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Logarithmic mean inequality for generalized trigonometric and hyperbolic functions

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Abstract. In this paper we study the convexity and concavity properties of generalized trigonometric and hyperbolic functions in case of Logarithmic mean.

1 Introduction

Recently, the study of the generalized trigonometric and generalized hyperbolic functions has got huge attention of numerous authors, and has appeared the huge number of papers involving the equalities and inequalities and basis properties of these function, e.g. see [7, 8, 9, 6, 10, 13, 14, 18, 23] and the references therein. These generalized trigonometric and generalized hyperbolic functions p-functions depending on the parameter $p > 1$ were introduced by Lindqvist [19] in 1995. These functions coincides with the usual functions for $p = 2$. Thereafter Takesheu took one further step and generalized these function for two parameters $p, q > 1$, so-called (p, q)-functions. In [8], some convexity and concavity properties of p-functions were studied. Thereafter those results were

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extended in [5] for two parameters in the sense of Power mean inequality. In this paper we study the convexity and concavity property of p-function with respect Logarithmic mean. Before we formulate our main result we will define generalized trigonometric and hyperbolic functions customarily.

The eigenfunction $\sin_p$ of the so-called one-dimensional p-Laplacian problem [12]

$$-\Delta_p u = -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u,\ u(0) = u(1) = 0, \quad p > 1,$$

is the inverse function of $F : (0,1) \to \left(0, \frac{\pi_p}{2}\right)$, defined as

$$F(x) = \mathrm{arcsin}_p(x) = \int_0^x (1-t^p)^{-\frac{1}{p}}\,dt,$$

where

$$\pi_p = 2\mathrm{arcsin}_p(1) = \frac{2}{p}\int_0^1 (1-s)^{-1/p}s^{1/p-1}\,ds = \frac{2}{p}B\left(1-\frac{1}{p},\frac{1}{p}\right) = \frac{2\pi}{p\,\sin\left(\frac{\pi}{p}\right)},$$

here $B(.,.)$ denotes the classical beta function.

The function arcsin_p is called the generalized inverse sine function, and coincides with usual inverse sine function for $p = 2$. Similarly, the other generalized inverse trigonometric and hyperbolic functions $\mathrm{arccos}_p : (0,1) \to (0, \pi_p/2)$, $\mathrm{arctan}_p : (0,1) \to (0, b_p)$, $\mathrm{arcsinh}_p : (0,1) \to (0, c_p)$, $\mathrm{arctanh}_p : (0,1) \to (0,\infty)$, where

$$b_p = \frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right) - \psi\left(\frac{1}{2p}\right)\right) = 2^{-\frac{1}{p}}F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{1}{2}\right),$$
$$c_p = \left(\frac{1}{2}\right)^{\frac{1}{p}}F\left(1,\frac{1}{p};1+\frac{1}{p},\frac{1}{2}\right),$$

are defined as follows

$$\mathrm{arccos}_p(x) = \int_0^{(1-x^p)^{\frac{1}{p}}} (1-t^p)^{-\frac{1}{p}}\,dt, \quad \mathrm{arctan}_p(x) = \int_0^x (1+t^p)^{-1}\,dt,$$

$$\mathrm{arcsinh}_p(x) = \int_0^x (1+t^p)^{-\frac{1}{p}}\,dt, \quad \mathrm{arctanh}_p(x) = \int_0^x (1-t^p)^{-1}\,dt,$$

where $F(a,b;c;z)$ is *Gaussian hypergeometric function* [1].

The generalized cosine function is defined by

$$\frac{d}{dx}\sin_p(x) = \cos_p(x), \quad x \in [0, \pi_p/2]\,.$$

It follows from the definition that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p}\,,$$

and

$$|\cos_p(x)|^p + |\sin_p(x)|^p = 1, \quad x \in \mathbb{R}. \tag{1}$$

Clearly we get

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}.$$

The generalized tangent function $\tan_p$ is defined by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)},$$

and applying (1) we get

$$\frac{d}{dx}\tan_p(x) = 1 + \tan_p(x)^p.$$

For $x \in (0, \infty)$, the inverse of generalized hyperbolic sine function $\sinh_p(x)$ is defined by

$$\operatorname{arcsinh}_p(x) = \int_0^x (1 + t^p)^{-1/p}\,dt,$$

and generalized hyperbolic cosine and tangent functions are defined by

$$\cosh_p(x) = \frac{d}{dx}\sinh_p(x), \quad \tanh_p(x) = \frac{\sinh_p(x)}{\cosh_p(x)}\,,$$

respectively. It follows from the definitions that

$$|\cosh_p(x)|^p - |\sinh_p(x)|^p = 1. \tag{2}$$

From above definition and (2) we get the following derivative formulas,

$$\frac{d}{dx}\cosh_p(x) = \cosh_p(x)^{2-p}\sinh_p(x)^{p-1}, \quad \frac{d}{dx}\tanh_p(x) = 1 - |\tanh_p(x)|^p.$$

Note that these generalized trigonometric and hyperbolic functions coincide with usual functions for $p = 2$.

For two distinct positive real numbers x and y, the Arithmetic mean, Geometric mean, Logarithmic mean, Harmonic mean and the Power mean of order $p \in \mathbb{R}$ are respectively defined by

$$A(x,y) = \frac{x+y}{2}, \quad G(x,y) = \sqrt{xy},$$

$$L(x,y) = \frac{x-y}{\log(x)-\log(y)}, \quad x \neq y,$$

$$H(x,y) = \frac{1}{A(1/x, 1/y)},$$

and

$$M_t = \begin{cases} \left(\dfrac{x^t+y^t}{2}\right)^{1/t}, & t \neq 0, \\ \sqrt{xy}, & t = 0. \end{cases}$$

Let $f : I \to (0,\infty)$ be continuous, where I is a sub-interval of $(0,\infty)$. Let M and N be the means defined above, the we call that the function f is MN-convex (concave) if

$$f(M(x,y)) \leq (\geq) N(f(x), f(y)) \quad \text{for} \quad \text{all } x, y \in I.$$

Recently, Generalized convexity/concavity with respect to general mean values has been studied by Anderson et al. in [2]. We recall one of their results as follows

Lemma 1 [2, Theorem 2.4] *Let* I *be an open sub-interval of* $(0,\infty)$ *and let* $f : I \to (0,\infty)$ *be differentiable. Then* f *is* HH*-convex (concave) on* I *if and only if* $x^2 f'(x)/f(x)^2$ *is increasing (decreasing).*

In [4], Baricz studied that if the functions f is differentiable, then it is (a,b)-convex (concave) on I if and only if $x^{1-a} f'(x)/f(x)^{1-b}$ is increasing (decreasing).

It is important to mention that $(1,1)$-convexity means the AA-convexity, $(1,0)$-convexity means the AG-convexity, and $(0,0)$-convexity means GG-convexity.

Motivated by the results given in [2, 4], we contribute to the topic by giving the following result.

Theorem 1 *Let* $f : I \to (0,\infty)$ *be a continuous and* $I \subseteq (0,\infty)$*, then*

1. $L(f(x), f(y)) \geq (\leq) f(L(x, y))$,

2. $L(f(x), f(y)) \geq (\leq) f(A(x, y))$,

if f *is increasing and* log*-convex (concave).*

Theorem 2 *For* $x, y \in (0, \pi_p/2)$*, the following inequalities*

1. $L(\sin_p(x), \sin_p(y)) \leq \sin_p(L(x, y)), \quad p > 1$,

2. $L(\cos_p(x), \cos_p(y)) \leq \cos_p(L(x, y)), \quad p \geq 2$.

Theorem 3 *For* $p > 1$*, we have*

1. $L(1/\sin_p(x), 1/\sin_p(y)) \geq 1/\sin_p(A(x, y)), \quad x, y \in (0, \pi_p/2)$,

2. $L(1/\cos_p(x), 1/\cos_p(y)) \geq 1/\cos_p(L(x, y)), \quad x, y \in (0, \pi_p/2)$,

3. $L(\tanh_p(x), \tanh_p(y)) \leq \tanh_p(A(x, y)), \quad x, y \in (0, \infty)$,

4. $L(\mathrm{arcsinh}_p(x), \mathrm{arcsinh}_p(y)) \leq \mathrm{arcsinh}_p(A(x, y)), \quad x, y \in (0, 1)$,

5. $L(\arctan_p(x), \arctan_p(y)) \leq \arctan_p(A(x, y)), \quad x, y \in (0, 1)$.

2 Preliminaries and Proofs

We give the following lemmas which will be used in the proof of our main result.

Lemma 2 [22] *Let* $f, g : [a, b] \to R$ *be integrable functions, both increasing or both decreasing. Furthermore, let* $p : [a, b] \to R$ *be a positive, integrable function. Then*

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx. \tag{3}$$

If one of the functions f *or* g *is non-increasing and the other non-decreasing, then the inequality in* (3) *is reversed.*

Lemma 3 [17] *If* $f(x)$ *is continuous and convex function on* $[a, b]$*, and* $\varphi(x)$ *is continuous on* $[a, b]$*, then*

$$f\left(\frac{1}{b-a}\int_a^b \varphi(x)dx\right) \leq \frac{1}{b-a}\int_a^b f(\varphi(x))\,dx. \tag{4}$$

If function $f(x)$ *is continuous and concave on* $[a, b]$*, then the inequality in* (4) *reverses.*

Lemma 4 [3] *For two distinct positive real numbers* a, b, *we have* $L < A$.

Lemma 5 *For* $p > 1$, *the function* $\sin_p(x)$ *is* HH-*concave on* $(0, \pi_p/2)$.

Proof. Let $f(x) = f_1(x)f_2(x), x \in (0, \pi_p/2)$, where $f_1(x) = 1/\sin(x)$ and $f_2(x) = x^2 \cos_p(x)/\sin_p(x)$. Clearly, f_1 is decreasing, so it is enough to prove that f_2 is decreasing, then the proof follows from Lemma 1. We get

$$\begin{aligned} f_2'(x) &= \frac{\sin_p(x)(\cos_p(x) - x\cos_p(x)^{2-p}\sin_p(x)^{p-1}) - x\cos_p(x)^2}{\sin_p(x)^2} \\ &= \frac{\cos_p(x)^2((1 - x\tan_p(x)^{p-1})\tan_p(x) - x)}{\sin_p(x)^2} = f_3(x)\frac{\cos_p(x)^2}{\sin_p(x)^2}, \end{aligned}$$

where $f_3(x) = \tan_p(x) - x\tan_p(x)^p - 1$. Again, one has

$$f_3'(x) = p\tan_p(x)^{p-1}(1 + \tan_p(x)^p)x < 0.$$

Thus, f_3 is decreasing and $g(x) < g(0) = 0$. This implies that $f_2' < 0$, hence f_2 is strictly decreasing, the product of two decreasing functions is decreasing. This implies the proof. □

Proof of Theorem 1. We get

$$L(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} 1\,dt}{\int_{f(y)}^{f(x)} \frac{1}{t}dt} = \frac{\int_y^x f'(u)du}{\int_y^x \frac{f'(u)}{f(u)}du}. \tag{5}$$

It is assumed that the function $f(x)$ is increasing and $\log f$ is convex, this implies that $\frac{f'(x)}{f(x)}$ is increasing. Letting $p(x) = 1, f(x) = f(u)$ and $g(x) = f'(u)/f(u)$ in Lemma 2, we get

$$\int_y^x 1du \int_y^x f'(u)du \geq \int_y^x \frac{f'(u)}{f(u)}du \int_y^x f(u)du.$$

This is equivalent to

$$L(f(x), f(y)) = \frac{\int_y^x f'(u)du}{\int_y^x \frac{f'(u)}{f(u)}du} \geq \frac{\int_y^x f(u)du}{\int_y^x 1du}.$$

By Lemmas 3 and 4, and keeping in mind that log-convexity of f implies the convexity of f, we get

$$L(f(x), f(y)) \geq f\left(\frac{\int_y^x udu}{x - y}\right) = f\left(\frac{x + y}{2}\right) \geq f(L(x, y)).$$

The proof of converse follows similarly. If we repeat the lines of proof of part (1), and use the concavity of the function, and Lemmas 3 & 4 then we arrive at the proof of part (2).

Proof of Theorem 2. It is easy to see that the function $\sin_p(x)$ is increasing and log-concave. So the proof of part (1) follows easily from Theorem 1. We also offer another proof as follows:

It can be observed easily that

$$L(\sin_p(x), \sin_p(y)) = \frac{\int_y^x \cos_p(u)du}{\int_{\sin_p(y)}^{\sin_p(x)} \frac{1}{t}dt} = \frac{\int_y^x \cos_p u du}{\int_y^x \frac{\cos_p u}{\sin_p(u)} du},$$

and

$$\sin_p(L(x,y)) = \sin_p\left(\frac{x-y}{\log\frac{x}{y}}\right) = \sin_p\left(\frac{\int_y^x 1du}{\int_y^x \frac{1}{u}du}\right).$$

Clearly, $\cos_p(u)$ and $\sin_p(1/u)$, utilizing Chebyshev inequality, we have

$$\int_y^x \cos_p(u)du \int_y^x \sin_p(1/u)du \leq \int_y^x 1du \int_y^x \cos_p u \sin_p \frac{1}{u} du.$$

So, we get

$$\int_y^x \cos_p u du \int_y^x \sin_p(1/u)du < \int_y^x 1du \int_y^x \frac{\cos_p(u)}{\sin_p(u)} du.$$

Where we apply simple inequality $\sin_p\left(\frac{1}{u}\right) < \frac{1}{\sin_p(u)}$. In order to prove inequality (1), we only prove

$$\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du} \leq \sin_p\left(\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du}\right).$$

Consider a partition T of the interval $[y,x]$ into n equal length sub-interval by means of points $y = x_0 < x_1 < \cdots < x_n = x$ and $\Delta x_i = \frac{x-y}{n}$. Picking an arbitrary point $\xi_i \in [x_{i-1}, x_i]$ and using Lemma 1, we have

$$\frac{n}{\sum_{i=1}^{n} \sin_p \frac{1}{\xi_i}} \leq \sin_p\left(\frac{n}{\sum_{i=1}^{n} \frac{1}{\xi_i}}\right)$$

$\Leftrightarrow$

$$\frac{x-y}{\lim\limits_{n\to\infty}\left(\frac{x-y}{n}\sum\limits_{i=1}^{n}\sin_p\frac{1}{\xi_i}\right)}\le \sin_p\left(\frac{x-y}{\lim\limits_{n\to\infty}\left(\frac{x-y}{n}\sum\limits_{i=1}^{n}\frac{1}{\xi_i}\right)}\right)$$

$\Leftrightarrow$

$$\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du}\le \sin_p\left(\frac{\int_y^x 1du}{\int_y^x \sin_p(1/u)du}\right).$$

This completes the proof.

For (2), clearly $\cos_p(x)$ is decreasing and $\tan_p(x)^{p-1}$ is increasing. One has

$$(\cos_p(x))'' = \cos_p(x)\tan_p(x)^{p-2}\left(1-p+(2-p)\tan_p(x)^p\right)<0,$$

this implies that $\cos_p(x)$ is concave on $(0,\pi_p/2)$.

Using Tchebyshef inequality, we have

$$\int_y^x 1du\int_y^x \cos_p(u)\tan_p(u)^{p-1}du\le \int_y^x \cos_p(u)du\int_y^x \tan_p(u)^{p-1}du,$$

which is equivalent to

$$\frac{\int_y^x \cos_p(u)\tan_p(u)^{p-1}du}{\int_y^x \tan_p(u)^{p-1}du}\le\frac{\int_y^x \cos_p(u)du}{\int_y^x 1du}. \tag{6}$$

Substituting $t=\cos_p(u)$ in (6), we get

$$L(\cos_p(x),\cos_p(y))=\frac{\int_{\cos_p(y)}^{\cos_p(x)}1dt}{\int_{\cos_p(y)}^{\cos_p(x)}\frac{1}{t}dt}=\frac{\int_y^x \cos_p(u)\tan_p(u)^{p-1}du}{\int_y^x \tan_p(u)^{p-1}du}\le\frac{\int_y^x \cos_p(u)du}{\int_y^x 1du}.$$

Using Lemma 3 and concavity of $\cos_p(x)$, we obtain

$$L(\cos_p(x),\cos_p y)\le \cos_p\left(\frac{\int_y^x udu}{x-y}\right)=\cos_p\left(\frac{x+y}{2}\right)\le \cos_p\left(L(x,y)\right).$$

Proof of Theorem 3. Let $g_1(x)=1/\cos_p(x)$, $x\in(0,\pi_p/2)$ and $g_2(x)=\tanh_p(x)$, $x>0$. We get

$$(\log(g_1(x)))''=(p-1)\tan_p(x)^{p-2}(1+\tan_p(x)^p)>0,$$

and

$$(\log(g_2(x)))'' = \frac{1-\tanh_p(x)^p}{\tanh_p(x)^2}((1-p)\tanh_p(x)^p - 1) < 0.$$

This implies that g_1 and g_2 are log-convex, clearly both functions are increasing, and log-convexity implies the convexity, so g_1 and g_2 are convex functions. Now the proof follows easily from Theorem 1. The rest of proof follows similarly.

Corollary 1 *For $p > 1$, we have*

1. $L(\tan_p(x), \tan_p(y)) \geq \tan_p(L(x,y)), \quad x, y \in (s_p, \pi_p/2)$, *where s_p is the unique root of the equation $\tan_p(x) = 1/(p-1)^{1/p}$,*
2. $L(\operatorname{arctanh}_p(x), \operatorname{arctanh}_p(y)) \geq \operatorname{arctanh}_p(L(x,y)), \quad x, y \in (r_p, 1)$, *where r_p is the unique root of the equation $x^{p-1}\operatorname{arctanh}_p(y) = 1/p$.*

Proof. Write $f_1(x) = \tan_p(x)$. We get

$$\left(\frac{f_1'(x)}{f(x)}\right)' = \left(\frac{1+\tan_p^p(x)}{\tan_p(x)}\right)' = \frac{1+\tan_p^p(x)}{\tan_p^2(x)}\left[(p-1)\tan_p^p(x) - 1\right] > 0$$

on $\left(s_p, \frac{\pi_p}{2}\right)$. This implies that f_1 is log-convex, clearly f_1 is increasing, and the proof follows easily from Theorem 1. The proof of part (2) follows similarly. □

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The solutions of time and space conformable fractional heat equations with conformable Fourier transform

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Abstract. In this paper our aim is to find the solutions of time and space fractional heat differential equations by using new definition of fractional derivative called conformable fractional derivative. Also based on conformable fractional derivative definition conformable Fourier Transform is defined. Fourier sine and Fourier cosine transform definitions are given and space fractional heat equation is solved by conformable Fourier transform.

1 Introduction

Fractional differential equations which are the generalization of differential equations are successful models of real life events and have many applications in various fields in science [1]-[8]. So the subject becomes very captivating. Hence, many researchers have been trying to form a new definition of fractional derivative. Most of these definitions include integral form for fractional derivatives. Two of these definitions which are most popular:

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1. *Riemann-Liouville definition:* If n is a positive integer and $\alpha \in [n-1, n)$, α derivative of f is given by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

2. *Caputo definition:* If n is a positive integer and $\alpha \in [n-1, n)$, α derivative of f is given by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

In [9, 10] R. Khalil and et al. give a new definition of fractional derivative called “conformable fractional derivative”.

Definition 1 *Let* $f : [0, \infty) \to \mathbb{R}$ *be a function.* α^{th} *order conformable fractional derivative of* f *is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. *If* f*is* α*-differentiable in some* $(0, a)$, $a > 0$, *and* $\lim_{t \to 0^+} f^{(\alpha)}(t)$ *exists, then define*

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

This new definition satisfies the properties which are given in the following theorem [9, 10].

Theorem 1 *Let* $\alpha \in (0, 1]$ *and* f, g *be* $\alpha-$ *differentiable at point* $t > 0$. *Then*

(a) $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.

(b) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.

(c) $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$.

(d) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(e) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(g) - fT_\alpha(f)}{g^2}$.

(f) *If, in addition to f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

In Section 2, we will give the solution of fractional heat equation for $0 < \alpha < 1$ with the help of conformable fractional derivative definition. In Section 3, we will give conformable Fourier transform, conformable Fourier sine and cosine transform definitions and solve the space fractional heat equation with this transform.

2 Time fractional heat equation

General form for one dimension heat equation is

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

Heat equation has many fractional forms. In this paper we investigate the solution of time fractional heat differential equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0 \tag{1}$$

with conditions

$$u(0,t) = 0,\ t \geq 0 \tag{2}$$

$$u(L,t) = 0,\ t \geq 0 \tag{3}$$

$$u(x,0) = f(x),\ 0 \leq x \leq L \tag{4}$$

where the derivative is conformable fractional derivative and $0 < \alpha < 1$. Firstly we can mention conformable fractional linear differential equations with constant coefficients

$$\frac{\partial^\alpha y}{\partial t^\alpha} \pm \mu^2 y = 0. \tag{5}$$

From formula (f) in Theorem 1 we can obtain

$$\frac{\partial^\alpha y}{\partial t^\alpha} = t^{1-\alpha}\frac{dy}{dt}. \tag{6}$$

By substituting (6) in (5) it becomes following first order linear differential equation

$$t^{1-\alpha}\frac{dy}{dt} \pm \mu^2 y = 0. \tag{7}$$

One can easily see that the solution of equation (7)

$$y = ce^{\frac{\pm\mu^2}{\alpha} t^\alpha}. \tag{8}$$

Now we can use separation of variables method [11] for solution of our time fractional heat equation (1). Let $u = P(x)Q(t)$. Substituting this equation in Eq. (1), we have

$$\frac{d^\alpha Q(t)}{dt^\alpha} P(x) = \kappa \frac{d^2 P(x)}{dx^2} Q(t)$$

from which we obtain

$$\frac{d^\alpha Q(t)}{dt^\alpha} / \kappa Q(t) = \frac{d^2 P(x)}{dx^2} / P(x) = \omega.$$

As a result:

$$\frac{d^\alpha Q(t)}{dt^\alpha} - \omega\kappa Q(t) = 0$$

and

$$\frac{d^2 P(x)}{dx^2} - \omega P(x) = 0.$$

Now, we think about the equation

$$\frac{d^2 P(x)}{dx^2} - \omega P(x) = 0.$$

For this equation, there are three cases for values of ω to be evaluated. $\omega = 0, \ \omega = -\mu^2, \ \omega = \mu^2$.
Conditions (2) and (3) give

$$\mu = \frac{n\pi}{L} \text{ and } P_n(x) = a_n \sin\frac{n\pi x}{L}. \tag{9}$$

Equations (5) and (8) give,

$$Q_n(t) = b_n e^{-\left(\frac{n\pi}{L}\right)^2 \frac{\kappa}{\alpha} t^\alpha}. \tag{10}$$

Then, using the equations (9) and (10) the solution of the Cauchy problem which satisfies two boundary conditions obtained as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 \frac{\kappa}{\alpha} t^\alpha}. \tag{11}$$

With the help of condition (4)

$$c_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx. \tag{12}$$

Substituting (12) in (11) we find the solution as

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 \frac{\kappa}{\alpha} t^\alpha} \left[\frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx\right].$$

3 Conformable Fourier transform

In [12] Abdeljawad gave the definition of conformable Laplace transform and in [13] Negero made a study on application of Fourier transform to partial differential equations. Now in this section we define conformable Fourier transform, infinite and finite Fourier sine and cosine transform. We give some properties of this transforms. At the end we use finite Fourier sine transform to solve space fractional heat equation.

Definition 2 *Let* $0 < \alpha \leq 1$ *and* $h(x)$ *is real valued function defined on* $(-\infty,\infty)$. *The conformable Fourier transform of* $h(x)$ *which is denoted by* $F_\alpha\{h(t)\}(w)$ *is given by*

$$F_\alpha\{h(t)\}(w) = H_\alpha(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iw\frac{t^\alpha}{\alpha}}\, h(t)\, t^{\alpha-1}\, dt.$$

Theorem 2 *Let* $0 < \alpha \leq 1$ *and* $h(x)$ *is* $\alpha-$ *differentiable real valued function defined on* $(-\infty,\infty)$. *Then*

$$F_\alpha\{T_\alpha(h)(t)\}(w) = iwH_\alpha(w).$$

Proof. The proof followed by Theorem 1 (f) and known integration by parts. □

Lemma 1 *Let* $f:(-\infty,\infty)\to \mathrm{R}$ *be a function which satisfies* $F_\alpha\{h(t),w\} = H_\alpha(w)$ *property. Then,*

$$F_\alpha\{h(t)\}(w) = F\left\{h((\alpha t)^{\frac{1}{\alpha}})\right\}(w) \tag{13}$$

where $F\{h(t)\}(w) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-iwt} h(t)\ dt.$

Proof. One can prove it easily by setting $t = \frac{u^{\alpha}}{\alpha}$. □

Lemma 2 $F_{\alpha}\{h(t)\}(w)$ *Fourier transform is a linear operator.*

$$F_{\alpha}\{af + bg\} = aF_{\alpha}\{f\} + bF_{\alpha}\{g\}.$$

Theorem 3 (Convolution Theorem). *Let* $g(t)$ *and* $h(t)$ *be arbitrary functions. Then*

$$F_{\alpha}\{g * h\} = \sqrt{2\pi} F_{\alpha}\{g\} F_{\alpha}\{h\}$$

where $g*h$ *is the convolution of functions* $g(t)$ *and* $h(t)$ *defined as*

$$(g * h)(t) = \int\limits_{-\infty}^{\infty} g(x)h(t-x)dx = \int\limits_{-\infty}^{\infty} g(t-x)h(x)dx.$$

Proof. From Lemma 1, by using definition and changing the order of integration, we get

$$\begin{aligned}
F_{\alpha}\{(g * h)(t)\} &= F\left\{(g * h)((\alpha t)^{\frac{1}{\alpha}})\right\}, \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} g((\alpha x)^{\frac{1}{\alpha}}) h((\alpha(t-x))^{\frac{1}{\alpha}}) e^{-iwt} dx dt, \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} g((\alpha x)^{\frac{1}{\alpha}}) h((\alpha(t-x))^{\frac{1}{\alpha}}) e^{-iwt} dt dx.
\end{aligned}$$

By making substitution $t - x = v$, so $t = v + x$,

$$\begin{aligned}
F_{\alpha}\{(g * h)(t)\} &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} g((\alpha x)^{\frac{1}{\alpha}}) h((\alpha v)^{\frac{1}{\alpha}}) e^{-iw(v+x)} dv dx, \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} g((\alpha x)^{\frac{1}{\alpha}}) e^{-iwx} dx \int\limits_{-\infty}^{\infty} h((\alpha v)^{\frac{1}{\alpha}}) e^{-iwv} dv, \\
&= \sqrt{2\pi} F_{\alpha}\{g\} F_{\alpha}\{h\}.
\end{aligned}$$

□

Conformable Fourier transform of partial derivatives

Lemma 3 *For given* $u(x,t)$ *with* $-\infty < x < \infty$ *and* $t > 0$*, we have*

i. $F_\alpha\left\{\frac{\delta}{\delta t}(u(x,t))\right\}(w) = \frac{d}{dt}\widehat{u}(w,t)$.

ii. $F_\alpha\left\{\frac{\delta^n}{\delta t^n}(u(x,t))\right\}(w) = \frac{d^n}{dt^n}\widehat{u}(w,t), n = 1,2,3,...$

iii. $F_\alpha\{T_\alpha(u(x,t))\}(w) = iw\,\widehat{u}(w,t)$.

iv. $F_\alpha\left\{\underbrace{T_\alpha ... T_\alpha(u(x,t))}_{n \text{ times}}\right\}(w) = (iw)^n\,\widehat{u}(w,t), n = 1,2,3,...$

Fourier sine and cosine transform

In this subsection we shall discuss the Fourier sine and cosine transforms and some of their properties. These transforms are convenient for problems over semi-infinite and some of finite intervals in a spatial variable in which the function or its derivative is prescribed on the boundary.

Infinite Fourier sine and cosine transform

Definition 3 (Fourier cosine transform). *The Fourier cosine Transform of a function* $f : [0,\infty] \to \mathrm{R}$ *which is denoted by* $F_c^\alpha(f(t))$ *is defined as*

$$F_c^\alpha\{f(t)\} = \widehat{f}(w) = F_c^\alpha(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\left(w\frac{x^\alpha}{\alpha}\right)x^{\alpha-1}dx.$$

Definition 4 (Fourier sine transform). *The Fourier sine Transform of a function* $f : [0,\infty] \to \mathrm{R}$ *is defined as*

$$F_s^\alpha\{f(t)\} = \widehat{f}(w) = F_s^\alpha(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\left(w\frac{x^\alpha}{\alpha}\right)x^{\alpha-1}dx.$$

Lemma 4 F_s^α *and* F_c^α *are linear operators, i.e.,*

$$F_c^\alpha\{af + bg\} = aF_c^\alpha\{f\} + bF_c^\alpha\{g\},$$
$$F_s^\alpha\{af + bg\} = aF_s^\alpha\{f\} + bF_s^\alpha\{g\}.$$

Theorem 4 *Let* f *be a function defined for* $t > 0$ *and* $f(t) \to 0$ *as* $x \to \infty$*. Then*

1. $F_c^\alpha(T_\alpha(f)(t)) = wF_s^\alpha(f(t)) - \sqrt{\frac{2}{\pi}}f(0)$.
2. $F_s^\alpha(T_\alpha(f)(t)) = -wF_c^\alpha(f(t))$.

Proof. It can be easily proved by using Theorem 1 (f) and integration by parts. □

Finite Fourier sine and cosine transform

When the physical problem is defined on a finite domain, it is generally not suitable to use transformation with an infinite range of integration. In such cases usage of finite Fourier transform is very advantageous.

Definition 5 *The finite Fourier sine transform of* $f(t), 0 < t < L$ *defined as*

$$F_s^\alpha\{f(t)\} = F_s^\alpha(n) = \int_0^L f(t)\sin\left(\frac{n\pi t^\alpha}{L^\alpha}\right)t^{\alpha-1}dt$$

where $0 < \alpha < 1$.
The inverse Fourier sine transform is defined as follows,

$$f(x) = \frac{2\alpha}{L^\alpha}\sum_{n=1}^{\infty}F_s^\alpha(n)\sin\left(\frac{n\pi t^\alpha}{L^\alpha}\right).$$

Definition 6 *The finite Fourier cosine transform of* $f(t), 0 < t < L$ *defined as*

$$F_c^\alpha\{f(t)\} = F_c^\alpha(n) = \int_0^L f(t)\cos\left(\frac{n\pi t^\alpha}{L^\alpha}\right)t^{\alpha-1}dt$$

where $0 < \alpha < 1$.
The inverse Fourier cosine transform is defined as follows,

$$f(x) = \frac{\alpha}{L^\alpha}F_c^\alpha(0) + \frac{2\alpha}{L^\alpha}\sum_{n=1}^{\infty}F_c^\alpha(n)\cos\left(\frac{n\pi t^\alpha}{L^\alpha}\right).$$

In bounded domain, the Fourier sine and cosine transforms are useful to solve PDE's. Therefore we can give following calculations.

$$F_s^\alpha\left\{\frac{\delta^\alpha u}{\delta x^\alpha}\right\} = -\frac{n\pi\alpha}{L^\alpha}F_c^\alpha\{u(x,t)\},$$

$$F_s^\alpha\left\{\frac{\delta^\alpha}{\delta x^\alpha}\frac{\delta^\alpha u}{\delta x^\alpha}\right\} = -\frac{n\pi\alpha}{L^\alpha}F_c^\alpha\left\{\frac{\delta^\alpha u}{\delta x^\alpha}\right\}$$
$$= -\frac{n^2\pi^2\alpha^2}{L^{2\alpha}}F_s^\alpha\{u(x,t)\} - \frac{n\pi\alpha}{L^\alpha}\left[u(L,t)\cos n\pi - u(0,t)\right]. \tag{14}$$

And then,

$$F_c^\alpha\left\{\frac{\delta^\alpha u}{\delta x^\alpha}\right\} = \frac{n\pi\alpha}{L^\alpha}F_s^\alpha\{u(x,t)\} - \left[u(0,t) - u(L,t)\cos n\pi\right],$$
$$F_c^\alpha\left\{\frac{\delta^\alpha}{\delta x^\alpha}\frac{\delta^\alpha u}{\delta x^\alpha}\right\} = -\frac{n^2\pi^2\alpha^2}{L^{2\alpha}}F_c^\alpha\{u(x,t)\} - \frac{n\pi\alpha}{L^\alpha}\left[\frac{\delta^\alpha u(0,t)}{\delta x^\alpha} - \frac{\delta^\alpha u(L,t)}{\delta x^\alpha}\cos n\pi\right].$$

Now, let's apply this transform to solve space fractional heat equation,

$$\frac{\delta u}{\delta t} = \frac{\delta^\alpha}{\delta x^\alpha}\frac{\delta^\alpha u}{\delta x^\alpha}, 0 < x < L, t > 0 \tag{15}$$

with the conditions,

$$u(L,t) = u(0,t) = 0 \tag{16}$$

$$u(x,0) = f(x) \tag{17}$$

where $0 < \alpha < 1$.

When we apply the Fourier sine transform both sides of the equation, we have the following equality by using (14) and the conditions (16)

$$\frac{d\widehat{u}(n,t)}{dt} = -\frac{n^2\pi^2\alpha^2}{L^{2\alpha}}\widehat{u}(n,t).$$

Solving the above differential equation gives us,

$$\widehat{u}(n,t) = Ce^{-\frac{n^2\pi^2\alpha^2}{L^{2\alpha}}t}.$$

To evaluate C, we apply Fourier sine transform to the condition (17). At the end we have C as,

$$C = \widehat{u}(n,0) = \int_0^L f(x)\sin\left(\frac{n\pi x^\alpha}{L^\alpha}\right)x^{\alpha-1}dx.$$

Hence we get,

$$\widehat{u}(n,t) = \left[\int_0^L f(x)\sin\left(\frac{n\pi x^\alpha}{L^\alpha}\right)x^{\alpha-1}dx\right]e^{-\frac{n^2\pi^2\alpha^2}{L^{2\alpha}}t}.$$

At the end applying inverse Fourier sine transform, the solution of Eq. (15) obtained as

$$u(x,t) = \frac{2\alpha}{L^\alpha}\sum_{n=1}^{\infty}\left[\int_0^L f(x)\sin\left(\frac{n\pi x^\alpha}{L^\alpha}\right)x^{\alpha-1}dx\right]e^{-\frac{n^2\pi^2\alpha^2}{L^{2\alpha}}t}\sin\left(\frac{n\pi x^\alpha}{L^\alpha}\right).$$

4 Conclusion

In this paper we discuss about the solution of time and space fractional heat differential equations. Conformable fractional derivative definition is used for the solution time fractional heat equation. Conformable Fourier transform which will have very important role in fractional calculus like conformable Laplace transform is defined and given an application for space fractional heat equation. We can say that this definition has many advantages in the solution procedure of fractional differential equations. Some comparisons with classical fractional differential equations are given by Khalil and Abdeljawad before. This paper can help to see the researchers that given definitions are very helpful under the suitable conditions.

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Slowly changing function connected growth properties of wronskians generated by entire and meromorphic functions

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using generalised ${}_pL^*$-type with rate pand generalised ${}_pL^*$-weak type with rate p and wronskians generated by one of the factors.

1 Introduction, definitions and notations

Let $\mathbb{C}$ be the set of all finite complex numbers and f be a meromorphic function defined on $\mathbb{C}$. We will not explain the standard notations and definitions in the theory of entire and meromorphic functions as those are available in [4] and [9]. In the sequel we use the following notation : $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

The following definitions are well known:

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Definition 1 *A meromorphic function* $a \equiv a(z)$ *is called small with respect to* f *if* $T(r,a) = S(r,f)$.

