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A few results on generalized Janowski type functions associated with (j, k)-symmetrical functions

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Abstract. The aim of the present article is to introduce and study new subclass of Janowski type functions defined using notions of Janowski functions and (j, k)-symmetrical functions. Certain interesting coefficient inequalities, sufficiency criteria, distortion theorem, neighborhood property are investigated for this class.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

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which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ \text{ and } \ |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. For any $f \in \mathcal{A}$, ρ-neighborhood of $f(z)$ can be defined as:

$$\mathcal{N}_\rho(f) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \sum_{n=2}^{\infty} n|a_n - b_n| \leq \rho \right\}. \tag{2}$$

For $e(z) = z$, we can see that

$$\mathcal{N}_\rho(e) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \sum_{n=2}^{\infty} n|b_n| \leq \rho \right\}. \tag{3}$$

The idea of neighborhoods was first introduced by Goodman [9] which was further generalized by Ruscheweyh [5]. For f and g be analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function ω in $\mathcal{U}$ such that $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$, and we denote this by $f \prec g$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real parts.

Definition 1 [6] *Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\Re\{p(z)\} > 0$, $z \in \mathcal{U}$.*

Any function p in $\mathcal{P}$ has the representation $p(z) = \frac{1+w(z)}{1-w(z)}$ where $w \in \Omega$ and

$$\Omega = \{w \in \mathcal{A} \ \text{ and } \ w(0) = 0 \ \ |w(z)| < 1\}.$$

The class of functions $\mathcal{P}$ with positive real part plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points $\mathcal{S}_s^*$ have been defined by using the concept of class of functions with positive real part.

Definition 2 [1] *Let $\mathcal{P}[A, B]$, where $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \frac{1+Aw(z)}{1+Bw(z)}$, $z \in \mathcal{U}$, $w \in \Omega$. $p \in \mathcal{P}[A, B]$ if and only if $p(z) \prec \frac{1+Az}{1+Bz}$.*

Definition 3 *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$*

$$f(e^{\frac{2\pi i}{k}}z) = e^{\frac{2\pi i}{k}}f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get class of odd univalent functions.

The notion of (j, k)-symmetrical functions $(j = 0, 1, 2, \ldots, k-1;\ k = 2, 3, \ldots)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j, k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings [10].

Definition 4 *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0, 1, 2, \ldots, k-1$ where $k \geq 2$ is a natural number. A function $f : \mathcal{D} \mapsto \mathbb{C}$ where $\mathcal{D}$ is a k-fold symmetric set, is called (j, k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z), \qquad z \in \mathcal{U}.$$

We note that the family of all (j, k)-symmetric functions is denoted be $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ the classes of even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1 [10] *For every mapping $f : \mathcal{D} \mapsto \mathbb{C}$, where $\mathcal{D}$ is a k-fold symmetric set, there exists exactly the sequence of (j, k)- symmetrical functions $f_{j,k}$,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu j} f(\varepsilon^{\nu} z), \tag{4}$$

$$(f \in \mathcal{A};\ k = 1, 2, \ldots;\ j = 0, 1, 2, \ldots, k-1)$$

From (4) we can get

$$f_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu j} f(\varepsilon^{\nu} z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu j}\left(\sum_{n=1}^{\infty} a_n (\varepsilon^{\nu} z)^n\right),$$

then

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n, \quad a_1 = 1, \quad \delta_{n,j} = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-j)\nu} = \begin{cases} 1, & n = lk + j; \\ \\ 0, & n \neq lk + j. \end{cases} \tag{5}$$

Alsarari and Latha [4] introduced and studied the classes $\mathcal{S}^{(j,k)}(A, B)$ and $\mathcal{K}^{(j,k)}(A, B)$ which are starlike and convex with respect to (j, k)-symmetric points, respectively.

Definition 5 [4] *A function* f *in* $\mathcal{A}$ *is said to belong to the class* $\mathcal{S}^{(j,k)}(A, B)$*,* $(-1 \leq B < A \leq 1)$ *if*

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + Az}{1 + Bz}, \qquad z \in \mathcal{U},$$

where $f_{j,k}(z)$ *defined by* (5)*.*

This class is generalizes the classes studied by Ohsang and Yaungjae [7] and Sakaguchi [8].

We need the following lemma to prove our main results.

Lemma 1 [3] *Let* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}[A, B]$*, then for* $n \geq 1$*,*

$$|p_n| \leq (A - B).$$

2 Main results

Theorem 2 *If* $f \in \mathcal{S}^{(j,k)}(A, B)$*, then for* $n \geq 2$*,* $-1 \leq B < A \leq 1$*.*

$$|a_n| \leq \prod_{m=1}^{n-1} \frac{\delta_{m,j}[(A - B) - 1] + m}{m + 1 - \delta_{m+1,j}},$$

where $\delta_{n,j}$ *by* (5)*.*

Proof. By Definition (5) we have

$$\frac{zf'(z)}{f_{j,k}(z)} = p(z), \qquad p \in \mathcal{P}[A, B],$$

then we have

$$zf'(z) = [1 + \sum_{n=1}^{\infty} p_n z^n] f_{j,k}(z)$$

by (1) and (5), we have

$$(1 - \delta_{1,j})z + \sum_{n=2}^{\infty} [n - \delta_{n,j}] a_n z^n = \left[\sum_{n=1}^{\infty} p_n z^n\right] \left[\sum_{n=1}^{\infty} \delta_{n,j} a_n z^n\right].$$

Equating coefficients of z^n on both sides, we have

$$a_n = \frac{1}{[n - \delta_{n,j}]} \sum_{m=1}^{n-1} p_m \delta_{n-m,j} a_{n-m}, \qquad \delta_{1,j} = 1,$$

by Lemma 1, we have

$$|a_n| \leq \frac{(A - B)}{[n - \delta_{n,j}]} \sum_{m=1}^{n-1} \delta_{m,j} |a_m|.$$

Now we want to prove that

$$\frac{(A - B)}{[n - \delta_{n,j}]} \sum_{m=1}^{n-1} \delta_{m,j} |a_m| \leq \prod_{m=1}^{n-1} \frac{\delta_{m,j}[(A - B) - 1] + m}{[m + 1 - \delta_{m+1,j}]}. \tag{6}$$

For this, we use the induction method. The result is true for $n = 2$ and 3. Let the hypothesis be true for $n = m$, we have

$$\frac{(A - B)}{[m - \delta_{m,j}]} \sum_{r=1}^{m-1} \delta_{r,j} |a_r| \leq \prod_{r=1}^{m-1} \frac{\delta_{r,j}[(A - B) - 1] + r}{[r + 1 - \delta_{r+1,j}]}.$$

Multiplying both sides by $\frac{\delta_{m,j}[(A-B)-1]+m}{[m+1-\delta_{m+1,j}]}$, we get

$$\prod_{r=1}^{m} \frac{\delta_{r,j}[(A - B) - 1] + r}{[r + 1 - \delta_{r+1,j}]} \geq \frac{\delta_{m,j}[(A - B) - 1] + m}{[m + 1 - \delta_{m+1,j}]} \cdot \frac{(A - B)}{[m - \delta_{m,j}]} \sum_{r=1}^{m-1} \delta_{r,j} |a_r|,$$

since

$$
\begin{aligned}
& \frac{\delta_{m.j}[(A-B)-1]+m}{[m+1-\delta_{m+1,j}]} \cdot \frac{(A-B)}{[m-\delta_{m,j}]} \sum_{r=1}^{m-1} \delta_{r,j}|a_r| \\
= \quad & \frac{(A-B)}{[m+1-\delta_{m+1,j}]} \cdot \left[1+\frac{\delta_{m,j}(A-B)}{[m-\delta_{m,j}]}\right] \sum_{r=1}^{m-1} \delta_{r,j}|a_r|, \\
\geq \quad & \frac{(A-B)}{[m+1-\delta_{m+1,j}]} \cdot \left[\sum_{r=1}^{m-1} \delta_{r,j}|a_r| + \delta_{m,j}|a_m|\right], \\
= \quad & \frac{(A-B)}{[m+1-\delta_{m+1,j}]} \cdot \left[\sum_{r=1}^{m} \delta_{r,j}|a_r|\right].
\end{aligned}
$$

That is

$$
|a_{m+1}| \leq \frac{(A-B)}{[m-\delta_{m,j}]} \sum_{r=1}^{m} \delta_{r,j}|a_r| \leq \prod_{r=1}^{m} \frac{\delta_{r,j}[(A-B)-1]+r}{[r+1-\delta_{r+1,j}]},
$$

which shows that inequality (6) is true for $n = m+1$. This completes the proof. □

Theorem 3 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *be analytic in* $\mathcal{U}$,*for* $(-1 \leq B < A \leq 1)$, *we have*

$$
\sum_{n=2}^{\infty} \{(n-\delta_{n,j}) + |A\delta_{n,j} - Bn|\} |a_n| \leq (A-B).
$$

Then, $f(z) \in \mathcal{S}^{(j,k)}(A,B)$.

Proof. For the proof of Theorem 3, it suffices to show that the values for $\frac{zf'(z)}{f_{j,k}(z)}$, satisfy

$$
\left| \frac{zf'(z) - f_{j,k}(z)}{Af_{j,k}(z) - Bzf'(z)} \right| \leq 1,
$$

we have

$$\begin{aligned}
\left|\frac{zf'(z)-f_{j,k}(z)}{Af_{j,k}(z)-Bzf'(z)}\right| &= \left|\frac{\sum_{n=2}^{\infty}(n-\delta_{n,j})a_n z^{n-1}}{(A-B)+\sum_{n=2}^{\infty}\{A\delta_{n,j}-Bn\}\,a_n z^{n-1}}\right| \\
&\leq \frac{\sum_{n=2}^{\infty}(n-\delta_{n,j})|a_n||z|^{n-1}}{(A-B)-\sum_{n=2}^{\infty}|A\delta_{n,j}-Bn|\,|a_n||z|^{n-1}} \\
&\leq \frac{\sum_{n=2}^{\infty}(n-\delta_{n,j})|a_n|}{(A-B)-\sum_{n=2}^{\infty}|A\delta_{n,j}-Bn|\,|a_n|}.
\end{aligned}$$

This last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\{(n-\delta_{n,j})+|A\delta_{n,j}-Bn|\}|a_n| \leq (A-B),$$

hence $\left|\frac{zf'(z)-f_{j,k}(z)}{Af_{j,k}(z)-Bzf'(z)}\right| \leq 1$, and Theorem 3 is proved. □

Theorem 4 *Let* $f(z) \in \mathcal{S}^{(j,k)}(A,B)$, *for* $(-1 < B < A \leq 1)$, *then*

$$|z| - \sum_{n=2}^{i}|a_n||z|^n - \tau_i|z|^{i+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{i}|a_n||z|^n + \tau_i|z|^{i+1},$$

where

$$\tau_i = \frac{(A-B)-\sum_{n=2}^{i}\{(n-\delta_{n,j})+|A\delta_{n,j}-Bn|\}|a_n|}{\{(i+1)(1-|B|)-[1-|A|]\delta_{i+1,j}\}}.$$

Proof. From Theorem 3 we have

$$\begin{aligned}
&\sum_{n=i+1}^{\infty}\{(n-\delta_{n,j})+|A\delta_{n,j}-Bn|\}|a_n| \\
&\qquad \leq (A-B)-\sum_{n=2}^{i}\{(n-\delta_{n,j})+|A\delta_{n,j}-Bn|\}|a_n|.
\end{aligned}$$

On the other hand

$$(n-\delta_{n,j})+|A\delta_{n,j}-Bn| \geq n(1-|B|)-[1-|A|]\delta_{n,j}$$

and hence $n(1-|B|)-[1-|A|]\delta_{n,j}$ is monotonically increasing with respect to n. So we can write

$$\{(i+1)(1-|B|)-[1-|A|]\delta_{i+1,j}\}\sum_{n=i+1}^{\infty}|a_n|$$

$$\leq (A-B) - \sum_{n=2}^{i} \{(n-\delta_{n,j}) + |A\delta_{n,j} - Bn|\} |a_n|$$

which implies that

$$\sum_{n=i+1}^{\infty} |a_n| \leq \tau_i,$$

hence we have

$$|f(z)| \leq |z| + \sum_{n=2}^{i} |a_n||z|^n + \tau_i |z|^{i+1},$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{i} |a_n||z|^n - \tau_i |z|^{i+1}.$$

This completes the proof of theorem. □

Theorem 5 *For* $(-1 < B < A \leq 1)$,

$$\mathcal{S}^{(j,k)}(A,B) \subseteq \mathcal{N}_\rho(e),$$

where

$$\rho = \left[\frac{(A-B)\{2(1-|B|) - (1-|A|)\delta_{2,j} + 1\}}{2(1-|B|) - (1-|A|)\delta_{2,j}} \right].$$

Proof. For function $f \in \mathcal{S}^{(j,k)}(A,B)$, by Theorem 3, immediately yields

$$\{2(1-|B|) - (1-|A|)\delta_{2,j}\} \sum_{n=2}^{\infty} |a_n| \leq (A-B),$$

so, that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(A-B)}{2(1-|B|) - (1-|A|)\delta_{2,j}}. \tag{7}$$

On the other hand, we also find from Theorem 3,

$$\sum_{n=2}^{\infty} (n-\delta_{n,j})|a_n| \leq (A-B),$$

also

$$\sum_{n=2}^{\infty} (n-1)|a_n| \leq (A-B),$$

or

$$\sum_{n=2}^{\infty} n|a_n| \leq (A-B) + \sum_{n=2}^{\infty} |a_n|,$$

that is,

$$\sum_{n=2}^{\infty} n|a_n| \leq \left[\frac{(A-B)\{2(1-|B|)-(1-|A|)\delta_{2,j}+1\}}{2(1-|B|)-(1-|A|)\delta_{2,j}}\right] = \rho,$$

which, in view of the definition (3), proves Theorem 5. □

Now we define the neighborhood for each of the class $\mathcal{S}^{(j,k)}(A,B)$. A function $f \in \mathcal{A}$ is said to be in class $\mathcal{S}^{(j,k)}(A,B,\eta)$ if there exists $g \in \mathcal{S}^{(j,k)}(A,B)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\eta.$$

Theorem 6 *Let* $g \in \mathcal{S}^{(j,k)}(A,B)$, *and suppose that*

$$\eta = 1 - \frac{\rho\{2(1-|B|)-(1-|A|)\delta_{2,j}\}}{2\{2(1-|B|)-(1-|A|)\delta_{2,j}-(A-B)\}}, \tag{8}$$

then

$$\mathcal{N}_\rho(g) \subseteq \mathcal{S}^{(j,k)}(A,B,\eta).$$

Proof. Suppose that $f \in \mathcal{N}_\rho(g)$. We then find from (2), that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \rho,$$

which readily implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\rho}{2}.$$

Next, since $g \in \mathcal{S}^{(j,k)}(A,B)$, from (7), we have

$$\sum_{n=2}^{\infty} |b_n| \leq \frac{(A-B)}{2(1-|B|)-(1-|A|)\delta_{2,j}},$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - 1\right| &\leq \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} |b_n|} \\ &\leq \frac{\rho\{2(1-|B|) - (1-|A|)\delta_{2,j}\}}{2\{2(1-|B|) - (1-|A|)\delta_{2,j}| - (A-B)\}} \\ &= 1 - \eta.\end{aligned}$$

That shows that $f \in \mathcal{S}^{(j,k)}(A, B, \alpha, \eta)$ for η given by (8), which completes the proof. □

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A study of the absence of arbitrage opportunities without calculating the risk-neutral probability

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Abstract. In this paper, we establish the property of conditional full support for two processes: the Ornstein Uhlenbeck and the stochastic integral in which the Brownian Bridge is the integrator and we build the absence of arbitrage opportunities without calculating the risk-neutral probability.

1 Introduction

Stochastic portfolio theory is a section of mathematical finance. It is introduced by Fernholz [1, 2], and then further developed by Fernholz, Karatzas and Kardaras [3]. It analyses the results of portfolio by a new and different structure.

The conditional full support (CFS) is a simple condition on asset prices which specifies that from any time, the asset price path can continue arbitrarily close to any given path with positive conditional probability. The conditional full support's notion is introduced by Guasoni et al. (2008) [16] who proves

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that the fractional Brownian motion with arbitrary Hurst parameter has a desired property.

This latter is generalized by Cherny (2008) [17] who proves that any Brownian moving average satisfies the conditional full support condition. Then, the (CSF) property is established for Gaussian processes with stationary increments by Gasbarra (2011) [18].

Let's note that, by the main result of Guasoni et al. (2008) [16], the CFS generates the consistent price systems which admit a martingale measure. In 2014, Attila Herczegh et al. provide a new result on conditional full support in higher dimensions [19].

By the main result of Guasoni, Résonyi, and Schachermayer [16], the CFS generates the consistent price systems which admit a martingale measure.

M. S. Pakkanan in 2009 [7] presents conditions that imply the conditional full support for the process $Z := H + K * W$, where W is a Brownian motion and H is a continuous process.

This paper is organized as follows. Section 2 presents some basic concepts from stochastic portfolio theory and some results on consistent price system. In section 3, we present the conditions that imply the conditional full support (CFS) property for processes $Z := H+K*W$. In section 4, we establish our main result on the conditional full support for the processes: the Ornstein Uhlenbeck and the stochastic integral such that the Brownian Bridge is the integrator and we build the absence of arbitrage opportunities without calculating the risk-neutral probability in the case of existence of the consistent price systems. Finnaly we give a conclusion.

2 Reminder

2.1 Markets and portfolios

We shall place ourselves in a model M for a financial market of the form

$$\begin{aligned} dB(t) &= B(t)r(t)dt, \qquad B(0) = 1, \\ dS_i(t) &= S_i(t)\left(b_i(t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)dW_\nu(t)\right), \\ S_i(0) &= s_i > 0;\ i = 1, \dots, n, \end{aligned} \tag{1}$$

consisting of a money-market $B(.)$ and of n stocks, whose prices $S_1(.);\dots;S_n(.)$ are driven by the d-dimensional Brownian motion $W(.) = W_1(.);\dots;W_d(.))'$ with $d \geq n$.

The following notations are adopted; The interest-rate process $r(.)$ for the money-market, the vector-valued process $b(.) = (b_1(.);\dots;b_n(.))'$ of rates of return for the various stocks, and the (n*d)-matrix-valued process $\sigma(.) = (\sigma_{i\nu}(.))_{1<i<n,1\leq\nu\leq d}$ of stock-price volatilities.

Definition 1 *A portfolio $\pi(.) = (\pi_1(.),\dots,\pi_n(.))'$ is an $\mathbb{F}$-progressively measurable process, bounded uniformly in (t,w), with values in the set*

$$\bigcup_{k\in\mathbb{N}}\{(\pi_1,\dots,\pi_n)\in\mathbb{R}^n|\pi_1^2+\dots+\pi_n^2\leq k^2,\pi_1+\dots+\pi_n=1\}.$$

The Market Portfolio

The stock price $S_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$S(t) = S_1(t)+\dots+S_n(t) \quad \text{and} \quad \mu_i(t) = \frac{S_i(t)}{S(t)}, \quad i = 1,\dots,n \tag{2}$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively.

Clearly, $0<\mu_i(t)<1$, $\forall\, i=1,\dots,n$ and $\sum_{i=1}^n \mu_i(t) = 1$.

The resulting wealth process $V^{w,\mu}(.)$ satisfies

$$\frac{dV^{w,\mu}(t)}{V^{w,\mu}(t)} = \sum_{i=1}^n \mu_i(t)\frac{dS_i(t)}{S_i(t)} = \sum_{i=1}^n \frac{dS_i(t)}{S(t)} = \frac{dS(t)}{S(t)}.$$

2.2 Conditional full support

Definition 2 *Let $\mathcal{O}\subset\mathbb{R}^n$ be an open set and $(S(t))_{t\in[0,T]}$ be a continuous adapted process taking values in $\mathcal{O}$. We say that S has conditional full support in $\mathcal{O}$ if for all $t\in[0,T]$ and for all open set $G\subset C([0,T],\mathcal{O})$,*

$$\mathbb{P}(S\in G|\mathfrak{F}_t)>0, \quad \text{a.s. on the event} \quad (S|_{[0,t]}\in\{g|_{[0,t]}:g\in G\}). \tag{3}$$

We will also say that S has full support in $\mathcal{O}$, or simply full support when $\mathcal{O}=\mathbb{R}^n$, if (3) holds for $t=0$ and for all open subset of $C([0,T],\mathcal{O})$.

Recall also, the notion of consistent price system.

Definition 3 *Let $\varepsilon > 0$. An $\varepsilon-\mathrm{consistent}$ price system to S is a pair $(\widetilde{S}, \mathbf{Q})$, where $\mathbf{Q}$ is a probability measure equivalent to $\mathbf{P}$ and $\widetilde{S}$ is a $\mathbf{Q}-\mathrm{martingale}$ in the filtration $\mathfrak{F}$, such that*

$$\frac{1}{1+\varepsilon} \leq \frac{\widetilde{S}_i(t)}{S_i(t)} \leq 1+\varepsilon, \quad \text{almost surely for all } t \in [0,T] \text{ and } i = 1, \dots, n.$$

Note that $\widetilde{S}$ is a martingale under $\mathbf{Q}$, hence we may asuume that it is càdlàg, but it is not required in the definition that $\widetilde{S}$ is continuous.

Theorem 1 [14] *Let $\mathcal{O} \subset (0,\infty)^n$ be the open set defined by*

$$\mathcal{O} = \mathcal{O}(\delta) = \left\{ x \in (0,\infty)^n : \max_j \frac{x_j}{x_1 + \ldots + x_n} < 1 - \delta \right\} \tag{4}$$

and assume that the price process takes values and has conditional full support in $\mathcal{O}$. Then for any $\varepsilon > 0$ there is an $\varepsilon-\mathrm{consistent}$ price system $(\widetilde{S}, \mathbf{Q})$ such that $\widetilde{S}$ takes values in $\mathcal{O}$.

To check the condition of Theorem 2.1 we apply the next Theorem. In comparison to the existing results, we mention that our findings seem to be new in the sense that we do not assume that our process solves a stochastic differential equation as it is done in Stroock and Varadhan [11] and it is not only for one dimensional processes as it is in Pakkanen [7].

Theorem 2 [14] *Let X be a n-dimensional Itô proces on $[0,T]$, such that*

$$dX_i(t) = \mu_i(t)dt + \sum_{\nu=1}^{n} \sigma_{i\nu}(t)dW_\nu(t).$$

Assume that $|\mu|$ is bounded and σ satisfies

$$\varepsilon|\xi|^2 \leq |\sigma'(t)\xi|^2 \leq M|\xi|^2, \quad \text{a.s. for all } t \in [0,T] \text{ and } \xi \in \mathbb{R}^n \text{ and } \varepsilon, M > 0.$$

Then X has conditional full support.

2.3 Consistent Price System and Conditional Full support

Theorem 3 [14] *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and $(S(t))_{t\in[0,T]}$ be an $\mathcal{O}$-valued, continuous adapted process having conditional full support in $\mathcal{O}$.*

Besides, let $(\varepsilon_t)_{t\in[0,T]}$ *be a continuous positive process, that satisfies*

$$|\varepsilon_t - \varepsilon_s| \leq L_s \sup_{s\leq u\leq t}|S(u) - S(s)|, \quad \text{for all } 0 \leq s \leq t \leq T,$$

with some progressively measurable finite valued $(L_s)_{s\in[0,T]}$.

Then S *admits an* ε*-consistent price system in the sense that, there is an equivalent probability* Q *on* $\mathfrak{F}_t$ *and a process* $(\widetilde{S}(t))_{t\in[0,T]}$ *taking values in* $\mathcal{O}$ *such that* $\widetilde{S}$ *is* Q *martingale, bounded in* $L^2(Q)$ *and finally* $|S(t) - \widetilde{S}(t)| \leq \varepsilon_t$ *almost surely for all* $t \in [0, T]$.

Lemma 1 [14] *Under the assumption of theorem 3.1 there is a sequence of stopping times* $(\tau_n)_{n\geq1}$*, a sequence of random variables* $(X_n)_{n\geq0}$ *and an equivalenty* Q *such that*

1. $\tau_0 = 0$, (τ_n) *is increasing and* $\bigcup_n\{\tau_n = T\}$ *has full probability,*
2. $(X_n)_{n\geq0}$ *is a* Q *martingale in the discrete time filtration* $(\mathfrak{g}_n = \mathfrak{F}_{\tau_n})_{n\geq0}$*, bounded in* $L^2(Q)$*,*
3. *if* $\tau_n \leq t \leq \tau_{n+1}$ *then* $|S_t - X_{n+1}| \leq \varepsilon_t$.

Corollary 1 [14] *Assume that the continuous adapted process* S *evoling in* $\mathcal{O}$ *has conditional full support in* $\mathcal{O}$*. Let* τ *be a stopping time and denote by* $Q_{S|\mathfrak{F}_\tau}$ *the regular version of the conditional distribution of S given* $\mathfrak{F}_\tau$.

Then the support of the random measure $Q_{S|\mathfrak{F}_\tau}$ *is*

$$\operatorname{supp} Q_{S|\mathfrak{F}_\tau} = \left\{ g \in C([0,T], \mathcal{O}) : g|_{[0,\tau]} = S|_{[0,\tau]} \right\}, \qquad \text{almost surely.}$$

3 Conditional full support for stochastic integrals

We shall establish the CFS for processes of the form

$$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0, T],$$

where H is a continuous process, the integrator W is a Brownian motion, and the integrand k satisfies some varying assumptions (to be clarified below). We focus on three cases, each of which requires a separate treatment (see [7]).

First, we study the case:

(1) **Independent integrands and Brownian integrators**

Theorem 4 [7] *Let us define*

$$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0, T]$$

Suppose that

- $(H_t)_{t\in[0,T]}$ *is a continuous process*
- $(k_t)_{t\in[0,T]}$ *is a measurable process s.t.* $\int_0^T K_s^2 ds < \infty$
- $(W_t)_{t\in[0,T]}$ *is a standard Brownian motion independent of* H *and* k.

If we have

$$meas(t \in [0, T] : k_t = 0) = 0 \qquad \mathbf{P} - a.s$$

then Z *has CFS.*

As an application of this result, we show that several popular stochastic volatility models have the CFS property.

