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On a new subclass of bi-univalent functions satisfying subordinate conditions

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Abstract. In the present investigation, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $\mathcal{S}_{\Sigma}(\lambda, \phi)$. The results presented in this paper improve or generalize the recent work of Magesh and Yamini [15].

1 Introduction and definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [8] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

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and

$$f\left(f^{-1}(w)\right)=w\ ,\quad \left(|w|<r_0(f)\ ,\ r_0(f)\geq \frac{1}{4}\right),$$

where

$$f^{-1}(w)=w\ -a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots .$$

A function $f(z)\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z)\prec g(z)\,,\qquad (z\in U)$$

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0)=0\ \text{ and }\quad |w(z)|<1\qquad (z\in U)$$

such that

$$f(z)=g(w(z))\qquad (z\in U)\,.$$

Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, (see [20]).

Lewin [14] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2|\leq\sqrt{2}$ for $f\in\Sigma$. Netanyahu [16] showed that $\max|a_2|=\frac{4}{3}$ if $f(z)\in\Sigma$.

Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $S^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α $(0<\alpha\leq 1)$ respectively (see [16]). Thus, following Brannan and Taha [4], a function $f(z)\in A$ is the class $S_{\Sigma}^{\star}(\alpha)$ of strongly bi-starlike functions of order α $(0<\alpha\leq 1)$ if each of the following conditions is satisfied:

$$f\in\Sigma,\quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ z\in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ w\in U)$$

where g is the extension of f^{-1} to U. The classes $S_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^{\star}(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [3], [7], [9], [13], [15], [20], [21], [22]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [6], [10], [11], [12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\}$; $\mathbb{N}=\{1,2,3,...\}$) is still an open problem.

In this paper, by using the method [17] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Magesh and Yamini and get more accurate estimates than that given in [15].

2 Coefficient estimates

In the following, let ϕ be an analytic function with positive real part in U, with $\phi(0)=1$ and $\phi'(0)>0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots \quad (B_1>0). \tag{2}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk U with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, and suppose that

$$u(z)=b_1z+\sum_{n=2}^{\infty}b_nz^n, \quad v(w)=c_1w+\sum_{n=2}^{\infty}c_nw^n \quad (|z|<1). \tag{3}$$

It is well known that

$$|b_1|\leq 1, \quad |b_2|\leq 1-|b_1|^2, \quad |c_1|\leq 1, \quad |c_2|\leq 1-|c_1|^2. \tag{4}$$

Next, the equations (2) and (3) lead to

$$\phi(u(z))=1+B_1b_1z+\left(B_1b_2+B_2b_1^2\right)z^2+\cdots, \quad |z|<1 \tag{5}$$

and

$$\phi(v(w))=1+B_1c_1w+\left(B_1c_2+B_2c_1^2\right)w^2+\cdots, \quad |w|<1. \tag{6}$$

Definition 1 *A function* $f \in \Sigma$ *is said to be* $S_\Sigma(\lambda, \phi)$, $0 \leq \lambda \leq 1$, *if the following subordination hold*

$$\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4\left(\lambda - \lambda^2\right) z + \left(2\lambda^2 - \lambda\right) zf'(z) + \left(2\lambda^2 - 3\lambda + 1\right) f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w) + \left(2\lambda^2 - \lambda\right) w^2 g''(w)}{4\left(\lambda - \lambda^2\right) w + \left(2\lambda^2 - \lambda\right) wg'(w) + \left(2\lambda^2 - 3\lambda + 1\right) g(w)} \prec \phi(w)$$

where $g(w) = f^{-1}(w)$.

Theorem 1 *Let* f *given by (1) be in the class* $S_\Sigma(\lambda, \phi)$. *Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|\left(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1\right) B_1^2 - \left(1 + 3\lambda - 2\lambda^2\right)^2 B_2\right| + \left(1 + 3\lambda - 2\lambda^2\right)^2 B_1}} \tag{7}$$

and

$$|a_3| \leq \begin{cases} \frac{B_1}{2(2\lambda^2+1)}; & \text{if } B_1 \leq \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)} \\ \frac{\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2 B_2\right| B_1+2\left(2\lambda^2+1\right)B_1^3}{2(2\lambda^2+1)\left[\left|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2 B_2\right|+(1+3\lambda-2\lambda^2)^2 B_1\right]}; & \\ & \text{if } B_1 > \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)}. \end{cases} \tag{8}$$

Proof. Let $f \in S_\Sigma(\lambda, \phi)$, $0 \leq \lambda \leq 1$. Then there are analytic functions $u, v : U \to U$ given by (3) such that

$$\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4\left(\lambda - \lambda^2\right) z + \left(2\lambda^2 - \lambda\right) zf'(z) + \left(2\lambda^2 - 3\lambda + 1\right) f(z)} = \phi(u(z)) \tag{9}$$

and

$$\frac{wg'(w) + \left(2\lambda^2 - \lambda\right) w^2 g''(w)}{4\left(\lambda - \lambda^2\right) w + \left(2\lambda^2 - \lambda\right) wg'(w) + \left(2\lambda^2 - 3\lambda + 1\right) g(w)} = \phi(v(w)) \tag{10}$$

where $g(w) = f^{-1}(w)$. Since

$$\begin{aligned} &\frac{zf'(z) + \left(2\lambda^2 - \lambda\right) z^2 f''(z)}{4\left(\lambda - \lambda^2\right) z + \left(2\lambda^2 - \lambda\right) zf'(z) + \left(2\lambda^2 - 3\lambda + 1\right) f(z)} \\ = \;& 1 + \left(1 + 3\lambda - 2\lambda^2\right) a_2 z \\ &+ \left[\left(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1\right) a_2^2 + \left(4\lambda^2 + 2\right) a_3\right] z^2 + \cdots \end{aligned}$$

and

$$\frac{wg'(w)+\left(2\lambda^2-\lambda\right)w^2g''(w)}{4\left(\lambda-\lambda^2\right)w+\left(2\lambda^2-\lambda\right)wg'(w)+\left(2\lambda^2-3\lambda+1\right)g(w)}$$

$$\begin{aligned} &= 1-\left(1+3\lambda-2\lambda^2\right)a_2w \\ &\quad +\left[\left(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3\right)a_2^2-\left(4\lambda^2+2\right)a_3\right]w^2+\cdots, \end{aligned}$$

it follows from (5), (6), (9) and (10) that

$$\left(1+3\lambda-2\lambda^2\right)a_2 = B_1b_1, \tag{11}$$

$$\left(12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1\right)a_2^2+\left(4\lambda^2+2\right)a_3 = B_1b_2+B_2b_1^2, \tag{12}$$

and

$$-\left(1+3\lambda-2\lambda^2\right)a_2 = B_1c_1, \tag{13}$$

$$\left(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3\right)a_2^2-\left(4\lambda^2+2\right)a_3 = B_1c_2+B_2c_1^2. \tag{14}$$

From (11) and (13) we obtain

$$c_1 = -b_1. \tag{15}$$

By adding (14) to (12), further computations using (11) to (15) lead to

$$\left[2\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-2\left(1+3\lambda-2\lambda^2\right)^2B_2\right]a_2^2 = B_1^3\left(b_2+c_2\right). \tag{16}$$

(15) and (16), together with (4), give that

$$\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2B_2\right||a_2|^2 \le B_1^3\left(1-|b_1|^2\right). \tag{17}$$

From (11) and (17) we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2B_2\right|+\left(1+3\lambda-2\lambda^2\right)^2B_1}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (14) from (12), we obtain

$$4\left(2\lambda^2+1\right)a_3-4\left(2\lambda^2+1\right)a_2^2 = B_1\left(b_2-c_2\right)+B_2\left(b_1^2-c_1^2\right). \tag{18}$$

Then, in view of (4) and (15) , we have

$$2\left(2\lambda^2+1\right)B_1\left|a_3\right| \le \left[2\left(2\lambda^2+1\right)B_1-\left(1+3\lambda-2\lambda^2\right)^2\right]\left|a_2\right|^2+B_1^2.$$

Notice that (7), we get

$$\left|a_3\right| \le \begin{cases} \frac{B_1}{2(2\lambda^2+1)}; & \text{if } \ B_1 \le \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)} \\ \frac{\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2B_2\right|B_1+2\left(2\lambda^2+1\right)B_1^3}{2(2\lambda^2+1)\left[\left|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2B_2\right|+(1+3\lambda-2\lambda^2)^2B_1\right]}; & \\ \qquad \text{if } \quad B_1 > \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)}. & \end{cases}$$

□

Putting $\lambda = 0$ in Theorem 1, we have the following corollary.

Corollary 1 *Let* f *given by (1) be in the class* $S_{\Sigma}^{*}(\phi)$. *Then*

$$\left|a_2\right| \le \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2-B_2\right|+B_1}}$$

and

$$\left|a_3\right| \le \begin{cases} \frac{B_1}{2}; & \text{if } \ B_1 \le \frac{1}{2} \\ \frac{\left|B_1^2-B_2\right|B_1+2B_1^3}{2\left[\left|B_1^2-B_2\right|+B_1\right]}; & \text{if } \quad B_1 > \frac{1}{2}. \end{cases}$$

The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollary 1 are improvement of the estimates obtained in Corollary 2.1 in [19].

Corollary 2 *If let*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1+2\alpha z+2\alpha^2z^2+\ldots \qquad (0<\alpha\le 1),$$

then inequalities (7) and (8) become

$$\left|a_2\right| \le \frac{2\alpha}{\sqrt{\left|20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1\right|\alpha+\left(1+3\lambda-2\lambda^2\right)^2}} \tag{19}$$

and

$$|a_3| \le \begin{cases} \frac{\alpha}{2\lambda^2+1}; & \text{if } \ 0<\alpha\le \frac{\left(1+3\lambda-2\lambda^2\right)^2}{4(2\lambda^2+1)} \\ \frac{\left[|20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1|+4\left(2\lambda^2+1\right)\right]\alpha^2}{(2\lambda^2+1)\left[|20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1|\alpha+(1+3\lambda-2\lambda^2)^2\right]}; & \text{if } \ \frac{\left(1+3\lambda-2\lambda^2\right)^2}{4(2\lambda^2+1)}<\alpha\le 1. \end{cases} \tag{20}$$

The bounds on $|a_2|$ and $|a_3|$ given by (19) and (20) are more accurate than that given in Theorem 2.1 in [15].

We note that for $\lambda = 0$, the class $\mathcal{S}_\Sigma(\lambda, \phi)$ reduces to the class of strongly bi-starlike functions of order α $(0 < \alpha \le 1)$ and denoted by $S^\star_\Sigma(\alpha)$.

Putting $\lambda = 0$ in Corollary 2, we have the following corollary.

Corollary 3 *Let* f *given by (1) be in the class* $S^*_\Sigma(\alpha)$, $(0<\alpha\le 1)$. *Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \tag{21}$$

and

$$|a_3| \le \begin{cases} \alpha; & \text{if } \ 0<\alpha\le \frac{1}{4} \\ \frac{5\alpha^2}{\alpha+1}; & \text{if } \ \frac{1}{4}<\alpha\le 1. \end{cases} \tag{22}$$

The bounds on $|a_3|$ given by (22) is more accurate than that given by Remark 2.2 in [17] and Theorem 2.1 in [4].

Remark 1 *The bounds on $|a_3|$ given by (22) is more accurate than that given in Corollary 2.3 in [18].*

Corollary 4 *If let*

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots \qquad (0<\alpha\le 1),$$

then inequalities (7) and (8) become

$$|a_2| \le \frac{2(1-\alpha)}{\sqrt{\left|2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2\right|+(1+3\lambda-2\lambda^2)^2}} \tag{23}$$

and

$$|a_3| \leq \begin{cases} \frac{1-\alpha}{2\lambda^2+1}; & \text{if} \quad \frac{4(2\lambda^2+1)-(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)} \leq \alpha < 1 \\ \frac{\left[\left|2(1-\alpha)\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)-\left(1+3\lambda-2\lambda^2\right)^2\right|+4(1-\alpha)\left(2\lambda^2+1\right)\right](1-\alpha)}{(2\lambda^2+1)\left[\left|2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2\right|+(1+3\lambda-2\lambda^2)^2\right]}; & \\ & \text{if} \quad 0 \leq \alpha < \frac{4(2\lambda^2+1)-(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)}. \end{cases} \tag{24}$$

The bounds on $|a_2|$ and $|a_3|$ given by (23) and (24) are more accurate than that given in Theorem 3.1 in [15].

Putting $\lambda = 0$ in Corollary 4, we have the following corollary.

Corollary 5 *Let f given by (1) be in the class $S^{\star}_{\Sigma}(\alpha)$, $(0 \leq \alpha < 1)$. Then*

$$|a_2| \leq \frac{2(1-\alpha)}{\sqrt{1+|1-2\alpha|}} \tag{25}$$

and

$$|a_3| \leq \begin{cases} 1-\alpha; & \text{if} \quad \frac{3}{4} \leq \alpha < 1 \\ \frac{(1-\alpha)|1-2\alpha|+4(1-\alpha)^2}{1+|1-2\alpha|}; & \text{if} \quad 0 \leq \alpha < \frac{3}{4}. \end{cases} \tag{26}$$

The bounds on $|a_3|$ given by (26) is more accurate than that given by Remark 2.2 in [17] and Theorem 3.1 in [4].

Remark 2 *The bounds on $|a_3|$ given by (26) is more accurate than that given in Corollary 3.3 in [18].*

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Nonhomogeneous linear differential equations with entire coefficients having the same order and type

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Abstract. In this paper we will investigate the growth of solutions of certain class of nonhomogeneous linear differential equations with entire coefficients having the same order and type. This work improves and extends some previous results in [1], [7] and [9].

1 Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [6]). We denote by $\sigma(f)$ the order of growth of f that defined by

$$\sigma(f) = \limsup_{r \to +\infty} \frac{\log T(r,f)}{\log r};$$

and the type of a meromorphic function f of finite order σ is defined by

$$\tau(f) = \limsup_{r \to +\infty} \frac{T(r,f)}{r^{\sigma}},$$

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where $T(r,f)$ is the Nevanlinna characteristic function of f. We remark that if f is an entire function then we have

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\tau_M(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^\sigma}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Consider the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_0(z)f = H(z), \tag{1}$$

where $A_0 \not\equiv 0, A_1, \ldots, A_{k-1}, H \not\equiv 0$ are entire functions. It is well known that all solutions of (1) are entire functions. The case when the coefficients are polynomials has been studied by Gundersen, Steinbart and Wang in [5] and if p is the largest integer such that A_p is transcendental, Frei proved in [3] that there exist at most p linearely independent finite order solutions of the corresponding homogeneous equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_0(z)f = 0. \tag{2}$$

Several authors studied the case when the coefficients have the same order. In 2008, Tu and Yi investigated the growth of solutions of the homogeneous equation (2) when most coefficients have the same order, see [8]. Next, in 2009, Wang and Laine improved this work to nonhomogeneous equation (1) by proving the following result.

Theorem 1 [9] *Suppose that* $A_j(z) = h_j(z)e^{P_j(z)}$ $(j = 0, \ldots, k-1)$, *where* $P_j(z) = a_{jn}z^n + \ldots.. + a_{j0}$ *are polynomials with degree* $n \geq 1$, $h_j(z)$ *are entire functions of order less than* n, *not all vanishing, and that* $H(z) \not\equiv 0$ *is an entire function of order less than* n. *If* a_{jn} $(j = 0, \ldots, k-1)$ *are distinct complex numbers, then every solution of* (1) *is of infinite order.*

Now how about the case when a_{jn} $(j = 0, \ldots, k-1)$ are equals? we will answer this question in this paper. For the homogeneous equation case, Huang and Sun proved the following result.

Theorem 2 [7] *Let* $A_j(z) = B_j(z)e^{P_j(z)}$ $(j = 0, \ldots, k-1)$, *where* $B_j(z)$ *are entire functions,* $P_j(z)$ *are non constant polynomials with*
$\deg(P_j(z) - P_i(z)) \geq 1$ *and* $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_j(z) - P_i(z))$ $(i \neq j)$. *Then every transcendental solution* f *of* (2) *satisfies* $\sigma(f) = \infty$.

The nonhomogeneous case of this result is improved later in Theorem 4. Recentely, in [1] the authors investigated the order and hyper-order of solutions of the linear differential equation

$$f^{(k)}+\left(A_{k-1}(z)e^{P_{k-1}(z)}e^{\lambda z^m}+B_{k-1}(z)\right)f^{(k-1)}+...+\left(A_0(z)e^{P_0(z)}e^{\lambda z^m}+B_0(z)\right)f=0,$$

where $\lambda \in \mathbb{C}-\{0\}$, $m \geq 2$ is an integer and $\max_{j=0,\dots,k-1}\{\deg P_j(z)\} < m$, A_j, B_j $(j=0,...,k-1)$ are entire functions of order less than m.

In this paper we will investigate certain class of nonhomogeneous linear differential equations with entire coefficients having the same order and type. In fact we will prove the following results.

Theorem 3 *Consider the linear differential equation*

$$f^{(k)}+B_{k-1}(z)\,e^{P_{k-1}(z)}e^{\lambda z^m}f^{(k-1)}+...+B_0(z)\,e^{P_0(z)}e^{\lambda z^m}f=H(z), \qquad (3)$$

where $\lambda \neq 0$ *is a complex constant,* $m \geq 2$ *is an integer,* $P_j(z) = a_{jn}z^n + ... + a_{j0}$ $(j=0,...,k-1)$ *be non constant polynomials such that* $n<m$; $B_0 \not\equiv 0, B_1, ..., B_{k-1}, H \not\equiv 0$ *are entire functions of order smaller than* n. *If one of the following occurs:*

(1) a_{jn} $(j=0,...,k-1)$ *are distinct complex numbers;*

(2) *there exist* $s,t \in \{0,1,...,k-1\}$ *such that* $\arg a_{sn} \neq \arg a_{tn}$ *and for* $j \neq s,t$ $a_{jn} = c_j a_{sn}$ *or* $a_{jn} = c_j a_{tn}$ *with* $0<c_j<1$, $B_sB_t \not\equiv 0$;

then every solution of (3) *is of infinite order.*

Corollary 1 *Consider the linear differential equation*

$$f^{(k)}+B_{k-1}(z)\,e^{\lambda z^3+a_{k-1}z^2+b_{k-1}z}f^{(k-1)}+...+B_0(z)\,e^{\lambda z^3+a_0z^2+b_0z}f=H(z)$$

where $\lambda \in \mathbb{C}-\{0\}$, a_j *are distinct complex numbers (or satisfy the condition (2) of Theorem 3) and* $B_0 \not\equiv 0, B_1, ..., B_{k-1}, H \not\equiv 0$ *are entire functions of order smaller than* 2. *Then every solution* f *of this differential equation is of infinite order.*

Theorem 4 *Let* $A_j(z) = B_j(z)\,e^{P_j(z)}$ $(j=0,...,k-1)$, *where* $B_j(z)$ *are entire functions,* $P_j(z)$ *be non constant polynomials with*
$\deg(P_j(z)-P_i(z)) \geq 1$ *and* $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_j(z)-P_i(z))$ $(i \neq j)$, *and let* $H(z) \not\equiv 0$ *be an entire function of order less than* 1. *Then every solution of* (1) *is of infinite order.*

Example 1 *Consider the linear differential equation*

$$f^{(4)} + B_3(z)\, e^{z^2+z} f^{(3)} + B_2(z)\, e^{2z^2+z} f'' + B_1(z)\, e^{2z^2+iz} f' + B_0(z)\, e^{z^2+iz} f = H(z),$$

where $B_0 \not\equiv 0, B_1, B_2, H \not\equiv 0$ *are entire functions of order less than* 1. *By Theorem 4, every solution of this differential equation is of infinite order.*

Theorem 5 *Let* $A_j(z) = B_j(z)\, P_j\left(e^{R(z)}\right) + G_j(z)\, Q_j(e^{-R(z)})$ *for* $j = 0, 1, ..., k-1$ *where* $P_j(z)$, $Q_j(z)$ *and* $R(z) = c_d z^d + ... + c_1 z + c_0$ $(d \geq 1)$ *are polynomials; and let* $B_j(z)$, $G_j(z)$, $H(z) \not\equiv 0$ *be entire functions of order less than* d. *Suppose that* $B_0(z)\, P_0(z) + G_0(z)\, Q_0(z) \not\equiv 0$ *and there exists* s $(0 \leq s \leq k-1)$ *such that for* $j \neq s$, $\deg P_s > \deg P_j$ *and* $\deg Q_s > \deg Q_j$. *Then every solution* f *of* (1) *is of infinite order.*

Example 2 *By Theorem 5, every solution of the differential equation*

$$f'' + \sin\left(2z^2\right) f' + \cos\left(z^2\right) f = \sin z$$

is of infinite order.

2 Preliminaries Lemmas

We need the following lemmas for our proofs.

Lemma 1 [4] *Let* $f(z)$ *be a transcendental meromorphic function of finite order* σ, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E \subset [0, 2\pi)$ *of linear measure zero such that for all* $z = re^{i\theta}$ *with* $|z|$ *sufficiently large and* $\theta \in [0, 2\pi) \setminus E$, *and for all* k, j, $0 \leq j \leq k$, *we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2 [2] *Let* $P(z) = a_n z^n + ... + a_0$, $(a_n = \alpha + i\beta \neq 0)$ *be a polynomial with degree* $n \geq 1$ *and* $A(z)$ $(\not\equiv 0)$ *be entire function with* $\sigma(A) < n$. *Set* $f(z) = A(z)\, e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E \subset [0, 2\pi)$ *that has linear measure zero, such that for any* $\theta \in [0, 2\pi) \setminus E \cup H$, *where* $H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, there is* $R > 0$ *such that for* $|z| = r > R$, *we have*

(i) *if* $\delta(P,\theta) > 0$, *then*

$$\exp\{(1-\varepsilon)\,\delta(P,\theta)\,r^n\} \le |f(z)| \le \exp\{(1+\varepsilon)\,\delta(P,\theta)\,r^n\},$$

(ii) *if* $\delta(P,\theta) < 0$, *then*

$$\exp\{(1+\varepsilon)\,\delta(P,\theta)\,r^n\} \le |f(z)| \le \exp\{(1-\varepsilon)\,\delta(P,\theta)\,r^n\}.$$

Lemma 3 [7] *Let* $n \ge 2$ *and* $A_j(z) = B_j(z)\,e^{P_j(z)}$ $(1 \le j \le n)$, *where each* $B_j(z)$ *is an entire function and* $P_j(z)$ *is a nonconstant polynomial. Suppose that* $\deg(P_j(z) - P_i(z)) \ge 1$, $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_j(z) - P_i(z))$ *for* $i \ne j$. *Then there exists a set* $H_1 \subset [0, 2\pi)$ *that has linear measure zero, such that for any given constant* $M > 0$ *and* $z = re^{i\theta}$, $\theta \in [0, 2\pi) - (H_1 \cup H_2)$, *we have some integer* $s = s(\theta) \in \{1, 2, ..., n\}$, *for* $j \ne s$,

$$\frac{\left|A_j\left(re^{i\theta}\right)\right| |z|^M}{|A_s(re^{i\theta})|} \to 0, \quad \text{as } r \to \infty,$$

where $H_2 = \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0 \text{ or } \delta(P_i - P_j, \theta) = 0,\ i \ne j\}$ *is a finite set.*

Lemma 4 [9] *Let* $f(z)$ *be an entire function and suppose that*

$$G(z) = \frac{\log^+ \left|f^{(k)}(z)\right|}{|z|^\sigma}$$

is unbounded on some ray $\arg z = \theta$ *with constant* $\sigma > 0$. *Then there exists an infinite sequence of points* $z_n = r_n e^{i\theta}$ $(n = 1, 2, ...)$, *where* $r_n \to \infty$, *such that* $G(z_n) \to \infty$ *and*

$$\frac{\left|f^{(j)}(z_n)\right|}{\left|f^{(k)}(z_n)\right|} \le \frac{1}{(k-j)!}\,(1 + o(1))\, r_n^{k-j}, \qquad j = 0, 1, ..., k-1$$

as $n \to \infty$.