Definition 2 *Let* $a_1, a_2, ...a_k$ *be linearly independent meromorphic functions and small with respect to* f. *We denote by* $L(f) = W(a_1, a_2, ...a_k; f)$ *the Wronskian determinant of* $a_1, a_2, ..., a_k$, f *i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 3 *If* $a \in \mathbb{C} \cup \{\infty\}$, *the quantity*

$$\begin{aligned} \delta(a;f) &= 1 - \limsup_{r \to \infty} \frac{N(r,a;f)}{T(r,f)} \\ &= \liminf_{r \to \infty} \frac{m(r,a;f)}{T(r,f)} \end{aligned}$$

is called the Nevanlinna deficiency of the value 'a*'.*

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a;f) + \delta(\infty;f) \leq 2$ (cf [4], p. 43). If in particular, $\sum_{a \neq \infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [7] defined it in the following way:

Definition 4 [7] *A positive continuous function* $L(r)$ *is called a slowly changing function if for* $\varepsilon (> 0)$,

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \text{ for } r \geq r(\varepsilon) \text{ and}$$

uniformly for $k (\geq 1)$.

Somasundaram and Thamizharasi [8] introduced the notions of L-order and L-lower order for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalized concept for L-order and L-lower order for entire function are L^*-order and L^*-lower order. Their definitions are as follows:

Definition 5 [8] *The* L^**-order* $\rho_f^{L^*}$ *and the* L^**-lower order* $\lambda_f^{L^*}$ *of an entire function* f *are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} .$$

When f *is meromorphic, the above definition reduces to*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}.$$

In the line of Somasundaram and Thamizharasi [8], for any two positive integers m and p, Datta and Biswas [1] introduced the following definition:

Definition 6 [1] *The* m*-th generalized* ${}_pL^*$*-order with rate* p *denoted by* ${}^{(m)}_{(p)}\rho_f^{L^*}$ *and the* m*-th generalized* ${}_pL^*$*-lower order with rate* p *denoted as* ${}^{(m)}_{(p)}\lambda_f^{L^*}$ *of an entire function* f *are defined in the following way:*

$${}^{(m)}_{(p)}\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[m+1]} M(r,f)}{\log\left[r\exp^{[p]} L(r)\right]} \text{ and } {}^{(m)}_{(p)}\lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[m+1]} M(r,f)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where both m *and* p *are positive integers.*

When f *is meromorphic, it can be easily verified that*

$${}^{(m)}_{(p)}\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[r\exp^{[p]} L(r)\right]} \text{ and } {}^{(m)}_{(p)}\lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where both m *and* p *are positive integers.*

To compare the relative growth of two entire or meromorphic functions having same non zero finite generalized ${}_pL^*$-order with rate p, one may introduce the definitions of generalised ${}_pL^*$-type with rate p and generalised ${}_pL^*$-lower type with rate p of entire and meromorphic functions having finite positive generalised ${}_pL^*$-order with rate p in the following manner:

Definition 7 *The* m*-th generalised* ${}_pL^*$*-type with rate* p *denoted by* ${}_{(p)}^{(m)}\sigma_f^{L^*}$ *and* m*-th generalised* ${}_pL^*$*-lower type with rate* p *of an entire function* f *denoted by* ${}_{(p)}^{(m)}\overline{\sigma}_f^{L^*}$ *are respectively defined as follows:*

$$
\begin{aligned}
{}_{(p)}^{(m)}\sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}} \text{ and} \\
{}_{(p)}^{(m)}\overline{\sigma}_f^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}},\ 0 < {}_{(p)}^{(m)}\rho_f^{L^*} < \infty,
\end{aligned}
$$

where m *and* p *are any two positive integers.*
For meromorphic f*,*

$$
\begin{aligned}
{}_{(p)}^{(m)}\sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}} \text{ and} \\
{}_{(p)}^{(m)}\overline{\sigma}_f^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\rho_f^{L^*}}},\ 0 < {}_{(p)}^{(m)}\rho_f^{L^*} < \infty,
\end{aligned}
$$

where both m *and* p *are positive integers.*

Analogously to determine the relative growth of two entire or meromorphic functions having same non zero finite generalized ${}_pL^*$-lower order with rate p one may introduce the definition of generalised ${}_pL^*$-weak type with rate p of entire and meromorphic functions having finite positive generalized ${}_pL^*$-lower order with rate p in the following way:

Definition 8 *The* m*-th generalised* ${}_pL^*$*-weak type with rate* p *denoted by* ${}_{(p)}^{(m)}\tau_f^{L^*}$ *of an entire function* f *is defined as follows:*

$$
{}_{(p)}^{(m)}\tau_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\lambda_f^{L^*}}}\ ,\ 0 < {}_{(p)}^{(m)}\lambda_f^{L^*} < \infty,
$$

where both m *and* p *are positive integers.*

Also one may define the growth indicator ${}_{(p)}^{(m)}\overline{\tau}_f^{L^*}$ *of an entire function* f *in the following manner:*

$$
{}_{(p)}^{(m)}\overline{\tau}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[r \exp^{[p]} L(r)\right]^{{}_{(p)}^{(m)}\lambda_f^{L^*}}}\ ,\ 0 < {}_{(p)}^{(m)}\lambda_f^{L^*} < \infty,
$$

where m *and* p *are any two positive integers.*
For meromorphic f,

$$
\begin{aligned}
{}^{(m)}_{(p)}\overline{\tau}_f^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r,f)}{\left[r\exp^{[p]} L(r)\right]^{{}^{(m)}_{(p)}\lambda_f^{L^*}}} \text{ and} \\
{}^{(m)}_{(p)}\tau_f^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r,f)}{\left[r\exp^{[p]} L(r)\right]^{{}^{(m)}_{(p)}\lambda_f^{L^*}}},\ 0 < {}^{(m)}_{(p)}\lambda_f^{L^*} < \infty,
\end{aligned}
$$

where both m *and* p *are positive integers.*

Lakshminarasimhan [5] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [6] worked on the entire functions of L-bounded index and of non uniform L-bounded index. Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using generalised ${}_pL^*$-order with rate p, generalised ${}_pL^*$- type with rate p and generalised ${}_pL^*$-weak type with rate p and wronskians generated by one of the factors which extend some results of [2].

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [3] *Let* f *be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$
\begin{aligned}
\text{(i) } (p)^{(m)}\sigma_{L(f)}^{L^*} &= \{1+k-k\delta(\infty;f)\}\cdot(p)^{(m)}\sigma_f^{L^*} \text{ for } m=1 \text{ and} \\
(p)^{(m)}\sigma_{L(f)}^{L^*} &= {}^{(m)}_{(p)}\sigma_f^{L^*} \text{ otherwise}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(ii) } (p)^{(m)}\overline{\sigma}_{L(f)}^{L^*} &= \{1+k-k\delta(\infty;f)\}\cdot(p)^{(m)}\overline{\sigma}_f^{L^*} \text{ for } m=1 \text{ and} \\
(p)^{(m)}\overline{\sigma}_{L(f)}^{L^*} &= {}^{(m)}_{(p)}\overline{\sigma}_f^{L^*} \text{ otherwise.}
\end{aligned}
$$

Lemma 2 [3] *Let* f *be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\begin{aligned}(i)\ (p)^{(m)}\tau^{L^*}_{L(f)} &= \{1+k-k\delta(\infty;f)\}\cdot(p)^{(m)}\tau^{L^*}_{f} \text{ for } m=1 \text{ and}\\ (p)^{(m)}\tau^{L^*}_{L(f)} &= {}^{(m)}_{(p)}\tau^{L^*}_{f} \text{ otherwise}\end{aligned}$$

and

$$\begin{aligned}(ii)\ (p)^{(m)}\overline{\tau}^{L^*}_{L(f)} &= \{1+k-k\delta(\infty;f)\}\cdot {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f} \text{ for } m=1 \text{ and}\\ (p)^{(m)}\overline{\tau}^{L^*}_{L(f)} &= {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f} \text{ otherwise.}\end{aligned}$$

3 Theorems

In this section we present the main results of the paper.

Theorem 1 *If* f *be transcendental meromorphic and* g *be entire such that* $0 < {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f} \leq {}^{(n)}_{(p)}\sigma^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L^*}_{f}$ *and* $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ *where* m, n *and* p *are any three positive integers, then*

$$\begin{aligned}\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot(p)\sigma^{L^*}_{f}} &\leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))}\\ &\leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{f}}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))}\\ &\leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{f}}\end{aligned}$$

and

$$\begin{aligned}\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}} &\leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}}\end{aligned}$$

for $n>1$.

Proof. From the definition of ${}^{(n)}_{(p)}\sigma^{L^*}_{L(f)}$, ${}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}$ and in view of Lemma 1, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[m-1]} T(r, f\circ g) \geq \left({}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}}, \tag{1}$$

$$\begin{aligned} \log^{[n-1]} T(r, L(f)) &\leq \left({}^{(n)}_{(p)}\sigma^{L^*}_{L(f)} + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{L(f)}} \\ \text{i.e., } \log^{[n-1]} T(r, L(f)) &\leq \left({}^{(n)}_{(p)}\sigma^{L^*}_{f} + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{f}} \end{aligned} \tag{2}$$

for $n > 1$ and

$$T(r, L(f)) \leq \{1 + k - k\delta(\infty; f)\}\cdot\left({}_{(p)}\sigma^{L^*}_{f} + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}_{(p)}\rho^{L^*}_{f}}. \tag{3}$$

Now from (1), (2) and the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L}_{f}$, it follows for all sufficiently large values of r that,

$$\frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))} \geqslant \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} - \varepsilon}{{}^{(n)}_{(p)}\sigma^{L^*}_{f} + \varepsilon} \quad \text{for } n > 1.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))} \geqslant \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}} \quad \text{for } n > 1. \tag{4}$$

Similarly from (1), (3) and in view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}_{(p)}\rho^{L}_{f}$, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(f))} \geqslant \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty; f)\}\cdot {}_{(p)}\sigma^{L^*}_{f}}. \tag{5}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m-1]} T(r, f\circ g) \leq \left({}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}} \tag{6}$$

and for all sufficiently large values of r,

$$\begin{aligned} \log^{[n-1]} T(r, L(f)) &\geq \left({}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{L(f)} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{L(f)}} \\ \text{i.e., } \log^{[n-1]} T(r, L(f)) &\geq \left({}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f} - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{f}} \end{aligned} \tag{7}$$

for $n > 1$ and

$$T(r, L(f)) \geq \{1 + k - k\delta(\infty; f)\} \cdot \left({}_{(p)}\overline{\sigma}_f^{L^*} - \varepsilon\right) \left[r \exp^{[p]} L(r)\right]^{{}_{(p)}\rho_f^{L^*}}. \quad (8)$$

Combining (6) and (7) and the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L}_{f}$, we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T(r, f \circ g)}{\log^{[n-1]} T(r, L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} + \varepsilon}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f} - \varepsilon} \quad \text{for } n > 1.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f \circ g)}{\log^{[n-1]} T(r, L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}} \quad \text{for } n > 1. \quad (9)$$

Likewise from (6) and (8) and in view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}_{(p)}\rho^{L}_{f}$, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f \circ g)}{T(r, L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}\overline{\sigma}^{L^*}_{f}}. \quad (10)$$

Also for a sequence of values of r tending to infinity it follows that

$$\begin{aligned} \log^{[n-1]} T(r, L(f)) &\leq \left({}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{L(f)}\} + \varepsilon\right) \left[r \exp^{[p]} L(r)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{L(f)}} \\ \text{i.e., } \log^{[n-1]} T(r, L(f)) &\leq \left({}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f} + \varepsilon\right) \left[r \exp^{[p]} L(r)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{f}} \quad (11) \end{aligned}$$

for $n > 1$ and

$$T(r, L(f)) \leq \{1 + k - k\delta(\infty; f)\} \cdot \left({}_{(p)}\overline{\sigma}_f^{L^*} + \varepsilon\right) \left[r \exp^{[p]} L(r)\right]^{{}_{(p)}\rho_f^{L^*}}. \quad (12)$$

Now from (1), (11) and the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L}_{f}$, we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T(r, f \circ g)}{\log^{[n-1]} T(r, L(f))} \geq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} - \varepsilon}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f} + \varepsilon} \quad \text{for } n > 1.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{\log^{[n-1]}T\left(r,L(f)\right)}\geq\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}}\text{ for }n>1. \tag{13}$$

Analogously from (1), (12) and in view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}={}_{(p)}\rho^{L}_{f}$, we get that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{T\left(r,L(f)\right)}\geq\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1+k-k\delta\left(\infty;f\right)\}\cdot\ {}_{(p)}\overline{\sigma}^{L^*}_{f}}. \tag{14}$$

Also for all sufficiently large values of r,

$$\log^{[m-1]}T\left(r,f\circ g\right)\leq\left({}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}+\varepsilon\right)\left[r\exp^{[p]}L\left(r\right)\right]^{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}}. \tag{15}$$

In view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}={}^{(n)}_{(p)}\rho^{L}_{f}$, it follows from (7) and (15) for all sufficiently large values of r that

$$\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{\log^{[n-1]}T\left(r,L(f)\right)}\leq\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}+\varepsilon}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}-\varepsilon}\text{ for }n>1.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{\log^{[n-1]}T\left(r,L(f)\right)}\leq\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}}\text{ for }n>1. \tag{16}$$

Similarly from (8) and (15) and in view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}={}_{(p)}\rho^{L}_{f}$, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{T\left(r,L(f)\right)}\leq\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{\{1+k-k\delta\left(\infty;f\right)\}\cdot\ {}_{(p)}\overline{\sigma}^{L^*}_{f}}. \tag{17}$$

Thus the theorem follows from (4), (5), (9), (10), (13), (14), (16) and (17). □

The following theorem can be proved in the line of Theorem 1 and so its proof is omitted.

Theorem 2 *If* f *be meromorphic and* g *be transcendental entire with* $0 < {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g} \leq {}^{(n)}_{(p)}\sigma^{L^*}_{g} < \infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L^*}_{g}$ *and* $\sum\limits_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;g)\}\cdot {}_{(p)}\sigma^{L^*}_{g}} \leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;g)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{g}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;g)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{g}}$$

and

$$\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{g}} \leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(g))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(g))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}}$$

for $n > 1$.

Theorem 3 *If* f *be transcendental meromorphic and* g *be entire such that* $0 < {}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\sigma^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L^*}_{f}$ *and* $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\sigma^{L^*}_{f}} \leq \limsup_{r\to\infty}\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))}$$

and

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))}$$

for $n > 1$.

Proof. From the definition of ${}^{(n)}_{(p)}\sigma^{L^*}_{L(f)}$ and in view of Lemma 1, we get for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[n-1]} T\left(r, L(f)\right) &\geq \left({}^{(n)}_{(p)}\sigma^{L^*}_{L(f)}\} - \varepsilon\right)\left[r\exp^{[p]} L\left(r\right)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{L(f)}} \\ \text{i.e., } \log^{[n-1]} T\left(r, L(f)\right) &\geq \left({}^{(n)}_{(p)}\sigma^{L^*}_{f} - \varepsilon\right)\left[r\exp^{[p]} L\left(r\right)\right]^{{}^{(n)}_{(p)}\rho^{L^*}_{f}} \qquad (18) \end{aligned}$$

for $n > 1$ and

$$T\left(r, L(f)\right) \geq \{1 + k - k\delta\left(\infty; f\right)\} \cdot \left({}_{(p)}\sigma^{L^*}_{f} - \varepsilon\right)\left[r\exp^{[p]} L\left(r\right)\right]^{{}_{(p)}\rho^{L^*}_{f}}. \qquad (19)$$

Now from (15), (18) and the condition ${}^{(m)}_{(p)}\rho^{L^*}_{fog} = {}^{(n)}_{(p)}\rho^{L^*}_{f}$, it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T\left(r, f\circ g\right)}{\log^{[n-1]} T\left(r, L(f)\right)} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog} + \varepsilon}{{}^{(n)}_{(p)}\sigma^{L^*}_{f} - \varepsilon} \text{ for } n > 1\ .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[m-1]} T\left(r, f\circ g\right)}{\log^{[n-1]} T\left(r, L(f)\right)} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}} \text{ for } n > 1\ . \qquad (20)$$

Similarly from (15), (19) and in view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{fog} = {}_{(p)}\rho^{L}_{f}$, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[m-1]} T\left(r, f\circ g\right)}{T\left(r, L(f)\right)} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{\{1 + k - k\delta\left(\infty; f\right)\} \cdot {}_{(p)}\sigma^{L^*}_{f}}\ . \qquad (21)$$

Again for a sequence of values of r tending to infinity that

$$\log^{[m-1]} T\left(r, f\circ g\right) \geqslant \left({}^{(m)}_{(p)}\sigma^{L^*}_{fog} - \varepsilon\right)\left[r\exp^{[p]} L\left(r\right)\right]^{{}^{(m)}_{(p)}\rho^{L^*}_{fog}}. \qquad (22)$$

So combining (2) and (22) and in view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{fog} = {}^{(n)}_{(p)}\rho^{L^*}_{f}$, we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]} T\left(r, f\circ g\right)}{\log^{[n-1]} T\left(r, L(f)\right)} \geqslant \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog} - \varepsilon}{{}^{(n)}_{(p)}\sigma^{L^*}_{f} + \varepsilon} \text{ for } n > 1\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{\log^{[n-1]}T\left(r,L(f)\right)}\geqslant\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}}\text{ for } n>1\ . \tag{23}$$

Analogously from (3) and (22) and in view of the condition ${}^{(m)}_{(p)}\rho^{L^*}_{fog}={}_{(p)}\rho^{L}_{f}$ we get that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{T\left(r,L(f)\right)}\geqslant\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{\left\{1+k-k\delta\left(\infty;f\right)\right\}\cdot\ {}_{(p)}\sigma^{L^*}_{f}}\ . \tag{24}$$

Thus the theorem follows from (20), (21), (23) and (24). □

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 *If* f *be meromorphic and* g *be transcendental entire with* $0<{}^{(m)}_{(p)}\sigma^{L^*}_{fog}<\infty$, $0<{}^{(n)}_{(p)}\sigma^{L^*}_{g}<\infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{fog}={}^{(n)}_{(p)}\rho^{L^*}_{g}$ *and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ *where* m, n *and* p *are any three positive integers, then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)}&\leq\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{\left\{1+k-k\delta\left(\infty;g\right)\right\}\cdot\ {}_{(p)}\sigma^{L^*}_{g}}\\&\leq\limsup_{r\to\infty}\liminf_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{T\left(r,L(g)\right)}\end{aligned}$$

and

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{\log^{[n-1]}T\left(r,L(g)\right)}\leq\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{{}^{(n)}_{(p)}\sigma^{L^*}_{g}}\leq\limsup_{r\to\infty}\frac{\log^{[m-1]}T\left(r,f\circ g\right)}{\log^{[n-1]}T\left(r,L(g)\right)}$$

for $n>1$.

The following theorem is a natural consequence of Theorem 1 and Theorem 3.

Theorem 5 *If* f *be transcendental meromorphic and* g *be entire such that* $0<{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{fog}\leq{}^{(m)}_{(p)}\sigma^{L^*}_{fog}<\infty$, $0<{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}\leq{}^{(n)}_{(p)}\sigma^{L^*}_{f}<\infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{fog}={}^{(n)}_{(p)}\rho^{L^*}_{f}$ *and* $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ *where* m, n *and* p *are any three positive integers,*

then

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))}\leq\min\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{A\cdot\ {}_{(p)}\overline{\sigma}^{L^*}_{f}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{A\cdot\ {}_{(p)}\sigma^{L^*}_{f}}\right\}$$
$$\leq\max\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{A\cdot\ {}_{(p)}\overline{\sigma}^{L^*}_{f}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{A\cdot\ {}_{(p)}\sigma^{L^*}_{f}}\right\}$$
$$\leq\limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))}$$

where $A=\{1+k-k\delta(\infty;f)\}$ *and*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))}\leq\min\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}}\right\}$$
$$\leq\max\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}}\right\}$$
$$\leq\limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))}$$

for $n>1$.

Analogously one may state the following theorem without its proof.

Theorem 6 *If* f *be meromorphic and* g *be transcendental entire with* $0<{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}\leq{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}<\infty$*,* $0<{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}\leq{}^{(n)}_{(p)}\sigma^{L^*}_{g}<\infty$*,* ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}={}^{(n)}_{(p)}\rho^{L^*}_{g}$ *and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ *where* m*,* n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}\leq\min\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{B\cdot\ {}_{(p)}\overline{\sigma}^{L^*}_{g}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{B\cdot\ {}_{(p)}\sigma^{L^*}_{g}}\right\}$$
$$\leq\max\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{B\cdot\ {}_{(p)}\overline{\sigma}^{L^*}_{g}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{B\cdot\ {}_{(p)}\sigma^{L^*}_{g}}\right\}$$
$$\leq\limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}$$

where $B = \{1 + k - k\delta(\infty; g)\}$ *and*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f \circ g)}{\log^{[n-1]} T(r, L(g))} \le \min\left\{ \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{fog}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{{}^{(n)}_{(p)}\sigma^{L^*}_{g}} \right\}$$
$$\le \max\left\{ \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{fog}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{fog}}{{}^{(n)}_{(p)}\sigma^{L^*}_{g}} \right\}$$
$$\le \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f \circ g)}{\log^{[n-1]} T(r, L(g))}$$

for $n > 1$.

Now in the line of Theorem 1, Theorem 3, Theorem 5 and Theorem 2, Theorem 4, Theorem 6 respectively and with the help of Lemma 2 one can easily prove the following six theorems using the notion of generalised ${}_pL^*$-weak type with rate p and therefore their proofs are omitted.

Theorem 7 *If* f *be transcendental meromorphic and* g *be entire such that* $0 < {}^{(m)}_{(p)}\tau^{L^*}_{fog} \le {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog} < \infty$, $0 < {}^{(n)}_{(p)}\tau^{L^*}_{f} \le {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{fog} = {}^{(n)}_{(p)}\lambda^{L^*}_{f}$ *and* $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{\{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}\overline{\tau}^{L^*}_{f}} \le \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f \circ g)}{T(r, L(f))}$$
$$\le \frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{\{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}\tau^{L^*}_{f}}$$
$$\le \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f \circ g)}{T(r, L(f))}$$
$$\le \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog}}{\{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}\tau^{L^*}_{f}}$$

and

$$\frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \le \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f \circ g)}{\log^{[n-1]} T(r, L(f))} \le \frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}}$$

for $n > 1$.

Theorem 8 *If* f *be transcendental meromorphic and* g *be entire with* $0 < {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\lambda^{L^*}_{f}$ *and* $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty; f)\} \cdot {}_{(p)}\overline{\tau}^{L^*}_{f}}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(f))}$$

and

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))}$$

for $n > 1$.

Theorem 9 *If* f *be transcendental meromorphic and* g *be entire such that* $0 < {}^{(m)}_{(p)}\tau^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\tau^{L^*}_{f} \leq {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\lambda^{L^*}_{f}$ *and* $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(f))} \leq \min\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\overline{\tau}^{L^*}_{f}}\right\}$$
$$\leq \max\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\overline{\tau}^{L^*}_{f}}\right\}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(f))}$$

where $A = \{1 + k - k\delta(\infty; f)\}$ *and*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))} \le \min\left\{ \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \right\}$$

$$\le \max\left\{ \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \right\} \le \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))}$$

for $n > 1$.

Theorem 10 *If* f *be meromorphic and* g *be transcendental entire with* $0 < {}^{(m)}_{(p)}\tau^{L^*}_{f\circ g} \le {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\tau^{L^*}_{g} \le {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g} < \infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\lambda^{L^*}_{g}$ *and* $\sum\limits_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty; g)\} \cdot {}_{(p)}\overline{\tau}^{L^*}_{g}} \le \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))}$$

$$\le \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty; g)\} \cdot {}_{(p)}\tau^{L^*}_{g}}$$

$$\le \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))}$$

$$\le \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty; g)\} \cdot {}_{(p)}\tau^{L^*}_{g}}$$

and

$$\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g}} \le \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} \le \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{g}}$$

$$\le \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} \le \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{g}}$$

for $n > 1$.

Theorem 11 *If* f *be meromorphic and* g *be transcendental entire such that* $0 < {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog} < \infty$, $0 < {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g} < \infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{fog} = {}^{(n)}_{(p)}\lambda^{L^*}_{g}$ *and* $\sum\limits_{a \neq \infty} \delta(a;g) + \delta(\infty;g) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))} \leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog}}{\{1 + k - k\delta(\infty; g)\} \cdot {}_{(p)}\overline{\tau}^{L^*}_{g}} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))}$$

and

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} \leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g}} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))}$$

for $n > 1$.

Theorem 12 *If* f *be meromorphic and* g *be transcendental entire with* $0 < {}^{(m)}_{(p)}\tau^{L^*}_{fog} \leq {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog} < \infty$, $0 < {}^{(n)}_{(p)}\tau^{L^*}_{g} \leq {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g} < \infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{fog} = {}^{(n)}_{(p)}\lambda^{L^*}_{g}$ *and* $\sum\limits_{a \neq \infty} \delta(a;g) + \delta(\infty;g) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))} &\leq \min\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{B \cdot {}_{(p)}\tau^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog}}{B \cdot {}_{(p)}\overline{\tau}^{L^*}_{g}}\right\} \\ &\leq \max\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{B \cdot {}_{(p)}\tau^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog}}{B \cdot {}_{(p)}\overline{\tau}^{L^*}_{g}}\right\} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))}\end{aligned}$$

where $B = \{1 + k - k\delta(\infty; g)\}$ *and*

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} &\leq \min\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{{}^{(n)}_{(p)}\tau^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g}}\right\} \\ &\leq \max\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{fog}}{{}^{(n)}_{(p)}\tau^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{fog}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g}}\right\} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))}\end{aligned}$$

for $n > 1$.

We may now state the following theorems without their proofs based on generalised ${}_pL^*$- type with rate p and generalised ${}_pL^*$-weak type with rate p.

Theorem 13 *If* f *be transcendental meromorphic and* g *be entire such that* $0 < {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\tau^{L^*}_{f} \leq {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\lambda^{L^*}_{f}$ *and* $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\overline{\tau}^{L^*}_{f}} \leq \liminf_{r\to\infty}\frac{\log^{[m-1]} T(r,f\circ g)}{T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\tau^{L^*}_{f}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]} T(r,f\circ g)}{T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\tau^{L^*}_{f}}$$

and

$$\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \leq \liminf_{r\to\infty}\frac{\log^{[m-1]} T(r,f\circ g)}{\log^{[n-1]} T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]} T(r,f\circ g)}{\log^{[n-1]} T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}}$$

for $n > 1$.

Theorem 14 *If* f *be transcendental meromorphic and* g *be entire with* $0 < {}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\lambda^{L^*}_{f}$ *and* $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]} T(r,f\circ g)}{T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\overline{\tau}^{L^*}_{f}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]} T(r,f\circ g)}{T(r,L(f))}$$

and

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))} \leq \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))}$$

for $n > 1$.

Theorem 15 *Let* f *be transcendental meromorphic and* g *be entire such that* $0 < {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\tau^{L^*}_{f} \leq {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\lambda^{L^*}_{f}$ *and* $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(f))} &\leq \min\left\{ \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\overline{\tau}^{L^*}_{f}} \right\} \\
&\leq \max\left\{ \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{A \cdot {}_{(p)}\tau^{L^*}_{f}} \right\} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(f))}
\end{aligned}$$

where $A = \{1 + k - k\delta(\infty; f)\}$ *and*

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))} &\leq \min\left\{ \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \right\} \\
&\leq \max\left\{ \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{f}} \right\} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(f))}
\end{aligned}$$

for $n > 1$.

Theorem 16 *If* f *be transcendental meromorphic and* g *be entire with* $0 < {}^{(m)}_{(p)}\tau^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g} < \infty$, $0 < {}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f} \leq {}^{(n)}_{(p)}\sigma^{L^*}_{f} < \infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L^*}_{f}$ *and* $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ *where* m, n *and* p *are any three positive integers,*

then

$$\begin{aligned}\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\sigma^{L^*}_{f}} &\leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))} \\ &\leq \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{f}} \\ &\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))} \\ &\leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{f}}\end{aligned}$$

and

$$\begin{aligned}\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}} &\leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}} \\ &\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}}\end{aligned}$$

for $n>1$.

Theorem 17 *If* f *be transcendental meromorphic and* g *be entire such that* $0<{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}<\infty$, $0<{}^{(n)}_{(p)}\sigma^{L^*}_{f}<\infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g}={}^{(n)}_{(p)}\rho^{L^*}_{f}$ *and* $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ *where* m, n *and* p *are any three positive integers, then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))} &\leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;f)\}\cdot {}_{(p)}\sigma^{L^*}_{f}} \\ &\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))}\end{aligned}$$

and

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))} \leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))}$$

for $n>1$.

Theorem 18 *If* f *be transcendental meromorphic and* g *be entire with* $0 < {}^{(m)}_{(p)}\tau^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g} < \infty$*,* $0 < {}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f} \leq {}^{(n)}_{(p)}\sigma^{L^*}_{f} < \infty$*,* ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\rho^{L^*}_{f}$ *and* $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$ *where* m*,* n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))} \leq \min\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{A\cdot {}_{(p)}\overline{\sigma}^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{A\cdot {}_{(p)}\sigma^{L^*}_{f}}\right\}$$

$$\leq \max\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{A\cdot {}_{(p)}\overline{\sigma}^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{A\cdot {}_{(p)}\sigma^{L^*}_{f}}\right\}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(f))}$$

where $A = \{1 + k - k\delta(\infty;f)\}$ *and*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))} \leq \min\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}}\right\}$$

$$\leq \max\left\{\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{f}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{f}}\right\}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))}$$

for $n > 1$.

Theorem 19 *If* f *be meromorphic and* g *be transcendental entire such that* $0 < {}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g} \leq {}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g} < \infty$*,* $0 < {}^{(n)}_{(p)}\tau^{L^*}_{g} \leq {}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g} < \infty$*,* ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} = {}^{(n)}_{(p)}\lambda^{L^*}_{g}$ *and* $\sum_{a\neq\infty}\delta(a;g) + \delta(\infty;g) = 2$ *where* m*,* n *and* p *are any three positive integers, then*

$$\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty;g)\}\cdot {}_{(p)}\overline{\tau}^{L^*}_{g}} \leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}$$

$$\leq \frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{\{1 + k - k\delta(\infty;g)\}\cdot {}_{(p)}\tau^{L^*}_{g}}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}$$

$$\leq \frac{{}_{(p)}^{(m)}\sigma_{f\circ g}^{L^*}}{\{1+k-k\delta\,(\infty;g)\}\cdot\ {}_{(p)}\tau_{g}^{L^*}}$$

and

$$\frac{{}_{(p)}^{(m)}\overline{\sigma}_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\overline{\tau}_{g}^{L^*}} \leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T\,(r,f\circ g)}{\log^{[n-1]}T\,(r,L(g))} \leq \frac{{}_{(p)}^{(m)}\overline{\sigma}_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\tau_{g}^{L^*}}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T\,(r,f\circ g)}{\log^{[n-1]}T\,(r,L(g))} \leq \frac{{}_{(p)}^{(m)}\sigma_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\tau_{g}^{L^*}}$$

for $n>1$.

Theorem 20 *If* f *be meromorphic and* g *be transcendental entire with* $0<{}_{(p)}^{(m)}\sigma_{f\circ g}^{L^*}<\infty$, $0<{}_{(p)}^{(n)}\overline{\tau}_{g}^{L^*}<\infty$, ${}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}={}_{(p)}^{(n)}\lambda_{g}^{L^*}$ *and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ *where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T\,(r,f\circ g)}{T\,(r,L(g))} \leq \frac{{}_{(p)}^{(m)}\sigma_{f\circ g}^{L^*}}{\{1+k-k\delta\,(\infty;g)\}\cdot\ {}_{(p)}\overline{\tau}_{g}^{L^*}}$$
$$\leq \limsup_{r\to\infty}\liminf_{r\to\infty}\frac{\log^{[m-1]}T\,(r,f\circ g)}{T\,(r,L(g))}$$

and

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T\,(r,f\circ g)}{\log^{[n-1]}T\,(r,L(g))} \leq \frac{{}_{(p)}^{(m)}\sigma_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\overline{\tau}_{g}^{L^*}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T\,(r,f\circ g)}{\log^{[n-1]}T\,(r,L(g))}$$

for $n>1$.

Theorem 21 *If* f *be meromorphic and* g *be transcendental entire such that* $0<{}_{(p)}^{(m)}\overline{\sigma}_{f\circ g}^{L^*}\leq{}_{(p)}^{(m)}\sigma_{f\circ g}^{L^*}<\infty$, $0<{}_{(p)}^{(n)}\tau_{g}^{L^*}\leq{}_{(p)}^{(n)}\overline{\tau}_{g}^{L^*}<\infty$, ${}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}={}_{(p)}^{(n)}\lambda_{g}^{L^*}$ *and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ *where* m, n *and* p *are any three positive integers,*

then

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}\leq \min\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{B\cdot {}_{(p)}\tau^{L^*}_{g}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{B\cdot {}_{(p)}\overline{\tau}^{L^*}_{g}}\right\}$$

$$\leq \max\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{B\cdot {}_{(p)}\tau^{L^*}_{g}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{B\cdot {}_{(p)}\overline{\tau}^{L^*}_{g}}\right\}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}$$

where $B=\{1+k-k\delta(\infty;g)\}$ *and*

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(g))}\leq \min\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{g}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g}}\right\}$$

$$\leq \max\left\{\frac{{}^{(m)}_{(p)}\overline{\sigma}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\tau^{L^*}_{g}},\frac{{}^{(m)}_{(p)}\sigma^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\tau}^{L^*}_{g}}\right\}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(g))}$$

for $n>1$.

Theorem 22 *If* f *be meromorphic and* g *be transcendental entire with* $0<{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}\leq {}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}<\infty$, $0<{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}\leq {}^{(n)}_{(p)}\sigma^{L^*}_{g}<\infty$, ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g}={}^{(n)}_{(p)}\rho^{L^*}_{g}$ *and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ *where* m, n *and* p *are any three positive integers, then*

$$\frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;g)\}\cdot {}_{(p)}\sigma^{L^*}_{g}}\leq \liminf_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}$$

$$\leq \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;g)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{g}}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{T(r,L(g))}$$

$$\leq \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{\{1+k-k\delta(\infty;g)\}\cdot {}_{(p)}\overline{\sigma}^{L^*}_{g}}$$

and

$$\frac{{}_{(p)}^{(m)}\tau_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\sigma_g^{L^*}} \leq \liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} \leq \frac{{}_{(p)}^{(m)}\tau_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\overline{\sigma}_g^{L^*}}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} \leq \frac{{}_{(p)}^{(m)}\overline{\tau}_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\overline{\sigma}_g^{L^*}}$$

for $n > 1$.

Theorem 23 *If* f *be meromorphic and* g *be transcendental entire such that* $0 < {}_{(p)}^{(m)}\overline{\tau}_{f\circ g}^{L^*} < \infty$, $0 < {}_{(p)}^{(n)}\sigma_g^{L^*} < \infty$, ${}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} = {}_{(p)}^{(n)}\rho_g^{L^*}$ *and* $\sum\limits_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ *where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))} \leq \frac{{}_{(p)}^{(m)}\overline{\tau}_{f\circ g}^{L^*}}{\{1 + k - k\delta(\infty; g)\} \cdot {}_{(p)}\sigma_g^{L^*}}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))}$$

and

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} \leq \frac{{}_{(p)}^{(m)}\overline{\tau}_{f\circ g}^{L^*}}{{}_{(p)}^{(n)}\sigma_g^{L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))}$$

for $n > 1$.

