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On submersion and immersion submanifolds of a quaternionic projective space

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Abstract. We study submanifolds of a quaternionic projective space, it is of great interest how to pull down some formulae deduced for submanifolds of a sphere to those for submanifolds of a quaternionic projective space.

1 Introduction

It is well known that an odd-dimensional sphere is a circle bundle over the quaternionic projective space. Consequently, many geometric properties of the quaternionic projective space are inherited from those of the sphere.

Let M be a connected real n-dimensional submanifold of real codimension p of a quaternionic Kähler manifold $\overline{M}^{n+p}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r-dimensional normal distribution ν of the normal bundle $TM^{\perp}$ such that

$$\begin{aligned} F\nu_x \subset \nu_x, \quad & G\nu_x \subset \nu_x, \quad & H\nu_x \subset \nu_x, \\ F\nu_x^{\perp} \subset T_xM, \quad & G\nu_x^{\perp} \subset T_xM, \quad & H\nu_x^{\perp} \subset T_xM, \end{aligned}$$

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at each point x in M, then M is called a QR-submanifold of r QR-dimension, where $\nu^{\perp}$ denotes the complementary orthogonal distribution to ν in TM [2, 14, 16].

Equivalently, there exists distributions $(D_x, D_x^{\perp})$ of the tangent bundle TM, such that

$$\begin{array}{lll} FD_x \subset D_x, & GD_x \subset D_x, & HD_x \subset D_x, \\ FD_x^{\perp} \subset T_xM^{\perp}, & GD_x^{\perp} \subset T_xM^{\perp}, & HD_x^{\perp} \subset T_xM^{\perp}, \end{array}$$

where $D_x^{\perp}$ denotes the complementary orthogonal distribution to D_x in TM. Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated by many authors [3, 9, 14, 16, 18, 20] in connection with the shape operator and the induced almost contact 3-structure. Recently, Kwon and Pak have studied QR-submanifolds of $(p-1)$ QR-dimension isometrically immersed in a quaternionic projective space $QP^{\frac{n+p}{4}}$ [14, 16].
Pak and Sohn studied n-dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{\frac{(n+p)}{4}}$ [19].
Kim and Pak studied n-dimensional QR-submanifold of maximal QR-dimension isometrically immersed in a quaternionic projective space [13].

2 Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type $(1,1)$ over $\overline{M}$ satisfying the following conditions (a), (b) and (c):
(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{array}{lll} F^2 = -I, & G^2 = -I, & H^2 = -I, \\ FG = -GF = H, & GH = -HG = F, & HF = -FH = G. \end{array} \tag{1}$$

(b) There is a Riemannian metric g which is hermite with respect to all of F, G and H.
(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{2}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a canonical local basis of the bundle V in $\overline{\mathcal{U}}$ (cf. [11, 12]).

For canonical local basis $\{F, G, H\}$ and $\{F', G', H'\}$ of V in coordinate neighborhoods of $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}'$, it follows that in $\overline{\mathcal{U}} \cap \overline{\mathcal{U}}'$

$$\begin{pmatrix} F' \\ G' \\ H' \end{pmatrix} = \left(s_{xy}\right) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \qquad (x, y = 1, 2, 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (1). As is well known [11, 12], every quaternionic Kähler manifold is orientable.

Now let M be an n-dimensional QR-submanifold of maximal QR-dimension, that is, of $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field ξ such that $\nu_x^{\perp} = span\{\xi\}$ at each point x in M. We set

$$F\xi = -U, \qquad G\xi = -V, \qquad H\xi = -W. \tag{3}$$

Denoting by D_x the maximal quaternionic invariant subspace

$$T_xM \cap FT_xM \cap GT_xM \cap HT_xM,$$

of T_xM, we have $D_x^{\perp} \supset Span\{U, V, W\}$, where $D_x^{\perp}$ means the complementary orthogonal subspace to D_x in T_xM. But, using (1), we can prove that $D_x^{\perp} = Span\{U, V, W\}$ [2, 16]. Thus we have

$$T_xM = D_x \oplus Span\{U, V, W\}, \qquad \forall x \in M,$$

which together with (1) and (3) imply

$$FT_xM, GT_xM, HT_xM \subset T_xM \oplus Span\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ $(\xi_1 := \xi)$ of normal vectors to M, we have

$$FX = \varphi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \quad HX = \theta X + \omega(X)\xi, \tag{4}$$

$$\begin{aligned} F\xi_\alpha &= -U_\alpha + P_1\xi_\alpha, \quad G\xi_\alpha = -V_\alpha + P_2\xi_\alpha, \\ H\xi_\alpha &= -W_\alpha + P_3\xi_\alpha, \quad (\alpha = 1, \dots, p). \end{aligned} \tag{5}$$

Then it is easily seen that $\{\varphi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively.

Also, from the hermitian properties

$$\begin{aligned} g(FX, \xi_\alpha) &= -g(X, F\xi_\alpha), \quad g(GX, \xi_\alpha) = -g(X, G\xi_\alpha), \\ g(HX, \xi_\alpha) &= -g(X, H\xi_\alpha), \quad (\alpha = 1, \dots, p). \end{aligned}$$

It follows that

$$g(X, U_\alpha) = u(X)\delta_{1\alpha}, \;\; g(X, V_\alpha) = v(X)\delta_{1\alpha}, \;\; g(X, W_\alpha) = w(X)\delta_{1\alpha},$$

and hence

$$\begin{aligned} &g(X, U_1) = u(X), \;\; g(X, V_1) = v(X), \;\; g(X, W_1) = w(X), \\ &U_\alpha = 0, \;\; V_\alpha = 0, \;\; W_\alpha = 0, \;\; (\alpha = 2, \dots p). \end{aligned} \tag{6}$$

On the other hand, comparing (3) and (5) with $\alpha = 1$, we have $U_1 = U, V_1 = V, W_1 = W$, which together with (3) and (6) imply

$$\begin{array}{llll} g(X, U) = u(X), & g(X, V) = v(X), & g(X, W) = w(X), & \\ u(U) = 1, & v(V) = 1, & w(W) = 1, & \\ F\xi = -U, & G\xi = -V, & H\xi = -W & \\ F\xi_{\alpha = P_1 \xi_\alpha}, & G\xi_{\alpha = P_2 \xi_\alpha} & H\xi_{\alpha = P_3 \xi_\alpha}, & (\alpha = 2, \dots, p). \end{array}$$

Now, let ∇ be the Levi-Civita connection on M and $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. The Gauss and Weingarten formula are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \;\; \overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla_X^\perp \xi_\alpha, \;\; (\alpha = 1, \dots, p), \tag{7}$$

for any $X, Y \in \chi(M)$ and $\xi_\alpha \in \Gamma^\infty(T(M)^\perp)$, $(\alpha = 1, \dots, p)$. h is the second fundamental form and A_α are shape operator corresponding to ξ_α.
We have the following Gauss equation

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) \\ &\quad - \sum_{i=1}^{p} \{g(A_a Y, Z)g(A_a X, W) - g(A_a X, Z)g(A_a Y, W)\}, \end{aligned} \tag{8}$$

and Codazzi and Ricci equations

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)Z, \xi_a) &= (\nabla_X A_a)Y - (\nabla_Y A_a)X \\ &\quad - \sum_{b=1}^{p} \{s_{ba}(X)g(A_a Y, Z) - s_{ba}(Y)g(A_a X, Z)\}, \\ \overline{g}(\overline{R}(X, Y)\xi_a, \xi_b) &= g([A_b, A_a]X, Y) + \overline{g}(R^\perp(X, Y)\xi_a, \xi_b), \end{aligned} \tag{9}$$

where $\overline{R}$ and R are the curvature tensors of $\overline{M}$ and M, respectively. s_{ab} are called the coefficients of the third fundamental form of M in $\overline{M}$, such that satisfy

$$s_{ab} = -s_{ba}.$$

3 The principal circle bundle $S^{4n+3}(QP^n, S^3)$

Let Q^{n+1} be the $(n+1)$-dimensional quaternionic space with natural quaternionic Kähler structure $(\{F', G', H'\}, \langle,\rangle)$ and let S^{4n+3} be the unit sphere defined by

$$S^{4n+3} = \{(w^1, \dots, w^{n+1}) \in Q^{n+1} | \sum_{i=1}^{n+1} w^i (w^i)^* = 1\}$$
$$= \{(x_1^1, x_2^1, x_3^1, x_4^1, \dots, x_1^{n+1}, x_2^{n+1}, x_3^{n+1}, x_4^{n+1}) \in \mathbb{R}^{4n+4} |$$
$$\sum_{i=1}^{n+1} (x_1^i)^2 + (x_2^i)^2 + (x_3^i)^2 + (x_4^i)^2 = 1\}.$$

such that $w^* = (x_1, -x_2, -x_3, -x_4)$.
The unit normal vector field ξ to S^{4n+3} is given by

$$\xi = -\sum_{i=1}^{n+1} (x_1^i \frac{\partial}{\partial x_1^i} + x_2^i \frac{\partial}{\partial x_2^i} + x_3^i \frac{\partial}{\partial x_3^i} + x_4^i \frac{\partial}{\partial x_4^i}).$$

From

$$\langle F'\xi, \xi\rangle = \langle F'^2\xi, F'\xi\rangle = -\langle \xi, F'\xi\rangle,$$
$$\langle G'\xi, \xi\rangle = \langle G'^2\xi, G'\xi\rangle = -\langle \xi, G'\xi\rangle,$$
$$\langle H'\xi, \xi\rangle = \langle H'^2\xi, H'\xi\rangle = -\langle \xi, H'\xi\rangle,$$

it follows $\langle F'\xi, \xi\rangle = 0, \langle G'\xi, \xi\rangle = 0, \langle H'\xi, \xi\rangle = 0$, that is, $F'\xi, G'\xi, H'\xi \in T(S^{4n+3})$. We put

$$F'\xi = -\iota U', \quad G'\xi = -\iota V', \quad H'\xi = -\iota W', \tag{10}$$

where ι denotes the immersion of S^{4n+3} into Q^{n+1}. From the Hermitian property of $\langle,\rangle$, it is easily seen that U', V', W' are unit tangent vector field of S^{4n+3}.

We put

$$H_p(S^{4n+3}) = \{X' \in T_p(S^{4n+3}) | u'(X') = 0, v'(X') = 0, w'(X') = 0\}.$$

Then u', v', w' define a connection form of the principal bundle $S^{4n+3}(QP^n, S^3)$ and we have

$$T_p(S^{4n+3}) = H_p(S^{4n+3}) \oplus span\{U'_p, V'_p, W'_p\}.$$

We call $H_p(S^{4n+3})$ and $span\{U'_p, V'_p, W'_p\}$ the horizontal subspace and vertical subspace of $T_p(S^{4n+3})$, respectively. By definition, the horizontal subspace $H_p(S^{4n+3})$ is isomorphic to $T_{\pi(p)}(QP^n)$, where π is the natural projection from S^{4n+3} onto QP^n. Therefore, for a vector field X on QP^n, there exists unique horizontal vector field X' of S^{4n+3} such that $\pi(X') = X$. The vector field X' is called the horizontal lift of X and we denote it by X^*.

Proposition 1 *As a subspace of* $T_p(Q^{n+1})$, $H_p(S^{4n+3})$ *is* $\{F', G', H'\}$*-invariant subspace.*

Proof. By definition (10) of the vertical vector field $\{U', V', W'\}$, for $X' \in H_p(S^{4n+3})$, it follows

$$\begin{aligned}
\langle F'\iota X', \xi\rangle &= -\langle \iota X', F'\xi\rangle = \langle \iota X', \iota U'\rangle = 0,\\
\langle G'\iota X', \xi\rangle &= -\langle \iota X', G'\xi\rangle = \langle \iota X', \iota V'\rangle = 0,\\
\langle H'\iota X', \xi\rangle &= -\langle \iota X', H'\xi\rangle = \langle \iota X', \iota W'\rangle = 0.
\end{aligned}$$

This shows that $F'\iota X', G'\iota X', H'\iota X' \in T_p(S^{4n+3})$. In entirely the same way we compute

$$\begin{aligned}
\langle F'\iota X', \iota U'\rangle &= -\langle \iota X', F'\iota U'\rangle = \langle \iota X', -\xi\rangle = 0,\\
\langle G'\iota X', \iota V'\rangle &= -\langle \iota X', G'\iota V'\rangle = \langle \iota X', -\xi\rangle = 0,\\
\langle H'\iota X', \iota W'\rangle &= -\langle \iota X', H'\iota W'\rangle = \langle \iota X', -\xi\rangle = 0.
\end{aligned}$$

By use from relations

$$\begin{aligned}
F'V' &= -H'\xi, \quad F'W' = G'\xi, \quad G'U' = H'\xi,\\
G'W' &= F'\xi, \quad H'U' = -G'\xi, \quad H'V' = F'\xi,
\end{aligned}$$

we have

$$\begin{aligned}
\langle F'\iota X', \iota V'\rangle &= -\langle \iota X', F'\iota V'\rangle = \langle \iota X', H'\xi\rangle = -\langle \iota X', \iota W'\rangle = 0,\\
\langle F'\iota X', \iota W'\rangle &= -\langle \iota X', F'\iota W'\rangle = \langle \iota X', G'\xi\rangle = -\langle \iota X', \iota V'\rangle = 0,\\
\langle G'\iota X', \iota U'\rangle &= -\langle \iota X', G'\iota U'\rangle = \langle \iota X', H'\xi\rangle = -\langle \iota X', \iota W'\rangle = 0,\\
\langle G'\iota X', \iota W'\rangle &= -\langle \iota X', G'\iota W'\rangle = \langle \iota X', F'\xi\rangle = -\langle \iota X', \iota U'\rangle = 0,\\
\langle H'\iota X', \iota V'\rangle &= -\langle \iota X', H'\iota V'\rangle = \langle \iota X', F'\xi\rangle = -\langle \iota X', \iota U'\rangle = 0,\\
\langle H'\iota X', \iota U'\rangle &= -\langle \iota X', H'\iota U'\rangle = \langle \iota X', G'\xi\rangle = -\langle \iota X', \iota V'\rangle = 0.
\end{aligned}$$

and hence $F'\iota X', G'\iota X', H'\iota X' \in H_p(S^{4n+3})$, which completes the proof. □

Therefore, the almost quaternionic structure $\{F, G, H\}$ can be induced on $T_{\pi(p)}$ (QP^n) and we set

$$(FX)^* = F'\iota X^*, \quad (GX)^* = G'\iota X^*, \quad (HX)^* = H'\iota X^*. \tag{11}$$

Next, using the Gauss formula (7) for the vertical vector field $\{U', V', W'\}$ and a horizontal vector fields X' of $T_p(S^{4n+3})$, we compute

$$\nabla^E_{X'}U' = \nabla'_{X'}U' + g'(A'X', U')\xi = \nabla'_{X'}U' + \langle X', U'\rangle\xi = \nabla'_{X'}U', \tag{12}$$

by similar computation for vector fields $\{V', W'\}$ we have

$$\begin{aligned} \nabla^E_{X'}V' &= \nabla'_{X'}V', \\ \nabla^E_{X'}W' &= \nabla'_{X'}W', \end{aligned} \tag{13}$$

where ∇^E denotes the Euclidean connection of E^{4n+4}, ∇' denotes the connection of S^{4n+3} and A' denotes the shape operator with respect to ξ. Now, using relations (10), (12) and (13) and the Weingarten formula (7), we conclude

$$\begin{aligned} \nabla'_{X'}U' &= -\nabla^E_{X'}F'\xi' = -(\nabla^E_{X'}F')\xi - F'\nabla^E_{X'}\xi \\ &= -(r(X')G'\xi - q(X')H'\xi) - F'\nabla^E_{X'}\xi \\ &= r(X')V' - q(X')W' + F'(\iota A'X') \\ &= r(X')V' - q(X')W' + F'\iota X' \end{aligned} \tag{14}$$

by similar computation for vector fields $\{V', W'\}$ we have

$$\begin{aligned} \nabla'_{X'}V' &= -\nabla^E_{X'}G'\xi' = -r(X')U' + p(X')W' + G'\iota X', \\ \nabla'_{X'}W' &= -\nabla^E_{X'}H'\xi' = q(X')U' - p(X')V' + H'\iota X'. \end{aligned} \tag{15}$$

Consequently, according to notation (11), relations (14) and (15) can be written as

$$\begin{aligned} \nabla'_{X^*}U' &= r(X^*)V' - q(X^*)W' + (FX)^*, \\ \nabla'_{X^*}V' &= -r(X^*)U' + p(X^*)W' + (GX)^*, \\ \nabla'_{X^*}W' &= q(X^*)U' - p(X^*)V' + (HX)^*. \end{aligned} \tag{16}$$

We note that, since by definition, the Lie derivative of a horizontal lift of a vector field with respect to a vertical vector field is zero, it follows

$$\begin{aligned} 0 &= L_{U'}X^* = [U', X^*] = \nabla'_{U'}X^* - \nabla'_{X^*}U', \\ 0 &= L_{V'}X^* = [V', X^*] = \nabla'_{V'}X^* - \nabla'_{X^*}V' \\ 0 &= L_{W'}X^* = [W', X^*] = \nabla'_{W'}X^* - \nabla'_{X^*}W' \end{aligned}$$

and using (16), we conclude

$$\begin{aligned} \nabla'_{U'}X^* &= r(X^*)V' - q(X^*)W' + (FX)^*, \\ \nabla'_{V'}X^* &= -r(X^*)U' + p(X^*)W' + (GX)^*, \\ \nabla'_{W'}X^* &= q(X^*)U' - p(X^*)V' + (HX)^*. \end{aligned} \tag{17}$$

We define a Riemannian metric g and a connection ∇ in QP^n respectively by

$$g(X, Y) = g'(X^*, Y^*), \tag{18}$$

$$\nabla_X Y = \pi(\nabla'_{X^*} Y^*). \tag{19}$$

Then $(\nabla'_X Y)^*$ is the horizontal part of $\nabla'_{X^*}Y^*$ and therefore

$$\begin{aligned} \nabla'_{X^*}Y^* = (\nabla'_X Y)^* &+ g'(\nabla'_{X^*}Y^*, U')U' \\ &+ g'(\nabla'_{X^*}Y^*, V')V' + g'(\nabla'_{X^*}Y^*, W')W'. \end{aligned} \tag{20}$$

Using relations (16) and (18), we compute

$$\begin{aligned} g'(\nabla'_{X^*}Y^*, U') &= -g'(Y^*, \nabla'_{X^*}U') \\ &= -g'(Y^*, r(X^*)V' - q(X^*)W' + (FX)^*) = -g(Y, FX), \\ g'(\nabla'_{X^*}Y^*, V') &= -g(Y, GX), \\ g'(\nabla'_{X^*}Y^*, V') &= -g(Y, HX), \end{aligned}$$

and, using (20), we conclude

$$\nabla'_{X^*}Y^* = (\nabla_X Y)^* - g(Y, FX)U' - g(Y, GX)V' - g(Y, HX)W'. \tag{21}$$

Proposition 2 *∇ is the Levi-Civita connection for g.*

Proof. Let T be the torsion tensor field of ∇. Then we have

$$\begin{aligned} T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] = \pi(\nabla'_{X^*}Y^*) - \pi(\nabla'_{Y^*}X^*) - [\pi X^*, \pi Y^*] \\ &= \pi(\nabla'_{X^*}Y^* - \nabla'_{Y^*}X^* - [X^*, Y^*]) = \pi(T'(X^*, Y^*)) = 0, \end{aligned}$$

hence ∇ is torsion free. We now show that ∇ is a metric connection.

$$\begin{aligned} (\nabla_X g)(Y, Z) &= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &= X^*(g'(Y^*, Z^*)) - g'((\nabla_X Y)^*, Z^*) - g'(Y^*, (\nabla_X Z)^*). \end{aligned}$$

Since Z^* is horizontal vector field, from relation (20), it follows that

$$\begin{aligned} g'((\nabla_X Y)^*, Z^*) &= g'(\nabla'_{X^*} Y^*, Z^*) - g'(\nabla'_{X^*} Y^*, U') g'(U', Z^*) \\ &\quad - g'(\nabla'_{X^*} Y^*, V') g'(V', Z^*) - g'(\nabla'_{X^*} Y^*, U') g'(U', Z^*) \\ &= g'(\nabla'_{X^*} Y^*, Z^*) \end{aligned}$$

and we compute

$$\begin{aligned} (\nabla_X g)(Y, Z) &= X^*(g'(Y^*, Z^*)) - g'(\nabla'_{X^*} Y^*, Z^*) - g'(\nabla'_{X^*} Z^*, Y^*) \\ &= (\nabla'_{X^*} g')(Y^*, Z^*), \end{aligned}$$

where we have used the fact that ∇' is the Levi-Civita connection for g'. Thus ∇ is the Levi-Civita connection for g and the proof is complete. □

Now, by (21), it follows

$$\begin{aligned} [X^*, Y^*] &= [X, Y]^* + g'([X^*, Y^*], U')U' \\ &\quad + g'([X^*, Y^*], V')V' + g'([X^*, Y^*], W')W' \\ &= [X, Y]^* + g'(\nabla'_{X^*} Y^* - \nabla'_{Y^*} X^*, U')U' \\ &\quad + g'(\nabla'_{X^*} Y^* - \nabla'_{Y^*} X^*, V')V' + g'(\nabla'_{X^*} Y^* - \nabla'_{Y^*} X^*, W')W' \\ &= [X, Y]^* + g'((\nabla'_X Y)^* - g(Y, FX)U' - g(Y, GX)V' \\ &\quad - g(Y, HX)W', U')U' - g'((\nabla_Y X)^* - g(FY, X)U' \\ &\quad - g(GY, X)V' - g(HY, X)W', U')U' + g'((\nabla_X Y)^* \\ &\quad - g(Y, FX)U' - g(Y, GX)V' - g(Y, HX)W', V')V' \\ &\quad - g'((\nabla_Y X)^* - g(FY, X)U' - g(GY, X)V' - g(HY, X)W', V')V' \\ &\quad + g'((\nabla_X Y)^* - g(Y, FX)U' - g(Y, GX)V' - g(Y, HX)W', W')W' \\ &\quad - g'((\nabla_Y X)^* - g(FY, X)U' - g(GY, X)V' - g(HY, X)W', W')W' \\ &= [X, Y]^* - 2g(Y, FX)U' - 2g(Y, GX)V' - 2g(Y, HX)W'. \end{aligned} \tag{22}$$

Consequently, using (16), (17), (21) and (22), the curvature tensor R of QP^n is calculated as follows:

$$\begin{aligned} R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \pi\{\nabla'_{X^*}(\nabla_Y Z)^*) - \nabla'_{Y^*}(\nabla_X Z)^*) - \nabla'_{[X,Y]^*} Z^*)\} \\ &= \pi\{\nabla'_{X^*}(\nabla'_{Y^*} Z^* + g(Z, FY)U' + g(Z, GY)V' + g(Z, HY)W') \\ &\quad - \nabla'_{Y^*}(\nabla'_{X^*} Z^* + g(Z, FX)U' + g(Z, GX)V' + g(Z, HX)W') \\ &\quad - \nabla'_{[X^*,Y^*]+2g(Y,FX)U'+2g(Y,GX)V'+2g(Y,HX)W'} Z^*)\} \end{aligned}$$

$$
\begin{aligned}
&= \pi\{\nabla'_{X^*}\nabla'_{Y^*}Z^* + g(Z,FY)\nabla'_{X^*}U' + g(Z,GY)\nabla'_{X^*}V' \\
&\quad + g(Z,HY)\nabla'_{X^*}W' - \nabla'_{Y^*}\nabla'_{X^*}Z^* - g(Z,FX)\nabla'_{Y^*}U' \\
&\quad - g(Z,GX)\nabla'_{Y^*}V' - g(Z,HX)\nabla'_{X^*}W' - \nabla'_{[X^*,Y^*]}Z^* \\
&\quad - 2g(Y,FX)\nabla'_{U'}Z^* - 2g(Y,GX)\nabla'_{V'}Z^* - 2g(Y,HX)\nabla'_{W'}Z^*\} \\
&= \pi\{R'(X^*,Y^*)Z^* + g(Z,FY)(r(X^*)V' - q(X^*)W' + (FX)^*) \\
&\quad + g(Z,GY)(-r(X^*)U' + p(X^*)W' + (GX)^*) \\
&\quad + g(Z,HY)(q(X^*)U' - p(X^*)V' + (HX)^*) \\
&\quad + g(Z,FX)(r(Y^*)V' - q(Y^*)W' + (FY)^*) \\
&\quad + g(Z,GX)(-r(Y^*)U' + p(Y^*)W' + (GY)^*) \\
&\quad + g(Z,HX)(q(Y^*)U' - p(Y^*)V' + (HY)^*) \\
&\quad + 2g(Y,FX)(r(Z^*)V' - q(Z^*)W' + (FZ)^*) \\
&\quad + 2g(Y,GX)(-r(Z^*)U' + p(Z^*)W' + (GZ)^*) \\
&\quad + 2g(Y,HX)q(Z^*)U' - p(Z^*)V' + (HZ)^*\}.
\end{aligned}
$$

Since the curvature tensor R' of S^{4n+3} satisfies

$$
R'(X^*,Y^*)Z^* = g'(Y^*,Z^*)X^* - g'(X^*,Z^*)Y^* = g(Y,Z)X^* - g(X,Z)Y^*, \tag{23}
$$

we conclude that the curvature tensor of QP^n is given by

$$
\begin{aligned}
R(X,Y)Z = g(Y,Z)X &- g(X,Z)Y \\
&+ g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ \\
&+ g(GY,Z)GX - g(GX,Z)GY - 2g(GX,Y)GZ \\
&+ g(HY,Z)HX - g(HX,Z)HY - 2g(HX,Y)HZ.
\end{aligned} \tag{24}
$$

4 Submanifolds of quaternionic manifolds

Let M be an n-dimensional submanifold of $QP^{\frac{n+p}{4}}$ and $\pi^{-1}(M)$ be the circle bundle over M which is compatible with the Hopf map

$$
\pi : S^{n+p+3} \longrightarrow QP^{\frac{n+p}{4}}.
$$

Then $\pi^{-1}(M)$ is a submanifold of S^{n+p+3}. The compatibility with the Hopf map is expressed by $\pi o \iota' = \iota o \pi$ where ι' and ι are the immersions of M into $QP^{\frac{n+p}{4}}$ and $\pi^{-1}(M)$ into S^{n+p+3}, respectively.
Let ξ_a, $a = 1,\dots,p$ be orthonormal normal local fields to M in $QP^{\frac{n+p}{4}}$ and

ξ_a^* be the horizontal lifts of ξ_a. Then ξ_a^* are mutually orthonormal normal local fields to $\pi^{-1}(M)$ in S^{n+p+3}. At each point $y \in \pi^{-1}(M)$ we compute

$$\begin{aligned} g^S(\iota' X^*, \xi_a^*) &= g^S((\iota X)^*, \xi_a^*) = \overline{g}(\iota X, \xi_a) = 0, \\ g^S(\iota' U, \xi_a^*) &= g^S(U', \xi_a^*) = 0, \\ g^S(\iota' V, \xi_a^*) &= g^S(V', \xi_a^*) = 0, \\ g^S(\iota' W, \xi_a^*) &= g^S(W', \xi_a^*) = 0, \\ g^S(\xi_a^* \xi_b^*) &= \overline{g}(\xi_a, \xi_b) = \delta_{ab}, \end{aligned}$$

where g^S and $\overline{g}$ denote the Riemannian metric on S^{n+p+3} and $QP^{\frac{n+p}{4}}$, respectively. Here $U' = \iota' U, V' = \iota' V, W' = \iota' W$ are unit tangent vector field of S^{n+p+3} defined by relation (10).
Now, let $\nabla^S, \nabla', \overline{\nabla}$ and ∇ be the Riemannian connections of S^{n+p+3}, $\pi^{-1}(M)$, $QP^{\frac{n+p}{4}}$ and M, respectively. By means of the Gauss formula (7) and relations (4) and (21), we compute

$$\begin{aligned} \nabla^S_{X^*} \iota' Y^* = \nabla^S_{X^*} (\iota Y)^* &= (\overline{\nabla}_X \iota Y)^* - \overline{g}(F\iota X, \iota Y)\iota' U \\ &\quad - \overline{g}(G\iota X, \iota Y)\iota' V - \overline{g}(H\iota X, \iota Y)\iota' W \\ &= (\iota \nabla_X Y + h(X,Y))^* - \overline{g}(\iota \varphi X, \iota Y) U' \\ &\quad - g(\iota \psi X, \iota Y) V' - g(\iota \theta X, \iota Y) W' \\ &= \iota' (\nabla_X Y)^* + (h(X,Y))^* \\ &\quad - g(\varphi X, Y) U' - g(\psi X, Y) V' - g(\theta X, Y) W' \end{aligned} \tag{25}$$

where g is the metric on M. On the other hand, we also have

$$\begin{aligned} \nabla^S_{X^*} \iota' Y^* = \iota' \nabla'_{X^*} Y^* + h'(X^*, Y^*) &= \iota'((\nabla_X Y)^* - g(\varphi X, Y) U \\ &\quad - g(\psi X, Y) V - g(\theta X, Y) W) + h'(X^*, Y^*) \end{aligned} \tag{26}$$

where h and h' denote the second fundamental form of M and $\pi^{-1}(M)$, respectively. Comparing the vertical part and horizontal part of relations (25) and (26), we conclude

$$(h(X,Y))^* = h'(X^*, Y^*), \tag{27}$$

that is,

$$\sum_{a=1}^{p} g'(A'_a X^*, Y^*) \xi_a^* = (\sum_{a=1}^{p} g(A_a X, Y) \xi_a)^* = \sum_{a=1}^{p} g(A_a X, Y) \xi_a^*,$$

where A_a and A'_a are the shape operators with respect to normal vector fields ξ_a and ξ^*_a of M and $\pi^{-1}(M)$, respectively. Consequently, we have

$$g'(A'_aX^*, Y^*) = g(A_aX, Y), \quad (a = 1, ..., p).$$

Next, using the weingarten formula, we calculate $\nabla^S_{X^*}\xi^*_a$ as follows

$$\nabla^S_{X^*}\xi^*_a = -\iota' A'_aX^* + \nabla'^{\perp}_{X^*}\xi^*_a = -\iota' A'_aX^* + \sum_{b=1}^{p} s'_{ab}(X^*)\xi^*_b. \tag{28}$$

where $\nabla'^{\perp}$ is normal connection $\pi^{-1}(M)$ in S^{n+p+3}.
On the other hand, from relation (21), it follows

$$\begin{aligned}\nabla^S_{X^*}\xi^*_a &= (\overline{\nabla}_X\xi_a)^* - \overline{g}(F\iota X, \xi_a)\iota' U - \overline{g}(G\iota X, \xi_a)\iota' V - \overline{g}(H\iota X, \xi_a)\iota' W \\ &= (-\iota A_aX + \nabla^{\perp}_X\xi_a)^* - \sum_{b=1}^{p}\{u^b(X)\overline{g}(\xi_a, \xi_b)U' \\ &\quad + v^b(X)\overline{g}(\xi_a, \xi_b)V' + w^b(X)\overline{g}(\xi_a, \xi_b)W'\} \\ &= -\iota'(A_aX)^* + \sum_{b=1}^{p}(s_{ab}(X^*)\xi_b)^* - u^a(X)U' \\ &\quad - v^a(X)V' - w^a(X)W',\end{aligned} \tag{29}$$

where $\nabla^{\perp}$ is normal connection M in $QP^{\frac{n+p}{4}}$.
We have put

$$\begin{aligned}F\iota x &= \iota\varphi X + \sum_{a=1}^{p} u^a(X)\xi_a, \\ G\iota x &= \iota\psi X + \sum_{a=1}^{p} v^a(X)\xi_a, \\ H\iota x &= \iota\theta X + \sum_{a=1}^{p} w^a(X)\xi_a.\end{aligned} \tag{30}$$

Comparing relations (28) and (29), we obtain

$$\begin{aligned}A'_aX^* &= (A_aX)^* + u^a(X)U' + v^a(X)V' + w^a(X)W' \\ &= (A_aX)^* + g(U_a, X)U' + g(V_a, X)V + g(W_a, X)W', \\ \nabla'^{\perp}_{X^*}\xi^*_a &= (\nabla^{\perp}_X\xi_a)^*,\end{aligned} \tag{31}$$

that is, $s'_{ab}(X^*) = s_{ab}(X)^*$, where U_a, V_a, W_a are defined by

$$F\xi_a = -U_a + \sum_{b=1}^{p} P_{1_{ab}}\xi_b,$$
$$G\xi_a = -V_a + \sum_{b=1}^{p} P_{2_{ab}}\xi_b,$$
$$H\xi_a = -W_a + \sum_{b=1}^{p} P_{3_{ab}}\xi_b. \tag{32}$$

where, $\sum_{b=1}^{p} P_{i_{ab}}\xi_b = P_i\xi_a, \quad (i = 1, 2, 3)$. Now, we consider $\nabla^S_U \xi^*_a$ and using relations (17) and (32) imply

$$\nabla^S_U \xi^*_a = (F\xi_a)^* = -\iota U_{a^*} + P_1\xi^*_a,$$
$$\nabla^S_V \xi^*_a = (G\xi_a)^* = -\iota V_{a^*} + P_2\xi^*_a,$$
$$\nabla^S_W \xi^*_a = (H\xi_a)^* = -\iota W_{a^*} + P_3\xi^*_a. \tag{33}$$

On the other hand, from the Weingarten formula, it follows

$$\nabla^S_U \xi^*_a = -\iota' A'_a U + \nabla'^{\perp}_U \xi^*_a = -\iota' A'_a U + \sum_{b=1}^{p} s'_{ab}(U)\xi^*_b,$$
$$\nabla^S_V \xi^*_a = -\iota' A'_a V + \nabla'^{\perp}_V \xi^*_a = -\iota' A'_a V + \sum_{b=1}^{p} s'_{ab}(V)\xi^*_b,$$
$$\nabla^S_W \xi^*_a = -\iota' A'_a W + \nabla'^{\perp}_W \xi^*_a = -\iota' A'_a W + \sum_{b=1}^{p} s'_{ab}(W)\xi^*_b. \tag{34}$$

Consequently, using (33) and (34), we obtain

$$A'_a U = U^*_a, \quad A'_a V = V^*_a, \quad A'_a W = W^*_a,$$
$$s'_{ab}(U) = P_1, \quad s'_{ab}(V) = P_2, \quad s'_{ab}(W) = P_3, \tag{35}$$
$$\nabla'^{\perp}_U \xi^*_a = (FX)^* + \iota U_{a^*},$$
$$\nabla'^{\perp}_V \xi^*_a = (GX)^* + \iota V_{a^*},$$
$$\nabla'^{\perp}_W \xi^*_a = (HX)^* + \iota W_{a^*}. \tag{36}$$

The first relations of (31) and (35), we get

$$g'(A'_a A'_b X^*, Y^*) = g(A_a A_b X, Y) + u^b(X)u^a(Y) + v^b(X)v^a(Y) + w^b(X)w^a(Y), \tag{37}$$

and especially,

$$g'(A'^2_a X^*, Y^*) = g(A^2_a X, Y) + u^a(X)u^a(Y) + v^a(X)v^a(Y) + w^a(X)w^a(Y), \tag{38}$$

For $x \in M$, let $\{e_1, \ldots, e_n\}$be an orthonormal basis of $T_x M$ and y be a point of $\pi^{-1}(M)$ such that $\pi(y) = x$. We take an orthonormal basis $\{e^*_1, \ldots, e^*_n, U, V, W\}$ of $T_y(\pi^{-1}(M))$. Then, using the first relations (35) and (39), we compute

$$\begin{aligned}\sum_{a=1}^{p} \operatorname{trace} A'^2_a &= \sum_{a=1}^{p} \Big\{ \sum_{i=1}^{n} g'(A'^2_a e^*_i, e^*_i) + g'(A'^2_a U, U) \\ &\quad + g'(A'^2_a V, V) + g'(A'^2_a W, W)\} \\ &= \sum_{a=1}^{p} \Big\{ \sum_{i=1}^{n} g(A'^2_a e_i, e_i) + u^a(e_i)u^a(e_i) \\ &\quad + v^a(e_i)v^a(e_i) + w^a(e_i)w^a(e_i) \\ &\quad + g'(A'_a U, A'_a U) + g'(A'_a V, A'_a V) + g'(A'_a W, A'_a W)\} \\ &= \sum_{a=1}^{p} \{\operatorname{trace} A^2_a + 2g(U_a, U_a) + 2g(V_a, V_a) + 2g(W_a, W_a)\},\end{aligned} \tag{39}$$

we conclude

Proposition 3 *Under the above assumptions, the following inequality*

$$\sum_{a=1}^{p} \operatorname{trace} A'^2_a \geq \sum_{a=1}^{p} \operatorname{trace} A^2_a$$

is always valid. The equality holds, if and only if M *is a* $\{F, G, H\}$- *invariant submanifold.*

Corollary 1 [8] M *is a totally geodesic submanifold if and only if relation* $A_\xi = 0$ *holds for any normal vector field* ξ *of* M. *Particularly,* M *is totally geodesic if and only if* $A_1 = \ldots = A_p = 0$ *for an orthonormal frame field* $\xi_1, \ldots, \xi_p$ *of* $T^\perp(M)$

Proposition 4 *Under the condition stated above, if* $\pi^{-1}(M)$ *is a totally geodesic submanifold of* S^{n+p+3}, *then* M *is a totally geodesic* $\{F, G, H\}$- *invariant submanifold.*

Proof. Since $\pi^{-1}(M)$ is a totally geodesic submanifold of S^{n+p+3}, using Corollary (26), it follows $A'_a = 0$. The first Relation (31) then implies $A_a = 0$ and $U_a = V_a = W_a = 0$, which, using relation (32), completes the proof. □

Further, for the normal curvature of M in $QP^{\frac{n+p}{4}}$, using relation (24) and the second relation (9), we obtain

$$\begin{aligned}\bar{g}(R^{\perp}(X,Y)\xi_a,\xi_b) = g([A_a,A_b]X,Y) &+ u^b(X)u^a(Y) \\ &- u^a(X)u^b(Y) + v^b(X)v^a(Y) - v^a(X)v^b(Y) \\ &+ w^b(X)w^a(Y) - w^a(X)w^b(Y) \\ &- 2g(\varphi X,Y)P_1 - 2g(\psi X,Y)P_2 - 2g(\theta X,Y)P_3\end{aligned} \tag{40}$$

Therefore, if M be a totally geodesic submanifold of $\{F,G,H\}$- invariant submanifold, we conclude

$$\bar{g}(R^{\perp}(X,Y)\xi_a,\xi_b) = -2g(\varphi X,Y)P_1 - 2g(\psi X,Y)P_2 - 2g(\theta X,Y)P_3 \tag{41}$$

In this case the normal space $T_x^{\perp}(M)$ is also $\{F,G,H\}$- invariant and P_1, P_2, P_3 never vanish. We have thus proved

Proposition 5 *The normal curvature of a totally geodesic submanifold of* $\{F,G,H\}$- *invariant submanifold of a quaternionic projective space never vanishes.*

This proposition show that the normal connection of the quaternionic projective space which is immersed standardly in a higher dimensional quaternionic projective space not flat.

Finally, we give a relation between the normal curvature $R^{\perp}$ and $R'^{\perp}$ of M and $\pi^{-1}(M)$, respectively, where M is a n-dimensional submanifold of $QP^{\frac{n+p}{4}}$ and $\pi^{-1}(M)$ is the circle bundle over M which is compatible with the Hopf map π. Using relation (37), we obtain

$$\begin{aligned} g'([A'_a,A'_b]X^*,Y^*) &= g([A_a,A_b]X,Y) + u^b(X)u^a(Y) - u^a(X)u^b(Y) \\ &+ v^b(X)v^a(Y) - v^a(X)v^b(Y) + w^b(X)w^a(Y) - w^a(X)w^b(Y),\end{aligned}$$

and therefore, from the second relation (9), it follows

$$\begin{aligned}&-g^S(R'^S(\iota'X^*,\iota'Y^*)\xi_a^*,\xi_b^*) + g^S(R'^{\perp}(X^*,Y^*)\xi_a^*,\xi_b^*) \\ &\quad = -\bar{g}(\bar{R}(\iota X,\iota Y)\xi_a,\xi_b) + \bar{g}(R^{\perp}(X,Y)\xi_a,\xi_b) + u^b(X)u^a(Y) - u^a(X)u^b(Y) \\ &\quad\quad + v^b(X)v^a(Y) - v^a(X)v^b(Y) + w^b(X)w^a(Y) - w^a(X)w^b(Y).\end{aligned}$$

Using the expression (23) and (24) , for curvature of S^{n+p+3} and $QP^{\frac{n+p}{4}}(C)$, respectively and relations (30) and (32) imply

$$\begin{aligned} g^S(R^\perp(X^*,Y^*)\xi_a^*,\xi_b^*) &= \overline{g}(R^\perp(X,Y)\xi_a,\xi_b) + 2g(\varphi X,Y)P_1 \\ &+ 2g(\psi X,Y)P_2 + 2g(\theta X,Y)P_3 \end{aligned}$$

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Lah numbers, Laguerre polynomials of order negative one, and the nth derivative of $\exp(1/x)$

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Abstract. In this note we point out interesting connections among Lah numbers, Laguerre polynomials of order negative one, and exponential polynomials. We also discuss several different expressions for the nth derivative of $\exp(1/x)$. A new representation of this derivative is given in terms of exponential polynomials.

1 Introduction

The Lah numbers $L(n,k)$ (named after Ivo Lah, a Slovenian mathematician) can be defined by the formula

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}, \quad 1 \le k \le n, \ L(0,0) = 1, \tag{1}$$

or, by the generating function

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}.$$

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The Lah numbers convert the falling factorial to the rising factorial and vise-versa

$$x(x+1)\dots(x+n-1) = \sum_{k=1}^{n} L(n,k)\, x(x-1)\dots(x-k+1),$$
$$x(x-1)\dots(x-n+1) = \sum_{k=1}^{n} (-1)^{n-k} L(n,k)\, x(x+1)\dots(x+k-1)$$

(these are the fundamental identities obtained by Ivo Lah).

