such that* $\lambda_h^{[l]}(g) > 0$, $\rho_h^{[l]}(f) < \infty$, $\overline{\sigma}_g < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf \frac{\log^{[l]} T_h^{-1} T_{f \circ g}(r)}{\log^{[l]} T_h^{-1} T_g(\exp r^{\rho_g})} \leq \frac{\overline{\sigma}_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(g)}.$$

Using the concept of the growth indicators τ_g and $\overline{\tau}_g$ of an entire function g, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 1, Theorem 2, Theorem 3 and Theorem 4 respectively.

Theorem 5 *Let* f *be meromorphic,* g *and* h *be any two entire functions such that* $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, $\overline{\tau}_g < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\limsup \frac{\log^{[l]} T_h^{-1} T_{f \circ g}(r)}{\log^{[l]} T_h^{-1} T_f\left(\exp r^{\lambda_g}\right)} \leq \frac{\overline{\tau}_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(f)}.$$

Theorem 6 *Let* f *be a meromorphic function,* g *and* h *be any two entire functions such that* $\lambda_h^{[l]}(g) > 0$, $\rho_h^{[l]}(f) < \infty$, $\overline{\tau}_g < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\limsup \frac{\log^{[l]} T_h^{-1} T_{f \circ g}(r)}{\log^{[l]} T_h^{-1} T_g\left(\exp r^{\lambda_g}\right)} \leq \frac{\overline{\tau}_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(g)}.$$

Theorem 7 *Let* f *be meromorphic,* g *and* h *be any two entire functions such that* $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, $\tau_g < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf \frac{\log^{[l]} T_h^{-1} T_{f \circ g}(r)}{\log^{[l]} T_h^{-1} T_f\left(\exp r^{\lambda_g}\right)} \leq \frac{\tau_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(f)}.$$

Theorem 8 *Let* f *be a meromorphic function,* g *and* h *be any two entire functions such that* $\lambda_h^{[l]}(g) > 0$, $\rho_h^{[l]}(f) < \infty$, $\tau_g < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l]} T_h^{-1} T_g\left(\exp r^{\lambda_g}\right)} \leq \frac{\tau_g \cdot \rho_h^{[l]}(f)}{\lambda_h^{[l]}(g)}.$$

Theorem 9 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \rho_h^{[l]}(f) < \infty$, (ii) $\rho_h^{[l]}(f) = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $0 < \sigma_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{[l]}(f) \cdot \sigma_g}{\sigma_h^{[l]}(f)}.$$

Proof. From (1), we get for all sufficiently large values of r that

$$\log^{[l]} T_h^{-1} T_{fog}(r) \leqslant \left(\rho_h^{[l]}(f) + \varepsilon\right) \log M_g(r) + O(1). \tag{3}$$

Using Definition 2 we obtain from (3) for all sufficiently large values of r that

$$\log^{[l]} T_h^{-1} T_{fog}(r) \leqslant \left(\rho_h^{[l]}(f) + \varepsilon\right) (\sigma_g + \varepsilon) \cdot r^{\rho_g} + O(1). \tag{4}$$

Now in view of condition (ii) we obtain from (4) for all sufficiently large values of r that

$$\log^{[l]} T_h^{-1} T_{fog}(r) \leqslant \left(\rho_h^{[l]}(f) + \varepsilon\right) (\sigma_g + \varepsilon) \cdot r^{\rho_h^{[l]}(f)} + O(1). \tag{5}$$

Again in view of Definition 6 we get for a sequence of values of r tending to infinity that

$$\log^{[l-1]} T_h^{-1} T_f(r) \geq \left(\sigma_h^{[l]}(f) - \varepsilon\right) r^{\rho_h^{[l]}(f)}. \tag{6}$$

Now from (5) and (6), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\left(\rho_h^{[l]}(f) + \varepsilon\right) (\sigma_g + \varepsilon) \cdot r^{\rho_h^{[l]}(f)} + O(1)}{\left(\sigma_h^{[l]}(f) - \varepsilon\right) r^{\rho_h^{[l]}(f)}}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{[l]}(f) \cdot \sigma_g}{\sigma_h^{[l]}(f)}.$$

Hence the theorem follows. □

Using the notion of lower type and relative lower type, we may state the following theorem without proof as it can be carried out in the line of Theorem 9:

Theorem 10 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \rho_h^{[l]}(f) < \infty$, (ii) $\rho_h^{[l]}(f) = \rho_g$, (iii) $\overline{\sigma}_g < \infty$, (iv) $0 < \overline{\sigma}_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{[l]}(f) \cdot \overline{\sigma}_g}{\overline{\sigma}_h^{[l]}(f)} .$$

Similarly using the notion of type and relative lower type one may state the following two theorems without their proofs because those can also be carried out in the line line of Theorem 9:

Theorem 11 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, (ii) $\rho_h^{[l]}(f) = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $0 < \overline{\sigma}_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\lambda_h^{[l]}(f) \cdot \sigma_g}{\overline{\sigma}_h^{[l]}(f)}.$$

Theorem 12 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \rho_h^{[l]}(f) < \infty$, (ii) $\rho_h^{[l]}(f) = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $0 < \overline{\sigma}_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{[l]}(f) \cdot \sigma_g}{\overline{\sigma}_h^{[l]}(f)}.$$

Similarly, using the concept of weak type and relative weak type, we may state next four theorems without their proofs as those can be carried out in the line of Theorem 9, Theorem 10, Theorem 11 and Theorem 12 respectively.

Theorem 13 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, (ii) $\lambda_h^{[l]}(f) = \lambda_g$, (iii) $\overline{\tau}_g < \infty$, (iv) $0 < \overline{\tau}_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{[[l]]}(f) \cdot \overline{\tau}_g}{\overline{\tau}_h^{[l]}(f)}.$$

Theorem 14 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, (ii) $\lambda_h^{[l]}(f) = \lambda_g$, (iii) $\tau_g < \infty$, (iv) $0 < \tau_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{[l]}(f) \cdot \tau_g}{\tau_h^{[l]}(f)}.$$

Theorem 15 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \lambda_h^{[l]}(f) < \infty$, (ii) $\lambda_h^{[l]}(f) = \lambda_g$, (iii) $\overline{\tau}_g < \infty$, (iv) $0 < \tau_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\lambda_h^{[l]}(f) \cdot \overline{\tau}_g}{\tau_h^{[l]}(f)}.$$

Theorem 16 *Let* f *be meromorphic and* g, h *be any two entire functions such that* (i) $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \infty$, (ii) $\lambda_h^{[l]}(f) = \lambda_g$, (iii) $\overline{\tau}_g < \infty$, (iv) $0 < \tau_h^{[l]}(f) < \infty$ *and* h *satisfy the Property (A) where* $l > 1$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{[l]}(f) \cdot \overline{\tau}_g}{\tau_h^{[l]}(f)}.$$

Theorem 17 *Let* f *be meromorphic* g, h *and* l *be any three entire functions such that* $0 < \overline{\sigma}_h^{[m]}(f \circ g) \leq \sigma_h^{[m]}(f \circ g) < \infty$, $0 < \overline{\sigma}_l^{[n]}(f) \leq \sigma_l^{[n]}(f) < \infty$ *and* $\rho_h^{[m]}(f \circ g) = \rho_l^{[n]}(f)$ *where* m *and* n *any positive integers* > 1. *Then*

$$\frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\sigma[n]_l(f)} \leq \liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\overline{\sigma}_l^{[n]}(f)}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\sigma_h^{[m]}(f \circ g)}{\overline{\sigma}_l^{[n]}(f)}.$$

Proof. From the definition of $\sigma_l(f)$ and $\overline{\sigma}_h(f \circ g)$, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[m-1]} T_h^{-1} T_{fog}(r) \geqslant \left(\overline{\sigma}_h^{[m]}(f \circ g) - \varepsilon\right) r^{\rho_h^{[m]}(fog)} \tag{7}$$

and

$$\log^{[n-1]} T_l^{-1} T_f(r) \leq \left(\sigma_l^{[n]}(f) + \varepsilon\right) r^{\rho_l^{[n]}(f)}. \tag{8}$$

Now from (7), (8) and the condition $\rho_h^{[m]}(f \circ g) = \rho_l^{[n]}(f)$, it follows for all large values of r that,

$$\frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \geqslant \frac{\left(\overline{\sigma}_h^{[m]}(f \circ g) - \varepsilon\right)}{\left(\sigma_l^{[n]}(f) + \varepsilon\right)}.$$

As $\varepsilon (> 0)$ is arbitrary , we obtain that

$$\liminf_{r \to \infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \geqslant \frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\sigma_l^{[n]}(f)}. \tag{9}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m-1]} T_h^{-1} T_{fog}(r) \leq \left(\overline{\sigma}_h^{[m]}(f \circ g) + \varepsilon\right) r^{\rho_h^{[m]}(fog)} \tag{10}$$

and for all sufficiently large values of r,,

$$\log^{[n-1]} T_l^{-1} T_f(r) \geqslant \left(\overline{\sigma}_l^{[n]}(f) - \varepsilon\right) r^{\rho_l^{[n]}(f)}. \tag{11}$$

Combining the condition $\rho_h(f \circ g) = \rho_l(f)$, (10) and (11) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\left(\overline{\sigma}_h^{[m]}(f \circ g) + \varepsilon\right)}{\left(\overline{\sigma}_l^{[n]}(f) - \varepsilon\right)}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\overline{\sigma}_l^{[n]}(f)}. \tag{12}$$

Also for a sequence of values of r tending to infinity that

$$\log^{[n-1]} T_l^{-1} T_f(r) \leq \left(\overline{\sigma}_l^{[n]}(f) + \varepsilon\right) r^{\rho_l^{[n]}(f)}. \tag{13}$$

Now from (7), (13) and the condition $\rho_h(f \circ g) = \rho_l(f)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \geq \frac{\left(\overline{\sigma}_h^{[m]}(f \circ g) - \varepsilon\right)}{\left(\overline{\sigma}_l^{[n]}(f) + \varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T_h^{-1}T_{fog}(r)}{\log^{[n-1]}T_l^{-1}T_f(r)}\geq\frac{\overline{\sigma}_h^{[m]}(f\circ g)}{\overline{\sigma}_l^{[n]}(f)}. \tag{14}$$

Also for all sufficiently large values of r,,

$$\log^{[n-1]}T_h^{-1}T_{fog}(r)\leq\left(\overline{\sigma}_h^{[m]}(f\circ g)+\varepsilon\right)r^{\rho_h^{[m]}(fog)}. \tag{15}$$

As the condition $\rho_h(f\circ g)=\rho_l(f)$, it follows from (11) and (15) for all sufficiently large values of r that

$$\frac{\log^{[m-1]}T_h^{-1}T_{fog}(r)}{\log^{[n-1]}T_l^{-1}T_f(r)}\leq\frac{\left(\overline{\sigma}_h^{[m]}(f\circ g)+\varepsilon\right)}{\left(\overline{\sigma}_l^{[n]}(f)-\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T_h^{-1}T_{fog}(r)}{\log^{[n-1]}T_l^{-1}T_f(r)}\leq\frac{\overline{\sigma}_h^{[m]}(f\circ g)}{\overline{\sigma}_l^{[n]}(f)}\,. \tag{16}$$

Thus the theorem follows from (9), (12), (14) and (16). □

The following theorem can be proved in the line of Theorem 17 and so the proof is omitted.

Theorem 18 *Let f be meromorphic, g,h and k be any three entire functions such that $0<\overline{\sigma}_h^{[m]}(f\circ g)\leq\sigma_h^{[m]}(f\circ g)<\infty$, $0<\overline{\sigma}_k^{[n]}(g)\leq\sigma_k^{[n]}(g)<\infty$ and $\rho_h^{[m]}(f\circ g)=\rho_k^{[n]}(g)$ where $\min\{m,n\}>1$. Then*

$$\begin{aligned}\frac{\overline{\sigma}_h^{[m]}(f\circ g)}{\sigma_k^{[n]}(g)}&\leq\liminf_{r\to\infty}\frac{\log^{[m-1]}T_h^{-1}T_{fog}(r)}{\log^{[n-1]}T_k^{-1}T_g(r)}\leq\frac{\overline{\sigma}_h^{[m]}(f\circ g)}{\overline{\sigma}_k^{[n]}(g)}\\&\leq\limsup_{r\to\infty}\frac{\log^{[m-1]}T_h^{-1}T_{fog}(r)}{\log^{[n-1]}T_k^{-1}T_g(r)}\leq\frac{\sigma_h^{[m]}(f\circ g)}{\overline{\sigma}_k^{[n]}(g)}.\end{aligned}$$

Theorem 19 *Let f be meromorphic g, h and l be any three entire functions such that $0<\sigma_h^{[m]}(f\circ g)<\infty$, $0<\sigma_l^{[n]}(f)<\infty$ and $\rho_h^{[m]}(f\circ g)=\rho_l^{[n]}(f)$ where m and n are any positive integers with $m>1$ and $n>1$ respectively. Then*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T_h^{-1}T_{fog}(r)}{\log^{[n-1]}T_l^{-1}T_f(r)}\leq\frac{\sigma_h^{[m]}(f\circ g)}{\sigma_l^{[n]}(f)}\leq\limsup_{r\to\infty}\frac{\log^{[m-1]}T_h^{-1}T_{fog}(r)}{\log^{[n-1]}T_l^{-1}T_f(r)}.$$

Proof. From the definition of $\sigma_l^{[n]}(f)$, we get for a sequence of values of r tending to infinity that

$$\log^{[n-1]} T_l^{-1} T_f(r) \geqslant \left(\sigma_l^{[n]}(f) - \varepsilon\right) r^{\rho_l^{[n]}(f)}. \tag{17}$$

Now from (15), (17) and the condition $\rho_h^{[m]}(f \circ g) = \rho_l^{[b]}(f)$, it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\left(\sigma_h^{[m]}(f \circ g) + \varepsilon\right)}{\left(\sigma_l^{[n]}(f) - \varepsilon\right)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\sigma_h^{[m]}(f \circ g)}{\sigma_l^{[n]}(f)}. \tag{18}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m-1]} T_h^{-1} T_{fog}(r) \geqslant \left(\sigma_h^{[m]}(f \circ g) - \varepsilon\right) r^{\rho_h^{[m]}(fog)}. \tag{19}$$

So combining the condition $\rho_h(f \circ g) = \rho_l(f)$, (8) and (19), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \geqslant \frac{\left(\sigma_h^{[m]}(f \circ g) - \varepsilon\right)}{\left(\sigma_l^{[n]}(f) + \varepsilon\right)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \geqslant \frac{\sigma_h^{[m]}(f \circ g)}{\sigma_l^{[n]}(f)}. \tag{20}$$

Thus the theorem follows from (18) and (20). □

The following theorem can be carried out in the line of Theorem 19 and therefore we omit its proof.

Theorem 20 *Let f be meromorphic, g, h and k be any three entire functions such that $0 < \sigma_h^{[m]}(f \circ g) < \infty$, $0 < \sigma_k^{[n]}(g) < \infty$ and $\rho_h^{[m]}(f \circ g) = \rho_k^{[n]}(g)$ where m and n are any positive integers > 1. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_k^{-1} T_g(r)} \leq \frac{\sigma_h^{[m]}(f \circ g)}{\sigma_k^{[n]}(g)} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_k^{-1} T_g(r)}.$$

The following theorem is a natural consequence of Theorem 17 and Theorem 19.

Theorem 21 *Let* f *be meromorphic* g, h *and* l *be any three entire functions such that* $0 < \overline{\sigma}_h^{[m]}(f \circ g) \leq \sigma_h^{[m]}(f \circ g) < \infty$, $0 < \overline{\sigma}_l^{[n]}(f) \leq \sigma_l^{[n]}(f) < \infty$ *and* $\rho_h^{[m]}(f \circ g) = \rho_l^{[n]}(f)$ *where* m *and* n *are any positive integers with* $m > 1$ *and* $n > 1$ *respectively. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \min\left\{\frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\overline{\sigma}_l^{[n]}(f)}, \frac{\sigma_h^{[m]}(f \circ g)}{\sigma_l^{[n]}(f)}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\overline{\sigma}_l^{[n]}(f)}, \frac{\sigma_h^{[m]}(f \circ g)}{\sigma_l^{[n]}(f)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} .$$

The proof is omitted.

Analogously one may state the following theorem without its proof as it is also a natural consequence of Theorem 18 and Theorem 20.

Theorem 22 *Let* f *be meromorphic*, g, h *and* k *be any three entire functions such that* $0 < \overline{\sigma}_h^{[m]}(f \circ g) \leq \sigma_h^{[m]}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{[n]}(g) \leq \sigma_k^{[n]}(g) < \infty$ *and* $\rho_h^{[m]}(f \circ g) = \rho_k^{[n]}(g)$ *where* m *and* n *are any positive integers* > 1. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_k^{-1} T_g(r)} \leq \min\left\{\frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\overline{\sigma}_k^{[n]}(g)}, \frac{\sigma_h^{[m]}(f \circ g)}{\sigma_k^{[n]}(g)}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}_h^{[m]}(f \circ g)}{\overline{\sigma}_k^{[n]}(g)}, \frac{\sigma_h^{[m]}(f \circ g)}{\sigma_k^{[n]}(g)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_k^{-1} T_g(r)}.$$

In the same way , using the concept of relative weak type, we may state next two theorems without their proofs as those can be carried out in the line of Theorem 17 and Theorem 19 respectively.

Theorem 23 *Let* f *be meromorphic* g, h *and* l *be any three entire functions such that* $0 < \tau_h^{[m]}(f \circ g) \leq \overline{\tau}_h^{[m]}(f \circ g) < \infty$, $0 < \tau_l^{[n]}(f) \leq \overline{\tau}_l^{[n]}(f) < \infty$ *and* $\lambda_h^{[m]}(f \circ g) = \lambda_l^{[n]}(f)$ *where* m *and* n *any positive integers* > 1. *Then*

$$\frac{\tau_h^{[m]}(f \circ g)}{\overline{\tau}_l^{[n]}(f)} \leq \liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\tau_h^{[m]}(f \circ g)}{\tau_l^{[n]}(f)}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \leq \frac{\overline{\tau}_h^{[m]}(f \circ g)}{\tau_l^{[n]}(f)}.$$

Theorem 24 *Let* f *be meromorphic* g, h *and* l *be any three entire functions such that* $0 < \overline{\tau}_h^{[m]}(f \circ g) < \infty$, $0 < \overline{\tau}_l^{[n]}(f) < \infty$ *and* $\lambda_h^{[m]}(f \circ g) = \lambda_l^{[n]}(f)$ *where* m *and* n *are any positive integers with* $m > 1$ *and* $n > 1$ *respectively. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} \le \frac{\overline{\tau}_h^{[m]}(f \circ g)}{\overline{\tau}_l^{[n]}(f)} \le \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)}.$$

The following theorem is a natural consequence of Theorem 23 and Theorem 24:

Theorem 25 *Let* f *be meromorphic* g, h *and* l *be any three entire functions such that* $0 < \tau_h^{[m]}(f \circ g) \le \overline{\tau}_h^{[m]}(f \circ g) < \infty$, $0 < \tau_l^{[n]}(f) \le \overline{\tau}_l^{[n]}(f) < \infty$ *and* $\lambda_h^{[m]}(f \circ g) = \lambda_l^{[n]}(f)$ *where* m *and* n *any positive integers* > 1. *Then*

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)} &\le \min\left\{ \frac{\overline{\tau}_h^{[m]}(f \circ g)}{\overline{\tau}_l^{[n]}(f)}, \frac{\tau_h^{[m]}(f \circ g)}{\tau_l^{[n]}(f)} \right\} \\ &\le \max\left\{ \frac{\overline{\tau}_h^{[m]}(f \circ g)}{\overline{\tau}_l^{[n]}(f)}, \frac{\tau_h^{[m]}(f \circ g)}{\tau_l^{[n]}(f)} \right\} \le \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[n-1]} T_l^{-1} T_f(r)}. \end{aligned}$$

The following two theorems can be proved in the line of Theorem 23 and Theorem 24 respectively and therefore their proofs are omitted.

Theorem 26 *Let* f *be meromorphic,* g, h *and* k *be any three entire functions such that* $0 < \tau_h^{[m]}(f \circ g) \le \overline{\tau}_h^{[m]}(f \circ g) < \infty$, $0 < \tau_k^{[n]}(g) \le \overline{\tau}_k^{[n]}(g) < \infty$ *and* $\lambda_h^{[m]}(f \circ g) = \lambda_k^{[n]}(g)$ *where* m *and* n *are any positive integers* > 1. *Then*

$$\begin{aligned} \frac{\tau_h^{[m]}(f \circ g)}{\overline{\tau}_k^{[n]}(g)} &\le \liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[m-1]} T_k^{-1} T_g(r)} \le \frac{\tau_h^{[m]}(f \circ g)}{\tau_k^{[n]}(g)} \\ &\le \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[m-1]} T_k^{-1} T_g(r)} \le \frac{\overline{\tau}_h^{[m]}(f \circ g)}{\tau_k^{[n]}(g)}. \end{aligned}$$

Theorem 27 *Let* f *be meromorphic,* g, h *and* k *be any three entire functions such that* $0 < \overline{\tau}_h^{[m]}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{[n]}(g) < \infty$ *and* $\lambda_h^{[m]}(f \circ g) = \lambda_k^{[n]}(g)$ *where* m *and* n *any positive integers* > 1. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[m-1]} T_k^{-1} T_g(r)} \le \frac{\overline{\tau}_h^{[m]}(f \circ g)}{\overline{\tau}_k^{[n]}(g)} \le \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[m-1]} T_k^{-1} T_g(r)}.$$

The following theorem is a natural consequence of Theorem 26 and Theorem 27.

Theorem 28 *Let* f *be meromorphic,* g, h *and* k *be any three entire functions such that* $0 < \tau_h^{[m]}(f \circ g) \leq \overline{\tau}_h^{[m]}(f \circ g) < \infty$, $0 < \tau_k^{[n]}(g) \leq \overline{\tau}_k^{[n]}(g) < \infty$ *and* $\lambda_h^{[m]}(f \circ g) = \lambda_k^{[n]}(g)$ *where* m *and* n *are any positive integers* > 1. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[m-1]} T_k^{-1} T_g(r)} \leq \min\left\{\frac{\overline{\tau}_h^{[m]}(f\circ g)}{\overline{\tau}_k^{[n]}(g)}, \frac{\tau_h^{[m]}(f\circ g)}{\tau_k^{[n]}(g)}\right\}$$
$$\leq \max\left\{\frac{\overline{\tau}_h^{[m]}(f\circ g)}{\overline{\tau}_k^{[n]}(g)}, \frac{\tau_h^{[m]}(f\circ g)}{\tau_k^{[n]}(g)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T_h^{-1} T_{fog}(r)}{\log^{[m-1]} T_k^{-1} T_g(r)}.$$

Theorem 29 *Let* f *be meromorphic,* g *and* h *be any two entire functions such that* $0 < \lambda_h^{[l]}(f) \leq \rho_h^{[l]}(f) < \rho_g \leq \infty$ *and* $\sigma_h^{[l]}(f) < \infty$ *where* $l > 1$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \geq \frac{\lambda_h^{[l]}(f)}{\sigma_h^{[l]}(f)}.$$

Proof. Since $\rho_h^{[l]}(f) < \rho_g$ and $T_h^{-1}(r)$ is a increasing function of r, we get from Lemma 2 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[l]} T_h^{-1} T_{fog}(r) &\geq \log^{[l]} T_h^{-1} T_f(\exp(r^{\mu})) \\ \text{i.e., } \log^{[l]} T_h^{-1} T_{fog}(r &\geq \left(\lambda_h^{[l]}(f) - \varepsilon\right) \cdot r^{\mu} \\ \text{i.e., } \log^{[l]} T_h^{-1} T_{fog}(r) &\geq \left(\lambda_h^{[l]}(f) - \varepsilon\right) \cdot r^{\rho_h^{[l]}(f)}. \end{aligned} \tag{21}$$

Again in view of Definition 6, we get for all sufficiently large values of r that

$$\log^{[l-1]} T_h^{-1} T_f(r) \leq \left(\sigma_h^{[l]}(f) + \varepsilon\right) r^{\rho_h^{[l]}(f)}. \tag{22}$$

Now from (21) and (22), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \geq \frac{\left(\lambda_h^{[l]}(f) - \varepsilon\right) \cdot r^{\rho_h^{[l]}(f)}}{\left(\sigma_h^{[l]}(f) + \varepsilon\right) r^{\rho_h^{[l]}(f)}}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{fog}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \geq \frac{\lambda_h^{[l]}(f)}{\sigma_h^{[l]}(f)}.$$

Thus the theorem follows. □

Now we state the following theorem without its proof as it can be carried out in the line of Theorem 29 and with the help of Definition 7:

Theorem 30 *Let* f *be meromorphic,* g *and* h *be any two entire functions such that* $0 < \lambda_h^{[l]}(f) < \rho_g \leq \infty$ *and* $\overline{\tau}_h^{[l]}(f) < \infty$ *where* $l > 1$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_f(r)} \geq \frac{\lambda_h^{[l]}(f)}{\overline{\tau}_h^{[l]}(f)}.$$

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Trace inequalities of Cassels and Grüss type for operators in Hilbert spaces

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Abstract. Some trace inequalities of Cassels type for operators in Hilbert spaces are provided. Applications in connection to Grüss inequality and for convex functions of selfadjoint operators are also given.

1 Introduction

Let $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ be two positive n-*tuples* with

$$0 < m_1 \le a_i \le M_1 < \infty \text{ and } 0 < m_2 \le b_i \le M_2 < \infty; \tag{1}$$

for each $i \in \{1, \ldots, n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

a) *Pólya-Szegö's inequality* [44]:

$$\frac{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}{\left(\sum_{k=1}^{n} a_k b_k\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2.$$

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b) *Shisha-Mond's inequality* [48]:

$$\frac{\sum_{k=1}^{n} a_k^2}{\sum_{k=1}^{n} a_k b_k} - \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2} \leq \left[\left(\frac{M_1}{m_2} \right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2} \right)^{\frac{1}{2}} \right]^2 .$$

If $\overline{\mathbf{w}} = (w_1, \ldots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

c) *Cassels' inequality* [15]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \ldots, n\}, \tag{2}$$

then

$$\frac{\left(\sum_{k=1}^{n} w_k a_k^2 \right) \left(\sum_{k=1}^{n} w_k b_k^2 \right)}{\left(\sum_{k=1}^{n} w_k a_k b_k \right)^2} \leq \frac{(M+m)^2}{4mM}.$$

d) *Greub-Reinboldt's inequality* [34]. We have

$$\left(\sum_{k=1}^{n} w_k a_k^2 \right) \left(\sum_{k=1}^{n} w_k b_k^2 \right) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \left(\sum_{k=1}^{n} w_k a_k b_k \right)^2 ,$$

provided $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfy the condition (1) .

For other recent results providing discrete reverse inequalities, see the monograph online [15].

The following reverse of Schwarz's inequality in inner product spaces holds [16].

Theorem 1 (Dragomir, 2003, [16]) *Let* $A, a \in \mathbb{C}$ *and* $x, y \in H$, *a complex inner product space with the inner product* $\langle \cdot, \cdot \rangle$. *If*

$$\operatorname{Re} \langle Ay - x, x - ay \rangle \geq 0, \tag{3}$$

or equivalently,

$$\left\| x - \frac{a+A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \, \|y\| , \tag{4}$$

holds, then we have the inequality

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4 . \tag{5}$$

The constant $\frac{1}{4}$ *is sharp in* (5).

In 1935, G. Grüss [35] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{6}$$
$$\leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma),$$

where $f,\ g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi,\ \gamma \leq g(x) \leq \Gamma \tag{7}$$

for each $x \in [a, b]$, where ϕ, Φ, γ, Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [18], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

Theorem 2 (Dragomir, 1999, [18]) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R},\mathbb{C})$ *and* $e \in H$, $\|e\| = 1$. *If* φ, γ, Φ, Γ *are real or complex numbers and* x, y *are vectors in* H *such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{8}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4}|\Phi - \varphi|\,|\Gamma - \gamma|. \tag{9}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it can not be replaced by a smaller constant.*

For other results of this type, see the recent monograph [21] and the references therein.

For other Grüss type results for integral and sums see the papers [1]-[3], [8]-[10], [17]-[24], [31], and the references therein.

In order to state some reverses of Schwarz and Grüss type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

2 Some facts on trace of operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{10}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{11}$$

showing that the definition (10) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{12}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A| x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (11) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{13}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{14}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (14) converges absolutely and it is independent from the choice of basis.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [45]

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{15}$$

where $\alpha \in (0,1)$ and $A, B \in \mathcal{B}(H)$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{16}$$

with $A, B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [49].

For some classical trace inequalities see [11], [13], [42] and [53], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [11], [32], [36], [37], [39], [46] and [50].

We denote by

$$\mathcal{B}_1^+(H) := \{P : P \in \mathcal{B}_1(H), P \text{ and is selfadjoint and } P \ge 0\}.$$

We obtained recently the following result [29]:

Theorem 3 *For any $A, C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H)\setminus\{0\}$ we have the inequality*

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\le \inf_{\lambda\in\mathbb{C}}\|A - \lambda\cdot 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\le \inf_{\lambda\in\mathbb{C}}\|A - \lambda\cdot 1_H\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},
\end{aligned} \tag{17}$$

where $\|\cdot\|$ is the operator norm.

In the following we establish other similar results for trace that generalize the classical Cassels' inequality stated in the introduction.

3 Cassels type trace inequalities

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$\mathcal{C}_{\alpha,\beta}(T,U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform

$$\mathcal{C}_{\alpha,\beta}(T) := (T^* - \bar{\alpha}1_H)(\beta 1_H - T) = \mathcal{C}_{\alpha,\beta}(T, 1_H),$$

where 1_H is the identity operator, which has been introduced in [27] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned}
\operatorname{Re}\langle\mathcal{C}_{\alpha,\beta}(T,U)x,x\rangle &= \operatorname{Re}\langle\mathcal{C}_{\beta,\alpha}(T,U)x,x\rangle \\
&= \frac{1}{4}|\beta-\alpha|^2\|Ux\|^2 - \left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\|^2 \\
&= \frac{1}{4}|\beta-\alpha|^2\left\langle|U|^2 x,x\right\rangle - \left\langle\left|T-\frac{\alpha+\beta}{2}\cdot U\right|^2 x,x\right\rangle
\end{aligned} \tag{18}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

Lemma 1 *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) *The transform $\mathcal{C}_{\alpha,\beta}(T,U)$ (or, equivalently, $\mathcal{C}_{\beta,\alpha}(T,U)$) is accretive;*

(ii) *We have the norm inequality*

$$\left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\| \leq \frac{1}{2}|\beta-\alpha|\,\|Ux\| \tag{19}$$

for any $x \in H$;

(iii) *We have the following inequality in the operator order*

$$\left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2 \leq \frac{1}{4}|\beta-\alpha|^2|U|^2.$$

As a consequence of the above lemma we can state:

Corollary 1 *Let* α, $\beta \in \mathbb{C}$ *and* T, $U \in B(H)$. *If* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive, then*

$$\left\| T - \frac{\alpha+\beta}{2} \cdot U \right\| \leq \frac{1}{2} |\beta - \alpha| \, \|U\| . \tag{20}$$

Remark 1 *In order to give examples of linear operators* T, $U \in B(H)$ *and numbers* α, $\beta \in \mathbb{C}$ *such that the transform* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive, it suffices to select two bounded linear operator* S *and* V *and the complex numbers* z, w $(w \neq 0)$ *with the property that* $\|Sx - zVx\| \leq |w| \, \|Vx\|$ *for any* $x \in H$, *and, by choosing* $T = S$, $U = V$, $\alpha = \frac{1}{2}(z + w)$ *and* $\beta = \frac{1}{2}(z - w)$ *we observe that* T *and* U *satisfy* (19), *i.e.,* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive.*

The following result also holds:

Lemma 2 *Let, either* $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ *and* γ, $\Gamma \in \mathbb{C}$. *Then*

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right) \geq 0 \tag{21}$$

if and only if

$$\operatorname{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right) \leq \frac{1}{4}|\Gamma - \gamma|^2 \operatorname{tr}\left(P|B|^2\right). \tag{22}$$

To simplify the writing, we the say that (A, B) *satisfies the* P-(γ, Γ)*-trace property.*

Proof. We have the equalities

$$\begin{aligned}
&\frac{1}{4}|\Gamma - \gamma|^2 P|B|^2 - P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2 \\
&= P\left[\frac{1}{4}|\Gamma - \gamma|^2 |B|^2 - \left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right] \\
&= P\left[\frac{1}{4}|\Gamma - \gamma|^2 |B|^2 - \left(A - \frac{\gamma+\Gamma}{2}B\right)^*\left(A - \frac{\gamma+\Gamma}{2}B\right)\right]
\end{aligned} \tag{23}$$

$$= P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2 - |A|^2 + \overline{\frac{\gamma+\Gamma}{2}}B^*A + \frac{\gamma+\Gamma}{2}A^*B - \left|\frac{\gamma+\Gamma}{2}\right|^2|B|^2\right]$$
$$= P\left[-|A|^2 + \overline{\frac{\gamma+\Gamma}{2}}B^*A + \frac{\gamma+\Gamma}{2}A^*B + \left(\frac{1}{4}|\Gamma-\gamma|^2 - \left|\frac{\gamma+\Gamma}{2}\right|^2\right)|B|^2\right]$$
$$= P\left[-|A|^2 + \overline{\frac{\gamma+\Gamma}{2}}B^*A + \frac{\gamma+\Gamma}{2}A^*B - \operatorname{Re}(\Gamma\overline{\gamma})|B|^2\right]$$

for any bounded operators A, B, P and the complex numbers γ, $\Gamma \in \mathbb{C}$.

Taking the trace in (23) we get

$$\frac{1}{4}|\Gamma-\gamma|^2 \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right) \tag{24}$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \overline{\frac{\gamma+\Gamma}{2}}\operatorname{tr}(PB^*A) + \frac{\gamma+\Gamma}{2}\operatorname{tr}(PA^*B)$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \overline{\frac{\gamma+\Gamma}{2}}\operatorname{tr}(PB^*A) + \frac{\gamma+\Gamma}{2}\overline{\operatorname{tr}(PB^*A)}$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \overline{\frac{\gamma+\Gamma}{2}}\operatorname{tr}(PB^*A) + \overline{\overline{\frac{\gamma+\Gamma}{2}}\operatorname{tr}(PB^*A)}$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + 2\operatorname{Re}\left[\overline{\frac{\gamma+\Gamma}{2}}\operatorname{tr}(PB^*A)\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}(PB^*A)\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\overline{\overline{\Gamma}}\operatorname{tr}(PB^*A)}\right]$$
$$= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}\right].$$

Since

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right)$$
$$= \operatorname{Re}\left[\operatorname{tr}\left(\Gamma PA^*B + \overline{\gamma}PB^*A - \overline{\gamma}\Gamma PB^*B - PA^*A\right)\right]$$
$$= \operatorname{Re}\left[\Gamma\operatorname{tr}\left(PA^*B\right) + \overline{\gamma}\operatorname{tr}(PB^*A)\right] - \overline{\gamma}\Gamma\operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}\left(P|A|^2\right)\right]$$
$$= \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)} + \overline{\gamma}\operatorname{tr}(PB^*A)\right] - \operatorname{tr}\left(P|B|^2\right)\operatorname{Re}(\overline{\gamma}\Gamma) - \operatorname{tr}\left(P|A|^2\right),$$

then we get

$$\frac{1}{4}|\Gamma-\gamma|^2 \operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right) \tag{25}$$
$$= \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right),$$

which proves the desired equivalence. □

Corollary 2 *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$. *If the transform* $\mathcal{C}_{\gamma,\Gamma}(A,B)$ *is accretive, then* (A,B) *satisfies the* P-(γ,Γ)*-trace property.*

We have the following result:

Theorem 4 *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$ *with* $\mathrm{Re}(\Gamma\overline{\gamma}) = \mathrm{Re}(\Gamma)\mathrm{Re}(\gamma) + \mathrm{Im}(\Gamma)\mathrm{Im}(\gamma) > 0$.

(i) *If* (A,B) *satisfies the* P-(γ,Γ)*-trace property, then we have*

$$\begin{aligned}
&\mathrm{tr}\left(P|A|^2\right)\mathrm{tr}\left(P|B|^2\right) \\
&\leq \frac{1}{4}\cdot\frac{[\mathrm{Re}(\gamma+\Gamma)\mathrm{Re}\,\mathrm{tr}(PB^*A) + \mathrm{Im}(\gamma+\Gamma)\mathrm{Im}\,\mathrm{tr}(PB^*A)]^2}{\mathrm{Re}(\Gamma)\mathrm{Re}(\gamma) + \mathrm{Im}(\Gamma)\mathrm{Im}(\gamma)} \\
&\leq \frac{1}{4}\cdot\frac{|\gamma+\Gamma|^2}{\mathrm{Re}(\Gamma\overline{\gamma})}|\mathrm{tr}(PB^*A)|^2.
\end{aligned} \tag{26}$$

(ii) *If the transform* $\mathcal{C}_{\gamma,\Gamma}(A,B)$ *is accretive, then the inequality* (26) *also holds.*

Proof. (i) If (A,B) satisfies the P-(γ,Γ)-trace property, then, on utilizing the calculations above, we have

$$\begin{aligned}
0 &\leq \frac{1}{4}|\Gamma-\gamma|^2\,\mathrm{tr}\left(P|B|^2\right) - \mathrm{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right) \\
&= -\mathrm{tr}\left(P|A|^2\right) - \mathrm{Re}(\Gamma\overline{\gamma})\,\mathrm{tr}\left(P|B|^2\right) + \mathrm{Re}[\overline{\gamma}\mathrm{tr}(PB^*A)] + \mathrm{Re}\left[\Gamma\overline{\mathrm{tr}(PB^*A)}\right] \\
&= -\mathrm{tr}\left(P|A|^2\right) - \mathrm{Re}(\Gamma\overline{\gamma})\,\mathrm{tr}\left(P|B|^2\right) + \mathrm{Re}[\overline{\gamma}\mathrm{tr}(PB^*A)] + \mathrm{Re}\left[\overline{\overline{\Gamma}\mathrm{tr}(PB^*A)}\right] \\
&= -\mathrm{tr}\left(P|A|^2\right) - \mathrm{Re}(\Gamma\overline{\gamma})\,\mathrm{tr}\left(P|B|^2\right) + \mathrm{Re}[\overline{\gamma}\mathrm{tr}(PB^*A)] + \mathrm{Re}\left[\overline{\Gamma}\mathrm{tr}(PB^*A)\right] \\
&= -\mathrm{tr}\left(P|A|^2\right) - \mathrm{Re}(\Gamma\overline{\gamma})\,\mathrm{tr}\left(P|B|^2\right) + \mathrm{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\mathrm{tr}(PB^*A)\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathrm{tr}\left(P|A|^2\right) + \mathrm{Re}(\Gamma\overline{\gamma})\,\mathrm{tr}\left(P|B|^2\right) \leq \mathrm{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\mathrm{tr}(PB^*A)\right] \\
&= \mathrm{Re}(\gamma+\Gamma)\mathrm{Re}\,\mathrm{tr}(PB^*A) + \mathrm{Im}(\gamma+\Gamma)\mathrm{Im}\,\mathrm{tr}(PB^*A).
\end{aligned} \tag{27}$$

Making use of the elementary inequality

$$2\sqrt{pq} \leq p + q, \ \ p, q \geq 0,$$

we also have

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)} \le \operatorname{tr}\left(P|A|^2 big\right) + \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right). \tag{28}$$

Utilising (27) and (28) we get

$$\begin{aligned}&\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}\\ &\le \frac{\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\operatorname{tr}(PB^*A)+\operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\operatorname{tr}(PB^*A)}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\end{aligned} \tag{29}$$

that is equivalent with the first inequality in (26).

The second inequality in (26) is obvious by Schwarz inequality

$$(ab+cd)^2 \le \left(a^2+c^2\right)\left(b^2+d^2\right), \; a,b,c,d \in \mathbb{R}.$$

The (ii) is obvious from (i). □

Remark 2 *We observe that the inequality between the first and last term in* (26) *is equivalent to*

$$0 \le \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right) - |\operatorname{tr}(PB^*A)|^2 \le \frac{1}{4}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|\operatorname{tr}(PB^*A)|^2. \tag{30}$$

Corollary 3 *Let, either* $P \in \mathcal{B}_+(H)$, $A \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ *and* γ, $\Gamma \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma}) = \operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma) > 0$.

(i) *If* A *satisfies the* P-(γ,Γ)*-trace property, namely*

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right) \ge 0 \tag{31}$$

or, equivalently

$$\operatorname{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}1_H\right|^2\right) \le \frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}(P), \tag{32}$$

then we have

$$\begin{aligned}&\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} \le \frac{1}{4}\cdot\frac{\left[\operatorname{Re}(\gamma+\Gamma)\frac{\operatorname{Re}\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \operatorname{Im}(\gamma+\Gamma)\frac{\operatorname{Im}\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2}{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma)+\operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)}\\ &\le \frac{1}{4}\cdot\frac{|\gamma+\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2.\end{aligned} \tag{33}$$

(ii) *If the transform* $\mathcal{C}_{\gamma,\Gamma}(A)$ *is accretive, then the inequality* (26) *also holds.*

(iii) *We have*

$$0 \leq \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4} \cdot \frac{|\gamma - \Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2. \tag{34}$$

Remark 3 *The case of selfadjoint operators is as follows.*

Let A, B *be selfadjoint operators and either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $m, M \in \mathbb{R}$ *with* $mM > 0$.

(i) *If* (A, B) *satisfies the* P-(m, M)*-trace property, then we have*

$$\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) \leq \frac{(m+M)^2}{4mM}\left[\operatorname{tr}(PBA)\right]^2 \tag{35}$$

or, equivalently

$$0 \leq \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}(PBA)\right]^2 \leq \frac{(m-M)^2}{4mM}\left[\operatorname{tr}(PBA)\right]^2. \tag{36}$$

(ii) *If the transform* $\mathcal{C}_{m,M}(A, B)$ *is accretive, then the inequality* (35) *also holds.*

(iii) *If* $(A - mB)(MB - A) \geq 0$, *then* (35) *is valid.*

We observe that the inequality (35) *is the operator trace inequality version of Cassels' inequality from Introduction.*

4 Trace inequalities of Grüss type

Let P be a selfadjoint operator with $P \geq 0$. The functional $\langle \cdot, \cdot \rangle_{2,P}$ defined by

$$\langle A, B \rangle_{2,P} := \operatorname{tr}(PB^*A) = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$, i.e. $\langle \cdot, \cdot \rangle_{2,P}$ satisfies the properties:

(h) $\langle A, A \rangle_{2,P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

(hh) $\langle \cdot, \cdot \rangle_{2,P}$ is linear in the first variable;

(hhh) $\langle B, A \rangle_{2,P} = \overline{\langle A, B \rangle_{2,P}}$ for any $A, B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations

$$\|A\|_{2,P}^2 := \operatorname{tr}\left(P|A|^2\right) = \operatorname{tr}\left(APA^*\right) = \operatorname{tr}\left(|A|^2 P\right)$$

and

$$\langle A, B \rangle_{2,P} = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

for any $A, B \in \mathcal{B}_2(H)$.

The same definitions can be considered if $P \in \mathcal{B}_1^+(H)$ and $A, B \in \mathcal{B}(H)$.

We have the following Grüss type inequality:

Theorem 5 *Let, either* $P \in \mathcal{B}_+(H)$, A, B, $C \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, A, B, $C \in \mathcal{B}(H)$ *with* $P|A|^2$, $P|B|^2$, $P|C|^2 \neq 0$ *and* λ, Γ, δ, $\Delta \in \mathbb{C}$ *with* $\mathrm{Re}(\Gamma\overline{\gamma})$, $\mathrm{Re}(\Delta\overline{\delta}) > 0$. *If* (A, C) *has the trace* P-(λ, Γ)*-property and* (B, C) *has the trace* P-(δ, Δ)*-property, then*

$$\left|\frac{\mathrm{tr}(PB^*A)\,\mathrm{tr}\left(P|C|^2\right)}{\mathrm{tr}(PC^*A)\,\mathrm{tr}(PB^*C)} - 1\right| \leq \frac{1}{4} \cdot \frac{|\gamma-\Gamma|\,|\delta-\Delta|}{\sqrt{\mathrm{Re}(\Gamma\overline{\gamma})\,\mathrm{Re}\left(\Delta\overline{\delta}\right)}}. \tag{37}$$

Proof. We prove in the case that $P \in \mathcal{B}_+(H)$ and A, B, $C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$|\langle A, B\rangle_{2,P}|^2 \leq \langle A, A\rangle_{2,P}\,\langle B, B\rangle_{2,P}$$

for any A, $B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\,\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\,\langle C, B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\begin{aligned}
&\left|\langle A, B\rangle_{2,P}\,\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\,\langle C, B\rangle_{2,P}\right|^2 \\
&\leq \left[\|A\|_{2,P}^2\,\|C\|_{2,P}^2 - |\langle A, C\rangle_{2,P}|^2\right]\left[\|B\|_{2,P}^2\,\|C\|_{2,P}^2 - |\langle B, C\rangle_{2,P}|^2\right]
\end{aligned}$$

for any A, $B \in \mathcal{B}_2(H)$, namely

$$\begin{aligned}
&\left|\mathrm{tr}(PB^*A)\,\mathrm{tr}\left(P|C|^2\right) - \mathrm{tr}(PC^*A)\,\mathrm{tr}(PB^*C)\right|^2 \\
&\leq \left[\mathrm{tr}\left(P|A|^2\right)\mathrm{tr}\left(P|C|^2\right) - |\mathrm{tr}(PC^*A)|^2\right] \\
&\times \left[\mathrm{tr}\left(P|B|^2\right)\mathrm{tr}\left(P|C|^2\right) - |\mathrm{tr}(PB^*C)|^2\right],
\end{aligned} \tag{38}$$

where for the last term we used the equality $|\langle B, C\rangle_{2,P}|^2 = |\langle C, B\rangle_{2,P}|^2$.

Since (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ) -property, then by (30) we have

$$0 \leq \mathrm{tr}\left(P|A|^2\right)\mathrm{tr}\left(P|C|^2\right) - |\mathrm{tr}(PC^*A)|^2 \leq \frac{1}{4} \cdot \frac{|\gamma-\Gamma|^2}{\mathrm{Re}(\Gamma\overline{\gamma})}\,|\mathrm{tr}(PC^*A)|^2 \tag{39}$$

and

$$0 \le \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2 \le \frac{1}{4} \cdot \frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)} |\operatorname{tr}(PB^*C)|^2 . \tag{40}$$

If we multiply the inequalities (39) and (40) we get

$$\begin{aligned}
&\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right] \\
&\times \left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2\right] \\
&\le \frac{1}{16} \cdot \frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)} |\operatorname{tr}(PC^*A)|^2 |\operatorname{tr}(PB^*C)|^2 .
\end{aligned} \tag{41}$$

If we use (38) and (41) we get

$$\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\
&\le \frac{1}{16} \cdot \frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)} |\operatorname{tr}(PC^*A)|^2 |\operatorname{tr}(PB^*C)|^2 .
\end{aligned} \tag{42}$$

Since P, A, B, $C \neq 0$ then by (39) and (40) we get $\operatorname{tr}(PC^*A) \neq 0$ and $\operatorname{tr}(PB^*C) \neq 0$. Now, if we take the square root in (42) and divide by $|\operatorname{tr}(PC^*A)$ $\operatorname{tr}(PB^*C)|$ we obtain the desired result (37). □

Corollary 4 *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *with* $P|A|^2$, $P|B|^2 \neq 0$ *and* $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma})$, $\operatorname{Re}\left(\Delta\overline{\delta}\right) > 0$. *If* A *has the trace* P-(λ, Γ)*-property and* B *has the trace* P-(δ, Δ)*-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB^*)} - 1\right| \le \frac{1}{4} \cdot \frac{|\gamma-\Gamma||\delta-\Delta|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{Re}\left(\Delta\overline{\delta}\right)}}. \tag{43}$$

The case of selfadjoint operators is useful for applications.

Remark 4 *Assume that* A, B, C *are selfadjoint operators. If, either* $P \in \mathcal{B}_+(H)$, $A, B, C \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B, C \in \mathcal{B}(H)$ *with* PA^2, PB^2, $PC^2 \neq 0$ *and* $m, M, n, N \in \mathbb{R}$ *with* $mM, nN > 0$. *If* (A, C) *has the trace* P-(m, M)*-property and* (B, C) *has the trace* P-(n, N)*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(PCA)\operatorname{tr}(PBC)} - 1\right| \le \frac{1}{4} \cdot \frac{(M-m)(N-n)}{\sqrt{mnMN}}. \tag{44}$$

If A *has the trace* P-(k,K)*-property and* B *has the trace* P-(l,L)*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB)}-1\right|\le\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}},\tag{45}$$

where $kK,\ lL>0$.

We observe that, if $0<k1_H\le A\le K1_H$ *and* $0<l1_H\le B\le L1_H$, *then by* (46) *we have*

$$|\operatorname{tr}(PBA)\operatorname{tr}(P)-\operatorname{tr}(PA)\operatorname{tr}(PB)|\le\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}}\operatorname{tr}(PA)\operatorname{tr}(PB)\tag{46}$$

or, equivalently

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|\le\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}.\tag{47}$$

5 Applications for convex functions

In the paper [30] we obtained amongst other the following reverse of the Jensen trace inequality:

$$\begin{aligned}
0&\le\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\\
&\le\frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\\
&\le\begin{cases}\frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\\[2ex]
\frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\end{cases}\\
&\le\begin{cases}\frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\\[2ex]
\frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\end{cases}\\
&\le\frac{1}{4}\left[f'(M)-f'(m)\right](M-m).
\end{aligned}\tag{48}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A)\subseteq[m,M]$

for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$, then by taking $P = I_n$ in (48) we get

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\le \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n} \\
&\le \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \dfrac{\operatorname{tr}\left(\left|A - \frac{\operatorname{tr}(A)}{n} 1_H\right|\right)}{n} \\ \\ \frac{1}{2}(M - m) \dfrac{\operatorname{tr}\left(\left|f'(A) - \frac{\operatorname{tr}(f'(A))}{n} 1_H\right|\right)}{n} \end{cases} \\
&\le \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}([f'(A)]^2)}{n} - \left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}\left[f'(M) - f'(m)\right](M - m).
\end{aligned} \tag{49}
$$

The following reverse inequality also holds:

Proposition 1 *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars* m, M *with* $0 < m < M$. *If* f *is a continuously differentiable convex function on* $[m, M]$ *with* $f'(m) > 0$ *and* $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$, *then we have*

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\
&\le \frac{1}{4} \cdot \frac{(M - m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}.
\end{aligned} \tag{50}
$$

The proof follows by the inequality (47) and the details are omitted.

Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ with $f'(m) > 0$ then by taking $P = I_n$ in (50) we get

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\le \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n} \\
&\le \frac{1}{4} \cdot \frac{(M-m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(A)}{n} \frac{\operatorname{tr}(f'(A))}{n}.
\end{aligned} \tag{51}
$$

We consider the power function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1, \infty)$, f is convex while for $r \in (0, 1)$, f is concave.

Let $r \ge 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \\
&\le r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right] \\
&\le \frac{1}{4} r \frac{(M-m)\left(M^{r-1} - m^{r-1}\right)}{m^{r/2} M^{r/2}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}.
\end{aligned} \tag{52}
$$

If we take the first and last term in (52) we get the inequality:

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(P)\operatorname{tr}(PA^r)}{\operatorname{tr}(PA)\operatorname{tr}(PA^{r-1})} - \frac{\operatorname{tr}(P)[\operatorname{tr}(PA)]^{r-1}}{\operatorname{tr}(PA^{p-1})[\operatorname{tr}(P)]^{r-1}} \\
&\le \frac{1}{4} r \frac{(M-m)\left(M^{r-1} - m^{r-1}\right)}{m^{r/2} M^{r/2}}.
\end{aligned} \tag{53}
$$

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Split equality monotone variational inclusions and fixed point problem of set-valued operator

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Abstract. In this paper, we are concerned with the split equality problem of finding an element in the zero point set of the sum of two monotone operators and in the common fixed point set of a finite family of quasi- nonexpansive set-valued mappings. Strong convergence theorems are established under suitable condition in an infinite dimensional Hilbert spaces. Some applications of the main results are also provided.

1 Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) was recently introduced by Censor and Elfving [1] and is formulated as

$$\text{to finding} \quad x^* \in C \quad \text{such that} \quad \mathcal{A}x^* \in Q, \tag{1}$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Such models were successfully developed for instance in radiation therapy treatment planning, sensor

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networks, resolution enhancement and so on [2, 3, 4]. Initiated by SFP, several split type problems have been investigated and studied, for example, the split common fixed point problem (SCFP) [5], the split variational inequality problem (SVIP) [6], and the split null point problem (SCNP) [7]. Many authors have studied the SFP in infinite-dimensional Hilbert spaces, see, for example, [8-13] and some of the references therein.

Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators, see [14-17]. The central problem is to iteratively find a zero point of the sum of two monotone operators, that is, $0 \in (A + B)(x)$. Many real world problems can be formulated as a problem of the above form. For instance, a stationary solution to the initial value problem of the evolution equation

$$\begin{cases} 0 \in Fu + \frac{\partial u}{\partial t}, \\ u_0 = u(0), \end{cases} \tag{2}$$

can be recast as the inclusion problem when the governing maximal monotone F is of the form $F = A+B$; for more details, see [14] and the references therein.

Let $F : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $G : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be set-valued mappings with nonempty values, and let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be mappings. Then, inspired by the work in [6], Moudafi [18] introduced the following split monotone variational inclusion problem (SMVIP):

$$\begin{cases} \text{find} \qquad x^* \in \mathcal{H}_1 \qquad \text{such that} \qquad 0 \in f(x^*) + F(x^*), \\ \text{ and such that} \qquad y^* = \mathcal{A}x^* \in \mathcal{H}_1 \qquad \text{solves} \qquad g(y^*) + G(y^*). \end{cases} \tag{3}$$

Moudafi [18], present an algorithm for solving the SMVIP and obtain a weak convergence theorem for the algorithm.

Very recently, Moudafi [19] introduced the following split equality problem. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be real Hilbert spaces. Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be two bounded linear operators, let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$. The split equality problem (SEP) is to find

$$x \in C, \quad y \in Q \qquad \text{such that} \quad \mathcal{A}x = \mathcal{B}y, \tag{4}$$

Obviously, if $\mathcal{B} = I$ and $\mathcal{H}_2 = \mathcal{H}_3$ then (SEP) reduces to (SFP). This kind of split equality problem allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, such as decomposition methods for PDEs, applications in game theory, and intensity-modulated radiation therapy, (see [20, 21]).

Each nonempty closed convex subset of a Hilbert space can be regarded as a set of fixed points of a projection. In [22], Moudafi introduced the following split equality fixed point problem:

Let $\mathcal{A}:\mathcal{H}_1\to\mathcal{H}_3$, $\mathcal{B}:\mathcal{H}_2\to\mathcal{H}_3$ be two bounded linear operators, let $S:\mathcal{H}_1\to\mathcal{H}_1$ and $T:\mathcal{H}_2\to\mathcal{H}_2$ be two nonlinear operators such that $\mathrm{Fix}(S)\neq\emptyset$ and $\mathrm{Fix}(T)\neq\emptyset$. The split equality fixed point problem (SEFP) is to find

$$x\in\mathrm{Fix}(S),\quad y\in\mathrm{Fix}(T)\qquad\text{such that}\quad \mathcal{A}x=\mathcal{B}y. \tag{5}$$

Moudafi [22], proposed some algorithms for solving the split equality fixed point problem. In these algorithms we need to compute norm of the operators, which is difficult. To solve the split equality fixed point problem for quasi-nonexpansive mappings, Zhao [23] proposed the following iteration algorithm which does not require any knowledge of the operator norms:

Theorem 1 *Let $\mathcal{H}_1,\mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A}:\mathcal{H}_1\to\mathcal{H}_3$ and $\mathcal{B}:\mathcal{H}_2\to\mathcal{H}_3$ be bounded linear operators. Let $S:\mathcal{H}_1\to\mathcal{H}_1$ and $T:\mathcal{H}_2\to\mathcal{H}_2$ be quasi-nonexpansive mappings such that $S-I$ and $T-I$ are demiclosed at 0. Suppose $\Omega=\{x\in\mathrm{Fix}(S),y\in\mathrm{Fix}(T):\mathcal{A}x=\mathcal{B}y\}\neq\emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0\in\mathcal{H}_1,\quad y_0\in\mathcal{H}_2$ and by*

$$\begin{cases} u_n=x_n-\gamma_n\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\\ x_{n+1}=\beta_n u_n+(1-\beta_n)\,S(u_n),\\ w_n=y_n+\gamma_n\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\\ y_{n+1}=\beta_n w_n+(1-\beta_n)\,T(w_n),\qquad \forall n\geq 0. \end{cases} \tag{6}$$

Assume that the step-size γ_n is chosen in such a way that

$$\gamma_n\in\Big(\epsilon,\frac{2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2}-\epsilon\Big),n\in\Pi$$

otherwise $\gamma_n=\gamma$ (γ being any nonnegative value), where the index set $\Pi=\{n:\mathcal{A}x_n-\mathcal{B}y_n\neq 0\}$. Let $\{\alpha_n\}\subset(\delta,1-\delta)$ and $\{\beta_n\}\subset(\eta,1-\eta)$ for small enough $\delta,\eta>0$. Then, the sequences $\{(x_n,y_n)\}$ converges weakly to $(x^\star,y^\star)\in\Omega$.

On the other hand, in the last years, many authors studied the problems of finding a common element of the set of zero point of the sum of two monotone operators and the set of fixed points of nonlinear operators, see [24, 25]. The motivation for studying such a problem is in its possible application to mathematical models whose constraints can be expressed as fixed-point problems and/or variational inclusion problem: see, for instance, [26, 27].

Now, we consider the following split equality monotone variational inclusions and fixed point problem:

Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $F : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $G : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be set-valued mappings with nonempty values, and let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be mappings. Let for $i \in \{1, 2, ..., m\}$, $T_i : \mathcal{H}_1 \to CB(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ be two finite family of set valued mappings. We find a point

$$x \in \bigcap_{i=1}^{m} Fix(T_i) \bigcap (f + F)^{-1}(0), \quad \text{and}$$
$$y \in \bigcap_{i=1}^{m} Fix(S_i) \bigcap (g + G)^{-1}(0) \quad \text{such that} \quad \mathcal{A}x = \mathcal{B}y.$$

Motivated by the above works, the purpose of this paper is to introduce a new algorithm for the split equality problem for finding an element in the zero point set of the sum of two operators which are inverse-strongly monotone and a maximal monotone and in the common fixed point set of a finite family of quasi-nonexpansive set-valued mappings. Under suitable conditions, we prove that the sequences generated by the proposed new algorithm converges strongly to a solution of the split equality problem in Hilbert spaces. Our results improve and generalize the result of Takahashi et al. [11], Moudafi [18, 22], Censor et al. [6], Zhao [23], and many others.

2 Preliminaries

A subset $E \subset \mathcal{H}$ is called proximal if for each $x \in \mathcal{H}$, there exists an element $y \in E$ such that

$$\| x - y \| = dist(x, E) = \inf\{\| x - z \| : z \in E\}.$$

We denote by $CB(E), CC(E), K(E)$ and $P(E)$ the collection of all nonempty closed bounded subsets, nonempty closed convex subsets, nonempty compact subsets, and nonempty proximal bounded subsets of E respectively. The Hausdorff metric $\mathfrak{h}$ on $CB(\mathcal{H})$ is defined by

$$\mathfrak{h}(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(\mathcal{H})$.

Let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping. An element $x \in \mathcal{H}$ is said to be a fixed point of T, if $x \in Tx$. We use $\mathrm{Fix}(T)$ to denote the set of all fixed points of T. An element $x \in \mathcal{H}$ is said to be an endpoint of a set-valued mapping T if x is a fixed point of T and $T(x) = \{x\}$. We say that T satisfies the endpoint condition if each fixed point of T is an endpoint of T. We also say that a family of set-valued mapping $T_i, (i = 1, 2, ..., m)$ satisfies the common endpoint condition if $T_i(x) = \{x\}$ for all $x \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$.

Definition 1 *A set-valued mapping* $T : \mathcal{H} \to CB(\mathcal{H})$ *is called*

(i) *nonexpansive if*

$$\mathfrak{h}(Tx, Ty) \leq \|x - y\|, \quad x, y \in \mathcal{H}.$$

(ii) *quasi-nonexpansive if* $\mathrm{Fix}(T) \neq \emptyset$ *and* $\mathfrak{h}(Tx, Tp) \leq \|x - p\|$ *for all* $x \in \mathcal{H}$ *and all* $p \in \mathrm{Fix}(T)$.

(iii) *generalized nonexpansive* [28] *if*

$$\mathfrak{h}(Tx, Ty) \leq \mu \,\mathrm{dist}(x, Tx) + \|x - y\|, \quad x, y \in \mathcal{H},$$

for some $\mu > 0$.

It is obvious that every generalized nonexpansive set- valued mapping with nonempty fixed point set $\mathrm{Fix}(T)$ is quasi-nonexpansive.

We use the following notion in the sequel:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

Definition 2 *Let* E *be a nonempty subset of a real Hilbert space* $\mathcal{H}$ *and let* $T : E \to CB(E)$ *be a set-valued mapping. The mapping* $I - T$ *is said to be demiclosed at zero if for any sequence* $\{x_n\}$ *in* E*, the conditions* $x_n \rightharpoonup x^*$ *and* $\lim_{n\to\infty} \mathrm{dist}(x_n, Tx_n) = 0$*, imply* $x^* \in \mathrm{Fix}(T)$.

The proof of the following result is similar to the proof of Theorem 3.4 in [29], and so is not included.

Lemma 1 *Let* E *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$*. Let* $T : E \to K(E)$ *be a generalized nonexpansive set- valued mapping. Then* $I - T$ *is demiclosed in zero.*

Lemma 2 [30] *Let* E *be a closed convex subset of a real Hilbert space* $\mathcal{H}$*. Let* $T : E \to CB(E)$ *be a quasi-nonexpansive set-valued mapping satisfies the endpoint condition. Then* $\mathrm{Fix}(T)$ *is closed and convex.*

Given a nonempty closed convex set $C \subset \mathcal{H}$, the mapping that assigns every point $x \in \mathcal{H}$, to its unique nearest point in C is called the metric projection onto C and is denoted by P_C; i.e., $P_C \in C$ and $\|x - P_C x\| = \inf_{y \in C} \|x - y\|$. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x) \rangle \le 0, \qquad \forall x \in \mathcal{H}, y \in C.$$

The metric projection, P_C, satisfies the nonexpansivity condition with $\mathrm{Fix}(P_C) = C$.

Let $f : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator. It is well known that the Variational Inequality Problem is to find $u \in E$ such that

$$\langle fu, v - u \rangle \ge 0, \qquad \forall v \in E. \tag{7}$$

We denote by $VI(E, f)$ the solution set of (7). The operator $f : \mathcal{H} \to \mathcal{H}$ is called Inverse strongly monotone with constant $\beta > 0$, $(\beta - ism)$ if

$$\langle f(x) - f(y), x - y \rangle \ge \beta \|f(x) - f(y)\|^2, \quad \forall x, y \in E.$$

It is known that if f is β- inverse strongly monotone, and $\lambda \in (0, 2\beta)$ then $P_E(I - \lambda f)$ is nonexpansive, where P_E is the metric projection onto E.

Let F be a mapping of $\mathcal{H}$ into $2^{\mathcal{H}}$. The effective domain of F is denoted by $dom(F)$, that is, $dom(F) = \{x \in \mathcal{H} : Fx \neq \emptyset\}$. A multi-valued mapping F is said to be a monotone operator on $\mathcal{H}$ if $\langle u - v, x - y \rangle \ge 0$, for all $x, y \in dom(F)$, $u \in Fx$ and $v \in Fy$. Classical examples of monotone operators are subdifferential operators of functions that are convex, lower semicontinuous, and proper; linear operators with a positive symmetric part. See, e.g. [31, 32]. A monotone operator F on $\mathcal{H}$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $\mathcal{H}$. For a maximal monotone operator F on $\mathcal{H}$ and $r > 0$, the resolvent of F for r is $J_r^F = (I + rF)^{-1} : \mathcal{H} \to dom(F)$. This operator enjoys many important properties that make it a central tool in monotone operator theory and its applications. In particular, it is single-valued, firmly nonexpansive in the sense that

$$\|J_r^F x - J_r^F y\|^2 \le \langle x - y, J_r^F x - J_r^F y \rangle, \qquad \forall x, y \in \mathcal{H}.$$

Finally, the set $\mathrm{Fix}(J_r^F) = \{x \in \mathcal{H} : J_r^F x = x\}$ of fixed points of J_r^F coincides with $F^{-1}(0)$.

Lemma 3 [33] *For each* $x_1, \cdots, x_m \in \mathcal{H}$ *and* $\alpha_1, \cdots, \alpha_m \in [0, 1]$ *with* $\sum_{i=1}^m \alpha_i = 1$ *the equality*

$$\|\alpha_1 x_1 + ... + \alpha_m x_m\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1 \le i < j \le m} \alpha_i \alpha_j \|x_i - x_j\|^2,$$

holds.

Lemma 4 [34] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\vartheta_n)a_n + \vartheta_n\delta_n, \qquad n \geq 0,$$

where $\{\vartheta_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\vartheta_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 [35] *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $(\Gamma_{n_j})_{j\geq 0}$ *of* (Γ_n) *such that* $\Gamma_{n_j} < \Gamma_{n_j+1}$ *for all* $j \geq 0$*. Also consider the sequence of integers* $(\tau(n))_{n\geq n_0}$ *defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $(\tau(n))_{n\geq n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty} \tau(n) = \infty$, *and, for all* $n \geq n_0$*, the following two estimates hold:*

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \qquad \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

3 Algorithm and convergence theorem

The main result of this paper is the following.

Theorem 2 *Let* $\mathcal{H}_1, \mathcal{H}_2$ *and* $\mathcal{H}_3$*, be real Hilbert spaces,* $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ *and* $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ *be bounded linear operators. Let* $f : \mathcal{H}_1 \to \mathcal{H}_1$ *and* $g : \mathcal{H}_2 \to \mathcal{H}_2$ *be respectively* α *and* β*- inverse strongly monotone operators and* F, G *two maximal monotone operators on* $\mathcal{H}_1, \mathcal{H}_2$*. Let for* $i \in \{1,2,...,m\}$*,* $T_i : \mathcal{H}_1 \to CB(\mathcal{H}_1)$ *and* $S_i : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ *be two finite families of quasi-nonexpansive set valued mappings such that* $S_i - I$ *and* $T_i - I$ *are demiclosed at* 0*, and* S_i *and* T_i *satisfies the common endpoint condition. Suppose* $\Omega = \{x \in \bigcap_{i=1}^{m} Fix(T_i) \bigcap (f+F)^{-1}(0), \quad y \in \bigcap_{i=1}^{m} Fix(S_i) \bigcap (g+G)^{-1}(0) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ *and by*

$$\begin{cases} z_n = x_n - \gamma_n\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = J_{\lambda_n}^F(I - \lambda_n f)z_n, \\ x_{n+1} = \alpha_n\vartheta + \beta_n u_n + \sum_{i=1}^{m} \delta_{n,i}\nu_{n,i} \\ w_n = y_n + \gamma_n\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = J_{\mu_n}^G(I - \mu_n g)w_n, \\ y_{n+1} = \alpha_n\zeta + \beta_n t_n + \sum_{i=1}^{m} \delta_{n,i}s_{n,i} \qquad \forall n \geq 0, \end{cases} \tag{8}$$

where $\nu_{n,i} \in T_i u_n$, $s_{n,i} \in S_i t_n$ *and the step-size* γ_n *is chosen in such a way that*

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon), n \in \Pi$$

otherwise $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the index set* $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. *Let the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\lambda_n\}$ *and* $\{\mu_n\}$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \sum_{i=1}^{m} \delta_{n,i} = 1$, *and* $\liminf_n \beta_n \delta_{n,i} > 0$ *for each* $i \in \{1, 2, ..., m\}$,

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ *and* $\{\mu_n\} \subset [c, d] \subset (0, 2\beta)$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequences $\{(x_n, y_n)\}$ *converges strongly to* $(x^\star, y^\star) \in \Omega$.

Proof. Firstly, we prove that $\{x_n\}$ and $\{y_n\}$ are bounded. Take $(x^\star, y^\star) \in \Omega$. It is obvious that $J^F_{\lambda_n}(x^\star - \lambda_n f x^\star) = x^\star$. Since the operator $J^F_{\lambda_n}$ is nonexpansive and f is $\alpha-$ inverse strongly monotone we have

$$\begin{aligned} \|u_n - x^\star\|^2 &= \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - J^F_{\lambda_n}(x^\star - \lambda_n f x^\star)\|^2 \\ &\leq \|(z_n - \lambda_n f z_n) - (x^\star - \lambda_n f x^\star)\|^2 \\ &= \|(z_n - x^\star) - \lambda_n(f z_n - f x^\star)\|^2 \\ &= \|z_n - x^\star\|^2 - 2\lambda_n\langle z_n - x^\star, f z_n - f x^\star\rangle + \lambda_n^2\|f z_n - f x^\star\|^2 \\ &\leq \|z_n - x^\star\|^2 - 2\lambda_n \alpha\|f z_n - f x^\star\|^2 + \lambda_n^2\|f z_n - f x^\star\|^2 \\ &= \|z_n - x^\star\|^2 + \lambda_n(\lambda_n - 2\alpha)\|f z_n - f x^\star\|^2. \end{aligned} \tag{9}$$

Similarly, we obtain that

$$\|t_n - y^\star\|^2 \leq \|w_n - y^\star\|^2 + \mu_n(\mu_n - 2\beta)\|g w_n - g y^\star\|^2. \tag{10}$$

By Lemma 3 and inequality (9), we have

$$
\begin{aligned}
\|x_{n+1}-x^\star\|^2 &= \|\alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^{m} \delta_{n,i} v_{n,i} - x^\star\|^2 \\
&\leq \alpha_n \|\vartheta - x^\star\|^2 + \beta_n \|u_n - x^\star\|^2 \\
&\quad + \sum_{i=1}^{m} \delta_{n,i} \|v_{n,i} - x^\star\|^2 - \sum_{i=1}^{m} \beta_n \delta_{n,i} \|v_{n,i} - u_n\|^2 \\
&= \alpha_n \|\vartheta - x^\star\|^2 + \beta_n \|u_n - x^\star\|^2 \\
&\quad + \sum_{i=1}^{m} \delta_{n,i} \mathrm{dist}(v_{n,i}, T_i x^\star)^2 - \sum_{i=1}^{m} \beta_n \delta_{n,i} \|v_{n,i} - u_n\|^2 \\
&\leq \alpha_n \|\vartheta - x^\star\|^2 + \beta_n \|u_n - x^\star\|^2 \\
&\quad + \sum_{i=1}^{m} \delta_{n,i} \mathfrak{h}(T_i u_n, T_i x^\star)^2 - \sum_{i=1}^{m} \beta_n \delta_{n,i} \|v_{n,i} - u_n\|^2 \\
&\leq \alpha_n \|\vartheta - x^\star\|^2 + \beta_n \|u_n - x^\star\|^2 \\
&\quad + \sum_{i=1}^{m} \delta_{n,i} \|u_n - x^\star\|^2 - \sum_{i=1}^{m} \beta_n \delta_{n,i} \|v_{n,i} - u_n\|^2 \\
&\leq \alpha_n \|\vartheta - x^\star\|^2 + (1-\alpha_n) \|z_n - x^\star\|^2 \\
&\quad - \sum_{i=1}^{m} \beta_n \delta_{n,i} \|v_{n,i} - u_n\|^2 \\
&(1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|f z_n - f x^\star\|^2.
\end{aligned} \tag{11}
$$

Similarly, from inequality (10) we have

$$
\begin{aligned}
\|y_{n+1}-y^\star\|^2 &= \|\alpha_n \zeta + \beta_n t_n + \sum_{i=1}^{m} \delta_{n,i} s_{n,i} - y^\star\|^2 \\
&\leq \alpha_n \|\zeta - y^\star\|^2 + \beta_n \|t_n - y^\star\|^2 \\
&\quad + \sum_{i=1}^{m} \delta_{n,i} \|s_{n,i} - y^\star\|^2 - \sum_{i=1}^{m} \beta_n \delta_{n,i} \|s_{n,i} - t_n\|^2 \\
&\leq \alpha_n \|\zeta - y^\star\|^2 + (1-\alpha_n) \|w_n - y^\star\|^2 \\
&\quad - \sum_{i=1}^{m} \beta_n \delta_{n,i} \|s_{n,i} - t_n\|^2 \\
&\quad + (1-\alpha_n)\mu_n(\mu_n - 2\beta)\|g w_n - g y^\star\|^2.
\end{aligned} \tag{12}
$$

From algorithm (8) we have that

$$
\begin{aligned}
\|z_n - x^\star\|^2 &= \|x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) - x^\star\|^2 \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&\quad - 2\gamma_n \langle x_n - x^\star, \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&\quad - 2\gamma_n \langle \mathcal{A}x_n - \mathcal{A}x^\star, (\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n \|\mathcal{A}x_n - \mathcal{A}x^\star\|^2 \\
&\quad - \gamma_n \|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n \|\mathcal{B}y_n - \mathcal{A}x^\star\|^2.
\end{aligned} \tag{13}
$$

By similar way we obtain that

$$
\begin{aligned}
\|w_n - y^\star\|^2 &= \|y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) - y^\star\|^2 \\
&= \|y_n - y^\star\|^2 + \gamma_n^2 \|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n \|\mathcal{B}y_n - \mathcal{B}y^\star\|^2 \\
&\quad - \gamma_n \|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n \|\mathcal{A}x_n - \mathcal{B}y^\star\|^2.
\end{aligned} \tag{14}
$$

By adding the two last inequalities and by taking into account the fact that $\mathcal{A}x^\star = \mathcal{B}y^\star$ we obtain

$$
\begin{aligned}
\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2 &= \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2 \\
&\quad - \gamma_n [2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 \\
&\quad - \gamma_n (\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&\quad + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)] \\
&\leq \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2.
\end{aligned} \tag{15}
$$

This implies that

$$
\begin{aligned}
&\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2 \leq (1 - \alpha_n)(\|z_n - x^\star\|^2 \\
&\qquad + \|w_n - y^\star\|^2) + \alpha_n (\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\quad \leq (1 - \alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) + \alpha_n (\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\quad \leq \max\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\} \\
&\quad \vdots \\
&\quad \leq \max\{\|x_0 - x^\star\|^2 + \|y_0 - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\}.
\end{aligned} \tag{16}
$$

Thus $\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2$ is bounded. Therefore $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently $\{z_n\}, \{w_n\}, \{u_n\}$ and $\{v_n\}$ are all bounded. From (11),

(12) and (15) we have that

$$
\begin{aligned}
&\|x_{n+1}-x^\star\|^2+\|y_{n+1}-y^\star\|^2\\
&\quad\le(1-\alpha_n)(\|z_n-x^\star\|^2+\|w_n-y^\star\|^2)+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2)\\
&\qquad-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|s_{n,i}-t_n\|^2\\
&\qquad-(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|fz_n-fx^\star\|^2\\
&\qquad-(1-\alpha_n)\mu_n(2\beta-\mu_n)\|gw_n-gy^\star\|^2\\
&\quad\le(1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2)+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2) &(17)\\
&\qquad-(1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2-\gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2\\
&\qquad+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)]\\
&\qquad-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|s_{n,i}-t_n\|^2\\
&\qquad-(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|fz_n-fx^\star\|^2\\
&\qquad-(1-\alpha_n)\mu_n(2\beta-\mu_n)\|gw_n-gy^\star\|^2.
\end{aligned}
$$

From above inequality we have that

$$
\begin{aligned}
(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|fz_n-fx^\star\|^2\le{}&(1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2)\\
&-\|x_{n+1}-x^\star\|^2-\|y_{n+1}-y^\star\|^2 \qquad (18)\\
&+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2).
\end{aligned}
$$

By our assumption that

$$\gamma_n\in(\epsilon,\frac{2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2}-\epsilon),$$

we have that

$$(\gamma_n+\epsilon)\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2\le2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2.$$

From above inequality and inequality (17) we have that

$$
\begin{aligned}
&(1-\alpha_n)\gamma_n\epsilon(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)\\
&\quad\le(1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2-\gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2\\
&\qquad+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)] &(19)\\
&\quad\le(1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2)-\|x_{n+1}-x^\star\|^2-\|y_{n+1}\\
&\qquad-y^\star\|^2+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2).
\end{aligned}
$$

Put $\Gamma_n = \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2$ for all $n \in \mathbb{N}$. We finally analyze the inequalities (18) and (19) by considering the following two cases.

Case A. Suppose that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq n_0$ (for n_0 large enough). In this case, since Γ_n is bounded, the limit $\lim_{n\to\infty} \Gamma_n$ exists. Since $\lim_{n\to\infty} \alpha_n = 0$, from (19) and by our assumption that on $\{\gamma_n\}$ we have

$$\lim_{n\to\infty} (\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2) = 0.$$

So we obtain that $\lim_{n\to\infty} \|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\| = 0$ and $\lim_{n\to\infty} \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\| = 0$. This implies that $\lim_{n\to\infty} \|\mathcal{A}x_n - \mathcal{B}y_n\| = 0$. Also from (18) we deduce

$$\lim_{n\to\infty} (1 - \alpha_n)\lambda_n(2\alpha - \lambda_n)\|fz_n - fx^\star\|^2 = 0$$

By our assumption that $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$, we obtain that

$$\lim_{n\to\infty} \|fz_n - fx^\star\| = 0. \tag{20}$$

By similar argument, from inequality (17) we get that

$$\lim_{n\to\infty} \|gw_n - gy^\star\| = \lim_{n\to\infty} \|v_{n,i} - u_n\| = \lim_{n\to\infty} \|s_{n,i} - t_n\| = 0, \tag{21}$$

Since $\mathtt{dist}(u_n, T_i u_n) \leq \|v_{n,i} - u_n\|$ we have

$$\lim_{n\to\infty} \mathtt{dist}(u_n, T_i u_n) = 0, \qquad i \in \{1, 2, \dots, m\}. \tag{22}$$

Similarly, from (21) we arrive at

$$\lim_{n\to\infty} \mathtt{dist}(t_n, S_i t_n) = 0, \qquad i \in \{1, 2, \dots, m\}. \tag{23}$$

By using the firm nonexpansivity of $J^F_{\lambda_n}$ and noticing that $J^F_{\lambda_n}(x^\star - \lambda_n fx^\star) = x^\star$ we obtain

$$\begin{aligned}
\|u_n - x^\star\|^2 &= \|J^F_{\lambda_n}(z_n - \lambda_n fz_n) - J^F_{\lambda_n}(x^\star - \lambda_n fx^\star)\|^2 \\
&\leq \langle (z_n - \lambda_n fz_n) - (x^\star - \lambda_n fx^\star), J^F_{\lambda_n}(z_n - \lambda_n fz_n) - J^F_{\lambda_n}(x^\star - \lambda_n fx^\star)\rangle \\
&= \frac{1}{2}(\|(z_n - \lambda_n fz_n) - (x^\star - \lambda_n fx^\star)\|^2 + \|J^F_{\lambda_n}(z_n - \lambda_n fz_n) - x^\star)\|^2 \\
&\quad - \|(z_n - \lambda_n fz_n) - (x^\star - \lambda_n fx^\star) - (J^F_{\lambda_n}(z_n - \lambda_n fz_n) - x^\star)\|^2)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}(\|z_n - x^\star\|^2 + \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - x^\star\|^2 \\
&\quad - \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n) - \lambda_n(f z_n - f x^\star)\|^2) \\
&= \frac{1}{2}(\|z_n - x^\star\|^2 + \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - x^\star)\|^2 \\
&\quad - \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2) \\
&\quad + 2\lambda_n\langle z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n), f z_n - f x^\star\rangle - \lambda_n^2\|f z_n - f x^\star\|^2).
\end{aligned} \tag{24}
$$

Which implies that

$$
\begin{aligned}
&\|u_n - x^\star\|^2 \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - x^\star\|^2 \\
&\quad \le \|z_n - x^\star\|^2 - \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 \\
&\quad\quad + 2\lambda_n\langle z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n), f z_n - f x^\star\rangle - \lambda_n^2\|f z_n - f x^\star\|^2.
\end{aligned} \tag{25}
$$

Utilizing Lemma 3 and inequality (25) we get

$$
\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|\alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^{m} \delta_{n,i} v_{n,i} - x^\star\|^2 \\
&\le \alpha_n\|\vartheta - x^\star\|^2 + \beta_n\|u_n - x^\star\|^2 + \sum_{i=1}^{m} \delta_{n,i}\|v_{n,i} - x^\star\|^2 \\
&\le \alpha_n\|\vartheta - x^\star\|^2 + \beta_n\|u_n - x^\star\|^2 + \sum_{i=1}^{m} \delta_{n,i}\|u_n - x^\star\|^2 \\
&\le \alpha_n\|\vartheta - x^\star\|^2 + (1 - \alpha_n)\|u_n - x^\star\|^2 \\
&\le \alpha_n\|\vartheta - x^\star\|^2 + (1 - \alpha_n)\|z_n - x^\star\|^2 \\
&\quad - (1 - \alpha_n)\|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 \\
&\quad + 2(1 - \alpha_n)\lambda_n\langle z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n), f z_n - f x^\star\rangle \\
&\quad - (1 - \alpha_n)\lambda_n^2\|f z_n - f x^\star\|^2 \\
&\le \alpha_n\|\vartheta - x^\star\|^2 + (1 - \alpha_n)\|z_n - x^\star\|^2 \\
&\quad - (1 - \alpha_n)\|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 \\
&\quad + 2(1 - \alpha_n)\lambda_n\|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|\|f z_n - f x^\star\|.
\end{aligned} \tag{26}
$$

By similar argument we obtain

$$
\begin{aligned}
\|y_{n+1}-y^\star\|^2 &= \|\alpha_n \zeta + \beta_n t_n + \sum_{i=1}^{m} \delta_{n,i} s_{n,i} - y^\star\|^2 \\
&\le \alpha_n \|\zeta - y^\star\|^2 + (1-\alpha_n)\|w_n - y^\star\|^2 \\
&\quad - (1-\alpha_n)\|w_n - J^G_{\mu_n}(w_n - \mu_n g w_n)\|^2 \\
&\quad + 2(1-\alpha_n)\mu_n \|w_n - J^G_{\mu_n}(w_n - \mu_n g w_n)\| \|g w_n - g y^\star\|.
\end{aligned} \tag{27}
$$

By adding the inequality (26) and the inequality (27) we get

$$
\begin{aligned}
&\|x_{n+1}-x^\star\|^2 + \|y_{n+1}-y^\star\|^2 \\
&\quad \le (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\qquad - (1-\alpha_n)\|(z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 \\
&\qquad - (1-\alpha_n)\|(w_n - J^G_{\mu_n}(w_n - \mu_n g w_n)\|^2 \\
&\qquad + 2(1-\alpha_n)\lambda_n \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\| \|f z_n - f x^\star\| \\
&\qquad + 2(1-\alpha_n)\mu_n \|w_n - J^G_{\mu_n}(w_n - \mu_n g w_n)\| \|g w_n - g y^\star\|.
\end{aligned} \tag{28}
$$

Consequently,

$$
\begin{aligned}
(1-\alpha_n)\|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 &\le \Gamma_n - \Gamma_{n+1} + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\quad + 2(1-\alpha_n)\lambda_n \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\| \|f z_n - f x^\star\| \\
&\quad + 2(1-\alpha_n)\mu_n \|w_n - J^G_{\mu_n}(w_n - \mu_n g w_n)\| \|g w_n - g y^\star\|.
\end{aligned} \tag{29}
$$

This implies that

$$
\lim_{n\to\infty} \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\| = 0. \tag{30}
$$

By similar argument we obtain

$$
\lim_{n\to\infty} \|w_n - J^G_{\mu_n}(w_n - \mu_n g w_n)\| = 0. \tag{31}
$$

Since $\|z_n - x_n\| = \gamma_n \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|$ and $\{\gamma_n\}$ is bounded, we have

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{32}
$$

From (30) and (32) we have

$$
\|x_n - u_n\| \le \|x_n - z_n\| + \|z_n - u_n\| \to 0, \quad \text{as} \quad n \to \infty.
$$

Therefore

$$
\|x_{n+1} - x_n\| \le \alpha_n \|\vartheta - x_n\| + \beta_n \|u_n - x_n\| + \sum_{i=1}^{m} \delta_{n,i} \|v_{n,i} - x_n\| \to 0, \text{ as } n \to \infty. \tag{33}
$$

Similarly we have that $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$.

Now we claim that $(\omega_w(x_n),\omega_w(y_n))\subset\Omega$, where

$$\omega_w(x_n)=\{x\in\mathcal{H}_1 : x_{n_i}\rightharpoonup x \text{ for some subsequence} \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded we have $\omega_w(x_n)$ and $\omega_w(y_n)$ are nonempty. Now, take $\widehat{x}\in\omega_w(x_n)$ and $\widehat{y}\in\omega_w(y_n)$. Thus, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\widehat{x}$. Without loss of generality, we can assume that $x_n\rightharpoonup\widehat{x}$. Now, we are in a position to show that $\widehat{x}\in(f+F)^{-1}(0)$. Since $\lim_{n\to\infty}\|z_n-x_n\|=0$, we have $z_n\rightharpoonup\widehat{x}$. By our assumption that f is α- inverse strongly monotone mapping we have

$$\langle z_n-\widehat{x}, fz_n-f\widehat{x}\rangle\geq\alpha\|fz_n-f\widehat{x}\|^2.$$

Now, from $z_n\rightharpoonup\widehat{x}$ we deduce $fz_n\to f\widehat{x}$. From $u_n=J^F_{\lambda_n}(z_n-\lambda_n fz_n)$, we have $z_n-\lambda_n fz_n\in(I+\lambda_n F)u_n$, hence $\frac{z_n-u_n}{\lambda_n}-fz_n\in Fu_n$. Since F is monotone, we get, for any $(u,v)\in F$ that

$$\langle u_n-u, \frac{z_n-u_n}{\lambda_n}-fz_n-v\rangle\geq 0.$$

Since $\lim_{n\to\infty}\|z_n-u_n\|=0$, we have $u_n\rightharpoonup\widehat{x}$. Now above inequality implies that

$$\langle\widehat{x}-u,-f\widehat{x}-v\rangle\geq 0.$$

This gives that $-f\widehat{x}\in F\widehat{x}$, that is $0\in(f+F)\widehat{x}$. This proves that $\widehat{x}\in(f+F)^{-1}(0)$. By similar argument we can obtain that $\widehat{y}\in(g+G)^{-1}(0)$. Next we show that $\widehat{x}\in\bigcap_{i=1}^m \mathrm{Fix}(T_i)$ and $\widehat{y}\in\bigcap_{i=1}^m \mathrm{Fix}(S_i)$. Since $\lim_{n\to\infty}\mathrm{dist}(T_i u_n, u_n)=0$ and $u_n\rightharpoonup\widehat{x}$, noticing the demiclosedness of T_i-I in 0, we get that $\widehat{x}\in\mathrm{Fix}(T_i)$ (for each $i\in\{1,2,...,m\}$). By similar argument we obtain that $\widehat{y}\in\bigcap_{i=1}^m\mathrm{Fix}(S_i)$. On the other hand, $\mathcal{A}\widehat{x}-\mathcal{B}\widehat{y}\in\omega_w(\mathcal{A}x_n-\mathcal{B}y_n)$ and weakly lower semi continuity of the norm imply that

$$\|\mathcal{A}\widehat{x}-\mathcal{B}\widehat{y}\|\leq\liminf_{n\to\infty}\|\mathcal{A}x_n-\mathcal{B}y_n\|=0.$$

Thus $(\widehat{x},\widehat{y})\in\Omega$. We also have the uniqueness of the weak cluster point of $\{x_n\}$ are $\{y_n\}$, (see [23] for details) which implies that the whole sequences $\{(x_n,y_n)\}$ weakly convergence to a point $(\widehat{x},\widehat{y})\in\Omega$. Put $C=\bigcap_{i=1}^m\mathrm{Fix}(T_i)\bigcap(f+F)^{-1}(0)$ and $Q=\bigcap_{i=1}^m\mathrm{Fix}(S_i)\bigcap(g+G)^{-1}(0)$. Next we prove that the sequences $\{(x_n,y_n)\}$ converges strongly to $(\vartheta^\star,\zeta^\star)$ where $\vartheta^\star=P_C\,\vartheta$ and $\zeta^\star=P_Q\,\zeta$. First we show that

$$\limsup_{n\to\infty}\langle\vartheta-\vartheta^\star, x_n-\vartheta^\star\rangle\leq 0. \tag{34}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle\vartheta-\vartheta^\star, x_{n_k}-\vartheta^\star\rangle = \limsup_{n\to\infty}\langle\vartheta-\vartheta^\star, x_n-\vartheta^\star\rangle.$$

Since $\{x_{n_k}\}$ converges weakly to $\widehat{x}$, it follows that

$$\limsup_{n\to\infty}\langle\vartheta-\vartheta^\star, x_n-\vartheta^\star\rangle = \lim_{k\to\infty}\langle\vartheta-\vartheta^\star, x_{n_k}-\vartheta^\star\rangle = \langle\vartheta-\vartheta^\star, \widehat{x}-\vartheta^\star\rangle \le 0. \quad (35)$$

By similar argument we obtain that

$$\limsup_{n\to\infty}\langle\zeta-\zeta^\star, y_n-\zeta^\star\rangle \le 0. \quad (36)$$

From the inequality, $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \qquad (\forall x, y \in \mathcal{H}_1)$, we find that

$$\begin{aligned}
\|x_{n+1}-\vartheta^\star\|^2 &\le \|\beta_n u_n + \sum_{i=1}^m \delta_{n,i}v_{n,i} - (1-\alpha_n)\vartheta^\star\|^2 + 2\alpha_n\langle\vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&= (1-\alpha_n)^2\|\frac{\beta_n}{(1-\alpha_n)}u_n + \frac{\sum_{i=1}^m\delta_{n,i}}{(1-\alpha_n)}v_{n,i} - \vartheta^\star\|^2 \\
&\quad + 2\alpha_n\langle\vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&\le \beta_n(1-\alpha_n)\|u_n-\vartheta^\star\|^2 + \sum_{i=1}^m \delta_{n,i}(1-\alpha_n)\|v_{n,i}-\vartheta^\star\|^2 \\
&\quad + 2\alpha_n\langle\vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&= (1-\alpha_n)(\beta_n + \sum_{i=1}^m\delta_{n,i})\|u_n-\vartheta^\star\|^2 + 2\alpha_n\langle\vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&\le (1-\alpha_n)^2\|u_n-\vartheta^\star\|^2 + 2\alpha_n\langle\vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle.
\end{aligned}$$

Similarly we obtain that

$$\|y_{n+1}-\zeta^\star\|^2 \le (1-\alpha_n)^2\|t_n-\zeta^\star\|^2 + 2\alpha_n\langle\zeta-\zeta^\star, y_{n+1}-\zeta^\star\rangle. \quad (37)$$

By adding the two last inequalities we have that

$$\begin{aligned}
\|x_{n+1}-\vartheta^\star\|^2 &+ \|y_{n+1}-\zeta^\star\|^2 \\
&\le (1-\alpha_n)^2(\|x_n-\vartheta^\star\|^2 + \|y_n-\zeta^\star\|^2) \\
&+ 2\alpha_n(\langle\vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle + \langle\zeta-\zeta^\star, y_{n+1}-\zeta^\star\rangle).
\end{aligned} \quad (38)$$

It immediately follows that

$$\begin{aligned}\Gamma_{n+1} &\le (1-\alpha_n)^2\Gamma_n + 2\alpha_n\eta_n \\ &= (1-2\alpha_n)\Gamma_n + \alpha_n^2\Gamma_n + 2\alpha_n\eta_n \\ &\le \quad (1-2\alpha_n)\Gamma_n + 2\alpha_n\{\tfrac{\alpha_n N}{2} + \eta_n) \\ &\le \quad (1-\rho_n)\Gamma_n + \rho_n\delta_n,\end{aligned} \tag{39}$$

where $\eta_n = \langle\vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle + \langle\zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle$, $N = \sup\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2 : n \ge 0\}$, $\rho_n = 2\alpha_n$ and $\delta_n = \frac{\alpha_n N}{2} + \eta_n$. It is easy to see that $\rho_n \to 0, \sum_{n=1}^{\infty}\rho_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Hence, all conditions of Lemma 4 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty}\Gamma_n = 0$. Consequently $\lim_{n\to\infty}\|x_n - \vartheta^\star\| = \lim_{n\to\infty}\|y_n - \zeta^\star\| = 0$, that is $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$.

Case B. Assume that $\{\Gamma_n\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$, $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. From (17), we deduce

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \le \alpha_n(\|\vartheta - \vartheta^\star\|^2 + \|\upsilon - \zeta^\star\|^2) - \alpha_n(\|x_n - \vartheta^\star\|^2 + \|y_n - \zeta^\star\|^2). \tag{40}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\{y_n\}$ and $\{x_n\}$ are bounded, we derive that

$$\lim_{n\to\infty}(\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)}) = 0. \tag{41}$$

Following an argument similar to that in $\mathsf{Case}\ A$ we have

$$\Gamma_{\tau(n)+1} \le (1 - \rho_{\tau(n)})\Gamma_{\tau(n)} + \rho_{\tau(n)}\delta_{\tau(n)},$$

where $\limsup_{n\to\infty}\delta_{\tau(n)} \le 0$. Since $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have

$$\rho_{\tau(n)}\Gamma_{\tau(n)} \le \rho_{\tau(n)}\delta_{\tau(n)}.$$

Since $\rho_{\tau(n)} > 0$ we deduce that

$$\Gamma_{\tau(n)} \le \delta_{\tau(n)}.$$

Hence $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. This together with (41), implies that $\lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$. Applying Lemma 5 to get

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_n\} \le \Gamma_{\tau(n)+1}. \tag{42}$$

Therefore $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$. This completes the proof.

□

As a consequence of our main result we have the following theorem for single valued mappings.

Theorem 3 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be respectively α and β- inverse strongly monotone operators and F, G two maximal monotone operators on $\mathcal{H}_1, \mathcal{H}_2$. Let for $i \in \{1, 2, ..., m\}$, $T_i : \mathcal{H}_1 \to \mathcal{H}_1$ and $S_i : \mathcal{H}_2 \to \mathcal{H}_2$ be two finite families of quasi-nonexpansive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Suppose $\Omega = \{x \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap (f + F)^{-1}(0), \quad y \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) \bigcap (g + G)^{-1}(0) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = J^F_{\lambda_n}(I - \lambda_n f) z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} T_i u_n \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = J^G_{\mu_n}(I - \mu_n g) w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} S_i t_n \qquad \forall n \geq 0. \end{cases} \tag{43}$$

Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$,$\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.1. Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Now, let $T : \mathcal{H} \to P(\mathcal{H})$ be a set- valued mapping and let

$$P_T(x) = \{y \in Tx : \|x - y\| = \mathrm{dist}(x, Tx)\}, \quad x \in \mathcal{H}.$$

It can be easily seen $\mathrm{Fix}(T) = \mathrm{Fix}(P_T)$. From this we have the following theorem.

Theorem 4 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be respectively α and β- inverse strongly monotone operators and F, G two maximal monotone operators on $\mathcal{H}_1, \mathcal{H}_2$. Let for $i \in \{1, 2, ..., m\}$, $T_i : \mathcal{H}_1 \to CC(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to CC(\mathcal{H}_2)$ be two finite families of set valued mappings such that $P_{S_i} : \mathcal{H}_1 \to \mathcal{H}_1$ and $P_{T_i} : \mathcal{H}_2 \to \mathcal{H}_2$ are generalized nonexpansive. Suppose $\Omega = \{x \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap (f + F)^{-1}(0), \quad y \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) \bigcap (g + G)^{-1}(0) : \mathcal{A}x =$*

$\mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = J^F_{\lambda_n}(I - \lambda_n f)z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} P_{T_i} u_n \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = J^G_{\mu_n}(I - \mu_n g)w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} P_{S_i} t_n \qquad \forall n \geq 0. \end{cases} \tag{44}$$

Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$,$\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.1. Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Remark 1 *In [11], Takahashi et al. present some algorithms for generalized split feasibility problem for finding fixed point of nonlinear single valued mappings and the zero point of a maximal monotone operator. They proved some weak convergence theorems for finding a solution of the generalized split feasibility problem. In this paper we present an algorithm for solving split equality problem for finding common fixed point of a finite family of quasi-nonexpansive set-valued mappings and the zero point of the sum of two monotone operators. Our algorithm do not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.*

Remark 2 *In [23], Zhao present a weak convergence theorem for solving split equality fixed point problem of quasi-nonexpansive mapping (see theorem 1.1 of this paper). In this paper we extend the result for solving split equality common fixed problem of a finite family of quasi-nonexpansive set valued mappings. We also present a strong convergence theorem which is more desirable than weak convergence.*

Remark 3 *Moudafi [18] and Censor et al. [6] present some algorithms for solving the split monotone variational inclusion problem. They establish some weak convergence theorems for these algorithms. In this paper we present an algorithm for split equality monotone variational inclusion problem. Our algorithm do not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.*

4 Application

In this section, using Theorem 3.1, we can obtain well-known and new strong convergence theorems in a Hilbert space.

Variational inequality

Let $\mathcal{H}$ be a Hilbert space, and let h be a proper lower semicontinuous convex function of $\mathcal{H}$ into $\mathbb{R}$. Then the subdifferential ∂h of h is defined as follows:

$$\partial h(x) = \{z \in \mathcal{H} : h(x) + \langle z, u - x\rangle \leq h(u), \forall u \in \mathcal{H}\}$$

for all $x \in \mathcal{H}$. From Rockafellar [31] , we know that ∂h is amaximal monotone operator. Let C be a nonempty closed convex subset of $\mathcal{H}$, and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases} \tag{45}$$

Then, i_C is a proper lower semicontinuous convex function on $\mathcal{H}$. So, we can define the resolvent operator $J_r^{\partial i_C}$ of i_C for $r > 0$, i.e.,

$$J_r^{\partial i_C}(x) = (I + r\partial i_C)^{-1}(x), \quad x \in \mathcal{H}.$$

We know that $J_r^{\partial i_C}(x) = P_C x$ for all $x \in \mathcal{H}$ and $r > 0$; see [32]. Moreover, for the single valued operator $f : \mathcal{H} \to \mathcal{H}$ we have

$$x \in (\partial i_C + f)^{-1}(0) \Leftrightarrow x \in VI(C, f).$$

Theorem 5 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, and let C and Q, be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be respectively α and β- inverse strongly monotone operators. Let for $i \in \{1, 2, \ldots, m\}$, $T_i : \mathcal{H}_1 \to K(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to K(\mathcal{H}_2)$ be two finite families of generalized nonexpansive set-valued mappings such that S_i and T_i satisfies the common endpoint condition. Suppose $\Omega = \{x \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap VI(C, f), \quad y \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) \bigcap VI(Q, g) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let*

$\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = P_C(I - \lambda_n f) z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = P_Q(I - \mu_n g) w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} s_{n,i} \qquad \forall n \geq 0, \end{cases} \tag{46}$$

where $v_{n,i} \in T_i u_n$, $s_{n,i} \in S_i t_n$. Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.1. Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Equilibrium problem

Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$. Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$. The equilibrium problem for Φ is to find $x^\star \in C$ such that

$$\Phi(x^\star, y) \geq 0, \qquad \forall y \in C. \tag{47}$$

The set of such solutions $x^\star$ is denoted by $\mathrm{EP}(\Phi)$.

It has been a connection between the equilibrium problem and the related problems in applied sciences such as variational inequalities, optimal theory, complementarity problems, Nash equilibrium in game theory and so on (see [36, 37]). In other words, numerous problems in physics, optimization, and economics can be nicely reduced to find a solution of (47) as well. In the recent years iterative algorithms for finding a common element of the set of solutions of equilibrium problem and the set of fixed points of nonlinear mappings have been studied by many authors (see, e.g., [38-42]).

For solving the equilibrium problem, let us assume that the bifunction Φ satisfies the following conditions:

(A1) $\Phi(x, x) = 0$ for all $x \in C$,

(A2) Φ is monotone, i.e., $\Phi(x, y) + \Phi(y, x) \leq 0$, for any $x, y \in C$,

(A3) for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Phi(tz + (1-t)x, y) \leq \Phi(x, y),$$

(A4) for each $x \in C$, $y \to \Phi(x, y)$ is convex and lower semi-continuous.