Lemma 5 [9] *Let* $f(z)$ *be an entire function with finite order* $\sigma(f)$. *Suppose that there exists a set* $E \subset [0, 2\pi)$ *which has linear measure zero, such that* $\log^+|f\left(re^{i\theta}\right)| \le Mr^\sigma$ *for any ray* $\arg z = \theta \in [0, 2\pi) \setminus E$, *where* M *is a positive constant depending on* θ, *while* σ *is a positive constant independent of* θ. *Then* $\sigma(f) \le \sigma$.

Lemma 6 [10] *Suppose that* $f_1(z), f_2(z), ..., f_n(z)$ $(n \geq 2)$ *are linearly independent meromorphic functions and* $g_1(z), g_2(z), ..., g_n(z)$ *are entire fuctions satisfying the following conditions*

(i) $\sum\limits_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0.$

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n.$

(iii) For $1 \leq i \leq n,\ 1 \leq j < k \leq n,$

$$T(r, f_j) = o\left\{T\left(r, e^{g_j(z)-g_k(z)}\right)\right\}, \quad (r \to \infty,\ r \notin E)$$

where E *is a set with finite linear measure.*

Then $f_j \equiv 0,\ 1 \leq j \leq n.$

3 Proof of main results

Proof. [Proof of Theorem 3] We will prove the two cases together. If we suppose that f is a solution of (3) of finite order $\sigma(f) = \sigma < \infty$, (contrary to the assertion), then $\sigma \geq n$. Indeed, if $\sigma < n$ then we get the following contradiction. From (3), we can write

$$\left(B_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + B_0(z) e^{P_0(z)} f\right) e^{\lambda z^m} = H(z) - f^k. \qquad (4)$$

Now for the condition (1), if $B_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + B_0(z) e^{P_0(z)} f \equiv 0$, then by Lemma 6, we have $B_0(z) f \equiv 0$, and since $B_0(z) \not\equiv 0$, then $f \equiv 0$, which implies that $H(z) \equiv 0$, a contradiction. So

$$B_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + B_0(z) e^{P_0(z)} f \not\equiv 0.$$

Then the order of growth of the left side of (4) is equal m and the order of the right side is smaller than n, a contradiction. So, we have $\sigma(f) = \sigma \geq n$. And for the condition (2), to apply Lemma 6 we may collecte terms of the same power, and we have at least two terms linearly independents: if

$$B_s(z) f^{(s)} e^{P_s(z)} + B_t(z) f^{(t)} e^{P_t(z)} + \sum_{u=1}^{p} G_u e^{c_{j_u} P_s(z)} + \sum_{v=1}^{q} L_v e^{c_{i_v} P_t(z)} \equiv 0$$

by Lemma 6, $B_s(z) f^{(s)} \equiv 0$, and since $B_s(z) \not\equiv 0$, then $f^{(s)} \equiv 0$ and so $f^{(k)} \equiv 0$, which implies that $H(z) \equiv 0$, a contradiction. So

$$B_s(z) f^{(s)} e^{P_s(z)} + B_t(z) f^{(t)} e^{P_t(z)} + \sum_{u=1}^{p} G_u e^{c_{j_u} P_s(z)} + \sum_{v=1}^{q} L_v e^{c_{i_v} P_t(z)} \not\equiv 0.$$

By similar reasoning as above we get $\sigma(f) = \sigma \geq n$.
By Lemma 1, for any given ε $(0 < \varepsilon < 1)$, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero , such that if $\psi \in [0, 2\pi) \setminus E_1$, then

$$\frac{\left|f^{(j)}(z)\right|}{\left|f^{(i)}(z)\right|} \leq |z|^{k\sigma}, \quad 0 \leq i < j \leq k \tag{5}$$

as $z \to \infty$ along $\arg z = \psi$. Denote $E_2 = \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0,\ 0 \leq j \leq k\} \cup \{\theta \in [0, 2\pi) : \delta(P_j - P_i, \theta) = 0,\ 0 \leq i < j \leq k\} \cup \{\theta \in [0, 2\pi) : \delta(\lambda z^m, \theta) = 0\}$, so E_2 is a finite set. Suppose that $H_j \subset [0, 2\pi)$ is the exceptional set applying lemma 2 to $A_j(z) = B_j(z) e^{\lambda z^m + P_j(z)}$ $(j = 0, \ldots, k-1)$. Then $E_3 = \cup_{j=0}^{k-1} H_j$ has linear measure zero. Set $E = E_1 \cup E_2 \cup E_3$. Take $\arg z = \psi \in [0, 2\pi) - E$. We need to treat two principal cases:
Case (i): $\delta = \delta(\lambda z^m, \psi) < 0$. By lemma 2, for a given $0 < \varepsilon < 1$, we have

$$|A_j(z)| \leq \exp\{(1-\varepsilon)\delta r^m\}. \tag{6}$$

Now we prove that $\dfrac{\log^+ \left|f^{(k)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z = \psi_0$. Suppose that it is not the case. By Lemma 4, there is a sequence of points $z_i = r_i e^{i\theta}$ $(i = 1, 2, \ldots)$, such that $r_i \to \infty$ as $i \to \infty$, and that

$$\frac{\log^+ \left|f^{(k)}(z_i)\right|}{|z_i|^{\sigma(H)+\varepsilon}} \to \infty \tag{7}$$

and

$$\frac{\left|f^{(j)}(z_i)\right|}{\left|f^{(k)}(z_i)\right|} \leq (1 + o(1)) r_i^{t-j}, \qquad j = 0, 1, \ldots, k-1. \tag{8}$$

From (7) and the definition of the order $\sigma(H)$, it is easy to see that

$$\left|\frac{H(z_i)}{f^{(k)}(z_i)}\right| \to 0 \tag{9}$$

as $z_i \to \infty$. From (3), we obtain

$$1 \leq |A_{k-1}(z_i)| \left|\frac{f^{(k-1)}(z_i)}{f^{(k)}(z_i)}\right| + \ldots + |A_0(z_i)| \left|\frac{f(z_i)}{f^{(k)}(z_i)}\right| + \left|\frac{H(z_i)}{f^{(k)}(z_i)}\right|. \tag{10}$$

Using (5)-(9) in (10), we get

$$1 \leq r_i^k \exp\{(1-\varepsilon)\delta r_i^m\}.$$

This is impossible since $\delta<0$. Therefore $\frac{\log^{+}|f^{(k)}(z_i)|}{|z_i|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z=\psi$. Assume that $\frac{\log^{+}|f^{(k)}(z_i)|}{|z_i|^{\sigma(H)+\varepsilon}}\leq M_1$ (M_1 is a constant) and so

$$|f^{(k)}(z)|\leq M_1\exp\{r^{\sigma(H)+\varepsilon}\}. \tag{11}$$

Using the elementary triangle inequality for the well know equality

$$f(z)=f(0)+f'(0)z+...+\frac{1}{(k-1)!}f^{(k-1)}(0)z^{k-1}+\int_0^z...\int_0^{\xi_1}f^{(k)}(\xi)\,d\xi d\xi_1...d\xi_{k-1},$$

and (11), we obtain

$$|f(z)|\leq(1+o(1))r^k|f^{(k)}(z)|\leq(1+o(1))M_1r^k\exp\{r^{\sigma(H)+\varepsilon}\}\leq\exp\{r^{\sigma(H)+2\varepsilon}\}, \tag{12}$$

on any ray $\arg z=\psi\in[0,2\pi)-E$.

Case (ii): $\delta=\delta(\lambda z^m,\psi)>0$. Now we pass to $\delta_j=\delta(P_j,\psi)$. For the condition (1), since a_{jn} $(j=0,...,k-1)$ are distinct complex numbers, then there exists some $s\in\{0,1,2...,k-1\}$ such that $\delta_s>\delta_j$ for all $j\neq s$. For the condition (2), set $\delta'=\max\{\delta_s,\delta_t\}$ and without loss of generality we may assume that $\delta'=\delta_s$. In both cases, we have

$$\left|\frac{A_j(z)}{A_s(z)}\right||z|^M\to 0,\quad\text{and}\quad\frac{|z|^M}{|A_s(z)|}\to 0, \tag{13}$$

as $|z|\to\infty$, for any $M>0$. Suppose that $\frac{\log^{+}\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is unbounded on the ray $\arg z=\psi$. Then by lemma 4 there is a sequence of points $z_i=r_ie^{i\psi}$, such that $r_i\to\infty$, and

$$\frac{\log^{+}\left|f^{(s)}(z_i)\right|}{|z_m|^{\sigma(H)+\varepsilon}}\to\infty, \tag{14}$$

and

$$\frac{\left|f^{(j)}(z_i)\right|}{\left|f^{(s)}(z_i)\right|}\leq(1+o(1))r_i^{s-j},\qquad j=0,1,...,s-1. \tag{15}$$

From (14) and the definition of order, it is easy to see that

$$\left|\frac{H(z_i)}{f^{(s)}(z_i)}\right|\to 0 \tag{16}$$

as $r_i \to \infty$. From (3), we can write

$$1 \leq \frac{1}{|A_s(z_i)|}\left|\frac{f^{(k)}(z_i)}{f^{(s)}(z_i)}\right| + \left|\frac{f^{(k-1)}(z_i)}{f^{(s)}(z_i)}\right| \frac{|A_{k-1}(z_i)|}{|A_s(z_i)|} + \dots \\ + \left|\frac{f^{(s+1)}(z_i)}{f^{(s)}(z_i)}\right| \frac{|A_{s+1}(z_i)|}{|A_s(z_i)|} + \left|\frac{f^{(s-1)}(z_i)}{f^{(s)}(z_i)}\right| \frac{|A_{s-1}(z_i)|}{|A_s(z_i)|} + \dots \\ + \left|\frac{f(z_i)}{f^{(s)}(z_i)}\right| \frac{|A_0(z_i)|}{|A_s(z_i)|} + \frac{1}{|A_s(z_i)|}\left|\frac{H(z_i)}{f^{(s)}(z_i)}\right|; \tag{17}$$

and by using (5), (13), (15) and (16) in (17) a contradiction follows as $z_i \to \infty$. Then $\dfrac{\log^+|f^{(s)}(z_i)|}{|z_i|^{\sigma(H)+\varepsilon}}$ is bounded and we have $|f^{(s)}(z)| \leq M_2 \exp\{r^{\sigma(H)+\varepsilon}\}$ on the ray $\arg z = \psi$. This implies, as in Case (i), that

$$|f(z)| \leq \exp\{r^{\sigma(H)+2\varepsilon}\}.$$

We conclude that in all cases we have

$$|f(z)| \leq \exp\{r^{\sigma(H)+2\varepsilon}\}$$

on any ray $\arg z = \psi \in [0, 2\pi) - E$, provided that r is large enough. Then by Lemma 5, $\sigma(f) \leq \sigma(H) + 2\varepsilon < n$ $(0 < 2\varepsilon < n - \sigma(H))$, a contradiction. Hence, every solution of (3) must be of infinite order. □

Proof. [Proof of Theorem 4] We suppose contrary to the assertion that f is a solution of (1) of finite order $\sigma(f) = \sigma < \infty$. First we prove that $\sigma \geq 1$. Indeed, if $\sigma < 1$ then we will have the following contradiction. From (1), we can write

$$B_{k-1}(z)\, e^{P_{k-1}(z)} f^{(k-1)} + \dots + B_0(z)\, e^{P_0(z)} f = H(z) - f^{(k)}. \tag{18}$$

By the same rasoning as in the proof of Theorem 3, we get that the order of the left side of (18) is greather than or equal to 1 and the order of the right side of (18) is smaller than 1, a contradiction. Therefore $\sigma \geq 1$.
Take $\arg z = \psi \in [0, 2\pi) - E$ where E has linear measure zero and set $\delta_j = \delta(P_j, \psi)$ $(j = 0, \dots, k-1)$. By Lemma 3, there exists some $s \in \{0, 1, 2\dots, k-1\}$ such that for $j \neq s$, $M > 0$, we have

$$\left|\frac{A_j(z)}{A_s(z)}\right| |z|^M \to 0, \quad \text{as } z \to \infty. \tag{19}$$

We need to treat two cases:
Case (i): $\delta_s > 0$. In this case we have also

$$\frac{1}{|A_s(z)|}|z|^M \to 0, \quad \text{as } z \to \infty. \tag{20}$$

We prove that $\dfrac{\log^+|f^{(s)}(z)|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z = \psi$. Suppose that it is not the case. Then by lemma 4 there is a sequence of points $z_i = r_i e^{i\psi_0}$, such that $r_i \to \infty$, and (14), (15), (16) hold. As in the proof of Theorem 3, by using (17) we get a contradiction. Therefore, $\dfrac{\log^+|f^{(s)}(z)|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded and so we conclude that

$$|f(z)| \le \exp\{r^{\sigma(H)+2\varepsilon}\}. \tag{21}$$

Case (ii): $\delta_s < 0$. Obsiouly in this case $\delta_j < 0$ for all j and we have

$$|A_j(z)| \le \exp\left\{(1-\varepsilon)\,\delta_j r^{d_j}\right\},$$

where $d_j = \deg(P_j)$; which implies that

$$|A_j(z)|\,|z|^M \to 0, \quad \text{as } z \to \infty.$$

We use the same reasoning as in Case (i) in the proof of Theorem 3, we prove that $\dfrac{\log^+|f^{(s)}(z)|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z = \psi$ and we conclude that

$$|f(z)| \le \exp\{r^{\sigma(H)+2\varepsilon}\}.$$

Then by Lemma 5, $\sigma(f) \le \sigma(H) + 2\varepsilon < 1$ $(0 < 2\varepsilon < 1 - \sigma(H))$, a contradiction. So, every solution of (1) must be of infinite order. □

Proof. [Proof of Theorem 5] Suppose that f is a solution of (1) of finite order $\sigma(f) = \sigma < \infty$. By the same reasoning as in the proof of Theorem 4 and taking account the assumption that $B_0(z)P_0(z) + G_0(z)Q_0(z) \not\equiv 0$ and there exists s $(0 \le s \le k-1)$ such that for $j \ne s$, $\deg P_s > \deg P_j$ and $\deg Q_s > \deg Q_j$, we can prove that $\sigma \ge d$.
Set $\delta(R,\theta) = \text{Real}\left(c_d e^{id\theta}\right)$ and

$$P_j\left(e^{R(z)}\right) = a_{jm_j} e^{m_j R(z)} + a_{j(m_j-1)} e^{(m_j-1)R(z)} + \ldots + a_{j1} e^{R(z)} + a_{j0},$$

$$Q_j\left(e^{-R(z)}\right)=b_{jn_j}e^{-n_jR(z)}+b_{j(n_j-1)}e^{-(n_j-1)R(z)}+...+b_{j1}e^{-R(z)}+b_{j0}.$$

By Lemma 2, it is easy to get the following
(i) If $\delta(R,\theta)>0$, then

$$\exp\left\{(1-\varepsilon)m_j\delta(R,\theta)r^d\right\}\leq|A_j(z)|\leq\exp\left\{(1+\varepsilon)m_j\delta(R,\theta)r^d\right\}, \tag{22}$$

(ii) if $\delta(R,\theta)<0$, then

$$\exp\left\{-(1-\varepsilon)n_j\delta(R,\theta)r^d\right\}\leq|A_j(z)|\leq\exp\left\{-(1+\varepsilon)n_j\delta(R,\theta)r^d\right\}. \tag{23}$$

Take $\arg z=\psi\in[0,2\pi)-E$ where E has linear measure zero. We prove that $\frac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z=\psi$. Suppose that it is not the case. Then by lemma 4 there is a sequence of points $z_i=r_ie^{i\psi_0}$, such that $r_i\to\infty$, and (14), (15), (16) hold. From (1) we can write

$$\begin{aligned}|A_s(z_i)| \leq\ & \left|\frac{f^{(k)}(z_i)}{f^{(s)}(z_i)}\right|+|A_{k-1}(z_i)|\left|\frac{f^{(k-1)}(z_i)}{f^{(s)}(z_i)}\right|+... \\ &+|A_{s+1}(z_i)|\left|\frac{f^{(s+1)}(z_i)}{f^{(s)}(z_i)}\right|+|A_{s-1}(z_i)|\left|\frac{f^{(s-1)}(z_i)}{f^{(s)}(z_i)}\right|+... \\ &+|A_0(z_m)|\left|\frac{f(z_i)}{f^{(s)}(z_i)}\right|+\left|\frac{H(z_i)}{f^{(s)}(z_i)}\right|.\end{aligned} \tag{24}$$

If $\delta(R,\theta)>0$, then by using (14), (15), (16) and (22) in (24), we obtain

$$\exp\left\{(1-\varepsilon)m_s\delta(R,\theta)r_i^d\right\}\leq r_i^M\exp\left\{(1+\varepsilon)(m_s-1)\delta(R,\theta)r_i^d\right\},$$

where $M>0$ is a constant. A contradiction follows by taking $0<\varepsilon<\frac{1}{2m_s-1}$. Now if $\delta(R,\theta)<0$, by using (23) instead of (22) in (24), we obtain

$$\exp\left\{-(1-\varepsilon)n_s\delta(R,\theta)r^d\right\}\leq r_i^M\exp\left\{-(1+\varepsilon)(n_s-1)\delta(R,\theta)r_i^d\right\},$$

a contradiction follows by taking $0<\varepsilon<\frac{1}{2n_s-1}$.

Therefore, $\frac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on any ray $\arg z=\psi\in[0,2\pi)-E$ and so as the previous reasoning we conclude that

$$|f(z)|\leq\exp\{r^{\sigma(H)+2\varepsilon}\}.$$

Then by Lemma 5, $\sigma(f)\leq\sigma(H)+2\varepsilon<d$ $(0<2\varepsilon<d-\sigma(H))$, a contradiction. So, every solution of (1) must be of infinite order. $\square$

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On the maximal exponent of the prime power divisor of integers

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Abstract. The largest exponent of the prime powers function is investigated on the set of numbers of form one plus squares of primes.

1 Introduction

1.1. Notation. Let, as usual, $\mathcal{P}$, $\mathbb{N}$ be the set of primes, positive integers, respectively. For a prime divisor p of n let $\nu_p(n)$ be defined by $p^{\nu_p(n)} \| n$. Then $n = \prod_{p|n} p^{\nu_p(n)}$. Let

$$H(n) = \max_{p|n} \nu_p(n) \quad \text{and} \quad h(n) = \min_{p|n} \nu_p(n).$$

We denote by $\pi(x)$ the number of primes $p \le x$ and by $\pi(x, k, \ell)$ the number of primes $p \le x, p \equiv \ell \pmod{k}$.

1.2. Preliminaries. A. Niven proved in [7] that

$$\sum_{n \le x} h(n) = x + \frac{\zeta(3/2)}{\zeta(3)} \sqrt{x} + o(\sqrt{x}) \quad (x \to \infty) \tag{1}$$

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and that

$$\frac{1}{x}\sum_{n\le x} H(n) \to B \quad (x\to\infty), \quad \text{where} \quad B = 1+\sum_{k=2}^{\infty}\Big(1-\frac{1}{\zeta(k)}\Big). \tag{2}$$

W. Schwarz and J. Spilker showed in [8] that

$$\sum_{n\le x} H(n) = \mathcal{M}(H)x + O\Big(x^{3/4}\exp(-\gamma\sqrt{\log x})\Big) \quad (x\to\infty), \tag{3}$$

$$\sum_{n\le x} \frac{1}{H(n)} = \mathcal{M}\Big(\frac{1}{H}\Big)x + O(x^{3/4}\exp(-\gamma\sqrt{\log x}) \quad (x\to\infty), \tag{4}$$

where $\gamma>0$ is a suitable constant, $\mathcal{M}(H)=B$, $\mathcal{M}(\frac{1}{H})$ are suitable positive numbers.

D. Suryanayana and Sita Ramachandra Rao [9] proved that the error term in (3) and (4) can be improved to

$$O(\sqrt{x}\exp(-\gamma(\log x)^{3/5}(\log\log x)^{-1/5})).$$

They proved furthermore that

$$\sum_{n\le x} h(n) = c_1x + c_2x^{1/2} + c_3x^{1/3} + c_4x^{1/4} + c_5x^{1/5} + O(x^{1/6}), \tag{5}$$

$$\sum_{n\le x} \frac{1}{h(n)} = d_1x + d_2x^{1/2} + d_3x^{1/3} + d_4x^{1/4} + d_5x^{1/5} + O(x^{1/6}). \tag{6}$$

Gu Tongxing and Cao Huizhong announced in [4] that they can improve the error term in (3) to

$$O(\sqrt{x}\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})).$$

I. Kátai and M. V. Subbarao [5] investigated the asymptotic of

$$A_x(r) := \natural\{n\in[x,x+Y] \mid H(n)=r\}, \quad Y = x^{\frac{1}{2r+1}}\log x,$$

and

$$B_x(r) := \natural\{p\in\mathcal{P}, p\in[x,x+Y] \mid H(p+1)=r\}, \quad Y = x^{\frac{7}{12}+\epsilon}$$

for fixed $r\ge 1$.

Namely, they proved that

$$A_x(r) = Y\Big(\eta(r+1) - \eta(r)\Big) + O\Big(\frac{Y}{\log x}\Big), \ \eta(s) = \frac{1}{\zeta(s)} - 1 \quad (s = 1, 2, \cdots)$$

and

$$B_x(r) = e(r)\frac{Y}{\log x} + O\Big(\frac{Y}{(\log x)^2}\Big),$$

where

$$e(1) = \prod_{p \in \mathcal{P}} \Big(1 - \frac{1}{p(p-1)}\Big),$$

and for $r \geq 2$

$$e(r) = \prod_{p \in \mathcal{P}} \Big(1 - \frac{1}{(p-1)p^r}\Big) - \prod_{p \in \mathcal{P}} \Big(1 - \frac{1}{(p-1)p^{r-1}}\Big).$$

In [6] we can read some results on (5) assuming the Riemann conjecture.

Our main interest now is to give the asymptotic of the number of those $n \leq x, n \in \mathcal{B}$, for which $H(n) = r$ uniformly as $1 \leq r \leq \kappa(x)$, where $\kappa(x)$ is as large as it is possible. We shall investigate it when $\mathcal{B} =$ set of shifted primes.