The Lah numbers have many other interesting applications in analysis and combinatorics (see [1, 2, 9, 12, 16]). They have appeared recently in several papers concerning the consecutive derivatives of the function $\exp(1/x)$. In [10] five proofs were given of the following formula:

$$D^n e^{1/x} = (-1)^n e^{1/x} x^{-n} \sum_{k=1}^{n} L(n,k)\, x^{-k}. \tag{2}$$

where $D = \frac{d}{dx}$ and $n \geq 1$. This formula was obtained also by Feng Qi (see [13] and the remarks in Section 5 there). The formula is a nice application of Lah numbers to a problem in analysis.

At the same time, entry 1.1.3.2 on p. 4 in Brychkov's handbook [6] says that

$$\frac{d^n}{dx^n}[x^\lambda e^{-a/x}] = (-1)^n n! e^{-a/x} x^{\lambda-n} L_n^{(-\lambda-1)}(a/x)$$

where $L_n^{(\alpha)}(x)$ are the generalized (or associated) Laguerre polynomials of order α (see [14], [16]). The same formula appears as entry 18.5.6. on page 446 in the handbook [15]. With $\lambda = 0$ and $a = -1$ this becomes

$$D^n e^{1/x} = (-1)^n n! e^{1/x} x^{-n} L_n^{(-1)}(-1/x). \tag{3}$$

As a matter of fact, the derivatives $D^n e^{1/x}$ have been evaluated long time ago. For example, the nth derivative can be found in the nice little book of Schwatt [18], first published in 1924. The formula on top of page 22 in [18] reads

$$D^n e^{cx^p} = n! e^{cx^p} x^{-n} \sum_{k=1}^{n} \frac{(-1)^k}{k!} c^k x^{pk} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \binom{pj}{n} \tag{4}$$

where c, p are arbitrary parameters. The same formula appears also on page 27. With $c = 1$ and $p = -1$ this becomes

$$D^n e^{1/x} = (-1)^n n! e^{1/x} x^{-n} \sum_{k=1}^{n} \frac{(-1)^k}{k!} x^{-k} \left\{ \sum_{j=1}^{k} (-1)^j \binom{k}{j} \binom{n+j-1}{n} \right\} \quad (5)$$

since $\binom{-j}{n} = (-1)^n \binom{n+j-1}{n}$.

In the next three sections we discuss the relations among the three formulas for $D^n e^{1/x}$, namely, equations (2), (3), and (5). This will reveal interesting connections of Lah numbers to Laguerre polynomials and also to Stirling numbers. In Section 4 we present a new formula for $D^n e^{cx^p}$ in terms of the exponential polynomials $\varphi_n(x)$ considered in [4] and [5].

2 Laguerre polynomials

The generalized Laguerre polynomials can be defined by the generating function

$$\frac{1}{(1-t)^{\alpha+1}} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n, \; |t| < 1,$$

or by the Rodriguez formula

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} D^n (e^{-x} x^{n+\alpha}) = \frac{x^{-\alpha}}{n!} (D-1)^n x^{n+\alpha}, \;\; n = 0, 1, \ldots$$

(see [14]). When $\alpha = 0$ these are the usual Laguerre polynomials $L_n^{(0)}(x) = L_n(x)$. Usually, in the theory of $L_n^{(\alpha)}(x)$ the restriction $\mathrm{Re}\, \alpha > -1$ is imposed. In fact, the case $\alpha = -1$ is very interesting and most of the theory holds true for $\alpha = -1$. We shall focus here on the polynomials $L_n^{(-1)}(x)$ defined by

$$L_n^{(-1)}(x) = \frac{x e^x}{n!} D^n (e^{-x} x^{n-1}) = \frac{x}{n!} (D-1)^n x^{n-1}, \; n = 0, 1, \ldots$$

or, by the generating function, $|t| < 1$

$$e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(-1)}(x)\, t^n. \quad (6)$$

We have

$$L_0^{(-1)}(x) = 1,$$
$$L_1^{(-1)}(x) = -x,$$
$$L_2^{(-1)}(x) = \frac{x^2}{2} - x,$$
$$L_3^{(-1)}(x) = \frac{-x^3}{6} + x^2 - x,$$

etc. The coefficients of these polynomials are very close to the Lah number and we can see the exact connection when we compare (2) to (3). However, we shall give an independent proof of this connection in order to justify the value $\alpha = -1$ in (3).

Proposition 1 *For any* $n \geq 0$*,*

$$L_n^{(-1)}(x) = \frac{1}{n!} \sum_{k=0}^{n} L(n, k) (-x)^k. \tag{7}$$

This reveals the connection between the Lah numbers and the Laguerre polynomials $L_n^{(-1)}(x)$and it becomes clear now that formulas (2) and (3) are the same. We also notice that (2) is true also for $n = 0$ with the summation starting from $k = 0$, that is,

$$D^n e^{1/x} = (-1)^n e^{1/x} x^{-n} \sum_{k=0}^{n} L(n, k) x^{-k}.$$

Proof. From the Rodriguez formula for $L_n^{(\alpha)}(x)$ one derives easily the representation

$$L_n^{(\alpha)}(x) = \Gamma(n + \alpha + 1) \sum_{k=0}^{n} \frac{(-x)^k}{\Gamma(k + \alpha + 1)k!(n - k)!}$$

where we cannot set $\alpha = -1$ directly. However, when $n = 0$ this becomes

$$L_0^{(\alpha)}(x) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} = 1$$

and any restriction on α can be dropped. For $n \geq 1$ we separate the first term with $k = 0$ and write

$$L_n^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)} + \Gamma(n + \alpha + 1) \sum_{k=1}^{n} \frac{(-x)^k}{\Gamma(k + \alpha + 1) k! (n - k)!}.$$

Setting $\alpha \to -1$ we find for $n \geq 1$

$$L_n^{(-1)}(x) = \Gamma(n) \sum_{k=1}^{n} \frac{(-x)^k}{\Gamma(k)\, k!\, (n-k)!},$$

since

$$\lim_{\alpha \to -1} \frac{1}{\Gamma(\alpha+1)} = 0.$$

This representation can be written in the form

$$L_n^{(-1)}(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(-x)^k}{k!} \tag{8}$$

which is (7). The proof is completed. □

The representation (7) also shows an important difference between $L_n^{(-1)}(x)$ and $L_n^{(\alpha)}(x)$ for $n \geq 1$. While

$$L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}$$

is different from zero when $\alpha \neq -1$, we have $L_n^{(-1)}(0) = 0$. At the same time, many properties of $L_n^{(\alpha)}(x)$ are shared also by $L_n^{(-1)}(x)$. For example, we have the orthogonally relation ([14, p. 84], [17, p. 204–205])

$$\int_0^\infty x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\, \delta_{n,m}$$

for all $n, m \geq 0$ and $\alpha > -1$. Analyzing the proof of this equation in [17] we conclude that it extends to $\alpha = -1$ when $n, m \geq 1$,

$$\int_0^\infty x^{-1} e^{-x} L_n^{(-1)}(x) L_m^{(-1)}(x)\, dx = \frac{\delta_{n,m}}{n}\ . \tag{9}$$

This and other properties of $L_n^{(-1)}(x)$ can be used to derive properties for the Lah numbers. Here we have the following:

Proposition 2 *For any integers* $n, m \geq 1,\ n \neq m$

$$\sum_{k=1}^{n} (-1)^k L(n,k) \sum_{j=1}^{m} (-1)^j L(m,j)(k+j-1)! = 0 \tag{10}$$

and when $m = n$

$$\sum_{k=1}^{n}\sum_{j=1}^{n}(-1)^{k+j}L(n,k)\,L(n,j)\,(k+j-1)! \ = \frac{(n!)^2}{n}. \tag{11}$$

Proof. Substituting (7) in (9) we arrive at (10) and (11) after simple computation. □

3 The Todorov - Charalambides identity

Here we shall discuss equation (4). Let $s(n,k)$ and $S(n,k)$ be the Stirling numbers of the first kind and the second kind correspondingly (see [9]). It is known that these numbers satisfy the orthogonality relation

$$\sum_{k=0}^{n} s(n,k)\,S(k,m) = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

while the alternating sums are related to the Lah numbers (see [9, p. 156]):

$$L(n,m) = (-1)^n \sum_{k=0}^{n} s(n,k)\,S(k,m)\,(-1)^k. \tag{12}$$

The following identity extends this representation.

Proposition 3 *For any two nonnegative integers* n, m, *and every complex number* z *we have*

$$\frac{m!}{n!}\sum_{k=0}^{n} s(n,k)\;S(k,m)\,z^k = (-1)^m \sum_{j=0}^{m} \binom{m}{j}(-1)^j \binom{zj}{n} \tag{13}$$

This identity was obtained by Todorov [19], who showed that both sides equal Taylor's coefficients of the function $f(t) = ((1+t)^z - 1)^m$. It was also found independently by Charalambides in his study of the generalized factorial coefficients (see [7] and [8]). A short proof of (13) is given in the recent paper [3].

Now we show that equation (12) follows from (13). Setting $z = -1$ in (13) we find

$$\frac{m!}{n!}\sum_{k=0}^{n} s(n,k)\;S(k,m)\,(-1)^k = (-1)^m \sum_{j=0}^{m} \binom{m}{j}(-1)^j \binom{-j}{n}. \tag{14}$$

The RHS becomes

$$(-1)^{m+n}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\binom{n+j-1}{n}$$

since

$$\binom{-j}{n}=(-1)^n\binom{n+j-1}{n}.$$

Next we use a well-known identity from [11]

$$\sum_{j=0}^{m}\binom{m}{j}(-1)^j\binom{y+j}{n}=(-1)^m\binom{y}{n-m} \tag{15}$$

and choosing $y=n-1$ we find

$$\begin{aligned}(-1)^{m+n}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\binom{n+j-1}{n}&=(-1)^n\binom{n-1}{n-m}\\&=(-1)^n\binom{n-1}{m-1}.\end{aligned} \tag{16}$$

Now from (14)

$$\frac{m!}{n!}\sum_{k=0}^{n}s(n,k)\,S(k,m)\,(-1)^k=(-1)^n\binom{n-1}{m-1},$$

or

$$\sum_{k=0}^{n}s(n,k)\,S(k,m)\,(-1)^k=(-1)^nL(n,m)$$

which proves (12).

At the same time we can apply (16) to equation (5). This gives

$$\begin{aligned}D^n e^{1/x}&=(-1)^n n!e^{1/x}x^{-n}\sum_{k=1}^{n}\frac{(-1)^k}{k!}x^{-k}\left\{\sum_{j=1}^{k}(-1)^j\binom{k}{j}\binom{n+j-1}{n}\right\}\\&=(-1)^n n!e^{1/x}x^{-n}\sum_{k=1}^{n}\frac{(-1)^k}{k!}x^{-k}\left\{(-1)^k\frac{k!}{n!}L(n,k)\right\}\\&=(-1)^n e^{1/x}x^{-n}\sum_{k=1}^{n}L(n,k)x^{-k}\end{aligned}$$

which is exactly (2). We see that the formula for the derivatives $D^n e^{1/x}$ was practically found by Schwatt.

4 Schwatt's formula in terms of exponential polynomials

With the help of the Todorov - Charalambides identity, Schwatt's formula (4) can be written in terma of Stirling numbers and exponential polynomials.

The polynomials $\varphi_n(x)$, $n = 0, 1, \dots$, defined by

$$\varphi_m(x) = \sum_{k=0}^{m} S(m,k)\, x^k$$

are known as the exponential polynomials (or single-variable Bell polynomials) – see [4] and [5]. They have the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \varphi_n(x) \frac{t^n}{n!},$$

and can be defined also by the important property $(xD)^n e^x = \varphi_n(x)\, e^x$, $n = 0, 1, \dots$

Proposition 4 *For any* $n \geq 0$ *and any two numbers* c, p *we have*

$$D^n e^{cx^p} = e^{cx^p} x^{-n} \sum_{j=0}^{n} s(n,j) p^j \varphi_j(cx^p) \tag{17}$$

and in particular, when $c = 1$ *and* $p = -1$,

$$D^n e^{1/x} = e^{1/x} x^{-n} \sum_{j=0}^{n} s(n,j)\, (-1)^j \varphi_j(1/x). \tag{18}$$

Proof. Substituting (13) in (4) we obtain

$$\begin{aligned}
D^n e^{cx^p} &= n! e^{cx^p} x^{-n} \sum_{k=1}^{n} \frac{(-1)^k}{k!} c^k x^{pk} \left\{ (-1)^k \frac{k!}{n!} \sum_{j=0}^{k} s(n,j)\, S(j,k)\, p^j \right\} \\
&= e^{cx^p} x^{-n} \sum_{k=1}^{n} c^k x^{pk} \left\{ \sum_{j=0}^{k} s(n,j)\, S(j,k)\, p^j \right\} \\
&= e^{cx^p} x^{-n} \sum_{j=0}^{n} s(n,j)\, p^j \left\{ \sum_{k=1}^{j} c^k x^{pk} S(j,k) \right\} \\
&= e^{cx^p} x^{-n} \sum_{j=0}^{n} s(n,j) p^j \varphi_j(cx^p).
\end{aligned}$$

Comparing this to (2) we arrive at the identity

$$\sum_{k=1}^{n} L(n,k)\,x^k = (-1)^n \sum_{j=0}^{n} s\,(n,j)\,(-1)^j \varphi_j(x). \tag{19}$$

Also, from (8),

$$L_n^{(-1)}(x) = \frac{(-1)^n}{n!} \sum_{j=0}^{n} s\,(n,j)\,(-1)^j \varphi_j(-x). \tag{20}$$

With $p = 1/2$ in (17) we have

$$D^n e^{\,c\sqrt{x}} = e^{c\sqrt{x}} x^{-n} \sum_{j=0}^{n} \frac{1}{2^j} s\,(n,j)\,\varphi_j(c\,\sqrt{x}). \tag{21}$$

□

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On some classes of mixed-super quasi-Einstein manifolds

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Abstract. Quasi-Einstein manifold and generalized quasi-Einstein manifold are the generalizations of Einstein manifold. The purpose of this paper is to study the mixed super quasi-Einstein manifold which is also the generalizations of Einstein manifold satisfying some curvature conditions. We define both Riemannian and Lorentzian doubly warped product on this manifold. Finally, we study the completeness properties of doubly warped products on $MS(QE)_4$ for both the Riemannian and Lorentzian cases.

1 Introduction

The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [7]. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1}$$

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and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \forall \; X \in \chi(M), \tag{2}$$

where, U being a unit vector field and $\chi(M)$ is the set of all differentiable vector fields on M.

Here a and b are called the *associated scalars*, A is called the associated 1-form and U is called the *generator* of the manifold. Such an n-dimensional manifold will be denoted by $(QE)_n$.

As a generalization of quasi-Einstein manifold, in [8], U. C. De and G. C. Ghosh defined the *generalized quasi-Einstein manifold.* A non-flat Riemannian manifold is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{3}$$

where a, b and c are non-zero scalars and A, B are two 1-forms such that

$$g(X, U) = A(X) \quad \text{and} \quad g(X, V) = B(X), \tag{4}$$

U and V being unit vectors which are orthogonal, i.e.,

$$g(U, V) = 0. \tag{5}$$

The vector fields U and V are called the generators of the manifold. This type of manifold will be denoted by $G(QE)_n$.

In [6], M. C. Chaki introduced the super quasi-Einstein manifold, denoted by $S(QE)_n$, where the Ricci tensor is not identically zero and satisfies the condition

$$\begin{aligned} S(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) \\ + A(Y)B(X)] + dD(X, Y), \end{aligned} \tag{6}$$

where a, b, c and d are scalars such that b, c, d are nonzero, A, B are two nonzero 1-forms defined as (4) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$D(X, U) = 0, \quad \forall X \in \chi(M). \tag{7}$$

Here a, b, c and d are called the *associated scalars*, A, B are called the *associated main and auxiliary 1-forms* respectively, U, V are called the main and the *auxiliary generators* and D is called the *associated tensor* of the manifold.

In [3], A. Bhattacharyya and T. De introduced the notion of *mixed generalized quasi-Einstein manifold*. A non-flat Riemannian manifold is called *mixed generalized quasi-Einstein manifold* if its Ricci tensor is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ + d[A(X)B(Y) + A(Y)B(X)], \tag{8}$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = A(X) \quad \text{and} \quad g(X,V) = B(X), \qquad \forall \;\; X \in \chi(M), \tag{9}$$

and also

$$g(U,V) = 0. \tag{10}$$

A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold becomes to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

In [4], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of $MS(QE)_n$.

A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is called *mixed super quasi-Einstein manifold* if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) \\ + A(Y)B(X)] + eD(X,Y), \tag{11}$$

where a, b, c, d, e are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and A, B are two non zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \qquad \forall \;\; X \in \chi(M), \tag{12}$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \qquad \forall \;\; X \in \chi(M). \tag{13}$$

Here a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and the auxiliary generators and D is called the associated tensor of the manifold. We denote this type of manifold $MS(QE)_n$.

The notation of warped product generalizes that of a surface of revolution. Warped products were first defined by O'Neill and Bishop in [5]. They used this concept to construct Riemannian manifolds with negative sectional curvature. Then Beem, Ehrlich and Powell pointed out that many exact solutions in Einstein's field equation can be expressed in terms of Lorentzian warped products [2].

In general, doubly warped product was studied by Btilent Unal in [12], can be considered as a generalization of singly warped product. A doubly warped product (M, g) is a product manifold which is of the form $M =_f B \times_b F$ with the metric $g = f^2_{g_B} \oplus b^2_{g_F}$ where $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ are smooth map.

So if (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and also $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ be smooth functions, then the doubly warped product is the product manifold $B \times F$ furnished with the metric tensor $f^2_{g_B} \oplus b^2_{g_F}$ defined by

$$g = (f \circ \sigma)^2 \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F).$$

The functions $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ are called warping functions and $\pi : B \times F \longrightarrow B$ and $\sigma : B \times F \longrightarrow F$ are usual projections map.

If (F, g_F) and (B, g_B) are both Riemannian manifolds, then $({}_f B \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ is also a Riemannian manifold. We call $({}_f B \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ a Loretzian doubly warped product if (F, g_F) is Riemannian and either (B, g_B) is Lorentzian or else (B, g_B) is a one-dimensional manifold with a negative definite metric $-dt^2$. If neither b nor f is constant, then we have a proper doubly warped product.

Global hyperbolicity is the most important condition on Causality, which lies at the top of the so-called causal hierarchy of spacetimes and is involved in problems as Cosmic Censorship, predictability etc.

A connected Lorentzian manifold is called time-orientable iff it admits a nowhere-vanishing timelike vector field (defining future causal directions). A piecewise C^1 curve $c : I \longrightarrow M$ in a time-oriented manifold (M, g) is called future iff $c'(t)$ is future for every $t \in I$. For any point $p \in M$, the future of p (resp. past of p), denoted by $J^+(p)$ (resp. $J^-(p)$), is the set of all points q s.t.there is a future curve from p to q (resp. from q to p).

There are different alternative definitions of what global hyperbolicity means, but perhaps the most standard one is the following. A spacetime (M, g) is said globally hyperbolic if and only if it satisfies two conditions: (A) compactness of $J^+(p) \cap J^-(q)$ for all $p, q \in M$ (i.e. no "naked" singularity can exist) and (B) strong causality (no "almost closed" causal curve exists).

Global hyperbolicity is also discussed in the theorem (34), (36) and the corollary (36) in [13].

In this paper we find that a Riemannian manifold is a manifold of mixed super quasi constant curvature iff it is conformally flat $MS(QE)_n$. Also we have studied Ricci-pseudosymmetric $MS(QE)_n$. Next we have obtained some expressions for Riemannian curvature tensor when $MS(QE)_n$ satisfies the curvature conditions $C.S = 0$, $\tilde{C}.S = 0$ and $C_1.S = 0$, where C is the Weyl conformal curvature tensor, $\tilde{C}$ is the concircular curvature tensor and C_1 quasi-conformal curvature tensor. We have also proved that in a conformally flat $MS(QE)_n$ $(n \geq 3)$ with $R(X, Y).S = 0$, the vector fields U, V corresponding to 1-forms A, B are co-directional. Finally in the last two sections, we discuss about the doubly warped product on $MS(QE)_n$ and completeness of doubly warped products on $MS(QE)_4$ with examples.

2 Preliminaries

In this section we consider $MS(QE)_n$, $(n \geq 3)$ with associated scalars a, b, c, d, e, associated main and auxiliary 1-forms A, B, main and auxiliary generators U, V and associated symmetric $(0, 2)$ tensor D.

So (11), (12) and (13) will hold. Since U and V are mutually orthogonal unit vector fields, we have

$$g(U, U) = 1, \quad g(V, V) = 1 \quad \text{and} \quad g(U, V) = 0, \tag{14}$$

$$\operatorname{trace} D = 0 \tag{15}$$

$$D(X, U) = 0, \quad \forall\, X \in \chi(M). \tag{16}$$

Also using (14) in (12), we get

$$A(V) = B(U) = 0. \tag{17}$$

Now setting $X = Y = e_i$, where $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, in (11) and taking summation over i, $1 \leq i \leq n$, we obtain

$$r = na + b + c, \tag{18}$$

where r is the scalar curvature of the manifold.
Also, from (11), we have

$$S(U, U) = a + b \tag{19}$$

$$S(V, V) = a + c + eD(V, V) \tag{20}$$

$$S(U, V) = d. \tag{21}$$

If X is a unit vector field, then $S(X,X)$ is the Ricci-curvature in the direction of X. Hence from (19) and (20) we can state that $a+b$ and $a+c+eD(V,V)$ are the Ricci curvature in the directions of U and V respectively.
Let Q be the Ricci operator, i.e.,

$$g(QX,Y) = S(X,Y) \quad \forall\, X,Y \in \chi(M). \tag{22}$$

Also we have

$$g(lX,Y) = D(X,Y). \tag{23}$$

Another notion of curvature called mixed super quasi constant curvature was introduced in [4]. A Riemannian manifold is said to be a manifold of mixed super quasi-constant curvature if it is conformally flat and the curvature tensor R of type $(0,4)$ satisfies the condition

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = {} & m[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + p[g(X,W)A(Y)\\ & A(Z) - g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)\\ & A(Y)A(W)] + q[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)\\ & + g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] + s[\{A(Y)B(Z)\\ & + B(Y)A(Z)\}g(X,W) - \{A(X)B(Z) + B(X)\\ & A(Z)\}g(Y,W) + \{A(X)B(W) + B(X)A(W)\}g(Y,Z)\\ & - \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)] + t[g(Y,Z)D(X,W)\\ & - g(X,Z)D(Y,W) + g(X,W)D(Y,Z) - g(Y,W)D(X,Z)].\end{aligned} \tag{24}$$

An n-dimensional *Riemannian manifold* (M^n, g) is called *Ricci-pseudosymmetric* [9] if the tensors $R.S$ and $Q(g,S)$ are *linearly dependent*, where

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{25}$$

$$Q(g,S)(Z,W;X,Y) = -S((X\wedge Y)Z,W) - S(Z,(X\wedge Y)W) \tag{26}$$

and

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$$

for vector fields X,Y,Z,W on M^n, R denotes the curvature tensor of M^n. The condition of *Ricci-pseudosymmetry* is equivalent to the relation

$$(R(X,Y).S)(Z,W) = L_S Q(g,S)(Z,W;X,Y) \tag{27}$$

holding on the set

$$U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}, \tag{28}$$

where L_S is some function on U_S. If $R.S = 0$ then M^n is called *Ricci-semi-symmetric.* Every *Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true* [9].
The Weyl conformal curvature tensor C of type $(1,3)$ of an n-dimensional *Riemannian manifold* (M^n, g), $(n \geq 3)$ is defined by [15]

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{29}$$

The *concircular curvature tensor* $\tilde{C}$ of type $(1,3)$ of n-dimentional *Riemanian manifold* (M^n, g),$(n \geq 3)$ is defined by [15]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{30}$$

for any vector fields $X, Y, Z \in \chi(M)$.
The *quasi-conformal curvature tensor* was defined by Yano and Sawaki [14] as

$$C_1(X,Y)Z = \lambda R(X,Y)Z + \mu\{S(Y,Z)X + S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} - \frac{r}{n}\left[\frac{\lambda}{(n-1)} + 2\mu\right][g(Y,Z)X - g(X,Z)Y], \tag{31}$$

where λ and μ are nonzero constants. If $\lambda = 1$ and $\mu = \frac{1}{n-2}$, then quasi-conformal curvature tensor is reduced to the conformal curvature tensor.

3 Relation between manifold of mixed super quasi constant curvature and $MS(QE)_n$

Let M be a Riemannian manifold with mixed super quasi constant curvature and $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Taking $X = W = \{e_i\}$ and summing over i, $1 \leq i \leq n$ in (24) and using (23), we obtain

$$S(Y,Z) = m(n-2)g(Y,Z) + p(n-2)A(Y)A(Z) + q(n-2)B(Y)B(Z) + s(n-2)[A(Y)B(Z) + A(Z)B(Y)] + t(n-2)D(Y,Z), \tag{32}$$

which imply

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y), \tag{33}$$

where $a = m(n-2), b = p(n-2), c = q(n-2), d = s(n-2), e = t(n-2)$. So, (M^n, g) is a $MS(QE)_n$.
Conversely, suppose (M^n, g) is conformally flat $MS(QE)_n$. Then

$$\begin{aligned} R(X,Y)Z = &\frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\} \\ &- \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{34}$$

Now using (11), (18) and (19), we get

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = &[g(Y,Z)g(X,W) \\ &- g(X,Z)g(Y,W)]\left\{\frac{2a}{n-2} - \frac{na+b+c}{(n-1)(n-2)}\right\} \\ &+ [g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &+ g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]\left\{\frac{b}{n-2}\right\} \\ &+ [g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &+ g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)]\left\{\frac{c}{n-2}\right\} \\ &+ [\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) - \{A(X)B(Z) \\ &+ B(X)A(Z)\}g(Y,W) + \{A(X)B(W) + B(X)A(W)\}g(Y,Z) \\ &- \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)]\left\{\frac{d}{n-2}\right\} \\ &+ [g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ &+ g(X,W)D(Y,Z) - g(Y,W)D(X,Z)]\left\{\frac{e}{n-2}\right\}. \end{aligned} \tag{35}$$

So,

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = &m_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ p_1[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &+ g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ &+ q_1[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &+ g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] \\ &+ s_1[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) \\ &- \{A(X)B(Z) + B(X)A(Z)\}g(Y,W) + \{A(X)B(W) \\ &+ B(X)A(W)\}g(Y,Z) - \{A(Y)B(W) \end{aligned} \tag{36}$$

$$+ B(Y)A(W)\}g(X,Z)] + t[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) + g(X,W)D(Y,Z) - g(Y,W)D(X,Z)],$$

where, $m = \frac{a(n-2)-b-c}{(n-1)(n-2)}$, $p = \frac{b}{n-2}$, $q = \frac{c}{n-2}$, $s = \frac{d}{n-2}$, $t = \frac{e}{n-2}$.
So, (M^n, g) is a manifold of mixed super quasi constant curvature.
Then we have the following theorem:

Theorem 1 *A Riemannian manifold is a manifold of mixed super quasi constant curvature iff it is conformally flat* $MS(QE)_n$.

4 Ricci-pseudosymmetric $MS(QE)_n$

In this section we consider a *Ricci-pseudosymmetric* $MS(QE)_n$ and prove the following theorem:

Theorem 2 *Let* (M^n, g), $(n \geq 3)$, *be a* $MS(QE)_n$. *If* M^n *is Ricci-pseudosym metric then the following conditions hold on* M^n :

$$R(X, Y, U, V) = L_S\{A(Y)B(X) - A(X)B(Y)\} \tag{37}$$

$$D(R(X,Y)U, V) = L_S\{A(Y)D(X,V) - A(X)D(Y,V)\} \tag{38}$$

$$D(R(X,Y)V, V) = L_S\{B(Y)D(X,V) - B(X)D(Y,V)\} \tag{39}$$

for all vector fields X, Y *on* M^n, *where* U, V *are the generators of the manifold* M^n.

Proof. Assume that M^n is *Ricci-pseudosymmetric*. Then by the use of (25) to (28), we can obtain

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_S\{g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)\}. \tag{40}$$

Since M_n is also $MS(QE)_n$, using the well-known properties of the curvature tensor R we get

$$\begin{aligned} b[A(R(X,Y)Z)A(W) + A(Z)A(R(X,Y)W)] + c[B(R(X,Y)Z)B(W) \\ + B(Z)B(R(X,Y)W)] + d[A(R(X,Y)Z)B(W) + A(W)B(R(X,Y)Z) \\ + A(Z)B(R(X,Y)W) + A(R(X,Y)W)B(Z)] + e[D(R(X,Y)Z,W) \\ + D(Z,R(X,Y)W)] = L_S\{b[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ + g(Y,W)A(X)A(Z) - g(X,W)A(Y)A(Z)] + c[g(Y,Z)B(X)B(W) \\ - g(X,Z)B(Y)B(W) + g(Y,W)B(X)B(Z) - g(X,W)B(Y)B(Z)] \end{aligned} \tag{41}$$

$$+ d[g(Y,Z)A(X)B(W) + g(Y,Z)A(W)B(X) - g(X,Z)A(Y)B(W) - g(X,Z)A(W)B(Y) + g(Y,W)A(X)B(Z) + g(Y,W)A(Z)B(X) - g(X,W)A(Y)B(Z) - g(X,W)A(Z)B(Y)] + e[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) + g(Y,W)D(X,Z) - g(X,W)D(Y,Z)]\}.$$

Putting $Z = U$ and $W = V$ in (41), we get

$$\begin{aligned} b[R(X,Y,V,U) - L_S\{A(X)B(Y) - A(Y)B(X)\}] + c[R(X,Y,U,V) \\ - L_S\{A(Y)B(X) - A(X)B(Y)\}] + e[D(R(X,Y)U,V) \\ - L_S\{A(Y)D(X,V) - A(X)D(Y,V)\}] = 0. \end{aligned} \tag{42}$$

Taking $Z = W = U$ in (41), we get

$$d[R(X,Y,U,V) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0.$$

Since $d \neq 0$, we get

$$R(X,Y,U,V) - L_S\{A(Y)B(X) - A(X)B(Y)\} = 0. \tag{43}$$

Which gives (37). Similarly, if we take $Z = W = V$ in (41), we get

$$\begin{aligned} d[R(X,Y,V,U) - L_S\{A(X)B(Y) - A(Y)B(X)\}] + e[D(R(X,Y)V,V) \\ - L_S\{B(Y)D(X,V) - B(X)D(Y,V)\}] = 0. \end{aligned} \tag{44}$$

From (42) and (43), we get

$$e[D(R(X,Y)U,V) - L_S\{A(Y)D(X,V) - A(X)D(Y,V)\}] = 0.$$

Since $e \neq 0$,

$$D(R(X,Y)U,V) - L_S\{A(Y)D(X,V) - A(X)D(Y,V)\} = 0.$$

Which gives (38).
Again from (43) and (44), we obtain (39). So our theorem is proved. □

5 $MS(QE)_n$ satisfying the condition $C.S = 0$

In this section we consider a $MS(QE)_n)(n \geq 3)$ satisfying the condition $C.S = 0$. Then we have

$$S(C(X,Y)Z,W) + S(Z,C(X,Y)W) = 0. \tag{45}$$

Now using (11) in (45), we get,

$$\begin{aligned}&ag(C(X,Y)Z,W)+bA(C(X,Y)Z)A(W)+cB(C(X,Y)Z)B(W)\\&d[A(C(X,Y)Z)B(W)+B(C(X,Y)Z)A(W)]+eD(C(X,Y)Z,W)\\&ag(Z,C(X,Y)W)+bA(Z)A(C(X,Y)W)+cB(Z)B(C(X,Y)W)\\&d[A(Z)B(C(X,Y)W)+B(Z)A(C(X,Y)W)]+eD(Z,C(X,Y)W)=0.\end{aligned} \tag{46}$$

From (46),

$$\begin{aligned}&b[A(C(X,Y)Z)A(W)+A(Z)A(C(X,Y)W)]+c[B(C(X,Y)Z)B(W)\\&\quad+B(Z)B(C(X,Y)W)]+d[A(C(X,Y)Z)B(W)+B(C(X,Y)Z)A(W)\\&\quad+A(Z)B(C(X,Y)W)+B(Z)A(C(X,Y)W)]+e[D(C(X,Y)Z,W)\\&\quad+D(Z,C(X,Y)W)]=0.\end{aligned} \tag{47}$$

Putting $Z=W=U$ in (47), we get

$$2b[A(C(X,Y)U]+2d[B(C(X,Y)U]=0. \tag{48}$$

So, we obtain

$$2d[B(C(X,Y)U]=0$$

As $d\neq 0$, we get,

$$B(C(X,Y)U=0. \tag{49}$$

That is

$$C(X,Y,U,V)=0. \tag{50}$$

So, from (29), we obtain

$$\begin{aligned}R(X,Y,U,V)=&\frac{1}{n-2}[A(QY)B(X)-A(X)B(QY)+A(Y)B(QX)\\&-A(QX)B(Y)]-\frac{r}{(n-1)(n-2)}\{A(Y)B(X)-A(X)B(Y)\}\end{aligned} \tag{51}$$

So, we can state that

Theorem 3 *In a* $MS(QE)_n$ $(n\geq 3)$ *with* $C.S=0$*, the curvature tensor* R *of the manifold satisfies the relation* (51)*.*

6 $MS(QE)_n$ satisfying the condition $\tilde{C}.S = 0$

In this section we consider a $MS(QE)_n$ ($n \geq 3$) satisfying the condition $\tilde{C}.S = 0$.Then we have,

$$S(\tilde{C}(X, Y)Z, W) + S(Z, \tilde{C}(X, Y)W) = 0. \tag{52}$$

From (11) in (52), we get,

$$\begin{aligned}
&ag(\tilde{C}(X, Y)Z, W) + bA(\tilde{C}(X, Y)Z)A(W) + cB(\tilde{C}(X, Y)Z)B(W)\\
&d[A(\tilde{C}(X, Y)Z)B(W) + B(\tilde{C}(X, Y)Z)A(W)] + eD(\tilde{C}(X, Y)Z, W)\\
&ag(Z, \tilde{C}(X, Y)W) + bA(Z)A(\tilde{C}(X, Y)W) + cB(Z)B(\tilde{C}(X, Y)W)\\
&d[A(Z)B(\tilde{C}(X, Y)W) + B(Z)A(\tilde{C}(X, Y)W)] + eD(Z, \tilde{C}(X, Y)W) = 0.
\end{aligned} \tag{53}$$

Putting $Z = W = U$ in (53), we get

$$d[B(C(X, Y)U] = 0.$$

As $d \neq 0$,

$$B(\tilde{C}(X, Y)U = 0. \tag{54}$$

That is,

$$\tilde{C}(X, Y, U, V) = 0. \tag{55}$$

So, from (30), we get

$$R(X, Y, U, V) = \frac{r}{n(n-1)}[A(Y)B(X) - A(X)B(Y)]. \tag{56}$$

Thus, we have

Theorem 4 *In a $MS(QE)_n$ ($n \geq 3$) with $\tilde{C}.S = 0$, the curvature tensor R of the manifold satisfies the relation (56).*

7 $MS(QE)_n$ satisfying the condition $C_1.S = 0$

In this section we consider a $MS(QE)_n$ ($n \geq 3$) satisfying the condition $C_1.S = 0$. Then we have,

$$S(C_1(X, Y)Z, W) + S(Z, C_1(X, Y)W) = 0 \tag{57}$$

for any vector fields $X, Y, Z, W \in \chi(M)$. Then we have the following theorem:

Theorem 5 *Let* (M^n, g) $(n \geq 3)$ *be a* $MS(QE)_n$*. If the condition* $C_1.S = 0$ *holds on* M^n *then the curvature tensor R of* M^n *satisfies the following property:*

$$\lambda R(X, Y, U, V) = \left[\frac{na+b+c}{n}\left(\frac{\lambda}{n-1} + 2\mu\right) - \mu(2a+b+c)\right]\{A(Y)B(X) - A(X)B(X)\} - \mu e\{D(X, V)A(Y) - D(Y, V)A(X)\} \tag{58}$$

for all vector fields X, Y *on* M^n*, where* U, V *are the generators of the manifold* M^n*.*

Proof. Since, $C_1.S = 0$ holds on M^n we have,

$$S(C_1(X, Y)Z, W) + S(Z, C_1(X, Y)W) = 0.$$

Since M^n be a $MS(QE)_n$, using (11) in (57), we obtain

$$\begin{aligned} &ag(C_1(X, Y)Z, W) + bA(C_1(X, Y)Z)A(W) + cB(C_1(X, Y)Z)B(W) \\ &d[A(C_1(X, Y)Z)B(W) + B(C_1(X, Y)Z)A(W)] + eD(C_1(X, Y)Z, W) \\ &ag(Z, C_1(X, Y)W) + bA(Z)A(C_1(X, Y)W) + cB(Z)B(C_1(X, Y)W) \\ &d[A(Z)B(C_1(X, Y)W) + B(Z)A(C_1(X, Y)W)] + eD(Z, C_1(X, Y)W) = 0. \end{aligned} \tag{59}$$

From (59),

$$\begin{aligned} &b[A(C_1(X, Y)Z)A(W) + A(Z)A(C_1(X, Y)W)] + c[B(C_1(X, Y)Z)B(W) \\ &+ B(Z)B(C_1(X, Y)W)] + d[A(C_1(X, Y)Z)B(W) + B(C_1(X, Y)Z)A(W) \\ &+ A(Z)B(C_1(X, Y)W) + B(Z)A(C_1(X, Y)W)] + e[D(C_1(X, Y)Z, W) \\ &+ D(Z, C_1(X, Y)W)] = 0. \end{aligned} \tag{60}$$

Putting $Z = W = U$ in (60), we get

$$2b[A(C_1(X, Y)U] + 2d[B(C_1(X, Y)U] = 0. \tag{61}$$

So, we obtain

$$2d[B(C_1(X, Y)U] = 0.$$

As $d \neq 0$, we get

$$B(C_1(X, Y)U = 0. \tag{62}$$

That is

$$C_1(X, Y, U, V) = 0.$$

Now using (31), we obtain

$$\lambda R(X,Y,U,V) = \mu\{S(X,U)g(Y,V) - S(Y,U)g(X,V) - g(Y,U)S(X,V) + g(X,U)S(Y,V)\} + \frac{r}{n}\left[\frac{\lambda}{(n-1)} + 2\mu\right][g(Y,U)g(X,V) - g(X,U)g(Y,V)]. \tag{63}$$

Using (11) and (18) in (63), we get,

$$\lambda R(X,Y,U,V) = \left[\frac{na+b+c}{n}\left(\frac{\lambda}{n-1} + 2\mu\right) - \mu(2a+b+c)\right]\{A(Y)B(X) - A(X)B(X)\} - \mu e\{D(X,V)A(Y) - D(Y,V)A(X)\}.$$

Hence the proof. □

8 Conformally flat $MS(QE)_n$ $(n \geq 3)$ with $R(X,Y).S = 0$

Let us consider a *conformally flat* $MS(QE)_n$ $(n \geq 3)$. Then, from (29), we get

$$R(X,Y)Z = \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{64}$$

Since the manifold satisfies $R(X,Y).S = 0$, we get

$$S(R(X,Y)Z,W) + S(Z,C(X,Y)W) = 0. \tag{65}$$

Using (64) in (65) we obtain

$$g(Y,Z)S(QX,W) - g(X,Z)S(QY,W) + g(Y,W)S(QX,Z) - g(X,W)S(QY,Z) = \frac{r}{n-1}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)]. \tag{66}$$

Let λ be the eigen value of Q corresponding to the eigen vector X. Then $QX = \lambda X$, i.e., $S(X,W) = \lambda g(X,W)$ (where the manifold is not Einstein) and hence

$$S(QX,W) = \lambda S(X,W). \tag{67}$$

Now using (67) in (66) we get,

$$\left(\lambda - \frac{r}{n-1}\right)[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z) = 0.$$

Which gives

$$g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z) = 0, \quad (68)$$

provided $\lambda - \frac{r}{n-1} \neq 0$. Now using (11) in (68) we get

$$\begin{aligned} &g(Y,Z)[ag(X,W) + bA(X)A(W) + cB(X)B(W) + d\{A(X)B(W) \\ &\quad + B(X)A(W)\} + eD(X,W)] - g(X,Z)[ag(Y,W) + bA(Y)A(W) \\ &\quad + cB(Y)B(W) + d\{A(Y)B(W) + B(Y)A(W)\} + eD(Y,W)] + g(Y,W) \\ &\quad [ag(X,Z) + bA(X)A(Z) + cB(X)B(Z) + d\{A(X)B(Z) + B(X)A(Z)\} \\ &\quad + eD(X,Z)] - g(X,W)[ag(Y,Z) + bA(Y)A(Z) + cB(Y)B(Z) \\ &\quad + d\{A(Y)B(Z) + B(Y)A(Z)\} + eD(Y,Z)] = 0. \end{aligned} \quad (69)$$

Now putting $Z = W = U$ in (69), we obtain,

$$2d[A(Y)B(X) - B(Y)A(X)] = 0.$$

As $d \neq 0$, so

$$A(Y)B(X) - B(Y)A(X) = 0, \quad (70)$$

that is, the vector fields U and V are co-directional. Thus we can state the following:

Theorem 6 *If, in a conformally flat Ricci-semisymmetric* $MS(QE)_n$ $(n \geq 3)$ $\frac{r}{n-1}$ *is not an eigenvalue of the Ricci-operator* Q*, the vector fields* U *and* V *corresponding to the 1-forms* A *and* B *respectively are co-directional.*

9 Example of doubly warped product on $MS(QE)_n$

In [10], B. Pal, A. Bhattacharyya and M. Tarafdar defined warped product on $MS(QE)_4$. Here, we define doubly warped product on four dimensional $MS(QE)_n$. Let (M^4, g) be a 4-dimensional Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2], \quad (71)$$

where $p > 0$ is a smooth function and $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4.
In [11], A. A. Shaikh and S. K. Hui have shown that (71) becomes $G(QE)_n$. As it is non-Einstein metric, so one can easily show that (71) is $MS(QE)_n$.