Application to stochastic volatility model:

Let us consider the price process $(P_t)_{t\in[0,T]}$ in $\mathbb{R}_+$ given by :

$$dP_t = P_t(f(t, V_t)dt + \rho g(t, V_t)dB_t + \sqrt{1 - \rho^2} g(t, V_t)dW_t,$$

$P_0 = p_0 \in \mathbb{R}_+$, where

(a) $f, g \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$,

(b) (B, W) is a planar Brownian motion,

(c) $\rho \in (-1, 1)$,

(d) V is a (measurable) process in $\mathbb{R}^d$ s.t. $g(t, V_t) \neq 0$ a.s. for all $t \in [0, T]$,

(e) (B, V) is independent of W.

Write using Itô's formula:

$$\log P_t = \underbrace{\log P_0 + \int_0^t (f(s,V_s) - \frac{1}{2}g(s,V_s)^2)ds + \rho\int_0^t g(s,V_s)dB_s}_{=H_t}$$

$$\underbrace{+\sqrt{1-\rho^2}\int_0^t g(s,V_s)dW_s}_{=K_s}.$$

Since W is independent from B and V, the previous Theorem implies that $\log P$ has CFS, and from the next remark, it follows that P has CFS.

Remark 1 *If* $I \subset \mathbb{R}$ *is an open interval and* $f : \mathbb{R} \longrightarrow I$ *is a homeomorphism, then* $g \longmapsto f \circ g$ *is a homeomorphism between* $C_x([0,T])$ *and* $C_{f(x)}([0,T], I)$*.*

Hence, for $f(X)$*, understood as a process on* I*, we have*

$$f(X) \quad \text{has} \quad \mathbb{F}-\text{CFS} \Longleftrightarrow X \quad \text{has} \quad \mathbb{F}-\text{CFS}. \tag{5}$$

Next, we weaken the assumption about independence and consider the second case:

(2) **Progressive integrands and Brownian integrators**

Remark 2 *In general, the assumption about independence between* W *and* (H,k) *is necessary.*

Namely, if e.g.

$$H_t = 1; k_t := e^{W_t - \frac{1}{2}t}; t \in [0,T],$$

then $Z = k = \xi(W)$*, the Dolans exponential of* W*, which is strictly positive and thus does not have CFS if the process is considered in* $\mathbb{R}$*.*

Theorem 5 [7] *Suppose that*

- $(X_t)_{t\in[0,T]}$ *and* $(W_t)_{t\in[0,T]}$ *are continuous process,*
- h *and* k *are progressive* $[0,T] \times C([0,T])^2 \longrightarrow \mathbb{R}$*,*
- ε *is a random variable,*
- *and* $\mathcal{F}_t = \sigma\{\varepsilon, X_s, W_s : s \in [0,t]\}, t \in [0,T]$

If W *is an* $(\mathcal{F}_t)_{t\in[0,T]}$*-Brownian motion and*

- $E[e^{\lambda \int_0^T k_s^{-2} ds}] < \infty$ *for all* $\lambda > 0$,
- $E[e^{2\int_0^T k_s^{-2} h_s^2 ds}] < \infty$ *and*
- $\int_0^T k_s^2 ds \leq \overline{K}$ *a.s for some constant* $\overline{K} \in (0, \infty)$,

then the process

$$Z_t = \varepsilon + \int_0^t h_s ds + \int_0^t k_s dw_s, \qquad t \in [0, T]$$

has CFS.

(3) **Independent integrands and general integrators**

Since the Brownian motion has CFS, one might wonder if the previous results can be generalized to the case where the integrator is merely a continuous process with CFS. While the proofs of these results use quite heavily methods specific to Brownian motion (martingales, time changes), so in the case of independent integrands of finite variations, we are able to prove this conjecture.

Theorem 6 [7] *Suppose that*

- $(H_t)_{t\in[0,T]}$ *is a continuous process,*
- $(k_t)_{t\in[0,T]}$ *is a process of finite variation, and*
- $X = (X_t)_{t\in[0,T]}$ *is a continuous process independent of* H *and* k.

Let us define

$$Z_t := H_t + \int_0^t k_s dX_s, \qquad t \in [0, T].$$

If X *has CFS and*

$$\inf_{t\in[0,T]} |k_t| > 0 \qquad \mathbf{P} - \text{a.s.},$$

then Z *has CFS.*

4 Main result

In this part, we will use the following theorem to demonstrate the absence of arbitration without calculating the risk-neutral probability for the two models below.

Theorem 7 [4, theorem 1.2] *Let* (X_t) *be an* $\mathbb{R}^d_+$*-valued, continuous adapted process satisfying (CFS); then* X *admits an* ε*-consistent pricing system for all* $\varepsilon > 0$.

4.1 Ornstein-Uhlenbeck process driven by Brownian motion

The (one-dimensional) Gaussian Ornstein-Uhlenbeck process $X = (X_t)_t \geq 0$ can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t \qquad t > 0.$$

Where we see

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)}\, dW_s. \qquad t \geq 0.$$

It is readily seen that X_t is normally distributed. We have

$$X_t = \underbrace{X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})}_{H_t} + \int_0^t \underbrace{\sigma e^{\theta(s-t)}}_{K_s}\, dW_s. \qquad t \geq 0. \tag{6}$$

To establish the property of CFS for this process, the conditions of theorem 3.1 will be applied.

The processes (H_s) and (K_s) in (6) satisfy

1. Process (H_s) is a continuous process,
2. (K_s) is a measurable process such that $\int_0^T K_s^2 ds < \infty$, and
3. (W_t) is a standard Brownian motion independent of H and K.

Consequently, the process (X_t) has the property of CFS and there are the consistent price systems which can be seen as generalization of equivalent martingale measures.

This shows that this price process doesn't admit arbitrage opportunities under arbitrary small transaction and using it, we guarantee no-arbitrage without calculating the risk-neutral probability.

4.2 Independent integrands and Brownian Bridge integrators.

To state our main result for the application of CFC in which the Brownian Bridge is the integrator, we need to recall some facts of Brownian bridge.

Let us start with a Brownian motion $B = (B_t, t \geq 0)$ and its natural filtration $\mathbb{F}^B$. Define a new filtration as $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ with $\mathcal{G}_t = \mathcal{F}_t^{(B_1)} = \mathcal{F}_t^B \vee \sigma(B_1)$. In this filtration, the process $(B_t, t \geq 0)$ is no longer a martingale. It is easy to be convinced of this by looking at the process $(E(B_1 \mid \mathcal{F}_t^{(B_1)}), t \leq 1)$: this process is identically equal to B_1, not to B_t, hence $(B_t; t \geq 0)$ is not a $\mathbb{G}$-martingale. However, $(B_t, t \geq 0)$ is a $\mathbb{G}$-semi-martingale, as follows from the next proposition 1.

In general, if $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ is a filtration larger than $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$, i.e., $\mathcal{F}_t \subset \mathcal{H}_t, \forall t \geq 0$ (we shall write $\mathbb{F} \subset \mathbb{H}$), it is not true that an $\mathbb{F}$-martingale remains a martingale in the filtration $\mathbb{H}$. It is not even true that $\mathbb{F}$-martingales remain $\mathbb{H}$-semi-martingales.

Before giving this proposition, we recall the definition of Brownian bridge.

Definition 4 *The Brownian bridge* $(b_t; 0 \leq t \leq 1)$ *is defined as the conditioned process* $(B_t; t \leq 1 | B_1 = 0)$.

Note that $B_t = (B_t - tB_1) + tB_1$ *where, from the Gaussian property, the process* $(B_t - tB_1; t \leq 1)$ *and the random variable* B_1 *are independent. Hence*

$$(b_t; 0 \leq t \leq 1) \overset{law}{=} (B_t - tB_1;\ 0 \leq t \leq 1).$$

The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t); s \leq t$. *Moreover, it satisfies* $b_0 = b_1 = 0$.

Proposition 1 [15] *Let* $\mathcal{F}_t^{(B_1)} = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(B_1)$. *The process*

$$\beta_t = B_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1 - s} ds$$

is an $\mathbb{F}^{(B_1)}$*-martingale, and an* $\mathbb{F}^{(B_1)}$ *Brownian motion. In other words,*

$$B_t = \beta_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1 - s} ds$$

is the decomposition of B *as an* $\mathbb{F}^{(B_1)}$*-semi-martingale.*

Example of application: The following example was studied by Monique Jeanblanc et al. [15], we will later introduce our approach to this application, this approach is based on the conditional full support property. M.Jeanblanc et al. study within the problem occurring in insider trading: existence of arbitrage using strategies adapted w.r.t. the large filtration.

Our approach is to prove the existence of no arbitrage in the case $0 \leq t < 1$ without calculating the dynamics of wealth and risk neutral probability.

Let

$$dS_t = S_t(\mu dt + \sigma db_t),$$

where μ and σ are constants and S_t defines the price of a risky asset. Assume that the riskless asset has a constant interest rate r.

The wealth of an agent is

$$dX_t = rX_t dt + \widehat{\pi}_t(dS_t - rS_t dt) = rX_t dt + \pi_t \sigma X_t(dW_t + \theta dt); \quad X_0 = x,$$

where $\theta = \frac{\mu - r}{\sigma}$ and $\pi = (\widehat{\pi} S_t / X_t)$ assumed to be an $\mathbb{F}^B$-adapted process.

Here, $\widehat{\pi}$ is the number of shares of the risky asset, and π the proportion of wealth invested in the risky asset. It follows that

$$\ln(X_T^{\pi,x}) = \ln x + \int_0^T (r - \frac{1}{2}\pi_s^2 \sigma^2 + \theta \pi_s \sigma) ds + \int_0^T \sigma \pi_s dW_s$$

Then,

$$E(\ln(X_T^{\pi,x})) = \ln x + \int_0^T E\left(r - \frac{1}{2}\pi_s^2 \sigma^2 + \theta \pi_s \sigma\right) ds$$

The solution of $\max E(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\theta}{\sigma}$ and

$$\sup E(\ln(X_T^{\pi,x}) = \ln x + T\left(r + \frac{1}{2}\theta^2\right)$$

Note that, if the coefficients r, μ and σ are $\mathbb{F}$-adapted, the same computation leads to

$$\sup E(\ln(X_T^{\pi,x}) = \ln x + \int_0^T E\left(r_t + \frac{1}{2}\theta_t^2\right) dt,$$

where $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$.

We now enlarge the filtration with S_1.

In the enlarged filtration, setting, for $t < 1, \alpha_t = \frac{B_1 - B_t}{1-t}$, the dynamics of S are

$$dS_t = S_t((\mu + \sigma \alpha_t) dt + \sigma d\beta_t),$$

and the dynamics of the wealth are

$$dX_t = rX_t dt + \pi_t \sigma X_t(d\beta_t + \widetilde{\theta}_t dt), \quad X_0 = x$$

with $\widetilde{\theta}_t = \frac{\mu - r}{\sigma} + \alpha_t$.
The solution of $\max \mathsf{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\widetilde{\theta}_s}{\sigma}$.

Then, for $T < 1$,

$$\ln(X_T^{\pi,x,*}) = \ln x + \int_0^T (r + \frac{1}{2}\widetilde{\theta}_s^2) ds + \int_0^T \sigma \pi_s d\beta_s$$

$$\begin{aligned} \mathsf{E}(\ln(X_T^{\pi,x,*})) &= \ln x + \int_0^T (r + \frac{1}{2}(\theta^2 + \mathsf{E}(\alpha_s^2) + 2\theta \mathsf{E}(\alpha_s)) ds \\ &= \ln x + (r + \frac{1}{2}\theta^2)T + \frac{1}{2}\int_0^T \mathsf{E}(\alpha_s^2) ds, \end{aligned}$$

where we have used the fact that $\mathsf{E}(\alpha_t) = 0$ (if the coefficients r, μ and σ are $\mathbb{F}$-adapted, α is orthogonal to $\mathcal{F}_t$, hence $\mathsf{E}(\alpha_t \theta_t) = 0$).

Let

$$\begin{aligned} V^{\mathbb{F}}(x) &= \max \mathsf{E}(\ln(X_T^{\pi,x})); \pi \text{ is } \mathbb{F} \text{ admissible} \\ V^{\mathbb{G}}(x) &= \max \mathsf{E}(\ln(X_T^{\pi,x})); \pi \text{ is } \mathbb{G} \text{ admissible} \end{aligned}$$

Then $V^{\mathbb{G}}(x) = V^{\mathbb{F}}(x) + \frac{1}{2}\mathsf{E}\int_0^T \alpha_s^2 ds = V^{\mathbb{F}}(x) - \frac{1}{2}\ln(1 - T)$.

If $T = 1$, the value function is infinite: there is an arbitrage opportunity and there exists no an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \le 1)$ is a $\mathbb{G}$-martingale. However, for any $\epsilon \in]0;1]$, there exists a uniformly integrable $\mathbb{G}$-martingale L defined as

$$dL_t = \frac{\mu - r + \sigma\sigma_t}{\sigma} L_t d\beta_t, t \le 1 - \epsilon, \quad L_0 = 1,$$

such that, setting $d\mathbb{Q} \mid_{\mathcal{G}_t} = L_t d\mathbb{P} \mid_{\mathcal{G}_t}$, the process $(e^{-rt}S_t; t \le 1 - \epsilon)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.

This is the main point in the theory of insider trading where the knowledge of the terminal value of the underlying asset creates an arbitrage opportunity and this is effective at time 1.

Our approach to this example: We consider the previous example. Let

$$dS_t = S_t(\mu dt + \sigma db_t),$$

The standard Brownian bridge $b(t)$ is a solution of the following stochastic equation.

$$\begin{aligned} db_t &= -\frac{b_t}{1 - t} dt + dW_t; \qquad 0 \le t < 1 \\ b_0 &= 0. \end{aligned} \tag{7}$$

The solution of the above equation is

$$b_t = (1-t)\int_0^t \frac{1}{1-s} dW_s,$$

We may now verify that S has CFS.

By positivity of S, Itô's formula yields

$$\log S_t = \log S_0 + \left\{ \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\left(1-t\right)\int_0^t \frac{1}{1-s} dW_s \right\}, \qquad 0 \le t < 1.$$

We have

$$\log S_t = \underbrace{\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t}_{=:H_t} + \int_0^t \underbrace{\sigma\left(1-t\right)\frac{1}{1-s}}_{=:K_s} dW_s, \qquad 0 \le t < 1.$$

1. (H_t) is a continuous process,
2. $(K_s) = \sigma(1-t)\frac{1}{1-s}$ is a measurable process s.t. $\int_0^t K_s^2 ds < \infty$,
3. (W_t) is a standard Brownian motion independent of H and K,

which clearly satisfy the assumptions of theorem (3.1) and $\log S_t$ has CFS, then S has CFS for $0 \le t < 1$ and there is the consistent price systems and this is a martingale. Using it, we guarantee no-arbitrage without calculating the risk-neutral probability.

5 Conclusion

In this paper, we have investigated the conditional full support for two processes, the Ornstein Uhlenbeck and the Stochastic integral in which the Brownian Bridge is the integrator, and we have also built the absence of arbitrage opportunities without calculating the risk-neutral probability in the existence of the consistent price systems which admit a martingale measure.

Prospects: In mathematical finance, the CoxIngersollRoss model (or CIR model) describes the evolution of interest rates. It is a type of "one factor model" (short rate model) as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives. It was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model. The CIR model specifies that the instantaneous interest rate follows the stochastic differential equation, also named the CIR Process:

$$dX_t = \theta(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t \qquad t > 0,$$

where (W_t) is a Wiener process and θ, μ and σ are the parameters. The parameter θ corresponds to the speed of adjustment, μ to the mean and σ to volatility. The drift factor $\theta(\mu - X_t)$ is exactly the same as in the Vasicek model. It ensures a mean reversion of the interest rate towards the long run value μ, with speed of adjustment governed by the strictly positive parameter θ.

As prospects, we establish the condition of CFS for the Cox-Ingersoll-Ross model.

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Approximation properties of (p, q)-Bernstein type operators

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Abstract. We introduce a new generalization of the q-Bernstein operators involving (p, q)-integers, and we establish some direct approximation results. Further, we define the limit (p, q)-Bernstein operator, and we obtain its estimation for the rate of convergence. Finally, we introduce the (p, q)-Kantorovich type operators, and we give a quantitative estimation.

1 Introduction

The applications of q-calculus in the field of approximation theory have led to the discovery of new generalizations of the Bernstein operators. The first generalization involving q-integers was obtained by Lupaş [7] in 1987. Ten years later Phillips [12] gave another generalization of the Bernstein operators introducing the so-called q-Bernstein operators. In comparison with Phillips' generalization, the Lupaş' generalization gives rational functions rather than polynomials. Nowadays, q-Bernstein operators form an area of an intensive research. A survey of the obtained results and references in this area during the first decade of study can be found in [11]. After that several well-known positive linear operators and other new operators have been generalized to their q-variants, and their approximation behavior have been studied (see e.g. [1] and [3]).

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The (p, q)-calculus is a further new generalization of the q-calculus, its basic definitions and some properties may be found in the papers [6], [13], [14], [15]. The (p, q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q},$$

where $n = 0, 1, 2, \dots$ and $0 < q < p \leq 1$. For $p = 1$, we recover the well-known q-integers (see [5]). Obviously

$$[n]_{p,q} = p^{n-1}[n]_{q/p}. \tag{1}$$

The (p, q)-factorials $[n]_{p,q}!$ are defined by

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \dots [n]_{p,q}, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0, \end{cases}$$

and the (p, q)-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n.$$

Further, we set

$$(a - b)^n_{p,q} = \begin{cases} (a - b)(pa - qb) \dots (p^{n-1}a - q^{n-1}b), & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0. \end{cases}$$

By simple computations, using (1), we get

$$[n]_{p,q}! = p^{n(n-1)/2}[n]_{q/p}!, \tag{2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{\{n(n-1)-k(k-1)-(n-k)(n-k-1)\}/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p} \tag{3}$$

and

$$(a - b)^n_{p,q} = p^{n(n-1)/2}(a - b)^n_{q/p}, \tag{4}$$

where

$$(a - b)^n_q = \begin{cases} (a - b)(a - qb) \dots (a - q^{n-1}b), & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0 \end{cases}$$

in the case when $0 < q < 1$.

The goal of the paper is to introduce a new generalization of the q-Bernstein operators involving (p,q)-integers. These (p,q)-Bernstein operators approximate each continuous function uniformly on $[0,1]$, and some direct approximation results are established with the aid of the modulus of continuity given by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x - y| \le \delta\}, \quad \delta > 0, \tag{5}$$

where $f \in C[0,1]$. Further, we define the limit (p,q)-Bernstein operator and we estimate the rate of convergence by the modulus of continuiuty (5). The concept of limit q-Bernstein operator was introduced by Il'inskii and Ostrovska [4], and its rate of convergence was established by Wang and Meng in [16]. Finally, we define a (p,q)-Kantorovich variant of the (p,q)-Bernstein operators, and we give a quantitative estimation using (5).

2 (p,q)-Bernstein operators

For $0 < q < p \le 1$, $f \in C[0,1]$, $x \in [0,1]$ and $n = 1, 2, \ldots$, we define the (p,q)-Bernstein polynomials as follows:

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} p^{\{k(k-1)-n(n-1)\}/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n-k} f\left(p^n \frac{[k]_{p,q}}{[n]_{p,q}}\right). \tag{6}$$

For $p = 1$ and $0 < q < 1$, we recover the q-Bernstein polynomials (see [12]):

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} x^k (1-x)_{q}^{n-k} f\left(\frac{[k]_{q}}{[n]_{q}}\right). \tag{7}$$

Theorem 1 *If the sequences* (p_n) *and* (q_n) *satisfy* $0 < q_n < p_n \le 1$ *for* $n = 1, 2, \ldots$, *and* $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$ *as* $n \to \infty$, *then*

$$|B_{n,p_n,q_n}(f;x) - f(x)| \le 2\omega\left(f; \left(2(1-p_n^n)x^2 + \frac{x(1-x)}{[n]_{q_n/p_n}}\right)^{1/2}\right)$$

for all $f \in C[0,1]$ *and* $x \in [0,1]$.

Proof. By (6), (3)-(4) and (1), we have

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p} x^k (1-x)_{q/p}^{n-k} f\left(p^k \frac{[k]_{q/p}}{[n]_{q/p}}\right). \tag{8}$$

Hence, in view of [12, (13)], we obtain

$$B_{n,p,q}(1;x) = B_{n,q/p}(1;x) = 1. \tag{9}$$

By (8) and [12, (14)], we get

$$p^n x = p^n B_{n,q/p}(t;x) \le B_{n,p,q}(t;x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q/p} x^k(1-x)_{q/p}^{n-k} p^k \frac{[k]_{q/p}}{[n]_{q/p}}. \tag{10}$$

Analogously, by (8) and [12, (15)], we get

$$\begin{aligned} B_{n,p,q}(t^2;x) &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q/p} x^k(1-x)_{q/p}^{n-k} p^{2k} \frac{[k]_{q/p}^2}{[n]_{q/p}^2} \\ &\le B_{n,q/p}(t^2;x) = x^2 + \frac{x(1-x)}{[n]_{q/p}}. \end{aligned} \tag{11}$$

On the other hand, it is known for (5) that

$$\omega(f;\lambda\delta) \le (1+\lambda)\omega(f;\delta), \tag{12}$$

where $\lambda \ge 0$ and $\delta > 0$. Then, by (8), [12, (13)], Hölder's inequality and (9)-(11), we obtain

$$\begin{aligned} &|B_{n,p_n,q_n}(f;x) - f(x)| \\ &\le \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q_n/p_n} x^k(1-x)_{q_n/p_n}^{n-k} \left| f\left(p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}}\right) - f(x) \right| \\ &\le \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q_n/p_n} x^k(1-x)_{q_n/p_n}^{n-k} \omega\left(f; \left| p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - x \right|\right) \\ &\le \omega(f;\delta) \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q_n/p_n} x^k(1-x)_{q_n/p_n}^{n-k} \left(1 + \delta^{-1} \left| p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - x \right|\right) \\ &\le \omega(f;\delta) \left\{ 1 + \delta^{-1} \left(\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q_n/p_n} x^k(1-x)_{q_n/p_n}^{n-k} \right.\right. \\ &\quad \left.\left. \times \left(p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - x \right)^2 \right)^{1/2} \right\} \\ &= \omega(f;\delta) \left\{ 1 + \delta^{-1} (B_{n,p_n,q_n}(t^2;x) - 2xB_{n,p_n,q_n}(t;x) \right. \\ &\quad \left. + x^2 B_{n,p_n,q_n}(1;x))^{1/2} \right\} \end{aligned}$$

$$\begin{aligned} &\leq \omega(f;\delta)\left\{1+\delta^{-1}\left(x^2+\frac{x(1-x)}{[n]_{q_n/p_n}}-2p_n^n x^2+x^2\right)^{1/2}\right\} \\ &= \omega(f;\delta)\left\{1+\delta^{-1}\left(2(1-p_n^n)x^2+\frac{x(1-x)}{[n]_{q_n/p_n}}\right)^{1/2}\right\}. \end{aligned}$$

Choosing $\delta = \left(2(1-p_n^n)x^2+\frac{x(1-x)}{[n]_{q_n/p_n}}\right)^{1/2}$, we arrive at the statement of our theorem. □

Theorem 2 *If the sequences* (p_n) *and* (q_n) *satisfy* $0 < q_n < p_n \leq 1$ *for* $n = 1, 2, \ldots,$ *and* $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$ *as* $n \to \infty$, *then*

$$|B_{n,p_n,q_n}(f;x) - B_{n,q_n/p_n}(f;x)| \leq \omega(f;1-p_n^n)$$

for all $f \in C[0,1]$ *and* $x \in [0,1]$.

Proof. Because $\left| p^k \frac{[k]_{q/p}}{[n]_{q/p}} - \frac{[k]_{q/p}}{[n]_{q/p}} \right| \leq 1-p^k \leq 1-p^n$ for $k = 0, 1, \ldots, n$, we find from (8), (7) and [12, (13)], that

$$\begin{aligned} &|B_{n,p_n,q_n}(f;x) - B_{n,q_n/p_n}(f;x)| \\ &\leq \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n/p_n} x^k(1-x)_{q_n/p_n}^{n-k} \left| f\left(p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}}\right) - f\left(\frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}}\right) \right| \\ &\leq \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n/p_n} x^k(1-x)_{q_n/p_n}^{n-k} \omega\left(f; \left| p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} \right|\right) \\ &\leq \omega(f;1-p_n^n) B_{n,q_n/p_n}(1;x) = \omega(f;1-p_n^n), \end{aligned}$$

which is the required estimation. □

Remark 1 There exist sequences (p_n) and (q_n) with the properties enumerated in Theorem 1: $p_n = 1 - \frac{1}{(n+1)^2}$ and $q_n = 1 - \frac{1}{n+1}$, $n = 1, 2, \ldots$

We also mention, if $0 < q_n < p_n \leq 1$ for $n = 1, 2, \ldots,$ $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$, then $[n]_{q_n/p_n} \to \infty$ and $\frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}} \to 1$ as $n \to \infty$.

Remark 2 In [9] and [10] are introduced two different generalizations of the q-Bernstein polynomials (7) involving (p, q)-integers. The first one does not preserve even the constant functions, and the second one is a (q/p)-Bernstein polynomial. Our (p, q)-Bernstein polynomials defined by (6) are different from the above mentioned generalizations. The advantage of (6) is that it allows us to introduce the limit (p, q)-Bernstein operator.