Theorem 24 *If* f *be transcendental meromorphic and* g *be entire with* $0 < {}_{(p)}^{(m)}\tau_{f\circ g}^{L^*} \leq {}_{(p)}^{(m)}\overline{\tau}_{f\circ g}^{L^*} < \infty$, $0 < {}_{(p)}^{(n)}\overline{\sigma}_g^{L^*} \leq {}_{(p)}^{(n)}\sigma_g^{L^*} < \infty$, ${}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} = {}_{(p)}^{(n)}\rho_g^{L^*}$ *and* g *has the maximum deficiency sum where* m, n *and* p *are any three positive integers, then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))} \leq \min\left\{ \frac{{}_{(p)}^{(m)}\tau_{f\circ g}^{L^*}}{B \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}, \frac{{}_{(p)}^{(m)}\overline{\tau}_{f\circ g}^{L^*}}{B \cdot {}_{(p)}\sigma_g^{L^*}} \right\}$$

$$\leq \max\left\{ \frac{{}_{(p)}^{(m)}\tau_{f\circ g}^{L^*}}{B \cdot {}_{(p)}\overline{\sigma}_g^{L^*}}, \frac{{}_{(p)}^{(m)}\overline{\tau}_{f\circ g}^{L^*}}{B \cdot {}_{(p)}\sigma_g^{L^*}} \right\}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{T(r, L(g))}$$

where $B = \{1 + k - k\delta(\infty; g)\}$ *and*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} \le \min\left\{ \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{g}} \right\}$$

$$\le \max\left\{ \frac{{}^{(m)}_{(p)}\tau^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\overline{\sigma}^{L^*}_{g}}, \frac{{}^{(m)}_{(p)}\overline{\tau}^{L^*}_{f\circ g}}{{}^{(n)}_{(p)}\sigma^{L^*}_{g}} \right\}$$

$$\le \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))}$$

for $n > 1$.

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Numerical solution of time fractional Burgers equation

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Abstract. In this article, the time fractional order Burgers equation has been solved by quadratic B-spline Galerkin method. This method has been applied to three model problems. The obtained numerical solutions and error norms L_2 and L_∞ have been presented in tables. Absolute error graphics as well as those of exact and numerical solutions have been given.

1 Introduction

The Burgers equation is a nonlinear equation for diffusive waves in fluid dynamics. It exists various physical problems such as one-dimensional sound waves in a viscous medium, waves in fluid filled viscous elastic tubes, shock waves in a viscous medium and magnetohy-drodynamic waves in a medium with finite electrical conductivity, turbulence etc. [1]. Numerical solutions of the Burgers equation in the literature have been obtained using different methods and techniques [2, 3, 4, 5, 6, 7]. In addition, the fractional order Burgers equation has been solved by many authors [8, 9, 10, 11, 12, 13, 14].

The main idea underlying the finite element method, finite element nodes that are related to entire of the equivalent system can discretize the problem

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area and the most appropriate one will be a true physical behavioral model to choose the most appropriate type of element. Thus with the help of this method, an equation which is hard to solve can be turned into a few solvable set of equations. Finite element adjustable yet small enough and large enough to reduce computation load of the problem in available sizes[15].

Due to its capacity for non-integer order derivatives and integrals of fractional calculus have become an indispensable part of applied mathematics. Applications of differentiation and integration with non-integer orders can be traced back to premature in history, so it can be said that it is not new[16]. Many different techniques and methods of dealing with fractional differential equations resulting analytical and numerical solutions can be found in a wide variety of studies in the literature [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

In this paper, we consider the time fractional Burgers equation for $0 < \gamma < 1$

$$\frac{\partial^{\gamma} U(x,t)}{\partial t^{\gamma}} + U(x,t)\frac{\partial U(x,t)}{\partial x} - \nu \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t) \tag{1}$$

with the boundary conditions

$$U(a,t) = h_1(t)\ , \quad U(b,t) = h_2(t), \quad t \geq 0 \tag{2}$$

and the initial condition

$$U(x,0) = g(x)\ , \quad a \leq x \leq b, \tag{3}$$

where ν is a viscosity parameter and

$$\frac{\partial^{\gamma} U(x,t)}{\partial t^{\gamma}} = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \frac{\partial U(x,\tau)}{\partial \tau} d\tau$$

is the Caputo fractional derivative [32]. In this paper, to achieve a finite element layout of the time fractional Burgers equation, Caputo fractional derivative formulation can be discretizated through L1 formulae [17]:

$$\frac{\partial^{\gamma} f(t)}{\partial t^{\gamma}}\Big|_{t_m} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} \left[(k+1)^{1-\gamma} - k^{1-\gamma}\right] \left[f(t_{m-k}) - f(t_{m-1-k})\right].$$

2 Quadratic B-spline finite element Galerkin solutions

In this section, the time fractional Burgers equation has been solved by quadratic B-spline Galerkin method. For this firstly, Eq. (1) is multiplied with weigh function $W(x)$ and then integrated over the region, we get

$$\int_a^b \left(\frac{\partial^\gamma U}{\partial t^\gamma} + U\frac{\partial U}{\partial x} - \nu \frac{\partial^2 U}{\partial x^2} \right) W dx = \int_a^b W f(x,t) dx. \tag{4}$$

In Eq. (4), if we apply partial integration, we have weak form

$$\int_{x_m}^{x_{m+1}} \left(W\frac{\partial^\gamma U}{\partial t^\gamma} + WU\frac{\partial U}{\partial x} + \nu \frac{\partial W}{\partial x}\frac{\partial U}{\partial x} \right) dx = \nu W \frac{\partial U}{\partial x}\Big|_{x_m}^{x_{m+1}} + \int_{x_m}^{x_{m+1}} W f(x,t) dx. \tag{5}$$

which is on only one of the $[x_m, x_{m+1}]$ finite element of Eq. (1). To modify the global coordinate system to the local one we did made use of transformation $\xi = x - x_m$. So, Eq. (5) turns into the form

$$\int_0^h \left(W\frac{\partial^\gamma U}{\partial t^\gamma} + WU\frac{\partial U}{\partial \xi} + \nu \frac{\partial W}{\partial \xi}\frac{\partial U}{\partial \xi} \right) d\xi = \nu\, W \frac{\partial U}{\partial \xi}\Big|_0^h + \int_0^h W\tilde{f}(\xi,t) d\xi. \tag{6}$$

We describe quadratic B-spline base functions. Let us consider the interval $[a,b]$ is partitioned into N finite elements of uniformly equal length by the knots x_m, $m = 0,1,2,...,N$ such that $a = x_0 < x_1 \cdots < x_N = b$ and $h = x_{m+1} - x_m$. The quadratic B-splines $Q_m(x)$, $(m = -1(1)N)$, at the knots x_m are defined over the interval $[a,b]$ by [33]

$$Q_m(x) = \frac{1}{h^2} \begin{cases} (x - x_{m-1})^2, & x \in [x_{m-1}, x_m], \\ (x - x_{m-1})^2 - 3(x - x_m)^2, & x \in [x_m, x_{m+1}], \\ (x - x_{m-1})^2 - 3(x - x_m)^2 + 3(x - x_{m+1})^2, & x \in [x_{m+1}, x_{m+2}], \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

The set of splines $\{Q_{-1}(x), Q_0(x), \ldots, Q_N(x)\}$ forms a basis for the functions defined over $[a,b]$. For this reason, an approximation solution $U_N(x,t)$ may be written in terms of the quadratic B-splines trial functions as:

$$U_N(x,t) = \sum_{m=-1}^{N} \delta_m(t) Q_m(x) \tag{8}$$

where $\delta_m(t)$'s are time dependent parameters. Each quadratic B-spline involves three elements therefore every element of $[x_m, x_{m+1}]$ is coated with three quadratic B-splines. In this problem, the finite elements are described on the interval $[x_m, x_{m+1}]$ and the elements knots x_m, x_{m+1}. Using the nodal values U_m and $U_m^{'}$ supplied in terms of the parameter $\delta_m(t)$

$$\begin{aligned} U_N(x_m) &= U_m = \delta_{m-1} + \delta_m, \\ U_N'(x_m) &= U_m' = 2(-\delta_{m-1} + \delta_m)/h \end{aligned}$$

the variation of $U_N(x, t)$ over the typical element $[x_m, x_{m+1}]$ is presented by

$$U_N(\xi, t) = \sum_{j=m-1}^{m+1} \delta_j(t) Q_j(\xi).$$

The Eq. (6) is the element equation for a typical element "e". Eq. (7) can be written as follows

$$\begin{matrix} Q_{m-1} \\ Q_m \\ Q_{m+1} \end{matrix} = \frac{1}{h^2} \left\{ \begin{matrix} (h-\xi)^2, \\ h^2 + 2h\xi - 2\xi^2, \\ \xi^2. \end{matrix} \right. \tag{9}$$

Inserting equations Eqs. (9) into Eq. (6), we have

$$\begin{aligned} \sum_{j=m-1}^{m+1} \left[\int_0^h Q_i Q_j d\xi \right] \dot{\delta} + \sum_{k=m-1}^{m+1} \sum_{j=m-1}^{m+1} \left[\int_0^h Q_i Q_k' Q_j d\xi \right] \delta \\ + \nu \sum_{j=m-1}^{m+1} \left[\int_0^h Q_i' Q_j' d\xi \right] \delta - \nu \sum_{j=m-1}^{m+1} \left[Q_i Q_j' \right] \delta \Big|_0^h \\ = \int_0^h Q_i \tilde{f}(\xi, t) d\xi, \quad i = m-1, m, m+1 \end{aligned} \tag{10}$$

where $\dot{\gamma}$ shows γ^{th} order fractional derivative with respect to t. If we take

$$A_{ij}^e = \int_0^h Q_i Q_j d\xi, \; B_{ikj}^e = \int_0^h Q_i Q_k' Q_j d\xi,$$

$$C_{ij}^e = \int_0^h Q_i' Q_j' d\xi, \; D_{ij}^e = Q_i Q_j' \Big|_0^h, \; E_i^e = \int_0^h Q_i \tilde{f}(\xi, t) d\xi$$

Eq. (10) can be written in the matrix form

$$A^e\dot{\delta}^e + B^e\delta^e + \nu C^e\delta^e - \nu D^e\delta^e = E^e \tag{11}$$

where $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1})$. When the above integrations are calculated by using quadratic B-spline functions, we have

$$A^e_{ij} = \int_0^h Q_i Q_j d\xi = \frac{h}{30}\begin{bmatrix} 6 & 13 & 1 \\ 13 & 54 & 13 \\ 1 & 13 & 6 \end{bmatrix},$$

$$B^e_{ikj} = \int_0^h Q_i Q_k' Q_j d\xi = \frac{1}{30}\begin{bmatrix} (-10,-19,-1)\delta^e & (8,12,0)\delta^e & (2,7,1)\delta^e \\ (-19,-54,-7)\delta^e & (12,0,-12)\delta^e & (7,54,19)\delta^e \\ (-1,-7,-2)\delta^e & (0,-12,-8)\delta^e & (1,19,10)\delta^e \end{bmatrix},$$

$$C^e_{ij} = \int_0^h Q_i' Q_j' d\xi = \frac{2}{3h}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

$$D^e_{ij} = Q_i Q_j'\Big|_0^h = \frac{2}{h}\begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

where $i, j, k = m-1, m, m+1$. By writing the matrices A, B, C, D and E which are obtained by combining element matrixes in Eq. (11), we have the following matrix form equation:

$$A\dot{\delta} + (B + \nu C - \nu D)\delta = E \tag{12}$$

where $\delta = (\delta_{-1}, \delta_0, \delta_1, ..., \delta_{N-1}, \delta_N)$. If we write L1 formula

$$\dot{\delta}_m = \frac{d^\gamma \delta}{dt^\gamma} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} \left[(k+1)^{1-\gamma} - k^{1-\gamma}\right]\left[\delta_m^{n-k} - \delta_m^{n-k-1}\right],$$

instead of $\dot{\delta}$ and Crank-Nicolson formula

$$\delta_m = \frac{1}{2}(\delta_m^n + \delta_m^{n+1})$$

instead of δ, We have the recurrence correlation between sequential time levels about the unknown parameters $\delta_m^{n+1}(t)$

$$\begin{aligned} &[A + (\Delta t)^\gamma \Gamma(2-\gamma)(B + \nu C - \nu D)/2]\,\delta^{n+1} \\ &= [A - (\Delta t)^\gamma \Gamma(2-\gamma)(B + \nu C - \nu D)/2]\,\delta^n \\ &\quad - A\sum_{k=1}^{n} \left[(k+1)^{1-\gamma} - k^{1-\gamma}\right]\left[\delta^{n-k} - \delta^{n-k-1}\right] + (\Delta t)^\gamma \Gamma(2-\gamma) E \end{aligned} \tag{13}$$

$\delta = (\delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$. The system (13) is composed of $N+2$ linear equations that include unknown parameters $N+2$. To achieve unique solution to these systems, we need two additional restrictions. These are obtained from the boundary conditions and can be used to eliminate δ_{-1} and δ_N from the systems. For this reason, we achieve a $N \times N$ solvable system of equations.

Initial state

The initial vector $\mathbf{d}^0 = (\delta_{-1}, \delta_0, \delta_1, \ldots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T$ is obtained by the initial and boundary conditions. Therefore, the approximation (8) can be rewritten for the initial condition as

$$U_N(x,0) = \sum_{m=-1}^{N} \delta_m(0) Q_m(x)$$

where the $\delta_m(0)$'s are unknown parameters. We need the initial numerical approximation $U_N(x,0)$ provides the conditions:

$$\begin{gathered} U_N(x,0) = U(x_m,0), \quad m = 0(1)N \\ U_N'(x_0,0) = U'(x_0,0). \end{gathered}$$

So, using these conditions leads to a matrix system of the form

$$W\mathbf{d}^0 = \mathbf{b}$$

where

$$W = \begin{bmatrix} \frac{-2}{h} & \frac{2}{h} & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & 1 & \\ & & & & & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{b} = (U'(x_0,0), U(x_0,0), U(x_1,0), \ldots, U(x_{N-2},0), U(x_{N-1},0), U(x_N,0))^T.$$

3 Numerical examples and results

In this section, we find the numerical solutions of problems which are obtained by quadratic B-spline Galerkin method. We calculate the accuracy of the method by the error norm L_2

$$L_2 = \left\|U^{exact} - U_N\right\|_2 \simeq \sqrt{\frac{b-a}{N}\sum_{j=0}^{N}\left|U_j^{exact} - (U_N)_j\right|^2}$$

and the error norm L_∞

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|.$$

Problem 1: Firstly, we consider the Eq. (1) with boundary conditions

$$U(0,t) = t^2\,, \quad U(1,t) = et^2, \quad t \geq 0$$

and the initial condition as

$$U(x,0) = 0\,, \quad 0 \leq x \leq 1.$$

The $f(x,t)$ is of the form

$$f(x,t) = \frac{2t^{2-\gamma}e^x}{\Gamma(3-\gamma)} + t^4e^{2x} - \nu t^2e^x.$$

The exact solution of the problem is given by

$$U(x,t) = t^2e^x.$$

The numerical solutions and the error norms for Problem 1 are given in Tables 1-3. If the results for $\gamma = 0.50$, $\Delta t = 0.00025$, $t = 1$, $\nu = 1$ and different number of partitions are examined in Table 1, one can see that when the number of partitions N are increased, the error norms L_2 and L_∞ decrease significantly. The results which are obtained for $\gamma = 0.50$, $N = 80$, $t = 1$, $\nu = 1$ and for different Δt time steps are given in Table 2. From this table it is clearly seen that when the Δt time steps decrease, the error norms L_2 and L_∞ decrease as it is expected. The results for different values of γ, $\Delta t = 0.00025$, $N = 40$, $t = 1$, $\nu = 1$ are given with the error norms L_2 and L_∞ in Table 3. The error distributions obtained by quadratic B-spline Galerkin method for $\Delta t = 0.00025$, $N = 80$, $t = 1$, $\nu = 1$ and different values of γ are given Fig. 1.

Table 1: Error norms and numerical solutions of Problem 1 for $\gamma = 0.50$, $\Delta t = 0.00025$, $t = 1$, $\nu = 1$.

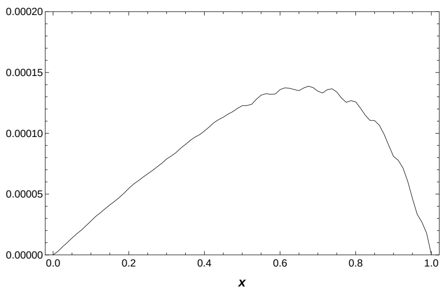
x	$N = 10$	$N = 20$	$N = 40$	$N = 80$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	1.105440	1.105287	1.105216	1.105197	1.105171
0.2	1.222203	1.221644	1.221493	1.221455	1.221403
0.3	1.351078	1.350217	1.349992	1.349935	1.349859
0.4	1.493437	1.492287	1.491996	1.491922	1.491825
0.5	1.650663	1.649270	1.648922	1.648838	1.648721
0.6	1.824294	1.822727	1.822342	1.822247	1.822119
0.7	2.016049	2.014378	2.013979	2.013882	2.013753
0.8	2.227650	2.226118	2.225747	2.225661	2.225541
0.9	2.461512	2.460020	2.459745	2.459680	2.459603
1.0	2.718282	2.718282	2.718282	2.718282	2.718282
$L_2 \times 10^3$	1.632995	0.447720	0.161833	0.092624	
$L_\infty \times 10^3$	2.296683	0.625018	0.227352	0.133125	

Table 2: Error norms and numerical solutions of Problem 1 for $\gamma = 0.50$, $N = 80$, $t = 1$, $\nu = 1$.

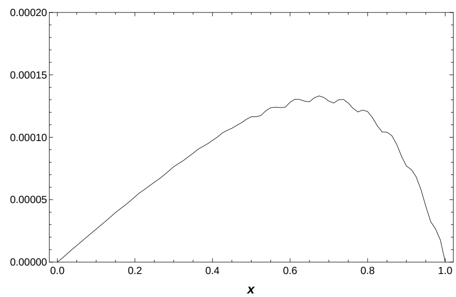
x	$\Delta t = 0.002$	$\Delta t = 0.001$	$\Delta t = 0.0005$	$\Delta t = 0.00025$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	1.105356	1.105276	1.105236	1.105216	1.105171
0.2	1.221768	1.221611	1.221533	1.221493	1.221403
0.3	1.350395	1.350164	1.350049	1.349992	1.349859
0.4	1.492516	1.492218	1.492070	1.491996	1.491825
0.5	1.649543	1.649188	1.649011	1.648922	1.648721
0.6	1.823031	1.822636	1.822440	1.822342	1.822119
0.7	2.014687	2.014282	2.014080	2.013979	2.013753
0.8	2.226387	2.226020	2.225837	2.225747	2.225541
0.9	2.460180	2.459931	2.459807	2.459745	2.459603
1.0	2.718282	2.718282	2.718282	2.718282	2.718282
$L_2 \times 10^3$	0.660788	0.375012	0.232768	0.092624	
$L_\infty \times 10^3$	0.936619	0.530231	0.328303	0.133125	

Table 3: Error norms and numerical solutions of Problem 1 for $\Delta t = 0.00025$, $N = 40$, $t = 1$, $\nu = 1$.

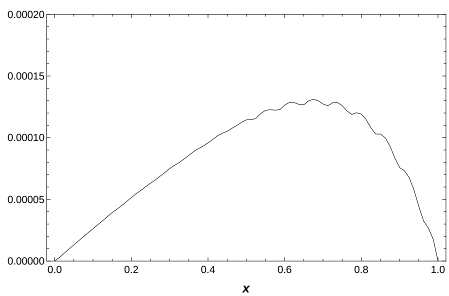
x	$\gamma = 0.10$	$\gamma = 0.25$	$\gamma = 0.75$	$\gamma = 0.90$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	1.105218	1.105217	1.105216	1.105219	1.105171
0.2	1.221497	1.221495	1.221493	1.221497	1.221403
0.3	1.349997	1.349995	1.349990	1.349996	1.349859
0.4	1.492001	1.492000	1.491993	1.492000	1.491825
0.5	1.648930	1.648928	1.648920	1.648928	1.648721
0.6	1.822351	1.822348	1.822339	1.822347	1.822119
0.7	2.013987	2.013984	2.013977	2.013985	2.013753
0.8	2.225751	2.225750	2.225744	2.225751	2.225541
0.9	2.459747	2.459747	2.459744	2.459749	2.459603
1.0	2.718282	2.718282	2.718282	2.718282	2.718282
$L_2 \times 10^3$	0.167077	0.165443	0.159924	0.166085	
$L_\infty \times 10^3$	0.235837	0.232645	0.224523	0.232565	



(a) $\gamma = 0.25$



(b) $\gamma = 0.50$



(c) $\gamma = 0.75$

Figure 1: Error distributions of Problem 1 for $\Delta t = 0.00025$, $N = 80$, $t = 1$, $\nu = 1$.

Problem 2: We secondly consider the Eq. (1), with boundary conditions

$$U(0,t) = t^2\,, \quad U(1,t) = -t^2, \quad t \geq 0$$

and the initial condition as

$$U(x,0) = 0\,, \quad 0 \leq x \leq 1.$$

The term $f(x,t)$ is of the form

$$f(x,t) = \frac{2t^{2-\gamma}\cos(\pi x)}{\Gamma(3-\gamma)} - \pi t^4 \cos(\pi x)\sin(\pi x) + \nu\pi^2 t^2 \cos(\pi x).$$

The exact solution of the problem is given by

$$U(x,t) = t^2 \cos(\pi x).$$

Numerical solutions and the error norms of Problem 2 which are achieved by the presented method for different values of division numbers, time steps, ν and γ are given in Tables 4-7, respectively. When the tables are analyzed, it is easily seen that the numerical solutions converge to exact solution and the error norms L_2 and L_∞ decrease considerably by increasing the number of division number, time step and decreasing the ν. We give the error distributions of this method for different values of γ, $\Delta t = 0.00025$, $N = 80$, $t = 1$, $\nu = 1$ in Fig. 2.

Table 4: Error norms and numerical solutions of Problem 2 for $\gamma = 0.50$, $\Delta t = 0.00025$, $t = 1$, $\nu = 1$.

x	$N = 10$	$N = 20$	$N = 40$	$N = 80$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.951278	0.950847	0.951005	0.951057	0.951057
0.2	0.808287	0.808744	0.808954	0.809019	0.809017
0.3	0.587257	0.587574	0.587738	0.587788	0.587785
0.4	0.308724	0.308910	0.308993	0.309019	0.309017
0.5	0.000000	0.000000	0.000000	0.000000	0.000000
0.6	-0.308724	-0.308909	-0.308996	-0.309020	-0.309017
0.7	-0.587257	-0.587574	-0.587741	-0.587787	-0.587785
0.8	-0.808286	-0.808744	-0.808957	-0.809017	-0.809017
0.9	-0.951277	-0.950847	-0.951008	-0.951060	-0.951057
1.0	-1.000000	-1.000000	-1.000000	-1.000000	-1.000000
$L_2 \times 10^3$	0.435334	0.183000	0.041977	0.001982	
$L_\infty \times 10^3$	0.731099	0.273318	0.063233	0.004192	

Table 5: Error norms and numerical solutions of Problem 2 for $\gamma = 0.50$, $N = 80$, $t = 1$, $\nu = 1$.

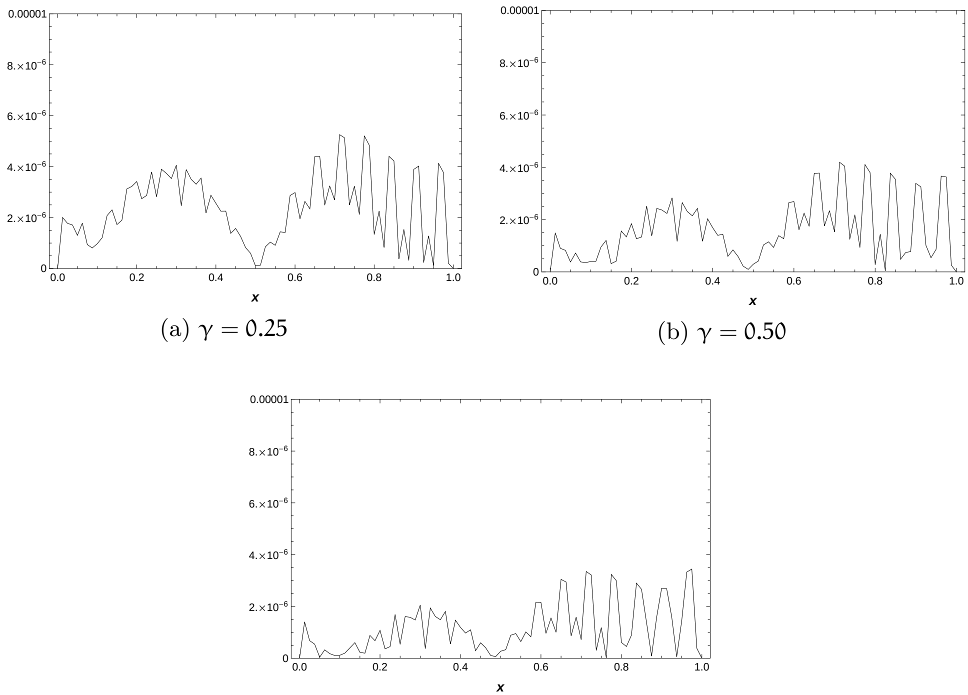
x	$\Delta t = 0.002$	$\Delta t = 0.001$	$\Delta t = 0.0005$	$\Delta t = 0.00025$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.951198	0.951117	0.951078	0.951057	0.951057
0.2	0.809192	0.809093	0.809044	0.809019	0.809017
0.3	0.587927	0.587848	0.587808	0.587788	0.587785
0.4	0.309094	0.309051	0.309030	0.309019	0.309017
0.5	0.000000	0.000000	0.000000	0.000000	0.000000
0.6	-0.309095	-0.309052	-0.309030	-0.309020	-0.309017
0.7	-0.587926	-0.587847	-0.587807	-0.587787	-0.587785
0.8	-0.809191	-0.809092	-0.809042	-0.809017	-0.809017
0.9	-0.951201	-0.951120	-0.951080	-0.951060	-0.951057
1.0	-1.000000	-1.000000	-1.000000	-1.000000	-1.000000
$L_2 \times 10^3$	0.124076	0.054112	0.019282	0.001982	
$L_\infty \times 10^3$	0.175640	0.077491	0.028460	0.004192	

Table 6: Error norms and numerical solutions of Problem 2 for $\gamma = 0.50$, $\Delta t = 0.0005$, $N = 80$, $t = 0.1$.

x	$\nu = 1$	$\nu = 0.5$	$\nu = 0.1$	Exact
0.0	0.010000	0.010000	0.010000	0.010000
0.1	0.009517	0.009517	0.009514	0.009511
0.2	0.008099	0.008098	0.008095	0.008090
0.3	0.005886	0.005885	0.005882	0.005878
0.4	0.003095	0.003094	0.003092	0.003090
0.5	0.000000	0.000000	0.000000	0.000000
0.6	-0.003095	-0.003094	-0.003092	-0.003090
0.7	-0.005886	-0.005885	-0.005882	-0.005878
0.8	-0.008099	-0.008098	-0.008095	-0.008090
0.9	-0.009517	-0.009517	-0.009514	-0.009511
1.0	-0.010000	-0.010000	-0.010000	-0.010000
$L_2 \times 10^3$	0.006442	0.005834	0.003115	
$L_\infty \times 10^3$	0.009009	0.008167	0.004425	

Table 7: Error norms and numerical solutions of Problem 2 for $\Delta t = 0.00025$, $N = 80$, $t = 1$, $\nu = 1$.

x	$\gamma = 0.10$	$\gamma = 0.25$	$\gamma = 0.75$	$\gamma = 0.90$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.951058	0.951058	0.951056	0.951057	0.951057
0.2	0.809021	0.809020	0.809018	0.809019	0.809017
0.3	0.587791	0.587789	0.587787	0.587788	0.587785
0.4	0.309021	0.309020	0.309018	0.309019	0.309017
0.5	0.000000	0.000000	0.000000	0.000000	0.000000
0.6	-0.309020	-0.309020	-0.309019	-0.309020	-0.309017
0.7	-0.587788	-0.587788	-0.587786	-0.587787	-0.587785
0.8	-0.809020	-0.809018	-0.809016	-0.809017	-0.809017
0.9	-0.951061	-0.951060	-0.951059	-0.951060	-0.951057
1.0	-1.000000	-1.000000	-1.000000	-1.000000	-1.000000
$L_2 \times 10^3$	0.003492	0.002733	0.001520	0.001886	
$L_\infty \times 10^3$	0.006455	0.005257	0.003443	0.004065	



(a) $\gamma = 0.25$

(b) $\gamma = 0.50$

(c) $\gamma = 0.75$

Figure 2: Error distributions of Problem 2 for $\Delta t = 0.00025$, $N = 80$, $t = 1$, $\nu = 1$.

Problem 3: Finally, we consider the Eq. (1) with boundary conditions

$$U(0,t)=0\ ,\quad U(1,t)=0,\quad t\geq 0$$

and the initial conditions as

$$U(x,0)=0\ ,\quad 0\leq x\leq 1.$$

The term $f(x,t)$ is of the form

$$f(x,t)=\frac{2t^{2-\gamma}\sin(2\pi x)}{\Gamma(3-\gamma)}+2\pi t^{4}\sin(2\pi x)\cos(2\pi x)+4\nu\pi^{2}t^{2}\sin(2\pi x).$$

The exact solution of the problem is given by

$$U(x,t)=t^{2}\sin(2\pi x).$$

Finally, error norms and numerical solutions for Problem 3 which calculated to test the accuracy of the solutions are given in Tables 8-11. The error norms and numerical solutions for different values of N, $\gamma=0.50$, $\Delta t=0.00025$, $t=1$, $\nu=1$ are presented in Table 8. From the table, it is understood that while the value of N is increasing, the error norms decrease. The results obtained for $\gamma=0.50$, $N=120$, $t=1$, $\nu=1$, different time steps by this method are given in Table 9. From the table, it canbe seen that as Δt time steps decrease, error norms decrease considerably. The tables show us that the numerical solutions are really close to the exact solutions. For $\Delta t=0.0005$, $N=120$, $t=1$, $\nu=1$ and different values of ν numerical solutions and error norms are given in Table 10. It shows us that while the value of ν is decreasing, the error norms decrease substantially. Again, for $\Delta t=0.0005$, $N=120$, $t=1$, $\nu=1$ and different values of γ, the result obtained by the presented method are given in Table 11. The error distributions achieved by the quadratic B-spline Galerkin method for $\Delta t=0.0005$, $N=120$, $t=1$, $\nu=1$ and different values of γ are presented in Fig. 3.

Table 8: Error norms and numerical solutions of Problem 3 for $\gamma = 0.50$, $\Delta t = 0.00025$, $t = 1$, $\nu = 1$.

x	$N = 40$	$N = 50$	$N = 80$	$N = 100$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.585106	0.586153	0.587257	0.587505	0.587785
0.2	0.947079	0.948617	0.950262	0.950638	0.951057
0.3	0.947320	0.948761	0.950310	0.950666	0.951057
0.4	0.585586	0.586434	0.587348	0.587562	0.587785
0.5	0.000001	-0.000002	0.000000	0.000007	0.000000
0.6	-0.585584	-0.586437	-0.587346	-0.587548	-0.587785
0.7	-0.947318	-0.948767	-0.950310	-0.950661	-0.951057
0.8	-0.947078	-0.948621	-0.950260	-0.950631	-0.951057
0.9	-0.585106	-0.586155	-0.587257	-0.587503	-0.587785
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	2.899412	1.774196	0.577143	0.305058	
$L_\infty \times 10^3$	4.063808	2.495647	0.813220	0.430014	

Table 9: Error norms and numerical solutions of Problem 3 for $\gamma = 0.50$, $N = 120$, $t = 1$, $\nu = 1$.

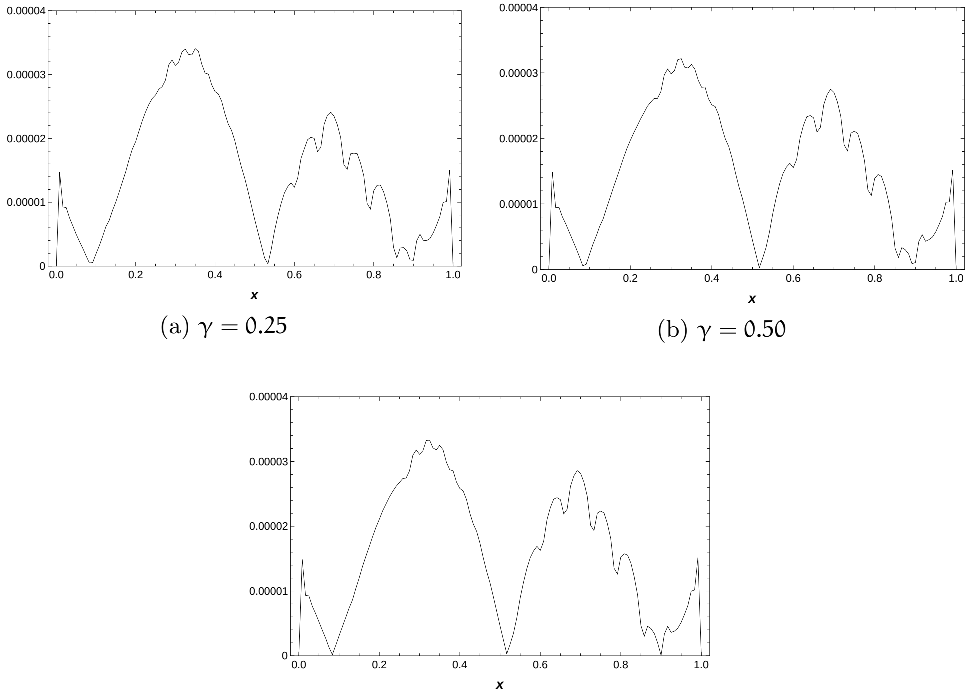
x	$\Delta t = 0.0025$	$\Delta t = 0.002$	$\Delta t = 0.001$	$\Delta t = 0.0005$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.588970	0.588675	0.588083	0.587788	0.587785
0.2	0.952952	0.952484	0.951545	0.951076	0.951057
0.3	0.952914	0.952458	0.951544	0.951086	0.951057
0.4	0.588914	0.588635	0.588087	0.587810	0.587785
0.5	0.000005	0.000005	0.000005	0.000004	0.000000
0.6	-0.588905	-0.588630	-0.588077	-0.587801	-0.587785
0.7	-0.952912	-0.952456	-0.951540	-0.951084	-0.951057
0.8	-0.952949	-0.952479	-0.951540	-0.951070	-0.951057
0.9	-0.588968	-0.588672	-0.588080	-0.587784	-0.587785
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	1.392372	1.048597	0.359489	0.017828	
$L_\infty \times 10^3$	1.974356	1.487805	0.512105	0.032162	

Table 10: Error norms and numerical solutions of Problem 3 for $\gamma = 0.50$, $\Delta t = 0.0005$, $N = 120$, $t = 0.1$.

x	$\nu = 1$	$\nu = 0.5$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.005$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.005902	0.005900	0.005890	0.005879	0.005878	0.005878
0.2	0.009550	0.009546	0.009531	0.009512	0.009510	0.009511
0.3	0.009550	0.009546	0.009531	0.009512	0.009510	0.009511
0.4	0.005902	0.005900	0.005890	0.005878	0.005877	0.005878
0.5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.6	-0.005902	-0.005900	-0.005890	-0.005878	-0.005876	-0.005878
0.7	-0.009550	-0.009546	-0.009531	-0.009512	-0.009510	-0.009511
0.8	-0.009550	-0.009546	-0.009531	-0.009512	-0.009510	-0.009511
0.9	-0.005902	-0.005900	-0.005890	-0.005879	-0.005878	-0.005878
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.029174	0.026666	0.015017	0.001045	0.000758	
$L_\infty \times 10^3$	0.041294	0.037739	0.021269	0.002001	0.002341	

Table 11: Error norms and numerical solutions of Problem 3 for $\Delta t = 0.0005$, $N = 120$, $t = 1$, $\nu = 1$.

x	$\gamma = 0.10$	$\gamma = 0.25$	$\gamma = 0.75$	$\gamma = 0.90$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.586505	0.587787	0.587788	0.587791	0.587785
0.2	0.950362	0.951076	0.951078	0.951082	0.951057
0.3	0.950933	0.951088	0.951088	0.951092	0.951057
0.4	0.587791	0.587813	0.587811	0.587814	0.587785
0.5	0.000000	0.000007	0.000005	0.000004	0.000000
0.6	-0.587833	-0.587798	-0.587802	-0.587804	-0.587785
0.7	-0.951333	-0.951080	-0.951085	-0.951089	-0.951057
0.8	-0.952217	-0.951068	-0.951072	-0.951076	-0.951057
0.9	-0.589827	-0.587784	-0.587785	-0.587788	-0.587785
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.879696	0.017780	0.018641	0.021398	
$L_\infty \times 10^3$	2.051516	0.034072	0.033291	0.037357	



(a) $\gamma = 0.25$

(b) $\gamma = 0.50$

(c) $\gamma = 0.75$

Figure 3: Error distributions of Problem 3 for $\Delta t = 0.0005$, $N = 120$, $t = 1$, $\nu = 1$.