We know the following lemma which appears implicitly in Blum et al. [36] and Combettes et al. [37].

Lemma 6 [36, 37] *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$ *and let* Φ *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* $(A1) - (A4)$. *Let* $r > 0$ *and* $x \in \mathcal{H}$. *Then, there exists* $z \in C$ *such that*

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \qquad \forall y \in C.$$

Further, if

$$U_r^{\Phi} x = \{z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

Then, the following hold:

(i) U_r^{Φ} *is single valued and firmly nonexpansive;*

(ii) $\mathrm{Fix}(U_r^{\Phi}) = \mathrm{EP}(\Phi)$;

(iii) $\mathrm{EP}(\Phi)$ *is closed and convex.*

We call such U_r^{Φ} the resolvent of Φ for $r > 0$. Using above lemma, we have the following lemma, see [24] for a more general result.

Lemma 7 [24] *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$ *and let* Φ *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfy* $(A1) - (A4)$. *Let* B_Φ *be a set-valued mapping of* $\mathcal{H}$ *into itself defined by*

$$B_\Phi(x) = \begin{cases} \{z \in \mathcal{H} : \Phi(x, y) + \langle y - x, z\rangle \geq 0, \quad \forall\, y \in C\}, & \forall\, x \in C \\ \emptyset, \quad \forall\, x \notin C. \end{cases} \tag{48}$$

Then $\mathrm{EP}(\Phi) = B_\Phi^{-1}(0)$ *and* B_Φ *is a maximal monotone operator with* $\mathrm{dom}(B_\Phi) \subset C$. *Furthermore, for any* $x \in \mathcal{H}$ *and* $r > 0$, *the resolvent* U_r^{Φ} *of* Φ *coincides with the resolvent of* B_Φ, *i.e.,*

$$U_r^{\Phi}(x) = (I + rB_\Phi)^{-1}(x).$$

Form Lemma 4.3 and Theorems 3.1 we have the following results.

Theorem 6 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, and let C and Q, be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ be functions satisfying conditions $(A1) - (A4)$. Let for $i \in \{1, 2, \dots, m\}$, $T_i : \mathcal{H}_1 \to K(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to K(\mathcal{H}_2)$ be two finite families of generalized nonexpansive set-valued mappings such that S_i and T_i satisfies the common endpoint condition. Suppose $\Omega = \{x \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap EP(\Phi), y \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) \bigcap EP(\Psi) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = U_{r_n}^{\Phi} z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = U_{\kappa_n}^{\Phi} w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} s_{n,i} \qquad \forall n \geq 0, \end{cases} \tag{49}$$

where $v_{n,i} \in T_i u_n$, $s_{n,i} \in S_i t_n$. Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ satisfy the conditions of Theorem 3.1. Assume that $\liminf_n r_n > 0$ *and* $\liminf_n \kappa_n > 0$. *Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.*

Proof. For the bifunctions $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ we can define B_Φ and B_Ψ in Lemma 7. Putting $F = B_\Phi$ and $G = B_\Psi$ in Theorem 3.1, we obtain from Lemma 7 that $U_{r_n}^{\Phi}(x) = (I + r_n B_\Phi)^{-1}(x)$ and $U_{\kappa_n}^{\Psi}(x) = (I + \kappa_n B_\Psi)^{-1}(x)$. Thus by setting $f = g = 0$, we obtain the desired result by Theorem 3.1. $\square$

Numerical example

Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{R}$. For each $x \in \mathbb{R}$ define set- valued mappings T_i and S_i as follows:

$$T_1 x = [0, \frac{x}{2}], \qquad T_2(x) = \begin{cases} 0, & x < 0 \\ [0, \frac{x}{3}], & 0 \leq x < 3 \\ [1, 2] & x \geq 3, \end{cases}$$

and

$$S_1 x = [0, \frac{x}{5}], \qquad S_2 x = [0, \frac{x}{2}].$$

It is easy to see that T_2 is generalized nonexpansive mapping and T_1, S_1, S_2 are nonexpansive mappings. We put $C = Q = [0, \infty)$ and define the bifunctions

$\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ as follows:

$$\Phi = y^2 + xy - 2x^2, \qquad \Psi = x(y - x).$$

We observe that the functions Φ and Ψ satisfying the conditions $(A1) - (A4)$. We also have $U_r^{\Phi} = \frac{x}{3r+1}$ and $U_r^{\Psi} = \frac{x}{r+1}$. Also we define $\mathcal{A}x = 2x$ and $\mathcal{B}x = 3x$, hence $\mathcal{A}^{\star}x = 2x$ and $\mathcal{B}^{\star}x = 3x$. Put $\alpha_n = \frac{1}{n+1}$, $\beta_n = \delta_{n,1} = \delta_{n,2} = \frac{n}{3n+3}$, $r_n = \kappa_n = 1$ and $\gamma_n = \frac{1}{6}$. Then these sequences satisfy the conditions of Theorem 4.4. We have the following algorithm:

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) = \frac{1}{3}x_n + y_n, \\ u_n = U_{r_n}^{\Phi} z_n = \frac{z_n}{4}, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \delta_{n,1} v_{n,1} + \delta_{n,2} v_{n,2} \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) = x_n - \frac{1}{2}y_n, \\ t_n = U_{\kappa_n}^{\Phi} w_n = \frac{w_n}{2}, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \delta_{n,1} s_{n,1} + \delta_{n,2} s_{n,2} \qquad \forall n \geq 0. \end{cases} \tag{50}$$

Taking $(x_0, y_0) = (1, 1), \vartheta = \zeta = 2$, $v_{n,1} = v_{n,2} = \frac{u_n}{5}$ and $s_{n,1} = s_{n,2} = \frac{t_n}{5}$, we have the following algorithm:

$$\begin{cases} z_n = \frac{1}{3}x_n + y_n, \\ u_n = \frac{z_n}{4} = \frac{x_n}{12} + \frac{y_n}{4}, \\ x_{n+1} = \frac{2}{n+1} + \frac{7n}{15n+15} u_n = \frac{2}{n+1} + \frac{(7n)\,x_n}{180n+180} + \frac{(7n)\,y_n}{60n+60}, \\ w_n = x_n - \frac{1}{2}y_n, \\ t_n = \frac{w_n}{2} = \frac{x_n}{2} - \frac{y_n}{4}, \\ y_{n+1} = \frac{2}{n+1} + \frac{7n}{15n+15} t_n = \frac{2}{n+1} + \frac{(7n)\,x_n}{30n+30} - \frac{(7n)\,y_n}{60n+60} \qquad \forall n \geq 0. \end{cases} \tag{51}$$

We observe that, $\{(x_n, y_n)\}$ is convergent to $(0, 0)$. We note that $\Omega = \{(0, 0)\}$.

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Study on subclasses of analytic functions

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Abstract. By making use of new linear fractional differential operator, we introduce and study certain subclasses of analytic functions associated with Symmetric Conjugate Points and defined in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Inclusion relationships are established and convolution properties of functions in these subclasses are discussed.

1 Introduction and preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

A function $f \in A$ is called starlike if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geq 0, (z \in \mathbb{U}).$$

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The class of starlike functions is denoted by S.

A function $f \in A$ is called convex if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0, (z \in \mathbb{U}).$$

The class of convex functions is denoted by K.

A function $f \in A$ is called starlike of order ρ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geq \rho, (\rho > 0, z \in \mathbb{U}). \tag{2}$$

The class of starlike functions of order ρ is denoted by $SV^{\star}(\rho)$.

Similarly a function $f \in A$ is called convex of order ρ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \rho, (\rho > 0, z \in \mathbb{U}). \tag{3}$$

The class of starlike functions of order ρ is denoted by $KV(\rho)$.

It follows from (2) and (3) that $f \in KV(\rho)$ if and only if $zf'(z) \in SV^{\star}(\rho)$.

Let $f \in A$ and $g \in SV^{\star}(\rho)$, then $f \in A$ is called close-to-convex of order θ and type ρ if and only if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) \geq \theta, (0 \leq \theta, \rho < 1, z \in \mathbb{U}).$$

The class of close-to-convex of order θ and type ρ is denoted by $CV(\theta, \rho)$.

In 1959, Sakaguchi [1] introduced the following class of analytic functions:

A function $f \in A$ is called starlike with respect to symmetrical points, and its class is denoted by SV_s, if it satisfies the analytic criterion

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, (z \in \mathbb{U}).$$

For more details we refer to study Shanmugam et al. [2], Chand and Singh [3] and Das and Singh [4] respectively.

In 1987, El-Ashwah and Thomas [5] introduced the following class of analytic functions:

A function $f \in A$ is called starlike with respect to symmetric conjugate points, and its class is denoted by SV_{sc}, if it satisfies the analytic criterion

$$\Re\left(\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right) > 0, \ (z \in \mathbb{U}).$$

For two functions f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec g(z)\,(z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$ which (by definition) is analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1,$$

such that

$$f(z) = g(w(z))\ z \in \mathbb{U}.$$

Indeed it is known that

$$f(z) \prec g(z)\ z \in \mathbb{U} \Rightarrow f(0) = g(0)\ and\ f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z)\ z \in \mathbb{U} \Leftrightarrow f(0) = g(0)\ and\ f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z) = z+\sum\limits_{k=2}^{\infty} a_k z^k$ and $g(z) = z+\sum\limits_{k=2}^{\infty} b_k z^k$ the Hadamard product (or convolution) $f * g$ is defined as usual by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let

$$\varphi(a, c; z) = z_2F(1, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k-1},\ (z \in \mathbb{U}; c \neq 0, -1, -2, .s),$$

where $(a)_k$ is the pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(k + a)}{\Gamma(a)} = \left\{ \begin{array}{ll} 1 & \text{if } k = 0 \\ a(a + 1)(a + 2).s(a + k - 1) & \text{if } k \in N \end{array} \right\}.$$

In 1987, Owa and Srivastava [7] introduced the operator as follow

$$\begin{aligned} \Omega^{\alpha} f(z) &= \Gamma(2 - \alpha) z^{\alpha} D_z^{\alpha} f(z) = \varphi(2, 2 - \alpha; z) * f(z), \\ \alpha &\neq 2, 3, 4, \ldots. \end{aligned} \tag{4}$$

Also note that $\Omega^0 f(z) = f(z)$ and $D_z^\alpha f(z)$ is the fractional derivative of order α given in [6].

For $f \in A$, we define the linear fractional differential operator as follow:

$$\begin{aligned}
I_\lambda^{0,\nu}(\alpha,\beta,\mu)f(z) &= f(z)\\
I_\lambda^{1,\nu}(\alpha,\beta,\mu)f(z) &= \left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right)\Omega^\alpha f(z) + \left(\frac{\mu+\lambda}{\nu+\beta}\right) z(\Omega^\alpha f(z))'\\
I_\lambda^{2,\nu}(\alpha,\beta,\mu)f(z) &= I_\lambda^\alpha(I_\lambda^{1,\nu}(\alpha,\beta,\mu)f(z))\\
&\;\;\vdots\\
I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z) &= I_\lambda^\alpha(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)f(z))\,.
\end{aligned} \tag{5}$$

If $f(z)$ is given by (1) then from (5) we have

$$\begin{aligned}
&I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z)\\
&= z + \sum_{k=2}^{\infty}\left(\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)^n a_k z^k.
\end{aligned} \tag{6}$$

Using (4) we conclude that

$$I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z) = \underbrace{[\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z).s\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)]} * f(z), \tag{7}$$

where

$$\begin{aligned}
g_{\beta,\lambda}^{\mu,\nu}(z) &= \frac{z-\left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right)z^2}{(1-z)^2}\\
&= \left(z - \frac{\nu-\mu+\beta-\lambda}{\nu+\beta}z^2\right)(1+2z+3z^2+\cdots)\\
&= z + \left(1+\frac{\mu+\lambda}{\nu+\beta}\right)z^2 + \left(1+2\frac{\mu+\lambda}{\nu+\beta}\right)z^3+\cdots\\
&\;\;\vdots\\
g_{\beta,\lambda}^{\mu,\nu}(z) &= z + \sum_{k=2}^{\infty}\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)z^k.
\end{aligned} \tag{8}$$

$$\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z).s\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z) = \text{n-times product}.$$

Similarly, we find the following:

$$g^{\mu}_{\beta,\lambda}(z) = z + \sum_{k=2}^{\infty} \left(\frac{\beta + (\mu+\lambda)(k-1)}{\beta} \right) z^k. \tag{9}$$

$$g^{\nu}_{\beta,\lambda}(z) = z + \sum_{k=2}^{\infty} \left(\frac{\nu + \lambda(k-1) + \beta}{\nu + \beta} \right) z^k. \tag{10}$$

$$g^{\nu}_{\beta}(z) = z + \sum_{k=2}^{\infty} \left(\frac{\nu + (k-1) + \beta}{\nu + \beta} \right) z^k. \tag{11}$$

$$g^{\mu}_{\lambda}(z) = z + \sum_{k=2}^{\infty} \left(1 + (\mu+\lambda)(k-1)\right) z^k. \tag{12}$$

$$g_{\lambda}(z) = z + \sum_{k=2}^{\infty} \left(1 + \lambda(k-1)\right) z^k. \tag{13}$$

Further, a straightforward calculation reveals that many differential operators introduced in other papers are special cases of the differential operator defined by (6) which generalizes some well known differential operators.

1. $\beta = 1$, $\mu = 0$, $\alpha = 0$, we obtain, Aouf et al. differential operator [8].
2. $\alpha = 0$, we obtain differential operator of Darus and Faisal [29].
3. $\beta = 0$, $\alpha = 0$, we obtain differential operator Darus and Faisal [30].
4. $\nu = 1$, $\beta = o$, $\mu = 0$, $\alpha = 0$, we obtain, Al-Oboudi differential operator [9].
5. $\nu = 1$, $\beta = o$, $\mu = 0$, $\lambda = 1$, $\alpha = 0$, we obtain, Sălăgean's operator [10].
6. $\beta = l$, $\mu = 0$, $\alpha = p$, we obtain, A. Catas operator [26].
7. $\nu = 1$, $\beta = o$, $\mu = 0$, we obtain, Al-Oboudi–Al-Amoudi operator [14, 25].
8. $\beta = 1$, $\lambda = 1$, $\mu = 0$, $\alpha = 0$, we obtain, Cho–Srivastava operator [12, 13].
9. $\nu = 1$, $\beta = 1$, $\lambda = 1$, $\mu = 0$, $\alpha = 0$, we obtain, Uralegaddi–Somanatha operator [11].
10. $\nu = 1$, $\beta = o$, $\mu = 0$, $\lambda = 0$, $n = 1$, we obtain, Owa–Srivastava operator [7].

11. $\beta = l$, $\mu = 0$, $\alpha = p$, $\lambda = 1$, we obtain, Kumar et al. and Srivastava et al. operators [27, 28].

For $f \in A$, we define f_m by

$$f_m(z) = \frac{1}{2m}\sum_{k=0}^{m-1}\left[\omega^{-k}f(\omega^k z) + \omega^k\overline{f(\omega^k\overline{z})}\right], \tag{14}$$

where m be a positive integer and $\omega = \exp(2\pi/(m))$.

A function $f \in A$ is called λ-starlike with respect to $2m$-symmetric conjugate points and its class is denoted by $SV_m(\lambda)$, if it satisfy the analytic criterion

$$\Re\left(\frac{(1-\lambda)zf'(z) + \lambda(zf'(z))'}{(1-\lambda)f_m(z) + \lambda zf'_m(z)}\right) > 0, \ (z \in U, \lambda \geq 0),$$

where f_m is given by (14). For details about $SV_m(\lambda)$, we refer to study [15, 16, 17, 18].

By using (14), we have

$$\begin{aligned}
f_m(z) &= \frac{1}{2m}\sum_{k=0}^{m-1}\left\{\omega^{-k}f(\omega^k z) + \omega^k\overline{f(\omega^k\overline{z})}\right\}, \text{ implies} \\
I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z) &= \frac{1}{2m}\sum_{k=0}^{m-1}\{\omega^{-k}I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k z) \\
&\quad + \omega^k\overline{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k\overline{z})}\}, \\
z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))' &= \frac{1}{2m}\sum_{k=0}^{m-1}\{\omega^{-k}z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k z))' \\
&\quad + \omega^k z(\overline{D_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k\overline{z})})'\}, \\
I_\lambda^{n,\nu}(\alpha,\beta,\mu)(zf'_m(z)) &= \frac{1}{2m}\sum_{k=0}^{m-1}\{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(zf'(\omega^k z)) \\
&\quad + \overline{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(zf'(\omega^k\overline{z})}\}, \\
I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(\omega^j z) &= \omega^j I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z), \\
I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(\overline{z}) &= \overline{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)}.
\end{aligned} \tag{15}$$

Next we introduce new subclasses of analytic functions in $\mathbb{U}$ associated with linear fractional differential operator $I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z)$, as follow;

Definition 1 *For* $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ *and* $\alpha, \beta, \lambda, \mu, \nu \geq 0$, *let* $\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ *denote the class of functions* f *defined by* (1) *and satisfying the analytic criterion*

$$\frac{z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f_m(z)} \prec h(z), \;\; z \in \mathbb{U},$$

where h *is a convex function in* $\mathbb{U}$ *with* $h(0) = 1$.

Definition 2 *Let* $\mathcal{KV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ *denote the class of functions* f *defined by* (1) *and satisfying the analytic criterion if* $\frac{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)g_m(z)}{z} \neq 0$ *and*

$$\frac{z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)g_m(z)} \prec h(z), \;\; z \in \mathbb{U},$$

for some $g \in \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$.

Remark 1 *In 2010, F. M. Al-oboudi* [25], *introduced certain subclasses of analytic functions which contains only functions of the form given in* (13), *but there were infinite analytic functions in the open unit disk* $\mathbb{U}$, *of the form given in* (8), (9), (10), (11) *and* (12) *respectively, which were out of range of the classes given in [25]. Therefore it was necessary to find out or to introduce a new differential operator of the form* (6). *We introduce the subclasses* $\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ *and* $\mathcal{KV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ *of analytic functions by using such differential operator, with a different approach that includes all functions of the form given in* (8) *to* (13).

2 Main results

In this section, we have discussed inclusion relations as well as convolution properties for the function belonging to the classes $\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ and $\mathcal{KV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ respectively.

Lemma 1 [19] *Let* f *and* g *be starlike functions of order* $1/2$ *then so is* $f * g$.

Lemma 2 [20] *Let* P *be a complex function in* $\mathbb{U}$ *with* $\Re(P(z)) > 0$ *for* $z \in \mathbb{U}$ *and let* h *be a convex function in* $\mathbb{U}$. *If* p *is analytic in* $\mathbb{U}$ *with* $p(0) = h(0)$ *and if*

$$p(z) + P(z)zp'(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Lemma 3 [21] *Let* $c > -1$ *and let* $I_c : A \to A$ *be the integral operator defined by* $F = I_c(f)$, *where*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt.$$

Let h *be a convex function, with* $h(0) = 1$ *and then* $\Re(h(z) + c) > 0$, $z \in \mathbb{U}$. *If* $f \in A$ *and* $\frac{zf'(z)}{f(z)} \prec h(z)$, *then*

$$\frac{zF'(z)}{F(z)} \prec q(z) \prec h(z),$$

where q *is univalent and satisfies the differential equation*

$$q(z) + \frac{zq'(z)}{q(z) + c} = h(z).$$

Lemma 4 [22] *Let* f *and* g, *respectively be in the classes* K *and* S, *then for every function* $F \in A$, *we have*

$$\frac{(f(z) * g(z)F(z))}{(f(z) * g(z))} \in \overline{co}(F(\mathbb{U})), \;\; z \in \mathbb{U},$$

where $\overline{co}$ *denotes the closed convex hull.*

Lemma 5 [22] *Let* f *and* g *be univalent starlike of order* $\frac{1}{2}$ *for every function* $F \in A$, *we have*

$$\frac{(f(z) * g(z)F(z))}{(f(z) * g(z))} \in \overline{co}(F(\mathbb{U})), \;\; z \in \mathbb{U},$$

where $\overline{co}$ *denotes the closed convex hull.*

Theorem 1 *Let* h *be a convex function in* $\mathbb{U}$ *with* $h(0) = 1$, $\overline{h(\overline{z})} = h(z)$ *and let* $\mu + \lambda \geq \nu + \beta$, *if* $f \in \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ *then*

$$\frac{z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu) f_m(z))'}{I^{n,\nu}_{\lambda}(\alpha, \beta, \mu) f_m(z)} \prec h(z), \; z \in \mathbb{U}. \tag{16}$$

Moreover, if $\Re\left(h(z) + \frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\right) > 0$ *in* $\mathbb{U}$ *then*

$$\frac{z\big(I^{n-1,\nu}_{\lambda}(\alpha, \beta, \mu((\Omega^{\alpha} f_m(z))\big)'}{I^{n-1,\nu}_{\lambda}(\alpha, \beta, \mu)(\Omega^{\alpha} f_m(z))} \prec q(z) \prec h(z), \; z \in \mathbb{U}. \tag{17}$$

Where q *is the univalent solution of the differential equation*

$$q(z)+\frac{zq'(z)}{h(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}}=h(z),\ q(0)=1. \tag{18}$$

Proof. Because

$$\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)=\left\{f\in A:\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z),\ z\in\mathbb{U}\right\}.$$

It is remaining to show that $f_m\in\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha,\beta,\mu)$, since

$$f_m(z)=\frac{1}{2m}\sum_{k=0}^{m-1}\left\{\omega^{-k}f(\omega^k z)+\omega^k\overline{f(\omega^k\overline{z})}\right\}.$$

As

$$\begin{aligned}f\in\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha,\beta,\mu)&\Rightarrow\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z)\\&\Rightarrow\frac{(D^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(z))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z).\end{aligned}$$

After replacing z by $\omega^j z$ and $\omega^j\overline{z}$, we get

$$\omega^{-j}\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j z))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z),\ \text{and}\ \omega^j\overline{\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j\overline{z}))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}}\prec h(z), \tag{19}$$

because

$$\overline{h(\overline{z})}=h(z),\ I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(\overline{z})=\overline{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)},\ \text{and}$$

$$I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(\omega^j z)=\omega^j I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z).$$

Using (19) we have

$$\frac{1}{2m}\sum_{k=0}^{k-1}\omega^{-j}\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j z))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}+\omega^j\overline{\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j\overline{z}))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}}\prec h(z),$$

implies

$$\frac{1}{2m}\sum_{k=0}^{k-1}\omega^{-j}\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(\omega^j z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}+\omega^j\overline{\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(\omega^j\overline{z}))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}}\prec h(z),$$

since we have

$$
\begin{aligned}
&z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))' \\
&\quad = \frac{1}{2m}\sum_{k=0}^{m-1}\left\{\omega^{-k}z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k z))' + \omega^k z(\overline{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k \overline{z}}))'\right\}
\end{aligned}
$$

and this implies

$$
\begin{aligned}
&\frac{1}{2m}\sum_{k=0}^{k-1}\omega^{-j}\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^j z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} + \omega^j\frac{\overline{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^j \overline{z}))'}}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \\
&\quad = \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)},
\end{aligned}
$$

therefore

$$
\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \prec h(z).
$$

Hence (16) is satisfied.

From (5) we have

$$
I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z) = I_\lambda^{\alpha}(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)f(z)),
$$

implies

$$
\begin{aligned}
I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z) = &\left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right)(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)) \\
&+\left(\frac{\mu+\lambda}{\nu+\beta}\right)z(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))',
\end{aligned}
$$

implies

$$
(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)) = \frac{\nu+\beta}{(\mu+\lambda)z^{(\frac{\mu+\lambda}{\nu+\beta})-1}}\int_0^z t^{(\frac{\mu+\lambda}{\nu+\beta})-1} I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(t)dt.
$$

Applying Lemma 3, we have

$$
f(z) = I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z),\ \ F(z) = (I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)),
$$

and

$$
\frac{zf'(z)}{f(z)} = \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \prec h(z)(\text{proved})
$$

with

$$\Re\big(h(z) + \frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\big) > 0.$$

Therefore

$$\frac{z\big(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu((\Omega^\alpha f_m(z))\big)'}{I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))} \prec q(z),$$

and q satisfied the equation

$$q(z) + \frac{zq'(z)}{h(z) + \frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}} = h(z).$$

Hence proved. $\square$

Corollary 1 *Let* $\Re\big(h(z)\big) > 0$, *if* $f \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$ *then* $I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m \in S$ *and hence* $I_\lambda^{n,\nu}(\alpha,\beta,\mu)f$ *is close to convex function.*

Theorem 2 *Let* $\Re\big(h(z)\big) > 0$ *and* $\overline{h(\overline{z})} = h(z)$ *then the following inclusions hold*

$$\mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h) \subseteq \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h) \subseteq \mathcal{SV}_{m,\lambda}^{n-1,\nu}(\alpha,\beta,\mu)(h).$$

Proof. Let $f \in \mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h)$. To prove $\mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h) \subseteq \mathcal{SV}_{m,\lambda}^{n,\nu}$ $(\alpha,\beta,\mu)(h)$, it is enough to show that $f \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$.

Applying Theorem 1, if $f \in \mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h)$ then

$$\frac{z(I_\lambda^{n+1,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n+1,\nu}(\alpha,\beta,\mu)f_m(z)} \prec h(z),\ z \in \mathbb{U},$$

if Moreover, if $\Re\big(h(z) + \frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\big) > 0$ in $\mathbb{U}$ then

$$\frac{z\big(I_\lambda^{n,\nu}(\alpha,\beta,\mu((\Omega^\alpha f_m(z))\big)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))} \prec q(z) \prec h(z),\ z \in \mathbb{U},$$

and $\Re\big(q(z) + \frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\big) > 0$ in $\mathbb{U}$. Let

$$p(z) = \frac{z\big(I_\lambda^{n,\nu}(\alpha,\beta,\mu((\Omega^\alpha f(z))\big)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}.$$

Then p is analytic in U and satisfies

$$p(z)+\frac{zp'(z)}{q(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}}\prec h(z),$$

by using Lemma 2 $p(z)\prec h(z)$, implies $\Omega^{\alpha}f(z)\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$, applying Theorem 4 implies $f(z)\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$. Hence

$$\mathcal{SV}^{n+1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h).$$

Similarly we can show that $\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)$, and $\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n-2,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)$ and so on, therefore

$$\mathcal{SV}^{n+1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h).s,$$

which generalized the Al-Amiri et al. [16] results. □

Corollary 2 *Taking* $h(z)=\frac{1+z}{1-z}$, *in Theorem 2, then*

$$\mathcal{SV}^{n+1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z})\subseteq\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z})\subseteq\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z}).$$

Implies that $\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z})$ *are starlike functions with respect to symmetric conjugate points.*

Theorem 3 *If* $f\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$ *then* $f*g\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$ *where* $\Re\big(h(z)\big)>0$ *and* g *is a convex function with real coefficients in* $\mathbb{U}$.

Proof. Since $f\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$, applying Theorem 1, we get $I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)$ $f_m(z)\in S$. Using the convolution properties we have

$$\begin{aligned}\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f*g)(z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f_m*g)(z)}&=\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f(z)*g(z)))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f_m(z)*g(z))}\\&=\frac{g(z)*(z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'}{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\\&=\frac{g(z)*(z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'/I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z))I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\\&=\frac{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)(z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'/I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z))}{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}.\end{aligned}$$

Implies that

$$\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(f*g)(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(f_m*g)(z)} \in \overline{co}\left(\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m}(U)\right) \subseteq h(U), \quad z \in U.$$

Hence

$$f * g \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h).$$

$\square$

Theorem 4 *If* $\Omega^\alpha f(z) \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$ *then* $f(z) \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$, *where* $\Re\big(h(z)\big) > 0, \overline{h(\overline{z})} = h(z)$.

Proof. Let $\Omega^\alpha f(z) \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$, applying Theorem 1, implies

$$\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m(z)} \prec h(z),$$

or

$$(I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m(z) \in S.$$

Because

$$\Omega^\alpha f(z) = \varphi(2, 2-\alpha; z) * f(z)$$

and

$$I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z) = \underbrace{[\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z).s\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)]} * f(z),$$

therefore we can write that

$$\begin{aligned} I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z) &= \varphi(2-\alpha,2;z) * D_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)) \\ z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))' &= \varphi(2-\alpha,2;z) * z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f(z))', \end{aligned} \tag{20}$$

where $\varphi(2-\alpha, 2; z) \in K$.

Using (20) we have

$$\begin{aligned} \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} &= \frac{\varphi(2-\alpha,2;z) * z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f(z)))'}{\varphi(2-\alpha,2;z) * I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}, \\ \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} & \\ = \frac{\varphi(2-\alpha,2;z) * I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))\frac{(z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f(z)))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}}{\varphi(2-\alpha,2;z) * I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}&, \\ \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \in \overline{co}\left(\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m}(\mathbb{U})\right) &\subseteq h(\mathbb{U}), \quad z \in \mathbb{U}. \end{aligned} \tag{21}$$

Hence

$$f(z) \in \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h).$$

$\square$

Theorem 5 *Let* $0 \le \alpha_1 < \alpha < 1$, *and* $\mathrm{Re}(h(z)) > \frac{1}{2}$, *then the following inclusions hold*

$$\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h) \subseteq \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha_1, \beta, \mu)(h)\,.$$

Proof. Let $f(z) \in \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$, from (7) we have

$$I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z) = \underbrace{[\varphi(2, 2-\alpha; z) * g^{\mu,\nu}_{\beta,\lambda}(z).s\varphi(2, 2-\alpha; z) * g^{\mu,\nu}_{\beta,\lambda}(z)]} * f(z),$$

implies

$$I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z) = \underbrace{[\varphi(2, 2-\alpha_1; z) * g^{\mu,\nu}_{\beta,\lambda}(z).s\varphi(2, 2-\alpha_1; z) * g^{\mu,\nu}_{\beta,\lambda}(z)]} * f(z),$$

$$\begin{aligned} I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z) = {} & \underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} \\ & * I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z), \end{aligned}$$

implies that

$$\begin{aligned} z(I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z))' = {} & \underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} \\ & * z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z))', \end{aligned}$$

applying same technique, we have

$$\begin{aligned} I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f_m(z) = {} & \underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} \\ & * I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f_m(z), \end{aligned}$$

since Y. Ling and S. Ding [24] already proved that $\varphi(2-\alpha, 2-\alpha_1; z) \in S(1/2)$, therefore by Lemma 1 we get $[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)] \in S(1/2)$.

From last two equations we get

$$\frac{z(I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f_m(z)} = \frac{\underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} * z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z))'}{\underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} * I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f_m(z)},$$

after simplification and using Lemma 5 we deduce that

$$\frac{z(I_\lambda^{n,\nu}(\alpha_1,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha_1,\beta,\mu)f_m(z)} \in \overline{co}\left(\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m}(\mathbb{U})\right) \subseteq h(\mathbb{U}), \quad z \in \mathbb{U},$$

therefore

$$f \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha_1,\beta,\mu)(h).$$

Hence proved. □

Remark 2

1. *When* $\nu = 1$, $\mu = \beta = 0$, $\mathcal{SV}_{m,\lambda}^{n,1}(\alpha,0,0)(h) = \mathcal{SV}_{m,\lambda}^{n}(\alpha)(h)$, *the classes of functions related to starlike functions with respect to symmetric conjugate points, defined and studied by M. K. Al-Oboudi* [25].

2. *For* $n = m = \nu = 1$, $\mu = \beta = 0$, *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{1,\lambda}^{1,1}(\alpha,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}^{\lambda,\alpha}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-starlike functions with respect to conjugate points, defined and studied by Radha* [23].

3. *For* $n = m = \nu = \lambda = 1$, $\alpha = \mu = \beta = 0$, *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{1,1}^{1,1}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}\left(\frac{1+z}{1-z}\right)$, *the class of starlike functions with respect to conjugate points, defined and studied by El-Ashwah and Thomas* [5].

4. *For* $n = \nu = 1$, $\alpha = \mu = \beta = 0$, *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{m,\lambda}^{1,1}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}^{m,\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-starlike functions with respect to* $2m$*-symmetric conjugate points, defined and studied by Al-Amiri et al.* [15, 16].

5. *For* $m = \nu = 1$, $n = \mu = \beta = 0$, $\mathcal{SV}_{1,\lambda}^{0,1}(\alpha,0,0)(h) = \mathcal{SV}(\alpha)(h)$, *the class of functions related to starlike functions with respect to conjugate points, defined and studied by Ravichandran* [31].

6. *For* $m = n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{1,\lambda}^{1,1}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}^{\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-starlike functions with respect to conjugate points, defined and studied by Radha* [23].

7. *For* $\lambda = n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{m,1}^{1,1}(0,0,0)(h) = \mathcal{SV}^{m}\left(\frac{1+z}{1-z}\right)$, *the class of starlike functions with respect to* $2m$*-symmetric conjugate points, defined by Al-Amiri et al.* [16].

8. *For* $\nu = 1$, $\mu = \beta = 0$, $\mathcal{KV}_{m,\lambda}^{n,1}(\alpha,0,0)(h) = \mathcal{KV}_{\lambda}^{m,n}(\alpha)(h)$, *the classes of functions related to starlike functions with respect to symmetric conjugate points, defined and studied by M.K. Al-Oboudi* [25].

9. *For* $n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{KV}^{1,1}_{m,\lambda}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{KV}^{m}_{\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-close to convex functions with respect to symmetric conjugate points, defined and studied by Al-Amiri et al.* [16].

10. *For* $m = n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{KV}^{1,1}_{1,\lambda}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{KV}^{\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-close to convex functions with respect to symmetric conjugate points, defined and studied by Radha* [23].

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MHD boundary layer flow and heat transfer characteristics of a nanofluid over a stretching sheet

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Abstract. The study of radiative heat transfer in a nanofluid with the influence of magnetic field over a stretching surface is investigated numerically. Physical mechanisms responsible for magnetic parameter, radiation parameter between the nanoparticles and the base fluid, such as Brownian motion and thermophoresis, are accounted for in the model. The parameters for Prandtl number Pr, Eckert number Ec , Lewis number Le, stretching parameter b/a and constant parameter m are examined. The governing partial differential equations were converted into nonlinear ordinary differential equations by using a suitable similarity transformation, which are solved numerically using the Nactsheim-Swigert shooting technique together with Runge-Kutta six order iteration scheme. The accuracy of the numerical method is tested by performing various comparisons with previously published work and the results are found to be in excellent agreement. Numerical results for velocity, temperature

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and concentration distributions as well as skin-friction coefficient, Nusselt number and Sherwood number are discussed at the sheet for various values of physical parameters.

1 Introduction

Nanofluid technology has been receiving a lot of attention as a research topic and takes important part for further development of higher performance due to effective applications in the field of electronics engineering, transportation, biomedical research etc. Choi [1] created 'Nanofluids' by suspending nanometresized metallic particles in common fluids and reported their highly enhanced thermal properties.

Nanofluids have novel properties that make them potentially useful in many applications in heat transfer, including microelectronics, fuel cells, pharmaceutical processes, and hybrid-powered engines. Nanoparticles are of great scientific interest as they are effectively a bridge between bulk materials and atomic or molecular structures. In the past decades, heat transfer enhancement technology has been developed and widely applied to heat exchanger applications; for example, refrigeration, automotives, process industry, chemical industry, etc. There are numerous biomedical applications that involve nanofluids such as magnetic cell separation, drug delivery, cancer therapeutics, cry preservation, Nan cryosurgery. The random motion of nanoparticles within the base fluid is called Brownian motion, and results from continuous collisions between the nanoparticles and the molecules of the base fluid. The nanoparticle concentration, base fluid, and particle size appear to be the most influential parameters for improving the heat transfer efficiency of nanofluid. Thermophoresis is important when the particle sizes are small and the temperature gradients are large.

Thermal radiation is important in some applications because of the manner in which radiant emission depends on temperature and nanoparticle volume fraction. Rad and Aghanajafi [2] studied the thermal analysis of single phase laminar flow nanofluid cooled rectangular microchannel heat sink (MCHS) subject to the uniform wall temperature condition. Afify et al. [3] studied the steady two-dimensional boundary layer flow past a wedge immersed in nanofluids under the effects of thermal radiation and non-uniform heat source (or sink).

The study of the magnetohydrodynamic (MHD) flow for electrically conducting fluid past a heated surface has attracted many researchers in view of its important applications in many engineering problems such as plasma studies, petroleum industries, magnetohydrodynamic power generators, cooling of nuclear reactors, the boundary layer control in aerodynamics, and crystal growth. The effect of the magnetic field over a stretching surface with or without heat and mass transfer was investigated by Vajravelu and Hadjincolaou [4], Pop and Na [5], Takhar et al. [6], Chamkha [7], Singh et al. [8], Ferdows et al. [9] and Afify [10, 11]. Recently, Turkyilmazoglu [12] investigated the magnetohydrodynamic slip flow of an electrically conducting, viscoelastic fluid past a stretching surface. All these studies were concerned the electrically conducting fluid with a low thermal conductivity. This, in turn, limits the enhancement of heat transfer in the enclosure particularly in the presence of the magnetic field. Nanofluids with enhanced thermal characteristics have widely been examined to improve the heat transfer performance of many engineering applications [1]. The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Masuda et al. [13]. This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems (Buongiorno and Hu [14]). Yu et al. [15] and Murshed et al. [16], provide a detailed literature review of nanofluids including synthesis, potential applications, experimental and analytical analysis of effective thermal conductivity, effective thermal diffusivity, and convective heat transfer.

The research on nanofluids is gaining a lot of attention in recent few years. The effect of various parameters on nanofluid thermal conductivity has been obtained by Jang and Choi [17]. The convective heat transfer in a nanofluid past a vertical plate using a model in which Brownian motion and Thermophoresis are accounted with the simplest possible boundary conditions have discussed by Kuznestov and Neild [18, 19], they have also studied the problem of natural convection past a vertical plate analytically in a porous medium saturated by a nanofluid. Khan and Pop [20] have investigated the problem of laminar fluid flow over the stretching surface in a nanofluid and they investigated it numerically.

Very recently, an analytical solution for boundary layer flow of a nanofluid past a stretching sheet was examined by Hassani et al. [21]. They compared the result with Khan and Pop [20]. Avramenko et al. [22] analyzed a self-similar analysis of fluid flow and boundary layer heat-mass transfer of nanofluids. Boundary layer flow of a nanofluid over a moving surface in a flowing fluid has been studied by Bachok et al. [23]. The problem of steady boundary layer shear flow over a stretching/shrinking sheet in a nanofluid was investigated by Yacob et al. [24]. They found that the heat transfer rate at the surface increases with increasing nanoparticles volume fraction while it decrease with convective parameter. Ferdows and Hamad [25, 26] studied a similarity solution of boundary layer stagnation-point of nanofluid flow; and investigate viscous flow of heat transfer of nanofluid over nonlinearly stretching sheet.

On the other hand, the study of Magnetohydrodynamics (MHD) boundary layer flow of a nanofluid over a stretching surface has become several industrial, scientific and engineering applications. Engineers employ MHD principle in the design of heat exchangers, pumps and flow matters, in space vehicle propulsion, thermal protection, controlling the rate of cooling etc. A number of technical processes concerning polymers involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. Further glass blowing, manufacture of plastic and rubber sheet, continuous casting of metals and spinning of flows involve the flow due to a stretching surface. MHD natural convection nanofluid flow over a linearly stretching sheet was analyzed by Hamad [27]. Aminossadati et al. [28] studied the heat transfer performance of water$-Al_2O_3$ nanofluid in a horizontal microchannel that is under the influence of a transverse magnetic field. Recently, Hamad and Pop [29] studied the unsteady magnetohydrodynamic flow of a nanofluid past an oscillatory moving vertical permeable semi-infinite flat plate with constant heat source in a rotating frame of reference.

The objective of the present study is to investigate the steady of MHD boundary layer flow, heat transfer and concentration distribution over a stretching surface in a nanofluid in the presence of thermal radiation with Brownian motion and thermophoresis. By using similarity transformations, the set of governing equations and the boundary conditions are reduced to nonlinear coupled ordinary differential equations with appropriate boundary conditions. Furthermore, the similarity equations are solved numerically by using the Nactsheim-Swigert shooting technique together with Runge-Kutta six order iteration schemes. The obtained results are shown graphically, and the physical aspects of the problem are discussed. Also, the skin-friction coefficient, Nusselt number and Sherwood number are discussed numerically for different variations of the physical parameters included in the analysis.

2 Mathematical formulation

Consider steady two-dimensional boundary layer flow of a nanofluid past a stretching sheet with a linear velocity variation with the distance x i.e. $u_w = ax$ where a is a constant and x is the coordinate measured along the stretching surface, as shown in figure 1. A steady uniform stress leading to equal and opposite forces is applied along the $x-$axis so that the sheet is stretched keeping

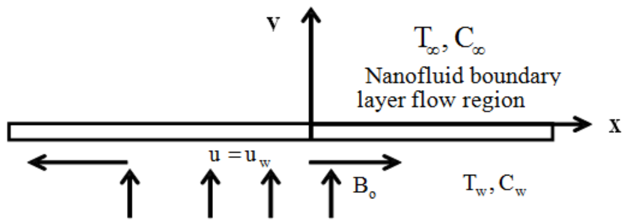

Figure 1: A sketch of the physical problem

the origin fixed. The flow is considered to be laminar, Newtonian, incompressible and electrically conducting fluid. The impressed electrical field is assumed to be zero and both the induced magnetic and electric fields of the flow are negligible in comparison with the applied magnetic field which corresponds to very small magnetic Reynolds number. The temperature and the concentration of the ambient fluid are T_∞ and C_∞, and those at the stretching surface are T_w and C_w respectively. The velocity of the plate (uniform velocity) considered as U and q_r is radiative heat flux in the $y-$direction. A magnetic field of uniform strength B_0 is applied in the $y-$direction normal to the plate surface. Also a and b are the linear stretching constant, l is the characteristics length and A_1, A_2 are the constants whose values depend on the properties of the fluid. Under these assumptions, the boundary layer form of the governing equations is given by Kuznetsov and Nield [18] and Nield and Kuznetsov [19]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}(U - u) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} - \frac{\alpha}{\kappa}\frac{\partial q_r}{\partial y} + \frac{\alpha}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \tau\left(D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right) \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

The boundary conditions for this problem can be written as

$$u = ax, v = 0, T = T_w = T_\infty + A_1\left(\frac{x}{l}\right)^m, C = C_w = C_\infty + A_2\left(\frac{x}{l}\right)^m \quad \text{at} \quad y = 0$$

$$u = U = bx, \quad T = T_\infty, \quad C = C_\infty \quad \text{as} \quad y \to \infty \tag{5}$$

where, α is the thermal diffusivity, κ is the thermal conductivity, D_B is the Brownian diffusion coefficient and D_T is the thermophoresis diffusion coefficient. The Rosseland approximation [30] has been considered for radiative heat flux which leads to:

$$q_r = -\frac{4\sigma}{3\kappa^*}\frac{\partial T^4}{\partial y} \tag{6}$$

where, σ is the Stefan-Boltzmann constant and κ^* is the mean absorption coefficient. The temperature difference with in the flow is sufficiently small such that T^4 may be expressed as a linear function of the temperature, then the Taylors series for T^4 about T_∞ after neglecting higher order terms, $T^4 = 4T_\infty^3 - 3T_\infty^4$.

In order to attains a similarity solution to equations (1) to (4) with the boundary conditions (5) the following dimensionless variables are used,

$$\eta = y\sqrt{\frac{a}{\nu}}, \psi = x\sqrt{a\nu}f(\eta), \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \varphi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$$

$$\text{and} \quad u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x} \tag{7}$$

From the above transformations the non dimensional, nonlinear, coupled ordinary differential equations are obtained as:

$$f''' + ff'' - f'^2 + M\left(\frac{b}{a} - f'\right) + \frac{b^2}{a^2} = 0 \tag{8}$$

$$(1 + R)\theta'' + EcPrf''^2 + Prf\theta' - mPrf'\theta + PrNb\theta'\varphi' + PrNt\theta'^2 = 0 \tag{9}$$

$$\varphi'' + Lef\varphi' + \frac{Nt}{Nb}\theta'' - mLef'\varphi = 0 \tag{10}$$

and the corresponding boundary conditions,

$$f = 0, \quad f' = 1, \quad \theta = 1, \quad \varphi = 1 \quad \text{at} \quad \eta = 0$$

$$f' = \frac{b}{a}, \quad \theta = 0, \quad \varphi = 0 \quad \text{as} \quad \eta \to \infty \tag{11}$$

where the notation primes denote differentiation with respect to η and the parameters are defined as Magnetic parameter $M = \frac{\sigma B_0^2}{\rho a}$, Radiation parameter $R = \frac{16\sigma T_\infty^3}{3\kappa\kappa^*}$, Prandtl number $Pr = \frac{\nu}{\alpha}$, Eckert number $Ec = \frac{u_w^2}{c_p(T_w - T_\infty)}$, Lewis number $Le = \frac{\nu}{D_B}$, Brownian parameter $Nb = \frac{(\rho c)_p D_B(\varphi_w - \varphi_\infty)}{\nu(\rho c)_f}$, Thermophoresis parameter $Nt = \frac{(\rho c)_p D_T (T_w - T\infty)}{\nu T_\infty (\rho c)_f}$, stretching parameter b/a and constant parameter m. The physical quantities of the skin-friction coefficient, the reduced Nusselt number and reduced Sherwood number are calculated respectively by the following equations,

$$C_f Re_x^{-1/2} = -f''(0) \tag{12}$$

$$Nu Re_x^{-1/2} = -(1+R)\theta'(0) \tag{13}$$

$$Sh Re_x^{-1/2} = -\varphi'(0) \tag{14}$$

where $Re_x = \frac{x u_w(x)}{\nu}$ is the local Reynolds number.

3 Numerical technique

The non dimensional, nonlinear, coupled ordinary differential equations (8) to (10) with boundary condition (11) are solved numerically using standard initially value solver the shooting method. For the purpose of this method, the Nactsheim-Swigert shooting iteration technique [31] together with Runge-Kutta six order iteration scheme is taken and determines the temperature and concentration as a function of the coordinate η. Extension of the iteration shell to above equation system of differential equation (11) is straight forward, there are three asymptotic boundary condition and hence three unknown surface conditions $f''(0), \theta'(0)$ and $\varphi'(0)$.

4 Results and discussion

The heat and mass transfer problem associated with laminar flow of the nanofluids over a stretching surface has been studied. In order to investigate the physical representation of the problem, the numerical values of velocity f', temperature θ and species concentration φ with the boundary layer have been computed for different parameters as the Magnetic parameter M, the Radiation parameter R, the Prandtl number Pr, the Eckert number Ec, the Lewis number Le, the Brownian motion parameter Nb, the Thermophoresis parameter Nt, the stretching parameter b/a and constant parameter m respectively.

Comparison: In order to assess the accuracy of the numerical results the present results are compared with the solution of Khan and Pop [20] and the values of the Magnetic parameter M, the Radiation parameter R, the Eckert number Ec, the stretching parameter b/a, and constant parameter m are considered zero (see figures 2−5 and table 1).

Table 1: Comparison of results for the reduced Nusselt number when M=0, R=0, Ec=0, b/a=0, Pr=Le=10, and m=0

Parameter Nb=Nb=0	Present results	Khan and Pop [20]
0.1	0.9524	0.9524
0.2	0.3653	0.3654
0.3	0.1351	0.1355
0.4	0.0490	0.0495
0.5	0.0178	0.0179

From table 1 it is observed that, with increasing the Brownian motion parameter and thermophoresis parameter Nusselt number decreases. Therefore the comparison shows a good agreement.

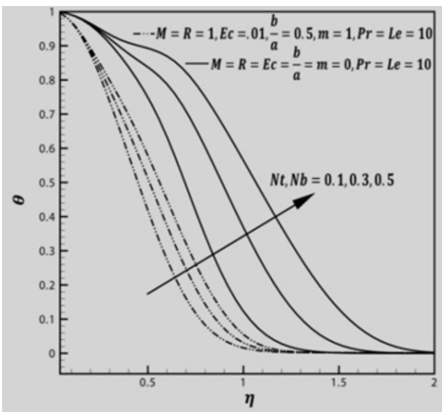

Figure 2: Effect of Nt and Nb on temperature profiles

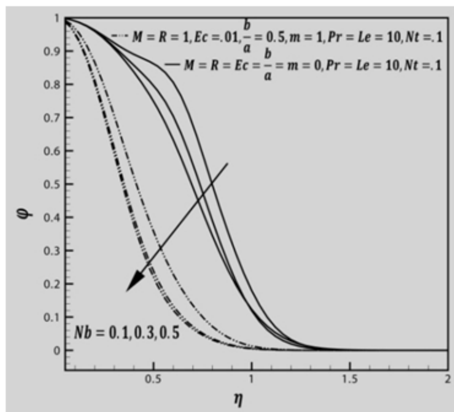

Figure 3: Effect of Nb on concentration profiles

Figure 2 represents the dimensionless temperature distribution $\theta(\eta)$ for different values of Nb where $Pr = Le = 10$, $Nt = 0.1$, $M = 1.0$, $R = 1.0$, $b/a = 0.5$, $Ec = 0.01$ and $m = 1.0$. For comparison, the values of Magnetic parameter M, the Radiation parameter R, the Eckert number Ec, the stretching parameter b/a and constant parameter m are considered zero. Then for above cases it is observed that, temperature increases as the thermophoresis parameter Nt and the Brownian motion parameter Nb increases. The qualitative agreement between these two results is good.

Figure 3 displays the dimensionless concentration distribution $\varphi(\eta)$ for dif-

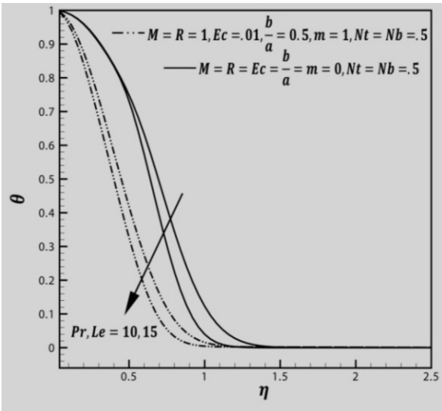

Figure 4: Effect of Pr and Le on temperature profiles

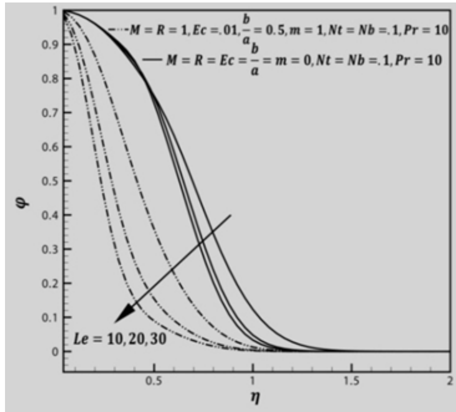

Figure 5: Effect of Le on concentration profiles

ferent values of Nb where $Pr = Le = 10$, $Nt = 0.1$, $M = 1.0$, $R = 1.0$, $b/a = 0.5$, $Ec = 0.01$ and $m = 1.0$. For comparison, the values of Magnetic parameter M, the Radiation parameter R, the Eckert number Ec, the stretching parameter b/a and constant parameter m are considered zero. Then it is examined for both cases, concentration decreases as the Brownian motion parameter increases. The qualitative agreement between these two results is good.