We know that $({}_fB \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ is a Lorentzian doubly warped product if (F, g_F) is Riemannian and either (B, g_B) is Lorentzian or else (B, g_B) is a one-dimensional manifold with a negative definite metric $-dt^2$. To define Lorentzian doubly warped product on$MS(QE)_n$, we take the line element on $R \times R^3$ where we consider R is the B and R^3 is the F. If we consider the above example, we have the metric g_F , where (F, g_F) is Riemannian and the metric g_B, where (B, g_B) is a one-dimensional manifold with a negative definite metric $ds^2_B = -(dx^4)^2$. Here, the metric g_F on R^3 is

$$ds^2_F = \frac{1}{1+2p}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

and the warping function

$$f : R^3 \longrightarrow (0, \infty)$$

is defined by

$$f(x^1, x^2, x^3) = \sqrt{(1+2p)}$$

and the other warping function is

$$b : R \longrightarrow (0, \infty),$$

which is defined by

$$b(x^4) = (1+2p).$$

Here, we see that the warping functions $f = \sqrt{(1+2p)} > 0$ and $b = (1+2p) > 0$, both are also smooth functions. Therefore the metric

$$ds^2_M = f^2 ds^2_B + b^2 ds^2_F$$

which is

$$ds^2 = g_{ij}dx^i dx^j = -(1+2p)(dx^4)^2 + (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2].$$

This is the example of Lorentzian doubly warped product on $MS(QE)_4$.

Next we consider the another example. Let (M^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{72}$$

where $0 < x^1 < \frac{\pi}{2}$ but $x^1 \neq \frac{\pi}{4}$ and $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it can be easily shown that it is a mixed super quasi-Einstein manifold with non-vanishing scalar curvature.

We know that $({}_f B \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ is a Riemannian doubly warped product if (F, g_F) and (B, g_B) are both Riemannian manifolds. To define Riemannian doubly warped product on $MS(QE)_4$, we take the line element on $L^2 \times L^2$, where $B = F = L^2 = R \times R$. If we consider the example (72), we have the metric g_B, where (B, g_B) is Riemannian and the metric g_F , where (F, g_F) is also Riemannian with metrices

$$ds_B^2 = (dx^1)^2 + \frac{1}{e^{2x^1}} \sin^2 x^1 (dx^2)^2,$$

$$ds_F^2 = \frac{1}{\sin^2 x^1}[(dx^3)^2 + (dx^4)^2]$$

and the warping function

$$f : L^2 \longrightarrow (0, \infty)$$

is defined by

$$f(x^1, x^2) = \sqrt{e^{2x^1}}$$

and the other warping function is

$$b : L^2 \longrightarrow (0, \infty),$$

which is defined by

$$b(x^3, x^4) = \sin^2 x^1.$$

Here, we see that the warping functions $f = \sqrt{e^{2x^1}} > 0$ and $b = \sin^2 x^1 > 0$ both are also smooth functions. Therefore the metric

$$ds_M^2 = f^2 ds_B^2 + b^2 ds_F^2$$

which is

$$ds_M^2 = e^{2x^1}\left[(dx^1)^2 + \frac{1}{\sin^2 x^1}(dx^2)^2\right] + \sin^4 x^1 \left[\frac{1}{\sin^2 x^1}(dx^3)^2 + \frac{1}{\sin^2 x^1}(dx^4)^2\right],$$

is the example of Riemannian doubly warped product on $MS(QE)_4$.

10 Completeness of doubly warped products on $MS(QE)_4$

In this section, we obtain some results on completeness properties of Riemannian doubly warped products and Lorentzian doubly warped products on $MS(QE)_4$.

The Riemannian case

In this subsection, we state some results about completeness of Riemannian doubly warped products. Here we want to investigate about the completeness properties of Riemannian doubly warped products with respect to the example (72), which is $MS(QE)_4$. Now it is clear that $\inf(f) > 0$ and $\inf(b) > 0$ and $B = F = L^2 = R \times R$. Therefore (B, g_B) and (F, g_F) are complete Riemannian manifolds. Hence by proposition (32) of [12], we can state that

Example 1 *Let* $M = B \times F$ *be a Riemannian doubly warped product on* $MS(QE)_4$ *endowed with the metric given by*

$$ds_M^2 = e^{2x^1}\left[(dx^1)^2 + \frac{1}{\sin^2 x^1}(dx^2)^2\right] + \sin^4 x^1\left[\frac{1}{\sin^2 x^1}(dx^3)^2 + \frac{1}{\sin^2 x^1}(dx^4)^2\right],$$

where, x^1, x^2, x^3, x^4 *are the standard coordinates of* M^4*. Then* (M^4, g) *is a complete Riemannian manifold.*

Here we want to discuss about global hyperbolicity of mixed super quasi-Einstein space-time with doubly warped product fibers by using [1]. Let us consider the example. Let (M_4, g) be a Riemannian manifold endowed with the metric given by

$$ds_M^2 = -(dx^4)^2 + x^1\left[(x^3)^4\{(dx^1)^2\} + \frac{2d}{(x^3)^4}\left\{(dx^2)^2 + \frac{(x^3)^4}{2dx^1}(dx^3)^2\right\}\right],$$

where, x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it can be easily shown that it is a mixed super quasi-Einstein manifold with non vanishing scalar curvature. Now this manifold is of the form

$$M = (c, d) \times_h (B_f \times_b F),$$

a Lorentzian singly warped product with the metric

$$g = -(dx^4)^2 \oplus h^2(f^2 g_B + b^2 g_F),$$

where $-\infty \leq c \leq d \leq \infty$,

$$h : (c, d) \longrightarrow (0, \infty)$$

is defined by $h = \sqrt{x^1}$, which is strictly positive and smooth. Also (B, g_B) and (F, g_F) are complete Riemannian manifolds and $\inf(b)$ that is $\inf(\frac{\sqrt{2d}}{(x^3)^2}) > 0$ or $\inf(f)$ that is $\inf((x^3)^2) > 0$. Then we have the following:

Example 2 *Let* $M = (c,d) \times_h (B_f \times_b F)$, *be a Lorentzian singly warped product on* $MS(QE)_4$ *endowed with the metric given by*

$$ds_M^2 = -(dx^4)^2 + x^1 \left[(x^3)^4 \{(dx^1)^2\} + \frac{2d}{(x^3)^4} \left\{ (dx^2)^2 + \frac{(x^3)^4}{2dx^1} (dx^3)^2 \right\} \right],$$

where, x^1, x^2, x^3, x^4 *are the standard coordinates of* M^4. *Then* (M^4, g) *is globally hyperbolic.*

Lorentzian case

We now consider the nonspacelike geodesic completeness of Lorentzian warped products of the form

$$M =_f (c,d) \times_b F$$

with the metric

$$g = f^2 dt^2 \oplus b^2 g_F,$$

where $-\infty \le c \le d \le \infty$. Here a space-time is said to be null (respectively, timelike) geodesically incomplete if some future directed null (respectively, timelike) geodesic can not be extended for arbitrary negative and positive values of an affine parameter. Let us consider (M^4, g) be a 4-dimensional Lorentzian manifold endowed with the metric given by

$$ds^2 = e^{2x^1}(dx^1)^2 + (\sin^2)x^1[(dx^2)^2 + (dx^3)^2 - (dx^4)^2],$$

where, x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it is clear that it is mixed super quasi-Einstein manifold with non vanishing scalar curvature. Now, this metric can be written as

$$ds^2 = e^{2x^1}\left[(dx^1)^2 + \frac{1}{\sin^2 x^1}(dx^2)^2\right] + \sin^4 x^1 \left[\frac{1}{\sin^2 x^1}(dx^3)^2 - \frac{1}{\sin^2 x^1}(dx^4)^2\right].$$

Take $B = F = L^2 = R \times R$ and define

$$f : L^2 \longrightarrow (0, \infty)$$

is defined by

$$f(x^1, x^2) = \sqrt{e^{2x^1}}$$

and the another function

$$b : L^2 \longrightarrow (0, \infty)$$

is defined by

$$b(x^3, x^4) = \sin^2 x^1.$$

Let us define

$$\alpha : (-\infty, \infty) \longrightarrow B$$

is defined by

$$\alpha(t) = (t, t)$$

and

$$\beta : (-\infty, \infty) \longrightarrow B$$

is defined by

$$\beta(t) = (t, t).$$

Clearly, α and β are complete null geodesics of B and F. Also, if $\gamma = (\alpha, \beta)$ then it is a null pre-geodesic in M and $\gamma'' = \gamma$ by equation in proposition 2.3 in [12]. Now using the example (3.8) in [12], we get γ is incomplete. Then we can state

Example 3 *If* (B, g_B) *and* (F, g_F) *are null complete pseudo-Riemannian manifolds then* $M =_f B \times_b f$ *is not a null complete pseudo-Riemannian doubly warped product with the metric* $g_M = f^2 g_B \oplus b^2 g_F$ *on* $MS(QE)_4$.

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Pseudo-slant submanifolds in cosymplectic space forms

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Abstract. In this paper, we study the geometry of the pseudo-slant submanifolds of a cosymplectic space form. Necessary and sufficient conditions are given for a submanifold to be a pseudo-slant submanifold, pseudo-slant product, mixed geodesic and totally geodesic in cosymplectic manifolds. Finally, we give some results for totally umbilical pseudo-slant submanifold in a cosymplectic manifold and cosymplectic space form.

1 Introduction

The differential geometry of slant submanifolds has shown an increasing development since B. Y. Chen [3, 4] defined slant submanifolds in complex manifolds as a natural generalization of both the invariant and anti-invariant submanifolds. Many research articles have been appeared on the existence of these submanifolds in different knows spaces. The slant submanifols of an almost contact metric manifolds were defined and studied by A. Lotta [2]. After, this type submanifolds were studied by J.L Cabrerizo et. al [7] of Sasakian manifolds. Recently, in [8] M. Atçeken studied slant and pseudo-slant submanifold

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in $(LCS)_n$-manifolds. The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by N. Papagiuc [12]. Recently, A. Carriazo [1] defined and studied bi-slant immersions in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifolds in almost Hermitian manifolds. The contact version of pseudo-slant submanifolds has been defined and studied by V. A. Khan and M. A. Khan [15].

In this paper, we study pseudo-slant submanifolds of a cosymplectic manifold. In section 2, we review basic formulas and definitions for a cosymplectic manifold and their submanifolds. In section 3, we recall the definition and some basic results of a pseudo-slant submanifold of an almost contact metric manifold. In section 4, we give some new results for totally umbilical pseudo-slant submanifold in a cosymplectic manifold $\widetilde{M}$ and cosymplectic space form $\widetilde{M}(c)$.

2 Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary fact and formulas from the theory of Cosymplectic manifolds and their submanifols.

Let $\widetilde{M}$ be a $(2m+1)$-dimensional almost contact metric manifold together with a metric tensor g, a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η on $\widetilde{M}$ which satisfy

$$\phi^2X = -X + \eta(X)\xi, \tag{1}$$

$$\phi\xi = 0,\ \eta(\phi X) = 0,\ \eta(\xi) = 1,\ \eta(X) = g(X,\xi) \tag{2}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y) \tag{3}$$

for any vector fields X, Y on $\widetilde{M}$. An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\widetilde{M}\times R$ given by

$$J(X,\ f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is the $C^\infty-$ function on $\widetilde{M}\times R$. The condition for normality in terms of ϕ, ξ and η is $[\phi,\phi] + 2d\eta\otimes\xi = 0$ on $\widetilde{M}$, where $[\phi,\phi](X,Y) = \phi^2 [X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ is the Nijenhuis tensor of ϕ. Finally the fundamental $2-$form Φ is defined by $\Phi(X,Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be cosymplectic, if it is normal and both Φ and are η closed. So we have on a cosymplectic manifold $\widetilde{M}$

$$(\widetilde{\nabla}_X \phi)Y = 0 \tag{4}$$

for any vector fields X, Y on $\widetilde{M}$. (4) implies that

$$\widetilde{\nabla}_X \xi = 0 \tag{5}$$

for any $X \in \Gamma(T\widetilde{M})$, that is ξ is a killing vector field.

Let $\widetilde{R}$ be the curvature tensor of the connection $\widetilde{\nabla}$. The sectional curvature of a ϕ- sectional is called a ϕ- sectional curvature. A cosymplectic manifold with constant ϕ- sectional curvature c is said to be a cosymplectic space form and it is denoted by $\widetilde{M}(c)$. The curvature tensor $\widetilde{R}$ of a cosymplectic space form $\widetilde{M}(c)$ is given by

$$\begin{aligned}\widetilde{R}(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(\phi Y, Z)\phi X \\ &+ g(X, \phi Z)\phi Y + 2g(X, \phi Y)\phi Z\}\end{aligned} \tag{6}$$

for any vector fields X, Y, Z tangent to $\widetilde{M}$[13].

Now, let M be an isometrical immersed submanifold of a contact metric manifold $\widetilde{M}$ and denote by the same symbol g the Riemanian metric induced on M. Let $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ be the diferential vector fields set tangent and normal to M, respectively. Also we denote by ∇ and $\nabla^{\perp}$ induced connections on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{7}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are the second fundamental form and shape operator for the immersed of M into $\widetilde{M}$, respectively. They are related as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{9}$$

We put

$$h_{ij}^r = g(h(e_i, e_j), e_r) \quad \text{and} \quad \|h\|^2 \ = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) \ ,$$

where, $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\Gamma(TM)$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ is also orthonormal basis of $\Gamma(T^{\perp}M)$.

The mean curvature vector H of M is given by

$$H = \frac{1}{n}\mathrm{trace}(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{10}$$

A submanifold M of an contact metric manifold $\widetilde{M}$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H, \tag{11}$$

where H is the mean curvature vector. A submanifold M is said to be totally geodesic submanifold if $h(X, Y) = 0$, for each $X, Y \in \Gamma(TM)$ and M is said to be minimal submanifold if $H = 0$.

For any submanifold M of a Riemannian manifold $\widetilde{M}$, the equation of Gauss is given by

$$\widetilde{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \tag{12}$$

where $\widetilde{R}$ and R denote the Riemannian curvature tensor of $\widetilde{M}$ and M, respectively, where the covariant derivative ∇h of h is defined by

$$(\nabla_X h)(Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y) \tag{13}$$

for any $X, Y, Z \in \Gamma(TM)$.

The normal component of (12) is said to be the Codazzi equation is given by

$$(\widetilde{R}(X, Y)Z)^{\perp} = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \tag{14}$$

where $(\widetilde{R}(X, Y)Z)^{\perp}$ denotes the normal part of $\widetilde{R}(X, Y)Z$. If $(\widetilde{R}(X, Y)Z)^{\perp} = 0$, then M is said to be curvature-invariant submanifold of $\widetilde{M}$. The Ricci equation is given by

$$g(\ \widetilde{R}(X, Y)V, U) = g(\widetilde{R}^{\perp}(X, Y)V, U) + g([A_U, A_V]X, Y), \tag{15}$$

for any $X, Y, \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$, where $\widetilde{R}^{\perp}$ denotes the Riemannian curvature tensor of the normal $T^{\perp}M$ and if $\widetilde{R}^{\perp} = 0$, then the normal connection of M is called flat.

A cosymplectic manifold $\widetilde{M}$ is said to be η-Einstein if its Ricci tensor S of type $(0, 2)$ is of the from

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{16}$$

where a, b are smooth functions on $\widetilde{M}$. If $b = 0$, then the manifold is called Einstein.

3 Pseudo-slant submanifolds of a cosymplectic manifold

In this section, we study pseudo-slant submanifolds in a cosymplectic manifold and we give some characterization results.

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$, we can set

$$\phi X = TX + NX, \tag{17}$$

where TX and NX denote the tangential and the normal components of ϕX, respectively. In the same way, for any $V \in \Gamma(T^{\perp}M)$, we can write

$$\phi V = tV + nV, \tag{18}$$

where $tV(resp.nV)$ are tangential(resp. normal) components of ϕV.

A submanifold M is said to be invariant if N is identically zero, that is, $\phi X \in \Gamma(TM)$ for all $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if T is identically zero, that is, $\phi X \in \Gamma(T^{\perp}M)$ for all $X \in \Gamma(TM)$.

Thus by using (1), (17) and (18), we obtain

$$T^2 = -I - tN + \eta \otimes \xi, \quad NT + nN = 0 \tag{19}$$

and

$$n^2 = -I - Nt, \quad Tt + tn = 0. \tag{20}$$

Furthermore, the covariant derivatives of the tensor field T, N, t and n are, respectively, defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y \tag{21}$$

$$(\nabla_X N)Y = \nabla_X^{\perp} NY - N\nabla_X Y \tag{22}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla_X^{\perp} V \tag{23}$$

and

$$(\nabla_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V. \tag{24}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(TX, Y) = -g(X, TY)$ and $V, U \in \Gamma(T^{\perp}M)$. By using (3), (17) and (18), we have $g(U, nV) = -g(nU, V)$. These show that T and n are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have

$$g(NX, V) = -g(X, tV), \tag{25}$$

which gives the relation between N and t.

Taking into account (6) and (15), we have

$$\begin{aligned} g(\widetilde{R}^{\perp}(X, Y)V, U) = \frac{c}{4}\{&g(X, tV)g(U, NY) - g(Y, tV)g(NX, U) \\ &+ 2g(X, TY)g(nV, U)\} + g([A_V, A_U]X, Y) \end{aligned} \tag{26}$$

for any $X, Y \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$.

By using (6) and (12), the Riemanian curvature tensor R of an immersed submanifold M of a cosymplectic space form $\widetilde{M}(c)$ is given by

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(X, \phi Z)\phi Y \\ &+ g(\phi Y, Z)\phi X + 2g(X, \phi Y)\phi Z\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y \\ &+ (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z). \end{aligned} \tag{27}$$

Comparing the tangential and normal parts of the both sides of this equation, we have, following equations of Gauss and Codazzi equation respectively:

$$\begin{aligned} (R(X, Y)Z)^{T} = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(X, TZ)TY \\ &+ g(TY, Z)TX + 2g(X, TY)TZ\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y \end{aligned} \tag{28}$$

and

$$\begin{aligned} (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \frac{c}{4}\{&g(X, TZ)NY + g(TY, Z)NX \\ &+ 2g(X, TY)NZ\}. \end{aligned} \tag{29}$$

By an easy computation, we obtain the following formulas

$$(\nabla_X T)Y = A_{NY}X + th(X, Y) \tag{30}$$

and

$$(\nabla_X N)Y = nh(X, Y) - h(X, TY). \tag{31}$$

Similarly, for any $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$, we obtain

$$(\nabla_X t)V = A_{nV}X - TA_V X \tag{32}$$

and

$$(\nabla_X n)V = -h(tV, X) - NA_V X. \tag{33}$$

Since M is tangent to ξ, making use of (5), (7), (9) and (17), we obtain

$$\nabla_X \xi = 0, \qquad h(X, \xi) = 0, \qquad A_V \xi = 0 \tag{34}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

Definition 1 *A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be slant submanifold if for any $x \in M$ and $X \in T_xM - \xi$ the angle between T_xM and ϕX is constant. The constant angle $[0, \frac{\pi}{2}]$ is then called slant angle of M. If $\theta = 0$, the submanifold is invariant submanifold, if $\theta = \frac{\pi}{2}$ then, it is anti-invariant submanifold, if $\theta \in (0, \frac{\pi}{2})$ then it is proper slant submanifold* [2].

In almost contact metric manifolds, J. L Cabrerizo [7] proved the following theorem.

Theorem 1 *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM)$. Then, M is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi) \tag{35}$$

furthermore, in this case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$ [7].

Corollary 1 *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(TM)$, we have*

$$g(TX, TY) = \cos^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{36}$$

and

$$g(NX, NY) = \sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}\,[7]. \tag{37}$$

Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ with slant angle θ. Then for any $X \in \Gamma(TM)$, from (19) and (35), we have

$$-\cos^2\theta(X-\eta(X)\xi) = -X+\eta(X)\xi - tNX$$

from which

$$tNX = -\sin^2\theta(X-\eta(X)\xi) \tag{38}$$

by using (37),

$$N^2X = -\sin^2\theta(X-\eta(X)\xi) \tag{39}$$

from (38) and (39) we, obtain

$$N^2 = tN.$$

It is well known that $th = 0$ plays an important role in the geometry of submanifolds. This means that the induced structure T is a cosymplectic structure on M.

By using (30) and (34), we obtain

$$\eta((\nabla_X T)Y) = 0,$$

for $X, Y \in \Gamma(D_\theta)$.

Definition 2 *We say that M is a pseudo-slant submanifold of an almost contact metric manifold $\widetilde{M}$ if there exist two orthogonal distributions D_θ and $D^\perp$ on M such that*

(a) TM *admits the orthogonal direct decomposition* $TM = D^\perp \oplus D_\theta$, $\xi \in \Gamma(D_\theta)$,

(b) *The distribution $D^\perp$ is anti-invariant(totally-real) i.e.,* $\phi D^\perp \subset (T^\perp M)$,

(c) *The distribution D_θ is a slant, that is, the slant between of D_θ and $\phi(D_\theta)$ is a constant* [15].

Let d_1 =dim$(D^\perp)$ and $d_2 = dim(D_\theta)$. We distinguish the following five cases.

(i) If $d_2 = 0$ or $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.

(ii) If $d_1 = 0$ and $\theta = 0$, then M is invariant submanifold.

(iii) If $d_1 = 0$ and $\theta \neq \{0, \frac{\pi}{2}\}$, then M is a proper slant submanifold.

(iv) If $d_2 d_1 \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold.

(v) If $d_2 d_1 \neq 0$ and $\theta \neq \{0, \frac{\pi}{2}\}$, then M is a proper pseudo-slant submanifold.

By μ we denote the orthogonal complementary of $\phi(TM)$ in $T^{\perp}M$, then we have the following sum

$$T^{\perp}M = N(D^{\perp}) \oplus N(D_{\theta}) \oplus \mu.$$

Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then for any $Z, W \in \Gamma(D^{\perp})$ and $U \in \Gamma(TM)$, also by using (4), (7) and (9), we have

$$\begin{aligned} g(A_{NZ}W - A_{NW}Z, U) &= g(h(W,U), NZ) - g(h(Z,U), NW) \\ &= g(\widetilde{\nabla}_U W, \phi Z) - g(\widetilde{\nabla}_U Z, \phi W) \\ &= g(\phi\widetilde{\nabla}_U Z, W) - g(\phi\widetilde{\nabla}_U W, Z) \\ &= g(\widetilde{\nabla}_U \phi Z - (\widetilde{\nabla}_U \phi)Z, W) + g((\widetilde{\nabla}_U \phi)W - \widetilde{\nabla}_U \phi W, Z) \\ &= g(\widetilde{\nabla}_U \phi Z, W) - g(\widetilde{\nabla}_U \phi W, Z) \\ &= -g(A_{NZ}U, W) + g(A_{NW}U, Z) \\ &= g(A_{NW}Z - A_{NZ}W, U). \end{aligned}$$

It follows that

$$A_{NZ}W = A_{NW}Z.$$

Theorem 2 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then the tensor N is parallel if and only if the tensor t is parallel.*

Proof. By using (9), (31) and (32), we have

$$\begin{aligned} g((\nabla_X N)Y, V) &= g(nh(X,Y), V) - g(h(X,TY), V) \\ &= -g(h(X,Y), nV) - g(A_V X, TY) \\ &= -g(A_{nV}X, Y) + g(TA_V X, Y) \\ &= g(-A_{nV}X + TA_V X, Y) = g((\nabla_X t)V, Y), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. This proves our assertion. □

Theorem 3 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then the tensor N is parallel if and only if*

$$A_V TY = -A_{nV}Y$$

for any $Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. By using (9) and (31), we have

$$\begin{aligned} g((\nabla_X N)Y, V) &= g(nh(X,Y), V) - g(h(X,TY), V) \\ &= -g(h(X,Y), nV) - g(A_V TY, X) \\ &= -g(A_{nV}X, Y) - g(A_V TY, X) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. This proves our assertion. □

Theorem 4 *Let* M *be a proper pseudo-slant submanifold of a cosymplectic manifold* $\widetilde{M}$. *The covariant derivation of* T *is skew-symmetric, that is*

$$g((\nabla_X T)Y, Z) = -g((\nabla_X T)Z, Y),$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. For any $X, Y, Z \in \Gamma(TM)$, by using (9), (25) and (30), we obtain

$$\begin{aligned} g((\nabla_X T)Y, Z) &= g(A_{NY}X + th(X,Y), Z) \\ &= g(h(X,Z), NY) - g(h(X,Y), NZ) \\ &= -g(th(X,Z), Y) - g(A_{NZ}X, Y) \\ &= -g(A_{NZ}X + th(X,Z), Y) \\ &= -g((\nabla_X T)Z, Y). \end{aligned}$$

This complete the proof. □

Theorem 5 *Let* M *be a proper pseudo-slant submanifold of a cosymplectic manifold* $\widetilde{M}$. *Then the tensor* T *is parallel if and only if*

$$A_{NY}X = A_{NX}Y$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y, Z \in \Gamma(TM)$, by using (9), (25) and (30), we obtain

$$\begin{aligned} g((\nabla_X T)Y, Z) &= g(A_{NY}X + th(X,Y), Z) \\ &= g(h(X,Z), NY) - g(h(X,Y), NZ) \\ &= g(A_{NY}Z, X) - g(A_{NZ}Y, X) \end{aligned}$$

This complete the proof. □

Theorem 6 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. The covariant derivation of n is skew-symmetric, that is,*

$$g((\nabla_X n)V, U) = -g((\nabla_X n)U, V),$$

for any $X \in \Gamma(TM)$ and $V, U \in \Gamma(T^\perp M)$.

Proof. For any $X \in \Gamma(TM)$ and $V, U \in \Gamma(T^\perp M)$, from (9), (25) and (33), we reach

$$\begin{aligned} g((\nabla_X n)V, U) &= g(-h(tV, X) - NA_V X, U) \\ &= g(-A_U X, tV) + g(A_V X, tU) \\ &= g(NA_U X, V) + g(h(X, tU), V) \\ &= -g(-NA_U X - h(X, tU), V) \\ &= -g((\nabla_X n)U, V). \end{aligned}$$

This proves our assertion. □

Theorem 7 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then the tensor n is parallel if and only if the shape operator A_V of M satisfies the condition*

$$A_V tU = A_U tV, \tag{40}$$

for all $U, V \in \Gamma(T^\perp M)$.

Proof. From (9), (25) and (33), we have

$$\begin{aligned} g((\nabla_X n)V, U) &= -g(h(tV, X), U) - g(NA_V X, U) \\ &= -g(A_U tV, X) + g(A_V X, tU) \\ &= g(A_V tU - A_U tV, X), \end{aligned}$$

for all $X \in \Gamma(TM)$. The proof is complete. □

Theorem 8 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. If tensor n is parallel then, M is totally geodesic submanifold of $\widetilde{M}$.*

Proof. Since n is parallel, from (33) and (17), we have

$$h(tV, X) + \phi A_V X = 0 \tag{41}$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. Applying ϕ to (41) and taking into account (1) and (34), we obtain

$$\begin{aligned} 0 &= \phi^2 A_V X + \phi h(tV, X) \\ &= -A_V X + \eta(A_V X)\xi + th(tV, X) + nh(tV, X). \end{aligned}$$

This yields to

$$-A_V X + th(tV, X) = 0.$$

On the other hand, also by using (9), (19), (25) and (40), we conclude that

$$\begin{aligned} g(A_V X, Z) = g(th(tV, X), Z) &= -g(h(tV, X), NZ) \\ = -g(A_{NZ} tV, X) &= -g(A_V tNZ, X), \end{aligned}$$

for $Z \in \Gamma(TM)$. Taking into account of $tNZ = -Z + \eta(Z)\xi - T^2 Z$, we obtain

$$\begin{aligned} g(A_V Z, X) &= -g(-A_V Z + \eta(Z)A_V\xi - A_V T^2 Z, X) \\ &= g(A_V Z, X) + g(A_V X, T^2 Z) \end{aligned}$$

that is,

$$g(T^2 A_V X, Z) = 0.$$

Here, by using (36), we conclude

$$0 = -g(TA_V X, TZ) = -\cos^2\theta g(A_V X, Z)$$

for all $Z \in \Gamma(TM)$. Since M is a proper pseudo-slant submanifold, we arrive at $A_V = 0$, that is, M is totally geodesic in $\widetilde{M}$. □

Definition 3 *A pseudo-slant submanifold M of cosymplectic manifold $\widetilde{M}$ is said to be D_θ-geodesic (resp. $D^{\perp}$-geodesic) if $h(X, Y) = 0$ for $X, Y \in \Gamma(D_\theta)$ (resp. $h(Z, W) = 0$ for $Z, W \in \Gamma(D^{\perp})$). If $h(X, Z) = 0$, M is called mixed geodesic submanifold, for any $X \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$.*

Theorem 9 *Let M be a proper pseudo-slant submanifold of a Cosymplectic manifold $\widetilde{M}$. If t is parallel, then either M is a mixed-geodesic or an anti-invariant submanifold.*

Proof. From Theorem 2 and (31) we obtain

$$nh(X, Y) = 0,$$

for any $X \in \Gamma(D_\theta)$ and $Y \in \Gamma(D^\perp)$. Also by using (31) and (34), we conclude that

$$nh(Y, TX) - h(Y, T^2X) = \cos^2\theta h(X, Y) = 0.$$

This proves our assertion. □

Theorem 10 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. If t is parallel, then either M is a $D^\perp$-geodesic or an anti-invariant submanifold of $\widetilde{M}$.*

Proof. If t is parallel, then making use of (32), we obtain

$$TA_{NY}Z = 0,$$

for any $Y, Z \in \Gamma(D^\perp)$. This implies that M is either anti-invariant or $A_{NY}Z = 0$. So we obtain

$$g(h(Z, W), NY) = 0,$$

for any $Y, Z, W \in \Gamma(D^\perp)$. Also by using (32), we conclude that

$$g(A_{nV}Z, Y) - g(TA_VZ, Y) = g(h(Y, Z), nV) = 0,$$

for any $V \in \Gamma(T^\perp M)$. This tells us that M is either $D^\perp$-geodesic or it is an anti-invariant submanifold. □

Given a proper pseudo-slant submanifold M of a Cosymplectic manifold $\widetilde{M}$, if the distributions D_θ and $D^\perp$ are totally geodesic in M, then M is said to be contact pseudo-slant product.

Theorem 11 *Let M be a pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then M is a contact pseudo-slant product if and only if the shape operator of M satisfies*

$$A_{ND^\perp}TD_\theta = A_{NTD_\theta}D^\perp.$$

Proof. Since the ambient space $\widetilde{M}$ is a cosymplectic manifold, for any $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^\perp)$, we have

$$\begin{aligned} g(\nabla_XY, Z) &= g(\widetilde{\nabla}_X\phi Y, \phi Z) = g(\widetilde{\nabla}_XTY, \phi Z) + g(\widetilde{\nabla}_XNY, \phi Z) \\ &= -g(\widetilde{\nabla}_X\phi TY, Z) + g(\nabla^\perp_X NY, NZ) \\ &= -g(\nabla_XT^2Y, Z) - g(\widetilde{\nabla}_XNTY, Z) + g(\nabla^\perp_X NY, NZ) \\ &= \cos^2\theta g(\nabla_XY, Z) + g(A_{NTY}X, Z) + g(N\nabla_XY, NZ) - g(h(X, TY), NZ), \end{aligned}$$

which implies that

$$\cos^2\theta g(\nabla_X Y, Z) = g(A_{NTY}Z - A_{NZ}TY, X). \tag{42}$$

On the other hand, for any $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D_\theta)$, we reach at

$$\begin{aligned}
g(\nabla_Z W, X) &= -g(\widetilde{\nabla}_Z X, W) = -g(\widetilde{\nabla}_Z \phi X, \phi W) \\
&= -g(\widetilde{\nabla}_Z TX, \phi W) - g(\widetilde{\nabla}_Z NX, \phi W) \\
&= g(\widetilde{\nabla}_Z \phi TX, W) - g(\nabla^\perp_Z NX, NW) \\
&= g(\nabla_Z T^2 X, W) + g(\nabla_Z NTX, W) - g(\nabla^\perp_Z NX, NW) \\
&= -\cos^2\theta g(\nabla_Z X, W) - g(A_{NTX}Z, W) - g(N\nabla_Z X, NW) \\
&\quad - g((\nabla_Z N)X, NW) \\
&= \cos^2\theta g(\nabla_Z W, X) - g(A_{NTX}Z, W) - g(N\nabla_Z X, NW) \\
&\quad + g(h(Z, TX), NW).
\end{aligned}$$

This implies that

$$\cos^2\theta g(\nabla_Z W, X) = g(A_{NTX}W - A_{NW}TX, Z). \tag{43}$$

From (42) and (43), we get desired result. □

4 Pseudo-slant submanifolds in cosymplectic space forms

In this section, we will study pseudo-slant submanifolds in a cosymplectic space form, give some characterization and submanifold will be characterized.

Theorem 12 *Let M be a pseudo-slant submanifold of a cosymplectic space form $\widetilde{M}(c)$ such that $c \neq 0$. If M is a curvature-invariant pseudo-slant submanifold, then M is either semi-invariant or anti-invariant submanifold.*

Proof. We suppose that M is a curvature-invariant pseudo-slant submanifold of a cosymplectic space form $\widetilde{M}(c)$ such that $c \neq 0$. Then from (29) and (14), we have

$$g(X, TZ)NY + g(TY, Z)NX + 2g(X, TY)NZ = 0, \tag{44}$$

for any $X, Y, Z \in \Gamma(TM)$. Taking $X = Z$ and $Y = TZ$ in (44), we have

$$g(TZ, TZ)NZ = 0.$$

Here, by using (36) and (37), we obtain

$$\cos^2\theta\sin^2\theta\left\{g(Z,Z)-\eta^2(Z)\right\}^2=0.$$

This implies that $\sin 2\theta\{g(Z,Z)-\eta^2(Z)\}=0$, that is, M is either a semi-invariant or an anti-invariant submanifold. Thus the proof is complete. □

Theorem 13 *Let M be a pseudo-slant submanifold of a cosymplectic space form $\widetilde{M}(c)$ with flat normal connection such that $c\neq 0$. If $TA_V=A_VT$ for any vector V normal to M, then M is either an anti- invariant or it is a generic submanifold of $\widetilde{M}(c)$.*

Proof. If the normal connection of M is flat, then from (26), we have

$$\begin{aligned} g([A_U,A_V]X,Y) &= \frac{c}{4}\{g(X,\phi V)g(U,\phi Y)-g(Y,\phi V)g(\phi X,U)\\ &\quad +2g(X,\phi Y)g(\phi V,U)\} \end{aligned}$$

for any $X,Y\in\Gamma(TM)$ and $U,V\in\Gamma(T^{\perp}M)$. Here, choosing $U=nV$ and $Y=TX$, by direct calculations, we can state

$$g([A_V,A_{nV}]X,TX)=-\frac{c}{2}\{g(TX,TX)g(nV,nV)\},$$

that is,

$$g(A_{nV}A_VTX-A_VA_{nV}TX,X)=-\frac{c}{2}\{g(TX,TX)g(nV,nV)\},$$

from which

$$tr(A_{nV}A_VT)-tr(A_VA_{nV}T)=\frac{c}{2}tr(T^2)g(nV,nV).$$

If $TA_V=A_VT$, then we conclude that $tr(A_{nV}A_VT)=tr(A_VA_{nV}T)$ and thus

$$\frac{c}{2}tr(T^2)g(nV,nV)=0,$$

from here $\dim(TM)=2q+q+1$, then we can easily to see that

$$(2p+q+1)\cos^2\theta g(nV,nV)=0.$$

Thus θ is either $\frac{\pi}{2}$ or $n=0$. This implies that M is either an anti-invariant or it is a generic submanifold. □

Theorem 14 *Let* M *be a proper pseudo-slant submanifold of a cosymplectic space form* $\widetilde{M}(c)$. *Then the Ricci tensor* S *of* M *is given by*

$$\begin{aligned} S(X,W) &= \frac{c}{4}\left\{2p+q-1+3\cos^2\theta\right\}(g(X,W)-\eta(X)\eta(W)) \\ &\quad +(2p+q+1)g(h(X,W),H)-\sum_{l=1}^{2p+q+1} g(h(e_l,W),h(X,e_l)) \end{aligned} \tag{45}$$

for any $X, W \in \Gamma(TM)$.

Proof. For any $X, Y, Z \in \Gamma(TM)$, by using (6) and (12), we have

$$\begin{aligned} g(R(X,Y)Z,W) = \frac{c}{4}\{&g(Y,Z)g(X,W)-g(X,Z)g(Y,W) \\ &+\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W) \\ &+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z) \\ &+g(X,\phi Z)g(\phi Y,W)-g(Y,\phi Z)g(\phi X,W) \\ &+2g(X,\phi Y)g(\phi Z,W)\}+g(h(X,W),h(Y,Z)) \\ &-g(h(Y,W),h(X,Z)). \end{aligned} \tag{46}$$

Now, let $e_1, e_2, \ldots, e_p, e_{p+1} = \sec\theta Te_1, e_{p+2} = \sec\theta Te_2, \ldots, e_{2p} = \sec\theta Te_p, e_{2p+1} = \xi, e_{2p+2}, e_{2p+3}, \ldots, e_{2p+q+1}$ be an orthonormal basis of $\Gamma(TM)$ such that $e_1, e_2, \ldots, e_p, e_{p+1} = \sec\theta Te_1, e_{p+2} = \sec\theta Te_2, \ldots, e_{2p} = \sec\theta Te_p, e_{2p+1} = \xi$ are tangent to $\Gamma(D_\theta)$ and $e_{2p+2}, e_{2p+3}, \ldots, e_{2p+q+1}$ are tangent to $\Gamma(D^\perp)$. Hence, from (46) taking $Y = Z = e_i, e_j, e_k$ and $1 \le i \le p, 1 \le j \le p, \xi, 2p+2 \le k \le 2p+q+1$ then, we obtain

$$\begin{aligned} S(X,W) &= \sum_{i=1}^{p} g(R(X,e_i)e_i,W) + \sum_{j=p+1}^{2p} g(R(X,\sec\theta Te_j)\sec\theta Te_j,W) \\ &\quad + g(R(X,\xi)\xi,W) + \sum_{k=2p+2}^{2p+q+1} g(R(X,e_k)e_k,W) \\ &= \frac{c}{4}\{(2p+q)g(X,W)\} - \frac{c}{4}\{(2p+q-1)\eta(X)\eta(W) \\ &\quad + 3\cos^2\theta[g(X,W)-\eta(X)\eta(W)] - g(X,W)\} \\ &\quad + (2p+q+1)g(h(X,W),H) - \sum_{i=1}^{p} g(h(e_i,W),h(X,e_i)) \end{aligned}$$

$$- \sum_{j=p+1}^{2p} g(h(\sec\theta Te_j, W), h(X, \sec\theta Te_j)) + g(h(\xi, W), h(X, \xi))$$
$$- \sum_{k=2p+2}^{2p+q+1} g(h(e_k, W), h(X, e_k)).$$

Here

$$\sum_{l=1}^{2p+q+1} g(h(e_l, W), h(X, e_l)) = \sum_{i=1}^{p} g(h(e_i, W), h(X, e_i))$$

$$+ \sum_{j=p+1}^{2p} g(h(\sec\theta Te_j, W), h(X, \sec\theta Te_j))$$
$$+ \sum_{k=2p+2}^{2p+q+1} g(h(e_k, W), h(X, e_k))$$

hance, we have

$$\begin{aligned} S(X, W) &= \frac{c}{4}\left\{2p + q - 1 + 3\cos^2\theta\right\}(g(X, W) - \eta(X)\eta(W)) \\ &+(2p + q + 1)g(h(X, W), H) - \sum_{l=1}^{2p+q+1} g(h(e_l, W), h(X, e_l)) \end{aligned}$$

the proof is complete. □

Theorem 15 *Let* M *be a pseudo-slant submanifold of a cosymplectic space form* $\widetilde{M}(c)$*. Then the scalar curvature* ρ *of* M *is given by*

$$\rho = \frac{c}{4}\{2p + q - 1 + 3\cos^2\theta\}(2p + q) + (2p + q + 1)^2 \|H)\|^2 - \|h\|^2. \quad (47)$$

Proof. By using (45), we have

$$\rho = \sum_{l=1}^{2p+q+1} S(e_l, e_l)$$

which gives (47). Thus the proof is complete. □

Theorem 16 *Let* M *be a proper pseudo-slant submanifold of a cosymplectic space form* $\widetilde{M}(c)$ *such that* $c \neq 0$*. Every totally umbilical pseudo-slant submanifold* M *in a cosymplectic space form* $\widetilde{M}(c)$ *is a semi-invariant or anti-invariant submanifold.*

Proof. We suppose that M is totally umbilical pseudo-slant submanifold in cosymplectic space form $\widetilde{M}(c)$. Since M is totally geodesic, we have

$$g(\widetilde{R}(X,Y)Z,\phi Z) = g((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \phi Z) = 0,$$

or

$$\begin{aligned} g(\widetilde{R}(X,Y)Z,\phi Z) &= g(\nabla^{\perp}_X g(Y,Z)H - g(\nabla_X Y,Z)H - g(\nabla_X Z,Y)H, \phi Z) \\ &\quad - g(\nabla^{\perp}_Y g(X,Z)H - g(\nabla_Y X,Z)H - g(\nabla_Y Z,X)H, \phi Z) = 0 \end{aligned}$$

for any $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$. Since the ambient space M is a cosymplectic space form, from (6) we infer

$$g(\widetilde{R}(X,Y)Z,\phi Z) = \frac{c}{2} g(X,\phi Y) g(NZ,NZ) = 0. \tag{48}$$

Taking $Y = TX$ in equation (48), we have

$$g(X,\phi TX)g(NZ,NZ) = 0.$$

Here, by using (36) and (37), we obtain

$$\cos^2\theta \sin^2\theta g(Z,Z)\{g(X,X) - \eta^2(X)\} = 0.$$

This implies that $\sin 2\theta = 0$, that is, M is either a semi-invariant or an anti-invariant submanifold. This proves our assertion. □

Theorem 17 *Let* M *be a totally umbilical pseudo-slant submanifold of a Cosymplectic space form* $\widetilde{M}(c)$*. Then the Ricci tensor* S *of* M *is given by*

$$S(X,W) = \frac{c}{4}\left\{2p + q - 1 + 3\cos^2\theta\right\}(g(X,W) - \eta(X)\eta(W)) \tag{49}$$

for any $X, W \in \Gamma(TM)$.