4 Conclusion

In this paper, quadratic B-spline Galerkin method has been applied to acquire the numerical solutions of three problems for the time fractional Burgers equation. The time fractional derivative operator is made allowance for the Caputo fractional derivative in these problems. It can be easily viewed from the numerical solutions and error norms in tables obtained that this is an extremely good method to achieve numerical solutions of time fractional partial differential equations arising in physics and engineering.

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Analytical and numerical results of fractional differential-difference equations

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Abstract. In this paper, we examine the fractional differential-difference equation (FDDE) by employing the proposed sensitivity approach (SA) and Adomian transformation method (ADTM). In SA the nonlinear differential-difference equation is converted to infinite linear equations which have a wide criterion to solve for the analytical solution. By ADTM, the FDDE is converted into ordinary differential-difference equation that can be solved. We test both the techniques through some test problems which are arising in nonlinear dynamical systems and found that ADTM is equivalently appropriate and simpler method to handle than SA.

1 Introduction

A differential-difference equation (DDEs), first studied by Fermi et al. [1] in the 1950s is of enormous significance in describing physical phenomena of various fields such as mechanical engineering, biophysics, condensed matter physics, and in different physical problems like currents flow in electrical networks [2], particle vibrations in lattices, and pulses in biological chains [3]. Many forms of DDEs are discovered to analyze the discrete nonlinear system.

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In recent decades fractional derivatives are introduced to deal with non differentiable functions. The theory of using integrals and derivatives of an arbitrary order, fractional calculus, discussed about 300 years ago, have applications in fractional control of engineering systems, acoustics, damping laws, bioengineering and biomedical applications, electromagnetism, hydrology, signal processing, and many others [4, 5, 6]. It is used for examining stochastic processes forced by fractional Brownian processes [7, 8], non-random fractional phenomena in physics [9, 10, 11], the study of porous systems, and quantum mechanics [12, 13].

Recently, there have been a number of schemes committed to the solution of fractional differential equations. The Adomian decomposition method [14], homotopy perturbation method [15, 16, 17], homotopy analysis method [18, 19], Taylor matrix method [20] and many others have been used to solve the fractional differential equations.

In the present paper, the sensitivity approach [21, 22, 23] which has been presented to solve various kinds of optimal control problems and analysis of time delay systems has used. In this approach, a sensitivity parameter has introduced which transform the original nonlinear fractional differential-difference equation (FDDE) to a linear sequence of FDDEs. The system of equations then now consists of a linear term and a nonlinear series terms. Iterations have been done only for nonlinear series terms, i.e., the result of a sequence of linear FDDEs leads to nonlinear terms for compensation is extended to solve FDDEs. Also, the Adomian decomposition method (ADM) has been modified by a special transformation. The transformation has converted the fractional order differential-difference equation to ordinary differential-difference equation which then solved by the Adomian decomposition method.

2 Preliminaries

The modified Riemann-Liouville derivative of order α is defined, for a function $f(x)$, by

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\eta)^{-1-\alpha} (f(\eta) - f(0)) d\eta; \quad \alpha < 0, \tag{1}$$

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\eta)^{-\alpha} (f(\eta) - f(0)) d\eta; \quad 0 < \alpha < 1, \tag{2}$$

$$D_x^{\alpha} f(x) = \left(f^{(n)}(x)\right)^{(\alpha - n)} d\eta; \quad n \le \alpha \le n+1, \ n \ge 1, \tag{3}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a continuous function. Properties of modified Riemann Liouville derivative are given as

$$D_x^{\alpha} x^{(\beta)} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}; \quad \beta > 0, \tag{4}$$

$$d^{\alpha} x(t) = \Gamma(\alpha+1) dx(t); \quad \beta > 0, \tag{5}$$

3 Implementation of the methods

Consider the nonlinear FDDEs in the form of:

$$D_t^{\alpha} U_n(t) = \mathcal{N}(\ldots, U_{n-1}(t), U_n(t), U_{n+1}(t), \ldots); \tag{6}$$

with initial conditions

$$U_n(t_0) = f(n); \tag{7}$$

where $\mathcal{N}$ is the non linear function,$U_n(t)$ is an unknown function, $f(n)$is the initial condition and t,n are independent variables. Assuming that a unique solution exist for Eq. (4). It is difficult to obtain exact solution of nonlinear FDDE (6). Only in some cases, we can obtain exact solution.

Sensitivity approach (SA)

In this approach, a sensitivity parameter Λ, which varies between zero and unity, is introduced into nonlinear terms of FDDE. When $\Lambda = 0$, the nonlinear problem is transformed to a simple problem, which can be solved through analytic method. When $\Lambda = 1$, the original nonlinear problem is obtained. This transformation leads to solving a sequence of linear FDDEs instead of solving nonlinear FDDEs. Now, we embedded a sensitivity parameter Λ in Eqs. (6)–(7) and form the following sensitized FDDEs:

$$D_t^{\alpha} U_n(t,\Lambda) = \mathcal{N}(..., U_{n-1}(t,\Lambda), U_n(t,\Lambda), U_{n+1}(t,\Lambda), \ldots) \tag{8}$$

with initial conditions

$$U_n(t,\Lambda)\Big|_{t=t_0} = f(n) \tag{9}$$

Where $0 \leq \Lambda \leq 1$. In the following explanation, we suppose that the solution of Eq. (6) is distinctively existed and $U_n(t,\Lambda)$ with Λ is infinitely differentiable with respect to the Λ in the region of $\Lambda = 0$, and its Maclaurin series is convergent at $\Lambda = 1$. Apparently when $\Lambda = 1$ Eq. (8) is equivalent to the original problem Eq. (6). According to the assumption we can write:

$$U_n(t,\Lambda) = \sum_{n=1}^{\infty} \Lambda^j U_n^{(j)}(t) \tag{10}$$

where $(*)^{(j)} = \frac{1}{j!}\frac{\partial^j}{\partial\Lambda^j}(*)\Big|_{\Lambda=0}$ Now, we substitute Eq. (10) into Eq. (8) and equating terms with the same order of Λ on each side we have:

$$\Lambda^0 : D_t^{\alpha} U_n^{(0)}(t) = \mathcal{N}(U_n^{(0)}(t)), \qquad U_n^{(0)}(t_0) = f(n), \tag{11}$$

$$\Lambda^1 : D_t^{\alpha} U_n^{(1)}(t) = \mathcal{N}(U_n^{(1)}(t)) + g_n^{(0)}(t), \qquad U_n^{(1)}(t_0) = 0, \tag{12}$$

$$\Lambda^j : D_t^{\alpha} U_n^{(j)}(t) = \mathcal{N}(U_n^{(j)}(t)) + g_n^{(j-1)}(t), \qquad U_n^{(j)}(t_0) = 0, \tag{13}$$

Where, $g^{(j-1)}(t)$ is the coefficient of $\Lambda^{(j-1)}$ in the expanding of $f(n)$ and can be resolve in the following manner:

$$g^{(j-1)}(t) = \frac{1}{(j-1)!}\frac{\mathcal{N}(..., U_{n-1}(t,\Lambda), U_n(t,\Lambda), U_{n+1}(t,\Lambda), \dots)}{\partial\Lambda(j-1)}\bigg|_{\Lambda=0}, \tag{14}$$

It should be noticed that Eq. (11) gives linear approximate and Eq. (6) gives correction term to linear approximate solution by keeping in consideration second order nonlinearity and so on. If the above process caries on, at each step, a system of non-homogeneous linear FDDEs is obtained in which non-homogeneous terms are known from the previous step. Hence, solving the presented sequence is a recursive process. After indentifying $U^j(t)$ for $j \geq 1$, it is obvious that $\Lambda = 1$ should be set in Eq. (8) and Eq. (9) so that they deform to the exact solution of Eq. (6) and so we have:

$$U_n(t,1) = \sum_{j=0}^{\infty} U_n^{(j)}(t) \tag{15}$$

In this way, the original nonlinear FDDE has been converted into a sequence of linear FDDEs, which should be solved in a recursive development and this will overcomes the complexity of working with nonlinear FDDEs. It is clear from the above procedure, a nonlinear FDDE is transformed into a sequence of linear FDDEs. In order to solve the Eq. (8), the following sensitized linear FDDE can be constructed:

$$D_t^{\alpha} U_n^{(0)}(t,\Lambda) = \Lambda\mathcal{N}(\dots, U_{n-1}^{(0)}(t,\Lambda), U_n^{(0)}(t,\Lambda), U_{n+1}^{(0)}(t,\Lambda), \dots) \tag{16}$$

$$U_n^{(0)}(t,\Lambda) = f(n) \tag{17}$$

Assuming the solution of Eq. (16) as

$$U_n(t,\Lambda) = \sum_{n=1}^{\infty} \Lambda^j U_n^{(0,j)}(t) \tag{18}$$

Now by substituting Eq. (18) into Eq. (16) and equating terms with the same order of Λ on each side we have:

$$\begin{aligned}
\Lambda^0 : D_t^{\alpha} U_n^{(0,0)}(t) &= \mathcal{N}(\dots U_{n-1}^{(0,0)}(t,\Lambda), U_n^{(0,0)}(t,\Lambda), U_{n+1}^{(0,0)}(t,\Lambda), \dots), &(19)\\
U_n^{(0,0)}(t_0) &= f(n), \\
\Lambda^1 : D_t^{\alpha} U_n^{(0,1)}(t) &= \mathcal{N}(\dots U_{n-1}^{(0,1)}(t,\Lambda), U_n^{(0,1)}(t,\Lambda), U_{n+1}^{(0,1)}(t,\Lambda), \dots), &(20)\\
U_n^{(0,1)}(t_0) &= 0, \\
&\vdots \\
\Lambda^j : D_t^{\alpha} U_n^{(0,j)}(t) &= \mathcal{N}(\dots U_{n-1}^{(0,j)}(t,\Lambda), U_n^{(0,j)}(t,\Lambda), U_{n+1}^{(0,j)}(t,\Lambda), \dots), &(21)\\
U_n^{(0,j)}(t_0) &= 0, \\
&\vdots
\end{aligned}$$

By taking the inverse operator of D_t^{α} on applying Eqs. (19)–(21), we get the solution of FDDE as

$$U_n^0(t) = U_n^{(0,1)}(t) = \sum_{j=0}^{\infty} U_n^{(0,j)}(t) \tag{22}$$

In the same way, Eq. (6) has to be solved for $U_n^0(t)$ for $j \geq 1$. After some similar calculation we have:

$$
\begin{aligned}
& U_n^{j,k}(t) = U_n^{(j,0)}(t) + U_n^{(j,1)}(t) + U_n^{(j,2)}(t) + U_n^{(j,3)}(t) + \dots && (23)\\
& U_n^{(j,0)}(t) = 0, \\
& \quad \vdots \\
& U_n^{(j,1)}(t) = -\mathcal{J}_1^{\alpha} g_n^{j-1}(t) && (24)\\
& \quad \vdots \\
& U_n^{(j,k)}(t) = -\mathcal{J}_1^{\alpha}\mathcal{N}(\dots U_{n-1}^{(j,k)}(t), U_n^{(j,k)}(t), U_{n+1}^{(j,k)}(t)\dots)^j.
\end{aligned}
$$

Since, the steps above are enough to find the analytical solution, however, only some iteration of sub-problems and the original problem are sufficient to get a satisfactory accurate solution. We can replace ∞ by any positive integers S and T in the above mentioned series which may help in obtaining an approximate closed-form solution

$$
U_n(t) = \sum_{j=0}^{S} \sum_{k=0}^{T} U_n^{(j,k)}(t) \tag{25}
$$

Adomian decomposition transformation method (ADTM)

Fractional complex transforms [24, 25] has now been become a useful tool to convert fractional differential equations to ordinary differential equations, which provides a very simple and easy solution approach. In the present method, the FDDE has transformed into ordinary DDE and then utilizing the Adomian decomposition method to solve for exact or analytic solutions. Duan [26, 27] has provided the efficient recurrence one variable formula to decompose the multivariable Adomian polynomials to solve the non-linear differential equation. Recalling the Eq. (6)

$$
D_t^{\alpha} U_n(t) = \mathcal{N}(\dots, U_{n-1}(t), U_n(t), U_{n+1}(t), \dots) \tag{26}
$$

with initial conditions

$$
U_n(t_0) = f(n); \tag{27}
$$

Let us suppose

$$
T = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{28}
$$

Differentiating Eq. (28) and making use of modified Riemann Liouville derivative

$$\frac{d^{\alpha}U_n(t)}{dt^{\alpha}} = U_n'(T) \tag{29}$$

Putting in Eq. (26), which transform the FDDE into ordinary DDE

$$U_{n,m}'(T) = \mathcal{L}U_{n,m}(T) = \mathcal{N}(\ldots, U_{n-1,m}(T), U_{n,m}(T), U_{n+1,m}(T), \ldots) \tag{30}$$

where $\mathcal{L} = \frac{d}{dT}$ and $\mathcal{L}^{-1} = \int_{T_0}^{T}(*)dT$ are the linear operator the inverse operators respectively. Applying the inverse operator on both sides of the Eq. (26) with Eq. (27) gives

$$U_{n,m+1}(T) = f(n) + \mathcal{L}^{-1}[\mathcal{N}(\ldots, U_{n-1,m}(T), U_{n,m}(T), U_{n+1,m}(T), \ldots)] \tag{31}$$

In this section, three examples have presented to demonstrate the applicability of the suggested methods to solve nonlinear fractional differential-difference equations.

4 Test problems

Problem 1

Consider the following Volterra equation

$$D_t^{\alpha}U_n(t) = U_n(t)(U_{n+1}(t) - U_{n-1}(t)) \tag{32}$$

with initial conditions

$$U_n(0) = n \tag{33}$$

The exact solution for $\alpha = 1$ can be written as

$$U_n(t) = \frac{n}{(1-2t)}$$

For solving this equation, the following new equation is constructed with sensitivity parameter:

$$D_t^{\alpha}U_n(t, \Lambda) = \Lambda U_n(t, \Lambda)(U_{n+1}(t, \Lambda) - U_{n-1}(t, \Lambda)) \tag{34}$$

with initial conditions

$$U_n(0, \Lambda) = n \tag{35}$$

Now assume:

$$U_n(t,\Lambda) = \sum_{j=0}^{\infty} \Lambda^j U_n^{(0,j)}(t) \tag{36}$$

$$U_{n+1}(t,\Lambda) = \sum_{j=0}^{\infty} \Lambda^j U_{n+1}^{(0,j)}(t) \tag{37}$$

$$U_{n-1}(t,\Lambda) = \sum_{j=0}^{\infty} \Lambda^j U_{n-1}^{(0,j)}(t) \tag{38}$$

Substitute Eq. (36)–(38) in Eq. (34), it has been seen that the nonlinear original FDDEs are changed into a set of linear recursive FDDEs by using Eq. (19)-(21), in which at each step, the non-homogeneous terms are calculated from the preceding steps and process can be handled very simply which can solved the equation, we get the following series solution.

$$U_n(t) = n + \frac{2nt^{\alpha}}{\Gamma(\alpha+1)} + \frac{8nt^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{8nt^{3\alpha}(4\Gamma(\alpha+1)^2 + \Gamma(2\alpha+1))}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \ldots \tag{39}$$

By putting $\alpha = 1$, we recover ref. [24].

Now for the Adomian decomposition transformation method, Let us suppose

$$\eta = \frac{1}{\Gamma(\alpha+1)} t^{\alpha} \tag{40}$$

From modified Riemann Liouville derivative, we have

$$\frac{d^{\alpha}U_n(t)}{dt^{\alpha}} = U_n'(\eta) \tag{41}$$

Using Eq. (41) in Eq. (32)

$$U_{n,m}' = U_{n,m}(\eta)[U_{n+1,m}(\eta) - U_{n-1,m}(\eta)]$$

$$L U_{n,m} = A(U_{n,m}, U_{n+1,m}) - B(U_{n,m}, U_{n-1,m}) \tag{42}$$

Operating L^{-1} on both sides gives:

$$U_{n,m+1} = U_n(0) + L^{-1}(A(U_{n,m}, U_{n+1,m}) - B(U_{n,m}, U_{n-1,m})) \tag{43}$$

Where the nonlinear terms$A(U_{n,m}, U_{n+1,m})$ and $B(U_{n,m}, U_{n-1,m})$ can be decomposed as follows:

$$A_{n,m} = \sum_{i=0}^{m-1} U_{n,i}U_{n+1,m-1-i} \qquad B_{n,m} = \sum_{i=0}^{m-1} U_{n,i}U_{n-1,m-1-i} \tag{44}$$

$$A_{n,1} = U_{n,0}U_{n+1,0}, \quad B_{n,1} = U_{n,0}U_{n-1,0},$$

$$A_{n,2} = U_{n,1}U_{n+1,0} + U_{n,0}U_{n+1,1}, \quad B_{n,2} = U_{n,1}U_{n-1,0} + U_{n,0}U_{n-1,1},$$

$$A_{n,3} = U_{n,2}U_{n+1,0} + U_{n,1}U_{n+1,1} + U_{n,0}U_{n+1,2},$$

$$B_{n,3} = U_{n,2}U_{n-1,0} + U_{n,1}U_{n-1,1} + U_{n,0}U_{n-1,2},$$

$$A_{n,4} = U_{n,3}U_{n+1,0} + U_{n,2}U_{n+1,1} + U_{n,1}U_{n+1,2} + U_{n,0}U_{n+1,3}$$

$$B_{n,4} = U_{n,3}U_{n-1,0} + U_{n,2}U_{n-1,1} + U_{n,1}U_{n-1,2} + U_{n,0}U_{n-1,3} \tag{45}$$

The solution of the transformed problem is

$$U_n = 2n\eta + 4n\eta^2 + 8n\eta^3 + 16n\eta^4 + 32n\eta^5 + 64n\eta^6 + \dots \tag{46}$$

Now, replacing

$$\eta = \frac{1}{\Gamma(\alpha+1)} t^{\alpha}$$

in Eq.(46), we get

$$\begin{aligned} U_n = {} & 2n\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 4n\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^2 + 8n\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^3 \\ & + 16n\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^4 + \dots \end{aligned} \tag{47}$$

Table 1: Numerical comparison of problem 1 at $t = 0.01$

n	$\alpha = 0.5$		$\alpha = 0.75$			$\alpha = 1$	
	SA	ADTM	SA	ADTM	SA	ADTM	Exact
-20	-27.1982	-25.8288	-21.5089	-21.4780	-20.4082	-20.4082	-20.4082
-10	-13.5991	-12.9144	-10.7544	-10.7390	-10.2041	-10.2041	-10.2041
10	13.5991	12.9144	10.7544	10.7390	10.2041	10.2041	10.2041
20	27.1982	25.8288	21.5089	21.4780	20.4082	20.4082	20.4082

Problem 2

Let us consider hybrid nonlinear difference equation of the Korteweg-de Vries (KdV) equations

$$D_t^{\alpha} U_n(t) = (1 - (U_n(t))^2)(U_{n+1}(t) - U_{n-1}(t)) \tag{48}$$

with initial conditions

$$U_n(0) = \tanh(k)\tanh(kn) \tag{49}$$

The exact solution of Eq. (48) for can be written as:

$$U_n(t) = \tanh(k)\tanh(kn + 2\tanh(k)t) \tag{50}$$

Eq. (48) can be simplified as follows:

$$D_t^{\alpha} U_n(t) = (U_{n+1}(t) - U_{n-1}(t)) - (U_n(t))^2 U_{n+1}(t) + (U_n(t))^2 U_{n-1}(t) \tag{51}$$

The solution of Eq. (48) by SA is given by

$$U_n(t) = \tanh(k)\tanh(kn) + \frac{1}{\Gamma(\alpha+1)} t^{\alpha}(\cosh(2k) + \cosh(2kn))\left(\frac{1}{\cosh(kn-k)}\right)$$

$$\left(\frac{1}{(\cosh(kn))^2}\right)\left(\frac{1}{\cosh(kn+k)}\right)(\tanh(k))^2)) + \dots \tag{52}$$

Now, applying the Adomian transformation method the solution of Eq. (48) with Eq. (49) is

$$U_n = \tanh(k)\tanh(kn) + \eta(\cosh(2k) + \cosh(2kn))\left(\frac{1}{\cosh(kn-k)}\right)\left(\frac{1}{(\cosh(kn))^2}\right)$$

$$\left(\frac{1}{\cosh(kn+k)}\right)(\tanh(k))^2\ldots \tag{53}$$

Hence, the solution of the original problem is given by

$$\begin{aligned} U_n &= \tanh(k)\tanh(kn) + \frac{1}{\Gamma(\alpha+1)}t^{\alpha}(\cosh(2k)+\cosh(2kn)) \\ &\left(\frac{1}{\cosh(kn-k)}\right)\left(\frac{1}{(\cosh(kn))^2}\right) \\ &\left(\frac{1}{\cosh(kn+k)}\right)(\tanh(k))^2\ldots \end{aligned} \tag{54}$$

Table 2: Numerical comparison of problem 2 for $k=0.1$ $t=0.01$

n	$\alpha=0.5$		$\alpha=0.75$			$\alpha=1$	
	SA	ADTM	SA	ADTM	SA	ADTM	Exact
-20	-0.291307	-0.291307	-0.291308	-0.291308	-0.291309	-0.291309	-0.291309
-10	-0.289080	-0.289134	-0.289502	-0.289509	-0.289695	-0.289694	-0.289694
0	0.060562	0.059690	0.032840	0.032701	0.016973	0.0169534	0.0169534
10	0.290275	0.290363	0.290150	0.290162	0.290030	0.290030	0.290030
20	0.291310	0.291310	0.291310	0.29131	0.291309	0.291309	0.291309

Problem 3

Consider the following fractional differential-difference problem

$$D_t^{\alpha}U_n(t) = (U_n(t))^2(U_{n+1}(t)-U_{n-1}(t)) \tag{55}$$

with initial condition

$$U_n(0) = 1-\frac{1}{n^2} \tag{56}$$

The exact solution of Eq. (55) for $\alpha=1$ can be written as:

$$U_n(t) = 1-\frac{1}{(n+2t)^2} \tag{57}$$

By the sensitivity approach and ADTM the solution of the problem are given by

$$U_n(t) = 1-\frac{1}{n^2}+\frac{4t^{\alpha}}{n^3\Gamma(\alpha+1)}-\frac{24t^2\alpha}{n^4\Gamma(2\alpha+1)}+\ldots \tag{58}$$

$$U_n(t) = 1-\frac{1}{n^2}+\frac{4t^{\alpha}}{n^3\Gamma(\alpha+1)}-\frac{12t^2\alpha}{n^4(\Gamma(\alpha+1))^2}+\ldots \tag{59}$$

Table 3: Numerical comparison of problem 3 at $t = 0.3$

n	$\alpha = 0.5$		$\alpha = 0.75$			$\alpha = 1$	
	SA	ADTM	SA	ADTM	SA	ADTM	Exact
-20	0.997137	0.99716	0.997259	0.997264	0.997343	0.997343	0.997343
-10	0.986425	0.98698	0.987878	0.987971	0.988683	0.988683	0.988683
10	0.991922	0.992079	0.991510	0.991555	0.988683	0.991100	0.991100
20	0.99777	0.997783	0.997707	0.997707	0.997644	0.997644	0.997644

5 Conclusions

In this study, we extend the sensitivity approach for solving the fractional differential difference equation and proposed a new Adomian decomposition transformation method, and obtain the analytical solution of Volterra and mKDV lattice equations. The solution shows that, both techniques are quite useful for solving a variety of linear and nonlinear fractional problems, but ADTM provides an easy and reliable scheme to be implemented on various problems. The comparison has also been done which is almost approximate to the exact solution. Numerical examples show that the suggested scheme is clearly quite efficient and potent technique in finding the solutions of the proposed equations (see table 1-3).

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Focal representation of k-slant Helices in $\mathbb{E}^{m+1}$

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Abstract. The focal representation of a generic regular curve γ in $\mathbb{E}^{m+1}$ consists of the centers of the osculating hyperplanes. A k-slant helix γ in $\mathbb{E}^{m+1}$ is a (generic) regular curve whose unit normal vector V_k makes a constant angle with a fixed direction $\overrightarrow{U}$ in $\mathbb{E}^{m+1}$. In the present paper we proved that if γ is a k-slant helix in $\mathbb{E}^{m+1}$, then the focal representation C_γ of γ in $\mathbb{E}^{m+1}$ is an $(m-k+2)$-slant helix in $\mathbb{E}^{m+1}$.

1 Introduction

Curves with constant slope, or so-called general helices (inclined curves), are well-known curves in the classical differential geometry of space curves. They are defined by the property that the tangent makes a constant angle with a fixed line (the axis of the general helix) (see, [1], [4], [7] and [8]). In [10], the definition is more restrictive: the fixed direction makes constant angle with these all the vectors of the Frenet frame. It is easy to check that the definition

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only works in the odd dimensional case. Moreover, in the same reference, it is proven that the definition is equivalent to the fact the ratios $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_4}{\kappa_3}, \ldots$, κ_i being the curvatures, are constant. Further, J. Monterde has considered the Frenet curves in $\mathbb{E}^m$ which have constant curvature ratios (i.e., $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3} \ldots$. are constant) [14]. The Frenet curves with constant curvature ratios are called ccr-curves. Obviously, ccr-curves are a subset of generalized helices in the sense of [10]. It is well known that *curves with constant curvatures* (W-curves) are well-known ccr-curves [12], [15].

Recently, Izumiya and Takeuchi have introduced the concept of slant helix in Euclidean 3-space $\mathbb{E}^3$ by requiring that the normal lines make a constant angle with a fixed direction [11]. Further in [3] Ali and Turgut considered the generalization of the concept of slant helix to Euclidean n-space $\mathbb{E}^n$, and gave some characterizations for a non-degenerate slant helix. As a future work they remarked that it is possible to define a slant helix of type-k as a curve whose unit normal vector V_k makes a constant angle with a fixed direction $\overrightarrow{U}$ [9].

For a smooth curve (a source of light) γ in $\mathbb{E}^{m+1}$, the caustic of γ (defined as the envelope of the normal lines of γ) is a singular and stratified hypersurface. The focal curve of γ , C_γ , is defined as the singular stratum of dimension 1 of the caustic and it consists of the centers of the osculating hyperspheres of γ . Since the center of any hypersphere tangent to γ at a point lies on the normal plane to γ at that point, the focal curve of γ may be parametrized using the Frenet frame $(t, n_1, n_2, dots, n_m)$ of γ as follows:

$$C_\gamma(\theta) = (\gamma + c_1 n_1 + c_2 n_2 + \cdots + c_m n_m)(\theta),$$

where the coefficients $c_1, \ldots, c_m$ are smooth functions that are called focal curvatures of γ [18].

This paper is organized as follows: Section 2 gives some basic concepts of the Frenet curves in $\mathbb{E}^{m+1}$. Section 3 tells about the focal representation of a generic curve given with a regular parametrization in $\mathbb{E}^{m+1}$. Further this section provides some basic properties of focal curves in $\mathbb{E}^{m+1}$ and the structure of their curvatures. In the final section we consider k-slant helices in $\mathbb{E}^{m+1}$. We prove that if γ is a k-slant helix in $\mathbb{E}^{m+1}$ then the focal representation C_γ of γ is an $(m-k+2)$-slant helix in $\mathbb{E}^{m+1}$.

2 Basic concepts

Let $\gamma = \gamma(s) : I \to \mathbb{E}^{m+1}$ be a regular curve in $\mathbb{E}^{m+1}$, (i.e., $\|\gamma'(s)\|$ is nowhere zero) where I is an interval in $\mathbb{R}$. Then γ is called a *Frenet curve of osculating* *order* d, $(2 \leq d \leq m+1)$ if $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(d)}(s)$ are linearly independent and $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(d+1)}(s)$ are no longer linearly independent for all s in I [18]. In this case, $Im(\gamma)$ lies in a d-dimensional Euclidean subspace of $\mathbb{E}^{m+1}$. To each Frenet curve of rank d there can be associated the orthonormal d-frame $\{t, n_1, \ldots, n_{d-1}\}$ along γ, the Frenet r-frame, and $d-1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$:I $\longrightarrow \mathbb{R}$, the Frenet curvatures, such that

$$\begin{bmatrix} t' \\ n_1' \\ n_2' \\ \ldots \\ n_{d-1}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \ldots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \ldots & 0 \\ 0 & -\kappa_2 & 0 & \ldots & 0 \\ \ldots & & & & \kappa_{d-1} \\ 0 & 0 & \ldots & -\kappa_{d-1} & 0 \end{bmatrix} \begin{bmatrix} t \\ n_1 \\ n_2 \\ \ldots \\ n_{d-1} \end{bmatrix} \tag{1}$$

where, v is the speed of γ. In fact, to obtain $t, n_1, \ldots, n_{d-1}$ it is sufficient to apply the Gram-Schmidt orthonormalization process to $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(d)}(s)$. Moreover, the functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are easily obtained as by-product during this calculation. More precisely, $t, n_1, \ldots, n_{d-1}$ and $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are determined by the following formulas:

$$\begin{aligned} v_1(s) &: = \gamma'(s) \quad ; t := \frac{v_1(s)}{\|v_1(s)\|}, \\ v_\alpha(s) &: = \gamma^{(\alpha)}(s) - \sum_{i=1}^{\alpha-1} < \gamma^{(\alpha)}(s), v_i(s) > \frac{v_i(s)}{\|v_i(s)\|^2}, \\ \kappa_{\alpha-1}(s) &: = \frac{\|v_\alpha(s)\|}{\|v_{\alpha-1}(s)\| \, \|v_1(s)\|}, \\ n_{\alpha-1} &: = \frac{v_\alpha(s)}{\|v_\alpha(s)\|}, \end{aligned} \tag{2}$$

where $\alpha \in \{2, 3, \ldots, d\}$ (see, [8]).

A Frenet curve of rank d for which $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are constant is called (generalized) screw line or helix [6]. Since these curves are trajectories of the 1-parameter group of the Euclidean transformations, so, F. Klein and S. Lie called them W-*curves* [12]. For more details see also [5]. γ is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \ldots, \frac{\kappa_i}{\kappa_{i-1}}$ $(1 \leq i \leq m-1)$ are constant [14], [15].

3 The focal representation of a curve in $\mathbb{E}^{m+1}$

The hyperplane normal to γ at a point is the union of all lines normal to γ at that point. The envelope of all hyperplanes normal to γ is thus a component of the focal set that we call the main component (the other component is the curve γ itself, but we will not consider it) [16].

Definition 1 *Given a generic curve (i.e., a Frenet curve of osculating order* $m+1$*)* $\gamma : \mathbb{R} \to \mathbb{E}^{m+1}$*, let* $F : \mathbb{E}^{m+1} \times \mathbb{R} \to \mathbb{R}$ *be the* $(m+1)$*-parameter family of real functions given by*

$$F(q, \theta) = \frac{1}{2} \left\| q - \gamma(\theta) \right\|^2 . \tag{3}$$

The caustic of the family F *is given by the set*

$$\left\{ q \in \mathbb{E}^{m+1} : \exists \theta \in \mathbb{R} : F'_q(\theta) = 0 \text{ and } F''_q(\theta) = 0 \right\} \tag{4}$$

[16].

Proposition 1 *[17] The caustic of the family* $F(q, \theta) = \frac{1}{2} \left\| q - \gamma(\theta) \right\|^2$ *coincides with the focal set of the curve* $\gamma : \mathbb{R} \to \mathbb{E}^{m+1}$.

Definition 2 *The center of the osculating hypersphere of* γ *at a point lies in the hyperplane normal to the* γ *at that point. So we can write*

$$C_\gamma = \gamma + c_1 n_1 + c_2 n_2 + \cdots + c_m n_m, \tag{5}$$

which is called the focal curve of γ*, where* $c_1, c_2, \ldots, c_m$ *are smooth functions of the parameter of the curve* γ*. We call the function* c_i *the* i^{th} *focal curvature of* γ*. Moreover, the function* c_1 *never vanishes and* $c_1 = \frac{1}{\kappa_1}$ *[18].*

The focal curvatures of γ, parametrized by arc length s, satisfy the following "scalar Frenet equations" for $c_m \neq 0$:

$$\begin{aligned}
1 &= \kappa_1 c_1 \\
c_1 &= \kappa_2 c_2 \\
c_2 &= -\kappa_2 c_1 + \kappa_3 c_3 \\
&\cdots \\
c_{m-1} &= -\kappa_{m-1} c_{m-2} + \kappa_m c_m \\
c_m - \frac{(R_m^2)'}{2c_m} &= -\kappa_m c_{m-1}
\end{aligned} \tag{6}$$

where R_m is the radius of the osculating m-sphere. In particular $R_m^2 = \|C_\gamma - \gamma\|^2$ [18].

Theorem 1 [16] *Let $\gamma : s \to \gamma(s) \in \mathbb{E}^{m+1}$ be a regular generic curve. Write $\kappa_1, \kappa_2, \dots, \kappa_m$ for its Euclidean curvatures and $\{t, n_1, n_2, \dots, n_m\}$ for its Frenet Frame. For each non-vertex $\gamma(s)$ of γ, write $\varepsilon(s)$ for the sign of $(c'_m + c_{m-1}\kappa_m)(s)$ and $\delta_\alpha(s)$ for the sign of $(-1)^\alpha \varepsilon(s)\kappa_m(s)$, $\alpha = 1, \dots, m$. Then the following holds:*

a) *The Frenet frame $\{T, N_1, N_2, \dots, N_m\}$ of C_γ at $C_\gamma(s)$ is well-defined and its vectors are given by $T = \varepsilon n_m$, $N_\alpha = \delta_\alpha n_{m-l}$, for $l = 1, \dots, m-1$, and $N_m = \pm t$. The sign in $\pm t$ is chosen in order to obtain a positive basis.*

b) *The Euclidean curvatures $K_1, K_2, \dots, K_m$ of the parametrized focal curve of γ, $C_\gamma : s \to C_\gamma(s)$, are related to those of γ by:*

$$\frac{K_1}{|\kappa_m|} = \frac{K_2}{\kappa_{m-1}} = \dots = \frac{|K_m|}{\kappa_1} = \frac{1}{|c'_m + c_{m-1}\kappa_m|}, \tag{7}$$

the sign of K_m is equal to δ_m times the sign chosen in $\pm t$.

That is the Frenet formulas of C_γ at $C_\gamma(s)$ are

$$\begin{aligned} T' &= \frac{1}{A}|\kappa_m| N_1 \\ N'_1 &= \frac{1}{A}(-|\kappa_m| T + \kappa_{m-1} N_2) \\ N'_2 &= \frac{1}{A}(-|\kappa_{m-1}| N_1 + \kappa_{m-2} N_3) \\ &\dots \\ N'_{m-1} &= \frac{1}{A}(-\kappa_2 N_{m-2} \mp \delta_m \kappa_1 N_m) \\ N'_m &= \frac{1}{A} \mp \delta_m \kappa_1 N_{m-1} \end{aligned} \tag{8}$$

where $A = |c'_m + c_{m-1}\kappa_m|$.

Corollary 1 *Let $\gamma = \gamma(s)$ be a regular generic curve in $\mathbb{E}^{m+1}$ and $C_\gamma : s \to C_\gamma(s)$ be the focal representation of γ. Then the Frenet frame of C_γ becomes as follows;*

i) *If m is even, then*

$$\begin{aligned} T &= n_m \\ N_1 &= -n_{m-1} \\ N_2 &= n_{m-2} \\ &\dots \\ N_{m-1} &= -n_1 \\ N_m &= t \end{aligned} \tag{9}$$

ii) *If* m *is odd, then*

$$\begin{aligned} T &= n_m \\ N_1 &= -n_{m-1} \\ N_2 &= n_{m-2} \\ &\ldots \\ N_{m-1} &= n_1 \\ N_m &= -t. \end{aligned} \tag{10}$$

Proof. By the use of (7) with (8) we get the result. □

4 k-Slant helices

Let $\gamma = \gamma(s) : I \to \mathbb{E}^{m+1}$ be a regular generic curve given with arclength parameter. Further, let $\overrightarrow{U}$ be a unit vector field in $\mathbb{E}^{m+1}$ such that for each $s \in I$ the vector $\overrightarrow{U}$ is expressed as the linear combinations of the orthogonal basis $\{V_1(s), V_2(s), \ldots, V_{m+1}(s)\}$ with

$$\overrightarrow{U} = \sum_{j=1}^{m+1} a_j(s)V_j(s). \tag{11}$$

where $a_j(s)$ are differentiable functions, $1 \leq j \leq m+1$.