1.3. Auxiliary results.

Lemma 1 (Brun-Titchmarsh inequality). *We have*

$$\pi(x, k, \ell) < C\frac{x}{\varphi(k)\log\frac{x}{k}}.$$

Lemma 2 (Siegel-Walfisz theorem). *We have*

$$\pi(x, k, \ell) = \frac{\mathrm{li}x}{\varphi(k)}\Big(1 + O(e^{-c\sqrt{\log x}})\Big)$$

uniformly as $(k, \ell) = 1$, $k \leq (\log x)^A$. *Here* A *is arbitrary,* $c > 0$ *is a fixed constant.*

Lemma 3 ([1]) *Let* q *be an odd prime,* $D = q^n$ $(n = 1, 2, \cdots)$, $\epsilon > 0$ *be an arbitrary small, and* M *be an arbitrary large positive number. Then the asymptotic law*

$$\pi(x, D, \ell) = \frac{\mathrm{li}x}{\varphi(D)}\Big(1 + O\Big((\log x)^{-M}\Big)\Big)$$

holds for $D \leq x^{3/8-\epsilon}$, $(\ell, D) = 1$.

Lemma 4 ([2]) *Let a be an integer, $a \geq 2$. If $A > 0$, then there is a $B > 0$ for which*

$$\sum_{\substack{d \leq \frac{x^{1/2}}{q}(\log x)^{-B} \\ (d,q)=1}} \max_{(r,qd)=1} \max_{y \leq x} \left| \pi(y,qd,r) - \frac{\mathrm{li}\,x}{\varphi(qd)} \right| \ll \frac{x}{\varphi(q)(\log x)^A}, \mathrm{li}\,x = \int_2^x \frac{du}{\log u}$$

uniformly for moduli $q \leq x^{1/3}\exp(-(\log\log x)^3)$ that are powers of a.

While the implicit constant in $\ll$ may depend upon a, B is a function of A alone. $B = A + 6$ is permissible.

We shall use a special consequence of this assertion:

Corollary. *Let a be an integer, $a \geq 2$, $D = a^n$ $(n = 1, 2, \cdots)$, $D \leq x^{1/3}\exp(-(\log\log x)^3)$. Let $A > 0$ be an arbitrary constant. Then*

$$\pi(x, D, \ell) = \frac{\mathrm{li}\,x}{\varphi(D)}\Big(1 + O\Big(\frac{1}{(\log x)^A}\Big)\Big), \quad (\ell, D) = 1$$

Lemma 5 ([3]) *Let $q = p^r$, p an odd prime, $qx^{\frac{3}{5}+\epsilon} \leq h \leq x$. Then*

$$\pi(x+h,q,\ell) - \pi(x,q,\ell) = (1 + o_x(1))\frac{h}{\varphi(q)\log x}$$

as $x \to \infty$, $(\ell, q) = 1$.

2 Formulation of the theorems

Let $(0 <)U, V$ be coprime integers, and let Q be the smallest prime for which

$$U(1+2m) + V \equiv 0 \pmod{Q}$$

has a solution, that is

$$Q = \begin{cases} 2 \text{ if } 2|U+V \\ \text{smallest prime for which } (Q2U) = 1, \text{ if } 2 \nmid U+V. \end{cases}$$

Let

$$M_{U,V}(x \mid k) = \natural\{p \leq x \mid H(Up+V) = k\}.$$

Theorem 1 *Assume that* $r(x) \to \infty$ *arbitrarily slowly. Then, in the interval* $r(x) < k < (\frac{1}{3} - \epsilon)\frac{\log x}{\log Q}$, *we have*

$$M_{U,V}(x \mid k) = \frac{\mathrm{li}x}{\varphi(Q^k)}\Big(1 - \frac{1}{Q}\Big) \cdot (1 + o_x(1)).$$

Let $P(n) = n^2 + 1$. Then $4 \nmid P(n)$, $3 \nmid P(n)$, $5 \mid P(2)$, $5 \mid P(3)$. For every k there exists $1 \leq \ell_k < \frac{5^k}{2}$, such that $P(\ell_k) \equiv 0 \pmod{5^k}$. The congruence $P(n) \equiv 0 \pmod{5^k}$ has exactly two solutions: ℓ_k and $5^k - \ell_k$. It obvious that $(\ell_k, 5) = 1$.

Let

$$E(x \mid k) = \natural\{p \leq x \mid H(p^2 + 1) = k\}.$$

Theorem 2 *Assume that* $r(x) \to \infty$ *arbitrarily slowly. Then, in the interval* $r(x) < k < (\frac{1}{3} - \epsilon)\frac{\log x}{\log 5}$, *we have*

$$E(x \mid k) = \frac{2}{5^k}\mathrm{li}x(1 + o_x(1)).$$

3 Proof of Theorem 1.

It is obvious that

$$M_{U,V}(x \mid k) \leq \sum_q{}^{*} \Big[Q(x, q^k, r_{q,k}) - Q(x, q^{k+1}, r_{q,k+1})\Big],$$

where q runs over all those primes for which $U(1 + 2m) + V \equiv 0 \pmod{q}$ has a solution, $r_{q,k} \equiv VU^{-1} \pmod{q^k}$, $r_{q,k+1} \equiv VU^{-1} \pmod{q^{k+1}}$.

By using Lemma 3 and Lemma 1 we obtain that

$$M_{U,V}(x \mid k) \leq \frac{\mathrm{li}x}{\varphi(Q^k)}\Big(1 - \frac{1}{Q}\Big) \cdot \Big(1 + O\Big(\frac{1}{(\log x)^M}\Big)\Big) + \\ + C \sum_{\substack{q > Q \\ q \in \mathcal{P}}} \frac{\mathrm{li}x}{\varphi(q^k)} + C \sum_{\substack{Q < q \\ q^k \geq \sqrt{x}}} \frac{x}{q^k}.$$

It is clear that

$$\sum_{\substack{q > Q \\ q \in \mathcal{P}}} \frac{1}{\varphi(q^k)} = \frac{o_x(1)}{\varphi(Q^k)}$$

and that

$$\sum_{\substack{q^k \geq \sqrt{x} \\ q \in \mathcal{P}}} \frac{1}{\varphi(q^k)} = O\Big(\frac{1}{x^{1/4}}\Big),$$

thus

$$M_{U,V}(x \mid k) \leq \Big(1 + o_x(1)\Big) \frac{\mathrm{li}\, x}{Q^k}.$$

On the other hand

$$M_{U,V}(x \mid k) \geq \Big[Q(x, Q^k, r_{Q,k}) - Q(x, Q^{k+1}, r_{Q,k+1})\Big] - \sum_{\substack{q > Q \\ q \in \mathcal{P}}} Q(x, Q^k q^k, r_{Qq,k}).$$

The sum on right hand side is less than

$$C \frac{\mathrm{li}\ x}{Q^k} \sum_{(Q<)q} \frac{1}{q^k} + O(x^{3/4}) \leq o_x(1) \frac{\mathrm{li}\, x}{Q^k}.$$

From Lemma 3 our theorem follows.

4 Proof of Theorem 2

We have

$$E(x \mid k) = S + O(T),$$

where

$$S = \natural\{p \leq x : \quad 5^k \| p^2 + 1\}$$

and

$$T = \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \natural\{p \leq x : \quad q^k \| p^2 + 1\}.$$

Thus, by using Lemma 1 and $k \geq \gamma(x)$,

$$T \leq \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \frac{2C \mathrm{li}\, x}{\varphi(q^k)} + \sum_{\substack{q^k > x \\ q \in \mathcal{P}}} \frac{x}{q^k} = o_x(1) \frac{\mathrm{li}\, x}{5^k}.$$

Hence we obtain that

$$E(x \mid k) \leq \frac{2}{5^k} \mathrm{li}\, x (1 + o_x(1)).$$

On the other hand

$$E(x \mid k) \geq S - \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \natural\{p \leq x : \quad 5^k \cdot q^k \| p^2 + 1\}.$$

By using Lemma 1, the sum on the right can be overestimated by

$$\frac{C \mathrm{li}\, x}{5^k} \sum_{q > 5} \frac{1}{\varphi(q^k)} + \frac{x}{5^k} \sum_{q^k > \sqrt{x}} \frac{1}{q^k},$$

which is clearly $o_x(1)S$.

This completes the proof of Theorem 2.

5 Further remarks

By using Lemma 5 we can prove short interval version of Theorem 1 and 2.

Theorem 3 *Let* $5^k x^{3/5+\epsilon} \leq h \leq x$, $k \geq g(x)$. *Then*

$$E(x + h \mid k) - E(x) = \frac{h}{5^k} \frac{1}{\log x} (1 + o_x(1)).$$

Theorem 4 *Let Let* U, V *be coprime integers,* $U > 0, U + V = odd$, Q *be the smallest prime which is not a divisor of* $2U$. *Let* $k \geq g(x)$, $Q^k x^{3/5+\epsilon} \leq h \leq x$. *Then*

$$M_{U,V}(x + h \mid k) - M_{U,V}(x) = (1 + o_x(1)) \frac{h}{Q^k} \frac{1}{\log x} \quad \textit{as} \quad x \to \infty.$$

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Certain non-linear differential polynomials sharing a non zero polynomial

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1 Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have same zeros with same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [6]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share

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$a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have same zeros with same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

Throughout this paper, we need the following definition.

$$\Theta(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, W. K. Hayman (see [6], Corollary of Theorem 9) proved the following theorem.

Theorem A *Let f be a transcendental meromorphic function and n (≥ 3) is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [3], Yang and Hua [16] got a unicity theorem respectively corresponding Theorem A.

Theorem B *Let f and g be two non-constant entire (meromorphic) functions, $n \geq 6$ (≥ 11) be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Noting that $f^n(z) f^{'}(z) = \frac{1}{n+1}(f^{n+1}(z))^{'}$, Fang [4] considered the case of k-th derivative and proved the following results.

Theorem C *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem D *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+8$. If $(f^n(z)(f(z)-1))^{(k)}$ and $(g^n(z)(g(z)-1))^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2008, X. Y. Zhang and W. C. Lin [21] proved the following result.

Theorem E *Let f and g be two non-constant entire functions, and let n, m and k be three positive integers with $n > 2k+m+4$. If $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM, then either $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1,\omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega - 1)^m$.*

In 2001 an idea of gradation of sharing of values was introduced in ([7], [8]) which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1 [7, 8] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\le k)$ if and only if it is an a-point of g with multiplicity m $(\le k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

If $a(z)$ is a small function with respect to $f(z)$ and $g(z)$, we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight l according as $f(z) - a(z)$ and $g(z) - a(z)$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$ respectively.

In 2008, L. Liu [12] proved the following.

Theorem F *Let f and g be two non-constant entire functions, and let n, m and k be three positive integers such that $n > 5k + 4m + 9$. If $E_0(1,[f^n(f-1)^m]^{(k)}) = E_0(1,[g^n(g-1)^m]^{(k)})$ then either $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1,\omega_2) = \omega_1^n(\omega_1-1)^m - \omega_2^n(\omega_2-1)^m$.*

Recently P. Sahoo [14] proved the following result.

Theorem G *Let f and g be two transcendental meromorphic functions and n (≥ 1), k (≥ 1), m (≥ 0) and $l(\ge 0)$ be four integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share (b,l) for a nonzero constant b. Then*

(1) *when $m = 0$, if $f(z) \ne \infty$, $g(z) \ne \infty$ and $l \ge 2$, $n > 3k + 8$ or $l = 1$, $n > 5k + 10$ or $l = 0$, $n > 9k + 14$, then either $f \equiv tg$, where t is a constant satisfying $t^n = 1$, or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$,*

(2) *when $m = 1$ and $\Theta(\infty;f) > \frac{2}{n}$ then either $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)} \equiv b^2$, except for $k = 1$ or $f \equiv g$, provided one of $l \ge 2$, $n > 3k + 11$ or $l = 1$, $n > 5k + 14$ or$l = 0$, $n > 9k + 20$ holds; and*

(3) *when $m \ge 2$, and $l \ge 2$, $n > 3k + m + 10$ or $l = 1$, $n > 5k + 2m + 12$ or $l = 0$, $n > 9k + 4m + 16$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv b^2$ except for $k = 1$ or $f \equiv g$ or f and g satisfying the algebraic equation*

$R(f,g)=0$, *where*

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m-\omega_2^n(\omega_2-1)^m.$$

It is quite natural to ask the following questions.
Question 1: Can lower bound of n be further reduced in Theorems F, G?
Question 2: Can one remove the condition $f\neq\infty$, $g\neq\infty$ when $m=0$ in Theorem G?

In this paper, taking the possible answer of the above questions into background we obtain the following results which improve and generalize Theorems F, G.

Theorem 1 *Let f and g be two transcendental meromorphic functions and let $p(z)$ be a nonzero polynomial with $\deg(p)=l$. Suppose $[f^n(f-1)^m]^{(k)}-p$ and $[g^n(g-1)^m]^{(k)}-p$ share $(0,k_1)$, where $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ are three integers. Now when one of the following conditions holds:*

(i) $k_1\geq 2$ *and* $n>3k+m+8(=s_2)$;

(ii) $k_1=1$ *and* $n>4k+\frac{3m}{2}+9(=s_1)$;

(iii) $k_1=0$ *and* $n>9k+4m+14(=s_0)$;

then the following conclusions occur

(1) *when $m=0$, then either $f\equiv tg$, where t is a constant satisfying $t^n=1$, or if $p(z)$ is not a constant and $n>\max\{s_i,2k+2l-1\}$, $i=0,1,2$, then $f(z)=c_1e^{cQ(z)}$, $g(z)=c_2e^{-cQ(z)}$, where $Q(z)=\int_0^z p(z)dz$, c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n=-1$, if $p(z)$ is a nonzero constant b, then $f(z)=c_3e^{dz}$, $g(z)=c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k}=b^2$;*

(2) *when $m=1$ and $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n}$, then either $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)}\equiv p^2$, except for $k=1$ or $f\equiv g$;*

(3) *when $m\geq 2$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv p^2$ except for $k=1$ or $f\equiv g$ or f and g satisfying the algebraic equation $R(f,g)=0$, where*

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m-\omega_2^n(\omega_2-1)^m.$$

In addition, when f and g share $(\infty,0)$, then the possibility $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv p^2$ does not occur for $m\geq 1$.

Remark 1 *When* f *and* g *share* ∞ *IM then the conditions (i), (ii) and (iii) of Theorem 1 will be replaced by respectively* $l \geq 2$ *and* $n > 3k + m + 7$, $l = 1$ *and* $n > 4k + \frac{3m}{2} + 8$ *and* $l = 0$ *and* $n > 9k + 4m + 13$.

Theorem 2 *Let* f *and* g *be two transcendental entire functions and let* $p(z)$ *be a nonzero polynomial with* $\deg(p) = l$. *Suppose* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0, k_1)$, *where* $n\ (\geq 1)$, $k\ (\geq 1)$, $m\ (\geq 0)$ *are three integers. Now when one of the following conditions holds:*

(i) $k_1 \geq 2$ *and* $n > 2k + m + 4 (= s_2)$;

(ii) $k_1 = 1$ *and* $n > \frac{5k+3m+9}{2} (= s_1)$;

(iii) $k_1 = 0$ *and* $n > 5k + 4m + 7 (= s_0)$;

then the following conclusions occur

(1) *when* $m = 0$, *then either* $f \equiv tg$, *where* t *is a constant satisfying* $t^n = 1$, *or if* $p(z)$ *is not a constant and* $n > \max\{s_i, k + 2l\}$, $i = 0, 1, 2$, *then* $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(z)dz$, c_1, c_2 *and* c *are constants such that* $(nc)^2(c_1c_2)^n = -1$,
if $p(z)$ *is a nonzero constant* b, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4 *and* d *are constants such that* $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$;

(2) *when* $m = 1$ then $f \equiv g$;

(3) *when* $m \geq 2$, *then either* $f \equiv g$ *or* f *and* g *satisfying the algebraic equation* $R(f, g) = 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m.$$

We now explain some definitions and notations which are used in the paper.

Definition 2 [10] *Let* p *be a positive integer and* $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ $(\overline{N}(r, a; f \mid\geq p))$ *denotes the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not less than* p.

(ii) $N(r, a; f \mid\leq p)$ $(\overline{N}(r, a; f \mid\leq p))$ *denotes the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not greater than* p.

Definition 3 {11, cf.[18]} *For* $a \in \mathbb{C} \cup \{\infty\}$ *and a positive integer* p *we denote by* $N_p(r, a; f)$ *the sum* $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots \overline{N}(r, a; f \mid\geq p)$. *Clearly* $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4 *Let* $a, b \in \mathbb{C} \cup \{\infty\}$. *Let* p *be a positive integer. We denote by* $\overline{N}(r, a; f \mid \geq p \mid g = b)$ *(*$\overline{N}(r, a; f \mid \geq p \mid g \neq b)$*) the reduced counting function of those* a*-points of* f *with multiplicities* $\geq p$, *which are the* b*-points (not the* b*-points) of* g.

Definition 5 {cf.[1], 2} *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p, *a* 1*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p > q$, *by* $N_E^{1)}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q = 1$ *and by* $\overline{N}_E^{(2}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q \geq 2$, *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.

Definition 6 {cf.[1], 2} *Let* k *be a positive integer. Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p, *a* 1*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_{f>k}(r, 1; g)$ *the reduced counting function of those* 1*-points of* f *and* g *such that* $p > q = k$. $\overline{N}_{g>k}(r, 1; f)$ *is defined analogously.*

Definition 7 *[7, 8] Let* f, g *share a value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

Lemma 1 [15] *Let* f *be a non-constant meromorphic function and let* $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ *be meromorphic functions such that* $T(r, a_i(z)) = S(r, f)$ *for* $i = 0, 1, 2, ..., n$. *Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 [20] *Let* f *be a non-constant meromorphic function, and* p, k *be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \leq T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2}$$

$$N_p\left(r,0;f^{(k)}\right) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{3}$$

Lemma 3 [9] *If* $N(r,0;f^{(k)} \mid f \neq 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f*, where a zero of* $f^{(k)}$ *is counted according to its multiplicity, then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid < k) + k\overline{N}(r,0;f \mid \geq k) + S(r,f).$$

Lemma 4 [11] *Let* f_1 *and* f_2 *be two non-constant meromorphic functions satisfying* $\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$ *for* $i = 1,2$*. If* $f_1^s f_2^t - 1$ *is not identically zero for arbitrary integers* s *and* $t(|s| + |t| > 0)$*, then for any positive* ε*, we have*

$$N_0(r,1;f_1,f_2) \leq \varepsilon T(r) + S(r;f_1,f_2),$$

where $N_0(r,1;f_1,f_2)$ *denotes the deduced counting function related to the common* 1*-points of* f_1 *and* f_2 *and* $T(r) = T(r,f_1) + T(r,f_2)$, $S(r;f_1,f_2) = o(T(r))$ *as* $r \longrightarrow \infty$ *possibly outside a set of finite linear measure.*

Lemma 5 [6] *Suppose that* f *is a non-constant meromorphic function,* $k \geq 2$ *is an integer. If*

$$N(r,\infty,f) + N(r,0;f) + N(r,0;f^{(k)}) = S(r,\frac{f'}{f}),$$

then $f(z) = e^{az+b}$*, where* $a \neq 0$*,* b *are constants.*

Lemma 6 [5] *Let* $f(z)$ *be a non-constant entire function and let* $k \geq 2$ *be a positive integer. If* $f(z)f^{(k)}(z) \neq 0$*, then* $f(z) = e^{az+b}$*, where* $a \neq 0, b$ *are constant.*

Lemma 7 [19] *Let* f *be a non-constant meromorphic function, and let* k *be a positive integer. Suppose that* $f^{(k)} \not\equiv 0$*, then*

$$N(r,0;f^{(k)}) \leq N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 8 *Let* f *and* g *be two non-constant meromorphic functions. Let* n (≥ 1), k (≥ 1) *and* m (≥ 0) *be three integers such that* $n > 3k+m+1$. *If* $[f^n(f-1)^m]^{(k)} \equiv [g^n(g-1)^m]^{(k)}$, *then* $f^n(f-1)^m \equiv g^n(g-1)^m$.

Proof. We have $[f^n(f-1)^m]^{(k)} \equiv [g^n(g-1)^m]^{(k)}$. Integrating we get

$$[f^n(f-1)^m]^{(k-1)} \equiv [g^n(g-1)^m]^{(k-1)} + c_{k-1}.$$

If possible suppose $c_{k-1} \neq 0$. Now in view of Lemma 2 for $p = 1$ and using second fundamental theorem we get

$$\begin{aligned}
& (n+m)T(r,f) \\
\leq\; & T(r,[f^n(f-1)^m]^{(k-1)}) - \overline{N}(r,0;[f^n(f-1)^m]^{(k-1)}) + N_k(r,0;f^n(f-1)^m) \\
& + S(r,f) \\
\leq\; & \overline{N}(r,0;[f^n(f-1)^m]^{(k-1)}) + \overline{N}(r,\infty;f) + \overline{N}(r,c_{k-1};[f^n(f-1)^m]^{(k-1)}) \\
& -\overline{N}(r,0;[f^n(f-1)^m]^{(k-1)}) + N_k(r,0;f^n(f-1)^m) + S(r,f) \\
\leq\; & \overline{N}(r,\infty;f) + \overline{N}(r,0;[g^n(g-1)^m]^{(k-1)}) + k\overline{N}(r,0;f) + N(r,0;(f-1)^m) \\
& + S(r,f) \\
\leq\; & (k+1+m)\ T(r,f) + (k-1)\overline{N}(r,\infty;g) + N_k(r,0;g^n(g-1)^m) + S(r,f) \\
\leq\; & (k+1+m)\ T(r,f) + k\ \overline{N}(r,\infty;g) + k\ \overline{N}(r,0;g) + N(r,0;(g-1)^m) \\
& +S\ (r,f) \\
\leq\; & (k+1+m)\ T(r,f) + (2k+m)\ T(r,g) + S(r,f) + S(r,g) \\
\leq\; & (3k+2m+1)\ T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)\ T(r,g) \leq (3k+2m+1)\ T(r) + S(r).$$

Combining these we get

$$(n-m-3k-1)\ T(r) \leq S(r),$$

which is a contradiction since $n > 3k+m+1$. Therefore $c_{k-1} = 0$ and so

$$[f^n(f-1)^m]^{(k-1)} \equiv [g^n(g-1)^m]^{(k-1)}.$$

Proceeding in this way we obtain

$$[f^n(f-1)^m]' \equiv [g^n(g-1)^m]'.$$

Integrating we get

$$f^n(f-1)^m \equiv g^n(g-1)^m + c_0.$$

If possible suppose $c_0 \neq 0$. Now using second fundamental theorem we get

$$\begin{aligned}
& (n+m)T(r,f) \\
\leq\ & \overline{N}(r,0;f^n(f-1)^m) + \overline{N}(r,\infty;f^n(f-1)^m) + \overline{N}(r,c_0;f^n(f-1)^m) + S(r,f) \\
\leq\ & \overline{N}(r,0;f) + mT(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g^n(g-1)^m) + S(r,f) \\
\leq\ & (m+1)\ T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + m\ T(r,g) + S(r,f) \\
\leq\ & (3+2m)\ T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)\ T(r,g) \leq (3+2m)\ T(r) + S(r).$$

Combining these we get

$$(n-3-m)\ T(r) \leq S(r),$$

which is a contradiction since $n > 4+m$. Therefore $c_0 = 0$ and so

$$f^n(f-1)^m \equiv g^n(g-1)^m.$$

This proves the Lemma. □

Lemma 9 *Let* f, g *be two transcendental meromorphic functions, let* $n(\geq 1)$, $m(\geq 0)$ *and* $k(\geq 1)$ *be three integers with* $n > k+2$. *If* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0,0)$, *where* $p(z)$ *is a non zero polynomial, then* $T(r,f) = O(T(r,g))$ *and* $T(r,g) = O(T(r,f))$.