Figure 4 shows the dimensionless temperature distribution $\theta(\eta)$ for different values of Pr and Le where $Nt = Nb = 0.5$, $M = 1.0$, $R = 1.0$, $b/a = 0.5$, $Ec = 0.01$ and $m = 1.0$. For comparison, the values of Magnetic parameter M, the Radiation parameter R, the Eckert number Ec, the stretching parameter b/a and constant parameter m are considered zero. Then for both cases it is noted that, temperature decreases as increases. The qualitative agreement between these two results is good.

Figure 5 exhibits the dimensionless concentration distribution $\varphi(\eta)$ for different values of Le where $Nt = Nb = 0.1$, $Pr = 10$, $M = 1.0$, $R = 1.0$, $b/a = 0.5$, $Ec = 0.01$ and $m = 1.0$. For the comparison, the values of Magnetic parameter M, Radiation parameter R, Eckert number Ec, stretching parameter b/a and constant parameter m are considered zero. Then it is detected for both cases, concentration profiles decreases as Le increases. The qualitative agreement between these two results is good.

Also an effect on thermal boundary layer thickness and concentration boundary layer thickness have been found in figures 2–5, when Magnetic parameter M, Radiation parameter R, the Eckert number Ec, stretching parameter b/a and constant parameter m are introduced.

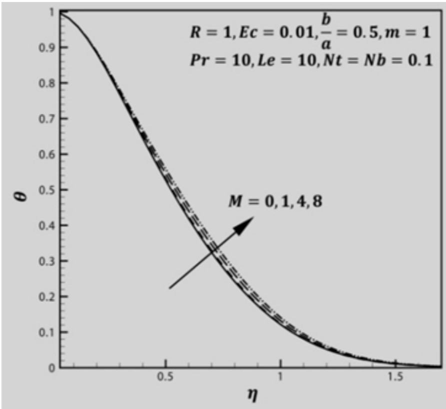

Figure 6: Effect of M on temperature profiles

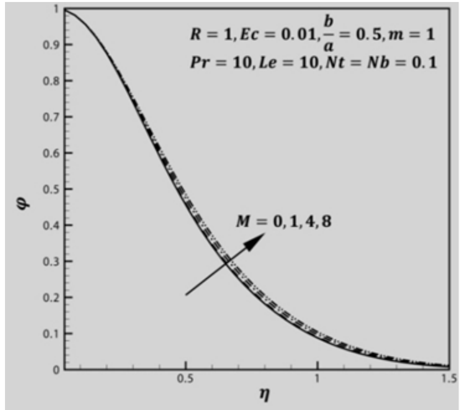

Figure 7: Effect of M on concentration profiles

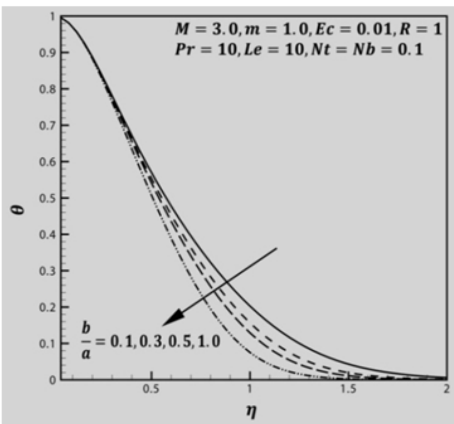

Figure 8: Effect of b/a on temperature profiles

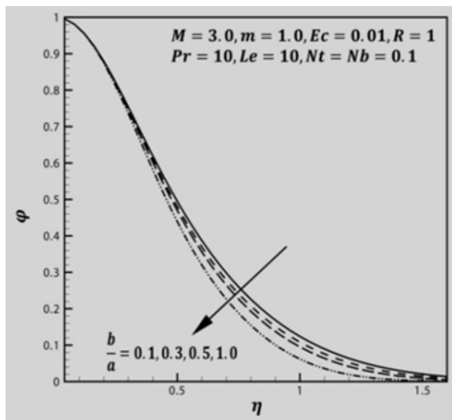

Figure 9: Effect of b/a on concentration profiles

In figures 6 and 7, the dimensionless temperature distribution $\theta(\eta)$ and the dimensionless concentration distribution $\varphi(\eta)$ are plotted respectively for the different values of Magnetic parameter M where $Nt = Nb = 0.1$, $Pr = 10$, $Le = 10$, $R = 1.0$, $b/a = 0.5$, $Ec = 0.01$ and $m = 1.0$. It is observed that, as the magnetic parameter M increases the temperature and concentration increases gradually.

In figures 8 and 9, the dimensionless temperature distribution $\theta(\eta)$ and the dimensionless concentration distribution $\varphi(\eta)$ are described respectively for the different values of stretching parameter b/a where $M = 3.0$, $Nt = Nb =$

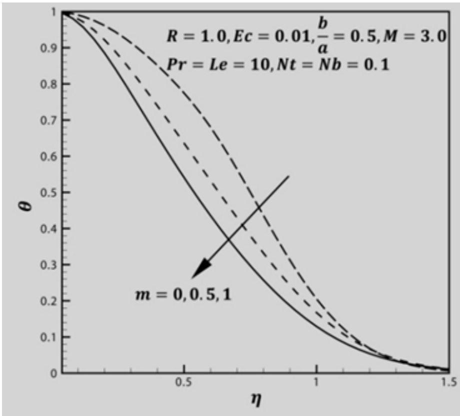

Figure 10: Effect of m on temperature profiles

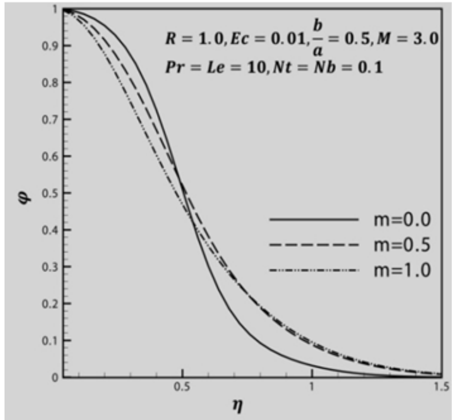

Figure 11: Effect of m on concentration profiles

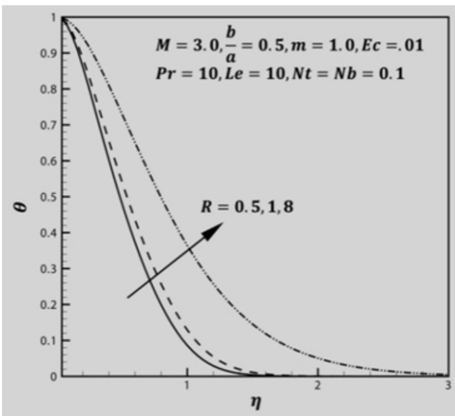

Figure 12: Effect of R on temperature profiles

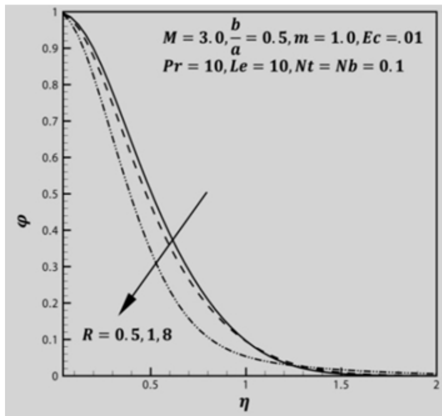

Figure 13: Effect of R on concentration profiles

0.1, $Pr = 10$, $Le = 10$, $R = 1.0$, $Ec = 0.01$ and $m = 1.0$. It is observed that, as the stretching parameter b/a increases the temperature and concentration decreases.

In figures 10 and 11 the dimensionless temperature distribution $\theta(\eta)$ and the dimensionless concentration distribution $\varphi(\eta)$ are plotted respectively for the different values of constant parameter m where $M = 3.0$, $Nt = Nb = 0.1$, $Pr = 10$, $Le = 10$, $R = 1.0$, $b/a = 0.5$ and $Ec = 0.01$. It is observed that, as the constant parameter m increases the temperature and concentration decreases gradually.

In figures 12 and 13 the dimensionless temperature distribution $\theta(\eta)$ and the dimensionless concentration distribution $\varphi(\eta)$ are plotted respectively for the different values of Radiation parameter R where $M = 3.0$, $Nt = Nb = 0.1$, $Pr = 10$, $Le = 10$, $b/a = 0.5$ and $Ec = 0.01$. There is an increase in the Radiation parameter R leads to an increase in the temperature while the reverse effects have been found for concentration.

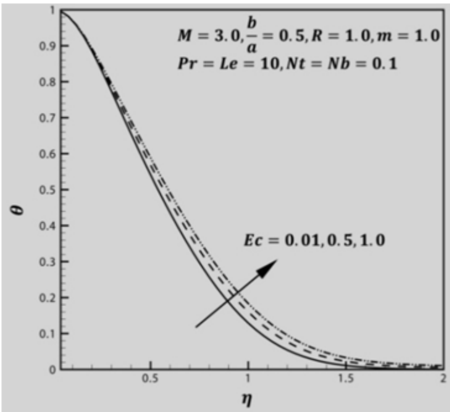

Figure 14: Effect of Ec on temperature profiles

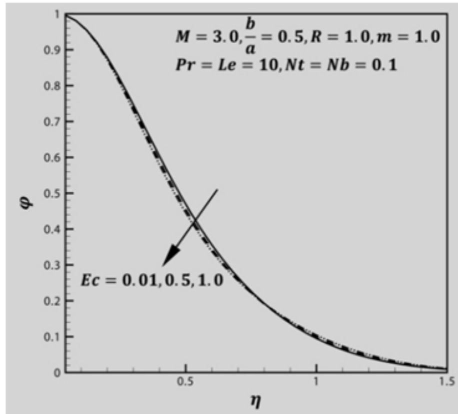

Figure 15: Effect of Ec on concentration profiles

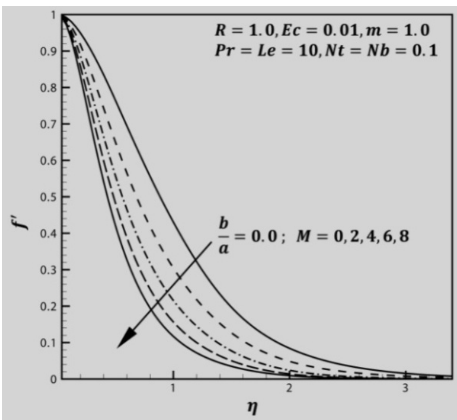

Figure 16: Effect of M on velocity profiles when b/a=0.0

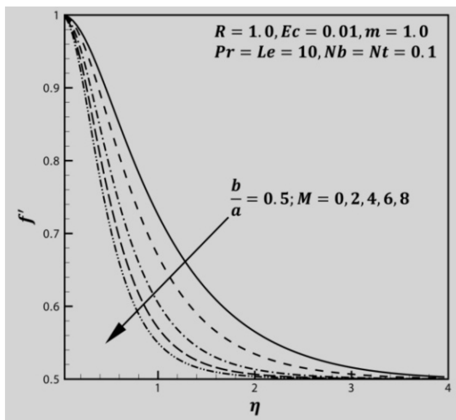

Figure 17: Effect of M on velocity profiles when b/a=0.5

In figures 14 and 15 the dimensionless temperature distribution $\theta(\eta)$ and the dimensionless concentration distribution $\varphi(\eta)$ are plotted respectively for the different values of Eckert number Ec where $M = 3.0$, $Nt = Nb = 0.1$, $Pr =$

10, $Le = 10$, $m = 1.0$, $R = 1.0$ and $b/a = 0.5$. It is noticed that, as increase in the Ec the temperature increases where as concentration decreases.

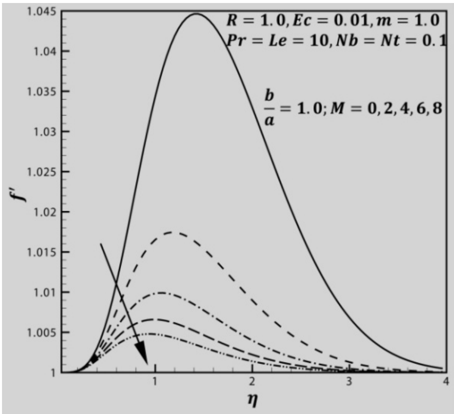

Figure 18: Effect of M on velocity profiles when b/a=1.0

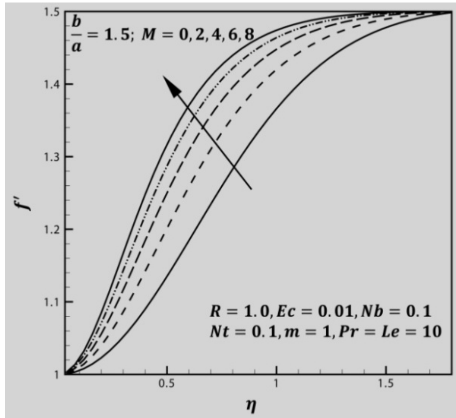

Figure 19: Effect of M on velocity profiles when b/a=1.5

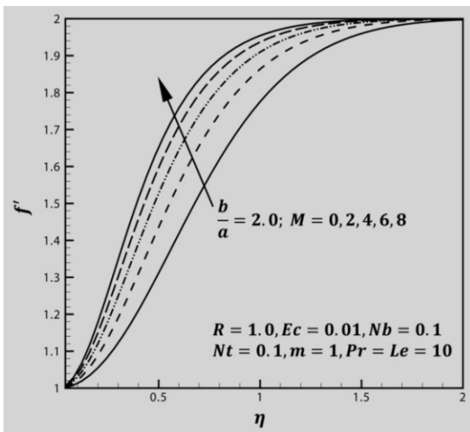

Figure 20: Effect of M on velocity profiles when b/a=2.0

In figures 16-20 the dimensionless velocity distribution $f'(\eta)$ plotted respectively for different values of Magnetic parameter M where $Nt = Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, Ec and $R = 1.0$. Consider here the values of stretching parameter $b/a = 0$, 0.5, 1.0, 1.5 and 2.0. It is observed that, as the Magnetic parameter M increases the velocity decreases gradually when b/a =0, 0.5 and 1 while the reverse effect have been found when $b/a = 1.5$ and 2.0. And the velocity profiles are converges with respect to boundary conditions.

Since the physical interest of the problem, the dimensionless skin-friction coefficient $(-f'')$, the dimensionless heat transfer rate $(-\theta')$ and the dimensionless mass transfer rate $(-\varphi')$ at the sheet are plotted against Thermophoresis parameter Nt and illustrated in figures 21–32.

Figure 21 shows the dimensionless heat transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of Magnetic parameter M where $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$ and $b/a = 0.5$. It is noted that there is an increase in heat transfer rate as Magnetic parameter M increases.

Figure 22 displays the dimensionless heat transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of Radiation parameter R where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$ and $b/a = 0.5$. It is observed that there is an increase in heat transfer rate as Radiation parameter R increases.

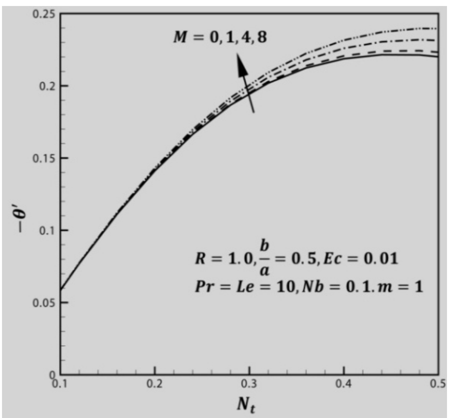

Figure 21: Effect of M on heat transfer rate

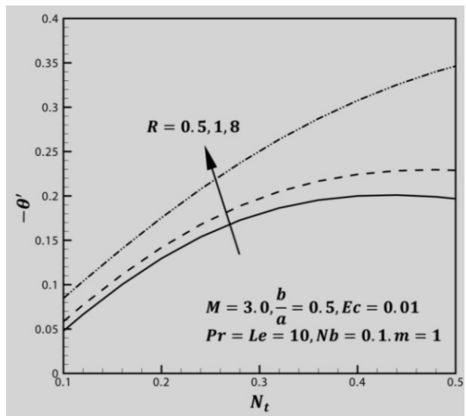

Figure 22: Effect of R on heat transfer rate

Figure 23 represents the dimensionless heat transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of stretching parameter b/a where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, Ec=0.01, $Le = 10$, $m = 1.0$ and $R = 1.0$. It is examined that there is a decrease in heat transfer rate as Stretching parameter b/a increases.

Figure 24 illustrates the dimensionless heat transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of constant parameter m where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $R = 1.0$, Ec=0.01 and $b/a = 0.5$. It is detected that there is a decrease in heat transfer rate as constant parameter m increases.

Figure 25 portrays the dimensionless heat transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of Eckert number Ec where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$ and $b/a = 0.5$. It is verified that there is an increase in heat transfer rate as Eckert number Ec increases.

Figure 26 depicts the dimensionless mass transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of Magnetic parameter M where $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$, Ec=0.01 and $b/a = 0.5$. It is noted that there is an increase in mass transfer rate as Magnetic parameter M increases.

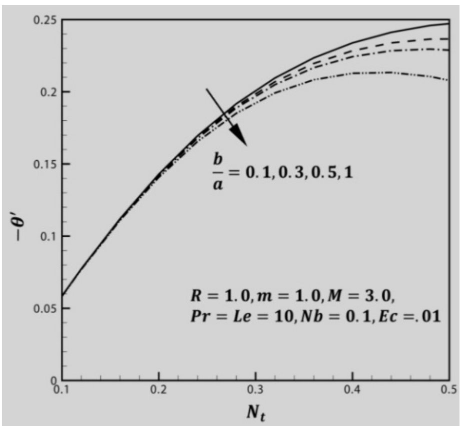

Figure 23: Effect of b/a on heat transfer rate

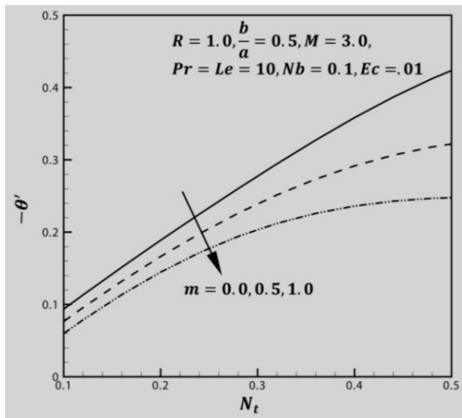

Figure 24: Effect of m on heat transfer rate

Figure 27 exhibits the dimensionless mass transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of Radiation parameter R where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $Ec = 0.01$ and $b/a = 0.5$. It is observed that there is a decrease in mass transfer rate as Radiation parameter R increases.

Figure 28 displays the dimensionless mass transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of stretching parameter b/a where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$ and $Ec = 0.01$. It is examined that there is a decrease in mass transfer rate as stretching parameter b/a increases.

Figure 29 represents the dimensionless mass transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of constant parameter where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$

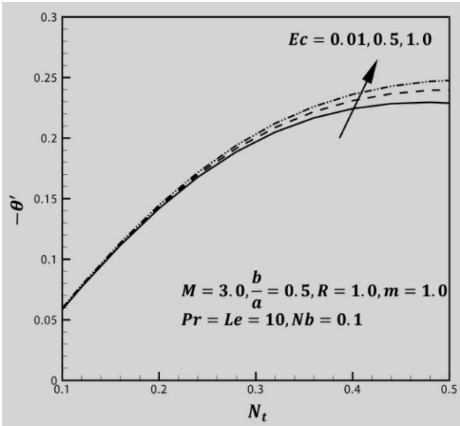

Figure 25: Effect of Ec on heat transfer rate

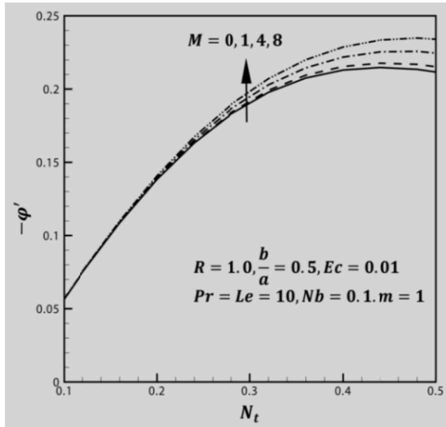

Figure 26: Effect of M on mass transfer rate

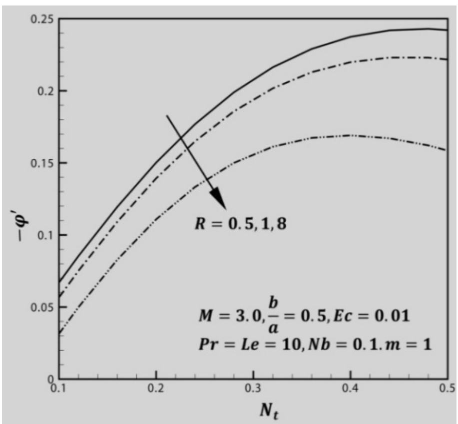

Figure 27: Effect of R on mass transfer rate

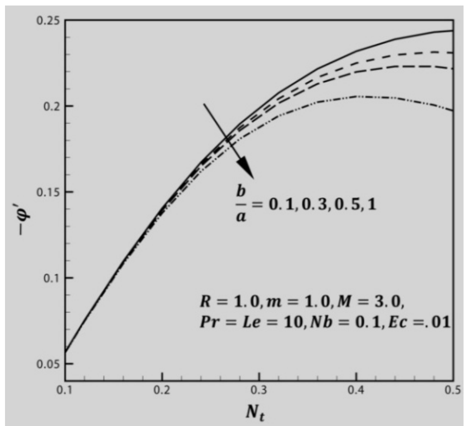

Figure 28: Effect of b/a on mass transfer rate

and $b/a = 0.5$. It is detected that there is a decrease in mass transfer rate as constant parameter m increases.

Figure 30 shows the dimensionless mass transfer rate $(-\theta')$ plotted against Thermophoresis parameter Nt for the different values of Eckert number Ec where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$ and $b/a = 0.5$. It is verified that there is a decrease in mass transfer rate as Eckert number Ec increases.

Figure 31 illustrates the dimensionless skin-friction coefficient $(-f'')$ plotted against Thermophoresis parameter Nt for the different values of Magnetic

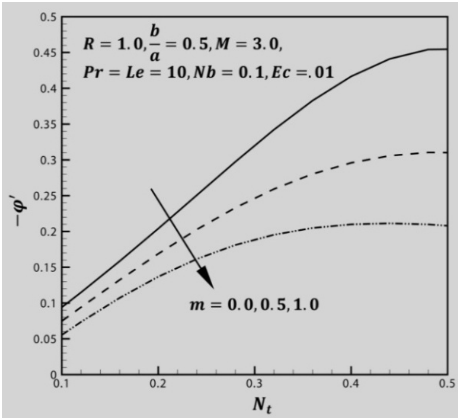

Figure 29: Effect of m on mass transfer rate

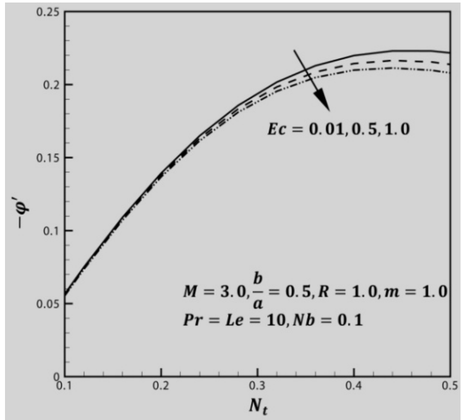

Figure 30: Effect of Ec on mass transfer rate

parameter M where $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$, $Ec = 0.01$, and $b/a = 0.5$. It is verified that there is an increase in skin-friction coefficient as Magnetic parameter M increases.

Figure 32 shows the dimensionless skin-friction coefficient $(-f'')$ plotted against Thermophoresis parameter Nt for the different values of stretching parameter b/a where $M = 3.0$, $Nb = 0.1$, $Pr = 10$, $Le = 10$, $m = 1.0$, $R = 1.0$ and $Ec = 0.01$. It is verified that there is an increase in skin-friction coefficient as stretching parameter b/a increases.

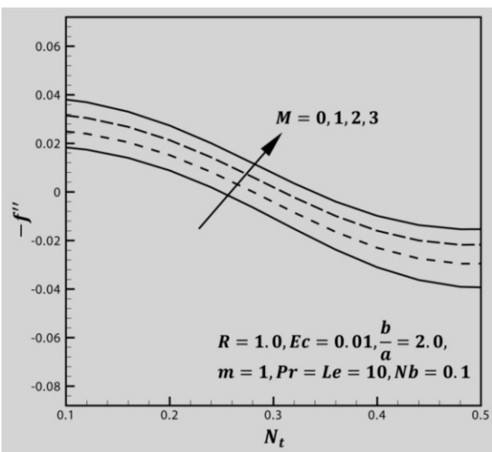

Figure 31: Effect of M on skin-friction coefficient

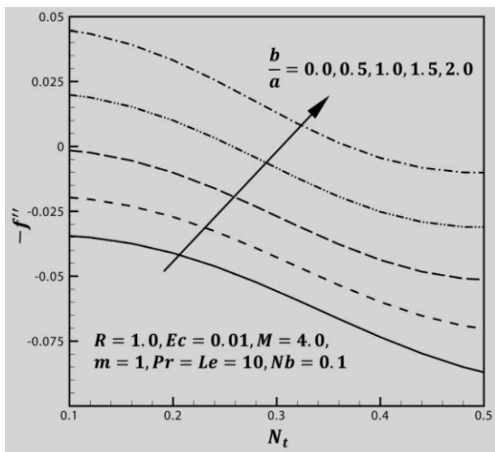

Figure 32: Effect of b/a on skin-friction coefficient

5 Conclusions

Laminar boundary layer flow of a nanofluid has been investigated for steady flow over the stretching surface with the influence of magnetic field and thermal radiation. The effects of Brownian motion and thermophoresis are incorporated into the model for nanofluids. The results are presented for various parameters. The velocity, temperature and concentration distributions also dimensionless skin-friction coefficient, heat and mass transfer rates at the sheet for different parameters are studied and shown graphically. We anticipate that our model would be useful to develop newer applications of magnetic fluids and magnetic flow. For the accuracy of the numerical results, a comparison with Khan and Pop [20] has been showed. The important findings of the investigation from graphical representation are listed below:

- The temperature and concentration boundary layer thickness increases as the magnetic parameter increases for nanofluids. Also the surface heat and mass transfer rate at the sheet are increased. It is interesting to note that the impacts of thermophoresis particle deposition in the presence of magnetic field with Brownian motion have a substantial effect on the flow field and, thus, on the skin-friction coefficient, heat transfer and concentration rate from the sheet to the fluid. Nanofluids are important because they can be used in numerous applications involving heat transfer and other applications such as in science and technology of magnetic fluids.

- As the Stretching parameter increases the temperature and concentration boundary layer thickness decreases gradually for nanofluids. There is also decrease in heat and mass transfer rate at the sheet have been found.

- The temperature and concentration boundary layer thickness decreases gradually as the constant parameter increases for nanofluids. Also the heat transfer rate and mass transfer rate at the sheet are decreased.

- As the Radiation parameter increases the temperature increases while the reverse effect have been found for concentration boundary layer thickness for nanofluids. There is an increase in heat transfer rate and decrease in mass transfer rate have also found.

- As the Eckert number increases the temperature boundary layer thickness increases for nanofluids while the reverse effect have been found for concentration boundary layer thickness. Also an increase in heat transfer rate at the sheet and decrease in mass transfer rate at the sheet have been found.

- The velocity profiles decreases for increasing Magnetic Parameter when the values of Stretching parameter is up to 1.0, but when the Stretching parameter take values > 1 (up to 2.0) then the velocity profiles increases. Also the skin-friction coefficient increases for increasing Magnetic Parameter as well as stretching parameter.

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Asymptotic normality of conditional distribution estimation in the single index model

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Abstract. This paper deals with the estimation of conditional distribution function based on the single-index model. The asymptotic normality of the conditional distribution estimator is established. Moreover, as an application, the asymptotic $(1-\gamma)$ confidence interval of the conditional distribution function is given for $0 < \gamma < 1$.

1 Introduction

The single functional index models have received a considerable attention because of their wide applications in many areas such as economics, medicine, financial econometric and so on. The study of these models has been developed rapidly, see Ait-Saidi *et al.* (2005, 2008a, 2008b). Recently, Attaoui *et*

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al. (2011) investigated the kernel estimator of the conditional density of a scalar response variable Y, given a Hilbertian random variable X when the observations are from a single functional index model. The pointwise and the uniform almost complete convergence of the estimator with rates in this model were obtained for independent observations. Furthermore, Ling *et al.* (2012) obtained the asymptotic normality of the conditional density estimator and the conditional mode estimator for the α-mixing dependence functional time series data. Ling *et al.* (2014) investigated the pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimation with rate for the conditional density in the setting of the α-mixing functional data, which extend the i.i.d case in Attaoui et al. (2011) to the dependence setting, the convergence rate of the kernel estimation for the conditional mode was also obtained.

The main contribution of this paper is to establish the asymptotic normality for the estimator of conditional distribution function in the i.i.d. case when the single functional index θ is fixed. As an application, the asymptotic $(1-\gamma)$ confidence interval for the conditional density function $F(\theta, y, x)$ is presented. The outline of the present paper is as follows. In section 2, we introduce the model as well as basic assumptions that are necessary in deriving the main result of this paper. In section 3, we state the main result of the paper; the asymptotic normality of the estimator for the conditional distribution function. As an application, the asymptotic $(1-\gamma)$ confidence interval of the conditional distribution function is given for $0 < \gamma < 1$. Finally, the technical proofs are related to section 4.

2 Model and some basic assumptions

Let $\{(X_i, Y_i), 1 \leq i \leq n\}$ be n random variables, identically distributed as the random pair (X, Y) with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\|.\|$ generated by an inner product $< ., . >$. Under such topological structure and for a fixed functional θ, we suppose that the conditional probability distribution of Y given $< X, \theta >=< x, \theta >$ exists and is given by

$$\forall y \in \mathbb{R}, F(\theta, y, x) = \mathbb{P}(Y \leq y| < X, \theta >=< x, \theta > . \quad (1)$$

The nonparametric kernel estimator $\widehat{F}(\theta, y, x)$ of $F(\theta, y, x)$ is defined as follows,

$$\widehat{F}(\theta, y, x) = \frac{\sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >))H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >))}, \tag{2}$$

where K is a kernel, H is a cumulative distribution function (cdf) and $h_K = h_{K,n}$(resp,$h_H = h_{H,n}$) is a sequence of positive real numbers which goes to zero as n tends to infinity, and with the convention $0/0 = 0$.

Let, for any $x \in \mathcal{H}$, $i = 1, \ldots, n$ and $y \in \mathbb{R}$

$$K_i(\theta, x) := K(h_K^{-1}| < x - X_i, \theta > |), \text{ and } H_i(y) := H(h_H^{-1}(y - Y_i)).$$

We denote by $B_\theta(x, h) = \{X \in \mathcal{H}/0 < | < x - X, \theta > | < h\}$ the ball centered at x with radius h, let $\mathcal{N}_x$ be a fixed neighborhood of x in $\mathcal{H}$, S_R will be a fixed compact subset of $\mathbb{R}$.

Now, we introduce the following basic assumptions that are necessary in deriving the main result of this paper.

(H1) $\mathbb{P}(X \in B_\theta(x, h_K)) =: \phi_{\theta,x}(h) > 0, \quad \phi_{\theta,x}(h) \to 0$ as $h \to 0$.

(H2) The conditional cumulative distribution $F(\theta, y, x)$ satisfies the Hölder condition, that is:

$\forall (y_1, y_2) \in S_R \times S_R\ ,\ \forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x.$

$|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \le C_{\theta,x}(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}),\ b_1 > 0,\ b_2 > 0.$

(H3) For $j = 0, 1$, $H^{(j)}$ satisfies the lipschitz conditions and

$$m := \inf_{t \in [0,1]} K(t)H'(t) > 0,$$

with

$\int H'(t)dt = 1, \quad \int H^2(t)dt < \infty$ and $\int |t|^{b_2} H^{(1)}(t)dt < \infty$

(H4) The kernel K is nonnegative, with compact support $[0, 1]$ of class $\mathcal{C}^1$ on $[0, 1)$ such that $K(1) > 0$ and its derivative K' exists on $[0, 1)$ and $K'(t) < 0$.

(H5) For all $u \in [0, 1]$, $\lim_{h \to 0} \frac{\phi_{\theta,x}(uh)}{\phi_{\theta,x}(h)} = \lim_{h \to 0} \xi_h^{\theta,x}(u) = \xi_0^{\theta,x}(u)$.

(H6) The bandwidth h_H satisfies,

(i) $\dfrac{\log n}{n\,\phi_{\theta,x}(h_K)} \to 0$, as $n \to \infty$.

(ii) $nh_H^2\phi_{\theta,x}^2(h_K) \longrightarrow \infty$, and $\frac{nh_H^3\phi_{\theta,x}(h_K)}{\log^2 n} \longrightarrow \infty$ as $n \to \infty$.

(iii) $nh_H^2\phi_{\theta,x}^3(h_K) \longrightarrow 0$, as $n \to \infty$.

(H7) (i) $\frac{\phi_{\theta,x}(h)}{n} + \phi_x(h) = \mathcal{O}(\frac{1}{n})$.

(ii) $\sqrt{n\phi_{\theta,x}(h)} \to 0$ as $n \to \infty$.

Comments on the assumptions. Assumption (H1) is the same as one given in Ferraty *et al.* (2005). Assumption (H2) is a regularity conditions which characterize the functional space of our model and is needed to evaluate the bias term of our asymptotic results. Assumptions (H3) and (H5) and (H6) are technical conditions for the proofs. Assumptions (H4) is classical in functional estimation for finite or infinite dimension spaces.

Remark 1 *Assumption (H5) is known as (for small* h*) the "concentration assumption acting on the distribution of* X*" in infinite dimensional spaces.*

The function $\xi_h^x(\cdot)$ *intervening in assumption (H9) is increasing for all fixed* h*. Its pointwise limit* $\xi_0^x(\cdot)$ *plays a determinant role. It is possible to specify this function (with* $\xi_0(u) := \xi_0^x(u)$ *in the above examples by:*

1. $\xi_0(u) = u^\gamma$,
2. $\xi_0(u) = \delta_1(u)$, *where* $\delta_1(\cdot)$ *is Dirac function,*
3. $\xi_0(u) = \mathbf{1}_{]0,1]}(u)$.

3 Main results: Asymptotic normality of the estimator $\widehat{F}(\theta, y, x)$

In this part of paper, we give the asymptotic normality of the conditional cumulative distribution function in the single functional index model. The main result is given in the following theorem.

Theorem 1 *Under Assumptions (H1)-(H7) we have*

$$\sqrt{\frac{n\phi_{\theta,x}(h_K)}{\sigma^2(\theta,y,x)}}(\widehat{F}(\theta,y,x) - F(\theta,y,x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Where

$$\sigma^2(\theta,y,x)=\frac{C_2(\theta,x)F(\theta,y,x)(1-F(\theta,y,x))}{C_1^2(\theta,x)},$$

with $C_j(\theta,x) = K^j(1)-\int_0^1 sK'(s)\beta_{\theta,x}(s)ds$ *for* $j=1,2$, " $\xrightarrow{\mathcal{D}}$ " *means the convergence in distribution.*

Proof. Consider, for $i=1,\dots,n$,

$$K_i(\theta,x)=K(h_K^{-1}(<x-X_i,\theta>)),\ H_i(y)=H\left(h_H^{-1}\left(y-Y_i\right)\right),$$
$$\widehat{F}_N(\theta,y,x)=\frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)H_i(y),$$
$$\widehat{F}_D(\theta,x)=\frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x),$$
$$\Delta_i(x,\theta)=\frac{K(h_K^{-1}(<x-X_i,\theta>))}{\mathbb{E}K_1(\theta,x)}.$$

In order to establish the asymptotic normality of $\widehat{F}(\theta,t,x)$ we have to consider the following decomposition

$$\begin{aligned}
\widehat{F}(\theta,y,x)-F(\theta,y,x) &= \frac{\widehat{F}_N(\theta,y,x)}{\widehat{F}_D(\theta,x)}-\frac{C_1(\theta,x)F(\theta,y,x)}{C_1(\theta,x)}\\
&= \frac{1}{\widehat{F}_D(\theta,x)}\left(\widehat{F}_N(\theta,y,x)-\mathbb{E}\widehat{F}_N(\theta,y,x)\right)\\
&\quad -\frac{1}{\widehat{F}_D(\theta,x)}\left(C_1(\theta,x)F(\theta,y,x)-\mathbb{E}\widehat{F}_N(\theta,y,x)\right)\\
&\quad +\frac{F(\theta,y,x)}{\widehat{F}_D(\theta,x)}\left(C_1(\theta,x)-\mathbb{E}\left[\widehat{F}_D(\theta,x)\right]\right)\\
&\quad -\frac{F(\theta,y,x)}{\widehat{F}_D(\theta,x)}\left(\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)\right)\\
&= \frac{1}{\widehat{F}_D(\theta,x)}A_n(\theta,y,x)+B_n(\theta,y,x) \qquad (3)
\end{aligned}$$

where

$$A_n(\theta,y,x)=\frac{1}{n\mathbb{E}K_1(x,\theta)}\sum_{i=1}^{n}\left\{\left(H_i(y)-F(\theta,y,x)\right)K_i(\theta,x)\right.$$

$$-\mathbb{E}\left[\left(H_i(y) - F(\theta, y, x)\right) K_i(\theta, x)\right] \Big\} = \frac{1}{n\mathbb{E}K_1(x,\theta)} \sum_{i=1}^{n} N_i(\theta, y, x),$$

and

$$N_i(\theta, y, x) = \left(H_i(y) - F(\theta, y, x)\right) K_i(\theta, x) - \mathbb{E}\left[\left(H_i(y) - F(\theta, y, x)\right) K_i(\theta, x)\right].$$

It follows that,

$$\begin{aligned} n\phi_{\theta,x}(h_K)Var\left(A_n(\theta, t, x)\right) &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(x,\theta)} Var(N_1) \\ &\quad + \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2 K_1(x,\theta)} \sum_{|i-j|>0}^{n}\sum Cov(N_i, N_j) \\ &= V_n(\theta, t, x) \end{aligned} \tag{4}$$

Then, the rest of the proof is based on the following Lemmas

Lemma 1 *Under hypotheses (H1)-(H3), (H5) and (H7), as $n \to \infty$ we have*

$$n\phi_{\theta,x}(h_K)Var\left(A_n(\theta, y, x)\right) \longrightarrow V(\theta, y, x),$$

where

$$V(\theta, y, x) = \frac{C_2(\theta, x)}{(C_1(\theta, x))^2} F(\theta, y, x)\left(1 - F(\theta, y, x)\right).$$

Lemma 2 *Under hypotheses (H1)-(H3) and (H5)-(H7), as $n \to \infty$ we have*

$$\left(\frac{n\phi_{\theta,x}(h_K)}{V(\theta, y, x)}\right)^{1/2} A_n(\theta, y, x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

Lemma 3 *Under assumptions (H1)-(H3) and (H5)-(H7); as $n \to \infty$ we have*

$$\sqrt{n\phi_{\theta,x}(h_K)} B_n(\theta, y, x) \longrightarrow 0 \quad \textit{in Probabilty.}$$

$\square$

Now, because the unknown functions $C_j(\theta, x)$ and $F(\theta, y, x)$ intervening in the expression of the variance, we need to estimate the quantities $C_1(\theta, x)$, $C_2(\theta, x)$ and $F(\theta, y, x)$, respectively.

By assumptions (H1)-(H4) we know that $a_j(\theta, x)$ can be estimated by $\widehat{C_j}(\theta, x)$ which is defined as

$$\widehat{C}_j(\theta, x) = \frac{1}{n\widehat{\phi}_{\theta,x}(h_K)} \sum_{i=1}^{n} K_i^j(\theta, x) \ , \ j = 1, 2$$

where

$$\widehat{\phi}_{\theta,x}(h_K) = \frac{1}{n} \sum_{i=1}^{n} I_{\{|<x-X_i,\theta>|<h_k\}}.$$

By applying the kernel estimator of $F(\theta, y, x)$ given above, the quantity $\sigma^2(\theta, x)$ can be estimated finally by:

$$\widehat{\sigma}^2(\theta, x) = \frac{\widehat{C}_2(\theta, x)\widehat{F}(\theta, y, x)}{\widehat{C}_1^2(\theta, x)} \int H^2(t)dt.$$

Next, we can derive the following corollary:

Corollary 1 *Under assumptions of Theorem 1, we have*

$$\sqrt{\frac{n\widehat{\phi}_{\theta,x}(h_K)}{\widehat{\sigma}^2(\theta, y, x)}}(\widehat{F}(\theta, y, x) - F(\theta, y, x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Thus, following this Corollary we can approximate $(1-\gamma)$ confidence interval of $F(\theta, y, x)$ by

$\widehat{F}(\theta, y, x) \pm t_{\gamma/2} \times \dfrac{\widehat{\sigma}(\theta, x)}{\sqrt{n\widehat{\phi}_{\theta,x}(h_K)}}$, where $t_{\gamma/2}$ is the upper $\gamma/2$ quantile of standard Normal $\mathcal{N}(0, 1)$.

4 Proofs of technical lemmas

Proof. [Proof of Lemma 1]

Let

$$\begin{aligned} V_n(\theta, y, x) &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(\theta, x)} \mathbb{E}\left[K_1^2(\theta, x)\left(H_1(y) - F(\theta, y, x)\right)^2\right] \\ &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(\theta, x)} \mathbb{E}\left[K_1^2(\theta, x)\mathbb{E}\left((H_1(y) - F(\theta, y, x))^2 \,|<\theta, X_1>\right)\right] \end{aligned} \tag{5}$$

Using the definition of conditional variance, we have

$$\mathbb{E}\left[\left(H(h_H^{-1}(y - Y_1)) - F(\theta, y, x)\right)^2 |<\theta, X_1>\right] = J_{1n} + J_{2n},$$

where

$$J_{1n} = Var\left(H(h_H^{-1}(y - Y_1))| < \theta, X_1 >\right),$$

and

$$J_{2n} = \left[\mathbb{E}\left(H(h_H^{-1}(y - Y_1))| < \theta, X_1 >\right) - F(\theta, y, x)\right]^2$$

$\rightsquigarrow$**Concerning** J_{1n}. Let

$$\begin{aligned} J_{1n} &= \mathbb{E}\left[H^2\left(\frac{y - Y_1}{h_H}\right)| < \theta, x >\right] - \left(\mathbb{E}\left[H\left(\frac{y - Y_1}{h_H}\right)| < \theta, X_1 >\right]\right)^2 \\ &= \mathcal{J}_1 + \mathcal{J}_2 \end{aligned}$$

- By the property of double conditional expectation, we get that

$$\begin{aligned} \mathcal{J}_1 &= \mathbb{E}\left[H^2\left(\frac{y - Y_1}{h_H}\right)| < \theta, X_1 >\right] \\ &= \int_{\mathbb{R}} H^2\left(\frac{y - v}{h_H}\right) dF(\theta, v, X_1) \\ &= \int_{\mathbb{R}} H^2(t) dF(\theta, y - h_H t, X_1). \end{aligned} \tag{6}$$

On the other hand, by integrating by part and under assumption (H3), we have

$$\begin{aligned} \mathcal{J}_1 &= \int_{\mathbb{R}} 2H(t)H'(t)F(\theta, y - h_H t, X_1) du \\ &= \int_{\mathbb{R}} 2H(t)H'(t)\left(F(\theta, y - h_H t, X_1) - F(\theta, y, x)\right) du \\ &\quad + \int_{\mathbb{R}} 2H(t)H'(t)F(\theta, y, x) du. \end{aligned}$$

Clearly, we have

$$\int_{\mathbb{R}} 2H(t)H'(t)F(\theta, y, x) du = \left[H^2(t)F(\theta, y, x)\right]_{-\infty}^{+\infty} = F(\theta, y, x), \tag{7}$$

thus

$$\int_{\mathbb{R}} H^2(t) dF(\theta, y - h_H t, X_1) = F(\theta, y, x) + \mathcal{O}(h_K^{b_1} + h_H^{b_2}). \tag{8}$$

$\rightsquigarrow$ **Concerning $\mathcal{J}_2$.** Let

$$\begin{aligned} I &= \mathbb{E}\left(H_i(y)| < X_1, \theta >\right) \\ \mathbb{E}\left(H\left(\frac{y-Y_1}{h_H}\right)| < X_1, \theta >\right) &= \int_{\mathbb{R}} H\left(\frac{y-u}{h_H}\right) f(\theta, y, X_1) du, \\ &= \int_{\mathbb{R}} H\left(\frac{y-u}{h_H}\right) dF(\theta, y, X_1), \\ &= \int_{\mathbb{R}} H'\left(\frac{y-u}{h_H}\right) F(\theta, u, X_1) du, \\ &= \int_{\mathbb{R}} H'(t)\left(F(\theta, y-h_H t, X_1) - F(\theta, y, x)\right) dt \\ &\quad + F(\theta, y, x) \int_{\mathbb{R}} H'(t) dt. \end{aligned}$$

Because H' is a probability density and by hypotheses (H2) and (H3), we can write:

$$\begin{aligned} I &\leq C_{x,\theta} \int_{\mathbb{R}} H'(t)\left(h_K^{b_1} + |t|^{b_2} h_H^{b_2}\right) dt + F(\theta, y, x) \\ &= \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + F(\theta, y, x). \end{aligned}$$

Finally, by hypothesis (H3) we get

$$\mathcal{J}_2 \longrightarrow F^2(\theta, y, x), \quad \text{as } n \to \infty. \tag{9}$$

The last equality is due to the fact that H' is a probability density, thus we have by hypothesis (H3)

$$\int_{\mathbb{R}} H'(t)\left(F(\theta, y-h_H t, X_1) - F(\theta, y, x)\right) dt \leq \int_{\mathbb{R}} H'(t)\left(|t|^{b_2} h_H^{b_2} + h_K^{b_1}\right) dt \underset{n\to\infty}{\longrightarrow} 0.$$

$\rightsquigarrow$ **Concerning J_{2n}.**

We have by integration by parts and changing variables

$$\begin{aligned} J_{2n} &= \mathbb{E}\left(H_1(y)| < \theta, X_1 >\right) \\ &= \mathbb{E}\left(H\left(\frac{y-Y_1}{h_H}\right)| < \theta, X_1 >\right) \\ &= \int H\left(\frac{y-v}{h_H}\right) f(\theta, v, X_1) dv \end{aligned}$$

$$
\begin{aligned}
&= \int H\left(\frac{y-v}{h_H}\right) dF(\theta, v, X_1) \\
&= \int H'(t)F(\theta, y-h_H t, X_1)dt \\
&= F(\theta, y, x)\int H'(t)dt + \int H'(t)\left(F(\theta, y-h_H t, x) - F(\theta, y, x)\right)dt,
\end{aligned}
$$

the last equality is due to the fact that H' is a probability density.
Thus, we have:

$$
J_{2n} = F(\theta, y, x) + \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) \tag{10}
$$

Finally, we obtain that $J_{2n} \underset{n\to\infty}{\longrightarrow} 0$.
Meanwhile, by (H1), (H2), (H4) and (H5), it follows that:

$$
\frac{\phi_{\theta,x}(h_K)\mathbb{E}K_1^2(\theta, x)}{\mathbb{E}^2K_1(\theta, x)} \underset{n\to\infty}{\longrightarrow} \frac{C_2(\theta, x)}{(C_1(\theta, x))^2},
$$

Then, by combining equations (5)-(10), it leads to

$$
V_n(\theta, y, x) \underset{n\to\infty}{\longrightarrow} \frac{C_2(\theta, x)}{(C_1(\theta, x))^2}F(\theta, y, x)\left(1 - F(\theta, y, x)\right). \tag{11}
$$

□

Proof. [Proof of Lemma 2]
We will establish the asymptotic normality of $A_n(\theta, t, x)$ suitably normalized.
We have

$$
\begin{aligned}
\sqrt{n\phi_{\theta,x}(h_K)}A_n(\theta, y, x) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}K_1(\theta, x)}\sum_{i=1}^{n} N_i(\theta, y, x) \\
&= \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\sqrt{n}\mathbb{E}K_1(\theta, x)}\sum_{i=1}^{n} N_i(\theta, y, x) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \Xi_i(\theta, y, x) = \frac{1}{\sqrt{n}}S_n
\end{aligned} \tag{12}
$$

Now, we can write,

$$
\Xi_i = \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\mathbb{E}K_1(\theta, x)}N_i,
$$

Thus
$$Var(\Xi_i) = \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(\theta,x)} Var(N_i) = V_n(\theta,y,x).$$
Note that by (11), we have $Var(\Xi_i) \longrightarrow V(\theta,y,x)$ as n goes to infinity. Obviously, we have
$$\sqrt{\frac{n\phi_{\theta,x}(h_K)}{V(\theta,y,x)}}\,(A_n(\theta,y,x)) = (nV(\theta,y,x))^{-1/2}\, S_n.$$

Thus, the asymptotic normality of $(nV(\theta,y,x))^{-1/2}\, S_n$, is deduced from the following results

$$\left|\mathbb{E}\left\{\exp\left(izn^{-1/2}S_n\right)\right\} - \prod_{j=0}^{n}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Xi_j\right)\right\}\right| \longrightarrow 0, \tag{13}$$

$$\frac{1}{n}\sum_{j=0}^{n}\mathbb{E}\left(\Xi_j^2\right) \longrightarrow V(\theta,y,x), \tag{14}$$

$$\frac{1}{n}\sum_{j=0}^{n}\mathbb{E}\left(\Xi_j^2 \mathbf{1}_{\{|\Xi_j|>\varepsilon\sqrt{nV(\theta,y,x)}\}}\right) \longrightarrow 0, \ \text{for every } \ \varepsilon > 0. \tag{15}$$

While equations (13) and (14) show that the Υ_j are asymptotically independent, verifying that the sum of their variances tends to $V(\theta,y,x)$. Expression (15) is the Lindeberg-Feller's condition for a sum of independent terms. Asymptotic normality of S_n is a consequence of equations (13)-(15).