Differentiating $\overrightarrow{U}$ and using the Frenet equations (1), one can get

$$\frac{d\overrightarrow{U}}{ds} = \sum_{i=1}^{m+1} P_i(s)V_i(s), \tag{12}$$

where

$$\begin{aligned} P_1(s) &= a_1^{'} - \kappa_1 a_2, \qquad (13) \\ P_i(s) &= a_i^{'} + \kappa_{i-1}a_{i-1} - \kappa_i a_{i+1}, \ 2 \leq i \leq m, \\ P_{m+1}(s) &= a_{m+1}^{'} + \kappa_m a_m. \end{aligned}$$

If the vector field $\overrightarrow{U}$ is constant then the following system of ordinary differential equations are obtained

$$\begin{aligned} 0 &= a_1^{'} - \kappa_1 a_2, \\ 0 &= a_2^{'} + \kappa_1 a_1 - \kappa_2 a_3, \\ 0 &= a_i^{'} + \kappa_{i-1}a_{i-1} - \kappa_i a_{i+1}, \ 3 \leq i \leq m, \\ 0 &= a_{m+1}^{'} + \kappa_m a_m. \end{aligned} \tag{14}$$

Definition 3 *Recall that a unit speed generic curve* $\gamma = \gamma(s) : I \rightarrow \mathbb{E}^{m+1}$ *is called a* k*-type slant helix if the vector field* V_k $(1 \leq k \leq m+1)$ *makes a constant angle* θ_k *with the fixed direction* $\overrightarrow{U}$ *in* $\mathbb{E}^{m+1}$*, that is*

$$< \overrightarrow{U}, V_k >= \cos\theta_k, \ \ \theta_k \neq \frac{\pi}{2}. \tag{15}$$

A 1-type slant helix is known as cylindrical helix [2] or generalized helix [13], [4]. For the characterization of generalized helices in $(n+2)$-dimensional Lorentzian space $\mathbb{L}^{n+2}$ see [19].

We give the following result;

Theorem 2 *Let* $\gamma = \gamma(s)$ *be a regular generic curve in* $\mathbb{E}^{m+1}$*. If* $C_\gamma : s \rightarrow C_\gamma(s)$ *is the focal representation of* γ *then the following statements are valid;*

i) If γ *is a* 1*-slant helix then the focal representation* C_γ *of* γ *is an* $(m+1)$*-slant helix in* $\mathbb{E}^{m+1}$.

ii) If γ *is an* $(m+1)$*-slant helix then the focal representation* C_γ *of* γ *is a* 1*-slant helix in* $\mathbb{E}^{m+1}$.

iii) If γ *is a k-slant helix* $(2 < k < m)$ *then the focal representation* C_γ *of* γ *is an* $(m-k+2)$*-slant helix in* $\mathbb{E}^{m+1}$.

Proof. i) Suppose γ is a 1-slant helix in $\mathbb{E}^{m+1}$. Then by Definition 3 the vector field V_1 makes a constant angle θ_1 with the fixed direction $\overrightarrow{U}$ defined in (11), that is

$$< \overrightarrow{U}, V_1 >= \cos\theta_1, \ \ \theta_1 \neq \frac{\pi}{2}. \tag{16}$$

For the focal representation $C_\gamma(s)$ of γ, we can choose the orthogonal basis

$$\{V_1(s) = t, V_2(s) = n_1, \ldots, V_{m+1}(s) = n_m\}$$

such that the equalities (9) or (10) is hold. Hence, we get,

$$< \overrightarrow{U}, V_1 >=< \overrightarrow{U}, t >=< \overrightarrow{U}, \pm N_m >= \text{cons}. \tag{17}$$

where $\{T, N_1, N_2, \ldots, N_m\}$ is the Frenet frame of C_γ at point $C_\gamma(s)$. From the equality (17) it is easy to see that C_γ is an (m+1)-slant helix of $\mathbb{E}^{m+1}$.

ii) Suppose γ is an $(m+1)$-slant helix in $\mathbb{E}^{m+1}$. Then by Definition 3 the vector field V_{m+1} makes a constant angle θ_{m+1} with the fixed direction $\overrightarrow{U}$ defined in (11), that is

$$< \overrightarrow{U}, V_{m+1} >= \cos\theta_{m+1}, \ \ \theta_{m+1} \neq \frac{\pi}{2}. \tag{18}$$

For the focal representation $C_\gamma(s)$ of γ, one can get

$$< \overrightarrow{U}, V_{m+1} >=< \overrightarrow{U}, n_m >=< \overrightarrow{U}, T >= cons. \tag{19}$$

where $\{V_1 = t, V_2 = n_1, \ldots, V_{m+1} = n_m\}$ and $\{T, N_1, N_2, \ldots, N_m\}$ are the Frenet frame of γ and C_γ, respectively. From the equality (19) it is easy to see that C_γ is a 1-slant helix of $\mathbb{E}^{m+1}$.

iii) Suppose γ is a k-slant helix in $\mathbb{E}^{m+1}$ $(2 \leq k \leq m)$. Then by Definition 3 the vector field V_k makes a constant angle θ_k with the fixed direction $\overrightarrow{U}$ defined in (11), that is

$$< \overrightarrow{U}, V_k >= \cos\theta_k,\ \theta_k \neq \frac{\pi}{2},\ 2 \leq k \leq m. \tag{20}$$

Let $C_\gamma(s)$ be the focal representation of γ. Then using the equalities (9) or (10) we get

$$< \overrightarrow{U}, V_k >=< \overrightarrow{U}, n_{k-1} >=< \overrightarrow{U}, N_{m-k+1} >= cons.,\ 2 \leq k \leq m \tag{21}$$

where

$$\{V_1 = t, V_2 = n_1, \ldots, V_{m+1} = n_m\}$$

and

$$\left\{\widetilde{V}_1 = T, \widetilde{V}_2 = N_1, \ldots, \widetilde{V}_{m-k+2} = N_{m-k+1}, \ldots, \widetilde{V}_{m+1} = N_m\right\}$$

are the Frenet frame of γ and C_γ, respectively. From the equality (21) it is easy to see that C_γ is an $(m-k+2)$-slant helix of $\mathbb{E}^{m+1}$. □

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Bounds on third Hankel determinant for close-to-convex functions

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Abstract. In this paper, we have obtained upper bound on third Hankel determinant for the functions belonging to the class of close-to-convex functions.

1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z : \ |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{H}(\mathbb{U})$ which are normalized by $f(0) = 0,\ f'(0) = 1$ and have the following form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \quad z \in \mathbb{U}. \tag{1}$$

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We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$. Let $\mathcal{P}$ be the class of all functions $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = 1$ and $\Re(p(z)) > 0$. The function $p \in \mathcal{P}$ have the following form:

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots, \quad z \in \mathbb{U}. \tag{2}$$

Further, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}. \tag{3}$$

Moreover, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{C}$ of close-to-convex functions in $\mathbb{U}$, if there exist a function $g \in \mathcal{S}^*$, such that the following inequality holds:

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{U}. \tag{4}$$

The class of close-to-convex functions was introduced by Kaplan [9]. In [16], Noonan and Thomas studied the q^{th} Hankel determinants $H_q(n)$ of functions $f \in \mathcal{A}$ of the form (1) for $q \geq 1$, which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \dots & \dots & a_{n+2(q-1)} \end{vmatrix} \qquad (a_1 = 1). \tag{5}$$

The Hankel determinants $H_q(n)$ have been investigated by several authors to study its rate of growth as $n \to \infty$ and to determine bounds on it for specific values of q and n. For example, Pommerenke [22] proved that the Hankel determinants of univalent functions satisfy $|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$ ($n = 1, 2, \dots,$ $q = 2, 3, \dots$), where $\beta > 1/4000$ and K depends only on q. Later, Hayman [8] proved that $|H_2(n)| < A\, n^{1/2}$ ($n = 1, 2, \dots;$ A is an absolute constant) for areally mean univalent functions. Pommerenke [21] investigated the Hankel determinant of areally mean p-valent functions, univalent functions as well as of starlike functions. Ehrenborg studied Hankel determinant of the exponential polynomials [6] and Noor studied Hankel determinant for Bazilevic functions in [18] and for functions with bounded boundary rotations in [17, 19] also for close-to-convex functions in [20].

A classical theorem of Fekete and Szegö [7] considered the second Hankel determinant $H_2(1) = a_3 - a_2^2$ for univalent functions. They made an early study for the estimate of well known *Fekete-Szegö functional* $\left|a_3 - \mu a_2^2\right|$ when μ is real. Jenteng [12] investigated the sharp upper bound for second Hankel determinant $|H_2(2)| = |a_2 a_4 - a_3^2|$ for univalent functions whose derivative has positive real part. Recently, Lee *et al.* [13] have obtained bounds on $|H_2(2)|$ for functions belonging to the subclasses of Ma-Minda starlike and convex functions. Further Bansal [2] have obtained bounds on $|H_2(2)|$ for some new class of analytic functions. Recently, Babalola [1], Raza and Malik [24] and Bansal *et al.* [3] have studied third Hankel determinant $H_3(1)$, for various classes of analytic and univalent functions. In the present paper we investigate the upper bound on $|H_3(1)|$ for the functions belonging to the class of close-to-convex functions $\mathcal{K}$ defined by (4). To derive our results, we shall need the following Lemmas:

Lemma 1 (Carathéodory's Lemma [4], see also [5, p. 41]). *Let the function* $p \in \mathcal{P}$ *be given by the series then the sharp estimate* $|c_n| \leq 2, \ n = 1, 2, \cdots$ *holds. The inequality is sharp for each* n.

Lemma 2 (cf. [14, p. 254], see also [15]). *Let the function* $p \in \mathcal{P}$ *be given by* (2)*, then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some x, $|x| \leq 1$, *and*

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some z, $|z| \leq 1$.

Lemma 3 ([5, p. 44]). *If* $f \in \mathcal{S}^*$ *be given by* (1)*, then* $|a_n| \leq n$ $(n = 2, 3, \dots)$. *Strict inequality holds for all* n *unless* f *is rotation of the Koebe function* $k(z) = z/(1 - z)^2$.

Lemma 4 ([23]). *If* $f \in \mathcal{C}$ *be given by* (1)*, then* $|a_n| \leq n$ $(n = 2, 3, \dots)$. *Equality holds for all* n *when* f *is rotation of the Koebe function.*

Lemma 5 ([10]). *If* $f \in \mathcal{S}^*$ *be given by* (1)*, then for any real number* μ*, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if} \quad \mu \leq \frac{1}{2} \\ 1, & \text{if} \quad \frac{1}{2} \leq \mu \leq 1 \\ 4\mu - 3, & \text{if} \quad \mu \geq 1. \end{cases}$$

Lemma 6 ([11]). *If* $f \in \mathcal{C}$ *be given by* (1)*, then* $|a_3 - a_2^2| \leq 1$. *There is a function in* $\mathcal{C}$ *such that equality holds.*

Lemma 7 ([12]). *If* $f \in \mathcal{S}^*$ *be given by* (1), *then* $|a_2a_4 - a_3^2| \le 1$. *Equality is attended for the the Koebe function.*

Lemma 8 ([1]). *If* $f \in \mathcal{S}^*$ *be given by* (1), *then* $|a_2a_3 - a_4| \le 2$. *Equality is attained by Koebe function.*

2 Main results

Our first main result is contained in the following theorem:

Theorem 1 *Let the function* $f \in \mathcal{C}$ *be given by* (1), *then*

$$|a_2a_3 - a_4| \le 3. \tag{6}$$

Proof. Let the function $f \in \mathcal{C}$ be given by (6), then from the definition, we have

$$zf'(z) = g(z)p(z), \quad z \in \mathbb{U}, \tag{7}$$

for $p(z) \in \mathcal{P}$. The function $g(z)$ in (7) is a starlike function and let it have the form $g(z) = z + b_2z^2 + b_3z^3 + \dots$. Substituting the valves of $f(z), g(z)$ and $p(z)$ and equating the coefficients, we get

$$2a_2 = b_2 + c_1 \tag{8}$$
$$3a_3 = b_3 + b_2c_1 + c_2 \tag{9}$$
$$4a_4 = b_4 + b_3c_1 + b_2c_2 + c_3. \tag{10}$$

Now

$$\begin{aligned} |a_2a_3 - a_4| &= \left| \frac{b_2 + c_1}{2} \frac{b_3 + b_2c_1 + c_2}{3} - \frac{b_4 + b_3c_1 + b_2c_2 + c_3}{4} \right| \\ &= \left| \frac{1}{4}(b_2b_3 - b_4) - \frac{c_1}{12}(b_3 - 2b_2^2) - \frac{1}{12}b_2b_3 + \frac{1}{6}b_2c_1^2 \right. \\ &\qquad \left. + \left(\frac{c_1}{6} - \frac{b_2}{12}\right) c_2 - \frac{c_3}{4} \right| \end{aligned} \tag{11}$$

Substituting values of c_2 and c_3 by Lemma 2 in (11), we get

$$\begin{aligned} |a_2a_3 - a_4| = &\left| \frac{1}{4}(b_2b_3 - b_4) - \frac{c_1}{12}(b_3 - 2b_2^2) - \frac{1}{12}b_2b_3 \right. \\ &+ \frac{1}{6}b_2c_1^2 + \left(\frac{c_1}{6} - \frac{b_2}{12}\right) \frac{c_1^2 + (4 - c_1^2)x}{2} \\ &\left. - \frac{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z}{16} \right| \end{aligned}$$

$$= \left| \frac{1}{4}(b_2b_3 - b_4) - \frac{c_1}{12}(b_3 - 2b_2^2) - \frac{1}{12}b_2b_3 + \frac{1}{48}c_1^3 - \frac{1}{24}c_1(4-c_1^2)x + \frac{1}{8}b_2c_1^2 \right.$$
$$\left. -\frac{1}{24}b_2(4-c_1^2)x + \frac{1}{16}c_1(4-c_1^2)x^2 - \frac{1}{8}(4-c_1^2)(1-|x|^2)z \right|$$

By Lemma 1, we have $|c_1| \leq 2$. For convenience of notation, we take $c_1 = c$ and we may assume without loss of generality that $c \in [0, 2]$. Applying the triangle inequality with $\mu = |x|$ and using Lemma 3, Lemma 5 and Lemma 8, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1}{4}|b_2b_3 - b_4| + \frac{1}{12}c|b_3 - 2b_2^2| + \frac{1}{12}|b_2||b_3| + \frac{1}{48}c^3 + \frac{1}{8}|b_2|c^2 \\ &\quad + \frac{1}{24}(4-c^2)(c+|b_2|)\mu + \frac{c}{16}(4-c^2)\mu^2 + \frac{1}{8}(4-c^2)(1-\mu^2) \\ &\leq \frac{3}{2} + \frac{5}{12}c + \frac{1}{8}c^2 + \frac{1}{48}c^3 + \frac{1}{24}(4-c^2)(c+2)\mu \\ &\quad + \frac{1}{16}(4-c^2)(c-2)\mu^2 = F_1(c,\mu). \end{aligned} \tag{12}$$

Differentiating $F_1(c, \mu)$ partially with respect to c, we have

$$\begin{aligned} \frac{\partial F_1}{\partial c} &= \frac{5}{12} + \frac{c}{4} + \frac{c^2}{16} + \frac{\mu}{24}(4 - 3c^2 - 4c) + \frac{\mu^2}{16}(4 - 3c^2 + 4c) \\ &= \frac{1}{12}(5 - \mu c^2) + \frac{c}{12}(3 - 2\mu) + \frac{c^2}{16} + \frac{\mu}{24}(4 - c^2) + \frac{\mu^2}{16}(2-c)(3c+2) > 0, \end{aligned}$$

for $c \in [0, 2]$ and for any fixed μ with $\mu \in [0, 1]$. Therefore $F_1(c, \mu)$ is an increasing function of c on the closed interval $[0, 2]$, and hence $F_1(c, \mu)$ attained its maximum value at $c = 2$. Thus

$$\max_{0 \leq c \leq 2} F_1(c,\mu) = F_1(2,\mu) = G_1(\mu) \ (say). \tag{13}$$

From (12) and (13), we get $G_1(\mu) = 3$, which is independent of μ. Hence, the sharp upper bound of the functional $|a_2a_3 - a_4|$ can be obtained by setting $c = 2$ in (12), therefore

$$|a_2a_3 - a_4| \leq 3.$$

This completes the proof of Theorem 1. □

Theorem 2 *Let the function* $f \in \mathcal{C}$ *be given by* (1), *then*

$$H_2(2) = |a_2a_4 - a_3^2| \leq \frac{85}{36}. \tag{14}$$

Proof. Let $f \in \mathcal{C}$ of the form (1), then following the proof of Theorem 1, we get values of a_2, a_3 and a_4 given in (8)-(10). Using these values, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \left| \frac{b_2+c_1}{2}.\frac{b_4+b_3c_1+b_2c_2+c_3}{4} - \left(\frac{b_3+b_2c_1+c_2}{3}\right)^2 \right| \\
&= \left| \frac{1}{8}b_2b_4 - \frac{7}{72}b_2b_3c_1 + \frac{1}{8}b_2^2c_2 + \frac{1}{8}b_2c_3 + \frac{1}{8}b_3c_1^2 - \frac{7}{72}b_2c_1c_2 \right. \\
&\quad \left. + \frac{1}{8}b_4c_1 + \frac{1}{8}c_1c_3 - \frac{1}{9}b_3^2 - \frac{1}{9}b_2^2c_1^2 - \frac{1}{9}c_2^2 - \frac{2}{9}b_3c_2 \right| \\
&= \left| \frac{1}{8}(b_4 - b_2b_3)c_1 + \frac{1}{8}\left(b_3 - \frac{8}{9}b_2^2\right)c_1^2 + \frac{1}{8}(b_2b_4 - b_3^2) \right. \\
&\quad -\frac{2}{9}\left(b_3 - \frac{9}{16}b_2^2\right)c_2 + \frac{1}{36}b_2b_3c_1 \\
&\quad \left. + \frac{1}{8}b_2c_3 - \frac{7}{72}b_2c_1c_2 + \frac{1}{8}c_1c_3 + \frac{1}{72}b_3^2 - \frac{1}{9}c_2^2 \right|
\end{aligned}$$

Substituting the values of c_2 and c_3 from Lemma 2 in above equation, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \left| \frac{1}{8}(b_4 - b_2b_3)c_1 + \frac{1}{8}(b_3 - \frac{8}{9}b_2^2)c_1^2 + \frac{1}{8}(b_2b_4 - b_3^2) \right. \\
&\quad - \frac{1}{9}(b_3 - \frac{9}{16}b_2^2)(c_1^2 + x(4-c_1^2)) + \frac{1}{36}b_2b_3c_1 + \frac{1}{72}b_3^2 \\
&\quad - \frac{7}{144}b_2c_1(c_1^2 + x(4-c_1^2)) - \frac{1}{36}(c_1^2 + x(4-c_1^2))^2 \\
&\quad + \frac{1}{32}(b_2+c_1)[c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 \\
&\quad \left. +2(1-|x|^2)(4-c_1^2)z] \right| \\
&= \left| \frac{1}{8}(b_4 - b_2b_3)c_1 + \frac{1}{8}(b_3 - \frac{8}{9}b_2^2)c_1^2 + \frac{1}{8}(b_2b_4 - b_3^2) \right. \\
&\quad - \frac{1}{9}(b_3 - \frac{9}{16}b_2^2)c_1^2 - \frac{1}{9}(b_3 - \frac{9}{16}b_2^2)(4-c_1^2)x + \frac{1}{36}b_2b_3c_1 \\
&\quad + \frac{1}{72}b_3^2 - \frac{5}{288}b_2c_1^3 + \frac{1}{288}{c_1}^4 + \frac{1}{72}b_2c_1(4-c_1^2)x + \frac{1}{144}c_1^2x(4-c_1^2) \\
&\quad -\frac{1}{36}x^2(4-c_1^2)^2 - \frac{1}{32}c_1b_2x^2(4-c_1^2) - \frac{1}{32}c_1^2(4-c_1^2)x^2 \\
&\quad \left. + \frac{1}{16}(b_2+c_1)(4-c_1^2)(1-|x|^2)z \right|
\end{aligned}$$

By Lemma 1, we have $|c_1| \leq 2$. For convenience of notation, we take $c_1 = c$ and we may assume without loss of generality that $c \in [0, 2]$. Applying the triangle inequality in above equation with $\mu = |x|$ and using Lemma 3, Lemma 5, Lemma 7 and Lemma 8, we obtain

$$
\begin{aligned}
|a_2a_4 - a_3^2| \leq\ & \frac{1}{8}|b_4 - b_2b_3|c + \frac{1}{8}|b_3 - \frac{8}{9}b_2^2|c^2 + \frac{1}{8}|b_2b_4 - b_3^2| + \frac{1}{9}|b_3 \\
& - \frac{9}{16}b_2^2|c^2 + \frac{1}{9}|b_3 - \frac{9}{16}b_2^2|(4-c^2)\mu + \frac{1}{36}|b_2||b_3|c + \frac{1}{72}|b_3|^2 \\
& + \frac{5}{288}|b_2|c^3 + \frac{1}{288}c^4 + \frac{1}{72}|b_2|c(4-c^2)\mu + \frac{1}{144}c^2(4-c^2)\mu \\
& + \frac{1}{36}(4-c^2)^2\mu^2 + \frac{1}{32}|b_2|c(4-c^2)\mu^2 + \frac{1}{32}c^2\mu^2(4-c^2) \\
& + \frac{1}{16}(|b_2| + c)(4-c^2)(1-\mu^2) \\
\leq\ & \frac{1}{4}c + \frac{1}{8}c^2 + \frac{1}{8} + \frac{1}{9}c^2 + \frac{1}{9}(4-c^2)\mu + \frac{1}{6}c + \frac{1}{8} + \frac{5}{144}c^3 \\
& + \frac{1}{288}c^4 + \frac{1}{36}c(4-c^2)\mu + \frac{1}{144}c^2(4-c^2)\mu + \frac{1}{36}(4-c^2)^2\mu^2 \\
& + \frac{1}{16}c(4-c^2)\mu^2 + \frac{1}{32}c^2\mu^2(4-c^2) + \frac{1}{16}(2+c)(4-c^2)(1-\mu^2) \\
=\ & \frac{3}{4} + \frac{2}{3}c + \frac{1}{9}c^2 - \frac{1}{36}c^3 + \frac{1}{288}c^4 + \mu(4-c^2)\left(\frac{1}{9} + \frac{1}{36}c + \frac{1}{144}c^2\right) \\
& + \frac{1}{288}(c^2-4)(4-c^2)\mu^2 = F_2(c,\mu)
\end{aligned}
\tag{15}
$$

Differentiating $F_2(c, \mu)$ in above equation with respect to μ, we get

$$
\begin{aligned}
\frac{\partial F_2}{\partial \mu} &= \left(\frac{1}{9} + \frac{1}{36}c + \frac{1}{144}c^2\right)(4-c^2) + \frac{1}{144}(c^2-4)(4-c^2)\mu \\
&= \left(\frac{1}{36}(4-\mu) + \frac{1}{36}c + \frac{1}{144}c^2 + \frac{1}{144}\mu c^2\right)(4-c^2) > 0 \quad \text{for} \quad 0 \leq \mu \leq 1.
\end{aligned}
$$

Therefore $F_2(c, \mu)$ is an increasing function of μ for $0 \leq \mu \leq 1$ and for any fixed c with $c \in [0, 2]$. Hence it attains maximum value at $\mu = 1$. Thus

$$
\max_{0 \leq \mu \leq 1} F_2(c,\mu) = F_2(c,1) = G_2(c) \ \ (\text{say}). \tag{16}
$$

Therefore from (15) and (16), we have

$$
G_2(c) = \frac{1}{144}(164 + 112c + 8c^2 - 8c^3 - c^4). \tag{17}
$$

Now

$$\begin{aligned} G_2'(c) &= \frac{1}{36}[28 + 4c - 6c^2 - c^3] \\ &= \frac{1}{36}[4 + (6 + c)(4 - c^2)] > 0 \quad \text{for} \quad c \in [0, 2]. \end{aligned}$$

This shows that $G_2(c)$ is an increasing function of c, hence it will attains maximum value at $c = 2$. Therefore

$$\max_{0 \le c \le 2} G_2(c) = G_2(2) = \frac{85}{36}.$$

Hence the upper bound on $|a_2a_4 - a_3^2|$ can bee obtained by setting $\mu = 1$ and $c = 2$ in (15) or $c = 2$ in (17), therefore

$$|a_2a_4 - a_3^2| \le \frac{85}{36}.$$

□

Theorem 3 *Let the function* $f \in \mathcal{C}$ *be given by* (1), *then*

$$|H_3(1)| \le \frac{289}{12}. \tag{18}$$

Proof. Let $f \in \mathcal{C}$ of the form (1), then by definition $H_3(1)$ is given by

$$\begin{aligned} H_3(1) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \end{aligned} \tag{19}$$

Using the triangle inequality in (19), we have

$$|H_3(1)| = |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{20}$$

Now applying Lemma 4, Lemma 6, Theorem 1 and Theorem 2 in (20), we finally have the bound on $H_3(1)$ as desired. □

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Consistency rates and asymptotic normality of the high risk conditional for functional data

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Abstract. The maximum of the conditional hazard function is a parameter of great importance in seismicity studies, because it constitutes the maximum risk of occurrence of an earthquake in a given interval of time. Using the kernel nonparametric estimates of the first derivative of the conditional hazard function, we establish uniform convergence properties and asymptotic normality of an estimate of the maximum in the context of independence data.

1 Introduction

The statistical analysis of functional data studies the experiments whose results are generally the curves. Under this supposition, the statistical analysis

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focuses on a framework of infinite dimension for the data under study. This field of modern statistics has received much attention in the last 20 years, and it has been popularised in the book of Ramsay and Silverman (2005). This type of data appears in many fields of applied statistics: environmetrics (Damon and Guillas, 2002), chemometrics (Benhenni et al., 2007), meteorological sciences (Besse et al., 2000), etc.

From a theoretical point of view, a sample of functional data can be involved in many different statistical problems, such as: classification and principal components analysis (PCA) (1986, 1991) or longitudinal studies, regression and prediction (Benhenni et al., 2007; Cardo et al., 1999). The recent monograph by Ferraty and Vieu (2006) summarizes many of their contributions to the nonparametric estimation with functional data; among other properties, consistency of the conditional density, conditional distribution and regression estimates are established in the i.i.d. case under dependence conditions (strong mixing). Almost complete rates of convergence are also obtained, and different techniques are applied to several examples of functional data samples. Related work can be seen in the paper of Masry (2005), where the asymptotic normality of the functional nonparametric regression estimate is proven, considering strong mixing dependence conditions for the sample data. For automatic smoothing parameter selection in the regression setting, see Rachdi and Vieu (2007).

Hazard and conditional hazard

The estimation of the hazard function is a problem of considerable interest, especially to inventory theorists, medical researchers, logistics planners, reliability engineers and seismologists. The non-parametric estimation of the hazard function has been extensively discussed in the literature. Beginning with Watson and Leadbetter (1964), there are many papers on these topics: Ahmad (1976), Singpurwalla and Wong (1983), etc. We can cite Quintela (2007) for a survey.

The literature on the estimation of the hazard function is very abundant, when observations are vectorial. Cite, for instance, Watson and Leadbetter (1964), Roussas (1989), Lecoutre and Ould-Saïd (1993), Estvez et al. (2002) and Quintela-del-Rio (2006) for recent references. In all these works the authors consider independent observations or dependent data from time series. The first results on the nonparametric estimation of this model, in functional statistics were obtained by Ferraty et al. (2008). They studied the almost complete convergence of a kernel estimator for hazard function of a real random variable dependent on a functional predictor. Asymptotic normality of the latter estimator was obtained, in the case of α- mixing, by Quintela-del-Rio (2008). We refer to Ferraty *et al.* (2010) and Mahhiddine *et al.* (2014) for uniform almost complete convergence of the functional component of this nonparametric model.

When hazard rate estimation is performed with multiple variables, the result is an estimate of the conditional hazard rate for the first variable, given the levels of the remaining variables. Many references, practical examples and simulations in the case of non-parametric estimation using local linear approximations can be found in Spierdijk (2008).

Our paper presents some asymptotic properties related with the non-parametric estimation of the maximum of the conditional hazard function. In a functional data setting, the conditioning variable is allowed to take its values in some abstract semi-metric space. In this case, Ferraty *et al.* (2008) define non-parametric estimators of the conditional density and the conditional distribution. They give the rates of convergence (in an almost complete sense) to the corresponding functions, in a independence and dependence (α-mixing) context. We extend their results by calculating the maximum of the conditional hazard function of these estimates, and establishing their asymptotic normality, considering a particular type of kernel for the functional part of the estimates. Because the hazard function estimator is naturally constructed using these two last estimators, the same type of properties is easily derived for it. Our results are valid in a real (one- and multi-dimensional) context.

If X is a random variable associated to a lifetime (ie, a random variable with values in $\mathbb{R}^+$, the hazard rate of X (sometimes called hazard function, failure or survival rate) is defined at point x as the instantaneous probability that life ends at time x. Specifically, we have:

$$h(x) = \lim_{dx \to 0} \frac{\mathbb{P}\left(X \leq x + dx | X \geq x\right)}{dx}, \quad (x > 0).$$

When X has a density f with respect to the measure of Lebesgue, it is easy to see that the hazard rate can be written as follows:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}, \text{ for all } x \text{ such that } F(x) < 1,$$

where F denotes the distribution function of X and $S = 1 - F$ the survival function of X.

In many practical situations, we may have an explanatory variable Z and the main issue is to estimate the conditional random rate defined as

$$h^Z(x) = \lim_{dx \to 0} \frac{\mathbb{P}(X \leq x + dx | X > x, \quad Z)}{dx}, \quad \text{for } \; x > 0,$$

which can be written naturally as follows:

$$h^Z(x) = \frac{f^Z(x)}{S^Z(x)} = \frac{f^Z(x)}{1 - F^Z(x)}, \text{ once } F^Z(x) < 1. \tag{1}$$

Study of functions h and h^Z is of obvious interest in many fields of science (biology, medicine, reliability , seismology, econometrics, ...) and many authors are interested in construction of nonparametric estimators of h.

In this paper we propose an estimate of the maximum risk, through the nonparametric estimation of the conditional hazard function.

The layout of the paper is as follows. Section 2 describes the non-parametric functional setting: the structure of the functional data, the conditional density, distribution and hazard operators, and the corresponding non-parametric kernel estimators. Section 3 states the almost complete convergence[1] (with rates of convergence[2]) for nonparametric estimates of the derivative of the conditional hazard and the maximum risk. In Section 4, we calculate the variance of the conditional density, distribution and hazard estimates, the asymptotic normality of the three estimators considered is developed in this Section. Finally, Section 5 includes some proofs of technical Lemmas.

2 Nonparametric estimation with dependent functional data

Let $\{(Z_i, X_i), \; i = 1, \ldots, n\}$ be a sample of n random pairs, each one distributed as (Z, X), where the variable Z is of functional nature and X is scalar. Formally, we will consider that Z is a random variable valued in some semi-metric functional space $\mathcal{F}$, and we will denote by $d(\cdot, \cdot)$ the associated semi-metric. The conditional cumulative distribution of X given Z is defined for any $x \in \mathbb{R}$

[1]Recall that a sequence $(T_n)_{n \in \mathbb{N}}$ of random variables is said to converge almost completely to some variable T, if for any $\epsilon > 0$, we have $\sum_n \mathbb{P}(|T_n - T| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, (1987)).

[2]Recall that a sequence $(T_n)_{n \in \mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon > 0$ for which $\sum_n \mathbb{P}(|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = \mathcal{O}(u_n)$, a.co. (or equivalently by $T_n = \mathcal{O}_{a.co.}(u_n)$).

and any $z \in \mathcal{F}$ by

$$\mathsf{F}^{Z}(x) = \mathbb{P}(X \leq x | Z = z),$$

while the conditional density, denoted by $f^{Z}(x)$ is defined as the density of this distribution with respect to the Lebesgue measure on $\mathbb{R}$. The conditional hazard is defined as in the non-infinite case (1).

In a general functional setting, f, F and h are not standard mathematical objects. Because they are defined on infinite dimensional spaces, the term operators may be a more adjusted in terminology.

The functional kernel estimates

We assume the sample data $(X_i, Z_i)_{1 \leq i \leq n}$ is i.i.d.

Following in Ferraty *et al.* (2008), the conditional density operator $f^{Z}(\cdot)$ is defined by using kernel smoothing methods

$$\widehat{f}^{Z}(x) = \frac{\displaystyle\sum_{i=1}^{n} h_H^{-1} K\left(h_K^{-1} d(z, Z_i)\right) H'\left(h_H^{-1}(x - X_i)\right)}{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

where k and H' are kernel functions and h_H and h_K are sequences of smoothing parameters. The conditional distribution operator $F^{Z}(\cdot)$ can be estimated by

$$\widehat{F}^{Z}(x) = \frac{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right) H\left(h_H^{-1}(x - X_i)\right)}{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

with the function $H(\cdot)$ defined by $H(x) = \int_{-\infty}^{x} H'(t)dt$. Consequently, the conditional hazard operator is defined in a natural way by

$$\widehat{h}^{Z}(x) = \frac{\widehat{f}^{Z}(x)}{1 - \widehat{F}^{Z}(x)}.$$

For $z \in \mathcal{F}$, we denote by $h^{Z}(\cdot)$ the conditional hazard function of X_1 given $Z_1 = z$. We assume that $h^{Z}(\cdot)$ is unique maximum and its high risk point is denoted by $\theta(z) := \theta$, which is defined by

$$h^{Z}(\theta(z)) := h^{Z}(\theta) = \max_{x \in \mathcal{S}} h^{Z}(x). \tag{2}$$

A kernel estimator of θ is defined as the random variable $\widehat{\theta}(z) := \widehat{\theta}$ which maximizes a kernel estimator $\widehat{h}^Z(\cdot)$, that is,

$$\widehat{h}^Z(\widehat{\theta}(z)) := \widehat{h}^Z(\widehat{\theta}) = \max_{x \in \mathcal{S}} \widehat{h}^Z(x), \tag{3}$$

where h^Z and $\widehat{h}^Z$ are defined above.

Note that the estimate $\widehat{\theta}$ is note necessarily unique and our results are valid for any choice satisfying (3). We point out that we can specify our choice by taking

$$\widehat{\theta}(z) = \inf \left\{ t \in \mathcal{S} \text{ such that } \widehat{h}^Z(t) = \max_{x \in \mathcal{S}} \widehat{h}^Z(x) \right\}.$$

As in any non-parametric functional data problem, the behavior of the estimates is controlled by the concentration properties of the functional variable Z.

$$\phi_z(h) = \mathbb{P}(Z \in B(z, h)),$$

where $B(z, h)$ being the ball of center z and radius h, namely $B(z, h) = \mathbb{P}\left(f \in \mathcal{F}, d(z, f) < h\right)$ (for more details, see Ferraty and Vieu (2006), Chapter 6).

In the following, z will be a fixed point in $\mathcal{F}$, $\mathcal{N}_z$ will denote a fixed neighborhood of z, $\mathcal{S}$ will be a fixed compact subset of $\mathbb{R}^+$. We will led to the hypothesis below concerning the function of concentration ϕ_z

(H1) $\forall\, h > 0,\ 0 < \mathbb{P}\left(Z \in B(z, h)\right) = \phi_z(h)$ and $\lim_{h \to 0} \phi_z(h) = 0$

Note that (H1) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* of Z.