Proof. In view of Lemmas 1, 2 for $p=1$ and using second fundamental theorem for small function (see [17]) we get

$$\begin{aligned}
& (n+m)T(r,f) = T(r,f^n(f-1)^m) + O(1) \\
\leq\ & T(r,[f^n(f-1)^m]^{(k)}) - \overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + N_{k+1}(r,0;f^n(f-1)^m) \\
& + S(r,f) \\
\leq\ & \overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + \overline{N}(r,\infty;f) + \overline{N}(r,p;[f^n(f-1)^m]^{(k)}) \\
& -\overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + N_{k+1}(r,0;f^n(f-1)^m) + (\varepsilon+o(1))T(r,f) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,p;[f^n(f-1)^m]^{(k)}) + (k+1)\overline{N}(r,0;f) + N(r,0;(f-1)^m) \\
& +(\varepsilon+o(1))T(r,f) \\
\leq\ & (k+2+m)\ T(r,f) + \overline{N}(r,p;[g^n(g-1)^m]^{(k)}) + (\varepsilon+o(1))T(r,f) \\
\leq\ & (k+2+m)\ T(r,f) + (k+1)(n+m)\ T(r,g) + (\varepsilon+o(1))T(r,f),
\end{aligned}$$

i.e.,

$$(n-k-2)\, T(r,f) \leq (k+1)(n+m)\, T(r,g) + (\varepsilon + o(1))T(r,f),$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$. Since $n > k+2$, we have $T(r,f) = O(T(r,g))$. Similarly we have $T(r,g) = O(T(r,f))$. This completes the proof. □

Lemma 10 *Let* f, g *be two transcendental meromorphic functions and let* $F = \frac{[f^n(f-1)^m]^{(k)}}{p}$, $G = \frac{[g^n(g-1)^m]^{(k)}}{p}$, *where* $p(z)$ *is a non zero polynomial and* $n(\geq 1)$, $k(\geq 1)$ *and* $m(\geq 0)$ *are three integers such that* $n > 3k+m+3$. *If* $H \equiv 0$, *then* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0,\infty)$ *as well as one of the following conclusions occur*

(i) $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2$;

(ii) $f^n(f-1)^m \equiv g^n(g-1)^m$.

Proof. Let $P(w) = (w-1)^m$. Then $F = \frac{[f^nP(f)]^{(k)}}{p}$ and $G = \frac{[g^nP(g)]^{(k)}}{p}$.
Since $H \equiv 0$, by integration we get

$$\frac{1}{F-1} \equiv \frac{BG + A - B}{G-1}, \tag{4}$$

where A, B are constants and $A \neq 0$. From (4) it is clear that F and G share $(1,\infty)$. We now consider following cases.
Case 1. Let $B \neq 0$ and $A \neq B$.
If $B = -1$, then from (4) we have

$$F \equiv \frac{-A}{G-A-1}.$$

Therefore

$$\overline{N}(r, A+1; G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + \overline{N}(r,0;p).$$

So in view of Lemmas 1, 2 and the second fundamental theorem we get

$$\begin{aligned}
& (n+m)\, T(r,g) \\
\leq\ & T(r,G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,A+1;G) + N_{k+1}(r,0;g^nP(g)) \\
& - \overline{N}(r,0;G) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^nP(g)) + \overline{N}(r,\infty;f) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n) + N_{k+1}(r,0;P(g)) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + S(r,g) \\
\leq\ & T(r,f) + (k+2+m)\, T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.
So for $r \in I$ we have

$$(n-k-3)\, T(r,g) \leq S(r,g),$$

which is a contradiction since $n > k+3$.
If $B \neq -1$, from (4) we obtain that

$$F - (1+\frac{1}{B}) \equiv \frac{-A}{B^2[G + \frac{A-B}{B}]}.$$

So

$$\overline{N}(r, \frac{(B-A)}{B}; G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + \overline{N}(r,0;p).$$

Using Lemmas 1, 2 and the same argument as used in the case when $B = -1$ we can get a contradiction.
Case 2. Let $B \neq 0$ and $A = B$.
If $B = -1$, then from (4) we have

$$FG \equiv 1,$$

i.e.,

$$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv p^2,$$

i.e.,

$$[f^n (f-1)^m][g^n (g-1)^m] \equiv p^2.$$

If $B \neq -1$, from (4) we have

$$\frac{1}{F} \equiv \frac{BG}{(1+B)G - 1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r,0;F).$$

So in view of Lemmas 1, 2 and the second fundamental theorem we get

$$\begin{aligned}
& (n+m)\, T(r,g) \\
\leq\ & T(r,G) + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{1}{1+B}; G) + N_{k+1}(r.0;g^n P(g)) \\
& - \overline{N}(r,0;G) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + \overline{N}(r,0;F) + S(r,g)
\end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+(k+1)\overline{N}(r,0;f)+T(r,P(f)) \\
&\quad +k\overline{N}(r,\infty;f)+S(r,f)+S(r,g) \\
&\leq (k+2+m)\,T(r,g)+(2k+1+m)\,T(r,f)+S(r,f)+S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$(n-3k-3-m)\,T(r,g) \leq S(r,g),$$

which is a contradiction since $n > 3k+3+m$.
Case 3. Let $B = 0$. From (4) we obtain

$$F \equiv \frac{G+A-1}{A}. \tag{5}$$

If $A \neq 1$, then from (5) we obtain

$$\overline{N}(r,1-A;G) = \overline{N}(r,0;F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $A = 1$ and from (5) we obtain

$$F \equiv G,$$

i.e.,

$$[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}.$$

Then by Lemma 8 we have

$$f^n P(f) \equiv g^n P(g), \tag{6}$$

i.e.,

$$f^n (f-1)^m \equiv g^n (g-1)^m.$$

□

Lemma 11 *Let f, g be two transcendental meromorphic functions, $p(z)$ be a non-zero polynomial with $\deg(p(z)) = l$, n, k be two positive integers. Let $[f^n]^{(k)} - p$ and $[g^n]^{(k)} - p$ share $(0,\infty)$. Suppose $[f^n]^{(k)}[g^n]^{(k)} \equiv p^2$,*

(i) *if $p(z)$ is not a constant and $n > 2k+2l-1$, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)dz$, c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$,*

(ii) *if* $p(z)$ *is a nonzero constant* b *and* $n > 2k$*, then* $f(z) = c_3e^{cz}$*,* $g(z) = c_4e^{-cz}$*, where* c_3*,* c_4 *and* c *are constants such that* $(-1)^k(c_3c_4)^n(nc)^{2k} = b^2$.

Proof. Suppose
$$[f^n]^{(k)}[g^n]^{(k)} \equiv p^2. \tag{7}$$
We consider the following cases.
Case 1: Let $deg(p(z)) = l(\geq 1)$.
Let z_0 be a zero of f with multiplicity q. Then z_0 be a zero of $[f^n]^{(k)}$ with multiplicity $nq - k$. Now one of the following possibilities holds.
(i) z_0 will be neither a zero of $[g^n]^{(k)}$ nor a pole of g; (ii) z_0 will be a zero of g; (iii) z_0 will be a zero of $[g^n]^{(k)}$ but not a zero of g and (iv) z_0 will be a pole of g.
We now explain only the above two possibilities (i) and (iv) because other two possibilities follow from these.
For the possibility (i): Note that since $n \geq 2k + 2l$, we must have
$$nq - k \geq n - k \geq k + 2l. \tag{8}$$
Thus z_0 must be a zero of $[f^n]^{(k)}$ with multiplicity at least $k + 2l$. But we see from (7) that z_0 must be a zero of $p^2(z)$ with multiplicity atmost $2l$. Hence we arrive at a contradiction and so f has no zero in this case.
For the possibility (iv): Let z_0 be a pole of g with multiplicity q_1. Clearly z_0 will be pole of $[g^n]^{(k)}$ with multiplicity $nq_1 + k$. Obviously $q > q_1$, or else z_0 is a pole of $p(z)$, which is a contradiction since $p(z)$ is a polynomial. Clearly $nq - k \geq nq_1 + k$. Now
$$nq - k = nq_1 + k$$
implies that
$$n(q - q_1) = 2k. \tag{9}$$
Since $n \geq 2k + 2l$, we get a contradiction from (9). Hence we must have
$$nq - k > nq_1 + k.$$
This shows that z_0 is a zero of $p(z)$ and we have $N(r, 0; f) = O(\log r)$. Similarly we can prove that $N(r, 0; g) = O(\log r)$. Thus in general we can take $N(r, 0; f) + N(r, 0; g) = O(\log r)$.
We know that
$$N(r, \infty; [f^n]^{(k)}) = n\, N(r, \infty; f) + k\, \overline{N}(r, \infty; f).$$

Also by Lemma 7 we have

$$\begin{aligned} N(r,0;[g^n]^{(k)}) &\leq n\,N(r,0;g)+k\,\overline{N}(r,\infty;g)+S(r,g)\\ &\leq k\,\overline{N}(r,\infty;g)+\ O(\log r)+S(r,g). \end{aligned}$$

From (7) we get

$$N(r,\infty;[f^n]^{(k)})=N(r,0;[g^n]^{(k)}),$$

i.e.,

$$n\,N(r,\infty;f)+k\,\overline{N}(r,\infty;f)\leq k\,\overline{N}(r,\infty;g)+O(\log r)+S(r,g). \tag{10}$$

Similarly we get

$$n\,N(r,\infty;g)+k\,\overline{N}(r,\infty;g)\leq k\,\overline{N}(r,\infty;f)+O(\log r)+S(r,f). \tag{11}$$

Since f and g are transcendental, it follows that

$$S(r,f)+O(\log r)=S(r,f),\quad S(r,g)+O(\log r)=S(r,g).$$

Combining (10) and (11) we get

$$N(r,\infty;f)+N(r,\infty;g)=S(r,f)+S(r,g).$$

By Lemma 9 we have $S(r,f)=S(r,g)$ and so we obtain

$$N(r,\infty;f)=S(r,f),\quad N(r,\infty;g)=S(r,g). \tag{12}$$

Let

$$F_1=\frac{[f^n]^{(k)}}{p},\quad G_1=\frac{[g^n]^{(k)}}{p}. \tag{13}$$

Note that $T(r,F_1)\leq n(k+1)T(r,f)+S(r,f)$ and so $T(r,F_1)=O(T(r,f))$. Also by Lemma 2 one can obtain $T(r,f)=O(T(r,F_1))$. Hence $S(r,F_1)=S(r,f)$. Similarly we get $S(r,G_1)=S(r,g)$. Also

$$F_1G_1\equiv 1. \tag{14}$$

If $F_1\equiv cG_1$, where c is a nonzero constant, then F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1\not\equiv cG_1$ and so in view of (14) we see that F_1 and G_1 share $(-1,0)$.

Now by Lemma 7 we have

$$N(r,0;F_1)\leq n\,N(r,0;f)+k\,\overline{N}(r,\infty;f)+S(r,f)\leq S(r,F_1).$$

Similarly we have

$$N(r,0;G_1) \leq n\, N(r,0;g) + k\, \overline{N}(r,\infty;g) + S(r,g) \leq S(r,G_1).$$

Also we see that

$$N(r,\infty;F_1) = S(r,F_1), \quad N(r,\infty;G_1) = S(r,G_1).$$

Here it is clear that $T(r,F_1) = T(r,G_1) + O(1)$. Let

$$f_1 \;=\; \frac{F_1}{G_1}.$$

and

$$f_2 = \frac{F_1 - 1}{G_1 - 1}.$$

Clearly f_1 is non-constant. If f_2 is a nonzero constant then F_1 and G_1 share (∞,∞) and so from (14) we conclude that F_1 and G_1 have no poles. Next we suppose that f_2 is non-constant. Also we see that

$$F_1 = \frac{f_1(1-f_2)}{f_1 - f_2}, \quad G_1 = \frac{1-f_2}{f_1-f_2}.$$

Clearly

$$T(r,F_1) \leq 2[T(r,f_1) + T(r,f_2)] + O(1)$$

and

$$T(r,f_1) + T(r,f_2) \leq 4T(r,F_1) + O(1).$$

These give $S(r,F_1) = S(r;f_1,f_2)$. Also we see that

$$\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$$

for $i = 1,2$.
Next we suppose $\overline{N}(r,-1;F_1) \neq S(r,F_1)$, otherwise F_1 will be a constant. Also we see that

$$\overline{N}(r,-1;F_1) \leq N_0(r,1;f_1,f_2).$$

Thus we have

$$T(r,f_1) + T(r,f_2) \leq 4\, N_0(r,1;f_1,f_2) + S(r,F_1).$$

Then by Lemma 4 there exist two integers s and $t(|s|+|t|>0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s \left[\frac{F_1-1}{G_1-1}\right]^t \equiv 1. \tag{15}$$

We now consider following cases.

Case (i) Let $s=0$ and $t \neq 0$. Then from (15) we get

$$(F_1-1)^t \equiv (G_1-1)^t.$$

This shows that F_1 and G_1 share (∞, ∞) and so from (14) we conclude that F_1 and G_1 have no poles.

Case (ii) Suppose $s \neq 0$ and $t=0$. Then from (15) we get

$$F_1^s \equiv G_1^s$$

and so we arrive at a contradiction from (14).

Case (iii): Suppose $s>0$ and $t=-t_1$, where $t_1>0$. Then we have

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1-1}{G_1-1}\right]^{t_1}. \tag{16}$$

If possible suppose F_1 has a pole. Let z_{p_1} be a pole of F_1 of multiplicity p_1. Then from (14) we see that z_{p_1} must be a zero of G_1 of multiplicity p_1. Now from (16) we get $2s=t_1$ and so

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1-1}{G_1-1}\right]^{2s}.$$

This implies that

$$F_1^{s-1}+F_1^{s-2}G_1+F_1^{s-3}G_1^2+\ldots+F_1G_1^{s-2}+G_1^{s-1} \equiv G_1^s \frac{(F_1-1)^{2s}-(G_1-1)^{2s}}{(G_1-1)^{2s}(F_1-G_1)}. \tag{17}$$

If z_p is a zero of F_1-1 with multiplicity p then the Taylor expansion of F_1-1 about z_p is

$$F_1-1=a_p(z-z_p)^p+a_{p+1}(z-z_p)^{p+1}+\ldots\ldots, \quad a_p \neq 0.$$

Since F_1-1 and G_1-1 share $(0,\infty)$,

$$G_1-1=b_p(z-z_p)^p+b_{p+1}(z-z_p)^{p+1}+\ldots\ldots, \quad b_p \neq 0.$$

Let

$$\Phi_1 = \frac{F_1'}{F_1} - \frac{G_1'}{G_1} \quad \text{and} \quad \Phi_2 = \left(\frac{F_1'}{F_1}\right)^{2s} - \left(\frac{G_1'}{G_1}\right)^{2s}. \tag{18}$$

Since $F_1 \not\equiv cG_1$, where c is a nonzero constant, it follows that $\Phi_1 \not\equiv 0$ and $\Phi_2 \not\equiv 0$. Also

$$T(r, \Phi_1) = S(r, F_1) \quad \text{and} \quad T(r, \Phi_2) = S(r, F_1).$$

From (18) we find

$$\overline{N}_{(2}(r, 1; F_1) = \overline{N}_{(2}(r, 1; G_1) \leq N(r, 0; \Phi_1) = S(r, F_1).$$

Let $p = 1$. If $a_1 = b_1$, then by an elementary calculation gives that $\Phi_1(z) = O((z - z_1)^k)$, where k is a positive integer. This proves that z_1 is a zero of Φ_1. Next we suppose $a_1 \neq b_1$, but $a_1^{2s} = b_1^{2s}$. Then by an elementary calculation we get $\Phi_2(z) = O((z - z_1)^q)$ where q is a positive integer. This proves that z_1 is a zero of Φ_2.

Finally we suppose $a_1 \neq b_1$ and $a_1^{2s} \neq b_1^{2s}$. Therefore from (17) we arrive at a contradiction. Hence

$$N_{1)}(r, 1; F_1) = N_{1)}(r, 1; G_1) = S(r, F_1).$$

But this is impossible as $\overline{N}(r, 1; F_1) \sim T(r, F_1)$ and $\overline{N}(r, 1; G_1) \sim T(r, G_1)$. Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.

Case (iv): Suppose either $s > 0$ and $t > 0$ or $s < 0$ and $t < 0$. Then from (15) one can easily prove that F_1 and G_1 have no poles. Consequently from (14) we see that F_1 and G_1 have no zeros. We deduce from (13) that both f and g have no pole.

Since F_1 and G_1 have no zeros and poles, we have

$$F_1 \equiv e^{\gamma_1} G_1,$$

i.e.,

$$[f^n]^{(k)} \equiv e^{\gamma_1} [g^n]^{(k)},$$

where γ_1 is a non-constant entire function. Then from (7) we get

$$[f^n]^{(k)} \equiv c e^{\frac{1}{2}\gamma_1} p, \quad [g^n]^{(k)} \equiv c e^{-\frac{1}{2}\gamma_1} p, \tag{19}$$

where $c = \pm 1$. Since $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$, so we can take

$$f(z) = P_1(z) e^{\alpha_1(z)}, \quad g(z) = Q_1(z) e^{\beta_1(z)}, \tag{20}$$

P_1, Q_1 are nonzero polynomials, α_1, β_1 are two non-constant entire functions. If possible suppose that $P_1(z)$ is not a constant. Let z_1 be a zero of f with multiplicity t. Then z_1 must be a zero of $[f^n]^{(k)}$ with multiplicity $nt-k$. Note that $nt-k \geq n-k \geq k+2l$, as $n \geq 2k+2l$. Clearly z_1 must be a zero of $p^2(z)$ with multiplicity at least $k+2l$, which is impossible since z_1 can be a zero of $p^2(z)$ with multiplicity at most $2l$. Hence $P_1(z)$ is a constant. Similarly we can prove that $Q_1(z)$ is a constant. So we can rewrite f and g as follows

$$f = e^{\alpha}, \quad g = e^{\beta}. \tag{21}$$

We deduce from (7) and (21) that either both α and β are transcendental entire functions or both α and β are polynomials. We now consider following cases.

Subcase 1.1: Let $k \geq 2$.

First we suppose both α and β are transcendental entire functions.

Note that

$$S(r, n\alpha) = S(r, \frac{[f^n]'}{f^n}), \quad S(r, n\beta) = S(r, \frac{[g^n]'}{g^n}).$$

Moreover we see that

$$N(r, 0; [f^n]^{(k)}) \leq N(r, 0; p^2) = O(\log r).$$

$$N(r, 0; [g^n]^{(k)}) \leq N(r, 0; p^2) = O(\log r).$$

From these and using (21) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; [f^n]^{(k)}) = S(r, n\alpha) = S(r, \frac{[f^n]'}{f^n}) \tag{22}$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; [g^n]^{(k)}) = S(r, n\beta) = S(r, \frac{[g^n]'}{g^n}). \tag{23}$$

Then from (22), (23) and Lemma 5 we must have

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{24}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But these types of f and g do not agree with the relation (7).

Next we suppose α and β are both polynomials.
Clearly $\alpha+\beta \equiv C$ and $\deg(\alpha) = \deg(\beta)$. Also $\alpha^{'} \equiv \beta^{'}$. If $\deg(\alpha) = \deg(\beta) = 1$, then we again get a contradiction from (7).
Next we suppose $\deg(\alpha) = \deg(\beta) \geq 2$.
We deduce from (21) that

$$\begin{aligned}
&(f^n)^{'} = n\alpha^{'}e^{n\alpha}\\
&(f^n)^{''} = [n^2(\alpha^{'})^2 + n\alpha^{''}]e^{n\alpha}\\
&(f^n)^{'''} = [n^3(\alpha^{'})^3 + 3n^2\alpha^{'}\alpha^{''} + n\alpha^{'''}]e^{n\alpha}\\
&(f^n)^{(iv)} = [n^4(\alpha^{'})^4 + 6n^3(\alpha^{'})^2\alpha^{''} + 3n^2(\alpha^{''})^2 + 4n^2\alpha^{'}\alpha^{'''} + n\alpha^{(iv)}]e^{n\alpha}\\
&(f^n)^{(v)} = [n^5(\alpha^{'})^5 + 10n^4(\alpha^{'})^3\alpha^{''} + 15n^3\alpha^{'}(\alpha^{''})^2 + 10n^3(\alpha^{'})^2\\
&\alpha^{'''} + 10n^2\alpha^{''}\alpha^{'''} + 5n^2\alpha^{'}\alpha^{(iv)} + n\alpha^{(v)}]e^{n\alpha}\\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
&[f^n]^{(k)} = [n^k(\alpha^{'})^k + K(\alpha^{'})^{k-2}\alpha^{''} + P_{k-2}(\alpha^{'})]e^{n\alpha},
\end{aligned}$$

where K is a suitably positive integer and $P_{k-2}(\alpha^{'})$ is a differential polynomial in $\alpha^{'}$.
Similarly we get

$$\begin{aligned}
[g^n]^{(k)} &= [n^k(\beta^{'})^k + K(\beta^{'})^{k-2}\beta^{''} + P_{k-2}(\beta^{'})]e^{n\beta}\\
&= [(-1)^k n^k(\alpha^{'})^k - K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''} + P_{k-2}(-\alpha^{'})]e^{n\beta}.
\end{aligned}$$

Since $\deg(\alpha) \geq 2$, we observe that $\deg((\alpha^{'})^k) \geq k \deg(\alpha^{'})$ and so $(\alpha^{'})^{k-2}\alpha^{''}$ is either a nonzero constant or $\deg((\alpha^{'})^{k-2}\alpha^{''}) \geq (k-1)\deg(\alpha^{'}) - 1$. Also we see that

$$\deg\left((\alpha^{'})^k\right) > \deg\left((\alpha^{'})^{k-2}\alpha^{''}\right) > \deg\left(P_{k-2}(\alpha^{'})\right) (\text{or } \deg\left(P_{k-2}(-\alpha^{'})\right)).$$

From (19), it is clear that the polynomials

$$n^k(\alpha^{'})^k + K(\alpha^{'})^{k-2}\alpha^{''} + P_{k-2}(\alpha^{'})$$

and

$$(-1)^k n^k(\alpha^{'})^k - K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''} + P_{k-2}(-\alpha^{'})$$

must be identical but this is impossible for $k \geq 2$. Actually the terms $n^k(\alpha^{'})^k + K(\alpha^{'})^{k-2}\alpha^{''}$ and $(-1)^k n^k(\alpha^{'})^k - K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''}$ can not be identical for

$k \geq 2$.