- **Proof of (13)** We make use of Volkonskii and Rozanov's lemma (see the appendix in Masry (2005) and the fact that the process (X_i) is i.i.d. Note that using that $V_j = \exp\left(izn^{-1/2}S_n\right)$, we have
$$\left|\mathbb{E}\left\{\exp\left(izn^{-1/2}S_n\right)\right\} - \prod_{j=0}^{n}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Xi_j\right)\right\}\right| \longrightarrow 0$$
as n goes to infinity.

- **Proof of (14)** Note that $Var(S_n) \longrightarrow V(\theta,y,x)$ by equation (11) and (12) (by the definition of the Ξ_i). Then because
$$\mathbb{E}\left(S_n\right)^2 = Var\left(S_n\right) = \sum_{j=0}^{n} Var\left(\Xi_j\right),$$

and, using the same arguments as those previously used in the proof of first term of equation (5), we obtain

$$\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\Xi_j^2\right) = \mathrm{Var}\left(\Xi_1\right),$$

as $\mathrm{Var}\left(\Xi_1\right) \longrightarrow V(\theta, y, x)$.

- **Proof of (15)** Recall that

$$\Xi_j = \sum_{i=0}^{n} \Upsilon_i.$$

Finally, to establish (15) it suffices to show that the set

$$\{|\Xi_j| > \varepsilon\sqrt{nV(\theta, y, x)}\}$$

is negligible for n large enough.

By using assumptions (H4) and (H5), we have

$$\left|\Upsilon_i\right| \leq C\left(\phi_{\theta,x}(h_K)\right)^{-1/2},$$

therefore

$$\left|\Xi_j\right| \leq Cn\left(\phi_{\theta,x}(h_K)\right)^{-1/2},$$

which goes to zero as n goes to infinity.

Since

$$|H_i(y) - F(\theta, y, x)| \leq 1.$$

Then for n large enough, the set $\left\{|\Xi_j| > \varepsilon\left(nV(\theta, y, x)\right)^{-1/2}\right\}$ becomes empty, this completes the proof and therefore that of the asymptotic normality of $\left(nV(\theta, y, x)\right)^{-1/2} S_n$ and the Lemma 2.

□

Proof. [Proof of Lemma 3]

We have

$$\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta, y, x) = \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta, x)}\Big\{\mathbb{E}\widehat{F}_N(\theta, y, x) - C_1(\theta, x)F(\theta, y, x)$$
$$+F(\theta, y, x)\left(C_1(\theta, x) - \mathbb{E}\widehat{F}_D(\theta, x)\right)\Big\}.$$

Firstly, observe that as $n \to \infty$

$$\frac{1}{\phi_{\theta,x}(h_K)}\mathbb{E}\left[K^l\left(\frac{< x - X_i, \theta >}{h_K}\right)\right] \longrightarrow C_l(\theta, x), \quad \text{for } l = 1, 2 \tag{16}$$

$$\mathbb{E}\left[\widehat{F}_D(\theta, x)\right] \longrightarrow C_1(\theta, x), \tag{17}$$

and

$$\mathbb{E}\left[\widehat{F}_N(\theta, y, x)\right] \longrightarrow C_1(\theta, x)F(\theta, y, x), \tag{18}$$

can be proved in the same way as in Ezzahrioui and Ould Said (2008) corresponding to their Lemmas 5.1 and 5.2. Then the proofs of (16)-(18) are omitted.

Secondly, making use of (16), (17) and (18), we have as $n \to \infty$

$$\left\{\mathbb{E}\widehat{F}_N(\theta, y, x) - C_1(\theta, x)F(\theta, y, x) + F(\theta, y, x)\left(C_1(\theta, x) - \mathbb{E}\widehat{F}_D(\theta, x)\right)\right\} \longrightarrow 0.$$

On other hand

$$\frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta, x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widehat{F}(\theta, y, x)}{\widehat{F}_D(\theta, x)\widehat{F}(\theta, y, x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widehat{F}(\theta, y, x)}{\widehat{F}_N(\theta, y, x)}. \tag{19}$$

Because $K(\cdot)H'(\cdot)$ is continuous with support on $[0, 1]$, then by hypotheses (H3) and (H4) $\exists\ m = \inf\limits_{t\in[0,1]} K(t)H'(t)$ such that

$$\widehat{F}_N(\theta, y, x) \geq \frac{m}{h_H\phi_{\theta,x}(h_K)}$$

which gives

$$\frac{n\phi_{\theta,x}(h_K)}{\widehat{F}_N(\theta, y, x)} \leq \frac{\sqrt{nh_H^2\phi_{\theta,x}(h_K)^3}}{m}$$

Finally, using (H6), the proof of Lemma 3 is completed.

$\square$

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Some conditions under which derivations are zero on Banach $*$-algebras

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Abstract. Let $\mathcal{A}$ be a Banach $*$-algebra. By $\mathcal{S}_{\mathcal{A}}$ we denote the set of all self-adjoint elements of $\mathcal{A}$ and by $\mathcal{O}_{\mathcal{A}}$ we denote the set of those elements in $\mathcal{A}$ which can be represented as finite real-linear combinations of mutually orthogonal projections. The main purpose of this paper is to prove the following result:
Suppose that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$ and $\{d_n\}$ is a sequence of uniformly bounded linear mappings satisfying $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$, where p is an arbitrary projection in $\mathcal{A}$. Then $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$ for each $n \geq 1$. In particular, if $\mathcal{A}$ is semi-prime and further, $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, then $d_n = 0$ for each $n \geq 1$.

1 Introduction and preliminaries

In this paper, $\mathcal{A}$ represents a Banach $*$-algebra over the complex field $\mathbb{C}$. If $\mathcal{A}$ is unital, then $\mathbf{1}$ will stand for its unit element. Moreover, $\mathcal{A}$ is called semi-prime if $a\mathcal{A}a = \{0\}$ implies that $a = 0$. A non-zero linear functional φ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in \mathcal{A}$. By $\Phi_{\mathcal{A}}$ we denote the set of all characters on $\mathcal{A}$. It is well known that, $\ker\varphi$ the kernel of φ is a maximal ideal of $\mathcal{A}$, where φ is an arbitrary element of $\Phi_{\mathcal{A}}$. We denote the set of all self-adjoint projections in $\mathcal{A}$ by $\mathcal{P}_{\mathcal{A}}$ (i.e., $\mathcal{P}_{\mathcal{A}} = \{p \in \mathcal{A} \mid p^2 = p,\ p^* = p\}$), and by $\mathcal{S}_{\mathcal{A}}$

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we denote the set of all self-adjoint elements of $\mathcal{A}$ (i.e., $\mathcal{S}_{\mathcal{A}} = \{a \in \mathcal{A} \mid a^* = a\}$). Next, the set of these elements in $\mathcal{A}$ which can be represented as finite real-linear combinations of mutually orthogonal self-adjoint projections, is denoted by $\mathcal{O}_{\mathcal{A}}$. Hence, we have $\mathcal{P}_{\mathcal{A}} \subseteq \mathcal{O}_{\mathcal{A}} \subseteq \mathcal{S}_{\mathcal{A}}$. Note that if $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{O}_{\mathcal{A}}$ is norm dense in $\mathcal{S}_{\mathcal{A}}$. More generally, the same is true for AW^*-algebras. Recall that a C^*-algebra is a Banach $*$-algebra in which, for every a, $\|a^*a\| = \|a\|^2$. A W^*-algebra is a weakly closed self-adjoint algebra of operators on a Hilbert space, and an AW^*-algebra is a C^*-algebra satisfying:
i) In the partially ordered set of projections, any set of orthogonal projections has a least upper bound (LUB),
ii) Any maximal commutative self-adjoint subalgebra is generated by its self-adjoint projections. That is, it is equal to the smallest closed subalgebra containing its self-adjoint projections.
When $\mathcal{A}$ is an AW^*-algebras it can be proved that each maximal commutative $*$-subalgebra of $\mathcal{A}$ is monotone complete and $\mathcal{A}$ is unital.
The above-mentioned definitions and results can all be found in [1], [5] and [10] and reader is referred to this sources for more general information on W^*-algebras and AW^*-algebras. In this paper, similar to Brešar [1], the author's attention is concentrated on Banach $*$-algebras in which $\mathcal{O}_{\mathcal{A}}$ is norm dense in $\mathcal{S}_{\mathcal{A}}$, i.e. $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$.

A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the Leibnitz's rule $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is called a Jordan derivation if $d(a^2) = d(a)a + ad(a)$ holds for all $a \in \mathcal{A}$. If we define a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{d^n}{n!}$, where I is the identity mapping on $\mathcal{A}$, then the Leibnitz's rule ensures us that d_n's satisfy the condition

$$d_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)d_k(b) \tag{1}$$

for each $a, b \in \mathcal{A}$ and each non-negative integer n. This motivates us to consider the sequences $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying (1). Such a sequence is called a higher derivation. A sequence $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$, where p is an arbitrary element of $\mathcal{P}_{\mathcal{A}}$, is called a pre-higher derivation. A pre-higher derivation $\{d_n\}$ is called uniformly bounded if there exists an $M > 0$ such that $\|d_n\| \leq M$ for each n. In current note, the focus of attention is on uniformly bounded pre-higher derivations. The question under which conditions all derivations are zero on a given $*$-algebra have attracted much attention of authors (for

instance, see [3], [4], [6], [8], [9], and [12]). In this paper, we also concentrate on this topic. Let us provide a background of our study. In 1955, Singer and Wermer [11] achieved a fundamental result which started investigation into the range of derivations on Banach algebras. The result states that if $\mathcal{A}$ is a commutative Banach algebra and $d : \mathcal{A} \to \mathcal{A}$ is a bounded derivation, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$, where $rad(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$. It is evident that if $\mathcal{A}$ is semi-simple, i.e. $rad(\mathcal{A}) = \{0\}$, then d is zero. In this paper, we prove that there is not any non-zero bounded derivation from $\mathcal{A}$ into $\mathcal{A}$ without considering the commutativity and semi-simplicity assumptions for $\mathcal{A}$. Indeed, we prove the following result:
Suppose that $\mathcal{A}$ is a semi-prime Banach $*$-algebra so that $\mathcal{O}_{\mathcal{A}}$ is norm dense in $\mathcal{S}_{\mathcal{A}}$, and $d : \mathcal{A} \to \mathcal{A}$ is a bounded derivation. If $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, then d is identically zero. In this case, it is possible that $rad(\mathcal{A}) \neq \{0\}$, and it means that $\mathcal{A}$ is not semi-simple.

Let $\{d_n\}$ be a uniformly bounded pre-higher derivation (i.e., $\|d_n\| \leq M$ for some positive number M) and p be an arbitrary element of $\mathcal{P}_{\mathcal{A}}$. Then, the function F given by $F(t) = \sum_{n=0}^{\infty} d_n(p)t^n$ is well defined for $|t| < 1$. Indeed,

$$\begin{aligned}\|\sum_{n=0}^{\infty} d_n(p)t^n\| &\leq \sum_{n=0}^{\infty} \|d_n(p)t^n\| = \sum_{n=0}^{\infty} \|d_n(p)\||t^n| \\ &\leq \sum_{n=0}^{\infty} \|d_n\|\|p\||t^n| \leq \sum_{n=0}^{\infty} M\|p\||t^n| < \infty.\end{aligned}$$

Moreover, the m-th derivative of F exists and is given by the formula $F^{(m)}(t) := \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} d_n(p)t^{n-m}$. There is a good match between $F(t)$ and the uniformly bounded pre-higher derivation $\{d_n\}$. Using $F(t)$ the following main result is proved:
Let $\mathcal{A}$ be a Banach $*$-algebra such that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$. Suppose that $\{d_n\}$ is a uniformly bounded pre-higher derivation. Then, $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$ for each $n \geq 1$. In particular, if $\mathcal{A}$ is semi-prime and further, $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, then $d_n = 0$ for each $n \geq 1$.

2 Results and proofs

Before proving the main results, we present the following lemma:

Lemma 1 [[1], Lemma 1] *Let $\mathcal{A}$ be a normed complex $*$-algebra. If a linear mapping δ of $\mathcal{A}$ into a normed $\mathcal{A}$-bimodule $\mathcal{M}$ satisfies* $\delta(p) = \delta(p)p + p\delta(p)$

for all $p \in \mathcal{P}_{\mathcal{A}}$*, then* $\delta(w^2) = \delta(w)w + w\delta(w)$ *holds for all* $w \in \mathcal{O}_{\mathcal{A}}$*. Moreover, if* $\mathcal{O}_{\mathcal{A}}$ *is dense in* $\mathcal{S}_{\mathcal{A}}$ *and* δ *is continuous, then* δ *is a Jordan derivation.*

Note that each member of $\Phi_{\mathcal{A}}$ is continuous (see [2]). Since the case $\Phi_{\mathcal{A}} = \emptyset$ makes everything trivial, so we will assume that $\Phi_{\mathcal{A}}$ is a non-empty set.

Theorem 1 *Let* $\mathcal{A}$ *be a Banach* $*$*-algebra such that* $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$*. Suppose that* $\{d_n\}$ *be a uniformly bounded pre-higher derivation. Then* $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$ *for each* $n \geq 1$*. In particular, if* $\mathcal{A}$ *is semi-prime and* $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$*, then* $d_n = 0$ *for each* $n \geq 1$*.*

Proof. Let p be an arbitrary element of $\mathcal{P}_{\mathcal{A}}$. We know that the function $F(t) = \sum_{n=0}^{\infty} d_n(p)t^n$ is well-defined for $|t| < 1$. Note that

$$\begin{aligned} F(t)F(t) &= \left(\sum_{n=0}^{\infty} d_n(p)t^n\right)\left(\sum_{n=0}^{\infty} d_n(p)t^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_{n-k}(p)d_k(p)\right)t^n \\ &= \sum_{n=0}^{\infty} d_n(p)t^n = F(t). \end{aligned}$$

Hence, $\varphi(F(t)) = 0$ or $\varphi(F(t)) = 1$, where φ is an arbitrary fixed element of $\Phi_{\mathcal{A}}$. Let $G(t) := \varphi(F(t))$. We have $G(t) = \varphi(\sum_{n=0}^{\infty} d_n(p)t^n) = \sum_{n=0}^{\infty} \varphi(d_n(p))t^n$. It is observed that $G(t)$ is a power series in $\mathbb{C}$. Thus, the m-th derivative of G exists and is given by the formula $G^{(m)}(t) := \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\varphi(d_n(p))t^{n-m}$. Since the function G is constant, we have $G^{(m)}(t) = 0$ for every $m \in \mathbb{N}\backslash\{0\}$ and every $|t| < 1$. We have $\varphi(d_1(p)) + 2\varphi(d_2(p))t + 3\varphi(d_3(p))t^2 + 4\varphi(d_4(p))t^3 + \ldots = G^{(1)}(t) = 0$. Putting $t = 0$ in the former equation, we obtain that $\varphi(d_1(p)) = 0$. Using an argument similar to what was described concerning $\varphi(d_1(p))$, we conclude that $\varphi(d_2(p)) = 0$. By continuing this procedure, we prove that $\varphi(d_n(p)) = 0$ for all $n \geq 1$. Our next task is to show that $\varphi(d_n(a)) = 0$ for every $a \in \mathcal{A}$. Let x be an arbitrary element of $\mathcal{O}_{\mathcal{A}}$. Hence, $x = \sum_{i=1}^{m} r_i p_i$, where $p_1, p_2, \ldots, p_m$ are mutually orthogonal self-adjoint projections and $r_1, r_2, \ldots, r_m$ are real numbers. We have $\varphi(d_n(x)) = \varphi(d_n(\sum_{i=1}^{m} r_i p_i)) = \sum_{i=1}^{m} r_i\varphi(d_n(p_i)) = 0$. Since $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$, $\varphi(d_n(a)) = 0$ for every $a \in \mathcal{S}_{\mathcal{A}}$. It is well-known that each a in $\mathcal{A}$ can be represented as $a = a_1 + ia_2$, $a_1, a_2 \in \mathcal{S}_{\mathcal{A}}$; therefore, $\varphi(d_n(a)) = \varphi(d_n(a_1 + ia_2)) = \varphi(d_n(a_1)) + i\varphi(d_n(a_2)) = 0$ for all $n \geq 1$, $a \in \mathcal{A}$ and $\varphi \in \Phi_{\mathcal{A}}$. It means that $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$. Now, suppose that $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$. It is obvious that if $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) = 0$, then $d_n(\mathcal{A}) = \{0\}$ for all $n \geq 1$. Assume that $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) = 1$. First we reduce our discussion to the case $d_1 = 0$. Since $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}}\ker\varphi) = 1$, there exists a non-zero element x_0 of $\mathcal{A}$ such that $\bigcap_{\varphi\in\Phi_{\mathcal{A}}}\ker\varphi = \{\alpha x_0 \mid \alpha \in \mathbb{C}\}$. Let a_0 be an element of $\mathcal{A}$ so that $d_1(a_0) \neq 0$. We have $d_1(a_0) = \psi(a_0)x_0$, where ψ is a function from $\mathcal{A}$ into the complex numbers. Having put $b = \frac{1}{\psi(a_0)}a_0$, we obtain $d_1(b) = d_1(\frac{1}{\psi(a_0)}a_0) = \frac{1}{\psi(a_0)}\psi(a_0)x_0 = x_0$ and it implies that $\psi(b) = 1$. First we will show $ax_0 + x_0a$ is a scalar multiple of x_0 for any a in $\mathcal{A}$. Let a be an element of $\mathcal{A}$. Then, $d_1(a^2) = \psi(a^2)x_0$ (*). Lemma 1 is just what we need to tell us that d_1 is a Jordan derivation, i.e. $d_1(a^2) = d_1(a)a + ad_1(a)$ for all $a \in \mathcal{A}$. Using the fact that d_1 is a Jordan derivation and the identity $ab+ba = (a+b)^2 - a^2 - b^2$, we get $d_1(ab+ba) = d_1(a)b + ad_1(b) + d_1(b)a + bd_1(a)$ for all $a, b \in \mathcal{A}$. Since d_1 is a Jordan derivation and $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}}\ker\varphi) = 1$, we have $d_1(a^2) = d_1(a)a + ad_1(a) = \psi(a)x_0a + a\psi(a)x_0 = \psi(a)(x_0a + ax_0)$ (**). Comparing (*) and (**) , we find that $\psi(a^2)x_0 = \psi(a)(ax_0 + x_0a)$. If $\psi(a) \neq 0$, then $ax_0 + x_0a = \frac{\psi(a^2)}{\psi(a)}x_0$. But if $\psi(a) = 0$, then we have

$$\begin{aligned}
\psi(ab+ba)x_0 &= d_1(ab+ba)\\
&= d_1(a)b + ad_1(b) + d_1(b)a + bd_1(a)\\
&= \psi(a)x_0b + a\psi(b)x_0 + \psi(b)x_0a + b\psi(a)x_0\\
&= ax_0 + x_0a.
\end{aligned}$$

It means that $ax_0 + x_0a$ is a scalar multiple of x_0 for any a in $\mathcal{A}$. Next, it will be shown that $x_0^2 = 0$. Suppose that $\psi(x_0) = 0$. We have $\psi(b^2)x_0 = d_1(b^2) = d_1(b)b + bd_1(b) = \psi(b)x_0b + b\psi(b)x_0 = x_0b + bx_0$. Applying d_1 on this equality and then using the fact that $d_1(x_0) = \psi(x_0)x_0 = 0$, we obtain that $x_0^2 = 0$. Now, suppose $\psi(x_0) \neq 0$. We therefore have

$$\psi(x_0^2)x_0 = d_1(x_0^2) = d_1(x_0)x_0 + x_0d_1(x_0) = 2\psi(x_0)x_0^2. \tag{2}$$

If $\psi(x_0^2) = 0$, then it follows from previous equality that $x_0^2 = 0$. Assume that $\psi(x_0^2) \neq 0$; so $x_0^2 = \frac{\psi(x_0^2)}{2\psi(x_0)}x_0$. Simplifying the notation, we put $\lambda = \frac{\psi(x_0^2)}{2\psi(x_0)}$. Replacing x_0^2 by λx_0 in $2\psi(x_0)x_0^2 = d_1(x_0^2)$, we have $2\psi(x_0)\lambda x_0 = \lambda d_1(x_0) = \lambda\psi(x_0)x_0$. Since $\psi(x_0) \neq 0$, $\lambda x_0 = 0$ and it implies that either $\lambda = 0$ or $x_0 = 0$, which is a contradiction. This contradiction shows that $\psi(x_0^2) = 0$ and by using (2) it is obtained that $x_0^2 = 0$. We know that $x_0a + ax_0 = \mu x_0$, where $\mu \in \mathbb{C}$. Multiplying the previous equality by x_0 and using the fact that $x_0^2 = 0$, we see that $x_0ax_0 = 0$ for any a in $\mathcal{A}$. Since $\mathcal{A}$ is semi-prime, $x_0 = 0$. From this contradiction we deduce that $d_1 = 0$. Hence, $d_2(p) = d_2(p)p + pd_2(p) + (d_1(p))^2 = d_2(p)p + pd_2(p)$ for every $p \in \mathcal{P}_{\mathcal{A}}$. Reusing

Lemma 1, we get d_2 is a Jordan derivation. Now, by a procedure similar to what was described concerning d_1, we obtain that $d_2 = 0$. Consequently, by continuing this procedure, we prove that $d_n = 0$ for all $n \geq 1$. $\square$

An immediate but noteworthy corollary to Theorem 1 is:

Corollary 1 *Let $\mathcal{A}$ be a semi-prime Banach $*$-algebra such that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$. If $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, then every bounded linear mapping $d : \mathcal{A} \to \mathcal{A}$ satisfying $d(p) = d(p)p + pd(p)$ for all $p \in \mathcal{P}_{\mathcal{A}}$, is identically zero.*

Proof. First, let us define a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{d^n}{n!}$, where I is the identity mapping on $\mathcal{A}$. A straightforward verification shows that $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$ for all $p \in \mathcal{P}_{\mathcal{A}}$. We have

$$\|d_n\| = \|\frac{d^n}{n!}\| \leq \frac{1}{n!}\|d\|^n < \sum_{n=0}^{\infty} \frac{\|d\|^n}{n!} = e^{\|d\|}$$

for each non-negative integer n. It means that $\{d_n\}$ is a uniformly bounded pre-higher derivation. It follows from Theorem 1 that $0 = d_1 = d$. Furthermore, Lemma 1 implies that every bounded Jordan derivation from $\mathcal{A}$ into $\mathcal{A}$ is zero. $\square$

Remark 1 *Let $\{d_n\}$ be a higher derivation on an algebra $\mathcal{A}$ with $d_0 = I$, where I is the identity mapping on $\mathcal{A}$. Based on Proposition 2.1 of [7] there is a sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$ such that*

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}$$

for each non-negative integer n. Therefore, we have

$$\begin{aligned}
&d_0 = I,\\
&d_1 = \delta_1,\\
&2d_2 = \delta_1 d_1 + \delta_2 d_0 = \delta_1\delta_1 + \delta_2,\\
&d_2 = \frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2,\\
&3d_3 = \delta_1 d_2 + \delta_2 d_1 + \delta_3 d_0 = \delta_1(\frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2) + \delta_2\delta_1 + \delta_3,\\
&d_3 = \frac{1}{6}\delta_1^3 + \frac{1}{6}\delta_1\delta_2 + \frac{1}{3}\delta_2\delta_1 + \frac{1}{3}\delta_3.
\end{aligned}$$

Now, assume that $\{d_n\}$ *is a bounded higher derivation (i.e.,* d_n *is a bounded linear map for every non-negative integer* n*). Obviously,* $\delta_1 = d_1$ *is bounded. Hence,* $\delta_2 = 2d_2 - \delta_1^2$ *is also bounded. Based on the* d_3 *formula, we have* $\delta_3 = 3d_3 - \frac{1}{2}\delta_1^3 - \frac{1}{2}\delta_1\delta_2 - \delta_2\delta_1$*. Using the boundedness of* d_3*,* δ_1 *and* δ_2*, we obtain that* δ_3 *is a bounded derivation. In the next step, we will show that every* δ_n *is a bounded derivation for every* $n \in \mathbb{N}$*. To reach this aim, we use induction on* n*. According to the above-mentioned discussion,* δ_1*,* δ_2 *and* δ_3 *are bounded derivations. Now, suppose that* δ_k *is a bounded derivation for* $k \leq n$*. We will show that* δ_{n+1} *is also a bounded derivation. Based on the proof of Theorem 2.3 in* [7]*, we have*

$$\delta_{n+1} = (n+1)d_{n+1} - \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n+1} (n+1)a_{r_1,\ldots,r_i}\delta_{r_1}\ldots\delta_{r_i}\right), \qquad (3)$$

where the inner summation is taken over all positive integers r_j *with* $\sum_{j=1}^{i} r_j = n+1$*. From* $\sum_{j=1}^{i} r_j = r_1 + r_2 + \ldots + r_i = n+1$ *along with the condition that* r_j *is a positive integer for every* $1 \leq j \leq i$*, we find that* $1 \leq r_j \leq n$ *for every* $1 \leq j \leq i$*. Since we are assuming* d_n *and* δ_k *are bounded linear mappings for all non-negative integer* n *and* $k \leq n$*, it follows from* (3) *that* δ_{n+1} *is a bounded derivation.*

We are now ready for Corollary 2.

Corollary 2 *Let* $\mathcal{A}$ *be a semi-prime Banach* $*$*-algebra such that* $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$*, and* $\{d_n\}$ *be a bounded higher derivation from* $\mathcal{A}$ *into* $\mathcal{A}$*.*
If $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$*, then* $d_n = 0$ *for all* $n \in \mathbb{N}$*.*

Proof. Let $\{d_n\}$ be the above-mentioned higher derivation. According to Theorem 2.3 of [7] there exists a sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$ such that

$$d_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_j + \ldots + r_i}\right)\delta_{r_1}\ldots\delta_{r_i}\right)$$

, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. It follows from Remark 1 that δ_n is a bounded derivation for every positive integer n. At this point, Corollary 1 completes the proof. □

Corollary 3 *Let* $\mathcal{A}$ *be a semi-prime Banach* $*$*-algebra such that* $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$*. If* $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$*, then* $\mathcal{A}$ *is commutative.*

Proof. Let x_0 be a non-zero arbitrary fixed element of $\mathcal{A}$. Define $d_{x_0} : \mathcal{A} \to \mathcal{A}$ by $d_{x_0}(a) = ax_0 - x_0 a$. Obviously, d_{x_0} is a bounded derivation. It follows from Corollary 1 that $d_{x_0}(a) = 0$, i.e. $ax_0 = x_0 a$ for all $a \in \mathcal{A}$. Since x_0 is arbitrary, $\mathcal{A}$ is commutative. $\square$

The above results lead us to the following conjecture:

Conjecture 1 *Let $\mathcal{A}$ be a semi-prime Banach $*$-algebra such that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$. If $\dim(\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi) < \infty$, then every bounded derivation from $\mathcal{A}$ into $\mathcal{A}$ is zero.*

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Some fixed point results for rational type and subrational type contractive mappings

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Abstract. In this paper, we introduce the concepts of rational type and subrational type contractive mappings. We expand and improve some fixed point theorems obtained by Alsulami *et al.* (Fixed Point Theory Appl., 2015, 2015: 97). Moreover, we give an example to support our results.

1 Introduction and preliminaries

Fixed point theory gains very large impetus due to its wide range of applications in various fields such as engineering, economics, computer science, and many others. It is well known that the contractive condition is indispensable in the study of fixed point theory. Banach fixed point theorem [1] is one of the pivotal results in mathematical analysis. Many authors (see, e.g., [2]-[8]) not only extend this theorem but also consider fixed points in various abstract spaces.

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In 2000, Branciari [2] introduced the notion of generalized metric space. A generalized metric is a semi-metric which does not satisfy the triangle inequality but satisfies a weaker condition called quadrilateral inequality.

Throughout this paper, denote $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}^* = \{1, 2, 3, \dots\}$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$. For the sake of author, we give some notations and notions as follows.

Definition 1 [2] *Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ be a mapping. If for all $x, y \in X$ and all distinct $u, v \in X$, each of which is different from x and y,*

(GMS1) $d(x, y) = 0$ *if and only if* $x = y$;

(GMS2) $d(x, y) = d(y, x)$;

(GMS3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then d is called a generalized metric and is abbreviated as GM. Here, the pair (X, d) is called a generalized metric space and is abbreviated as GMS.

Note that if d satisfies only (GMS1) and (GMS2), then it is called a semi-metric (see *e.g.*, [3]).

Definition 2 *Let (X, d) be a GMS. Then*

1. *a sequence $\{x_n\}$ in (X, d) is said to be GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$;*

2. *a sequence $\{x_n\}$ in (X, d) is said to be GMS Cauchy if and only if for every $\varepsilon > 0$, there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$, for all $n, m > N(\varepsilon)$;*

3. *(X, d) is said to be complete if every GMS Cauchy sequence in X is GMS convergent;*

4. *a mapping $T : X \to X$ is said to be continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.*

Definition 3 *Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be two mappings. We say that T is an α-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.*

Definition 4 [4] *Let (X, d) be a GMS and $\alpha : X \times X \to \mathbb{R}^+$. We say that X is called α-regular if, for any sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}^*$.*

Definition 5 *Let* X *be a nonempty set,* $T : X \to X$ *and* $\mu : X \times X \to \mathbb{R}^+$ *be two mappings. We say that* T *is a* μ*-subadmissible mapping if* $\mu(x, y) \leq 1$ *implies* $\mu(Tx, Ty) \leq 1$ *for all* $x, y \in X$.

Definition 6 *Let* (X, d) *be a GMS and* $\mu : X \times X \to \mathbb{R}^+$*. We say that* X *is called* μ*-subregular if, for any sequence* $\{x_n\}$ *in* X *such that* $x_n \to x$ *and* $\mu(x_n, x_{n+1}) \leq 1$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\mu(x_{n_k}, x) \leq 1$ *for all* $k \in \mathbb{N}^*$.

Definition 7 *A mapping* $H : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is called a function of subclass of type I if it is continuous and* $x \geq 1$ *implies* $H(1, y) \leq H(x, y)$ *for all* $y \in \mathbb{R}^+$.

Example 1 *We have the following functions of subclass of type I, for all* $x \in \mathbb{R}$ *and* $y \in \mathbb{R}^+$*:*

(1) $H(x, y) = (y + l)^x, \ l \geq 1$;
(2) $H(x, y) = (x + l)^y, \ l \geq 0$;
(3) $H(x, y) = xy^n, \ n \in \mathbb{N}$;
(4) $H(x, y) = x^n y, \ n \in \mathbb{N}$;
(5) $H(x, y) = \frac{(x+1)y}{2}$;
(6) $H(x, y) = \frac{(2x+1)y}{3}$;
(7) $H(x, y) = \frac{y}{n+1} \sum\limits_{i=0}^{n} x^i, \ n \in \mathbb{N}$;
(8) $H(x, y) = \left(\frac{1}{n+1} \sum\limits_{i=0}^{n} x^i + l \right)^y, \ l \geq 0, \ n \in \mathbb{N}$.

Definition 8 *Let* $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *and* $H : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *be two mappings. We say that the pair* $(\mathcal{F}, H)$ *is an upper class of type* I *if* H *is a function of subclass of type I and satisfies for all* $r, s, t \in \mathbb{R}^+$,

(1) $0 \leq s < 1 \Rightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$;
(2) $H(1, r) \leq \mathcal{F}(s, t) \Rightarrow r \leq st$.

Example 2 *We have the following upper classes of type I, for all* $x \in \mathbb{R}$, $y, t \in \mathbb{R}^+$, $s \in [0, 1)$*:*

(1) $H(x, y) = (y + l)^x$, $\mathcal{F}(s, t) = st + l$, $l \geq 1$;
(2) $H(x, y) = (x + l)^y$, $\mathcal{F}(s, t) = (1 + l)^{st}$, $l \geq 0$;
(3) $H(x, y) = xy^n$, $\mathcal{F}(s, t) = s^n t^n, \ n \in \mathbb{N}$;
(4) $H(x, y) = x^n y$, $\mathcal{F}(s, t) = st, \ n \in \mathbb{N}$;
(5) $H(x, y) = \frac{(2x+1)y}{3}$, $\mathcal{F}(s, t) = st$;
(6) $H(x, y) = \frac{(x+1)y}{2}$, $\mathcal{F}(s, t) = st$;

(7) $H(x,y)=\frac{y}{n+1}\sum_{i=0}^{n}x^i$, $\mathcal{F}(s,t)=st,\ n\in\mathbb{N}$;

(8) $H(x,y)=\left(\frac{1}{n+1}\sum_{i=0}^{n}x^i+l\right)^y$, $\mathcal{F}(s,t)=(1+l)^{st}$, $l\geq 0,\ n\in\mathbb{N}$.

Proposition 1 [6] Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS (X,d) with $\lim_{n\to\infty} d(x_n,u)=0$, where $u\in X$. Then $\lim_{n\to\infty} d(x_n,z)=d(u,z)$, for all $z\in X$. In particular, the sequence $\{x_n\}$ does not converge to z if $z\neq u$.

2 Main results

In this section, let $F(T)$ denote the set of fixed points of the mapping T. Let Ψ be a family of functions $\psi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the following properties:
(i) ψ is upper semi-continuous and nondecreasing;
(ii) $\lim_{n\to\infty}\psi^n(t)=0$ for all $t>0$.

By the above properties, we have $\psi(t)<t$, for each $t>0$. Indeed, if there exists $t_0>0$ such that $\psi(t_0)\geq t_0$, then by the monotonicity of ψ, it establishes that

$$\psi^n(t_0)\geq t_0,\qquad (n=1,2,\ldots),$$

thus

$$0=\lim_{n\to\infty}\psi^n(t_0)\geq t_0.$$

This is a contradiction.

Definition 9 *Let (X,d) be a GMS and $\alpha:X\times X\to\mathbb{R}^+$ a mapping. A mapping $T:X\to X$ is said to be an $(\alpha,\psi,\mathcal{F},H)$-rational type-I contractive mapping if there exists a function $\psi\in\Psi$, such that for all $x,y\in X$, the following condition holds:*

$$H\left(\alpha\left(x,y\right),d(Tx,Ty)\right)\leq\mathcal{F}\left(1,\psi(M(x,y))\right),\tag{1}$$

where $(\mathcal{F},H)$ is an upper class of type I and

$$M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)},\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}.\tag{2}$$

Theorem 1 *Let (X,d) be a complete GMS, $T:X\to X$ a mapping and $\alpha:X\times X\to\mathbb{R}^+$ a given function. Suppose that the following conditions are satisfied:*

(i) T *is an* α*-admissible mapping;*
(ii) T *is an* $(\alpha, \psi, \mathcal{F}, H)$*-rational type-I contractive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Let $x_0 \in X$ satisfy $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. We construct a sequence $\{x_n\}$ in X as $x_n = T^n x_0 = Tx_{n-1}$, for $n \in \mathbb{N}^*$. It is obvious that x_{n_0} is a fixed point of T if $x_{n_0} = x_{n_0+1}$, for some $n_0 \in \mathbb{N}$. Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is α-admissible, then

$$\begin{aligned}\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 &\Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1\\ &\Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1,\end{aligned}$$

and by induction, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

By similar argument, we get $\alpha(x_n, x_{n+2}) \geq 1$ for all $n \in \mathbb{N}$ from $\alpha(x_0, T^2x_0) \geq 1$. Consider (1) with $x = x_n$ and $y = x_{n+1}$, it follows that

$$\begin{aligned}H(1, d(x_{n+1}, x_{n+2})) &= H(1, d\left(Tx_n, Tx_{n+1}\right))\\ &\leq H(\alpha\left(x_n, x_{n+1}\right), d\left(Tx_n, Tx_{n+1}\right))\\ &\leq \mathcal{F}(1, \psi(M(x_n, x_{n+1}))),\end{aligned}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})),$$

where

$$\begin{aligned}M(x_n, x_{n+1}) = \max&\Bigg\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}),\\ &\frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})}\Bigg\}\\ = \max&\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\},\end{aligned}$$

If $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$ for some n, then

$$d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})) = \psi(d(x_{n+1}, x_{n+2})) < d(x_{n+1}, x_{n+2}),$$

which is impossible. Hence, $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi(M(x_n, x_{n+1})) = \psi(d\,(x_n,x_{n+1})) < d\,(x_n,x_{n+1}) \\ &\leq \psi(M(x_{n-1}, x_n)) = \psi(d\,(x_{n-1},x_n)) < d\,(x_{n-1},x_n)\,, \end{aligned} \tag{3}$$

and by the monotonicity of ψ, it is easy to see that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi(M(x_n, x_{n+1})) = \psi(d(x_n,x_{n+1})) \\ &\leq \psi(\psi(M(x_{n-1}, x_n))) = \psi(\psi(d(x_{n-1}, x_n))) = \psi^2(d(x_{n-1}, x_n)) \\ &\leq \psi^3(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^{n+1}(d(x_0, x_1)) \to 0 \ (n \to \infty), \end{aligned}$$

so

$$\lim_{n\to\infty} d\,(x_{n+1},x_{n+2}) = 0. \tag{4}$$

Putting $x = x_{n-1}$ and $y = x_{n+1}$ in (1), we have

$$\begin{aligned} H(1, d(x_n, x_{n+2})) &= H(1, d(Tx_{n-1}, Tx_{n+1})) \\ &\leq H(\alpha(x_{n-1}, x_{n+1}), d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \mathcal{F}(1, \psi(M(x_{n-1}, x_{n+1}))), \end{aligned}$$

which establishes that

$$d(x_n, x_{n+2}) \leq \psi(M(x_{n-1}, x_{n+1})), \tag{5}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1}), \\ &\quad \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_{n+1})} \Big\} \\ &= \max\Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\ &\quad \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+2})} \Big\} \\ &< \max\Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), \frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})}, \frac{d^2(x_{n-1}, x_n)}{1 + d(x_n, x_{n+2})} \Big\} \end{aligned}$$

based on (3).

If $M(x_{n-1}, x_{n+1}) < d(x_{n-1}, x_{n+1})$, then by (5), it is not hard to verify that

$$\begin{aligned} d(x_n, x_{n+2}) &\le \psi(M(x_{n-1}, x_{n+1})) \le \psi(d(x_{n-1}, x_{n+1})) \le \psi(\psi(M(x_{n-2}, x_n))) \\ &\le \psi(\psi(d(x_{n-2}, x_n))) = \psi^2(d(x_{n-2}, x_n)) \\ &\le \cdots \le \psi^n(d(x_0, x_2)) \to 0 \ (n \to \infty). \end{aligned}$$

Hence $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$.

If

$$M(x_{n-1}, x_{n+1}) < \max\left\{ d(x_{n-1}, x_n), \frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})}, \frac{d^2(x_{n-1}, x_n)}{1 + d(x_n, x_{n+2})} \right\},$$

then by (5), we arrive at

$$d(x_n, x_{n+2}) < \max\left\{ d(x_{n-1}, x_n), \frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})}, \frac{d^2(x_{n-1}, x_n)}{1 + d(x_n, x_{n+2})} \right\}. \tag{6}$$

Taking the limit as $n \to \infty$ from (6), we get $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$ because of (4).

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{7}$$

Now, we prove that $\{x_n\}$ is a GMS Cauchy sequence, that is, $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$, for all $k \in \mathbb{N}^*$. We have already proved the cases for $k = 1$ and $k = 2$ in (4) and (7), respectively. Take arbitrary $k \ge 3$. We discuss two cases.

Case 1. Suppose that $k = 2m + 1$, where $m \ge 1$. Using the quadrilateral inequality (GMS3), we have

$$\begin{aligned} d(x_n, x_{n+k}) = d(x_n, x_{n+2m+1}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + d(x_{n+2m}, x_{n+2m+1}) \\ &\le \sum_{p=n}^{n+2m} \psi^p(d(x_0, x_1)) \le \sum_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 0 \ (n \to \infty). \end{aligned}$$

Case 2. Suppose that $k = 2m$, where $m \ge 2$. Again, by applying the quadrilateral inequality (GMS3), we have

$$\begin{aligned} d(x_n, x_{n+k}) = d(x_n, x_{n+2m}) &\le d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + \ldots + d(x_{n+2m-1}, x_{n+2m}) \\ &\le d(x_n, x_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p(d(x_0, x_1)) \\ &\le d(x_n, x_{n+2}) + \sum_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 0 \ (n \to \infty). \end{aligned}$$

Uniting Case 1 and Case 2, we have $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$ for all $k \ge 3$. Thus, again by (4) and (7), we have $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$ for all $k \ge 1$. Hence we claim that $\{x_n\}$ is a GMS Cauchy sequence in (X, d). Since (X, d) is complete, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{8}$$

We shall show x^* is a fixed point of T. First, assume that T is continuous, then by (8), it is clear that

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

Due to Proposition 1, we conclude that $x^* = Tx^*$, that is, x^* is a fixed point of T.

Second, assume that X is α-regular. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \ge 1$ for all $k \in \mathbb{N}^*$. Choose $x = x_{n_k}$ and $y = x^*$ in (1), we get

$$\begin{aligned} H(1, d(x_{n_k+1}, Tx^*)) &= H(1, d(Tx_{n_k}, Tx^*)) \\ &\le H(\alpha(x_{n_k}, x^*), d(Tx_{n_k} Tx^*)) \\ &\le \mathcal{F}(1, \psi(M(x_{n_k}, x^*))), \end{aligned}$$

which follows that

$$d\left(x_{n_k+1}, Tx^*\right) \le \psi(M(x_{n_k}, x^*)), \tag{9}$$

where

$$\begin{aligned}
M(x_{n_k}, x^*) &= \max\Big\{ d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \\
&\qquad \frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1 + d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1 + d(Tx_{n_k}, Tx^*)} \Big\} \\
&= \max\Big\{ d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \\
&\qquad \frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1 + d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1 + d(x_{n_k+1}, Tx^*)} \Big\}.
\end{aligned}$$

Consider the upper semi-continuity of ψ, it derives from (9) that

$$\begin{aligned}
d(x^*, Tx^*) &= \limsup_{k\to\infty} d(x_{n_k+1}, Tx^*) \leq \limsup_{k\to\infty} \psi(M(x_{n_k}, x^*)) \\
&\leq \psi(\lim_{k\to\infty} M(x_{n_k}, x^*)) = \psi(d(x^*, Tx^*)),
\end{aligned}$$

which implies $x^* = Tx^*$ (otherwise, if $d(x^*, Tx^*) > 0$, then

$$d(x^*, Tx^*) \leq \psi(d(x^*, Tx^*)) < d(x^*, Tx^*)$$

is a contradiction).

Finally, assume that x^* and y^* are two different fixed points of T. Then by the hypothesis, $\alpha(x^*, y^*) \geq 1$. Hence, from (1) with $x = x^*$ and $y = y^*$ we conclude that

$$\begin{aligned}
H(1, d(x^*, y^*)) = H(1, d(Tx^*, Ty^*)) &\leq H(\alpha(x^*, y^*), d(Tx^*, Ty^*)) \\
&\leq \mathcal{F}(1, \psi(M(x^*, y^*))),
\end{aligned}$$

which establishes that

$$d(x^*, y^*) \leq \psi(M(x^*, y^*)),$$

where

$$\begin{aligned}
M(x^*, y^*) &= \max\Big\{ d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \\
&\qquad \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(x^*, y^*)}, \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(Tx^*, Ty^*)} \Big\} \\
&= d(x^*, y^*).
\end{aligned}$$

Thus $d(x^*, y^*) \leq \psi(d(x^*, y^*)) < d(x^*, y^*)$. This is a contradiction. Hence T has a unique fixed point. □

Corollary 1 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) *for all* $x, y \in X$,

$$(l+d(Tx, Ty))^{\alpha(x,y)} \leq \psi\left(M(x, y)\right) + l,$$

where $l \geq 1$ *is a constant and* $M(x, y)$ *is defined by* (2)*;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha\left(x_0, Tx_0\right) \geq 1$ *and* $\alpha\left(x_0, T^2x_0\right) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Choose $H(x, y) = (y + l)^x$ and $\mathcal{F}(s, t) = st + l$, then by Theorem 1, the desired proof is obtained. □

Corollary 2 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) *for all* $x, y \in X$,

$$(l + \alpha\left(x, y\right))^{d(Tx,Ty)} \leq (1 + l)^{\psi(M(x,y))},$$

where $l \geq 0$ *is a constant and* $M(x, y)$ *is defined by* (2)*;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Choose $H(x, y) = (x + l)^y$ and $\mathcal{F}(s, t) = (1 + l)^{st}$, then by Theorem 1, the proof is valid. □

Corollary 3 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) *for all* $x, y \in X$,

$$\alpha\left(x, y\right)(d(Tx, Ty))^n \leq (\psi(M(x, y)))^n, \ n \in \mathbb{N},$$

where $M(x, y)$ *is defined by* (2)*;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Take $H(x, y) = xy^n$ and $\mathcal{F}(s, t) = s^n t^n$, then by Theorem 1, the proof is completed. □

Corollary 4 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) *for all* $x, y \in X$*,*

$$\frac{k\alpha(x, y) + 1}{k + 1}(d(Tx, Ty))^n \leq (\psi(M(x, y)))^n, \ n \in \mathbb{N},$$

where $k \geq 0$ *is a constant and* $M(x, y)$ *is defined by* (2)*;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Take $H(x, y) = \frac{kx+1}{k+1} y^n$ and $\mathcal{F}(s, t) = s^n t^n$, then by Theorem 1, the claim holds. □

Remark 1 *Assume that* $H(x, y) = xy$ *and* $\mathcal{F}(s, t) = t$ *in Definition 9 and let (2) from Definition 8 be replaced by* $H(1, r) \leq \mathcal{F}(1, t) \Rightarrow r \leq t$*, then* $T : X \to X$ *is said to be an* (α, ψ)*-rational type-*I *contractive mapping. This is a definition from* [4]*. In this case, we easily get the following Corollary 5 from Theorem 1.*

Corollary 5 [4] *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) T *is an* (α, ψ)*-rational type-I contractive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Definition 10 *Let* (X, d) *be a GMS and* $\alpha : X \times X \to \mathbb{R}^+$ *a mapping. A mapping* $T : X \to X$ *is said to be* $(\alpha, \psi, \mathcal{F}, H)$*-rational type-II contractive mapping if there exists a function* $\psi \in \Psi$*, such that for all* $x, y \in X$*, the following condition holds:*

$$H(\alpha(x,y), d(Tx, Ty)) \leq \mathcal{F}(1, \psi(M(x,y))),$$

where $(\mathcal{F}, H)$ *is an upper class of type* I *and*

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Ty)d(x,y)}{1+d(x,Tx)+d(y,Tx)+d(y,Ty)}\right\}.$$

For this class of mappings we state a similar existence and uniqueness theorem.

Theorem 2 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *and* $\alpha : X \times X \to \mathbb{R}^+$ *be two mappings. Suppose that the following conditions are satisfied:*

(i) T *is an* α*-admissible mapping;*

(ii) T *is an* $(\alpha, \psi, \mathcal{F}, H)$*-rational type-II contractive mapping;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*

(iv) *either* T *is continuous, or* X *is* α*-regular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^* *. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x,y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. The proof can be done by the similar proof as the following Theorem 3. □

Definition 11 *Let* (X, d) *be a GMS and* $\mu : X \times X \to \mathbb{R}^+$ *a mapping. A mapping* $T : X \to X$ *is said to be a* $(\mu, \psi, \mathcal{F}, H)$*-subrational type-I contractive mapping if there exists a function* $\psi \in \Psi$*, such that for all* $x, y \in X$*, the following condition holds:*

$$H(1, d(Tx, Ty)) \leq \mathcal{F}(\mu(x,y), \psi(M(x,y))), \tag{10}$$

where $(\mathcal{F}, H)$ *is an upper class of type* I *and* $M(x,y)$ *is defined by* (2)*.*

Theorem 3 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\mu : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*

(i) T *is a* μ*-subadmissible mapping;*

(ii) T *is a* $(\mu, \psi, \mathcal{F}, H)$*-subrational type-I contractive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $\mu(x_0, Tx_0) \leq 1$ *and* $\mu(x_0, T^2x_0) \leq 1$*;*
(iv) *either* T *is continuous, or* X *is* μ*-subregular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**.Further, if for all* $x, y \in F(T)$*, we have* $\mu(x, y) \leq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Let $x_0 \in X$ satisfy $\mu(x_0, Tx_0) \leq 1$ and $\mu(x_0, T^2x_0) \leq 1$. We construct a sequence $\{x_n\}$ in X as $x_n = T^n x_0 = Tx_{n-1}$, for $n \in \mathbb{N}^*$. It is clear that x_{n_0} is a fixed point of T if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. As a result, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is μ-subadmissible, then

$$\begin{aligned}\mu(x_0, Tx_0) = \mu(x_0, x_1) \leq 1 &\Rightarrow \mu(Tx_0, Tx_1) = \mu(x_1, x_2) \leq 1\\ &\Rightarrow \mu(Tx_1, Tx_2) = \mu(x_2, x_3) \leq 1,\end{aligned}$$

and by induction, we get $\mu(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$.

By similar argument, on account of $\mu(x_0, T^2x_0) \leq 1$, we have $\mu(x_n, x_{n+2}) \leq 1$ for all $n \in \mathbb{N}$. Considering (10) with $x = x_n$ and $y = x_{n+1}$, we acquire that

$$\begin{aligned}H(1, d(x_{n+1}, x_{n+2})) &= H(1, d(Tx_n, Tx_{n+1}))\\ &\leq \mathcal{F}(\mu(x_n, x_{n+1}), \psi(M(x_n, x_{n+1})))\\ &\leq \mathcal{F}(1, \psi(M(x_n, x_{n+1}))),\end{aligned}$$

which follows that

$$d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})).$$

Similar to the process of proof of Theorem 1, we get (4). Next consider (10) with $x = x_{n-1}$ and $y = x_{n+1}$, it is easy to get (7). By similar proof of Theorem 1, we prove that there exists $x^* \in X$ such that (8) holds.