3 Limit (p, q)-Bernstein operator

For $q \in (0, 1)$, Il'inskii and Ostrovska proved in [4] that for each $f \in C[0, 1]$, the sequence $(B_{n,q}(f; x))$ converges to $B_{\infty,q}(f; x)$ as $n \to \infty$ uniformly for $x \in [0, 1]$, where

$$B_{\infty,q}(f;x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} f(1-q^k)\frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^s x), & \text{if} \quad 0 \le x < 1 \\ f(1), & \text{if} \quad x = 1 \end{cases}$$

is the limit q-Bernstein operator. Wang and Meng [16] proved for all $f \in C[0, 1]$ and $x \in [0, 1]$ that

$$|B_{n,q}(f;x) - B_{\infty,q}(f;x)| \le \left(2 + \frac{4}{q(1-q)} \ln \frac{1}{1-q}\right) \omega(f; q^n).$$

For $0 < q < p \le 1$, the limit (p, q)-Bernstein operator $B_{\infty,p,q} : C[0, 1] \to C[0, 1]$ is defined as follows:

$$B_{\infty,p,q}(f;x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} f(p^k - q^k)\frac{p^{(k+1)k/2}x^k}{(p-q)^k[k]_{p,q}!}\prod_{s=0}^{\infty}\frac{p^s - q^s x}{p^s}, & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1. \end{cases} \tag{13}$$

Theorem 3 *Let* $p, q \in (0, 1)$ *be given such that* $p^2 < q < p$. *Then, for every* $f \in C[0, 1]$, $x \in [0, 1]$ *and* $n = 1, 2, \dots,$ *we have*

$$|B_{n,p,q}(f;x) - B_{\infty,p,q}(f;x)| \le \left(4 + \frac{6p^2}{q(p-q)} \ln \frac{p}{p-q}\right) \omega\left(f; \left(\frac{q}{p}\right)^n\right).$$

Proof. Due to (13) and (2), we have

$$B_{\infty,p,q}(f;x) = \sum_{k=0}^{\infty} f(p^k - q^k)\frac{x^k}{\left(1-\frac{q}{p}\right)^k [k]_{q/p}!}\prod_{s=0}^{\infty}\left(1 - \left(\frac{q}{p}\right)^s x\right). \tag{14}$$

We set

$$w_{n,k}(q;x) = \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k(1-x)_q^{n-k} \text{ and } w_{\infty,k}(q;x) = \frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^s x).$$

Then, in view of (9) and [16, p. 154, (2.3)], we obtain

$$\sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right) = \sum_{k=0}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right) = 1. \tag{15}$$

Using (8), (14) and (15), we find

$$\begin{aligned}
&|B_{n,p,q}(f;x) - B_{\infty,p,q}(f;x)| \\
&= \quad \Bigg| \sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right)\left\{f\left(p^k\frac{[k]_{q/p}}{[n]_{q/p}}\right) - f(p^k - q^k)\right\} \\
&\quad + \sum_{k=0}^{n}\left\{w_{n,k}\left(\frac{q}{p};x\right) - w_{\infty,k}\left(\frac{q}{p};x\right)\right\}\{f(p^k - q^k) - f(p^n)\} \\
&\quad - \sum_{k=n+1}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right)\{f(p^k - q^k) - f(p^n)\}\Bigg| \\
&\leq \quad \sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right)\left| f\left(p^k\frac{[k]_{q/p}}{[n]_{q/p}}\right) - f(p^k - q^k)\right| \\
&\quad + \sum_{k=0}^{n}\left| w_{n,k}\left(\frac{q}{p};x\right) - w_{\infty,k}\left(\frac{q}{p};x\right)\right| |f(p^k - q^k) - f(p^n)| \\
&\quad + \sum_{k=n+1}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right)|f(p^k - q^k) - f(p^n)| \\
&=: \quad I_1 + I_2 + I_3.
\end{aligned} \tag{16}$$

The estimation of I_1 : by (1), we have

$$\begin{aligned}
\left| p^k\frac{[k]_{q/p}}{[n]_{q/p}} - (p^k - q^k)\right| &= \frac{[k]_{q/p}}{[n]_{q/p}}\left| p^k - (p^k - q^k)p^{k-n}\frac{[n]_{p,q}}{[k]_{p,q}}\right| \\
\leq \quad &\left| p^k - (p^k - q^k)p^{k-n}\frac{p^n - q^n}{p^k - q^k}\right| = p^k\left(\frac{q}{p}\right)^n \leq \left(\frac{q}{p}\right)^n
\end{aligned}$$

for $k = 0, 1, \dots, n$. Hence, by (15),

$$I_1 \leq \sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right)\omega\left(f;\left| p^k\frac{[k]_{q/p}}{[n]_{q/p}} - (p^k - q^k)\right|\right) \leq \omega\left(f;\left(\frac{q}{p}\right)^n\right). \tag{17}$$

The estimation of I_2 : for $k = 0, 1, \dots, n$, we have $|p^k - q^k - p^n| \le p^k(1 - p^{n-k}) + q^k \le p^k + q^k$. Hence, by (12),

$$\begin{aligned} |f(p^k - q^k) - f(p^n)| &\le \omega(f; |p^k - q^k - p^n|) \le \omega(f; p^k + q^k) \\ &= \omega\left(f; \frac{p^k + q^k}{(q/p)^n}\left(\frac{q}{p}\right)^n\right) \le \left(1 + \frac{p^k + q^k}{(q/p)^n}\right)\omega\left(f; \left(\frac{q}{p}\right)^n\right). \end{aligned} \tag{18}$$

But

$$\begin{aligned} \left(1 + \frac{p^k + q^k}{(q/p)^n}\right)\left(\frac{p}{q}\right)^k &= \left(\frac{q}{p}\right)^{-n}\left(\left(\frac{q}{p}\right)^n + p^k + q^k\right)\left(\frac{p}{q}\right)^k \\ &= \left(\frac{q}{p}\right)^{-n}\left(\left(\frac{q}{p}\right)^{n-k} + \left(\frac{p^2}{q}\right)^k + p^k\right) \le 3\left(\frac{q}{p}\right)^{-n}, \end{aligned}$$

because $p^2 < q < p$ and $k = 0, 1, \dots, n$. Then, by (18), we obtain

$$\begin{aligned} I_2 &\le \sum_{k=0}^{n}\left| w_{n,k}\left(\frac{q}{p}; x\right) - w_{\infty,k}\left(\frac{q}{p}; x\right)\right| 3\left(\frac{q}{p}\right)^{k-n}\omega\left(f; \left(\frac{q}{p}\right)^n\right) \\ &= 3\left(\frac{q}{p}\right)^{-n}\omega\left(f; \left(\frac{q}{p}\right)^n\right)\sum_{k=0}^{n}\left(\frac{q}{p}\right)^k\left| w_{n,k}\left(\frac{q}{p}; x\right) - w_{\infty,k}\left(\frac{q}{p}; x\right)\right|. \end{aligned}$$

Taking into account the estimation

$$\sum_{k=0}^{n} q^k |w_{n,k}(q; x) - w_{\infty,k}(q; x)| \le \frac{2q^n}{q(1-q)} \ln\frac{1}{1-q},$$

where $0 < q < 1$ (see [16, p. 156, (2.9)]), we find that

$$I_2 \le \frac{6p^2}{q(p-q)} \ln\frac{p}{p-q}\ \omega\left(f; \left(\frac{q}{p}\right)^n\right). \tag{19}$$

The estimation of I_3 : for $k \ge n+1$, we have $|p^k - q^k - p^n| \le p^n(1 - p^{k-n}) + q^k \le p^n + q^n$. Hence, by (12) and $p^2 < q < p$, we get

$$\begin{aligned} |f(p^k - q^k) - f(p^n)| &\le \omega(f; |p^k - q^k - p^n|) \le \omega(f; p^n + q^n) \\ &\le \left(1 + \frac{p^n + q^n}{(q/p)^n}\right)\omega\left(f; \left(\frac{q}{p}\right)^n\right) = \left(1 + \left(\frac{p^2}{q}\right)^n + p^n\right)\omega\left(f; \left(\frac{q}{p}\right)^n\right) \\ &\le 3\omega\left(f; \left(\frac{q}{p}\right)^n\right). \end{aligned}$$

Then, by (15),

$$I_3 \leq 3\omega\left(f;\left(\frac{q}{p}\right)^n\right)\sum_{k=n+1}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right) \leq 3\omega\left(f;\left(\frac{q}{p}\right)^n\right). \tag{20}$$

Combining (16)-(17) and (19)-(20), we obtain the statement of the theorem. □

4 (p, q)-Kantorovich operators

Our (p,q)-Kantorovich operators are defined as follows:

$$\begin{aligned} K_{n,p,q}(f;x) &= \frac{[n+1]_{p,q}}{p^n}\sum_{k=0}^{n} p^{\{k(k-1)-n(n-1)\}/2}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k(1-x)_{p,q}^{n-k} \\ &\quad \times q^{-k}\int_{p^{n+1}\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{p^n\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} f(u)\, d^R_{q/p}u, \end{aligned} \tag{21}$$

where $f \in C[0,1]$, $x \in [0,1]$, $n = 1,2,\ldots$, and the Riemann type q-integral of f over the interval $[a,b]$ $(0 \leq a < b;\ 0 < q < 1)$ is given by (see [2], [8])

$$\int_a^b f(u)\, d^R_q u = (1-q)(b-a)\sum_{j=0}^{\infty} q^j f(a+(b-a)q^j). \tag{22}$$

Remark 3 In [15] the (p,q)-integral of f over the interval $[0,a]$ is defined as

$$\int_0^a f(u)\, d_{p,q}u = (p-q)a\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}} f\left(a\frac{q^j}{p^{j+1}}\right),$$

where $0 < q < p \leq 1$. But $\frac{1}{p}a \notin [0,a]$ for $0 < p < 1$ (in the sum the case $j=0$), thus the function f is not defined at $\frac{1}{p}a$. For this reason we use the Riemann type (q/p)-integral in (21).

Theorem 4 *If the sequences* (p_n) *and* (q_n) *satisfy* $0 < q_n < p_n \leq 1$ *for* $n = 1,2,\ldots,$ *and* $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$ *as* $n \to \infty$, *then*

$$|K_{n,p_n,q_n}(f;x) - f(x)| \leq 2\omega(f;\sqrt{\delta_n(x)})$$

for all $f \in C[0,1]$ *and* $x \in [0,1]$, *where*

$$\delta_n(x) = \left\{2(1-p_n^n)\frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}} + \left(1 - \frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}}\right)^2 - \frac{[n]_{q_n/p_n}}{[n+1]^2_{q_n/p_n}}\right\} \times x^2 + 3\frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}} x + \frac{1}{[n+1]^2_{q_n/p_n}}.$$

Proof. By (21), (3)-(4) and (1), we have

$$K_{n,p,q}(f;x) = [n+1]_{q/p} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p} x^k (1-x)^{n-k}_{q/p} q^{-k} \int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} f(u)\, {}^{R}_{q/p} u. \tag{23}$$

By simple computations, using (22), we obtain

$$\int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} 1\, d^R_{q/p} u = p^k \frac{[k+1]_{q/p} - [k]_{q/p}}{[n+1]_{q/p}} = \frac{q^k}{[n+1]_{q/p}}, \tag{24}$$

$$\int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} u\, d^R_{q/p} u = \frac{q^k}{[n+1]_{q/p}} \left(p^k \frac{[k]_{q/p}}{[n+1]_{q/p}} + \frac{p}{p+q}\frac{q^k}{[n+1]_{q/p}}\right) \tag{25}$$

and

$$\int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} u^2\, d^R_{q/p} u = \frac{q^k}{[n+1]_{q/p}} \left(p^{2k} \frac{[k]^2_{q/p}}{[n+1]^2_{q/p}} + \frac{2p}{p+q} p^k \frac{[k]_{q/p}}{[n+1]_{q/p}} \times \frac{q^k}{[n+1]_{q/p}} + \frac{p^2}{p^2+pq+q^2}\frac{q^{2k}}{[n+1]^2_{q/p}}\right). \tag{26}$$

In what follows, taking into account (23)-(26), the proof is similar to the proof of Theorem 1, therefore we omit the details. □

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Generalized pairing strategies-a bridge from pairing strategies to colorings

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Abstract. In this paper we define a bridge between pairings and colorings of the hypergraphs by introducing a generalization of pairs called t-cakes for $t \in \mathbb{N}, t \geq 2$. For $t = 2$ the 2-cakes are the same as the well-known pairs of system of distinct representatives, that can be turned to pairing strategies in Maker-Breaker hypergraph games, see Hales and Jewett [12]. The two-colorings are the other extremity of t-cakes, in which the whole ground set of the hypergraph is one big cake that we divide into two parts (color classes). Starting from the pairings (2-cake placement) and two-colorings we define the generalized t-cake placements where we pair p elements by q elements ($p, q \in \mathbb{N}, 1 \leq p, q < t, p + q = t$).

The method also gives bounds on the condition of winnings in certain biased Chooser-Picker games, which can be introduced similarly to Beck [3]. We illustrate these ideas on the k-in-a-row games for different values of k on the infinite chessboard.

1 Introduction

Pairings and colorings

Let us start by recalling pairings and two-colorings of hypergraphs. Given a hypergraph $\mathcal{H} = (V, E)$, where $V = V(\mathcal{H})$ and $E = E(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H}) = \{S : S \subseteq V\}$

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are the set of vertices and edges, respectively. A bijection $\rho : X \to Y$, where $X, Y \subset V(\mathcal{H})$ and $X \cap Y = \emptyset$ is a **pairing** on the hypergraph $\mathcal{H}$. An $(x, \rho(x))$ pair **blocks** an $A \in E(\mathcal{H})$ edge, if A contains both elements of the pair. If the pairs of ρ block all $A \in E(\mathcal{H})$ edges, we say that ρ is a **good pairing** of $\mathcal{H}$.

Another, somehow related fundamental notion of hypergraphs are the two-colorings. A **two-coloring** of the hypergraph $\mathcal{H} = (V, E)$ is a partition of V into two disjoint (color) classes. We can also define blocking in the following way. A two-coloring of $\mathcal{H}$ **blocks** an $A \in E$ edge, if A has a non-empty intersection with both color classes. A two-coloring is a **good two-coloring**, if it blocks all $A \in E$ edges. Now, we underline a close relation between pairings and two-colorings that is important for us.

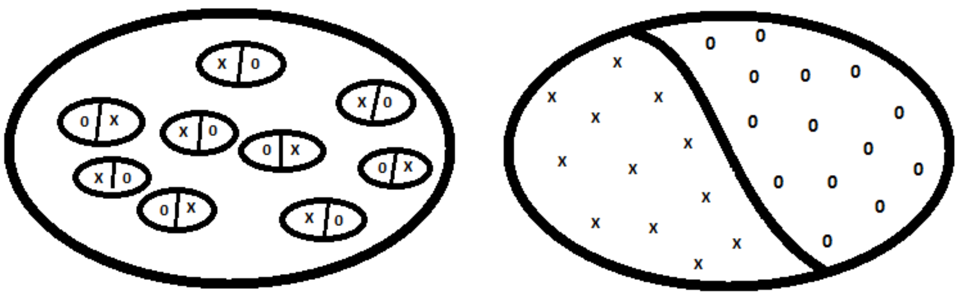

Figure 1: Pairings and two-colorings

Main connection between pairings and two-colorings

Note that a pairing induces a family of two-coloring of the hypergraph in the following sense. *Instead of coloring all vertices in one step, we only color a pair in each step*, one element by color 1, the other by color 2, until all pairs are colored (see Figure 1). The colorings of different pairs are independent. Finally we color unmatched vertices arbitrarily. If the initial pairing is good, then our coloring procedure is guaranteed to produce a good two-coloring.

From the other side, *instead of pairing one vertex by one vertex, we pair a subset of vertices (color class 1) by another subset of vertices (color class 2).*

Our main goal is to extend this relation into larger sets, namely when not pairs of elements, but pairs of subsets of vertices are given. We will call these pairs *cakes*, as these are motivated by the various cake-cutting problems.

A generalization of pairs and pairings

Let us call a subset of the hypergraph $\mathcal{H} = (V, E)$ **t-cake** if it consists of *exactly* t vertices of $\mathcal{H}$, with a previously given bipartition p, q of its elements, where $t \in \mathbb{N}, t \geq 2$ and $1 \leq p, q \in \mathbb{N}, p, q < t, p + q = t$. A t-cake is **balanced** if the two parts contain equal number of elements, i. e. $p = q$.

In case of infinite hypergraphs, for example, where the vertices are the squares of the infinite chessboard, t can be infinitely large, too.

Definition 1 *A* ***t-cake placement*** $\mathcal{T}$*, briefly a* ***t-placement****, on the* $\mathcal{H} = (V, E)$ *hypergraph is a non-overlapping placement of cakes on the hypergraph, where the size of every cake is* ***at most*** t*. If a* t*-placement for an even* t *contains only balanced* t*-cakes, we are talking about* ***p-pairing****, where* $p = t/2$*.*

In a t-placement, the different cakes can have different sized parts, so p depends on the actual cake. However, in most of our examples we deal with only p-pairings.

Clearly, a pairing is a 1-pairing and also a 2-placement ($t = 2, p = 1$). In fact, any pair has a unique way to be viewed as a cake. A two-coloring is also a special t-placement, where $t = |V|$ and the parts of the cake are the two color classes. Similarly to pairings and colorings, the concept of blocking is also crucial for t-placements.

Definition 2 *For a hypergraph* $\mathcal{H} = (V, E)$ *a* t*-cake* ***blocks an*** $A \in E$*, if both parts of the cake have a non-empty intersection with* A*. A* t*-placement* $\mathcal{T}$ *is a* ***good t-placement of*** $\mathcal{H}$*, if all edges of* $\mathcal{H}$ *are blocked by a cake of* $\mathcal{T}$*. An* $A \in E$ *of* $\mathcal{H}$ *is an* ***unblocked edge****, if there is no cake in* $\mathcal{T}$ *blocks* A*.*

We see a kind of monotonicity: if there is a good t-placement for a hypergraph $\mathcal{H}$, then there is a good $(t + 1)$-placement for $\mathcal{H}$, by definition.

Similarly to pairings, t-placements can be considered as a step by step coloring, too. Instead of coloring a pair (as by pairings), we color a cake in each step, one part of the cake by color 1 and the other one by color 2. In the case of a good t-placement, the step by step coloring produces a good coloring.

Let us recall now some definitions of hypergraph games.

Hypergraph games

We can define several games on a given hypergraph $\mathcal{H} = (V, E)$, where V can be infinite, but an $A \in E$ edge is always finite. The first and second players take elements of V in turns. In the **Maker-Maker** (M-M) version of the game, the player who is first to take all elements of some edge $A \in E$ wins the game. In the **Maker-Breaker** (M-B) version, Maker wins by taking every element of some $A \in E$. The other (usually the second) player, *Breaker*, wins by taking at least one vertex of every edge in E. Clearly, there is no draw in this game. The M-M and M-B games are closely related, since if Breaker wins as a second player, then the M-M game is a draw. On the other hand, if the first player has a winning strategy for the M-M game, then Maker also wins the M-B version. For more on these, see Berlekamp, Conway and Guy [5] or Beck [4].

There are biased (p, q) M-B games, where Maker takes p and Breaker takes q elements in each turn. In Chvátal and Erdős [6], Hefetz et al. [13], Krivelevich [14] and Pluhár [15] one can find more examples and applications of biased games.

Beck introduced Picker-Chooser and Chooser-Picker versions of M-B games in Beck [2], where the two players are **Picker** and **Chooser**. Picker selects a pair of elements, neither of which had been selected previously, and Chooser keeps one of these elements and gives the other one back to Picker. In the **Chooser-Picker** (C-P) version Chooser plays as Maker and Picker plays as Breaker, while the roles are swapped in the **Picker-Chooser** (P-C) game.

Similarly to the M-B games, we define a possible version of the biased Chooser-Picker game. Note that another version is used by Csernenszky [8].

Definition 3 *In the **biased(t) Chooser-Picker games** Picker as Breaker takes maximum t elements instead of one pair, and divides this set into two nonempty, disjoint parts. Chooser keeps one of these parts and gives the other part back to Picker.*

Note that the results on the t-placements give winning conditions on the biased(t) C-P games right away.

Since in our paper the k-in-a-row type games play an important role, we define $\mathcal{H}_k$, the hypergraph of the k-in-a-row games.

Definition 4 *The vertices of the **k-in-a-row hypergraph** $\mathcal{H}_k$ are the squares of the infinite (chess)board, i. e. the infinite square grid. The edges of the hypergraph $\mathcal{H}_k$ are the k-element sets of consecutive squares in a row horizontally, vertically or diagonally.*

Obviously, a good t-placement for $\mathcal{H}_k$ is a good t-placement for $\mathcal{H}_{k+1}$, too.

2 Some facts about pairings and t-placements

2.1 Pairings

Pairings are one way to show that Breaker has a winning strategy in hypergraph games. A good pairing ρ for a hypergraph $\mathcal{H}$ can be turned to a winning pairing strategy for Breaker in the M-B game on $\mathcal{H}$. Following ρ on $\mathcal{H}$ in a M-B game, for every $x \in X \cup Y$ element chosen by Maker, Breaker chooses $\rho(x)$ or in case of $x \in Y$ vice versa. If $x \notin X \cup Y$ then Breaker can choose an arbitrary vertex. Hence Breaker can block all edges and wins the game. Since we will study the $\mathcal{H}_k$ hypergraphs, our illustration is the k-in-a-row Maker-Breaker game.

The best upper bound for the value of k, where Breaker has a winning strategy in the k-in-a-row game is given by T. G. L. Zetters (alias A. Brouwer) [11]. He showed that Breaker can win the 8-in-a-row game. The best lower bound is given by Allis [1], he has shown that Maker wins the 5-in-a-row game on the boards of sizes 19×19 and 15×15. The following result is due to Hales and Jewett [5], which yields only a little weaker upper bound:

Theorem 1 *Breaker wins the 9-in-a-row M-B game by a pairing strategy, i. e. there exists a good pairing for the 9-in-a-row.*

Proof. Figure 2 is an extension of a pairing of an 8×8 torus (framed), where the pairs have a periodicity 8 in every line. Hence, the pairing blocks all 9-in-a-row edges. $\square$

A pairing is a **domino pairing** on the square grid, if all pairs consist of neighboring cells (horizontally, vertically or diagonally). Note that the pairing on Figure 2 is a domino pairing.

Csernenszky et al. [9] showed that to decide whether there exists a pairing strategy for an arbitrary hypergraph, is an NP-complete problem. A counting type proposition showed [9] that there is no good pairing strategy for $\mathcal{H}_k$, if $k < 9$. Our main purpose to introduce the notion of t-placement was to extend the pairing strategy to a wider class of hypergraph games (for example $\mathcal{H}_k$ where $k < 9$). Since the previous proposition plays an important role in our discussion, we formulate the exact statement.

For a hypergraph $\mathcal{H}$ let $d_2(\mathcal{H})$ (briefly d_2) be the greatest number, that many edges can be blocked by two vertices of $\mathcal{H}$, i. e. d_2 is the maximal co-degree.

Proposition 1 [9] *If there is a good pairing ρ for the hypergraph $\mathcal{H} = (V, E)$, then $d_2|X|/2 \geq |\mathcal{G}|$ must hold for all $X \subset V$, where $\mathcal{G} = \{A : A \in E, A \subset X\}$.*

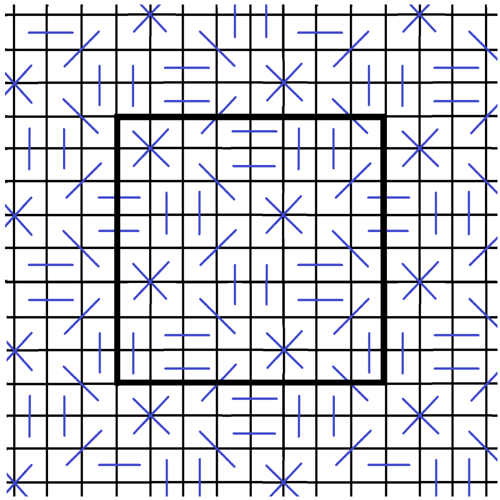

Figure 2: Hales-Jewett pairing against 9-in-a-row (Berlekamp et al. [5])

Proof. We will refer to X as a subboard. The edges of $\mathcal{G}$ can be blocked only by pairs coming from X. There are at most $|X|/2$ such pairs of ρ on the subboard of size $|X|$. Since a pair blocks maximum d_2 edges, $|X|/2$ pairs can block maximum $d_2|X|/2$. So, if there are more edges on the subboard, there can not be a good pairing. □

With the help of Proposition 1 we can conclude that there is no pairing strategy for $\mathcal{H}_k$ if $k < 9$. In the hypergraph $\mathcal{H}_k$, $d_2 = k-1$, because a pair can block at most $k-1$ edges. This can happen if and only if the pair is a domino. If X is an $n \times n$ subboard for sufficiently large n, then $|\mathcal{G}| = 4n^2 + O(n)$ because from every square four edges start (a vertical, a horizontal and two diagonal, except of the boarders). By Proposition 1, we have $(k-1)n^2/2 \geq 4n^2 + O(n)$; that is, $k \geq 9 + O(1/n)$.

2.2 t-placements

As pairings can help Breaker in the M-B and Picker in the normal C-P games, t-cakes can also help Picker in the biased(t) C-P games. If there is a good t-placement blocking a hypergraph, then for the corresponding biased(t) C-P hypergraph game Picker as Breaker has a winning strategy by just giving the cakes to Chooser in any order.

Let us count the number of hypergraph edges can be blocked by a given cake, that is the **blocking number of the cake**. In order to generalize the co-degree argument, we also have to take into consideration the shape and bipartition of the cake. For a given $t \in \mathbb{N}$ let d_t be the greatest blocking number among all t-cakes. A t-cake with blocking number d_t is called a best t-cake. Note that d_t is a monotonic function of t, because if a t-cake blocks d_t edges in a hypergraph, then adding any part by a single vertex the obtained $(t+1)$-cake blocks all previously blocked edges, too. However, it is easy to see that the ratio d_t/t — the best average blocking number per vertex — is not necessarily monotonic in t. Still, a case analysis shows that for the $\mathcal{H}_k$ hypergraph that the ratio d_t/t is monotonic in t at least if $t \leq 8$, as we will see later (for $t \geq 9$ it is undecided yet). With all these we can spell out the following generalization of Proposition 1.

Proposition 2 *If there is a good* t*-placement of* $\mathcal{H} = (V, E)$ *such that* $d_2/2 \leq d_3/3 \leq \cdots \leq d_t/t$, *then* $\frac{d_t}{t}|X| \geq |\mathcal{G}|$ *for every* $X \subset V$, *where* $\mathcal{G} = \{A : A \in E, A \subset X\}$.