Subcase 2: Let $k = 1$. Then from (7) we get

$$AB\alpha^{'}\beta^{'}e^{n(\alpha+\beta)} \equiv p^2, \tag{25}$$

where $AB = n^2$. Let $\alpha + \beta = \gamma$. Suppose that α and β are both transcendental entire functions. From (25) we know that γ is not a constant since in that case we get a contradiction. Then from (25) we get

$$AB\alpha^{'}(\gamma^{'} - \alpha^{'})e^{n\gamma} \equiv p^2. \tag{26}$$

We have $T(r, \gamma^{'}) = m(r, \gamma^{'}) \leq m(r, \frac{(e^{n\gamma})^{'}}{e^{n\gamma}}) + O(1) = S(r, e^{n\gamma})$. Thus from (26) we get

$$\begin{aligned} T(r, e^{n\gamma}) &\leq T(r, \frac{p^2}{\alpha^{'}(\gamma^{'} - \alpha^{'})}) + O(1) \\ &\leq T(r, \alpha^{'}) + T(r, \gamma^{'} - \alpha^{'}) + O(\log r) + O(1) \\ &\leq 2\,T(r, \alpha^{'}) + S(r, \alpha^{'}) + S(r, e^{n\gamma}), \end{aligned}$$

which implies that $T(r, e^{n\gamma}) = O(T(r, \alpha^{'}))$ and so $S(r, e^{n\gamma})$ can be replaced by $S(r, \alpha^{'})$. Thus we get $T(r, \gamma^{'}) = S(r, \alpha^{'})$ and so $\gamma^{'}$ is a small function with respect to $\alpha^{'}$. In view of (26) and by the second fundamental theorem for small functions we get

$$\begin{aligned} T(r, \alpha^{'}) &\leq \overline{N}(r, \infty; \alpha^{'}) + \overline{N}(r, 0; \alpha^{'}) + \overline{N}(r, 0; \alpha^{'} - \gamma^{'}) + S(r, \alpha^{'}) \\ &\leq O(\log r) + S(r, \alpha^{'}), \end{aligned}$$

which shows that $\alpha^{'}$ is a polynomial and so α is a polynomial, which contradicts that α is a transcendental entire function. Next suppose without loss of generality that α is a polynomial and β is a transcendental entire function. Thus γ is transcendental. So in view of (26) we can obtain

$$\begin{aligned} nT(r, e^{\gamma}) &\leq T(r, \frac{p^2}{\alpha^{'}(\gamma^{'} - \alpha^{'})}) + O(1) \\ &\leq T(r, \alpha^{'}) + T(r, \gamma^{'} - \alpha^{'}) + S(r, e^{\gamma}) \\ &\leq T(r, \gamma^{'}) + S(r, e^{\gamma}) = S(r, e^{\gamma}), \end{aligned}$$

which leads a contradiction. Thus α and β are both polynomials. Also from (25) we can conclude that $\alpha + \beta \equiv C$ for a constant C and so $\alpha^{'} + \beta^{'} \equiv 0$. Again from (25) we get $n^2e^{nC}\alpha^{'}\beta^{'} \equiv p^2$. By computation we get

$$\alpha^{'} = cp, \quad \beta^{'} = -cp. \tag{27}$$

Hence

$$\alpha = cQ + b_1, \quad \beta = -cQ + b_2, \tag{28}$$

where $Q(z) = \int_0^z p(z)dz$ and b_1, b_2 are constants. Finally f and g take the form

$$f(z) = c_1 e^{cQ(z)}, \quad g(z) = c_2 e^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$.
Case 2: Let $p(z)$ be a nonzero constant b. Since $n > 2k$, one can easily prove that f and g have no zeros. Now proceeding in the same way as done in proof of **Case 1** we get $f = e^{\alpha}$ and $g = e^{\beta}$, where α and β are two non-constant entire functions.
We now consider following two subcases:
Subcase 2.1: Let $k \geq 2$.
We see that $f^n(z)[f^n(z)]^{(k)} \neq 0$ and $g^n(z)[g^n(z)]^{(k)} \neq 0$. Then by Lemma 6 we must have

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{29}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But from (7) we see that $a + c = 0$.
Subcase 2.1: Let $k = 1$.
Considering **Subcase 1.2** one can easily get

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{30}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. Finally f and g take the form

$$f(z) = c_3 e^{dz}, \quad g(z) = c_4 e^{-dz},$$

where c_3, c_4 and d are nonzero constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$. This completes the proof. □

Lemma 12 *Let* f, g *be two transcendental meromorphic functions, let* n, m *and* k *be three positive integers such that* $n > k$. *If* f *and* g *share* $(\infty, 0)$ *then* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2$, *where* $p(z)$ *is a non zero polynomial.*

Proof. Suppose

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2. \tag{31}$$

Since f and g share $(\infty, 0)$ we have from (31) that f and g are transcendental entire functions. So we can take

$$f(z) = h(z)e^{\alpha(z)}, \tag{32}$$

where h is a nonzero polynomial and α is a non-constant entire function. We know that $(w-1)^m = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_0$, where $a_i = (-1)^{m-i}\,{}^mC_{m-i}$, $i = 0, 1, 2, \ldots, m$. Since $f = he^{\alpha}$, then by induction we get

$$(a_i f^{n+i})^{(k)} = t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)})e^{(n+i)\alpha}, \tag{33}$$

where $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)})$ $(i = 0, 1, 2, \ldots, m)$ are differential polynomials in
$\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)}$. Obviously

$$t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)}) \not\equiv 0,$$

for $i = 0, 1, 2, \ldots, m$ and $[f^n(f-1)^m]^{(k)} \not\equiv 0$. Now from (31) and (33) we obtain

$$\overline{N}(r, 0; t_m e^{m\alpha(z)} + \ldots + t_0) \leq N(r, 0; p^2) = S(r, f). \tag{34}$$

Since α is an entire function, we obtain $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, \ldots, k$. Hence $T(r, t_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, m$. So from (34) and using second fundamental theorem for small functions (see [17]), we obtain

$$\begin{aligned} mT(r, f) &= T(r, t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + S(r, f) \\ &\leq \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha} + t_0) \\ &\quad + S(r, f) \\ &\leq \overline{N}(r, 0; t_m e^{(m-1)\alpha} + \ldots + t_1) + S(r, f) \\ &\leq (m-1)T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction. This completes the Lemma. □

Lemma 13 *Let f and g be two non-constant meromorphic functions and $\alpha(\not\equiv 0, \infty)$ be small function of f and g. Let n, m and k be three positive integers such that $n \geq m + 3$. Then*

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv \alpha^2, \quad \text{for} \quad k = 1.$$

Proof. We omit the proof since it can be proved in the line of the proof of Lemma 3 [14]. □

Lemma 14 [1] *If f, g be two non-constant meromorphic functions such that they share $(1, 1)$. Then*

$$2\overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2}(r, 1; f) - \overline{N}_{f>2}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g).$$

Lemma 15 [2] *Let* f, g *share* $(1,1)$. *Then*

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f^{'}) + S(r,f),$$

where $N_0(r,0;f^{'})$ *is the counting function of those zeros of* $f^{'}$ *which are not the zeros of* $f(f-1)$.

Lemma 16 [2] *Let* f *and* g *be two non-constant meromorphic functions sharing* $(1,0)$. *Then*

$$\begin{aligned} & \overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \\ \leq\ & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Lemma 17 [2] *Let* f, g *share* $(1,0)$. *Then*

$$\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$$

Lemma 18 [2] *Let* f, g *share* $(1,0)$. *Then*

$$\begin{aligned} &(i) \quad \overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f^{'}) + S(r,f) \\ &(ii) \quad \overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g^{'}) + S(r,g). \end{aligned}$$

3 Proof of the Theorem

Proof of Theorem 1. Let $F = \frac{[f^nP(f)]^{(k)}}{p}$ and $G = \frac{[g^nP(g)]^{(k)}}{p}$, where $P(w) = (w-1)^m$. It follows that F and G share $(1,k_1)$ except for the zeros of $p(z)$.
Case 1 Let $H \not\equiv 0$.
Subcase 1.1 $k_1 \geq 1$.
From (1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.
Since H has only simple poles we get

$$\begin{aligned} N(r,\infty;H) \leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F| \geq 2) \\ & + \overline{N}(r,0;G| \geq 2) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}), \end{aligned} \tag{35}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{36}$$

While $k_1\geq 2$, using (35) and (36) we get

$$\begin{aligned}&\overline{N}(r,1;F)\\ &\leq N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2)\leq \overline{N}(r,\infty;f)\\ &\quad+\ \overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{*}(r,1;F,G)\\ &\quad+\ \overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g).\end{aligned} \tag{37}$$

Now in view of Lemma 3 we get

$$\begin{aligned}&\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\,|\geq 2)+\overline{N}_{*}(r,1;F,G)\\ &\leq \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F|\geq 2)+\overline{N}(r,1;F|\geq 3)\\ &\quad=\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;G|\geq 2)+\overline{N}(r,1;G|\geq 3)\\ &\leq \overline{N}_0(r,0;G^{'})+N(r,1;G)-\overline{N}(r,1;G)\\ &\leq N(r,0;G^{'}\mid G\neq 0)\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;g)+S(r,g),\end{aligned} \tag{38}$$

Hence using (37), (38), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}&(n+m)T(r,f)\\ &\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)\\ &\quad-N_0(r,0;F^{'})\\ &\leq 2\,\overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))\\ &\quad+\overline{N}(r,0;F|\geq 2)+\ \overline{N}(r,0;G|\geq 2)+\overline{N}(r,1;F|\geq 2)+\overline{N}_{*}(r,1;F,G)\\ &\quad+\overline{N}_0(r,0;G^{'})-N_2(r,0;F)+S(r,f)+S(r,g)\\ &\leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)\\ &\quad+S(r,f)+S(r,g)\end{aligned} \tag{39}$$

$$\leq 2\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+N_{k+2}(r,0;f^{n}P(f))+k\,\overline{N}(r,\infty;g)$$
$$+\ N_{k+2}(r,0;g^{n}P(g))+S(r,f)+S(r,g)$$
$$\leq 2\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))$$
$$+(k+2)\,\overline{N}(r,0;g)+T(r,P(g))+k\,\overline{N}(r,\infty;g)+S(r,f)+S(r,g)$$
$$\leq (k+4+m)\,T(r,f)+(2k+4+m)\,T(r,g)+S(r,f)+S(r,g)$$
$$\leq (3k+8+2m)\,T(r)+S(r).$$

In a similar way we can obtain

$$(n+m)\,T(r,g)\leq (3k+8+2m)\;T(r)+S(r). \tag{40}$$

Combining (39) and (40) we see that

$$(n+m)\,T(r)\leq (3k+8+2m)\;T(r)+S(r),$$

i.e.,

$$(n-3k-8-m)\;T(r)\leq S(r). \tag{41}$$

Since $n>3k+8+m$, (41) leads to a contradiction.
While $k_1=1$, using Lemmas 3, 14, 15, (35) and (36) we get

$$\begin{aligned}
&\overline{N}(r,1;F) \qquad\qquad (42)\\
\leq\;& N(r,1;F|=1)+\overline{N}_{L}(r,1;F)+\overline{N}_{L}(r,1;G)+\overline{N}_{E}^{(2}(r,1;F)\\
\leq\;& \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{*}(r,1;F,G)\\
&+\overline{N}_{L}(r,1;F)+\overline{N}_{L}(r,1;G)+\overline{N}_{E}^{(2}(r,1;F)+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})\\
&+S(r,f)+S(r,g)\\
\leq\;& \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_{L}(r,1;F)\\
&+2\overline{N}_{L}(r,1;G)+\overline{N}_{E}^{(2}(r,1;F)+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g)\\
\leq\;& \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{F>2}(r,1;G)\\
&+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g)\\
\leq\;& \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\geq 2)\\
&+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_{0}(r,0;G^{'})+\overline{N}_{0}(r,0;F^{'})+S(r,f)+S(r,g)\\
\leq\;& \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\geq 2)\\
&+N(r,0;G^{'}|G\neq 0)+\overline{N}_{0}(r,0;F^{'})+S(r,f)+S(r,g)\\
\leq\;& \frac{3}{2}\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\frac{1}{2}\,\overline{N}(r,0;F)+N_{2}(r,0;G)\\
&+\overline{N}_{0}(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Hence using (42), Lemmas 1 and 2 we get from second fundamental theorem that

$$
\begin{aligned}
&(n+m)T(r,f)\\
&\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)\\
&\quad -N_0(r,0;F^{'})\\
&\leq \frac{5}{2}\,\overline{N}(r,\infty,f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\frac{1}{2}\overline{N}(r,0;F)\\
&\quad +N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leq \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+\frac{1}{2}\overline{N}(r,0;F)\\
&\quad +N_2(r,0;G)+S(r,f)+S(r,g)\\
&\leq \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g) \qquad (43)\\
&\quad +N_{k+2}(r,0;g^nP(g))+\frac{1}{2}\{k\overline{N}(r,\infty;f)\\
&\quad +N_{k+1}(r,0;f^nP(f))\}+S(r,f)+S(r,g)\\
&\leq \frac{5+k}{2}\,\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+\frac{3k+5}{2}\,\overline{N}(r,0;f)\\
&\quad +\frac{3}{2}\,T(r,P(f))+(k+2)\,\overline{N}(r,0;g)+T(r,P(g))+S(r,f)+S(r,g)\\
&\leq \left(2k+5+\frac{3m}{2}\right)\,T(r,f)+(2k+4+m)\,T(r,g)+S(r,f)+S(r,g)\\
&\leq \left(4k+9+\frac{5m}{2}\right)\,T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$(n+m)\,T(r,g)\leq \left(4k+9+\frac{5m}{2}\right)\,T(r)+S(r). \tag{44}$$

Combining (43) and (44) we see that

$$\left(n-4k-9-\frac{3m}{2}\right)\,T(r)\leq S(r). \tag{45}$$

Since $n>4k+9+\frac{3m}{2}$, (45) leads to a contradiction.
Subcase 1.2 $k_1=0$. Here (36) changes to

$$N_E^{1)}(r,1;F\mid=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G). \tag{46}$$

Using Lemmas 3, 16, 17, 18, (35) and (46) we get

$$\begin{aligned}
&\overline{N}(r,1;F)\\
&\le N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\le \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)\\
&\quad+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})\\
&\quad+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\le \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)\\
&\quad+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\quad+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\le \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)\\
&\quad+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F)+\overline{N}_L(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)\\
&\quad+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_0(r,0;F^{'})\\
&\quad+S(r,f)+S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,0;G^{'}|G\ne 0)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\le 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+\overline{N}(r,0;G)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned}\tag{47}$$

Hence using (47), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f)\\
&\le T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\le \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)\\
&\quad-N_0(r,0;F^{'})\\
&\le 4\overline{N}(r,\infty,f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+2\,\overline{N}(r,0;F)\\
&\quad+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)+\overline{N}(r,0;G)-N_2(r,0;F)\\
&\quad+S(r,f)+S(r,g)\\
&\le 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+2\,\overline{N}(r,0;F)
\end{aligned}$$

$$
\begin{aligned}
&+N_2(r,0;G)+\overline{N}(r,0;G)+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+2k\overline{N}(r,\infty;f)\\
&+2\ N_{k+1}(r,0;f^nP(f))+k\ \overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&+k\overline{N}(r,\infty;g)+N_{k+1}(r,0;g^nP(g))+S(r,f)+S(r,g)\\
&\leq (2k+4)\ \overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+(3k+4)\overline{N}(r,0;f)\\
&+3T(r,P(f))+(2k+3)\ \overline{N}(r,0;g)+2T(r,P(g))+S(r,f)+S(r,g)\\
&\leq (5k+8+3m)\ T(r,f)+(4k+6+2m)\ T(r,g)+S(r,f)+S(r,g)\\
&\leq (9k+14+5m)\ T(r)+S(r).
\end{aligned}
\tag{48}
$$

In a similar way we can obtain

$$(n+m)\ T(r,g)\leq (9k+14+5m)\ \ T(r)+S(r). \tag{49}$$

Combining (48) and (49) we see that

$$(n-9k-14-4m)\ \ T(r)\leq S(r). \tag{50}$$

Since $n>9k+14+4m$, (50) leads to a contradiction.
Case 2. Let $H\equiv 0$. Then by Lemma 10 we get either

$$f^n(f-1)^m\equiv g^n(g-1)^m \tag{51}$$

or

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv p^2. \tag{52}$$

We now consider following two subcases.
Subcase 2.1: Let $m=0$.
Now from (51) we get $f^n\equiv g^n$ and so $f\equiv tg$, where t is a constant satisfying $t^n=1$.
Also from (52) we get

$$[f^n]^{(k)}[g^n]^{(k)}\equiv p^2.$$

Then by Lemma 11 we get the conclusion (1).
Subcase 2.2: Let $m\geq 1$.
Applying Lemma 13, from (52) we see that

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\not\equiv p^2,$$

for $k = 1$.
In addition, when f and g share $(\infty, 0)$, then by Lemma 12 we must have

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2.$$

Next we consider the relation (51) and let $h = \frac{g}{f}$.
First we suppose that h is non-constant.
For $m = 1$: Then from (51) we get $f \equiv \frac{1-h^n}{1-h^{n+1}}$, i.e.,

$$f \equiv \Big(\frac{h^n}{1+h+h^2+\ldots+h^n} - 1\Big).$$

Hence by Lemma 1 we get

$$T(r,f) = T(r, \sum_{j=0}^{n} \frac{1}{h^j}) + O(1) = n\ T(r, \frac{1}{h}) + S(r,h) = n\ T(r,h) + S(r,h).$$

Similarly we have $T(r,g) = nT(r,h) + S(r,h)$. Therefore $S(r,f) = S(r,g) = S(r,h)$.
Also it is clear that

$$\sum_{j=1}^{n} \overline{N}(r, u_j; h) \le \overline{N}(r, \infty; f),$$

where $u_j = \exp(\frac{2j\pi i}{n+1})$ and $j = 1, 2, \ldots, n$.
Then by the second fundamental theorem we get

$$(n-2)\ T(r,h) \le \sum_{j=1}^{n} \overline{N}(r, u_j; h) + S(r,f) \le \overline{N}(r, \infty; f) + S(r,f).$$

Similarly we have

$$(n-2)\ T(r,h) \le \overline{N}(r, \infty; g) + S(r,g).$$

Adding and simplifying these we get

$$2(n-2)T(r,h) \le n(2 - \Theta(\infty; f) - \Theta(\infty; g) + \varepsilon)T(r,h) + S(r,h),$$

where $0 < \varepsilon < \Theta(\infty; f) + \Theta(\infty; g)$. This leads to a contradiction as $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$.

For $m \geq 2$: Then from (51) we can say that f and g satisfying the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m.$$

Next we suppose that h is a constant.
Then from (51) we get

$$f^n \sum_{i=0}^{m} (-1)^i\, {}^mC_{m-i}\, f^{m-i} \equiv g^n \sum_{i=0}^{m} (-1)^i\, {}^mC_{m-i} g^{m-i}. \tag{53}$$

Now substituting $g = fh$ in (53) we get

$$\sum_{i=0}^{m} (-1)^i\, {}^mC_{m-i}\, f^{n+m-i}(h^{n+m-i} - 1) \equiv 0,$$

which implies that $h = 1$. Hence $f \equiv g$. This completes the proof. □

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Applications of double lacunary sequences to n-norm

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Abstract. In the present paper we define some classes of double lacunary sequence spaces over n-normed spaces by means of a Musielak- Orlicz function. We study some relevant algebraic and topological properties. Further some inclusion relation among the classes are also examined.

1 Introduction and preliminaries

The initial work on double sequences is found in Bromwich [4]. Out of the definitions of convergence commonly employed for double series, only that due to Pringsheim permits a series to converge conditionally. Therefore, in spite of any disadvantages which it may possess, this definition is better adapted than others for the study of many problems in double sequences and series. Chief among the reasons why the theory of double sequences, under the Pringsheim definition of convergence, presents difficulties not encountered in the theory of simple sequences is the fact that a double sequence $\{x_{ij}\}$ may converge without x_{ij} being a bounded function of i and j. Thus it is not surprising that many authors in dealing with the convergence of double sequences should have restricted themselves to the class of bounded sequences, or in dealing with the

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summability of double series, to the class of series for which the function whose limit is the sum of the series is a bounded function of i and j. Without such a restriction, peculiar things may sometimes happen; for example, a double power series may converge with partial sum $\{S_{ij}\}$ unbounded at a place exterior to its associated circles of convergence. Nevertheless there are problems in the theory of double sequences and series where this restriction of boundedness as it has been applied is considerably more stringent than need be. After Bromwich, the study of double sequences was initiated by Hardy [11], Moricz [26], Moricz and Rhoades [19], Tripathy ([35], [36]), Basarir and Sonalcan [2] and many others. Hardy [11] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [38] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences.

In order to extend the notion of convergence of sequences, statistical convergence was introduced by Schoenberg [34] and the idea depends on the notion of density [31] of subset of $\mathbb{N}$. Mursaleen and Edely [23] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [21] and Mursaleen and Edely [24] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{mn})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\nu)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Basar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l > n$ see [27]. We shall write more briefly as P-convergent. The double sequence $x = (x_{kl})$ is bounded if there exists a positive number M such that $|x_{kl}| < M$ for all k and l.
The notion of difference sequence spaces was introduced by Kizmaz [12], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was

further generalized by Et. and Colak [5] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, s be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in Z\},$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \begin{pmatrix} m \\ \nu \end{pmatrix} x_{k+\nu}.$$

Taking $m = 1$, we get the spaces which were introduced and studied by Kizmaz [12].

An orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [14] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [14] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,

2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [37], Theorem 10.4.2, pp. 183). For more details about sequence spaces see ([17], [22], [25], [28], [29], [30], [32], [33]) and reference therein.
Let ℓ_∞, c and c_0 denotes the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [13], it was shown that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s \right\}.$$

In ([15], [16]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number L such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $g_r = (i_r - i_{r-1}) \to \infty$ $(r \to \infty)$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by n_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman [6] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L \right\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, g_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \overline{g}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Let $k_{r,s} = k_r l_s$, $g_{r,s} = g_r \overline{g}_s$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \ \& \ l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\overline{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \overline{q}_s$.
The concept of 2-normed spaces was initially developed by Gähler [7] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [20]. Since then, many others have studied this concept and obtained various results, see Gunawan ([8],[9]) and Gunawan and Mashadi [10]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and

4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{R}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
Let $(X, \|\cdot, \cdots, \cdot\|)$ be a real n-normed space and $w(n-X)$ denotes the space of X-valued sequences. Let $p = (p_{k,l})$ be any bounded sequence of positive real numbers, $d = (d_{k,l})$ be any sequence of strictly positive real numbers and $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions. In this paper we define the following sequence spaces:

$$[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|\,]^{\theta}(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{k,l}} = 0,$$
$$\text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X, \text{ for some } L \text{ and } \rho > 0\Big\},$$

$$[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|\,]_0^{\theta}(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{k,l}} = 0,$$
$$\text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X \text{ and } \rho > 0\Big\} \text{ and}$$

$$[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|\,]_\infty^{\theta}(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \sup_{r,s} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{k,l}} < \infty,$$
$$\text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X \text{ and } \rho > 0\Big\}.$$

When $\mathcal{M}(x) = x$, we get

$$[c^2, p, d, \|\cdot, \cdots, \cdot\|\,]^{\theta}(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_{k,l}} = 0,$$
$$\text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X \text{ for some } L \text{ and } \rho > 0\Big\},$$

$$[c^2, p, d, \|\cdot, \cdots, \cdot\|\,]_0^{\theta}(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_{k,l}} = 0,$$
$$\text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X \text{ and } \rho > 0\Big\}$$

and

$$[c^2, p, d, \|\cdot, \cdots, \cdot\|]^\theta_\infty(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \\ \sup_{r,s} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left(\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}} < \infty, \\ \text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X \text{ and } \rho > 0\ \Big\}.$$

If we take $p = (p_{k,l}) = 1$ and $d = (d_{k,l}) = 1$ for all k, l then we get

$$[c^2, \mathcal{M}, \|\cdot, \cdots, \cdot\|]^\theta(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \\ \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] = 0, \\ \text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X, \text{ for some } L \text{ and } \rho > 0\Big\},$$

$$[c^2, \mathcal{M}, \|\cdot, \cdots, \cdot\|]^\theta_0(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \\ \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] = 0, \\ \text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X \text{ and } \rho > 0\Big\} \text{ and}$$

$$[c^2, \mathcal{M}, \|\cdot, \cdots, \cdot\|]^\theta_\infty(\Delta^m) = \Big\{x = (x_{k,l}) \in w(n-X) : \\ \sup_{r,s} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] < \infty, \\ \text{uniformly in } u \text{ and } v,\ \ z_1, \cdots, z_{n-1} \in X \text{ and } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$ and let $K = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}). \tag{1}$$

The aim of this paper is to introduce some new type of lacunary double sequence spaces defined by a sequence of Orlicz function $\mathcal{M} = (M_{k,l})$ over n-normed spaces and to establish some topological properties and some inclusion relation between above defined sequence spaces.

2 Main results

Theorem 1 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $d = (d_{k,l})$ be a sequence of strictly positive real numbers. Then the sequence spaces $[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]^{\theta}(\Delta^m)$, $[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_0^{\theta}(\Delta^m)$ and $[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_{\infty}^{\theta}(\Delta^m)$ are linear spaces over the field of real numbers $\mathbb{R}$.*

Proof. Let $x = (x_{k,l}), y = (y_{k,l}) \in [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_0^{\theta}(\Delta^m)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0,$$

uniformly in u and v, and

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m y_{k+u,l+v}}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0,$$

uniformly in u and v.

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing and convex function so by using inequality (1), we have

$$\begin{aligned}
&\frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m(\alpha x_{k+u,l+v} + \beta y_{k+u,l+v})}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&= \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\alpha\Delta^m(x_{k+u,l+v})}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right.\right. \\
&\quad \left.\left. + \left\|\frac{d_{k,l}\beta\Delta^m(y_{k+u,l+v})}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&\le K\frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \frac{1}{2^{p_{k,l}}}\left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m(x_{k+u,l+v})}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&\quad + K\frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \frac{1}{2^{p_{k,l}}}\left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m(y_{k+u,l+v})}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}
\end{aligned}$$

$$\begin{aligned}
&\le K\frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m(x_{k+u,l+v})}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&\quad + K\frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m(y_{k+u,l+v})}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}
\end{aligned}$$

$\longrightarrow 0$ as $r \longrightarrow \infty, s \longrightarrow \infty$ uniformly in u and v.
Thus, we have $\alpha x + \beta y \in [\ c^2, \mathcal{M}, p, d,\ \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m)$. Hence $[c^2, \mathcal{M}, p, d,$ $\|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m)$ is a linear space. Similarly, we can prove that $[c^2, \mathcal{M}, p, d,$ $\|\cdot, \cdots, \cdot\|]^\theta(\Delta^m)$ and $[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m)$ are linear spaces. □

Theorem 2 *Suppose $\mathcal{M} = (M_{k,l})$ is a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $d = (d_{k,l})$ be a sequence of strictly positive real numbers, then $[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m)$ is a topological linear space paranormed by*

$$g(x) = \inf\left\{\rho^{\frac{p_{r,s}}{H}} : \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \right.$$
$$\left. \leq 1, r, s \in \mathbb{N}\right\},$$

where $H = \max(1, \sup_{k,l} p_{k,l}) < \infty$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_{k,l}) \in [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m)$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$\inf\left\{\rho^{\frac{p_{r,s}}{H}} : \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \right.$$
$$\left. \leq 1, r, s \in \mathbb{N}\right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1.$$

Thus,

$$\left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\epsilon}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}}$$
$$\leq \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1,$$

for each r, s, u and v. Suppose that $x_{k,l} \neq 0$ for each $k, l \in \mathbb{N}$. This implies that $d_{k,l}\Delta^m x_{k+u,l+v} \neq 0$, for each $k, l, u, v \in \mathbb{N}$. Let $\epsilon \to 0$, then $\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\epsilon}, z_1, \cdots, z_{n-1} \right\| \to \infty$. It follows that

$$\left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\epsilon}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \to \infty$$

which is a contradiction. Therefore, $d_{k,l}\Delta^m x_{k+u,l+v} = 0$ for each k, l, u and v and thus $x_{k,l} = 0$ for each $k, l \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \leq 1$$

for each r, s, u and v. Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m (x_{k+u,l+v} + y_{k+u,l+v})}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \\
&\leq \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v} + d_{k,l}\Delta^m y_{k+u,l+v}}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \\
&\leq \Bigg(\sum_{k,l \in I_{r,s}} \Bigg[\frac{\rho_1}{\rho_1 + \rho_2} M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \\
&\quad + \frac{\rho_2}{\rho_1 + \rho_2} M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m y_{k+u,l+v}}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \Bigg]^{p_{k,l}} \Bigg)^{\frac{1}{H}} \\
&\leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \\
&\quad + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\| \frac{d_{k,l}\Delta^m y_{k+u,l+v}}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \\
&\leq 1.
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$g(x+y) = \inf\Bigg\{\rho^{\frac{p_{r,s}}{H}} : \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m(x_{k+u,l+v}+y_{k+u,l+v})}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1,\ \ r,s,u,v \in \mathbb{N}\Bigg\},$$

$$\leq \inf\Bigg\{\rho_1^{\frac{p_{r,s}}{H}} : \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1,\ \ r,s,u,v \in \mathbb{N}\Bigg\}$$

$$+ \inf\Bigg\{\rho_2^{\frac{p_{r,s}}{H}} : \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m y_{k+u,l+v}}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right) g\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1,\ \ r,s,u,v \in \mathbb{N}\Bigg\}.$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Bigg\{\rho^{\frac{p_{r,s}}{H}} : \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m \lambda x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1, r,s,u,v \in \mathbb{N}\Bigg\}.$$

Then

$$g(\lambda x) = \inf\Bigg\{(|\lambda| t)^{\frac{p_{r,s}}{H}} : \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{t}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1, r,s,u,v \in \mathbb{N}\Bigg\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_{r,s}} \leq \max(1, |\lambda|^{\sup p_{r,s}})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_{r,s}})$$

$$\inf\Big\{t^{\frac{p_{r,s}}{H}}:\Big(\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{t},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{H}}$$
$$\leq 1, r,s,u,v\in\mathbb{N}\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Proposition 1 *Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions. If* $\sup_{k,l}[M_{k,l}(x)]^{p_{k,l}}<\infty$ *for all fixed* $x>0$*, then* $[\,c^2,\mathcal{M},p,d,\|\cdot,\cdots,\cdot\|\,]_0^\theta(\Delta^m)\subset[\,c^2,\mathcal{M},p,d,\|\cdot,\cdots,\cdot\|\,]_\infty^\theta(\Delta^m)$.

Proof. Let $x=(x_{k,l})\in[\,c^2,\mathcal{M},p,d,\|\cdot,\cdots,\cdot\|\,]_0^\theta(\Delta_m)$, then there exists some positive ρ_1 such that

$$\lim_{r,s\to\infty}\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho_1},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}=0,$$

uniformly in u and v. Define $\rho=2\rho_1$. Since $\mathcal{M}=(M_{k,l})$ is non-decreasing and convex, by using inequality (1), we have

$$\begin{aligned}
&\sup_{r,s}\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}\\
&\quad=\sup_{r,s}\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}-L+L}{\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}\\
&\quad\leq K\sup_{r,s}\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[\frac{1}{2}M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}-L}{\rho_1},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}\\
&\qquad+K\sup_{r,s}\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[\frac{1}{2}M_{k,l}\Big(\Big\|\frac{L}{\rho_1},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}\\
&\quad\leq K\sup_{r,s}\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}-L}{\rho_1},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}\\
&\qquad+K\sup_{r,s}\frac{1}{g_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(\Big\|\frac{L}{\rho_1},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{k,l}}<\infty.
\end{aligned}$$

Hence $x=(x_{k,l})\in[\,c^2,\mathcal{M},p,d,\|\cdot,\cdots,\cdot\|\,]_\infty^\theta$. □

Theorem 3 *Let $0 < \inf p_{k,l} = h \leq p_{k,l} \leq \sup p_{k,l} = H < \infty$ and $\mathcal{M} = (M_{k,l})$, $\mathcal{M}' = (M'_{k,l})$ be two sequences of Orlicz functions satisfying Δ_2-condition, then we have*
(i) $[c^2, \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_0(\Delta^m) \subset [c^2, \mathcal{M} \circ \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_0(\Delta^m)$,

(ii) $[c^2, \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^{\theta}(\Delta^m) \subset [c^2, \mathcal{M} \circ \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^{\theta}(\Delta^m)$ *and*

(iii) $[c^2, \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_\infty(\Delta^m) \subset [c, \mathcal{M} \circ \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_\infty(\Delta^m)$.

Proof. Let $x = (x_{k,l}) \in [c^2, \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_0(\Delta_m)$. Then we have

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M'_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{k,l}} = 0,$$

uniformly in u and v.
Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_{k,l}(t) < \epsilon$ for $0 \leq t \leq \delta$. Let

$$y_{k+u,l+v} = M'_{k,l}\Big(\Big\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big) \text{ for all } k, l \in \mathbb{N}.$$

We can write

$$\begin{aligned}\frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} [M_{k,l}(y_{k+u,l+v})]^{p_{k,l}} &= \frac{1}{g_{r,s}} \sum_{\substack{k,l\in I_{r,s},\\ y_{k+u,l+v}\leq\delta}} [M_{k,l}(y_{k+u,l+v})]^{p_{k,l}} \\ &+ \frac{1}{g_{r,s}} \sum_{\substack{k,l\in I_{r,s},\\ y_{k+u,l+v}\leq\delta}} [M_{k,l}(y_{k+u,l+v})]^{p_{k,l}}.\end{aligned}$$

Since $\mathcal{M} = (M_{k,l})$ satisfying Δ_2-condition, we have

$$\begin{aligned}&\frac{1}{g_{r,s}} \sum_{\substack{k,l\in I_{r,s},\\ y_{k+u,l+v}\leq\delta}} [M_{k,l}(y_{k+u,l+v})]^{p_{k,l}} \\ &\leq [M_{k,l}(1)]^H \frac{1}{g_{r,s}} \sum_{\substack{k,l\in I_{r,s},\\ y_{k+u,l+v}\leq\delta}} [M_{k,l}(y_{k+u,l+v})]^{p_{k,l}} \\ &\leq [M_{k,l}(2)]^H \frac{1}{g_{r,s}} \sum_{\substack{k,l\in I_{r,s},\\ y_{k+u,l+v}\leq\delta}} [M_{k,l}(y_{k+u,l+v})]^{p_{k,l}}.\end{aligned} \tag{2}$$

For $y_{k+u,l+v} > \delta$, we have

$$y_{k+u,l+v} < \frac{y_{k+u,l+v}}{\delta} < 1 + \frac{y_{k+u,l+v}}{\delta}.$$

Since $\mathcal{M} = (M_{k,l})$ is non-decreasing and convex, it follows that

$$M_{k,l}(y_{k+u,l+v}) < M_{k,l}\Big(1 + \frac{y_{k+u,l+v}}{\delta}\Big) < \frac{1}{2}M_{k,l}(2) + \frac{1}{2}M_{k,l}\Big(\frac{2y_{k+u,l+v}}{\delta}\Big).$$

Since $(M_{k,l})$ satisfies Δ_2-condition, we can write

$$\begin{aligned} M_{k,l}(y_{k+u,l+v}) &< \frac{1}{2}T\frac{y_{k+u,l+v}}{\delta}M_{k,l}(2) + \frac{1}{2}T\frac{y_{k+u,l+v}}{\delta}M_{k,l}(2) \\ &= T\frac{y_{k+u,l+v}}{\delta}M_{k,l}(2). \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{g_{r,s}}\sum_{\substack{k,l\in I_{r,s},\\ y_{k+u,l+v}\le\delta}}[M_{k,l}(y_{k+u,l+v})]^{p_{k,l}} \\ &\le \max\Big(1, \Big(\frac{TM_{k,l}(2)}{\delta}\Big)^H\Big)\frac{1}{g_{r,s}}\sum_{\substack{k,l\in I_{r,s},\\ y_{k+u,l+v}\le\delta}}[(y_{k+u,l+v})]^{p_{k,l}} \end{aligned} \tag{3}$$

from equations (2) and (3), we have

$$x = (x_{k,l}) \in [c^2, \mathcal{M}\circ\mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m).$$

This completes the proof of (i).
Similarly, we can prove that

$$[\, c^2, \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|]^\theta(\Delta^m) \subset [\, c^2, \mathcal{M}\circ\mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|\,]^\theta(\Delta^m)$$

and

$$[\, c^2, \mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|\,]_\infty^\theta(\Delta^m) \subset [\, c^2, \mathcal{M}\circ\mathcal{M}', p, d, \|\cdot, \cdots, \cdot\|\,]_\infty^\theta(\Delta^m).$$

□

Corollary 1 *Let* $0 < \inf p_{k,l} = h \leq p_{k,l} \leq \sup p_{k,l} = H < \infty$ *and* $\mathcal{M} = (M_{k,l})$ *be a sequence of Orlicz functions satisfying* Δ_2*-condition, then we have*

$$[c^2, p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m) \subset [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m)$$

and

$$[c^2, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m) \subset [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m).$$

Proof. Taking $\mathcal{M}'(x) = x$ in the above theorem, we get the required result.□

Theorem 4 *Let* $\mathcal{M} = (M_{k,l})$ *be a sequence of Orlicz functions. Then the following statements are equivalent:*
(i) $[c^2, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m) \subset [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m)$,

(ii) $[c^2, p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m) \subset [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m)$ *and*

(iii) $\sup_r \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} [M_{k,l}(\frac{t}{\rho})]^{p_{k,l}} < \infty \;\; (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) The proof is obvious in view of the fact that

$$[\, c^2, p, d, \|\cdot, \cdots, \cdot\|\,]_0^\theta(\Delta^m) \subset [c^2, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m).$$

(ii) $\Rightarrow$(iii) Let $[c^2, p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m) \subset [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]_\infty^\theta(\Delta^m)$. Suppose that (iii) does not hold. Then for some t, there exists $\rho > 0$ such that

$$\sup_{r,s} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} [M_{k,l}(\frac{t}{\rho})]^{p_{k,l}} = \infty$$

and therefore we can find a subinterval $I_{r,s(j)}$ of the set of interval $I_{r,s}$ such that

$$\frac{1}{g_{r,s(j)}} \sum_{k,l \in I_{r,s(j)}} \Big[M_{k,l}\Big(\frac{j^{-1}}{\rho}\Big)\Big]^{p_{k,l}} > j, \; j = 1, 2. \tag{4}$$

Define the sequence $x = (x_{k,l})$ by

$$d_{k,l}\Delta^m x_{k+u,l+v} = \begin{cases} j^{-1}, & k, l \in I_{r,s(j)} \\ 0, & k, l \notin I_{r,s(j)} \end{cases} \quad \text{for all } u \text{ and } v \in \mathbb{N}.$$

Then $x = (x_{k,l}) \in [c^2, p, d, \|\cdot, \cdots, \cdot\|]_0^\theta(\Delta^m)$ but by equation (4),

$x = (x_{k,l}) \notin [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|\,]^{\theta}_{\infty}(\Delta^m)$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and $x = (x_{k,l}) \in [c^2, p, d, \|\cdot, \cdots, \cdot\|\,]^{\theta}_{\infty}(\Delta^m)$. Suppose that
$x = (x_{k,l}) \notin [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|\,]^{\theta}_{\infty}(\Delta^m)$.
Then

$$\sup_{r,s} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} = \infty. \tag{5}$$

Let $t = \|d_{k,l}\Delta^m x_{k+u,l+v}, z_1, \cdots, z_{n-1}\|$ for each k, l and fixed u, v, then by equations (5)

$$\sup_{r,s} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{t}{\rho}\Big)\Big] = \infty,$$

which contradicts (iii). Hence (i) must hold. □

Theorem 5 *Let* $1 \le p_{k,l} \le \sup p_{k,l} < \infty$ *and* $\mathcal{M} = (M_{k,l})$ *be a sequence of Orlicz functions. Then the following statements are equivalent:*
(i) $[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_{0}(\Delta^m) \subset [c^2, p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_{0}(\Delta^m)$,
(ii) $[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_{0}(\Delta^m) \subset [c^2, p, d, \|\cdot, \cdots, \cdot\|]^{\theta}_{\infty}(\Delta^m)$ *and*
(iii) $\inf_{r,s} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{t}{\rho}\Big)\Big]^{p_{k,l}} > 0 \;\; (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) It is trivial.

(ii) $\Rightarrow$ (iii) Let (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_{r,s} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{t}{\rho}\Big)\Big]^{p_{k,l}} = 0 \;\; (t, \rho > 0),$$

so we can find a subinterval $I_{r,s(j)}$ of the set of interval $I_{r,s}$ such that

$$\frac{1}{g_{r,s(j)}} \sum_{k,l \in I_{r,s(j)}} \Big[M_{k,l}\Big(\frac{j}{\rho}\Big)\Big]^{p_{k,l}} < j^{-1}, \;\; j = 1, 2, \cdots \tag{6}$$

Define the sequence $x = (x_{k,l})$ by

$$d_{k,l}\Delta^m x_{k+u,l+v} = \begin{cases} j, & k, l \in I_{r,s(j)} \\ 0, & k, l \notin I_{r,s(j)} \end{cases} \quad \text{for all } u \text{ and } v \in \mathbb{N}.$$

Thus by equation (6), $x=(x_{k,l}) \in [\, c^2, \mathcal{M}, p, d, \|\cdot,\cdots,\cdot\|]_0^{\theta}(\Delta^m)$, but $x=(x_{k,l}) \notin [c^2, p, d, \|\cdot,\cdots,\cdot\|\,]_{\infty}^{\theta}(\Delta^m)$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and suppose that $x=(x_{k,l}) \in [\, c^2, \mathcal{M}, p, d, \|\cdot,\cdots,\cdot\|\,]_0^{\theta}(\Delta^m)$, i.e,

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0, \quad \text{uniformly in } u \text{ and } v, \ \text{ for some } \rho>0. \tag{7}$$

Again, suppose that $x=(x_{k,l}) \notin [\, c^2, p, d, \|\cdot,\cdots,\cdot\|\,]_0^{\theta}(\Delta^m)$. Then, for some number $\epsilon>0$ and a subinterval $I_{r,s(j)}$ of the set of interval $I_{r,s}$, we have $\|d_{k,l}\Delta^m x_{k+u,l+v}, z_1,\cdots,z_{n-1}\| \geq \epsilon$ for all $k\in\mathbb{N}$ and some $u\geq u_0, v\geq v_0$. Then, from the properties of the Orlicz function, we can write

$$M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}} \geq M_{k,l}\left(\frac{\epsilon}{\rho}\right)^{p_{k,l}}$$

and consequently by (7)

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\frac{\epsilon}{\rho}\right)\right]^{p_{k,l}} = 0,$$

which contradicts (iii). Hence (i) must hold. □

Proposition 2 *Let* $0<p_{k,l}\leq q_{k,l}$ *for all* $k,l\in\mathbb{N}$ *and* $\left(\frac{q_{k,l}}{p_{k,l}}\right)$ *be bounded. Then,*
$[c^2, \mathcal{M}, q, d, \|\cdot,\cdots,\cdot\|]^{\theta}(\Delta^m) \subset [c^2, \mathcal{M}, p, d, \|\cdot,\cdots,\cdot\|]^{\theta}(\Delta^m)$.

Proof. Let $x\in[c^2, \mathcal{M}, q, d, \|\cdot,\cdots,\cdot\|]^{\theta}(\Delta^m)$. Write

$$t_{k,l} = \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v}-L}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right]^{q_{k,l}}$$

and $\mu_{k,l}=\frac{p_{k,l}}{q_{k,l}}$ for all $k,l\in\mathbb{N}$. Then $0<\mu_{k,l}\leq 1$ for $k,l\in\mathbb{N}$. Take $0<\mu<\mu_{k,l}$ for $k,l\in\mathbb{N}$. Define the sequences $(a_{k,l})$ and $(b_{k,l})$ as follows: For $t_{k,l}\geq 1$, let $a_{k,l}=t_{k,l}$ and $b_{k,l}=0$ and for $t_{k,l}<1$, let $a_{k,l}=0$ and $b_{k,l}=t_{k,l}$. Then clearly for all $k,l\in\mathbb{N}$, we have

$$t_{k,l}=a_{k,l}+b_{k,l}, \qquad t_{k,l}^{\mu_{k,l}} = a_{k,l}^{\mu_{k,l}} + b_{k,l}^{\mu_{k,l}}.$$

Now it follows that $a_{k,l}^{\mu_{k,l}} \leq a_{k,l} \leq t_{k,l}$ and $b_{k,l}^{\mu_{k,l}} \leq b_{k,l}^{\mu}$. Therefore,

$$\begin{aligned}\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} t_{k,l}^{\mu_{k,l}} &= \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} (a_{k,l}^{\mu_{k,l}} + b_{k,l}^{\mu_{k,l}}) \\ &\leq \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} t_{k,l} + \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} b_{k,l}^{\mu}.\end{aligned}$$

Now for each k and l,

$$\begin{aligned}\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} b_{k,l}^{\mu} &= \sum_{k,l \in I_{r,s}} \Big(\frac{1}{g_{r,s}} b_{k,l}\Big)^{\mu} \Big(\frac{1}{g_{r,s}}\Big)^{1-\mu} \\ &\leq \left(\sum_{k,l \in I_{r,s}} \Big[\Big(\frac{1}{g_{r,s}} b_{k,l}\Big)^{\mu}\Big]^{\frac{1}{\mu}} \right)^{\mu} \left(\sum_{k,l \in I_{r,s}} \Big[\Big(\frac{1}{g_{r,s}}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}} \right)^{1-\mu} \\ &= \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} b_{k,l} \right)^{\mu}\end{aligned}$$

and so

$$\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} t_{k,l}^{\mu_{k,l}} \leq \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} t_{k,l} + \left(\frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} b_{k,l} \right)^{\mu}.$$

Hence $x = (x_{k,l}) \in [\, c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\| \,]^{\theta}(\Delta^m)$. □

Theorem 6 (a) *If* $0 < \inf p_{k,l} \leq p_{k,l} \leq 1$ *for all* $k, l \in \mathbb{N}$, *then*

$$[\, c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\| \,]^{\theta}(\Delta^m) \subset [\, c^2, \mathcal{M}, d, \|\cdot, \cdots, \cdot\| \,]^{\theta}(\Delta^m).$$

(b) *If* $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, *for all* $k, l \in \mathbb{N}$. *Then*

$$[c^2, \mathcal{M}, d, \|\cdot, \cdots, \cdot\|]^{\theta}(\Delta^m) \subset [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]^{\theta}(\Delta^m).$$

Proof. (a) Let $x = (x_{k,l}) \in [\, c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\| \,]^{\theta}(\Delta^m)$, then

$$\lim_{r,s \to \infty} \frac{1}{g_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\Big\| \frac{d_{k,l} \Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1} \Big\| \Big) \Big]^{p_{k,l}} = 0.$$

Since $0 < \inf p_{k,l} \leq p_{k,l} \leq 1$. This implies that

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m_{x_{k+u,l+v}} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]$$
$$\leq \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}.$$

Therefore, $\displaystyle\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] = 0.$

This shows that $x = (x_{k,l}) \in [c^2, \mathcal{M}, d, \|\cdot, \cdots, \cdot\|]^\theta(\Delta^m)\cdot$
Therefore,

$$[c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]^\theta(\Delta^m) \subset [c^2, \mathcal{M}, d, \|\cdot, \cdots, \cdot\|\,]^\theta(\Delta^m).$$

This completes the proof.