Proof. From by using (11) and (45), we obtain

$$\begin{aligned} S(X,W) &= \frac{c}{4}\left\{2p + q - 1 + 3\cos^2\theta\right\}(g(X,W) - \eta(X)\eta(W)) \\ &\quad + (2p+q+1)g(g(X,W)H,H) - \sum_{l=1}^{2p+q+1} g(g(e_l,W)H, g(X,e_l)H) \end{aligned}$$

this complete the proof. Thus we have the following corollary. □

Corollary 2 *Every totally umbilical pseudo-slant submanifold* M *of a cosymplectic space form* $\widetilde{M}(c)$ *is an* η*-Einstein submanifold.*

Theorem 18 *Let* M *be a totally umbilical pseudo-slant submanifold of a cosymplectic space form* $\widetilde{M}(c)$*. Then the scalar curvature* ρ *of* M *is given by*

$$\rho = \frac{c}{4}\{2p + q - 1 + 3\cos^2\theta\}(2p + q). \tag{50}$$

Proof. By using (49), we have

$$\rho = \sum_{l=1}^{2p+q+1} S(e_l, e_l)$$

which gives (50). Thus the proof is complete. □

Example 1 *Let* M *be a submanifold of* $\mathbb{R}^9$ *defined by*

$$x(u, v, s, w, z) = (u, -\sqrt{2}v, v\sin\alpha, v\cos\alpha, s\cos w, -\cos w, s\sin w, -\sin w, z).$$

We can easily to see that the tangent bundle of M *is spanned by the tangent vectors*

$$\begin{aligned}
e_1 &= \frac{\partial}{\partial x_1}, e_5 = \xi = \frac{\partial}{\partial z},\\
e_2 &= -\sqrt{2}\frac{\partial}{\partial y_1} + \sin\alpha\frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial y_2},\\
e_3 &= \cos w\frac{\partial}{\partial x_3} + \sin w\frac{\partial}{\partial x_4},\\
e_4 &= -s\sin w\frac{\partial}{\partial x_3} + \sin w\frac{\partial}{\partial y_3} + s\cos w\frac{\partial}{\partial x_4} - \cos w\frac{\partial}{\partial y_4}.
\end{aligned}$$

We define the almost contact structure ϕ *of* $\mathbb{R}^9$*, by*

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \phi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \le i, j \le 4.$$

For any vector field $X = \lambda_i\frac{\partial}{\partial x_i} + \mu_j\frac{\partial}{\partial y_i} + \nu\frac{\partial}{\partial z} \in \Gamma(T\mathbb{R}^9)$*, then we have*

$$g(X, X) = \lambda_i^2 + \mu_j^2 + \nu^2, \quad g(\phi X, \phi X) = \lambda_i^2 + \mu_j^2$$

and

$$\phi^2 X = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_i} = -X + \eta(X)\xi,$$

for any $i, j = 1, 2, 3, 4$. *It follows that* $g(\phi X, \phi X) = g(X, X) - \eta^2(X)$. *Thus* (ϕ, η, ξ, g) *is an almost contact metric structure on* $\mathbb{R}^9$. *Thus we have*

$$\phi e_1 = \frac{\partial}{\partial y_1},$$
$$\phi e_2 = \sqrt{2}\frac{\partial}{\partial x_1} + \sin\alpha \frac{\partial}{\partial y_2} - \cos\alpha \frac{\partial}{\partial x_2},$$

$$\phi e_3 = \cos w \frac{\partial}{\partial y_3} + \sin w \frac{\partial}{\partial y_4},$$
$$\phi e_4 = -s \sin w \frac{\partial}{\partial y_3} - \sin w \frac{\partial}{\partial x_3} + s \cos w \frac{\partial}{\partial y_4} + \cos w \frac{\partial}{\partial x_4}.$$

By direct calculations, we can infer $D_\theta = \mathrm{span}\{e_1, e_2\}$ *is a slant distribution with slant angle* $\cos\theta = \frac{g(e_1, \phi e_2)}{\|e_1\|\|\phi e_2\|} = \frac{\sqrt{6}}{3}$, $\theta = \cos^{-1}(\frac{\sqrt{6}}{3})$. *Since* $g(\phi e_3, e_i) = 0$, $i = 1, 2, 4, 5$ *and* $g(\phi e_4, e_j) = 0$, $j = 1, 2, 3, 5$, ϕe_3, ϕe_4 *are orthogonal to* M, $D^\perp = \mathrm{span}\{e_3, e_4\}$ *is an anti-invariant distribution. Thus* M *is a* 5-*dimensional proper pseudo-slant submanifold of* $\mathbb{R}^9$ *with its usual almost contact metric structure.*

Let ∇ *be the Levi-Civita connection on* $\mathbb{R}^9$. *Then we have*

$$\begin{aligned} 0 &= [e_1, e_1] = [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = [e_5, e_5] \\ &= [e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_1, e_5] = [e_2, e_3] \\ &= [e_2, e_4] = [e_2, e_5] = [e_3, e_5] = [e_4, e_5], \end{aligned}$$

$$\begin{aligned} [e_3, e_4] = &\left(\frac{\cos 2w}{\sin w}\right)\frac{\partial}{\partial x_3} + \left(\frac{1}{s \sin w}\right)\frac{\partial}{\partial y_3} - \left(\frac{1}{\cos w}\right)\frac{\partial}{\partial x_4} \\ &+ \left(\frac{(s-1)}{s}\cos w\right)\frac{\partial}{\partial y_4}, \end{aligned}$$

and

$$g(e_1, e_1) = g(e_3, e_3) = 1, g(e_2, e_2) = 3, g(e_4, e_4) = s^2 + 1, g(e_5, e_5) = 1,$$

$$\begin{aligned} g(e_1, e_2) &= g(e_1, e_3) = g(e_1, e_4) = g(e_1, e_5) = 0, \\ g(e_2, e_3) &= g(e_2, e_4) = g(e_2, e_5) = 0, \\ g(e_3, e_4) &= g(e_3, e_5) = g(e_4, e_5) = 0. \end{aligned}$$

Using Koszul's formula, the Riemannian connection ∇ *of the metric* g *is given by*

$$\begin{aligned} 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Z) \\ - g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [X, Y]) \end{aligned}$$

Koszul's formula yields, we can find

$$\begin{aligned} 0 &= \nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_1} e_5 \\ &= \nabla_{e_2} e_2 = \nabla_{e_2} e_4 = \nabla_{e_2} e_5 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 \\ &= \nabla_{e_3} e_5 = \nabla_{e_4} e_5 = \nabla_{e_5} e_5, \quad \nabla_{e_4} e_4 = -s e_3, \quad \nabla_{e_3} e_3 = \frac{1}{s} e_3, \end{aligned}$$

$$\begin{aligned} \nabla_{e_3} e_4 = \frac{s}{s^2+1} \left(1 - s^2 + (1 - s - s^2)\cos^2 w + s^2 \sin^2 w\right) e_4 \\ + \left(\frac{\cos 2w - \tan^2 w}{\tan w}\right) e_3. \end{aligned}$$

Thus we can say that M *is* D_θ*-geodesic and mixed- geodesic. But it is not* $D^\perp$*-geodesic.*

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Some vector inequalities for two operators in Hilbert spaces with applications

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Abstract. In this paper we establish some vector inequalities for two operators related to Schwarz and Buzano results. We show amongst others that in a Hilbert space H we have the inequality

$$\frac{1}{2}\left[\left\langle\frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}+\left|\left\langle\frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|\right] \geq |\langle \operatorname{Re}(B^*A)\,x,y\rangle|$$

for A, B two bounded linear operators on H such that $\operatorname{Re}(B^*A)$ is a nonnegative operator and any vectors $x, y \in H$.

Applications for norm and numerical radius inequalities are given as well.

1 Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\,\|y\| \geq |\langle x, y\rangle| \text{ for any } x, y \in H. \tag{1}$$

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The equality case holds in (1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [5] (see also [24]) established the following refinement of (1):

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \geq |\langle x, y\rangle| \tag{2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \geq |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle|$$

and by (2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \\ &\geq 2 |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle|, \end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2}\left[\, \|x\| \|y\| + |\langle x, y\rangle| \,\right] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

A family $\{e_j\}_{j \in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j, k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j, k \in J.$$

If the *linear span* of the family $\{e_j\}_{j \in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j \in J}$ we have *Bessel's inequality*

$$\sum_{j \in J} |\langle x, e_j\rangle|^2 \leq \|x\|^2 \text{ for any } x \in H.$$

This becomes *Parseval's identity*

$$\sum_{j \in J} |\langle x, e_j\rangle|^2 = \|x\|^2 \text{ for any } x \in H,$$

when $\{e_j\}_{j \in J}$ an othonormal basis in H.

For an othonormal family $\mathcal{E} = \{e_j\}_{j \in J}$ we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j \in J} \langle x, e_j\rangle e_j \,, \; x \in H. \tag{4}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}}x, y\rangle = \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle, \; x, y \in H \text{ and } \langle P_{\mathcal{E}}x, x\rangle = \sum_{j\in J} |\langle x, e_j\rangle|^2, \; x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}$, $\|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle e$, $x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle, \; x, y \in H \tag{5}$$

and Buzano's inequality can be written as

$$\frac{1}{2}\left[\, \|x\| \, \|y\| + |\langle x, y\rangle| \,\right] \geq |\langle P_e x, y\rangle| \tag{6}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

In an effort to generalize the inequality (6) for general projection, in [21] we obtained the following result

$$\frac{1}{2}\left[\, \|x\| \, \|y\| + |\langle x, y\rangle| \,\right] \geq |\langle Px, y\rangle| \tag{7}$$

for any $x, y \in H$ and $P : H \to H$ a projection on H.

In particular, we then have the inequality

$$\frac{1}{2}\left[\, \|x\| \, \|y\| + |\langle x, y\rangle| \,\right] \geq \left|\left\langle \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle \right\rangle\right| \tag{8}$$

for any orthonormal family $\{e_j\}_{j\in J}$ and any $x, y \in H$.

Motivated by the above results we establish in this paper some vector inequalities for two operators A, B for which the operator $\mathrm{Re}\,(B^*A)$ is nonnegative in the operator order that are related to the inequality (6). Applications for norm and numerical radius inequalities are provided as well.

For other Schwarz and Buzano related inequalities in inner product spaces, see [1]-[4], [5]-[14], [22]-[26], [30]-[39], and the monographs [16], [17] and [18].

2 Vector inequalities for two operators

For a bounded linear operator T we use the concepts of *absolute value* and *real part* of T defined as

$$|T| = (T^*T)^{1/2} \text{ and } \mathrm{Re}\,(T) = \frac{T + T^*}{2}. \tag{9}$$

We have the following vector inequality:

Theorem 1 *Let* A, B *two bounded linear operators on* H *such that* $\operatorname{Re}(B^*A)$ *is a nonnegative operator. Then for any* $x, y \in H$ *we have the inequality*

$$\begin{aligned} &\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle^{1/2} \left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle^{1/2} \\ &\geq \langle \operatorname{Re}(B^*A)x, x\rangle^{1/2} \langle \operatorname{Re}(B^*A)y, y\rangle^{1/2} \\ &+ \left| \left\langle \frac{|A|^2+|B|^2}{2}x, y \right\rangle - \langle \operatorname{Re}(B^*A)x, y\rangle \right|. \end{aligned} \tag{10}$$

Proof. Using Schwarz inequality we have

$$\|Ax - Bx\|^2 \|Ay - By\|^2 \geq |\langle Ax - Bx, Ay - By\rangle|^2 \tag{11}$$

for any $x, y \in H$.

Observe that

$$\begin{aligned} \|Ax - Bx\|^2 &= \langle Ax, Ax\rangle - \langle Ax, Bx\rangle - \langle Bx, Ax\rangle + \langle Bx, Bx\rangle \\ &= \langle A^*Ax, x\rangle - \langle B^*Ax, x\rangle - \langle A^*Bx, x\rangle + \langle B^*Bx, x\rangle \\ &= \left\langle |A|^2 x, x\right\rangle + \left\langle |B|^2 x, x\right\rangle - \langle (B^*A + A^*B)x, x\rangle \\ &= 2\left[\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle - \langle \operatorname{Re}(B^*A)x, x\rangle\right] \geq 0 \end{aligned} \tag{12}$$

and, similarly,

$$\|Ay - By\|^2 = 2\left[\left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle - \langle \operatorname{Re}(B^*A)y, y\rangle\right] \geq 0 \tag{13}$$

for any $x, y \in H$.

We also have

$$\langle Ax - Bx, Ay - By\rangle = 2\left[\left\langle \frac{|A|^2+|B|^2}{2}x, y \right\rangle - \langle \operatorname{Re}(B^*A)x, y\rangle\right] \tag{14}$$

for any $x, y \in H$.

Using the inequality (11) and the equalities (12)-(14) we get

$$\begin{aligned} &\left[\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle - \langle \operatorname{Re}(B^*A)x, x\rangle\right] \\ &\times \left[\left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle - \langle \operatorname{Re}(B^*A)y, y\rangle\right] \\ &\geq \left| \left\langle \frac{|A|^2+|B|^2}{2}x, y \right\rangle - \langle \operatorname{Re}(B^*A)x, y\rangle \right|^2 \end{aligned} \tag{15}$$

for any $x, y \in H$.

Since $\operatorname{Re}(B^*A) \geq 0$, then we have

$$\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle \geq \langle \operatorname{Re}(B^*A)\, x, x\rangle \geq 0$$

and

$$\left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle \geq \langle \operatorname{Re}(B^*A)\, y, y\rangle \geq 0$$

for any $x, y \in H$.

Using the elementary inequality that holds for any real numbers a, b, c, d

$$(ac - bd)^2 \geq (a^2 - b^2)(c^2 - d^2),$$

we have

$$\begin{aligned}
&\left(\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle^{1/2} \left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle^{1/2} \right. \\
&\left. - \langle \operatorname{Re}(B^*A)x, x\rangle^{1/2} \langle \operatorname{Re}(B^*A)y, y\rangle^{1/2}\right)^2 \\
&\geq \left[\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle - \langle \operatorname{Re}(B^*A)x, x\rangle\right] \\
&\times \left[\left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle - \langle \operatorname{Re}(B^*A)y, y\rangle\right]
\end{aligned} \tag{16}$$

for any $x, y \in H$.

Making use of (15) and (16) we get

$$\begin{aligned}
&\left(\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle^{1/2} \left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle^{1/2} \right. \\
&\left. - \langle \operatorname{Re}(B^*Ax, x\rangle^{1/2} \langle \operatorname{Re}(B^*A)y, y\rangle^{1/2}\right)^2 \\
&\geq \left|\left\langle \frac{|A|^2+|B|^2}{2}x, y \right\rangle - \langle \operatorname{Re}(B^*A)\, x, y\rangle\right|^2
\end{aligned} \tag{17}$$

for any $x, y \in H$.

Since

$$\left\langle \frac{|A|^2+|B|^2}{2}x, x \right\rangle^{1/2} \left\langle \frac{|A|^2+|B|^2}{2}y, y \right\rangle^{1/2} \geq \langle \operatorname{Re}(B^*A)x, x\rangle^{1/2} \langle \operatorname{Re}(B^*A)y, y\rangle^{1/2}$$

for any $x, y \in H$, then by taking the square root in (17) we get the desired result from (10). □

Corollary 1 *With the assumptions in Theorem 1 we have*

$$\begin{aligned}&\left\langle \frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}-\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right| \\ &\geq \langle \mathrm{Re}(B^*A)x,x\rangle^{1/2}\langle \mathrm{Re}(B^*A)y,y\rangle^{1/2}-|\langle \mathrm{Re}(B^*A)x,y\rangle|\geq 0\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\left\langle \frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}+\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right| \\ &\geq \langle \mathrm{Re}(B^*A)x,x\rangle^{1/2}\langle \mathrm{Re}(B^*A)y,y\rangle^{1/2}+|\langle \mathrm{Re}(B^*A)x,y\rangle|\end{aligned} \tag{19}$$

for any $x,y\in H$.

Proof. From the triangle inequality we have

$$\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle-\langle \mathrm{Re}(B^*A)x,y\rangle\right|\geq\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|-|\langle \mathrm{Re}(B^*A)x,y\rangle|$$

and

$$\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle-\langle \mathrm{Re}(B^*A)x,y\rangle\right|\geq|\langle \mathrm{Re}(B^*A)x,y\rangle|-\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|$$

for any $x,y\in H$, which together with (10) produce the inequalities (18) and (19). □

Remark 1 *With the assumptions in Theorem 1 we have*

$$\begin{aligned}&\frac{1}{2}\left[\left\langle \frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}\right. \\ &\left.+\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|\right]\geq|\langle \mathrm{Re}(B^*A)x,y\rangle|\end{aligned} \tag{20}$$

for any $x,y\in H$.

If we assume that A is a bounded linear operator such that $\mathrm{Re}\left(A^2\right)\geq 0$, then by taking $B=A^*$ above, we have the inequalities

$$\begin{aligned}&\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2} \\ &\geq \langle \mathrm{Re}(A^2)x,x\rangle^{1/2}\langle \mathrm{Re}(A^2)y,y\rangle^{1/2} \\ &+\left|\left\langle \frac{|A|^2+|A^*|^2}{2}x,y\right\rangle-\langle \mathrm{Re}(A^2)x,y\rangle\right|,\end{aligned} \tag{21}$$

$$\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2}-\left|\left\langle \frac{|A|^2+|A^*|^2}{2}x,y\right\rangle\right| \tag{22}$$
$$\geq \langle \mathrm{Re}(A^2)x,x\rangle^{1/2}\langle \mathrm{Re}(A^2)y,y\rangle^{1/2}-|\langle \mathrm{Re}(A^2)x,y\rangle|\geq 0,$$

$$\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2}+\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right| \tag{23}$$
$$\geq \langle \mathrm{Re}(A^2)x,x\rangle^{1/2}\langle \mathrm{Re}(A^2)y,y\rangle^{1/2}+|\langle \mathrm{Re}(A^2)x,y\rangle|$$

and

$$\frac{1}{2}\left[\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2}\right.$$
$$\left.+\left|\left\langle \frac{|A|^2+|A^*|^2}{2}x,y\right\rangle\right|\right]\geq |\langle \mathrm{Re}(A^2)x,y\rangle| \tag{24}$$

for any $x,y\in H$.

Assume that A is invertible, then by selecting $B=(A^{-1})^*$ above and taking into account that

$$|B|^2=B^*B=A^{-1}(A^{-1})^*=A^{-1}\left(A^*\right)^{-1}=\left(A^*A\right)^{-1}=|A|^{-2}$$

then from the above we get the inequalities

$$\left\langle \frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2}$$
$$\geq \|x\|\,\|y\|+\left|\left\langle \frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle-\langle x,y\rangle\right|, \tag{25}$$

$$\left\langle \frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2}$$
$$-\left|\left\langle \frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle\right|\geq \|x\|\,\|y\|-|\langle x,y\rangle|\geq 0, \tag{26}$$

$$\left\langle \frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2}$$
$$+\left|\left\langle \frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle\right|\geq \|x\|\,\|y\|+|\langle x,y\rangle| \tag{27}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\left\langle\frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle\frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2}\right.\\&\left.+\left|\left\langle\frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle\right|\right]\geq|\langle x,y\rangle|\end{aligned}\tag{28}$$

for any $x, y \in H$.

If $A, B \geq 0$ with $AB = BA$, then from (10) we have

$$\begin{aligned}&\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2}\\&\geq\langle ABx,x\rangle^{1/2}\langle ABy,y\rangle^{1/2}+\left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle-\langle ABx,y\rangle\right|,\end{aligned}\tag{29}$$

$$\begin{aligned}&\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2}-\left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle\right|\\&\geq\langle ABx,x\rangle^{1/2}\langle ABy,y\rangle^{1/2}-|\langle ABx,y\rangle|\geq 0,\end{aligned}\tag{30}$$

$$\begin{aligned}&\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2}+\left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle\right|\\&\geq\langle ABx,x\rangle^{1/2}\langle ABy,y\rangle^{1/2}+|\langle ABx,y\rangle|\end{aligned}\tag{31}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2}\right.\\&\left.+\left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle\right|\right]\geq|\langle ABx,y\rangle|\end{aligned}\tag{32}$$

for any $x, y \in H$.

We observe that if $A = 1_H$ and $B = P$, with P a projection on H, then we obtain from (32)

$$\frac{1}{2}\left[\left\langle\frac{1_H+P}{2}x,x\right\rangle^{1/2}\left\langle\frac{1_H+P}{2}y,y\right\rangle^{1/2}+\left|\left\langle\frac{1_H+P}{2}x,y\right\rangle\right|\right]\geq|\langle Px,y\rangle|\tag{33}$$

for any $x, y \in H$.

If $e \in H$, $\|e\| = 1$ then by taking $P = P_e$ defined in the introduction, we get the inequality

$$\begin{aligned}&\frac{1}{4}\left[\left[\|x\|^2+|\langle x,e\rangle|^2\right]^{1/2}\left[\|y\|^2+|\langle y,e\rangle|^2\right]^{1/2}+|\langle x,y\rangle+\langle x,e\rangle\langle e,y\rangle|\right]\\&\geq|\langle x,e\rangle\langle e,y\rangle|\end{aligned}\tag{34}$$

for any $x, y \in H$.

Since

$$|\langle x, y\rangle + \langle x, e\rangle\langle e, y\rangle| \le |\langle x, y\rangle| + |\langle x, e\rangle\langle e, y\rangle|$$

then by (34) we have

$$\frac{1}{4}\left[\left[\,\|x\|^2+|\langle x, e\rangle|^2\,\right]^{1/2}\left[\,\|y\|^2+|\langle y, e\rangle|^2\,\right]^{1/2}+|\langle x, y\rangle|+|\langle x, e\rangle\langle e, y\rangle|\right] \ge |\langle x, e\rangle\langle e, y\rangle|,$$

which implies that

$$\frac{1}{3}\left(\left[\|x\|^2 + |\langle x, e\rangle|^2\right]^{1/2}\left[\|y\|^2 + |\langle y, e\rangle|^2\right]^{1/2} + |\langle x, y\rangle|\right) \ge |\langle x, e\rangle\langle e, y\rangle| \tag{35}$$

for any $x, y \in H$.

We recall that $U : H \to H$ is a *unitary operator* if $U^*U = UU^* = 1_H$. If U and V are unitary operators with $\mathrm{Re}\,(V^*U) \ge 0$, then by (20) we have

$$\frac{1}{2}\left[\,\|x\|\,\|y\| + |\langle x, y\rangle|\,\right] \ge \left|\langle \mathrm{Re}\,(V^*U)\,x, y\rangle\right| \tag{36}$$

for any $x, y \in H$.

In particular, if U is a unitary operator with $\mathrm{Re}\,(U) \ge 0$ then by taking $V = 1_H$ in (36) we get

$$\frac{1}{2}\left[\|x\|\|y\| + |\langle x, y\rangle|\right] \ge \left|\langle \mathrm{Re}(U)x, y\rangle\right| \tag{37}$$

for any $x, y \in H$.

3 Inequalities for norm and numerical radius

Let $\left(H; \langle\cdot,\cdot\rangle\right)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [27, p. 1]:

$$W(T) = \left\{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\right\}.$$

The *numerical radius* $w\,(T)$ of an operator T on H is defined by [27, p. 8]:

$$w\,(T) = \sup\left\{\,|\lambda|\,, \lambda \in W\,(T)\,\right\} = \sup\left\{|\langle Tx, x\rangle|, \|x\| = 1\right\}.$$

It is well known that $w\,(\cdot)$ is a norm on the Banach algebra $B\,(H)$ and the following inequality holds true

$$w\,(T) \le \|T\| \le 2w\,(T)\,,\ \text{for any } T \in B\,(H)\,.$$

Utilising Buzano's inequality (3) we obtained the following inequality for the numerical radius [13] or [14]:

Theorem 2 *Let* $(\mathsf{H};\langle\cdot,\cdot\rangle)$ *be a Hilbert space and* $\mathsf{T}:\mathsf{H}\to\mathsf{H}$ *a bounded linear operator on* H. *Then*

$$w^2(\mathsf{T})\le\frac{1}{2}\big[w(\mathsf{T}^2+\|\mathsf{T}\|^2\big]. \tag{38}$$

The constant $\frac{1}{2}$ *is best possible in (38).*

The following general result for the product of two operators holds [27, p. 37]:

Theorem 3 *If* U,V *are two bounded linear operators on the Hilbert space* $(\mathsf{H},\langle\cdot,\cdot\rangle)$, *then* $w(\mathsf{UV})\le 4w(\mathsf{U})\,w(\mathsf{V})$. *In the case that* $\mathsf{UV}=\mathsf{VU}$, *then* $w(\mathsf{UV})\le 2w(\mathsf{U})\,w(\mathsf{V})$. *The constant* 2 *is best possible here.*

The following results are also well known [27, p. 38].

Theorem 4 *If* U *is a unitary operator that commutes with another operator* V, *then*

$$w(\mathsf{UV})\le w(\mathsf{V}). \tag{39}$$

If U *is an isometry and* $\mathsf{UV}=\mathsf{VU}$, *then (39) also holds true.*

We say that U and V *double commute* if $\mathsf{UV}=\mathsf{VU}$ and $\mathsf{UV}^*=\mathsf{V}^*\mathsf{U}$. The following result holds [27, p. 38].

Theorem 5 *If the operators* U *and* V *double commute, then*

$$w(\mathsf{UV})\le w(\mathsf{V})\,\|\mathsf{U}\|. \tag{40}$$

As a consequence of the above, we have [27, p. 39]:

Corollary 2 *Let* U *be a normal operator commuting with* V. *Then*

$$w(\mathsf{UV})\le w(\mathsf{U})\,w(\mathsf{V}). \tag{41}$$

A related problem with the inequality (40) is to find the best constant c for which the inequality

$$w(\mathsf{UV})\le c\,w(\mathsf{U})\,\|\mathsf{V}\|$$

holds for any two commuting operators $\mathsf{U},\mathsf{V}\in B(\mathsf{H})$. It is known that $1.064<c<1.169$, see [3], [35] and [36].

In relation to this problem, it has been shown in [25] that:

Theorem 6 *For any* $U, V \in B(H)$ *we have*

$$w\left(\frac{UV+VU}{2}\right) \leq \sqrt{2}w(U)\|V\|. \tag{42}$$

For other numerical radius inequalities see the recent monograph [18] and the references therein.

Theorem 7 *Let* A, B *two bounded linear operators on* H *such that* $\mathrm{Re}(B^*A)$ *is a nonnegative operator. Then for any* $U, V \in B(H)$ *we have*

$$\begin{aligned}\|V\mathrm{Re}(B^*A)U\| \leq \frac{1}{2}&\left\|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right\|\left\|V\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}\right\| \\ &+\frac{1}{2}\left\|V\left(\frac{|A|^2+|B|^2}{2}\right)U\right\|,\end{aligned} \tag{43}$$

$$\begin{aligned}w(V\mathrm{Re}(B^*A)U) \leq \frac{1}{2}&\left\|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right\|\left\|V\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}\right\| \\ &+\frac{1}{2}w\left(V\left(\frac{|A|^2+|B|^2}{2}\right)U\right)\end{aligned} \tag{44}$$

and

$$\begin{aligned}w(V\mathrm{Re}(B^*A)U) \leq \frac{1}{4}&\left\|\left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right|^2+\left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}V^*\right|^2\right\| \\ &+\frac{1}{2}w\left(V\left(\frac{|A|^2+|B|^2}{2}\right)U\right).\end{aligned} \tag{45}$$

Proof. From the inequality (20) we have

$$\begin{aligned}\left|\langle \mathrm{Re}(B^*A)Ux, V^*y\rangle\right| \leq \frac{1}{2}&\left[\left\langle\frac{|A|^2+|B|^2}{2}Ux, Ux\right\rangle^{1/2}\left\langle\frac{|A|^2+|B|^2}{2}V^*y, V^*y\right\rangle^{1/2}\right. \\ &\left.+\left|\left\langle\frac{|A|^2+|B|^2}{2}Ux, V^*y\right\rangle\right|\right]\end{aligned}$$

for any $x, y \in H$, which is equivalent to

$$\begin{aligned}&\left|\langle V\mathrm{Re}(B^*A)Ux, y\rangle\right| \\ &\leq \frac{1}{2}\left[\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2}\left\langle V\frac{|A|^2+|B|^2}{2}V^*y, y\right\rangle^{1/2}\right. \\ &\left.+\left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, y\right\rangle\right|\right]\end{aligned} \tag{46}$$

for any $x, y \in H$.

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ we have

$$\begin{aligned}
\|V\mathrm{Re}(B^*A)U\| &= \sup_{\|x\|=\|y\|=1} \left|\langle V\mathrm{Re}(B^*A)Ux, y\rangle\right| \\
&\leq \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[\left\langle U^* \frac{|A|^2+|B|^2}{2} Ux, x \right\rangle^{1/2} \left\langle V \frac{|A|^2+|B|^2}{2} V^* y, y \right\rangle^{1/2} \right. \\
&\quad \left. + \left| \left\langle V \frac{|A|^2+|B|^2}{2} Ux, y \right\rangle \right| \right] \\
&\leq \frac{1}{2} \left[\sup_{\|x\|=1} \left\langle U^* \frac{|A|^2+|B|^2}{2} Ux, x \right\rangle^{1/2} \sup_{\|y\|=1} \left\langle V \frac{|A|^2+|B|^2}{2} V^* y, y \right\rangle^{1/2} \right. \\
&\quad \left. + \sup_{\|x\|=\|y\|=1} \left| \left\langle V \frac{|A|^2+|B|^2}{2} Ux, y \right\rangle \right| \right] \\
&= \frac{1}{2} \left[\left\| U^* \frac{|A|^2+|B|^2}{2} U \right\|^{1/2} \left\| V \frac{|A|^2+|B|^2}{2} V^* \right\|^{1/2} + \left\| V \frac{|A|^2+|B|^2}{2} U \right\| \right].
\end{aligned} \tag{47}$$

Since

$$U^* \frac{|A|^2+|B|^2}{2} U = \left| \left(\frac{|A|^2+|B|^2}{2} \right)^{1/2} U \right|^2$$

and

$$V \frac{|A|^2+|B|^2}{2} V^* = \left| \left(\frac{|A|^2+|B|^2}{2} \right)^{1/2} V^* \right|^2$$

then

$$\left\| U^* \frac{|A|^2+|B|^2}{2} U \right\|^{1/2} = \left\| \left(\frac{|A|^2+|B|^2}{2} \right)^{1/2} U \right\|$$

and

$$\left\| V \frac{|A|^2+|B|^2}{2} V^* \right\|^{1/2} = \left\| \left(\frac{|A|^2+|B|^2}{2} \right)^{1/2} V^* \right\| = \left\| V \left(\frac{|A|^2+|B|^2}{2} \right)^{1/2} \right\|.$$

Using (46) we also have

$$\begin{aligned}
\left|\langle V\mathrm{Re}(B^*A)Ux, x\rangle\right| \leq \frac{1}{2} &\left[\left\langle U^* \frac{|A|^2+|B|^2}{2} Ux, x \right\rangle^{1/2} \left\langle V \frac{|A|^2+|B|^2}{2} V^* x, x \right\rangle^{1/2} \right. \\
&\left. + \left| \left\langle V \frac{|A|^2+|B|^2}{2} Ux, x \right\rangle \right| \right]
\end{aligned} \tag{48}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ we have

$$\begin{aligned} w(V\mathrm{Re}(B^*A)U) &= \sup_{\|x\|=1} \left|\langle V\mathrm{Re}(B^*A)Ux, x\rangle\right| \\ &\le \frac{1}{2}\Bigg[\sup_{\|x\|=1}\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2} \sup_{\|x\|=1}\left\langle V\frac{|A|^2+|B|^2}{2}V^*x, x\right\rangle^{1/2} \\ &\quad + \sup_{\|x\|=1}\left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle\right|\Bigg] \\ &= \frac{1}{2}\left[\left\|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right\|\left\|V\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}\right\| + w\left(V\frac{|A|^2+|B|^2}{2}U\right)\right] \end{aligned} \tag{49}$$

and the inequality (44) is proved.

By the arithmetic mean – geometric mean inequality we have

$$\begin{aligned} &\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2}\left\langle V\frac{|A|^2+|B|^2}{2}V^*x, x\right\rangle^{1/2} \\ &\le \frac{1}{2}\left[\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle + \left\langle V\frac{|A|^2+|B|^2}{2}V^*x, x\right\rangle\right] \\ &= \frac{1}{2}\left\langle\left[\left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right|^2 + \left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}V^*\right|^2\right]x, x\right\rangle \end{aligned} \tag{50}$$

for any $x \in H$, $\|x\| = 1$.

From (48) we have

$$\begin{aligned} &\left|\langle V\mathrm{Re}(B^*A)Ux, x\rangle\right| \\ &\le \frac{1}{4}\left\langle\left[\left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right|^2 + \left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}V^*\right|^2\right]x, x\right\rangle \\ &\quad + \frac{1}{2}\left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle\right| \end{aligned} \tag{51}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (51) we get the desired inequality (45). □

Corollary 3 *If* $A, B \geq 0$ *with* $AB = BA$, *then for any* $U, V \in B(H)$ *we have*

$$\begin{aligned} \|VABU\| \leq \frac{1}{2}\left\| \left(\frac{A^2+B^2}{2}\right)^{1/2} U\right\| \left\| V\left(\frac{A^2+B^2}{2}\right)^{1/2}\right\| \\ + \frac{1}{2}\left\| V\left(\frac{A^2+B^2}{2}\right) U\right\|, \end{aligned} \tag{52}$$

$$\begin{aligned} w(VABU) \leq \frac{1}{2}\left\| \left(\frac{A^2+B^2}{2}\right)^{1/2} U\right\| \left\| V\left(\frac{A^2+B^2}{2}\right)^{1/2}\right\| \\ + \frac{1}{2} w\left(V\left(\frac{A^2+B^2}{2}\right) U\right) \end{aligned} \tag{53}$$

and

$$\begin{aligned} w(VABU) \leq \frac{1}{4}\left\| \left| \left(\frac{A^2+B^2}{2}\right)^{1/2} U\right|^2 + \left| \left(\frac{A^2+B^2}{2}\right)^{1/2} V^*\right|^2 \right\| \\ + \frac{1}{2} w\left(V\left(\frac{A^2+B^2}{2}\right) U\right). \end{aligned} \tag{54}$$

Remark 2 *If we take in Corollary 3* $A = 1_H$ *and* $B = P$, *a projection on* H, *then we get*

$$\begin{aligned} \|VPU\| \leq \frac{1}{2}\left\| \left(\frac{1_H+P}{2}\right)^{1/2} U\right\| \left\| V\left(\frac{1_H+P}{2}\right)^{1/2}\right\| \\ + \frac{1}{2}\left\| V\left(\frac{1_H+P}{2}\right) U\right\|, \end{aligned} \tag{55}$$

$$\begin{aligned} w(VPU) \leq \frac{1}{2}\left\| \left(\frac{1_H+P}{2}\right)^{1/2} U\right\| \left\| V\left(\frac{1_H+P}{2}\right)^{1/2}\right\| \\ + \frac{1}{2} w\left(V\left(\frac{1_H+P}{2}\right) U\right) \end{aligned} \tag{56}$$

and

$$\begin{aligned} wVPU) \leq \frac{1}{4}\left\| \left| \left(\frac{1_H+P}{2}\right)^{1/2} U\right|^2 + \left| \left(\frac{1_H+P}{2}\right)^{1/2} V^*\right|^2 \right\| \\ + \frac{1}{2} w\left(V\left(\frac{1_H+P}{2}\right) U\right). \end{aligned} \tag{57}$$

Finally, we have:

Corollary 4 *Let* T *be a unitary operator with* $\mathrm{Re}\,(T) \geq 0$. *Then for any* $U, V \in B\,(H)$ *we have*

$$\|V\mathrm{Re}\,(T)\,U\| \leq \frac{1}{2}\left[\,\|U\|\,\|V\| + \|VU\|\,\right], \tag{58}$$

$$w\,(V\mathrm{Re}\,(T)\,U) \leq \frac{1}{2}\left[\,\|U\|\,\|V\| + w\,(VU)\,\right] \tag{59}$$

and

$$w\,(V\mathrm{Re}\,(T)\,U) \leq \frac{1}{4}\left\|\,|U|^2 + |V^*|^2\,\right\| + \frac{1}{2}w\,(VU)\,. \tag{60}$$

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Logarithmically complete monotonicity of a function related to the Catalan-Qi function

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Abstract. In the paper, the authors find necessary and sufficient conditions such that a function related to the Catalan-Qi function, which is an alternative generalization of the Catalan numbers, is logarithmically complete monotonic.

1 Introduction

It is stated in [11, 40] that the Catalan numbers C_n for $n \geq 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

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$$\frac{2}{1+\sqrt{1-4x}}=\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}C_nx^n=1+x+2x^2+5x^3+14x^4+\cdots.$$

One of explicit formulas of C_n for $n\ge 0$ reads that

$$C_n=\frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)},$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,d\,t,\quad \Re(z)>0$$

is the classical Euler gamma function. In [8, 11, 40, 43], it was mentioned that there exists an asymptotic expansion

$$C_x\sim\frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}}-\frac{9}{8}\frac{1}{x^{5/2}}+\frac{145}{128}\frac{1}{x^{7/2}}+\cdots\right) \tag{1}$$

for the Catalan function C_x.

A generalization of the Catalan numbers C_n was defined in [9, 10, 16] by

$${}_pd_n=\frac{1}{n}\binom{pn}{n-1}=\frac{1}{(p-1)n+1}\binom{pn}{n}$$

for $n\ge 1$. The usual Catalan numbers $C_n={}_2d_n$ are a special case with $p=2$.

In combinatorics and statistics, the Fuss-Catalan numbers $A_n(p,r)$ are defined [6, 45] as numbers of the form

$$A_n(p,r)=\frac{r}{np+r}\binom{np+r}{n}=r\frac{\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}.$$

It is easy to see that

$$A_n(2,1)=C_n,\quad n\ge 0\quad\text{and}\quad A_{n-1}(p,p)={}_pd_n,\quad n\ge 1.$$

There have existed some literature, such as [2, 4, 5, 7, 12, 14, 18, 19, 20, 21, 41, 42, 45], on the investigation of the Fuss-Catalan numbers $A_n(p,r)$.

In [31, Remark 1], an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced by

$$C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(z+a)}{\Gamma(z+b)},\quad \Re(a),\Re(b)>0,\quad \Re(z)\ge 0.$$

For the uniqueness and convenience of referring to the quantity $C(a,b;x)$, we call the quantity $C(a,b;x)$ the Catalan-Qi function and, when taking

$x = n \geq 0$, call $C(a, b; n)$ the Catalan-Qi numbers. In the recent papers [13, 15, 22, 24, 25, 29, 30, 31, 32, 33, 34, 39], among other things, some properties, including the general expression and a generalization of the asymptotic expansion (1), the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities, of the Catalan numbers C_n, the Catalan function C_x, the Catalan-Qi numbers $C(a, b; n)$, the Catalan-Qi function $C(a, b; x)$, and the Fuss-Catalan numbers $A_n(p, r)$ were established. Very recently, we discovered in [25, Theorem 1.1] a relation between the Fuss-Catalan numbers $A_n(p, r)$ and the Catalan-Qi numbers $C(a, b; n)$, which reads that

$$A_n(p, r) = r^n \frac{\prod_{k=1}^{p} C\left(\frac{k+r-1}{p}, 1; n\right)}{\prod_{k=1}^{p-1} C\left(\frac{k+r}{p-1}, 1; n\right)}$$

for integers $n \geq 0$, $p > 1$, and $r > 0$.

Recall from [3, 26, 28, 38] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if it satisfies $0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$ on I for all $k \in \mathbb{N}$.

From the viewpoint of analysis, motivated by the idea in the papers [27, 35, 36, 37] and closely-related references cited therein, the author considered in [23] the function $\mathcal{C}_{a,b;x}(t) = C(a + t, b + t; x)$ for $t, x \geq 0$ and $a, b > 0$ and obtained the following conclusions:

1. the function $\mathcal{C}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \leq x \leq 1$ and $a \leq b$ or $x \geq 1$ and $a \geq b$,
2. the function $\frac{1}{\mathcal{C}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \leq x \leq 1$ and $a \geq b$ or $x \geq 1$ and $a \leq b$.

This implies the logarithmically complete monotonicity of $[\mathcal{C}_{a,b;x}(t)]^{\pm 1}$ in $t \geq 0$ along with the ray $\begin{cases} u(t) = a + t \\ v(t) = b + t \end{cases}$ on the plane (u, v), where $x \geq 0$ and $a, b > 0$. Then one may ask a question: how about its logarithmically complete monotonicity along the ray $\begin{cases} u(t) = a + \alpha t \\ v(t) = b + \beta t \end{cases}$ for $\alpha, \beta \geq 0$ with $(\alpha, \beta) \neq (0, 0)$ when $x, t \geq 0$ and $a, b > 0$? In other words, is the function

$$\mathcal{C}_{a,b;x;\alpha,\beta}(t) = C(a + \alpha t, b + \beta t; x), \quad x \geq 0, \quad a, b > 0$$

of logarithmically complete monotonicity in $t \in [0, \infty)$? When $\alpha = \beta \neq 0$, this question has been answered essentially by the above-mentioned conclusions in [23]; when $\alpha = 0$ or $\beta = 0$, this question has been answered virtually by [34, Theorem 1.2] which states that the function $[C(a, b; x)]^{\pm 1}$ is logarithmically completely monotonic

1. with respect to $a > 0$ if and only if $x \gtreqless 1$,
2. with respect to $b > 0$ if and only if $x \lesseqgtr 1$.