From now on we assume that V is the set of squares of the infintie square grid. In this case a t-cake has geometrical shape. In the next section we consider some possible small t-cakes for $\mathcal{H}_k$, and we will list the best t-cakes for all $t \leq 8$.

3 Some t-cakes and the maximal blocking numbers

The most interesting and treatable examples are the 4-cakes, we start with those.

3.1 4-cakes in general

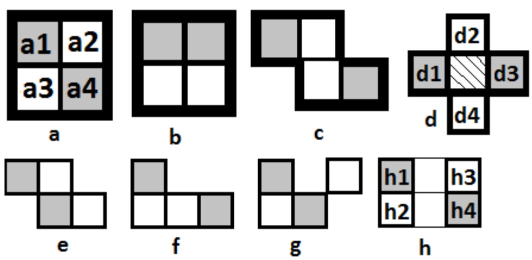

Figure 3: Some 4-cakes

On Figure 3 there are some 4-cakes. The partitions of the cakes are the white and gray squares. In the first row on Figure 3, all 4-cakes block $4k-4$

edges of the k-in-a-row. However, the cakes in the second row block fewer than $4k-4$ edges.

For example, *4-cake* a blocks in two vertical and two horizontal directions: along the lines $(a1, a3)$ and $(a2, a4)$ vertically, while along $(a1, a2)$ and $(a3, a4)$ horizontally. In detail, $(a1, a3)$ blocks $k-1$ edges in its direction as a single domino, because $a1$ and $a3$ are neighboring cells and they are in different parts of the cake a. The same is true for the pairs $(a2, a4)$, $(a1, a2)$ and $(a3, a4)$. Adding up, the cake a blocks $4(k-1)$ edges. The pairs $(a1, a4)$ and $(a2, a3)$ do not block any k-in-a-row edges because those are in the same parts of the cake a, hence a does not block any diagonal edges.

Cakes b and c (or their rotated copies) block edges on two diagonal and on two horizontal (or rotatedly vertical) lines, while d (which does not contain the central square) blocks in four diagonal directions. Since on each line there are neighboring cells (dominoes) blocking, those block $k-1$ edges per line, which adds up $4(k-1)$.

Cakes e and f block only in three directions, adding up only to $3(k-1)$ edges. The cakes g and h block in four directions, but not only by neighboring cells. Obviously, cake g blocks $4k-5$ and cake h blocks $4k-6$ edges.

3.2 The maximal blocking numbers of t-cakes for $2 \le t \le 8$

The 2-cakes are the pairs. One pair blocks maximum $k-1$ edges of the k-in-a-row. Hence, the edges of $\mathcal{H}_k$ can be blocked by pairs only for $k \ge 9$, see Proposition 1.

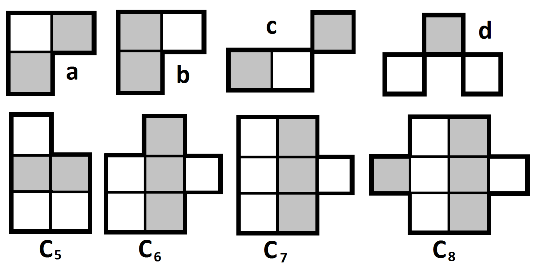

Figure 4: The best 3-cakes (first row) and the best 5-, 6-, 7- and 8-cakes

It is easy to see that a 3-cake (with a 1+2 bipartition) cannot block more than $2k-2$ edges of $\mathcal{H}_k$. On the first row of Figure 4 we listed all best 3-cakes. 3-cake a blocks $(k-1)+(k-1)$ edges in a horizontal and a vertical, d in two diagonal, b and c in a diagonal and in a vertical (rotatedly in a horizontal) direction. Since all other 3-cakes block fewer edges than the previous cakes, $d_t = 2k-2$. Applying the Proposition 2 on an $X = n \times n$ board for sufficiently large n, we get $(2k-2)n^2/3 \geq 4n^2 + O(n)$, so $k \geq 7 + O(1/n)$, hence a 3-placement cannot block $\mathcal{H}_6$, but theoretically it can block $\mathcal{H}_7$ or $\mathcal{H}_8$. However, it is still an open question if there exists any good 3-placement for 7- or 8-in-a-row.

In the previous subsection on the first row of Figure 3 we listed 4-cakes blocking $4k-4$ edges. We note that this is the maximum number of edges that can be blocked by any 4-cake. Actually, the first row of Figure 3 contains the complete list of the best 4-cakes. Hence, $d_4 = 4k-4$. Applying the Proposition 2 we get $(4k-4)n^2/4 \geq 4n^2+O(n)$, so $k \geq 5+O(1/n)$. Therefore, theoretically there can exist a good 4-placement for $\mathcal{H}_5$ and $\mathcal{H}_6$, but these questions are undecided yet.

In the second row of the Figure 4, there are the best t-cakes for $5 \leq t \leq 8$, named C_5, C_6, C_7 and C_8. In the following table we summarize the values of d_t, d_t/t and the lower bounds of k drawn from Proposition 2, for that a t-placement can block $\mathcal{H}_k$.

t	2	3	4	5	6	7	8
d_t	$k-1$	$2k-2$	$4k-4$	$5k-4$	$7k-6$	$9k-8$	$11k-9$
$\frac{d_t}{t}$	$\frac{1}{2}k - \frac{1}{2}$	$\frac{2}{3}k - \frac{2}{3}$	$k-1$	$k - \frac{4}{5}$	$\frac{7}{6}k - 1$	$\frac{9}{7}k - \frac{8}{7}$	$\frac{11}{8}k - \frac{9}{8}$
$k \geq$	9	7	5	4.8	4.29	4	3.73

Hereafter we can spell out results on $\mathcal{H}_k$. Namely, for a given k which is the smallest value of t for that there exists a good t-placement for $\mathcal{H}_k$.

4 k-in-a-row

At first we list our results in a table. Columns and rows stand for the values of k and t, respectively. "Yes" designates the existence of a good placement, "No" means that there is no good placement, while the case of "?" is undecided yet.

Some of the "No's" are the simple consequences of the previous propositions. The rest of the paper is devoted to prove the remaining claims which are summarized in the table. In the rest we will use X and O for the colors (as usually in the Tic-Tac-Toe game).

k \ t	2	3	4	5	6	7	8	t ≥ 9	∞
2	No	No	No	No	No	No	No	No	No
3	No	No	No	No	No	No	No	No	Yes
4	No	No	No	No	No	No	No	?	Yes
5	No	No	?	?	?	?	Yes	Yes	Yes
6	No	No	?	?	Yes	Yes	Yes	Yes	Yes
7	No	?	Yes	Yes	Yes	Yes	Yes	Yes	Yes
8	No	?	Yes	Yes	Yes	Yes	Yes	Yes	Yes
9	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

4.1 The case 2-in-a-row

Remark 1 *There exists no good two-coloring of the infinite board, which can block all $\mathcal{H}_2$ edges on the board.*

Proof. Assume that we have a good coloring. Let us choose an arbitrary square having the color say X. If there is an other X among its neighbors, we can not get a good coloring. But if all of its eight neighbors have color O, then among the neighbors there are two monochromatic cells next to each other. □

4.2 The case 3-in-a-row

The next theorem is in fact a special case of the Theorem 2 in Dumitrescu and Radoičić [10]. We give the sketch of their proofs.

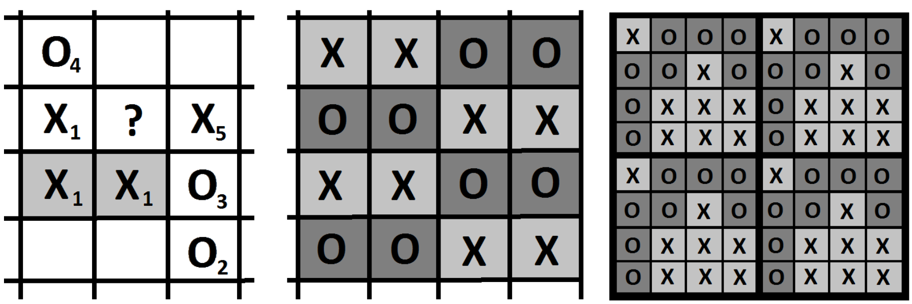

Figure 5: Colorings blocking 3- and 4-in-a-row edges

Theorem 2 *There is a unique good two-coloring of the board for $\mathcal{H}_3$.*

Proof. Following the left hand side of the Figure 5, there are at least two edge-neighboring cells on the board, which have the same color (colored by light gray on Figure 5). If not, we would get the infinite chessboard coloring with arbitrary long monochromatic diagonal lines.

Assume the three X's with index 1 in our coloring. O_2, O_3, O_4 and X_5 are forced if we want to exclude monochromatic 3-in-a-row edges. The square marked by an "?" gives a contradiction.

This proves that if we take two edge-neighboring X's (let us say in a row), then any square right above or under them must contain an O. This ensures that the global coloring must be the middle coloring of Figure 5. □

Corollary 1 *There is no good* t*-placement with finite* t *for* $\mathcal{H}_3$*.*

Proof. This follows from the uniqueness of the coloring of the Theorem 2. If there were a good t-placement with a finite t, then the colorings of some cake-parts could be switched, which would result in infinite many good colorings. □

4.3 The case 4-in-a-row

Of course by the monotonicity in k the previous coloring of $\mathcal{H}_3$ blocks also $\mathcal{H}_4$. For $\mathcal{H}_4$ there is an other good two-coloring in the paper of Dumitrescu and Radoičić [10], see on the right hand side of the Figure 5. Note that this coloring blocks four consecutive monochromatic squares on all lines, including every rational slope. One may ask, how many good two-colorings are for $\mathcal{H}_4$?

It is also an open question if there is a good t-placement with finite t for $\mathcal{H}_4$. From Proposition 2 we know a lower bound, that there are no good t-placement for $\mathcal{H}_4$ for $t \leq 6$. On the other hand, Observation 2 shows that there are no t-placement for t=7 and 8, too.

Remark 2 *There are no good* t*-placement for* t*=7 and 8 for* $\mathcal{H}_4$*.*

Proof. Consider the following. For $t = 7, 8$ a good placement must contain at least one cake C_7 and/or C_8 of Figure 4. (Any other 7- and 8-cakes block fewer edges than it would be enough to block $\mathcal{H}_4$.) But C_7 and C_8 contain three consecutive squares in a vertical row in the same part of the cakes. The color of the two squares neighboring the three consecutive squares in C_7 and C_8 is uncontrolled and results in an unblocked 4-in-a-row. □

4.4 The cases 9- or more-in-a-row

Theorem 1 shows "Yes" for $t = 2, k = 9$. Because of monotonicity, it also holds for every $k \geq 9$. Moreover, it follows from Proposition 1, that pairings cannot block all k-in-a-row edges on the board for $k < 9$.

4.5 The cases 7-in-a-row and 5-in-a-row

Theorem 3 *There is a 6-placement that blocks all 7-in-a-row edges on the board.*

Proof. Let us consider the 6-placement on Figure 6, where the different parts of the cakes are colored by gray and white. That placement clearly blocks horizontally three, vertically four and diagonally seven consecutive squares in every row. □

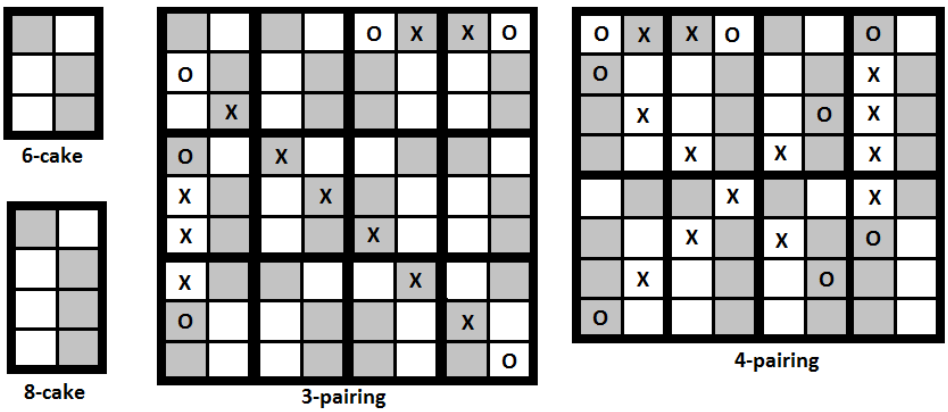

Figure 6: 6- and 8-cakes blocking 7- and 5-in-a-row

Theorem 4 *There is an 8-placement that blocks all 5-in-a-row edges on the board.*

Proof. Modifying the previous 6-cakes by two additional squares, we obtain the 8-placement on the right of Figure 6. That placement blocks all 5-in-a-row edges on the board, even more, it blocks every three consecutive squares horizontally. □

Corollary 2 *Theorem 4 implies that Picker as Breaker wins the biased(8) C-P 5-in-a-row game.*

4.6 The case 7-in-a-row by 4-cakes

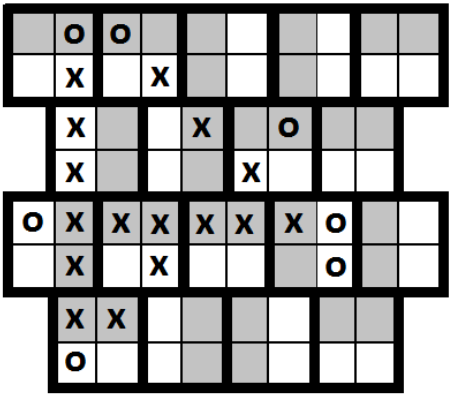

Figure 7: 2-pairing blocks 7-in-a-row edges

Theorem 5 *There is a good 4-placement for the 7-in-a-row.*

Proof. The 4-placement on Figure 7 blocks every 7-in-a-row edges. □

Note that the same placement blocks all 8-in-a-row edges by monotonicity.

Corollary 3 *Theorem 5 implies that Picker as Breaker wins the biased(4) C-P 7-in-a-row game.*

Even more, from Csernenszky [7] we know that Picker wins the normal C-P 7-in-a-row game, too.

So far all t-placements we gave were also a $t/2$-pairing by definition. Our last example for $\mathcal{H}_6$ shows that there are other interesting good t-placements.

4.7 The case 6-in-a-row

Theorem 6 *There is a good 6-placement for $\mathcal{H}_6$.*

Proof. Let us consider the two types of 6-cakes placed alternately in the 6-placement of Figure 8. That configuration obviously provides the appropriate blocking. □

Corollary 4 *Theorem 6 implies that Picker as Breaker wins the biased(6) C-P 6-in-a-row game.*

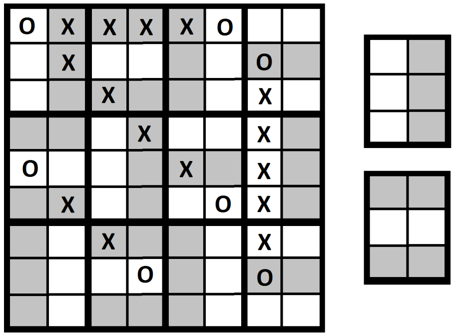

Figure 8: 6-cakes block 6-in-a-row

5 Conclusion and open problems

In this paper we built a bridge between the pairings and the two-colorings of hypergraphs. We have introduced a new object, the t-cakes, and illustrated their use in the game k-in-a-row for some $k \in \mathbb{N}$.

The 9-in-a-row Maker-Maker game is a draw, even more, Breaker wins the Maker-Breaker version of it by a pairing strategy. The best result is due to Brouwer (see T.G.L. Zetters [11]) who proved some 35 years ago that Breaker can win the 8-in-a-row, while Csernenszky et al. [9] showed that it cannot be done by a pairing strategy. On the other direction, Allis [1] showed that the first player wins the Maker-Maker 5-in-a-row game on the boards of sizes 19×19 and 15×15. The cases of the infinite board and $k = 6$ and $k = 7$ are still open.

Picker (as Breaker) wins the Chooser-Picker version of the 7-in-a-row game (Csernenszky [7]); the cases for $k \leq 6$ are open. In the biased versions — in which Picker selects t elements and divides it into two parts, then Chooser takes one of those parts — Picker wins even if $k \leq 6$.

While there are no good pairings for the 7-in-a-row game, there is a good 2-pairing (4-placement) which is a generalization of pairings. Obviously, by monotonicity, it works for 8-in-a-row, too. In the case of 6-in-a-row, there is a good 6-placement, that means Picker as Breaker wins the biased(6) 6-in-a-row game. To win the 5-in-a-row, Picker has a good 4-pairing (8-placement). It is still open if there are good t-placements for $4 \leq t \leq 7$.

We think that no finite cakes can block the winning sets of the hypergraph 4-in-row. Still, we can prove this only in the case when the sizes of the cakes are not larger than eight. There are more than one good two-colorings (∞-

placements) blocking all 4-in-a-row edges on the board. The number and the structure of those good colorings are still open questions.

The 3-in-a-row case is very simple, there is a unique good two-coloring. Then the biased(∞) k-in-a-row game is that Picker divides the infinite square grid into two parts, neither of those containing three consecutive squares in a row. Finally, there are no two-colorings blocking all 2-in-a-row edges.

Acknowledgements

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On packing density of growing size circles

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Abstract. For any given natural number an arrangement of growing size circles, a packing of the plane will be constructed such that its packing density coincides – in the asymptotical sense – with that of the 'classical' hexagonal circle packing!

1 The construction

For a natural $n \geq 3$ let us define a special circle packing as follows. First, we circumscribe the unit circle with n circles of the same radius $r_{n,1}$ such that they also touch their both neighbours. Thus we get zone one, $Z_{n,1}$.

Then we draw n circles of the same radius $r_{n,2}$ such that they touch two circles from $Z_{n,1}$ and also their both neighbours with radius $r_{n,2}$, getting this way $Z_{n,2}$, etc.

Denote by $S_{n,k}$ the set of circles of the first k zones:

$$S_{n,k} = \cup_{i=1}^{k} Z_{n,i},$$

and let

$$S_n = \cup_{k=1}^{\infty} S_{n,k}.$$

Then S_n is a packing of the plane, an infinite set of circles with pairwise disjoint interiors, and a natural problem is to find the fraction of the plane filled by the circles making up this packing.

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Key words and phrases: circle packing, packing density

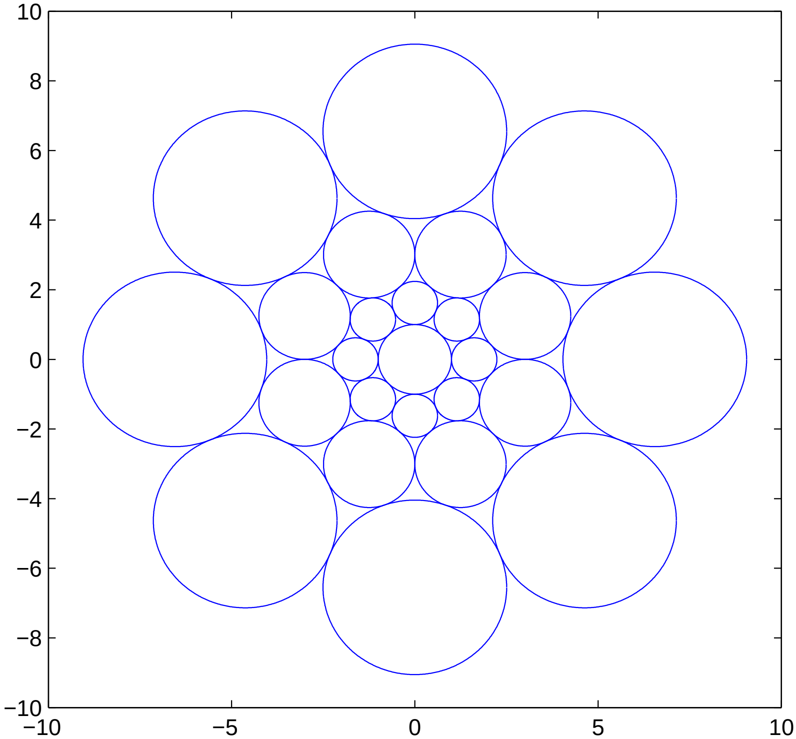

Figure 1. The set $S_{8,3}$, i.e. the first three zones for $n = 8$.

2 The main theorem

The packing density of the arrangement S_n related to a bounded domain $D \subset \mathbb{R}^2$ is the ratio

$$\frac{\sum |C \cap D|}{|D|}, \; C \in S_n,$$

where $|\cdot|$ denotes the area of its argument. It is customary to define (see e.g. Kuperberg [1]) the packing density in an Euclidean space by means of a limit, taking e.g. balls B_r of radius r centered at the origin:

$$\lim_{r\to\infty} \frac{\sum |C \cap B_r|}{|B_r|}, \; C \in S_n.$$

However, in our case taking polygons is more capable. Denote by $P_{n,k}$ the regular $n-$gon with vertices at the centres of circles in $Z_{n,k}$, and let

$$\delta_{n,k} = \frac{|S_n \cap P_{n,k}|}{|P_{n,k}|} \equiv \frac{|S_{n,k} \cap P_{n,k}|}{|P_{n,k}|}.$$

Then,

$$\delta_n = \lim_{k\to\infty} \delta_{n,k}$$

is the packing density of S_n, and

$$\delta^* = \lim_{n\to\infty} \delta_n$$

is the quantity we are interested in.

Theorem 1 *With the notations above we have*

$$\delta^* = \frac{\pi}{2\sqrt{3}}.$$

Remark 1 *As is known (see e.g. the survey on the first page in [2], showing the contributions of Lagrange, A. Thue, L. Fejes Tóth to the subject), the optimal packing density for circles is just this quantity - a curious coincidence!*

Remark 2 *The interested reader should also consult [3] and [4] for further information.*

3 The proof

Let $n, k \in \mathbb{N},\ n \geq 3$ be given. Assume that the centre of one of the circles belonging to $Z_{n,1}$ lies on the x-axis, i.e. at $A_{n,1} := (1 + r_{n,1}, 0)$.

Denote by $B_{n,1}$ the point, where the half-line $y = \tan(\frac{\pi}{n})x$ is tangent to the circle chosen. It suffices to consider the 'basic' sector $B_{n,1}OA_{n,1}$, as is seen on Figure 2 for the case $n = 8,\ k = 3$.

It is easy to see that the centre $A_{n,i}$ of the $i-th$ circle in the basic sector lies on the x-axis for i odd, and on the line $y = \tan(\frac{\pi}{n})x$ for i even, and just reversely for the $B'_{n,i}s$. Also note that the angles at the $B'_{n,i}s$ are rectangles. Introduce now the notations

$$s_n = \sin\left(\frac{\pi}{n}\right), \quad t_n = \tan\left(\frac{\pi}{n}\right).$$

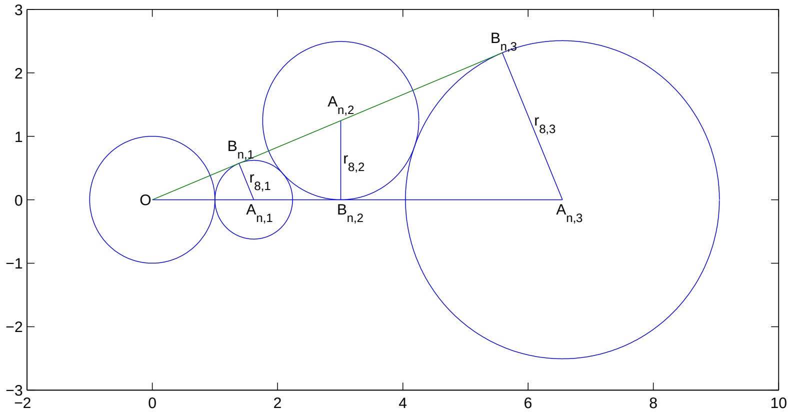

Figure 2. The basic sector for $S_{8,3}$

The radius $r_{n,1}$ can be obtained from the triangle $B_{n,1}OA_{n,1}$:

$$s_n = \frac{r_{n,1}}{1+r_{n,1}} \quad \Rightarrow \quad r_{n,1} = \frac{s_n}{1-s_n}$$

Since $A_{n,1}B_{n,2}A_{n,2}B_{n,1}$ and $A_{n,2}B_{n,3}A_{n,3}B_{n,2}$ are similar quadrilaterals (in fact, both are inscribed quadrilaterals with two rectangles), it follows that $r_{n,2}/r_{n,1} = r_{n,3}/r_{n,2}$, giving in general

$$r_{n,k} = r_{n,1}\, q_n^{k-1}, \quad q_n = \frac{r_{n,2}}{r_{n,1}}.$$

The area of the polygon $P_{n,k}$ is $2n$ times the area of triangle $OA_{n,k}B_{n,k}$, i.e.

$$|P_{n,k}| = \frac{n}{t_n} r_{n,k}^2 = \frac{n r_{n,1}^2}{t_n} q_n^{2k-2}.$$

The sum of areas of the circles in $S_{n,k}$ is

$$n\pi \sum_{i=1}^{k} r_{n,i}^2 = n\pi r_{n,1}^2 \sum_{i=1}^{k} q_n^{2k-2} = n\pi r_{n,1}^2 \frac{q_n^{2k}-1}{q_n^2-1}.$$

However, the contribution of the $k-$th zone to $|Z_{n,k} \cap P_{n,k}|$ is only $\frac{n-2}{2}\pi r_{n,k}^2$, instead of $n\pi r_{n,k}^2$. Consequently we have

$$|S_{n,k} \cap P_{n,k}| = \pi r_{n,1}^2 q_n^{2k-2}\Big(\frac{n}{q_n^2-1} + \frac{n-2}{2}\Big) - \frac{\pi n r_{n,1}^2}{q_n^2-1}.$$

When calculating the limit of $\delta_{n,k}$ for $k \to \infty$, the magnitude of the quotient $q_n = \frac{r_{n,2}}{r_{n,1}}$ is decisive. Introducing the new variable

$$t = \tan\left(\frac{\pi}{2n}\right)$$

(cf. the standard trigonometric substitution $t = \tan(\frac{x}{2})$ in calculus) we have

$$s_n = \frac{2t}{1+t^2}, \quad t_n = \frac{2t}{1-t^2},$$

and also

$$r_{n,1} = \frac{2t}{(1-t)^2}, \quad r_{n,2} = \frac{2t(u+2t\sqrt{v})}{(1+t)^2(1-t)^4},$$

where

$$u = 1+4t^2-t^4, \quad v = (3-t^2)(1+t^2).$$

The radius $r_{n,2}$ can be calculated by considering the triangles $A_{n,1}B_{n,2}A_{n,2}$ and $OB_{n,2}A_{n,2}$. Analysing the function $t \to \frac{r_{n,2}}{r_{n,1}}$ we see that it is greater than one for $0 < t < 1$, or equivalently, that the relation $q_n > 1$ holds for $n \geq 3$. Therefore, in case of $k \to \infty$ the second (negative) term in $|S_{n,k} \cap P_{n,k}|$ can be omitted to get

$$\delta_n = \frac{\pi t_n}{r_{n,2}^2 - r_{n,1}^2}\left(r_{n,1}^2 + \frac{n-2}{2n}(r_{n,2}^2 - r_{n,1}^2)\right).$$

With the notation $\omega = \frac{n-2}{2n}$ we obviously have $0 < \omega < 1$ for $n \geq 3$, which yields in a natural way the lower and upper bounds

$$\delta_n^0 < \delta_n < \delta_n^1.$$

Since the difference of these bounds is simply

$$\delta_n^1 - \delta_n^0 = \pi t_n = O(t) \quad (t \to 0),$$

we can replace δ_n e.g. by its lower bound

$$\delta_n^0 = \frac{\pi t_n r_{n,1}^2}{r_{n,2}^2 - r_{n,1}^2} = \frac{\pi(1-t^2)^3}{2(2tv + u\sqrt{v})}.$$

Having gotten rid of the singularity, we can immediately substitute $t = 0$ (corresponding to $n \to \infty$) to get the desired result

$$\delta^* = \lim_{n\to\infty} \delta_n.$$

□

Considering this surprising coincidence, one puts the question: is there a more general principle, this conclusion can be drawn from?