We shall show that the limit x^* of $\{x_n\}$ is a fixed point of T. First, we assume that T is continuous, then by (8), it is clear that

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

Due to Proposition 1, we obtain $x^* = Tx^*$, that is, x^* is a fixed point of T.

Now, we suppose that X is μ-subregular. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\mu(x_{n_k}, x^*) \leq 1$ for all $k \in \mathbb{N}^*$. Choosing $x = x_{n_k}$ and $y = x^*$ in (10) , we arrive at

$$\begin{aligned}H(1, d(x_{n_k+1}, Tx^*)) &= H(1, d(Tx_{n_k}, Tx^*))\\ &\leq \mathcal{F}(\mu(x_{n_k}, x^*), \psi(M(x_{n_k}, x^*)))\\ &\leq \mathcal{F}(1, \psi(M(x_{n_k}, x^*))),\end{aligned}$$

which implies that (9). Similar argument of Theorem 1, we claim that x^* is a fixed point of T.

Finally, assume that x^* and y^* are two different fixed points of T. Then by the hypothesis, $\mu(x^*, y^*) \leq 1$. Hence, from (10) with $x = x^*$ and $y = y^*$ we conclude that

$$\begin{aligned} H(1, d(x^*, y^*)) &= H(1, d(Tx^*, Ty^*)) \\ &\leq \mathcal{F}(\mu(x^*, y^*), \psi(M(x^*, y^*))) \\ &\leq \mathcal{F}(1, \psi(M(x^*, y^*))), \end{aligned}$$

which establish that

$$d(x^*, y^*) \leq \psi(M(x^*, y^*)).$$

Hence by the same proof of as one in Theorem 1, T has a unique fixed point. □

Corollary 6 *Let (X, d) be a complete GMS, $T : X \to X$ a self mapping and $\mu : X \times X \to \mathbb{R}^+$ a given function. Suppose that the following conditions are satisfied:*

(i) *T is a μ-subadmissible mapping;*

(ii) *for all $x, y \in X$,*

$$d(Tx, Ty) \leq \mu(x, y)\psi(M(x, y)),$$

where $M(x, y)$ is defined by (2);

(iii) *there exists $x_0 \in X$ such that $\mu(x_0, Tx_0) \leq 1$ and $\mu(x_0, T^2x_0) \leq 1$;*

(iv) *either T is continuous, or X is μ-subregular.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\mu(x, y) \leq 1$, then T has a unique fixed point in X.

Proof. Choose $H(x, y) = xy^n$ and $\mathcal{F}(s, t) = s^n t^n$, $n \in \mathbb{N}$, then by Theorem 3, the desired proof is completed. □

The following example illustrates that Theorem 3 is inspired by [7].

Example 3 *Let $X = \{1, 2, 3, 4\}$ and define a mapping $d : X \times X \to \mathbb{R}^+$ by*

$$\begin{aligned} &d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0, \\ &d(1,2) = d(2,1) = 3, \\ &d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1, \\ &d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4. \end{aligned}$$

Clearly, d *is not a metric on* X *in view of*

$$3 = d(1,2) \geq d(1,3) + d(3,2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, d *is a GM on* X*, further,* (X, d) *is a complete GMS. Define* $T : X \to X$ *by*

$$T1 = T2 = 1, \quad T3 = 2, \quad T4 = 3,$$

$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = \frac{9t}{10}$ *and* $\mu : X \times X \to \mathbb{R}^+$ *by*

$$\mu(x,y) = \begin{cases} 1, & \textit{if } x, y \in \{1,2\}, \\ 10, & \textit{others.} \end{cases}$$

Then, for $x, y \in \{1,2\}$*, we have*

$$d(Tx, Ty) = 0 \leq \mu(x,y)\psi(M(x,y)).$$

On the one hand, for $x \in \{1,2\}$ *and* $y = 3$ *we obtain that*

$$d(Tx, T3) = d(1,2) = 3$$

and

$$\begin{aligned} M(1,3) &= \max\left\{ d(1,3), d(1,T1), d(3,T3), \frac{d(1,T1)d(3,T3)}{1 + d(1,3)}, \frac{d(1,T1)d(3,T3)}{1 + d(T1,T3)} \right\} \\ &= 1, \\ M(2,3) &= \max\left\{ d(2,3), d(2,T2), d(3,T3), \frac{d(2,T2)d(3,T3)}{1 + d(2,3)}, \frac{d(2,T2)d(3,T3)}{1 + d(T2,T3)} \right\} \\ &= 3. \end{aligned}$$

Thus

$$\begin{aligned} d(T1, T3) = 3 \leq 10 \cdot \frac{9}{10} \cdot 1 = 9 = \mu(1,3)\psi(M(1,3)), \\ d(T2, T3) = 3 \leq 10 \cdot \frac{9}{10} \cdot 3 = 27 = \mu(2,3)\psi(M(2,3)). \end{aligned}$$

On the other hand, for $x \in \{1,2\}$ *and* $y = 4$ *we obtain*

$$d(Tx, T4) = d(1,3) = 1$$

and

$$
\begin{aligned}
M(1,4) &= \max\left\{d(1,4), d(1,T1), d(4,T4), \frac{d(1,T1)d(4,T4)}{1+d(1,4)}, \frac{d(1,T1)d(4,T4)}{1+d(T1,T4)}\right\} \\
&= 4, \\
M(2,4) &= \max\left\{d(2,4), d(2,T2), d(4,T4), \frac{d(2,T2)d(4,T4)}{1+d(2,4)}, \frac{d(2,T2)d(4,T4)}{1+d(T2,T4)}\right\} \\
&= 6.
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
d(T1,T4) = 1 \le 10 \cdot \frac{9}{10} \cdot 4 = 36 = \mu(1,4)\psi(M(1,4)), \\
d(T2,T4) = 1 \le 10 \cdot \frac{9}{10} \cdot 6 = 54 = \mu(2,4)\psi(M(2,4)).
\end{aligned}
$$

For $x, y \in \{3,4\}$*, the contraction condition is obvious. Clearly,* T *satisfies the conditions of Theorem 3 (or Corollary 6) and hence* T *has a unique fixed point* $x = 1$.

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Solvable and nilpotent right loops

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Abstract. In this paper the notions of solvable right transversal and nilpotent right transversal are defined. Further, it is proved that if a core-free subgroup has a generating solvable transversal, then the whole group is solvable.

1 Introduction

Let G be a group and H a proper subgroup of G. A *normalized right transversal* is a subset of G obtained by selecting one and only one element from each right coset of H in G, including the identity from the coset H. Now we will call it a *transversal* in place of normalized right transversal. Suppose that S is a transversal of H in G. We define an operation $\circ$ on S as follows: for $x, y \in S$, $\{x \circ y\} := S \cap Hxy$. It is easy to check that $(S, \circ)$ is a right loop, that is the equation of the type $X \circ a = b$, X is unknown and $a, b \in S$ has a unique solution in S, and $(S, \circ)$ has a two-sided identity. In [5], it has been shown that for each right loop there exists a pair (G, H) such that H is a core-free subgroup of the group G and the given right loop can be identified with a transversal of H in G. Not all transversals of a subgroup generate the group. But for finite groups, it is proved by Cameron in [2], that if a subgroup is core-free, then always there exists a transversal which generates the whole group. We call such a transversal a *generating transversal.*

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Let $(S, \circ)$ be a right loop (identity denoted by 1). Let $x, y, z \in S$. Define a map $R(y, z)$ from S to S as follows: $R(y, z)(x)$ is the unique solution of the equation $X \circ (y \circ z) = (x \circ y) \circ z$, where X is unknown. It is easy to verify that $R(y, z)$ is a bijective map. For a set X, let $Sym(X)$ denote the symmetric group on X. We denote by $RInn(S)$ the subgroup of $Sym(S)$, generated by the set $\{R(y, z) \mid y, z \in S\}$. This group is called the *right inner mapping group* of the right loop S. It measures the deviation of a right loop from being a group. Also note that right multiplication R_s by an element $s \in S$ gives a bijective map from S to S. The subgroup generated by $\{R_s \mid s \in S\}$ is the *right multiplication group* $RMlt(S)$ of the right loop S. One should note that the right multiplication group $RMlt(S)$ factorizes as $RInn(S)R(S)$ in $Sym(S)$, where $R(S) = \{R_s \mid s \in S\}$. We will follow the right action convention for the map, that is the image of an element x under a map f is denoted by xf. Note that if H is a core-free subgroup of a group G and S is a generating transversal of H in G, then $G \cong RMlt(S)$ such that $H \cong RInn(S)$ (see [6, Lemma, p. 1343]).

A non-empty subset T of right loop S is called a *right subloop* of S, if it is right loop with respect to induced binary operation on T ([7, Definition 2.1, p. 2683]). An equivalence relation R on a right loop S is called a *congruence* in S, if it is a right subloop of $S \times S$. Also an *invariant right subloop* of a right loop S is precisely the equivalence class of the identity of a congruence in S ([7, Definition 2.8, p. 2689]). It is observed in the proof of [7, Proposition 2.10, p. 2690] that if T is an invariant right subloop of S, then the set $S/T = \{T \circ x | x \in S\}$ becomes right loop called as *quotient of S mod T*. Let R be the congruence associated to an invariant right subloop T of S. Then we also denote S/T by S/R.

2 Some properties of right loops

In this section, we will recall some basic facts about right loops and also prove some of the results which will be used in next sections. Let $(S, \circ, /, 1)$ be a right loop, with right division $/$ and two-sided identity 1. Then $(S, \circ, /, 1)$ is Mal'tsev algebra with Mal'tsev term $P(x, y, z) = (x/y) \circ z$.

The proof of the following Fundamental Theorem of homomorphism for right loops is as usual.

Proposition 1 *Let $\rho : S \to S'$ be a homomorphism of right loops. Then there exists a unique injective homomorphism $\bar{\rho} : S/Ker\rho \to S'$ such that $\bar{\rho} \circ \nu = \rho$, where $\nu : S \to S/Ker\rho$ is the natural homomorphism.*

Lemma 1 *Let* G *be a group,* H *a subgroup of* G *and* S *a transversal of* H *in* G*. Suppose that* $N \trianglelefteq G$ *containing* H*. Then*

$$G/N = HS/N \cong S/N \cap S.$$

Proof. Suppose that $\circ$ denotes the induced right loop operation on S. Consider the map $\psi : S \to HS/N$ defined as $x\psi = xN$. This is a homomorphism, for

$$\begin{aligned}(x \circ y)\psi &= (x \circ y)N \\ &= hxyN \text{ for some } h \in H \\ &= (x)\psi(y)\psi \quad (H \subseteq N).\end{aligned}$$

Also, $\mathrm{Ker}\psi = \{x \in S |\ xN = N\} = S \cap N$. Since for $h \in H$ and $x \in S$, we have $hxN = xN$ and $x\psi = xN$, ψ is onto and so by Proposition 1, $S/N \cap S \cong HS/N$. □

Let G be a group, H a subgroup and S a transversal of H in G. Suppose that $\circ$ is the induced right loop structure on S. We define a map $f : S \times S \to H$ as: for $x, y \in S$, $f(x, y) := xy(x \circ y)^{-1}$. We further define the action θ of H on S as $\{x\theta h\} := S \cap Hxh$ where $h \in H$ and $x \in S$. Identifying S with the set $H\backslash G$ of all right cosets of H in G, we get a transitive permutation representation $\chi_S : G \to Sym(S)$ defined by $\{(x)(g)\chi_S\} = S \cap Hxg$, $g \in G, x \in S$. The kernel $\ker\chi_S$ of this action is $\mathrm{Core}_G(H)$, the core of H in G.
One can check that $(\langle S\rangle \cap H)\chi_S \cong RInn(S)$, where $\langle S\rangle$ denotes the subgroup of G generated by S. Since χ_S is injective on S and if we identify S with $(S)\chi_S$, then $(\langle S\rangle)\chi_S \cong RMlt(S)$. One can also verify that $\ker(\chi_S|_{\langle S\rangle} : \langle S\rangle \to RMlt(S)) = \ker(\chi_S|_{\langle S\rangle \cap H} : \langle S\rangle \cap H \to RInn(S)) = \mathrm{Core}_{\langle S\rangle}(\langle S\rangle \cap H)$ and $\chi_S|_S$=the identity map on S.

With these notations it is easy to prove following lemma.

Lemma 2 *For* $x, y, z \in S$*, we have* $xR(y, z) = x\theta f(y, z)$*.*

Lemma 3 *Let* H *be a subgroup of a group* G *and* S *a transversal of* H *in* G*. Let* U *be a congruence on* S *considered as a right loop such that* $\{(x, x\theta h) \mid h \in H, x \in S\} \subseteq U$*. Let* T *be the equivalence class of* 1 *under* U*. Then* S/U *is a group. Moreover,* $N = HT \trianglelefteq HS = G$ *(and so* $H \leq N$ *and* $N \cap S = T$*) and* $G/N \cong S/U$*.*

Proof. Let R be a congruence on S generated by $\{(x, xR(y, z))|x, y, z \in S\}$. Then, clearly $R \subseteq U$ and S/U is a group. Let $\phi : G \to S/U$ be the map defined by $(hx)\phi = T \circ x,\ h \in H,\ x \in S$. This is a homomorphism, because for all $h_1,\ h_2 \in H$ and $x_1,\ x_2 \in S$,

$$\begin{aligned}(h_1x_1h_2x_2)\phi &= (h_1h(x_1\theta h_2 \circ x_2))\phi \text{ for some } h \in H\\ &= T \circ (x_1\theta h_2 \circ x_2)\\ &= (T \circ x_1)\ \circ\ (T \circ x_2) \qquad (\text{for } (x_1, x_1\theta h_2) \in U)\\ &= (h_1x_1)\phi(h_2x_2)\phi.\end{aligned}$$

Let $h \in H$ and $x \in S$. Then $hx \in \mathrm{Ker}\phi$ if and only if $x \in T$. Hence $\mathrm{Ker}\phi = HT = N(say)$. This proves the lemma. □

3 Solvable right loops

In this section, we will define a solvable right loop and obtain some of its properties.

Definition 1 *A right loop S is said to be a solvable right loop if it has a finite composition series with abelian group factors.*

Definition 2 *Let S be a transversal of a subgroup H of G. We call S a solvable transversal if it is solvable with respect to the induced right loop structure.*

We define $S^{(1)}$ to be the smallest invariant right subloop of S such that $S/S^{(1)}$ is an abelian group. We define $S^{(n)}$ by induction. Suppose $S^{(n-1)}$ is defined. Then $S^{(n)}$ is an invariant right subloop of S such that $S^{(n)} = (S^{(n-1)})^{(1)}$.

Theorem 1 *If a group has a solvable generating transversal with respect to a core-free subgroup, then the group is solvable.*

Proof. Let G be a group and H a core-free subgroup of it. Suppose that S is a generating transversal of H in G. Then the group G can be written as HS. By Lemma 1, $G/HG^{(1)} \cong S/S \cap HG^{(1)}$. So,

$$S^{(1)} \subseteq S \cap HG^{(1)}. \tag{1}$$

By Lemma 3, $HS^{(1)}$ is a normal subgroup of G. Thus $G/HS^{(1)} = S/S^{(1)}$ (Lemma 1). Since $S/S^{(1)}$ is abelian, $G^{(1)} \subseteq HS^{(1)}$. Thus

$$S \cap HG^{(1)} \subseteq S^{(1)}. \tag{2}$$

From (2) and (1), it is clear that

$$S \cap HG^{(1)} = S^{(1)} \tag{3}$$

and

$$HG^{(1)} = HS^{(1)}. \tag{4}$$

We will use induction to prove that $HS^{(n)} = H(HS^{(n-1)})^{(1)}$ for $n \geq 1$. Define $S^{(0)} = S$. For $n = 1$, $HS^{(1)} = HG^{(1)} = H(HS^{(0)})^{(1)}$ (by (4)). By induction, suppose that $HS^{(n-1)} = H(HS^{(n-2)})^{(1)}$.

Since $S^{(n-1)}/S^{(n)} \cong HS^{(n-1)}/HS^{(n)}$ is an abelian group, $(HS^{(n-1)})^{(1)} \subseteq HS^{(n)}$. Thus $H(HS^{(n-1)})^{(1)} \subseteq HS^{(n)}$.

Further, $HS^{(n-1)}/H(HS^{(n-1)})^{(1)} \cong S^{(n-1)}/(S^{(n-1)} \cap H(HS^{(n-1)})^{(1)})$ by Lemma 1. So $S^{(n)} \subseteq S^{(n-1)} \cap H(HS^{(n-1)})^{(1)} = S \cap H(HS^{(n-1)})^{(1)}$. That is,

$$HS^{(n)} = H(HS^{(n-1)})^{(1)} \text{ for all } n \geq 1. \tag{5}$$

Now (5) implies that

$$HS^{(n)} = H(HS^{(n-1)})^{(1)} \supseteq H(H(HS^{(n-2)})^{(1)})^{(1)} \supseteq H(HS^{(n-2)})^{(2)}. \tag{6}$$

Proceeding inductively, we have $HS^{(n)} \supseteq H(HS)^{(n)} = HG^{(n)}$. Suppose that S is a solvable right loop, that is there exists $n \in \mathbb{N}$ such that $S^{(n)} = \{1\}$. Then $G^{(n)} \subseteq H$. Since $G^{(n)}$ is a normal subgroup of G contained in H, so $G^{(n)} = \{1\}$. This proves the theorem. □

The converse of the above theorem is not true. For example take G to be the symmetric group on three symbols and H to be any two order subgroup of it. Then H has no solvable generating transversal but we know that G is solvable. Following is an easy consequence of the above theorem.

Corollary 1 *The right multiplication group of a solvable right loop is a solvable group.*

4 Nilpotent right loops

In this section, we define nilpotent right loops as a special case of the nilpotent Mal'tsev algebras defined in [8]. We will obtain some properties of nilpotent right loops. This will generalize a result of [1].

Definition 3 [8, Definition 211, p. 24] *Let β and γ be congruences on a right loop S. Let $(\gamma|\beta)$ be a congruence on β. Then γ is said to centralize β by means of the centering congruence $(\gamma|\beta)$ such that following conditions are satisfied:*

(i) $(x, y)\ (\gamma|\beta)\ (u, v) \Rightarrow x\ \gamma\ u$, *for all* $(x, y), (u, v) \in \beta$.

(ii) *For all* $(x, y) \in \beta$, *the map* $\pi : (\gamma|\beta)_{(x,y)} \to \gamma_x$ *defined by* $(u, v) \mapsto u$ *is a bijection, where for a set* X *and an equivalence relation* δ *on* X, δ_w *denotes the equivalence class of* $w \in X$ *under* δ.

(iii) *For all* $(x, y) \in \gamma$, $(x, x)\ (\gamma|\beta)\ (y, y)$.

(iv) $(x, y)\ (\gamma|\beta)\ (u, v) \Rightarrow (y, x)\ (\gamma|\beta)\ (v, u)$, *for all* $(x, y), (u, v) \in \beta$.

(v) $(x, y)\ (\gamma|\beta)\ (u, v)$ *and* $(y, z)\ (\gamma|\beta)\ (v, w) \Rightarrow (x, z)\ (\gamma|\beta)\ (u, w)$, *for all* $(x, y), (u, v), (y, z)$ *and* (v, w) *in* β.

By (i) and (iv), we observe that $(x, y)\ (\gamma|\beta)\ (u, v) \Rightarrow y\ \gamma\ v$.

Let S be a right loop. If a congruence α on S is centralized by $S \times S$, then it is called a *central congruence* (see [8, p. 42]). By [8, Proposition 221, p. 34] and [8, Proposition 226, p. 38], there exists a unique maximal central congruence $\zeta(S)$ on S, called as the *center congruence* of S. For a right loop, it is product of all centralizing congruences. The *center* $\mathcal{Z}(S)$ of S is defined as ζ_1, the equivalence class of the identity 1. In [4, Proposition 3.3, p. 6], it is observed that if $x \in \mathcal{Z}(S)$, then $x \circ (y \circ z) = (x \circ y) \circ z$ for all $y, z \in S$. In [4, Proposition 3.4, p. 6], it is observed that if $x \in \mathcal{Z}(S)$, then $x \circ y = y \circ x$ for all $y \in S$. This means that the center $\mathcal{Z}(S)$ is an abelian group.

Definition 4 *A right loop* S *is said to be nilpotent if it has a central series*

$$\{1\} = \mathcal{Z}_0 \leq \mathcal{Z}_1 \leq \cdots \leq \mathcal{Z}_n = S$$

for some $n \in \mathbb{N}$, *where*

$$\mathcal{Z}_{i+1}/\mathcal{Z}_i = \mathcal{Z}(S/\mathcal{Z}_i) \text{ and } \mathcal{Z}_1 = \mathcal{Z}(S).$$

On can observe that $\mathcal{Z}_i$ $(0 \leq i \leq n)$ is an invariant right subloop of S. We call a transversal S of a subgroup H of a group G to be *nilpotent*, if it is nilpotent with respect to the induced right loop structure.

Lemma 4 *Every nilpotent right loop is a solvable right loop.*

Proof. It follows from the fact that the central series of a nilpotent right loop is a composition series with abelian group factors. □

Since a nilpotent right loop S is solvable, by Corollary 1, $RMlt(S)$ is solvable. But in this proof we do not know much about the structure of $RInn(S)$. We will obtain that if S is a nilpotent right loop of prime power order, then the order of $RMlt(S)$ will be a prime power.

Proposition 2 *Let S be a right loop. Let $\theta : \mathrm{RInn}(S) \to \mathrm{RInn}(S/\mathcal{Z}(S))$ be the onto homomorphism induced by the natural projection $\nu : S \to S/\mathcal{Z}(S)$. Then $\mathrm{Ker}\theta$ is isomorphic to a subgroup of an abelian group $\prod_{\mathcal{A}} \mathcal{Z}(S)$ for some indexing set $\mathcal{A}$.*

Proof. Let $\mathcal{A} = \{x_1, \cdots, x_i, \cdots\}$ be a set obtained by choosing one element from each right coset of $\mathcal{Z}(S)$ in S, with $x_1 = 1 \in \mathcal{Z}(S)$. Then $S = \sqcup_{x_i \in \mathcal{A}}(\mathcal{Z}(S) \circ x_i)$. Let $h \in \mathrm{Ker}\theta$. Then $x_i h = z \circ x_i$ for some $z \in \mathcal{Z}(S)$. If $y = u_i \circ x_i$, where $u_i \in \mathcal{Z}(S)$, then

$(y)h = (u_i \circ x_i)h = u_i \circ (x_i)h$ (by condition (C7) of [5, Definition 2.1, p. 71] and [4, Proposition 3.3, p. 6])

$= u_i \circ (z \circ x_i)$

$= (u_i \circ z) \circ x_i$ (for $u_i \in \mathcal{Z}(S)$)

$= (z \circ u_i) \circ x_i$

$= z \circ (u_i \circ x_i)$ (for $z \in \mathcal{Z}(S)$)

$= z \circ y.$

Thus $h \in \mathrm{Ker}\theta$ is completely determined by $x_i h$ ($x_i \in \mathcal{A}$). Therefore, it defines a map $\eta : \mathrm{Ker}\theta \to \prod_{\mathcal{A}} \mathcal{Z}(S)$ by $(h)\eta = (z_i)_{\mathcal{A}}$, where $(x_i)h = z_i \circ x_i$. One can check that η is injective homomorphism. □

Following is the finite version of above proposition.

Corollary 2 *Let S be a finite right loop with $|S/\mathcal{Z}(S)| = k$. Let $\theta : \mathrm{RInn}(S) \to \mathrm{RInn}(S/\mathcal{Z}(S))$ be the onto homomorphism induced by natural projection $\nu : S \to S/\mathcal{Z}(S)$. Then $\mathrm{Ker}\theta$ is isomorphic to a subgroup of abelian group $\mathcal{Z}(S) \times \cdots \times \mathcal{Z}(S)$ ($k-1$ times).*

Let S be a nilpotent right loop with central series

$$\{1\} = \mathcal{Z}_0 \leq \mathcal{Z}_1 \leq \cdots \leq \mathcal{Z}_n = S. \tag{7}$$

Let $\theta_j : \mathrm{RInn}(S) \to \mathrm{RInn}(S/\mathcal{Z}_j)$ $(0 \leq j \leq n-1)$ be onto homomorphism induced by the natural projection $\nu_j : S \to S/\mathcal{Z}_j$. Then this will give a series

$$\{1\} = \mathrm{Ker}\theta_0 \leq \cdots \leq \mathrm{Ker}\theta_{n-1} = \mathrm{RInn}(S).$$

Let $\theta : \mathrm{RInn}(S/\mathcal{Z}_j) \to \mathrm{RInn}((S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j))$ be onto homomorphism induced by the natural projection $\nu : S/\mathcal{Z}_j \to (S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j)$. By Proposition 2, $\mathrm{Ker}\theta$ is isomorphic to a subgroup of $\prod_{\mathcal{B}} \mathcal{Z}_{j+1}/\mathcal{Z}_j$ for some indexing set $\mathcal{B}$.

We now observe that each member of $\mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$ induces a member of $\mathrm{Ker}\theta$. For this, we will see the action of an element of $\mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$ on the elements of $(S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j)$. Let $h_{j+1}\mathrm{Ker}\theta_j \in \mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$, where $h_{j+1} \in \mathrm{Ker}\theta_{j+1}$ and $(\mathcal{Z}_{j+1}/\mathcal{Z}_j) \circ (\mathcal{Z}_j \circ x) \in (S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j)$. By the definition of θ_j, each element of $\mathrm{Ker}\theta_j$ acts trivially on the cosets of $\mathcal{Z}_j$. Since $\mathrm{RInn}(S/\mathcal{Z}_{j+1}) \cong \mathrm{RInn}((S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j))$, by definition of θ_{j+1}, h_{j+1} also acts trivially on $(\mathcal{Z}_{j+1}/\mathcal{Z}_j) \circ (\mathcal{Z}_j \circ x)$. Thus, we have proved the following:

Proposition 3 *Let S be a nilpotent right loop with central series 7. Then there exists a series*

$$\{1\} = \mathrm{Ker}\theta_0 \leq \cdots \leq \mathrm{Ker}\theta_{n-1} = \mathrm{RInn}(S)$$

such that $\mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$ is isomorphic to a subgroup of $\prod_{\mathcal{B}} \mathcal{Z}_{j+1}/\mathcal{Z}_j$ for some indexing set $\mathcal{B}$.

Corollary 3 *Let S be a nilpotent right loop. Then the right inner mapping group $\mathrm{RInn}(S)$ is a solvable group.*

Proof. By Proposition 3, central series of S gives a series of $\mathrm{RInn}(S)$ with abelian quotients. □

Corollary 4 *If a group G has a nilpotent generating transversal with respect to a core-free subgroup H, then H is solvable.*

Corollary 5 *Let S be a nilpotent generating transversal with respect to a core-free subgroup H of a finite group G such that $|S| = p^n$ for some prime p and $n \in \mathbb{N}$. Then both H and G are p-groups.*

5 Some examples

In this section, we will observe some examples and counterexamples. We have seen that the concepts of solvability and nilpotency of a right loop can be transferred in term of a generating transversal of a core-free subgroup of a group. There are examples of groups where no non-trivial subgroup is core-free. Following is an example of such a group:

Example 1 *Consider the group* $G = \langle x_1, x_2, x_3, x_4 | x_1^{p^n} = x_2^{p^3} = x_3^{p^2} = x_4^{p^2} = 1, [x_1, x_2] = x_2^{p^2}, [x_1, x_3] = x_3^{p}, [x_1, x_4] = x_4^{p}, [x_2, x_3] = x_1^{p^{n-1}}, [x_2, x_4] = x_2^{p^2}, [x_3, x_4] = x_4^{p}\rangle$ *where p is an odd prime and n is the natural number greater than 2. The above example has been taken from [3]. This is a nilpotent group of class 2 having no nontrivial core-free subgroup.*

We now observe that a solvable right loop which is not a group need not be a nilpotent right loop.

Example 2 *Let* $G = \mathrm{Alt}(4)$*, the alternating group of degree* 4 *and* $H = \{I, (1,2)(3,4)\}$*, where* I *denotes the identity permutation. Consider a right transversal* $S = \{I, (1,3)(2,4), (1,2,3), (1,3,2), (2,3,4), (1,3,4)\}$ *of* H *in* G*. Note that* $\langle S\rangle = G$ *and* H *is core-free. Then* $\mathrm{RMlt}(S) \cong G$ *and* $\mathrm{RInn}(S) \cong H$*. Also note that* $S \cap N_G(H) = \{I, (1,3)(2,4)\}$*, where* $N_G(H)$ *denotes the normalizer of* H *in* G*. By* [4, Proposition 3.3, p. 6]*,* $\mathcal{Z}(S) \subseteq S \cap N_G(H)$*. Let* $\circ$ *be the induced binary operation on* S *as defined in the Section* 1*. Observe that* $(1,3)(2,4) \circ (1,3,4) \neq (1,3,4) \circ (1,3)(2,4)$*. This implies that* $\mathcal{Z}(S) = \{I\}$*. Hence* S *can not be nilpotent.*

Now by Lemma 1, $S/(S \cap N_G(H))$ *is isomorphic to the cyclic group of order* 3*. This implies that* S *is solvable.*

Now, we observe that, unlike for the case of groups, a right loop of prime power order need not be nilpotent.

Example 3 *Let* $G =$

$$\langle (1,3)(2,4)(5,7,6,8), (1,4)(2,3)(5,8,6,7), (1,5)(2,6)(3,7)(4,8)\rangle \leq \mathit{Sym}(8),$$

where $\mathit{Sym}(n)$ *denotes the symmetric group of degree* n*. Let* H *be the stabilizer of* 1 *in* G*. Consider* $S = \{I, (1,2)(3,4), (1,3)(2,4)(5,7,6,8), (1,4)(2,3)(5,8,6,7), (1,5)(2,6)(3,7)(4,8), (1,6)(2,5)(3,8)(4,7), (1,7)(2,8)(3,6,4,5), (1,8)(2,7)(3,5,4,6)\}$*. Clearly* S *is right transversal of* H *in* G*. Note that the center* $Z(G) = \{I, (1,2)(3,4)(5,6)(7,8)\}$ *and* $N_G(H) = HZ(G)$*. Since* H *is core-free and* $\langle S\rangle = G$*,* $G \cong \mathrm{RMlt}(S)$ *and* $H \cong \mathrm{RInn}(S)$*. Observe that* $S \cap N_G(H) = \{I, (1,2)(3,4)\}$*. By* [4, Proposition 3.3, p. 6]*,* $\mathcal{Z}(S) \subseteq S \cap N_G(H)$*. Let* $\circ$ *be the induced binary operation on* S *as defined in the section* 1*. Observe that* $(1,2)(3,4) \circ (1,5)(2,6)(3,7)(4,8) \neq (1,5)(2,6)(3,7)(4,8) \circ (1,2)(3,4)$*. This implies that* $\mathcal{Z}(S) = \{I\}$*. Hence* S *cannot be nilpotent.*

Next, we will show that there are core-free subgroups of a nilpotent group which has none of its generating transversals nilpotent. But before proceeding to further examples, we need to prove the following results.

Proposition 4 *Suppose that* G *is a nilpotent group of class* 2*,* H *is a core-free subgroup of* G *and* S *is generating transversal of* H *in* G*. Then* $\mathcal{Z}(G) \cap S = \mathcal{Z}(S)$*.*

Proof. Take $x \in \mathcal{Z}(S)$. Then $x \circ y = y \circ x$ for all $y \in S$. This implies $xyx^{-1}y^{-1} \in H$ for all $y \in S$. Since group is nilpotent of class 2, so all commutators are central. For H is core-free, so H will not contain any commutator element. This implies $xyx^{-1}y^{-1} = 1$ or $xy = yx$ for all $y \in S$. This proves that $\mathcal{Z}(S) \subseteq \mathcal{Z}(G) \cap S$ (for S generates G). Converse is obvious. This proves the lemma. $\square$

Proposition 5 *For some prime* p*, suppose that* G *is a* p*-group of nilpotent class* 2*,* H *is a core-free subgroup of* G *and* S *is generating transversal of* H *in* G*. Then* $\mathcal{Z}(G) \cap \Phi(G) \cap S = \{1\}$ *where* $\Phi(G)$ *is the Frattini subgroup of* G*.*

Proof. Suppose that $1 \neq x \in \mathcal{Z}(G) \cap \Phi(G) \cap S$. Then by Proposition 4, $x \in \mathcal{Z}(S)$. Also $\mathcal{Z}(S)$ is an invariant right subloop, so $|\mathcal{Z}(S)|$ divides $|S|$. Consider $S' = S \setminus \{x\} \cup \{hx\}$ for some $1 \neq h \in H$. Note that S' also generates G. Then by Proposition 4, order of center of S' is one less than the order of center of S and also $|\mathcal{Z}(S')|$ divides $|S|$. This is not possible for order of S is p power. This proves the lemma. $\square$

Example 4 *Consider the group* $G = \langle x_1, x_2, x_3, x_4 | x_1^{p^n} = x_2^{p^2} = x_3^{p^2} = x_4^{p^4} = 1, [x_1, x_2] = x_2^p, [x_1, x_3] = x_3^p, [x_1, x_4] = x_3^p, [x_2, x_3] = x_1^{p^{n-1}}, [x_2, x_4] = x_2^p, [x_3, x_4] = 1\rangle$ *where* p *is an odd prime. The above example has been taken from* [3]*. By the Lemma 2.1 of* [3]*, this group is a nilpotent group of of class* 2 *and its center and Frattini subgroup are equal. By Propositions* 4 *and* 5*, it follows that center of each generating transversal is trivial. So none of the generating transversal is nilpotent.*

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Tournaments, oriented graphs and football sequences

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Abstract. Consider the result of a soccer league competition where n teams play each other exactly once. A team gets three points for each win and one point for each draw. The total score obtained by each team v_i is called the f-score of v_i and is denoted by f_i. The sequences of all f- scores $[f_i]_{i=1}^{n}$ arranged in non-decreasing order is called the f-score sequence of the competition. We raise the following problem: Which sequences of non-negative integers in non-decreasing order is a football sequence, that is the outcome of a soccer league competition. We model such a competition by an oriented graph with teams represented by vertices in which the teams play each other once, with an arc from team u to team v if and only if u defeats v. We obtain some necessary conditions for football sequences and some characterizations under restrictions.

1 Introduction

Ranking of objects is a typical practical problem. One of the popular ranking methods is the pairwise comparison of the objects. Many authors describe

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different applications: e.g., biological, chemical, network modeling, economical, human relation modeling, and sport applications.

A tournament is an irreflexive, complete, asymmetric digraph, and the score s_v of a vertex v in a tournament is the number of arcs directed away from that vertex. We interpret a tournament as the result of a competition between n teams with teams represented by vertices in which the teams play each other once (ties not allowed), with an arc from team u to team v if and only if u defeats v. A team receives one point for each win. With this scoring system, team v receives a total of s_v points. We call the sequence $S = [s_1, s_2, \cdots, s_n]$ as the score sequence, if s_i is the score of some vertex v_i. Thus a sequence $S = [s_1, s_2, \cdots, s_n]$ of non-negative integers in non-decreasing order is a score sequence if it realizes some tournament. Landau [21] in 1953 characterized the score sequences of a tournament.

Theorem 1 [21] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} s_i \geq \binom{|I|}{2}, \tag{1}$$

with equality when $|I| = n$, where $|I|$ is the cardinality of the set $|I|$.

Since $s_1 \leq \cdots \leq s_n$, the inequality (1), called Landau inequalities, are equivalent to $\sum_i^k s_i \geq \binom{k}{2}$, for $k = 1, 2, \cdots, n-1$, and equality for $k = n$.

There are now several proofs of this fundamental result in tournament theory, clever arguments involving gymastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization, a constructive argument utilizing network flows, another one involving systems of distinct representatives. Landau's original proof appeared in 1953 [21], Matrix considerations by Fulkerson [15] (1960) led to a proof, discussed by Brauldi and Ryser [10] in (1991). Berge [7] in (1960) gave a network flow proof and Alway [3] in (1962) gave another proof. A constructive proof via matrices by Fulkerson [16] (1965), proof of Ryser (1964) appears in the monograph of Moon (1968). An inductive proof was given by Brauer, Gentry and Shaw [8] (1968). The proof of Mahmoodian [23] given in (1978) appears in the textbook by Behzad, Chartrand and Lesnik-Foster [6](1979). A proof by contradiction was given by Thomassen [33] (1981) and was adopted by Chartrand and Lesniak [13] in subsequent revisions of their 1979 textbook, starting with their 1986 revision. A nice proof was given by Bang and Sharp [5](1979) using systems of distinct representatives. Three years later in 1982, Achutan, Rao and Ramachandra-Rao [1] obtained a proof as result of some slightly more general work. Bryant [12] (1987) gave a proof via a slightly different use of distinct representatives. Partially ordered sets were employed in a proof by Aigner [2] in 1984 and described by Li [22] in 1986 (his version appeared in 1989). Two proofs of sufficiency appeared in a paper by Griggs and Reid [17] (1996) one a direct proof and the second is self contained. Again two proofs appeared in 2009 one by Brauldi and Kiernan [11] using Rado's theorem from Matroid theory, and another inductive proof by Holshouser and Reiter [19] (2009). More recently Santana and Reid [32] (2012) have given a new proof in the vein of the two proofs by Griggs and Reid (1996).

The following is the recursive method to determine whether or not a sequence is the score sequence of some tournament. It also provides an algorithm to construct the corresponding tournament.

Theorem 2 [21] *Let S be a sequence of n non-negative integers not exceeding $n-1$, and let S' be obtained from S by deleting one entry s_k and reducing $n-1-s_k$ largest entries by one. Then S is the score sequence of some tournament if and only if S' is the score sequence.*

Brauldi and Shen [9] obtained stronger inequalities for scores in tournaments. These inequalities are individually stronger than Landau's inequalities, although collectively the two sets of inequalities are equivalent.

Theorem 3 [9] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} s_i \geq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{2} \binom{|I|}{2} \tag{2}$$

with equality when $|I| = n$

It can be seen that equality can often occur in (2), for example, equality hold for regular tournaments of odd order n whenever $|I| = k$ and $I = \{n-k+1, \cdots, n\}$. Further Theorem 2 is best possible in the sense that, for any real $\epsilon > 0$, the inequality

$$\sum_{i \in I} s_i \geq (\frac{1}{2} + \epsilon) \sum_{i \in I} (i-1) + (\frac{1}{2} - \epsilon) \binom{|I|}{2}$$

fails for some I and some tournaments, for example, regular tournaments. Brauldi and Shen [9] further observed that while an equality appears in (2), there are implications concerning the strong connectedness and regularity of every tournament with the score sequence S. Brauldi and Shen also obtained the upper bounds for scores in tournaments.

Theorem 4 [9] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each subset* $I \subseteq [n] = \{1, 2, \cdots, n\}$,

$$\sum_{i \in I} s_i \leq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{4} |I|(2n - |I| - 1),$$

with equality when $|I| = n$

An oriented graph is a digraph with no symmetric pairs of directed arcs and without self loops. If D is an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and if $d^+(v)$ and $d^-(v)$ are respectively, the outdegree and indegree of a vertex v, then $a_v = n - 1 + d^+(v) - d^-(v)$ is called the score of v. Clearly, $0 \leq a_v \leq 2n - 2$. The score sequence $A(D)$ of D is formed by listing the scores in non-decreasing order. One of the interpretations of an oriented graph is a competition between n teams in which each team competes with every other exactly once, with ties allowed. A team receives two points for each win and one point for each tie. For any two vertices u and v in an oriented graph D, we have one of the following possibilities.
(i). An arc directed from u to v, denoted by $u(1-0)v$, (ii). An arc directed from v to u, denoted by $u(0-1)v$, (iii). There is no arc from u to v and there is no arc from v to u, and is denoted by $u(0-0)v$.

If $d^*(v)$ is the number of those vertices u in D which have $v(0-0)u$, then $d^+(v) + d^-(v) + d^*(v) = n - 1$. Therefore, $a_v = 2d^+(v) + d^*(v)$. This implies that each vertex u with $v(1-0)u$ contributes two to the score of v. Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$, and each $v(0-0)u$ contributes two(one each at u and v) to scores, therefore the sum total of all the scores is $2\binom{n}{2}$. With this scoring system, player v receives a total of a_v points.

Avery [4] obtained the following characterization of score sequences in oriented graphs.

Theorem 5 [4] *A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of an oriented graph if and only if for each*

$I \subseteq [n] = \{1, 2, \cdots, n\}$,

$$\sum_{i \in I} a_i \geq 2\binom{|I|}{2}, \tag{3}$$

with equality when $|I| = n$.

Since $a_1 \leq a_2 \leq \cdots \leq a_n$, the inequality (3) are equivalent to

$$\sum_{i}^{k} a_i \geq 2\binom{k}{2}, \text{ for } \; k = 1, 2, \cdots, n-1$$

with equality for $k = n$.

A constructive proof of Avery's theorem can be seen in Pirzada, Merajuddin and Samee [29] and another proof in Pirzada et. al [28]. A recursive characterization of score sequences in oriented graphs also appears in Avery [4].

Theorem 6 [4] *Let* A *be a sequence of integers between 0 and* $2n-2$ *inclusive and let* A' *be obtained from* A *by deleting the greatest entry* $2n-2-r$ *say, and reducing each of the greatest* r *remaining entries in* A *by one. Then* A *is a score sequence if and only if* A' *is a score sequence.*

Theorem 6 provides an algorithm for determining whether a given non-decreasing sequence A of non-negative integers is a score sequence of an oriented graph and for constructing a corresponding oriented graph. Pirzada, Merajuddin and Samee (2008) obtained the stronger inequalities for oriented graph scores.

An r-digraph is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let D be an r-digraph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of a vertex v_i. Define p_{v_i} (or simply p_i)$= r(n-1) + d^+_{v_i} - d^-_{v_i}$ as the mark (or r-score) of v_i, so that $0 \leq p_{v_i} \leq 2r(n-1)$. Then the sequence $P = [p_i]_1^n$ in non-decreasing order is called the mark sequence of D.

An analogous result to Landau$'$s theorem on tournament scores [21] is the following characterization of marks in r-digraphs and is due to Pirzada [27].

Theorem 7 [27] *A sequence* $P = [p_i]_1^n$ *of non-negative integers in non-decreasing order is the mark sequence of an* r*-digraph if and only*

$$\textstyle\sum_{i=1}^{t} p_i \geq rt(t-1),$$

for $1 \leq t \leq n$, *with equality when* $t = n$.

Various results on mark sequences in digraphs are given in [25, 27] and we can find certain stronger inequalities of marks for digraphs in [26] and for multidigraphs in [30].

2 Football sequences

If D is an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and if $d^+(v_i)$ and $d^-(v_i)$ are respectively the outdegree and indegree of a vertex v_i, define f_{v_i} (or briefly f_i) as

$$f_i = n - 1 + 2d^+(v_i) - d^-(v_i)$$

and call f_i as the football score(or briefly f-score) of v_i. Clearly

$$0 \leq f_{v_i} \leq 3(n-1).$$

The f-score sequence $F(D)$ (or briefly F) of D is formed by listing the f-scores in non-decreasing or non-increasing order. For any two vertices u and v in an oriented graph D, we have one of the following possibilities.

(i). An arc directed from u to v, denoted by $u \to v$ and we write this as $u(1 - -0)v$.

(ii). An arc directed from v to u, denoted by $u \leftarrow v$ and we write this as $u(0 - -1)v$.

(iii). There is no arc directed from u to v and there is no arc directed from v to u, denoted by $u \sim v$ and we write this as $u(0 - -0)v$.

If $d^*(v)$ is the number of those vertices u in D for which we have $v(0 - -0)u$, then

$$d^+(v) - d^-(v) + d^*(v) = n - 1.$$

Therefore,

$$f_v = d^+(v) - d^-(v) + d^*(v) + 2d^+(v) - d^-(v) = 3d^+(v) + d^*(v).$$

This implies that each vertex u with $v(1 - -0)u$ contributes three to the f-score of v, and each vertex u with $v(0 - -0)u$ contributes one to the f-score of v.

Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$, and each $v(0--0)u$ contributes two (one each at u and v) to f-scores, therefore

$$2\binom{n}{2} \leq \sum_{i=1}^{n} f_i \leq 3\binom{n}{2}.$$

We interpret an oriented graph as the result of a football tournament with teams represented by vertices in which the teams play each other once, with an arc from team u to team v if and only if u defeats v. A team receives three points for each win and one point for each draw (tie). With this f-scoring system, team v receives a total of f_v points.

We call the sequence $F = [f_1, f_2, \cdots, f_n]$ as the football sequence, if f_i is the f-score of some vertex v_i. Thus a sequence $F = [f_1, f_2, \cdots, f_n]$ of non-negative integers in non-decreasing order is a football sequence if it realizes some oriented graph. Several results on football sequences can be found in Ivanyi [20].

In an oriented graph the vertex of indegree zero is called a transmitter. This means that the transmitter represents that team in the game which does not lose any match.

Theorem 8 *If the sequence* $F = [f_1, f_2, \cdots, f_n]$ *of non-negative integers in non-decreasing order is a football sequence then for* $1 \leq k \leq n-1$ *and* $2\binom{k}{2} \leq x_k \leq 3\binom{k}{2}$,

$$\sum_{i=1}^{k} f_i \geq x_k,$$

and for $2\binom{n}{2} \leq x_n \leq 3\binom{n}{2}$

$$\sum_{i=1}^{n} f_i = x_n.$$

Lemma 1 *There is no oriented graph with* n *vertices whose* f*-score of some vertex is* $3n-4$.

Proof. Let D be an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Let v_i be the vertex with f-score f_i. In case $v_i(1-0)v$ to all $v \in V - \{v_i\}$, then f-score of v_i is $3(n-1)$. If $v_i(1-0)v$ for all $v \in V - \{v_i, v_j\}$, for some $v_j \in V$ and $i \neq j$, then f-score of v_i is $3(n-2)+1 = 3n-5$. We note that the possible f score can be $3(n-1)$ or $3(n-2)+1$. Thus the f-score f_i is either $3(n-1)$ or $f_i \leq 3(n-2)+1 = 3n-5$. These imply that the f-score cannot be $3n-4$. □

Lemma 2 *In an oriented graph with n vertices if the f-score f_i and n are of the same parity, then the vertex v_i with f-score f_i is not the transmitter.*

Proof. Let $D(V, A)$ be an oriented graph with $V = \{v_1, v_2, \cdots, v_n\}$ so that $f_{v_i} = f_i$. Let n and f_i be of same parity, that is either (a) n and f_i both are even or (b) n and f_i both are odd.

In D, let $v_i(1-0)u$, $v_i(0-0)w$ and $v_i(0-1)z$ with $u \in U$, $w \in W$, $z \in Z$ and $V = U \cup W \cup Z \cup \{v_i\}$. Further let $|U| = x$, $|W| = y$ and $|Z| = t$. Clearly

$$x + y + t = n - 1. \tag{4}$$

Case (a) $n-1$ is odd and f_i is even. We have $f_i = 3x + y$. Since f_i is even, $3x + y$ is even. Thus either (i) x is odd and y is odd, or (ii) x is even and y is even. In both cases, it follows from (4) that t is odd.

Case (b) $n-1$ is even and f_i is odd. So $3x + y$ is odd. This is possible if (iii) x is even and y is odd, or (ii) x is odd and y is even. In both cases, again it follows from (4) that t is odd.

Thus in all cases we have $|Z| = t = \text{ odd}$, which implies that $|Z| \neq \phi$ so that there is at least one vertex z such that $z(1-0)v_n$. Hence v_i is not a transmitter. □

Lemma 2 shows that if the number of teams n and the f-score f_i are both odd or both even, then the team represented by v_i with f-score is not the transmitter, meaning it loses at least once in the competition.

Theorem 9 *In an oriented graph with n vertices the vertex with f-score f_i is a transmitter if (1) n and f_i are of different parity and (2) $f_i \equiv (n-1) (\text{mod } 2)$ and $f_i \equiv 3(n-1)(\text{mod } 2)$.*

Proof. Let $D(V, A)$ be the oriented graph with n vertices whose vertex set is $V = \{v_1, v_2, \cdots, v_n\}$. Let f-score of v_i be f_i and let v_i be the transmitter. Then in D, we have either $v_i(1-0)v_j$ or $v_i(0-0)v_j$ for all all $j \neq i$. Let U be the set of vertices for which $v_i(1-0)u$ and W be the set of vertices for which $v_i(1-0)w$ and let $|U| = x$ and $|W| = y$. Clearly

$$x + y = n - 1 \tag{5}$$

and

$$f_i = 3x + y. \tag{6}$$

Two cases can arise, (a) n is odd or (b) n is even.

Case (a) n is odd. Then $n-1$ is even so that $x+y$ is even. This is possible if either (i) x odd and y odd or (ii) x even and y even. In case of (i) $f_i = 3x+y = \mathrm{odd}+\mathrm{odd} = \mathit{even}$ and in case of (ii) $f_i = 3x+y = \mathit{even}+\mathit{even} = \mathit{even}$. Thus we see that n and f_i are of different parity.

Case (a) n is even, so that $n-1$ is odd and $x+y$ is odd. This is possible if either (iii) x odd and y even or (ii) x even and y odd. In both cases we observe that f_i is odd. Therefore again we obtain that n and f_i are of different parity.

Solving (5) and (6) together for x and y, we get

$$x = \frac{1}{2}[f_i - (n-1)] \tag{7}$$

$$y = \frac{1}{2}[3(n-1) - f_i]. \tag{8}$$

Clearly x and y are positive integers, thus the right hand sides of (7) and (8) are positive integers. This implies that $f_i - (n-1)$ and $3(n-1) - f_n$ are both divisible by 2. Hence $f_n \equiv (n-1)(\mathrm{mod}\ 2)$ and $f_n \equiv 3(n-1)(\mathrm{mod}\ 2)$. □

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Unbounded solutions of an iterative-difference equation

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Abstract. Unbounded solutions for the iterative-difference equation

$$f^2(x) = \lambda f(x+a) + \mu x, \ \ x \in \mathbb{R},$$

have been considered in [Continuous solutions of an iterative-difference equation and Brillouët problem, Publ. Math. Debrecen, 78 (2011), 613–624], where λ, μ, a are real constants. In this paper, we continue to study the solutions not being included there, and further give the convex and concave ones. Finally, continuous solutions of this equation with an extra item were also given, which continuously depend on the parameter a.