In this paper, we will discuss the rest cases $\alpha, \beta > 0$ and $\alpha \neq \beta$ of the above question. Our main results can be formulated as the following theorem.

Theorem 1 *If and only if $\alpha = 0$ and $\beta > 0$, or $\alpha > 0$ and $\beta = 0$, or $\alpha = \beta > 0$, the function $\mathcal{C}_{a,b;x;\alpha,\beta}(t)$ is of some logarithmically complete monotonicity. Concretely speaking,*

1. *the function $[C(a, b; x)]^{\pm 1}$ is logarithmically completely monotonic*
   (a) *with respect to $a > 0$ if and only if $x \gtreqless 1$,*
   (b) *with respect to $b > 0$ if and only if $x \lesseqgtr 1$,*
2. *the function $\mathcal{C}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \le x \le 1$ and $a \le b$ or $x \ge 1$ and $a \ge b$,*
3. *the function $\frac{1}{\mathcal{C}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \le x \le 1$ and $a \ge b$ or $x \ge 1$ and $a \le b$.*

2 Proof of Theorem 1

Taking the logarithm of $\mathcal{C}_{a,b;x;\alpha,\beta}(t)$ and differentiating with respect to t give

$$[\ln \mathcal{C}_{a,b;x;\alpha,\beta}(t)]' = \psi(\beta t + b) - \psi(\alpha t + a) + x\left(\frac{1}{\beta t + b} - \frac{1}{\alpha t + a}\right) + \psi(\alpha t + x + a) - \psi(\beta t + x + b).$$

Making use of

$$\psi(z) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}}\right) \mathrm{d}u, \quad \Re(z) > 0$$

in [1, p. 259, 6.3.21] leads to

$$\begin{aligned}
[\ln \mathcal{C}_{a,b;x;\alpha,\beta}(t)]' &= \int_0^\infty \frac{e^{-(a+\alpha t)u} - e^{-(b+\beta t)u}}{1-e^{-u}} \,\mathrm{d}\, u \\
&\quad + x \int_0^\infty \Big[e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}\Big] \,\mathrm{d}\, u \\
&\quad + \int_0^\infty \frac{e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}}{1-e^{-u}} e^{-xu} \,\mathrm{d}\, u \\
&= \int_0^\infty \big[e^{-xu} - 1 + x\big(1-e^{-u}\big)\big] \frac{e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}}{1-e^{-u}} \,\mathrm{d}\, u \\
&= x \int_0^\infty \bigg(\frac{1-e^{-u}}{u} - \frac{1-e^{-xu}}{xu}\bigg) \frac{e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}}{1-e^{-u}} u \,\mathrm{d}\, u.
\end{aligned}$$

It is easy to see that the function $\frac{1-e^{-u}}{u}$ is positive and strictly decreasing on $(0,\infty)$. Hence,

$$\frac{1-e^{-u}}{u} - \frac{1-e^{-xu}}{xu} \gtreqless 0 \tag{2}$$

for $u \in (0,\infty)$ if and only if $x \lesseqgtr 1$.

Recall from [17, Chapter XIII], [38, Chapter 1], and [44, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \le (-1)^k f^{(k)}(x) < \infty$ on I for all $k \ge 0$. It is not difficult to see that a positive function f is logarithmically completely monotonic if and only if the function $-(\ln f)'$ is completely monotonic. The famous Bernstein-Widder theorem, [44, p. 160, Theorem 12a], states that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \le x < \infty$ is that $f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\, \alpha(t)$, where α is bounded and non-decreasing and the above integral converges for $0 \le x < \infty$. Therefore, it is sufficient to find necessary and sufficient conditions on $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha \ne \beta$ for the function

$$\begin{aligned}
e^{-(b+\beta t)u} - e^{-(a+\alpha t)u} &= \int_{(b+\beta t)u}^{(a+\alpha t)u} e^{-v} \,\mathrm{d}\, v \\
&= \int_0^1 [(a-b) + (\alpha-\beta)t] u e^{-[(1-s)(b+\beta t)+s(a+\alpha t)]u} \,\mathrm{d}\, s \\
&= \int_0^1 [(a-b) + (\alpha-\beta)t] e^{-[(1-s)\beta+s\alpha]ut} u e^{-[(1-s)b+sa]u} \,\mathrm{d}\, s
\end{aligned}$$

to be completely monotonic in $t \in [0,\infty)$ for all $u \in (0,\infty)$.

By induction, we obtain

$$\left[(A+Bt)e^{-Dt}\right]^{(k)}=(-1)^k D^{k-1}(BDt+AD-kB)e^{-Dt},\quad k\ge 0,$$

where A,B,D are real constants. Accordingly, the function $(A+Bt)e^{-Dt}$ is completely monotonic in $t\in[0,\infty)$ if and only if $A,B\ge 0$, $D>0$, and

$$D^{k-1}(BDt+AD-kB)\ge 0,\quad k\ge 0,\quad t\in[0,\infty). \tag{3}$$

Simply speaking, the function $(A+Bt)e^{-Dt}$ is completely monotonic in $t\in[0,\infty)$ if and only if $A\ge 0$, $B=0$, and $D>0$. Applying A to $a-b$, B to $\alpha-\beta$, and D to $[(1-s)\beta+s\alpha]u$ yields that the function $e^{-(b+\beta t)u}-e^{-(a+\alpha t)u}$ is completely monotonic in $t\in[0,\infty)$ if and only if $a\ge b$, $\alpha=\beta$, and $\alpha,\beta\ge 0$ with $(\alpha,\beta)\ne(0,0)$. Combining this result with the inequality (2) and with the proofs of [23, Theorem 1.1] and [34, Theorem 1.2] concludes that, if and only if $\alpha=0$ and $\beta>0$, or $\alpha>0$ and $\beta=0$, or $\alpha=\beta>0$, the function $\mathcal{C}_{a,b;x;\alpha,\beta}(t)$ is of some logarithmically complete monotonicity. The proof of Theorem 1 is thus complete.

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Generalizations of Steffensen's inequality via some Euler-type identities

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Abstract. Using Euler-type identities some new generalizations of Steffensen's inequality for $n-$convex functions are obtained. Moreover, the Ostrowski-type inequalities related to obtained generalizations are given. Furthermore, using inequalities for the Čebyšev functional in terms of the first derivative some new bounds for the remainder in identities related to generalizations of Steffensen's inequality are proven.

1 Introduction

Firstly, we recall the well-known Steffensen inequality which reads (see [11]):

Theorem 1 *Suppose that* f *is nonincreasing and* g *is integrable on* $[a, b]$ *with* $0 \leq g \leq 1$ *and* $\lambda = \int_a^b g(t)dt$. *Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt. \qquad (1)$$

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The inequalities are reversed for f *nondecreasing.*

Mitrinović stated in [8] that the inequalities in (1) follow from the identities

$$\begin{aligned} &\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &= \int_a^{a+\lambda} [f(t) - f(a+\lambda)][1-g(t)]dt + \int_{a+\lambda}^b [f(a+\lambda) - f(t)]g(t)dt \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\ &= \int_a^{b-\lambda} [f(t) - f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b [f(b-\lambda) - f(t)][1-g(t)]dt. \end{aligned} \tag{3}$$

In [4] Dedić, Matić and Pečarić derived Euler-type identities which extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials.

Theorem 2 *Let* $f : [a, b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is continuous function of bounded variation on* $[a, b]$ *for some* $n \geq 1$*. Then for every* $x \in [a, b]$ *we have*

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + T_n(x) + R_n^1(x) \tag{4}$$

and

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + T_{n-1}(x) + R_n^2(x), \tag{5}$$

where $T_0(x) = 0$*, and for* $1 \leq m \leq n$

$$\begin{aligned} T_m(x) &= \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right], \\ R_n^1(x) &= -\frac{(b-a)^{n-1}}{n!} \int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t), \\ R_n^2(x) &= -\frac{(b-a)^{n-1}}{n!} \int_{[a,b]} \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t). \end{aligned}$$

Here, $B_k(x)$, $k \geq 0$ are the Bernoulli polynomials, B_k, $k \geq 0$ are the Bernoulli numbers and $B_k^*(x)$, $k \geq 0$ are periodic functions of period one, related to the Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \leq x < 1$$

and

$$B_k^*(x+1) = B_k^*(x), \quad x \in \mathbb{R}.$$

Let us recall some properties of the Bernoulli polynomials. The first three Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$

and

$$B_n'(x) = nB_{n-1}(x), \; n \in \mathbb{N}.$$

$B_0^*(x)$ is a constant equal to 1, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer. For $k \geq 2$, $B_k^*(x)$ is a continuous function.

For more details on Bernoulli polynomials and Bernoulli numbers see [1] or [7].

Next, let us recall the definition of the divided difference.

Definition 1 *Let* f *be a real-valued function defined on the segment* $[a, b]$*. The* $n-$th order divided difference *of the function* f *at distinct points* $x_0, \ldots, x_n \in [a, b]$*, is defined recursively by*

$$[x_i; f] = f(x_i), \quad (i = 0, \ldots, n)$$

and

$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}.$$

The value $[x_0, \ldots, x_n; f]$ is independent of the order of the points $x_0, \ldots, x_n$. The previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq \cdots \leq x_n$ and letting

$$[\underbrace{x, \ldots, x}_{(j+1) \text{ times}}; f] = \frac{f^{(j)}(x)}{j!}, \tag{6}$$

provided that $f^{(j)}$ exists.

In this paper we use Euler-type identities given in Theorem 2 to obtain some new identities related to Steffensen's inequality. Using these new identities we obtain new generalizations of Steffensen's inequality for $n-$convex functions. In Section 3 we give the Ostrowski-type inequalities related to obtained generalizations. In Section 4 we prove some new bounds for the remainder in obtained identities using inequalities for the Čebyšev functional in terms of the first derivative. Further, in Section 5 we give mean value theorems for functionals related to obtained new generalizations of Steffensen's inequality for $n-$convex functions. In Section 6 we use previously defined functionals to construct $n-$exponentially convex functions. We conclude this paper with the applications to Stolarsky-type means.

Throughout the paper, it is assumed that all integrals under consideration exist and that they are finite.

2 Generalizations of Steffensen's inequality via Euler-type identities

The aim of this section is to obtain generalizations of Steffensen's inequality for $n-$convex functions using the identities (4) and (5). We begin with the following result:

Theorem 3 *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a, b]$ for some $n \geq 2$ and let $g : [a, b] \to \mathbb{R}$ be an integrable function. Let $\lambda = \int_a^b g(t)dt$ and let the function G_1 be defined by*

$$G_1(x) = \begin{cases} \int_a^x (1 - g(t))dt, & x \in [a, a + \lambda], \\ \int_x^b g(t)dt, & x \in [a + \lambda, b]. \end{cases} \tag{7}$$

Then

$$\begin{aligned} &\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) \times [f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &= \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b G_1(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx \right) f^{(n)}(t)dt. \end{aligned} \tag{8}$$

Proof. Applying integration by parts and then using the definition of the function G_1, the identity (2) becomes

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\
&\qquad = -\int_a^{a+\lambda}\left(\int_a^x (1-g(t)dt\right)df(x) - \int_{a+\lambda}^b\left(\int_x^b g(t)dt\right)df(x) \\
&\qquad = -\int_a^b G_1(x)f'(x)dx.
\end{aligned}$$

Now applying the identity (4) on the function f' we obtain

$$\begin{aligned}
f'(x) &= \frac{f(b)-f(a)}{b-a} + \sum_{k=1}^{n}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)[f^{(k)}(b)-f^{(k)}(a)] \\
&\quad - \frac{(b-a)^{n-1}}{n!}\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)f^{(n+1)}(t)dt \\
&= \sum_{k=0}^{n}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)[f^{(k)}(b)-f^{(k)}(a)] \\
&\quad - \frac{(b-a)^{n-1}}{n!}\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)f^{(n+1)}(t)dt.
\end{aligned} \tag{9}$$

Hence, using (9) we obtain

$$\begin{aligned}
&\int_a^b G_1(x)f'(x)dx \\
&= \sum_{k=0}^{n}\frac{(b-a)^{k-1}}{k!}\left(\int_a^b G_1(x)B_k\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k)}(b)-f^{(k)}(a)] \\
&\quad - \frac{(b-a)^{n-1}}{n!}\int_a^b G_1(x)\left(\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)f^{(n+1)}(t)dt\right)dx.
\end{aligned} \tag{10}$$

Applying Fubini's theorem on the last term in (10) and replacing n with $n-1$ we obtain (8). This identity is valid for $n-1 \geq 1$, i.e. $n \geq 2$. □

Similarly, using the identity (5) the following theorem holds.

Theorem 4 *Let* $f : [a, b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is continuous function of bounded variation on* $[a, b]$ *for some* $n \geq 2$ *and let* $g : [a, b] \to \mathbb{R}$ *be an* *integrable function. Let* $\lambda = \int_a^b g(t)dt$ *and let the function* G_1 *be defined by (7). Then*

$$\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] = \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b G_1(x) \left[B^*_{n-1}\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right) \right] dx \right) f^{(n)}(t)dt. \quad (11)$$

We continue with the results related to the identity (3).

Theorem 5 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is continuous function of bounded variation on* $[a,b]$ *for some* $n \geq 2$ *and let* $g : [a,b] \to \mathbb{R}$ *be an integrable function. Let* $\lambda = \int_a^b g(t)dt$ *and let the function* G_2 *be defined by*

$$G_2(x) = \begin{cases} \int_a^x g(t)dt, & x \in [a, b-\lambda], \\ \int_x^b (1 - g(t))dt, & x \in [b-\lambda, b]. \end{cases} \quad (12)$$

Then

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) \times [f^{(k-1)}(b) - f^{(k-1)}(a)] = \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b G_2(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx \right) f^{(n)}(t)dt. \quad (13)$$

Proof. Similar to the proof of Theorem 3 applying integration by parts on the identity (3) and then using the identity (4) on the function f'. □

Theorem 6 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is continuous function of bounded variation on* $[a,b]$ *for some* $n \geq 2$ *and let* $g : [a,b] \to \mathbb{R}$ *be an integrable function. Let* $\lambda = \int_a^b g(t)dt$ *and let the function* G_2 *be defined by*

(12). Then

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt$$
$$+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]$$
$$= \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b G_2(x) \left[B^*_{n-1}\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right) \right] dx \right) f^{(n)}(t)dt. \tag{14}$$

Proof. Similar to the proof of Theorem 5 using the identity (5) on the function f'. □

Using previously obtained identities we can obtain the following generalizations of Steffensen's inequality for $n-$convex functions.

Theorem 7 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geq 2$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function. Let $\lambda = \int_a^b g(t)dt$ and let the function G_1 be defined by (7).*

(i) *If f is $n-$convex and*

$$\int_a^b G_1(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx \geq 0, \quad t \in [a,b], \tag{15}$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt$$
$$+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]. \tag{16}$$

(ii) *If f is $n-$convex and*

$$\int_a^b G_1(x) \left[B^*_{n-1}\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right) \right] dx \geq 0, \quad t \in [a,b], \tag{17}$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt$$
$$+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]. \tag{18}$$

Proof. If the function f is n-convex, without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$ see [10, p. 16 and p. 293]. Now we can apply Theorem 3 to obtain (16) and Theorem 4 to obtain (18). □

Similarly, applying Theorems 5 and 6 we obtain the following generalizations of Steffensen's inequality for $n-$convex functions.

Theorem 8 *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a, b]$ for some $n \geq 2$ and let $g : [a, b] \to \mathbb{R}$ be an integrable function. Let $\lambda = \int_a^b g(t)dt$ and let the function G_2 be defined by (12).*

(i) *If f is $n-$convex and*

$$\int_a^b G_2(x) B_{n-1}^* \left(\frac{x-t}{b-a} \right) dx \geq 0, \quad t \in [a, b], \tag{19}$$

then

$$\begin{aligned} \int_a^b f(t)g(t)dt &\geq \int_{b-\lambda}^b f(t)dt \\ &- \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1} \left(\frac{x-a}{b-a} \right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]. \end{aligned} \tag{20}$$

(ii) *If f is $n-$convex and*

$$\int_a^b G_2(x) \left[B_{n-1}^* \left(\frac{x-t}{b-a} \right) - B_{n-1} \left(\frac{x-a}{b-a} \right) \right] dx \geq 0, \quad t \in [a, b], \tag{21}$$

then

$$\begin{aligned} \int_a^b f(t)g(t)dt &\geq \int_{b-\lambda}^b f(t)dt \\ &- \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1} \left(\frac{x-a}{b-a} \right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]. \end{aligned} \tag{22}$$

3 Ostrowski-type inequalities

In this section we give the Ostrowski-type inequalities related to generalizations obtained in the previous section.

Theorem 9 *Suppose that all assumptions of Theorem 3 hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. *Let* $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \geq 2$. *Then we have*

$$\left| \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \right.$$
$$\left. + \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \right| \tag{23}$$
$$\leq \frac{(b-a)^{n-2}}{(n-1)!} \left\|f^{(n)}\right\|_p \left(\int_a^b \left| \int_a^b G_1(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx \right|^q dt \right)^{\frac{1}{q}}.$$

The constant on the right-hand side of (23) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. Let us denote

$$C(t) = \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b G_1(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx.$$

Using the identity (8) and applying Hölder's inequality we obtain

$$\left| \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \right.$$
$$\left. + \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \right|$$
$$= \left| \int_a^b C(t) f^{(n)}(t) dt \right| \leq \left\|f^{(n)}\right\|_p \left(\int_a^b |C(t)|^q \, dt \right)^{\frac{1}{q}}.$$

For the proof of the sharpness of the constant $\left(\int_a^b |C(t)|^q \, dt\right)^{\frac{1}{q}}$ let us find a function f for which the equality in (23) is obtained.
For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = \operatorname{sgn} C(t) \, |C(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $f^{(n)}(t) = \operatorname{sgn} C(t)$.
For $p = 1$ we prove that

$$\left| \int_a^b C(t) f^{(n)}(t) dt \right| \leq \max_{t \in [a,b]} |C(t)| \left(\int_a^b \left| f^{(n)}(t) \right| dt \right) \tag{24}$$

is the best possible inequality. Suppose that $|C(t)|$ attains its maximum at $t_0 \in [a,b]$. First we assume that $C(t_0) > 0$. For ε small enough we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} 0, & a \le t \le t_0, \\ \frac{1}{\varepsilon\, n!}(t-t_0)^n, & t_0 \le t \le t_0+\varepsilon, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\varepsilon \le t \le b. \end{cases}$$

Then for ε small enough

$$\left|\int_a^b C(t) f^{(n)}(t)dt\right| = \left|\int_{t_0}^{t_0+\varepsilon} C(t)\frac{1}{\varepsilon}dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt.$$

Now from the inequality (24) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt \le C(t_0)\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}dt = C(t_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt = C(t_0)$$

the statement follows. In the case $C(t_0) < 0$, we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t-t_0-\varepsilon)^{n-1},, & a \le t \le t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \le t \le t_0+\varepsilon, \\ 0, & t_0+\varepsilon \le t \le b, \end{cases}$$

and the rest of the proof is the same as above. □

Using the identity (11) we obtain the following result.

Theorem 10 *Suppose that all assumptions of Theorem 4 hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p,q \le \infty$, $1/p+1/q=1$. *Let* $|f^{(n)}|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \ge 2$. *Then we have*

$$\begin{aligned} &\left|\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \right. \\ &\left. + \sum_{k=1}^{n-1}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\right| \\ &\le \frac{(b-a)^{n-2}}{(n-1)!}\|f^{(n)}\|_p\left(\int_a^b\left|\int_a^b G_1(x)\left[B^*_{n-1}\left(\frac{x-t}{b-a}\right)\right.\right.\right. \\ &\left.\left.\left. - B_{n-1}\left(\frac{x-a}{b-a}\right)\right]dx\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{25}$$

The constant on the right-hand side of (25) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Similarly, we obtain the following Ostrowski-type inequalities related to results given in Theorems 5 and 6.

Theorem 11 *Suppose that all assumptions of Theorem 5 hold. Assume* (p, q) *is a pair of conjugate exponents, that is* $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. *Let* $\left|f^{(n)}\right|^p : [a, b] \to \mathbb{R}$ *be an R-integrable function for some* $n \geq 2$. *Then we have*

$$\left| \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \right.$$
$$\left. + \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \right| \tag{26}$$
$$\leq \frac{(b-a)^{n-2}}{(n-1)!} \left\| f^{(n)} \right\|_p \left(\int_a^b \left| \int_a^b G_2(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx \right|^q dt \right)^{\frac{1}{q}}.$$

The constant on the right-hand side of (26) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Theorem 12 *Suppose that all assumptions of Theorem 6 hold. Assume* (p, q) *is a pair of conjugate exponents, that is* $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. *Let* $\left|f^{(n)}\right|^p : [a, b] \to \mathbb{R}$ *be an R-integrable function for some* $n \geq 2$. *Then we have*

$$\left| \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \right.$$
$$\left. + \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \right| \tag{27}$$
$$\leq \frac{(b-a)^{n-2}}{(n-1)!} \left\| f^{(n)} \right\|_p \left(\int_a^b \left| \int_a^b G_2(x) \left[B^*_{n-1}\left(\frac{x-t}{b-a}\right) \right. \right. \right.$$
$$\left. \left. \left. - B_{n-1}\left(\frac{x-a}{b-a}\right) \right] dx \right|^q dt \right)^{\frac{1}{q}}.$$

The constant on the right-hand side of (27) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

4 Generalizations related to the bounds for the Čebyšev functional

Let $f, h : [a, b] \to \mathbb{R}$ be two Lebesgue integrable functions. By $T(f, h)$ we denote the Čebyšev functional

$$T(f, h) := \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b h(t)dt.$$

In [3] Cerone and Dragomir proved the following bound for the Čebyšev functional.

Theorem 13 *Let* $f : [a, b] \to \mathbb{R}$ *be a Lebesgue integrable function and* $h : [a, b] \to \mathbb{R}$ *be an absolutely continuous function with* $(\cdot - a)(b - \cdot)[h']^2 \in L[a, b]$. *Then we have the inequality*

$$|T(f, h)| \le \frac{1}{\sqrt{2}}[T(f, f)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (x-a)(b-x)[h'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{28}$$

The constant $\frac{1}{\sqrt{2}}$ *in* (28) *is the best possible.*

Also, Cerone and Dragomir [3] proved the following inequality of Grüss type.

Theorem 14 *Assume that* $h : [a, b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a, b]$ *and* $f : [a, b] \to \mathbb{R}$ *is absolutely continuous with* $f' \in L_\infty[a, b]$. *Then we have the inequality*

$$|T(f, h)| \le \frac{1}{2(b-a)}\|f'\|_\infty \int_a^b (x-a)(b-x)dh(x). \tag{29}$$

The constant $\frac{1}{2}$ *in* (29) *is the best possible.*

In the sequel we use the aforementioned bound for the Čebyšev functional to obtain generalizations of the results proved in Section 2.

Firstly, let us denote

$$H_1(t) = \int_a^b G_1(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx. \tag{30}$$

Theorem 15 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n)}$ *is absolutely continuous function for some* $n \geq 2$ *with* $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$ *and let* g *be an integrable function on* $[a,b]$. *Let* $\lambda = \int_a^b g(t)dt$ *and let the functions* G_1 *and* H_1 *be defined by* (7) *and* (30). *Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &- \frac{(b-a)^{n-3}[f^{(n-1)}(b) - f^{(n-1)}(a)]}{(n-1)!}\int_a^b H_1(t)dt = S_n^1(f;a,b)\end{aligned} \tag{31}$$

where the remainder $S_n^1(f;a,b)$ *satisfies the estimation*

$$\left|S_n^1(f;a,b)\right| \leq \frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!}[T(H_1,H_1)]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{32}$$

Proof. Applying Theorem 13 for $f \to H_1$ and $h \to f^{(n)}$ we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b H_1(t)f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b H_1(t)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\ &\leq \frac{1}{\sqrt{2}}[T(H_1,H_1)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.\end{aligned} \tag{33}$$

Hence, if we subtract

$$\begin{aligned}&\frac{(b-a)^{n-1}}{(n-1)!} \cdot \frac{1}{b-a}\int_a^b H_1(t)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt \\ &\qquad = \frac{(b-a)^{n-3}}{(n-1)!}[f^{(n-1)}(b) - f^{(n-1)}(a)]\int_a^b H_1(t)dt\end{aligned}$$

from both side of the identity (8) and use the inequality (33) we obtain the representation (31). □

Similarly, using the identity (11) we obtain the following result. Let us denote

$$\Phi_1(t) = \int_a^b G_1(x)\left[B_{n-1}^*\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right]dx. \tag{34}$$

Theorem 16 *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function for some $n \geq 2$ with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a, b]$ and let g be an integrable function on $[a, b]$. Let $\lambda = \int_a^b g(t)dt$ and let the functions G_1 and Φ_1 be defined by (7) and (34). Then*

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\
&+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \\
&- \frac{(b-a)^{n-3}[f^{(n-1)}(b) - f^{(n-1)}(a)]}{(n-1)!} \int_a^b \Phi_1(t)dt = S_n^2(f; a, b)
\end{aligned} \tag{35}$$

where the remainder $S_n^2(f; a, b)$ satisfies the estimation

$$\left| S_n^2(f; a, b) \right| \leq \frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!} [T(\Phi_1, \Phi_1)]^{\frac{1}{2}} \left| \int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}.$$

We continue with the results related to the identities (13) and (14). Let us denote

$$H_2(t) = \int_a^b G_2(x) B_{n-1}^*\left(\frac{x-t}{b-a}\right) dx \tag{36}$$

and

$$\Phi_2(t) = \int_a^b G_2(x) \left[B_{n-1}^*\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right) \right] dx. \tag{37}$$

Theorem 17 *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function for some $n \geq 2$ with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a, b]$ and let g be an integrable function on $[a, b]$. Let $\lambda = \int_a^b g(t)dt$ and let the functions G_2, H_2 and Φ_2 be defined by (12), (36) and (37) respectively. Then*

(i)

$$\begin{aligned}
&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\
&+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \\
&- \frac{(b-a)^{n-3}[f^{(n-1)}(b) - f^{(n-1)}(a)]}{(n-1)!} \int_a^b H_2(t)dt = S_n^3(f; a, b)
\end{aligned} \tag{38}$$

where the remainder $S_n^3(f;a,b)$ *satisfies the estimation*

$$\left|S_n^3(f;a,b)\right| \leq \frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!}\left[T(H_2,H_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.$$

(ii)

$$\begin{aligned}&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\ &+\sum_{k=1}^{n-1}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)] \\ &-\frac{(b-a)^{n-3}[f^{(n-1)}(b)-f^{(n-1)}(a)]}{(n-1)!}\int_a^b \Phi_2(t)dt = S_n^4(f;a,b)\end{aligned} \tag{39}$$

where the remainder $S_n^4(f;a,b)$ *satisfies the estimation*

$$\left|S_n^4(f;a,b)\right| \leq \frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!}\left[T(\Phi_2,\Phi_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 15. □

The following Grüss type inequalities also hold.

Theorem 18 *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ $(n\geq 2)$ *is absolutely continuous function and* $f^{(n+1)}\geq 0$ *on* $[a,b]$. *Let the function* H_1 *be defined by* (30). *Then we have the representation* (31) *and the remainder* $S_n^1(f;a,b)$ *satisfies the bound*

$$\left|S_n^1(f;a,b)\right| \leq \frac{(b-a)^{n-1}}{(n-1)!}\|H_1'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}-\left[a,b;f^{(n-2)}\right]\right\}. \tag{40}$$

Proof. Applying Theorem 14 for $f\to H_1$ and $h\to f^{(n)}$ we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b H_1(t)f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b H_1(t)dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\ &\leq \frac{1}{2(b-a)}\|H_1'\|_\infty\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt.\end{aligned} \tag{41}$$

Since

$$\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt = \int_a^b [2t-(a+b)]f^{(n)}(t)dt$$
$$= (b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - 2\left(f^{(n-2)}(b)-f^{(n-2)}(a)\right).$$

Using the representation (8) and the inequality (41) we deduce (40). □

Theorem 19 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ $(n \geq 2)$ is absolutely continuous function and $f^{(n+1)} \geq 0$ on $[a,b]$. Let H_1, Φ_1 and Φ_2 be defined by (30), (34) and (37), respectively. Then we have the representations (35), (38) and (39) where the remainders $S_n^i(f;a,b), i=2,3,4$ satisfy the bounds*

$$\left|S_n^2(f;a,b)\right| \leq \frac{(b-a)^{n-1}}{(n-1)!}\|\Phi_1'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \left[a,b;f^{(n-2)}\right]\right\},$$

$$\left|S_n^3(f;a,b)\right| \leq \frac{(b-a)^{n-1}}{(n-1)!}\|H_2'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \left[a,b;f^{(n-2)}\right]\right\}$$

and

$$\left|S_n^4(f;a,b)\right| \leq \frac{(b-a)^{n-1}}{(n-1)!}\|\Phi_2'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \left[a,b;f^{(n-2)}\right]\right\}.$$

5 Mean value theorems

Motivated by inequalities (16), (18), (20) and (22), under the assumptions of Theorems 7 and 8 we define the following linear functionals:

$$L_1(f) = \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt$$
$$+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)] \tag{42}$$

$$L_2(f) = \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt$$
$$+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)] \tag{43}$$

$$L_3(f) = \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \tag{44}$$

$$L_4(f) = \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]. \tag{45}$$

Remark 1 We have $L_i(f) \geq 0$, $i = 1, \ldots, 4$ for all $n-$convex functions f.

Now, we give the Lagrange-type mean value theorem related to defined functionals.

Theorem 20 *Let* $f : [a, b] \to \mathbb{R}$ *be such that* $f \in C^n[a, b]$. *If the inequalities in* (15) $(i = 1)$, (17) $(i = 2)$, (19) $(i = 3)$ *and* (21) $(i = 4)$ *hold, then there exist* $\xi_i \in [a, b]$ *such that*

$$L_i(f) = f^{(n)}(\xi_i) L_i(\varphi), \quad i = 1, \ldots, 4 \tag{46}$$

where $\varphi(x) = \frac{x^n}{n!}$ *and* L_i, $i = 1, \ldots, 4$ *are defined by* (42)-(45).

Proof. Let us denote

$$m = \min_{x \in [a,b]} f^{(n)}(x) \quad \text{and} \quad M = \max_{x \in [a,b]} f^{(n)}(x).$$

For a given function $f \in C^n[a, b]$ we define the functions $F_1, F_2 : [a, b] \to \mathbb{R}$ with

$$F_1(x) = M\varphi(x) - f(x) \quad \text{and} \quad F_2(x) = f(x) - m\varphi(x).$$

Now $F_1^{(n)}(x) = M - f^{(n)}(x) \geq 0$, so from Remark 1 we conclude $L_i(F_1) \geq 0$, $i = 1, \ldots, 4$ and then $L_i(f) \leq M \cdot L_i(\varphi)$. Similarly, from $F_2^{(n)}(x) = f^{(n)}(x) - m \geq 0$ we conclude $m \cdot L_i(\varphi) \leq L_i(f)$. Hence, $m \cdot L_i(\varphi) \leq L_i(f) \leq M \cdot L_i(\varphi)$, $i = 1, \ldots, 4$. If $L_i(\varphi) = 0$, then (46) holds for all $\xi_i \in [a, b]$. Otherwise,

$$m \leq \frac{L_i(f)}{L_i(\varphi)} \leq M, \quad i = 1, \ldots 4.$$

Since $f^{(n)}$ is continuous on $[a, b]$ there exist $\xi_i \in [a, b]$, $i = 1, \ldots, 4$ such that (46) holds and the proof is complete. □

We continue with the Cauchy-type mean value theorem.

Theorem 21 *Let* $f, F : [a, b] \to \mathbb{R}$ *be such that* $f, F \in C^n[a, b]$ *and* $F^{(n)} \neq 0$. *If the inequalities in* (15) $(i = 1)$, (17) $(i = 2)$, (19) $(i = 3)$ *and* (21) $(i = 4)$ *hold, then there exist* $\xi_i \in [a, b]$ *such that*

$$\frac{L_i(f)}{L_i(F)} = \frac{f^{(n)}(\xi)}{F^{(n)}(\xi)}, \quad i = 1, \ldots, 4 \tag{47}$$

where L_i, $i = 1, \ldots, 4$ *are defined by* (42)-(45).

Proof. We define functions $\phi_i(x) = f(x)L_i(F) - F(x)L_i(f)$, $i = 1, \ldots, 4$. According to Theorem 20 there exist $\xi_i \in [a, b]$ such that

$$L_i(\phi_i) = \phi_i^{(n)}(\xi_i)L_i(\varphi), \quad i = 1, \ldots, 4.$$

Since $L_i(\phi_i) = 0$ it follows that $f^{(n)}(\xi_i)L_i(F) - F^{(n)}(\xi_i)L_i(f) = 0$ and (47) is proved. □

6 $n-$exponential convexity

Let us begin by recalling some definitions and results related to $n-$exponential convexity. For more details see e.g. [2], [6] and [9].

Definition 2 *A function* $\psi : I \to \mathbb{R}$ *is said to be* n-exponentially convex in the Jensen sense *on* I *if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

holds for all choices of $\xi_i \in \mathbb{R}$ *and* $x_i \in I$, $i = 1, \ldots, n$.

A function $\psi : I \to \mathbb{R}$ *is said to be* n-exponentially convex *if it is* n-*exponentially convex in the Jensen sense and continuous on* I.

Remark 2 It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

Definition 3 *A function* $\psi : I \to \mathbb{R}$ *is said to be* exponentially convex in the Jensen sense *on* I *if it is* n*-exponentially convex in the Jensen sense for all* $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ *is said to be* exponentially convex *if it is exponentially convex in the Jensen sense and continuous.*

Remark 3 It is known that $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Proposition 1 *If* f *is a convex function on* I *and if* $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, *then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f *is concave, the inequality is reversed.*

We use defined functionals L_i, $i = 1, \ldots, 4$ to construct exponentially convex functions. An elegant method of producing $n-$ exponentially convex and exponentially convex functions is given in [9]. In the sequel the notion log denotes the natural logarithm function.

Theorem 22 *Let* $\Omega = \{f_p : p \in J\}$, *where J is an interval in* $\mathbb{R}$, *be a family of functions defined on an interval* I *in* $\mathbb{R}$ *such that the function* $p \mapsto [x_0, \ldots, x_m; f_p]$ *is* $n-$*exponentially convex in the Jensen sense on* J *for every* $(m+1)$ *mutually different points* $x_0, \ldots, x_m \in I$. *Let* L_i, $i = 1, \ldots, 4$ *be linear functionals defined by* (42) $-$ (45). *Then* $p \mapsto L_i(f_p)$ *is* $n-$*exponentially convex function in the Jensen sense on* J.
If the function $p \mapsto L_i(f_p)$ *is continuous on* J, *then it is* $n-$*exponentially convex on* J.

Proof. For $\xi_j \in \mathbb{R}$ and $p_j \in J$, $j = 1, \ldots, n$, we define the function

$$\Psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j+p_k}{2}}(x).$$

Using the assumption that the function $p \mapsto [x_0, \ldots, x_m; f_p]$ is n-exponentially convex in the Jensen sense, we have

$$[x_0, \ldots, x_m, \Psi] = \sum_{j,k=1}^{n} \xi_j \xi_k [x_0, \ldots, x_m; f_{\frac{p_j+p_k}{2}}] \geq 0,$$

which in turn implies that Ψ is a m-convex function on J, so $L_i(\Psi) \geq 0$, $i = 1, \ldots, 4$. Hence

$$\sum_{j,k=1}^{n} \xi_j \xi_k L_i \left(f_{\frac{p_j+p_k}{2}} \right) \geq 0.$$

We conclude that the function $p \mapsto L_i(f_p)$ is n-exponentially convex on J in the Jensen sense.

If the function $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is n-exponentially convex by definition. □

As an immediate consequence of the above theorem we obtain the following corollary:

Corollary 1 *Let* $\Omega = \{f_p : p \in J\}$, *where* J *is an interval in* $\mathbb{R}$, *be a family of functions defined on an interval* I *in* R, *such that the function* $p \mapsto [x_0, \ldots, x_m; f_p]$ *is exponentially convex in the Jensen sense on* J *for every* $(m+1)$ *mutually different points* $x_0, \ldots, x_m \in I$. *Let* L_i, $i = 1, \ldots, 4$, *be linear functionals defined by* (42)-(45). *Then* $p \mapsto L_i(f_p)$ *is an exponentially convex function in the Jensen sense on* J. *If the function* $p \mapsto L_i(f_p)$ *is continuous on J, then it is exponentially convex on* J.

Corollary 2 *Let* $\Omega = \{f_p : p \in J\}$, *where* J *is an interval in* $\mathbb{R}$, *be a family of functions defined on an interval* I *in* $\mathbb{R}$, *such that the function* $p \mapsto [x_0, \ldots, x_m; f_p]$ *is* 2*-exponentially convex in the Jensen sense on* J *for every* $(m+1)$ *mutually different points* $x_0, \ldots, x_m \in I$. *Let* L_i, $i = 1, \ldots, 4$ *be linear functionals defined by* (42)-(45). *Then the following statements hold:*

(i) *If the function* $p \mapsto L_i(f_p)$ *is continuous on* J, *then it is* 2*-exponentially convex function on* J. *If* $p \mapsto L_i(f_p)$ *is additionally strictly positive, then it is also log-convex on* J. *Furthermore, the following inequality holds true:*

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1, \ldots, 4$$

for every choice $r, s, t \in J$, *such that* $r < s < t$.

(ii) *If the function* $p \mapsto L_i(f_p)$ *is strictly positive and differentiable on J, then for every* $p, q, u, v \in J$, *such that* $p \le u$ *and* $q \le v$, *we have*

$$\mu_{p,q}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega), \tag{48}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \ne q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)}\right), & p = q, \end{cases} \tag{49}$$

for $f_p, f_q \in \Omega$.

Proof.

(i) This is an immediate consequence of Theorem 22 and Remark 3.

(ii) Since $p \mapsto L_i(f_p)$ is positive and continuous, by (i) we have that $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Applying Proposition 1 we get

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p - q} \le \frac{\log L_i(f_u) - \log L_i(f_v)}{u - v}, \tag{50}$$

for $p \le u, q \le v, p \ne q, u \ne v$. Hence, we conclude that

$$\mu_{p,q}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega).$$

Cases $p = q$ and $u = v$ follow from (50) as limit cases.

□

Remark 4 Results from the above theorem and corollaries still hold when two of the points $x_0, \ldots, x_m \in I$ coincide, say $x_1 = x_0$, for a family of differentiable functions f_p such that the function $p \mapsto [x_0, \ldots, x_m; f_p]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $m + 1$ points coincide for a family of n differentiable functions with the same property. The proofs use (6) and suitable characterization of convexity.

7 Applications to Stolarsky type means

In this section, we present some families of functions which fulfil the conditions of Theorem 22, Corollary 1, Corollary 2 and Remark 4. This enables us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [5].

Example 1 Let us consider a family of functions

$$\Omega_1 = \{f_p : \mathbb{R} \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^n}, & p \neq 0, \\ \frac{x^n}{n!}, & p = 0. \end{cases}$$

Since $\frac{d^n f_p}{dx^n}(x) = e^{px} > 0$, the function f_p is n-convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^n f_p}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 22 we also have that $p \mapsto [x_0, \ldots, x_n; f_p]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 1 we conclude that $p \mapsto L_i(f_p), i = 1, \ldots, 4$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $p \mapsto f_p$ is not continuous for $p = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(L_i, \Omega_1)$, $i = 1, \ldots, 4$, from (49), becomes

$$\mu_{p,q}(L_i, \Omega_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{L_i(id \cdot f_p)}{L_i(f_p)} - \frac{n}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{L_i(id \cdot f_0)}{L_i(f_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. By Corollary 2 $\mu_{p,q}(L_i, \Omega_1)$, $i = 1, \ldots, 4$ are monotonic functions in parameters p and q.

Since

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}} (\log x) = x,$$

using Theorem 21 it follows that:

$$M_{p,q}(L_i, \Omega_1) = \log \mu_{p,q}(L_i, \Omega_1), \quad i = 1, \ldots, 4$$

satisfy

$$a \leq M_{p,q}(L_i, \Omega_1) \leq b, \quad i = 1, \ldots, 4.$$

So, $M_{p,q}(L_i, \Omega_1)$, $i = 1, \ldots, 4$ are monotonic means.

Example 2 Let us consider a family of functions

$$\Omega_2 = \{g_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \dfrac{x^p}{p(p-1)\cdots(p-n+1)}, & p \notin \{0,1,\ldots,n-1\}, \\ \dfrac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & p = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since $\frac{d^n g_p}{dx^n}(x) = x^{p-n} > 0$, the function g_p is $n-$convex for $x > 0$ and $p \mapsto \frac{d^n g_p}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mappings $p \mapsto L_i(g_p), i = 1,\ldots,4$ are exponentially convex. Hence, for this family of functions $\mu_{p,q}(L_i,\Omega_2)$, $i = 1,\ldots,4$, from (49), is equal to

$$\mu_{p,q}(L_i,\Omega_2) = \begin{cases} \left(\dfrac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\dfrac{L_i(g_0 g_p)}{L_i(g_p)} + \displaystyle\sum_{k=0}^{n-1}\dfrac{1}{k-p}\right), & p = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\dfrac{L_i(g_0 g_p)}{2L_i(g_p)} + \displaystyle\sum_{\substack{k=0 \\ k\neq p}}^{n-1}\dfrac{1}{k-p}\right), & p = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 21 we conclude that

$$a \leq \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}} \leq b, \quad i = 1,\ldots,4. \tag{51}$$

So, $\mu_{p,q}(L_i,\Omega_2)$, $i = 1,\ldots,4$ are means and by (48) they are monotonic.

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Consistency rates and asymptotic normality of the high risk conditional for functional data

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Abstract. The maximum of the conditional hazard function is a parameter of great importance in seismicity studies, because it constitutes the maximum risk of occurrence of an earthquake in a given interval of time. Using the kernel nonparametric estimates of the first derivative of the conditional hazard function, we establish uniform convergence properties and asymptotic normality of an estimate of the maximum in the context of independence data.