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Study of periodic and nonnegative periodic solutions of nonlinear neutral functional differential equations via fixed points

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Abstract. In this paper, we study the existence of periodic and nonnegative periodic solutions of the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)\,h(x(t)) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))).$$

We invert this equation to construct a sum of a completely continuous map and a large contraction which is suitable for applying the modification of Krasnoselskii's theorem. The Caratheodory condition is used for the functions Q and G.

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Key words and phrases: Krasnoselskii-Burton's theorem, large contraction, neutral differential equation, integral equation, periodic solution, positive solution

1 Introduction

Theory of functional differential equations with delay has undergone a rapid development in the previous fifty years. We refer the readers to [1]-[6], [8]-[15] and references therein for a wealth of reference materials on the subject. More recently researchers have given special attentions to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations. In particular, qualitative analysis such as periodicity and positivity of solutions of neutral differential equations has been studied extensively by many authors.

Recently, in [1], the authors discussed the existence and positivity of periodic solutions for the first-order delay differential equation

$$x'(t) = -a(t)h(x(t)) + G(t, x(t - \tau(t))), \tag{1}$$

by employing the Krasnoselskii-Burton's fixed point theorem, the authors obtained existence results for periodic and positive periodic solutions.

In [14], the Krasnoselskii-Burton's fixed point theorem was used to establish the existence of periodic solutions for the first-order nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)h(x(t)) + c(t)x'(t - \tau(t)) + G(t, x(t), x(t - \tau(t))). \tag{2}$$

In [8], the authors used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions for the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t - \tau(t))) + G(t, x(t), x(t - \tau(t))). \tag{3}$$

Also, the authors used the contraction mapping principle to show the uniqueness of periodic solutions and stability of the zero solutions of (3).

In the current paper, we are interested in the analysis of qualitative theory of periodic and nonnegative periodic solutions of neutral differential equations. Inspired and motivated by the works mentioned above and the papers [1]-[6], [8]-[15] and the references therein, we study the existence of periodic and nonnegative periodic solutions of the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)h(x(t)) + \frac{d}{dt}Q(t, x(t - \tau(t))) + G(t, x(t), x(t - \tau(t))), \tag{4}$$

where a is a positive continuous real-valued function. The function $h : \mathbb{R} \to \mathbb{R}$ is continuous, $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the

Caratheodory condition. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [7], Theorem 3) to show the existence and nonnegativity of periodic solutions for equation (4). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [7], the added term destroys the contraction but it replaces with a so called large contraction which is suitable for fixed point theory. During the process we have to transform (4) into an integral equation written as a sum of two mappings, one is a large contraction and the other is completely continuous. After that, we use a variant of Krasnoselskii's fixed point theorem, to show the existence and nonnegativity of a periodic solution.

Note that in our consideration the neutral term $\frac{d}{dt}Q(t, x(t-\tau(t)))$ of (4) produces nonlinearity in the derivative term $\frac{d}{dt}x(t-\tau(t))$. The neutral term $\frac{d}{dt}x(t-\tau(t))$ of (2) in [14] enters linearly. As a consequence, our analysis is different from that in [14].

The organization of this paper is as follows. In Section 2, we present the inversion of totally nonlinear neutral differential equation (4), some definitions and Krasnoselskii-Burton's fixed point theorem. For details on Krasnoselskii-Burton's theorem we refer the reader to [7]. In Sections 3 and 4, we present our main results on existence of periodic and nonnegative periodic solutions of (4).

2 Preliminaries

For $T > 0$ define $P_T = \{\phi : \phi \in C(\mathbb{R}, \mathbb{R}),\ \phi(t+T) = \phi(t)\}$ where $C(\mathbb{R}, \mathbb{R})$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{t\in[0,T]} |x(t)|.$$

In this paper we assume that

$$a(t-T) = a(t),\ \tau(t-T) = \tau(t),\ \tau(t) \geq \tau^* > 0, \tag{5}$$

with τ continuously and τ^* is constant, a is positive and

$$1 - e^{-\int_{t-T}^{t} a(s)ds} \equiv \frac{1}{\eta} \neq 0. \tag{6}$$

The functions $Q(t,x)$ and $G(t,x,y)$ are periodic in t of period T. That is

$$Q(t-T,x)=Q(t,x),\; G(t-T,x,y)=G(t,x,y). \tag{7}$$

The following lemma is fundamental to our results.

Lemma 1 *Suppose (5)–(7) hold. If $x\in P_T$, then x is a solution of equation (4) if and only if*

$$\begin{aligned}
&x(t)\\
&=\eta\int_{t-T}^{t}\kappa(t,u)\,a(u)\,[x(u)-h(x(u))]\,du+Q(t,x(t-\tau(t)))\\
&+\eta\int_{t-T}^{t}\kappa(t,u)\,[-a(u)\,Q(u,x(u-\tau(u)))+G(u,x(u),x(u-\tau(u)))]\,du,
\end{aligned} \tag{8}$$

where

$$\kappa(t,u)=e^{-\int_u^t a(s)ds}. \tag{9}$$

Proof. Let $x\in P_T$ be a solution of (4). Rewrite the equation (4) as

$$\begin{aligned}
&\frac{d}{dt}[x(t)-Q(t,x(t-\tau(t)))]+a(t)[x(t)-Q(t,x(t-\tau(t)))]\\
&=a(t)x(t)-a(t)h(x(t))-a(t)Q(t,x(t-\tau(t)))+G(t,x(t),x(t-\tau(t)))\\
&=a(t)[x(t)-h(x(t))]-a(t)Q(t,x(t-\tau(t)))+G(t,x(t),x(t-\tau(t))).
\end{aligned}$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned}
&\int_{t-T}^{t}\left[(x(u)-Q(u,x(u-\tau(u))))\,e^{\int_0^u a(s)ds}\right]'du\\
&=\int_{t-T}^{t}a(u)[x(u)-h(x(u))]\,e^{\int_0^u a(s)ds}du\\
&+\int_{t-T}^{t}[-a(u)\,Q(u,x(u-\tau(u)))+G(u,x(u),x(u-\tau(u)))]\,e^{\int_0^u a(s)ds}du.
\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&(x(t) - Q(t, x(t - \tau(t))))\, e^{\int_0^t a(s)ds} \\
&- (x(t - T) - Q(t - T, x(t - T - \tau(t - T))))\, e^{\int_0^{t-T} a(s)ds} \\
&= \int_{t-T}^{t} a(u)\,[x(u) - h(x(u))]\, e^{\int_0^u a(s)ds} du \\
&+ \int_{t-T}^{t} [G(u, x(u), x(u - \tau(u))) - a(u)\, Q(u, x(u - \tau(u)))]\, e^{\int_0^u a(s)ds} du.
\end{aligned}$$

By dividing both sides of the above equation by $\exp(\int_0^t a(s)\, ds)$ and using the fact that $x(t) = x(t - T)$, we obtain

$$\begin{aligned}
&x(t) - Q(t, x(t - \tau(t))) \\
&= \eta \int_{t-T}^{t} a(u)\,[x(u) - h(x(u))]\, e^{-\int_u^t a(s)ds} du \\
&+ \eta \int_{t-T}^{t} [G(u, x(u), x(u - \tau(u))) - a(u)\, Q(u, x(u - \tau(u)))]\, e^{\int_0^u a(s)ds} du.
\end{aligned} \tag{10}$$

The converse implication is easily obtained and the proof is complete. □

Now, we give some definitions which we are going to use in what follows.

Definition 1 *The map* $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is said to satisfy Carathéodory conditions with respect to* $L^1[0, T]$ *if the following conditions hold.*

(i) *For each* $z \in \mathbb{R}^n$, *the mapping* $t \mapsto f(t, z)$ *is Lebesgue measurable.*

(ii) *For almost all* $t \in [0, T]$, *the mapping* $z \mapsto f(t, z)$ *is continuous on* $\mathbb{R}^n$.

(iii) *For each* $r > 0$, *there exists* $\alpha_r \in L^1([0, T], \mathbb{R})$ *such that for almost all* $t \in [0, T]$ *and for all* z *such that* $|z| < r$, *we have* $|f(t, z)| \leq \alpha_r(t)$.

T. A. Burton observed that Krasnoselskii's result (see [12]) can be more attractive in applications with certain changes and formulated Theorem 1 below (see [7] for the proof).

Definition 2 *Let* $(\mathcal{M}, d)$ *be a metric space and assume that* $B : \mathcal{M} \to \mathcal{M}$. B *is said to be a large contraction, if for* φ, $\psi \in \mathcal{M}$, *with* $\varphi \neq \psi$, *we have* $d(B\varphi, B\psi) < d(\varphi, \psi)$, *and if* $\forall \epsilon > 0$, $\exists \delta < 1$ *such that*

$$[\varphi, \psi \in \mathcal{M},\ d(\varphi, \psi) \geq \epsilon] \Longrightarrow d(B\varphi, B\psi) < \delta d(\varphi, \psi).$$

It is proved in [7] that a large contraction defined on a closed bounded and complete metric space has a unique fixed point.

Theorem 1 (Krasnoselskii-Burton) *Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathcal{B}, \|.\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that*
(i) *A is completely continuous,*
(ii) *B is large contraction,*
(ii) *$x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$.*
Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

3 Existence of periodic solutions

To apply Theorem 1, we need to define a Banach space $\mathcal{B}$, a closed bounded convex subset $\mathcal{M}$ of $\mathcal{B}$ and construct two mappings; one is a completely continuous and the other is large contraction. So, we let $(\mathcal{B}, \|.\|) = (P_T, \|.\|)$ and

$$\mathcal{M} = \{\varphi \in P_T, \ \|\varphi\| \leq L\} \tag{11}$$

with $L \in (0, 1]$. For $x \in \mathcal{M}$, let the mapping H be defined by

$$H(x) = x - h(x), \tag{12}$$

and by (8), define the mapping $S : P_T \to P_T$ by

$$\begin{aligned}(S\varphi)(t) &\\ = \eta \int_{t-T}^{t} &\kappa(t,u)\, a(u)\, H(\varphi(u))\, du + Q(t, \varphi(t - \tau(t))) \\ + \eta \int_{t-T}^{t} &\kappa(t,u)\left[-a(u)\, Q(u, \varphi(u - \tau(u))) + G(u, \varphi(u), \varphi(u - \tau(u)))\right] du.\end{aligned} \tag{13}$$

Therefore, we express the above equation as

$$(S\varphi)(t) = (A\varphi)(t) + (B\varphi)(t),$$

where $A, B : P_T \to P_T$ are given by

$$\begin{aligned}(A\varphi)(t) &\\ = Q(t, \varphi&(t - \tau(t))) \\ + \eta \int_{t-T}^{t} &\kappa(t,u)\left[-a(u)\, Q(u, \varphi(u - \tau(u))) + G(u, \varphi(u), \varphi(u - \tau(u)))\right] du.\end{aligned} \tag{14}$$

and

$$(B\varphi)(t) = \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du. \tag{15}$$

We will assume that the following conditions hold.

(H1) $a \in L^1[0,T]$ is bounded.

(H2) Q, G satisfies Carathéodory conditions with respect to $L^1[0,T]$.

(H3) There exists periodic functions q_1, $q_2 \in L^1[0,T]$, with period T, such that

$$|Q(t,x)| \leq q_1(t)|x| + q_2(t).$$

(H4) There exists periodic functions g_1, g_2, $g_3 \in L^1[0,T]$, with period T, such that

$$|G(t,x,y)| \leq g_1(t)|x| + g_2(t)|y| + g_3(t).$$

Now, we need the following assumptions

$$q_1(t)L + q_2(t) \leq \frac{\gamma_1}{2} L, \tag{16}$$

$$g_1(t)L + g_2(t)L + g_3(t) \leq \gamma_2 L a(t), \tag{17}$$

$$J(\gamma_1 + \gamma_2) \leq 1, \tag{18}$$

where γ_1, γ_2 and J are positive constants with $J \geq 3$.

Lemma 2 *For A defined in (14), suppose that (5)–(7), (16)–(18) and (H1)–(H4) hold. Then $A : \mathcal{M} \to \mathcal{M}$.*

Proof. Let A be defined by (14). Obviously, $A\varphi$ is continuous. First by (5) and (7), a change of variable in (14) shows that $(A\varphi)(t+T) = (A\varphi)(t)$. That is, if $\varphi \in P_T$ then $A\varphi$ is periodic with period T. Next, let $\varphi \in \mathcal{M}$, by (16)–(18)

and (H1)–(H4) we have

$$\begin{aligned}
&|(A\varphi)(t)| \\
&\leq |Q(t, \varphi(t-\tau(t)))| \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\left(a(u)|Q(u, \varphi(u-\tau(u)))| + |G(u, \varphi u, \varphi(u-\tau(u)))|\right) du \\
&\leq q_1(t)|\varphi(t-\tau(t))| + q_2(t) \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\left[q_1(u)|\varphi(u-\tau(u))| + q_2(u)\right] du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\left[g_1(u)|\varphi(u)| + g_2(u)|\varphi(u-\tau(u))| + g_3(u)\right] du \\
&\leq \gamma_1 L + \gamma_2 L \leq \frac{L}{J} \leq L.
\end{aligned}$$

That is $A\varphi \in \mathcal{M}$. □

Lemma 3 *For $A : \mathcal{M} \to \mathcal{M}$ defined in (14), suppose that (5)–(7), (16)–(18) and (H1)–(H4) hold. Then A is completely continuous.*

Proof. We show that A is continuous in the supremum norm, Let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&|(A\varphi_n)(t) - (A\varphi)(t)| \\
&\leq |Q(t, \varphi_n(t-\tau(t))) - Q(t, \varphi(t-\tau(t)))| \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, |Q(u, \varphi_n(u-\tau(u))) - Q(u, \varphi(u-\tau(u)))| du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, |G(u, \varphi_n(u), \varphi_n(u-\tau(u))) - G(u, \varphi(u), \varphi(u-\tau(u)))| du
\end{aligned}$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty} |(A\varphi_n)(t) - (A\varphi)(t)| = 0$. Then A is continuous.

We next show that A is completely continuous. Let $\varphi \in \mathcal{M}$, then, by Lemma 2, we see that

$$\|A\varphi\| \leq L.$$

And so the family of functions $A\varphi$ is uniformly bounded. Again, let $\varphi \in \mathcal{M}$. Without loss of generality, we can pick $\omega < t$ such that $t - \omega < T$. Then

$$\begin{aligned}
&|(A\varphi)(t) - (A\varphi)(\omega)| \\
&\leq |Q(t, \varphi(t - \tau(t))) - Q(\omega, \varphi(\omega - \tau(\omega)))| \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t, u)\, a(u)\, Q(u, \varphi(u - \tau(u)))\, du \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega, u)\, a(u)\, Q(u, \varphi(u - \tau(u)))\, du \right| \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t, u)\, G(u, \varphi(u), \varphi(u - \tau(u)))\, du \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega, u)\, G(u, \varphi(u), \varphi(u - \tau(u)))\, du \right| \\
&\leq |Q(t, \varphi(t - \tau(t))) - Q(\omega, \varphi(\omega - \tau(\omega)))| \\
&+ 2\eta\kappa_0 \int_{\omega-T}^{t-T} \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du \\
&+ \eta \int_{\omega-T}^{\omega} |\kappa(t, u) - \kappa(\omega, u)| \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du \\
&\leq |Q(t, \varphi(t - \tau(t))) - Q(\omega, \varphi(\omega - \tau(\omega)))| \\
&+ 2\eta\kappa_0 \int_{\omega}^{t} \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du \\
&+ \eta \int_{0}^{T} |\kappa(t, u) - \kappa(\omega, u)| \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du,
\end{aligned}$$

where $\kappa_0 = \max_{u \in [t-T, t]} \{\kappa(t, u)\}$, then by the Dominated Convergence Theorem $|(A\varphi)(t) - (A\varphi)(\omega)| \to 0$ as $t - \omega \to 0$ independently of $\varphi \in \mathcal{M}$. Thus $(A\varphi)$ is equicontinuous. Hence by Ascoli-Arzela's theorem A is completely continuous. □

Now, we state an important result see [1, Theorem 3.4] and for convenience we present below its proof, we deduce by this theorem that the following are sufficient conditions implying that the mapping H given by (12) is a large contraction on the set $\mathcal{M}$.

(H5) $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,

(H6) the function h is strictly increasing on $[-L, L]$,

(H7) $\sup_{t \in (-L, L)} h'(t) \leq 1$.

Theorem 2 *Let* $h : \mathbb{R} \to \mathbb{R}$ *be a function satisfying (H5)–(H7). Then the mapping* H *in (12) is a large contraction on the set* $\mathcal{M}$.

Proof. Let $\varphi, \psi \in \mathcal{M}$ with $\varphi \neq \psi$. Then $\varphi(t) \neq \psi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\varphi, \psi)$, i.e.,

$$D(\varphi, \psi) = \{t \in \mathbb{R} : \varphi(t) \neq \psi(t)\}.$$

For all $t \in D(\varphi, \psi)$, we have

$$\begin{aligned} &|(H\varphi)(t) - (H\psi)(t)| \\ &\leq |\varphi(t) - \psi(t) - h(\varphi(t)) + h(\psi(t))| \\ &\leq |\varphi(t) - \psi(t)| \left| 1 - \frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} \right|. \end{aligned} \tag{19}$$

Since h is a strictly increasing function we have

$$\frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} > 0 \text{ for all } t \in D(\varphi, \psi). \tag{20}$$

For each fixed $t \in D(\varphi, \psi)$ define the interval $I_t \subset [-L, L]$ by

$$I_t = \begin{cases} (\varphi(t), \psi(t)) & \text{if } \varphi(t) < \psi(t), \\ (\psi(t), \varphi(t)) & \text{if } \psi(t) < \varphi(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\varphi, \psi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} = h'(c_t).$$

By (H6) and (H7) we have

$$0 \leq \inf_{u \in (-L,L)} h'(u) \leq \inf_{u \in I_t} h'(u) \leq h'(c_t) \leq \sup_{u \in I_t} h'(u) \leq \sup_{u \in (-L,L)} h'(u) \leq 1. \tag{21}$$

Hence, by (19)–(21) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \leq |\varphi(t) - \psi(t)| \left| 1 - \inf_{u \in (-L,L)} h'(u) \right|, \tag{22}$$

for all $t \in D(\varphi, \psi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\epsilon \in (0, 1)$ and assume that φ and ψ are two functions in $\mathcal{M}$ satisfying

$$\epsilon \leq \sup_{t \in (-L,L)} |\varphi(t) - \psi(t)| = \|\varphi - \psi\|.$$

If $|\varphi(t) - \psi(t)| \le \frac{\epsilon}{2}$ for some $t \in D(\varphi, \psi)$, then we get by (21) and (22) that

$$|(H\varphi)(t) - (H\psi)(t)| \le |\varphi(t) - \psi(t)| \le \frac{1}{2} \|\varphi - \psi\|. \tag{23}$$

Since h is continuous and strictly increasing, the function $h\left(u + \frac{\epsilon}{2}\right) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\epsilon}{2} \le |\varphi(t) - \psi(t)|$ for some $t \in D(\varphi, \psi)$, then by (H6) and (H7) we conclude that

$$1 \ge \frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2L} \min\left\{h\left(u + \frac{\epsilon}{2}\right) - h(u) : u \in [-L, L]\right\} > 0.$$

Hence, (19) implies

$$|(H\varphi)(t) - (H\psi)(t)| \le (1 - \lambda) \|\varphi - \psi\|. \tag{24}$$

Consequently, combining (23) and (24) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \le \delta \|\varphi - \psi\|, \tag{25}$$

where

$$\delta = \max\left\{\frac{1}{2}, 1 - \lambda\right\}.$$

The proof is complete. □

The next result shows the relationship between the mappings H and B in the sense of large contractions. Assume that

$$\max\{|H(-L)|, |H(L)|\} \le \frac{2L}{J}. \tag{26}$$

Lemma 4 *Let B be defined by (15), suppose (H5)–(H6) hold. Then $B : \mathcal{M} \to \mathcal{M}$ is a large contraction.*

Proof. Let B be defined by (15). Obviously, $B\varphi$ is continuous and it is easy to show that $(B\varphi)(t + T) = (B\varphi)(t)$. Let $\varphi \in \mathcal{M}$

$$\begin{aligned} |(B\varphi)(t)| &\le \int_{t-T}^{t} \kappa(t, u)\, a(u) \max\{|H(-L)|, |H(L)|\}\, du \\ &\le \frac{2L}{J} < L, \end{aligned}$$

which implies $B : \mathcal{M} \to \mathcal{M}$.

By Theorem 2, H is large contraction on $\mathcal{M}$, then for any $\varphi, \psi \in \mathcal{M}$, with $\varphi \neq \psi$ and for any $\epsilon > 0$, from the proof of that Theorem, we have found a $\delta < 1$, such that

$$\begin{aligned}|(B\varphi)(t) - (B\psi)(t)| &= \left|\eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, [H(\varphi(u)) - H(\psi(u))]\, du\right| \\ &\leq \|\varphi - \psi\| \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, du \leq \delta \|\varphi - \psi\|.\end{aligned}$$

The proof is complete. □

Theorem 3 *Suppose the hypothesis of Lemmas 2, 3 and 4 hold. Let $\mathcal{M}$ defined by (11). Then the equation (4) has a T-periodic solution in $\mathcal{M}$.*

Proof. By Lemma 2, 3, A is continuous and $A(\mathcal{M})$ is contained in a compact set. Also, from Lemma 4, the mapping B is a large contraction. Next, we show that if $\varphi, \psi \in \mathcal{M}$, we have $\|A\psi + B\varphi\| \leq L$. Let $\varphi, \psi \in \mathcal{M}$ with $\|\varphi\|, \|\psi\| \leq L$. By (16)–(18)

$$\begin{aligned}\|A\psi + B\varphi\| &\leq (\gamma_1 + \gamma_2) L + \frac{2}{J} L \\ &\leq \frac{L}{J} + \frac{2L}{J} \leq L.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (4). Hence (4) has a T-periodic solution. □

4 Existence of nonnegative periodic solutions

In this section we obtain the existence of a nonnegative periodic solution of (4). By applying Theorem 1, we need to define a closed, convex, and bounded subset $\mathbb{M}$ of P_T. So, let

$$\mathbb{M} = \{\phi \in P_T : 0 \leq \phi \leq K\}. \tag{27}$$

where K is positive constant. To simplify notation, we let

$$m = \min_{u \in [t-T,t]} e^{-\int_u^t a(s)ds}, \; M = \max_{u \in [t-T,t]} e^{-\int_u^t a(s)ds}. \tag{28}$$

It is easy to see that for all $(t, u) \in [0, 2T]^2$,

$$m \leq \kappa(t, u) \leq M. \tag{29}$$

Then we obtain the existence of a nonnegative periodic solution of (4) by considering the two cases;

(1) $Q(t, y) \geq 0 \ \forall t \in [0, T], \ y \in \mathbb{M}$.

(2) $Q(t, y) \leq 0 \ \forall t \in [0, T], \ y \in \mathbb{M}$.

In the case one, we assume for all $t \in [0, T]$, $x, y \in \mathbb{M}$, that there exist a positive constant c_1 such that

$$0 \leq Q(t, y) \leq c_1 y, \tag{30}$$

$$c_1 < 1, \tag{31}$$

$$0 \leq -a(t) Q(t, y) + G(t, x, y) \tag{32}$$

$$a(t) H(\varphi(t)) - a(t) Q(t, y) + G(t, x, y) \leq \frac{K(1 - c_1)}{M\eta T}. \tag{33}$$

Lemma 5 *Let* A, B *given by (14), (15) respectively, assume (30)–(33) hold. Then* $A, B : \mathbb{M} \to \mathbb{M}$.

Proof. Let A defined by (15). So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned}
0 &\leq (A\varphi)(t) \leq Q(t, \varphi(t - \tau(t))) \\
&\quad + \eta \int_{t-T}^{t} \kappa(t, u) \left[-a(u) Q(u, \varphi(u - \tau(u))) + G(u, \varphi(u), \varphi(u - \tau(u)))\right] du \\
&\leq \eta \int_{t-T}^{t} M \frac{K(1 - c_1)}{M\eta T} du + c_1 K = K,
\end{aligned}$$

That is $A\varphi \in \mathbb{M}$.