1 Introduction

The iterative-difference equation

$$f^2(x) = f(x+a) - x, \quad x \in \mathbb{R} \tag{1}$$

proposed by N. Brillouët-Belluot [2], was deduced from the equation $x+f(y+f(x)) = y+f(x+f(y))$, a special form of the functional equation

$$x + g(y + f(x)) = y + f(x + g(y))$$

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investigated in [8, 12]. Although the nonexistence of continuous solutions and existence of special solutions were discussed [1, 7, 11], it is still treated as a difficult problem to find continuous solutions of Eq.(1). In 2010, as a generalized form of (1), the equation

$$f^2(x) = \lambda f(x+a) + \mu x, \tag{2}$$

where λ, μ, a are real and $a\lambda \neq 0$ was considered on a compact interval by N. Brillouët-Belluot and W. Zhang ([3]). In fact, when $a = 0$ all continuous solutions were presented in 1974 by S. Nabeya ([10]) and continuous solutions defined on a real Hausdorff topological linear space were studied by J. Dhombres ([5]) in the case that $\lambda + \mu = 1$. The authors in [3] not only searched all affine solutions, but also constructed piecewise continuous solutions of Eq.(2). Later, Y. Zeng and W. Zhang [17] investigated the solutions on the whole $\mathbb{R}$ and gave the unbounded continuous solutions of Eq.(2) in some cases, where the following cases are still open: **(E1)** $|\lambda| \in (0, 1]$; **(E2)** $|\lambda| \in (1, 2]$ and $|\mu| \in [|\lambda| - 1, +\infty)$; **(E3)** $|\lambda| \in (2, +\infty)$ and $|\mu| \in (\lambda^2/4, +\infty)$. Although when $\lambda = 1$ and $\mu \leq -1$, a special case of **(E1)**, was solved (see Theorem 1 in [17]), the existence or nonexistence of solutions almost remains unknown for the cases **(E1)**-**(E3)**. This paper is a continuation of studying Eq.(2) on the whole $\mathbb{R}$. We first consider the continuous solutions with the form of

$$f(x) = \alpha x + f_1(x), \tag{3}$$

where α is a real constant and $f_1 : \mathbb{R} \to \mathbb{R}$ is continuous. By using the Banach fixed point principle, the existence and uniqueness of solutions including convex and concave ones, are given for cases **(E2)**-**(E3)**. Finally, in order to show the characterization of solutions for case **(E1)**, we investigate the continuous solutions of Eq.(2) with an extra item and prove that those solutions continuously depend on the parameter a.

2 Existence of unbounded solutions

By replacing the function $f(x)$ with $g(x) := \frac{1}{a} f(ax)$, it suffices to consider the solutions of Eq.(2) with $a = 1$, i.e.,

$$f^2(x) = \lambda f(x+1) + \mu x, \quad x \in \mathbb{R}. \tag{4}$$

Further, substituting (3) into (4) we obtain

$$\begin{aligned} f^2(x-1) &= \alpha(\alpha(x-1) + f_1(x-1)) + f_1(\alpha(x-1) + f_1(x-1)) \\ &= \lambda\alpha x + \lambda f_1(x) + \mu(x-1), \end{aligned}$$

and then

$$f_1(x) = \frac{\alpha^2 - \lambda\alpha - \mu}{\lambda} x + \frac{\alpha}{\lambda} f_1(x-1) + \frac{1}{\lambda} f_1(f_1(x-1) + \alpha x - \alpha) - \frac{\alpha^2 - \mu}{\lambda}.$$

Let

$$\Phi(\mathbb{R}; L) = \{f : \mathbb{R} \to \mathbb{R} \mid \mathrm{Lip} f \le L\}$$

for a given real constant $L > 0$. We note that $\Phi(\mathbb{R}; L)$ is a complete metric space equipped with the metric $d(f, g) := \sup_{x\in\mathbb{R}} |f(x) - g(x)|$.

Theorem 1 *For a given* $\alpha \in \mathbb{R}$, *Eq.* (4) *has a unique solution* $f(x) = \alpha x + f_1(x)$, *where* $f_1 \in \Phi(\mathbb{R}; L)$ *provided that* $L > 0$ *and*

$$|\frac{\alpha^2 - \lambda\alpha - \mu}{\lambda}| + 2|\frac{\alpha}{\lambda}|L + \frac{1}{|\lambda|}L^2 \le L, \tag{5}$$

$$\frac{|\alpha| + 1 + L}{|\lambda|} < 1. \tag{6}$$

Proof. For a given $L > 0$, define a mapping $\mathcal{F} : \Phi(\mathbb{R}; L) \to C^0(\mathbb{R})$ by

$$\mathcal{F} f_1(x) = \tfrac{\alpha^2 - \lambda\alpha - \mu}{\lambda} x + \tfrac{\alpha}{\lambda} f_1(x-1) + \tfrac{1}{\lambda} f_1(f_1(x-1) + \alpha x - \alpha) - \tfrac{\alpha^2 - \mu}{\lambda}, \tag{7}$$

where $f_1 \in \Phi(\mathbb{R}; L)$. It follows from (5) that for every $x, y \in \mathbb{R}$,

$$\begin{aligned}
|\mathcal{F} f_1(x) - \mathcal{F} f_1(y)| &= \Bigg| \frac{\alpha^2 - \lambda\alpha - \mu}{\lambda} x + \frac{\alpha}{\lambda} f_1(x-1) + \frac{1}{\lambda} f_1(f_1(x-1) + \alpha x - \alpha) \\
&\quad - \frac{\alpha^2 - \lambda\alpha - \mu}{\lambda} y + \frac{\alpha}{\lambda} f_1(y-1) + \frac{1}{\lambda} f_1(f_1(y-1) + \alpha y - \alpha) \Bigg| \\
&\le \left| \frac{\alpha^2 - \lambda\alpha - \mu}{\lambda} \right| |x - y| + \left| \frac{\alpha}{\lambda} \right| L|x - y| \\
&\quad + \frac{1}{|\lambda|} L|\alpha x + f_1(x-1) - \alpha y - f_1(y-1)| \\
&\le \left(\left| \frac{\alpha^2 - \lambda\alpha - \mu}{\lambda} \right| + 2\left| \frac{\alpha}{\lambda} \right| L + \frac{1}{|\lambda|} L^2 \right) |x - y| \le L|x - y|.
\end{aligned}$$

This implies that $\mathcal{F}$ maps $\Phi(\mathbb{R}; L)$ into itself. Furthermore, for every $f_1, f_2 \in \Phi(\mathbb{R}; L)$ we have

$$
\begin{aligned}
&|\mathcal{F}f_1(x) - \mathcal{F}f_2(x)| \\
&= |\tfrac{\alpha}{\lambda} f_1(x-1) + \tfrac{1}{\lambda} f_1(f_1(x-1) + \alpha x - \alpha) - \tfrac{\alpha}{\lambda} f_2(x-1) - \tfrac{1}{\lambda} f_2(f_2(x-1) + \alpha x - \alpha)| \\
&\le |\tfrac{\alpha}{\lambda}| d(f_1, f_2) + \tfrac{1}{|\lambda|} |f_1(f_1(x-1) + \alpha x - \alpha) - f_2(f_2(x-1) + \alpha x - \alpha)| \\
&\le |\tfrac{\alpha}{\lambda}| d(f_1, f_2) + \tfrac{1}{|\lambda|} |f_1(f_1(x-1) + \alpha x - \alpha) - f_2(f_1(x-1) + \alpha x - \alpha)| \\
&+ \tfrac{1}{|\lambda|} |f_2(f_1(x-1) + \alpha x - \alpha) - f_2(f_2(x-1) + \alpha x - \alpha)| \\
&\le \tfrac{(|\alpha| + 1 + L)}{|\lambda|} d(f_1, f_2), \forall x \in \mathbb{R},
\end{aligned}
$$

i.e.,

$$
d(\mathcal{F}f_1, \mathcal{F}f_2) \le \frac{(|\alpha| + 1 + L)}{|\lambda|} d(f_1, f_2).
$$

Hence, $\mathcal{F} : \Phi(\mathbb{R}; L) \to \Phi(\mathbb{R}; L)$ is a contraction if condition (6) holds. Therefore, by Banach's fixed point theorem, $\mathcal{F}$ has a unique fixed point in the class $\Phi(\mathbb{R}; L)$. □

Remark 1 *In view of Theorem 1, the assumptions on* α, L, λ *and* μ *are given by* (5)-(6)*. Compared with Theorem 2 in* [17]*, although our conditions seem complicated because of the inequalities between* α, λ, μ*, cases* **(E2)**-**(E3)** *in Theorem 1 were not considered in* [17]*.*

Example 1 *Take* $\alpha = 0.4$*,* $\lambda = -2$*,* $\mu = 1$ *and* $L = 0.28$*.*

It is easy to see that λ, μ satisfy **(E2)**. Moreover,

$$
\left|\frac{\alpha^2 - \lambda\alpha - \mu}{\lambda}\right| + 2\left|\frac{\alpha}{\lambda}\right| L + \frac{1}{|\lambda|} L^2 = 0.1712 < 0.28, \quad \frac{|\alpha| + 1 + L}{|\lambda|} = 0.84 < 1,
$$

which implies that (5)-(6) hold. Therefore, Eq. (4) has a unique solution $f(x) = \frac{2}{5}x + f_1(x)$, where $f_1 \in \Phi(\mathbb{R}; \frac{7}{25})$.

Example 2 *Choose* $\alpha = 0.5$*,* $\lambda = -2.2$*,* $\mu = 1.25$*, and* $L = 0.3$*.*

One can check that λ, μ fulfills case **(E3)**. Furthermore,

$$|\frac{\alpha^2 - \lambda\alpha - \mu}{\lambda}| + 2|\frac{\alpha}{\lambda}|L + \frac{1}{|\lambda|}L^2 < 0.224 < L, \quad \frac{|\alpha| + 1 + L}{|\lambda|} < 0.82 < 1.$$

Thus, conditions (5)-(6) in Theorem 1 are satisfied and Eq. (4) has a unique solution $f(x) = \frac{1}{2}x + f_1(x)$ for $f_1 \in \Phi(\mathbb{R}; \frac{3}{10})$.

In what follows, we turn to consider the convex and concave solutions of Eq. (4). A function f is convex (concave) if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$ (if the reverse of the inequality holds). Let

$$\Phi^{cv}(L) = \{f \in \Phi(\mathbb{R}; L) : f \ \text{ is } \ \text{convex} \ \text{ and } \ \text{increasing}\}.$$

$$\Phi^{cc}(L) = \{f \in \Phi(\mathbb{R}; L) : f \ \text{ is } \ \text{concave} \ \text{ and } \ \text{increasing}\}.$$

Clearly, $\Phi^{cv}(L)$ and $\Phi^{cc}(L)$ are closed subsets of $\Phi(\mathbb{R}; L)$, which are also complete metric spaces.

Theorem 2 *Replace* $\Phi(\mathbb{R}; L)$ *in Theorem 1 by* $\Phi^{cv}(L)(\Phi^{cc}(L))$ *and suppose that the conditions of Theorem 1 hold. If* $\lambda > 0$, $\alpha > 0$ *and* $\alpha^2 - \lambda\alpha - \mu > 0$, *then Eq.* (4) *has a unique convex (concave) solution* $f(x) = \alpha x + f_1(x)$, *where* $f_1 \in \Phi^{cv}(L)(\Phi^{cc}(L))$.

Proof. It suffices to prove the case of convex, the proof for concave is similar. Firstly, we give some useful facts by the convexity of f_1. Note that for every $x, y \in \mathbb{R}$ and $t \in [0, 1]$,

$$\begin{aligned} f_1(tx + (1-t)y - 1) &= f_1(t(x-1) + (1-t)(y-1)) \\ &\leq tf_1(x-1) + (1-t)f_1(y-1), \end{aligned} \tag{8}$$

it follows that

$$t(f_1(x-1) + \alpha x - \alpha) + (1-t)(f_1(y-1) + \alpha y - \alpha)$$

$$= t\alpha x + \alpha(1-t)y - \alpha + tf_1(x-1) + (1-t)f_1(y-1)$$

$$\geq t\alpha x + \alpha(1-t)y - \alpha + f_1(t(x-1) + (1-t)(y-1))$$

$$= t\alpha x + \alpha(1-t)y - \alpha + f_1(tx + (1-t)y - 1).$$

Thus,

$$\begin{aligned}
&f_1(t\alpha x + \alpha(1-t)y - \alpha + f_1(tx + (1-t)y - 1)\\
&\le f_1(t(f_1(x-1) + \alpha x - \alpha) + (1-t)(f_1(y-1) + \alpha y - \alpha)) \qquad (9)\\
&\le tf_1(f_1(x-1) + \alpha x - \alpha) + (1-t)f_1(f_1(y-1) + \alpha y - \alpha)
\end{aligned}$$

since f_1 is increasing. From (8)-(9), for the function $\mathcal{F}f_1$ defined in (7) we have

$$\begin{aligned}
&\mathcal{F}f_1(tx + (1-t)y)\\
&= \tfrac{\alpha^2-\lambda\alpha-\mu}{\lambda}(tx + (1-t)y) + \tfrac{\alpha}{\lambda}f_1(tx + (1-t)y - 1)\\
&\quad + \tfrac{1}{\lambda}f_1(\alpha(tx + (1-t)y) - \alpha + f_1(tx + (1-t)y - 1) - 1) - \tfrac{\alpha^2+\alpha-\mu}{\lambda}\\
&\le t\mathcal{F}f_1(x) + (1-t)\mathcal{F}f_1(y),
\end{aligned}$$

which implies that $\mathcal{F}f_1$ is convex and increasing if $\lambda > 0$, $\alpha > 0$, $\alpha^2-\lambda\alpha-\mu > 0$. Therefore, $\mathcal{F}$ maps $\Phi^{cv}(L)(\Phi^{cc}(L))$ into itself, and the rest proof is same as that of Theorem 1. $\square$

Example 3 *Take* $\alpha = 0.8$, $\lambda = 2$, $\mu = -0.96$ *and* $L = 0.09$.

It is easy to see that

$$|\frac{\alpha^2-\lambda\alpha-\mu}{\lambda}| + 2|\frac{\alpha}{\lambda}|L + \frac{1}{|\lambda|}L^2 < 0.077 < 0.09, \quad |\frac{\alpha}{\lambda}| + \frac{1}{|\lambda|} + L = 0.99 < 1,$$

and then the conditions in Theorem 2 are satisfied. Therefore, Eq. (4) has a unique solution $f(x) = \frac{4}{5}x + f_1(x)$ where $f_1 \in \Phi^{cv}(L)$.

3 Unbounded solutions with $\varepsilon(a)$

In this section we discuss the continuous solutions of Eq. (2) with an extra item, that is,

$$f^2(x) = \lambda f(x + a) + \mu x + \varepsilon(a) \qquad (10)$$

for $\lambda \in (0, 1]$, where $\varepsilon : \mathbb{R} \to \mathbb{R}$ is continuous with respect to the parameter a. Clearly, Eq. (2) becomes a second order iterative equation when $a = 0$, which has been extensively studied [4, 6, 9, 13, 14, 15, 16, 18, 19]. In what follows, our aim is to find a suitable continuous function ε in order to give a construction of solutions. Here, we consider Eq. (10) under the hypothesis

$(\mathcal{H})$ $\lambda \in (0,1]$, $\lambda + \mu = 1$ and $\varepsilon(a) := (4-3\lambda)a$.

The following Theorems are our main results in this section.

Theorem 3 *Under hypothesis $(\mathcal{H})$, for $a < 0$ and each $x_0 \in (-\infty, +\infty)$, Eq.(10) has a continuous solution in $(-\infty, x_0]$. Moreover, the solution depends on arbitrarily chosen orientation-preserving homeomorphism $f_1 : [x_1, x_0] \to [x_2, x_1]$, where $x_1 := x_0 + 2a$ and $x_2 := \lambda(x_1 + a) + \mu x_0 + \varepsilon(a)$.*

Theorem 4 *Under hypothesis $(\mathcal{H})$, for $a > 0$ and each $x_0 \in (-\infty, +\infty)$, Eq. (10) has a continuous solution in $[x_0, +\infty)$. Moreover, the solution depends on arbitrarily chosen orientation-preserving homeomorphism $f_1 : [x_0, x_1] \to [x_1, x_2]$, where $x_1 := x_0 + 2a$ and $x_2 := \lambda(x_1 + a) + \mu x_0 + \varepsilon(a)$.*

We only give a proof to Theorem 3. Theorem 4 can be proved similarly.

Proof. Let $x_1 := x_0 + 2a$. Substituting x_1 and $\varepsilon(a)$ defined in $(\mathcal{H})$ into $x_2 := \lambda(x_1 + a) + \mu x_0 + \varepsilon(a)$, we obtain $x_2 = x_1 + 2a$. This gives the fact that $x_2 < x_1 < x_0$. Then we extend these points to a sequence $(x_n)_{n=2}^{\infty}$ by the recurrence formula

$$x_n = \lambda(x_{n-1} + a) + \mu x_{n-2} + \varepsilon(a). \tag{11}$$

We assert that $(x_n)_{n=2}^{\infty}$ is a strictly decreasing sequence in $(-\infty, x_0]$ satisfying

$$x_n = x_{n-1} + 2a. \tag{12}$$

In fact, (12) is trivial for $n = 2$. Suppose that this claim holds for all positive integers $n \le k$, where $k \ge 2$ is a certain integer. Then

$$x_{k+1} = \lambda(x_k + a) + \mu x_{k-1} + \varepsilon(a) = \lambda(x_k + a) + \mu(x_k - 2a) + (4-3\lambda)a = x_k + 2a$$

by the fact $\lambda + \mu = 1$. Thus, (12) is proved by induction.

The monotonicity of $(x_n)_{n=2}^{\infty}$ implies that this sequence is divergent, i.e.,

$$(-\infty, x_0] = \bigcup_{n=1}^{\infty} I_n, \tag{13}$$

where $I_n := [x_n, x_{n-1}]$.

Choose an orientation-preserving homeomorphism $f_1 : I_1 \to I_2$ arbitrarily such that

$$f_1(x_0) = x_1, \quad f_1(x_0 + a) = x_1 + a, \quad f_1(x_1) = x_2.$$

Then for all $n \geq 2$ we recursively define

$$f_n(x) := \lambda f_{n-1}(f_{n-1}^{-1}(x) + a) + \mu f_{n-1}^{-1}(x) + \varepsilon(a), \quad \forall x \in I_n. \tag{14}$$

We assert that for each positive integer $n \geq 2$ mapping $f_n : I_n \to I_{n+1}$ is an orientation-preserving homeomorphism fulfilling

$$f_n(x_{n-1}) = x_n, \quad f_n(x_{n-1} + a) = x_n + a, \quad f_n(x_n) = x_{n+1}. \tag{15}$$

It is trivial for $n = 2$. Actually,

$$f_2(x_1) = \lambda f_1(f_1^{-1}(x_1) + a) + \mu f_1^{-1}(x_1) + \varepsilon(a) = \lambda f_1(x_0 + a) + \mu x_0 + \varepsilon(a) = x_2,$$

$$\begin{aligned} f_2(x_1 + a) &= \lambda f_1(f_1^{-1}(x_1 + a) + a) + \mu f_1^{-1}(x_1 + a) + \varepsilon(a) \\ &= \lambda f_1(x_0 + 2a) + \mu(x_0 + a) + \varepsilon(a) \\ &= \lambda x_2 + \mu(x_0 + a) + (4 - 3\lambda)a \\ &= x_2 + a, \end{aligned}$$

and

$$f_2(x_2) = \lambda f_1(f_1^{-1}(x_2) + a) + \mu f_1^{-1}(x_2) + \varepsilon(a) = \lambda f_1(x_1 + a) + \mu x_1 + \varepsilon(a) = x_3.$$

Suppose that the assertion is true for all integers $n \leq k$, where $k \geq 2$ is a certain integer. It is easy to see that f_k is an orientation-preserving homeomorphism. It has an inverse f_k^{-1}, which is strictly increasing defined on I_k such that $f_k^{-1}(x_k) = x_{k-1}$ and $f_k^{-1}(x_{k+1}) = x_k$. Thus, by (14) $f_{k+1}(x)$ is well defined on I_{k+1}. Furthermore, by (15) we have

$$\begin{aligned} f_{k+1}(x_k) &= \lambda f_k(f_k^{-1}(x_k) + a) + \mu f_k^{-1}(x_k) + \varepsilon(a) \\ &= \lambda(x_k + a) + \mu x_{k-1} + \varepsilon(a) = x_{k+1}. \end{aligned}$$

Similarly, we also get $f_{k+1}(x_{k+1}) = x_{k+2}$. Thus,

$$\begin{aligned} f_{k+1}(x_k + a) &= \lambda f_k(f_k^{-1}(x_k + a) + a) + \mu f_k^{-1}(x_k + a) + \varepsilon(a) \\ &= \lambda f_k(x_{k-1} + 2a) + \mu(x_{k-1} + a) + \varepsilon(a) \\ &= \lambda x_{k+1} + \mu(x_{k-1} + a) + (4 - 3\lambda)a \\ &= x_{k+1} + a \end{aligned}$$

and (15) is proved by induction.

Finally, for arbitrary $x \in (-\infty, x_0]$ by (13) there exists $n \in \mathbb{N}$ such that $x \in I_n$, where $\mathbb{N}$ denotes the set of all positive integers. Define

$$f(x) := f_n(x). \tag{16}$$

Therefore, (14) and (16) lead to the fact

$$f^2(x) = f_{n+1} \circ f_n(x) = \lambda f_n(x+a) + \mu x = \lambda f(x+a) + \mu x,$$

which implies that the function defined by (16) is a continuous solution of Eq. (10) on $(-\infty, x_0]$. This completes the proof. $\square$

Theorems 3-4 present a manner of construction for the continuous and unbounded solutions of Eq. (10) with $\varepsilon(a) = (4-3\lambda)a$. Clearly, this construction is not unique, which depends on the chosen of $\varepsilon(a)$. $\square$

Example 4 *Consider the equation*

$$f^2(x) = \frac{2}{3} f(x+a) + \frac{1}{3} x + 2a,$$

where $\lambda = \frac{2}{3}$, $\mu = \frac{1}{3}$ *and* $\varepsilon(a) = 2a$.

All conditions in Theorems 3-4 can be verified respectively and therefore the equation has unbounded solutions on $(-\infty, +\infty)$.

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Maximal perpendicularity in certain Abelian groups

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Abstract. We define perpendicularity in an Abelian group G as a binary relation satisfying certain five axioms. Such a relation is maximal if it is not a subrelation of any other perpendicularity in G. A motivation for the study is that the poset $(\mathcal{P}, \subseteq)$ of all perpendicularities in G is a lattice if G has a unique maximal perpendicularity, and only a meet-semilattice if not. We study the cardinality of the set of maximal perpendicularities and, on the other hand, conditions on the existence of a unique maximal perpendicularity in the following cases: $\mathsf{G} \cong \mathbb{Z}^n$, G is finite, G is finitely generated, and $\mathsf{G} = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$. A few such conditions are found and a few conjectured. In studying $\mathbb{R}^n$, we encounter perpendicularity in a vector space.

1 Introduction

Over the years, the concept of "perpendicular" has been considered axiomatically from several different perspectives. Perhaps the most well-known axiomatic description of perpendicularity is presented in the classical textbook

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[1] by Bachmann. This approach is designed for the construction of plane geometry and it is based on studying reflections in the metric plane which is a notion to serve as a common basis of Euclidean, hyperbolic and elliptic planes.

Davis [2, 3] studied rings and Abelian groups with orthogonality relations. In his approach, the aim of defining an orthogonality relation on an Abelian group was to generalize the concept of a disjointness relation on a linear space introduced earlier by Veksler [7].

A more recent axiomatization of perpendicularity and parallelism is given in [4]. This axiom system was originally constructed for educational purposes and it is applicable enough for the examination of the geometry of perpendicular and parallel lines in the Euclidean plane, and certain other non-trivial planar or numeric models, too.

The present approach to defining algebraic perpendicularity was originally laid down in [5]; this article is a sequel to that. Our definition is based on the idea of describing the additive properties of the elements of an inner product space for which the inner product is zero in terms of the binary operation of an Abelian group.

Following the notation of [5], let $\mathsf{G} = (\mathsf{G}, +)$ be an Abelian group, $\mathsf{G} \neq \{0\}$, and let $\perp$ be a *perpendicularity* in G, that is, a binary relation satisfying

(A1) $\forall a \in \mathsf{G} : \exists b \in \mathsf{G} : a \perp b$,

(A2) $\forall a \in \mathsf{G} \setminus \{0\} : a \not\perp a$,

(A3) $\forall a, b \in \mathsf{G} : a \perp b \Rightarrow b \perp a$,

(A4) $\forall a, b, c \in \mathsf{G} : a \perp b \wedge a \perp c \Rightarrow a \perp (b + c)$,

(A5) $\forall a, b \in \mathsf{G} : a \perp b \Rightarrow a \perp -b$.

The *trivial perpendicularity*

$$a \perp b \iff a = 0 \vee b = 0$$

always exists. A perpendicularity $\perp$ is *minimal* if it is not a superrelation of any other perpendicularity in G. This clearly happens if and only if $\perp$ is trivial; hence, minimal perpendicularity is always unique. Similarly, a perpendicularity is *maximal* if it is not a subrelation of any other perpendicularity in G.

A few results on minimal and maximal perpendicularities follow easily.

Proposition 1 *If* G *is cyclic, then it has a unique maximal perpendicularity. If* G *is cyclic and infinite, then it has only the trivial perpendicularity.*

Proof. See [5, Theorem 14] and [5, Example 8]. □

Maximal perpendicularity is not necessarily unique even if G is finite. For example [5, Example 7], the Klein four group has three nontrivial perpendicularities, all of them maximal.

Proposition 2 *A maximal perpendicularity always exists.*

Proof. If $\perp_1 \subseteq \perp_2 \subseteq \ldots$ are perpendicularities in G, then $\cup_{i=1}^{\infty} \perp_i$ is clearly a perpendicularity in G. So, the claim follows from Zorn's lemma. □

Let $(\mathcal{P}, \subseteq)$ be the poset (partially ordered set) of all perpendicularities in G. (In fact, every nonempty family of sets is a poset under subset relation.)

Proposition 3 *A perpendicularity in G is maximal if and only if it is a maximal element of $\mathcal{P}$. There is a unique maximal perpendicularity in G if and only if there is a largest element in $\mathcal{P}$. The trivial perpendicularity is the unique minimal perpendicularity of G, in other words, the smallest element of $\mathcal{P}$.*

Proof. Easy and omitted. □

A motivation for the present study is that $\mathcal{P}$ is a lattice if G has a unique maximal perpendicularity, and only a meet-semilattice if not. Below we survey the uniqueness of maximal perpendicularity in the following cases: $G \cong \mathbb{Z}^n$ (Section 2), G is finite (Section 3), G is finitely generated (Section 4), and $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \cong (\mathbb{Q}_+, \cdot)$ (Sections 5 and 6). In addition to solving the question about the uniqueness in certain cases, we shall conjecture a few equivalent conditions for the existence of a unique maximal perpendicularity. We complete our paper by regarding $\mathbb{R}^n$ both as an additive group and as a vector space.

2 $G \cong \mathbb{Z}^n$, $n > 1$

If $G \cong \mathbb{Z}$, then it has only the trivial perpendicularity by Proposition 1. The case of $G \cong \mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n copies, $n > 1$) is hence more interesting.

Let us choose $g_1, \ldots, g_n \in G$ such that

$$\begin{aligned} g_1 &= (1, 0, 0, 0, \ldots, 0), \\ g_2 &= (\gamma_{21}, 1, 0, 0, \ldots, 0), \\ g_3 &= (\gamma_{31}, \gamma_{32}, 1, 0, \ldots, 0), \\ &\vdots \\ g_n &= (\gamma_{n1}, \gamma_{n2}, \ldots, \gamma_{n.n-1}, 1), \end{aligned} \tag{1}$$

where the γ_{ij}'s are integers. Denote by $\langle\cdot\rangle$ the generated subgroup.

Lemma 1 *If* $G \cong \mathbb{Z}^n$ *and* $g_1, \ldots, g_n$ *are as in* (1)*, then*

$$G = \langle g_1 \rangle \oplus \cdots \oplus \langle g_n \rangle. \tag{2}$$

Proof. For any $x \in G$, there obviously are unique $\xi_1, \ldots, \xi_n \in \mathbb{Z}$ satisfying

$$x = \xi_1 g_1 + \cdots + \xi_n g_n.$$

□

Let $g_1, \ldots, g_n$, $n > 1$, be as above. Also choose $g_1', \ldots, g_n' \in G$ as in (1) such that $g_i' \neq g_i$ for at least one $i \in N = \{1, \ldots, n\}$. So, there is $m \in N$ with

$$g_1 = g_1', \ldots, g_{m-1} = g_{m-1}', \, g_m \neq g_m'. \tag{3}$$

Let $a, b \in G$. Then, by Lemma 1,

$$a = a_1 + \cdots + a_n = a_1' + \cdots + a_n', \quad b = b_1 + \cdots + b_n = b_1' + \cdots + b_n', \tag{4}$$

where $a_i, b_i \in \langle g_i \rangle$ and $a_i', b_i' \in \langle g_i' \rangle$ for all $i \in N$.

Define now the relations $\perp_0$ and $\perp_0'$ by

$$\begin{aligned} a \perp_0 b &\iff \forall i \in N : a_i = 0 \vee b_i = 0, \\ a \perp_0' b &\iff \forall i \in N : a_i' = 0 \vee b_i' = 0. \end{aligned} \tag{5}$$

These relations are clearly perpendicularities in G.

Lemma 2 *Let* $\perp_0$ *and* $\perp_0'$ *be as in* (5)*. A maximal perpendicularity* $\perp_{\max}$ *in* $G \cong \mathbb{Z}^n$*,* $n > 1$*, cannot contain both of them.*

Proof. We proceed by contradiction. Suppose that

$$\perp_{\max} \supseteq \perp_0 \cup \perp_0' . \tag{6}$$

We have $g_m \perp_0 g_1, \ldots, g_{m-1}$ and $g_m' \perp_0' g_1', \ldots, g_{m-1}'$ implying that $g_m' \perp_0' g_1, \ldots, g_{m-1}$ by (3). Therefore

$$g_m, g_m' \perp_{\max} g_1, \ldots, g_{m-1}$$

by (6). Now, applying (A3), (A4) and (A5) yields that

$$(g_m - g_m') \perp_{\max} (\xi_1 g_1 + \cdots + \xi_{m-1} g_{m-1})$$

for all $\xi_1, \ldots, \xi_{m-1} \in \mathbb{Z}$.

But $d = g_m - g_m' = (\delta_1, \ldots, \delta_n)$ has $\delta_m = \cdots = \delta_n = 0$, which implies that there are $\xi_1, \ldots, \xi_{m-1} \in \mathbb{Z}$ such that $d = \xi_1 g_1 + \cdots + \xi_{m-1} g_{m-1}$. So, $d \perp_{\max} d$ violating (A2) because $d \neq 0$ by (3). □

Theorem 1 *There are infinitely many maximal perpendicularities in* $G \cong \mathbb{Z}^n$*,* $n > 1$*.*

Proof. There are infinitely many choices of the g_i's in (1). Different choices give different $\perp_0$'s in (5). Hence, the claim follows from Lemma 2. □

Is $\perp_0$ defined by (5) maximal? The answer is negative. Namely, let $a, b \in G$ and write them as

$$a = \alpha_1 g_1 + \cdots + \alpha_n g_n, \quad b = \beta_1 g_1 + \cdots + \beta_n g_n,$$

where the α_i's and β_i's are integers. Define $\perp_1$ by

$$a \perp_1 b \iff \alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 0. \tag{7}$$

Obviously $\perp_1$ is a perpendicularity and $\perp_0$ is its proper subset. But then, is $\perp_1$ maximal? This question remains open, yet we conjecture as follows.

Conjecture 1 *A perpendicularity in* $G \cong \mathbb{Z}^n$*,* $n > 1$*, is maximal if and only if it is of the form* (7)*.*

We encounter another open question concerning the cardinality of the set S of maximal perpendicularities in $G \cong \mathbb{Z}^n$, $n > 1$. Denoting by $|\cdot|$ the cardinality, Theorem 1 yields that $|S| \geq \aleph_0$. On the other hand, $|\mathbb{Z}^n \times \mathbb{Z}^n| = \aleph_0$, so, the cardinality of the set of all binary relations in $G \cong \mathbb{Z}^n$ is $2^{\aleph_0}$. Consequently, $|S| \leq 2^{\aleph_0}$.

But which of these inequalities is equality? The following proposition tells what we already know.

Proposition 4 *If Conjecture 1 is true, then the set of maximal perpendicularities in* $G \cong \mathbb{Z}^n$*,* $n > 1$*, has cardinality* $\aleph_0$*.*

Proof. A maximal perpendicularity is of the form (5) by the conjecture. Because there are countably infinite choices of each g_i, $i > 1$, in (1), there are also countably infinite choices of the sequence $(g_1, \ldots, g_n)$. □

3 Finite G

In this section, we assume that G is also finite (in addition to being Abelian). If G is cyclic, then it has a unique maximal perpendicularity by Proposition 1. So, in the rest of this section, suppose that G be noncyclic if not mentioned otherwise. We begin by describing its structure.

Theorem 2 *If* G *is noncyclic and finite, then it has cyclic subgroups* $H_1, \ldots, H_r$*,* $r > 1$*, of prime power order such that*

$$G = H_1 \oplus \cdots \oplus H_r. \tag{8}$$

These orders are unique. All decompositions (8) have the same number of summands of each order.

Proof. See [6, p. 394, Theorem 1]. □

Let $a, b \in G$, and let $H_1, \ldots, H_r$ be as in (8). Analogously to (4),

$$a = a_1 + \cdots + a_r, \quad b = b_1 + \cdots + b_r,$$

where $a_i, b_i \in H_i$ for all $i = 1, \ldots, r$. Similarly as in (5), we now define

$$a \perp_0 b \iff \forall i \in \{1, \ldots, r\} : a_i = 0 \vee b_i = 0. \tag{9}$$

Further, if $\emptyset \neq A, B \subseteq G$, we write $A \perp B$ denoting that $x \perp y$ for all $x \in A$, $y \in B$.

Lemma 3 *Let* $\perp$ *be a perpendicularity in* G*, and let* $a, b \in G$*. If* $a \perp b$*, then* $\langle a \rangle \perp \langle b \rangle$ *and* $\langle a \rangle \cap \langle b \rangle = \{0\}$*.*

Proof. Let $\xi, \eta \in \mathbb{Z}$. Then $a \perp \eta b$ by (A4) and (A5). Further, applying also (A3), we get $\xi a \perp \eta b$. This proves the first claim. If $z \in \langle a \rangle \cap \langle b \rangle$, then $z = \xi a = \eta b$ for some $\xi, \eta \in \mathbb{Z}$. Now, the first claim implies that $z \perp z$; hence, $z = 0$ by (A1) verifying the second claim. □

Theorem 3 *Let* G *be noncyclic and finite. If* $|G|$ *is square-free, then* G *has a unique maximal perpendicularity which, in fact, is* $\perp_0$ *defined in (9).*

Proof. Let $\perp$ be a perpendicularity in G. We claim that $\perp \subseteq \perp_0$. We can omit the trivial perpendicularity; so, we suppose that $0 \neq x, y \in G$ and $x \perp y$.

By Theorem 2 and square-freeness, G has cyclic subgroups $H_1, \ldots, H_r$ with prime orders $p_1, \ldots, p_r$, $r > 1$, respectively, such that

$$G = H_1 \oplus \cdots \oplus H_r.$$

Clearly,

$$H_i = \{x \in G \mid |x| = p_i\}, \quad i = 1, \ldots, r,$$

where $|\cdot|$ denotes the order. Hence, this decomposition is unique (up to the ordering). Therefore, if H is a subgroup of G, then

$$H = H_{t_1} \oplus \cdots \oplus H_{t_s}$$

for certain indices $t_1, \dots, t_s \in \{1, \dots, r\}$. In particular, there are indices $i_1, \dots, i_k$ and $j_1, \dots, j_l$ such that

$$\langle x \rangle = H_{i_1} \oplus \cdots \oplus H_{i_k}, \quad \langle y \rangle = H_{j_1} \oplus \cdots \oplus H_{j_l}.$$

Since $\langle x \rangle \cap \langle y \rangle = \{0\}$ by Lemma 3, we have $H_{i_u} \neq H_{j_v}$ for all u, v. Therefore, $x \perp_0 y$, and the claim follows. □

We conjecture that also the converse holds.

Conjecture 2 *Let* G *be as in Theorem* 2. *The following conditions are equivalent:*

(a) G *has a unique maximal perpendicularity,*

(b) $|G|$ *is square-free,*

(c) (8) *is unique* (*up to the ordering of the* H_i*'s*).

Theorem 3 states that (b)⇒(a). The following proposition states that (a)⇒(c). The part (c)⇒(b) remains open.

Proposition 5 *Let* G *be as above. If* G *has a unique maximal perpendicularity, then* (8) *is unique.*

Proof. Contrary to the uniqueness of (8), we suppose that there are decompositions

$$G = H_1 \oplus \cdots \oplus H_r = H_1' \oplus \cdots \oplus H_r'$$

such that $\{H_1, \dots, H_r\} \neq \{H_1', \dots, H_r'\}$. Let $H_i = \langle g_i \rangle$ and $H_i' = \langle g_i' \rangle$, $i = 1, \dots, r$. We define $\perp_0$ as we did in (9) and $\perp_0'$ in an analogous manner applying $G = H_1' \oplus \cdots \oplus H_r'$. As in the proof of Lemma 2, we can show that no maximal perpendicularity contains both $\perp_0$ and $\perp_0'$. (In this lemma, $g_1 = g_1'$, but without any role in the proof.) The uniqueness of maximal perpendicularity is thus violated. □

4 Finitely generated G

Next, we assume that G is finitely generated. In Proposition 1 and Theorem 2, we already studied the cases G is cyclic and finite, respectively. Therefore, let G now be noncyclic and infinite. Its structure is described in Theorem 4 which follows immediately from [6, p. 411, Theorem 3].

Theorem 4 *If G is noncyclic and infinite but finitely generated, then it has cyclic subgroups $H_1, \ldots, H_r$ of prime power order and a subgroup $H_0 \cong \mathbb{Z}^n$, $n \geq 1$, such that*

$$G = H_0 \oplus H_1 \oplus \cdots \oplus H_r = H_0 \oplus K. \tag{10}$$

These orders are unique. All decompositions (10) have the same number of summands of each order.

Applying our previous results, it is now easy to study maximal perpendicularities in G.

Theorem 5 *Let G be as in Theorem 4. If $n > 1$, then G has infinitely many maximal perpendicularities.*

Proof. Decompose H_0 as in (2) and define perpendicularities $\perp_0$ and $\perp_0'$ in H_0 as in (5). By Lemma 2, a maximal perpendicularity in H_0 cannot contain both of them. Therefore, regarding them also as relations in G, a maximal perpendicularity in G cannot either contain both of them. Because there are infinitely many $\perp_0$'s, the claim follows. □

The proof of the next theorem is very similar to that of Theorem 3. Actually, the proof applies also when one subgroup is infinite (but cyclic).

Theorem 6 *Let G be as above. If $n = 1$ and $|K|$ is square-free, then G has a unique maximal perpendicularity.*

By the similarity between the above results and those in Section 3, we present an analogy to Conjecture 2.

Conjecture 3 *Let G be as above. The following conditions are equivalent:*

(a) *G has a unique maximal perpendicularity,*

(b) *$n = 1$ and $|K|$ is square-free,*

(c) (10) *is unique (up to the ordering of the* H_i*'s).*

Applying an analogous argument as in the proof of Proposition 5, we also get the following proposition.

Proposition 6 *Let* G *be as above. If* G *has a unique maximal perpendicularity, then* (10) *is unique.*

5 $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$

Now, let $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ (i.e., the set of infinite integer sequences with only finitely many nonzero terms). We begin the examination of this case by recording a result corresponding to Theorem 1.

Theorem 7 *There are infinitely many maximal perpendicularities in* $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ *.*

Proof. Analogously to (1), choose $g_1, g_2, \dots \in G$ such that $g_1 = (1, 0, 0, \dots)$ and

$$g_i = (\gamma_{i1}, \dots, \gamma_{i,i-1}, 1, 0, 0, \dots), \quad i = 1, 2, \dots,$$

and $g_1', g_2', \dots$ similarly. A simple modification of the proof of Theorem 1 applies. □

Let $a, b \in G$. Write them as

$$a = \alpha_1 g_1 + \alpha_2 g_2 + \dots, \quad b = \beta_1 g_1 + \beta_2 g_2 + \dots .$$

Analogously to (7), we define

$$a \perp_1 b \iff \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots = 0. \tag{11}$$

(The sum is finite, because only finitely many α_i's and β_i's are nonzero.) Analogously to Conjecture 1, we state as follows.

Conjecture 4 *Let* G *be as in Theorem 7. A perpendicularity in* G *is maximal if and only if it is of the form* (11).

The question about the cardinality of the set of all perpendicularities in $G \cong \mathbb{Z}^n$, $n > 1$, remained open in Proposition 4 since the answer depends on Conjecture 1. However, we can solve this question in the case of $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$.

Proposition 7 *Let* G *be as above. The set of its maximal perpendicularities has cardinality* $2^{\aleph_0}$*.*

Proof. Let S denote the set of maximal perpendicularities in G. The set of all possible sequences $(g_1, g_2, \dots)$ has cardinality $2^{\aleph_0}$. Therefore, $|S| \geq 2^{\aleph_0}$.

On the other hand,

$$G = A_0 \cup A_1 \cup A_2 \cup \cdots,$$

where $A_0 = \{(0, 0, \dots)\}$ and

$$A_i = \{(x_0, x_1, \dots, x_i, 0, 0, \dots) \,|\, x_0, \dots, x_i \in \mathbb{Z}, x_i \neq 0\}, \quad i = 1, 2, \dots .$$

Clearly, $|A_1| = |A_2| = \cdots = \aleph_0$. Hence, $|G| = \aleph_0$ and, further, $|G \times G| = \aleph_0$. This implies that the cardinality of the set of all binary relations in G is $2^{\aleph_0}$. Consequently, $|S| \leq 2^{\aleph_0}$ and the claim follows. □

6 $G = (\mathbb{Q}_+, \cdot)$

As a sequel to the previous section, we consider the multiplicative group $\mathbb{Q}_+$ of the set of positive rational numbers. Let $\mathbb{P}$ denote the set of primes. Every $x \in \mathbb{Q}_+$ can be uniquely expressed as

$$x = \prod_{p \in \mathbb{P}} p^{\nu(p,x)}, \tag{12}$$

where $\nu(p, x) \in \mathbb{Z}$ for each $p \in \mathbb{P}$, and only finitely many of them are nonzero. For example,

$$\nu(2, 45) = 0,\ \nu(3, 45) = 2,\ \nu(5, 45) = 1,\ \nu(7, 45) = \nu(11, 45) = \cdots = 0,$$
$$\nu(2, 1) = \nu(3, 1) = \cdots = 0,$$
$$\nu(2, \tfrac{8}{25}) = 3,\ \nu(3, \tfrac{8}{25}) = 0,\ \nu(5, \tfrac{8}{25}) = -2,\ \nu(7, \tfrac{8}{25}) = \nu(11, \tfrac{8}{25}) = \cdots = 0.$$

If the nonzero values of $\nu(p, x)$ are $\nu(p_1, x) = \nu_1, \dots, \nu(p_k, x) = \nu_k$, then (12) reads

$$x = p_1^{\nu_1} \cdots p_k^{\nu_k}.$$

Define the mapping $F : \mathbb{Q}_+ \to \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ by $F(x) = (\nu(2, x), \nu(3, x), \dots)$. For example

$$F(45) = F(2^0 \cdot 3^2 \cdot 5^1 \cdot 7^0 \cdot \cdots) = (0, 2, 1, 0, 0, \dots),$$
$$F(1) = F(2^0 \cdot 3^0 \cdot \cdots) = (0, 0, \dots),$$
$$F(\tfrac{8}{25}) = F(2^3 \cdot 3^0 \cdot 5^{-2} \cdot 7^0 \cdot \cdots) = (3, 0, -2, 0, 0, \dots).$$

It is easy to see that F is an isomorphism. Thus all results of Section 5 are valid in $\mathbb{Q}_+$; see also [5, Section 5].

7 $G = (\mathbb{R}^n, +)$, $n > 1$

Finally, let us consider $G = (\mathbb{R}^n, +)$, $n > 1$. If $a = (\alpha_1, \dots, \alpha_n), b = (\beta_1, \dots, \beta_n) \in G$, we can define, analogously to (5), the perpendicularity $\perp_0$:

$$a \perp_0 b \iff \forall i \in N : \alpha_i = 0 \vee \beta_i = 0.$$

However, we cannot define, analogously to (7), the perpendicularity $\perp_1$:

$$a \perp_1 b \iff \alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 0,$$

because we are allowed to use only addition (and multiplication by an integer) in this group. Therefore it is reasonable to regard $\mathbb{R}^n$ as a vector space rather than as a group, but perpendicularity in a vector space is beyond our scope. However, we take a small step to it and, more generally, to perpendicularity in a module.

Let $M \neq \{0\}$ be a module over a ring R. We say that a relation $\perp$ in M is a perpendicularity in M if it satisfies (A1)–(A5) and

(A6) $\forall a, b \in M, \gamma \in R : a \perp b \Rightarrow a \perp \gamma b$.

Since an Abelian group G is a module over $\mathbb{Z}$, a perpendicularity in G is also a perpendicularity in this module.

We define in the vector space $V = \mathbb{R}^n$, $n > 1$, the Euclidean inner product

$$\langle a, b \rangle = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n.$$

The relation $\perp_1$:

$$a \perp_1 b \iff \langle a, b \rangle = 0$$

is clearly a perpendicularity in V. We show that it is maximal.

We proceed by contradiction. Suppose that $\perp_1$ is a proper subset of a perpendicularity $\perp$. Then there are $a, b \in V$ with $a \perp b$ and $a \not\perp_1 b$. Since the orthogonal complement $\{a\}^{\perp_1}$ is an $(n-1)$-dimensional subspace of V, it is spanned by a linearly independent set $S = \{c_1, \dots, c_{n-1}\}$. Here $a \perp_1 c_1, \dots, c_{n-1}$ and, because $\perp_1 \subset \perp$, also $a \perp c_1, \dots, c_{n-1}$. Since $b \notin \{a\}^{\perp_1} = \operatorname{span} S$, the set $S_b = S \cup \{b\}$ is linearly independent; so it spans V. Now $a \perp c_1, \dots, c_{n-1}, b$;

hence $a \in (S_b)^{\perp} = V^{\perp} = \{0\}$, which implies that $a = 0$. But then $a \perp_1 b$, contradicting $a \not\perp_1 b$.

Let Q be a real symmetric positive definite $n \times n$ matrix. Define in V the inner product

$$[a, b] = \langle Qa, b\rangle.$$

The above proof applies also to the perpendicularity $\perp'$:

$$a \perp' b \iff [a, b] = 0;$$

so $\perp'$ is maximal. Conversely, we conjecture that all maximal perpendicularities in V are obtained in this way.

8 Summary

For $G \cong \mathbb{Z}^n$, G finite, G finitely generated, and $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, we were able to only partially answer the question how many maximal perpendicularities G has. Nevertheless, these results may assist us in characterizing all maximal perpendicularities in the case $G \cong \mathbb{Z}^n$ or $G = \mathbb{Z}\oplus\mathbb{Z}\oplus\cdots$ (Conjectures 1 and 4) or, in the case G is finite or finitely generated, to typify those Abelian groups that have a unique maximal perpendicularity (Conjectures 2 and 3). However, more effort is needed to know whether our suppositions were correct or not. In studying $\mathbb{R}^n$, we encountered perpendicularity in a vector space and, more generally, in a module. This topic is interesting for further research.

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Polyphase matrix characterization of framelets on local fields of positive characteristic

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Abstract. An important tool for the construction of framelets on local fields of positive characteristic using unitary extension principle was presented by Shah and Debnath [Tight wavelet frames on local fields, Analysis, 33 (2013), 293-307]. In this article, we continue the study of framelets on local fields and present a polyphase matrix characterization of framelets generated by the extension principle.

1 Introduction

Along with the study of wavelet bases, there had been a continuing research effort in the study of tight wavelet frames (framelets) and have gained considerable popularity in recent times, primarily due to their substantiated applications in diverse and widespread fields of science and engineering. A framelet is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system. By sacrificing orthonormality and allowing redundancy, the framelets become much easier to construct than the orthonormal wavelets. The main tool for construction and characterization of wavelet frames are

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the several extension principles, the unitary extension principle and oblique extension principle as well as their generalized versions, the mixed unitary extension principle and the mixed oblique extension principle. They provide sufficient conditions for constructing tight and dual wavelet frames for any given refinable function which generates a multiresolution analysis (MRA). These essential methods were firstly introduced by Ron and Shen in [8] and in the fundamental work of Daubechies et al. [2] for scalar refinable functions. The resulting tight wavelet frames are based on a multiresolution analysis, and the generators are often called *framelets*. To mention only a few references on tight wavelet frames, the reader is referred to [3], [5]-[7] and many references therein.

A field K equipped with a topology is called a local field if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different.

The local field K is a natural model for the structure of wavelet frame systems, as well as a domain upon which one can construct wavelet basis functions. There is a substantial body of work that has been concerned with the construction of wavelets on local fields or more generally on local fields of positive characteristic. For example, L. Benedetto and J. Benedetto [1] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Jiang et al. [4] pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(K)$. Subsequently, tight wavelet frames on local fields of positive characteristic were constructed by Shah and Debnath [15] using extension principles. More precisely, they provide a sufficient condition for finite number of functions $\{\psi_1, \psi_2, \ldots, \psi_L\}$ to form a tight wavelet frame for $L^2(K)$ and established a complete characterization of tight wavelet frames on local fields by virtue of the modulation matrix $\mathcal{M}(\xi) = \left\{h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\right\}_{\ell,k=0}^{q-1}$ formed by the framelet symbols $m_\ell(\xi), \ell = 0, 1, \ldots, L$ associated with the scaling function $\phi(x)$ and basic wavelets $\psi_\ell(x), 1 \leq \ell \leq L$. The characterizations of tight wavelet frames on local fields were completely established by Shah and Abdullahb [12] by virtue of two basic equations in the Fourier domain. These studies were continued by Shah and his colleagues in [9]–[14], where they have provided some algorithms for constructing wave packet frames, framelet packets, semi-orthogonal wavelet frames and periodic wavelet frames on local fields of positive characteristic.