1 Introduction

The statistical analysis of functional data studies the experiments whose results are generally the curves. Under this supposition, the statistical analysis

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focuses on a framework of infinite dimension for the data under study. This field of modern statistics has received much attention in the last 20 years, and it has been popularised in the book of Ramsay and Silverman (2005). This type of data appears in many fields of applied statistics: environmetrics (Damon and Guillas, 2002), chemometrics (Benhenni et al., 2007), meteorological sciences (Besse et al., 2000), etc.

From a theoretical point of view, a sample of functional data can be involved in many different statistical problems, such as: classification and principal components analysis (PCA) (1986,1991) or longitudinal studies, regression and prediction (Benhenni et al., 2007; Cardo et al., 1999). The recent monograph by Ferraty and Vieu (2006) summarizes many of their contributions to the nonparametric estimation with functional data; among other properties, consistency of the conditional density, conditional distribution and regression estimates are established in the i.i.d. case under dependence conditions (strong mixing). Almost complete rates of convergence are also obtained, and different techniques are applied to several examples of functional data samples. Related work can be seen in the paper of Masry (2005), where the asymptotic normality of the functional nonparametric regression estimate is proven, considering strong mixing dependence conditions for the sample data. For automatic smoothing parameter selection in the regression setting, see Rachdi and Vieu (2007).

Hazard and conditional hazard

The estimation of the hazard function is a problem of considerable interest, especially to inventory theorists, medical researchers, logistics planners, reliability engineers and seismologists. The non-parametric estimation of the hazard function has been extensively discussed in the literature. Beginning with Watson and Leadbetter (1964), there are many papers on these topics: Ahmad (1976), Singpurwalla and Wong (1983), etc. We can cite Quintela (2007) for a survey.

The literature on the estimation of the hazard function is very abundant, when observations are vectorial. Cite, for instance, Watson and Leadbetter (1964), Roussas (1989), Lecoutre and Ould-Saïd (1993), Estvez et al. (2002) and Quintela-del-Rio (2006) for recent references. In all these works the authors consider independent observations or dependent data from time series. The first results on the nonparametric estimation of this model, in functional statistics were obtained by Ferraty et al. (2008). They studied the almost complete convergence of a kernel estimator for hazard function of a real random variable dependent on a functional predictor. Asymptotic normality of the latter estimator was obtained, in the case of α- mixing, by Quintela-del-Rio (2008). We refer to Ferraty *et al.* (2010) and Mahhiddine *et al.* (2014) for uniform almost complete convergence of the functional component of this nonparametric model.

When hazard rate estimation is performed with multiple variables, the result is an estimate of the conditional hazard rate for the first variable, given the levels of the remaining variables. Many references, practical examples and simulations in the case of non-parametric estimation using local linear approximations can be found in Spierdijk (2008).

Our paper presents some asymptotic properties related with the non-parametric estimation of the maximum of the conditional hazard function. In a functional data setting, the conditioning variable is allowed to take its values in some abstract semi-metric space. In this case, Ferraty *et al.* (2008) define non-parametric estimators of the conditional density and the conditional distribution. They give the rates of convergence (in an almost complete sense) to the corresponding functions, in a independence and dependence (α-mixing) context. We extend their results by calculating the maximum of the conditional hazard function of these estimates, and establishing their asymptotic normality, considering a particular type of kernel for the functional part of the estimates. Because the hazard function estimator is naturally constructed using these two last estimators, the same type of properties is easily derived for it. Our results are valid in a real (one- and multi-dimensional) context.

If X is a random variable associated to a lifetime (ie, a random variable with values in $\mathbb{R}^+$, the hazard rate of X (sometimes called hazard function, failure or survival rate) is defined at point x as the instantaneous probability that life ends at time x. Specifically, we have:

$$h(x) = \lim_{dx \to 0} \frac{\mathbb{P}\left(X \leq x + dx | X \geq x\right)}{dx}, \quad (x > 0).$$

When X has a density f with respect to the measure of Lebesgue, it is easy to see that the hazard rate can be written as follows:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}, \text{ for all } x \text{ such that } F(x) < 1,$$

where F denotes the distribution function of X and $S = 1 - F$ the survival function of X.

In many practical situations, we may have an explanatory variable Z and the main issue is to estimate the conditional random rate defined as

$$h^Z(x) = \lim_{dx \to 0} \frac{\mathbb{P}(X \le x + dx | X > x, \quad Z)}{dx}, \quad \text{for} \quad x > 0,$$

which can be written naturally as follows:

$$h^Z(x) = \frac{f^Z(x)}{S^Z(x)} = \frac{f^Z(x)}{1 - F^Z(x)}, \text{ once } F^Z(x) < 1. \tag{1}$$

Study of functions h and h^Z is of obvious interest in many fields of science (biology, medicine, reliability , seismology, econometrics, ...) and many authors are interested in construction of nonparametric estimators of h.

In this paper we propose an estimate of the maximum risk, through the nonparametric estimation of the conditional hazard function.

The layout of the paper is as follows. Section 2 describes the non-parametric functional setting: the structure of the functional data, the conditional density, distribution and hazard operators, and the corresponding non-parametric kernel estimators. Section 3 states the almost complete convergence[1] (with rates of convergence[2]) for nonparametric estimates of the derivative of the conditional hazard and the maximum risk. In Section 4, we calculate the variance of the conditional density, distribution and hazard estimates, the asymptotic normality of the three estimators considered is developed in this Section. Finally, Section 5 includes some proofs of technical Lemmas.

2 Nonparametric estimation with dependent functional data

Let $\{(Z_i, X_i), \; i = 1, \ldots, n\}$ be a sample of n random pairs, each one distributed as (Z, X), where the variable Z is of functional nature and X is scalar. Formally, we will consider that Z is a random variable valued in some semi-metric functional space $\mathcal{F}$, and we will denote by $d(\cdot,\cdot)$ the associated semi-metric. The conditional cumulative distribution of X given Z is defined for any $x \in \mathbb{R}$

[1] Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable T, if for any $\epsilon > 0$, we have $\sum_n \mathbb{P}(|T_n - T| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, (1987)).

[2] Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon > 0$ for which $\sum_n \mathbb{P}(|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = \mathcal{O}(u_n)$, a.co. (or equivalently by $T_n = \mathcal{O}_{a.co.}(u_n)$).

and any $z \in \mathcal{F}$ by

$$F^Z(x) = \mathbb{P}(X \leq x | Z = z),$$

while the conditional density, denoted by $f^Z(x)$ is defined as the density of this distribution with respect to the Lebesgue measure on $\mathbb{R}$. The conditional hazard is defined as in the non-infinite case (1).

In a general functional setting, f, F and h are not standard mathematical objects. Because they are defined on infinite dimensional spaces, the term operators may be a more adjusted in terminology.

The functional kernel estimates

We assume the sample data $(X_i, Z_i)_{1 \leq i \leq n}$ is i.i.d.

Following in Ferraty *et al.* (2008), the conditional density operator $f^Z(\cdot)$ is defined by using kernel smoothing methods

$$\widehat{f}^Z(x) = \frac{\sum_{i=1}^{n} h_H^{-1} K\left(h_K^{-1} d(z, Z_i)\right) H'\left(h_H^{-1}(x - X_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

where k and H' are kernel functions and h_H and h_K are sequences of smoothing parameters. The conditional distribution operator $F^Z(\cdot)$ can be estimated by

$$\widehat{F}^Z(x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right) H\left(h_H^{-1}(x - X_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

with the function $H(\cdot)$ defined by $H(x) = \int_{-\infty}^{x} H'(t)dt$. Consequently, the conditional hazard operator is defined in a natural way by

$$\widehat{h}^Z(x) = \frac{\widehat{f}^Z(x)}{1 - \widehat{F}^Z(x)}.$$

For $z \in \mathcal{F}$, we denote by $h^Z(\cdot)$ the conditional hazard function of X_1 given $Z_1 = z$. We assume that $h^Z(\cdot)$ is unique maximum and its high risk point is denoted by $\theta(z) := \theta$, which is defined by

$$h^Z(\theta(z)) := h^Z(\theta) = \max_{x \in \mathcal{S}} h^Z(x). \tag{2}$$

A kernel estimator of θ is defined as the random variable $\widehat{\theta}(z) := \widehat{\theta}$ which maximizes a kernel estimator $\widehat{h}^Z(\cdot)$, that is,

$$\widehat{h}^Z(\widehat{\theta}(z)) := \widehat{h}^Z(\widehat{\theta}) = \max_{x \in \mathcal{S}} \widehat{h}^Z(x), \tag{3}$$

where h^Z and $\widehat{h}^Z$ are defined above.

Note that the estimate $\widehat{\theta}$ is note necessarily unique and our results are valid for any choice satisfying (3). We point out that we can specify our choice by taking

$$\widehat{\theta}(z) = \inf\left\{ t \in \mathcal{S} \text{ such that } \widehat{h}^Z(t) = \max_{x \in \mathcal{S}} \widehat{h}^Z(x) \right\}.$$

As in any non-parametric functional data problem, the behavior of the estimates is controlled by the concentration properties of the functional variable Z.

$$\phi_z(h) = \mathbb{P}(Z \in B(z, h)),$$

where $B(z, h)$ being the ball of center z and radius h, namely $B(z, h) = \mathbb{P}\,(f \in \mathcal{F}, d(z, f) < h)$ (for more details, see Ferraty and Vieu (2006), Chapter 6).

In the following, z will be a fixed point in $\mathcal{F}$, $\mathcal{N}_z$ will denote a fixed neighborhood of z, $\mathcal{S}$ will be a fixed compact subset of $\mathbb{R}^+$. We will led to the hypothesis below concerning the function of concentration ϕ_z

(H1) $\forall\, h > 0,\ 0 < \mathbb{P}\,(Z \in B(z, h)) = \phi_z(h)$ and $\lim_{h \to 0} \phi_z(h) = 0$

Note that (H1) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* of Z.

Our nonparametric models will be quite general in the sense that we will just need the following simple assumption for the marginal distribution of Z, and let us introduce the technical hypothesis necessary for the results to be presented. The non-parametric model is defined by assuming that

(H2) $\begin{cases} \forall\,(x_1, x_2) \in \mathcal{S}^2, \forall\,(z_1, z_2) \in \mathcal{N}_z^2, \text{ for some } b_1 > 0,\ b_2 > 0 \\ |F^{z_1}(x_1) - F^{z_2}(x_2)| \le C_z(d(z_1, z_2)^{b_1} + |x_1 - x_2|^{b_2}), \end{cases}$

(H3) $\begin{cases} \forall\,(x_1, x_2) \in \mathcal{S}^2, \forall\,(z_1, z_2) \in \mathcal{N}_z^2, \text{ for some } j = 0, 1,\ \nu > 0,\ \beta > 0 \\ |f^{z_1\,(j)}(x_1) - f^{z_2\,(j)}(x_2)| \le C_z(d(z_1, z_2)^{\nu} + |x_1 - x_2|^{\beta}), \end{cases}$

(H4) $\exists\,\gamma < \infty, f'^Z(x) \le \gamma,\ \forall\,(z, x) \in \mathcal{F} \times \mathcal{S}$, (H5) $\exists\, \tau > 0, F^Z(x) \leq 1 - \tau, \ \ \forall\, (z, x) \in \mathcal{F} \times \mathcal{S}.$

(H6) H' is twice differentiable such that

$$\begin{cases} \text{(H6a) } \forall\,(t_1,t_2) \in \mathbb{R}^2;\ |H^{(j)}(t_1) - H^{(j)}(t_2)| \leq C|t_1 - t_2|, \ \text{ for } j = 0,1,2 \\ \text{and } H^{(j)} \text{are bounded for } j = 0,1,2; \\ \text{(H6b)} \displaystyle\int_{\mathbb{R}} t^2 H'^2(t)dt < \infty; \\ \text{(H6c)} \displaystyle\int_{\mathbb{R}} |t|^{\beta}(H''(t))^2 dt < \infty. \end{cases}$$

(H7) The kernel K is positive bounded function supported on $[0, 1]$ and it is of class $\mathcal{C}^1$ on $(0, 1)$ such that $\exists\, C_1,\ C_2,\ -\infty < C_1 < K'(t) < C_2 < 0$ for $0 < t < 1$.

(H8) There exists a function $\zeta_0^z(\cdot)$ such that for all $t \in [0, 1]$

$$\lim_{h_K \to 0} \frac{\phi_z(t h_K)}{\phi_z(h_K)} = \zeta_0^z(t) \ \text{ and } \ n h_H \phi_x(h_K) \to \infty \ \text{ as } n \to \infty.$$

(H9) The bandwidth h_H and h_K and small ball probability $\phi_z(h)$ satisfying

$$\begin{cases} \text{(H9a)} \displaystyle\lim_{n\to\infty} h_K = 0, \ \lim_{n\to\infty} h_H = 0; \\ \text{(H9b)} \displaystyle\lim_{n\to\infty} \frac{\log n}{n\phi_x(h_K)} = 0; \\ \text{(H9c)} \displaystyle\lim_{n\to\infty} \frac{\log n}{n h_H^{2j+1} \phi_x(h_K)} = 0, \ j = 0, 1. \end{cases}$$

Remark 1 *Assumption (H1) plays an important role in our methodology. It is known as (for small h) the "concentration hypothesis acting on the distribution of X" in infi- nite-dimensional spaces. This assumption is not at all restrictive and overcomes the problem of the non-existence of the probability density function. In many examples, around zero the small ball probability$\phi_z(h)$ can be written approximately as the product of two independent functions $\psi(z)$ and $\varphi(h)$ as $\phi_z(h) = \psi(z)\varphi(h) + o(\varphi(h))$. This idea was adopted by Masry (2005) who reformulated the Gasser et al. (1998) one. The increasing proprety of $\phi_z(\cdot)$ implies that $\zeta_h^z(\cdot)$ is bounded and then integrable (all the more so $\zeta_0^z(\cdot)$ is integrable).*

Without the differentiability of $\phi_z(\cdot)$, this assumption has been used by many authors where $\psi(\cdot)$ is interpreted as a probability density, while $\varphi(\cdot)$ may be *interpreted as a volume parameter. In the case of finite-dimensional spaces, that is* $\mathcal{S} = \mathbb{R}^d$*, it can be seen that* $\phi_z(h) = C(d)h^d\psi(z) + oh^d)$*, where* $C(d)$ *is the volume of the unit ball in* $\mathbb{R}^d$*. Furthermore, in infinite dimensions, there exist many examples fulfilling the decomposition mentioned above. We quote the following (which can be found in Ferraty et al. (2007)):*

1. $\phi_z(h) \approx \psi(h)h^\gamma$ *for som* $\gamma > 0$.
2. $\phi_z(h) \approx \psi(h)h^\gamma \exp\{C/h^p\}$ *for som* $\gamma > 0$ *and* $p > 0$.
3. $\phi_z(h) \approx \psi(h)/|\ln h|$.

The function $\zeta_h^z(\cdot)$ *which intervenes in Assumption (H9) is increasing for all fixed* h*. Its pointwise limit* $\zeta_0^z(\cdot)$ *also plays a determinant role. It intervenes in all asymptotic properties, in particular in the asymptotic variance term. With simple algebra, it is possible to specify this function (with* $\zeta_0(u) := \zeta_0^z(u)$ *in the above examples by:*

1. $\zeta_0(u) = u^\gamma$,
2. $\zeta_0(u) = \delta_1(u)$ *where* $\delta_1(\cdot)$ *is Dirac function,*
3. $\zeta_0(u) = \mathbf{1}_{]0,1]}(u)$.

Remark 2 *Assumptions (H2) and (H3) are the only conditions involving the conditional probability and the conditional probability density of* Z *given* X*. It means that* $F(\cdot|\cdot)$ *and* $f(\cdot|\cdot)$ *and its derivatives satisfy the Hölder condition with respect to each variable. Therefore, the concentration condition (H1) plays an important role. Here we point out that our assumptions are very usual in the estimation problem for functional regressors (see, e.g., Ferraty et al. (2008)).*

Remark 3 *Assumptions (H6), (H7) and (H9) are classical in functional estimation for finite or infinite dimension spaces.*

3 Nonparametric estimate of the maximum of the conditional hazard function

Let us assume that there exists a compact $\mathcal{S}$ with a unique maximum θ of h^Z on $\mathcal{S}$. We will suppose that h^Z is sufficiently smooth (at least of class $\mathcal{C}^2$) and verifies that $h'^Z(\theta) = 0$ and $h''^Z(\theta) < 0$.

Furthermore, we assume that $\theta \in \mathcal{S}^\circ$, where $\mathcal{S}^\circ$ denotes the interior of $\mathcal{S}$, and that θ satisfies the uniqueness condition, that is; for any $\varepsilon > 0$ and $\mu(z)$, there exists $\xi > 0$ such that $|\theta(z) - \mu(z)| \geq \varepsilon$ implies that $|h^Z(\theta(z)) - h^Z(\mu(z))| \geq \xi$.

We can write an estimator of the first derivative of the hazard function through the first derivative of the estimator. Our maximum estimate is defined by assuming that there is some unique $\widehat{\theta}$ on $\mathcal{S}^\circ$.

It is therefore natural to try to construct an estimator of the derivative of the function h^Z on the basis of these ideas. To estimate the conditional distribution function and the conditional density function in the presence of functional conditional random variable Z.

The kernel estimator of the derivative of the function conditional random functional h^Z can therefore be constructed as follows:

$$\widehat{h'}^Z(x) = \frac{\widehat{f'}^Z(x)}{1 - \widehat{F}^Z(x)} + (\widehat{h}^Z(x))^2, \tag{4}$$

the estimator of the derivative of the conditional density is given in the following formula:

$$\widehat{f'}^Z(x) = \frac{\sum_{i=1}^{n} h_H^{-2} K(h_K^{-1} d(Z, Z_i)) H''(h_H^{-1}(x - X_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(Z, Z_i))}. \tag{5}$$

Later, we need assumptions on the parameters of the estimator, ie on K, H, H', h_H and h_K are little restrictive. Indeed, on one hand, they are not specific to the problem estimate of h^Z (but inherent problems of F^Z, f^Z and f'^Z estimation), and secondly they consist with the assumptions usually made under functional variables.

We state the almost complete convergence (withe rates of convergence) of the maximum estimate by the following results:

Theorem 1 *Under assumption (H1)-(H7) we have*

$$\widehat{\theta} - \theta \to 0 \quad a.co. \tag{6}$$

Remark 4 *The hypothesis of uniqueness is only established for the sake of clarity. Following our proofs, if several local estimated maxima exist, the asymptotic results remain valid for each of them.*

Proof. Because $h'^Z(\cdot)$ is continuous, we have, for all $\epsilon > 0$. $\exists\, \eta(\epsilon) > 0$ such that

$$|x - \theta| > \epsilon \Rightarrow |h'^Z(x) - h'^Z(\theta)| > \eta(\epsilon).$$

Therefore,

$$\mathbb{P}\{|\widehat{\theta} - \theta| \geq \epsilon\} \leq \mathbb{P}\{|h'^Z(\widehat{\theta}) - h'^Z(\theta)| \geq \eta(\epsilon)\}.$$

We also have

$$|h'^Z(\widehat{\theta}) - h'^Z(\theta)| \leq |h'^Z(\widehat{\theta}) - \widehat{h}'^Z(\widehat{\theta})| + |\widehat{h}'^Z(\widehat{\theta}) - h'^Z(\theta)| \leq \sup_{x \in \mathcal{S}} |\widehat{h}'^Z(x) - h'^Z(x)|, \tag{7}$$

because $\widehat{h}'^Z(\widehat{\theta}) = h'^Z(\theta) = 0$.

Then, uniform convergence of h'^Z will imply the uniform convergence of $\widehat{\theta}$. That is why, we have the following lemma.

Lemma 1 *Under assumptions of Theorem 1, we have*

$$\sup_{x \in \mathcal{S}} |\widehat{h}'^Z(x) - h'^Z(x)| \longrightarrow 0 \quad a.co. \tag{8}$$

□

The proof of this lemma will be given in Appendix.

Theorem 2 *Under assumption (H1)-(H7) and (H9a) and (H9c), we have*

$$\sup_{x \in \mathcal{S}} |\widehat{\theta} - \theta| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n h_H^3 \phi_z(h_K)}}\right). \tag{9}$$

Proof. By using Taylor expansion of the function h'^Z at the point $\widehat{\theta}$, we obtain

$$h'^Z(\widehat{\theta}) = h'^Z(\theta) + (\widehat{\theta} - \theta) h''^Z(\theta_n^*), \tag{10}$$

with θ^* a point between θ and $\widehat{\theta}$. Now, because $h'^Z(\theta) = \widehat{h}'^Z(\widehat{\theta})$

$$|\widehat{\theta} - \theta| \leq \frac{1}{h''^Z(\theta_n^*)} \sup_{x \in \mathcal{S}} |\widehat{h}'^Z(x) - h'^Z(x)|. \tag{11}$$

The proof of Theorem will be completed showing the following lemma.

Lemma 2 *Under the assumptions of Theorem 2, we have*

$$\sup_{x \in \mathcal{S}} |\widehat{h}'^Z(x) - h'^Z(x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n h_H^3 \phi_z(h_K)}}\right). \tag{12}$$

The proof of lemma will be given in the Appendix.

□

4 Asymptotic normality

To obtain the asymptotic normality of the conditional estimates, we have to add the following assumptions:

(H6d) $\int_{\mathbb{R}} (H''(t))^2 dt < \infty$,

(H10) $0 = \widehat{h'}^Z(\widehat{\theta}) < |\widehat{h'}^Z(x)|), \forall x \in \mathcal{S},\ x \neq \widehat{\theta}$

The following result gives the asymptotic normality of the maximum of the conditional hazard function. Let

$$\mathcal{A} = \left\{ (z,x) : (z,x) \in \mathcal{S} \times \mathbb{R},\ a_2^x F^Z(x) \left(1 - F^Z(x)\right) \neq 0 \right\}.$$

Theorem 3 *Under conditions (H1)-(H10) we have* $(\theta \in \mathcal{S}/f^Z(\theta), 1 - F^Z(\theta) > 0)$

$$\left(n h_H^3 \phi_z(h_K)\right)^{1/2} \left(\widehat{h}'^Z(\theta) - h'^Z(\theta)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma^2_{h'}(\theta)\right)$$

where $\rightarrow^{\mathcal{D}}$ *denotes the convergence in distribution,*

$$a_l^x = K^l(1) - \int_0^1 \left(K^l(u)\right)' \zeta_0^x(u) du \quad \textit{for } l = 1, 2$$

and

$$\sigma^2_{h'}(\theta) = \frac{a_2^x h^Z(\theta)}{\left(a_1^x\right)^2 (1 - F^Z(\theta))} \int (H''(t))^2 dt.$$

Proof. Using again (17), and the fact that

$$\frac{\left(1 - F^Z(x)\right)}{(1 - \widehat{F}^Z(x))\,(1 - F^Z(x))} \longrightarrow \frac{1}{1 - F^Z(x)};$$

and

$$\frac{\widehat{f'}^Z(x)}{\left(1 - \widehat{F}^Z(x)\right)(1 - F^Z(x))} \longrightarrow \frac{f'^Z(x)}{(1 - F^Z(x))^2}.$$

The asymptotic normality of $\left(n h_H^3 \phi_z(h_K)\right)^{1/2} \left(\widehat{h'}^Z(\theta) - h'^Z(\theta)\right)$ can be deduced from both following lemmas,

Lemma 3 *Under Assumptions (H1)-(H2) and (H6)-(H8), we have*

$$(n\phi_z(h_K))^{1/2}\left(\widehat{F}^Z(x)-F^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{F^Z}(x)\right), \tag{13}$$

where

$$\sigma^2_{F^Z}(x)=\frac{a_2^x F^Z(x)\left(1-F^Z(x)\right)}{\left(a_1^x\right)^2}.$$

Lemma 4 *Under Assumptions (H1)-(H3) and (H5)-(H9), we have*

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{h}^Z(x)-h^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{h^Z}(x)\right), \tag{14}$$

where

$$\sigma^2_{h^Z}(x)=\frac{a_2^x h^Z(x)}{\left(a_1^x\right)^2(1-F^Z(x))}\int_{\mathbb{R}}(H'(t))^2dt.$$

Lemma 5 *Under Assumptions of Theorem 3, we have*

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{f'}^Z(x)-f'^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{f'^Z}(x)\right); \tag{15}$$

where

$$\sigma^2_{f'^Z(x)}=\frac{a_2^x f^Z(x)}{\left(a_1^x\right)^2}\int_{\mathbb{R}}(H''(t))^2dt.$$

Lemma 6 *Under the hypotheses of Theorem 3, we have*

$$\mathrm{Var}\left[\widehat{f'}^Z_N(x)\right]=\frac{\sigma^2_{f'^Z(x)}}{nh_H^3\phi_z(h_K)}+o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right),$$

$$\mathrm{Var}\left[\widehat{F}^Z_N(x)\right]=o\left(\frac{1}{nh_H\phi_z(h_K)}\right);$$

and

$$\mathrm{Var}\left[\widehat{F}^Z_D\right]=o\left(\frac{1}{nh_H\phi_z(h_K)}\right).$$

Lemma 7 *Under the hypotheses of Theorem 3, we have*

$$\mathrm{Cov}(\widehat{f'}^Z_N(x),\widehat{F}^Z_D)=o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right),$$

$$\mathrm{Cov}(\widehat{f'}^Z_N(x), \widehat{F}^Z_N(x)) = o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right)$$

and

$$\mathrm{Cov}(\widehat{F}^Z_D, \widehat{F}^Z_N(x)) = o\left(\frac{1}{nh_H\phi_z(h_K)}\right).$$

Remark 5

It is clear that, the results of lemmas, Lemma 6 and Lemma 7 allows to write

$$\mathrm{Var}\left[\widehat{F}^Z_D - \widehat{F}^Z_N(x)\right] = o\left(\frac{1}{nh_H\phi_z(h_K)}\right)$$

The proofs of lemmas, Lemma3 can be seen in Belkhir *et al.* (2015), Lemma lem2-4 and Lemma lem3-4 see Rabhi *et al.* (2015).

□

Finally, by this last result and (10), the following theorem follows:

Theorem 4 *Under conditions (H1)-(H10), we have* $(\theta \in \mathcal{S}/f^Z(\theta), 1-F^Z(\theta) > 0)$

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}(\widehat{\theta} - \theta) \xrightarrow{\mathcal{D}} N\left(0, \frac{\sigma^2_{h'}(\theta)}{(h''^Z(\theta))^2}\right);$$

with $\sigma^2_{h'}(\theta) = h^Z(\theta)\left(1 - F^Z(\theta)\right)\int (H''(t))^2 dt.$

5 Proofs of technical lemmas

Proof. Proof of Lemma 1 and Lemma 2. Let

$$\widehat{h}'^Z(x) = \frac{\widehat{f}'^Z(x)}{1 - \widehat{F}^Z(x)} + (\widehat{h}^Z(x))^2, \tag{16}$$

with

$$\widehat{h}'^Z(x) - h'^Z(x) = \underbrace{\left\{\left(\widehat{h}^Z(x)\right)^2 - \left(h^Z(x)\right)^2\right\}}_{\Gamma_1} + \underbrace{\left\{\frac{\widehat{f}'^Z(x)}{1 - \widehat{F}^Z(x)} - \frac{f'^Z(x)}{1 - F^Z(x)}\right\}}_{\Gamma_2}; \tag{17}$$

for the first term of (17) we can write

$$\left|\left(\widehat{h}^Z(x)\right)^2 - \left(h^Z(x)\right)^2\right| \leq \left|\widehat{h}^Z(x) - h^Z(x)\right| . \left|\widehat{h}^Z(x) + h^Z(x)\right|, \tag{18}$$

because the estimator $\widehat{h}^Z(\cdot)$ converge a.co. to $h^Z(\cdot)$ we have

$$\sup_{x\in\mathcal{S}}\left|\left(\widehat{h}^Z(x)\right)^2-\left(h^Z(x)\right)^2\right|\leq 2\left|h^Z(\theta)\right|\sup_{x\in\mathcal{S}}\left|\widehat{h}^Z(x)-h^Z(x)\right|; \tag{19}$$

for the second term of (17) we have

$$\begin{aligned}\frac{\widehat{f}'^Z(x)}{1-\widehat{F}^Z(x)}-\frac{f'^Z(x)}{1-F^Z(x)} &= \frac{1}{(1-\widehat{F}^Z(x))(1-F^Z(x))}\left\{\widehat{f}'^Z(x)-f'^Z(x)\right\}\\ &+\frac{1}{(1-\widehat{F}^Z(x))(1-F^Z(x))}\left\{f'^Z(x)\left(\widehat{F}^Z(x)-F^Z(x)\right)\right\}\\ &+\frac{1}{(1-\widehat{F}^Z(x))(1-F^Z(x))}\left\{F^Z(x)\left(\widehat{f}'^Z(x)-f'^Z(x)\right)\right\}.\end{aligned}$$

Valid for all $x\in\mathcal{S}$. Which for a constant $C<\infty$, this leads

$$\sup_{x\in\mathcal{S}}\left|\frac{\widehat{f}'^Z(x)}{1-\widehat{F}^Z(x)}-\frac{f'^Z(x)}{1-F^Z(x)}\right|\leq$$

$$C\frac{\left\{\sup_{x\in\mathcal{S}}\left|\widehat{f}'^Z(x)-f'^Z(x)\right|+\sup_{x\in\mathcal{S}}\left|\widehat{F}^Z(x)-F^Z(x)\right|\right\}}{\inf_{x\in\mathcal{S}}\left|1-\widehat{F}^Z(x)\right|}. \tag{20}$$

Therefore, the conclusion of the lemma follows from the following results:

$$\sup_{x\in\mathcal{S}}|\widehat{F}^Z(x)-F^Z(x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_z(h_K)}}\right), \tag{21}$$

$$\sup_{x\in\mathcal{S}}|\widehat{f}'^Z(x)-f'^Z(x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H^3\phi_z(h_K)}}\right), \tag{22}$$

$$\sup_{x\in\mathcal{S}}|\widehat{h}^Z(x)-h^Z(x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H\phi_z(h_K)}}\right), \tag{23}$$

$$\exists\,\delta>0\ \text{ such that }\ \sum_1^\infty\mathbb{P}\left\{\inf_{y\in\mathcal{S}}|1-\widehat{F}^Z(x)|<\delta\right\}<\infty. \tag{24}$$

The proofs of (21) and (22) appear in Ferraty *et al.* (2006), and (23) is proven in Ferraty *et al.* (2008).

- Concerning (24) by equation (21), we have the almost complete convergence of $\widehat{F}^Z(x)$ to $F^Z(x)$. Moreover,

$$\forall \varepsilon > 0 \qquad \sum_{n=1}^{\infty} \mathbb{P}\left\{|\widehat{F}^Z(x) - F^Z(x)| > \varepsilon\right\} < \infty.$$

On the other hand, by hypothesis we have $F^Z < 1$, i.e.

$$1 - \widehat{F}^Z \geq F^Z - \widehat{F}^Z,$$

thus,

$$\inf_{y\in\mathcal{S}} |1-\widehat{F}^Z(x)| \leq (1-\sup_{x\in\mathcal{S}} F^Z(x))/2 \Rightarrow \sup_{x\in\mathcal{S}} |\widehat{F}^Z(x)-F^Z(x)| \geq (1-\sup_{x\in\mathcal{S}} F^Z(x))/2.$$

In terms of probability is obtained

$$\begin{aligned} &\mathbb{P}\left\{\inf_{x\in\mathcal{S}} |1 - \widehat{F}^Z(x)| < (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2\right\} \\ &\leq \mathbb{P}\left\{\sup_{x\in\mathcal{S}} |\widehat{F}^Z(x) - F^Z(x)| \geq (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2\right\} < \infty. \end{aligned}$$

Finally, it suffices to take $\delta = (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2$ and apply the results (21) to finish the proof of this Lemma.

□

Proof. Proof of Lemma 4. We can write for all $x \in \mathcal{S}$

$$\begin{aligned} \widehat{h}^Z(x) - h^Z(x) &= \frac{\widehat{f}^Z(x)}{1 - \widehat{F}^Z(x)} - \frac{f^Z(x)}{1 - F^Z(x)} \\ &= \frac{1}{\widehat{D}^Z(x)}\Big\{ \left(\widehat{f}^Z(x) - f^Z(x)\right) + f^Z(x)\left(\widehat{F}^Z(x) - F^Z(x)\right) \\ &\quad -F^Z(x)\left(\widehat{f}^Z(x) - f^Z(x)\right) \Big\}, \qquad (25) \\ &= \frac{1}{\widehat{D}^Z(x)}\Big\{ \left(1 - F^Z(x)\right)\left(\widehat{f}^Z(x) - f^Z(x)\right) \\ &\quad -f^Z(x)\left(\widehat{F}^Z(x) - F^Z(x)\right) \Big\}; \end{aligned}$$

with $\widehat{D}^Z(x) = (1 - F^Z(x))\left(1 - \widehat{F}^Z(x)\right)$.

As a direct consequence of the Lemma 3, the result (26) (see Belkhir *et al.* (2015)) and the expression (25), permit us to obtain the asymptotic normality for the conditional hazard estimator.

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{f}^Z(x)-f^Z(x)\right)\xrightarrow{\mathcal{D}} N\left(0,\sigma^2_{f^Z}(x)\right); \tag{26}$$

where

$$\sigma^2_{f^Z(x)}=\frac{a_2^x f^Z(x)}{\left(a_1^x\right)^2}\int_{\mathbb{R}}(H'(t))^2dt.$$

□

Proof. Proof of Lemma 5. For $i=1,\ldots,n$, we consider the quantities $K_i=K\left(h_K^{-1}d(z,Z_i)\right)$, $H_i''(x)=H''\left(h_H^{-1}(x-X_i)\right)$ and let $\widehat{f'}_N^Z(x)$ (resp. $\widehat{F}_D^Z$) be defined as

$$\widehat{f'}_N^Z(x)=\frac{h_H^{-2}}{n\,\mathbb{E}K_1}\sum_{i=1}^{n}K_iH_i''(x)\qquad\left(\text{resp. }\widehat{F}_D^Z=\frac{1}{n\,\mathbb{E}K_1}\sum_{i=1}^{n}K_i\right).$$

This proof is based on the following decomposition

$$\begin{aligned}\widehat{f'}^Z(x)-f'^Z(x) &= \frac{1}{\widehat{F}_D^Z}\left\{\left(\widehat{f'}_N^Z(x)-\mathbb{E}\widehat{f'}_N^Z(x)\right)-\left(f'^Z(x)-\mathbb{E}\widehat{f'}_N^Z(x)\right)\right\}+\\ &\quad \frac{f'^Z(x)}{\widehat{F}_D^Z}\left\{\mathbb{E}\widehat{F}_D^Z-\widehat{F}_D^Z\right\},\end{aligned} \tag{27}$$

and on the following intermediate results.

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}_N^Z(x)-\mathbb{E}\widehat{f'}_N^Z(x)\right)\xrightarrow{\mathcal{D}} N\left(0,\sigma^2_{f'^Z}(x)\right); \tag{28}$$

where $\sigma^2_{f'^Z}(x)$ is defined as in Lemma 5.

$$\lim_{n\to\infty}\sqrt{nh_H^3\phi_z(h_K)}\left(\mathbb{E}\widehat{f'}_N^Z(x)-f'^Z(x)\right)=0. \tag{29}$$

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{F}_D^Z(x)-1\right)\xrightarrow{\mathbb{P}}0,\quad\text{as}\quad n\to\infty. \tag{30}$$

- Concerning (28).

By the definition of $\widehat{f'}^Z_N(x)$, it follows that

$$\begin{aligned}
\Omega_n &= \sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}^Z_N(x) - \mathbb{E}\widehat{f'}^Z_N(x)\right) \\
&= \sum_{i=1}^n \frac{\sqrt{\phi_z(h_K)}}{\sqrt{nh_H}\mathbb{E}K_1}\left(K_iH_i'' - \mathbb{E}K_iH_i''\right) \\
&= \sum_{i=1}^n \Delta_i,
\end{aligned}$$

which leads

$$Var(\Omega_n) = nh_H^3\phi_z(h_K)Var\left(\widehat{f'}^Z_N(x) - \mathbb{E}\left[\widehat{f'}^Z_N(x)\right]\right). \tag{31}$$

Now, we need to evaluate the variance of $\widehat{f'}^Z_N(x)$. For this we have for all $1 \le i \le n$, $\Delta_i(z,x) = K_i(z)H_i''(x)$, so we have

$$\begin{aligned}
Var(\widehat{f'}^Z_N(x)) &= \frac{1}{\left(nh_H^2\mathbb{E}[K_1(z)]\right)^2}\sum_{i=1}^n\sum_{j=1}^n Cov\left(\Delta_i(z,x), \Delta_j(z,x)\right) \\
&= \frac{1}{n\left(h_H^2\mathbb{E}[K_1(z)]\right)^2} Var\left(\Delta_1(z,x)\right).
\end{aligned}$$

Therefore

$$Var\left(\Delta_1(z,x)\right) \le \mathbb{E}\left(H_1''^2(x)K_1^2(z)\right) \le \mathbb{E}\left(K_1^2(z)\mathbb{E}\left[H_1''^2(x)|Z_1\right]\right).$$

Now, by a change of variable in the following integral and by applying (H4) and (H7), one gets

$$\begin{aligned}
\mathbb{E}\left(H_1''^2(y)|Z_1\right) &= \int_{\mathbb{R}} H''^2\left(\frac{d(x-u)}{h_H}\right) f^Z(u)du \\
&\le h_H\int_{\mathbb{R}} H''^2(t)\left(f^Z(x-h_Ht,z) - f^Z(x)\right)dt \\
&\quad + h_Hf^Z(x)\int_{\mathbb{R}} H''^2(t)dt \qquad (32) \\
&\le h_H^{1+b_2}\int_{\mathbb{R}} |t|^{b_2}H''^2(t)dt + h_Hf^Z(x)\int_{\mathbb{R}} H''^2(t)dt \\
&= h_H\left(o(1) + f^Z(x)\int_{\mathbb{R}} H''^2(t)dt\right).
\end{aligned}$$

By means of (32) and the fact that, as $n \to \infty$, $\mathbb{E}\left(K_1^2(z)\right) \longrightarrow a_2^x \phi_z(h_K)$, one gets

$$\mathrm{Var}\left(\Delta_1(z,x)\right) = a_2^x \phi_z(h_K) h_H \left(o(1) + f^Z(x) \int_{\mathbb{R}} H''^2(t)dt \right).$$

So, using (H8), we get

$$\begin{aligned} &\frac{1}{n\left(h_H^2 \mathbb{E}[K_1(z)]\right)^2} \mathrm{Var}\left(\Delta_1(z,x)\right) \\ &= \frac{a_2^x \phi_z(h_K)}{n\left(a_1^x h_H^2 \phi_z(h_K)\right)^2} h_H \left(o(1) + f^Z(x) \int_{\mathbb{R}} H''^2(t)dt \right) \\ &= o\left(\frac{1}{n h_H^3 \phi_z(h_K)}\right) + \frac{a_2^x f^Z(x)}{(a_1^x)^2 n h_H^3 \phi_z(h_K)} \int_{\mathbb{R}} H''^2(t)dt. \end{aligned}$$

Thus as $n \to \infty$ we obtain

$$\frac{1}{n\left(h_H^2 \mathbb{E}[K_1(z)]\right)^2} \mathrm{Var}\left(\Delta_1(z,x)\right) \longrightarrow \frac{a_2^x f^Z(x)}{(a_1^x)^2 n h_H^3 \phi_z(h_K)} \int_{\mathbb{R}} H''^2(t)dt. \quad (33)$$

Indeed

$$\sum_{i=1}^{n} \mathbb{E}\Delta_i^2 = \frac{\phi_z(h_K)}{h_H \mathbb{E}^2 K_1} \mathbb{E}K_1^2 (H_1'')^2 - \frac{\phi_z(h_K)}{h_H \mathbb{E}^2 K_1} \left(\mathbb{E}K_1 H_1''\right)^2 = \Pi_{1n} - \Pi_{2n}. \quad (34)$$

As for Π_{1n}, by the property of conditional expectation, we get

$$\Pi_{1n} = \frac{\phi_z(h_K)}{\mathbb{E}^2 K_1} \mathbb{E}\left\{ K_1^2 \int H''^2(t) \left(f'^Z(x - t h_H) - f'^Z(x) + f'^Z(x) \right) dt \right\}.$$

Meanwhile, by (H1), (H3), (H7) and (H8), it follows that:

$$\frac{\phi_z(h_K)\mathbb{E}K_1^2}{\mathbb{E}^2 K_1} \underset{n\to\infty}{\longrightarrow} \frac{a_2^x}{(a_1^x)^2},$$

which leads

$$\Pi_{1n} \underset{n\to\infty}{\longrightarrow} \frac{a_2^x f^Z(x)}{(a_1^x)^2} \int (H''(t))^2 dt, \quad (35)$$

Regarding Π_{2n}, by (H1), (H3) and (H6), we obtain

$$\Pi_{2n} \underset{n\to\infty}{\longrightarrow} 0. \tag{36}$$

This result, combined with (34) and (35), allows us to get

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}\Delta_i^2 = \sigma_{f'Z}^2(x) \tag{37}$$

Therefore, combining (33) and (36)-(37), (28) is valid.

- Concerning (29).

  The proof is completed along the same steps as that of Π_{1n}. We omit it here.