Now, let B defined by (15). So, for any $\varphi \in \mathbb{M}$, we have

$$0 \leq (B\varphi)(t) \leq \eta \int_{t-T}^{t} M \frac{K(1 - c_1)}{M\eta T} du \leq \eta M T \frac{K}{M\eta T} = K.$$

That is $B\varphi \in \mathbb{M}$. $\square$

Theorem 4 *Suppose the hypothesis of Lemmas 3, 4 and 5 hold. Then equation (4) has a nonnegative* T*-periodic solution* x *in the subset* $\mathbb{M}$.

Proof. By Lemma 3, A is completely continuous. Also, from Lemma 4, the mapping B is a large contraction. By Lemma 5, $A, B : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $0 \leq A\psi + B\varphi \leq K$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \leq \varphi, \psi \leq K$. By (30)–(33)

$$\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&= \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du + Q(t, \psi(t-\tau(t))) \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) \left[-a(u)\, Q(u, \psi(u-\tau(u))) + G(u, \psi(u), \psi(u-\tau(u)))\right] du \\
&\leq \eta \int_{t-T}^{t} \kappa(t,u) \frac{K(1-c_1)}{M\eta T} du + c_1 K \\
&\leq \eta \int_{t-T}^{t} M \frac{K(1-c_1)}{M\eta T} du + c_1 K = K.
\end{aligned}$$

On the other hand,

$$(A\psi)(t) + (B\varphi)(t) \geq 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (4) and the proof is complete. □

In the case two, we substitute conditions (30)–(33) with the following conditions respectively. We assume that there exist a negative constant c_2 such that

$$c_2 y \leq Q(t, y) \leq 0, \tag{34}$$

$$-c_2 < 1, \tag{35}$$

$$\frac{-c_2 K}{M\eta T} \leq a(t) H(\varphi(t)) - a(t)\, Q(t, y) + G(t, x, y). \tag{36}$$

$$a(t) H(\varphi(t)) - a(t)\, Q(t, y) + G(t, x, y) \leq \frac{K}{M\eta T}. \tag{37}$$

Theorem 5 *Suppose (34)–(37) and the hypothesis of Lemmas 2, 3 and 4 hold. Then equation (4) has a nonnegative* T*-periodic solution* x *in the subset* $\mathbb{M}$.

Proof. By Lemma 2, 3, A is completely continuous. Also, from Lemma 4, the mapping B is a large contraction. To see that, it is easy to show as in Lemma 5 $A, B : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $0 \leq A\psi + B\varphi \leq K$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \leq \varphi, \psi \leq K$. By (34)–(37)

$$\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&= \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du + Q(t, \psi(t - \tau(t))) \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\left[-a(u)\, Q(u, \psi(u - \tau(u))) + G(u, \psi(u), \psi(u - \tau(u)))\right] du \\
&\leq \eta \int_{t-T}^{t} \kappa(t,u) \frac{K}{M\eta T} du = \eta \int_{t-T}^{t} M \frac{K}{M\eta T} du = K.
\end{aligned}$$

On the other hand,

$$(A\psi)(t) + (B\varphi)(t) \geq \eta \int_{t-T}^{t} M \frac{-c_2 K}{M\eta T} du + c_2 K = 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (4) and the proof is complete. $\square$

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Ricci solitons on QR-hypersurfaces of a quaternionic space form $\mathbb{Q}^n$

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Abstract. The purpose of this paper is to study Ricci solitons on QR-hypersurfaces M of a quaternionic space form $\mathbb{Q}^n$ such that the shape operator A with respect to N has one eigenvalue. We prove that Ricci soliton on QR- hypersurfaces M with eigenvalue zero is steady and for eigenvalue nonzero is shrinking.

1 Introduction

A Ricci soliton is defined on a Riemannian manifold (M, g) by

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} - \lambda g = 0 \tag{1}$$

where $L_V g$ is the Lie-derivative of the metric tensor g with respect to V and λ is a constant on M. The Ricci soliton is a natural generalization of an Einstein metric. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Compact Ricci solitons are the fixed points of the Ricci flow:

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) \tag{2}$$

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projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds. We denote a Ricci soliton by $(M, g, V; \lambda)$ and call the vector field V the potential vector field of the Ricci soliton. A trivial Ricci soliton is one for which V is Killing or zero. If its potential vector field $V = \nabla f$ such that f is some smooth function on M then a Ricci soliton $(M, g, V; \lambda)$ is called a gradient Ricci soliton and the smooth function f is called the potential function. It was proved by Grigory Perelman in [15] that any compact Ricci soliton is the sum of a gradient of some smooth function f up to the addition of a Killing field. Thus compact Ricci solitons are gradient Ricci solitons. In particular, Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904.

Hamilton [7] and Ivey [10] proved that a Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3, respectively. In [11], Ki proved that there are no real hypersurfaces with parallel Ricci tensor in a complex space form $\widetilde{M}^n(c)$ with $c \neq 0$ when $n \geq 3$. Kim [12] proved that when $n = 2$, this is also true. In particular, these results give that there is not any Einstein real hypersurfaces in a non-flat complex space form.

In [13], Chen studied important results on Ricci solitons which occur obviously on some Riemannian submanifolds. He presented several recent new criterions of trivial compact shrinking Ricci solitons.

Cho and Kimura [3] studied on Ricci solitons of real hypersurfaces in a non-flat complex space form and showed that a real hypersurface M in a non-flat complex space form $\widetilde{M}^n(c \neq 0)$ does not admit a Ricci soliton such that the Reeb vector field ξ is potential vector field. They defined so called η-Ricci soliton, such that satisfies

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} - \lambda g - \mu\eta \otimes \eta = 0 \tag{3}$$

where λ, μ are constants. They first proved that a real hypersurface M of a non-flat complex space form $\widetilde{M}^n(c)$ which accepts an η-Ricci soliton is a Hopf-hypersurface and classified that η-Ricci soliton real hypersurfaces in a non-flat complex space form.

We study Ricci solitons on QR-hypersurfaces M of a quaternionic space form $\mathbb{Q}^n$ such that the shape operator A with respect to N has one eigenvalues . We prove that Ricci soliton on QR- hypersurfaces M with eigenvalue zero is steady and for eigenvalue nonzero is shrinking.

2 Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type $(1,1)$ over $\overline{M}$ satisfying the following conditions (a), (b) and (c):
(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{aligned} &F^2 = -I, && G^2 = -I, && H^2 = -I, \\ &FG = -GF = H, && MGH = -HG = F, && HF = -FH = G. \end{aligned} \tag{4}$$

(b) There is a Riemannian metric g which is hermite with respect to all of F, G and H.
(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{5}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a canonical local basis of the bundle V in $\overline{\mathcal{U}}$ [9].

For canonical local basis $\{F, G, H\}$ and $\{F', G', H'\}$ of V in coordinate neighborhoods of $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}'$, it follows that in $\overline{\mathcal{U}} \cap \overline{\mathcal{U}}'$

$$\begin{pmatrix} F' \\ G' \\ H' \end{pmatrix} = \left(s_{xy}\right) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \qquad (x, y = 1, 2, 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (4). As is well known [9], every quaternionic Kähler manifold is orientable. Let $\overline{M}$ be quaternion Kaehler manifold and M be a real submanifold of $\overline{M}$. Then, M is said QR-submanifold if there exists a vector subbundle ν of the normal bundle such that we have

$$\begin{aligned} &F\nu_x = \nu_x, && G\nu_x = \nu_x, && H\nu_x = \nu_x, \\ &F\nu_x^{\perp}, && G\nu_x^{\perp}, && H\nu_x^{\perp} \subset T_xM, \end{aligned}$$

for $x \in M$, where $\nu^{\perp}$ is the complementary orthogonal bundle to ν in $TM^{\perp}$. We denote by D the complementary orthogonal distribution to $D^{\perp}$ in TM. Then D is invariant with respect to the action of $\{F, G, H\}$ i.e. we have

$$\begin{aligned} &FD_x = D_x, && GD_x = D_x, && HD_x = D_x, \\ &FD_x^{\perp}, && GD_x^{\perp}, && HD_x^{\perp} \subset T_x^{\perp}M, \end{aligned}$$

for any $x \in M$, where $TM = D \oplus D^{\perp}$ and $TM^{\perp} = \nu \oplus \nu^{\perp}$. D is called quaternion distribution.
Now let M be an n-dimensional QR-submanifold of maximal QR-dimension, that is, of $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field N such that $\nu_x^{\perp} = span\{N\}$ at each point x in M. We set

$$FN = -U, \quad GN = -V, \quad HN = -W. \tag{6}$$

Denoting by D_x the maximal quaternionic invariant subspace

$$T_xM \cap FT_xM \cap GT_xM \cap HT_xM,$$

of T_xM, we have $D_x^{\perp} \supset \mathrm{Span}\{U, V, W\}$, where $D_x^{\perp}$ means the complementary orthogonal subspace to D_x in T_xM. But, using (4), we can prove that $D_x^{\perp} = \mathrm{Span}\{U, V, W\}$ [13]. Thus we have

$$T_xM = D_x \oplus \mathrm{Span}\{U, V, W\}, \quad \forall x \in M,$$

which together with (4) and (6) imply

$$FT_xM, GT_xM, HT_xM \subset T_xM \oplus \mathrm{Span}\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\dots,p}$ $(N_1 := N)$ of normal vectors to M, we have

$$FX = \varphi X + u(X)N, \quad GX = \psi X + v(X)N, \quad HX = \theta X + \omega(X)N, \tag{7}$$

$$\begin{aligned} FN_\alpha &= -U_\alpha + P_1N_\alpha, \quad GN_\alpha = -V_\alpha + P_2N_\alpha, \\ HN_\alpha &= -W_\alpha + P_3N_\alpha, \quad (\alpha = 1, \dots, p). \end{aligned} \tag{8}$$

Then it is easily seen that $\{\varphi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively.
Moreover, the hermitian property of $[F, G, H\}$ implies

$$\begin{aligned} &g(X, \varphi U_\alpha) = -u(X)g(N_1, P_1N_\alpha), \\ &g(X, \psi V_\alpha) = -v(X)g(N_1, P_2N_\alpha), \\ &g(X, \theta W_\alpha) = -w(X)g(N_1, P_3N_\alpha), \quad (\alpha = 1, \dots, p). \end{aligned} \tag{9}$$

Also, from the hermitian properties

$$\begin{aligned} g(FX, N_\alpha) &= -g(X, FN_\alpha), \quad g(GX, N_\alpha) = -g(X, GN_\alpha), \\ g(HX, N_\alpha) &= -g(X, HN_\alpha), \quad (\alpha = 1, \dots, p). \end{aligned}$$

It follows that

$$g(X,U_\alpha)=u(X)\delta_{1\alpha},\quad g(X,V_\alpha)=v(X)\delta_{1\alpha},\quad g(X,W_\alpha)=w(X)\delta_{1\alpha},$$

and hence

$$\begin{aligned}&g(X,U_1)=u(X),\quad g(X,V_1)=v(X),\quad g(X,W_1)=w(X),\\&U_\alpha=0,\quad V_\alpha=0,\quad W_\alpha=0,\quad (\alpha=2,...p).\end{aligned}\tag{10}$$

On the other hand, comparing (6) and (8) with $\alpha=1$, we have $U_1=U, V_1=V, W_1=W$, which together with (6) and (10) imply

$$\begin{array}{llll} g(X,U)=u(X), & g(X,V)=v(X), & g(X,W)=w(X), & \\ u(U)=1, & v(V)=1, & w(W)=1, & \\ FN=-U, & GN=-V, & HN=-W & \\ FN_{\alpha=P_1N_\alpha}, & GN_{\alpha=P_2N_\alpha} & HN_{\alpha=P_3N_\alpha}, & (\alpha=2,...,p). \end{array}$$

from which, taking account of the skew-symmetry of P_1, P_2 and P_3 and using (9), we also have

$$\begin{array}{lll} u(\varphi X)=0, & v(\psi X)=0, & w(\theta X)=0,\\ \varphi U=0, & \psi V=0, & \theta W=0,\\ P_1N=0, & P_2N=0, & P_3N=0, \end{array}\tag{11}$$

From the equations of (6), we also have

$$\begin{array}{llll} \psi U=-W, & v(U)=0, & \theta U=V, & w(U)=0,\\ \varphi V=W, & u(V)=0, & \theta V=-U, & w(V)=0,\\ \varphi W=-V, & u(W)=0, & \psi W=U, & v(W)=0. \end{array}\tag{12}$$

Now, let ∇ be the Levi-Civita connection on M and $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. The Gauss and Weingarten formula are given by

$$\begin{aligned}&\overline{\nabla}_XY=\nabla_XY+h(X,Y),\\&\overline{\nabla}_XN_\alpha=-A_\alpha X+\nabla^\perp_XN_\alpha,\quad(\alpha=1,\dots,p),\end{aligned}\tag{13}$$

for any $X,Y\in\chi(M)$ and $N_\alpha\in\Gamma^\infty(T(M)^\perp)$, $(\alpha=1,\dots,p)$. h is the second fundamental form and A_α are shape operator corresponding to N_α.

Next, differentiating the equations of (6) covariantly and comparing the tangential and normal parts, we have

$$\begin{aligned}
\nabla_Y U &= r(Y)V - q(Y)W + \varphi A_1 Y,\\
\nabla_Y V &= -r(Y)U + p(Y)W + \psi A_1 Y,\\
\nabla_Y W &= q(Y)U - p(Y)V + \theta A_1 Y,
\end{aligned} \tag{14}$$

For QR-hypersurfaces M in a quaternionic space form $\overline{M}$ of quaternionic sectional curvature $4k$ the Gauss and Codazzi equations are written as follow:

$$\begin{aligned}
g(R(X,Y)Z,W) = k\{&g(Y,Z)X - g(X,Z)Y\\
&+ g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z\\
&+ g(\psi Y,Z)\psi X - g(\psi X,Z)GY - 2g(\psi X,Y)\psi Z\\
&+ g(\theta Y,Z)\theta X - g(\theta X,Z)\theta Y - 2g(\theta X,Y)\theta Z\}\\
&+ g(AY,Z)AX - g(AX,Z)AY,
\end{aligned} \tag{15}$$

$$\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = k\{&u(X)\varphi Y - u(Y)\varphi X - 2g(\varphi X,Y)U\\
&+ v(X)\psi Y - v(Y)\psi X - 2g(\psi X,Y)V\\
&+ w(X)\theta Y - w(Y)\theta X - 2g(\theta X,Y)W\},
\end{aligned} \tag{16}$$

hence the Ricci tensor is obtained as

$$\begin{aligned}
\mathrm{Ric}(X,Y) = k\{&(4n+7)g(X,Y) - 3\{u(X)u(Y) + v(X)v(Y) + w(X)w(Y)\}\}\\
&+ (\operatorname{trace} A)g(AX,Y) - g(AX,AY).
\end{aligned} \tag{17}$$

for any tangent vector fields X, Y, Z on M, where R and Ric are the curvature and Ricci tensors of M, respectively.

3 Ricci soliton on QR hypersurfaces

Let M be a QR-hypersurface of a quaternionic space form $\overline{M}$ such that the shape operator A for unit normal vector field N has only one eigenvalue and let $\{e_1, \dots, e_{4n-4}, U, V, W\}$ be a local orthonormal fram field such that $D^{\perp} = \operatorname{span}\{U, V, W\}$ and $D = \operatorname{span}\{e_1, \dots, e_{n-1}, e_n = \varphi e_1, \dots, e_{2n-2} = \varphi e_{n-1}, e_{2n-1} = \psi e_1, \dots, e_{3n-3} = \psi e_{n-1}, e_{3n-2} = \theta e_1, \dots, e_{4n-4} = \theta e_{n-1}\}$.
We first prove

Theorem 1 *If the shape operator A with respect to unit normal vector field N of M has only one eigenvalue, then $\overline{M}$ is a quaternionic Euclidean space.*

Proof. According to the assumption, it follows that $A = 0$ or $AX = \alpha X$ for all $X \in T(M)$.
In both cases the Codazzi equation (16), we obtain

$$\begin{aligned}(X\alpha)Y - (Y\alpha)X = k\{&u(X)\varphi Y - u(Y)\varphi X - 2g(\varphi X, Y)U \\ &+ v(X)\psi Y - v(Y)\psi X - 2g(\psi X, Y)V \\ &+ w(X)\theta Y - w(Y)\theta X - 2g(\theta X, Y)W\},\end{aligned} \tag{18}$$

for all $X, Y \in TM$. Putting $Y = U$, the equation (21) reduces to

$$(X\alpha)U - (U\alpha)X \quad = \quad k\{-\varphi X + v(X)W - w(X)V\}, \tag{19}$$

also by putting $Y = V$ and $Y = W$, we have

$$\begin{aligned}(X\alpha)V - (V\alpha)X = k\{-\psi X + w(X)U - u(X)W\}, \\ (X\alpha)W - (W\alpha)X = k\{-\theta X + u(X)V - v(X)U\}.\end{aligned} \tag{20}$$

since dim $M \geq 7$, we can use $X, \varphi X, \psi X, \theta X, U, V$ and W in such a way that they are linearly independent and thus $k = 0$. □

Let $AX = \alpha X$, therefore by the relation (17), we obtain

$$\begin{aligned}&Ric(e_i, e_j) = \{(4n-2)\alpha^2\}\delta_{ij}, \qquad (i, j = 1, \ldots, 4n-4), \\ &Ric(U, U) = (4n-2)\alpha^2, \\ &Ric(V, V) = (4n-2)\alpha^2, \\ &Ric(W, W) = (4n-2)\alpha^2, \\ &Ric(U, V) = 0, \\ &Ric(U, W) = 0, \\ &Ric(W, V) = 0, \\ &Ric(e_i, U) = 0, \\ &Ric(e_i, V) = 0, \\ &Ric(e_i, W) = 0, \qquad (i = 1, ..., 4n-4).\end{aligned} \tag{21}$$

We consider QR-hypersurface M of a quaternionic space form $\mathbb{Q}^n$ satisfying Ricci soliton equation

$$\frac{1}{2}\mathcal{L}_{\widetilde{V}}g + Ric - \lambda g = 0 \tag{22}$$

with respect to potential vector field $\widetilde{V}$ on M for constant λ.
First Put

$$\widetilde{V} := fU, \qquad (f : M \to \mathbb{R}, f \neq 0) \tag{23}$$

Then definition of the Lie derivative and the first relation (14) imply

$$\begin{aligned}(\mathcal{L}_{fU}g)(X,Y) = {} & df(X)u(Y) + df(Y)u(X) \\ & + f\{r(X)v(Y) - q(X)w(Y) + r(Y)v(X) - q(Y)w(X) \\ & + g((\varphi A - A\varphi)Y, X)\}\end{aligned} \tag{24}$$

We compute

$$\begin{aligned}
&(\mathcal{L}_{fU}g)(U,U) = 2df(U), \\
&(\mathcal{L}_{fU}g)(V,V) = 2fr(V), \\
&(\mathcal{L}_{fU}g)(W,W) = -2fq(W), \\
&(\mathcal{L}_{fU}g)(U,V) = df(V) + fr(U), \\
&(\mathcal{L}_{fU}g)(U,W) = df(W) - fq(U), \\
&(\mathcal{L}_{fU}g)(W,V) = f\{-q(V) + r(W)\}, \\
&(\mathcal{L}_{fU}g)(U,e_i) = df(e_i), \\
&(\mathcal{L}_{fU}g)(V,e_i) = fr(e_i), \\
&(\mathcal{L}_{fU}g)(W,e_i) = -fq(e_i), \qquad (i = 1,\dots,4n-4), \\
&(\mathcal{L}_{f\xi}g)(e_i,e_j) = 0 \qquad (i,j = 1,\dots,4n-4).
\end{aligned} \tag{25}$$

Using relations (21) and (25), Ricci soliton equation (22) is equivalent to

$$\begin{aligned}
&df(U) = \lambda - (4n-2)\alpha^2, \\
&fr(V) = \lambda - (4n-2)\alpha^2, \\
&fq(W) = -\lambda + (4n-2)\alpha^2, \\
&df(V) = -fr(U), \\
&df(W) = fq(U), \\
&q(V) = r(W), \\
&df(e_i) = 0, \qquad (i = 1,\dots,4n-4), \\
&r(e_i) = 0, \\
&q(e_i) = 0, \\
&\{(4n-2)\alpha^2 - \lambda\}\delta_{ij} = 0, \qquad (i,j = 1,\dots,4n-4).
\end{aligned} \tag{26}$$

By the last relation (26), we have $\lambda = (4n-2)\alpha^2$ and thus the following theorem holds:

Theorem 2 *Let* M *be a* QR*-hypersurface of quternionic space form* $\mathbb{Q}^n$ *with* $AX = \alpha X$. *Then a Ricci soliton* $(M, g, \widetilde{V}, \lambda)$ *with potential vector field* $\widetilde{V} := fU$ *is shrinking Ricci soliton.*

Now, let $A = 0$, using relation (17),it follows that

$$\mathrm{Ric} = 0 \tag{27}$$

QR-hypersurface M $(n \geq 2)$ is considered in a quaternionic space form $\mathbb{Q}^n$ satisfying Ricci soliton equation.
By relations (27) and (25), Ricci soliton equation (22) is equivalent to

$$\begin{aligned}
&df(U) = \lambda, \\
&fr(V) = \lambda, \\
&fq(W) = -\lambda, \\
&df(V) = -fr(U), \\
&df(W) = fq(U), \\
&q(V) = r(W), \\
&df(e_i) = 0, \qquad (i = 1, \dots, 4n-4), \\
&r(e_i) = 0, \\
&q(e_i) = 0, \\
&\lambda\delta_{ij} = 0, \qquad (i, j = 1, \dots, 4n-4).
\end{aligned} \tag{28}$$

Using the last relation (28), it follows $\lambda = 0$ and hence

Theorem 3 *Let* M *be a* QR*-hypersurface of quaternionic space form* $\mathbb{Q}^n$ *with* $A = 0$. *Then a Ricci soliton* $(M, g, \widetilde{V}, \lambda)$ *with potential vector field* $V := fU$ *is steady Ricci soliton.*

hence, similar results were obtained when each structural vector fields $\{V, W\}$ of structure quaternionic $\{U, V, W\}$ be the potential vector field.

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Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers

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Abstract. In the paper, the authors find several identities, including a new recurrence relation for the Stirling numbers of the first kind, involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers.

1 Notation and main results

It is known that, for $x \in \mathbb{R}$, the quantities

$$\langle x\rangle_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases} = \prod_{\ell=0}^{n-1}(x-\ell) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

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and

$$\langle x\rangle_n=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge 1\\ 1, & n=0\end{cases}=\prod_{\ell=0}^{n-1}(x+\ell)=\frac{\Gamma(x+n)}{\Gamma(x)}$$

are respectively called the falling and rising factorials, where

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$$

is the classical gamma function, see [1, p. 255, 6.1.2]. For removable singularities of the ratio $\frac{\Gamma(x+m)}{\Gamma(x+n)}$ for $x\in\mathbb{R}$ and $m,n\in\mathbb{Z}$, please read [23, Theorem 1.1] and closely related references therein.

According to [4, pp. 293–294], there are two kinds of Cauchy numbers which may be defined respectively by

$$C_n=\int_0^1\langle x\rangle_n\,\mathrm{d}\,x\quad\text{and}\quad c_n=\int_0^1(x)_n\,\mathrm{d}\,x. \tag{1}$$

The Cauchy numbers C_n and c_n play important roles in some fields, such as approximate integrals, the Laplace summation formula, and difference-differential equations, and are also related to some famous numbers such as the Stirling, Bernoulli, and harmonic numbers. For recent conclusions on the Cauchy numbers, please read the papers [17, 18, 21, 30].

It is known that the coefficients expressing rising factorials $(x)_n$ in terms of falling factorials $\langle x\rangle_k$ are called the Lah numbers, denoted by $L(n,k)$. Precisely speaking,

$$(x)_n=\sum_{k=1}^{n}L(n,k)\langle x\rangle_k\quad\text{and}\quad\langle x\rangle_n=\sum_{k=1}^{n}(-1)^{n-k}L(n,k)(x)_k. \tag{2}$$

They can be computed by

$$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}$$

and have an interesting meaning in combinatorics: they count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets. For more and recent results on the Lah numbers $L(n,k)$, please refer to [13, 15, 16].

The Stirling numbers of the first kind $s(n,k)$ may be generated by

$$\langle x\rangle_n=\sum_{k=0}^{n}s(n,k)x^k \quad \text{and} \quad (x)_n=\sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k. \tag{3}$$

The combinatorial meaning of the unsigned Stirling numbers of the first kind $(-1)^{n-k}s(n,k)$ can be interpreted as the number of permutations of $\{1,2,\dots,n\}$ with k cycles. Recently there are some new results on the Stirling numbers of the first kind $s(n,k)$ obtained in [17, 20, 21, 22].

An infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^n f^{(n)}(x)\ge 0$ for $x\in I$ and $n\ge 0$. See [38, Definition 1.3] and [40, Chapter XII]. An infinitely differentiable function $f: I\subseteq(-\infty,\infty)\to[0,\infty)$ is called a Bernstein function on I if its derivative $f'(t)$ is completely monotonic on I. See [38, Definition 3.1].

The class of completely monotonic functions may be characterized by [40, Theorem 12b] which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is that $f(x)=\int_0^\infty e^{-xt}\,d\,\alpha(t)$, where $\alpha(t)$ is non-decreasing and the integral converges for $0<x<\infty$. The Bernstein functions on $(0,\infty)$ can be characterized by the assertion that a function $f:(0,\infty)\to\mathbb{R}$ is a Bernstein function if and only if it admits the representation

$$f(x)=a+bx+\int_0^\infty\left(1-e^{-xt}\right)d\,\mu(t), \tag{4}$$

where $a,b\ge 0$ and μ is a Radon measure on $(0,\infty)$ satisfying $\int_0^\infty \min\{1,t\}\,d\,\mu(t)<\infty$. See [38, Theorem 3.2]. The triplet (a,b,μ) determines f uniquely and vice versa. The representing measure μ and the characteristic triplet (a,b,μ) from the expression (4) are often called the Lévy measure and the Lévy triplet of the Bernstein function f. The formula (4) is called the Lévy-Khintchine representation of f.