Our nonparametric models will be quite general in the sense that we will just need the following simple assumption for the marginal distribution of Z, and let us introduce the technical hypothesis necessary for the results to be presented. The non-parametric model is defined by assuming that

(H2) $\begin{cases} \forall\,(x_1, x_2) \in \mathcal{S}^2, \forall\,(z_1, z_2) \in \mathcal{N}_z^2, \text{ for some } b_1 > 0,\ b_2 > 0 \\ |F^{z_1}(x_1) - F^{z_2}(x_2)| \le C_z\left(d(z_1, z_2)^{b_1} + |x_1 - x_2|^{b_2}\right), \end{cases}$

(H3) $\begin{cases} \forall\,(x_1, x_2) \in \mathcal{S}^2, \forall\,(z_1, z_2) \in \mathcal{N}_z^2, \text{ for some } j = 0, 1,\ \nu > 0,\ \beta > 0 \\ |f^{z_1\,(j)}(x_1) - f^{z_2\,(j)}(x_2)| \le C_z\left(d(z_1, z_2)^{\nu} + |x_1 - x_2|^{\beta}\right), \end{cases}$

(H4) $\exists\,\gamma < \infty, f'^Z(x) \le \gamma,\ \ \forall (z, x) \in \mathcal{F} \times \mathcal{S}$,

(H5) $\exists\,\tau > 0, F^Z(x) \le 1 - \tau,\ \ \forall (z, x) \in \mathcal{F} \times \mathcal{S}$.

(H6) H' is twice differentiable such that

$$\begin{cases} \text{(H6a)}\ \forall (t_1,t_2)\in\mathbb{R}^2;\ |H^{(j)}(t_1)-H^{(j)}(t_2)|\le C|t_1-t_2|,\ \text{ for } j=0,1,2 \\ \text{and } H^{(j)} \text{are bounded for } j=0,1,2; \\ \text{(H6b)} \displaystyle\int_{\mathbb{R}} t^2 H'^2(t)dt<\infty; \\ \text{(H6c)} \displaystyle\int_{\mathbb{R}} |t|^{\beta}(H''(t))^2 dt<\infty. \end{cases}$$

(H7) The kernel K is positive bounded function supported on $[0,1]$ and it is of class $\mathcal{C}^1$ on $(0,1)$ such that $\exists C_1,\ C_2,\ -\infty<C_1<K'(t)<C_2<0$ for $0<t<1$.

(H8) There exists a function $\zeta_0^z(\cdot)$ such that for all $t\in[0,1]$

$$\lim_{h_K\to 0}\frac{\phi_z(th_K)}{\phi_z(h_K)}=\zeta_0^z(t)\ \text{ and }\ nh_H\phi_x(h_K)\to\infty\ \text{ as } n\to\infty.$$

(H9) The bandwidth h_H and h_K and small ball probability $\phi_z(h)$ satisfying

$$\begin{cases} \text{(H9a)} \displaystyle\lim_{n\to\infty} h_K=0,\ \lim_{n\to\infty} h_H=0; \\ \text{(H9b)} \displaystyle\lim_{n\to\infty}\frac{\log n}{n\phi_x(h_K)}=0; \\ \text{(H9c)} \displaystyle\lim_{n\to\infty}\frac{\log n}{nh_H^{2j+1}\phi_x(h_K)}=0,\ j=0,1. \end{cases}$$

Remark 1 *Assumption (H1) plays an important role in our methodology. It is known as (for small h) the "concentration hypothesis acting on the distribution of X" in infi- nite-dimensional spaces. This assumption is not at all restrictive and overcomes the problem of the non-existence of the probability density function. In many examples, around zero the small ball probability $\phi_z(h)$ can be written approximately as the product of two independent functions $\psi(z)$ and $\varphi(h)$ as $\phi_z(h)=\psi(z)\varphi(h)+o(\varphi(h))$. This idea was adopted by Masry (2005) who reformulated the Gasser et al. (1998) one. The increasing proprety of $\phi_z(\cdot)$ implies that $\zeta_h^z(\cdot)$ is bounded and then integrable (all the more so $\zeta_0^z(\cdot)$ is integrable).*

Without the differentiability of $\phi_z(\cdot)$, this assumption has been used by many authors where $\psi(\cdot)$ is interpreted as a probability density, while $\varphi(\cdot)$ may be interpreted as a volume parameter. In the case of finite-dimensional spaces, that is $\mathcal{S}=\mathbb{R}^d$, it can be seen that $\phi_z(h)=C(d)h^d\psi(z)+oh^d)$, where $C(d)$ is *the volume of the unit ball in* $\mathbb{R}^d$*. Furthermore, in infinite dimensions, there exist many examples fulfilling the decomposition mentioned above. We quote the following (which can be found in Ferraty et al. (2007)):*

1. $\phi_z(h) \approx \psi(h)h^\gamma$ *for some* $\gamma > 0$.

2. $\phi_z(h) \approx \psi(h)h^\gamma \exp\{C/h^p\}$ *for some* $\gamma > 0$ *and* $p > 0$.

3. $\phi_z(h) \approx \psi(h)/|\ln h|$.

The function $\zeta_h^z(\cdot)$ *which intervenes in Assumption (H9) is increasing for all fixed* h*. Its pointwise limit* $\zeta_0^z(\cdot)$ *also plays a determinant role. It intervenes in all asymptotic properties, in particular in the asymptotic variance term. With simple algebra, it is possible to specify this function (with* $\zeta_0(u) := \zeta_0^z(u)$ *in the above examples by:*

1. $\zeta_0(u) = u^\gamma$,

2. $\zeta_0(u) = \delta_1(u)$ *where* $\delta_1(\cdot)$ *is Dirac function,*

3. $\zeta_0(u) = \mathbf{1}_{]0,1]}(u)$.

Remark 2 *Assumptions (H2) and (H3) are the only conditions involving the conditional probability and the conditional probability density of* Z *given* X*. It means that* $F(\cdot|\cdot)$ *and* $f(\cdot|\cdot)$ *and its derivatives satisfy the Hölder condition with respect to each variable. Therefore, the concentration condition (H1) plays an important role. Here we point out that our assumptions are very usual in the estimation problem for functional regressors (see, e.g., Ferraty et al. (2008)).*

Remark 3 *Assumptions (H6), (H7) and (H9) are classical in functional estimation for finite or infinite dimension spaces.*

3 Nonparametric estimate of the maximum of the conditional hazard function

Let us assume that there exists a compact $\mathcal{S}$ with a unique maximum θ of h^Z on $\mathcal{S}$. We will suppose that h^Z is sufficiently smooth (at least of class $\mathcal{C}^2$) and verifies that $h'^Z(\theta) = 0$ and $h''^Z(\theta) < 0$.

Furthermore, we assume that $\theta \in \mathcal{S}^\circ$, where $\mathcal{S}^\circ$ denotes the interior of $\mathcal{S}$, and that θ satisfies the uniqueness condition, that is; for any $\varepsilon > 0$ and $\mu(z)$, there exists $\xi > 0$ such that $|\theta(z) - \mu(z)| \geq \varepsilon$ implies that $|h^Z(\theta(z)) - h^Z(\mu(z))| \geq \xi$.

We can write an estimator of the first derivative of the hazard function through the first derivative of the estimator. Our maximum estimate is defined by assuming that there is some unique $\widehat{\theta}$ on $\mathcal{S}^{\circ}$.

It is therefore natural to try to construct an estimator of the derivative of the function h^Z on the basis of these ideas. To estimate the conditional distribution function and the conditional density function in the presence of functional conditional random variable Z.

The kernel estimator of the derivative of the function conditional random functional h^Z can therefore be constructed as follows:

$$\widehat{h'}^Z(x) = \frac{\widehat{f'}^Z(x)}{1 - \widehat{F}^Z(x)} + (\widehat{h}^Z(x))^2, \tag{4}$$

the estimator of the derivative of the conditional density is given in the following formula:

$$\widehat{f'}^Z(x) = \frac{\sum_{i=1}^{n} h_H^{-2} K(h_K^{-1} d(Z, Z_i)) H''(h_H^{-1}(x - X_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(Z, Z_i))}. \tag{5}$$

Later, we need assumptions on the parameters of the estimator, ie on K, H, H', h_H and h_K are little restrictive. Indeed, on one hand, they are not specific to the problem estimate of h^Z (but inherent problems of F^Z, f^Z and f'^Z estimation), and secondly they consist with the assumptions usually made under functional variables.

We state the almost complete convergence (withe rates of convergence) of the maximum estimate by the following results:

Theorem 1 *Under assumption (H1)-(H7) we have*

$$\widehat{\theta} - \theta \to 0 \quad \text{a.co.} \tag{6}$$

Remark 4 *The hypothesis of uniqueness is only established for the sake of clarity. Following our proofs, if several local estimated maxima exist, the asymptotic results remain valid for each of them.*

Proof. Because $h'^Z(\cdot)$ is continuous, we have, for all $\epsilon > 0$. $\exists\, \eta(\epsilon) > 0$ such that

$$|x - \theta| > \epsilon \Rightarrow |h'^Z(x) - h'^Z(\theta)| > \eta(\epsilon).$$

Therefore,

$$\mathbb{P}\{|\widehat{\theta}-\theta| \geq \epsilon\} \leq \mathbb{P}\{|h'^{Z}(\widehat{\theta}) - h'^{Z}(\theta)| \geq \eta(\epsilon)\}.$$

We also have

$$|h'^{Z}(\widehat{\theta})-h'^{Z}(\theta)| \leq |h'^{Z}(\widehat{\theta})-\widehat{h}'^{Z}(\widehat{\theta})|+|\widehat{h}'^{Z}(\widehat{\theta})-h'^{Z}(\theta)| \leq \sup_{x\in\mathcal{S}} |\widehat{h}'^{Z}(x)-h'^{Z}(x)|, \tag{7}$$

because $\widehat{h}'^{Z}(\widehat{\theta}) = h'^{Z}(\theta) = 0$.

Then, uniform convergence of h'^{Z} will imply the uniform convergence of $\widehat{\theta}$. That is why, we have the following lemma.

Lemma 1 *Under assumptions of Theorem 1, we have*

$$\sup_{x\in\mathcal{S}} |\widehat{h}'^{Z}(x) - h'^{Z}(x)| \to 0 \quad \text{a.co.} \tag{8}$$

□

The proof of this lemma will be given in Appendix.

Theorem 2 *Under assumption (H1)-(H7) and (H9a) and (H9c), we have*

$$\sup_{x\in\mathcal{S}} |\widehat{\theta}-\theta| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log n}{n h_H^3 \phi_z(h_K)}}\right). \tag{9}$$

Proof. By using Taylor expansion of the function h'^{Z} at the point $\widehat{\theta}$, we obtain

$$h'^{Z}(\widehat{\theta}) = h'^{Z}(\theta) + (\widehat{\theta}-\theta)h''^{Z}(\theta_n^*), \tag{10}$$

with θ^* a point between θ and $\widehat{\theta}$. Now, because $h'^{Z}(\theta) = \widehat{h}'^{Z}(\widehat{\theta})$

$$|\widehat{\theta}-\theta| \leq \frac{1}{h''^{Z}(\theta_n^*)} \sup_{x\in\mathcal{S}} |\widehat{h}'^{Z}(x) - h'^{Z}(x)|. \tag{11}$$

The proof of Theorem will be completed showing the following lemma.

Lemma 2 *Under the assumptions of Theorem 2, we have*

$$\sup_{x\in\mathcal{S}} |\widehat{h}'^{Z}(x) - h'^{Z}(x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log n}{n h_H^3 \phi_z(h_K)}}\right). \tag{12}$$

The proof of lemma will be given in the Appendix.

□

4 Asymptotic normality

To obtain the asymptotic normality of the conditional estimates, we have to add the following assumptions:

(H6d) $\int_{\mathbb{R}} (H''(t))^2 dt < \infty$,

(H10) $0 = \widehat{h'}^{Z}(\widehat{\theta}) < |\widehat{h'}^{Z}(x)|), \forall x \in \mathcal{S},\ x \neq \widehat{\theta}$

The following result gives the asymptotic normality of the maximum of the conditional hazard function. Let

$$\mathcal{A} = \left\{(z, x) : (z, x) \in \mathcal{S} \times \mathbb{R},\ a_2^x F^Z(x)\left(1 - F^Z(x)\right) \neq 0\right\}.$$

Theorem 3 *Under conditions (H1)-(H10) we have* $(\theta \in \mathcal{S}/f^Z(\theta), 1 - F^Z(\theta) > 0)$

$$\left(nh_H^3 \phi_z(h_K)\right)^{1/2} \left(\widehat{h}'^Z(\theta) - h'^Z(\theta)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma^2_{h'}(\theta)\right)$$

where $\rightarrow^{\mathcal{D}}$ *denotes the convergence in distribution,*

$$a_l^x = K^l(1) - \int_0^1 \left(K^l(u)\right)' \zeta_0^x(u) du \quad \textit{for } l = 1, 2$$

and

$$\sigma^2_{h'}(\theta) = \frac{a_2^x h^Z(\theta)}{\left(a_1^x\right)^2 (1 - F^Z(\theta))} \int (H''(t))^2 dt.$$

Proof. Using again (17), and the fact that

$$\frac{\left(1 - F^Z(x)\right)}{(1 - \widehat{F}^Z(x))\,(1 - F^Z(x))} \longrightarrow \frac{1}{1 - F^Z(x)};$$

and

$$\frac{\widehat{f}'^Z(x)}{\left(1 - \widehat{F}^Z(x)\right)(1 - F^Z(x))} \longrightarrow \frac{f'^Z(x)}{(1 - F^Z(x))^2}.$$

The asymptotic normality of $\left(nh_H^3 \phi_z(h_K)\right)^{1/2} \left(\widehat{h'}^Z(\theta) - h'^Z(\theta)\right)$ can be deduced from both following lemmas,

Lemma 3 *Under Assumptions (H1)-(H2) and (H6)-(H8), we have*

$$(n\phi_z(h_K))^{1/2}\left(\widehat{F}^Z(x)-F^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{F^Z}(x)\right), \tag{13}$$

where

$$\sigma^2_{F^Z}(x)=\frac{a_2^x F^Z(x)\left(1-F^Z(x)\right)}{\left(a_1^x\right)^2}.$$

Lemma 4 *Under Assumptions (H1)-(H3) and (H5)-(H9), we have*

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{h}^Z(x)-h^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{h^Z}(x)\right), \tag{14}$$

where

$$\sigma^2_{h^Z}(x)=\frac{a_2^x h^Z(x)}{\left(a_1^x\right)^2(1-F^Z(x))}\int_{\mathbb{R}}(H'(t))^2dt.$$

Lemma 5 *Under Assumptions of Theorem 3, we have*

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{f'}^Z(x)-f'^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{f'^Z}(x)\right); \tag{15}$$

where

$$\sigma^2_{f'^Z(x)}=\frac{a_2^x f^Z(x)}{\left(a_1^x\right)^2}\int_{\mathbb{R}}(H''(t))^2dt.$$

Lemma 6 *Under the hypotheses of Theorem 3, we have*

$$\mathrm{Var}\left[\widehat{f'}^Z_N(x)\right]=\frac{\sigma^2_{f'^Z(x)}}{nh_H^3\phi_z(h_K)}+o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right),$$

$$\mathrm{Var}\left[\widehat{F}^Z_N(x)\right]=o\left(\frac{1}{nh_H\phi_z(h_K)}\right);$$

and

$$\mathrm{Var}\left[\widehat{F}^Z_D\right]=o\left(\frac{1}{nh_H\phi_z(h_K)}\right).$$

Lemma 7 *Under the hypotheses of Theorem 3, we have*

$$\mathrm{Cov}(\widehat{f'}^Z_N(x),\widehat{F}^Z_D)=o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right),$$

$$\mathrm{Cov}(\widehat{f'}^{Z}_{N}(x), \widehat{F}^{Z}_{N}(x)) = o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right)$$

and

$$\mathrm{Cov}(\widehat{F}^{Z}_{D}, \widehat{F}^{Z}_{N}(x)) = o\left(\frac{1}{nh_H\phi_z(h_K)}\right).$$

Remark 5

It is clear that, the results of lemmas, Lemma 6 and Lemma 7 allows to write

$$\mathrm{Var}\left[\widehat{F}^{Z}_{D} - \widehat{F}^{Z}_{N}(x)\right] = o\left(\frac{1}{nh_H\phi_z(h_K)}\right)$$

The proofs of lemmas, Lemma 3 can be seen in Belkhir *et al.* (2015), Lemma 2-4 and Lemma 3-4 see Rabhi *et al.* (2015).

□

Finally, by this last result and (10), the following theorem follows:

Theorem 4 *Under conditions (H1)-(H10), we have* $(\theta \in \mathcal{S}/f^Z(\theta), 1-F^Z(\theta) > 0)$

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{\theta} - \theta\right) \xrightarrow{\mathcal{D}} N\left(0, \frac{\sigma^2_{h'}(\theta)}{(h''^Z(\theta))^2}\right);$$

with $\sigma^2_{h'}(\theta) = h^Z(\theta)\left(1 - F^Z(\theta)\right)\int (H''(t))^2 dt.$

5 Proofs of technical lemmas

Proof. [Proof of Lemma 1 and Lemma 2] Let

$$\widehat{h}'^Z(x) = \frac{\widehat{f}'^Z(x)}{1 - \widehat{F}^Z(x)} + (\widehat{h}^Z(x))^2, \tag{16}$$

with

$$\widehat{h}'^Z(x) - h'^Z(x) = \underbrace{\left\{\left(\widehat{h}^Z(x)\right)^2 - \left(h^Z(x)\right)^2\right\}}_{\Gamma_1} + \underbrace{\left\{\frac{\widehat{f}'^Z(x)}{1 - \widehat{F}^Z(x)} - \frac{f'^Z(x)}{1 - F^Z(x)}\right\}}_{\Gamma_2}; \tag{17}$$

for the first term of (17) we can write

$$\left|\left(\widehat{h}^Z(x)\right)^2 - \left(h^Z(x)\right)^2\right| \leq \left|\widehat{h}^Z(x) - h^Z(x)\right| . \left|\widehat{h}^Z(x) + h^Z(x)\right|, \tag{18}$$

because the estimator $\widehat{h}^Z(\cdot)$ converge a.co. to $h^Z(\cdot)$ we have

$$\sup_{x\in\mathcal{S}}\left|\left(\widehat{h}^Z(x)\right)^2-\left(h^Z(x)\right)^2\right|\leq 2\left|h^Z(\theta)\right|\sup_{x\in\mathcal{S}}\left|\widehat{h}^Z(x)-h^Z(x)\right|; \tag{19}$$

for the second term of (17) we have

$$\begin{aligned}\frac{\widehat{f}'^Z(x)}{1-\widehat{F}^Z(x)}-\frac{f'^Z(x)}{1-F^Z(x)} &= \frac{1}{(1-\widehat{F}^Z(x))(1-F^Z(x))}\left\{\widehat{f}'^Z(x)-f'^Z(x)\right\}\\ &+\frac{1}{(1-\widehat{F}^Z(x))(1-F^Z(x))}\left\{f'^Z(x)\left(\widehat{F}^Z(x)-F^Z(x)\right)\right\}\\ &+\frac{1}{(1-\widehat{F}^Z(x))(1-F^Z(x))}\left\{F^Z(x)\left(\widehat{f}'^Z(x)-f'^Z(x)\right)\right\}.\end{aligned}$$

Valid for all $x\in\mathcal{S}$. Which for a constant $C<\infty$, this leads

$$\sup_{x\in\mathcal{S}}\left|\frac{\widehat{f}'^Z(x)}{1-\widehat{F}^Z(x)}-\frac{f'^Z(x)}{1-F^Z(x)}\right|\leq$$

$$C\frac{\left\{\sup_{x\in\mathcal{S}}\left|\widehat{f}'^Z(x)-f'^Z(x)\right|+\sup_{x\in\mathcal{S}}\left|\widehat{F}^Z(x)-F^Z(x)\right|\right\}}{\inf_{x\in\mathcal{S}}\left|1-\widehat{F}^Z(x)\right|}. \tag{20}$$

Therefore, the conclusion of the lemma follows from the following results:

$$\sup_{x\in\mathcal{S}}|\widehat{F}^Z(x)-F^Z(x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_z(h_K)}}\right), \tag{21}$$

$$\sup_{x\in\mathcal{S}}|\widehat{f}'^Z(x)-f'^Z(x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H^3\phi_z(h_K)}}\right), \tag{22}$$

$$\sup_{x\in\mathcal{S}}|\widehat{h}^Z(x)-h^Z(x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H\phi_z(h_K)}}\right), \tag{23}$$

$$\exists\,\delta>0\ \text{ such that }\ \sum_1^{\infty}\mathbb{P}\left\{\inf_{y\in\mathcal{S}}|1-\widehat{F}^Z(x)|<\delta\right\}<\infty. \tag{24}$$

The proofs of (21) and (22) appear in Ferraty *et al.* (2006), and (23) is proven in Ferraty *et al.* (2008).

- Concerning (24) by equation (21), we have the almost complete convergence of $\widehat{F}^Z(x)$ to $F^Z(x)$. Moreover,

$$\forall \varepsilon > 0 \qquad \sum_{n=1}^{\infty} \mathbb{P}\left\{|\widehat{F}^Z(x) - F^Z(x)| > \varepsilon\right\} < \infty.$$

On the other hand, by hypothesis we have $F^Z < 1$, i.e.

$$1 - \widehat{F}^Z \geq F^Z - \widehat{F}^Z,$$

thus,

$$\inf_{y\in\mathcal{S}} |1-\widehat{F}^Z(x)| \leq (1-\sup_{x\in\mathcal{S}} F^Z(x))/2 \Rightarrow \sup_{x\in\mathcal{S}} |\widehat{F}^Z(x)-F^Z(x)| \geq (1-\sup_{x\in\mathcal{S}} F^Z(x))/2.$$

In terms of probability is obtained

$$\begin{aligned} &\mathbb{P}\left\{\inf_{x\in\mathcal{S}} |1 - \widehat{F}^Z(x)| < (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2\right\} \\ &\leq \mathbb{P}\left\{\sup_{x\in\mathcal{S}} |\widehat{F}^Z(x) - F^Z(x)| \geq (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2\right\} < \infty. \end{aligned}$$

Finally, it suffices to take $\delta = (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2$ and apply the results (21) to finish the proof of this Lemma.

□

Proof. [Proof of Lemma 4] We can write for all $x \in \mathcal{S}$

$$\begin{aligned} \widehat{h}^Z(x) - h^Z(x) &= \frac{\widehat{f}^Z(x)}{1 - \widehat{F}^Z(x)} - \frac{f^Z(x)}{1 - F^Z(x)} \\ &= \frac{1}{\widehat{D}^Z(x)}\Big\{ \left(\widehat{f}^Z(x) - f^Z(x)\right) + f^Z(x)\left(\widehat{F}^Z(x) - F^Z(x)\right) \\ &\qquad - F^Z(x)\left(\widehat{f}^Z(x) - f^Z(x)\right)\Big\}, \\ &= \frac{1}{\widehat{D}^Z(x)}\Big\{ \left(1 - F^Z(x)\right)\left(\widehat{f}^Z(x) - f^Z(x)\right) \\ &\qquad - f^Z(x)\left(\widehat{F}^Z(x) - F^Z(x)\right)\Big\}; \end{aligned} \tag{25}$$

with $\widehat{D}^Z(x) = (1 - F^Z(x))\left(1 - \widehat{F}^Z(x)\right)$.

As a direct consequence of the Lemma 3, the result (26) (see Belkhir *et al.* (2015)) and the expression (25), permit us to obtain the asymptotic normality for the conditional hazard estimator.

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{f}^Z(x)-f^Z(x)\right)\xrightarrow{\mathcal{D}} N\left(0,\sigma^2_{fZ}(x)\right); \tag{26}$$

where

$$\sigma^2_{f^Z(x)}=\frac{a_2^x f^Z(x)}{(a_1^x)^2}\int_{\mathbb{R}}(H'(t))^2dt.$$

□

Proof. [Proof of Lemma 5] For $i=1,\ldots,n$, we consider the quantities $K_i = K\left(h_K^{-1}d(z,Z_i)\right)$, $H_i''(x)=H''\left(h_H^{-1}(x-X_i)\right)$ and let $\widehat{f'}^Z_N(x)$ (resp. $\widehat{F}^Z_D$) be defined as

$$\widehat{f'}^Z_N(x) \;=\; \frac{h_H^{-2}}{n\,\mathbb{E}K_1}\sum_{i=1}^n K_iH_i''(x) \qquad \text{(resp. } \widehat{F}^Z_D \;=\; \frac{1}{n\,\mathbb{E}K_1}\sum_{i=1}^n K_i).$$

This proof is based on the following decomposition

$$\begin{aligned}\widehat{f'}^Z(x)-f'^Z(x)=&\frac{1}{\widehat{F}^Z_D}\left\{\left(\widehat{f'}^Z_N(x)-\mathbb{E}\widehat{f'}^Z_N(x)\right)-\left(f'^Z(x)-\mathbb{E}\widehat{f'}^Z_N(x)\right)\right\}\\&+\frac{f'^Z(x)}{\widehat{F}^Z_D}\left\{\mathbb{E}\widehat{F}^Z_D-\widehat{F}^Z_D\right\},\end{aligned} \tag{27}$$

and on the following intermediate results.

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}^Z_N(x)-\mathbb{E}\widehat{f'}^Z_N(x)\right)\xrightarrow{\mathcal{D}} N\left(0,\sigma^2_{f'Z}(x)\right); \tag{28}$$

where $\sigma^2_{f'Z}(x)$ is defined as in Lemma 5.

$$\lim_{n\to\infty}\sqrt{nh_H^3\phi_z(h_K)}\left(\mathbb{E}\widehat{f'}^Z_N(x)-f'^Z(x)\right)=0. \tag{29}$$

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{F}^Z_D(x)-1\right)\xrightarrow{\mathbb{P}}0,\quad\text{as}\quad n\to\infty. \tag{30}$$

- Concerning (28). By the definition of $\widehat{f'}_N^Z(x)$, it follows that

$$\begin{aligned}\Omega_n &= \sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}_N^Z(x) - \mathbb{E}\widehat{f'}_N^Z(x)\right)\\ &= \sum_{i=1}^{n} \frac{\sqrt{\phi_z(h_K)}}{\sqrt{nh_H}\mathbb{E}K_1}\left(K_iH_i'' - \mathbb{E}K_iH_i''\right)\\ &= \sum_{i=1}^{n}\Delta_i,\end{aligned}$$

which leads

$$\mathrm{Var}(\Omega_n) = nh_H^3\phi_z(h_K)\mathrm{Var}\left(\widehat{f'}_N^Z(x) - \mathbb{E}\left[\widehat{f'}_N^Z(x)\right]\right). \tag{31}$$

Now, we need to evaluate the variance of $\widehat{f'}_N^Z(x)$. For this we have for all $1 \le i \le n$, $\Delta_i(z,x) = K_i(z)H_i''(x)$, so we have

$$\begin{aligned}\mathrm{Var}(\widehat{f'}_N^Z(x)) &= \frac{1}{\left(nh_H^2\mathbb{E}[K_1(z)]\right)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}\left(\Delta_i(z,x),\Delta_j(z,x)\right)\\ &= \frac{1}{n\left(h_H^2\mathbb{E}[K_1(z)]\right)^2}\mathrm{Var}\left(\Delta_1(z,x)\right).\end{aligned}$$

Therefore

$$Var\left(\Delta_1(z,x)\right) \le \mathbb{E}\left(H_1''^2(x)K_1^2(z)\right) \le \mathbb{E}\left(K_1^2(z)\mathbb{E}\left[H_1''^2(x)|Z_1\right]\right).$$

Now, by a change of variable in the following integral and by applying (H4) and (H7), one gets

$$\begin{aligned}\mathbb{E}\left(H_1''^2(y)|Z_1\right) &= \int_{\mathbb{R}} H''^2\left(\frac{d(x-u)}{h_H}\right)f^Z(u)du\\ &\le h_H\int_{\mathbb{R}} H''^2(t)\left(f^Z(x-h_Ht,z) - f^Z(x)\right)dt\\ &\quad + h_Hf^Z(x)\int_{\mathbb{R}} H''^2(t)dt \qquad (32)\\ &\le h_H^{1+b_2}\int_{\mathbb{R}}|t|^{b_2}H''^2(t)dt + h_Hf^Z(x)\int_{\mathbb{R}} H''^2(t)dt\\ &= h_H\left(o(1) + f^Z(x)\int_{\mathbb{R}} H''^2(t)dt\right).\end{aligned}$$

By means of (32) and the fact that, as $n \to \infty$, $\mathbb{E}\left(K_1^2(z)\right) \longrightarrow a_2^x \phi_z(h_K)$, one gets

$$\operatorname{Var}\left(\Delta_1(z,x)\right) = a_2^x \phi_z(h_K) h_H \left(o(1) + f^Z(x) \int_{\mathbb{R}} H''^2(t)dt \right).$$

So, using (H8), we get

$$\begin{aligned}
&\frac{1}{n\left(h_H^2 \mathbb{E}[K_1(z)]\right)^2} \operatorname{Var}\left(\Delta_1(z,x)\right) \\
&\quad = \frac{a_2^x \phi_z(h_K)}{n\left(a_1^x h_H^2 \phi_z(h_K)\right)^2} h_H \left(o(1) + f^Z(x) \int_{\mathbb{R}} H''^2(t)dt \right) \\
&\quad = o\left(\frac{1}{n h_H^3 \phi_z(h_K)}\right) + \frac{a_2^x f^Z(x)}{(a_1^x)^2 n h_H^3 \phi_z(h_K)} \int_{\mathbb{R}} H''^2(t)dt.
\end{aligned}$$

Thus as $n \to \infty$ we obtain

$$\frac{1}{n\left(h_H^2 \mathbb{E}[K_1(z)]\right)^2} \operatorname{Var}\left(\Delta_1(z,x)\right) \longrightarrow \frac{a_2^x f^Z(x)}{(a_1^x)^2 n h_H^3 \phi_z(h_K)} \int_{\mathbb{R}} H''^2(t)dt. \quad (33)$$

Indeed

$$\sum_{i=1}^{n} \mathbb{E}\Delta_i^2 = \frac{\phi_z(h_K)}{h_H \mathbb{E}^2 K_1} \mathbb{E}K_1^2 (H_1'')^2 - \frac{\phi_z(h_K)}{h_H \mathbb{E}^2 K_1} \left(\mathbb{E}K_1 H_1''\right)^2 = \Pi_{1n} - \Pi_{2n}. \quad (34)$$

As for Π_{1n}, by the property of conditional expectation, we get

$$\Pi_{1n} = \frac{\phi_z(h_K)}{\mathbb{E}^2 K_1} \mathbb{E}\left\{ K_1^2 \int H''^2(t) \left(f'^Z(x - t h_H) - f'^Z(x) + f'^Z(x) \right) dt \right\}.$$

Meanwhile, by (H1), (H3), (H7) and (H8), it follows that:

$$\frac{\phi_z(h_K) \mathbb{E}K_1^2}{\mathbb{E}^2 K_1} \underset{n\to\infty}{\longrightarrow} \frac{a_2^x}{(a_1^x)^2},$$

which leads

$$\Pi_{1n} \underset{n\to\infty}{\longrightarrow} \frac{a_2^x f^Z(x)}{(a_1^x)^2} \int (H''(t))^2 dt, \quad (35)$$

Regarding Π_{2n}, by (H1), (H3) and (H6), we obtain

$$\Pi_{2n} \underset{n\to\infty}{\longrightarrow} 0. \tag{36}$$

This result, combined with (34) and (35), allows us to get

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}\Delta_i^2 = \sigma^2_{f'Z}(x) \tag{37}$$

Therefore, combining (33) and (36)-(37), (28) is valid.

- Concerning (29).

  The proof is completed along the same steps as that of Π_{1n}. We omit it here.

- Concerning (30). The idea is similar to that given by Belkhir *et al.* (2015). By definition of $\widehat{F}_D^Z(x)$, we have

$$\sqrt{nh_H^3\phi_z(h_K)}(\widehat{F}_D^Z(x) - 1) = \Omega_n - \mathbb{E}\Omega_n,$$

where $\Omega_n = \frac{\sqrt{nh_H^3\phi_z(h_K)}\sum_{i=1}^n K_i}{n\mathbb{E}K_1}$. In order to prove (30), similar to Belkhir *et al.* (2015), we only need to proov $\mathrm{Var}\ \Omega_n \to 0$, as $n \to \infty$. In fact, since

$$\begin{aligned}
\mathrm{Var}\ \Omega_n &= \frac{nh_H^3\phi_z(h_K)}{n\mathbb{E}^2K_1}(n\mathrm{Var}K_1) \\
&\leq \frac{nh_H^3\phi_z(h_K)}{\mathbb{E}^2K_1}\mathbb{E}K_1^2 \\
&= \Psi_1,
\end{aligned}$$

then, using the boundedness of function K allows us to get that:

$$\Psi_1 \leq Ch_H^3\phi_z(h_K) \to 0, \quad \text{as } n \to \infty.$$

It is clear that, the results of (21), (22), (24) and Lemma 6 permits us

$$\mathbb{E}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + F^Z(x)\right) \longrightarrow 0,$$

and

$$\mathrm{Var}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + F^Z(x)\right) \longrightarrow 0;$$

then

$$\widehat{F}_D^x - \widehat{F}_N^Z(x) - 1 + F^Z(x) \xrightarrow{\mathbb{P}} 0.$$

Moreover, the asymptotic variance of $\widehat{F}_D^Z - \widehat{F}_N^Z(x)$ given in Remark 5 allows to obtain

$$\frac{n h_H \phi_z(h_K)}{\sigma^2_{F^Z}(x)} \mathrm{Var}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + \mathbb{E}\left(\widehat{F}_N^Z(x)\right)\right) \longrightarrow 0.$$

By combining result with the fact that

$$\mathbb{E}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + \mathbb{E}\left(\widehat{F}_N^Z(x)\right)\right) = 0,$$

we obtain the claimed result.

Therefore, the proof of this result is completed.

Therefore, the proof of this Lemma is completed.

□

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Unified theory for certain generalized types of closed sets and some separation axioms

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Abstract. In this paper the notion of $\mu\nu g$-closed sets and certain characterizations of such sets have been given. As an application of $\mu\nu g$-closed sets, the notion of (μ, ν)-regular spaces and (μ, ν)-normal spaces have been introduced and some characterizations of such spaces are also given.

1 Introduction

For the last one decade or so, a new area of study has emerged and has been rapidly growing. The area is concerned with the investigations of generalized topological spaces and several classes of generalized types of open sets. Recently, a significant contribution to the theory of generalized open sets, was extended by A. Császár [1, 2, 3]. It is observed that a large number of papers are devoted to the study of generalized open sets, containing the class of open sets and possessing properties more or less similar to those of open sets.

We recall some notions defined in [2]. Let X be a non-empty set and $\exp X$ denote the power set of X. We call a class $\mu \subseteqq \exp X$ a generalized topology [2], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belong to μ. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-open

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sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [2, 3]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where the operator $\gamma : \exp X \to \exp X$ is said to be idempotent if $A \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $A \subseteq B \subseteq X$ implies $\gamma(A) \subseteq \gamma(B)$. It is known from [3, 1] that if μ is a GT on X, $x \in X$ and $A \subseteq X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \varnothing$. It is also well known from [3, 1] that $x \in i_\mu(A)$ if and only if there exists $U \in \mu$ with $x \in U$ such that $x \in U \subseteq A$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$ and $i_\mu(X \setminus A) = X \setminus c_\mu(A)$.

2 $\mu\nu g$-closed sets and $\mu\nu g$-open sets

Definition 1 *Let μ and ν be two GT's on a set X. Then $A \subseteq X$ is called $\mu\nu g$-closed [4] if $c_\nu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. The complement of a $\mu\nu g$-closed set is called a $\mu\nu g$-open set.*

Proposition 1 *Let μ and ν be two GT's on a set X. Then for $A, B \subseteq X$ the following holds:*

(i) *If A is ν-closed then A is $\mu\nu g$-closed.*

(ii) *If A is $\mu\nu g$-closed and μ-open then A is ν-closed.*

(iii) *If A is $\mu\nu g$-closed and $A \subseteq B \subseteq c_\nu(A)$, then B is $\mu\nu g$-closed.*

Proof. (i) Let A be a ν-closed subset of X and $A \subseteq U \in \mu$. Then $c_\nu(A) = A \subseteq U$ and thus A is $\mu\nu g$-closed.