- Concerning (30). The idea is similar to that given by Belkhir *et al.* (2015). By definition of $\widehat{F}_D^Z(x)$, we have

$$\sqrt{nh_H^3\phi_z(h_K)}(\widehat{F}_D^Z(x) - 1) = \Omega_n - \mathbb{E}\Omega_n,$$

where $\Omega_n = \frac{\sqrt{nh_H^3\phi_z(h_K)}\sum_{i=1}^n K_i}{n\mathbb{E}K_1}$. In order to prove (30), similar to Belkhir *et al.* (2015), we only need to proov $\mathrm{Var}\ \Omega_n \to 0$, as $n \to \infty$. In fact, since

$$\begin{aligned}\mathrm{Var}\ \Omega_n &= \frac{nh_H^3\phi_z(h_K)}{n\mathbb{E}^2K_1}(n\mathrm{Var}K_1)\\ &\le \frac{nh_H^3\phi_z(h_K)}{\mathbb{E}^2K_1}\mathbb{E}K_1^2\\ &= \Psi_1,\end{aligned}$$

then, using the boundedness of function K allows us to get that:

$$\Psi_1 \le Ch_H^3\phi_z(h_K) \to 0, \quad \text{as } n \to \infty.$$

It is clear that, the results of (21), (22), (24) and Lemma 6 permits us

$$\mathbb{E}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + F^Z(x)\right) \longrightarrow 0,$$

and

$$\mathrm{Var}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + F^Z(x)\right) \longrightarrow 0;$$

then

$$\widehat{F}_D^x - \widehat{F}_N^Z(x) - 1 + F^Z(x) \xrightarrow{\mathbb{P}} 0.$$

Moreover, the asymptotic variance of $\widehat{F}_D^Z - \widehat{F}_N^Z(x)$ given in Remark 5 allows to obtain

$$\frac{nh_H\phi_z(h_K)}{\sigma_{F^Z}^2(x)} Var\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + \mathbb{E}\left(\widehat{F}_N^Z(x)\right)\right) \longrightarrow 0.$$

By combining result with the fact that

$$\mathbb{E}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + \mathbb{E}\left(\widehat{F}_N^Z(x)\right)\right) = 0,$$

we obtain the claimed result.

Therefore, the proof of this result is completed.

Therefore, the proof of this Lemma is completed.

□

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Stability of the solutions of nonlinear third order differential equations with multiple deviating arguments

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Abstract. In this paper, with use of Lyapunov functional, we investigate asymptotic stability of solutions of some nonlinear differential equations of third order with delay. Our results include and improve some well-known results in the literature.

1 Introduction

The investigation of qualitative behavior of solutions such as stability, convergence, boundedness, asymptotic behavior to mention few, are very important problems in the theory and applications of differential equations. For instance, in applied sciences some practical problems concerning mechanics, engineering technique fields, economy, control theory, physical sciences and so on are associated with third, fourth and higher order nonlinear differential equations. In recent years, there has been increasing interest in obtaining sufficient conditions for the asymptotic stability and boundedness of solutions of the nonlinear third order differential equations. Many results relative to the stability, boundedness of solutions of third order differential equations with delays or without

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delays have been obtained. We refer the reader to the papers (Burton [1, 2], Swick [10] and Yoshizawa [16] and references therein) to discuss the qualitative properties of various form of nonlinear differential equations without delay.

The Lyapunov second method had also been found useful and applicable to study the qualitative properties of the equation with delay. Many interesting results, on the qualitative behavior of solutions of the third order differential equations have been obtained by Omeike [4, 5], Remili and Oudjedi [7], Sadek [8, 9], Tunç [11, 12, 13, 14] and Zhu [17] and references therein.

In 2009, the author [5] adapted [10] and used a suitable Lyapunov function to establish criteria which guarantee asymptotic stability of solution of non-autonomous delay differential equation of the third order that is bounded together with its derivatives on the real line, and boundedness under explicit conditions on the nonlinear terms of the equation

$$x''' + a(t)x'' + b(t)g(x') + c(t)h(x(t-r)) = p(t).$$

Recently, in 2013 Tunç and Gözen [15] considered the non autonomous differential equation of the third order with multiple deviating arguments:

$$x''' + a(t)x'' + nb(t)g(x') + c(t)\sum_{i=1}^{n} h_i(x(t-r)) = p(t).$$

He discussed the stability and boundedness of solutions of this equation.

Our aim in this paper, by using Lyapunov second method is to study the asymptotic stability of third-order nonlinear differential equation with multiple deviating arguments

$$\left[\psi(x'(t))x'(t)\right]'' + a(t)x''(t) + nb(t)g(x'(t)) + c(t)\sum_{i=1}^{n} h_i(x(t-r_i)) = 0, \quad (1)$$

and the boundedness of solutions of the equation

$$\left[\psi(x'(t))x'(t)\right]'' + a(t)x''(t) + nb(t)g(x'(t)) + c(t)\sum_{i=1}^{n} h_i(x(t-r_i)) = q(t), \quad (2)$$

where r_i are certain positive constants. It is supposed that the derivatives, $a'(t), b'(t), c'(t)$, $\psi'(y) = \dfrac{d\psi}{dy}$, and $h_i'(x) = \frac{dh_i}{dx}$, exist and are continuous.

In this work, we want to adopt the approach in Omeike [5] and Tunç [15] to extend the result in Swick [10] to the equation (1) and give sufficient criteria which guarantee the existence of uniform asymptotic stability of the solution with their derivatives on the real line. Obviously, the equations discussed in [5] and [15], are particular cases of our equation (2). Here, by this work, we improve the boundedness result obtained in [5, 15].

2 Preliminaries

First, we will give some basic definitions and important stability criteria for the general non-autonomous delay differential system. Consider the general non-autonomous delay differential system

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta)\ , \quad -r \leq \theta \leq 0, \quad t \geq 0, \tag{3}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$, $C_H := \{\phi \in C([-r, 0],\ \mathbb{R}^n) : \|\phi\| \leq H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t, \phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 1 [2] *An element* $\psi \in C$ *is in the* $\omega - \mathrm{limit}$ *set of* ϕ*, say* $\Omega(\phi)$*, if* $x(t, 0, \phi)$ *is defined on* $[0, +\infty)$ *and there is a sequence* $\{t_n\}, t_n \to \infty$*, as* $n \to \infty$*, with* $\|x_{t_n}(\phi) - \psi\| \to 0$ *as* $n \to \infty$ *where* $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \leq \theta \leq 0$.

Definition 2 [2] *A set* $Q \subset C_H$ *is an invariant set if for any* $\phi \in Q$*, the solution of (3),* $x(t, 0, \phi)$*, is defined on* $[0, \infty)$ *and* $x_t(\phi) \in Q$ *for* $t \in [0, \infty)$.

Lemma 1 [1] *If* $\phi \in C_H$ *is such that the solution* $x_t(\phi)$ *of (3) with* $x_0(\phi) = \phi$ *is defined on* $[0, \infty)$ *and* $\|x_t(\phi)\| \leq H_1 < H$ *for* $t \in [0, \infty)$*, then* $\Omega(\phi)$ *is a non-empty, compact, invariant set and*

$$\mathrm{dist}(x_t(\phi), \Omega(\phi)) \to 0 \quad \mathrm{as} \quad t \to \infty.$$

Lemma 2 [1] *let* $V(t, \phi) : I \times C_H \to \mathbb{R}$ *be a continuous functional satisfying a local Lipschitz condition.* $V(t, 0) = 0$*, and such that:*

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_3(\|\phi\|_2)$ *where* $\|\phi\|_2 = \left(\int_{t-r}^{t} \|\phi(s)\|^2 ds\right)^{\frac{1}{2}}$.

(ii) $\dot{V}_{(3)}(t, \phi) \leq -W_4(|\phi(0)|)$*, where,* W_i $(i = 1, 2, 3, 4)$ *are wedges. Then the zero solution of* (3) *is uniformly asymptotically stable.*

3 Assumptions and main results

The following assumptions will be needed throughout the paper. Let a_0, b_0, c_0, $d, m, d_0, d_1, A, B, C, L, M$, and $\varepsilon, \delta_i, \rho_i$ be an arbitrary but fixed positives numbers and suppose that $a(t), b(t), c(t) \in C^1(\mathbb{R}_+)$, $h \in C^1(\mathbb{R})$, $g \in C(\mathbb{R})$ and let ψ be a twice continuously differential function on I R, such that the following assumptions are satisfied:

i) $0 < a_0 \le a(t) \le A;\ 0 < b_0 \le b(t) \le B; 0 < c_0 \le c(t) \le C.$

ii) $c(t) \le b(t),\ b'(t) \le c'(t) \le 0$ for $t \in [0, \infty)$.

iii) $0 < m \le \psi(u) \le M;\ \ 0 < d_0 \le \dfrac{g(y)}{y} \le d_1$ for $y \neq 0$.

iv) $h_i(0) = 0, \dfrac{h_i(x)}{x} \ge \delta_i > 0\ (x \neq 0)$, and $|h_i'(x)| \le \rho_i$ for all x.

v) $\dfrac{M\rho_i}{d_0} < d < a_0.$

vi) $\dfrac{1}{2}da'(t) - b_0(dd_0 - M\displaystyle\sum_{i=1}^{n}\rho_i) \le -\varepsilon < 0.$

vii) $\displaystyle\int_{-\infty}^{+\infty} \left|\psi'(u)\right| du < \infty.$

viii) $\displaystyle\inf_{u\in\mathbb{R}} u\Psi'(u) = \eta > -m.$

ix) $Q(t) = \displaystyle\int_0^t |q(s)|\, ds < \infty.$

For ease of exposition throughout this paper we will adopt the following notation

$$P(t) = \psi(x'(t)), \qquad R(t) = \frac{\psi'(x'(t))}{\psi^2(x'(t))}x''(t).$$

Theorem 1 *In addition to conditions* (i)-(vii) *being satisfied, suppose that the following is also satisfied*

$$\sum_{i=1}^{n} r_i < \min\{\alpha_i,\ \beta_i\},$$

where

$$\alpha_i = \frac{2(a_0 - d)}{MC\rho_i}, \text{ and } \beta_i = \frac{2m^3\varepsilon}{C\rho_i M^2(d + dm^2 + m)}.$$

Then every solution of (1) is uniformly asymptotically stable.

Proof. We write the equation (1) as the following equivalent system

$$\begin{aligned}
x' &= \frac{1}{P(t)}y \\
y' &= z \\
z' &= -\frac{a(t)}{P(t)}z + a(t)R(t)y - nb(t)g\left(\frac{y}{P(t)}\right) - c(t)\sum_{i=1}^{n} h_i(x) \\
&\quad + c(t)\sum_{i=1}^{n}\int_{t-r_i}^{t}\frac{y(s)}{P(t)}h_i'(x(s))ds.
\end{aligned} \tag{4}$$

Note that the continuity of the functions $a(t), b(t), c(t), q(t)$ on $[0, +\infty[$, and $\psi(x'), g(x'), h_i(x)$ in their respective arguments on $\mathbb{R}$ with $h(0) = g(0) = 0$, guarantee the existence of the solution of (4) (see [3]). It is assumed that the right hand side of the system (4) satisfies a Lipschitz condition in $x(t), x'(t), x''(t)$ and $x(t - r_i)$. This assumption guarantees the uniqueness of solutions of (4) (see [3], pp.15).

We shall use as a tool to prove our main results a Lyapunov function $U = U(t, x_t, y_t, z_t)$ defined by

$$U(t, x_t, y_t, z_t) = \exp\left(-\frac{\gamma(t)}{\mu}\right)V(t, x_t, y_t, z_t) = \exp\left(-\frac{\gamma(t)}{\mu}\right)V, \tag{5}$$

where

$$\gamma(t) = \int_0^t |R(s)|\,ds,$$

and

$$\begin{aligned}
V &= dc(t)H(x) + c(t)y\sum_{i=1}^{n} h_i(x) + nb(t)P(t)G\left(\frac{y}{P(t)}\right) + \frac{1}{2}z^2 \\
&\quad + \frac{d}{P(t)}yz + \frac{da(t)}{2P^2(t)}y^2 + \sum_{i=1}^{n}\lambda_i\int_{-r_i}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds,
\end{aligned} \tag{6}$$

where $H(x) = \sum_{i=1}^{n}\int_0^x h_i(u)du$ and $G(y) = \int_0^y g(u)du$. μ and λ_i are certain positive constants, which will be specified later in the proof. From the definition of V in (6), we observe that the above Lyapunov functional can be rewritten as follows

$$V = V_1 + V_2 + \sum_{i=1}^{n} \lambda_i \int_{-r_i}^{0} \int_{t+s}^{t} y^2(\xi)d\xi ds,$$

with

$$V_1 = dc(t)H(x) + c(t)y \sum_{i=1}^{n} h_i(x) + nb(t)P(t)G(\frac{y}{P(t)}),$$

and

$$V_2 = \frac{1}{2}z^2 + \frac{d}{P(t)}yz + \frac{da(t)}{2P^2(t)}y^2.$$

First consider

$$\begin{aligned} V_2 &= \frac{1}{2}\left\{z^2 + \frac{2d}{P(t)}yz + \frac{da(t)}{P^2(t)}y^2\right\} \\ &= \frac{1}{2}\left(z + \frac{d}{P(t)}y\right)^2 + \frac{d(a(t)-d)}{2P^2(t)}y^2. \end{aligned}$$

Using the conditions on $a(t)$ in (v), $\dfrac{d(a(t)-d)}{2P^2(t)} \geq \dfrac{d(a_0-d)}{2P^2(t)} > 0$, it follows that there exists sufficiently small positive constant δ_2 such that

$$V_2 \geq \delta_2(y^2 + z^2). \tag{7}$$

$$V_1 \geq dc(t)H(x) + c(t)y \sum_{i=1}^{n} h_i(x) + \frac{nd_0 b(t)}{2P(t)}y^2,$$

since $\dfrac{g(y)}{y} \geq d_0 > 0$ implies that $G\left(\dfrac{y}{P(t)}\right) \geq \dfrac{d_0}{2P^2(t)}y^2$. We wish to arrange V_1, and using the assumptions (i)-(v), we get,

$$\begin{aligned} V_1 &\geq dc(t)H(x) + \frac{d_0 b(t)}{2P(t)} \sum_{i=1}^{n} \left\{y + \frac{c(t)h_i(x)P(t)}{d_0 b(t)}\right\}^2 \\ &\quad - \sum_{i=1}^{n} \frac{c^2(t)P(t)h_i^2(x)}{2d_0 b(t)} \\ &\geq dc(t) \sum_{i=1}^{n} \int_0^x \left(1 - \frac{c(t)P(t)h_i'(u)}{dd_0 b(t)}\right) h_i(u)du \\ &\geq dc(t) \sum_{i=1}^{n} \int_0^x \left(1 - \frac{M\rho_i}{dd_0}\right) h_i(u)du \end{aligned}$$

$$
\begin{aligned}
&\geq dc(t)\sum_{i=1}^{n}\int_0^x\left(1-\frac{M\rho_i}{dd_0}\right)\frac{h_i(u)}{u}udu\\
&\geq dc(t)\sum_{i=1}^{n}\int_0^x\left(1-\frac{M\rho_i}{dd_0}\right)\delta_i udu\\
&\geq \frac{dc(t)}{2}\sum_{i=1}^{n}\left(1-\frac{M\rho_i}{dd_0}\right)\delta_i x^2,
\end{aligned}
$$

so that

$$V_1 \geq \frac{\delta_3}{2}x^2, \tag{8}$$

where $\delta_3 = dc_0\sum_{i=1}^{n}\delta_i\left(1-\frac{M\rho_i}{dd_0}\right) > dc_0\sum_{i=1}^{n}\delta_i\left(1-\frac{d}{d}\right) = 0$. From (8), (7) and (6), It is easy to check that

$$V \geq \delta_2 y^2 + \delta_2 z^2 + \frac{\delta_3}{2}x^2 + \sum_{i=1}^{n}\lambda_i\int_{-r_i}^{0}\int_{t+s}^{t}y^2(\xi)d\xi ds.$$

Subject to the conditions of Theorem 1, $V(0,0,0) = 0$ and there exists sufficiently small positive constant k such that

$$V \geq k(x^2+y^2+z^2), \tag{9}$$

since the integral $\int_{t+s}^{t} y^2(\xi)d\xi$ is positive, where $k = \min\left(\delta_2, \frac{\delta_3}{2}\right)$.
Assumptions (iii) and (vii) imply the following:

$$
\begin{aligned}
\gamma(t) &= \int_0^t |R(s)|\,ds\\
&\leq \int_{\alpha_1(t)}^{\alpha_2(t)}\frac{|\psi'(\tau)|}{\psi^2(\tau)}d\tau\\
&\leq \frac{1}{m^2}\int_{-\infty}^{+\infty}\left|\psi'(\tau)\right|d\tau \leq N < \infty,
\end{aligned}
$$

where $\alpha_1(t) = \min\{x'(0), x'(t)\}$, and $\alpha_2(t) = \max\{x'(0), x'(t)\}$. Now, we can deduce that there exists a continuous function W_1 with $W_1(|\Phi(0)|) \geq 0$ such that $W_1(|\Phi(0)|) \leq U(t,\Phi)$.
The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $U(t,\phi) \leq W_2(\|\phi\|)$, is easily verified.

Now, let $(x, y, z) = (x(t), y(t), z(t))$ be any solution of differential system (4).
Differentiating the function V, defined in (6), along system (4) with respect to the independent variable t, we have

$$\begin{aligned}\frac{d}{dt}V &= dc'(t)H(x) + c'(t)y\sum_{i=1}^{n} h_i(x) + nb'(t)P(t)G\left(\frac{y}{P(t)}\right) + \frac{d-a(t)}{P(t)}z^2\\&+ R(t)\left[(a(t)-d)zy - nb(t)P(t)\left(g\left(\frac{y}{P(t)}\right)y - P(t)G\left(\frac{y}{P(t)}\right)\right)\right] + \sum_{i=1}^{n}\lambda_i r_i y^2\\&+ \left[\frac{da'(t) + 2c(t)P(t)\sum_{i=1}^{n} h_i'(x)}{2P^2(t)}y^2 - ndb(t)\frac{y}{P(t)}g\left(\frac{y}{P(t)}\right)\right]\\&+ c(t)\left(z + \frac{dy}{P(t)}\right)\sum_{i=1}^{n}\int_{t-r_i}^{t}\frac{y(s)}{P(s)}h_i'(x(s))ds - \sum_{i=1}^{n}\lambda_i\int_{t-r_i}^{t} y^2(\xi)d\xi.\end{aligned}$$

Consequently by the hypothesis (i)-(vi), it follows that

$$\begin{aligned}\frac{d}{dt}V \le\ & dc'(t)H(x) + c'(t)y\sum_{i=1}^{n} h_i(x) + \frac{nd_0b'(t)}{2P(t)}y^2 - \left(\frac{\varepsilon}{M^2} - \sum_{i=1}^{n}\lambda_i r_i\right)y^2\\&+ |R(t)|\left[(A-d)\,|zy| + \frac{3}{2}nBd_1y^2\right] - \frac{1}{M}(a_0-d)z^2\\&+ c(t)\left(z + \frac{dy}{P(t)}\right)\sum_{i=1}^{n}\int_{t-r_i}^{t}\frac{y(s)}{P(s)}h_i'(x(s))ds - \sum_{i=1}^{n}\lambda_i\int_{t-r_i}^{t} y^2(\xi)d\xi.\end{aligned}$$

We claim that

$$\theta(t, x, y) = dc'(t)H(x) + c'(t)y\sum_{i=1}^{n} h_i(x) + \frac{nd_0b'(t)}{2P(t)}y^2 \le 0,$$

for all x, y and $t \ge 0$. First suppose that $c'(t) = 0$, then

$$\theta(t, x, y) = \frac{nd_0b'(t)}{2P(t)}y^2 \le 0.$$

Finally, suppose that $c'(t) < 0$, the quantity in the brackets above can be written as,

$$
\begin{aligned}
\theta(t,x,y) &= dc'(t)\left[H(x)+\frac{1}{d}y\sum_{i=1}^{n}h_i(x)+\frac{nd_0b'(t)}{2dc'(t)P(t)}y^2\right]\\
&= dc'(t)\left[H(x)+\frac{d_0b'(t)}{2dc'(t)P(t)}\sum_{i=1}^{n}\left\{y+\frac{c'(t)P(t)h_i(x)}{d_0b'(t)}\right\}^2\right]\\
&\quad - dc'(t)\left[\sum_{i=1}^{n}\frac{c'(t)P(t)h_i^2(x)}{2dd_0b'(t)}\right],
\end{aligned}
$$

moreover, assumption (ii) implies $\dfrac{c'(t)}{b'(t)} \leq 1$, thus

$$
\begin{aligned}
\theta(t,x,y) &\leq dc'(t)\sum_{i=1}^{n}\int_0^x(1-\frac{P(t)h_i'(u)}{dd_0})h_i(u)du\\
&\leq dc'(t)\sum_{i=1}^{n}\int_0^x(1-\frac{M\rho_i}{dd_0})h_i(u)du\\
&\leq c'(t)\frac{\delta_3}{2c_0}x^2 \leq 0.
\end{aligned}
$$

Hence, on combining the two cases, we have $\theta(t,x,y) \leq 0$ for all $t \geq 0$, x and y. Utilizing the assumptions of theorem and Schwartz inequality $|uv| \leq \frac{1}{2}(u^2+v^2)$, the following inequalities are obtained

$$
\begin{aligned}
\frac{dc(t)}{P(t)}y\sum_{i=1}^{n}\int_{t-r_i}^{t}\frac{y(s)}{P(s)}h_i'(x(s))ds &\leq \sum_{i=1}^{n}\frac{dC\rho_i r_i}{2m}y^2+\frac{Cd}{2m^3}\sum_{i=1}^{n}\int_{t-r_i}^{t}\rho_i y^2(\xi)d\xi\\
&\leq \sum_{i=1}^{n}\frac{dC\rho_i r_i}{2m}y^2+\frac{Cd\rho_i}{2m^3}\sum_{i=1}^{n}\int_{t-r_i}^{t}y^2(\xi)d\xi,\\
c(t)z\sum_{i=1}^{n}\int_{t-r_i}^{t}\frac{y(s)}{P(s)}h_i'(x(s))ds &\leq \sum_{i=1}^{n}\frac{C\rho_i r_i}{2}z^2+\frac{C}{2m^2}\sum_{i=1}^{n}\int_{t-r_i}^{t}\rho_i y^2(\xi)d\xi\\
&\leq \sum_{i=1}^{n}\frac{C\rho_i r_i}{2}z^2+\frac{C\rho_i}{2m^2}\sum_{i=1}^{n}\int_{t-r_i}^{t}y^2(\xi)d\xi,
\end{aligned}
$$

and

$$\begin{aligned} W_1 &= |R(t)|\left[(A-d)\,|zy| + \frac{3}{2}nBd_1y^2\right] \\ &\le |R(t)|\left[\frac{A-d}{2}z^2 + \frac{A-d+3nBd_1}{2}y^2\right] \\ &\le k_1\,|R(t)|\,(y^2+z^2), \end{aligned}$$

where $k_1 = \dfrac{A-d+3nBd_1}{2}$. These estimates imply that

$$\begin{aligned} \frac{d}{dt}V \le &-\left[\frac{\varepsilon}{M^2} - \sum_{i=1}^{n}\left(\lambda_i + \frac{dC\rho_i}{2m}\right)r_i\right]y^2 \\ &-\left[\frac{a_0-d}{M} - \sum_{i=1}^{n}\frac{C\rho_i r_i}{2}\right]z^2 \\ &+\sum_{i=1}^{n}\left[\frac{C\rho_i}{2m^2}\left(1+\frac{d}{m}\right)-\lambda_i\right]\int_{t-r_i}^{t} y^2(\xi)d\xi \\ &+k_1|R(t)|(y^2+z^2). \end{aligned}$$

If we take $\dfrac{C\rho_i}{2m^2}\left(1+\dfrac{d}{m}\right) = \lambda_i$, the last inequality becomes

$$\begin{aligned} \frac{d}{dt}V \le &-\left[\frac{\varepsilon}{M^2} - \sum_{i=1}^{n}\frac{C\rho_i}{2m}\left(d+\frac{1}{m}+\frac{d}{m^2}\right)r_i\right]y^2 \\ &-\left[\frac{a_0-d}{M} - \sum_{i=1}^{n}\frac{C\rho_i r_i}{2}\right]z^2 + k_1|R(t)|(y^2+z^2\Big). \end{aligned}$$

Using (9), (5) and taking $\mu = \dfrac{k}{k_1}$ we obtain:

$$\begin{aligned} \frac{d}{dt}U &= \exp\left(-\frac{k_1\gamma(t)}{k}\right)\left(\frac{d}{dt}V - \frac{k_1|R(t)|}{k}V\right) \\ &\le \exp\left(-\frac{k_1\gamma(t)}{k}\right)\left[-\left(\frac{\varepsilon}{M^2} - \sum_{i=1}^{n}\frac{C\rho_i r_i}{2m}\left(d+\frac{1}{m}+\frac{d}{m^2}\right)\right)y^2 \right. \\ &\left. -\left(\frac{a_0-d}{M} - \sum_{i=1}^{n}\frac{C\rho_i r_i}{2})z^2\right]. \end{aligned} \tag{10}$$

Provided that

$$\sum_{i=1}^{n} r_i < \min\left\{\frac{2(a_0-d)}{MC\rho_i}, \frac{2m^3\varepsilon}{C\rho_i M^2(d+dm^2+m)}\right\}.$$

The inequality (10) becomes

$$\frac{d}{dt}U(t,x_t,y_t,z_t) \leq -\beta \exp\left(-\frac{k_1 N}{k})(y^2+z^2\right), \quad \text{for some} \quad \beta > 0.$$

It is clear that the largest invariant set in Z is $Q = \{0\}$, where

$$Z = \left\{\phi \in C_H : \frac{d}{dt}U(\phi) = 0\right\}.$$

Namely, the only solution of system (4) for which $\frac{d}{dt}U(t,x_t,y_t,z_t) = 0$ is the solution $x = y = z = 0$. Thus, we conclude that every solution of system (4) is uniformly asymptotically stable. Now from (4) we have

$$x'(t)\Psi(x'(t)) = y(t), \tag{11}$$

Furthermore, it follows from (iii) that

$$\frac{|y(t)|}{M} \leq \left|x'(t)\right| = \frac{|y(t)|}{\Psi(x'(t))} \leq \frac{|y(t)|}{m},$$

which implies that $\lim_{t\to\infty} x'(t) = 0$. Differentiating (11) we obtain

$$x''(t)\left[\Psi(x'(t)) + \Psi'(x'(t))x'(t)\right] = z(t), \tag{12}$$

then $\lim_{t\to\infty} x''(t) = 0$ since $\lim_{t\to\infty} \Psi(x'(t)) + \Psi'(x'(t))x'(t) = \Psi(0)$.
Thus, under the above discussion, we conclude that every solution of equation (1) is uniformly asymptotically stable. □

For the case $q(t) \neq 0$, we consider the equivalent system of (2)

$$\begin{aligned} x' &= \frac{1}{P(t)}y \\ y' &= z \\ z' &= -\frac{a(t)}{P(t)}z + a(t)R(t)y - nb(t)g\left(\frac{y}{P(t)}\right) - c(t)\sum_{i=1}^{n} h_i(x) \\ &\quad + c(t)\sum_{i=1}^{n}\int_{t-r_i}^{t}\frac{y(s)}{P(t)}h_i'(x(s))ds + q(t). \end{aligned} \tag{13}$$

The following result is introduced.

Theorem 2 *In addition to the assumptions of Theorem 1, we assume that* (viii) *and* (ix) *hold. Then, there exists a finite positive constant* C *such that every solution* $x(t)$ *of equation* (2) *defined by the initial functions*

$$x(0) = \phi(t), \qquad x'(0) = \phi'(t), \qquad x''(0) = \phi''(t),$$

satisfies the inequalities

$$|x(t)| \leq C, \quad |x'(t)| \leq C, \quad |x''(t)| \leq C \quad \forall t \geq 0,$$

where $\phi \in C^2([-r, 0], \mathbb{R})$.

Proof. An easy calculation from (13) and (5) yields that

$$\frac{d}{dt}U_{(11)} = \frac{d}{dt}U_{(4)} + (z + \frac{d}{P(t)}y)q(t).$$

Since $\frac{d}{dt}U_{(4)} \leq 0$, then it follows that

$$\frac{d}{dt}U_{(11)} \leq \left(|z| + \frac{d}{P(t)}|y|\right)|q(t)|.$$

Noting that $|x| \leq 1 + x^2$, which implies that

$$\begin{aligned}\left(|z| + \frac{d}{P(t)}|y|\right)|q(t)| &\leq k_2(|z| + |y|)|q(t)| \\ &\leq k_2(2 + z^2 + y^2)|q(t)| \\ &\leq k_2\|X\|^2|q(t)| + 2k_2|q(t)| \\ &\leq \frac{k_2}{\delta e^{-\frac{N}{\mu}}}|q(t)|U + 2k_2|q(t)|,\end{aligned}$$

where $k_2 = \max\left\{1, \frac{d}{m}\right\}$, recalling that

$$\delta e^{-\frac{N}{\mu}}\|X\|^2 \leq U(t, x_t, y_t, z_t).$$

Let $\eta = \max\left\{2k_2, \frac{k_2}{\delta e^{-\frac{N}{\mu}}}\right\}$, then

$$\frac{d}{dt}U_{(11)} \leq \eta|q(t)| + \eta|q(t)|U.$$

Multiplying each side of this inequality by the integrating factor $e^{-\eta Q(t)}$, we get

$$e^{-\eta Q(t)}\frac{d}{dt}U_{(11)} \le e^{-\eta Q(t)}\eta Q'(t) + e^{-\eta Q(t)}\eta Q'(t)U.$$

Integrating each side of this inequality from 0 to t, we get, where $X_0 = (x(0), y(0), z(0))$,

$$e^{-\eta Q(t)}U - U(0, X_0) \le 1 - e^{-\eta Q(t)}.$$

Since $Q(t) \le L$ for all t, we have

$$U(t, x_t, y_t, z_t) \le U(0, X_0)e^{\eta L} + [e^{\eta L} - 1] \quad \text{for} \quad t \ge 0.$$

Now, since the right-hand side is a constant, and since $U(t, x_t, y_t, z_t) \to \infty$ as $x^2 + y^2 + z^2 \to \infty$, it follows that there exists a $D > 0$ such that

$$|x(t)| \le D, \ |y(t)| \le D, \ |z(t)| \le D \quad \forall t \ge 0.$$

From (11) and (iii) we obtain

$$|x'| = \left|\frac{y}{\Psi(x')}\right| \le \frac{D}{m},$$

it follows from condition (viii) that

$$K(x') = \psi(x') + x'\psi'(x') \ge m + \eta,$$

but (12) implies

$$|x''| = \frac{|z|}{K(x')} \le \frac{D}{\eta + m},$$

thus we can deduce

$$|x(t)| \le C, \ |x'(t)| \le C, \ |x''(t)| \le C \quad \forall t \ge 0,$$

where $C = \sup\left(D, \dfrac{D}{m}, \dfrac{D}{\eta + m}\right)$. This completes the proof of theorem. □

Example 1

$$\begin{aligned}
&\left(\left(\frac{x'}{1 + x'^2} + n(n+1)\right)x'\right)'' + \left(4n^2(n+1)^2 - \frac{1}{2}e^{-2t} + \frac{1}{2}\right)x'' \\
&\quad + n\left(\frac{1}{1+t} + 1\right)\left(2x' + \frac{x'}{1 + x'^2}\right) \\
&\quad + \left(\frac{1}{2(1+t)} + \frac{1}{2}\right)\sum_{i=1}^{n}\left[ix(t - r_i) + \frac{ix(t - r_i)}{1 + |x(t - r_i)|}\right] = e^{-t}.
\end{aligned} \tag{14}$$

We can simply verify that

i) $4n^2(n+1)^2 = a_0 \leq a(t) = 4n^2(n+1)^2 - \frac{1}{2}e^{-2t} + \frac{1}{2} \leq 4n^2(n+1)^2 + \frac{1}{2},\ t \geq 0,$
$c_0 = \frac{1}{2} \leq c(t) = \frac{1}{2(1+t)} + \frac{1}{2} \leq C = 1 = b_0 \leq b(t) = \frac{1}{1+t} + 1 \leq 2,\ t \geq 0,$

ii) *From* (i) *we have* $b(t) > c(t)$ *and* $b'(t) \leq c'(t) \leq 0, \forall\, t \geq 0,$

iii) $\psi(x') = \frac{x'}{1+x'^2} + n(n+1)$. *Now, it is easy to see that*

$$\inf_{u\in\mathbb{R}} \Psi(u) = -\frac{1}{2} + n(n+1) > m = -1 + n(n+1),$$
$$\sup_{u\in\mathbb{R}} \Psi(u) = \frac{1}{2} + n(n+1) < M = 1 + n(n+1),$$
$$d_0 = 2 \leq \frac{g(y)}{y} = 2 + \frac{1}{1+y^2} \leq 3 = d_1 \text{ with } y \neq 0.$$

Also

iv) $\delta_i = i \leq \frac{h_i(x)}{x} = \left(i + \frac{i}{1+|x|}\right)$ *with* $x \neq 0$, *and* $|h_i'(x)| \leq \rho_i = 2i$,
then $\displaystyle\sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} 2i = n(n+1).$

v) *For* $d = 2Mn(n+1)$ *we have*

$$Mi = \frac{M\rho_i}{d_0} < Mn < d < a_0 = 4n^2(n+1)^2,$$

vi) $a'(t) = e^{-2t} \leq 1$, *and*

$$\frac{1}{2}da'(t) - b_0\left(dd_0 - M\sum_{i=1}^{n}\rho_i\right) \leq -\frac{3}{2}d + Mn(n+1) < 0.$$

vii) *An explicit calculation shows that*

$$\int_{-\infty}^{+\infty} |\psi'(u)|\, du = \int_{-\infty}^{+\infty} \left|\frac{u^2+1-2}{(1+u^2)^2}\right| du \leq \int_{-\infty}^{+\infty} \left[\left|\frac{1}{1+u^2}\right| + \left|\frac{2}{(1+u^2)^2}\right|\right] du$$
$$\leq 2\pi,$$

viii) $\inf_{u\in\mathbb{R}} u\Psi'(u) = \eta = -\frac{1}{4} > -n(n+1) + 1,$

ix) $Q(t) = \int_0^t e^{-s} ds < \infty$.

If we take $r_i = \frac{2k}{\pi^2 i^2}$, *with* $k = \min\{\alpha_n, \beta_n\}$. *Then*

$$\sum_{i=1}^{i=n} r_i < \sum_{i=1}^{\infty} \frac{2k}{\pi^2 i^2} = k < \min\{\alpha_i, \beta_i\}.$$

All the assumptions (i) *through* (ix) *are satisfied, we can conclude using Theorem 3.2 that every solution of* (14) *is uniformly bounded.*

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The time-varying shortest path problem with fuzzy transit costs and speedup

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Abstract. In this paper, we focus on the time-varying shortest path problem, where the transit costs are fuzzy numbers. Moreover, we consider this problem in which the transit time can be shortened at a fuzzy speedup cost. Speedup may also be a better decision to find the shortest path from a source vertex to a specified vertex.

1 Introduction

Time-varying shortest path problem may arise in the applications of mathematics such as transportation, telecommunication and computer networks. The problem is to find the shortest path from a source vertex to a target vertex, so that the total costs of path is minimized subject to the total times of path is at most T, where T is the time horizon and a given positive integer. The shortest path problem with fuzzy numbers has been studied by Kelvin [7], where a new model based on fuzzy number was presented. Lin and Chern [10] and Li and Gen [9] surveyed this subject, separately. Gent et al. in [3] solved the shortest path problem by genetic algorithm. Shirdel and Rezapour in [15] studied a k-objective time-varying shortest path problem, which cannot

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be combined into a single overall objective. Okada and Gen [13] concentrated on the problem with interval numbers. Then, Okada and Soper maintained their work on the shortest path in network with fuzzy number in [14]. We encourage the reader to consult [2, 4, 6, 8, 11, 12] for historical background, computational techniques and mathematical properties of the fuzzy shortest path problem. In this paper, we consider time-varying shortest path, where transit costs are triangular fuzzy numbers. Moreover, we assume that the transit times and the transit costs are dependent on discrete time T, where T is the time horizon. The preliminary and definitions are given in Section 2. The problem is discussed in Section 3 and two theorems are proved for solving of problem. An algorithm is presented in Section 4 for the mentioned problem.

2 Preliminary

Consider a time-varying network $G(V, A, b, c)$, where V and A are the set of vertices and the set of arcs, respectively, with $|V| = n$, $|A| = m$. The transit time $b(i, j, t)$ and the fuzzy transit cost $\tilde{c}(i, j, t)$ are associated with each arc (i, j), respectively, such that t is the departure time of a vertex i. Moreover, $\tilde{c}(i, j, t)$ is assumed to be triangular fuzzy number. The transit time $b(i, j, t)$ and the fuzzy transit cost $\tilde{c}(i, j, t)$ are the functions of discrete time $t = 0, 1, ..., T$, where T is a given positive integer. The waiting time at vertex i from t to $t + 1$ is shown by $w(i)$ and the associated fuzzy waiting cost is presented by $\tilde{c}(i, t)$.

Definition 1 [5] *The membership function* $\mu_A(x) : X \to [0, 1]$ *allocates a value between 0 or 1 to each member in* X*, where* X *is a universal set and* $A \subseteq X$*. The assigned values point out the membership grade of the element in the set* A*, and moreover the set* $\left\{ \left(x, \mu_A(x)\right) : x \in X \right\}$ *is named fuzzy set.*

Definition 2 [5] *A fuzzy number* $\tilde{A} = (\alpha, \beta, \gamma)$ *is called to be a triangular fuzzy number, when it has the following membership function:*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-\alpha}{\beta-\alpha} & \alpha \le x \le \beta \\ 1 & x = \beta \\ \dfrac{\gamma-x}{\gamma-\beta} & \beta \le x \le \gamma \\ 0 & \text{otherwise} \end{cases}$$

where, $\alpha \in R$, $\beta \in R$ *and* $\gamma \in R$.

Definition 3 [6] *Let* $\tilde{A} = (\alpha_1, \beta_1, \gamma_1)$ *be a triangular fuzzy number, then its ranking function* $\tilde{A}$ *is a function* $\mathfrak{R} : \mathfrak{R}(\tilde{A}) \to \mathbb{R}$, *where* $\mathfrak{R}(\tilde{A})$ *is the set of all fuzzy numbers. For a triangular fuzzy number* $\tilde{A} = (\alpha_1, \beta_1, \gamma_1)$, *the ranking function* $\mathfrak{R}$ *is calculated by* $\mathfrak{R}(\tilde{A}) = \frac{1}{4}(\alpha_1 + 2\beta_1 + \gamma_1)$.

Definition 4 [6] *Assume* $\tilde{A} = (\alpha_1, \beta_1, \gamma_1)$ *and* $\tilde{B} = (\alpha_2, \beta_2, \gamma_2)$ *are two triangular fuzzy numbers, then:*

- $\tilde{A} \oplus \tilde{B} = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2)$,
- $\tilde{A} > \tilde{B}$ *if and only if* $\mathfrak{R}(\tilde{A}) > \mathfrak{R}(\tilde{B})$,
- $\tilde{A} = \tilde{B}$ *if and only if* $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$,
- $\tilde{A} < \tilde{B}$ *if and only if* $\mathfrak{R}(\tilde{A}) < \mathfrak{R}(\tilde{B})$,
- *A triangular fuzzy number* $\tilde{A}_k$ *is the maximum triangular fuzzy numbers* $\tilde{A}_i$ *such that* $\mathfrak{R}(\tilde{A}_k \geq)\mathfrak{R}(\tilde{A}_i)$ *for all* $1 \leq i \leq n$,
- *Minimum triangular fuzzy numbers* $\tilde{A}_i$ *is similarly defined,*
- *Moreover, let* $\tilde{0} = \tilde{A} \Leftrightarrow \alpha_1 = 0, \beta_1 = 0, \gamma_1 = 0$ *and* $\tilde{A} = \tilde{\infty} \Leftrightarrow \mathfrak{R}(\tilde{A}) = \tilde{\infty}$.

Definition 5 [1] *Suppose a time-varying path from* i_1 *to* i_k *is specified by* $P\,(i_1 - i_2 - \cdots - i_k)$. *Consider* $\alpha(i_r)$ *be the arrival time of a vertex* i_r *on* P *such that* $\alpha(i_1) = t_1 \geqslant 0$ *and we have:*

$$\alpha(i_r) = \alpha(i_{r-1}) + w(i_{r-1}) + b\big(i_{r-1}, i_r, \tau(i_{r-1})\big) \quad \textit{for } 2 \leq r \leq k$$

where, $\tau(i_{r-1})$ *is the departure time of a vertex* i_{r-1} *for* $\mathsf{for} 2 \leq r \leq k$ *on* P *and we have:*

$$\tau(i_{r-1}) = \alpha(i_{r-1}) + w(i_{r-1}) \quad \textit{for } 2 \leq r \leq k.$$

Moreover, let $\alpha(s) = 0$ *for the source vertex* s.

Definition 6 [1] *Let* $P\,(i_1 = s - i_2 - \cdots - i_k)$ *be a time-varying path from* s *to* i_k, *then the time of time-varying path* P *is determined by* $\alpha(i_k) + w(i_k)$.

Definition 7 *The fuzzy cost of the time-varying path* $P\,(i_1 - i_2 - \cdots - i_k)$ *is defined as follow:*

$$\tilde{\zeta}(P) = \tilde{\zeta}(i_k) = \tilde{\zeta}(i_{k-1}) + \tilde{c}\big(i_{k-1}, i_k, \tau(i_{k-1})\big) + \Sigma_{t'=0}^{w(i_k)-1}\tilde{c}\big(i_k, t' + \alpha(i_k)\big)$$

Moreover, the path P *is the shortest path within time* t *if for each path* P' *within time* t, *we have:* $\zeta(P) \leq \zeta(P')$.

3 The fuzzy shortest path problem with speed up

Consider that the transit time $b(i,j,t)$ can be reduced at a fuzzy speedup cost $\tilde{c_\gamma}(i,j,t)$ i.e. an arc (i,j) is traversed in shorter time and $b(i,j,t)$ is rebated by paying the speedup cost $\tilde{c_\gamma}(i,j,t)$. Speedup on one or several arcs may be leaded to a better solution; especially it may be necessary when the deadline T is tight. Let $\gamma(i,j,t)$ be the amount of time reduced from the transit time $b(i,j,t)$ with fuzzy speedup cost $\tilde{c_\gamma}(i,j,t)$, such that $b(i,j,t)-\gamma(i,j,t)>0$.