It was obtained inductively in [32, Lemma 2.1] that the derivatives of the functions

$$h_\alpha(t)=\left(1+\frac{1}{t}\right)^{\alpha}, \quad t>0, \quad \alpha\in(-1,1)$$

and

$$H_\alpha(t)=\frac{h_\alpha(t)}{\alpha}-\frac{h_{\alpha-1}(t)}{\alpha-1}$$

may be computed by

$$h_\alpha^{(i)}(t)=\frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^{\alpha}\sum_{k=0}^{i-1}k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k}t^k \tag{5}$$

and

$$H_\alpha^{(i)}(t)=\frac{(-1)^i}{t^i(1+t)^{i+1}}\left(1+\frac{1}{t}\right)^\alpha\sum_{k=0}^{i-1}k!\binom{i+1}{k}\binom{i-1}{k}(\alpha)_{i-k}t^k,$$

for $i\in\mathbb{N}$. Consequently,

1. if $\alpha\in(0,1)$, the function $h_\alpha(t)$ is completely monotonic on $(0,\infty)$;
2. if $\alpha\in(-1,0)$, the function $h_\alpha(t)$ is a Bernstein function on $(0,\infty)$;
3. if $\alpha\in(0,1)$, the function $H_\alpha(t)$ is completely monotonic on $(0,\infty)$.

With the help of [32, Lemma 2.1], it was derived in [32, Theorem 1.1] that the weighted geometric mean

$$G_{x,y;\lambda}(t)=(x+t)^\lambda(y+t)^{1-\lambda}$$

is a Bernstein function of $t>-\min\{x,y\}$, where $\lambda\in(0,1)$ and $x,y\in\mathbb{R}$ with $x\ne y$. For more and detailed information on this topic, please refer to [2, 12, 22, 26, 32, 33, 34, 35, 36, 42] and closely related references therein.

In combinatorics, the Bell polynomials of the second kind (also called the partial Bell polynomials) $B_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ are defined by

$$B_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n \ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n\ge k\ge0$, see [4, p. 134, Theorem A]. The Faà di Bruno formula may be described in terms of the Bell polynomials of the second kind $B_{n,k}$ by

$$\frac{d^n}{d\,x^n}f\circ g(x)=\sum_{k=0}^n f^{(k)}(g(x))B_{n,k}\big(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\big),\qquad(6)$$

see [4, p. 139, Theorem C].

The aims of this paper are, by virtue of the famous Faà di Bruno formula (6), to find a new form for derivatives of the function $h_\alpha(t)$, and then, by comparing this new form with (5), to derive some identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers.

Our main results may be summarized up as the following theorem.

Theorem 1 *For* $i \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}$,

$$h_\alpha^{(i)}(t) = \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{k=0}^{i-1}\left[\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m\right]t^k. \tag{7}$$

Consequently, the identities

$$(\alpha)_n = \frac{1}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{m=1}^{n}\binom{n+k-m}{k}L(n+k,m)\langle\alpha\rangle_m, \tag{8}$$

$$c_n = \frac{1}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{m=1}^{n}\binom{n+k-m}{k}L(n+k,m)C_m, \tag{9}$$

$$c_n = \frac{1}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{\ell=1}^{n}\sum_{m=\ell}^{n}(-1)^{m-\ell}\binom{n+k-m}{k}L(n+k,m)L(m,\ell)c_\ell, \tag{10}$$

and

$$s(n,\ell) = \frac{(-1)^{n-\ell}}{k!\binom{n+k}{k}\binom{n+k-1}{k}}\sum_{m=\ell}^{n}\binom{n+k-m}{k}L(n+k,m)s(m,\ell) \tag{11}$$

hold for all $k, \ell \geq 0$ *and* $n \in \mathbb{N}$.

In next section, we will give a proof of Theorem 1. In the final section, we will list some remarks for explaining and interpreting the significance of identities obtained in Theorem 1.

2 Proof of Theorem 1

Now we are in a position to prove formulas or identities listed in Theorem 1.

The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy

$$B_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{12}$$

and

$$B_{n,k}(1!, 2!, 3!, \dots, (n-k+1)!) = L(n,k), \tag{13}$$

see [4, p. 135], where a and b are any complex numbers.

Taking in (6) $f(u) = (1+u)^\alpha$ and $u = g(t) = \frac{1}{t}$, employing (12) and (13), interchanging the order of the double sum, and simplifying yield

$$\begin{aligned}
h_\alpha^{(i)}(t) &= \sum_{k=1}^{i} \langle\alpha\rangle_k (1+u)^{\alpha-k} B_{i,k}\left(-\frac{1}{t^2}, \frac{2!}{t^3}, \dots, (-1)^{i-k+1}\frac{(i-k+1)!}{t^{i-k+2}}\right) \\
&= \sum_{k=1}^{i} \langle\alpha\rangle_k \left(1+\frac{1}{t}\right)^{\alpha-k} \frac{1}{t^k}\left(-\frac{1}{t}\right)^i B_{i,k}(1!, 2!, \dots, (i-k+1)!) \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{k=1}^{i} L(i,k)\langle\alpha\rangle_k (1+t)^{i-k} \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{m=0}^{i-1} L(i,i-m)\langle\alpha\rangle_{i-m} (1+t)^{m} \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{k=0}^{i-1} L(i,i-k)\langle\alpha\rangle_{i-k} \sum_{j=0}^{k}\binom{k}{j} t^j \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{j=0}^{i-1}\sum_{k=j}^{i-1} L(i,i-k)\langle\alpha\rangle_{i-k} \binom{k}{j} t^j \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{k=0}^{i-1}\sum_{q=k}^{i-1} L(i,i-q)\langle\alpha\rangle_{i-q} \binom{q}{k} t^k \\
&= \frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha \sum_{k=0}^{i-1}\sum_{m=1}^{i-k} L(i,m)\langle\alpha\rangle_{m} \binom{i-m}{k} t^k.
\end{aligned}$$

Comparing this with the formula (5) reveals

$$k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k} = \sum_{m=1}^{i-k} L(i,m)\langle\alpha\rangle_m \binom{i-m}{k}. \tag{14}$$

From this, the identity (8) follows immediately.

Integrating with respect to $\alpha \in (0,1)$ on both sides of (14) gives

$$\sum_{m=1}^{i-k}\binom{i-m}{k} L(i,m)\int_0^1 \langle\alpha\rangle_m \, d\alpha = k!\binom{i}{k}\binom{i-1}{k}\int_0^1 (\alpha)_{i-k}\, d\,\alpha,$$

that is, by (1),

$$\sum_{m=1}^{i-k}\binom{i-m}{k} L(i,m) C_m = k!\binom{i}{k}\binom{i-1}{k} c_{i-k}.$$

This can be rearranged as the identity (9).

Employing the second formula in (2) and the identity (14) acquires

$$\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m=\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=1}^{m}(-1)^{m-p}L(m,p)(\alpha)_p,$$

$$\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=1}^{m}(-1)^{m-p}L(m,p)(\alpha)_p=k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k},$$

$$\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}(-1)^{m-p}L(i,m)L(m,p)(\alpha)_p=k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k}.$$

Integrating on both sides of the above equality with respect to $\alpha\in(0,1)$ brings out

$$\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}(-1)^{m-p}L(i,m)L(m,p)\int_0^1(\alpha)_p\,\mathrm{d}\,\alpha \\ =k!\binom{i}{k}\binom{i-1}{k}\int_0^1(\alpha)_{i-k}\,\mathrm{d}\,\alpha,$$

that is,

$$\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}(-1)^{m-p}L(i,m)L(m,p)c_p=k!\binom{i}{k}\binom{i-1}{k}c_{i-k}.$$

This may be rearranged as (10).

Utilizing the formulas in (3) and (14) results in

$$\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m=\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=0}^{m}s(m,p)\alpha^p$$

$$=\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\sum_{p=1}^{m}s(m,p)\alpha^p=\sum_{p=1}^{i-k}\sum_{m=p}^{i-k}\binom{i-m}{k}L(i,m)s(m,p)\alpha^p$$

and

$$\begin{aligned}
k!\binom{i}{k}\binom{i-1}{k}(\alpha)_{i-k} &= k!\binom{i}{k}\binom{i-1}{k}\sum_{p=0}^{i-k}(-1)^{i-k-p}s(i-k,p)\alpha^p \\
&= k!\binom{i}{k}\binom{i-1}{k}\sum_{p=1}^{i-k}(-1)^{i-k-p}s(i-k,p)\alpha^p \\
&= \sum_{p=1}^{i-k}k!\binom{i}{k}\binom{i-1}{k}(-1)^{i-k-p}s(i-k,p)\alpha^p.
\end{aligned}$$

Equating coefficients of α^p in the above equations leads to

$$\sum_{m=p}^{i-k}\binom{i-m}{k}L(i,m)s(m,p) = k!\binom{i}{k}\binom{i-1}{k}(-1)^{i-k-p}s(i-k,p)$$

which may be reformulated as (11). The proof of Theorem 1 is complete.

3 Remarks

For explaining and interpreting the significance of formulas or identities obtained in Theorem 1, we are now list several remarks as follows.

Remark 1 *Because the sign of*

$$\sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle\alpha\rangle_m$$

can not be made clear easily, the formula (7) *is much more complicated than the formula* (5). *Concretely speaking, by virtue of the formula* (7), *we can not obviously see the properties that* $h_\alpha(t)$ *for* $\alpha\in(0,1)$ *is a completely monotonic function on* $(0,\infty)$ *and that* $h_\alpha(t)$ *for* $\alpha\in(-1,0)$ *is a Bernstein function on* $(0,\infty)$. *This implies that [32, Lemma 2.1] is much more useful and significant.*

Remark 2 *The recurrence relation* (11) *is a new "horizontal" recurrence relation for the Stirling numbers of the first kind* $s(n,k)$, *because it is different from those "triangular", "horizontal", "vertical", and "diagonal" recurrence relations, listed or obtained in [4, pp. 214–215, Theorems A, B, and C] and [19, 20], for the Stirling numbers of the first kind* $s(n,k)$.

Remark 3 *It is a very interesting phenomenon that the variable* $k \geq 0$ *only appears in the right hand sides of* (8) *to* (11) *and that* k *can change anyway.*

Remark 4 *Comparing* (2) *with* (8) *reveals*

$$L(n,m) = \frac{\binom{n+k-m}{k}}{k!\binom{n+k}{k}\binom{n+k-1}{k}} L(n+k,m).$$

Remark 5 *When letting* $\alpha \to -1^+$, *the identity* (14) *becomes*

$$k!\binom{i}{k}\binom{i-1}{k}(-1)_{i-k} = \sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)\langle -1\rangle_m,$$

$$0 = \sum_{m=1}^{i-k}\binom{i-m}{k}L(i,m)(-1)^m m!.$$

In other words, the identity

$$\sum_{m=1}^{i-k}(-1)^m m!\binom{i-m}{k}L(i,m) = 0,$$

which may be reformulated as

$$\sum_{m=1}^{n}(-1)^m\binom{n+k-m}{k}\binom{n+k-1}{m-1} = 0,$$

holds for all $i > k+1 \geq 1$ *and* $n \in \mathbb{N}$.

Taking $\alpha = \pm\frac{1}{2}$ *in* (14) *respectively reveals*

$$\sum_{m=1}^{i-k}(-1)^{m+1}\frac{(2m-3)!!}{2^m}\binom{i-m}{k}L(i,m) = k!\frac{(2(i-k)-1)!!}{2^{i-k}}\binom{i}{k}\binom{i-1}{k}$$

and

$$\sum_{m=1}^{i-k}(-1)^{m+1}\frac{(2m-1)!!}{2^m}\binom{i-m}{k}L(i,m) = k!\frac{(2(i-k)-3)!!}{2^{i-k}}\binom{i}{k}\binom{i-1}{k},$$

which are equivalent to

$$\sum_{m=1}^{n}(-1)^m\frac{(2m-3)!!}{2^m}\binom{n+k-m}{k}L(n+k,m) \\ = -\frac{(2n-1)!!k!}{2^n}\binom{n+k}{k}\binom{n+k-1}{k}$$

and

$$\sum_{m=1}^{n}(-1)^m\frac{(2m-1)!!}{2^m}\binom{n+k-m}{k}L(n+k,m) \\ =-\frac{(2n-3)!!k!}{2^n}\binom{n+k}{k}\binom{n+k-1}{k},$$

holds for all $i+1>k\geq 0$ *and* $n\in\mathbb{N}$.

Remark 6 *Let* $u=u(x)$ *and* $v=v(x)\neq 0$ *be differentiable functions. In [3, p. 40], the formula*

$$\frac{d^n}{dx^n}\left(\frac{u}{v}\right)=\frac{(-1)^n}{v^{n+1}}\begin{vmatrix} u & v & 0 & \dots & 0\\ u' & v' & v & \dots & 0\\ u'' & v'' & 2v' & \dots & 0\\ \hdotsfor{5} \\ u^{(n-1)} & v^{(n-1)} & \binom{n-1}{1}v^{(n-2)} & \dots & v\\ u^{(n)} & v^{(n)} & \binom{n}{1}v^{(n-1)} & \dots & \binom{n}{n-1}v' \end{vmatrix} \tag{15}$$

for the n*th derivative of the ratio* $\frac{u(x)}{v(x)}$ *was listed. For easy understanding and convenient availability, we now reformulate the formula (15) as*

$$\frac{d^n}{dx^n}\left(\frac{u}{v}\right)=\frac{(-1)^n}{v^{n+1}}\begin{vmatrix}A_{(n+1)\times 1} & B_{(n+1)\times n}\end{vmatrix}_{(n+1)\times(n+1)}, \tag{16}$$

where $|\cdot|_{(n+1)\times(n+1)}$ *denotes a determinant and the matrices*

$$A_{(n+1)\times 1}=(a_{i,1})_{0\leq i\leq n}$$

and

$$B_{(n+1)\times n}=(b_{i,j})_{0\leq i\leq n,0\leq j\leq n-1}$$

satisfy

$$a_{i,1}=u^{(i)}(x)\quad \text{and}\quad b_{i,j}=\binom{i}{j}v^{(i-j)}(x)$$

under the conventions that $v^{(0)}(x)=v(x)$ *and that* $\binom{p}{q}=0$ *and* $v^{(p-q)}(x)\equiv 0$ *for* $p<q$. *See [39, Lemma 2.1].*

Applying $u(x)=(1+t)^\alpha$ *and* $v(x)=t^\alpha$ *into (16) yields*

$$a_{i,1}=[(1+t)^\alpha]^{(i)}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}(1+t)^{\alpha-i}$$

and

$$b_{i,j} = \binom{i}{j}(t^{\alpha})^{(i-j)} = \binom{i}{j}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+j+1)}t^{\alpha-i+j}$$

for $0 \le i \le n$ *and* $0 \le j \le n-1$*. As a result, a new and alternative form for derivatives of the functions* $h_\alpha(t)$ *and* $H_\alpha(t)$ *may be established.*

Remark 7 *In recent years, the first author and his coauthors obtained some new properties of the Bell, Bernoulli, Euler, Genocchi, Lah, Stirling numbers or polynomials in [6, 7, 8, 9, 10, 11, 14, 27, 37, 41].*

Remark 8 *In recent years, the first author and other mathematicians together considered the complete monotonicity and the Bernstein function properties in [5, 12, 24, 25, 28, 29, 31].*

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Common fixed point theorems for contractive mappings satisfying Φ-maps in S-metric spaces

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Abstract. In this paper we prove the existence of the unique fixed point for the pair of weakly compatible self-mappings satisfying some Φ-type contractive conditions in the framework of S-metric spaces. Our results generalize, extend, unify, complement and enrich recently fixed point results in existing literature.

1 Introduction and preliminaries

In 1922. Banach [2] proposed a theorem, which is well-known as Banach's Fixed Point Theorem (or Banach's Contraction Principle, BCP for short) to establish the existence of solutions for nonlinear operator equations and integral equations. Since then, because of simplicity and usefulness, it has become

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a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Later, a huge amount of literature is witnessed on applications, generalizations and extensions of this theorem. They are carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups. Considering different mappings etc. Many mathematic problems require one to find a distance between two or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric structure. Due to the need to construct a suitable framework to model several distinguished problems of practical nature, the study of metric spaces has attracted and continues to attract the interest of many authors. Over last few decades, a numbers of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, partial metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [4, 5, 6, 8, 9, 13, 20]. Sedghi et al [18] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

In this paper, we prove a coupled coincidence fixed point theorem in the setting of a generalized metric space. First, we present some basic properties of S-metric spaces.

Following is the definition of generalized metric spaces or S-metric spaces.

Definition 1 [19] *Let X be a nonempty set. An S-metric on X is a function* $S : X\times X\times X \to [0,\infty)$ *that satisfies the following conditions, for each* $x, y, z, a \in X$,

(S1) $S(x, y, z) \geq 0$,

(S2) $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ *for all* $x, y, z, a \in X$.

The pair (X, S) *is called an* S*-metric space.*

Some examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X.

(2) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

(3) Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, y) + d(y, z)$ is an S-metric on X.

Lemma 1 [19], [7] *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z) \text{ and } S(x, x, z) \leq 2S(x, x, y) + S(z, z, y) \text{ for all } x, y, z \in X.$$

Also, $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Definition 2 [19] *Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x, r)$ and closed ball $B_S\,[x, r]$ with center x and radius r as follows respectively:*

$$\begin{aligned} B_s(x, r) &= \{y \in X : S(y, y, x) < r\}, \\ B_s[x, r] &= \{y \in X : S(y, y, x) \leq r\}. \end{aligned}$$

Example 1 [19] *Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Thus $B_s(1, 2) = \{y \in \mathbb{R} : S(y, y, 1) < 2\} = (0, 2)$.*

Definition 3 [19] *Let (X, S) be an S-metric space, and $A \subseteq X$.*

(1) If for every $x \in A$ there exists $r > 0$ such that $B_S(x, r) \subseteq A$, then the subset A is called open subset of X.

(2) Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x) < \varepsilon$ whenever $n \geq n_0$ and we denote this $\lim\limits_{n \longrightarrow \infty} x_n = x$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.

(5) The S-metric spaces (X, S) is said to be complete if every Cauchy sequence is convergent.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X(induced by the S-metric S).

Definition 4 [1] *Let f and g be single-valued self mappings on a set X. If $\omega = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and ω is called a point of coincidence of f and g.*

Definition 5 [10] *Let* f *and* g *be a single-valued self mappings on a set* X*. Mappings* f *and* g *are said to be weakly compatible if* $fx = gx$ *implies* $fgx = gfx$*,* $x \in X$*.*

Proposition 1 [1] *Let* f *and* g *be weakly compatible self mappings on a set* X*. If* f *and* g *have a unique point of coincidence* $\omega = fx = gx$*, then* ω *is the unique common fixed point of* f *and* g*.*

2 Common fixed point theorems

In 1977, Matkowski [12] introduced the Φ-maps as the following : let Φ be the set of all functions ϕ such that $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a nondecreasing function satisfying $\lim_{n\longrightarrow\infty} \phi^n(t) = 0$ for all $t \in (0,\infty)$. If $\phi \in \Phi$, then ϕ is called a $\Phi-$map. Furthermore, if ϕ is a Φ-map, then

(i) $\phi(t) < t$ for all $t \in (0,\infty)$,

(ii) $\phi(0) = 0$.

From now on, unless otherwise stated, ϕ is meant the Φ-map.

Lemma 2 [15], [16] *Let* (X,S) *be a* $S-$*metric space and let* $\{x_n\}$ *be a sequence in it such that*

$$\lim_{n\to\infty} S\left(x_{n+1}, x_{n+1}, x_n\right) = 0.$$

If $\{x_n\}$ *is not a Cauchy sequence, then there exist an* $\varepsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$*,* $n_k > m_k > k$ *of positive integers such that the following sequences tend to* ε *when* $k \to \infty$ *:*

$$S\left(x_{m_k}, x_{m_k}, x_{n_k}\right), S\left(x_{m_k}, x_{m_k}, x_{n_k+1}\right), S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k}\right),$$
$$S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}\right), S\left(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}\right), \ldots.$$

Proof. Suppose that the sequence $\{x_n\}$ is not a Cauchy. Then, there exists $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$, $\{x_{n_k}\}$, such that for every $k \in \mathbb{N}$ and $n_k > m_k > k$ the following is satisfied:

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } S(x_{m_k}, x_{m_k}, x_{n_k-1}) < \varepsilon.$$

Then, using Lemma 1 and (S3) we have

$$\begin{aligned}\varepsilon &\leq S(x_{m_k}, x_{m_k}, x_{n_k})\\ &= S(x_{n_k}, x_{n_k}, x_{m_k})\\ &\leq 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1})\\ &< 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + \varepsilon,\end{aligned}$$

and

$$\varepsilon \leq \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) \leq \varepsilon.$$

Therefore $\lim\limits_{k\to\infty} S(x_{n_k}, x_{n_k}, x_{m_k}) = \lim\limits_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon$. Further, as

$$|S(x_{n_k}, x_{n_k}, x_{m_k}) - S(x_{n_{k+1}}, x_{n_{k+1}}, x_{m_k})| \leq 2S(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k})$$

we obtain that

$$\lim_{k\to\infty} S(x_{n_{k+1}}, x_{n_{k+1}}, x_{m_k}) = \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_{k+1}}) = \varepsilon.$$

Analogous, it can be proved that

$$S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k}\right), S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}\right), S\left(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}\right), \ldots.$$

tend to ε. □

Theorem 1 *Let* (X, S) *be a* S*-metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq \phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}), \qquad (1)$$

for all $x, y, z \in X$. *If the range of* g *contains the range of* f, *and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X, *then* f *and* g *have a unique point of coincidence in* X. *Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Assume that f and g satisfy the condition (1). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that

$gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
S(gx_n, gx_n, gx_{n+1}) &= S(fx_{n-1}, fx_{n-1}, fx_n) \\
&\leq \phi(\max\{S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), \\
&\quad S(gx_n, gx_n, fx_n)\}) \\
&\leq \phi(\max\{S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_n, gx_n, fx_n)\}) \\
&= \phi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_n, gx_n, gx_{n+1})\}).
\end{aligned}$$

If $\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_n, gx_n, gx_{n+1})\} = S(gx_n, gx_n, gx_{n+1})$, then

$$S(gx_n, gx_n, gx_{n+1}) \leq \phi(S(gx_n, gx_n, gx_{n+1})) < S(gx_n, gx_n, gx_{n+1}),$$

which leads to a contradiction. This implies that

$$S(gx_n, gx_n, gx_{n+1}) \leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)).$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
S(gx_n, gx_n, gx_{n+1}) &= S(fx_{n-1}, fx_{n-1}, fx_n) \\
&\leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) \\
&\leq \phi^2(S(gx_{n-2}, gx_{n-2}, gx_{n-1})) \\
&\ \ \vdots \\
&\leq \phi^n(S(gx_0, gx_0, gx_1)).
\end{aligned}$$

So we have $\lim_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in $S-$metric space (X, S), then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S\left(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}\right) \text{ and } S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right), \tag{2}$$

Putting now in (1) $x = y = x_{m_k}, z = x_{n_k}$ we obtain

$$\begin{aligned}
&S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) \\
= \ &S(fx_{m_k}, fx_{m_k}, fx_{n_k}) \\
\leq \ &\phi(\{\max\{S(gx_{m_k}, gx_{m_k}, fx_{m_k}), S(gx_{m_k}, gx_{m_k}, fx_{m_k}), S(gx_{n_k}, gx_{n_k}, fx_{n_k})\}\}) \\
= \ &\phi(\{\max\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\}\}).
\end{aligned}$$

If $\max\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\} = S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})$, and since $S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) > 0$ we have

$$\begin{aligned} S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) &\leq \phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})) \\ &< S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}). \end{aligned}$$

Letting $k \to \infty$ we obtain

$$\varepsilon \leq \lim_{k\to\infty} \phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})) \leq 0.$$

A contradiction.

Analogous, if $\max\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\} = S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})$ we got a contradiction.

So, it follows that $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (1), we have

$$\begin{aligned} S(gx_n, gx_n, fp) &= S(fx_{n-1}, fx_{n-1}, fp) \\ &\leq \phi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, gx_n), \\ &\quad S(gp, gp, fp)\}) \\ &= \phi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\}). \end{aligned}$$

Case 1.

$$\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\} = S(gx_{n-1}, gx_{n-1}, gx_n),$$

we obtain that

$$S(gx_n, gx_n, fp) \leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) < S(gx_{n-1}, gx_{n-1}, gx_n).$$

By taking $n \to \infty$, we have $S(gp, gp, fp) = 0$ and so $gp = fp$.

Case 2.

$$\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\} = S(gp, gp, fp),$$

we obtain that

$$S(gx_n, gx_n, fp) \leq \phi(S(gp, gp, fp)).$$

By taking $n \to \infty$, we have $S(gp, gp, fp) \leq \phi(S(gp, gp, fp)) < S(gp, gp, fp)$, which leads to a contradiction. Therefore $gp = fp$. We now show that f and g have a unique point of coincidence. Suppose that $fl = gl$ for some $l \in X$. By applying (1), it follows that

$$\begin{aligned} S(gp, gp, gl) &= S(fp, fp, fl) \\ &\leq \phi(max\{S(gp, gp, fp), S(gp, gp, fp), S(gl, gl, fl)\}) \\ &= 0. \end{aligned}$$

Therefore $gp = gl$. This implies that f and g have a unique point of coincidence. By Proposition 1, we can conclude that f and g have a unique common fixed point. □

Corollary 1 *Let* (X, S) *be a* S*-metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq k \max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\},$$

for all $x, y, z \in X$ *where* $0 \leq k < 1$*. If the range of* g *contains the range of* f *and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Putting $\phi(t) = kt, t \geq 0, 0 \leq k < 1$ in (1), the result follows.