(b) Let $p_{k,l} \geq 1$ for each k, l and $\sup p_{k,l} < \infty$. Let $x = (x_{k,l}) \in [c^2, \mathcal{M}, d, \|\cdot, \cdots, \cdot\|]^\theta(\Delta^m)$. Then for each $\epsilon > 0$ there exists a positive integer N such that

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\right|\right)\right] = 0 < 1.$$

Since $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, we have

$$\lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq \lim_{r,s\to\infty} \frac{1}{g_{r,s}} \sum_{k,l\in I_{r,s}} \left[M_{k,l}\left(\left\|\frac{d_{k,l}\Delta^m x_{k+u,l+v} - L}{\rho}, z_1, \cdots, z_{n-1}\right|\right)\right]$$
$$= 0 < 1.$$

Therefore $x = (x_{k,l}) \in [c^2, \mathcal{M}, p, d, \|\cdot, \cdots, \cdot\|]^\theta(\Delta^m)$. □

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Convergence of three-step iterations for Ciric-quasi contractive operator in CAT(0) spaces

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Abstract. In this paper, we study three-step iteration process for Ciric-quasi contractive operator and establish strong convergence theorems for above mentioned operator and schemes in the setting of **CAT(0)** spaces. Our result extends and generalizes some previous work from the existing literature (see, e.g., [4, 30] and some others).

1 Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [8].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [22, 23]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed

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point. Since, then the fixed point theory for various mappings and iteration schemes in a CAT(0) space has been rapidly developed and a lot of papers appeared (see, [3, 11, 13, 14, 20, 21, 24, 25, 27, 31, 32]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k′) space for every $k' \geq k$ (see,e.g., [8]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and let $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [8]).

CAT(0) space

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \quad \leq \quad d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{1}$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [19]). If x, y_1, y_2 are points of a CAT(0) space and y_0 is the mid point of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \quad \leq \quad \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{2}$$

The inequality (2) is the (CN) inequality of Bruhat and Tits [9].

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [[8], p.163]). Moreover, if X is a CAT(0) metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z,x) + (1-\alpha)d(z,y), \tag{3}$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset C of a $\mathrm{CAT}(0)$ space X is convex if for any $x, y \in C$, we have $[x,y] \subset C$.

Algorithm 1. The sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$x_{n+1} = a_n T x_n + (1-a_n)x_n, \quad n \geq 1, \tag{4}$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence in (0,1) is called a Mann iterative sequence (see [26]).

Algorithm 2. The sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$\begin{aligned} y_n &= b_n T x_n + (1-b_n)x_n, \\ x_{n+1} &= a_n T y_n + (1-a_n)x_n, \quad n \geq 1, \end{aligned} \tag{5}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1] is called an Ishikawa iterative sequence (see [17]).

Algorithm 3. The sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$\begin{aligned} z_n &= c_n T x_n + (1-c_n)x_n, \\ y_n &= b_n T z_n + (1-b_n)x_n, \\ x_{n+1} &= a_n T y_n + (1-a_n)x_n, \quad n \geq 1, \end{aligned} \tag{6}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1) is called Noor iterative sequence (see [28]).

Algorithm 4. The sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$\begin{aligned} y_n &= b_n T x_n + (1-b_n)x_n, \\ x_{n+1} &= a_n T y_n + (1-a_n)T x_n, \quad n \geq 1, \end{aligned} \tag{7}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1) is called S-iterative sequence (see [2]).

Recently, Abbas and Nazir [1] introduced the following iterative process:

Algorithm 5. The sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= (1-a_n)Ty_n + a_nTz_n, \\ y_n &= (1-b_n)Tx_n + b_nTz_n, \\ z_n &= (1-c_n)x_n + c_nTx_n,\ n \geq 1 \end{aligned} \tag{8}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. They showed that this process converges faster than the Agarwal et al. [2].

Very recently, Thakur et al. [33] introduced the following iterative process:

Algorithm 6. The sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= (1-a_n)Tx_n + a_nTy_n, \\ y_n &= (1-b_n)z_n + b_nTz_n, \\ z_n &= (1-c_n)x_n + c_nTx_n,\ n \geq 1 \end{aligned} \tag{9}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. They showed that this process converges faster than all of the Picard, the Mann, the Ishikawa, the Noor, the Agarwal et al. and the Abbas et al. processes for contractions in the sense of Berinde [5] and in support gave analytic proof by a numerical example.

We now modify (9) in a CAT(0) space as follows.

Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T\colon C \to C$ be a mapping. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{aligned} x_{n+1} &= (1-a_n)Tx_n \oplus a_nTy_n, \\ y_n &= (1-b_n)z_n \oplus b_nTz_n, \\ z_n &= (1-c_n)x_n \oplus c_nTx_n,\ n \geq 1 \end{aligned} \tag{10}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$.

If we put $c_n = 0$ for all $n \geq 1$, then (10) reduces to the following iteration process

$$\begin{aligned} x_{n+1} &= (1-a_n)Tx_n \oplus a_nTy_n, \\ y_n &= (1-b_n)x_n \oplus b_nTx_n, \end{aligned} \tag{11}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$ is called modified S-iteration process.

We recall the following.

Let (X, d) be a metric space and $T\colon X \to X$ be a mapping. A mapping $T\colon X \to X$ is called an a-contraction if

$$d(Tx, Ty) \quad \leq \quad a\, d(x, y), \tag{12}$$

where $a \in (0, 1)$ and for all $x, y \in X$.

The mapping T is called Kannan mapping [18] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \quad \leq \quad b\, [d(x, Tx) + d(y, Ty)], \tag{13}$$

for all $x, y \in X$.

The mapping T is called Chatterjea mapping [12] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \quad \leq \quad c\, [d(x, Ty) + d(y, Tx)], \tag{14}$$

for all $x, y \in X$.

In 1972, combining these three definitions, Zamfirescu [34] proved the following important result.

Theorem Z. Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number a, b and c satisfying $a \in (0, 1)$, b, $c \in (0, \frac{1}{2})$ such that for any pair x, $y \in X$, at least one of the following conditions holds:

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$,

(z_2) $d(Tx, Ty) \leq b\, [d(x, Tx) + d(y, Ty)]$,

(z_3) $d(Tx, Ty) \leq c\, [d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ converges to p for any arbitrary but fixed $x_0 \in X$.

The conditions $(z_1) - (z_3)$ can be written in the following equivalent form

$$d(Tx, Ty) \quad \leq \quad h \max\left\{d(x, y),\ \frac{d(x, Tx) + d(y, Ty)}{2},\right.$$
$$\left.\frac{d(x, Ty) + d(y, Tx)}{2}\right\} \tag{15}$$

for all $x, y \in X$ and $0 < h < 1$, has been obtained by Ciric [10] in 1974.

A mapping satisfying (15) is called Ciric quasi-contraction. It is obvious that each of the conditions $(z_1) - (z_3)$ implies (15).

An operator T satisfying the contractive conditions $(z_1)-(z_3)$ in the theorem Z is called Z-operator.

In 2000, Berinde [4] introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|Tx - x\|, \quad (*)$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

He proved that this class is wider than the class of Zamfirescu operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem.

Theorem B. Let C be a nonempty closed convex subset of a normed space E. Let $T\colon C \to C$ be an operator satisfying $(*)$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by: for $x_1 = x \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by (5) where $\{a_n\}$ is a sequence in [0,1]. If $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

In this paper, inspired and motivated by [33, 34], we study an iteration process (10) and establish strong convergence theorems to approximate the fixed point for Ciric quasi contractive operator in the framework of CAT(0) spaces.

We need the following useful lemmas to prove our main result in this paper.

Lemma 1 (See [27]) *Let X be a CAT(0) space.*

(i) *For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = t\, d(x, y) \quad \textit{and} \quad d(y, z) = (1 - t)\, d(x, y). \quad (A)$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (A).

(ii) *For $x, y \in X$ and $t \in [0, 1]$, we have*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

Lemma 2 (See [6]) *Let $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying the following condition:*

$$p_{n+1} \leq (1 - s_n)p_n + q_n + r_n, \ \forall\, n \geq 0,$$

where $\{s_n\}_{n=0}^{\infty} \subset [0, 1]$. If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} q_n = O(s_n)$ and $\sum_{n=0}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n = 0$.

2 Strong convergence theorems in CAT(0) Space

In this section, we establish strong convergence result of iteration process (10) to approximate a fixed point for Ciric quasi contractive operator in the framework of CAT(0) spaces.

Theorem 1 *Let* C *be a nonempty closed convex subset of a complete CAT*(0) *space* X *and let* $T\colon C \to C$ *be an operator satisfying the condition* (15)*. Let* $\{x_n\}$ *be defined by the iteration scheme* (10)*. If* $\sum_{n=1}^{\infty} a_n b_n = \infty$*, then* $\{x_n\}$ *converges strongly to the unique fixed point of* T*.*

Proof. By Theorem Z, we know that T has a unique fixed point in C, say u. Consider x, $y \in C$. Since T is a operator satisfying (15), then if

$$\begin{aligned} d(Tx, Ty) &\le \frac{h}{2}[d(x, Tx) + d(y, Ty)] \\ &\le \frac{h}{2}[d(x, Tx) + d(y, x) + d(x, Tx) + d(Tx, Ty)], \end{aligned}$$

implies

$$\left(1 - \frac{h}{2}\right) d(Tx, Ty) \le \frac{h}{2} d(x, y) + h d(x, Tx),$$

which yields (using the fact that $0 < h < 1$)

$$d(Tx, Ty) \le \left(\frac{h/2}{1 - h/2}\right) d(x, y) + \left(\frac{h}{1 - h/2}\right) d(x, Tx). \tag{16}$$

If

$$\begin{aligned} d(Tx, Ty) &\le \frac{h}{2}[d(x, Ty) + d(y, Tx)] \\ &\le \frac{h}{2}[d(x, Tx) + d(Tx, Ty) + d(y, x) + d(x, Tx)], \end{aligned}$$

implies

$$\left(1 - \frac{h}{2}\right) d(Tx, Ty) \le \frac{h}{2}\, d(x, y) + h\, d(x, Tx),$$

which also yields (using the fact that $0 < h < 1$)

$$d(Tx, Ty) \le \left(\frac{h/2}{1 - h/2}\right) d(x, y) + \left(\frac{h}{1 - h/2}\right) d(x, Tx). \tag{17}$$

Denote

$$L_1 = \max\left\{h, \frac{h/2}{1 - h/2}\right\} = h,$$

$$L_2 = \max\left\{\frac{h}{1-h/2}, \frac{h}{1-h/2}\right\} = \frac{h}{1-h/2}.$$

Thus, in all cases,

$$\begin{aligned} d(Tx, Ty) &\leq L_1\, d(x, y) + L_2\, d(x, Tx) \\ &= h\, d(x, y) + \left(\frac{h}{1-h/2}\right) d(x, Tx) \end{aligned} \tag{18}$$

holds for all x, $y \in C$.

Also from (15) with $y = u = Tu$, we have

$$\begin{aligned} d(Tx, u) &\leq h \max\left\{d(x, u),\ \frac{d(x, Tx)}{2},\ \frac{d(x, u) + d(u, Tx)}{2}\right\} \\ &\leq h \max\left\{d(x, u),\ \frac{d(x, Tx)}{2},\ \frac{d(x, u) + d(u, Tx)}{2}\right\} \\ &\leq h \max\left\{d(x, u),\ \frac{d(x, u) + d(u, Tx)}{2},\ \frac{d(x, u) + d(u, Tx)}{2}\right\}. \end{aligned} \tag{19}$$

Since for non-negative real numbers a and b, we have

$$\frac{a+b}{2} \quad \leq \quad \max\{a,\ b\}. \tag{20}$$

Using (20) in (19), we have

$$d(Tx, u) \quad \leq \quad h\, d(x, u). \tag{21}$$

Now (21) gives

$$d(Tx_n, u) \quad \leq \quad h\, d(x_n, u) \tag{22}$$

$$d(Ty_n, u) \quad \leq \quad h\, d(y_n, u) \tag{23}$$

and

$$d(Tz_n, u) \quad \leq \quad h\, d(z_n, u). \tag{24}$$

Using (10), (21) and Lemma 1(ii), we have

$$\begin{aligned} d(z_n, u) &= d((1-c_n)x_n \oplus c_n Tx_n, u) \\ &\leq (1-c_n) d(x_n, u) + c_n d(Tx_n, u) \\ &\leq (1-c_n) d(x_n, u) + c_n h d(x_n, u) \\ &= [1-(1-h)c_n] d(x_n, u). \end{aligned} \tag{25}$$

Again using (10), (24), (25) and Lemma 1(ii), we have

$$
\begin{aligned}
d(y_n, u) &= d((1-b_n)z_n \oplus b_n T z_n, u) \\
&\leq (1-b_n)d(z_n, u) + b_n d(Tz_n, u) \\
&\leq (1-b_n)d(z_n, u) + b_n h\, d(z_n, u) \\
&= [(1-(1-h)b_n)]d(z_n, u) \\
&\leq [(1-(1-h)b_n)][(1-(1-h)c_n)]d(x_n, u) \\
&\leq [1-(1-h)b_n]d(x_n, u).
\end{aligned} \tag{26}
$$

Now using (10), (22), (23), (26) and Lemma 1(ii), we have

$$
\begin{aligned}
d(x_{n+1}, u) &= d((1-a_n)Tx_n \oplus a_n T y_n, u) \\
&\leq (1-a_n)d(Tx_n, u) + a_n d(Ty_n, u) \\
&\leq (1-a_n)h\, d(x_n, u) + a_n h\, d(y_n, u) \\
&\leq (1-a_n)h\, d(x_n, u) + a_n h[1-(1-h)b_n]d(x_n, u) \\
&= [(1-a_n)h + a_n h(1-(1-h)b_n)]d(x_n, u) \\
&= h[1-(1-h)a_n b_n]d(x_n, u) \\
&\leq [1-(1-h)a_n b_n]d(x_n, u) \\
&= (1-g_n)d(x_n, u)
\end{aligned} \tag{27}
$$

where $g_n = (1-h)a_n b_n$, since $0 < h < 1$ and by assumption of the theorem $\sum_{n=1}^{\infty} a_n b_n = \infty$, it follows that $\sum_{n=1}^{\infty} g_n = \infty$, therefore by Lemma 2, we get that $\lim_{n\to\infty} d(x_n, u) = 0$. Thus $\{x_n\}$ converges strongly to a fixed point of T.

To show uniqueness of the fixed point u, assume that $w_1, w_2 \in F(T)$ and $w_1 \neq w_2$.

Applying (15) and using the fact that $0 < h < 1$, we obtain

$$
\begin{aligned}
d(w_1, w_2) &= d(Tw_1, Tw_2) \\
&\leq h \max\Big\{d(w_1, w_2), \frac{d(w_1, Tw_1) + d(w_2, Tw_2)}{2}, \\
&\qquad \frac{d(w_1, Tw_2) + d(w_2, Tw_1)}{2}\Big\} \\
&= h \max\Big\{d(w_1, w_2), \frac{d(w_1, w_1) + d(w_2, w_2)}{2}, \\
&\qquad \frac{d(w_1, w_2) + d(w_2, w_1)}{2}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= h \max\Big\{d(w_1, w_2),\, 0,\, d(w_1, w_2)\Big\} \\
&\leq h\, d(w_1, w_2) \\
&< d(w_1, w_2), \text{ since } 0 < h < 1,
\end{aligned}
$$

which is a contradiction. Therefore $w_1 = w_2$. Thus $\{x_n\}$ converges strongly to the unique fixed point of T. This completes the proof. □

Theorem 2 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the condition (15). Let $\{x_n\}$ be defined by the iteration scheme (11). If $\sum_{n=1}^{\infty} a_n b_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Theorem 2 immediately follows from Theorem 1 by taking $c_n = 0$ for all $n \geq 1$. This completes the proof. □

The contraction condition (12) makes T continuous function on X while this is not the case with contractive conditions (13), (14) and (18).

The contractive conditions (13) and (14) both included in the class of Zamfirescu operators and so their convergence theorems for iteration process (10) are obtained in Theorem 1 in the setting of CAT(0) space.

Remark 1 *Our result extends the corresponding result of [30] to the case of three-step iteration process (10) and from uniformly convex Banach space to the setting of CAT(0) spaces.*

Remark 2 *Theorem 1 also extends Theorem B to the case of three-step iteration process (10) and from normed space to the setting of CAT(0) spaces.*

3 Application to contraction of integral type

Theorem 3 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the following condition:*

$$
\int_0^{d(Tx,Ty)} \mu(t)dt \leq h \int_0^{\max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}} \mu(t)dt \tag{28}
$$

for all x, $y \in X$ and $0 < h < 1$, where $\mu\colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt > 0$. Let $\{x_n\}$ be defined by the iteration process (10). If $\sum_{n=1}^{\infty} a_n b_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. The proof of Theorem 3 follows from Theorem 1 by taking $\mu(t) = 1$ over $[0, +\infty)$ since the contractive condition of integral type transforms into a general contractive condition (15) not involving integrals. This completes the proof. $\square$

Example 1 *Let* $X = \{0, 1, 2, 3, 4, 5\}$ *and* d *be the usual metric of reals. Let* $T\colon X \to X$ *be given by*

$$\begin{cases} T(x) = 5, & \text{if} \quad x = 0 \\ \quad\;\;\, = 3, & \text{otherwise.} \end{cases}$$

Again let $\mu\colon [0, +\infty) \to [0, +\infty)$ *be given by* $\mu(t) = 1$ *for all* $t \in [0, +\infty)$. *Then* $\mu\colon [0, +\infty) \to [0, +\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0, +\infty)$*, nonnegative, and such that for each* $\varepsilon > 0$*,* $\int_0^\varepsilon \mu(t)dt > 0$.

Let us take $x = 0$*,* $y = 1$*. Then from condition (28), we have*

$$\begin{aligned} 2 = \int_0^{d(Tx,Ty)} \mu(t)dt \quad &\leq \quad h \int_0^{\max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}} \mu(t)dt \\ &= \quad h \max\left\{1, \frac{7}{2}, \frac{7}{2}\right\} \end{aligned}$$

which implies $h \geq \frac{4}{7}$*. Now if we take* $0 < h < 1$*, then condition (28) is satisfied and* 3 *is of course a unique fixed point of* T.

The following corollaries are special cases of Theorem 3.

Corollary 1 *Let* C *be a nonempty closed convex subset of a complete CAT*(0) *space* X *and let* $T\colon C \to C$ *be an operator satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \mu(t)dt \quad \leq \quad h \int_0^{d(x,y)} \mu(t)dt \tag{29}$$

for all $x, y \in X$ *and* $h \in (0, 1)$*, where* $\mu\colon [0, +\infty) \to [0, +\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0, +\infty)$*, nonnegative, and such that for each* $\varepsilon > 0$*,* $\int_0^\varepsilon \mu(t)dt > 0$*. Let* $\{x_n\}$ *be defined by the iteration process* (10)*. If* $\sum_{n=1}^\infty a_n b_n = \infty$*, then* $\{x_n\}$ *converges strongly to the unique fixed point of* T.

Condition (29) is called Branciari [7] contractive condition of integral type.

Putting $\mu(t) = 1$ in the condition (29), we get Banach contraction condition.

Proof. The proof of corollary 1 immediately follows from Theorem 1 by taking $\mu(t) = 1$ over $[0, +\infty)$ and

$$\max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,Ty) + d(y,Tx)}{2}\right\} = d(x,y)$$

since the contractive condition of integral type transforms into a general contractive condition (12) not involving integrals. This completes the proof. □

Corollary 2 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \mu(t)dt \quad \leq \quad b \int_0^{[d(x,Tx)+d(y,Ty)]} \mu(t)dt \tag{30}$$

for all $x, y \in X$ and $b \in (0, \frac{1}{2})$, where $\mu\colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt > 0$. Let $\{x_n\}$ be defined by the iteration process (10). If $\sum_{n=1}^{\infty} a_n b_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.

Condition (30) is called Kannan contractive condition [18] of integral type.

Putting $\mu(t) = 1$ in the condition (30), we get Kannan contraction condition.

Proof. The proof of corollary 2 immediately follows from Theorem 1 by taking $\mu(t) = 1$ over $[0, +\infty)$ and

$$\max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,Ty) + d(y,Tx)}{2}\right\} = \frac{d(x,Tx) + d(y,Ty)}{2}$$

since the contractive condition of integral type transforms into a general contractive condition (13) not involving integrals. This completes the proof. □

Corollary 3 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \mu(t)dt \quad \leq \quad c \int_0^{[d(x,Ty)+d(y,Tx)]} \mu(t)dt \tag{31}$$

for all $x, y \in X$ *and* $c \in (0, \frac{1}{2})$*, where* $\mu\colon [0, +\infty) \to [0, +\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0, +\infty)$*, nonnegative, and such that for each* $\varepsilon > 0$*,* $\int_0^\varepsilon \mu(t)dt > 0$*. Let* $\{x_n\}$ *be defined by the iteration process* (10)*. If* $\sum_{n=1}^{\infty} a_n b_n = \infty$*, then* $\{x_n\}$ *converges strongly to the unique fixed point of* T*.*

Condition (31) is called Chatterjae contractive condition [12] of integral type.