(ii) Let A be a $\mu\nu g$-closed, μ-open subset of X. Then $c_\nu(A) \subseteq A$ and hence A is ν-closed.

(iii) Let $B \subseteq U$ where U is a μ-open set. Then $A \subseteq U$ and hence by $\mu\nu g$-closedness of A, $c_\nu(A) \subseteq U$. Now $c_\nu(A) \subseteq c_\nu(B) \subseteq c_\nu(c_\nu(A)) = c_\nu(A)$. Hence $c_\nu(A) = c_\nu(B)$. Therefore $c_\nu(B) \subseteq U$ and hence B is $\mu\nu g$-closed. □

Theorem 1 *Let μ and ν be two GT's on X. Then $A \subseteq X$ is $\mu\nu g$-closed if and only if $c_\nu(A) \cap F = \varnothing$ whenever $A \cap F = \varnothing$ and F is μ-closed.*

Proof. Let A be a $\mu\nu g$-closed subset of X and F be μ-closed with $A \cap F = \varnothing$. Then $A \subseteq X \setminus F$ where $X \setminus F$ is μ-open. Thus $c_\nu(A) \subseteq X \setminus F$. Therefore we have $c_\nu(A) \cap F = \varnothing$.

Conversely, let $A \subseteqq U$ and U be μ-open. Then $A \cap (X \setminus U) = \varnothing$ where $X \setminus U$ is μ-closed. Thus by hypothesis, $c_\nu(A) \cap (X \setminus U) = \varnothing$ and hence $c_\nu(A) \subseteqq U$ showing A to be $\mu\nu g$-closed. $\square$

Definition 2 [5] *Let* (X, μ) *be a GTS and* $A \subseteqq X$. *Then the subset* $\bigwedge_\mu(A)$ *is defined by*

$$\bigwedge_\mu(A) = \begin{cases} \bigcap\{G : A \subseteqq G, G \in \mu\}, & \textit{if there exists } G \in \mu \textit{ such that } A \subseteqq G; \\ X, & \textit{otherwise.} \end{cases}$$

Theorem 2 *Let* μ *and* ν *be two GT's on* X. *Then* $A(\subseteqq X)$ *is* $\mu\nu g$*-closed if and only if* $c_\nu(A) \subseteqq \bigwedge_\mu(A)$.

Proof. Suppose that A is $\mu\nu g$-closed. Let $x \notin \bigwedge_\mu(A)$. Then there exists a μ-open set G with $x \notin G$ and $A \subseteqq G$. Then $x \notin c_\nu(A)$ (as A is $\mu\nu g$-closed). Thus $c_\nu(A) \subseteqq \bigwedge_\mu(A)$.

Conversely, suppose that $c_\nu(A) \subseteqq \bigwedge_\mu(A)$. Let $A \subseteqq U$ where U is μ-open. Then $c_\nu(A) \subseteqq \bigwedge_\mu(A) \subseteqq \bigwedge_\mu(U) = U$. Thus A is $\mu\nu g$-closed. $\square$

Theorem 3 *Let* μ *and* ν *be two GT's on* X. *Then* $A(\subseteqq X)$ *is called* $\mu\nu g$*-closed if and only if* $c_\mu(\{x\}) \cap A \neq \varnothing$ *for each* $x \in c_\nu(A)$.

Proof. Suppose that A is $\mu\nu g$-closed and $c_\mu(\{x\}) \cap A = \varnothing$ for some $x \in c_\nu(A)$. Then $A \subseteqq X \setminus c_\mu(\{x\})$ where $X \setminus c_\mu(\{x\})$ is μ-open. Thus $c_\nu(A) \subseteqq X \setminus c_\mu(\{x\}) \subseteqq X \setminus \{x\}$. This contradicts the fact that $x \in c_\nu(A)$.

Conversely, suppose that A be not $\mu\nu g$-closed. Then $c_\nu(A) \setminus U \neq \varnothing$ for some μ-open set U with $A \subseteqq U$. Let $x \in c_\nu(A) \setminus U$. Then $x \in c_\nu(A)$ and $x \notin U$. Then $c_\mu(\{x\}) \cap U = \varnothing$ and hence $c_\mu(\{x\}) \cap A \subseteqq c_\mu(\{x\}) \cap U = \varnothing$. This shows that $c_\mu(\{x\}) \cap A = \varnothing$ for some $x \in c_\nu(A)$. $\square$

Theorem 4 *Let* μ *and* ν *be two GT's on* X. *Then a subset* $A(\subseteqq X)$ *is* $\mu\nu g$*-open if and only if* $F \subseteqq i_\nu(A)$ *whenever* $F \subseteqq A$ *and* F *is* μ*-closed.*

Proof. Suppose that A is $\mu\nu g$-open. Let $F \subseteqq A$ and F be μ-closed. Then $X \setminus A \subseteqq X \setminus F \in \mu$ and $X \setminus A$ is $\mu\nu g$-closed. Thus $X \setminus i_\nu(A) = c_\nu(X \setminus A) \subseteqq X \setminus F$ and hence $F \subseteqq i_\nu(A)$.

Conversely, let $X \setminus A \subseteqq U$ where U is μ-open. Then $X \setminus U \subseteqq A$ and $X \setminus U$ is μ-closed. Thus by the hypothesis, $X \setminus U \subseteqq i_\nu(A)$ and thus $c_\nu(X \setminus A) = X \setminus i_\nu(A) \subseteqq U$. Hence A is $\mu\nu g$-open. $\square$

Definition 3 *Let μ and ν be two GT's on X. Then μ and ν is said to have the property $(*)$ if $A \in \mu$, $B \in \nu$ implies that $A \cup B \in \mu$.*

Theorem 5 *Let μ and ν be two GT's on X satisfying the property $(*)$. Then the following are equivalent:*

(1) *A is $\mu\nu g$-closed.*

(2) *$c_\nu(A) \setminus A$ does not contain any non-empty μ-closed set.*

(3) *$c_\nu(A) \setminus A$ is $\mu\nu g$-open.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is a $\mu\nu g$-closed set. Let $F \subseteqq c_\nu(A) \setminus A$ and F be μ-closed. Then $A \subseteqq X \setminus F$ where $X \setminus F$ is μ-open and hence, $c_\nu(A) \subseteqq X \setminus F$. Therefore, we have $F \subseteqq X \setminus c_\nu(A)$ and hence, $F \subseteqq c_\nu(A) \cap (X \setminus c_\nu(A)) = \varnothing$. Thus $F = \varnothing$.

(2) $\Rightarrow$ (3): Let us assume that $F \subseteqq c_\nu(A) \setminus A$ and F be μ-closed. By (2), we have $F = \varnothing$ and $F \subseteqq i_\nu[c_\nu(A) \setminus A]$. Hence by Theorem 4, $c_\nu(A) \setminus A$ is $\mu\nu g$-open.

(3) $\Rightarrow$ (1): Suppose that $A \subseteqq U$ and U is μ-open. Then, $c_\nu(A) \setminus U \subseteqq c_\nu(A) \setminus A$. By (3), $c_\nu(A) \setminus A$ is $\mu\nu g$-open. Since μ and ν have the property $(*)$, $c_\nu(A) \setminus U$ is μ-closed (as $c_\nu(A)$ is ν-closed and $X \setminus U$ is μ-closed). By Theorem 4, we have $c_\nu(A) \setminus U \subseteqq i_\nu(c_\nu(A) \setminus A) = \varnothing$. [In fact if $i_\nu(c_\nu(A) \setminus A) \neq \varnothing$, then there exists some $x \in i_\nu(c_\nu(A) \setminus A)$. Then, there exists $G \in \nu$ such that $x \in G \subseteqq c_\nu(A) \setminus A$. Since $G \subseteqq X \setminus A$, we have $G \cap A = \varnothing$ and $G \in \nu$. Thus $G \cap c_\nu(A) = \varnothing$ and $G \subseteqq X \setminus c_\nu(A)$. Therefore, we obtain $G \subseteqq c_\nu(A) \cap (X \setminus c_\nu(A)) = \varnothing$.] Therefore, we have $c_\nu(A) \subseteqq U$ and hence A is $\mu\nu g$-closed. □

Theorem 6 *Let μ and ν be two GT's on X satisfying the property $(*)$. A subset A of X is $\mu\nu g$-open if and only if $G = X$ whenever G is μ-open and $i_\nu(A) \cup (X \setminus A) \subseteqq G$.*

Proof. Let A be a $\mu\nu g$-open set and G be μ-open with $i_\nu(A) \cup (X \setminus A) \subseteqq G$. Then $X \setminus G \subseteqq c_\nu(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $\mu\nu g$-closed and $X \setminus G$ is μ-closed, by Theorem 5, $X \setminus G = \varnothing$ and hence $G = X$.

Conversely let us assume that $F \subseteqq A$ and F be μ-closed. Since μ and ν have the property $(*)$, we have $i_\nu(A) \cup (X \setminus A) \subseteqq i_\nu(A) \cup (X \setminus F)$ and $i_\nu(A) \cup (X \setminus F)$ is μ-open. Thus by the hypothesis, $X = i_\nu(A) \cup (X \setminus F)$. Hence, $F = F \cap (i_\nu(A) \cup (X \setminus F)) = F \cap i_\nu(A) \subseteqq i_\nu(A)$. Thus from Theorem 4 it follows that A is $\mu\nu g$-open. □

Theorem 7 *Let μ and ν be two GT's on X. For any $x \in X$, $\{x\}$ is μ-closed or $\mu\nu g$-open.*

Proof. Suppose that $\{x\}$ is not μ-closed. Then $X\setminus\{x\}$ is not μ-open. Then either there does not exist any μ-open set containing $X \setminus \{x\}$ or the only μ-open set containing $X \setminus \{x\}$ is X itself. Therefore, $c_\nu(X \setminus \{x\}) \subseteqq X$ and hence, $X \setminus \{x\}$ is $\mu\nu g$-closed. Thus $\{x\}$ is $\mu\nu g$-open. □

3 (μ, ν)-regular space and (μ, ν)-normal space

Definition 4 *Let μ and ν be two GT's on X. Then (X, μ, ν) is said to be (μ, ν)-regular if for each μ-closed set F of X not containing x, there exist disjoint ν-open sets U and V such that $x \in U$ and $F \subseteqq V$.*

Theorem 8 *Let μ and ν be two GT's on X. Then the followings are equivalent:*

(i) X *is (μ, ν)-regular.*

(ii) *For each $x \in X$ and each $U \in \mu$ containing x there exists $V \in \nu$ containing x such that $x \in V \subseteqq c_\nu(V) \subseteqq U$.*

(iii) *For each μ-closed set F of X, $\cap\{c_\nu(V) : F \subseteqq V \in \nu\} = F$.*

(iv) *For each subset A of X and each $U \in \mu$ with $A \cap U \neq \varnothing$, there exists a $V \in \nu$ such that $A \cap V \neq \varnothing$ and $c_\nu(V) \subseteqq U$.*

(v) *For each non-empty subset A of X and each μ-closed subset F of X with $A \cap F = \varnothing$, there exist $V, W \in \nu$ such that $A \cap V \neq \varnothing$, $F \subseteqq W$ and $W \cap V = \varnothing$.*

(vi) *For each μ-closed set F with $x \notin F$ there exists $U \in \mu$ and a $\mu\nu g$-open set V such that $x \in U$, $F \subseteqq V$ and $U \cap V = \varnothing$.*

(vii) *For each $A \subseteqq X$ and each μ-closed set F with $A \cap F = \varnothing$ there exists $U \in \mu$ and a $\mu\nu g$-open set V such that $A \cap U \neq \varnothing$, $F \subseteqq V$ and $U \cap V = \varnothing$.*

(viii) *For each μ-closed set F of X, $F = \cap\{c_\nu(V) : F \subseteqq V, V$ is $\mu\nu g$-open$\}$.*

Proof. (i) $\Rightarrow$ (ii): Let U be a μ-open set containing x. Then $x \notin X \setminus U$, where $X \setminus U$ is μ-closed. Then by (i) there exist $G, V \in \nu$ such that $X \setminus U \subseteqq G$ and $x \in V$ and $G \cap V = \varnothing$. Thus $V \subseteqq X \setminus G$ and so $x \in V \subseteqq c_\nu(V) \subseteqq X \setminus G \subseteqq U$.

(ii) $\Rightarrow$ (iii): Let $X\setminus F \in \mu$ be such that $x \notin F$. Then by (ii) there exists $U \in \nu$ containing x such that $x \in U \subseteqq c_\nu(U) \subseteqq X\setminus F$. So, $F \subseteqq X\setminus c_\nu(U) = V$ (say)$\in \nu$ and $U \cap V = \varnothing$. Thus $x \notin c_\nu(V)$. Thus $F \supseteq \cap\{c_\nu(V) : F \subseteqq V \in \nu\}$.

(iii) $\Rightarrow$ (iv): Let $U \in \mu$ with $x \in U \cap A$. Then $x \notin X \setminus U$ and hence by (iii) there exists a ν-open set W such that $X \setminus U \subseteqq W$ and $x \notin c_\nu(W)$. We put

$V = X \setminus c_\nu(W)$, which is a ν-open set containing x and hence $A \cap V \neq \varnothing$ (as $x \in A \cap V$). Now $V \subseteqq X \setminus W$ and so $c_\nu(V) \subseteqq X \setminus W \subseteqq U$.

(iv) $\Rightarrow$ (v): Let F be a μ-closed set as in the hypothesis of (v). Then $X \setminus F$ is a μ-open set and $(X \setminus F) \cap A \neq \varnothing$. Then there exists $V \in \nu$ such that $A \cap V \neq \varnothing$ and $c_\nu(V) \subseteqq X \setminus F$. If we put $W = X \setminus c_\nu(V)$, then $F \subseteqq W$ and $W \cap V = \varnothing$.

(v) $\Rightarrow$ (i): Let F be a μ-closed set not containing x. Then by (v), there exist $W, V \in \nu$ such that $F \subseteqq W$ and $x \in V$ and $W \cap V = \varnothing$.

(i) $\Rightarrow$ (vi): Obvious as every ν-open set is $\mu\nu g$-open (by Proposition 1).

(vi) $\Rightarrow$ (vii): Let F be a μ-closed set such that $A \cap F = \varnothing$ for any subset A of X. Thus for $a \in A$, $a \notin F$ and hence by (vi), there exists $U \in \mu$ and a $\mu\nu g$-open set V such that $a \in U$, $F \subseteqq V$ and $U \cap V = \varnothing$ and $A \cap U \neq \varnothing$.

(vii) $\Rightarrow$ (i): Let $x \notin F$, where F be μ-closed. Since $\{x\} \cap F = \varnothing$, by (vii) there exist $U \in \mu$ and a $\mu\nu g$-open set W such that $x \in U$, $F \subseteqq W$ and $U \cap W = \varnothing$. Now put $V = i_\nu(W)$. Then $F \subseteqq V$ (by Theorem 4) and $U \cap V = \varnothing$.

(iii) $\Rightarrow$ (viii): We have $F \subseteqq \cap\{c_\nu(V) : F \subseteqq V$ and V is $\mu\nu g$-open$\} \subseteqq \cap\{c_\nu(V) : F \subseteqq V$ and V is ν-open$\} = F$.

(viii) $\Rightarrow$ (i): Let F be a μ-closed set in X not containing x. Then by (viii) there exists a $\mu\nu g$-open set W such that $F \subseteqq W$ and $x \in X \setminus c_\nu(W)$. Since F is μ-closed and W is $\mu\nu g$-open, $F \subseteqq i_\nu(W)$ (by Theorem 4). Take $V = i_\nu(W)$. Then $F \subseteqq V$, $x \in X \setminus c_\nu(V) = U$ (say) (as $(X \setminus F) \cap V = \varnothing$) and $U \cap V = \varnothing$. $\square$

Definition 5 *Let μ and ν be two GT's on a set X. Then (X, μ, ν) is said to be (μ, ν)-normal if for disjoint μ-closed sets F_1 and F_2, there exist $U_1, U_2 \in \nu$ such that $F_1 \subseteqq U_1, F_2 \subseteqq U_2$ with $U_1 \cap U_2 = \varnothing$.*

Theorem 9 *Let μ and ν be two GT's on X. Then the following properties are equivalent:*

(i) (X, μ, ν) *is (μ, ν)-normal;*

(ii) *for any two disjoint μ -closed sets F_1, F_2, there exist $\mu\nu g$-open sets V_1, V_2 such that $F_1 \subseteqq V_1$, $F_2 \subseteqq V_2$ and $V_1 \cap V_2 = \varnothing$;*

(iii) *for any μ-closed set F and any μ-open set U containing F, there exists a $\mu\nu g$-open set V such that $F \subseteqq V \subseteqq c_\nu(V) \subseteqq U$;*

(iv) *for any μ-closed set F and any μ-open set U containing F, there exists a ν-open set G such that $F \subseteqq G \subseteqq c_\nu(G) \subseteqq U$;*

(v) *for any disjoint μ-closed sets F_1, F_2, there exists a $\mu\nu g$-open set V such that $F_1 \subseteqq V$ and $c_\nu(V) \cap F_2 = \varnothing$;*

(vi) *for any disjoint μ-closed sets F_1, F_2, there exists a ν-open set G such that $F_1 \subseteqq G$ and $c_\nu(G) \cap F_2 = \varnothing$.*

Proof. (i)$\Rightarrow$ (ii): Follows from (i) as, every ν-open set is $\mu\nu g$-open.

(ii) $\Rightarrow$ (iii): Let F be a μ-closed set and U be any μ-open set containing F. Then F and $X \setminus U$ are disjoint μ-closed sets and by (ii) there exist $\mu\nu g$-open sets V_1, V_2 such that $F \subseteqq V_1$, $X \setminus U \subseteqq V_2$ with $V_1 \cap V_2 = \varnothing$. Since V_2 is $\mu\nu g$-open, by Theorem 4, $X \setminus U \subseteqq i_\nu(V_2)$. Hence, $c_\nu(V_1) \cap i_\nu(V_2) = \varnothing$. Therefore, we obtain $F \subseteqq V_1 \subseteqq c_\nu(V_1) \subseteqq X \setminus i_\nu(V_2) \subseteqq U$. Put $V = V_1$, then we obtain $F \subseteqq V \subseteqq c_\nu(V) \subseteqq U$.

(iii) $\Rightarrow$ (iv): Let F be a μ -closed set and U be any μ-open set containing F. Then by (iii) there exists a $\mu\nu g$-open set V such that $F \subseteqq V \subseteqq c_\nu(V) \subseteqq U$. By Theorem 4, $F \subseteqq i_\nu(V)$. Put $G = i_\nu(V)$. Then G is a ν-open set. Furthermore, we obtain $F \subseteqq G \subseteqq c_\nu(G) \subseteqq c_\nu(V) \subseteqq U$.

(iv) $\Rightarrow$ (v): Let F_1, F_2 be any two disjoint μ-closed sets. Since $X \setminus F_2$ is a μ-open set containing F_1, by (iv) there exists a ν-open set V such that $F_1 \subseteqq V \subseteqq c_\nu(V) \subseteqq X \setminus F_2$. By Proposition 1, V is $\mu\nu g$-open. Furthermore, we have $F_1 \subseteqq V$ and $c_\nu(V) \cap F_2 = \varnothing$.

(v) $\Rightarrow$ (vi): Let F_1, F_2 be any disjoint μ-closed sets. Then there exists a $\mu\nu g$-open set V such that $F_1 \subseteqq V$ and $c_\nu(V) \cap F_2 = \varnothing$. By Theorem 4, $F_1 \subseteqq i_\nu(V)$. Set $G = i_\nu(V)$. Then $G \in \nu$, $F_1 \subseteqq G$ and $c_\nu(G) \cap F_2 = \varnothing$.

(vi) $\Rightarrow$ (i): Let F_1, F_2 be any two disjoint μ-closed sets. Then by (vi) there exists $G \in \nu$ such that $F_1 \subseteqq G$ and $c_\nu(G) \cap F_2 = \varnothing$. Now, put $U_1 = G$ and $U_2 = X \setminus c_\nu(G)$. Then U_1 and U_2 are disjoint ν-open sets, $F_1 \subseteqq U_1$ and $F_2 \subseteqq U_2$. This shows that (X, μ, ν) is (μ, ν)-normal. □

4 Conclusion

Interchanging μ and ν by different weak forms of open sets we can characterize different weak forms of generalized open sets and different weak forms of regular and normal spaces. If $\mu = \nu$, then we get the results obtained in [6].

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Some generalization of integral inequalities for twice differentiable mappings involving fractional integrals

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Abstract. In this paper, a general integral identity involving Riemann-Liouville fractional integrals is derived. By use this identity, we establish new some generalized inequalities of the Hermite-Hadamard's type for functions whose absolute values of derivatives are convex.

1 Introduction

The following definition for convex functions is well known in the mathematical literature:

The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

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Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications(see, e.g.,[12, p.137], [6]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [6, 8, 9, 12], [14]-[16], [22], [23]) and the references cited therein.

In [16], Sarikaya et. al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1 *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a,b]$, then*

$$\begin{aligned}&\tfrac{1}{b-a}\int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right)\\ &\qquad = \tfrac{(b-a)^2}{2}\int_0^1 m(t)\left[f''(ta+(1-t)b)+f''(tb+(1-t)a)\right]dt,\end{aligned} \tag{2}$$

where

$$m(t) := \begin{cases} t^2, & t \in [0, \frac{1}{2}) \\ (1-t)^2, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Also, the main inequalities in [16], pointed out as follows:

Theorem 1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then*

$$\left|\tfrac{1}{b-a}\int_a^b f(x)dx - f(\tfrac{a+b}{2})\right| \leq \tfrac{(b-a)^2}{24}\left[\tfrac{|f''(a)|+|f''(b)|}{2}\right]. \tag{3}$$

Theorem 2 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$, then*

$$\left|\tfrac{1}{b-a}\int_a^b f(x)dx - f(\tfrac{a+b}{2})\right| \leq \tfrac{(b-a)^2}{8(2p+1)^{1/p}}\left[\tfrac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{1/q} \tag{4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [7, 10, 11, 13].

Definition 1 *Let* $f \in L_1[a,b]$. *The Riemann-Liouville integrals* $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x>a$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x<b$$

respectively. Here, $\Gamma(\alpha)$ *is the Gamma function and* $J^{0}_{a+}f(x)=J^{0}_{b-}f(x)=f(x)$.

Meanwhile, Sarikaya et al. [19] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2 *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 \leq a < b$. *If* $f' \in L[a,b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{b-a}{4}\left\{\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}\end{aligned} \tag{5}$$

with $\alpha > 0$.

It is remarkable that Sarikaya et al. [19] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3 *Let* $f:[a,b]\to\mathbb{R}$ *be a positive function with* $0 \leq a < b$ *and* $f \in L_1[a,b]$. *If* f *is a convex function on* $[a,b]$, *then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{6}$$

with $\alpha > 0$.

For some recent results connected with fractional integral inequalities see ([1, 2, 3, 4, 5], [17], [18], [20], [21], [24])

In this paper, we expand the Lemma 2 to the case of including a twice differentiable function involving Riemann-Liouville fractional integrals and some other integral inequalities using the generalized identity is obtained for fractional integrals.

2 Main results

For our results, we give the following important fractional integrtal identity:

Lemma 3 *Let* $f : [a, b] \to \mathbb{R}$ *be twice differentiable mapping on* (a, b) *with* $0 \leq a < b$. *If* $f'' \in L[a, b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned}
&(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha} f(\lambda a+(1-\lambda)b) \\
&\quad -\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)-} f(a)+(1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)+} f(b)\right] \\
&= -(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\int_0^1 t^{\alpha+1} f''[t(\lambda a+(1-\lambda)b)+(1-t)a]\,dt\right. \qquad (7) \\
&\quad \left. +\lambda\int_0^1 (1-t)^{\alpha+1} f''[tb+(1-t)(\lambda a+(1-\lambda)b)]\,dt\right\}
\end{aligned}$$

where $\lambda \in (0, 1)$ *and* $\alpha > 0$.

Proof. Integrating by parts

$$\begin{aligned}
&\int_0^1 t^{\alpha+1} f''[t(\lambda a+(1-\lambda)b)+(1-t)a]\,dt \\
&= \left.\frac{t^{\alpha+1} f'[t(\lambda a+(1-\lambda)b)+(1-t)a]}{(1-\lambda)(b-a)}\right|_0^1 \\
&\quad -\frac{\alpha+1}{(1-\lambda)(b-a)}\int_0^1 t^{\alpha} f'[t(\lambda a+(1-\lambda)b)+(1-t)a]\,dt \\
&= \frac{f'(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)}-\frac{\alpha+1}{(1-\lambda)(b-a)}
\end{aligned}$$

$$\times\left[\frac{f(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)}-\frac{\alpha}{(1-\lambda)(b-a)}\int_0^1 t^{\alpha-1}f\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt\right]$$
$$=\frac{f'(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)}-\frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{(1-\lambda)^2(b-a)^2}$$
$$+\frac{(\alpha+1)\alpha}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\int_a^{\lambda a+(1-\lambda)b}(x-a)^{\alpha-1}f(x)dx$$
$$=\frac{f'(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)}-\frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{(1-\lambda)^2(b-a)^2}$$
$$+\frac{(\alpha+1)\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)$$

that is,

$$\begin{aligned}&-\int_0^1 t^{\alpha+1}f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt\\&=-\frac{f'(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)}+\frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{(1-\lambda)^2(b-a)^2}\\&\quad-\frac{(\alpha+1)\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)\end{aligned}\tag{8}$$

and similarly we have

$$\begin{aligned}&-\int_0^1 (1-t)^{\alpha+1}f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]dt\\&=\frac{f'(\lambda a+(1-\lambda)b)}{\lambda(b-a)}+\frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{\lambda^2(b-a)^2}\\&\quad-\frac{(\alpha+1)\alpha}{\lambda^{\alpha+2}(b-a)^{\alpha+2}}\int_{\lambda a+(1-\lambda)b}^{b}(b-x)^{\alpha-1}f(x)dx\\&=\frac{f'(\lambda a+(1-\lambda)b)}{\lambda(b-a)}+\frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{\lambda^2(b-a)^2}\\&\quad-\frac{(\alpha+1)\Gamma(\alpha+1)}{\lambda^{\alpha+2}(b-a)^{\alpha+2}}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b).\end{aligned}\tag{9}$$

Adding (8) and (9) we have (7). This completes the proof. □

Corollary 1 *Under the assumptions Lemma 3 with $\lambda = \frac{1}{2}$, then it follows that*

$$\frac{-(b-a)^2}{8}\left\{\int_0^1 t^{\alpha+1} f''\left[t\left(\frac{a+b}{2}\right)+(1-t)a\right]dt\right.$$
$$\left.+\int_0^1 (1-t)^{\alpha+1} f''\left[tb+(1-t)\frac{a+b}{2}\right]dt\right\}$$
$$=(\alpha+1) f\left(\frac{a+b}{2}\right)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha} 2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right].$$

Remark 1 *If we choose $\alpha = 1$ in Corollary 1, we have*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{-(b-a)^2}{16}\left\{\int_0^1 t^2 f''\left[t\left(\frac{a+b}{2}\right)+(1-t)a\right]dt\right.$$
$$\left.+\int_0^1 (1-t)^2 f''\left[tb+(1-t)\frac{a+b}{2}\right]dt\right\}.$$

Theorem 4 *Let f:$[a,b] \to \mathbb{R}$ be twice differentiable mapping on (a,b) with $0 \le a < b$. If $|f''|^q$, $q \ge 1$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha} f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\right.$$
$$\left.\times\left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a)+(1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\right|$$
$$\le \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\frac{(\alpha+2)|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{\alpha+3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\frac{(\alpha+2)|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{\alpha+3}\right)^{\frac{1}{q}}\right\}. \tag{10}$$

where $\lambda \in (0,1)$ and $\alpha > 0$.

Proof. Firstly, we suppose that $q = 1$. Using Lemma 3 and convexity of $|f''|^q$, we find that

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\\
&\quad\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{+}}f(b)\right]\Bigg|\\
&\leq (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\int_0^1 t^{\alpha+1}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right|dt\right.\\
&\quad\left.+\lambda\int_0^1(1-t)^{\alpha+1}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right|dt\right\}\\
&\leq (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\int_0^1 t^{\alpha+1}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f''(a)\right|\right]dt\right.\\
&\quad\left.+\lambda\int_0^1(1-t)^{\alpha+1}\left[t\left|f''(b)\right|+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|\right]dt\right\}\\
&=\frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{\alpha+2}\left\{(1-\lambda)\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(a)\right|}{\alpha+3}\right)\right.\\
&\quad\left.+\lambda\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(b)\right|^q}{\alpha+3}\right)\right\}.
\end{aligned}$$

Secondly, we suppose that $q > 1$. Using Lemma 3 and power mean inequality, we have

$$\begin{aligned}
&\left\{(1-\lambda)\int_0^1 t^{\alpha+1}f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt\right.\\
&\quad\left.+\lambda\int_0^1(1-t)^{\alpha+1}f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]dt\right\}\\
&\leq(1-\lambda)\left(\int_0^1 t^{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha+1}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\lambda\left(\int_0^1(1-t)^{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha+1}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{11}$$

Hence, using convexity of $|f''|^q$ and (11) we obtain

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\\
&\quad\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)+}f(b)\right]\Bigg|\\
&\leq\frac{(b-a)^{2}(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\int_0^1 t^{\alpha+1}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f''(a)\right|\right]dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\lambda\left(\int_0^1(1-t)^{\alpha+1}\left[t\left|f''(b)\right|+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|\right]dt\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{(b-a)^{2}(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(a)\right|}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\lambda\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(b)\right|^{q}}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. $\square$

Corollary 2 *Under assumption Theorem 4 with $\lambda=\frac{1}{2}$, we obtain*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)\right]\right|\\
&\leq\frac{(b-a)^{2}}{8(\alpha+1)(\alpha+2)^{1-\frac{1}{q}}}\left\{\left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''(a)\right|^{q}}{\alpha+3}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''(b)\right|^{q}}{\alpha+3}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2 *If we choose $\alpha=1$ in Corollary 2, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leq\frac{(b-a)^{2}}{16\times 3^{1-\frac{1}{q}}}\left\{\left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''(a)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 5 *Let* $f:[a,b]\to\mathbb{R}$ *be twice differentiable mapping on* (a,b) *with* $0\le a<b$. *If* $|f''|^q$ *is convex on* $[a,b]$ *for same fixed* $q>1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\\
&\quad\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{+}}f(b)\right]\Bigg|\\
&\le\frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left\{(1-\lambda)\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\lambda\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}\tag{12}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\lambda\in(0,1)$ *and* $\alpha>0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder's inequality, we have

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\\
&\quad\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{+}}f(b)\right]\Bigg|\\
&\le(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\int_0^1 t^{p(\alpha+1)}\right)^{\frac{1}{p}}\right.\\
&\quad\left(\int_0^1|f''[t(\lambda a+(1-\lambda)b)+(1-t)a]|^q\,dt\right)^{\frac{1}{q}}\\
&\quad\left.+\lambda\left(\int_0^1(1-t)^{p(\alpha+1)}\right)^{\frac{1}{p}}\left(\int_0^1|f''[tb+(1-t)(\lambda a+(1-\lambda)b)]|^q\,dt\right)^{\frac{1}{q}}\right\}\\
&\le(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\\
&\quad\times\left\{(1-\lambda)\frac{1}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left(\int_0^1\left[t|f''(\lambda a+(1-\lambda)b)|^q+(1-t)|f''(a)|^q\right]dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\lambda\frac{1}{\left(p\left(\alpha+1\right)+1\right)^{\frac{1}{p}}}\left(\int_{0}^{1}\left[t\left|f''\left(b\right)\right|^{q}+\left(1-t\right)\left|f''(\lambda a+(1-\lambda)b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{\left(b-a\right)^{2}\left(1-\lambda\right)^{\alpha+1}\lambda^{\alpha+1}}{\left(p\left(\alpha+1\right)+1\right)^{\frac{1}{p}}}\left\{\left(1-\lambda\right)\left(\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^{q}+\left|f''\left(a\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\lambda\left(\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^{q}+\left|f''\left(b\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right\}.$$

□

Corollary 3 *Under assumption Theorem 5 with* $\lambda=\frac{1}{2}$, *we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]\right|$$

$$\leq\frac{\left(b-a\right)^{2}}{8\left(\alpha+1\right)\left(p\left(\alpha+1\right)+1\right)^{\frac{1}{p}}}\left\{\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''\left(a\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''\left(b\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right\}.$$

Remark 3 *If we choose* $\alpha=1$ *in Corollary 3, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$

$$\leq\frac{\left(b-a\right)^{2}}{16\left(2p+1\right)^{\frac{1}{p}}}\left\{\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''\left(a\right)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''\left(b\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right\}.$$

Theorem 6 *Let f:*$[a,b]\to\mathbb{R}$ *be twice differentiable mapping on* (a,b) *with* $0\leq a<b$. *If* $|f''|^{q}$ *is convex on* $[a,b]$ *for same fixed* $q>1$, *then the following inequality for fractional integrals holds:*

$$\left|\left(\alpha+1\right)\left(1-\lambda\right)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{\left(\alpha+1\right)\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\right.$$

$$\left.\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(a)+\left(1-\lambda\right)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{+}}f(b)\right]\right|$$

$$\leq\left(b-a\right)^{2}\left(1-\lambda\right)^{\alpha+1}\lambda^{\alpha+1}$$

$$\left\{(1-\lambda)\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right\}. \tag{13}$$

where $\lambda\in(0,1)$ *and* $\alpha>0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder's inequality, we have

$$\Bigg|(\alpha+1)(1-\lambda)^\alpha\lambda^\alpha f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha}$$
$$\times\left[\lambda^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)-}f(a)+(1-\lambda)^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)+}f(b)\right]\Bigg|$$
$$\leq(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\int_0^1 1^p\right)^{\frac{1}{p}}\left(\int_0^1 t^{q(\alpha+1)}\left|f''[t(\lambda a+(1-\lambda)b)+(1-t)a]\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\int_0^1 1^p\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^{q(\alpha+1)}\left|f''[tb+(1-t)(\lambda a+(1-\lambda)b)]\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\int_0^1 t^{q(\alpha+1)}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\int_0^1(1-t)^{q(\alpha+1)}\left[t\left|f''(b)\right|^q+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}$$
$$=(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right\}.$$

□

Corollary 4 *Under assumption Theorem 6 with* $\lambda=\frac{1}{2}$, *we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha 2^{1-\alpha}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)+}f(b)\right]\right|$$
$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\frac{(q(\alpha+1)+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left(q\left(\alpha+1\right)+1\right)\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''\left(b\right)\right|^{q}}{\left(q\left(\alpha+1\right)+1\right)\left(q\left(\alpha+1\right)+2\right)}\right)^{\frac{1}{q}}\Bigg\}.$$

Remark 4 *If we choose* $\alpha = 1$ *in Corollary 4, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left\{\left(\frac{\left(2q+1\right)\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''\left(a\right)\right|^{q}}{\left(2q+1\right)\left(2q+2\right)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left(2q+1\right)\left|f''\left(\frac{a+b}{2}\right)\right|^{q}+\left|f''\left(b\right)\right|^{q}}{\left(2q+1\right)\left(2q+2\right)}\right)^{\frac{1}{q}}\right\}.$$

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On certain upper bounds for the sum of divisors function $\sigma(n)$

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Abstract. Upper bounds for $\sigma(n)$ are provided in terms of other arithmetic functions as $\varphi(n)$, $d(n)$, $\psi(n)$, $P(n)$, etc. Comparision of older results are given, too.

1 Introduction

Let $n > 1$ be written in its canonical form

$$n = p_1^{a_1} \cdots p_r^{a_r} \tag{1}$$

where p_i are distinct primes, $a_i \geq 1$ integers, $i = 1, 2, \ldots, r$.