□

Example 2 *Let* $X = [0, 2]$ *and* $S(x, y, z) = max\{|x - y|, |y - z|, |x - z|\}$ *and* $\phi \in \Phi$*. Define* $f, g : X \to X$ *by*

$$fx = 1 \qquad \text{and} \qquad gx = 2 - x.$$

We obtain that f *and* g *satisfy (1) in Theorem 1. Indeed, we have*

$$S(fx, fy, fz) = 0,$$
$$\phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}) = \phi(\max\{|1 - x|, |1 - y|, |1 - z|\}).$$

It is obvious that the range of g *contains the range of* f *and* $g(X)$ *is a complete subspace of* (X, S)*. Furthermore,* f *and* g *are weakly compatible. Thus all assumptions in Theorem 1 are satisfied. This implies that* f *and* g *have a unique common fixed point fixed point which is* $x = 1$*.*

Theorem 2 *Let* (X, S) *be a* S*-metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq \max\{\phi(S(gx, gx, fx)), \phi(S(gy, gy, fy)), \phi(S(gz, gz, fz))\},$$

for all $x, y, z \in X$*. If the range of* g *contains the range of* f*, and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. The proof is very similar to the proof of Theorem 1 so we omitted it. □

Theorem 3 *Let* (X, S) *be a* S*-metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq \phi(S(gx, gy, gz)), \tag{3}$$

for all $x, y, z \in X$*, where* ϕ *satisfies* $\lim_{s \to t+} \phi(s) < t$ *for all* $t > 0$*. If the range of* g *contains the range of* f*, and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
S(gx_n, gx_n, gx_{n+1}) &= S(fx_{n-1}, fx_{n-1}, fx_n) \\
&\leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) \\
&\leq \phi^2(S(gx_{n-2}, gx_{n-2}, gx_{n-1})) \\
&\vdots \\
&\leq \phi^n(S(gx_0, gx_0, gx_1)).
\end{aligned}$$

This implies that $\lim_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in $S-$metric space (X, S), then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S\left(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}\right) \text{ and } S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right), \tag{4}$$

Putting now in (3) $x = y = x_{m_k}, z = x_{n_k}$, and since $S(gx_{m_k}, gx_{m_k}, gx_{n_k}) > 0$ we obtain

$$\begin{aligned} S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) &= S(fx_{m_k}, fx_{m_k}, fx_{n_k}) \\ &\leq \phi(S(gx_{m_k}, gx_{m_k}, gx_{n_k})). \end{aligned}$$

Letting $k \to \infty$ and using the assumption of the mapping ϕ we obtain

$$\begin{aligned} \varepsilon \leq \lim_{k\to\infty} \phi\left(S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right)\right) &= \lim_{S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right)\to\varepsilon^+} \phi\left(S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right)\right) \\ &= \lim_{t\to\varepsilon^+} \phi\left(t\right) < \varepsilon. \end{aligned}$$

A contradiction. Therefore, the sequences $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. By (3) we have

$$\begin{aligned} S(gp, gp, fp) &\leq 2S(gp, gp, gx_{n+1}) + S(gx_{n+1}, gx_{n+1}, fp) \\ &\leq 2S(gp, gp, gx_{n+1}) + \phi(S(gx_n, gx_n, gp)) \\ &\leq 2S(gp, gp, gx_{n+1}) + S(gx_n, gx_n, gp). \end{aligned}$$

By taking $n \to \infty$, we have $S(gp, gp, fp) = 0$ and so $gp = fp$. We now show that f and g have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. Assume that $gp \neq gq$. By applying (3), it follows that

$$\begin{aligned} S(gp, gp, gq) &= S(fp, fp, fq) \\ &\leq \phi(S(gp, gp, gq)) \\ &< S(gp, gp, gq), \end{aligned}$$

which leads to a contradiction. Therefore $gp = gq$. This implies that f and g have a unique point of coincidence. By Proposition 1, we can conclude that f and g have a unique common fixed point. □

By setting g to be the identity function on X, we immediately have the following corollary. This result extends and generalizes Boyd-Wong theorem from the metric spaces to the S-metric spaces. We do not need upper semi-continuity of the comparison function, we only use $\phi \in \Phi$ with $\lim_{s\to t^+} \phi\left(s\right) < t$, $t > 0$.

Corollary 2 *Let* (X, S) *be a complete* S*-metric space. Suppose that the mapping* $f : X \to X$ *satisfies*

$$S(fx, fy, fz) \leq \phi(S(x, y, z)),$$

for all $x, y, z \in X$*. Then* f *has a unique fixed point.*

Theorem 4 *Let* (X, S) *be a* S*-metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq k_1\phi(S(gx, gx, fx)) + k_2\phi(S(gy, gy, fy)) + k_3\phi(S(gz, gz, fz)) \quad (5)$$

for all $x, y, z \in X$, $k_1 + k_2 + k_3 < 1$*. If the range of* g *contains the range of* f*, and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Assume that f and g satisfy the condition (5). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&S(gx_n, gx_n, gx_{n+1}) = S(fx_{n-1}, fx_{n-1}, fx_n)\\
&\leq k_1\phi(S(gx_{n-1}, gx_{n-1}, fx_{n-1})) + k_2\phi(S(gx_{n-1}, gx_{n-1}, fx_{n-1}))\\
&\quad + k_3\phi(S(gx_n, gx_n, fx_n))\\
&= k_1\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_2\phi(S(gx_{n-1}, gx_{n-1}, gx_n))\\
&\quad + k_3\phi(S(gx_n, gx_n, gx_{n+1}))\\
&< (k_1 + k_2)\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_3 S(gx_n, gx_n, gx_{n+1}).
\end{aligned}$$

Now we have,

$$S(gx_n, gx_n, gx_{n+1}) < \frac{k_1 + k_2}{1 - k_3}\phi(S(gx_{n-1}, gx_{n-1}, gx_n)).$$

Let $r = \frac{k_1+k_2}{1-k_3} < 1$. Then

$$\begin{aligned}
S(gx_n, gx_n, gx_{n+1}) &< r\phi(S(gx_{n-1}, gx_{n-1}, gx_n))\\
&< \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) < \cdots < \phi^n S(gx_0, gx_0, gx_1)
\end{aligned}$$

This implies that $\lim\limits_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in $S-$metric space (X, S), then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S\left(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}\right) \text{ and } S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right). \quad (6)$$

Putting now in (3) $x = y = x_{m_k}, z = x_{n_k}$, and using the fact that $S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) > 0$ and $S(gx_{n_k}, gx_{n_k}, gx_{n_k+1}) > 0$ we obtain

$$\begin{aligned}
&S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1})\\
&= S(fx_{m_k}, fx_{m_k}, fx_{n_k})\\
&\leq k_1\phi(S(gx_{m_k}, gx_{m_k}, fx_{m_k})) + k_2\phi(S(gx_{m_k}, gx_{m_k}, fx_{m_k}))\\
&\quad + k_3\phi(S(gx_{n_k}, gx_{n_k}, fx_{n_k}))\\
&= k_1\phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})) + k_2\phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}))\\
&\quad + k_3\phi(S(gx_{n_k}, gx_{n_k}, gx_{n_k+1}))\\
&< k_1 S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) + k_2 S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})\\
&\quad + k_3 S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})
\end{aligned}$$

Letting $k \to \infty$ we obtain $\varepsilon \leq 0$.

A contradiction. So, the sequences $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (5), we have

$$\begin{aligned}
&S(gx_n, gx_n, fp) = S(fx_{n-1}, fx_{n-1}, fp)\\
&\leq k_1\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_2\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_3\phi(S(gp, gp, fp)).
\end{aligned}$$

Letting $n \to \infty$ we have

$$S(gp, gp, fp) \leq k_3\phi(S(gp, gp, fp)) < k_3 S(gp, gp, fp) < S(gp, gp, fp)$$

we got a contradiction. So, $gp = fp$. The proof that f and g have a unique point of coincidence is as in Theorem 1 so we omitted it. □

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Some Hermite-Hadamard type integral inequalities for operator AG-preinvex functions

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Abstract. In this paper, we introduce the concept of operator AG-preinvex functions and prove some Hermite-Hadamard type inequalities for these functions. As application, we obtain some unitarily invariant norm inequalities for operators.

1 Introduction and preliminaries

The following Hermite-Hadamard inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}. \tag{1}$$

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It was firstly discovered by Hermite in 1881 in the journal Mathesis (see [8]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [10].

Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893 [2]. In 1974, Mitrinovič found Hermites note in Mathesis [8]. Since (1) was known as Hadamards inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [10].

Definition 1 [13] *A continuous function* $f : I \subset \mathbb{R} \to \mathbb{R}^+$ *is said to be an AG-convex function (arithmetic-geometrically or log-convex function) on the interval* I *if*

$$f(\lambda a + (1-\lambda)b) \leq f(a)^{\lambda} f(b)^{1-\lambda}. \tag{2}$$

for $a, b \in I$ *and* $\lambda \in [0, 1]$, *i.e.,* $\log f$ *is convex.*

Theorem 1 [13] *Let* f *be an AG-convex function defined on* $[a, b]$. *Then, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)} \\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \log(f(u))du\right) \\ &\leq \sqrt{f\left(\frac{a+b}{2}\right)}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \\ &\leq \sqrt{f(a)f(b)}, \end{aligned} \tag{3}$$

where $u = \log t$.

Let $B(H)$ stands for the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in B(H)$ is positive and write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Let $B(H)_{sa}$ stand for the set of all self-adjoint elements of $B(H)$.

Let A be a self-adjoint operator in $B(H)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted by $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows:

for any $f, g \in C(Sp(A)))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t\in \mathrm{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define

$$f(A) = \Phi(f) \text{ for all } \ f \in C(\mathrm{Sp}(A))$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \ \text{ for any } \ t \in \mathrm{Sp}(A) \ \text{ implies that } \ f(A) \geq g(A), \tag{4}$$

in the operator order of $B(H)$, see [14].

Definition 2 *A real valued continuous function* f *on an interval* I *is said to be operator convex function if*

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B),$$

in the operator order, for all $\lambda \in [0, 1]$ *and self-adjoint operators* A *and* B *in* $B(H)$ *whose spectra are contained in* I*.*

In [4] Dragomir investigated the operator version of the Hermite-Hadamard inequality for operator convex functions. Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I then, for any self-adjoint operators A and B with spectra in I, the following inequalities holds

$$\begin{aligned}
f\left(\frac{A+B}{2}\right) &\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} f(tA + (1-t)B)dt \\
&\leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\
&\leq \int_0^1 f\left((1-t)A + tB\right) dt \\
&\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \\
&\leq \frac{f(A)+f(B)}{2},
\end{aligned}$$

for the first inequality in above, see [12].

In [5], Ghazanfari et al. gave the concept of operator preinvex function and obtained Hermite-Hadamard type inequality for operator preinvex function.

Definition 3 [5] *Let* X *be a real vector space, a set* $S \subseteq X$ *is said to be invex with respect to the map* $\eta : S \times S \to X$*, if for every* $x, y \in S$ *and* $t \in [0, 1]$*,*

$$x + t\eta(x, y) \in S.$$

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex (see [1]).

Let $S \subseteq X$ be an invex set with respect to η. For every $x, y \in S$. the η-path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y, x), t \in [0, 1]\}.$$

The mapping η is said to satisfy the condition (C) if for every $x, y \in S$ and $t \in [0, 1]$,

$$\eta(y, y + t\eta(y, x)) = -t\eta(x, y), \quad \eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y).$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0, 1]$, from conditions in (C), we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y), \tag{5}$$

see [9] for details.

Definition 4 *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$*. Then, the continuous* $f : \mathbb{R} \to \mathbb{R}$ *is said to be operator preinvex with respect to* η *on* S*, if for every* $A, B \in S$ *and* $t \in [0, 1]$*,*

$$f(A + t\eta(B, A)) \leq (1 - t)f(A) + tf(B), \tag{6}$$

in the operator order in $B(H)$*.*

Every operator convex function is operator preinvex with respect to the map $\eta(A, B) = A - B$, but the converse does not hold (see [5]).

Theorem 2 [5] *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$ *and* η *satisfies condition* (C)*. If for every* $A, B \in S$ *and* $V = A+\eta(B, A)$ *the function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is operator preinvex with respect to* η *on* η*-path* P_{AV} *with spectra of* A *and spectra of* V *in the interval* I*. Then we have the inequality*

$$f\left(\frac{A + V}{2}\right) \leq \int_0^1 f((A + t\eta(B, A))dt \leq \frac{f(A) + f(B))}{2}.$$

Throughout this paper, we introduce the concept of operator AG-preinvex functions and obtain some Hermite-Hadamard type inequalities for these class of functions. These results lead us to obtain some inequalities unitarily invariant norm inequalities for operators.

2 Some inequalities for operator AG-preinvex functions

In this section, we prove some Hermite-Hadamard type inequalities for operator AG-preinvex functions.

Definition 5 [13] *A continuous function* $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ *is said to be operator AG-convex (concave) if*

$$f(\lambda A + (1-\lambda)B) \leq \ (\geq)\ f(A)^{\lambda} f(B)^{1-\lambda}$$

for $0 \leq \lambda \leq 1$ *and self-adjoint operators* A *and* B *in* $B(H)$ *whose spectra are contained in* I.

Example 1 [6, Corollary 7.6.8] *Let* A *and* B *be to positive definite* $n \times n$ *complex matrices. For* $0 < \alpha < 1$*, we have*

$$|\alpha A + (1-\alpha)B| \geq |A|^{\alpha}|B|^{1-\alpha} \tag{7}$$

where $|\cdot|$ *denotes determinant of a matrix.*

Let f be an operator AG-convex function, for commutative positive operators $A, B \in B(H)$ whose spectra are contained in I, then we have

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \int_0^1 \sqrt{f(\alpha A + (1-\alpha)B) f((1-\alpha)A + \alpha B)} d\alpha \\ &\leq \sqrt{f(A)f(B)}, \end{aligned} \tag{8}$$

(see [13] for more inequalities).

Definition 6 *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$*. A continuous function* $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ *is called operator AG-preinvex with respect to* η *on* S *if*

$$f(A + t\eta(B, A)) \leq f(A)^{1-t} f(B)^t$$

for $t \in [0, 1]$ *such that spectra of* A *and* B *are contained in* I.

Remark 1 *Let* f *be an operator AG-preinvex function, in a commutative case, we then get*

$$\begin{aligned} f(A+t\eta(B,A)) &\leq f(A)^{1-t}f(B)^{t} \\ &\leq (1-t)f(A)+tf(B) \\ &\leq \max\{f(A),f(B)\} \end{aligned}$$

It means that f *is operator quasi preinvex i.e.,* $f(A+t\eta(B,A)) \leq \max\{f(A),f(B)\}$.

We need the following lemma for giving Hermite-Hadamard type inequalities for operator preinvex function.

Lemma 1 *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$ *and* $f : I \subseteq \mathbb{R} \to \mathbb{R}^{+}$ *be a continuous function on the interval* I. *Suppose that* η *satisfies condition (C). Then for every* $A, B \in S$ *and* $V = A + \eta(B,A)$ *and for some fixed* $s \in [0,1]$ *the function* f *is operator AG-preinvex with respect to* η *on* η*-path* P_{AV} *with spectra of* A *and* V *in the interval* I *if and only if the function* $\varphi_{A,B}$ *defined by*

$$\varphi_{A,B}(t) = f(A+t\eta(B,A)) \tag{9}$$

is a log-convex function on $[0,1]$.

Proof. Let φ be a log-convex function on $[0,1]$, we should prove that f is operator AG-preinvex with respect to η.
For every $C_1 := A + t_1\eta(B,A) \in P_{AV}$, $C_2 := A + t_2\eta(B,A) \in P_{AV}$, fixed $\lambda \in [0,1]$, by (9) we have

$$\begin{aligned} f(C_1+\lambda\eta(C_2,C_1)) &= f(A+t_1\eta(B,A)+\lambda\eta(A+t_2\eta(B,A),A+t_1\eta(B,A))) \\ &= f(A+t_1\eta(B,A)+\lambda(t_2-t_1)\eta(B,A)) \\ &= f(A+(t_1+\lambda t_2-\lambda t_1)\eta(B,A)) \\ &= f(A+((1-\lambda)t_1+\lambda t_2)\eta(B,A)) \\ &= \varphi((1-\lambda)t_1+\lambda t_2) \\ &\leq \varphi(t_1)^{1-\lambda}\varphi(t_2)^{\lambda} \\ &= \left(f(A+t_1\eta(B,A))\right)^{1-\lambda}\left(f(A+t_2\eta(B,A))\right)^{\lambda}. \end{aligned}$$

Conversely, let f be operator AG-preinvex, then, by (6)

$$\begin{aligned}
\varphi((1-\lambda)t_1+\lambda t_2) &= f(A+((1-\lambda)t_1+\lambda t_2)\eta(B,A)) \\
&= f(A+t_1\eta(B,A)+\lambda(t_2-t_1)\eta(B,A)) \\
&= f(A+t_1\eta(B,A)+\lambda\eta(A+t_2\eta(B,A),A+t_1\eta(B,A))) \\
&\le f(A+t_1\eta(B,A))^{1-\lambda}f(A+t_2\eta(B,A))^{\lambda} \\
&= \varphi(t_1)^{1-\lambda}\varphi(t_2)^{\lambda}.
\end{aligned}$$

□

Theorem 3 *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S\times S \to B(H)_{sa}$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a continuous function on the interval I. Suppose that η satisfies condition (C). Then for the operator AG-preinvex function f with respect to η on η-path P_{AV} such that spectra of A and V are in I, we have*

$$\begin{aligned}
f\left(\frac{A+V}{2}\right) &\le \sqrt{f\left(\frac{3A+V}{4}\right) f\left(\frac{A+3V}{4}\right)} \\
&\le \exp\left(\int_0^1 \log(f(A+t\eta(B,A)))dt\right) \\
&\le \sqrt{f\left(\frac{A+V}{2}\right)}\sqrt[4]{f(A)}\sqrt[4]{f(V)} \\
&\le \sqrt{f(A)f(V)} \\
&\le \frac{f(A)+f(V)}{2}
\end{aligned}$$

where $A, B \in S$ and $V = A+\eta(B,A)$ and for some fixed $s \in (0,1]$

Proof. Since f is an operator AG-preinvex function, so by Lemma 1 we have $\varphi(t) = f(A+t\eta(B,A))$ is log-convex on $[0,1]$.

On the other hand, in [11] we obtained the following inequalities for log-convex function φ on $[0,1]$:

$$\begin{aligned}
\varphi\left(\frac{1}{2}\right) &\le \sqrt{\varphi\left(\frac{1}{4}\right)\varphi\left(\frac{3}{4}\right)} \\
&\le \exp\left(\int_0^1 \log(\varphi(u))du\right)
\end{aligned}$$

$$\leq \sqrt{\varphi\left(\frac{1}{2}\right).\sqrt[4]{\varphi(0)}.\sqrt[4]{\varphi(1)}}$$
$$\leq \sqrt{\varphi(0)\varphi(1)}. \tag{10}$$

By knowing that

$$\begin{aligned}
\varphi(0) &= f(A) \\
\varphi\left(\frac{1}{4}\right) &= f\left(A+\frac{1}{4}\eta(B,A)\right) = f\left(\frac{3A+V}{4}\right) \\
\varphi\left(\frac{1}{2}\right) &= f\left(A+\frac{1}{2}\eta(B,A)\right) = f\left(\frac{A+V}{2}\right) \\
\varphi(1) &= f(V),
\end{aligned}$$

we obtain

$$\begin{aligned}
f\left(\frac{A+V}{2}\right) &\leq \sqrt{f\left(\frac{3A+V}{4}\right) f\left(\frac{A+3V}{4}\right)} \\
&\leq \exp\left(\int_0^1 \log(f(A+t\eta(B,A)))dt\right) \\
&\leq \sqrt{f\left(\frac{A+V}{2}\right)\sqrt[4]{f(A)}\,\sqrt[4]{f(V)}} \\
&\leq \sqrt{f(A)f(V)}.
\end{aligned}$$

□

3 Some unitarily invariant norm inequalities for operator AG-preinvex functions

In this section we prove some unitarily invariant norm inequalities for operators.

We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$ and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $B(H)$.

For $1 \leq p < \infty$, the Schatten p-norm of a compact operator A is defined by $\|A\|_p = (\operatorname{Tr}|A|^p)^{1/p}$, where Tr is the usual trace functional. Note that for compact operator A we have, $\|A\| = s_1(A)$, and if A is a Hilbert-Schmidt operator, then $\|A\|_2 = (\sum_{j=1}^{\infty} s_j^2(A))^{1/2}$. These norms are special examples of the more general class of the Schatten p-norms which are unitarily invariant [3].

Remark 2 *The author of [7] proved that if $A, B, X \in B(H)$ such that A, B are positive operators, then for $0 \leq \nu \leq 1$ we have*

$$|||A^{\nu} X B^{1-\nu}||| \leq |||AX|||^{\nu} |||XB|||^{1-\nu}. \tag{11}$$

Let $X = I$ in above inequality, we then get

$$|||A^{\nu} B^{1-\nu}||| \leq |||A|||^{\nu} |||B|||^{1-\nu}. \tag{12}$$

Lemma 2 *Let f be an operator AG-preinvex function and η satisfies the condition (C). Then the function $\varphi_{A,B} : [0,1] \to \mathbb{R}$ defined as follows*

$$\varphi(t) = |||f(A + t\eta(B, A))|||$$

is log-convex.

Proof. Let $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
\varphi((1-\lambda)t_1 + \lambda t_2) &= |||f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B, A))||| \\
&= |||f\left(A + t_1\eta(B, A) + \lambda(t_2 - t_1)\eta(B, A)\right)||| \\
&= |||f\left(A + t_1\eta(B, A) + \lambda\eta(A + t_2\eta(B, A), A + t_1\eta(B, A))\right)||| \\
&\leq |||f(A + t_1\eta(B, A))^{1-\lambda} f(A + t_2\eta(B, A))^{\lambda}||| \\
&\leq |||f(A + t_1\eta(B, A))|||^{1-\lambda} |||f(A + t_2\eta(B, A))|||^{\lambda} \quad \text{by (12)} \\
&= \varphi(t_1)^{1-\lambda}\varphi(t_2)^{\lambda}.
\end{aligned}$$

□

Theorem 4 *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a continuous function on the interval I. Suppose that η satisfies condition (C). Then for the operator AG-preinvex function f with* *respect to* η *on* η*-path* P_{AV} *such that spectra of* A *and* V *are in* I*, we have*

$$\begin{aligned}
\left|\left|\left|f\left(\frac{A+V}{2}\right)\right|\right|\right| &\leq \sqrt{\left|\left|\left|f\left(\frac{3A+V}{4}\right)\right|\right|\right| \left|\left|\left|f\left(\frac{A+3V}{4}\right)\right|\right|\right|} \\
&\leq \exp\left(\int_0^1 \log(|||f(A+t\eta(B,A))|||)dt\right) \\
&\leq \sqrt{\left|\left|\left|f\left(\frac{A+V}{2}\right)\right|\right|\right|}\sqrt[4]{|||f(A)|||}\sqrt[4]{|||f(V)|||} \\
&\leq \sqrt{|||f(A)|||\,|||f(V)|||} \\
&\leq \frac{|||f(A)|||+|||f(V)|||}{2}.
\end{aligned}$$

where $A, B \in S$ *and* $V = A + \eta(B, A)$ *and for some fixed* $s \in (0, 1]$

Proof. Since f is an operator AG-preinvex function, so by Lemma 2 we have $\varphi(t) = |||f(A + t\eta(B, A))|||$ is log-convex on $[0, 1]$.

On the other hand, in [11] we obtained the following inequalities for log-convex function φ on $[0, 1]$:

$$\begin{aligned}
\varphi\left(\frac{1}{2}\right) &\leq \sqrt{\varphi\left(\frac{1}{4}\right)\varphi\left(\frac{3}{4}\right)} \\
&\leq \exp\left(\int_0^1 \log(\varphi(u))du\right) \\
&\leq \sqrt{\varphi\left(\frac{1}{2}\right)}.\sqrt[4]{\varphi(0)}.\sqrt[4]{\varphi(1)} \\
&\leq \sqrt{\varphi(0)\varphi(1)}. \qquad (13)
\end{aligned}$$

By knowing that

$$\begin{aligned}
\varphi(0) &= |||f(A)||| \\
\varphi\left(\frac{1}{4}\right) &= \left|\left|\left|f\left(A+\frac{1}{4}\eta(B,A)\right)\right|\right|\right| = \left|\left|\left|f\left(\frac{3A+V}{4}\right)\right|\right|\right| \\
\varphi\left(\frac{1}{2}\right) &= \left|\left|\left|f\left(A+\frac{1}{2}\eta(B,A)\right)\right|\right|\right| = \left|\left|\left|f\left(\frac{A+V}{2}\right)\right|\right|\right| \\
\varphi(1) &= |||f(V)|||,
\end{aligned}$$

we obtain

$$
\begin{aligned}
\left|\left|\left| f\left(\frac{A+V}{2}\right) \right|\right|\right| &\leq \sqrt{\left|\left|\left| f\left(\frac{3A+V}{4}\right) \right|\right|\right| \left|\left|\left| f\left(\frac{A+3V}{4}\right) \right|\right|\right|} \\
&\leq \exp\left(\int_0^1 \log(|||f(A+t\eta(B,A))|||)dt\right) \\
&\leq \sqrt{\left|\left|\left| f\left(\frac{A+V}{2}\right) \right|\right|\right|} \sqrt[4]{|||f(A)|||} \sqrt[4]{|||f(V)|||} \\
&\leq \sqrt{|||f(A)|||\,|||f(V)|||}.
\end{aligned}
$$

□

Let $\eta(B,A) = B - A$ in the above theorem, then we obtain the following inequalities:

$$
\begin{aligned}
\left|\left|\left| f\left(\frac{A+B}{2}\right) \right|\right|\right| &\leq \sqrt{\left|\left|\left| f\left(\frac{3A+B}{4}\right) \right|\right|\right| \left|\left|\left| f\left(\frac{A+3B}{4}\right) \right|\right|\right|} \\
&\leq \exp\left(\int_0^1 \log(|||f((1-t)A+tB)|||\, dt\right) \\
&\leq \sqrt{\left|\left|\left| f\left(\frac{A+B}{2}\right) \right|\right|\right|} \sqrt[4]{|||f(A)|||} \sqrt[4]{|||f(B)|||} \\
&\leq \sqrt{|||f(A)|||\,|||f(B)|||} \\
&\leq \frac{|||f(A)||| + |||f(B)|||}{2}. \qquad (14)
\end{aligned}
$$

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