Theorem 1 *Define* $d_A^s(j,t)$ *as the fuzzy cost of a time-varying shortest path from* s *to* j *of time exactly* t *with speed up. Then* $d_A^s(j,0)=\tilde{\infty}$ *for all* $j\neq s$, $d_A^s(s,0)=\tilde{0}$ *and if* $t>0$ *have:*

$$d_A^s(j,t)=\min\Big\{d_A^s(j,t-1)+\tilde{c}(j,t-1),\min_{(i,j)\in A}\min_{u+b(i,j,u)-\gamma(i,j,t)=t}\left\{d_A^s(i,u)+\tilde{c}(i,j,u)+\tilde{c}_\gamma(i,j,u)\right\}\Big\}$$

Proof. It is clear that $d_A^s(j,0)=\tilde{\infty}$ for all $j\neq s$ and $d_A^s(s,0)=\tilde{0}$, since all transit times are positives. The theorem is proved by induction on $t>0$. Consider $t=1$, for $j=s$ the theorem clearly holds. If $j\neq s$, for $(s,j)\in A$ and $b(s,j,0)=1$, the theorem holds with $d_A^s(s,0)+\tilde{c}(s,j,0)+\tilde{c}_\gamma(s,j,0)$. Assume that the theorem is correct for $t'<t$ and $d_A^s(j,t)$ is finite. If $d_A^s(j,t)=d_A^s(j,t-1)+\tilde{c}(j,t-1)$, by induction, there is a path from s to j within time $t-1$, by waiting at j one unit of time more, the time of path is exactly t . If $d_A^s(j,t)=d_A^s(i,u)+\tilde{c}(i,j,u)+\tilde{c}_\gamma(i,j,u)$, since $b(i,j,t)-\gamma(i,j,t)>0$, then $u<t$, therefore by induction, there is a path from s to i within time u and cost $d_A^s(i,u)$. We can extend this path to j, obtaining a path from s to j within time $u+b(i,j,u)-\gamma(i,j,t)=t$ and cost $d_A^s(j,t)=d_A^s(i,u)+\tilde{c}(i,j,u)+\tilde{c}_\gamma(i,j,u)$. It is easy to see that $d_A^s(j,t)$ is the fuzzy cost of shortest path from s to j. □

Theorem 2 *Define* $d_A^{s^*}(j)$ *as the cost of a time-varying shortest path form* s *to* j *of time at most* T *with speed-up, then we have:*

$$d_A^{s^*}(j)=\min_{0\leq t\leq T}d_A^s(j,t).$$

Proof. The proof is Straightforward. □

4 The algorithm for solving fuzzy shortest path problem with speed up

The key idea in the below algorithm is to first sort the values of $u+b(i,j,u)-\gamma(i,j,t)=t$ for all $u=0,1,2,\ldots,T$ and all arcs $(i,j)\in A$, before the recursive relation as given in theorem 1 is applied to compute $d^s_A(j,t)$ for all $j \in V$ and $t = 0, 1, 2, \dots, T$.

Algorithm

1. Begin
2. Let $d^s_A(j,0) = \tilde{\infty}$ for all $j \neq s$, $d^s_A(s,0) = \tilde{0}$;
3. Sort all values $u + b(i,j,u) - \gamma(i,j,t) = t$ for all $u = 0, 1, 2, \dots, T$ and all arcs $(i,j) \in A$;
4. For $t = 0, 1, 2, \dots, T$, do;

   For $j \in V$, do;

   For each $(i,j) \in A$, and each u and γ, do;

$$d^s_A(j,t) = \min\left\{d^s_A(j,t-1) + \tilde{c}(j,t-1), \min_{(i,j)\in A} \min_{u+b(i,j,t)-\gamma(i,j,t)=t} \left\{d^s_A(i,u) + \tilde{c}(i,j,u) + \tilde{c}_\gamma(i,j,u)\right\}\right\}$$

5. Let $d^{s^*}_A(j) = \min_{0\le t\le T} d^s_A(j,t)$;
6. End

Example 1 *Consider a given time-varying network* G *in Figure 1, where* $T = 6$.

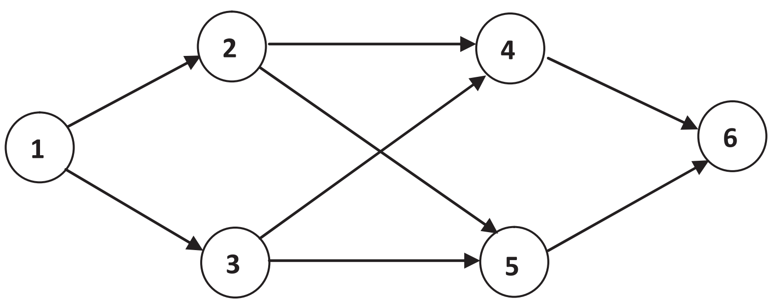

Figure1. *A network for example 1*

Assume that the waiting at vertices are not allowed, i.e. $w(i) = 0$ *for all* $i \in V$. *Furthermore, for arcs* $(1,2)$, $(1,3)$ *and* $(2,4)$ *and for each time* $t = 0, 1, \dots, 6$ *let:*

$$b(i,j,t) = 3,\ \tilde{c}(i,j,t) = (2,3,4)$$

for arcs $(4,6)$ *and* $(5,6)$ *and for each time* $t = 0,1,\ldots,6$ *let:*

$$b(i,j,t) = 3,\ \tilde{c}(i,j,t) = (1,3,5)$$

Other transit times and fuzzy transit costs are shown in Table 1.

Table 1. *Information for network G*

Arcs	**(2,5)**		**(3,4)**		**(3,5)**	
t \ **b,** $\tilde{c}$	**b**	$\tilde{c}$	**b**	$\tilde{c}$	**b**	$\tilde{c}$
0	1	(1,2,3)	2	(2,4,6)	3	(1,4,5)
1	4	(1,3,4)	2	(2,4,5)	2	(1,4,6)
2	3	(1,3,4)	1	(3,4,5)	3	(2,3,4)
3	3	(2,3,5)	4	(3,4,6)	5	(2,4,6)
4	2	(1,3,6)	3	(1,3,5)	4	(3,5,6)
5	3	(1,3,5)	2	(1,3,4)	3	(4,5,6)
6	4	(1,3,4)	3	(1,2,4)	2	(2,5,7)

There is not any feasible path from source vertex 1 to vertex 6 with $T = 6$*, because each path has time more than time horizon* $T = 6$*. Let* $\gamma(i,j,t) = 1$*, corresponding to each* γ*, consider that there is a speedup cost* $\tilde{c}_{\gamma}(i,j,t) = (2,4,6)$*. After applying the described algorithm to find the shortest path between the vertex 1 and the vertex 6, we can obtain a path 1-2-5-6 with fuzzy cost* $d_A^{s^*}(j) = (10,20,31)$.

5 Conclusion

In the time-varying shortest path problem, speedup may be a better decision for the solution, although it incurs an extra cost. In particular, speedup may become necessary when the deadline T is tight. In this paper, we have considered one class of the time-varying shortest path, where the transit costs are fuzzy numbers and speedups on all arcs along the path are decision variables. Moreover, we have presented an algorithm for solving the problem.

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Some common random fixed point theorems for contractive type conditions in cone random metric spaces

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Abstract. In this paper, we establish some common random fixed point theorems for contractive type conditions in the setting of cone random metric spaces. Our results unify, extend and generalize many known results from the current existing literature.

1 Introduction

Random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is needed for the study of various classes of random equations. The study of random fixed point theory was initiated by the Prague school of Probabilities in the 1950s [9, 10, 24]. Common random fixed point theorems are stochastic generalization of classical common fixed point theorems. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see e.g. [19]) and references mentioned therein. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [7] attracted the attention of several

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mathematicians and gave wings to the theory. Itoh [14] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become the full fledged research area and various ideas associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [4, 5, 6, 11, 22]). Papageorgiou [17, 18], Beg [2, 3] studied common random fixed points and random coincidence points of a pair of compatible random operators and proved fixed point theorems for contractive random operators in Polish spaces.

In 2007, Huang and Zhang [12] introduced the concept of cone metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 13, 21, 23] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

In 2008, Rezapour and Hamlbarani [21] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. Recently, Mehta et al. [16] introduced the concept of cone random metric space and proved an existence of random fixed point under weak contraction condition in the setting of cone random metric spaces.

In this paper, we establish some common random fixed point theorems for contractive type conditions in the setting of cone random metric spaces. Our results extend the corresponding results of [16] and some others from the current existing literature.

2 Preliminaries

Definition 1 (See [16]) *Let* (E, τ) *be a topological vector space. A subset* P *of* E *is called a cone whenever the following conditions hold:*

($\mathbf{c_1}$) P *is closed, nonempty and* $P \neq \{0\}$*;*

($\mathbf{c_2}$) $a, b \in R$*,* $a, b \geq 0$ *and* $x, y \in P$ *imply* $ax + by \in P$*;*

($\mathbf{c_3}$) *If* $x \in P$ *and* $-x \in P$ *implies* $x = 0$*.*

For a given cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P.

Definition 2 (See [12, 25]) *Let* X *be a nonempty set. Suppose that the mapping* $d \colon X \times X \to E$ *satisfies:*

($\mathbf{d_1}$) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*;*

($\mathbf{d_2}$) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

($\mathbf{d_3}$) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d *is called a cone metric* [12] *or* K*-metric* [25] *on* X *and* (X, d) *is called a cone metric space* [12].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1 (See [12]) *Let* $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ *and* $d \colon X \times X \to E$ *defined by* $d(x, y) = (|x - y|, \alpha|x - y|)$, *where* $\alpha \geq 0$ *is a constant. Then* (X, d) *is a cone metric space with normal cone* P *where* $K = 1$.

Example 2 (See [20]) *Let* $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \textit{for all}\, n\}$, (X, ρ) *a metric space, and* $d \colon X \times X \to E$ *defined by* $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq 1}$. *Then* (X, d) *is a cone metric space.*

Clearly, the above examples show that the class of cone metric spaces contains the class of metric spaces.

Definition 3 (See [12]) *Let* (X, d) *be a cone metric space. We say that* $\{x_n\}$ *is:*

(i) *a Cauchy sequence if for every* ε *in* E *with* $0 \ll \varepsilon$, *then there is an* N *such that for all* $n, m > N$, $d(x_n, x_m) \ll \varepsilon$;

(ii) *a convergent sequence if for every* ε *in* E *with* $0 \ll \varepsilon$, *then there is an* N *such that for all* $n > N$, $d(x_n, x) \ll \varepsilon$ *for some fixed* x *in* X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

In the following (X, d) will stands for a cone metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P.

Definition 4 (**Measurable function**) (See [16]) *Let* (Ω, Σ) *be a measurable space with* Σ*-a sigma algebra of subsets of* Ω *and* M *be a nonempty subset of a metric space* $X = (X, d)$. *Let* 2^M *be the family of nonempty subsets of* M *and* $C(M)$ *the family of all nonempty closed subsets of* M. *A mapping* $G \colon \Omega \to 2^M$ *is called measurable if for each open subset* U *of* M, $G^{-1}(U) \in \Sigma$, *where* $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\}$.

Definition 5 (**Measurable selector**) (See [16]) *A mapping* $\xi\colon \Omega \to M$ *is called a measurable selector of a measurable mapping* $G\colon \Omega \to 2^M$ *if* ξ *is measurable and* $\xi(\omega) \in G(\omega)$ *for each* $\omega \in \Omega$.

Definition 6 (**Random operator**) (See [16]) *The mapping* $T\colon \Omega \times M \to X$ *is said to be a random operator if and only if for each fixed* $x \in M$, *the mapping* $T(.,x)\colon \Omega \to X$ *is measurable.*

Definition 7 (**Continuous random operator**) (See [16]) *A random operator* $T\colon \Omega \times M \to X$ *is said to be continuous random operator if for each fixed* $x \in M$ *and* $\omega \in \Omega$, *the mapping* $T(\omega, .)\colon M \to X$ *is continuous.*

Definition 8 (**Random fixed point**) (See [16]) *A measurable mapping* $\xi\colon \Omega \to M$ *is a random fixed point of a random operator* $T\colon \Omega \times M \to X$ *if and only if* $T(\omega, \xi(\omega)) = \xi(\omega)$ *for each* $\omega \in \Omega$.

Definition 9 (**Cone Random Metric Space**) *Let* M *be a nonempty set and let the mapping* $d\colon \Omega \times M \to P$, *where* P *is a cone,* $\omega \in \Omega$ *be a selector, satisfy the following conditions:*

(i) $d(x(\omega), y(\omega)) \geq 0$ *and* $d(x(\omega), y(\omega)) = 0$ *if and only if* $x(\omega) = y(\omega)$ *for all* $x(\omega)$, $y(\omega) \in \Omega \times M$,

(ii) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$ *for all* $x, y \in M$, $\omega \in \Omega$ *and* $x(\omega)$, $y(\omega) \in \Omega \times M$,

(iii) $d(x(\omega), y(\omega)) \leq d(x(\omega), z(\omega)) + d(z(\omega), y(\omega))$ *for all* x, y, $z \in M$ *and* $\omega \in \Omega$ *be a selector,*

(iv) *for any* x, $y \in M$, $\omega \in \Omega$, $d(x(\omega), y(\omega))$ *is non-increasing and left continuous.*

Then d *is called cone random metric on* M *and* (M, d) *is called a cone random metric space.*

Definition 10 *Let* (X, d) *be a metric space. A mapping* $T\colon X \to X$ *is called an* a*-contraction if*

$$d(Tx, Ty) \leq a\, d(x, y) \text{ for all } x,\ y \in X, \tag{1}$$

where $a \in (0, 1)$.

Definition 11 *The mapping* T *is called Kannan contraction mapping* [15] *if there exists* $b \in (0, \frac{1}{2})$ *such that*

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)] \text{ for all } x,\ y\ \in X. \tag{2}$$

Definition 12 *The mapping* T *is called Chatterjea contraction mapping [8] if there exists* $c \in (0, \frac{1}{2})$ *such that*

$$d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)] \text{ for all } x,\ y\ \in X. \tag{3}$$

Generalized contraction condition for two mappings

Let (X, d) be a metric space and let $S,\ T\colon X \to X$ be two mappings satisfying the condition:

$$\begin{aligned} d(Sx, Ty) \leq{}& a\,d(x, y) + b\,[d(x, Sx) + d(y, Ty)] \\ &+ c\,[d(x, Ty) + d(y, Sx)], \end{aligned} \tag{4}$$

for all $x, y \in X$ and $a + 2b + 2c < 1$, where $a, b, c > 0$ are constants.

Remark 1 (i) *If we take* $S = T$ *and* $b = c = 0$*, then condition* (4) *reduces to the contraction condition* (1)*.*

(ii) *If we take* $S = T$ *and* $a = c = 0$*, then condition* (4) *reduces to the Kannan contraction condition* (2)*.*

(iii) *If we take* $S = T$ *and* $a = b = 0$*, then condition* (4) *reduces to the Chatterjea contraction condition* (3)*.*

Thus it is clear from Remark 1 that the generalized contraction condition for one or two mappings is weaker than Banach contraction, Kannan contraction and Chatterjea contraction conditions.

3 Main results

In this section we shall prove some common random fixed point theorems under generalized contractive condition (4) in the setting of cone random metric spaces.

Theorem 1 *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X*. Let* S *and*

T *be two continuous random operators defined on* M *such that for* $\omega \in \Omega$, $S(\omega, .)$, $T(\omega, .)\colon \Omega \times M \to M$ *satisfying the condition:*

$$\begin{aligned} d(S(x(\omega)), T(y(\omega))) \le a(\omega)\, d(x(\omega), y(\omega)) \\ + b(\omega)\,[d(x(\omega), S(x(\omega))) + d(y(\omega), T(y(\omega)))] \\ + c(\omega)\,[d(x(\omega), T(y(\omega))) + d(y(\omega), S(x(\omega)))] \end{aligned} \tag{5}$$

for all $x, y \in M$, $a(\omega) + 2b(\omega) + 2c(\omega) < 1$, *where* $a(\omega)$, $b(\omega)$, $c(\omega) > 0$ *and* $\omega \in \Omega$. *Then* S *and* T *have a unique common random fixed point in* X.

Proof. For each $x_0(\omega) \in \Omega \times M$ and $n = 0, 1, 2, \dots$, we choose $x_1(\omega), x_2(\omega) \in \Omega \times M$ such that $x_1(\omega) = S(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we define sequence of elements of M such that $x_{2n+1}(\omega) = S(x_{2n}(\omega))$ and $x_{2n+2}(\omega) = T(x_{2n+1}(\omega))$. Then from (5), we have

$$\begin{aligned} d(x_{2n+1}(\omega), x_{2n}(\omega)) &= d(S(x_{2n}(\omega)), T(x_{2n-1}(\omega))) \\ &\le a(\omega)\, d(x_{2n}(\omega), x_{2n-1}(\omega)) + b(\omega)\,[d(x_{2n}(\omega), S(x_{2n}(\omega))) \\ &\quad + d(x_{2n-1}(\omega), T(x_{2n-1}(\omega)))] \\ &\quad + c(\omega)\,[d(x_{2n}(\omega), T(x_{2n-1}(\omega))) \\ &\quad + d(x_{2n-1}(\omega), S(x_{2n}(\omega)))] \\ &\le a(\omega)\, d(x_{2n}(\omega), x_{2n-1}(\omega)) + b(\omega)\,[d(x_{2n}(\omega), x_{2n+1}(\omega)) \\ &\quad + d(x_{2n-1}(\omega), x_{2n}(\omega))] + c(\omega)\,[d(x_{2n}(\omega), x_{2n}(\omega)) \\ &\quad + d(x_{2n-1}(\omega), x_{2n+1}(\omega))] \\ &= a(\omega)\, d(x_{2n}(\omega), x_{2n-1}(\omega)) + b(\omega)\,[d(x_{2n}(\omega), x_{2n+1}(\omega)) \\ &\quad + d(x_{2n-1}(\omega), x_{2n}(\omega))] + c(\omega)\, d(x_{2n-1}(\omega), x_{2n+1}(\omega)) \\ &\le a(\omega)\, d(x_{2n}(\omega), x_{2n-1}(\omega)) + b(\omega)\,[d(x_{2n}(\omega), x_{2n+1}(\omega)) \\ &\quad + d(x_{2n-1}(\omega), x_{2n}(\omega))] + c(\omega)\,[d(x_{2n-1}(\omega), x_{2n}(\omega)) \\ &\quad + d(x_{2n}(\omega), x_{2n+1}(\omega))] \\ &= \big(a(\omega) + b(\omega) + c(\omega)\big)\, d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &\quad + (b(\omega) + c(\omega))\, d(x_{2n+1}(\omega), x_{2n}(\omega)) \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_{2n+1}(\omega), x_{2n}(\omega)) &\le \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - b(\omega) - c(\omega)}\right) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &= \lambda\, d(x_{2n}(\omega), x_{2n-1}(\omega)), \end{aligned} \tag{6}$$

where
$$\lambda = \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - b(\omega) - c(\omega)}\right).$$
By the assumption of the theorem
$$\begin{aligned} a(\omega) + 2b(\omega) + 2c(\omega) < 1 &\Rightarrow a(\omega) + b(\omega) + c(\omega) < 1 - b(\omega) - c(\omega) \\ &\Rightarrow \lambda = \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - b(\omega) - c(\omega)}\right) < 1. \end{aligned}$$

Similarly, we have
$$d(x_{2n}(\omega), x_{2n-1}(\omega)) \leq \lambda\, d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$
Hence
$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq \lambda^2\, d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$
On continuing this process, we get
$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq \lambda^{2n}\, d(x_1(\omega), x_0(\omega)).$$
Also for $n > m$, we have
$$\begin{aligned} d(x_n(\omega), x_m(\omega)) &\leq d(x_n(\omega), x_{n-1}(\omega)) + d(x_{n-1}(\omega), x_{n-2}(\omega)) + \dots \\ &\quad + d(x_{m+1}(\omega), x_m(\omega)) \\ &\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m)\, d(x_1(\omega), x_0(\omega)) \\ &\leq \left(\frac{\lambda^m}{1-\lambda}\right) d(x_1(\omega), x_0(\omega)). \end{aligned}$$

Let $0 \ll \varepsilon$ be given. Choose a natural number N such that $\left(\frac{\lambda^m}{1-\lambda}\right) d(x_1(\omega), x_0(\omega)) \ll \varepsilon$ for every $m \geq N$. Thus
$$d(x_n(\omega), x_m(\omega)) \leq \left(\frac{\lambda^m}{1-\lambda}\right) d(x_1(\omega), x_0(\omega)) \ll \varepsilon,$$
for every $n > m \geq N$. This shows that the sequence $\{x_n(\omega)\}$ is a Cauchy sequence in $\Omega \times M$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$ as $n \to \infty$. Choose a natural number N_1 such that
$$d(x_{2n+1}(\omega), x_{2n+2}(\omega)) \ll \frac{\varepsilon\,(1 - b(\omega) - c(\omega))}{2(a(\omega) + b(\omega) + c(\omega))}, \tag{7}$$

and

$$d(z(\omega), x_{2n+2}(\omega)) \ll \frac{\varepsilon\,(1 - b(\omega) - c(\omega))}{2(1 + a(\omega) + 2c(\omega))}, \tag{8}$$

for every $n \geq N_1$. Hence for $n \geq N_1$, we have

$$\begin{aligned}
d(z(\omega), S(z(\omega))) &\leq d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+2}(\omega), S(z(\omega)) \\
&= d(z(\omega), x_{2n+2}(\omega)) + d(S(z(\omega), T(x_{2n+1}(\omega)) \\
&\leq d(z(\omega), x_{2n+2}(\omega)) + a(\omega)\, d(z(\omega), x_{2n+1}(\omega)) \\
&\quad + b(\omega)\, [d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), T(x_{2n+1}(\omega)))] \\
&\quad + c(\omega)\, [d(z(\omega), T(x_{2n+1}(\omega))) + d(x_{2n+1}(\omega), S(z(\omega)))] \\
&= d(z(\omega), x_{2n+2}(\omega)) + a(\omega)\, d(z(\omega), x_{2n+1}(\omega)) \\
&\quad + b(\omega)\, [d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))] \\
&\quad + c(\omega)\, [d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), S(z(\omega)))] \\
&\leq d(z(\omega), x_{2n+2}(\omega)) \\
&\quad + a(\omega)\, [d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+2}(\omega), x_{2n+1}(\omega))] \\
&\quad + b(\omega)\, [d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))] \\
&\quad + c(\omega)\, [d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), x_{2n+2}(\omega)) \\
&\quad + d(x_{2n+2}(\omega), z(\omega)) + d(z(\omega), S(z(\omega)))] \\
&= (1 + a(\omega) + 2c(\omega))\, d(z(\omega), x_{2n+2}(\omega)) \\
&\quad + (b(\omega) + c(\omega))\, d(z(\omega), S(z(\omega))) \\
&\quad + (a(\omega) + b(\omega) + c(\omega))\, d(x_{2n+1}(\omega), x_{2n+2}(\omega)).
\end{aligned}$$

The above inequality gives

$$\begin{aligned}
d(z(\omega), S(z(\omega))) \leq &\left(\frac{1 + a(\omega) + 2c(\omega)}{1 - b(\omega) - c(\omega)}\right) d(z(\omega), x_{2n+2}(\omega)) \\
&+ \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - b(\omega) - c(\omega)}\right) d(x_{2n+1}(\omega), x_{2n+2}(\omega)).
\end{aligned} \tag{9}$$

Using (7) and (8) in (9), we get

$$d(z(\omega), S(z(\omega))) \ll \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{10}$$

Thus $d(z(\omega), S(z(\omega))) \ll \frac{\varepsilon}{m}$ for all $m \geq 1$. So $\frac{\varepsilon}{m} - d(z(\omega), S(z(\omega))) \in P$ for all $m \geq 1$. Since $\frac{\varepsilon}{m} \to 0$ as $m \to \infty$ and P is closed, $-d(z(\omega), S(z(\omega))) \in P$.

But $d(z(\omega), S(z(\omega))) \in P$. Therefore by definition 1($\mathbf{c_3}$), $d(z(\omega), S(z(\omega))$ $= 0$ and so $S(z(\omega)) = z(\omega)$.

In an exactly the similar way we can prove that for all $\omega \in \Omega$, $T(z(\omega)) = z(\omega)$. Hence $S(z(\omega)) = T(z(\omega)) = z(\omega)$. This shows that $z(\omega)$ is a common random fixed point of S and T.

Uniqueness

Let $v(\omega)$ be another random fixed point common to S and T, that is, for $\omega \in \Omega$, $S(v(\omega)) = T(v(\omega)) = v(\omega)$. Then for $\omega \in \Omega$, we have

$$\begin{aligned} d(z(\omega), v(\omega)) &= d(S(z(\omega)), T(v(\omega))) \\ &\leq a(\omega)\, d(z(\omega), v(\omega)) + b(\omega)\, [d(z(\omega), S(z(\omega))) \\ &\quad + d(v(\omega), T(v(\omega)))] + c(\omega)\, [d(z(\omega), T(v(\omega)) \\ &\quad + d(v(\omega), S(z(\omega)))] \\ &\leq (a(\omega) + 2c(\omega))\, d(z(\omega), v(\omega)) \\ &< d(z(\omega), v(\omega)), \text{ since } 0 < a(\omega) + 2c(\omega) < 1, \end{aligned}$$

a contradiction. Hence $z(\omega) = v(\omega)$ and so $z(\omega)$ is a unique common random fixed point of S and T. This completes the proof. □

Corollary 1 *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X. *Let* T *be a continuous random operator defined on* M *such that for* $\omega \in \Omega$, $T(\omega, .) \colon \Omega \times M \to M$ *satisfying the condition:*

$$\begin{aligned} d(T(x(\omega)), T(y(\omega))) \leq\, & a(\omega)\, d(x(\omega), y(\omega)) \\ & + b(\omega)\, [d(x(\omega), T(x(\omega))) + d(y(\omega), T(y(\omega)))] \\ & + c(\omega)\, [d(x(\omega), T(y(\omega))) + d(y(\omega), T(x(\omega)))] \end{aligned}$$

for all $x, y \in M$, $a(\omega) + 2b(\omega) + 2c(\omega) < 1$, *where* $a(\omega)$, $b(\omega)$, $c(\omega) > 0$ *and* $\omega \in \Omega$. *Then* T *has a unique random fixed point in* X.

Proof. The proof of the corollary immediately follows by putting $S = T$ in Theorem 1. This completes the proof. □

If we take $S = T$ and $b(\omega) = c(\omega) = 0$ in Theorem 1, then we obtain the following result as corollary.

Corollary 2 *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X*. Let* T *be a random operator defined on* M *such that for* $\omega \in \Omega$*,* $T(\omega, .) \colon \Omega \times M \to M$ *satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq a(\omega)\, d(x(\omega), y(\omega)),$$

for all $x, y \in M$*,* $a(\omega) \in (0, 1)$ *and* $\omega \in \Omega$*. Then* T *has a unique random fixed point in* X*.*

If we take $S = T$ and $a(\omega) = c(\omega) = 0$ in Theorem 1, then we obtain the following result as corollary.

Corollary 3 ([16], Corollary 3.2) *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X*. Let* T *be a continuous random operator defined on* M *such that for* $\omega \in \Omega$*,* $T(\omega, .) \colon \Omega \times M \to M$ *satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq b(\omega)\, [d(x(\omega), T(x(\omega))) + d(y(\omega), T(y(\omega)))]$$

for all $x, y \in M$*,* $b(\omega) \in (0, \frac{1}{2})$ *and* $\omega \in \Omega$*. Then* T *has a unique random fixed point in* X*.*

If we take $S = T$ and $a(\omega) = b(\omega) = 0$ in Theorem 1, then we obtain the following result as corollary.

Corollary 4 ([16], Corollary 3.3) *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X*. Let* T *be a continuous random operator defined on* M *such that for* $\omega \in \Omega$*,* $T(\omega, .) \colon \Omega \times M \to M$ *satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq c(\omega)\, [d(x(\omega), T(y(\omega))) + d(y(\omega), T(x(\omega)))]$$

for all $x, y \in M$*,* $c(\omega) \in (0, \frac{1}{2})$ *and* $\omega \in \Omega$*. Then* T *has a unique random fixed point in* X*.*

Theorem 2 *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X*. Let* S *and* T *be two continuous random operators defined on* M *such that for* $\omega \in \Omega$*,* $S(\omega, .)$*,* $T(\omega, .) \colon \Omega \times M \to M$ *satisfying the condition:*

$$\begin{aligned} d(S(x(\omega)), T(y(\omega))) \leq h(\omega) \max\Big\{ & d(x(\omega), y(\omega)),\ d(x(\omega), S(x(\omega))), \\ & d(y(\omega), T(y(\omega))),\ d(x(\omega), T(y(\omega))), \\ & d(y(\omega), S(x(\omega)))\Big\} \end{aligned} \tag{11}$$

for all $x, y \in M$, $0 < h(\omega) < 1$ *and* $\omega \in \Omega$. *Then* S *and* T *have a unique common random fixed point in* X.

Proof. For each $x_0(\omega) \in \Omega \times M$ and $n = 0, 1, 2, \ldots$, we choose $x_1(\omega), x_2(\omega) \in \Omega \times M$ such that $x_1(\omega) = S(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we define sequence of elements of M such that $x_{2n+1}(\omega) = S(x_{2n}(\omega))$ and $x_{2n+2}(\omega) = T(x_{2n+1}(\omega))$. Then from (11), we have

$$\begin{aligned}
d(x_{2n+1}(\omega), x_{2n}(\omega)) &= d(S(x_{2n}(\omega)), T(x_{2n-1}(\omega))) \\
&\leq h(\omega) \max\Big\{d(x_{2n}(\omega), x_{2n-1}(\omega)), \\
&\quad d(x_{2n}(\omega), S(x_{2n}(\omega))),\ d(x_{2n-1}(\omega), T(x_{2n-1}(\omega))), \\
&\quad d(x_{2n}(\omega), T(x_{2n-1}(\omega))),\ d(x_{2n-1}(\omega), S(x_{2n}(\omega)))\Big\} \\
&= h(\omega) \max\Big\{d(x_{2n}(\omega), x_{2n-1}(\omega)), \\
&\quad d(x_{2n}(\omega), x_{2n+1}(\omega)),\ d(x_{2n-1}(\omega), x_{2n}(\omega)), \qquad (12) \\
&\quad d(x_{2n}(\omega), x_{2n}(\omega)),\ d(x_{2n-1}(\omega), x_{2n+1}(\omega))\Big\} \\
&= h(\omega) \max\Big\{d(x_{2n}(\omega), x_{2n-1}(\omega)), \\
&\quad d(x_{2n}(\omega), x_{2n+1}(\omega)),\ d(x_{2n-1}(\omega), x_{2n}(\omega)), \\
&\quad d(x_{2n-1}(\omega), x_{2n+1}(\omega))\Big\} \\
&\leq h(\omega)\, d(x_{2n}(\omega), x_{2n-1}(\omega)).
\end{aligned}$$

Similarly, we have

$$d(x_{2n}(\omega), x_{2n-1}(\omega)) \leq h(\omega)\, d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

Hence

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq h(\omega)^2\, d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

On continuing this process, we get

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq h(\omega)^{2n}\, d(x_1(\omega), x_0(\omega)).$$

Also for $n > m$, we have

$$\begin{aligned}
d(x_n(\omega), x_m(\omega)) &\leq d(x_n(\omega), x_{n-1}(\omega)) + d(x_{n-1}(\omega), x_{n-2}(\omega)) + \ldots \\
&\quad + d(x_{m+1}(\omega), x_m(\omega)) \\
&\leq (h(\omega)^{n-1} + h(\omega)^{n-2} + \cdots + h(\omega)^m)\, d(x_1(\omega), x_0(\omega))
\end{aligned}$$

$$\leq \left(\frac{h(\omega)^m}{1-h(\omega)}\right) d(x_1(\omega), x_0(\omega)).$$

Let $0 \ll \varepsilon$ be given. Choose a natural number N such that $\left(\frac{h(\omega)^m}{1-h(\omega)}\right) d(x_1(\omega), x_0(\omega)) \ll \varepsilon$ for every $m \geq N$. Thus

$$d(x_n(\omega), x_m(\omega)) \leq \left(\frac{h(\omega)^m}{1-h(\omega)}\right) d(x_1(\omega), x_0(\omega)) \ll \varepsilon,$$

for every $n > m \geq N$.

This shows that the sequence $\{x_n(\omega)\}$ is a Cauchy sequence in $\Omega \times M$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$ as $n \to \infty$. Hence, we have

$$\begin{aligned} d(z(\omega), S(z(\omega))) &\leq d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+2}(\omega), S(z(\omega)) \\ &= d(z(\omega), x_{2n+2}(\omega)) + d(S(z(\omega)), T(x_{2n+1}(\omega))) \\ &\leq d(z(\omega), x_{2n+2}(\omega)) + h(\omega) \max\Big\{d(z(\omega), x_{2n+1}(\omega)), \\ &\quad d(z(\omega), S(z(\omega))),\ d(x_{2n+1}(\omega), T(x_{2n+1}(\omega))), \\ &\quad d(z(\omega), T(x_{2n+1}(\omega))),\ d(x_{2n+1}(\omega), S(z(\omega)))\Big\} \\ &= d(z(\omega), x_{2n+2}(\omega)) + h(\omega) \max\Big\{d(z(\omega), x_{2n+1}(\omega)), \\ &\quad d(z(\omega), S(z(\omega))),\ d(x_{2n+1}(\omega), x_{2n+2}(\omega)), \\ &\quad d(z(\omega), x_{2n+2}(\omega)),\ d(x_{2n+1}(\omega), S(z(\omega)))\Big\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$d(z(\omega), S(z(\omega))) \leq h(\omega)\, d(z(\omega), S(z(\omega)))$$

or,

$$(1-h(\omega)) d(z(\omega), S(z(\omega))) \leq 0$$

$$\Rightarrow d(z(\omega), S(z(\omega))) \leq 0, \text{ since } 0 < (1-h(\omega)) < 1.$$

Thus $-d(z(\omega), S(z(\omega))) \in P$. But $d(z(\omega), S(z(\omega))) \in P$. Therefore by definition 1($\mathbf{c_3}$), we have $d(z(\omega), S(z(\omega)) = 0$ and so $S(z(\omega)) = z(\omega)$.

In an exactly the similar way we can prove that for all $\omega \in \Omega$, $T(z(\omega)) = z(\omega)$. Hence $S(z(\omega)) = T(z(\omega)) = z(\omega)$. This shows that $z(\omega)$ is a common random fixed point of S and T. Rest of the proof is same as that of Theorem 1. This completes the proof. □

If we take $S = T$ in Theorem 2 we get the following result as corollary.

Corollary 5 *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X*. Let* T *be a continuous random operators defined on* M *such that for* $\omega \in \Omega$, $T(\omega, .) \colon \Omega \times M \to M$ *satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq h(\omega) \max\Big\{d(x(\omega), y(\omega)),\ d(x(\omega), T(x(\omega))),\ d(y(\omega), T(y(\omega))),\ d(x(\omega), T(y(\omega))),\ d(y(\omega), T(x(\omega)))\Big\} \tag{13}$$

for all $x, y \in M$, $0 < h(\omega) < 1$ *and* $\omega \in \Omega$*. Then* T *has a unique random fixed point in* X*.*

Proof. The proof of corollary 5 immediately follows by putting $S = T$ in Theorem 2. This completes the proof. □

The following corollary is a special case of Corollary 5.

Corollary 6 *Let* (X, d) *be a complete cone random metric space with respect to a cone* P *and let* M *be a nonempty separable closed subset of* X*. Let* T *be a continuous random operators defined on* M *such that for* $\omega \in \Omega$, $T(\omega, .) \colon \Omega \times M \to M$ *satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq h(\omega)\, d(x(\omega), y(\omega)) \tag{14}$$

for all $x, y \in M$, $0 < h(\omega) < 1$ *and* $\omega \in \Omega$*. Then* T *has a unique random fixed point in* X*.*

Condition (14) is called Banach contractive condition.

Proof. (Proof of corollary 6) The proof of corollary 6 immediately follows from Corollary 5 by taking

$$\max\Big\{d(x(\omega), y(\omega)),\ d(x(\omega), T(x(\omega))),\ d(y(\omega), T(y(\omega))),\ d(x(\omega), T(y(\omega))),\ d(y(\omega), T(x(\omega)))\Big\} = d(x(\omega), y(\omega)).$$

This completes the proof. □

Example 3 *Let* $\Omega = [0, 1]$ *and* Σ *be the sigma algebra of Lebesgue's measurable subset of* $[0, 1]$*. Take* $X = R$ *with* $d(x, y) = |x - y|$ *for* $x, y \in R$*. Define random mapping* T *from* $\Omega \times X$ *to* X *as* $T(\omega, x) = \omega - x$*. Then a measurable mapping* $\xi \colon \Omega \to X$ *defined as* $\xi(\omega) = \frac{\omega}{2}$ *for all* $\omega \in \Omega$*, serve as a unique random fixed point of* T*.*

Example 4 *Let* $M = R$ *and* $P = \{x \in M : x \geq 0\}$*, also* $\Omega = [0,1]$ *and* Σ *be the sigma algebra of Lebesgue's measurable subset of* $[0,1]$*. Let* $X = [0,\infty)$ *and define a mapping* $d\colon (\Omega\times X)\times(\Omega\times X) \to M$ *by* $d(x(\omega), y(\omega)) = |x(\omega)-y(\omega)|$*. Then* (X, d) *is a cone random metric space. Define random operator* T *form* $(\Omega \times X)$ *to* X *as* $T(\omega, x) = \frac{1-\omega^2+2x}{3}$*. Also sequence of mapping* $\xi_n\colon \Omega \to X$ *is defined by* $\xi_n(\omega) = (1-\omega^2)^{1+(1/n)}$ *for every* $\omega \in \Omega$ *and* $n \in N$*. Define measurable mapping* $\xi\colon \Omega \to X$ *as* $\xi(\omega) = (1-\omega^2)$ *for every* $\omega \in \Omega$*. Hence* $(1-\omega^2)$ *is the random fixed point of the random operator* T*.*

Example 5 *Let* $M = R$ *and* $P = \{x \in M : x \geq 0\}$*, also* $\Omega = [0,1]$ *and* Σ *be the sigma algebra of Lebesgue's measurable subset of* $[0,1]$*. Let* $X = [0,\infty)$ *and define a mapping* $d\colon (\Omega\times X)\times(\Omega\times X) \to M$ *by* $d(x(\omega), y(\omega)) = |x(\omega)-y(\omega)|$*. Then* (X, d) *is a cone random metric space. Define random operators* S *and* T *form* $(\Omega \times X)$ *to* X *as* $S(\omega, x) = \frac{1-\omega^2+x}{2}$ *and* $T(\omega, x) = \frac{1-\omega^2+2x}{3}$*. Also sequence of mapping* $\xi_n\colon \Omega \to X$ *is defined by* $\xi_n(\omega) = (1-\omega^2)^{1+(1/n)}$ *for every* $\omega \in \Omega$ *and* $n \in N$*. Define measurable mapping* $\xi\colon \Omega \to X$ *as* $\xi(\omega) = (1-\omega^2)$ *for every* $\omega \in \Omega$*. Hence* $(1-\omega^2)$ *is a common random fixed point of the random operators* S *and* T*.*

Example 6 *Let* $E = \{0, 1, 2, 3, 4\} \subset R$ *with the usual metric* d*. Consider* $\Omega = \{0, 1, 2, 3, 4\}$ *and let* Σ *be the sigma algebra of Lebesgue's measurable subset of* Ω*. Define* $S, T\colon \Omega \times E \to E$ *by*

$$\begin{cases} S(\omega, x) = 3, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ \qquad\quad\;\; = 1, & \text{otherwise,} \end{cases}$$

and

$$\begin{cases} T(\omega, x) = 2, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ \qquad\quad\;\; = 1, & \text{otherwise.} \end{cases}$$

Let us take $x(\omega) = 0$*,* $y(\omega) = 1$*. Then from condition* (11)*, we have*

$$\begin{aligned} 2 &= d(S(x(\omega)), T(y(\omega))) \\ &\leq h(\omega) \max\Big\{d(x(\omega), y(\omega)),\ d(x(\omega), S(x(\omega))), \\ &\qquad d(y(\omega), T(y(\omega))),\ d(x(\omega), T(y(\omega))), \\ &\qquad d(y(\omega), S(x(\omega)))\Big\} \\ &= h(\omega) \max\{1, 3, 0, 1, 2\} \end{aligned}$$

which implies $h(\omega) \geq \frac{2}{3}$. *Now if we take* $0 < h(\omega) < 1$, *then condition* (11) *is satisfied. The measurable function* $\xi\colon \Omega \to E$ *with* $\xi(\omega) = 1$ *is a unique common random fixed point of* S *and* T, *that is,* $S(\omega, x) = T(\omega, x) = 1 = \xi(\omega)$.

Example 7 *Let* $E = \{0, 1, 2, 3, 4\} \subset R$ *with the usual metric* d. *Consider* $\Omega = \{0, 1, 2, 3, 4\}$ *and let* Σ *be the sigma algebra of Lebesgue's measurable subset of* Ω. *Define* $S, T\colon \Omega \times E \to E$ *by*

$$\begin{cases} S(\omega, x) = 4, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ \qquad\quad\;\; = 3, & \text{otherwise,} \end{cases}$$

and

$$\begin{cases} T(\omega, x) = 2, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ \qquad\quad\;\; = 3, & \text{otherwise.} \end{cases}$$

Let us take $x(\omega) = 0$ *and* $y(\omega) = 1$. *Then condition (5) of Theorem 3.1 is satisfied with* $a(\omega) = b(\omega) = c(\omega) = \frac{1}{12}$ *and* $a(\omega) + 2b(\omega) + 2c(\omega) = \frac{5}{12} \in (0, 1)$. *The measurable function* $\xi\colon \Omega \to E$ *with* $\xi(\omega) = 3$ *is a unique common random fixed point of* S *and* T, *that is,* $S(\omega, x) = T(\omega, x) = 3 = \xi(\omega)$.

Remark 2 *Our results extend and generalize many known results from the current existing literature.*

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