Then it is well-known the following representations formula for the sum of divisors function $\sigma(n)$, and number of divisors function $d(n)$:

$$\sigma(n) = \prod_{i=1}^{r} \left(\frac{p_i^{a_i+1} - 1}{p_i - 1}\right),\ d(n) = \prod_{i=1}^{r} (a_i + 1) \tag{2}$$

Similarly, for the Euler's totient $\varphi(n)$, and Dedekind's totient $\psi(n)$ one has:

$$\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right),\ \psi(n) = n \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right) \tag{3}$$

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Here $r = \omega(r)$ usually denotes the number of distinct divisors of n, in contrast with the total number of prime factors of n, which is $a_1+a_2+\cdots+a_n = \Omega(n)$.

In what follows, let $P(n)$ denote the greatest prime factor of n. This definition applies for $n > 1$, but in (2) and (3) it is obvious the completion for the case $n = 1$, namely:

$$\sigma(1) = d(1) = \varphi(n) = \psi(1) = 1 \tag{4}$$

There are many inequalities for these arithmetical functions; for a survey of results, see the monograph [4], or the recent papers [5], [6], [8], [9], [10], [11].

Partcularly, the following upper bounds for the function $\sigma(n)$ are known:

$$\sigma(n) \leq \frac{n^2}{\varphi(n)} \text{ for } n \geq 1, \tag{5}$$

$$\sigma(n) \leq \varphi(n)\big(d(n)\big)^2 \text{ for } n \geq 2, \tag{6}$$

$$\sigma(n) \leq nd(n) - \varphi(n) \text{ for } n \geq 2, \tag{7}$$

$$\sigma(n) \leq \varphi(n) + d(n)\big(n - \varphi(n)\big) \text{ for } n \geq 1, \tag{8}$$

$$\sigma(n) \leq \Big(\frac{n+1}{2}\Big)d(n) \text{ for } n \geq 1. \tag{9}$$

We note that inequality (5) has been rediscovered many times in the literature. In a slightly different form it appeared in a paper by O. Meissener from 1907 (see [4], p. 77). Inequality (6) is due to A. Makowski (1974, see [4], p. 11); (7) is due to A. Makowski (1960, see [4], p. 11), while (8) is due to the first author (1989, see [4], p. 10). Finally, (9) is due to E.S. Langford (1978, see [4], p. 86).

An improvement of (6) for odd values of n, is due to first author (1988, see [4], p. 10):

$$\sigma(n) \leq \varphi(n)d(n) \text{ for } n \geq 1 \text{ odd}. \tag{10}$$

It is easy to see that (10) implies for even values:

$$\sigma(n) < 2\varphi(n)d(n) \text{ for } n \geq 2 \text{ even}. \tag{11}$$

In the same year, K.T. Atanassov (see [4], p. 88) proved the upper bounds:

$$\sigma(n) \leq \varphi(n)P(n) \text{ for } n \text{ odd}, \tag{12}$$

$$\sigma(n) < 4\varphi(n)P(n) \text{ for } n \text{ even}. \tag{13}$$

Here, as above, relation (13) is an immediate consequence of (12).

Remark 1 *As* $d(n)$ *and* $P(n)$ *are not generally comparable, inequalities* (10) *and* (11) *are independent of each other. For any* $n = p =$ *prime, one has* $d(n) = 2 < P(n)$*, so* (10) *is better than* (12)*. Also for* $n = p^2$*, when* $d(n) = 3 \leq P(n)$ *for* n *odd. However, even for prime powers* $n = p^a$*, when* $p > a+1$*, clearly* (12) *will be stronger than* (10)*.*

2 Main results

I. New inequalities

One of the aims of this paper is to offer an improvement of (12) and (13); as follows:

Theorem 1 *One has*

$$\sigma(n) < \frac{3}{4}\varphi(n)P(n) \text{ for } n \geq 3 \text{ odd,} \tag{14}$$

and

$$\sigma(n) < 3\varphi(n)P(n) \text{ for } n \text{ even.} \tag{15}$$

Proof. The following auxiliary result by R. A. Rankin (1963, see [1], p. 193) will be used:

Lemma 1 *For all* $n \geq 1$ *one has*

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1))}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \sqrt{\frac{3/4}{2n+1}} \tag{16}$$

Now, as a consequence, we can deduce an upper bound for $\left(\frac{n}{\varphi(n)}\right)^2 = \prod_{p|n}\left(\frac{p}{p-1}\right)^2 \leq \frac{3^2}{2^2} \cdot \frac{5^2}{4^2} \cdots \frac{(2m+1)^2}{(2m)^2}$, where we have denoted the greatest prime divisor of n as $2m+1$. Now, remark that by (16) one has $\frac{3 \cdot 5 \cdots (2m+1)}{2 \cdot 4 \cdots (2m)} \leq \sqrt{\frac{3}{4}(2m+1)} = \sqrt{\frac{3}{4}P(n)}$, which implies relation (14), by remarking that by (5) one has $\sigma(n) < \left(\frac{n}{\varphi(n)}\right)^2 \cdot \varphi(n) \leq \frac{3}{4}\varphi(n)P(n)$, for $n \geq 3$ odd, since in (5) there is equality only for $n = 1$. If $n = 2^k N$ $(k \geq 1, N \text{ odd})$ is an even integer, then $P(n) = P(N)$, $\varphi(n) = 2^{k-1}\varphi(N)$ and $\sigma(n) = (2^{k+1}-1)\sigma(N)$, so (15) follows from (13) by $2^{k+1} - 1 < 2^{k+1}$ and $4\frac{3}{4} = 3$. $\square$

Remark 2 *From the proof of Theorem 1 we can remark that*

$$\left(\frac{n}{\varphi(n)}\right)^2 \le \frac{3}{4}P(n) \text{ for } n \ge 3 \text{ odd,} \tag{17}$$

and

$$\left(\frac{n}{\varphi(n)}\right)^2 \le 3P(n) \text{ for } n \text{ even,} \tag{18}$$

which improve the classical inequality (see [5])

$$\frac{n}{\varphi(n)} \le P(n) \textit{ for all } n \ge 2. \tag{19}$$

This follows by $\left(\frac{n}{\varphi(n)}\right)^2 \ge \frac{n}{\varphi(n)}$. *Clearly,* (17) *improves* (19) *for all odd* n, *while* (18) *improves* (19) *for all* $n \neq 2^k$ *(i.e. powers of* 2*). Indeed,* $3P(n) \le P^2(n)$ *only if* $P(n) \ge 3$, *and for even* n *this is true for* $n \neq 2^k$.

Theorem 2 *One has*

$$\sigma(n) < \psi(n) + \sigma(n) \cdot \frac{3}{8}P(n) \textit{ for } n \textit{ odd,} \tag{20}$$

and

$$\sigma(n) < \psi(n) + \sigma(n) \cdot \frac{3}{2}P(n) \textit{ for } n \textit{ even.} \tag{21}$$

Proof.

The proof of the following auxiliary result may be found in [5]:

Lemma 2 *For all* $n \ge 1$ *one has*

$$2 \cdot \frac{\psi(n)}{n} \ge 1 + \frac{n}{\varphi(n)} \tag{22}$$

Now, by (5) and (22) one can write: $\sigma(n) - \psi(n) < \frac{n^2}{\varphi(n)} - \frac{n}{2} - \frac{n^2}{2\varphi(n)}$, so

$$\frac{\sigma(n) - \psi(n)}{\varphi(n)} < \frac{n}{2\varphi(n)} \cdot \left(\frac{n}{\varphi(n)} - 1\right) \text{ for } n > 1. \tag{23}$$

Now, (20) and (21) are consequences of (17) and (18) as applications to (23). □

Theorem 3 *One has*

$$\sigma(n) < \psi(n) + \frac{k \cdot n^2}{\varphi(n)}, \tag{24}$$

where $k = 1 - \frac{6}{\pi^2} = 0.392\ldots$

For all odd $n \geq 3$ *one has*

$$\sigma(n) < \psi(n) + \frac{3}{10}\varphi(n)P(n). \tag{25}$$

Proof. We shall use the following inequality from [6]:

$$\psi(n) > \frac{c \cdot n^2}{\varphi(n)}, \; n \geq 1, \tag{26}$$

where $c = 6/\pi^2$. Now, by (5) and (26) one has $\sigma(n) - \psi(n) < \frac{n^2}{\varphi(n)} - \frac{cn^2}{\varphi(n)} = (1-c)\frac{n^2}{\varphi(n)} = \frac{kn^2}{\varphi(n)}$, which proves relation (24). Now, as $\frac{n^2}{\varphi(n)} = \varphi(n) \cdot \left(\frac{n}{\varphi(n)}\right)^2$, and by (17) we get (18), by remarking that $\frac{3}{4}k = 0.294 \cdots < 0.3 = \frac{3}{10}$. □

Remark 3 *Relation* (25) *improves slightly* (20), *as* $\frac{3}{10} < \frac{3}{8}$.

Theorem 4 *One has*

$$\sigma(n) < \frac{\pi^2}{6} \cdot \psi(n), \; n \geq 1. \tag{27}$$

For all odd n *one has*

$$\sigma(n) < \psi(n) + a \cdot \varphi(n) \cdot 2^{\omega(n)}, \tag{28}$$

where $a = \pi^2/6 - 1$

Proof. For inequality (27) see paper [6]. For (28) use (27) and the remark that $\frac{\psi(n)}{\varphi(n)} = \prod_{p|n} \frac{p+1}{p-1} \leq 2^{\omega(n)}$ since $\frac{p+1}{p-1} \leq 2$ for $p \geq 3$ (i.e. $n =$ odd). Therefore, we can write $\sigma(n) - \psi(n) < a \cdot \psi(n) = a \cdot \varphi(n) \cdot \left(\frac{\psi(n)}{\varphi(n)}\right) < a\varphi(n) \cdot 2^{\omega(n)}$. □

Remark 4 *As* $0 < a < 1$, *from* (28) *we get also*

$$\sigma(n) < \psi(n) + \varphi(n) \cdot 2^{\omega(n)}, \; n \text{ odd}. \tag{29}$$

When n is squarefull, this improves the following inequality by K.T. Atanassov (see [12]):

$$\sigma(n) < \psi(n) + \varphi(n) \cdot 2^{\Omega(n)-\omega(n)}, \; n \geq 1. \tag{30}$$

Indeed, if n is squarefull (i.e., when in (1) all $a_i \geq 2$ for $i = 1, 2, \ldots, r$), we get $\Omega(n) = a_1 + \cdots + a_r \geq 2r = 2\omega(n)$, so $\omega(n) \leq \Omega(n) - \omega(n)$, and (29) refines (30).

II. Comparison of upper bounds for $\sigma(n)$

Many times, there have been published various inequalities containing also other arithmetic functions, but without comparison to each other. For example, it is not remarked in the literature that, inequality (5) is stronger than (6):

Theorem 5 *For all $n \geq 1$*

$$\sigma(n) \leq \frac{n^2}{\varphi(n)} \leq \varphi(n)\big(d(n)\big)^2, \tag{31}$$

i.e., inequality (5) implies inequality (6).

Proof. The second inequality of (31) may be rewritten as

$$\varphi(n)d(n) \geq n, \tag{32}$$

which is a known inequality of R. Sivaramakrishnan (1967, see [4], p. 10). The following improvement of (32) is due to the first author (1989, see [4], p. 10):

$$\varphi(n)d(n) \geq \varphi(n) + n - 1, n \geq 1. \tag{33}$$

□

Inequality (10) improves also (6) for odd values of n. The following result improves (10):

Theorem 6

$$\sigma(n) \leq \frac{\psi(n) \cdot d(n)}{2^{\omega(n)}}, \; n \geq 1. \tag{34}$$

For odd n, one has

$$\sigma(n) \leq \frac{\psi(n) \cdot d(n)}{2^{\omega(n)}} \leq \varphi(n)d(n). \tag{35}$$

Proof. Inequality (34) is due to the first author (1988, see [2]). Now, the socond inequality of (35) can be written as $\frac{\psi(n)}{\varphi(n)} \leq 2^{\omega(n)}$, n odd, which is proved earlier (see the proof of Theorem 4). $\square$

Theorem 7 *For all* $n \geq 1$,

$$\sigma(n) \leq n \cdot [\omega(n) + 1]. \tag{36}$$

For $n \neq$ *prime one has*

$$\sigma(n) \leq n \cdot [\omega(n) + 1] \leq n \cdot \Omega(n). \tag{37}$$

Proof. Inequality (36) appears in the first author's paper [3] from 1989, and it improves the better-known inequality due to R. L. Duncan (1967, see [4], p. 79):

$$\sigma(n) < n \cdot [\frac{7\omega(n) + 10}{6}],\ n \geq 1. \tag{38}$$

Indeed, it is easy to see that, $\omega(n) + 1 < \frac{7\omega(n)+10}{6}$.

We shall offer here a simple proof of (36). Assume that in the prime factorization (1) one has $p_1 < \cdots < p_r$. Then $p_1 \geq 2$, $p_2 \geq 3$, $\cdots$, $p_r \geq r+1$, so we get by (3) $\varphi(n) \geq n \cdot \left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{r+1}\right) = n \cdot \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{r}{r+1} = \frac{n}{r+1}$, giving:

$$\varphi(n) \geq \frac{n}{\omega(n) + 1},\ n \geq 1. \tag{39}$$

Now, inequality (36) is a consequence of (5) combined with (39). The second inequality of (37) is true, if $\Omega(n) - \omega(n) \geq 1$. This holds only if in the prime factorization (1) one has that $\Omega(n) \neq \omega(n)$, i.e. if $n \neq$ squarefree (i.e. $n = p_1 \cdots p_r$). The inequality

$$\sigma(n) \leq n \cdot \Omega(n),\ n \neq \text{ prime}, \tag{40}$$

is due to first author (1988, see [4], p. 87). In fact, a new proof of (40) will be offered here, if we prove that, it is true for any $n = p_1 \cdot p_2 \cdots p_r$ (p_i distinct primes), for $r \geq 2$. Equivalently,

$$(p_1 + 1) \cdots (p_r + 1) \leq p_1 \cdots p_r \cdot r,\ r \geq 2. \tag{41}$$

As $\left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_r}\right) \leq \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{r}\right) = \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{r+1}{r} = \frac{r+2}{2} \leq r$ for $r \geq 2$, inequality (40) is proved. $\square$

Remark 5 *The above proof shows that in fact*

$$\frac{\psi(n)}{n} \leq \frac{\omega(n)+2}{2}, \; n \geq 2. \tag{42}$$

Theorem 8 *If $\omega(n) \geq 3$, then*

$$\sigma(n) < n[\omega(n)+1] < d(n) \cdot \left(\frac{n+1}{2}\right), \tag{43}$$

(i.e. (36) is stronger than (9)). If $\omega(n) = 1$ and $n = p^a$ (p prime, $a \geq 1$), then for $a \geq 3$, (43) is true. If $\omega(n) = 2$ and n not squarefree, then (43) is again true. If $\omega(n) = 1$ and $n = p^a$ with $a \in \{1, 2\}$ or $n =$ squarefree, one has

$$\sigma(n) \leq d(n) \cdot \left(\frac{n+1}{2}\right) < n[\omega(n)+1]. \tag{44}$$

Proof. As $d(n) \geq 2^{\omega(n)}$, it is sufficient to prove that

$$2^{\omega(n)-1} \cdot (n+1) \geq n[\omega(n)+1]. \tag{45}$$

The inequality

$$2^{k-1} \geq k+1, \; k \geq 3, \tag{46}$$

can be proved immediately by induction. By letting $k = \omega(n)$, since $n+1 > n$, the proof of (41) is complete. Clearly, all inequalities are strict.

If $n = p^a$ (p prime), then $d(n) = a+1 \geq 4$, for $a \geq 3$ and $d(n) \geq 4 > \frac{4n}{n+1}$, so again (41) is true with strict inequality.

If $n \neq pq$ then $n = p^a \cdot q^b$, where at least one of a and $b \geq 2$. In this case $d(n) = (a+1)(b+1) \geq 2 \cdot 3 = 6$. On the other hand, one has $6 > \frac{6n}{n+1}$, so again get the strict inequality.

For $n = p$ one has $2 \cdot \left(\frac{p+1}{2}\right) < p \cdot 2$, while for $n = p^2$, $3 \cdot \left(\frac{p^2+1}{2}\right) < p^2 \cdot 2$ by $3 < p^2 (p \geq 2)$.

Finally, for $n = pq$, we have $4 \cdot \left(\frac{pq+1}{2}\right) < 3pq$ by $pq > 2$. □

Remark 6 *Therefore (43) is true for all $n > 1$ which are not primes, or square of primes, or which are not the product of two distinct primes.*

As a comparison of (8) and (9), the following holds true:

Theorem 9 *For all $n > 2$ even numbers one has*

$$\sigma(n) < d(n) \cdot \left(\frac{n+1}{2}\right) < \varphi(n) + d(n)(n - \varphi(n)). \tag{47}$$

When n is odd, generally (8) and (9) are not comparable.

Proof. The first inequality of (47) is strict as in (9) there is equality only for $n =$ prime. The second inequality may be written also as

$$d(n) \cdot \left(\frac{2\varphi(n) - n + 1}{2}\right) < \varphi(n). \tag{48}$$

Now, if n is even number, it is well-known (and it easily follows by the first relation of (2)) that $\varphi(n) \leq \frac{n}{2}$. This implies that

$$2\varphi(n) - n + 1 \leq 1. \tag{49}$$

Now, if eventually $2\varphi(n) - n + 1 \leq 0$, then (48) is trivially true. Otherwise, we will use besides (47), the following known inequality (see [4], p. 11):

$$d(n) < \varphi(n) \text{ for } n > 30. \tag{50}$$

Then inequality (48) holds true for all even $n > 30$. A particular verification shows that, in fact (48) holds true for all even numbers $4 \leq n \leq 30$. This proves the first part of the theorem.

Let $n = p^2$, where $p \geq 5$ is prime. As $d(p^2) = 3$ and $\varphi(p^2) = p^2 - p$, it is immediate that the second inequality of (47) holds in reverse order.

The same can be verified for $n = 5p$, where $p \geq 7$ is a prime.

On the other hand, for $n = 3p$ ($p \geq 5$ prime), the inequality holds in this order. Therefore, there are infinitely many odd values of n for which the inequality is true in both senses. □

Remark 7 *Since (8) may be written also as*

$$\sigma(n) + \varphi(n) \leq nd(n) + \varphi(n)(2 - d(n)), \tag{51}$$

by $2 - d(n) \leq 0$, clearly this inequality strongly refines relation (7). Another refinement of (7), namely

$$\sigma(n) + \varphi(n) \leq n \cdot 2^{\omega(n)}, \tag{52}$$

is due to C.A. Nicol (1996, see [4], p. 10).

When n is squarefree (i.e., a product of distinct primes), then, as $nd(n) = 2^{\omega(n)}$ and $2 \leq d(n)$, clearly (51) is stronger than (52).

It is easy to verify that for $n = 2^k (k \geq 1)$, (52) is stronger than (51).

An inequality refining (9) for all n, will be provided by

Theorem 10 *One has*

$$\sigma(n) \leq \frac{d(n)\sigma^*(n)}{2^{\omega(n)}} \leq d(n) \cdot (\frac{n+1}{2}),\ n \geq 1, \tag{53}$$

where $\sigma^*(n) = \prod_{i=1}^{r}(p_i^{a_i}+1)$ *(for the prime factorization (1) of* $n > 1$*),* $\sigma^*(1) = 1$*, denotes the sum of unitary divisors of* n *(see [4]).*

Proof. The first inequality of (53) is published in first author's paper [7] (1994), as an application of an inequality of Klamkin (see also [8])

For the second relation of (53), apply the following auxiliary result:

Lemma 3 *For* $x_i \geq 1$ *real numbers* $(i = 1, 2, \dots, r)$ *we have*

$$\prod_{i=1}^{r}(x_i+1) \leq 2^{r-1} \cdot (\prod_{i=1}^{r} x_i + 1). \tag{54}$$

This result is well-known, see e.g. [1].

Apply now (54) for $x_i = p_i^{a_i}$, $r = \omega(n)$, where n has prime factorization (1). Then we get

$$\sigma^*(n) \leq 2^{\omega(n)-1} \cdot (n+1),\ n > 1, \tag{55}$$

which gives the second inequality of (53). □

Remark 8 *As* $\psi(n) = \prod_{i=1}^{r}(p_i^{a_i} + p_i^{a_i-1})$*, clearly* $\sigma^*(n) \leq \psi(n)$*, so the first inequality of* (53) *offers a refinement of inequality* (34).

It is a natural question if (34) and (53) may be further compared. The following result answers this question:

Theorem 11 *For* $\omega(n) \geq 2$ *one has*

$$\frac{\psi(n)d(n)}{2^{\omega(n)}} < d(n) \cdot \frac{n+1}{2}. \tag{56}$$

Proof. By inequality (42) it will be sufficient to study

$$n \cdot \frac{\omega(n)+2}{2} \le 2^{\omega(n)} \cdot \frac{n+1}{2}. \tag{57}$$

For $\omega(n) = 1$ this becomes $3n \le 2(n+1)$, which is false.

Assume $\omega(n) \ge 2$. Then, as $2^k \ge k+2$ for any $k \ge 2$, by letting $k = \omega(n)$, and remarking that $n+1 > n$, (57) immediately follows. □

Remark 9 *Therefore, one has the following completion to Theorem 10:*

$$\sigma(n) \le \frac{d(n) \cdot \sigma^*(n)}{2^{\omega(n)}} \le \frac{d(n) \cdot \psi(n)}{2^{\omega(n)}} < d(n) \cdot \frac{n+1}{2}, \text{ for } \omega(n) \ge 2. \tag{58}$$

Remark 10 *In 2010 the first author (see [9]) proved a refinement of a new type of inequality (7):*

$$\sigma(n) \le \frac{nd(n) - \varphi(n)}{\omega(n)}, \text{ for } n \ge 2 \text{ and distinct from } 6, \tag{59}$$

which clearly gives another improvement of (7), related to (58):

$$\sigma(n) \le \frac{nd(n) - \varphi(n)}{2} < \frac{nd(n)}{2}, \text{ for } \omega(n) \ge 2 \text{ and } n \text{ distinct from } 6. \tag{60}$$

For inequalities related to the weaker relation of (60), see also [5].

Theorem 12 *1) The following improvement of (9) holds true:*

$$\sigma(n) < \frac{nd(n) - \varphi(n)}{2} < \frac{n-1}{2} \cdot d(n) \text{ for } \omega(n) \ge 2 \text{ and } n > 30; \tag{61}$$

2) The inequality

$$\sigma(n) \le \frac{n-1}{2} \cdot d(n), \tag{62}$$

holds true for i) $\omega(n) \ge 2$ *and* n *distinct from* 6 *. There is equality only for* $n = 10$*. ii) if* $\omega(n) = 1$*, let* $n = p^k$ *(*p *prime,* $k \ge 1$*). Then (62) is true if: a)* $k = 2$ *and* $p \ge 5$*; b)* $k \ge 3$ *and* $p \ge 5$*; c)* $k \ge 3$ *and* $p = 3$*; d)* $k \ge 4$ *and* $p = 2$*.*

Proof. 1) Applying inequality (59) for $\omega(n) \geq 2$ and n distinct from 6, the first inequality of (61) follows. The inequality is strict, since in [9] it is proved that in (59) there is equality only for $n = 10$. The second inequality follows by remarking that one can apply relation (50) for $n > 30$.

2) i) By (61), relation (62) holds true for $\omega(n) \geq 2$ and $n > 30$. A simple verification for $n = 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28$, which are the n with $\omega(n) = 2$, and $n < 30$ shows that (62) is not true only for $n = 6$. Also, it is true for $n = 10$, with equality.

ii) For $n = p^k$ inequality (62) becomes

$$\frac{p^{k+1}-1}{p-1} < \frac{p^k-1}{2} \cdot (k+1). \tag{63}$$

This inequality is not true for $k = 1$ (i.e. $n = p$=prime). Let $k = 2$. Then (63) becomes after a simple transformation: $p(p-2) > 5$. This is clearly true only for $p \geq 5$, so case a) is proved.

Apply now the Cauchy mean-value theorem of differential calculus to the functions $f(x) = x^{k+1}$ and $g(x) = x^k$ on the interval $[1, p]$, by obtaining: $\frac{f(p)-f(1)}{g(p)-g(1)} = \frac{f(c)}{g(c)}$, where c is in $(1, p)$. We get in this particular case: $\frac{p^{k+1}-1}{p^k-1} = \frac{(k+1)\cdot c^k}{k \cdot c^{k-1}} = \frac{k+1}{k} \cdot c < (k+1) \cdot \frac{p}{k}$.

Now, we have that $\frac{p}{k} < \frac{p-1}{2}$ for all $k \geq 3$ and $p \geq 5$, so case b) follows.

For $k \geq 3$ clearly we have to consider only the remaining cases $p = 2$ and $p = 3$. For $p = 2$ we have $n = 2^k$, and the inequality can be written equivalently as $2^k \cdot (k-3) > k$. This is true only for $k \geq 4$ (mathematical induction). Let now $p = 3$. Then we get the inequality $3^k \cdot (k-2) > k$, and this holds only for $k \geq 3$. Therefore, cases c) and d) are completely proved. □

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Properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative

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Abstract. In the paper are studied the properties of the image of a class of analytic functions defined by the Ruscheweyh derivative trough the Bernardi operator.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $g \in \mathcal{A}$ where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{2}$$

The Hadamard product is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

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Ruscheweyh [4] defined the derivative $D^\gamma : A \to A$ by

$$D^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), (\gamma \geq -1). \tag{4}$$

In the particular case $n \in \mathbb{N}_0 = \{0, 1, 2 \dots\}$

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}. \tag{5}$$

The symbol $D^n f(z) (n \in \mathbb{N}_0)$ was called the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. It is easy to see that

$$D^0 f(z) = f(z), D^1 f(z) = z f'(z)$$

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k \tag{6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{7}$$

Definition 1 *Let* f *and* g *be analytic functions in* U*. We say that the function* f *is subordinate to the function* g*, if there exist a function* w*, which is analytic in* U *and* $w(0) = 0; |w(z)| < 1; z \in U$*, such that* $f(z) = g(w(z)); \forall z \in U$*. We denote by* $\prec$ *the subordination relation.*

Attiya and Aouf defined in [2] the class $Q(n, \lambda, A, B)$ this way:

Definition 2 [2], [3] *For* $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ *let* $Q(n, \lambda, A, B)$ *denote the subclass of* $\mathcal{A}$ *which contain functions* $f(z)$ *of the form* (1) *such that*

$$(1-\lambda)(D^n f(z))' + \lambda (D^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{8}$$

Definition 3 [5] *A function* $f(z)$ *of the form* (1) *is said to be in the class* $V(\theta_k)$ *if* $f \in A$ *and* $\arg(a_k) = \theta_k$ *,*$\forall k \geq 2$*. If* $\exists \delta \in \mathbb{R}$ *such that* $\theta_k + (k-1)\delta \equiv \pi (\text{mod } 2\pi), \forall k \geq 2$ *then* $f(z)$ *is said to be in the class* $V(\theta_k, \delta)$*. The union of* $V(\theta_k, \delta)$ *taken over all possible sequences* $\{\theta_k\}$ *and all possible real numbers* δ *is denoted by* V*. Let* $VQ(n, \lambda, A, B)$ *denote the subclass of* V *consisting of functions* $f(z) \in Q(n, \lambda, A, B)$*.*

Theorem 1 [3] *Let the function* f *defined by* (1) *be in V. Then* $f \in VQ(n, \lambda, A, B)$, *if and only if*

$$E(f) = \sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \leq (B-A)(n+1) \tag{9}$$

where

$$C_k = (1+B)[n+1+\lambda(k-1)].$$

The extremal functions are

$$f_k(z) = z + \frac{(B-A)(n+1)}{kC_k\delta(n,k)} e^{i\theta_k} z^k, (k \geq 2).$$

Main results

Theorem 2 *Let*

$$F(z) = I_c f(z) = \frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt, c \in \mathbb{N}^*$$

If $f \in VQ(n, \lambda, 2\alpha - 1, B)$ *then* $F \in VQ(n, \lambda, 2\beta - 1, B)$, *where*

$$\beta = \beta(\alpha) = \frac{B+1+2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

The result is sharp.
Remark: The operator I_c *is the well-known Bernardi operator.*

Proof.
Let $f \in VQ(n, \lambda, 2\alpha - 1, B)$ and suppose it has the form (1). Then

$$F(z) = \frac{c+1}{z^c}\int_0^z \left(t + \sum_{k=2}^{\infty} a_k t^k\right) t^{c-1}dt =$$

$$= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k.$$

Since $f \in VQ(n, \lambda, 2\alpha - 1, B)$ we have

$$\sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \leq [B - (2\alpha - 1)](n+1)$$

or equivalently

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k|}{B-2\alpha+1} \le n+1. \tag{10}$$

We know from Theorem 1 that $F \in VQ(n,\lambda,2\beta-1,B)$ if and only if

$$\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|b_k| \le [B-(2\beta-1)](n+1)$$

or

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\frac{c+1}{c+k}\,|a_k|}{B-2\beta+1} \le n+1. \tag{11}$$

We note that the inequalities

$$\frac{k\delta(n,k)C_k\frac{c+1}{c+k}\,|a_k|}{B-2\beta+1} \le \frac{k\delta(n,k)C_k\,|a_k|}{B-2\alpha+1}, \forall\ k \ge 2 \tag{12}$$

imply (11). From (12) we have

$$\frac{c+1}{(c+k)(B-2\beta+1)} \le \frac{1}{B-2\alpha+1}$$

$$(c+1)(B-2\alpha+1) \le (c+k)(B-2\beta+1), \forall\ k \ge 2$$

$$\beta \le \frac{(k-1)(B+1)+2\alpha(c+1)}{2(c+k)}.$$

Let us consider the function

$$E(x) = \frac{(x-1)(B+1)+2\alpha(c+1)}{2(c+x)},$$

then its derivative is:

$$E'(x) = \frac{1}{2}\frac{(c+1)(B+1-2\alpha)}{(c+x)^2} > 0.$$

$E(x)$ is an increasing function. In our case we need $\beta \le E(k)$ and for this reason we choose $\beta = \beta(\alpha) = E(2) = \dfrac{B+1+2\alpha(c+1)}{2(c+2)}$.

$$\beta(\alpha) > \alpha \Leftrightarrow B+1+2\alpha c+2\alpha > 2\alpha c+4\alpha \Leftrightarrow B+1-2\alpha > 0.$$

The result is sharp, because if

$$f_2(z) = z + \frac{(B - 2\alpha + 1)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2,$$

then

$$F_2 = I_c f_2$$

belongs to $VQ(n, \lambda, 2\beta - 1, B)$ and its coefficients satisfy the corresponding inequality (9) with equality. Indeed,

$$F_2(z) = z + \frac{(B - 2\alpha + 1)(n+1)}{2C_2\delta(n,2)} \frac{c+1}{c+2} e^{i\theta_2} z^2 = z + \frac{(B - 2\beta(\alpha) + 1)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2$$

and

$$E(F_2) = 2\delta(n,2)C_2 \frac{(B - 2\beta(\alpha) + 1)(n+1)}{2C_2\delta(n,2)} = (B - 2\beta(\alpha) + 1)(n+1).$$

□

Theorem 3 *If* $f \in VQ(n, \lambda, A, B)$ *then* $F \in VQ(n, \lambda, A^*, B)$, *where* $A^* = \dfrac{B + A(c+1)}{c+2} > A$. *The result is sharp.*

Proof. Let $f \in VQ(n, \lambda, A, B)$ and suppose it has the form (1). Then

$$F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k.$$

Since $f \in VQ(n, \lambda, A, B)$ we have $\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \leq (B - A)(n+1)$ or equivalently

$$\frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k |a_k|}{B - A} \leq n + 1.$$

We know from Theorem 1 that $F \in VQ(n, \lambda, A^*, B)$ if and only if

$$\frac{\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k \frac{c+1}{c+k} |a_k|}{B - A^*} \leq n + 1, \forall k. \tag{13}$$

We note that
$$\frac{k\delta(n,k)C_k\frac{c+1}{c+k}|a_k|}{B-A^*} \le \frac{k\delta(n,k)C_k|a_k|}{B-A} \tag{14}$$
implies (13). From (14) we have
$$\frac{c+1}{(c+k)(B-A^*)} \le \frac{1}{B-A}$$
$$(c+1)(B-A) \le (c+k)(B-A^*), \forall\ k \ge 2$$
$$A^* \le \frac{B(k-1)+A(c+1)}{(c+k)}.$$

Let us consider the function
$$E(x) = \frac{B(x-1)+A(c+1)}{x+c};$$
its derivative is:
$$E'(x) = \frac{(B-A)(c+1)}{(x+c)^2} > 0.$$
$E(x)$ is an increasing function.

In our case we need $A^* \le E(k), \forall\ k \ge 2$ and for this reason we choose $A^* = E(2) = \dfrac{B+A(c+1)}{c+2}$.

We note that $A^* > A$, because
$$B + A(c+1) > A(c+2) \Leftrightarrow B > A.$$

The result is sharp, because if
$$f_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)}e^{i\theta_2}z^2,$$
then
$$F_2 = I_c f_2$$
belongs to $VQ(n,\lambda,A^*,B)$ and its coefficients satisfy the corresponding inequality (9) with equality. Indeed,
$$F_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)}\frac{c+1}{c+2}e^{i\theta_2}z^2 = z + \frac{(B-A^*)(n+1)}{2C_2\delta(n,2)}e^{i\theta_2}z^2$$
and
$$E(F_2) = 2\delta(n,2)C_2\frac{(B-A^*)(n+1)}{2C_2\delta(n,2)} = (B-A^*)(n+1).$$

□

Theorem 4 *If* $f \in VQ(n, \lambda, A, B)$ *then* $F \in VQ(n, \lambda, A, B^*)$, *where*

$$B^* = \frac{B(c+1)+A}{c+2} < B.$$

The result is sharp.

Proof. Let $f \in VQ(n, \lambda, A, B)$ and suppose it has the form (1).

Since $f \in VQ(n, \lambda, A, B)$ we have $\sum\limits_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \le (B-A)(n+1)$ or equivalently

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k|}{B-A} \le n+1.$$

We know from *Theorem 1* that $F \in VQ(n, \lambda, A, B^*)$ if and only if

$$\sum_{k=2}^{\infty} k\delta(n,k)C_k |b_k| \le (B^*-A)(n+1)$$

or

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k \frac{c+1}{c+k} |a_k|}{B^*-A} \le n+1. \tag{15}$$

We note that

$$\frac{k\delta(n,k)C_k \frac{c+1}{c+k} |a_k|}{B^*-A} \le \frac{k\delta(n,k)C_k |a_k|}{B-A}, \forall k \tag{16}$$

implies (15). From (16) we have

$$\frac{c+1}{(c+k)(B^*-A)} \le \frac{1}{B-A}$$
$$(c+1)(B-A) \le (c+k)(B^*-A), \forall k \ge 2$$
$$\frac{B(c+1)+A(k-1)}{c+k} \le B^*.$$

Let

$$E(x) = \frac{B(c+1)+A(x-1)}{x+c}$$

its derivative is:

$$E'(x) = \frac{(c+1)(A-B)}{(x+c)^2} < 0.$$

$E(x)$ is a decreasing function. In our case we need $E(k) \leq B^*$ and for this reason we choose $B^* = E(2) = \dfrac{A + B(c+1)}{c+2}$

$$B^* < B \Leftrightarrow A + Bc + B < Bc + 2B \Leftrightarrow A < B.$$

The result is sharp, because if

$$f_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2,$$

then

$$F_2 = I_c f_2$$

belongs to $VQ(n, \lambda, A, B^*)$ and its coefficients satisfy the corresponding inequality (9) with equality. Indeed,

$$F_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)} \frac{c+1}{c+2} e^{i\theta_2} z^2 = z + \frac{(B^*-A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2$$

and

$$E(F_2) = 2\delta(n,2)C_2 \frac{(B^*-A)(n+1)}{2C_2\delta(n,2)} = (B^*-A)(n+1).$$

□

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