Putting $\mu(t) = 1$ in the condition (31), we get Chatterjae contraction condition.

Proof. The proof of corollary 3 immediately follows from Theorem 1 by taking $\mu(t) = 1$ over $[0, +\infty)$ and

$$\max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\} = \frac{d(x,Ty)+d(y,Tx)}{2}$$

since the contractive condition of integral type transforms into a general contractive condition (14) not involving integrals. This completes the proof. □

Now, we give the examples in support of above corollaries.

Example 2 *Let* X *be the real line with the usual metric* d *and suppose* $C = [0, 1]$*. Define* $T\colon C \to C$ *by* $T(x) = \frac{x+1}{2}$ *for all* $x \in C$*. Obviously* T *is self-mapping with a unique fixed point* 1*. Again let* $\mu\colon [0, +\infty) \to [0, +\infty)$ *be given by* $\mu(t) = 1$ *for all* $t \in [0, +\infty)$*. Then* $\mu\colon [0, +\infty) \to [0, +\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0, +\infty)$*, nonnegative, and such that for each* $\varepsilon > 0$*,* $\int_0^\varepsilon \mu(t)dt > 0$*.*

If $x, y \in [0, 1]$*, then we have*

$$d(Tx, Ty) = \left|\frac{x-y}{2}\right|.$$

Let us take $x = 0$*,* $y = 1$*. Then from condition (29), we have*

$$\frac{1}{2} = \int_0^{d(Tx,Ty)} \mu(t)dt \quad \leq \quad h.1 = h\int_0^{d(x,y)} \mu(t)dt$$

which implies $h \geq \frac{1}{2}$*. Now if we take* $0 < h < 1$*, then condition* (29) *is satisfied and* 1 *is of course a unique fixed point of* T*.*

Example 3 *Let* X *be the real line with the usual metric* d *and suppose* $C = [0,1]$. *Define* $T\colon C \to C$ *by* $T(x) = \frac{x}{4}$ *for all* $x \in C$. *Obviously* T *is self-mapping with a unique fixed point* 0. *Again let* $\mu\colon [0,+\infty) \to [0,+\infty)$ *be given by* $\mu(t) = 1$ *for all* $t \in [0,+\infty)$. *Then* $\mu\colon [0,+\infty) \to [0,+\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,+\infty)$, *nonnegative, and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt > 0$.

If $x, y \in [0,1]$, *then we have*

$$d(Tx,Ty) = \Big|\frac{x-y}{4}\Big|.$$

Let us take $x = 0$, $y = 1$. *Then from condition* (30), *we have*

$$\frac{1}{4} = \int_0^{d(Tx,Ty)} \mu(t)dt \quad \le \quad b.\frac{3}{4} = b\int_0^{[d(x,Tx)+d(y,Ty)]} \mu(t)dt$$

which implies $b \ge \frac{1}{3}$. *Now if we take* $0 < b < \frac{1}{2}$, *then condition* (30) *is satisfied and* 0 *is of course a unique fixed point of* T.

Example 4 *Let* X *be the real line with the usual metric* d *and suppose* $C = [0,1]$. *Define* $T\colon C \to C$ *by* $T(x) = \frac{x}{4}$ *for all* $x \in C$. *Obviously* T *is self-mapping with a unique fixed point* 0. *Again let* $\mu\colon [0,+\infty) \to [0,+\infty)$ *be given by* $\mu(t) = 1$ *for all* $t \in [0,+\infty)$. *Then* $\mu\colon [0,+\infty) \to [0,+\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,+\infty)$, *nonnegative, and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt > 0$.

If $x, y \in [0,1]$, *then we have*

$$d(Tx,Ty) = \Big|\frac{x-y}{4}\Big|.$$

Let us take $x = 0$, $y = 1$. *Then from condition (31), we have*

$$\frac{1}{4} = \int_0^{d(Tx,Ty)} \mu(t)dt \quad \le \quad c\cdot\frac{5}{4} = c\int_0^{[d(x,Ty)+d(y,Tx)]} \mu(t)dt$$

which implies $c \ge \frac{1}{5}$. *Now if we take* $0 < c < \frac{1}{2}$, *then condition (31) is satisfied and* 0 *is of course a unique fixed point of* T.

4 Conclusion

The Ciric quasi contractive operator [10] is more general than Banach contraction, Kannan contraction, Chatterjea contraction and Zamfirescu operators. Thus the results obtained in this paper are improvement and generalization of several known results from the existing literature (see, e.g., [4, 30] and some others).

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All maximal idempotent submonoids of $\mathsf{Hyp}_{\mathsf{G}}(2)$

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Abstract. The purpose of this paper is to determine all maximal idempotent submonoids and some maximal compatible idempotent submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (2)$.

1 Introduction

In Universal Algebra, identities are used to classify algebras into collections, called varieties and hyperidentities are use to classify varieties into collections, called hypervarities. The concept of a hypersubstitution is a tool to study hyperidentities and hypervarities. The notion of a hypersubstitution originated by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [3]. In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [4]. The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid. There are several published papers on algebraic properties of this monoid and its submonoids.

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The concept of regular subsemigroup plays an important role in the theory of semigroup. The concept of an idempotent submonoid is an example of a regular subsemigroup. In 2013, W. Puninagool and S. Leeratanavalee studied the natural partial order on the set $E(\mathsf{Hyp}_G(2))$ of all idempotent elements of $\mathsf{Hyp}_G(2)$, see [6]. In 2012, the authors studied the natural partial order on $\mathsf{Hyp}_G(2)$, see [7]. In this paper we determine all maximal idempotent submonoids and give some maximal compatible idempotent submonoids of $\mathsf{Hyp}_G(2)$ under this partial order.

2 Generalized hypersubstitutions

Let $n \in \mathbb{N}$ be a natural number and $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-element set. Let $\{f_i \mid i \in I\}$ be a set of n_i-ary operation symbols indexed by the set I. We call the sequence $\tau = (n_i)_{i\in I}$ of arities of f_i, the *type*. An n-ary term of type τ is defined inductively by the following.

(i) Every $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, t_2, \ldots, t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an n-ary term of type τ.

We denote the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite number of applications of (ii) by $W_\tau(X_n)$ and let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ.

A mapping σ from $\{f_i \mid i \in I\}$ into $W_\tau(X)$ which does not necessarily preserve the arity is called a *generalized hypersubstitution of type* τ. The set of all generalized hypersubstitutions of type τ is denoted by $\mathsf{Hyp}_G(\tau)$. In general, to combine two mappings together we use a composition of mappings. But in this case to combine two generalized hypersubstitutions we need the concept of a generalized superposition of terms and the extension of a generalized hypersubstitution which are defined by the following.

Definition 1 A generalized superposition of terms is a mapping $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ where

(i) $S^m(x_j, t_1, \ldots, t_m) := t_j, 1 \leq j \leq m,$

(ii) $S^m(x_j, t_1, \ldots, t_m) := x_j, m < j \in \mathbb{N},$

(iii) $S^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m)).$

Definition 2 Let $\sigma \in Hyp_G(\tau)$. The extension of σ is a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ where

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Proposition 1 ([4]) *For arbitrary* $t, t_1, t_2, \ldots, t_n \in W_\tau(X)$ *and for arbitrary generalized hypersubstitution* $\sigma, \sigma_1, \sigma_2$ *we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

The binary operation of two generalized hypersubstitutions σ_1, σ_2 is defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. It turns out that $Hyp_G(\tau)$ together with the identity element σ_{id} where $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ is a monoid under $\circ_G$, see [4].

3 All Maximal idempotent submonoids of $Hyp_G(2)$

We recall first the definition of an idempotent element of a semigroup. Let S be a semigroup. An element $a \in S$ is called *idempotent* if $aa = a$. We denote the set of all idempotent elements of a semigroup S by $E(S)$. Let $E(S) \neq \emptyset$. Define $a \leq b (a, b \in E(S))$ iff $a = ab = ba$. Then $\leq$ is a partial order on $E(S)$. We call $\leq$ a natural partial order on $E(S)$. A natural partial order $\leq$ on a semigroup S is said to be a *compatible* if $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. Throughout this paper, let f be a binary operation symbol of type $\tau = (2)$. By σ_t we denote a generalized hypersubstitution which maps f to the term $t \in W_{(2)}(X)$. For $t \in W_{(2)}(X)$ we introduce the following notation:

(i) $leftmost(t)$:= the first variable (from the left) occurring in t,

(ii) $rightmost(t)$:= the last variable occurring in t,

(iii) $var(t)$:= the set of all variables occurring in t.

Let $\sigma_t \in Hyp_G(2)$, we denote $R_1 := \{\sigma_t \mid t = f(x_1, t') \text{ where } t' \in W_{(2)}(X) \text{ and } x_2 \notin var(t')\}, R_2 := \{\sigma_t \mid t = f(t', x_2) \text{ where } t' \in W_{(2)}(X) \text{ and } x_1 \notin var(t')\}, R_3 := \{\sigma_t \mid t \in \{x_1, x_2, f(x_1, x_2)\}\}$ and $R_4 := \{\sigma_t \mid var(t) \cap \{x_1, x_2\} = \emptyset\}$.

In 2008, W. Puninagool and S. Leeratanavalee [5] proved that: $\bigcup_{i=1}^{4} R_i = E(\mathsf{Hyp}_G(2))$.

Example 1 Let $\sigma_s \in R_1$ and $\sigma_t \in R_2$ such that $s = f(x_1, s')$ and $t = f(t', x_2)$ where $s' = f(x_4, x_1)$ and $t' = f(x_2, x_6)$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(f(x_2, x_6), x_2)]\\ &= S^2(f(x_1, f(x_4, x_1)), \widehat{\sigma}_s[f(x_2, x_6)], \widehat{\sigma}_s[x_2])\\ &= S^2(f(x_1, f(x_4, x_1), f(x_2, f(x_4, x_2)), x_2)\\ &= f(f(x_2, f(x_4, x_2)), f(x_4, f(x_2, f(x_4, x_2)))).\end{aligned}$$

So $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{4} R_i$.

By the previous example, we have $\bigcup_{i=1}^{4} R_i$ is not a subsemigroup of $\mathsf{Hyp}_G(2)$.

Let $\sigma_t \in \mathsf{Hyp}_G(2)$, we denote $R_1' := \{\sigma_t \mid t = f(x_1, t')$ where $t' \in W_{(2)}(X), x_2 \notin var(t')$ and $rightmost(t') \neq x_1\}$ and $R_2' := \{\sigma_t \mid t = f(t', x_2)$ where $t' \in W_{(2)}(X), x_1 \notin var(t')$ and $leftmost(t') \neq x_2\}$.

We denote $(MI)_{\mathsf{Hyp}_G(2)} = R_1' \cup R_2' \cup R_3 \cup R_4$, $(MI_1)_{\mathsf{Hyp}_G(2)} = R_1 \cup R_3 \cup R_4$ and $(MI_2)_{\mathsf{Hyp}_G(2)} = R_2 \cup R_3 \cup R_4$.

Proposition 2 $(MI)_{\mathsf{Hyp}_G(2)}$ *is an idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. It is clear that $(MI)_{\mathsf{Hyp}_G(2)} \subseteq \mathsf{Hyp}_G(2)$ and every element in $(MI)_{\mathsf{Hyp}_G(2)}$ is idempotent. Next, we show that $(MI)_{\mathsf{Hyp}_G(2)}$ is a submonoid of $\mathsf{Hyp}_G(2)$.
Case 1: $\sigma_t \in R_1'$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$. Let $\sigma_s \in (MI)_{\mathsf{Hyp}_G(2)}$.

Case 1.1: $\sigma_s \in R_1'$. Then $s = f(x_1, s')$ where $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, s')]\\ &= S^2(f(x_1, t'), x_1, \widehat{\sigma}_t[s'])\\ &= f(x_1, t') \quad \text{since } x_2 \notin var(t').\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_1' \subseteq (MI)_{\mathsf{Hyp}_G(2)}$.

Case 1.2: $\sigma_s \in R_2'$. Then $s = f(s', x_2)$ where $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_1, t')] = S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t'])$

$= f(S^2(s', x_1, w), S^2(x_2, x_1, w))$, where $w = \widehat{\sigma}_s[t']$. Since $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$, then $x_1, x_2 \notin var(w)$. Since $x_1 \notin var(s')$ and $x_1, x_2 \notin var(w)$, then $x_1, x_2 \notin var(S^2(s', x_1, w))$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s', x_2)] = S^2(f(x_1, t'), \widehat{\sigma}_t[s'], x_2) = f(S^2(x_1, u, x_2), S^2(t', u, x_2,))$, where $u = \widehat{\sigma}_t[s']$. Since $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$, we have $x_1, x_2 \notin var(u)$. Since $x_2 \notin var(t')$ and $x_1, x_2 \notin var(u)$, we have $x_1, x_2 \notin var(S^2(t', u, x_2))$. Then $\sigma_s \circ_G \sigma_t, \sigma_t \circ_G \sigma_s \in R_4' \subseteq (MI)_{Hyp_G(2)}$.

Case 1.3: $\sigma_s \in R_3$. Then $s = x_1$ or $s = x_2$ or $s = f(x_1, x_2)$.

If $s = x_1$, then $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_1] = x_1$ and $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_1}[f(x_2, t')] = S^2(x_1, x_2, \widehat{\sigma}_{x_1}[t']) = x_2$.

If $s = x_2$, then $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_2] = x_2$ and $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_2}[f(x_2, t')] = S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t'])$.

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, then $S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t']) = x_i \notin \{x_1, x_2\}$.

If $s = f(x_1, x_2)$, then $\sigma_s = \sigma_{id}$ such that $\sigma_t \circ_G \sigma_{id} = \sigma_t = \sigma_{id} \circ_G \sigma_t$.

Therefore $\sigma_s \circ_G \sigma_t, \sigma_s \circ_G \sigma_t \in (MI)_{Hyp_G(2)}$.

Case 1.4: $\sigma_s \in R_4$. Then $s = f(s_1, s_2)$ where $x_1, x_2 \notin var(s)$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s_1, s_2)] = S^2(f(x_2, t'), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) = f(S^2(x_2, w_1, w_2), S^2(t', w_1, w_2))$, where $w_1 = \widehat{\sigma}_t[s_1]$ and $w_2 = \widehat{\sigma}_t[s_2]$. Then $x_1, x_2 \notin var(w_1) \cup var(w_2)$. The consequence is $x_1, x_2 \notin var(S^2(t', w_1, w_2))$.
Since $x_1, x_2 \notin var(w_2) \cup var(S^2(t', w_1, w_2))$, so that $\sigma_t \circ_G \sigma_s \in R_4' \subseteq (MI)_{Hyp_G(2)}$.
Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_2, t')] = S^2(f(s_1, s_2), x_2, \widehat{\sigma}_s[t']) = f(s_1, s_2)$ since $x_1, x_2 \notin var(s)$. So that $\sigma_s \circ_G \sigma_t \in R_4 \subseteq (MI)_{Hyp_G(2)}$.
Case 2: $\sigma_t \in R_2'$ and $\sigma_s \in R_2' \cup R_3 \cup R_4$. It can be proved similarly as in Case 1. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (MI)_{Hyp_G(2)}$.
Case 3: $\sigma_t \in R_3$ and $\sigma_s \in R_3 \cup R_4$. It can be proved similarly as in Case 1.3. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (MI)_{Hyp_G(2)}$.
Case 4: $\sigma_t \in R_4$ and $\sigma_s \in R_4$. Then $\sigma_t \circ_G \sigma_s = \sigma_t \in R_4 \subseteq (MI)_{Hyp_G(2)}$.

Therefore $(MI)_{Hyp_G(2)}$ is a submonoid of $Hyp_G(2)$. □

Corollary 1 *$(MI_1)_{Hyp_G(2)}$ and $(MI_2)_{Hyp_G(2)}$ are idempotent submonoids of $Hyp_G(2)$.*

Proposition 3 *$(MI)_{Hyp_G(2)}$ is a maximal idempotent submonoid of $Hyp_G(2)$.*

Proof. Let K be a proper idempotent submonoid of $Hyp_G(2)$ such that $(MI)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$. Then σ_t is an idempotent element.
Case 1: $\sigma_t \in R_1 \backslash R_1'$. Then $t = f(x_1, t')$ where $x_2 \notin var(t')$ and $rightmost(t') =$

x_1. Choose $\sigma_s \in R_2' \subseteq K$, then $s = f(s', x_2)$ such that $x_1 \notin var(s')$ and $\mathsf{leftmost}(s') \neq x_2$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_1, t')] = S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t'])$ $= f(S^2(s', x_1, w), S^2(x_2, x_1, w))$ where $w = \widehat{\sigma}_s[t']$. Since $x_2 \in var(s)$ and $\mathsf{rightmost}(t') = x_1$, we have $x_1 \in var(w)$ and $S^2(s', x_1, w) \in W_{(2)}(X) \setminus X$. Since $x_1 \in var(w)$, $\sigma_s \circ_G \sigma_t$ is not idempotent. So $\sigma_t \in R_1'$.

Case 2: $\sigma_t \in R_2 \setminus R_2'$. Then $t = f(t', x_2)$ where $x_1 \notin var(t')$ and $\mathsf{leftmost}(t') = x_2$. Choose $\sigma_s \in R_1' \subseteq K$, then $s = f(x_1, s')$ such that $x_2 \notin var(s')$ and $\mathsf{rightmost}(s') \neq x_1$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t', x_2)] = S^2(f(x_1, s'), \widehat{\sigma}_s[t'], x_2) = f(S^2(x_1, w, x_2), S^2(s', w, x_2))$, where $w = \widehat{\sigma}_s[t']$. Since $x_1 \in var(s)$ and $\mathsf{leftmost}(t') = x_2$, we have $x_2 \in var(w)$ and $S^2(s', w, x_2) \in W_{(2)}(X) \setminus X$. Since $x_2 \in var(w)$, $\sigma_s \circ_G \sigma_t$ is not idempotent. So $\sigma_t \in R_2'$. Then $\sigma_t \in (MI)_{\mathsf{Hyp}_G(2)}$. Therefore $K \subseteq (MI)_{\mathsf{Hyp}_G(2)}$ and thus $K = (MI)_{\mathsf{Hyp}_G(2)}$. □

Proposition 4 $(MI_1)_{\mathsf{Hyp}_G(2)}$ *is a maximal idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. Let K be a proper idempotent submonoid of $\mathsf{Hyp}_G(2)$ such that $(MI_1)_{\mathsf{Hyp}_G(2)} \subseteq K \subset \mathsf{Hyp}_G(2)$. Let $\sigma_t \in K$. Then σ_t is an idempotent element. If $\sigma_t \in R_2$. Then $t = f(t', x_2)$ where $x_1 \notin var(t')$. Choose $\sigma_s \in R_1$ such that $s = f(x_1, s')$ where $x_2 \notin var(s')$, $s' \in W_{(2)}(X) \setminus X$ and $\mathsf{rightmost}(s') = x_1$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(x_1, s')] = S^2(f(t', x_2), x_1, \widehat{\sigma}_t[s']) = f(S^2(t', x_1, w), S^2(x_2, x_1, w))$, where $w = \widehat{\sigma}_t[s']$. Since $x_2 \in var(t)$, we have $x_1 \in var(w)$ and $S^2(t', x_1, w) \in W_{(2)}(X) \setminus X$. Since $x_1 \in var(w)$, $\sigma_t \circ_G \sigma_s$ is not idempotent, so $\sigma_t \in (MI_1)_{\mathsf{Hyp}_G(2)}$. Therefore $K = (MI_1)_{\mathsf{Hyp}_G(2)}$. □

Proposition 5 $(MI_2)_{\mathsf{Hyp}_G(2)}$ *is a maximal idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. The proof is similar to the proof of Proposition 4. □

Corollary 2 $\{(MI)_{\mathsf{Hyp}_G(2)}, (MI_1)_{\mathsf{Hyp}_G(2)}, (MI_2)_{\mathsf{Hyp}_G(2)}\}$ *is the set of all maximal idempotent submonoids of* $\mathsf{Hyp}_G(2)$.

Proposition 6 ([6]) *Let* σ_t *be an idempotent element. Then* $\sigma_{x_1} \leq \sigma_t$ *if and only if* $\mathsf{leftmost}(t) = x_1$.

Proposition 7 ([6]) *Let* σ_t *be an idempotent element. Then* $\sigma_{x_2} \leq \sigma_t$ *if and only if* $\mathsf{rightmost}(t) = x_2$.

Proposition 8 *For each* $t \in W_{(2)}(X)$ *where* $x_2 \notin var(t)$, $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ *is a maximal compatible idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. By using Proposition 6, $\sigma_{x_1} \leq \sigma_{f(x_1,t)}$. Then $\sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_{x_1}$ and σ_{id} is the identity element. We have $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is an idempotent submonoid of $Hyp_G(2)$. Since

$$\sigma_{f(x_1,t)} \circ_G \sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,t)} = \sigma_{x_1} \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,t)}.$$

We have $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is a compatible idempotent submonoid of $Hyp_G(2)$.

Let K be a proper compatible idempotent submonoid of $Hyp_G(2)$ such that $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\} \subseteq K \subset Hyp_G(2)$. Let $\sigma_s \in K$. Then σ_s is an idempotent element.

Case 1: $\sigma_s \in R_1 \setminus \{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$. Then $s = f(x_1, s')$ where $x_2 \notin var(s')$. Since K is a compatible idempotent submonoid and $\sigma_{f(x_1,t)} \leq \sigma_{id}$, we have $\sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,s')} = \sigma_{f(x_1,t)} \leq \sigma_{f(x_1,s')} = \sigma_{id} \circ_G \sigma_{f(x_1,s')}$ which is a contradiction.

Case 2: $\sigma_s \in R_2$. Then $s = f(s', x_2)$ where $x_1 \notin var(s')$. Since K is a compatible idempotent submonoid and $\sigma_{f(s',x_2)} \leq \sigma_{id}$, we have $\sigma_{x_1} \circ_G \sigma_{f(s',x_2)} = \sigma_{leftmost(s')} \leq \sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,s')}$. So $leftmost(s') = x_1$ which is a contradiction.

Case 3: $\sigma_s = \sigma_{x_2}$. Since K is a compatible idempotent submonoid and $\sigma_{x_1} \leq \sigma_{id}$, we have $\sigma_{x_2} \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_{x_2} = \sigma_{x_2} \circ_G \sigma_{id}$ which is a contradiction.

Case 4: $\sigma_s \in R_4$. Then $s = f(s_1, s_2) \in W_{(2)}X \setminus X$ where $x_1, x_2 \notin var(s)$. Since K is a compatible idempotent submonoid and $\sigma_{x_1} \leq \sigma_{id}$, we have $\sigma_s \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_s = \sigma_s \circ_G \sigma_{id}$ which is a contradiction.

Therefore $K = \{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is a maximal compatible idempotent submonoid of $Hyp_G(2)$. □

Proposition 9 *For each* $t \in W_{(2)}X$ *where* $x_1 \notin var(t)$, $\{\sigma_{x_2}, \sigma_{id}, \sigma_{f(t,x_2)}\}$ *is a maximal compatible idempotent submonoid of* $Hyp_G(2)$.

Proof. The proof is similar to the proof of Proposition 8. □

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