Drawing inspiration from the construction of framelets on local fields of positive characteristic, in this article, we firstly provide the polyphase representation of the framelet symbols $m_\ell(\xi), \ell = 0, 1, \ldots, L$ and then, establish a complete characterization of framelets on local fields in terms of the polyphase matrix $\mathcal{P}(\xi) = \left\{f_r^\ell\big(\xi\big)\right\}_{\ell,k=0}^{q-1}$ formed by the polyphase components $f_r^\ell, r = 0, 1, \ldots, q-1$ of the framelet symbols $m_\ell(\xi)$.

The rest of this paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and review some major concepts concerning framelets on local fields. In Section 3, we prove the main result of our paper, shows that a unitary polyphase matrix leads to a tight wavelet frame on local fields of positive characteristic.

2 Preliminaries on local fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. We use the notation of the book by Taibleson [16]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x, y \in K$;
(c) $|x + y| \le \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \le 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [16], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [16]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [16], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

The properties of the Fourier transform on the local field K are quite similar to those of the Fourier analysis on the real line (See Taibleson [16]). In particular, if $f \in L^1(K) \cap L^2(K)$, then $\hat{f} \in L^2(K)$ and $\|\hat{f}\|_2 = \|f\|_2$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1=\zeta_0,\zeta_1,\zeta_2,\dots,\zeta_{c-1}\}\subset\mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1}\cong GF(q)$. For $n\in\mathbb{N}_0$ satisfying

$$0\le n<q,\;\; n=a_0+a_1p+\cdots+a_{c-1}p^{c-1},\;\; 0\le a_k<p,\;\text{ and } k=0,1,\dots,c-1,$$

we define

$$u(n)=(a_0+a_1\zeta_1+\cdots+a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2}$$

Also, for $n=b_0+b_1q+b_2q^2+\cdots+b_sq^s,\; n\in\mathbb{N}_0,\; 0\le b_k<q, k=0,1,2,\dots,s$, we set

$$u(n)=u(b_0)+u(b_1)\mathfrak{p}^{-1}+\cdots+u(b_s)\mathfrak{p}^{-s}. \tag{3}$$

This defines $u(n)$ for all $n\in\mathbb{N}_0$. In general, it is not true that $u(m+n)=u(m)+u(n)$. But, if $r,k\in\mathbb{N}_0$ and $0\le s<q^k$, then $u(rq^k+s)=u(r)\mathfrak{p}^{-k}+u(s)$. Further, it is also easy to verify that $u(n)=0$ if and only if $n=0$ and $\{u(\ell)+u(k):k\in\mathbb{N}_0\}=\{u(k):k\in\mathbb{N}_0\}$ for a fixed $\ell\in\mathbb{N}_0$. Hereafter we use the notation $\chi_n=\chi_{u(n)}$, $n\ge 0$.

Let the local field K be of characteristic $p>0$ and $\zeta_0,\zeta_1,\zeta_2,\dots,\zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu\mathfrak{p}^{-j})=\begin{cases}\exp(2\pi i/p), & \mu=0 \text{ and } j=1,\\ 1, & \mu=1,\dots,c-1 \text{ or } j\neq 1.\end{cases} \tag{4}$$

Since $\bigcup_{j\in\mathbb{Z}}\mathfrak{p}^{-j}\mathfrak{D}=K$, we can regard $\mathfrak{p}^{-1}$ as the dilation and since $\{u(n):n\in\mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\Lambda=\{u(n):n\in\mathbb{N}_0\}$ can be treated as the translation set. Note that Λ is a subgroup of K^+ and unlike the standard wavelet theory on the real line, the translation set is not a group.

For given $\Psi:=\{\psi_1,\dots,\psi_L\}\subset L^2(K)$, define the wavelet system

$$\mathcal{W}(\psi,j,k):=\left\{\psi_{j,k}^{\ell}:1\le\ell\le L, j\in\mathbb{Z}, k\in\mathbb{N}_0\right\} \tag{5}$$

where $\psi_{j,k}^{\ell}=q^{j/2}\psi^{\ell}\big(\mathfrak{p}^{-j}\cdot-u(k)\big)$. The wavelet system $\mathcal{W}(\psi,j,k)$ is called a *framelet system*, if there exist positive numbers $0<A\le B<\infty$ such that

$$A\|f\|_2^2\le\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\left\langle f,\psi_{j,k}^{\ell}\right\rangle\right|^2\le B\|f\|_2^2,\quad\text{for all } f\in L^2(K). \tag{6}$$

The largest A and the smallest B for which (6) holds are called *wavelet frame bounds*. A wavelet frame is a *tight wavelet frame* if A and B are chosen such that

$A = B$ and then generators $\psi_1, \psi_2, \ldots, \psi_L$ are often referred as *tight framelets*. If only the right-hand inequality in (6) holds, then the system $\mathcal{W}(\psi, j, k)$ is called a Bessel sequence.

The construction of framelet systems often starts with the construction of a multiresolution analysis (MRA), which is built on refinable functions. A function $\phi \in L^2(K)$ is called *refinable* if it satisfies a refinement equation:

$$\phi(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} h_k \, \phi\big(\mathfrak{p}^{-1}x - u(k)\big), \tag{7}$$

for some $\{h_k : k \in \mathbb{N}_0\} \in l^2(\mathbb{N}_0)$. In the frequency domain, (7) can be written as

$$\hat{\phi}\big(\xi\big) = m_0(\mathfrak{p}\xi)\hat{\phi}(\mathfrak{p}\xi), \tag{8}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} h_k \, \overline{\chi_k(\xi)}, \tag{9}$$

is an integral periodic function in $L^2(\mathfrak{D})$ and is often called the *refinement symbol* of ϕ.

For a refinable function $\phi \in L^2(K)$ with $\hat{\phi}(0) \neq 0$, let V_0 be the closed shift invariant space generated by $\{\phi(\cdot - u(k) : k \in \mathbb{N}_0\}$ and $V_j = \big\{\phi\big(\mathfrak{p}^{-j} - u(k)\big) : k \in \mathbb{N}_0\big\}, j \in \mathbb{Z}$. Then, it is proved in [4] that the closed subspaces $\big\{V_j : j \in \mathbb{Z}\big\}$ constitutes an MRA for $L^2(K)$. Recall that an MRA is a family of closed subspaces $\big\{V_j : j \in \mathbb{Z}\big\}$ of $L^2(K)$ that satisfies (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$; (ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(K)$ and (iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$. We further assume that

$$\big|\hat{\phi}(0)\big|^2 = 1 \quad \text{and} \quad \lim_{j \to \infty} \Big|\hat{\phi}\big(\mathfrak{p}^{-j}\xi\big)\Big| = 0 \quad \text{for a.e. } \xi \in K. \tag{10}$$

Given an MRA generated by the refinable function $\phi(x)$, one can construct a set of basic tight framelets $\Psi := \big\{\psi_1, \ldots, \psi_L\big\} \subset V_1$ satisfying

$$\hat{\psi}^{\ell}(\xi) = m_{\ell}(\mathfrak{p}\xi)\hat{\phi}(\mathfrak{p}\xi), \tag{11}$$

where

$$m_{\ell}(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} h_k^{\ell} \, \overline{\chi_k(\xi)}, \quad \ell = 1, \ldots, L \tag{12}$$

are the integral periodic functions in $L^2(\mathfrak{D})$ and are called the *framelet symbols* or *wavelet masks* (see [5]).

With $m_\ell(\xi), \ell = 0, 1, \ldots L, L \geq q-1$, as the wavelet masks, we formulate the matrix $\mathcal{M}(\xi)$ as:

$$\mathcal{M}(\xi) = \begin{pmatrix} m_0\big(\xi + \mathfrak{p}u(0)\big) & m_0\big(\xi + \mathfrak{p}u(1)\big) & \ldots & m_0\big(\xi + \mathfrak{p}u(q-1)\big) \\ m_1\big(\xi + \mathfrak{p}u(0)\big) & m_1\big(\xi + \mathfrak{p}u(1)\big) & \ldots & m_1\big(\xi + \mathfrak{p}u(q-1)\big) \\ \vdots & \vdots & \ddots & \vdots \\ m_L\big(\xi + \mathfrak{p}u(0)\big) & m_L\big(\xi + \mathfrak{p}u(1)\big) & \ldots & m_L\big(\xi + \mathfrak{p}u(q-1)\big) \end{pmatrix}. \quad (13)$$

The matrix $\mathcal{M}(\xi)$ is called the *modulation matrix*. Shah and Debnath [15] gave a complete characterization of tight wavelet frames on local fields via extension principles and established a sufficient condition on $\Psi = \{\psi_1, \ldots, \psi_L\}$ such that the resulting wavelet system $\mathcal{W}(\psi, j, k)$ given by (5) forms a tight frame for $L^2(K)$. More precisely, the framlet system $\mathcal{W}(\psi, j, k)$ forms a tight framelet frame for $L^2(K)$ if

$$\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_q, \quad \text{for} \quad a.e.\ \xi \in \sigma(V_0) \quad (14)$$

where $\sigma(V_0) := \big\{\xi \in \mathfrak{D} : \sum_{k\in\mathbb{Z}} |\hat{\phi}\big(\xi + u(k)\big)|^2 \neq 0\big\}$.

3 Polyphase matrix characterization of framelets on local fields

In this section, we shall first drive the polyphase representation of framelet symbols $m_\ell(\xi), \ell = 0, 1, \ldots, L$ and then establish a complete characterization of tight framelets by means of their polyphase components.

The *polyphase representation* of the refinement mask $m_0(\xi)$ can be derived as

$$\begin{aligned} m_0(\xi) &= \frac{1}{\sqrt{q}} \sum_{k\in\mathbb{N}_0} h_k \overline{\chi_k(\xi)} \\ &= \frac{1}{\sqrt{q}} \sum_{r=0}^{q-1} \sum_{k\in\mathbb{N}_0} h_{u(r)+qk} \overline{\chi_{u(r)+qk}(\xi)} \\ &= \frac{1}{\sqrt{q}} \sum_{r=0}^{q-1} \overline{\chi_{u(r)}(\xi)} \sum_{k\in\mathbb{N}_0} h_{u(r)+qk} \overline{\chi_{qk}(\xi)} \\ &= \frac{1}{\sqrt{q}} \sum_{r=0}^{q-1} \overline{\chi_{u(r)}(\xi)} f_r^0 \left(\overline{\chi(q\xi)}\right), \end{aligned}$$

where

$$f_r^0(\xi) = \sum_{k\in\mathbb{N}_0} h_{u(r)+qk}\,\overline{x(k)}, \quad r = 0, 1, \ldots, q-1, x \in K. \tag{15}$$

Similarly, the framelet symbols $m_\ell(\xi), \ell = 1, 2, \ldots, L$ as defined in equation (12) can be splitted into polyphase components as

$$m_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{r=0}^{q-1} \overline{\chi_{u(r)}(\xi)} f_r^\ell\left(\overline{\chi(q\xi)}\right),$$

where

$$f_r^\ell(\xi) = \sum_{k\in\mathbb{N}_0} h_{u(r)+qk}\,\overline{x(k)}, \quad r = 0, 1, \ldots, q-1, x \in K. \tag{16}$$

With the polyphase components given by equations (15) and (16), we formulate the *polyphase matrix* $\mathcal{P}(\xi)$ as:

$$\mathcal{P}\left(\overline{\chi(q\xi)}\right) = \begin{pmatrix} f_0^0\left(\overline{\chi(q\xi)}\right) & f_0^1\left(\overline{\chi(q\xi)}\right) & \ldots & f_0^L\left(\overline{\chi(q\xi)}\right) \\ f_1^0\left(\overline{\chi(q\xi)}\right) & f_1^1\left(\overline{\chi(q\xi)}\right) & \ldots & f_1^L\left(\overline{\chi(q\xi)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ f_{q-1}^0\left(\overline{\chi(q\xi)}\right) & f_{q-1}^1\left(\overline{\chi(q\xi)}\right) & \ldots & f_{q-1}^L\left(\overline{\chi(q\xi)}\right) \end{pmatrix}. \tag{17}$$

Then, it is clear that

$$\mathcal{M}(\xi) = \mathcal{S}\left(\overline{\chi(\xi)}\right)\mathcal{P}\left(\overline{\chi(q\xi)}\right), \tag{18}$$

where

$$\mathcal{S}\left(\overline{\chi(\xi)}\right) = \begin{pmatrix} \overline{\chi}_{u(0)}(\xi+u(0)) & \overline{\chi}_{u(1)}(\xi+u(0)) & \ldots & \overline{\chi}_{u(q-1)}(\xi+u(0)) \\ \overline{\chi}_{u(0)}(\xi+u(1)) & \overline{\chi}_{u(1)}(\xi+u(1)) & \ldots & \overline{\chi}_{u(q-1)}(\xi+u(1)) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\chi}_{u(0)}(\xi+u(q-1)) & \overline{\chi}_{u(1)}(\xi+u(q-1)) & \ldots & \overline{\chi}_{u(q-1)}(\xi+u(q-1)) \end{pmatrix}.$$

Since $\mathcal{S}(\overline{\chi(\xi)})$ is unitary matrix, therefore condition (14) is equivalent to

$$\mathcal{P}\left(\overline{\chi(q\xi)}\right)\mathcal{P}\left(\overline{\chi(q\xi)}\right)^* = I_q. \tag{19}$$

For convenience, let $\overline{\chi(q\xi)} = \zeta$, then the matrix (17) can be rewritten as

$$\mathcal{P}(\zeta) = \begin{pmatrix} f_0^0(\zeta) & f_0^1(\zeta) & \ldots & f_0^L(\zeta)) \\ f_1^0(\zeta) & f_1^1(\zeta) & \ldots & f_1^L(\zeta)) \\ \vdots & \vdots & \ddots & \vdots \\ f_{q-1}^0(\zeta) & f_{q-1}^1(\zeta) & \ldots & f_{q-1}^L(\zeta) \end{pmatrix}. \tag{20}$$

The polyphase matrix $\mathcal{P}(\xi)$ is called a *unitary matrix* if condition (19) holds which is equivalent to

$$\sum_{\ell=0}^{L}\overline{f_r^\ell(\zeta)}f_{r'}^\ell(\zeta)=\delta_{r,r'}\Leftrightarrow\sum_{\ell=1}^{L}\overline{f_{r'}^\ell(\zeta)}f_r^\ell(\zeta)$$
$$=\delta_{r,r'}-\overline{f_r^0(\zeta)}f_{r'}^0(\zeta),\ 0\le r,r'\le q-1. \tag{21}$$

The following theorem, the main result of this article shows that a unitary polyphase matrix leads to a tight wavelet frame on local fields of positive characteristic.

Theorem 1 *Suppose that the refinable function* ϕ *and the framelet symbols* $m_0,m_1,\dots,m_L$ *satisfy equations* (8)–(11)*. Moreover, if the polyphase matrix* $\mathcal{P}(\zeta)$ *given by* (20) *satisfy UEP condition* (19)*, then the framelet system* $\mathcal{W}(\psi,j,k)$ *given by* (5) *constitutes a tight frame for* $L^2(K)$ *i.e.,*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle f,\psi_{j,k}^\ell\right\rangle\right|^2=\|f\|_2^2,\quad\text{for all } f\in L^2(K). \tag{22}$$

Proof. By Parseval's formula, we have

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle f,\psi_{j,k}^\ell\right\rangle\right|^2&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle f,\ q^{j/2}\psi^\ell\big(\mathfrak{p}^{-j}x-u(k)\big)\right\rangle\right|^2\\&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle \hat f,\ q^{j/2}\hat\psi^\ell\big(\mathfrak{p}^{-j}\xi\big)\chi_{\mathfrak{p}^j}(\xi)\right\rangle\right|^2\\&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}q^j\sum_{k\in\mathbb{N}_0}\left|\left\langle \hat f\big(\mathfrak{p}^{-j}\xi\big)\overline{\hat\psi^\ell(\xi)},\ \chi(\xi)\right\rangle\right|^2\\&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}q^j\int_K\left|\hat f(\mathfrak{p}^{-j}\xi)\right|^2\left|\hat\psi^\ell(\xi)\right|^2d\xi.\end{aligned}\tag{23}$$

Using the polyphase decomposition formula (16) of the framelet symbols $m_\ell(\xi),\ell=1,\dots,L$, we can write

$$\sum_{\ell=1}^{L}\left|\hat\psi^\ell(\xi)\right|^2=\sum_{\ell=1}^{L}\left|m_\ell(\mathfrak{p}\xi)\hat\phi(\mathfrak{p}\xi)\right|^2=\sum_{\ell=1}^{L}\overline{m_\ell(\mathfrak{p}\xi)}\ \overline{\hat\phi(\mathfrak{p}\xi)}\ m_\ell(\mathfrak{p}\xi)\hat\phi(\mathfrak{p}\xi)$$

$$
\begin{aligned}
&= \overline{\hat{\phi}(\mathfrak{p}\xi)} \sum_{\ell=1}^{L} \overline{\left\{ \frac{1}{\sqrt{q}} \sum_{r=0}^{q-1} \overline{\chi_{u(r)}(\mathfrak{p}\xi)} f_r^{\ell}(\zeta) \right\}} \left\{ \frac{1}{\sqrt{q}} \sum_{r'=0}^{q-1} \overline{\chi_{u(r')}(\mathfrak{p}\xi)} f_{r'}^{\ell}(\zeta) \right\} \hat{\phi}(\mathfrak{p}\xi) \\
&= \overline{\hat{\phi}(\mathfrak{p}\xi)} \, \frac{1}{q} \sum_{r=0}^{q-1} \sum_{r'=0}^{q-1} \chi_{u(r)-u(r')}(\mathfrak{p}\xi) \left\{ \sum_{\ell=1}^{L} \overline{f_r^{\ell}(\zeta)} \, f_{r'}^{\ell}(\zeta) \right\} \hat{\phi}(\mathfrak{p}\xi).
\end{aligned}
$$

Since the polyphase matrix $\mathcal{P}(\xi)$ is unitary, which is equivalent to condition (21), the above expression reduces to

$$
\begin{aligned}
\sum_{\ell=1}^{L} \left| \hat{\psi}^{\ell}(\xi) \right|^2 &= \overline{\hat{\phi}(\mathfrak{p}\xi)} \, \frac{1}{q} \sum_{r=0}^{q-1} \sum_{r'=0}^{q-1} \chi_{u(r)-u(r')}(\mathfrak{p}\xi) \left[\delta_{r,r'} - f_r^0(\zeta) \, f_{r'}^0(\zeta) \right] \hat{\phi}(\mathfrak{p}\xi) \\
&= \overline{\hat{\phi}(\mathfrak{p}\xi)} \, \hat{\phi}(\mathfrak{p}\xi) - \overline{\hat{\phi}(\mathfrak{p}\xi)} \, \frac{1}{q} \sum_{r=0}^{q-1} \sum_{r'=0}^{q-1} \chi_{u(r)-u(r')}(\mathfrak{p}\xi) \overline{f_r^0(\zeta)} \, f_{r'}^0(\zeta) \hat{\phi}(\mathfrak{p}\xi) \\
&= \left| \hat{\phi}(\mathfrak{p}\xi) \right|^2 - \overline{\hat{\phi}(\mathfrak{p}\xi)} \, \overline{m_0(\mathfrak{p}\xi)} \, m_0(\mathfrak{p}\xi) \hat{\phi}(\mathfrak{p}\xi) \\
&= \left| \hat{\phi}(\mathfrak{p}\xi) \right|^2 - \left| m_0(\mathfrak{p}\xi) \hat{\phi}(\mathfrak{p}\xi) \right|^2 \\
&= \left| \hat{\phi}(\mathfrak{p}\xi) \right|^2 - \left| \hat{\phi}(\xi) \right|^2 .
\end{aligned} \tag{24}
$$

Substituting equation (24) in (23), we obtain

$$
\begin{aligned}
\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \left| \left\langle f, \psi_{j,k}^{\ell} \right\rangle \right|^2 &= \sum_{j \in \mathbb{Z}} q^j \int_K \left| \hat{f}(\mathfrak{p}^{-j}\xi) \right|^2 \left\{ \left| \hat{\phi}(\mathfrak{p}\xi) \right|^2 - \left| \hat{\phi}(\xi) \right|^2 \right\} d\xi \\
&= \int_K \left| \hat{f}(\xi) \right|^2 \sum_{j \in \mathbb{Z}} \left\{ \left| \hat{\phi}(\mathfrak{p}^{j+1}\xi) \right|^2 - \left| \hat{\phi}(\mathfrak{p}^{j}\xi) \right|^2 \right\} d\xi.
\end{aligned} \tag{25}
$$

Using equation (10), the summand in the above equation can be reformatted as

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}} \left\{ \left| \hat{\phi}(\mathfrak{p}^{j+1}\xi) \right|^2 - \left| \hat{\phi}(\mathfrak{p}^{j}\xi) \right|^2 \right\} d\xi &= \lim_{j \to \infty} \left| \hat{\phi}(\mathfrak{p}^{j+1}\xi) \right|^2 - \lim_{j \to -\infty} \left| \hat{\phi}(\mathfrak{p}^{j}\xi) \right|^2 \\
&= \lim_{j \to \infty} \left| \hat{\phi}(\mathfrak{p}^{j}\xi) \right|^2 - \lim_{j \to \infty} \left| \hat{\phi}(\mathfrak{p}^{-j}\xi) \right|^2 \\
&= \left| \hat{\phi}(0) \right|^2 - \lim_{j \to \infty} \left| \hat{\phi}(\mathfrak{p}^{-j}\xi) \right|^2 \\
&= 1.
\end{aligned}
$$

Using the above estimate in equation (25), we obtain

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle f,\psi_{j,k}^{\ell}\right\rangle\right|^2=\int_K\left|\hat{f}(\xi)\right|^2d\xi=\left\|\hat{f}\right\|_2^2=\left\|f\right\|_2^2.$$

This completes the proof of the theorem. □

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ps-ro fuzzy strongly α-irresolute function

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Abstract. The prime objective of this paper is to introduce and characterize a new type of function in a fuzzy topological spaces called ps-ro fuzzy strongly α-irresolute function. The interrelations of this function with the parallel existing allied concepts are established. The independence of ps-ro fuzzy strongly α-irresolute and well known concept of fuzzy strongly α-irresolute function motivate authors to explore it. Also, this function is found to be stronger than ps-ro fuzzy continuity, ps-ro fuzzy semicontinuity, ps-ro fuzzy precontinuity and ps-ro fuzzy α-continuity. Further, several characterizations of these functions along with different conditions for their existence are obtained.

1 Introduction

Ever since the introduction of the concept of fuzzy logic and fuzzy sets by L. A. Zadeh [13] and fuzzy topological space by C. L. Chang [3], several concepts of general topology has been generalized successfully in fuzzy settings by different mathematicians in different directions. Fuzzy α-open sets and fuzzy α-continuity were introduced and studied in [2]. After the initiation of the idea of ps-ro fuzzy topology [8], several forms of fuzzy continuous type of functions viz ps-ro fuzzy continuous, ps-ro fuzzy semi continuous, ps-ro fuzzy α-continuous and ps-ro fuzzy precontinuous functions were introduced

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and explored in [9, 10], [4], [5] and [7] respectively. Here, the idea of ps-ro fuzzy strongly α-irresolute function is initiated. The interrelations of this function with the existing similar types of functions are explored. It is seen that this function neither implies nor implied by the existing concept of fuzzy strongly α-irresolute function. Also, ps-ro fuzzy continuity, ps-ro fuzzy semicontinuity, ps-ro fuzzy precontinuity and ps-ro fuzzy α-continuity are all found to be weaker than ps-ro fuzzy strongly α-irresoluteness.

2 Preliminaries

A fuzzy set A on a nonempty set X is a function from X to $I = [0, 1]$. For two fuzzy sets A and B, A is subset of B written as $A \leq B$ if $A(t) \leq B(t) \forall t \in X$. A fuzzy point x_t is a fuzzy set with value $t (0 < t \leq 1)$ at x, elsewhere the value is 0. x_t is said to be quasi-coincident (q-coincident, in short) with a fuzzy set A if $A(x) + t > 1$. A and B are said to be q-coincident, written as AqB if for some $x \in X$, $A(x) + B(x) > 1$ [11]. Throughout this paper, a fuzzy topological space(fts, for short) in the sense of Chang [3] is denoted by (X, τ) or simply by X. A fuzzy set A on X is called fuzzy α-open if $A \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$, where $\mathrm{int}A$ and $\mathrm{cl}A$ are fuzzy interior and closure of a fuzzy set A on X [2].

Corresponding to a fts (X, τ) one can establish a family of general topological spaces $(X, i_\alpha(\tau))$, where $i_\alpha(\tau) = \{A^\alpha : A \in \tau\}$ and $A^\alpha = \{x \in X : A(x) > \alpha\} \forall \alpha \in I_1 = [0, 1)$. Fuzzy regular openness of A in (X, τ) does not imply regular openness of A^α in $(X, i_\alpha(\tau))$ and also regular openness of A^α in $(X, i_\alpha(\tau))$ does not guarantee fuzzy regular openness of A in (X, τ). This gave birth of ps-ro fuzzy topology, which is a fuzzy topology on X and is generated by pseudo regular open fuzzy sets on (X, τ) which are defined as those members of τ whose corresponding crisp set on $(X, i_\alpha(\tau)) \forall \alpha \in I_1$ are regular open. Members of ps-ro fuzzy topology are called ps-ro open and their complements as ps-ro closed fuzzy sets on X [8, 9].

In a fts (X, τ), fuzzy ps-closure and ps-interior of A, denoted by $\text{ps-cl}(A)$ and $\text{ps-int}(A)$ are given by $\text{ps-cl}(A) = \wedge\{B : A \leq B,\ B \text{ is ps-ro closed fuzzy set on } X\}$ and $\text{ps-int}(A) = \vee\{B : B \leq A,\ B \text{ is ps-ro open fuzzy set on } X\}$ [9, 10]. A fuzzy set A on a fts (X, τ) is said to be ps-ro semiopen [4] (ps-ro α-open [5], ps-ro preopen [7]) fuzzy set if $A \leq \text{ps-cl}(\text{ps-int}(A))$(resp. $A \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(A)))$, $A \leq \text{ps-int}(\text{ps-cl}(A))$).

A function f between two fts (X, τ_1) and (Y, τ_2) is

(i) fuzzy strongly α-irresolute function if $f^{-1}(A) \in \tau_1$, for each fuzzy α-open A on Y [12].

(ii) ps-ro fuzzy continuous [9], [10] (ps-ro semicontinuous [4], ps-ro α-continuous [5], ps-ro precontinuous [7]) if $f^{-1}(A)$ is ps-ro open (resp. ps-ro semiopen, ps-ro α-open, ps-ro preopen)fuzzy set on X for each ps-ro open fuzzy set A on Y.

(iii) ps-ro fuzzy α-irresolute [6] if $f^{-1}(A)$ is ps-ro α-open on X, for each ps-ro α-open A on Y.

3 ps-ro fuzzy strongly α-irresolute function

Definition 1 *A function* f *between two* fts (X, τ_1) *and* (Y, τ_2) *is said to be* ps-ro *fuzzy strongly* α*-irresolute function if for each* ps-ro α*-open fuzzy set* U *on* Y*,* $f^{-1}(U)$ *is* ps-ro *open fuzzy set on* X.

Remark 1 *It follows directly from the definition that* ps-ro *fuzzy strongly* α*-irresoluteness implies* ps-ro *fuzzy* α*-irresoluteness,* ps-ro *fuzzy* α*-continuity,* ps-ro *fuzzy semicontinuity and* ps-ro *fuzzy precontinuity. But the converses are not true in general is given by the example below:*

Example 1 *Let* $X = \{a, b, c, d\}$ *and* $Y = \{w, x, y, z\}$*. Let* A*,* B *and* C *be fuzzy sets on* X *defined by* $A(a) = 0.2, A(b) = 0.2, A(c) = 0.2$ *and* $A(d) = 0.3$*;* $B(t) = 0.2,\ \forall\ t \in X$ *and* $C(t) = 0.5,\ \forall\ t \in X$*. Let* D*,* E*,* F *and* G *be fuzzy sets on* Y *defined by* $D(t) = 0.4\ \forall\ t \in Y$*;* $E(w) = 0.5, E(x) = 0.5, E(y) = 0.5$*, and* $E(z) = 0.6$*;* $F(t) = 0.3,\ \forall\ t \in Y$*;* $G(w) = 0.3, G(x) = 0.3, G(y) = 0.3, G(z) = 0.4$*. Clearly,* $\tau_1 = \{0, 1, A, B, C\}$ *and* $\tau_2 = \{0, 1, D, E, F, G\}$ *are fuzzy topologies on* X *and* Y *respectively. In the corresponding general topological space* $(X, i_\alpha(\tau_1))$*,* $\forall \alpha \in I_1 = [0, 1)$*, the open sets are* ϕ, X, A^α*,* B^α *and* C^α*, where* $A^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \{d\}, & \text{for } 0.2 \le \alpha < 0.3 \\ \phi, & \text{for } \alpha \ge 0.3 \end{cases}$*,* $B^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \phi, & \text{for } \alpha \ge 0.2 \end{cases}$ *and*

$$C^\alpha = \begin{cases} X, & \text{for } \alpha < 0.5 \\ \phi, & \text{for } \alpha \ge 0.5 \end{cases}$$

For $0.2 \le \alpha < 0.3$*, the closed sets on* $(X, i_\alpha(\tau_1))$ *are* ϕ, X *and* $X - \{d\}$*. Therefore,* $int(cl(A^\alpha)) = X$*. So,* A^α *is not regular open on* $(X, i_\alpha(\tau_1))$ *for* $0.2 \le \alpha < 0.3$*. Thus,* A *is not pseudo regular open fuzzy set on* (X, τ_1)*. So, the* ps-ro *fuzzy topology on* X *is* $\{0, 1, B, C\}$*. Similarly,* E *and* G *are not pseudo regular open fuzzy set on* Y *for* $0.5 \le \alpha < 0.6$ *and* $0.3 \le \alpha < 0.4$ *respectively and hence the* ps-ro *fuzzy topology on* Y *is* $\{0, 1, D, F\}$*. Define a function* f *from* (X, τ_1) *to* (Y, τ_2) *by* $f(a) = w$*,* $f(b) = x$*,* $f(c) = x$ *and* $f(d) = z$*.* G *is* ps-ro α*-open* *fuzzy set on* Y *but* $f^{-1}(G)$ *is not* ps-ro *open fuzzy set on* X*. Hence,* f *is not* ps-ro *fuzzy strongly* α*-irresolute.* $0, 1, D, F$ *and* U *satisfying* $F \leq U \leq D$ *are* ps-ro α*-open fuzzy sets on* Y*.* $f^{-1}(V)$ *is* ps-ro α*-open fuzzy set on* X *for all* ps-ro α*-open fuzzy set* V *on* Y *proving* f *to be* ps-ro *fuzzy* α*-irresolute function. Similarly, it can be verified that* f *is* ps-ro *fuzzy* α*-irresolute,* ps-ro *fuzzy* α*-continuous,* ps-ro *fuzzy semicontinuous and* ps-ro *fuzzy precontinuous.*

Remark 2 *Clearly,* ps-ro *fuzzy strongly* α*-irresoluteness implies* ps-ro *fuzzy continuity but the converse is not true is shown below:*

Example 2 *Let* $X = \{a, b, c, d\}$ *and* $Y = \{w, x, y, z\}$*. Let* A*,* B*,* C *and* D *be fuzzy sets on* X *given by* $A(a) = 0.2, A(b) = 0.2, A(c) = 0.3$ *and* $A(d) = 0.3$*;* $B(t) = 0.3,\ \forall\ t \in X$*;* $C(t) = 0.2\ \forall\ t \in X$ *and* $D(t) = 0.7\ \forall\ t \in X$*. Let* E*,* F *and* G *be fuzzy sets on* Y *defined by* $E(t) = 0.3\ \forall\ t \in Y$*;* $F(w) = 0.2, F(x) = 0.2, F(y) = 0.3$ *and* $F(z) = 0.3$ *and* $G(t) = 0.2\ \forall\ t \in Y$*.* $\tau_1 = \{0, 1, A, B, C, D\}$ *and* $\tau_2 = \{0, 1, E, F, G\}$ *are fuzzy topologies on* X *and* Y *respectively.* A *is not pseudo regular open fuzzy set on* X *for* $0.2 \leq \alpha < 0.3$*. So, the* ps-ro *fuzzy topology on* X *is* $\{0, 1, B, C, D\}$*. Again* F *is not pseudo regular open fuzzy set for* $0.2 \leq \alpha < 0.3$ *on* Y*. So, the* ps-ro *fuzzy topology on* Y *is* $\{0, 1, E, G\}$*. Let* f *be a function from* (X, τ_1) *to* (Y, τ_2) *given by* $f(a) = w$*,* $f(b) = w$*,* $f(c) = y$ *and* $f(d) = z$*.* $f^{-1}(U)$ *is* ps-ro *open fuzzy set on* X *for every* ps-ro *open fuzzy set* U *on* Y*, proving that* f *is* ps-ro *fuzzy continuous function.* F *is* ps-ro α*-open fuzzy set on* Y *but* $f^{-1}(F)$ *is not* ps-ro *open fuzzy set on* X*. Hence,* f *is not* ps-ro *fuzzy strongly* α*-irresolute.*

Now, we find the relation of ps-ro fuzzy strongly α-irresoluteness with well known existing concept of fuzzy strongly α-irresoluteness.

Remark 3 *In Example 2, fuzzy* α*-open sets on* Y *are* 0*,* 1*,* E*,* F*,* G *and* T *where* $G \leq T \leq E$*.* $f^{-1}(U) \in \tau_1$ *forall fuzzy* α*-open set* U *on* Y*. Therefore* f *is fuzzy strongly* α*-irresolute but* f *is not* ps-ro *fuzzy strongly* α*-irresolute.*

Example 3 *Let* $X = \{a, b, c, d\}$ *and* $Y = \{w, x, y, z\}$*. Let* A*,* B *and* C *be fuzzy sets on* X *given by* $A(t) = 0.3,\ \forall\ t \in X$*;* $B(t) = 0.4,\ \forall\ t \in X$ *and* $C(a) = 0.5, C(b) = 0.5, C(c) = 0.5, C(d) = 0.6$*. Let* E*,* F*,* G *and* H *be fuzzy sets on* Y *given by* $E(w) = 0.3, E(x) = 0.4, E(y) = 0.4, E(z) = 0.4$*;* $F(t) = 0.3\ \forall\ t \in Y$*;* $G(t) = 0.4\ \forall\ t \in Y$ *and* $H(w) = 0.1, H(x) = 0.1, H(y) = 0.1$ *and* $H(z) = 0.3$*.* $\tau_1 = \{0, 1, A, B, C\}$ *and* $\tau_2 = \{0, 1, E, F, G, H\}$ *are fuzzy topologies on* X *and* Y *respectively.* ps-ro *fuzzy topology on* X *and* Y *are* $\{0, 1, A, B\}$ *and* $\{0, 1, F, G\}$ *respectively. Let us define a function* f *from* (X, τ_1) *to* (Y, τ_2) *by* $f(a) = x$*,*

$f(b) = x$, $f(c) = y$ and $f(d) = z$. ps-ro α-open fuzzy set on Y are 0, 1, T, F and G where $F \leq T \leq G$. Also, $f^{-1}(U)$ is ps-ro open fuzzy set on X for all ps-ro α-open fuzzy set U on Y. Hence, f is ps-ro fuzzy strongly α-irresolute. H is fuzzy α-open set on Y, but $f^{-1}(H)$ is not fuzzy open set on X proving f is not fuzzy strongly α-irresolute.

Remark 4 *From Remark 3 and Example 3, it follows that ps-ro fuzzy strongly α-irresolute and fuzzy strongly α-irresolute functions are two independent concepts.*

Theorem 1 *If $f : X \to Y$ and $g : Y \to Z$ be two functions where X, Y and Z are three fts, then the following hold:*

(i) *if f and g are ps-ro fuzzy strongly α-irresolute function then gof is also so.*
(ii) *if f is ps-ro fuzzy strongly α-irresolute and g is ps-ro fuzzy α-continuous, then gof is ps-ro fuzzy continuous.*
(iii) *if f is ps-ro fuzzy continuous and g is ps-ro fuzzy strongly α-irresolute, then gof is ps-ro fuzzy strongly α-irresolute.*

Proof.

(i) Let A be a ps-ro α-open fuzzy set on Z. By given conditions, $g^{-1}(A)$ is ps-ro open and hence ps-ro α-open fuzzy set on Y. $f^{-1}(g^{-1}(A))$ is ps-ro open fuzzy set on X. As, $(gof)^{-1}(A) = f^{-1}(g^{-1}(A))$, $(gof)^{-1}(A)$ is ps-ro open fuzzy set on X, showing gof is ps-ro fuzzy strongly α-irresolute.
(ii) Let B be any ps-ro open fuzzy set on Z. By given hypothesis, $g^{-1}(B)$ is ps-ro α-open fuzzy set on Y and $f^{-1}(g^{-1}(B))$ is ps-ro open fuzzy set on X. As, $(gof)^{-1}(B) = f^{-1}(g^{-1}(B))$, $(gof)^{-1}(B)$ is ps-ro open fuzzy set on X and hence gof is ps-ro fuzzy continuous.
(iii) Let B be any ps-ro α-open fuzzy set on Z. $g^{-1}(B)$ is ps-ro open fuzzy set on Y and $f^{-1}(g^{-1}(B))$ is ps-ro open fuzzy set on X. Using $(gof)^{-1}(B) = f^{-1}(g^{-1}(B))$, $(gof)^{-1}(B)$ is ps-ro open fuzzy set on X. So, gof is a ps-ro fuzzy strongly α-irresolute.

□

Theorem 2 *A function f from a fts (X, τ_1) to another fts (Y, τ_2) is ps-ro fuzzy strongly α-irresolute iff for any fuzzy point x_α of X and any ps-ro α-open fuzzy set V on Y with $f(x_\alpha)qV$, there exist a ps-ro open fuzzy set U on X such that $x_\alpha qU \leq f^{-1}(V)$.*

Proof. Let f be ps-ro fuzzy strongly α-irresolute. Let x_α be a fuzzy point on X and V be ps-ro α-open fuzzy set on Y such that $f(x_\alpha)qV$. $f^{-1}(V)$ is ps-ro open fuzzy set on X and $Vf(x) + \alpha > 1$. So, $x_\alpha q f^{-1}(V)$. Taking $f^{-1}(V) = U$, the result follows.

Conversely, let V be a ps-ro α-open fuzzy set on Y and x_α be a fuzzy point on $f^{-1}(V)$. Then, $x_\alpha \leq f^{-1}(V)$, $f(x_\alpha) \leq f(f^{-1}(V)) \leq V$. Choosing a fuzzy point x'_α with $x'_\alpha(x) = 1 - x_\alpha(x)$, we have $V(y) + f(x'_\alpha)(y) = V(y) + f(1 - x_\alpha)(y) \geq V(y) + (1 - V)(y) = 1$. So, $f(x'_\alpha)qV$. Then there exists a ps-ro open fuzzy set U on X such that $x'_\alpha qV \leq f^{-1}(V)$. Since $x'_\alpha qV$, $x'_\alpha(x) + U(x) = 1 - x_\alpha(x) + U(x) > 1$. So, $x_\alpha \leq U$. Hence, $x_\alpha \leq U \leq f^{-1}(V)$. x_α being arbitrary, taking union of all such relations, $\vee\{U : x_\alpha \in f^{-1}(V)\} = f^{-1}(V)$, proving $f^{-1}(V)$ is ps-ro open fuzzy set on X and hence, f is ps-ro fuzzy strongly α-irresolute. □

Definition 2 *For any fuzzy set* A *on* fts (X, τ), *the smallest* ps-ro α-*closed fuzzy set containg* A *is called* ps-$\alpha cl(A)$ *and the largest* ps-ro α-*open fuzzy set contained in* A *is called* ps-$\alpha int(A)$.

Theorem 3 *For a function* f *from a* fts (X, τ_1) *to another* fts (Y, τ_2) *the following statements are equivalent.*

(a) f *is* ps-ro *fuzzy strongly* α-*irresolute.*
(b) *the inverse image of each* ps-ro α-*closed fuzzy set on* Y *is* ps-ro *closed fuzzy set on* X.
(c) *for each fuzzy point* x_α *on* X *and each* ps-ro α-*open fuzzy set* B *on* Y *and* $f(x_\alpha) \in B$, *there exists* ps-ro *open fuzzy set* A *on* X *such that* $x_\alpha \in A$ *and* $f(A) \leq B$.
(d) $f(\text{ps-}cl(A)) \leq \text{ps-}\alpha clf(A)$ $\forall$ *fuzzy set* A *on* X.
(e) $\text{ps-}cl(f^{-1}(B)) \leq f^{-1}(\text{ps-}\alpha cl(B))$ $\forall$ *fuzzy set* B *on* Y.
(f) $f^{-1}(\text{ps-}\alpha intB) \leq \text{ps-}int(f^{-1}(B))$ $\forall$ *fuzzy set* B *on* Y.

Proof. (a) ⇒ (b) Let f be ps-ro fuzzy strongly α-irresolute function. Let B be ps-ro α-closed fuzzy set on Y. Then $(1 - B)$ is ps-ro α-open fuzzy set on Y. Since f is ps-ro fuzzy strongly α-irresolute, $f^{-1}(1 - B)$ is ps-ro open fuzzy set on X. As $f^{-1}(1 - B) = 1 - f^{-1}(B)$, the result follows.
(b) ⇒ (a) Let B be ps-ro α-open fuzzy set on Y. Then $f^{-1}(1 - B)$ is ps-ro closed fuzzy set on X. As, $f^{-1}(1 - B) = 1 - f^{-1}(B)$, $f^{-1}(B)$ is ps-ro open fuzzy set on X. Therefore f is ps-ro fuzzy strongly α-irresolute.
(a) ⇒ (c) Let x_α be any fuzzy point on X and B be any ps-ro α-open fuzzy set on Y such that $f(x_\alpha) \in B$. Since f is ps-ro fuzzy strongly α-irresolute function,

$f^{-1}(B)$ is ps-ro open fuzzy set on X which contains x_α. Taking $f^{-1}(B) = A$, the result follows.
(c) $\Rightarrow$ (a) Let the given condition hold and B be any ps-ro α-open fuzzy set on Y. If $f^{-1}(B) = 0$, then the result is true. If $f^{-1}(B) \neq 0$, then there exist fuzzy point x_α on $f^{-1}(B)$. So, there exist ps-ro open fuzzy set U_{x_α} on X which contains x_α such that $x_\alpha \in U_{x_\alpha} \leq f^{-1}(B)$. Since x_α is arbitrary, taking union of all such relations, we get $f^{-1}(B) = \vee\{x_\alpha : x_\alpha \in f^{-1}(B)\} \leq \vee\{U_{x_\alpha} : x_\alpha \in f^{-1}(B)\} \leq f^{-1}(B)$. This shows that $f^{-1}(B) = \vee\{U_{x_\alpha} : x_\alpha \in f^{-1}(B)\}$ which imply that $f^{-1}(B)$ is ps-ro open fuzzy set on X. Therefore f is ps-ro fuzzy strongly α-irresolute.
(b) $\Rightarrow$ (d) Let A be fuzzy set on X. Then $A \leq f^{-1}(f(A)) \leq f^{-1}(\text{ps-}\alpha cl(f(A)))$. Since $\text{ps-}\alpha cl(f(A))$ is ps-ro α-closed fuzzy set on Y, $f^{-1}(\text{ps-}\alpha cl(f(A)))$ is ps-ro closed fuzzy set on X. Now, $\text{ps-}cl(A) \leq f^{-1}(\text{ps-}\alpha cl(f(A)))$ and $f(\text{ps-}cl(A)) \leq f(f^{-1}(\text{ps-}\alpha cl(f(A)))) \leq \text{ps-}\alpha cl(f(A))$. Thus, $f(\text{ps-}clA) \leq \text{ps-}\alpha clf(A)$.
(d) $\Rightarrow$ (e) For any fuzzy set B on Y, let $A = f^{-1}(B)$. By hypothesis, we have $f(\text{ps-}cl(f^{-1}(B))) \leq \text{ps-}\alpha clf(f^{-1}(B)) \leq \text{ps-}\alpha cl(B)$ and $\text{ps-}cl(f^{-1}(B)) \leq f^{-1}(f(\text{ps-}cl(f^{-1}(B)))) \leq f^{-1}(\text{ps-}\alpha cl(B))$. Thus $\text{ps-}cl(f^{-1}(B) \leq f^{-1}(\text{ps-}\alpha cl(B))$.
(e) $\Rightarrow$ (f) $f^{-1}(\text{ps-}\alpha intB) = f^{-1}(1 - \text{ps-}\alpha cl(1 - B))$
$= 1 - f^{-1}(\text{ps-}\alpha cl(1 - B))$
$\leq 1 - \text{ps-}cl(f^{-1}(1 - B))$
$= 1 - \text{ps-}cl(1 - f^{-1}(B))$
$= (1 - (1 - \text{ps-}int(f^{-1}(B))))$
$= \text{ps-}int(f^{-1}(B))$. So, $f^{-1}(\text{ps-}\alpha intB) \leq \text{ps-}int(f^{-1}(B))$.
(f) $\Rightarrow$ (a) Let B be any ps-ro α-open fuzzy set on Y. Then $B = \text{ps-}\alpha intB$. Then $f^{-1}(\text{ps-}\alpha intB) = f^{-1}(B) \leq \text{ps-}int(f^{-1}(B))$. Also, $\text{ps-}int(f^{-1}(B)) \leq f^{-1}(B)$. Therefore, $\text{ps-}int(f^{-1}(B)) = f^{-1}(B)$ which imply that $f^{-1}(B)$ is ps-ro open fuzzy set on X. Thus, f is ps-ro fuzzy strongly α-irresolute function. $\square$

Theorem 4 *A bijective function* f *from a* fts (X, τ_1) *to another* fts (Y, τ_2) *is* ps-ro *fuzzy strongly* α*-irresolute iff* $\text{ps-}\alpha int(f(A)) \leq f(\text{ps-}int(A))$*, for every fuzzy set* A *on* X*.*

Proof. Let f be ps-ro fuzzy strongly α-irresolute and A be any fuzzy set on X. $\text{ps-}\alpha int(f(A))$ being ps-ro α-open fuzzy set on Y, $f^{-1}(\text{ps}\alpha int(f(A))$ is ps-ro open fuzzy set on X. Now, $f^{-1}(\text{ps}\alpha int(f(A)) \leq \text{ps-}int(f^{-1}(f(A))) = \text{ps-}int(A)$, as f is one-to-one, which gives $f(f^{-1}(\text{ps}\alpha int(f(A))) \leq f(\text{ps-}int(A))$ and hence, $\text{ps}\alpha int(f(A)) \leq f(\text{ps-}int(A))$, as f is onto.

Conversely, let B be any ps-ro α-open fuzzy set on Y. Then, $B = \text{ps-}\alpha int(B) = \text{ps-}\alpha int(f(f^{-1}(B))$, (since f is onto). By given condition, ps-αint

$(f(f^{-1}(B)) \leq f(\text{ps-int}(f^{-1}(B))$ so, $f^{-1}(B) \leq f^{-1}(f(\text{ps-int}(f^{-1}(B)))) = \text{ps-int}(f^{-1}(B))$,(since f is one-to-one). But $\text{ps-int}(f^{-1}(B)) \leq f^{-1}(B)$. Therefore, $\text{ps-int}(f^{-1}(B)) = f^{-1}(B)$ proving that $f^{-1}(B)$ is ps-ro open fuzzy set on X. Hence, f is ps-ro fuzzy strongly α-irresolute function. □

Lemma 1 [1] *Let* $g : X \to X \times Y$ *be the graph of a function* $f : (X, \tau_1) \to (Y, \tau_2)$ *defined by* $g(x) = (x, f(x))$. *If* A *and* B *are fuzzy sets on* X *and* Y *respectively, then* $g^{-1}(A \times B) = A \wedge f^{-1}(B)$.

Theorem 5 *For a function* f *from a* fts (X, τ_1) *to another* fts (Y, τ_2), *if the graph* $g : X \to X \times Y$ *of* f *is* ps-ro *fuzzy strongly α-irresolute then* f *is also* ps-ro *fuzzy strongly α-irresolute.*

Proof. Let B be any ps-ro α-open fuzzy set on Y. Using Lemma 1 we get, $f^{-1}(B) = 1 \wedge f^{-1}(B) = g^{-1}(1 \times B)$. Now, $(1 \times B)$ is ps-ro α-open fuzzy set on $(X \times Y)$ and since g is ps-ro fuzzy strongly α-irresolute function, $g^{-1}(1 \times B)$ is ps-ro open fuzzy set on X. Thus, $f^{-1}(B)$ is ps-ro open fuzzy set on X. Hence f is ps-ro fuzzy strongly α-irresolute. □

Definition 3 *A fuzzy set* A *on a* fts (X, τ) *is called* ps-ro *fuzzy dense set if* $\text{ps-cl}(A) = 1$ *and* A *is called nowhere* ps-ro *fuzzy dense set if* $\text{ps-int}(\text{ps-cl}(A)) = 0$.

Theorem 6 *If a function* $f : (X, \tau_1) \to (Y, \tau_2)$ *is* ps-ro *fuzzy strongly α-irresolute, then* $f^{-1}(A)$ *is* ps-ro *closed fuzzy set on* X *for any nowhere* ps-ro *fuzzy dense set* A *on* Y.

Proof. Let A be any nowhere ps-ro fuzzy dense set on Y. Then, $1 - \text{ps-int}(\text{ps-cl}A) = 1$ i.e., $\text{ps-cl}(\text{ps-int}(1 - A)) = 1$. As $\text{ps-int}1 = 1$, $\text{ps-int}(\text{ps-cl}(\text{ps-int}(1-A))) = \text{ps-int}1 = 1$. $1-A \leq 1 = \text{ps-int}(\text{ps}-\text{cl}(\text{ps}-\text{int}(1-A)))$ which shows that $1 - A$ is ps-ro α-open fuzzy set on Y. f being ps-ro fuzzy α-irresolute, $f^{-1}(1 - A)$ is ps-ro open fuzzy set on X. Using $f^{-1}(1 - A) = 1 - f^{-1}(A)$, $f^{-1}(A)$ is ps-ro closed fuzzy set on X. □

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